MASTERARBEIT / MASTER'S THESIS

Titel der Masterarbeit / Title of the Master's Thesis

„The loop $O(n)$ model and Monte Carlo methods in statistical mechanics"

verfasst von / submitted by

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angestrebter akademischer Grad / in partial fulfilment of the requirements for the degree of

Master of Science (MSc)

Wien, 2023 / Vienna, 2023

Studienkennzahl lt. Studienblatt / degree programme code as it appears on the student record sheet:	UA 066821
Studienrichtung lt. Studienblatt / degree programme as it appears on the student record sheet:	Masterstudium Mathematik UG2002
Betreut von / Supervisor:	Ass.-Prof. Alexander Glazman, PhD

Abstract

This thesis is devoted to the loop $O(n)$ model and Markov-chain-Monte-Carlo methods aimed at simulating this model.

The loop $O(n)$ model is a two-parameter family of probability distributions supported on configurations of disjoint cycles on the hexagonal lattice. The loop $O(n)$ model is subject of active research: early works proposed detailed predictions about its properties and behaviour, a significant portion of which are yet to be mathematically confirmed.

Markov-chain-Monte-Carlo (MCMC) methods comprise a class of algorithms aimed at sampling probability distributions. While MCMC algorithms were initially limited by a lack of available computational resources, the rapid technological advances of the last decades have enabled MCMC methods to become firmly established tools in a diverse set of fields. In particular, by allowing for large-scale simulation studies of lattice models like the loop $O(n)$ model, Monte-Carlo methods have aided the development of modern statistical mechanics.

In Chapter 1, we seek to motivate the study of the loop $O(n)$ model. Starting with a historical account on the development of lattice models, we introduce the Ising model by means of which we introduce the notion of phase transitions.

Chapter 2 is devoted to the study of the loop $O(n)$ model. After defining the model, we state the main conjectures and results. In the rest of the chapter we present a selection of tools and mathematical results for the loop $O(n)$ model. There, the primary object of study will be the spin representation of the loop $O(n)$ model. Among other things, we will show that this spin representation – in a specific range of parameters – satisfies the Fortuin-Kasteleyn-Ginibre (FKG) inequality. This will eventually lead to the construction of thermodynamic limits. We conclude the chapter by showing a certain uniqueness result regarding Gibbs measures of the loop $O(n)$ model.

The final chapter of the thesis, Chapter 3, is dedicated to sampling methods, in particular Markov-chain-Monte-Carlo algorithms. After a preliminary discussion about finite Markov chains, we introduce the Metropolis-Hastings algorithm and give a brief account on the efficiency of MCMC algorithms. The remainder of the chapter will be spent developing a selection of MCMC algorithms for simulating the loop $O(n)$ model. The algorithms under consideration are the classical Glauber dynamics, a generalized Swendsen-Wang algorithm and a modern algorithm termed Worm dynamics. We also present a method inspired by the Chayes-Machta algorithm which can be used to extend sampling methods from the Ising model to the loop $O(n)$ model. In particular, the aforementioned generalized Swendsen-Wang algorithm is based on this extension methods. Along the way we will cover topics which are of independent interest beyond the realm of sampling methods. These include the random-cluster model, the Edwards-Sokal coupling and the high-temperature expansion of the Ising model.

Abstrakt (Deutsch)

Diese Arbeit ist dem Loop $O(n)$ Modell und Markov-Chain-Monte-Carlo-Verfahren zur Simulation dieses Modells gewidmet.

Das Loop $O(n)$ Modell ist eine zweiparametrige Familie von Wahrscheinlichkeitsverteilungen, die auf Konfigurationen von disjunkten Zyklen auf dem hexagonalen Graphen beruhen. Das Loop $O(n)$ Modell ist Gegenstand aktiver Forschung: In frühen Arbeiten wurden detaillierte Vorhersagen über seine Eigenschaften und sein Verhalten gemacht, von denen ein großer Teil noch mathematisch bestätigt werden muss.

Markov-Chain-Monte-Carlo (MCMC)-Methoden umfassen eine Klasse von Algorithmen, die auf das sogenannte Sampling von Wahrscheinlichkeitsverteilungen abzielen. Während MCMC-Algorithmen anfangs durch einen Mangel an verfügbaren Rechenressourcen eingeschränkt waren, haben die rasanten technologischen Fortschritte der letzten Jahrzehnte dazu geführt, dass sich MCMC-Methoden in einer Vielzahl von Bereichen fest etabliert haben.

In Kapitel 1 versuchen wir, das Studium des Loop $O(n)$ Modells zu motivieren. Ausgehend von einem historischen Überblick über die Entwicklung von Gittermodellen stellen wir das Ising-Modell vor, mit dessen Hilfe wir den Begriff der Phasenübergänge einführen.

Das Kapitel 2 ist der Untersuchung des Loop $O(n)$ Modells gewidmet. Nachdem wir das Modell definiert haben, formulieren wir die wichtigsten Vermutungen und Ergebnisse. Im weiteren Verlauf des Kapitels erarbeiten wir eine Auswahl dieser Ergebnissen. Das Hauptaugenmerk liegt dabei auf der Spin-Repräsentation des Loop $O(n)$ Modells. Unter anderem werden wir zeigen, dass diese Spin-Repräsentation – in einem bestimmten Bereich von Parametern – die Fortuin-Kasteleyn-Ginibre (FKG) Ungleichung erfüllt. Wir schließen das Kapitel ab, indem wir ein bestimmtes Eindeutigkeitsresultat in Bezug auf Gibbs-Maße des Loop $O(n)$ Modells beweisen.

Das letzte Kapitel der Arbeit, Kapitel 3, ist Sampling-Methoden gewidmet, insbesondere den Markov-Chain-Monte-Carlo-Algorithmen. Nach einer einleitenden Diskussion über endliche Markov-Ketten stellen wir den Metropolis-Hastings-Algorithmus vor und geben einen kurzen Überblick über die Effizienz von MCMC-Algorithmen. Im weiteren Verlauf des Kapitels wird eine Auswahl von MCMC-Algorithmen für die Simulation des Loop $O(n)$ Modells entwickelt. Die betrachteten Algorithmen sind die klassische Glauber-Dynamik, ein verallgemeinerter Swendsen-Wang-Algorithmus und ein moderner Algorithmus, der als Wurm-Algorithmus bezeichnet wird. Außerdem stellen wir eine vom Chayes-Machta-Algorithmus inspirierte Methode vor, mit der Sampling-Methoden vom Ising-Modell auf das Loop $O(n)$ Modell übertragen werden können.

Acknowledgements

First and foremost, I would like to express my heartfelt gratitude to my supervisor, Alexander Glazman, who introduced me to this exiting field and, with remarkable patience, guided me through this rather lengthy project.

I also owe particular thanks to Nathanaël Berestycki and Marcin Lis, from whose diligent teaching I have greatly benefited and whose passion has inspired me. Their excellent courses are, in no small part responsible for my fondness for probability theory.

Last but not least, I would like to thank my family and friends, whose continued support and encouragement over all these years have allowed me to pursue this path. I am immensely grateful for your unwavering presence and friendship.

Contents

1 Introduction

1.1 Historical background

In an experiment performed in 1895, Pierre Curie showed that ferromagnetic materials, when heated above a specific temperature, termed the *Curie temperature* of that material, suddenly loose their inherent magnetization. This phenomenon of a so-called *phase transition* is not specific to magnetism. Other examples which can be observed in everyday life include the freezing or boiling of water, where the density of the fluid abruptly changes as its temperature surpasses a certain critical value. Stated differently, certain macroscopic properties of water, a large system of locally interacting H_2O molecules, do not depend continuously on the temperature of the fluid, but exhibit a drastic change exactly at some critical temperature.

Naturally, physicists were eager to introduce and study *models* reflecting this peculiar behaviour. That is, physicists searched for formal descriptions of magnetic materials which could be mathematically analysed and which they hoped would shed light on the mechanisms underlying these phase transitions observed in nature. Willhelm Lenz, a German physicist, proposed such a mathematical model in 1920 [Len20] and appointed his Ph.D. student Ernst Ising to study it. His idealized model sought to describe ferromagnets as a large collection of locally interacting spins placed on a lattice. Each *spin-configuration* – an assignment of ± 1 to each site of the lattice – is assigned a certain probability, depending on how well spins in the configuration align and on a temperature parameter of the model. A configuration in which spins are very organized corresponds to a state where the magnet induces its own magnetic field, while a disordered configuration corresponds to a state in which the magnet does not possess ferromagnetic properties. As part of his dissertation [Isi24] Ernst Ising showed that the one-dimensional version of

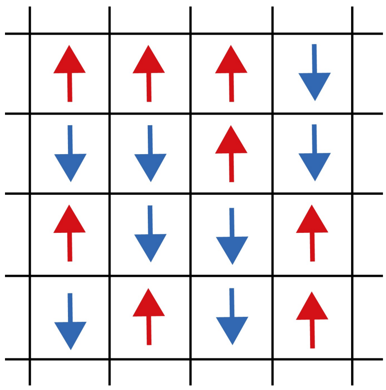

Figure 1: A configuration of Lenz's model on a two-dimensional square lattice. The red arrows correspond to $+1$ spins and the blue arrows to -1 spins.

Lenz's model does not exhibit a phase transition. Based on this disappointing result, Ising conjectured that the model would not undergo a phase transition in any dimension. This beliefe was widely shared among contemporary physicists. Contrary to these expectations, in 1936 the German-born British

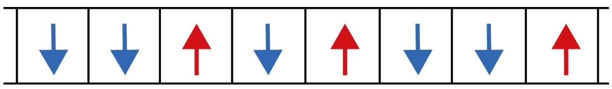

Figure 2: A spin-configuration of Lenz's model in one dimension. The red arrows correspond to $+1$ spins and the blue arrows to -1 spins. Ising showed that this version of the model does not exhibit a phase transition.

physicist Rudolf Peierls was able to show [Pei36] that the Ising model, in any but one dimension, does in fact undergo a phase transition. He proved this result using an elegant combinatorial argument, now well-recognized as *Peierls argument*. Ironically, it was Peierls in the aforementioned publication, who coined the term *Ising model*.

> 'In the meantime it was shown by Heisenberg that the forces leading to ferromagnetism are due to electron exchange. Therefore the energy function is of a more complicated nature than was assumed by Ising; it depends not only on the arrangement of the elementary magnets, but also on the speed with which they exchange their places.
>
> The Ising model is therefore now only of mathematical interest. Since, however, the problem of Ising's model in more than one dimension has led to a good deal of controversy and in particular since the opinion has often been expressed that the solution of the three-dimensional problem could be reduced to that of the linear model and would lead to similar results, it may be worth while to give its solution.' – Rudolf Peierls ([Pei36])

Only many years later, Ising, who had left academia shortly after completing his dissertation, learned that the model whose study he had pioneered, had become famous.

In response to Ising's 1925 publication, physicists considered generalized models, which they hoped would display richer behavior and in particular mirror the phase transitions that had been experimentally established by Pierre Curie. One later such generalization was the *spin $O(n)$* model, introduced by the American physicist Harry Eugene Stanley [Sta68]. In contrast to the Ising model, where spins could either point up or down, Stanley's model generally allowed spins to point in any of infinitely many possible directions[1]. In 1981 it was observed [DMNS81] that the spin $O(n)$ model on

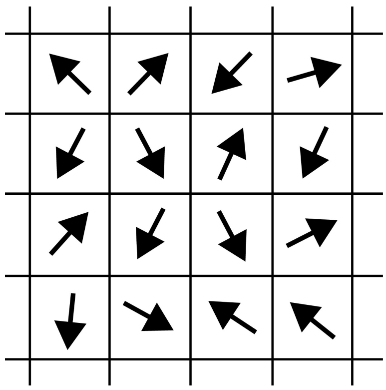

Figure 3: A configuration of Stanley's spin $O(n)$ model with $n = 2$, also called XY-model. The spins are free to point in any direction.

the hexagonal lattice induces a certain probability distribution on Eulerian subgraphs[2] of the hexagonal lattice, thereupon named the *loop $O(n)$ model*. Even though the spin $O(n)$ model is only defined for integers $n \geq 1$, the definition of the loop $O(n)$ model naturally extends to any real $n > 0$.

1.2 The Ising model and its phase transition

We will now expound on the behaviour of the Ising model and give a mathematical definition. Moreover, we will state a theorem summarizing the result about phase transitions in the Ising model alluded to in the above discussion.

The Ising model is defined as a probability distribution over spin-configurations on a finite graph G. Such a spin configuration $\sigma \in \{-1, +1\}^{V(G)}$ assigns one of two possible spins, -1 or $+1$, to each vertex in the graph. The *energy* $H(\sigma)$ associated to a spin-configuration σ is determined by number of nearest-neighbour vertex-pairs in G for which the two spins in σ disagree. The probability associated to a spin-configuration σ depends on its energy $H(\sigma)$ and on the inverse-temperature parameter β

[1] In the spin $O(n)$ model, spins take values in the $(n-1)$-sphere $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n \mid \|x\|_2 = 1\}$, where $n \in \mathbb{N}$ is an integer. For $n = 1$, this yields $\mathbb{S}^0 = \{-1, +1\}$ as the only possible spin values, and one recovers the Ising model. However for any $n \geq 2$, the $(n-1)$-sphere consists of a continuous range of possible spins.

[2] A graph is called Eulerian – after the Swiss mathematician Leonhard Euler – if every vertex in the graph has even degree, that is, if every vertex is incident to an even number of edges in the graph. As a result, Eulerian subgraphs of the hexagonal lattice consist of non-intersecting closed loops, isolated vertices and infinite simple paths.

of the model via the so-called *Boltzmann distribution*. To be specific, the probability assigned to a spin-configuration σ of the Ising model is proportional to $\exp\{-\beta H(\sigma)\}$. Additionally, one can restrict the Ising model by conditioning the spins on the boundary of the graph to coincide with some fixed spin-configuration, the so-called *boundary condition*.

Definition (Ising model with boundary conditions)**.** Let G be a finite graph. The *Hamiltonian*[3] of the Ising model on G is the map $H : \{-1,1\}^{V(G)} \to \mathbb{R}$ given by the formula

$$H(\sigma) \coloneqq -\sum_{uv\in E(G)} \sigma(u)\sigma(v).$$

Let $\beta \in (0,\infty)$ and $\xi \in \{-1,1\}^{V(G)}$. The Ising model on G with boundary conditions ξ at inverse temperature β is the probability distribution on

$$\mathsf{SpinConf}_G^{\xi} \coloneqq \{\sigma \in \{-1,1\}^{V(G)} \mid \sigma(v) = \xi(v),\ \forall v \in \partial_V G\},$$

given by the formula

$$\mu^{\xi}_{G,\beta}(\sigma) \coloneqq \frac{1}{Z_{G,\beta}}\, e^{-\beta H(\sigma)},$$

where the constant

$$Z^{\xi}_{G,\beta} \coloneqq \sum_{\sigma\in\mathsf{SpinConf}_G^{\xi}} e^{-\beta H(\sigma)}$$

is the *partition function* of the model.

Since each pair of disagreeing spins comes with an energetic cost, the probability distribution underlying the Ising model a priori favours spin-configurations which have few such disagreements. However, since the number of possible spin-configurations is exponential in the system size, with most of those configurations corresponding to rather disordered states, it is not guaranteed that the model will typically occupy an ordered state: energy has to compete with entropy.

The energetic cost associated to each spin disagreement is proportional to the inverse temperature parameter β. If β is close to zero, the energetic cost associated to spin disagreements becomes negligible. In that case, entropy wins over energy and configurations of the model will typically be disordered. If however β is chosen to be very large, the high energetic cost of spin disagreements will, with high probability, force the system to occupy an ordered state.

Denote the extremal boundary conditions given by the constant spin configurations $\xi \equiv +1$ and $\xi \equiv -1$ by $+$ and $-$ respectively. Formally, the phase transition of the Ising model is usually formulated as an abrupt change in the so-called *magnetization* of the model as the temperature parameter varies. This magnetization quantifies the impact an external magnetic field, modelled by $+$ boundary conditions, has on the spins at the center of the system.

Definition (Magnetization)**.** Consider $\beta \in (0,\infty)$ and integers $d, k \in \mathbb{N}$. Let Λ_k^d be the subgraph of $\mathbb{Z}^d$ induced by vertices with graph distance at most k from the origin. The magnetization of the Ising model at inverse temperature β on Λ_k^d is given by the expected value

$$M_k^d(\beta) \coloneqq \mu^{+}_{\Lambda_k^d,\beta}[\sigma_0],$$

where σ_0 is the random value of the spin at the origin $0 \in \mathbb{Z}^d$.

The following theorem summarizes the absence of a phase transition for the one-dimensional Ising model proven by Ising [Isi24], and the presence of a phase transition in dimensions two and higher, first shown by Peierls [Pei36].

[3]The Hamiltonian, named after the Irish mathematician and physicist Sir William Rowan Hamilton, quantifies the *energy* associated to a given configuration.

Theorem 1.1 (Phase transition of the Ising model)**.** *Let $d \in \mathbb{N}$ and $\beta \in (0, \infty)$. There exists a non-negative real number $M^d(\beta) \geq 0$, called spontaneous magnetization of the Ising model, such that*

$$M_k^d(\beta) \to M^d(\beta),$$

as $k \to \infty$. Moreover, for $d \geq 2$, there exists a positive real number $\beta_c(d) \in (0, \infty)$, such that for any $\beta \in (0, \beta_c(d))$, we have

$$M_d(\beta) = 0,$$

while for any $\beta \in (\beta_c(d), \infty)$, we have

$$M_d(\beta) > 0.$$

For $d = 1$, we have $M_d(\beta) = 0$ for any $\beta \in (0, \infty)$.

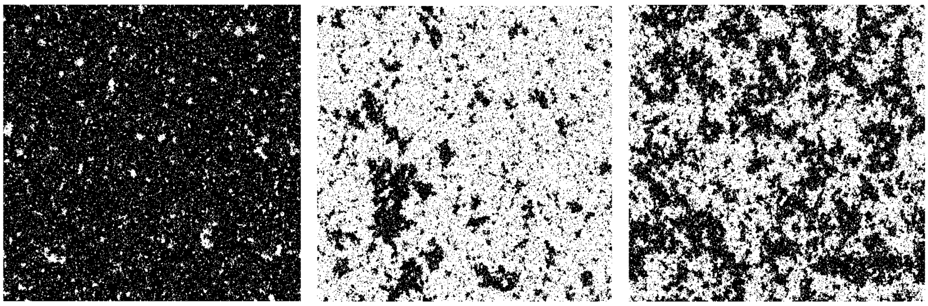

Figure 4: Approximate samples of the Ising model on a 512×512 square domain of the triangular lattice with periodic boundary conditions. The samples were generated using Glauber dynamics (see Section 3.2). Spins with value $+1$ are drawn in black, spins with value -1 are drawn in white. The figure shows samples of sub-critical (low temperature), critical, and super-critical (high-temperature) Ising model. The corresponding inverse temperatures are 0.37, $\beta_c \approx 0.274$ and 0.195, from left to right. The left configuration corresponds to an ordered state, where most spins are $+1$, only occasionally interrupted by -1 spin islands. The right configuration corresponds to a disordered state.

Theorem 1.1 admits the following physical interpretation. At low temperature (corresponding to large β), the high energetic cost of spin disagreements forces the system to adopt the $+1$ spins prescribed by the boundary conditions, thus braking the symmetry of the model. In this case, even on a large lattices, the spin at the origin will favor the value $+1$. At sufficiently high temperatures (corresponding to small β), the energetic cost associated to spin disagreements becomes negligible, and the spin at the origin *forgets* the $+1$ boundary conditions as the system size grows. In that case the plus-minus symmetry is preserved and the spin-value at the origin averages out to 0 in the infinite-volume limit. In the one-dimensional Ising model, boundary conditions are forgotten, in the sense just described, at any positive temperature.

2 The loop $O(n)$ model

Let us begin the discussion of the loop $O(n)$ model by introducing some terminology. The *triangular lattice* $\mathbb{T}$ is the undirected graph $\mathbb{T} = (V(\mathbb{T}), E(\mathbb{T}))$, where

$$V(\mathbb{T}) := \{r + s \cdot e^{i\frac{\pi}{3}} \mid r, s \in \mathbb{Z}\},$$

and $E(\mathbb{T})$ consists of all nearest-neighbour pairs in $V(\mathbb{T})$. The *hexagonal* (or *honeybomb*) *lattice* $\mathbb{H} = (V(\mathbb{H}), E(\mathbb{H}))$ is the dual graph of $\mathbb{T}$. Based on that duality we will sometimes refer to vertices of $\mathbb{T}$ as *hexagons* or *faces*.

A *loop* in $\mathbb{H}$ is a finite connected set $\lambda \subseteq E(\mathbb{H})$ such that every vertex in $(V(\mathbb{H}), \lambda)$ has even degree. A *domain* D of $\mathbb{H}$ is a subgraph of $\mathbb{H}$ which contains all edges and vertices along and inside some loop

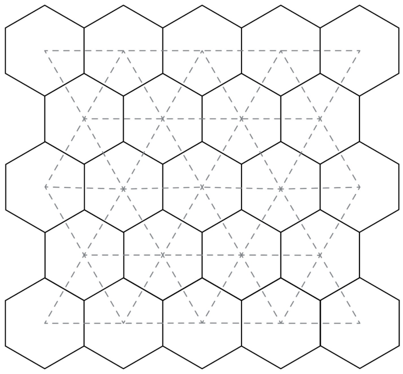

Figure 5: A subgraph of the hexagonal lattice and the corresponding dual subgraph of the triangular lattice. The former is drawn in black and the latter is drawn with dashed grey lines.

∂D. A *loop configuration* on $\mathbb{H}$ is a set $\omega \subseteq E(\mathbb{H})$ such that every vertex in $(V(\mathbb{H}), \omega)$ has even degree. We denote the set of such loop configurations by $\mathsf{LoopConf}_{\mathbb{H}}$. A *cylinder set* $C \subseteq \mathsf{LoopConf}_{\mathbb{H}}$ is an event depending only on finitely many edges in $\mathbb{H}$.

Definition (State-space of the loop $O(n)$ model)**.** Fix a loop configuration $\xi \in \mathsf{LoopConf}_{\mathbb{H}}$ and a domain D of $\mathbb{H}$. Define the set

$$\mathsf{LoopConf}_D^{\xi} := \left\{\omega \in \mathsf{LoopConf}_{\mathbb{H}} \mid \omega|_{E(\mathbb{H})\setminus E(D)} = \xi|_{E(\mathbb{H})\setminus E(D)}\right\},$$

and let $\mathcal{L}$ be the σ-algebra on $\mathsf{LoopConf}_{\mathbb{H}}$ given by

$$\mathcal{L} := \sigma(\{\pi_e^{\mathbb{H}} : e \in E(\mathbb{T})\}),$$

where $\pi_e^{\mathbb{H}} : \mathsf{LoopConf}_{\mathbb{H}} \to \{0,1\}, \omega \mapsto \omega(e)$ is the canonical projection on $e \in E(\mathbb{H})$. Similarly, for a domain D of $\mathbb{H}$, we define the σ-algebras

$$\mathcal{L}_D := \sigma(\{\pi_e^{\mathbb{H}} : e \in E(D)\}),$$
$$\mathcal{S}_D := \sigma(\{\pi_e^{\mathbb{H}} : e \notin E(D)\}).$$

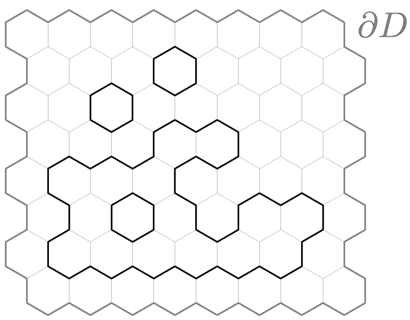

Figure 6: A domain D with a loop configuration $\omega \in \mathsf{LoopConfig}_D^{\emptyset}$ composed of four non-intersecting loops. The edges of ω are drawn with thick black lines. The edge boundary ∂_D is drawn in grey.

Definition (Loop O(n) model with boundary conditions)**.** Consider real parameters $n, x > 0$ and let D be a domain of $\mathbb{H}$. The loop O(n) model $\mathbb{P}_{D,n,x}^{\xi}$ on D with edge-weight x and boundary conditions ξ is the probability measure $\mathbb{P}_{D,n,x}^{\xi}$ on $(\mathsf{LoopConf}_{\mathbb{H}}, \mathcal{L})$ given by

$$\mathbb{P}_{D,n,x}^{\xi}(\omega) := \begin{cases} \frac{1}{Z_{D,n,x}^{\xi}} n^{l_D(\omega)} x^{|\omega \cap E(D)|} & \text{if } \omega \in \mathsf{LoopConf}_D^{\xi}, \\ 0 & \text{otherwise,} \end{cases}$$

where $l_D(\omega)$ denotes the number of distinct loops in ω which intersect D and

$$Z^{\xi}_{D,n,x} := \sum_{\omega \in \mathsf{LoopConf}^{\xi}_{D}} n^{l_D(\omega)} x^{|\omega \cap E(D)|}$$

is a normalizing constant, called *partition function* of the loop $O(n)$ model.

The boundary conditions given by the empty loop configuration $\emptyset \in \mathsf{LoopConf}_H$ play a special role. To lighten notation, we will denote the loop $O(n)$ model with empty (or *vacant*) boundary conditions by $\mathbb{P}_{D,n,x} := \mathbb{P}^{\emptyset}_{D,n,x}$.

To study phase transitions of the loop $O(n)$ model, we will need to consider infinite-volume variants of the model. We are particularly interested in a certain type of infinite-volume loop $O(n)$ models called *Gibbs measures*[4]. Gibbs measures satisfy a natural consistency property: when constricting a Gibbs measure of the loop $O(n)$ model to a finite domain D, we recover a finite volume loop $O(n)$ model. This formulation of Gibbs measure is due to the three mathematical physicists Roland Dobrushin [Dob73], Oscar Lanford and David Ruelle [IR69] (DLR).

Definition (DLR equation)**.** Consider parameters $n, x > 0$. A measure $\mathbb{P}_{n,x}$ on $(\mathsf{LoopConf}_{\mathbb{H}}, \mathcal{L})$ is called a *Gibbs measure* for the loop $O(n)$ model with edge-weight x, if for any domain D of $\mathbb{H}$, for $\mathbb{P}_{n,x}$-almost all loop configurations $\xi \in \mathsf{LoopConf}_{\mathbb{H}}$ and any event $A \in \mathcal{L}$ we have

$$\mathbb{P}_{n,x}[\mathbb{1}_A \mid \mathcal{S}_D](\xi) = \mathbb{P}^{\xi}_{D,n,x}(A).$$

For a loop configuration $\omega \in \mathsf{LoopConf}_{\mathbb{H}}$, denote by $\mathsf{R}(\omega)$ the diameter of the largest loop in ω surrounding the origin. For $k \in \mathbb{N}$, denote by Λ_k the domain of $\mathbb{H}$ induced by the ball of radius k (with respect to the graph distance) in $\mathbb{H}$ and denote by A_k the domain of $\mathbb{H}$ induced by vertices in $\Lambda_{2k} \setminus \Lambda_k$.

2.1 Results and conjectures

In the early 1980s Bernhard Nienhuis conjectured [Nie82] that, for $n \leq 2$, the loop $O(n)$ model exhibits a certain phase transition at edge-weight

$$x_c(n) = \frac{1}{\sqrt{2 + \sqrt{2 - n}}}.$$

Indeed, he predicted that the model is in a *sub-critical* regime when $x < x_c(n)$, where the probability that a given vertex is on a loop of length at least k decays exponentially fast in k, while in the *critical regime* $x \geq x_c(n)$ these probabilities decay according to a power law. Moreover, in the critical regime $n \leq 2$, $x \geq x_c(n)$, it is predicted that the model has a conformally invariant scaling limit described by $\mathsf{CLE}(\kappa)$ (*Conformal Loop Ensemble*) with parameter $\kappa = \kappa(n, x) \in [\frac{8}{3}, 8]$ given by

$$\kappa(n, x) = \begin{cases} \frac{4\pi}{2\pi - \arccos(-n/2)} & \text{if} x = x_c(n), \\ \frac{4\pi}{\arccos(-n/2)} & \text{if} x > x_c(n). \end{cases}$$

For $n > 2$, the model is expected to exhibit exponential decay of loop lengths for all $x > 0$. Mathematical progress on those conjectures is still rather limited. We now list some of the proven results.

Convergence to a conformally invariant scaling limit has only been established for the loop $O(1)$ model. For $n = x = 1$, the model is equivalent to an Ising model at infinite temperature, which in turn is equivalent to critical site percolation on the triangular lattice. In that case, Smirnov proved [Smi01]

[4]Named after the American physicist Josiah Willard Gibbs, one of the founders of *statistical mechanics* - a term he coined.

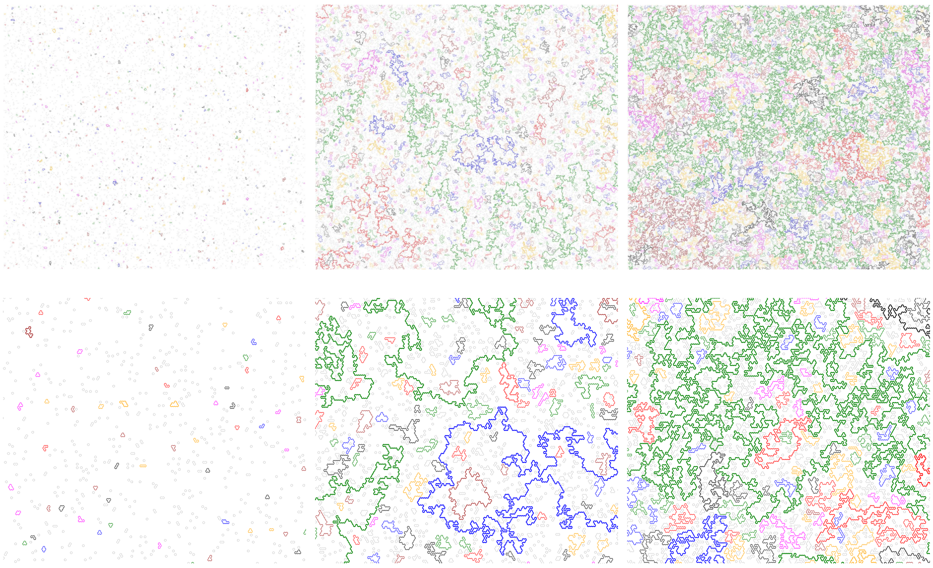

Figure 7: *Top row:* Approximate samples of the loop $O(1.5)$ model on a 512×512 square domain of the hexagonal lattice with periodic boundary conditions. *Bottom row:* The same configurations when magnified by a factor of 3.4.
The samples were generated using Glauber dynamics (see Section 3.2). The thickness of the lines is proportional to the respective loop length. Different loops were drawn in different colors, to help distinguishing them visually. The edge-weights corresponding to the three samples from left to right are $x_c(1.5) - 0.1,\ x_c(1.5)$ and $x_c(1.5) + 0.1$.

that certain crossing probabilites have a conformally invariant scaling limit and sketched a proof [Smi06] for showing convergence of the so-called *exploration path* to $\mathsf{SLE}(6)$ (Schramm-Löwner-Evolution). This was later shown by Camia and Newman [CN06a] together with the result that the full scaling limit is given by $\mathsf{CLE}(6)$ [CN06b; CN04]. For $n = 1$, $x = x_c(1) = \frac{1}{\sqrt{3}}$, the model is equivalent to a critical [Ons44] Ising model on the triangular lattice and it was shown [CS11; CDHKS13; HK11; Izy15; Smi07] that the exploration path converges to $\mathsf{SLE}(3)$ and that the full scaling limit is given by $\mathsf{CLE}(3)$ [BH18].

Duminil-Copin, Peled, Samotij and Spinka showed [DPSS17] that, for sufficiently large $n > 0$, loop lengths decay exponentially fast for a large class of domains.

Theorem 2.1 ([DPSS17, Theorem 1.4.])**.** *There exist $n_0, \alpha > 0$ such that for any $n \geq n_0$ and $x \in (0, \infty)$ the following holds. Let D be a domain satisfying*

$$\mathsf{LoopConf}_D^{\emptyset} = \{\omega \cap E(D) \mid \omega \in \mathsf{LoopConf}_D^{\tau}\},$$

for some fully-packed[5] *loop configuration $\tau \in \mathsf{LoopConf}_{\mathbb{H}}$ consisting only of trivial loops*[6]*. Then, for any $u \in V(D)$ and any integer $k > 6$, we have*

$$\mathbb{P}_{D,n,x}^{\emptyset}(\textit{there exists a loop of length } k \textit{ surrounding } u) \leq n^{-\alpha k}.$$

Glazman and Manolescu showed [GM18] that for $n \geq 1$ the loop $O(n)$ model exhibits exponential decay of loop sizes when the edge-weight x is sufficiently small.

Theorem 2.2 ([GM18, Theorem 1.1.])**.** *For any $n > 1$, there exists $\epsilon(n) > 0$ such that the loop $O(n)$ model exhibits exponential decay for all $x < \frac{1}{\sqrt{3}} + \epsilon(n)$.*

[5]We say that a loop configuration $\omega \subseteq E(\mathbb{H})$ is fully-packed if every vertex $u \in V(\mathbb{H})$ is incident to an edge $e \in \omega$.
[6]That is, loops of length 6.

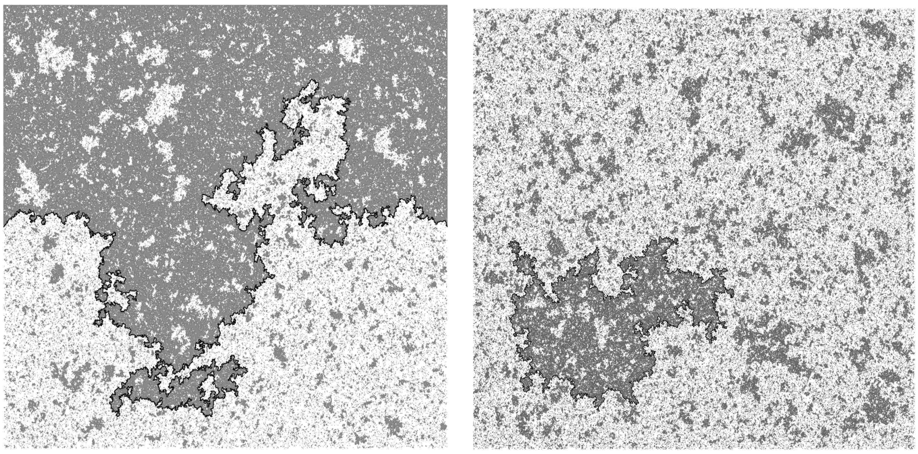

Figure 8: *Left:* Approximate sample of the Ising model at the critical temperature on a 512×512 domain of the triangular lattice with *Dobrushin boundary conditions*: the lower half of the domain boundary is conditioned to have -1 spin while the upper half is conditioned to have $+1$ spins. Hexagons with $+1$ spin are coloured grey and hexagons with -1 spin are coloured white. The interface between the two boundary clusters yields a unique path connecting the left and right sides of the domain. This so-called *exploration path* is drawn in black. It has been shown [CDHKS14] that the distribution of this random curve converges to $\mathsf{SLE}(3)$ in the infinite-volume limit. *Right:* infinite-volume of an approximate sample of the loop $O(1.5)$ model with critical edge-weight $x_c(1.5)$ on a 1024×1024 domain of the hexagonal lattice with vacant boundary conditions. Hexagons with $+1$ spin are coloured grey and hexagons with -1 spin are coloured white. The largest loop is drawn in black.

By studying the spin representation of the loop $O(n)$ model and establishing so-called *RSW* (*Russo-Seymour-Welsh*) estimates, Duminil-Copin, Glazman, Peled and Spinka [DGPS20] were able to show the following dichotomy theorem for $n \geq 1$ and $nx^2 \leq 1$. In this range of parameters the spin representation satisfies an essential positive association property, called FKG (*Fortuin–Kasteleyn–Ginibre*) inequality, on which the proof crucially relies. In particular, the FKG inequality allows to adapt previous arguments for the random-cluster model due to Duminil-Copin, Sidoravicius and Tassion [DST16] to the loop $O(n)$ setting.

Theorem 2.3 ([DGPS20, Theorem 2.]). *Consider parameters $n, x \in \mathbb{R}$ such that $n \geq 1$ and $nx^2 \leq 1$. Then there exists a unique periodic*[7] *Gibbs measure $\mathbb{P}_{n,x}$ for the loop $O(n)$ model with edge-weight x. Moreover, exactly one of the following occurs:*

1. *There exists $c > 0$ such that $\mathbb{P}_{n,x}(\mathsf{R} \geq k) \leq e^{-ck}$ for any $k \geq 1$.*
2. *There exists $c > 0$ such that for any $k > 1$ and any loop configuration ξ,*

$$c \leq \mathbb{P}_{A_k,n,x}(\textit{there is a loop in } A_k \textit{ surrounding } 0) \leq 1 - c.$$

Based on the above dichotomy theorem the authors furthermore showed that the model exhibits macroscopic loops at Nienhuis conjectured critical point

$$x_c(n) = \frac{1}{\sqrt{2 + \sqrt{2 - n}}}$$

by ruling out that loop lengths decay exponentially fast using so-called *parafermionic observables.*

[7] Here, a measure μ on $(\mathsf{LoopConf}_{\mathbb{H}}, \mathcal{L})$ is called periodic, if it is invariant under translations by two linearly independent vectors $a, b \in \mathbb{C}$. That is, $\mu(\tau_a E) = \mu(\tau_b E) = \mu(E)$ for any event $E \in \mathcal{L}$, where τ_a and τ_b denote the corresponding translations. In particular, any translation invariant measure is periodic.

Theorem 2.4 ([DGPS20, Theorem 1.]). *For $n \in [1,2]$ and $x = x_c(n)$, there exists $c > 0$ such that for any $k > 1$ and any loop configuration $\xi \in \mathsf{LoopConf}_{\mathbb{H}}$,*

$$c \leq \mathbb{P}_{A_k,n,x}(\textit{there is a loop in } A_k \textit{ surrounding } 0) \leq 1 - c.$$

Glazman and Manolescu showed [GM23], that the loop $O(n)$ model has a unique Gibbs measure in the FKG regime without assuming periodicity as in Theorem 2.3.

Theorem 2.5 ([GM23, Theorem 1.9.]). *Consider parameters $n, x \in \mathbb{R}$ such that $n \geq 1$ and $nx^2 \leq 1$. Then there exists a unique Gibbs measure $\mathbb{P}_{n,x}$ for the loop $O(n)$ model with edge-weight x. This measure is translation invariant and tail trivial. Moreover, $\mathbb{P}_{n,x}$ can be obtained as the weak limit of $(\mathbb{P}^{\xi_k}_{D_k,n,x})_{k\geq 1}$, for any sequence of domains $(D_k)_{k\geq 1}$ exhausting $\mathbb{H}$ and any sequence of loop configurations $(\xi_k)_{k\geq 1}$.*

Crawford, Glazman, Harel and Peled were the first to show [CGHP20] that the loop $O(n)$ model exhibits macroscopic loops in a regime of parameters with positive Lebesgue measure.

Theorem 2.6 ([CGHP20, Corollary 2.]). *There exist constants $\delta, c > 0$ for which the following holds. Let $1 \leq n \leq 1 + \delta$ and $1 - \delta \leq x \leq \frac{1}{\sqrt{n}}$. For any $k > 1$ and any loop configuration $\xi \in \mathsf{LoopConf}_{\mathbb{H}}$,*

$$c \leq \mathbb{P}_{A_k,n,x}(\textit{there is a loop in } A_k \textit{ surrounding } 0) \leq 1 - c.$$

Most recently Glazman and Lammers showed [GL23] that the loop $O(2)$ model exhibits macroscopic loops for $x \in [\frac{1}{\sqrt{2}}, 1]$. Their proof uses a double-spin representation of the loop $O(2)$ model which is shown to satisfy the FKG inequality.

Theorem 2.7 ([GM23, Theorem 1.]). *Let $n = 2$ and $x \in [\frac{1}{\sqrt{2}}, 1]$. Then the loop $O(n)$ model with edge weight x has a unique Gibbs measure $\mathbb{P}_{2,x}$. This measure is extremal and shift-invariant and ergodic with respect to the symmetries of $\mathbb{H}$, exhibits no bi-infinite paths almost surely, and satisfies*

$$\mathbb{P}_{2,x}(\textit{each face } f \in V(\mathbb{T}) \textit{ is surrounded by infinitely many loops}) = 1.$$

Moreover, $\mathbb{P}^{\emptyset}_{D_k,n,x}$ converges weakly to $\mathbb{P}_{2,x}$ for any increasing sequence of domains $(D_k)_{k\geq 1}$ exhausting $\mathbb{H}$.

Many conjectures regarding the loop $O(n)$ model still have to be mathematically established. For example, the predicted conformal invariance of scaling limits has so far only been shown for the critical Ising case $n = 1, x = \frac{1}{\sqrt{3}}$ and the critical percolation case $n = x = 1$. Moreover the existence of macroscopic loops for $x > x_c(n)$ remains largely open.

2.2 The loop $O(n)$ model in the FKG regime

In this section we will discuss some of the theory for the loop $O(n)$ model. In the process we will develop essential tools and arguments, which apply to many other lattice models and on which many of the sampling algorithms for the loop $O(n)$ rely. We will restrict our focus to the *FKG regime*, a range of parameters for which the spin representation of the loop $O(n)$ model satisfies certain positive association properties which greatly aid the analysis of the loop $O(n)$ model. Our goal is to prove the following theorem.

Theorem 2.8 (Loop $O(n)$ Gibbs measure). *Consider parameters $n, x \in \mathbb{R}$ such that $n \geq 1$ and $nx^2 \leq 1$. Then there exists a unique translation invariant Gibbs measure $\mathbb{P}_{n,x}$ for the loop $O(n)$ model with edge-weight x. The measure $\mathbb{P}_{n,x}$ is supported on loop configurations with no infinite paths and is invariant to all automorphisms of $\mathbb{H}$. Moreover, $\mathbb{P}_{n,x}$ can be obtained as a thermodynamic limit under empty boundary conditions. That is, for any sequence of domains $(D_k)_{k\geq 1}$ exhausting $\mathbb{H}$, we have*

$$\lim_{k\to\infty} \mathbb{P}^{\emptyset}_{D_k,n,x} = \mathbb{P}_{n,x},$$

in the sense of weak convergence.

Most of the results in the following exposition can be found in [DGPS20]. The presented proofs are either adapted from corresponding statements for the *random-cluster model* in [Gri03; DST16], or due to the author.

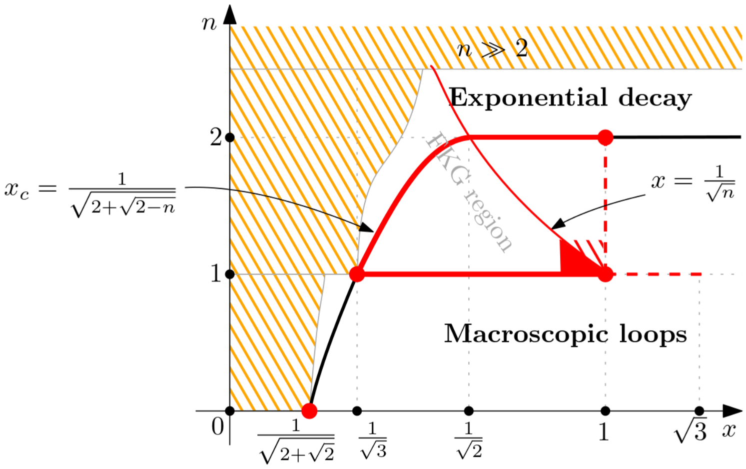

Figure 9: The predicted phase diagram for the loop O(n) model. The critical line x_c separating the regime of exponential decay from the regime of macroscopic loops is plotted in bold. The region $n \geq 1, x \leq \frac{1}{\sqrt{n}}$ where a dichotomy between the two behaviors is proved is denoted FKG region. Orange lines show regions where exponential decay is proved. Points or filled areas and lines in red mark regions where macroscopic loops are proven to occur. Exponential decay is ruled out in the red dashed area and lines. Picture adapted from Glazman–Manolescu [GM18].

2.2.1 Spin representation of the loop $O(n)$ model

An essential tool in the study of the loop $O(n)$ model is the so-called *spin representation*, a mapping between loop configurations on the hexagonal lattice and spin configurations on the triangular lattice. This correspondence allows us to apply techniques which have been developed for classical spin model like the Ising model to the loop $O(n)$ model.

In the following, a *spin configuration* on $\mathbb{T}$ is an element $\sigma \in \{-1,1\}^{V(\mathbb{T})}$. Furthermore, a cylinder set $E \subseteq \{-1,1\}^{V(\mathbb{T})}$ is an event which depends only on finitely many vertices in $\mathbb{T}$ and we denote by $\mathcal{F}$ the σ-algebra on $\{-1,1\}^{V(\mathbb{T})}$ generated by the family of cylinder sets in $\{-1,1\}^{V(\mathbb{T})}$.

Definition (Spin-Loop correspondence)**.** Let $\Phi : \{-1,1\}^{V(\mathbb{T})} \to \mathsf{LoopConf}_{\mathbb{H}}$ be the map defined by

$$\Phi(\sigma) := \{uv \in E(\mathbb{H}) \mid \sigma(u) \neq \sigma(v)\}.$$

That is, $\Phi(\sigma)$ is the set of all edges of the hexagonal lattice which separate faces of opposite spin in σ.

The reader can convince herself that this procedure indeed gives rise to a valid loop configuration and that the map Φ is 2 to 1.

Definition (State-space of the spin representation measure)**.** Let G be a *domain* of $\mathbb{T}$, that is, a finite connected subgraph of $\mathbb{T}$. For a spin configuration $\tau \in \{-1,1\}^{V(\mathbb{T})}$, we define

$$\mathsf{SpinConf}_G^\tau := \left\{\sigma \in \{-1,1\}^{V(\mathbb{T})} \mid \sigma|_{V(\mathbb{T})\setminus V(G)} = \tau|_{V(\mathbb{T})\setminus V(G)}\right\}.$$

Moreover, let $\mathcal{F}$ be the σ-algebra on $\{-1,1\}^V(\mathbb{T})$ given by

$$\mathcal{F} := \sigma(\{\pi_v^{\mathbb{T}} : v \in V(\mathbb{T})\},$$

where $\pi_v^{\mathbb{T}} : \{-1,1\}^V(\mathbb{T}) \to \{-1,1\}, \sigma \mapsto \sigma(v)$ is the canonical projection on $v \in V(\mathbb{T})$. Similarly for a domain G of $\mathbb{T}$, we define the σ-algebras

$$\mathcal{F}_G := \sigma(\{\pi_v^{\mathbb{T}} : v \in V(G)\},$$

$$\mathcal{T}_G := \sigma(\{\pi_v^{\mathbb{T}} : v \notin V(G)\}.$$

Definition (spin representation measure)**.** Fix a spin configuration $\tau \in \{-1,1\}^{V(\mathbb{T})}$, a finite subgraph G of $\mathbb{T}$ and real parameters $n, x > 0$. The spin representation measure $\nu^{\tau}_{G,n,x}$ with boundary conditions τ is the probability measure on $(\{-1,1\}^{V(\mathbb{T})}, \mathcal{F})$ given by

$$\nu^{\tau}_{G,n,x}(\sigma) := \begin{cases} \frac{1}{Z^{\tau}_{G,n,x}} n^{k_G(\sigma)} x^{e_G(\sigma)} & \text{if } \sigma \in \mathsf{SpinConf}^{\tau}_G, \\ 0 & \text{otherwise,} \end{cases}$$

where $k_G(\sigma) + 1$ is the number of spin clusters in σ that intersect G and

$$e_G(\sigma) := |\{uv \in \Phi(\sigma) \mid \{u, v\} \cap V(G) \neq \emptyset\}|,$$

is the number of edges in $\Phi(\sigma)$ which intersect G. The *partition function*

$$Z^{\tau}_{G,n,x} := \sum_{\sigma \in \mathsf{SpinConf}^{\tau}_G} n^{k(\sigma)} x^{e(\sigma)}$$

is the normalizing constant making $\nu^{\tau}_{G,n,x}$ a probability measure.

The spin representation measure with boundary conditions given by the two constant spin configurations, will be denoted by $\nu^{+}_{G,n,x}$ and $\nu^{-}_{G,n,x}$ respectively and the corresponding state-spaces will be denoted by $\mathsf{SpinConf}^{+}_G$, $\mathsf{SpinConf}^{-}_G$. By construction, the loop $O(n)$ model can be recovered as the

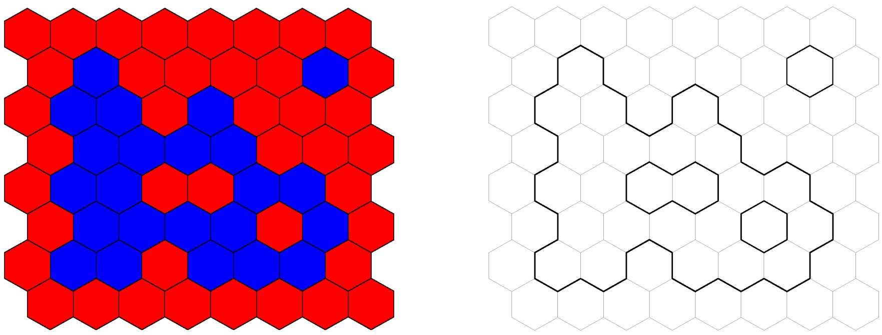

Figure 10: *Left:* A spin configuration $\sigma \in \mathsf{SpinConf}^{+}_G$. The blue faces correspond to -1 spins and the red faces to $+1$ spins. *Right:* The loop configuration $\Phi(\sigma)$.

pushforward of its corresponding spin representation measure under the map Φ.

Proposition 2.9. *Let $G \subseteq \mathbb{T}$ be a domain of $\mathbb{T}$ and let $D = D[G]$ be the domain of $\mathbb{H}$ which contains all edges and vertices which are part of a hexagon in G. Now let $\tau \in \{-1,1\}^{V(\mathbb{T})}$ and set $\xi := \Phi(\tau)$. Then*

$$\sigma \sim \nu^{\tau}_{G,n,x} \Longrightarrow \Phi(\sigma) \sim \mathbb{P}^{\xi}_{D,n,x}.$$

Proof. Let $\sigma \in \mathsf{SpinConf}^{\tau}_G$ and set $\omega = \Phi(\sigma)$. By definition, we have

$$e_G(\sigma) = |\omega \cap E(D)|, \qquad k_G(\sigma) = l_D(\omega).$$

Now, since the restricted map $\Phi : \mathsf{SpinConf}^{\tau}_G \to \mathsf{LoopConf}^{\xi}_D$ is a bijection, the result follows. □

Domain Markov property The spin representation model satisfies a natural consistency property, the *domain Markov property* (or *spatial Markov property*).

Proposition 2.10 (Domain Markov property)**.** *Fix parameters $n, x > 0$, a spin configuration $\tau \in \{-1,1\}^{V(\mathbb{T})}$ and let H, G be two finite induced subgraphs of $\mathbb{T}$ such that $H \subseteq G$. Then for any event $A \in \mathcal{F}$ and any $\tau' \in \mathsf{SpinConf}^{\tau}_H$ we have*

$$\nu^{\tau'}_{G,n,x}(A \mid \mathsf{SpinConf}^{\tau}_H) = \nu^{\tau}_{H,n,x}(A).$$

Proof of Proposition 2.10. Observe that since $\tau' \in \mathsf{SpinConf}_H^\tau$, we have $\nu_{G,n,x}^{\tau'}(\sigma) = \nu_{G,n,x}^{\tau}(\sigma)$ for any $\sigma \in \mathsf{SpinConf}_H^\tau$. Hence, for any $\sigma \in \mathsf{SpinConf}_H^\tau$, we have

$$\begin{aligned}
\nu_{G,n,x}^{\tau'}(\sigma \mid \mathsf{SpinConf}_H^\tau) &= \frac{\nu_{G,n,x}^{\tau'}(\{\sigma\} \cap \mathsf{SpinConf}_H^\tau)}{\nu_{G,n,x}^{\tau'}(\mathsf{SpinConf}_H^\tau)} \\
&= \frac{\nu_{G,n,x}^{\tau}(\sigma)}{\nu_{G,n,x}^{\tau}(\mathsf{SpinConf}_H^\tau)} \\
&= \frac{n^{k_G(\sigma)} x^{e_G(\sigma)}}{\sum_{\sigma' \in \mathsf{SpinConf}_H^\tau} n^{k_G(\sigma')} x^{e_G(\sigma')}} \\
&= \frac{n^{k_H(\sigma)} x^{e_H(\sigma)}}{\sum_{\sigma' \in \mathsf{SpinConf}_H^\tau} n^{k_H(\sigma')} x^{e_H(\sigma')}} = \nu_{H,n,x}^{\tau}(\sigma),
\end{aligned}$$

where the last equality follows from the fact that for spin configurations $\sigma, \sigma' \in \mathsf{SpinConf}_H^\tau$ we have

$$e_G(\sigma) - e_G(\sigma') = e_H(\sigma) - e_H(\sigma'),$$

$$k_G(\sigma) - k_G(\sigma') = k_H(\sigma) - k_H(\sigma').$$

Now the result follows, since

$$\begin{aligned}
\nu_{G,n,x}^{\tau'}(A \mid \mathsf{SpinConf}_H^\tau) &= \sum_{\sigma \in A \cap \mathsf{SpinConf}_H^\tau} \nu_{G,n,x}^{\tau'}(\sigma \mid \mathsf{SpinConf}_H^\tau) \\
&= \sum_{\sigma \in A \cap \mathsf{SpinConf}_H^\tau} \nu_{H,n,x}^{\tau}(\sigma) = \nu_{H,n,x}^{\tau}(A).
\end{aligned}$$

$\square$

We will now state an equivalent formulation of the spatial Markov property in terms of conditional expectations. This will be convenient when discussing Gibbs measures.

Corollary 2.11 (Domain Markov property reformulation). *Fix parameters $n, x > 0$, a spin configuration $\tau \in \{-1,1\}^{V(\mathbb{T})}$ and let H, G be two finite induced subgraphs of $\mathbb{T}$ such that $H \subseteq G$. Then for any event $A \in \mathcal{F}$ and any $\tau' \in \mathsf{SpinConf}_H^\tau$ we have*

$$\nu_{G,n,x}^{\tau'}[\mathbb{1}_A \mid \mathcal{T}_H](\cdot) = \nu_{H,n,x}^{\cdot}(A).$$

Proof. By Proposition 2.10, we know that for any $\tau' \in \mathsf{SpinConf}_H^\tau$

$$\nu_{G,n,x}^{\tau'}(A \mid \mathsf{SpinConf}_H^\tau) = \nu_{H,n,x}^{\tau}(A).$$

Hence, it suffices to show that

$$\nu_{G,n,x}^{\tau'}[\mathbb{1}_A \mid \mathcal{T}_H](\tau) = \nu_{G,n,x}^{\tau'}(A \mid \mathsf{SpinConf}_H^\tau),$$

for any $\tau' \in \mathsf{SpinConf}_H^\tau$. By the definition of conditional expectation, this amounts to verifying the equation

$$\nu_{G,n,x}^{\tau'}[\mathbb{1}_B(\cdot)\, \nu_{G,n,x}^{\tau'}(A \mid \mathsf{SpinConf}_H^{\cdot})] = \nu_{G,n,x}^{\tau'}[\mathbb{1}_A \mathbb{1}_B],$$

for all $B \in \mathcal{T}_H$. Indeed, let $B \in \mathcal{T}_H$, $\tau' \in \mathsf{SpinConf}_H^\tau$. Since B depends only on vertices outside of H we can choose finitely many distinct spin configurations $\{\tau_1, \dots, \tau_m\} \subseteq \mathsf{SpinConf}_G^{\tau'}$ such that

$$B \cap \mathsf{SpinConf}_G^{\tau'} = \bigsqcup_{k=1}^{m} \mathsf{SpinConf}_H^{\tau_k}.$$

Now,

$$\begin{aligned}
&\nu^{\tau'}_{G,n,x}[\mathbb{1}_B(\cdot)\, \nu^{\tau'}_{G,n,x}(A \mid \mathsf{SpinConf}^{\cdot}_H)] \\
&\quad = \sum_{\tau \in \mathsf{SpinConf}^{\tau'}_G \cap B} \nu^{\tau'}_{G,n,x}(\tau)\, \nu^{\tau'}_{G,n,x}(A \mid \mathsf{SpinConf}^{\tau}_H) \\
&\quad = \sum_{k=1}^{m} \sum_{\tau \in \mathsf{SpinConf}^{\tau_k}_H} \nu^{\tau'}_{G,n,x}(\tau)\, \nu^{\tau'}_{G,n,x}(A \mid \mathsf{SpinConf}^{\tau}_H) \\
&\quad = \sum_{k=1}^{m} \nu^{\tau'}_{G,n,x}(\mathsf{SpinConf}^{\tau_k}_H)\, \nu^{\tau'}_{G,n,x}(A \mid \mathsf{SpinConf}^{\tau_k}_H) \\
&\quad = \sum_{k=1}^{m} \nu^{\tau'}_{G,n,x}(A \cap \mathsf{SpinConf}^{\tau_k}_H) \\
&\quad = \nu^{\tau'}_{G,n,x}(A \cap B) = \nu^{\tau'}_{G,n,x}[\mathbb{1}_A \mathbb{1}_B],
\end{aligned}$$

and the result follows. $\square$

FKG inequality and comparison between boundary conditions In this section we will show that the spin representation measures satisfy a positive association property in a certain regime of parameters. As for other models like the Ising model, positive association properties have turned out to be an essential tool in the study of the loop $O(n)$ model.

In the following, for a spin configuration $\sigma \in \{-1,1\}^{V(\mathbb{T})}$ and a face $u \in V(\mathbb{T})$ we write σ^u, σ_u for the spin configurations which coincide with σ on $V(\mathbb{T}) \setminus \{u\}$ and satisfy $\sigma^u(u) = 1, \sigma_u(u) = -1$ respectively.

Definition (Partial order)**.** Define a partial order $\preceq$ on $\{-1,1\}^{V(\mathbb{T})}$ as follows. For $\sigma, \sigma' \in \{-1,1\}^{V(\mathbb{T})}$ write

$$\sigma \preceq \sigma' \quad \text{if and only if} \quad \sigma(u) \leq \sigma'(u), \ \ \forall v \in V(\mathbb{T}).$$

A function $f : \{-1,1\}^{V(\mathbb{T})} \to \mathbb{R}$ is called increasing (decreasing) if it is increasing (decreasing) with respect to the partial order $\preceq$. An event $A \in \mathcal{F}$ is called increasing (decreasing), if the function $\mathbb{1}_A$ is increasing (decreasing).

Theorem 2.12 (FKG inequality)**.** *Consider parameters $n, x \in \mathbb{R}$ such that $n \geq 1$ and $nx^2 \leq 1$ and let $G \subseteq \mathbb{T}$ be a finite induced subgraph. Fix a spin configuration $\tau \in \{-1,1\}^{V(\mathbb{T})}$. Then, the measure $\nu^{\tau}_{G,n,x}$ satisfies the FKG inequality. That is, for any increasing events $A, B \in \mathcal{F}$, and decreasing events $C, D \in \mathcal{F}$, we have*

$$\begin{aligned}
\nu^{\tau}_{G,n,x}(A \cap B) &\geq \nu^{\tau}_{G,n,x}(A)\, \nu^{\tau}_{G,n,x}(B), \\
\nu^{\tau}_{G,n,x}(C \cap D) &\geq \nu^{\tau}_{G,n,x}(C)\, \nu^{\tau}_{G,n,x}(D).
\end{aligned}$$

Remark 1 (Counterexample). The assumption $nx^2 \leq 1$ can not be weakened. Indeed, consider a domain G consisting of two adjacent hexagons $u, v \in V(\mathbb{T})$ and set $\tau \equiv 1$. Consider the events

$$A = \{\sigma \in \{-1,1\}^{V(\mathbb{T})} \mid \sigma(u) = 1\},\ B = \{\sigma \in \{-1,1\}^{V(\mathbb{T})} \mid \sigma(v) = 1\}.$$

Then, a quick calucation shows that

$$\nu^{\tau}_{G,n,x}(A) = \nu^{\tau}_{G,n,x}(B) = \frac{1 + nx^6}{Z_{G,n,x}},$$

$$\nu^{\tau}_{G,n,x}(A \cap B) = \frac{1}{Z_{G,n,x}},$$

where

$$Z_{G,n,x} = 1 + 2nx^6 + nx^{10}.$$

Hence

$$\begin{aligned}\frac{\nu^\tau_{G,n,x}(A)\cdot\nu^\tau_{G,n,x}(B)}{\nu^\tau_{G,n,x}(A\cap B)} &= \frac{(1+nx^6)^2}{1+2nx^6+nx^{10}} = \frac{1+2nx^6+n^2x^{12}}{1+2nx^6+nx^{10}}\\ &= 1+\frac{n^2x^{12}-nx^{10}}{1+2nx^6+nx^{10}} = 1+\frac{nx^{10}(nx^2-1)}{1+2nx^6+nx^{10}}.\end{aligned}$$

It follows that

$$\nu^\tau_{G,n,x}(A\cap B) \geq \nu^\tau_{G,n,x}(A)\,\nu^\tau_{G,n,x}(B),$$

if and only if $nx^2 \leq 1$.

Our proof of Theorem 2.12 is a variant of the standard argument - given for example in [Gri03, Theorem 2.19] - for proving the FKG inequality. Indeed, the reader can convince herself, that the *FKG lattice condition* [DGPS20, Proposition 8] implies Proposition 2.13.

Proposition 2.13. *Consider parameters $n, x \in \mathbb{R}$ such that $n \geq 1$ and $nx^2 \leq 1$ and let $G \subseteq \mathbb{T}$ be a finite induced subgraph. Fix a spin configuration $\tau \in \{-1,1\}^{V(\mathbb{T})}$ and a vertex $u \in V(G)$. Set $V_u = V(G)\setminus\{u\}$. Now consider two spin configurations $\sigma,\rho \in \mathsf{SpinConf}^\tau_G$ satisfying $\sigma \preceq \rho$. Then for a random spin configuration $\boldsymbol{\sigma} \sim \nu^\tau_{G,n,x}$, we have*

$$\mathbb{P}\left(\boldsymbol{\sigma}(u)=1 \mid \boldsymbol{\sigma}|_{V_u} = \sigma|_{V_u}\right) \leq \mathbb{P}\left(\boldsymbol{\sigma}(u)=1 \mid \boldsymbol{\sigma}|_{V_u} = \rho|_{V_u}\right).$$

Proof. Denote by $k^-(\sigma,u)$ and $k^+(\sigma,u)$ the number of distinct -1 and $+1$ spin clusters in $\sigma|_{V_u}$ adjacent to hexagon u and denote by $n^-(\sigma,u)$ and $n^+(\sigma,u)$ the number of hexagons adjacent to hexagon u with -1 and $+1$ spins respectively. We will begin with the following two claims.

Claim 2.13.1. *Consider a hexagon $u \in V(G)$ and a spin configuration $\sigma \in \mathsf{SpinConf}^\tau_D$. Then,*

$$k_G(\sigma^u) - k_G(\sigma_u) = k^-(\sigma,u) - k^+(\sigma,u).$$

Proof of Claim 2.13.1. The reader can easily convince herself of the validity of the claim. □

Claim 2.13.2. *Consider two spin configurations $\sigma,\rho \in \mathsf{SpinConf}^\tau_G$ satisfying $\sigma \preceq \rho$. Then,*

$$k^-(\sigma,u) - k^-(\rho,u) + k^+(\rho,u) - k^+(\sigma,u) \leq n^+(\rho,u) - n^+(\sigma,u).$$

Proof of Claim 2.13.2. We will show the claim by induction over the number of faces on which σ,ρ differ,

$$d_G(\sigma,\rho) := |\{v \in V(G) \mid \sigma(v) < \rho(v)\}|.$$

For $d_G(\sigma,\rho) = 0$ we have $\sigma = \rho$ and the inequality is trivial. Now let $j \in \mathbb{N}$ and assume that the inequality holds for any σ,ρ with $d(\sigma,\rho) = j$. We want to show that

$$k^-(\sigma,u) - k^-(\rho^v,u) + k^+(\rho^v,u) - k^+(\sigma,u) \leq n^+(\rho^v,u) - n^+(\sigma,u)$$

for any $v \in V(G)$. Using the induction assumption, it suffices to show that

$$k^-(\rho,u) - k^-(\rho^v,u) + k^+(\rho^v,u) - k^+(\rho,u) \leq n^+(\rho^v,u) - n^+(\rho,u)$$

for any $v \in V(G)$. Now distinguish the following two cases. If v is not adjacent to u, then $k^-(\rho,u) \leq k^-(\rho^v,u)$ and $k^+(\rho^v,u) \leq k^+(\rho,u)$ and the claim follows. If on the other hand v is adjacent to u, then $n^+(\rho^v,u) - n^+(\rho,u) = 1$ and it suffices to exclude that simultaneously

$$k^-(\rho,u) - k^-(\rho^v,u) = 1, \qquad k^+(\rho^v,u) - k^+(\rho,u) = 1.$$

However, $k^-(\rho,u) - k^-(\rho^v,u) = 1$ implies that the two faces adjacent to u and v have spin $+1$ in ρ which in turn implies that $k^+(\rho^v,u) - k^+(\rho,u) = 0$. □

These two claims imply the result. Indeed, consider two spin configurations $\sigma, \rho \in \mathsf{SpinConf}_D^\tau$ such that $\sigma \preceq \rho$. Using Claim 2.13.1, we get

$$\begin{aligned}\frac{\nu^\tau_{G,n,x}(\sigma^u)}{\nu^\tau_{G,n,x}(\sigma_u)} &= n^{k_G(\sigma^u)-k_G(\sigma_u)} \cdot x^{6-2\,n^+(\sigma,u)} \\ &= n^{k^-(\sigma,u)-k^+(\sigma,u)} \cdot x^{6-2\,n^+(\sigma,u)},\end{aligned}$$

and analogously

$$\frac{\nu^\tau_{G,n,x}(\rho^u)}{\nu^\tau_{G,n,x}(\rho)} = n^{k^-(\rho,u)-k^+(\rho,u)} \cdot x^{6-2\,n^+(\rho,u)}.$$

Now observe that our assumption $nx^2 \leq 1$ together with Claim 2.13.2 implies

$$\begin{aligned}&\frac{\nu^\tau_{G,n,x}(\sigma^u)}{\nu^\tau_{G,n,x}(\sigma_u)}\frac{\nu^\tau_{G,n,x}(\rho_u)}{\nu^\tau_{G,n,x}(\rho^u)} = \\ &\qquad = n^{k^-(\sigma,u)-k^+(\sigma,u)}\, n^{-(k^-(\sigma,u)-k^+(\sigma,u))} \cdot x^{2(n^+(\rho,u)-n^+(\sigma,u))} \\ &\qquad \leq n^{k^-(\sigma,u)-k^+(\sigma,u)}\, n^{-(k^-(\sigma,u)-k^+(\sigma,u))} \cdot n^{-(n^+(\rho,u)-n^+(\sigma,u))} \\ &\qquad = n^{k^-(\sigma,u)-k^+(\sigma,u)-(k^-(\sigma,u)-k^+(\sigma,u))-(n^+(\rho,u)-n^+(\sigma,u))} \\ &\qquad \leq n^0 = 1.\end{aligned}$$

Thus we find that

$$\frac{\nu^\tau_{G,n,x}(\sigma^u)}{\nu^\tau_{G,n,x}(\sigma_u)} \leq \frac{\nu^\tau_{G,n,x}(\rho^u)}{\nu^\tau_{G,n,x}(\rho_u)},$$

and hence

$$\begin{aligned}\mathbb{P}\left(\boldsymbol{\sigma}(u) = +1 \mid \boldsymbol{\sigma}|_{V_u} = \sigma\right) &= \frac{1}{1+\frac{\nu^\tau_{G,n,x}(\sigma_u)}{\nu^\tau_{G,n,x}(\sigma^u)}} \leq \frac{1}{1+\frac{\nu^\tau_{G,n,x}(\rho_u)}{\nu^\tau_{G,n,x}(\rho^u)}} \\ &= \mathbb{P}\left(\boldsymbol{\sigma}(u) = +1 \mid \boldsymbol{\sigma}|_{V_u} = \rho\right).\end{aligned}$$

□

Proof of Theorem 2.12. We show only the first inequality, the second one can be obtained by taking A^c, B^c. If $B \cap \mathsf{SpinConf}_G^\tau = \emptyset$, the result trivially holds. Henceforth assume that $B \cap \mathsf{SpinConf}_G^\tau \neq \emptyset$. We construct two coupled Markov chains $X^1 = (X^1_k)_{k\in\mathbb{N}}$ and $X^2 = (X^2_k)_{k\in\mathbb{N}}$ on $\mathsf{SpinConf}_G^\tau$ as follows. Pick some spin configuration $\sigma_0 \in B \cap \mathsf{SpinConf}_G^\tau$ and set

$$X^1_0 = X^2_0 = \sigma_0.$$

We define the Markov chains inductively. Indeed, assume that $X^1_k = \sigma, X^2_k = \rho$ for some spin configurations $\sigma, \rho \in \{-1,1\}^{V(\mathbb{T})}$. Consider two independent random variables $u \sim \mathrm{Unif}(V(G))$, $x \sim \mathrm{Unif}([0,1])$ and set

$$X^1_{k+1} = \begin{cases} \sigma^u, & \text{if } x < \mathbb{P}\left(\boldsymbol{\sigma}(u) = +1 \mid \boldsymbol{\sigma}|_{V_u} = \sigma|_{V_u}\right) \\ \sigma_u, & \text{otherwise.}\end{cases}$$

$$X^2_{k+1} = \begin{cases} \rho^u, & \text{if } x < \mathbb{P}\left(\boldsymbol{\sigma}(u) = +1 \mid \boldsymbol{\sigma}|_{V_u} = \rho|_{V_u}, \boldsymbol{\sigma} \in B\right) \text{ or } \rho_u \notin B \\ \rho_u, & \text{otherwise.}\end{cases}$$

Viewed as Markov chains on $\mathsf{SpinConf}_G^\tau$ and $\mathsf{SpinConf}_G^\tau \cap B$ respectively, X^1 and X^2 are irreducible and aperiodic. Moreover, by construction X^1 and X^2 are in detailed balance with $\nu^\tau_{G,n,x}$ and $\nu^\tau_{G,n,x}(\cdot \mid B)$. Thus, X^1 converges to $\nu^\tau_{G,n,x}$ and X^2 converges to $\nu^\tau_{G,n,x}(\cdot \mid B)$.

Now, Proposition 2.13 implies that the partial order $\preceq$ is preserved under these Markov chain dynamics. That is, for any $k \in \mathbb{N}$,

$$X_k^1 \preceq X_k^2 \Longrightarrow X_{k+1}^1 \preceq X_{k+1}^2.$$

It follows that

$$\mathbb{P}\left(X_k^1 \preceq X_k^2\right) = 1.$$

Now let $A \in \mathcal{F}$ be an increasing event. We have

$$\begin{aligned}
\nu_{G,n,x}^{\tau}(A) &= \lim_{k\to\infty} \mathbb{P}(X_k^1 \in A) = \lim_{k\to\infty} \mathbb{P}(X_k^1 \in A,\ X_k^1 \preceq X_k^2) \\
&\le \lim_{k\to\infty} \mathbb{P}(X_k^2 \in A,\ X_k^1 \preceq X_k^2) = \lim_{k\to\infty} \mathbb{P}(X_k^2 \in A) \\
&= \nu_{G,n,x}^{\tau}(A \mid B),
\end{aligned}$$

and hence

$$\nu_{G,n,x}^{\tau}(A \cap B) = \nu_{G,n,x}^{\tau}(A \mid B) \cdot \nu_{G,n,x}^{\tau}(B) \le \nu_{G,n,x}^{\tau}(A) \cdot \nu_{G,n,x}^{\tau}(B).$$

□

The FKG inequality allows us to establish a certain stochastic monotonicity between spin representation measures with different boundary conditions.

Proposition 2.14 (Comparison between boundary conditions)**.** *Consider parameters $n, x \in \mathbb{R}$ such that $n \ge 1$ and $nx^2 \le 1$ and let $G \subseteq \mathbb{T}$ be a finite induced subgraph. Consider $\tau, \tau' \in \mathbb{T}$ satisfying $\tau \preceq \tau'$. Then for any increasing event $A \in \mathcal{F}$, we have*

$$\nu_{G,n,x}^{\tau}(A) \le \nu_{G,n,x}^{\tau'}(A).$$

Proof. We first show the result under the assumption that

$$D := \{u \in V(\mathbb{T}) \mid \tau(u) < \tau'(u)\}$$

is finite. Let $A \in \mathcal{F}$ be an increasing event. By the domain Markov property

$$\begin{aligned}
\nu_{G,n,x}^{\tau} &= \nu_{G\cup D,n,x}^{\tau}(A \mid \mathsf{SpinConf}_G^{\tau}) \\
&= \nu_{G\cup D,n,x}^{\tau}(A \mid \{\sigma \in \{-1,1\}^{V(\mathbb{T})} \mid \sigma\big|_D \equiv -1\}) \\
&\le \nu_{G\cup D,n,x}^{\tau}(A),
\end{aligned}$$

where we used the FKG inequality in the last line. Analogously we obtain

$$\begin{aligned}
\nu_{G,n,x}^{\tau'} &= \nu_{G\cup D,n,x}^{\tau'}(A \mid \mathsf{SpinConf}_G^{\tau'}) \\
&= \nu_{G\cup D,n,x}^{\tau'}(A \mid \{\sigma \in \{-1,1\}^{V(\mathbb{T})} \mid \sigma\big|_D \equiv 1\}) \\
&\ge \nu_{G\cup D,n,x}^{\tau'}(A).
\end{aligned}$$

Finally since

$$\mathsf{SpinConf}_{G\cup D}^{\tau'} = \mathsf{SpinConf}_{G\cup D}^{\tau},$$

we get

$$\nu_{G,n,x}^{\tau}(A) \le \nu_{G\cup D,n,x}^{\tau}(A) = \nu_{G\cup D,n,x}^{\tau'}(A) \le \nu_{G,n,x}^{\tau'}(A),$$

finishing the proof under the above assumption.

Now pick a finite subgraph $H \subseteq \mathbb{T}$ containing G such that for any two spin configurations $\sigma_1, \sigma_2 \in \mathsf{SpinConf}_G^{\tau'}$ satisfying $\sigma_1\big|_H = \sigma_2\big|_H$ we have

$$e_G(\sigma_1) = e_G(\sigma_2),\ k_G(\sigma_1) = k_G(\sigma_2).$$

Now define

$$\tilde{\tau}(v) := \begin{cases} \tau'(v), & \text{if } v \in V(H) \\ \tau(v), & \text{otherwise.} \end{cases}$$

Then by the argument above we have

$$\nu_{G,n,x}^{\tau}(A) \leq \nu_{G,n,x}^{\tilde{\tau}}(A),$$

so it remains to show that

$$\nu_{G,n,x}^{\tilde{\tau}}(A) \leq \nu_{G,n,x}^{\tau'}(A).$$

We have

$$\nu_{G,n,x}^{\tilde{\tau}}(A) = \sum_{\sigma \in A} \nu_{G,n,x}^{\tilde{\tau}}(\sigma).$$

For any $\sigma \in A \cap \mathsf{SpinConf}_G^{\tilde{\tau}}$ contributing $\nu_{G,n,x}^{\tilde{\tau}}(\sigma)$ to the above sum, we find a unique corresponding $\sigma' \in A \cap \mathsf{SpinConf}_G^{\tau'}$ given by

$$\sigma'(v) = \begin{cases} \sigma(v), & \text{if } v \in V(H) \\ \tilde{\tau}(v), & \text{otherwise.} \end{cases}$$

which contributes $\nu_{G,n,x}^{\tau'}(\sigma') = \nu_{G,n,x}^{\tilde{\tau}}(\sigma)$ to the sum

$$\nu_{G,n,x}^{\tau'}(A) = \sum_{\sigma' \in A} \nu_{G,n,x}^{\tau'}(\sigma').$$

Thus

$$\nu_{G,n,x}^{\tilde{\tau}}(A) \leq \nu_{G,n,x}^{\tau'}(A)$$

and the result follows. □

The domain Markov property and the preceding Proposition allows us to compare spin representation measures on different graphs. This will be crucial in the construction of infinite-volume spin representation measures.

Corollary 2.15. *Consider parameters $n, x \in \mathbb{R}$ such that $n \geq 1$ and $nx^2 \leq 1$ and let $H, G \subseteq \mathbb{T}$ be two finite induced subgraphs such that $H \subseteq G$. Then, for any increasing event $A \in \mathcal{F}$ and any event $B \in \mathcal{T}_H$, we have*

$$\nu_{G,n,x}^{-}(A \mid B) \geq \nu_{H,n,x}^{-}(A), \qquad \nu_{G,n,x}^{+}(A \mid B) \leq \nu_{H,n,x}^{+}(A),$$

whenever the above conditioning is non-degenerate. In particular

$$\nu_{G,n,x}^{-}(A) \geq \nu_{H,n,x}^{-}(A), \qquad \nu_{G,n,x}^{+}(A) \leq \nu_{H,n,x}^{+}(A).$$

Proof. We show the first inequality, the second inequality can be obtained analogously. Since $B \in \mathcal{T}_H$ depends only on vertices outside of H, we can pick $\{\tau_1, \ldots, \tau_k\} \subseteq B$ such that

$$B \cap \mathsf{SpinConf}_G^{-} = \bigcup_{k=1}^{m} \mathsf{SpinConf}_H^{\tau_k}$$

is a partition. Now by the domain Markov property (Proposition 2.10) and Proposition 2.14 we get

$$\begin{aligned}
\nu^-_{G,n,x}(A \cap B) &= \sum_{k=1}^{m} \nu^-_{G,n,x}(A \cap \mathsf{SpinConf}^{\tau_k}_H) \\
&= \sum_{k=1}^{m} \nu^-_{G,n,x}(A \mid \mathsf{SpinConf}^{\tau_k}_H)\, \nu^-_{G,n,x}(\mathsf{SpinConf}^{\tau_k}_H) \\
&= \sum_{k=1}^{m} \nu^{\tau_k}_{H,n,x}(A)\, \nu^-_{G,n,x}(\mathsf{SpinConf}^{\tau_k}_H) \\
&\geq \nu^-_{H,n,x}(A) \sum_{k=1}^{m} \nu^-_{G,n,x}(\mathsf{SpinConf}^{\tau_k}_H) \\
&= \nu^-_{H,n,x}(A)\, \nu^-_{G,n,x}(B)
\end{aligned}$$

and dividing the inequality by $\nu^-_{G,n,x}(B)$ yields

$$\nu^-_{G,n,x}(A \mid B) \geq \nu^-_{H,n,x}(A).$$

The last two inequalities are obtained from the first two inequalities by setting $B \coloneqq \{-1,1\}^{V(\mathbb{T})}$. □

2.2.2 Thermodynamic limits of the spin representation model

In this section we will use the positive association properties of the spin representation measures established in the preceding section to construct *thermodynamic limits* of the spin representation model. In contrast to the spin representation measures on finite domains, we have so far considered, thermodynamic limits are supported on the whole lattice. We obtain these measures as limits of finite-volume measures in the sense of weak convergence. Since weak convergence makes use of the notion of continuity, we need to introduce topologies on the state-spaces under consideration.

Proposition 2.16. *The induced topologies*

$$\tau_{\mathcal{L}} \coloneqq \tau(\{\pi^{\mathbb{H}}_e : e \in E(\mathbb{H})\}),$$

$$\tau_{\mathcal{F}} \coloneqq \tau(\{\pi^{\mathbb{T}}_v : v \in V(\mathbb{T})\}),$$

on $\mathsf{LoopConfig}_{\mathbb{H}}$ *and* $\{-1,1\}^{V(\mathbb{T})}$ *respectively, are metrizable and compact.*[8] *Moreover,* $\mathcal{L}$ *is the Borel-σ-algebra corresponding to* $\tau_{\mathcal{L}}$ *and* $\mathcal{F}$ *is the Borel-σ-algebra corresponding to* $\tau_{\mathcal{F}}$.

We omit the proof here. In the following we will always consider the measurable spaces $(\{-1,1\}^{V(\mathbb{T})}, \mathcal{F})$ and $(\mathsf{LoopConfig}_{\mathbb{H}}, \mathcal{L})$ equipped with the corresponding product topologies $\tau_{\mathcal{F}}$ and $\tau_{\mathcal{L}}$, with respect to which $\mathcal{F}$ and $\mathcal{L}$ are Borel-σ-algebras. Moreover, for a sequence of measures $(\mu_n)_{n\geq 1}$ converging weakly to some measure μ, we will use the notation $\mu_n \xrightarrow{w} \mu$.

Definition (Thermodynamic limit)**.** Consider parameters $n, x > 0$. A measure ν on $(\{-1,1\}^{V(\mathbb{T})}, \mathcal{F})$ is called thermodynamic limit of the spin representation with parameters n and x, if there is a spin configuration $\tau \in \{-1,1\}^{V(\mathbb{T})}$ and an increasing sequence $(G_k)_{k\geq 1}$ of finite subgraphs of $\mathbb{T}$ satisfying $\bigcup_{k\geq 1} G_k = \mathbb{T}$, such that

$$\nu^{\tau}_{G_k,n,x} \to \nu$$

as $k \to \infty$ in the sense of weak convergence. We denote the set of such thermodynamic limits by $\mathcal{W}_{n,x}$.

Using the previously established positive association properties, we can construct two thermodynamic limits, corresponding to the two constant boundary conditions.

[8] Here we assume the image spaces of the canonical projection to be equipped with the discrete topology.

Proposition 2.17 (Existence of thermodynamic limits). *Consider parameters $n, x \in \mathbb{R}$ such that $n \geq 1$ and $nx^2 \leq 1$. Then, for any increasing sequence $(G_k)_{k\geq 1}$ of finite subgraphs of $\mathbb{T}$ satisfying $\bigcup_{k\geq 1} G_k = \mathbb{T}$, the sequences $(\nu^+_{G_k,n,x})_{k\geq 1}$ and $(\nu^-_{G_k,n,x})_{k\geq 1}$ converge weakly as $k \to \infty$ to measures $\nu^+_{n,x}$ and $\nu^-_{n,x}$ on $(\{-1,1\}^{V(\mathbb{T})}, \mathcal{F})$. In particular $\mathcal{W}_{n,x}$ is non-empty. Moreover, the measures $\nu^+_{n,x}$ and $\nu^-_{n,x}$ are extremal, in that any thermodynamic limit $\nu \in \mathcal{W}_{n,x}$ satisfies*

$$\nu^-_{n,x}(A) \leq \nu(A) \leq \nu^+_{n,x}(A),$$

for any increasing event $A \in \mathcal{F}$. In particular, if $\nu^-_{n,x} = \nu^+_{n,x}$, then $|\mathcal{W}_{n,x}| = 1$.

In the following, we will refer to the measures $\nu^+_{n,x}$ and $\nu^-_{n,x}$ as *extremal infinite-volume measures.*

Proof of Proposition 2.17. Consider an increasing sequence $(G_k)_{k\geq 1}$ of finite subgraphs of $\mathbb{T}$ such that $\bigcup_{k\geq 1} G_k = \mathbb{T}$. Consider any increasing event $A \in \mathcal{F}$ and fix integers $i < j$. Then, by Proposition 2.10 and Theorem 2.12,

$$\nu^+_{n,x,G_i}(A) = \nu^+_{n,x,G_j}(A \mid \mathsf{SpinConf}^+_{G_i}) \geq \nu^+_{n,x,G_j}(A).$$

Hence, $(\nu^+_{n,x,G_k}(A))_{k\geq 1}$ is decreasing and thus has a limit we will denote by $\nu^+_{n,x}(A) := \lim_{k\geq 1} \nu^+_{n,x,G_k}(A)$.

Since $\{-1,1\}^{V(\mathbb{T})}$ is compact with respect to its topology, we know that any family of probability measures is tight and thus by Prokhorov's Theorem (see for example [Bil17, Theorem 25.10.]) has a weak subsequential limit. Let $\nu = \lim_{i\to\infty} \nu^+_{n,x,G_{k_i}}$ be some such measure. Now, since the random variable $\mathbb{1}_A$ is continuous and bounded, we have

$$\nu(A) = \nu[\mathbb{1}_A] = \lim_{i\geq 1} \nu^+_{n,x,G_{k_i}}[\mathbb{1}_A] = \lim_{i\geq 1} \nu^+_{n,x,G_{k_i}}(A) = \nu^+_{n,x}(A).$$

Since the above holds for any increasing $A \in \mathcal{F}$, and any weak subsequential limit ν of $(\nu^+_{n,x,G_k})_{k\geq 1}$, all subsequential limits must coincide. It follows [Bil17, Corollary 25.10.] that $(\nu^+_{n,x,G_k})_{k\geq 1}$ converges weakly. □

Properties of thermodynamic limits We will now establish certain properties satisfied by thermodynamic limits. A direct corollary of the proof of Proposition 2.17, is that thermodynamic limits satisfy the FKG inequality.

Corollary 2.18 (FKG inequality). *Consider parameters $n, x \in \mathbb{R}$ such that $n \geq 1$ and $nx^2 \leq 1$. Then, for any $\nu \in \mathcal{W}_{n,x}$ and any increasing events $A, B \in \mathcal{F}$ depending on finitely many edges, we have*

$$\nu(A \cap B) \geq \nu(A)\nu(B).$$

Proof. Since $\nu \in \mathcal{W}_{n,x}$, there is a spin configuration $\tau \in \{-1,1\}^{V(\mathbb{T})}$ and a sequence of domains $(G_k)_{k\geq 1}$ exhausting $\mathbb{T}$, such that $\nu^\tau_{G_k,n,x} \xrightarrow{w} \nu$ as $k \to \infty$. Now let $A, B \in \mathcal{F}$ be increasing events depending on finitely many vertices. Then,

$$\nu_{n,x}(A \cap B) = \lim_{k\to\infty} \nu^\tau_{\Lambda_k,n,x}(A \cap B) = \lim_{k\to\infty} \nu^\tau_{\Lambda_k,n,x}(A)\nu^\tau_{\Lambda_k,n,x}(B) = \nu_{n,x}(A)\nu_{n,x}(B),$$

where the inequality follows from Theorem 2.12. □

The *finite-energy property* states that if we modify an event with positive probability on finitely many vertices, the resulting event has non-zero probability as well.

Definition (Finite-energy property). A measure ν on $(\{-1,1\}^{V(\mathbb{T})}, \mathcal{F})$ is said to satisfy the *finite-energy property* if for any $v \in V(\mathbb{T})$ we ν-almost-surely have

$$-1 < \nu[\,\pi_v \mid \mathcal{T}_v] < 1,$$

where

$$\mathcal{T}_v := \sigma(\{\pi_u \mid u \in V(G)\backslash\{v\}\}).$$

Proposition 2.19 (Finite-energy property)**.** *Consider parameters $n, x \in \mathbb{R}$ such that $n \geq 1$ and $nx^2 \leq 1$. Then, any $\nu \in \mathcal{W}_{n,x}$ has the finite-energy property.*

The upcoming proof of the finite-energy property is adapted from corresponding statements for the random-cluster model [Gri03, Theorem 4.17]. We begin by showing an analogue of Proposition 2.19 for finite domains.

Lemma 2.20. *Consider real parameters $n, x > 0$. There exists constants $-1 < c < C < 1$ such that for any a finite subgraph $G \subseteq \mathbb{T}$, any $v \in V(\mathbb{T})$ and any $\tau \in \{-1,1\}^{V(\mathbb{T})}$ we $\nu^{\tau}_{G,n,x}$-almost surely have*

$$c < \nu^{\tau}_{G,n,x}[\,\pi_v] < C.$$

Proof. Let $\boldsymbol{\sigma} \sim \nu^{\tau}_{G,n,x}$, then

$$\begin{aligned}
\mathbb{P}(\pi_v(\boldsymbol{\sigma}) = 1) &= \frac{1}{2} \sum_{\sigma \in \{-1,1\}^{V(\mathbb{T})}} \nu^{\tau}_{G,n,x}(\sigma^v) \\
&= \frac{1}{2} \sum_{\sigma \in \{-1,1\}^{V(\mathbb{T})}} \frac{\nu^{\tau}_{G,n,x}(\sigma^v)}{\nu^{\tau}_{G,n,x}(\sigma^v) + \nu^{\tau}_{G,n,x}(\sigma_v)} \\
&= \frac{1}{2} \sum_{\sigma \in \{-1,1\}^{V(\mathbb{T})}} \frac{1}{1 + \frac{\nu^{\tau}_{G,n,x}(\sigma_v)}{\nu^{\tau}_{G,n,x}(\sigma^v)}}.
\end{aligned}$$

Now one can easily check that for any $\sigma \in \{-1,1\}^{V(\mathbb{T})}$, we have that

$$\begin{aligned}
|k(\sigma_v) - k(\sigma^v)| &\leq 2, \\
|e(\sigma_v) - e(\sigma^v)| &\leq 6.
\end{aligned}$$

It follows that

$$\min(1,x)^6 \min(1,n)^2 \leq \frac{\nu^{\tau}_{G,n,x}(\sigma_v)}{\nu^{\tau}_{G,n,x}(\sigma^v)} \leq \max(1,x)^6 \max(1,n)^2.$$

Thus $\mathbb{P}(\pi_v(\boldsymbol{\sigma}) = 1)$ is bounded away from 0 and 1 uniformly in G. The same holds for $\mathbb{P}(\pi_v(\boldsymbol{\sigma}) = -1)$ and the result follows. □

Using the above Lemma and a martingale argument, we can now show that any thermodynamic limit has the finite-energy property.

Proof of Proposition 2.19. Since $\nu \in \mathcal{W}_{n,x}$, there is a spin configuration $\tau \in \{-1,1\}^{V(\mathbb{T})}$ and a sequence of domains $(G_k)_{k\geq 1}$ exhausting $\mathbb{T}$, such that $\nu^{\tau}_{G_k,n,x} \xrightarrow{w} \nu$ as $k \to \infty$. Let G^v_k be the induced subgraph of G_k satisfying $V(G^v_k) = V(G_k) \setminus \{v\}$. Then, $\sigma(\bigcup_{k\geq 1} \mathcal{F}_{G^v_k}) = \mathcal{T}_v$. Now define the closed discrete-time martingale

$$M_k := \nu[\,\pi_v \mid \mathcal{F}_{G^v_k}], \qquad k \in \mathbb{N}.$$

Since the random variable π_v is bounded, it follows that [Bil17, Theorem 35.6.] the martingale $(M_k)_{k\geq 1}$ converges almost surely and in L^1 to a limiting random variable $M \in L^1(\{-1,1\}^{V(\mathbb{T})}, \mathcal{T}_v, \nu)$. Moreover the reader can convince herself that the limit is given by

$$\lim_{k\to\infty} M_k = M := \nu[\,\pi_v \mid \mathcal{T}_v].$$

We thus have

$$\begin{aligned}
\nu[\,\pi_v \mid \mathcal{T}_v] &= \lim_{k\to\infty} \nu[\,\pi_v \mid \mathcal{F}_{G^v_k}] \\
&= \lim_{k\to\infty} \lim_{j\to\infty} \nu^{+}_{G_j,n,x}[\,\pi_v \mid \mathcal{F}_{G^v_k}] \\
&= \lim_{k\to\infty} \lim_{j\to\infty} \nu^{\cdot}_{G_j \setminus G^v_k,n,x}[\,\pi_v]
\end{aligned}$$

where we used that the random variable π_v is continuous in the second line and Proposition 2.10 in the last line. The result now follows from Lemma 2.20. □

Recall the *extremal infinite-volume* measures $\nu^{\pm}_{n,x} \in \mathcal{W}_{n,x}$ defined in Proposition 2.17.

Properties of the extremal infinite-volume measures So far we have shown that any thermodynamic limit of the spin representation model is positively associated and satisfies the finite-energy property. We will now establish further properties satisfied by the extremal infinite-volume measures $\nu^+_{n,x}$ and $\nu^-_{n,x}$. The first such property is *tail-triviality.* Informally, this property states that *tail-events*, that is, events which can only be decided by looking at the whole lattice, must be trivial. That is, tail-events must have probability 0 or 1. Such events include for example the existence of infinite spin clusters.

Definition. Consider an increasing sequence $(G_k)_{k\geq 1}$ of finite subgraphs of $\mathbb{T}$ such that $\bigcup_{k\geq 1} G_k = \mathbb{T}$. We define the *tail σ-algebra* $\mathcal{T}$ on $\{-1,1\}^{V(\mathbb{T})}$ as the intersection

$$\mathcal{T} \coloneqq \bigcap_{k\geq 1} \mathcal{T}_{G_k}.$$

A measure ν on $(\{-1,1\}^{V(\mathbb{T})}, \mathcal{F})$ is said to be *tail-trivial* if and only if for any tail event $A \in \mathcal{T}$ we have

$$\nu(A) \in \{0,1\}.$$

Remark 2. The reader can easily convince herself that the above definition does not depend on the choice of the sequence $(G_k)_{k\geq 1}$.

Proposition 2.21 (Tail triviality)**.** *Consider parameters $n, x \in \mathbb{R}$ such that $n \geq 1$ and $nx^2 \leq 1$. The extremal infinite-volume measures $\nu^+_{n,x}$ and $\nu^-_{n,x}$ are tail-trivial.*

We present an adapted proof of the corresponding statement for the random-cluster model [Gri03, Theorem 4.19].

Proof. Consider a finite induced subgraph $H \subseteq \mathbb{T}$ and an increasing sequence $(G_k)_{k\geq 1}$ of finite subgraphs of $\mathbb{T}$ such that $\bigcup_{k\geq 1} G_k = \mathbb{T}$. Let $A \in \mathcal{F}_H$ be increasing and pick $k \in \mathbb{N}$ large enough such that $H \subseteq G_k$. Then for any $B \in \mathcal{T}_H$ depending on finitely many edges we have

$$\nu^-_{G_k,n,x}(A \cap B) = \nu^-_{G_k,n,x}(A \mid B)\, \nu^-_{G_k,n,x}(B) \geq \nu^-_{H,n,x}(A)\, \nu^-_{G_k,n,x}(B)$$

by Corollary 2.15. Now since $\mathbb{1}_A, \mathbb{1}_B$ are continuous, we get

$$\begin{aligned} \nu^-_{n,x}(A \cap B) &= \lim_{k\to\infty} \nu^-_{G_k,n,x}(A \cap B) \\ &\geq \lim_{k\to\infty} \nu^-_{H,n,x}(A)\, \nu^-_{G_k,n,x}(B) \\ &= \nu^-_{H,n,x}(A)\, \nu^-_{n,x}(B). \end{aligned}$$

Since this inequality holds for any $B \in \mathcal{T}_H$ depending on finitely many edges it holds for any $B \in \mathcal{T}_H$ and hence for $B \in \mathcal{T}$. Now, passing to the limit $H \nearrow \mathbb{T}$ we obtain

$$\nu^-_{n,x}(A \cap B) \geq \nu^-_{n,x}(A)\, \nu^-_{n,x}(B)$$

for any $A \in \mathcal{F}, B \in \mathcal{T}$. Now since $B \in \mathcal{T}$ implies $B^c \in \mathcal{T}$ we also get

$$\nu^-_{n,x}(A \cap B^c) \geq \nu^-_{n,x}(A)\, \nu^-_{n,x}(B^c).$$

Summing the last two inequalities yields the tautology

$$\nu^-_{n,x}(A) \geq \nu^-_{n,x}(A),$$

which implies that

$$\nu^-_{n,x}(A \cap B) = \nu^-_{n,x}(A)\, \nu^-_{n,x}(B)$$

for any $A \in \mathcal{F}, B \in \mathcal{T}$. Finally since $\mathcal{T} \subseteq \mathcal{F}$ we see that

$$\nu^-_{n,x}(B) = \nu^-_{n,x}(B \cap B) = \nu^-_{n,x}(B)\, \nu^-_{n,x}(B) = \nu^-_{n,x}(B)^2$$

for any $B \in \mathcal{T}$ and hence $\nu^-_{n,x}(B) \in \{0,1\}$ as desired. ☐

Another property related to tail triviality is *ergodicity*. A measure is called ergodic, if any translation invariant event is trivial.

Definition (Translation invariance, ergodicity)**.** Let $\sigma \in \{-1,1\}^{V(\mathbb{T})}$, $A \in \mathcal{F}$ and $u \in V(\mathbb{T})$. Define the spin configuration $\tau_u\sigma$, $\tau_u^{-1}\sigma \in \{-1,1\}^{V(\mathbb{T})}$ as follows. For $v \in V(\mathbb{T})$, set

$$\tau_u\sigma(v) := \sigma(v-u), \qquad \tau_u^{-1}\sigma(v) := \sigma(v+u).$$

Similarly, define the event

$$\tau_u A := \{\sigma \in \{-1,1\}^{V(\mathbb{T})} \mid \tau_u^{-1}\sigma \in A\}.$$

The event $A \in \mathcal{F}$ is *invariant under translations* if for any $u \in V(\mathbb{T})$, we have

$$\tau_u A = A.$$

Let μ be a measure on $(\{-1,1\}^{V(\mathbb{T})}, \mathcal{F})$. The measure μ is *invariant under translations*, if for all $u \in V(\mathbb{T})$ and all $A \in \mathcal{F}$, we have

$$\mu(\tau_u A) = \mu(A).$$

The measure μ is called *ergodic*, if for any translation invariant event $A \in \mathcal{F}$, we have

$$\mu(A) \in \{0,1\}.$$

We will now show that the extremal infinite volume measures are translation invariant and ergodic. The proofs given are adapted from corresponding statement [Dum17a, Lemma 1.11] for the random-cluster model.

Proposition 2.22 (Translation invariance)**.** *Consider parameters $n, x \in \mathbb{R}$ such that $n \geq 1$ and $nx^2 \leq 1$. The measures $\nu^+_{n,x}, \nu^-_{n,x}$ are invariant under translations.*

Proof. We verify the statement for the measure $\nu^+_{n,x}$, the argument for $\nu^-_{n,x}$ is analogous. Let $A \in \mathcal{F}$ be an increasing event and $u \in V(\mathbb{T})$. Choose $k \in \mathbb{N}$, such that $u \in \Lambda_k$. Then, for any integer $i > k$, we have $\Lambda_{i-k} \subseteq \tau_u\Lambda_i \subseteq \Lambda_{i+k}$ and thus

$$\nu^+_{\Lambda_{i-k},n,x}(\tau_u A) \leq \nu^+_{\tau_u\Lambda_i,n,x}(\tau_u A) \leq \nu^+_{\Lambda_{i+k},n,x}(\tau_u A),$$

by the comparison between boundary conditions (Proposition 2.14). Since $\nu^+_{\tau_u\Lambda_i,n,x}(\tau_u A) = \nu^+_{\Lambda_i,n,x}(A)$, Proposition 2.17 implies that

$$\nu^+_{n,x}(A) = \lim_{i\to\infty} \nu^+_{\Lambda_i,n,x}(\tau_u A) = \lim_{i\to\infty} \nu^+_{\tau_u\Lambda_i,n,x}(\tau_u A) = \nu^+_{n,x}(\tau_u A).$$

Now, since the set of increasing events span the σ-algebra $\mathcal{F}$, it follows – by Dynkin's Lemma [Dur19, Theorem A.1.4.] – that the above equality indeed holds for any event $A \in \mathcal{F}$. □

Proposition 2.23 (Ergodicity)**.** *Consider parameters $n, x \in \mathbb{R}$ such that $n \geq 1$ and $nx^2 \leq 1$. The measures $\nu^+_{n,x}, \nu^-_{n,x}$ are ergodic.*

The upcoming proof of 2.23 is based on the so-called *mixing property*. This property states that any two events which depend on two finite sets of edges far apart from each other, are approximately independent. In Section 2.2.3 we present an alternative proof (see Proposition 2.31) of ergodicity, which only uses the already established translation invariance and tail triviality of the model.

Lemma 2.24 (Mixing property)**.** *Let $A, B \in \mathcal{F}$ be events depending on finitely many vertices. Then,*

$$\lim_{\|u\|\to\infty} \nu^\pm_{n,x}(A \cap \tau_u B) = \nu^\pm_{n,x}(A)\nu^\pm_{n,x}(B).$$

Proof of Lemma 2.24. We verify the mixing property for the measure $\nu^+_{n,x}$, the argument for $\nu^-_{n,x}$ is similar. We begin by showing the statement for increasing events $A, B \in \mathcal{F}$ depending on finitely many vertices.

Using the FKG inequality (Proposition 2.18) and translation invariance (Proposition 2.22) we get

$$\nu^+_{n,x}(A \cap \tau_u B) \geq \nu^+_{n,x}(A)\nu^+_{n,x}(\tau_u B) = \nu^+_{n,x}(A)\nu^+_{n,x}(B).$$

For the other direction, choose $k \in \mathbb{N}$ large enough, such that A and B depend only on vertices in Λ_k. Now, for any $N \geq 2k$, we can choose $u \in V(\mathbb{T})$ far enough from the origin, such that $\Lambda_N \cap \tau_u \Lambda_N = \emptyset$. Then, using Corollary 2.15, we get

$$\begin{aligned} \nu^+_{n,x}(A \cap \tau_u B) &= \nu^+_{n,x}(A \mid \tau_u B)\nu^+_{n,x}(\tau_u B) \\ &\leq \nu^+_{\Lambda_N,n,x}(A)\nu^+_{\tau_u\Lambda_N,n,x}(\tau_u B) \\ &= \nu^+_{\Lambda_N,n,x}(A)\nu^+_{\Lambda_N,n,x}(B). \end{aligned}$$

Taking the limit $\|u\| \to \infty$, we obtain the result for increasing events $A, B \in \mathcal{F}$.

Consider now the general case, $A, B \in \mathcal{F}$ depending on finitely many vertices. The reader can convince herself, that there are increasing events $(A^1_i)^N_{i=1}, (A^2_i)^N_{i=1}, (B^1_j)^M_{j=1}, (B^2_j)^M_{j=1}$ depending on finitely many vertices, such that

$$A^2_i \subseteq A^2_i, \qquad B^2_j \subseteq B^2_j, \qquad \forall 1 \leq i \leq N,\, 1 \leq j \leq M,$$

and

$$A = \bigsqcup_{i=1}^{N} A^1_i \setminus A^2_i, \qquad B = \bigsqcup_{j=1}^{M} B^1_j \setminus B^2_j.$$

Hence, we obtain

$$\begin{aligned} \lim_{\|u\|\to\infty} \nu^+_{n,x}(A \cap \tau_u B) &= \lim_{\|u\|\to\infty} \sum_{i=1}^{N}\sum_{j=1}^{M} \nu^+_{n,x}((A^1_i \setminus A^2_i) \cap (\tau_u B^1_j \setminus \tau_u B^2_j)) \\ &= \lim_{\|u\|\to\infty} \sum_{i=1}^{N}\sum_{j=1}^{M} \nu^+_{n,x}(A^1_i \cap \tau_u B^1_j) + \nu^+_{n,x}(A^2_i \cap \tau_u B^2_j) \\ &\qquad\qquad - \nu^+_{n,x}(A^1_i \cap \tau_u B^2_j) - \nu^+_{n,x}(A^2_i \cap \tau_u B^1_j) \\ &= \sum_{i=1}^{N}\sum_{j=1}^{M} \nu^+_{n,x}(A^1_i)\nu^+_{n,x}(B^1_j) + \nu^+_{n,x}(A^2_i)\nu^+_{n,x}(B^2_j) \\ &\qquad\qquad - \nu^+_{n,x}(A^1_i)\nu^+_{n,x}(B^2_j) - \nu^+_{n,x}(A^2_i)\nu^+_{n,x}(B^1_j) \\ &= \sum_{i=1}^{N}\sum_{j=1}^{M} \nu^+_{n,x}(A^1_i \setminus A^2_i)\nu^+_{n,x}(B^1_j \setminus B^2_j) \\ &= \nu^+_{n,x}(A)\nu^+_{n,x}(B), \end{aligned}$$

and the result follows. □

Proof of Proposition 2.23. The ergodicity of $\nu^+_{n,x}$ follows from the mixing property. Indeed, let $A \in \mathcal{F}$ be an translation invariant event. By a classic approximation result from measure-theory [Hal13, Theorem D.][9], we know that for any $\epsilon > 0$, there is a $A' \in \mathcal{F}$ depending on finitely many edges, such that $\nu^+_{n,x}(A \triangle A') < \epsilon$. Moreover, by Lemma 2.24, we can choose $u \in V(\mathbb{T})$, such that

$$|\nu^+_{n,x}(A')^2 - \nu^+_{n,x}(A' \cap \tau_u A')| < \epsilon.$$

[9]This result can be proven using Dynkin's Lemma [Dur19, Theorem A.1.4.].

We thus obtain

$$\begin{aligned}|\nu_{n,x}^+(A)^2 - \nu_{n,x}^+(A)| &\leq |\nu_{n,x}^+(A)^2 - \nu_{n,x}^+(A')^2| + |\nu_{n,x}^+(A')^2 - \nu_{n,x}^+(A' \cap \tau_u A')| \\ &\quad + |\nu_{n,x}^+(A' \cap \tau_u A') - \nu_{n,x}^+(A)| \\ &\leq (\nu_{n,x}^+(A) + \nu_{n,x}^+(A'))|\nu_{n,x}^+(A) - \nu_{n,x}^+(A')| + \epsilon \\ &\quad + \nu_{n,x}^+((A' \cap \tau_u A') \triangle A) \\ &\leq 2\nu_{n,x}^+(A \triangle A') + \epsilon + \nu_{n,x}^+((A' \triangle A) \cup (A \triangle \tau_u A)) \\ &\leq 3\epsilon + \nu_{n,x}^+(A' \triangle A) + \nu_{n,x}^+(\tau_u A' \triangle \tau_u A) \\ &\leq 5\epsilon,\end{aligned}$$

where the last inequality is due to the translation invariance of $\nu_{n,x}^+$. Since $\epsilon > 0$ was arbitrary, it follows that $\nu_{n,x}^+(A) = \nu_{n,x}^+(A)^2$ and thus $\nu_{n,x}^+(A) \in \{0, 1\}$ as desired.

$\square$

Now using the previously established properties, we show that the extremal infinite-volume measures satisfy the *0/1-infinite-cluster property*, which states that there can be at most one infinite spin-cluster of each sign.

Definition (0/1-infinite-cluster property). An infinite-volume measure ν on $(\{-1,1\}^{V(\mathbb{T})}, \mathcal{F})$ is said to satisfy the *0/1-infinite-cluster property* if

$$\nu(\text{there is at most one infinite spin-cluster of each sign}) = 1.$$

Theorem 2.25 (0/1-infinite-cluster property). *Let ν be measure on $(\{-1,1\}^{V(\mathbb{T})}, \mathcal{F})$. If ν is translation invariant, ergodic and has the finite-energy property, then ν satisfies the 0/1-infinite-cluster property. In particular, for parameters $n, x \in \mathbb{R}$ such that $n \geq 1$ and $nx^2 \leq 1$, the measures $\nu_{n,x}^+, \nu_{n,x}^-$ satisfy the 0/1-infinite-cluster property.*

We only sketch the proof, as the argument is best illustrated graphically. The proof for the case of percolation [Dum18, Theorem 3.1] can easily be modified to our setting.

Sketch of proof. By ergodicity we know that for both signs the number of infinite spin-clusters is almost surely constant. Using translation invariance and the finite-energy property, the so-called *Burton-Keane argument*[10] [BK89] allows us to exclude the existence of infinitely many such clusters. Now, assume that there is more than one infinite spin cluster of one sign. In that case, using continuity of measure [Dur19, Theorem 1.1.1.], we can pick a finite domain of $\mathbb{T}$ such that probability of having at least two infinite clusters intersecting this domain is positive. Then, the finite-energy property implies that the probability of having one infinite cluster less is positive, since we only need to modify the finite number of spins in the domain to connect two of the infinite spin clusters, thus reducing the number of infinite clusters by one. This however is a contradiction, since the number of infinite spin-clusters is almost surely constant. $\square$

Remark 3. The Burton-Keane argument only uses the translation invariance and finite-energy property of the measure. That is, the existence of infinitely many clusters can be excluded even if the measure is non-ergodic.

We proceed to show a stronger version of the 0/1-infinite-cluster property. Indeed, using *Zhang's argument*[11] we will now show that infinite clusters of opposite sign cannot coexist. In particular, there is at most one infinite cluster. The argument was first published in [Gri99, Lemma 11.12], where it is used to show that critical percolation threshold on $\mathbb{Z}^2$ is at least $\frac{1}{2}$.

[10] Named after the American mathematicians Robert Burton and Micheal Keane, who in 1989 presented an elegant argument, showing that there is at most one infinite percolation cluster.

[11] Named after the mathematician Y. Zhang who first presented the argument in 1988 in an unpublished work.

Proposition 2.26. *Let ν be measure on $(\{-1,1\}^{V(\mathbb{T})}, \mathcal{F})$. If ν satisfies the 0/1-infinite-cluster property and the FKG inequality, then,*

$$\nu(\textit{there is at most one infinite spin-cluster}) = 1.$$

In particular, for parameters $n, x \in \mathbb{R}$ such that $n \geq 1$ and $nx^2 \leq 1$, we have

$$\nu^{\pm}_{n,x}(\textit{there is at most one infinite spin-cluster}) = 1.$$

Proof. By the 0/1-infinite-cluster property, we know that, ν-almost-surely, there is at most one infinite spin-cluster for each sign. Assume by contradiction, that there is an infinite spin-cluster for each sign. Then, by continuity of measure [Dur19, Theorem 1.1.1.], for any $\epsilon > 0$ we can choose $N \in \mathbb{N}$ sufficiently large such that

$$\nu(\text{the two infinite clusters intersect } \Lambda_N) \geq 1 - \epsilon.$$

Now, consider the six sides of the vertex boundary $\partial\Lambda_N \subseteq V(\Lambda_N)$. Formally, define the upper side of the domain Λ_N as the set

$$\partial\Lambda_N^1 \coloneqq \{v \in \partial\Lambda_N \mid \mathrm{Im}(v) = \frac{\sqrt{3}}{2}N\},$$

and define the remaining sides of the hexagonal domain as

$$\partial\Lambda_N^k \coloneqq e^{i\frac{\pi}{3}} \cdot \partial\Lambda_N^{k-1}, \qquad k \in \{2,3,4,5,6\},$$

where to above product is to be understood pointwise. Then, by symmetry, we have for all $j, k \in \{1, \dots, 6\}$, that

$$\nu(\partial\Lambda_N^k \xleftrightarrow{+} \infty) = \nu(\partial\Lambda_N^j \xleftrightarrow{+} \infty),$$

Now, using the FKG inequality, we obtain

$$(1 - \nu(\partial\Lambda_N^1 \xleftrightarrow{+} \infty))^6 \leq \nu(\bigcap_{i=1}^{6}\{\partial\Lambda_N^i \xleftrightarrow{+} \infty\}^c) = 1 - \nu(\bigcup_{i=1}^{6}\{\partial\Lambda_N^i \xleftrightarrow{+} \infty\}),$$

and hence

$$\nu(\partial\Lambda_N^1 \xleftrightarrow{+} \infty) \geq 1 - (1 - \nu(\bigcup_{i=1}^{6}\{\partial\Lambda_N^i \xleftrightarrow{+} \infty\}))^{\frac{1}{6}}.$$

This is known as the *square root trick.* Now, since

$$\bigcup_{i=1}^{6}\{\partial\Lambda_N^i \xleftrightarrow{+} \infty\} = \{\partial\Lambda_N \xleftrightarrow{+} \infty\},$$

it follows that for sufficiently large $N \in \mathbb{N}$, we have

$$\nu(\partial\Lambda_N^k \xleftrightarrow{+} \infty) \geq 1 - \epsilon, \qquad \forall k \in \{1, \dots, 6\}.$$

Similar calculations show, that for sufficiently large $N \in \mathbb{N}$, we have

$$\nu(\partial\Lambda_N^k \xleftrightarrow{-} \infty) \geq 1 - \epsilon, \qquad \forall k \in \{1, \dots, 6\}.$$

Now, consider the events

$$A \coloneqq \{\partial\Lambda_N^1 \xleftrightarrow{+} \infty\} \cap \{\partial\Lambda_N^4 \xleftrightarrow{+} \infty\} \cap \{\partial\Lambda_N^2 \xleftrightarrow{-} \infty\} \cap \{\partial\Lambda_N^5 \xleftrightarrow{-} \infty\},$$

$$B \coloneqq \{\sigma \in \{-1,1\}^{V(\mathbb{T})} \mid \sigma(v) = 1,\ \forall v \in V(\Lambda_N)\}.$$

Using the union bound, we get

$$\nu(A) = 1 - \nu(A^c) \geq 1 - 4\epsilon > 0.$$

Hence, by the finite energy property, we obtain

$$\nu(A \cap B) > 0.$$

This however is a contradiction to the 0/1-infinite-cluster property, since

$$A \cap B \subseteq \{\text{there are two distinct infinite } -1 \text{ spin clusters}\}.$$

□

2.2.3 Gibbs measures of the spin representation model

As for the loop $O(n)$ model, we can define the notion of *Gibbs measures* for the spin representation model.

Definition (Gibbs measure for the spin representation model)**.** Consider parameters $n, x > 0$. A measure $\nu_{n,x}$ on $(\{-1,1\}^{V(\mathbb{T})}, \mathcal{F})$ is called *Gibbs measure* for the spin representation of the loop $O(n)$ model with edge-weight x, if for any finite subgraph $G \subseteq \mathbb{T}$, for $\nu_{n,x}$-almost all loop configurations $\tau \in \{-1,1\}^{V(\mathbb{T})}$ and any event $A \in \mathcal{F}$ we have

$$\nu_{n,x}[\mathbb{1}_A \mid \mathcal{T}_G](\tau) = \nu^{\tau}_{G,n,x}(A).$$

We write $\mathcal{G}_{n,x}$ for the set of such Gibbs measures.

Using the results established in the preceding section, we can now show that the two extremal infinite-volume measures are in fact Gibbs measures of the spin representation model.

Proposition 2.27. *Consider parameters $n, x \in \mathbb{R}$ such that $n \geq 1$ and $nx^2 \leq 1$. Then, for any $\nu \in \mathcal{W}_{n,x}$ satisfying the 0/1-infinite-cluster property, we have $\nu \in \mathcal{G}_{n,x}$. In particular, the extremal infinite-volume measures $\nu^+_{n,x}$, $\nu^-_{n,x}$ are Gibbs measures.*

Proof. Pick some $A \in \mathcal{F}$. By the definition of conditional expectation we need to show that for any event $B \in \mathcal{T}_G$,

$$\nu[\mathbb{1}_B(\cdot)\nu^{\cdot}_{G,n,x}(A)] = \nu[\mathbb{1}_B(\cdot)\mathbb{1}_A(\cdot)].$$

Recall that $\mathcal{F}$ is the σ-algebra corresponding to the topology introduced in Proposition 2.16. We begin by showing the following claim.

Claim 2.27.1. *Let $G \subseteq \mathbb{T}$ be a domain and $A \in \mathcal{F}$ be an event depending on finitely many vertices. Then, the random variable $h : \{-1,1\}^{V(\mathbb{T})} \to \mathbb{R}$, defined by*

$$h(\tau) := \nu^{\tau}_{G,n,x}(A)$$

is ν-almost-surely continuous.

Proof of Claim 2.27.1. Denote by $D_h \subseteq \{-1,1\}^{V(\mathbb{T})}$ the set of discontinuities of the random variable h. Observe that $\tau \in D_h$ implies that there are arbitrarily large domains $H \subseteq \mathbb{T}$, such that there is a configuration $\tau' \in \{-1,1\}^{V(\mathbb{T})}$ which satisfies $\tau'|_H = \tau|_H$ and

$$\nu^{\tau'}_{G,n,x}(A) \neq \nu^{\tau}_{G,n,x}(A).$$

The reader can convince herself that this again implies that in the restricted configuration $\tau|_{V(G)^c}$ there are at least two distinct infinite spin clusters of the same sign. Now, assume that $\nu(D_h) > 0$. Then, the finite energy property (Proposition 2.19) implies that

$$\nu(\text{There are at least two infinite clusters of the same sign}) > 0,$$

since we can modify the finite number of spins in $\tau|_{V(G)}$ to ensure that the two infinite clusters in $\tau|_{V(G)^c}$ are distinct. This however is a contradiction to the *0/1-infinite-cluster property* of ν. Hence $\nu(D_h) = 0$ and h is ν-almost-surely continuous. □

We now show that this claim implies the result. Since $\nu \in \mathcal{W}_{n,x}$, there is a spin configuration $\tau \in \{-1,1\}^{V(\mathbb{T})}$ and a sequence of domains $(G_k)_{k\geq 1}$ exhausting $\mathbb{T}$, such that $\nu^{\tau}_{G_k,n,x} \xrightarrow{w} \nu$ as $k \to \infty$. Hence, by the above claim, for any $A \in \mathcal{F}$ depending on finitely many vertices, we have

$$\nu[\mathbb{1}_B(\cdot)\nu^{\cdot}_{G,n,x}(A)] = \lim_{k\to\infty} \nu^{\tau}_{G_k,n,x}[\mathbb{1}_B(\cdot)\nu^{\cdot}_{G,n,x}(A)].$$

Now, since $G_k \nearrow \mathbb{T}$, the spatial Markov property (Corollary 2.11) implies that for large enough $k \in \mathbb{N}$, we have

$$\nu^{\cdot}_{G,n,x}(A) = \nu^{\tau}_{G_k,n,x}[\mathbb{1}_A \mid \mathcal{T}_G](\cdot).$$

Thus, using the above equality and the tower property of conditional expectation we obtain

$$\begin{aligned}
\nu[\mathbb{1}_B(\cdot)\nu^{\cdot}_{G,n,x}(A)] &= \lim_{k\to\infty} \nu^{\tau}_{G_k,n,x}[\mathbb{1}_B(\cdot)\nu^{\cdot}_{G,n,x}(A)] \\
&= \lim_{k\to\infty} \nu^{\tau}_{G_k,n,x}[\mathbb{1}_B(\cdot)\nu^{\tau}_{G_k,n,x}[\mathbb{1}_A \mid \mathcal{T}_G](\cdot)] \\
&= \lim_{k\to\infty} \nu^{\tau}_{G_k,n,x}[\mathbb{1}_B(\cdot)\mathbb{1}_A(\cdot)] \\
&= \nu[\mathbb{1}_B(\cdot)\mathbb{1}_A(\cdot)],
\end{aligned}$$

and hence

$$\nu[\mathbb{1}_A \mid \mathcal{T}_G](\cdot) = \nu^{\cdot}_{G,n,x}(A)$$

for any $A \in \mathcal{F}$ depending on finitely many vertices. Finally, since the $\mathcal{F}$ is generated by events depending on finitely many vertices, we get the corresponding statement for all $A \in \mathcal{F}$. $\square$

We will now work towards proving the following result.

Theorem 2.28. *Consider parameters $n, x \in \mathbb{R}$ such that $n \geq 1$ and $nx^2 \leq 1$. Then, $\nu^+_{n,x}, \nu^-_{n,x} \in \mathcal{G}_{n,x}$. Moreover for any translation invariant Gibbs measure $\nu \in \mathcal{G}_{n,x}$, there is a constant $c \in [0,1]$, such that*

$$\nu = c\,\nu^+_{n,x} + (1-c)\,\nu^-_{n,x}.$$

The upcoming proof of 2.28 is based on an argument by Georgii and Higuchi [GH00]. Our proof follows [DGPS20], in which the authors outline how the Georgii-Higuchi argument can be applied to the loop $O(n)$ setting. Before giving the proof, we will need to establish several results regarding Gibbs measures of the spin representation.

Proposition 2.29. *Consider parameters $n, x > 0$. Then, any $\nu_{n,x} \in \mathcal{G}_{n,x}$ has the finite-energy property.*

Proof. Consider the graph $G_v = (\{v\}, \emptyset)$. Since $\nu_{n,x}$ is Gibbs, we have

$$\begin{aligned}
\nu_{n,x}[\pi_v \mid \mathcal{T}_v] &= \nu_{n,x}[\mathbb{1}_{\pi(v)=1} \mid \mathcal{T}_v] - \nu_{n,x}[\mathbb{1}_{\pi(v)=-1} \mid \mathcal{T}_v] \\
&= \nu^{\cdot}_{G_v,n,x}(\{\pi(v)=1\}) - \nu^{\cdot}_{G_v,n,x}(\{\pi(v)=-1\}) \\
&= \nu^{\cdot}_{G_v,n,x}[\pi_v].
\end{aligned}$$

Hence the claim follows from Lemma 2.20. $\square$

Proposition 2.30. *Consider parameters $n, x \in \mathbb{R}$ such that $n \geq 1$ and $nx^2 \leq 1$. Then, any tail trivial $\nu_{n,x} \in \mathcal{G}_{n,x}$ satisfies the FKG inequality.*

Proof. Let $A, B \in \mathcal{F}$ be increasing events and $k \in \mathbb{N}$. Since $\nu_{n,x}$ is Gibbs, we have

$$\begin{aligned}
\nu_{n,x}(A \cap B) &= \nu_{n,x}[\nu_{n,x}[\mathbb{1}_A\mathbb{1}_B \mid \mathcal{T}_{\Lambda_k}]] \\
&= \nu_{n,x}[\nu^{\cdot}_{\Lambda_k,n,x}(A \cap B)] \\
&\geq \nu_{n,x}[\nu^{\cdot}_{\Lambda_k,n,x}(A)\nu^{\cdot}_{\Lambda_k,n,x}(B)] \\
&= \nu_{n,x}\left[\nu_{n,x}[\mathbb{1}_A \mid \mathcal{T}_{\Lambda_k}]\,\nu_{n,x}[\mathbb{1}_B \mid \mathcal{T}_{\Lambda_k}]\right],
\end{aligned}$$

where we used the FKG inequality (Proposition 2.12) in line three. Since $k \in \mathbb{N}$ was arbitrary, we can pass to the limit $k \to \infty$ and we get

$$\begin{aligned} \nu_{n,x}(A \cap B) &\geq \lim_{k\to\infty} \nu_{n,x}\left[\nu_{n,x}[\mathbb{1}_A \mid \mathcal{T}_{\Lambda_k}]\, \nu_{n,x}[\mathbb{1}_B \mid \mathcal{T}_{\Lambda_k}]\right] \\ &= \nu_{n,x}\left[\nu_{n,x}[\mathbb{1}_A \mid \mathcal{T}]\, \nu_{n,x}[\mathbb{1}_B \mid \mathcal{T}]\right], \end{aligned}$$

using dominated convergence [Bil17, Theorem 16.4.] and Doob's martingale convergence theorem [Bil17, Theorem 35.5.]. Now since $\nu_{n,x}$ is tail trivial we have

$$\nu_{n,x}[\mathbb{1}_A \mid \mathcal{T}] = \nu_{n,x}[\mathbb{1}_A], \qquad \nu_{n,x}[\mathbb{1}_B \mid \mathcal{T}] = \nu_{n,x}[\mathbb{1}_B],$$

and the result follows. □

Proposition 2.31. *Let ν be a probability measure on $(\{-1,1\}^{V(\mathbb{T})}, \mathcal{F})$. If ν is tail trivial and translation invariant, then ν is ergodic.*

Proof. For any translation invariant event $T \in \mathcal{F}$ we can choose a sequence of events $(B_k)_{k\geq 1}$, such that for any $k \geq 1$ we have, $B_k \in \mathcal{F}_{\Lambda_k}$ and

$$\nu(T \triangle B_k) < 2^{-k}.$$

Moreover for any $k \geq 1$, choose $u = u(k) \in V(\mathbb{T})$, such that $\Lambda_k \cap \tau_u \Lambda_k = \emptyset$ and define $\tilde{B}_k \coloneqq \tau_{u(k)} B_k$. Now, one can easily check, that $\tilde{B}_k \in \mathcal{T}_{\Lambda_k}$ and that $B \coloneqq \bigcap_{N\geq 1} \bigcup_{k\geq N} \tilde{B}_k \in \mathcal{T}$ is a tail event. Using the translation invariance of T and ν, it follows, that

$$\begin{aligned} \nu_{n,x}(T \triangle B) &\leq \nu\left(\bigcap_{N\geq 1}\bigcup_{k\geq N} T\triangle \tilde{B}_k\right) \leq \lim_{N\to\infty} \nu\left(\bigcup_{k\geq N} T\triangle\tilde{B}_k\right) \\ &\leq \lim_{N\to\infty}\sum_{k\geq N} \nu(T\triangle\tilde{B}_k) = \lim_{N\to\infty}\sum_{k\geq N}\nu(\tau_{u(k)}T\triangle\tau_{u(k)}B_k) \\ &= \lim_{N\to\infty}\sum_{k\geq N}\nu(T\triangle B_k) \leq \lim_{N\to\infty}\sum_{k\geq N} 2^{-k} = 0. \end{aligned}$$

Finally, since $B \in \mathcal{T}$, we have $\nu(B) \in \{0,1\}$ and hence $\nu(T) \in \{0,1\}$. Thus, if ν is extremal and translation invariant, then ν is ergodic. □

Remark 4. The assumption of translation invariance in Proposition 2.31 can be weakened. In fact, the above proof can easily modified to show that Proposition 2.31 also holds for periodic measures.

Definition. Consider parameters $n, x \geq 0$ and let $\nu_{n,x} \in \mathcal{G}_{n,x}$ be a Gibbs measure. The measure $\nu_{n,x}$ is called *extremal*, if for any $c \in (0,1)$ and any Gibbs measures $\nu^1_{n,x}, \nu^2_{n,x} \in \mathcal{G}_{n,x}$ satisfying

$$\nu_{n,x} = c\,\nu^1_{n,x} + (1-c)\,\nu^2_{n,x},$$

we have $\nu_{n,x} = \nu^1_{n,x} = \nu^2_{n,x}$. We write $\mathsf{ex}\mathcal{G}_{n,x} \subseteq \mathcal{G}_{n,x}$ for the set of extremal Gibbs measures.

Proposition 2.32. *Consider parameters $n, x \geq 0$. Then, any $\nu_{n,x} \in \mathsf{ex}\mathcal{G}_{n,x}$ is tail trivial. In particular, any translation invariant $\nu_{n,x} \in \mathsf{ex}\mathcal{G}_{n,x}$ is ergodic.*

Proof. We show that if $\nu_{n,x}$ is not tail trivial, then $\nu_{n,x}$ is not extremal. Indeed, assume that $\nu_{n,x}$ is not tail trivial. Then there exists a tail event $T \in \mathcal{T}$, such that $0 < \nu_{n,x}(T) < 1$ and we can define the measures

$$\nu^1_{n,x} \coloneqq \nu_{n,x}(\,\cdot \mid T), \qquad \nu^2_{n,x} \coloneqq \nu_{n,x}(\,\cdot \mid T^c).$$

Setting $c \coloneqq \frac{1}{\nu_{n,x}(T)}$, we thus get

$$\nu_{n,x} = c\,\nu^1_{n,x} + (1-c)\,\nu^2_{n,x}.$$

Moreover, for any $B \in \mathcal{T}_G$ and $A \in \mathcal{F}$, we have

$$\begin{aligned}\nu^1_{n,x}[\mathbb{1}_B(\cdot)\nu^{\cdot}_{G,n,x}(A)] &= \nu_{n,x}[\mathbb{1}_T\mathbb{1}_B(\cdot)\nu^{\cdot}_{G,n,x}(A)]\\ &= \nu_{n,x}[\mathbb{1}_{B\cap T}(\cdot)\nu^{\cdot}_{G,n,x}(A)]\\ &= \nu_{n,x}[\mathbb{1}_B(\cdot)\mathbb{1}_T(\cdot)\mathbb{1}_A(\cdot)]\\ &= \nu^1_{n,x}[\mathbb{1}_B(\cdot)\mathbb{1}_A(\cdot)],\end{aligned}$$

since $B \cap T \in \mathcal{T}$ and $\nu_{n,x}$ is Gibbs. It follows that $\nu^1_{n,x}$ is Gibbs and by a similar argument that $\nu^2_{n,x}$ is Gibbs. Hence $\nu_{n,x}$ is a mixture of $\nu^1_{n,x}$ and $\nu^2_{n,x}$. Thus, if $\nu_{n,x}$ is extremal, then $\nu_{n,x}$ is tail trivial. The last statement directly follows from Proposition 2.31. □

Remark 5. Observe, that the above proof of Proposition 2.32 in particular shows, that conditioning a Gibbs measure on a tail event – provided this conditioning is well defined – again yields a Gibbs measure.

Corollary 2.33. *Consider parameters $n, x \in \mathbb{R}$ such that $n \geq 1$ and $nx^2 \leq 1$. Then, for any translation invariant $\nu_{n,x} \in \mathsf{ex}\mathcal{G}_{n,x}$, we have*

$$\nu_{n,x}(\textit{there is at most one infinite spin-cluster}) = 1.$$

Proof. Since $\nu_{n,x} \in \mathcal{G}_{n,x}$ is translation invariant and extremal, Proposition 2.32 implies that $\nu_{n,x}$ ergodic and Proposition 2.29 implies that $\nu_{n,x}$ has the finite-energy property. Hence, the Burton-Keane argument (Theorem 2.25) implies that $\nu_{n,x}$ also satisfies the 0/1-infinite-cluster property. Finally, since $\nu_{n,x}$ moreover satisfies the FKG inequality by Proposition 2.30, Zhang's argument (Proposition 2.26) yields the result. □

Proposition 2.34. *Consider parameters $n \geq 1$ and $x > 0$ such that $nx^2 \leq 1$ and let $\nu_{n,x} \in \mathcal{G}_{n,x}$. If*

$$\nu_{n,x}(0 \xleftrightarrow{+} \infty) = 0,$$

then $\nu_{n,x} = \nu^-_{n,x}$. Similarly, if

$$\nu_{n,x}(0 \xleftrightarrow{-} \infty) = 0,$$

then $\nu_{n,x} = \nu^+_{n,x}$.

Lemma 2.35. *Consider parameters $n \geq 1$ and $x > 0$ such that $nx^2 \leq 1$. Let $H \subseteq G$ be two domains of $\mathbb{T}$ and $A \in \mathcal{F}_H$ be an increasing event. Define the events*

$$C^+ := \{\textit{there is a } +\textit{-circuit in } G \setminus H\},$$

$$C^- := \{\textit{there is a } -\textit{-circuit in } G \setminus H\}.$$

Then, for any Gibbs measure ν of the spin representation of the loop $O(n)$ model with edge-weight x we have

$$\nu(A \mid C^+) \geq \nu^+_{G,n,x}(A),$$

$$\nu(A \mid C^-) \leq \nu^-_{G,n,x}(A),$$

whenever the above conditioning is non-degenerate.

Proof. We show the statement for C^+. The corresponding statement for C^- can be shown analogously. There are finitely many possible *spin circuits* $(\gamma_k)_{k=1}^m$ (minimal connected set of vertices surrounding 0) in $G \setminus H$. We can choose domains $(H_k)_{k=1}^m$ such that for any $k \in \{1, \dots, m\}$ we have $H \subseteq H_k \subseteq G$ and $\gamma_k = \partial H_k$. Thus we obtain the partition

$$C = \bigsqcup_{k=1}^m C_k,$$

where

$$C_k \coloneqq \{\partial H_k \text{ is the outermost } -1 \text{ spin circuit in } G \setminus H\} \in \mathcal{T}_{H_k}.$$

Since ν is Gibbs, it follows that

$$\begin{aligned}\nu(A \cap C) = \sum_{k=1}^{m} \nu(A \cap C_k) = \sum_{k=1}^{m} \nu[\mathbb{1}_{C_k}(\cdot)\nu^{\cdot}_{H_k,n,x}(A)] = \nu^{+}_{H_k,n,x}(A) \sum_{k=1}^{m} \nu[\mathbb{1}_{C_k}(\cdot)] \\ \geq \nu^{+}_{G,n,x}(A) \sum_{k=1}^{m} \nu[\mathbb{1}_{C_k}(\cdot)] = \nu^{+}_{G,n,x}(A)\nu(C),\end{aligned}$$

that is

$$\nu(A \mid C) \geq \nu^{+}_{G,n,x}(A).$$

□

Lemma 2.36. *Consider parameters $n \geq 1$ and $x > 0$ such that $nx^2 \leq 1$. and let $\nu_{n,x}$ be a Gibbs measure of the spin representation of the loop $O(n)$ model with edge-weight x. Then for any increasing event $A \in \mathcal{F}$, we have*

$$\nu^{-}_{n,x}(A) \leq \nu_{n,x}(A) \leq \nu^{+}_{n,x}(A).$$

Proof. The reader can convince herself, that it suffices to show the statement for increasing events depending on finitely many edges. Indeed, let $G \subseteq \mathbb{T}$ be a domain and $A \in \mathcal{F}_G$. Since $\nu_{n,x}$ is Gibbs and $A \in \mathcal{F}_G$ is increasing, for sufficiently large $k \in \mathbb{N}$, we get

$$\nu_{n,x}(A) = \nu_{n,x}[\nu^{\cdot}_{\Lambda_k,n,x}(A)] \geq \nu_{n,x}[\nu^{-}_{\Lambda_k,n,x}(A)] = \nu^{-}_{\Lambda_k,n,x}(A),$$

where the inequality follows from Lemma 2.14. Now, taking the limit $k \to \infty$, we obtain

$$\nu_{n,x}(A) \geq \nu^{-}_{n,x}(A).$$

By a similar argument we obtain

$$\nu_{n,x}(A) \leq \nu^{-}_{n,x}(A),$$

and the result follows. □

Proof of Proposition 2.34. We show the first statement. The second statement can be shown analogously. Let G be a domain of $\mathbb{T}$ and $A \in \mathcal{F}_G$. Since

$$\nu_{n,x}(0 \overset{+}{\longleftrightarrow} \infty) = 0,$$

we have

$$\nu_{n,x}(\text{there is a } -1 \text{ spin circuit surrounding } 0) = 1.$$

and it follows that

$$\nu_{n,x}(A) = \lim_{k\to\infty} \nu(A \mid C_k),$$

where

$$C_k \coloneqq \{ \text{ there is a } -1 \text{ spin circuit in } \Lambda_k \setminus G\}.$$

Now using Lemma 2.35, we get

$$\nu_{n,x}(A) = \lim_{k\to\infty} \nu_{n,x}(A \mid C_k) \leq \lim_{k\to\infty} \nu^{-}_{\Lambda_k,n,x}(A) = \nu^{-}_{n,x}(A).$$

Moreover, by Lemma 2.36, we also have

$$\nu_{n,x}(A) \geq \nu^{-}_{n,x}(A),$$

and hence

$$\nu_{n,x}(A) = \nu^{-}_{n,x}(A).$$

Since G and $A \in \mathcal{F}_G$ were arbitrary, it follows that $\nu_{n,x} = \nu^{-}_{n,x}$ by uniqueness of measure. □

Definition (Half-plane, infinite butterfly)**.** For $k \in \{0, 1, 2\}$, define the three axes

$$A_k \coloneqq \{v \in \mathbb{T} \mid v = j \cdot e^{i\frac{k\cdot\pi}{3}} \text{ for some } j \in \mathbb{N}\}$$

of $\mathbb{T}$, and let π_k be the subgraph of $\mathbb{T}$ induced by the set

$$V(\pi_k) \coloneqq \{v \in \mathbb{T} \mid \exists u \in A_k, \text{ such that } \mathrm{Re}(v) = \mathrm{Re}(u) \text{ and } \mathrm{Im}(v) \geq \mathrm{Im}(u)\}.$$

An induced subgraph $\pi \subseteq \mathbb{T}$ is a called a *half-plane* in $\mathbb{T}$, if there exists a vertex $u \in V(\mathbb{T})$ and a $k \in \{0, 1, 2\}$, such that

$$V(\pi) = u + V(\pi_k).$$

For a half-plane π denote by $R(\pi)$ the half-plane induced by $V(\mathbb{T}) \setminus V(\pi)$.

A spin configuration $\sigma \in \{-1, 1\}^{V(\mathbb{T})}$ contains an *infinite butterfly* if there exist a pair of half-planes π and $R(\pi)$, such that σ contains infinite spin cluster of the same sign in both half-plane.

Lemma 2.37 (Partial Markov property)**.** *Let G be a domain of $\mathbb{T}$, let $A \cup B = \partial G$ be a partition of the vertex boundary of G into two connected sets of vertices and let τ be a spin configuration which is constant on A and B. Then for any spin configuration τ' coinciding with τ on ∂G, we have*

$$\nu^{\tau}_{G,n,x} = \nu^{\tau'}_{G,n,x},$$

for any $n, x > 0$.

Proof. This is straightfoward to verify. $\square$

Lemma 2.38 (Flip-reflection domination)**.** *Consider real parameters $n, x > 0$ such that $n \geq 1$ and $nx^2 \leq 1$. Let $\boldsymbol{\sigma}$ be sampled from a Gibbs measure $\nu \in \mathcal{G}_{n,x}$, let π be a half-plane in $\mathbb{T}$ and let R be the reflection through the boundary of π. Suppose that for every finite $\Delta \subseteq \pi$ there is a finite, connected, R-invariant $G = G(\boldsymbol{\sigma}, \Delta)$ with $\Delta \subseteq G$ such that $\boldsymbol{\sigma} \equiv 1$ on the part of the external vertex boundary of G which is in π. Then, ν stochastically dominates $\nu \circ R \circ T$.*

Proof. Let $\Delta \subseteq \pi$ be a finite domain of $\mathbb{T}$. Then, by assumption, for ν-almost-all $\sigma \in \{-1, 1\}^{V(\mathbb{T})}$, there is a finite, R-invariant domain $\boldsymbol{G}(\sigma) \supseteq \Delta$, such that $\sigma \geq R \circ T(\sigma)$ on $\partial\boldsymbol{G}(\sigma)$. Now, for any $\epsilon > 0$, we can choose a finite domain Λ, such that $\nu(\boldsymbol{G} \subseteq \Lambda) \geq 1 - \epsilon$ by continuity of measure. In the following, let us assume that $\boldsymbol{G}(\sigma)$ is chosen maximal in Λ if such a domain exists in Λ and set $\boldsymbol{G}(\sigma) = \emptyset$ otherwise. Observe, that maximality of $\boldsymbol{G}$ implies, that $\{\boldsymbol{G} = G\} \in \mathcal{T}_G$ for any domain $G \subseteq \Lambda$.

Now, let $G \subseteq \Lambda$ be an R-invariant domain, $A \in \mathcal{F}$ increasing and $\sigma \in \{\boldsymbol{G} = G\}$. Then, we have

$$\nu^{\sigma}_G(A) \geq \nu^{\sigma_-}_G(A) = \nu^{\tau}_G(A) = \nu^{R(T(\sigma))_+}_G(A) \geq \nu^{R(T(\sigma))}_G(A) = \nu^{\sigma}_G(T(R(A))),$$

where σ_- coincides with σ on π and is -1 elsewhere, $R(T(\sigma))_+$ coincides with $R(T(\sigma))$ on $R(\pi)$ and is $+1$ elsewhere, and τ is $+1$ on π and -1 on $R(\pi)$. Indeed, the two inequalities follows from the comparison between boundary conditions, the first two equalities follow from the partial Markov property (Lemma 2.37) and the last equality follows from the R-invariance of G.

Then, since ν is Gibbs and $\{\boldsymbol{G} = G\} \in \mathcal{T}_G$, we obtain

$$\begin{aligned}
\nu(A) \geq \nu(A \cap \{\boldsymbol{G} \neq \emptyset\}) &= \sum_{G \subseteq \Lambda} \nu(A \cap \{\boldsymbol{G} = G\}) = \sum_{G \subseteq \Lambda} \nu[\nu^{\cdot}_G(A) \mathbb{1}_{\{\boldsymbol{G}=G\}}(\cdot)] \\
&\geq \sum_{G \subseteq \Lambda} \nu[\nu^{R(T(\cdot))}_G(A) \mathbb{1}_{\{\boldsymbol{G}=G\}}(\cdot)] = \nu(R(T(A)) \cap \{\boldsymbol{G} \neq \emptyset\}) \geq \nu(R(T(A))) - \epsilon,
\end{aligned}$$

and the result follows, since ϵ was arbitrary. $\square$

Proposition 2.39. *Consider real parameters $n, x > 0$ such that $n \geq 1$ and $nx^2 \leq 1$ and let $\nu \in \mathcal{G}_{n,x}$ be a Gibbs measure. Then, ν-almost-surely there exists at least one infinite butterfly.*

Proof. Suppose the contrary. Then, there exists a Gibbs measure $\nu \in \mathcal{G}_{n,x}$ such that $\nu(B) < 1$, where

$$B := \{\sigma \in \{-1,1\}^{V(\mathbb{T})} \mid \text{there exists at least one infinite butterfly in } \sigma\}.$$

We can assume – without loss of generality – that ν is extremal in $\mathcal{G}_{n,x}$. Indeed, if ν is not extremal, then there are Gibbs measures $\nu_1, \nu_2 \in \mathcal{G}_{n,x}$ and a constant $c \in (0,1)$, such that $\nu = c\nu_1 + (1-c)\nu_2$. In particular, since $\nu(B) < 1$ we must have $\nu_1(B) < 1$ or $\nu_2(B) < 1$.

Now, suppose that there is an extremal Gibbs measure $\nu \in \mathsf{ex}\mathcal{G}_{n,x}$, such that $\nu(B) < 1$. Since extremality implies tail triviality (Proposition 2.32) and $B \in \mathcal{T}$, we find that $\nu(B) = 0$. We will show that this leads to a contradiction.

Claim 2.39.1. *The measure ν is $R \circ T$-invariant, for any reflection R. In particular, ν is periodic.*

Proof of claim. Let π and $R(\pi)$ be a pair of conjugate half-planes. Since $\nu(B) = 0$, we know that ν-almost-surely either π or $R(\pi)$ do not contain an infinite -1-cluster. Then, in light of tail triviality, we can assume that

$$\nu(\text{there is an infinite } -1\text{-cluster in } \pi) = 0.$$

This implies that for any finite domain $\Delta \subseteq \pi$, there ν-almost-surely exists a $+1$-semicircuit $\gamma \subseteq V(\pi)$. Then, $\gamma \cup R(\gamma)$ is an R-invariant circuit surrounding the R-invariant domain $\Delta \cup R(\Delta)$. Hence, Lemma 2.38 implies that ν stochastically dominates $\nu \circ R \circ T$. Since furthermore ν-almost-surely either π or $R(\pi)$ do not contain an infinite $+1$-cluster. We similarly obtain that ν stochastically dominated by $\nu \circ R \circ T$ and the claim follows. The periodicity follows from the composition of two reflections is a translation. $\square$

Now, since $\nu \in \mathsf{ex}\mathcal{G}_{n,x}$ is periodic, Propositions 2.26 and 2.31, imply[12] that ν-almost-surely there exists at most one infinite spin-cluster. Hence, by Proposition 2.34, we have $\nu \in \{\nu^+_{n,x}, \nu^-_{n,x}\}$. This however is a contradiction, since $\nu^{\pm}_{n,x}$ is not $R \circ T$ invariant. $\square$

Proof of Theorem 2.28. Let $\nu_{n,x} \in \mathcal{G}_{n,x}$ be a translation invariant Gibbs measure. It suffices to show that $\nu(E^+ \cap E^-) = 0$. Indeed, if this is the case, then the butterfly lemma (Lemma 2.39) implies that $\nu(E^+) + \nu(E^-) = 1$ and hence

$$\nu = \nu(E^+)\nu(\,\cdot \mid E^+) + \nu(E^-)\nu(\,\cdot \mid E^-) = \nu(E^+)\nu^+ + \nu(E^-)\nu^-,$$

where the second equality follows from Proposition 2.34 and Remark 5.

Now, assume by contradiction that $\nu(E^+ \cap E^-) > 0$. We can in fact assume that $\nu(E^+ \cap E^-) = 1$. Otherwise we replace ν by $\nu(\,\cdot \mid E^+ \cap E^-)$, which is also a translation invariant Gibbs measure (see Remark 5). Now, the butterfly lemma (Lemma 2.39) and the translation invariance of ν imply, that for some $k \in \{0,1,2\}$, say $k = 0$, and a sign, say $+$, there is a positive probability such that π_0 and $R(\pi_0)$ contain an infinite $+$-cluster. Moreover, since by assumption $\nu(E^-) = 1$, we can choose a radius $N \in \mathbb{N}$, such that Λ_N intersects both infinite $+$-clusters in Λ_N and the infinite $-$-cluster with positive probability. The finite-energy property then implies that $\nu(C_0) > 0$, where C_j, for some $j \in \mathbb{Z}$, denotes the event that $k \in V(\mathbb{T})$ is part of a two-sided infinite $+$-path with its two halves being contained in π_0 and $R(\pi_0)$, while $k+1 \in V(\mathbb{T})$ is part of an infinite $-$-cluster.

Claim 2.39.2. *The event $C := \{C_j$ happens for infinitely many $j \in \mathbb{N}\}$ has positive probability under ν.*

Proof of claim. First, observe that for any $j \in \mathbb{Z}$, we have $C_j = \tau_j C_0$ and hence $\nu(C_j) = \nu(C_0) > 0$ by the translation invariance of ν. Define the random variable $\alpha : \{-1,1\}^{V(\mathbb{T})} \to \mathbb{N}$ by setting

$$\alpha(\sigma) := \sup\{j \in \mathbb{N} \mid \sigma \in C_j\},$$

[12] Observe, that for the referenced propositions to apply directly, we would need translation invariance instead of periodicity. However, the proof of Proposition 2.31 can easily be adapted to periodic measures as already stated in Remark 5.

where we use the convention $\sup \emptyset = -1$. Then,

$$C \setminus C_0 \subseteq \{0 \leq \alpha < \infty\} = \bigcup_{i \in \mathbb{N}} \{\alpha = i\}.$$

It follows, that

$$\nu(C \setminus C_0) \leq \sum_{i \in \mathbb{N}} \nu(\alpha = i).$$

However, for any $i \in \mathbb{N}$, we have $\tau_1\{\alpha = i\} = \{\alpha = i+1\}$ and hence $\nu(\alpha = i) = \nu(\alpha = i+1)$. Thus, by the above inequality, we must have $\nu(\alpha = i) = 0$ and the claim follows. $\square$

Now, since

$$\{C_j \text{ happens for infinitely many } j \in \mathbb{N}\} \subseteq \{\text{there are infinitely many infinite spin clusters}\},$$

we arrive at contradiction, since the Burton-Keane (see the proof of Theorem 2.25 and Remark 3) shows that the latter event has probability 0. $\square$

2.2.4 Proof of Theorem 2.8

We finally proof the Theorem we set out to show.

Proof of Theorem 2.8. Consider the push-forward measure

$$\mathbb{P}_{n,x} := \Phi_{\#}(\nu^+_{n,x}) = \Phi_{\#}(\nu^-_{n,x}).$$

That is,

$$\mathbb{P}_{n,x}(A) = \nu^+_{n,x}(\Phi^{-1}(A)),$$

for $A \in \mathcal{S}$. To show that $\mathbb{P}_{n,x}$ is a Gibbs measure of the loop $O(n)$ model, we need to check that for $\mathbb{P}_{n,x}$-almost-all $\xi \in \mathsf{LoopConf}_{\mathbb{H}}$, we have

$$\mathbb{P}_{n,x}[\mathbb{1}_A \mid \mathcal{S}_D](\xi) = \mathbb{P}^{\xi}_{D,n,x}(A).$$

To that end let $B \in \mathcal{S}_D$. We have

$$\begin{aligned}
\mathbb{P}_{n,x}[\mathbb{1}_B(\cdot)\, \mathbb{P}^{\cdot}_{D,n,x}(A)] &= \nu^+_{n,x}[\mathbb{1}_B(\Phi(\cdot))\, \mathbb{P}^{\Phi(\cdot)}_{D,n,x}(A)] \\
&= \nu^+_{n,x}[\mathbb{1}_{\Phi^{-1}(B)}(\cdot)\, \nu^{\cdot}_{G,n,x}(\Phi^{-1}(A))] \\
&= \nu^+_{n,x}(\Phi^{-1}(B) \cap \Phi^{-1}(A))) = \mathbb{P}_{n,x}(A \cap B)
\end{aligned}$$

where we used Proposition 2.9 in the second line. Hence, $\mathbb{P}_{n,x}$ is a Gibbs measure. Now, consider the sequence of domains $(G_k)_{k \geq 1}$ induced by the corresponding set of hexagons in $(D_k)_{k \geq 1}$ and define the sequence of measures $(\mathbb{P}^{\emptyset}_{D_k,n,x})_{k \geq 1}$. By Proposition 2.9, we know that for any $k \in \mathbb{N}$, we have $\mathbb{P}^{\emptyset}_{D_k,n,x} = \Phi_{\#}(\nu^+_{G_k,n,x})$. We now show that as $k \to \infty$,

$$\mathbb{P}^{\emptyset}_{D_k,n,x} \to \mathbb{P}_{n,x},$$

in the sense of weak convergence. To that end, consider some bounded and continuous random variable $h : \mathsf{LoopConf}_{\mathbb{H}} \to \mathbb{R}$. By Proposition 2.17, we know that $\nu^+_{G_k,n,x} \xrightarrow{w} \nu^+_{n,x}$ as $k \to \infty$. Now, since the spin-to-loop map Φ is continuous, we obtain

$$\mathbb{P}^{\emptyset}_{D_k,n,x}[h] = \nu^+_{G_k,n,x}[h \circ \Phi] \to \nu^+_{n,x}[h \circ \Phi] = \mathbb{P}_{n,x}[h],$$

as $k \to \infty$, thus showing the claimed convergence.

Finally since by Proposition 2.26 $\nu^+_{n,x}$ has at most one infinite spin cluster, we conclude that

$$\mathbb{P}_{n,x}(\text{there exists an infinite path}) = 0.$$

Now assume that there is a Gibbs measure $\mathbb{P} \neq \mathbb{P}_{n,x}$ for the loop $O(n)$ model with edge-weight x. For a loop configuration $\omega \in \mathsf{LoopConf}_{\mathbb{H}}$ denote by $\Phi_+^{-1}(\omega)$ and $\Phi_-^{-1}(\omega)$ the unique spin configurations in $\Phi^{-1}(\{\omega\})$ satisfying $\Phi_+^{-1}(\omega)(0) = +1$ and $\Phi_-^{-1}(\omega)(0) = -1$. Similarly for an event $E \in \mathcal{F}$ and $s \in \{-,+\}$ we will use the notation $E_s := \{\sigma \in E \mid \sigma(0) = s\}$. Now define a random spin configuration $\boldsymbol{\sigma}$ as follows. Sample a random loop configuration $\boldsymbol{\omega} \sim \mathbb{P}$ and a random sign $s \sim \mathsf{Unif}(\{-,+\})$ and set $\boldsymbol{\sigma} = \Phi_s^{-1}(\boldsymbol{\omega})$. Denote by ν the law of $\boldsymbol{\sigma}$. Then, by construction, we have $\Phi_\#(\nu) = \mathbb{P}$, since

$$\nu(\Phi^{-1}(A)) = \nu(\Phi_+^{-1}(A)) + \nu(\Phi_-^{-1}(A)) = \frac{1}{2}\,\mathbb{P}(A) + \frac{1}{2}\,\mathbb{P}(A) = \mathbb{P}(A),$$

for $A \in \mathcal{L}$. Now let $G \subseteq \mathbb{T}$ be a domain, $A \in \mathcal{F}$ and $B \in \mathcal{T}_G$. Then for $s \in \{-,+\}$ we have

$$\begin{aligned}
\mathbb{P}\left[\mathbb{1}_B\left(\Phi_s^{-1}(\cdot)\right)\nu_{G,n,x}^{\Phi_s^{-1}(\cdot)}(A)\right] &= \mathbb{P}\left[\mathbb{1}_{\Phi(B_s)}(\cdot)\nu_{G,n,x}^{\Phi_s^{-1}(\cdot)}\left(A \cap \mathsf{SpinConf}_G^{\Phi_s^{-1}(\cdot)}\right)\right] \\
&= \mathbb{P}\left[\mathbb{1}_{\Phi(B_s)}(\cdot)\mathbb{P}_{D[G],n,x}^{\cdot}\left(\Phi\left(A \cap \mathsf{SpinConf}_G^{\Phi_s^{-1}(\cdot)}\right)\right)\right] \\
&= \mathbb{P}\left(\Phi(B_s) \cap \Phi\left(A_s\right)\right) = \mathbb{P}\left(\Phi(B_s \cap A_s)\right),
\end{aligned}$$

where we used that $\mathbb{P}$ is Gibbs and that Φ restricted to $\{\sigma \in \mathsf{SpinConf} \mid \sigma(0) = s\}$ is injective. It follows, that

$$\begin{aligned}
\nu\left[\mathbb{1}_B(\cdot)\nu_{G,n,x}^{\cdot}(A)\right] &= \frac{1}{2}\,\mathbb{P}\left[\mathbb{1}_B\left(\Phi_+^{-1}(\cdot)\right)\nu_{G,n,x}^{\Phi_+^{-1}(\cdot)}(A) + \mathbb{1}_B\left(\Phi_-^{-1}(\cdot)\right)\nu_{G,n,x}^{\Phi_-^{-1}(\cdot)}(A)\right] \\
&= \frac{1}{2}\,\mathbb{P}\left(\Phi(B_+ \cap A_+)\right) + \frac{1}{2}\,\mathbb{P}\left(\Phi(B_- \cap A_-)\right) \\
&= \nu(B_+ \cap A_+) + \nu(B_- \cap A_-) \\
&= \nu(B \cap A).
\end{aligned}$$

That is, $\nu \in \mathcal{G}_{n,x}$ is a Gibbs measure and hence, by Theorem 2.28, there is a constant $c \in [0,1]$, such that

$$\nu = c\,\nu_{n,x}^+ + (1-c)\,\nu_{n,x}^-.$$

It follows that

$$\begin{aligned}
\mathbb{P} &= \Phi_\#(\nu) \\
&= c\,\Phi_\#(\nu_{n,x}^+) + (1-c)\,\Phi_\#(\nu_{n,x}^-) \\
&= c\,\mathbb{P}_{n,x} + (1-c)\,\mathbb{P}_{n,x}^- = \mathbb{P}_{n,x},
\end{aligned}$$

which is a contradiction to the assumption $\mathbb{P} \neq \mathbb{P}_{n,x}$. Hence, $\mathbb{P}_{n,x}$ is the only translation invariant Gibbs measure of the loop $O(n)$ model with edge-weight x.

□

3 Sampling methods for lattice models

3.1 Markov-chain-Monte-Carlo sampling

All sampling methods try to achieve the same general task. Given a probability distribution μ on some state-space Ω, we want to generate random samples from μ. Sampling methods can be roughly divided into two categories; *direct sampling* and *Markov-chain-Monte-Carlo sampling*. The former refers to methods in which a sequence of *statistically independent* samples is generated. Unfortunately direct sampling is usually unfeasible when working with high-dimensional state-spaces, such as those encountered in statistical mechanics. Markov-chain-Monte-Carlo (MCMC) sampling on the other hand refers to sampling methods which involve simulating trajectories of Markov chains specifically designed to yield approximate samples from a given probability distribution as the chain approaches equilibrium. These algorithms rely on the theory of Markov processes and in particular build upon fundamental convergence theorems. We will cover the necessary definitions and results in the following section.

3.1.1 Finite Markov chains

A Markov chain is a stochastic process, which satisfies a certain *memorylessness* property. Conditional on the current state of the chain, the future is independent of the past.

Definition (Markov chain)**.** Consider a finite state-space Ω and a sequence $(X_k)_{k\in\mathbb{N}}$ of random variables taking values in Ω. We say that $(X_k)_{k\in\mathbb{N}}$ is a Markov chain in Ω, if and only if the chain satisfies the *Markov property.* That is, for any $n \in \mathbb{N}$ and any choice of states $\omega_0, \ldots, \omega_{n+1} \in \Omega$, we have

$$\mathbb{P}(X_{n+1} = \omega_{n+1} \mid X_0 = \omega_0, \ldots, X_n = \omega_n) = \mathbb{P}(X_{n+1} = \omega_{n+1} \mid X_n = \omega_n),$$

whenever the above conditioning is non-degenerate. If moreover for any $x, y \in \Omega$,

$$p_{xy} := \mathbb{P}(X_{n+1} = y \mid X_n = x)$$

is independent of n, then $(X_k)_{k\in\mathbb{N}}$ is said to be a *time-homogeneous* Markov chain with transition probability matrix $P = (p_{xy})_{x,y\in\Omega}$.

In the following we will only consider time-homogeneous Markov chains on finite state-spaces.

Irreducible and aperiodic Markov chains, play a special role in the theory of Markov processes. A Markov chain is called if it will eventually visit any element of the state-space on which it is defined. A Markov chain is called aperiodic, if - for any sufficiently large number $k \in \mathbb{N}$ - any state can be reached from any other state in exactly k steps with positive probability.

Definition (Irreducibility, aperiodicity)**.** Let $(X_k)_{k\in\mathbb{N}}$ be a Markov chain in Ω.

- The Markov chain is called *irreducible*, if for any $x, y \in \Omega$, there exists a $n \in \mathbb{N}$, such that

$$\mathbb{P}(X_n = y \mid X_0 = x) > 0.$$

- The Markov chain is called *aperiodic*, if for any $x \in \Omega$, we have

$$\mathsf{gcd}(\{n \in \mathbb{N} \mid \mathbb{P}(X_n = x \mid X_0 = x) > 0\}) = 1.$$

An equilibrium distribution of a Markov chain, is a probability distribution on the corresponding state-space, which is *stationary* with respect to the Markov chain dynamics.

Definition (Equilibrium distribution)**.** Let $X = (X_k)_{k\in\mathbb{N}}$ be a Markov chain in Ω with transition probability matrix $P = (p_{xy})_{x,y\in\Omega}$. A probability distribution $\pi : \Omega \to [0, 1]$, is called *equilibrium distribution* (or *stationary distribution*) of X, if for any $x \in \Omega$, we have

$$\pi(y) = \sum_{x\in\Omega} \pi(x)\, p_{x,y}.$$

In that case we say that π is stationary with respect to X.

A sufficient (but not necessary) criterion for verifying that a probability distribution is stationary with respect to a given Markov chain is given by the *detailed balance condition.*

Proposition 3.1 (Detailed balance condition)**.** *Let $X = (X_k)_{k\in\mathbb{N}}$ be a Markov chain in Ω with transition probability matrix $P = (p_{xy})_{x,y\in\Omega}$ and let $\pi : \Omega \to [0, 1]$ be some probability distribution. If for any $x, y \in \Omega$, we have*

$$\pi(x)\, p_{x,y} = \pi(y)\, p_{y,x},$$

then π is stationary with respect to X.

Proof. The detailed balance condition immediately implies

$$\sum_{x\in\Omega} \pi(x)\, p_{x,y} = \sum_{x\in\Omega} \pi(y)\, p_{y,x} = \pi(y) \sum_{x\in\Omega} p_{y,x} = \pi(y).$$

□

Theorem 3.2 (Existence and uniqueness of equilibrium distribution)**.** *Let $X = (X_k)_{k\in\mathbb{N}}$ be an irreducible Markov chain taking values in a finite state-space Ω. Then X has a unique stationary distribution $\pi : \Omega \to [0,1]$. Moreover, $\pi(x) > 0$ for all $x \in \Omega$.*

We now state two fundamental results for finite Markov chains. Proofs can be found for example in [KS76]. Corresponding statements hold also beyond the finite state-space setting, see for example [Nor97].

Theorem 3.3 (Convergence theorem)**.** *Consider an irreducible and aperiodic Markov chain $(X_n)_{n\in\mathbb{N}}$ taking values in a finite state-space Ω. If $(X_n)_{n\in\mathbb{N}}$ has equilibrium distribution $\pi : \Omega \to [0,1]$, then for any $x \in \Omega$, we have*

$$\lim_{n\to\infty} \mathbb{P}(X_n = x) = \pi(x).$$

Moreover, there exist constants a $C > 0$ and $0 < \alpha < 1$, such that for any $x, y \in \Omega$ and $n \geq 1$,

$$|\mathbb{P}(X_n = y \mid X_0 = x) - \pi(y)| \leq C\alpha^n.$$

As a corollary of the above convergence theorem we obtain that correlations of Markov chain observables decay exponentially with time.

Corollary 3.4 (Exponential decay of covariance)**.** *Consider an irreducible and aperiodic Markov chain $(X_n)_{n\in\mathbb{N}}$ taking values in a finite state-space Ω and let $f : \Omega \to \mathbb{R}$ be a random variable. There exist constants a $C > 0$ and $0 < \alpha < 1$, such that for any $n \geq 1$,*

$$\mathrm{Cov}(f(X_0), f(X_n)) \leq C\alpha^n.$$

Proof.

$$\begin{aligned}
\mathrm{Cov}(f(X_0),f(X_n)) &= \mathbb{E}[f(X_0)f(X_n)] - \mathbb{E}[f(X_0)]\mathbb{E}[f(X_n)] \\
&= \sum_{x,y\in\Omega} f(x)f(y)\,[\mathbb{P}(X_0 = x, X_n = y) - \mathbb{P}(X_0 = x)\mathbb{P}(X_n = y)] \\
&= \sum_{x,y\in\Omega} f(x)f(y)\mathbb{P}(X_0 = x)\,[\mathbb{P}(X_n = y \mid X_0 = x) - \mathbb{P}(X_n = y)] \\
&= \sum_{x,y\in\Omega} f(x)f(y)\mathbb{P}(X_0 = x)\,[\mathbb{P}(X_n = y \mid X_0 = x) - \pi(y)]
\end{aligned}$$

Now, by Theorem 3.3, there exist constants a $C > 0$ and $0 < \alpha < 1$, such that for any $n \geq 1$,

$$|\mathbb{P}(X_n = y \mid X_0 = x) - \pi(y)| \leq C\alpha^n.$$

Hence,

$$\begin{aligned}
\mathrm{Cov}(f(X_0),f(X_n)) &= \mathbb{E}[f(X_0)f(X_n)] - \mathbb{E}[f(X_0)]\mathbb{E}[f(X_n)] \\
&\leq \sum_{x,y\in\Omega} |f(x)f(y)|\mathbb{P}(X_0 = x)|\mathbb{P}(X_n = y \mid X_0 = x) - \pi(y)| \\
&\leq C\alpha^n \sum_{x,y\in\Omega} |f(x)f(y)|,
\end{aligned}$$

and the result follows, since Ω is finite. □

3.1.2 Metropolis-Hastings algorithm

The basic recipe underlying any Markov-chain-Monte-Carlo algorithm is the following. Given a finite state-space Ω and a target distribution $\mu : \Omega \to [0,1]$, construct an irreducible and aperiodic Markov chain $(X_n)_{n\in\mathbb{N}}$, such that μ is an equilibrium distribution of the chain. Then, simulate trajectories of the Markov chain through the state-space Ω. Once the chain is sufficiently equilibrated, record the states visited by $(X_n)_{n\in\mathbb{N}}$ to obtain approximate samples of the target distribution μ.

The Metropolis-Hastings scheme provides a flexible framework to construct Markov chains which are stationary with respect to a given target distribution. An arbitrarily chosen irreducible Markov chain is modified by including a rejection step after each transition in which the preceding transition may be revoked with a certain probability. The rejection probabilities are chosen in such a way that detailed balance with the target distribution is ensured.

Definition (Metropolis-Hastings scheme)**.** Let $X = (X_k)_{k\in\mathbb{N}}$ be a finite irreducible Markov chain in Ω with transition probability matrix $P = (p_{xy})_{x,y\in\Omega}$. Fix a strictly positive target distribution $\pi : \Omega \to (0,1]$, an initial state $\omega_0 \in \Omega$ and a function $h : [0,\infty) \to [0,1]$ satisfying the *Metropolis condition*:

$$h(\alpha) = \alpha \cdot h\left(\frac{1}{\alpha}\right), \qquad \forall \alpha \in \mathbb{R}.$$

Define the transition probability matrix $Q = (q_{xy})_{x,y\in\Omega}$ by setting

$$q_{xy} \coloneqq p_{xy}\, h\left(\frac{\pi(y)}{\pi(x)}\frac{p_{yx}}{p_{xy}}\right).$$

The Metropolis-Hastings chain $\mathsf{MH}(X,\pi,h,\omega_0)$ with proposal chain X and target distribution π is the Markov chain starting from state ω_0 defined by the transition probability matrix Q.

Proposition 3.5 (Stationarity of the Metropolis-Hastings chain)**.** *Let $X = (X_k)_{k\in\mathbb{N}}$ be a finite irreducible Markov chain in Ω with transition probability matrix $P = (p_{xy})_{x,y\in\Omega}$. Fix a strictly positive target distribution $\pi : \Omega \to (0,1]$, an initial state $\omega_0 \in \Omega$ and a function $h : [0,\infty) \to [0,1]$ satisfying the Metropolis condition. Then, π is stationary with respect to the Markov chain $\mathsf{MH}(X,\pi,h,\omega_0)$.*

Proof. We verify the detailed balance condition. Indeed, for $x, y \in \Omega$,

$$\begin{aligned}\pi(x)q_{xy} = \pi(x)p_{xy}\, h\left(\frac{\pi(y)}{\pi(x)}\frac{p_{yx}}{p_{xy}}\right) &= \pi(x)p_{xy}\frac{\pi(y)}{\pi(x)}\frac{p_{yx}}{p_{xy}}\, h\left(\frac{\pi(x)}{\pi(y)}\frac{p_{xy}}{p_{yx}}\right) \\ = \pi(y)p_{yx}\, h\left(\frac{\pi(x)}{\pi(y)}\frac{p_{xy}}{p_{yx}}\right) &= \pi(y)\, q_{yx}.\end{aligned}$$

□

Remark 6. Often when people refer to the Metropolis algorithm, it is implicitly assumed that $h : [0,\infty) \to [0,1]$ is chosen to be

$$h(\alpha) \coloneqq \min\{1, \alpha\},$$

which clearly satisfies

$$\alpha\, h\left(\frac{1}{\alpha}\right) = h(\alpha).$$

Other valid choices for h include for example

$$h(\alpha) \coloneqq \frac{\alpha}{1+\alpha}.$$

3.1.3 Efficiency of Markov-chain-Monte-Carlo methods

In this section we will discuss how the efficiency of Markov-chain-Monte-Carlo methods can be judged. The exposition is based on [Sok97].

Consider a finite state-space Ω with probability measure μ. The goal of Monte-Carlo methods is to obtain approximate values for observables $f : \Omega \to \mathbb{R}$ of the target distribution μ. The law of large numbers implies, that we can estimate $\mathbb{E}_\mu[f]$ by averaging independent realizations of the observable. That is, for independent random variables $X_1, X_2, \ldots \sim \mu$, we have

$$M_n^\mu(f) := \frac{1}{n}\sum_{k=1}^{n} f(X_k) \xrightarrow{a.s.} \mu[f],$$

as $n \to \infty$. The variance of this estimator gives us a grasp on the speed with which $M_n^\mu(f)$ approaches $\mu[f]$. Since the random variables are independent, we readily obtain

$$\mathrm{Var}[M_n^\mu(f)] = \frac{1}{n^2}\sum_{k=1}^{n} \mathrm{Var}[f(X_k)] = \frac{1}{n}\mathrm{Var}_\mu[f].$$

Samples obtained from Markov-chain-Monte-Carlo methods on the other hand are not independent but inherently correlated, which affects the speed of convergence at which $\mathbb{E}_\mu[f]$ is approached.

Definition (Autocorrelation functions)**.** Consider irreducible and aperiodic Markov Chain $X = (X_n)_{n\in\mathbb{N}}$ on a finite state-space Ω started from its equilibrium distribution μ and let $f : \Omega \to \mathbb{R}$ be a random variable satisfying $\mathrm{Var}_\mu[f] < \infty$. For $k \in \mathbb{N}$, define the (unnormalized) correlation function

$$C_f^X(k) := \mathrm{Cov}[f(X_0), f(X_k)],$$

and the normalized autocorrelation function

$$\rho_f^X(k) := \frac{C_f^X(k)}{C_f^X(0)}.$$

Lemma 3.6. *Consider irreducible and aperiodic Markov Chain $X = (X_n)_{n\in\mathbb{N}}$ on a finite state-space Ω started from its equilibrium distribution μ and let $f : \Omega \to \mathbb{R}$ be a random variable satisfying $\mathrm{Var}_\mu[f] < \infty$. Then, for any $n \in \mathbb{N}$, the variance of the sample mean*

$$M_n^X(f) := \frac{1}{n}\sum_{k=1}^{n} f(X_k)$$

is given by the following expression

$$\mathrm{Var}[M_n^X(f)] = \frac{\mathrm{Var}_\mu[f]}{n} \cdot 2\left(\frac{1}{2} + \sum_{k=1}^{n-1}(1 - \frac{k}{n})\ \rho_f^X(k)\right).$$

Proof.

$$\begin{aligned}
\mathrm{Var}[M_n^X(f)] &= \frac{1}{n^2}\mathrm{Var}[\sum_{k=1}^n f(X_k)] = \frac{1}{n^2}\sum_{i,j=1}^n \mathrm{Cov}[f(X_i), f(X_j)] \\
&= \frac{1}{n^2}\sum_{i,j=1}^n \mathrm{Cov}[f(X_0), f(X_{|i-j|})] \\
&= \frac{1}{n^2}\left(n\cdot \mathrm{Var}_\mu(f) + 2\sum_{k=1}^{n-1}(n-k)\,\mathrm{Cov}[f(X_0), f(X_k)]\right) \\
&= \frac{\mathrm{Var}_\mu[f]}{n} + \frac{2}{n^2}\sum_{k=1}^{n-1}(n-k)\,\mathrm{Cov}[f(X_0), f(X_k)] \\
&= \frac{\mathrm{Var}_\mu[f]}{n} + \frac{2}{n}\sum_{k=1}^{n-1}(1-\frac{k}{n})\,\mathrm{Cov}[f(X_0), f(X_k)] \\
&= \frac{\mathrm{Var}_\mu[f]}{n}\cdot 2\left(\frac{1}{2} + \sum_{k=1}^{n-1}(1-\frac{k}{n})\,\rho_f^X(k)\right)
\end{aligned}$$

□

Proposition 3.7. *Consider irreducible and aperiodic Markov Chain $X = (X_n)_{n\in\mathbb{N}}$ on a finite state-space Ω started from its equilibrium distribution μ and let $f : \Omega \to \mathbb{R}$ be a random variable satisfying* $\mathrm{Var}_\mu[f] < \infty$*. Then, there exists a constant $C > 0$, such that for all $n \in \mathbb{N}$,*

$$\Big|\sum_{k=1}^{n-1}(1-\frac{k}{n})\,\mathrm{Cov}[f(X_0), f(X_k)] - \sum_{k=1}^{\infty}\mathrm{Cov}[f(X_0), f(X_k)]\Big| \le \frac{C}{n}.$$

Proof. By Lemma 3.6 there exist constants a $C_1 > 0$ and $0 < \alpha < 1$, such that for any $n \ge 1$,

$$\mathrm{Cov}(f(X_0), f(X_n)) \le C\alpha^n.$$

Hence, we obtain

$$\begin{aligned}
&\Big|\sum_{k=1}^{n-1}(1-\frac{k}{n})\,\mathrm{Cov}[f(X_0), f(X_k)] - \sum_{k=1}^{\infty}\mathrm{Cov}[f(X_0), f(X_k)]\Big| \\
&\qquad\le \Big|\sum_{k=1}^{n-1}\frac{k}{n}\,\mathrm{Cov}[f(X_0), f(X_k)]\Big| + \Big|\sum_{k=n}^{\infty}\mathrm{Cov}[f(X_0), f(X_k)]\Big| \\
&\qquad\le \sum_{k=1}^{n-1}(\frac{k}{n}\,C_1\alpha^k) + C_2\alpha^n \le \frac{1}{n}C_3 + C_2\alpha^n \le \frac{C_4}{n},
\end{aligned}$$

for some appropriately chosen constants $C_2, C_3, C_4 > 0$. □

Definition (Integrated autocorrelation time)**.** Consider irreducible and aperiodic Markov Chain $X = (X_n)_{n\in\mathbb{N}}$ on a finite state-space Ω started from equilibrium and let $f : \Omega \to \mathbb{R}$ be a random variable satisfying $\mathrm{Var}_\mu[f] < \infty$. We define the *integrated autocorrelation time* of f as the constant

$$\tau_{\mathrm{int},f}^X := \frac{1}{2} + \sum_{k=1}^{\infty}\frac{\mathrm{Cov}[f(X_0), f(X_k)]}{\mathrm{Var}_\mu[f]}.$$

Corollary 3.8. *Consider irreducible and aperiodic Markov Chain $X = (X_n)_{n\in\mathbb{N}}$ on a finite state-space Ω started from its equilibrium distribution μ and let $f : \Omega \to \mathbb{R}$ be a random variable satisfying* $\mathrm{Var}_\mu[f] < \infty$*. Then, for all $n \in \mathbb{N}$,*

$$\mathrm{Var}[M_n^X(f)] = \mathrm{Var}[M_n^\mu(f)]\cdot 2\tau_{int,f}^X + \mathcal{O}(\frac{1}{n^2}).$$

Proof. Using Proposition 3.7, we obtain

$$\begin{aligned}
\mathrm{Var}[M_n^X(f)] &= \frac{\mathrm{Var}_\mu[f]}{n} \cdot 2\left(\frac{1}{2} + \sum_{k=1}^{n-1}(1-\frac{k}{n})\,\frac{\mathrm{Cov}[f(X_0), f(X_k)]}{\mathrm{Var}_\mu[f]}\right) \\
&= \frac{\mathrm{Var}_\mu[f]}{n} \cdot 2\left(\frac{1}{2} + \sum_{k=1}^{\infty}\frac{\mathrm{Cov}[f(X_0), f(X_k)]}{\mathrm{Var}_\mu[f]} + \mathcal{O}(\frac{1}{n})\right) \\
&= \frac{\mathrm{Var}_\mu[f]}{n} \cdot 2\left(\tau_{int,f}^X + \mathcal{O}(\frac{1}{n})\right) \\
&= \frac{\mathrm{Var}_\mu[f]}{n} \cdot 2\tau_{int,f}^X + \mathcal{O}(\frac{1}{n^2}) \\
&= \mathrm{Var}[M_n^\mu(f)] \cdot 2\tau_{int,f}^X + \mathcal{O}(\frac{1}{n^2}).
\end{aligned}$$

□

Critical slowing down A central difficulty in the statistical analysis of models like the Ising or loop $O(n)$ models and their phase transitions is a phenomenon called *critical slowing-down*. Unfortunately, the mathematical understanding of this phenomenon is still very limited and only few analytic results have been established. We conclude this chapter by giving a brief non-rigorous introduction to critical slowing down. Detailed discussions and further references can be found in [HH77],[Sok97].

Dynamic scaling theory predicts that the autocorrelation times of Markov-chain-Monte-Carlo algorithm diverge as the the critical parameter of the simulated system is approached. More precisely, consider a d-dimensional lattice model on a domain with diameter $L \in \mathbb{N}$ and a corresponding Markov-chain-Monte-Carlo algorithm X. Near the critical point of the model, the auto-correlation time of an observable f scales like

$$\tau_{int,f}^X(L) \sim L^{z_f^X + d},$$

with increasing linear system size $L \to \infty$. Here, z_f^X is the *dynamic critical exponent* corresponding the Markov chain X and observable f. For many classical models, including the Ising model, conventional local simulation methods like Glauber dynamics (see Section 3.2), non-rigorous arguments predict the largest dynamic critical exponents to be approximately 2. This has been confirmed by a large number of simulations. Modern algorithms like the Swendsen-Wang algorithm (Section 3.3) or the Worm algorithm (Section 3.5) have been able to achieve much lower dynamic critical exponents and have effectively eliminated the problem of critical slowing down. Estimated dynamic critical exponents for algorithms discussed in the underlying thesis can be found in references provided in the corresponding sections.

3.2 Glauber dynamics

Glauber dynamics is a classic sampling algorithm for the Ising model based on the Metropolis scheme. It was introduced by the American theoretical physicist Roy J. Glauber in 1963 [Gla63]. Glauber dynamics is considered a *local*, or *single-spin-flip* algorithm, because the approximating Markov chain moves through the state-space by flipping at most one spin at every step. Even though Glauber dynamics was originally introduced for the Ising model, it can be used to simulate a large number of spin models. In particular, using the infinite-volume of the loop $O(n)$, we obtain a direct analog of Glauber dynamics for simulating the loop $O(n)$ model.

Definition. Let G be domain of $\mathbb{T}$. For a face $f \in V(G)$ and a spin configuration $\sigma \in \{-1,1\}^{V(\mathbb{T})}$ define the spin configuration $s(\sigma, f) \in \{-1,1\}^{V(\mathbb{T})}$ by setting

$$s(\sigma, f)(f) \coloneqq -\sigma(f),$$

$$s(\sigma, f)(v) := \sigma(v) \qquad \text{for all } v \in V(\mathbb{T}) \setminus \{f\}.$$

That is, $s(\sigma, f)$ is the spin configuration obtained by flipping the spin of f in σ.

Definition (Glauber dynamics)**.** Let G be a domain of $\mathbb{T}$, fix $n, x > 0$ and let $\tau, \sigma_0 \in \{-1, 1\}^{V(\mathbb{T})}$. We construct a Markov chain $X = (X_k)_{k \in \mathbb{N}}$ as follows. For $k \in \mathbb{N}$ inductively define

$$X_{k+1} := s(X_k, \mathbf{u}),$$

where $\mathbf{u} \sim \mathsf{Unif}(V(G))$. Consider the map $h : [0, \infty) \to [0, 1], \alpha \mapsto \frac{\alpha}{1+\alpha}$, and define the *Glauber dynamics Markov chain*

$$\mathsf{GD}(G, n, x, \tau, \sigma_0) := \mathsf{MH}(X, \nu^{\tau}_{G,n,x}, h, \sigma_0).$$

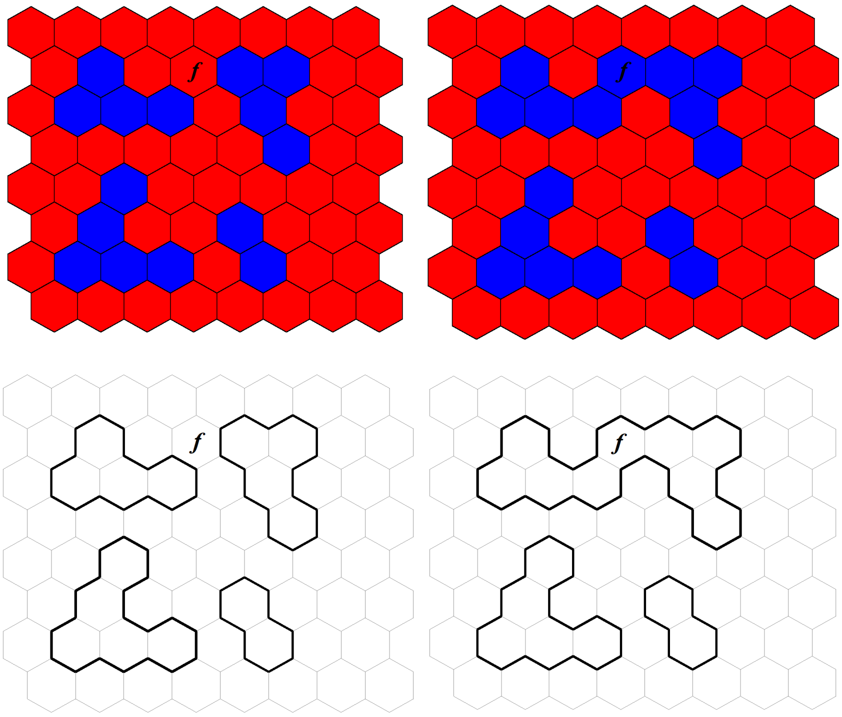

Figure 11: One step of Glauber dynamics illustrated. Loops are drawn in black, $+1$ spins are coloured red and -1 spins are coloured blue. *Top left:* A spin configuration σ on a domain $G \subseteq \mathbb{T}$ and a face $f \in V(G)$. *Top right:* The spin configuration $s(\sigma, f)$. *Bottom left:* The loop configuration $\Phi(\sigma)$ on $D[G] \leq \mathbb{H}$. *Bottom right:* The loop configuration $\Phi(s(\sigma, f))$.

Proposition 3.9 (Convergence of Glauber dynamics)**.** *Let G be a domain of $\mathbb{T}$, fix $n, x > 0$ and let $\tau, \sigma_0 \in \{-1, 1\}^{V(\mathbb{T})}$. The Markov chain $\mathsf{GD}(G, n, x, \tau, \sigma)$ converges to $\nu^{\tau}_{G,n,x}$.*

Proof. The Markov chain underlying $\mathsf{GD}(G, n, x, \tau, \sigma_0)$ is clearly irreducible and aperiodic. Moreover, $h : [0, \infty) \to [0, 1], \alpha \mapsto \frac{\alpha}{1+\alpha}$ satisfies $\alpha \cdot h(\frac{1}{\alpha}) = h(\alpha)$ for all $\alpha \in \mathbb{R}$. Hence the result follows from Proposition 3.5. $\square$

Remark 7. There is a natural probabilistic interpretation of the acceptance probability $\frac{\alpha}{1+\alpha}$ in the above algorithm. Indeed, fix a spin configuration $\sigma \in \{-1, 1\}^{V(\mathbb{T})}$ and a face $v \in V(\mathbb{T})$. The probability with which the spin of v in σ is flipped, is given by

$$\frac{\alpha}{1+\alpha} = \frac{\frac{\mu^{\tau}_{G,n,x}(\nu(\sigma,v))}{\mu^{\tau}_{G,n,x}(\sigma)}}{1+\frac{\mu^{\tau}_{G,n,x}(\nu(\sigma,v))}{\mu^{\tau}_{G,n,x}(\sigma)}} = \frac{\mu^{\tau}_{G,n,x}(\nu(\sigma,v))}{\mu^{\tau}_{G,n,x}(\sigma)+\mu^{\tau}_{G,n,x}(\nu(\sigma,v))}$$

$$= \frac{\mathbb{P}(\boldsymbol{\sigma}(v)=\nu(\sigma,v))}{\mathbb{P}\left(\boldsymbol{\sigma}|_{V(\mathbb{T})\setminus\{v\}} = \sigma|_{V(\mathbb{T})\setminus\{v\}}\right)}$$

$$= \mathbb{P}\left(\boldsymbol{\sigma}(v) = -\sigma(v) \mid \boldsymbol{\sigma}|_{V(\mathbb{T})\setminus\{v\}} = \sigma|_{V(\mathbb{T})\setminus\{v\}}\right).$$

Hence, Glauber dynamics moves through the state-space $\mathsf{SpinConf}^{\tau}_{G}$ by picking a face $v \in V(G)$ uniformly at random and resampling its spin according to $\mu^{\tau}_{G,n,x}$ conditional on the spin values outside of v.

Corollary 3.10. *Let $G \subseteq \mathbb{T}$ be a finite induced subgraph and let $D = D[G]$ be the domain of $\mathbb{H}$ which contains all edges and vertices which are part of a hexagon in G. Let $n, x > 0$, let $\sigma_0, \tau \in \{-1,1\}^{V(\mathbb{T})}$ and set $\xi := \Phi(\tau)$. Consider the Markov chain $X = \mathsf{GD}(G,n,x,\tau,\sigma)$. Then the Markov chain $(\Phi(X_k))_{k\in\mathbb{N}}$ converges to $\mathbb{P}^{\xi}_{D,n,x}$.*

3.3 Swendsen-Wang algorithm

In 1987 Robert H. Swendsen and Jian-Sheng Wang introduced the first so-called *cluster algorithm* [SW87] for simulating classical spin models. In contrast to the single-spin-flips used in Glauber dynamics, the Markov chain updates of cluster algorithms collectively flip spins of whole clusters. The introduction of this method – now known as the *Swendsen-Wang algorithm* – marks a milestone in the development of computational statistical physics as it was the first method to overcome the problem of critical slowing down (see Section 3.1.3) present in conventional local algorithms like Glauber dynamics.

The original formulation of the Swendsen-Wang algorithm was aimed at simulating the *Potts model*[13], which contains the Ising model as a special case. Since then, generalized versions of the method which apply to many different models have been developed. In the present section we will introduce the Swendsen-Wang algorithm for the Ising model. In Section 3.4, we will provide a general framework which can be used to adopt the Swendsen-Wang algorithm for simulating the loop $O(n)$ model.

The Swendsen-Wang algorithm is based on the *Edwards-Sokal coupling*, which relates the Ising model to the so-called *random-cluster model*.

The random-cluster model and the Edwards-Sokal coupling The analysis of the Ising model has lead to many interesting questions and results beyond the existence or absence of a phase transition. Much of that progress has been made by studying a so-called *percolation* model, the *random-cluster model*, also called *FK-percolation* after the two Dutch physicists Kees Fortuin and Piet Kasteleyn who originally introduced the model [FK72]. Even though the random-cluster model and the Ising model a priori have not much in common, there is a strong relationship between them, formalized by the so-called *Edwards-Sokal coupling*, named after Robert G. Edwards and Alan D. Sokal, who introduced the coupling [ES88].

The random-cluster model is defined as a probability distribution over *percolation-configurations* of a finite graph G. A percolation configuration is an element $\omega \in \{0,1\}^{E(G)}$ and hence assigns to each edge in the graph one of two possible values, 0 or 1. Edges which are assigned the value 1 are

[13]In the q-states Potts model, spins take values in $\{1,2,\ldots,q\}$. As for the Ising model, each spin disagreement comes with an energetic cost, determined by a temperature-like parameter.

called *occupied* edges and edges which are assigned the value 0 are called *open* edges. Apart from two model parameters, the probability of a given percolation configuration $\omega \in \{0,1\}^{E(G)}$ depends on the number of occupied edges and the connected components in the subgraph $(V(G), \omega)$, where. The way in which these connected components are counted is influenced by the *boundary conditions* chosen. In the random-cluster model boundary conditions are given by partitions of the boundary vertices $\partial_V G$, where vertices belonging to the same subset of the partition are effectively treated as being connected by occupied edges. The boundary conditions corresponding to the two trivial partitions $\{\{v\} \mid v \in V(G)\}$ and $\{V(G)\}$ are called *free* and *wired* boundary conditions, respectively.

Definition (Random-cluster model)**.** Let G be a finite graph, let ξ be a partition of $\partial_V G$ and fix parameters $p \in [0,1]$, $q \in (0,\infty)$. The random-cluster model on G with boundary conditions ξ is the probability distribution on $\{0,1\}^{E(G)}$ given by

$$\phi^{\xi}_{G,p,q}(\omega) \coloneqq \frac{1}{Z^{\xi}_{G,p,q}} p^{|\omega|}\, q^{k^{\xi}(\omega)}.$$

Here $k^{\xi}(\omega)$ denotes the number of connected components in $(V(G), \omega)$[14], where clusters which contain vertices of the same subset of the partition ξ are counted as one. The *partition function*

$$Z^{\xi}_{G,p,q} \coloneqq \sum_{\omega \in \{0,1\}^{E(G)}} p^{|\omega|}\, q^{k^{\xi}(\omega)}$$

of the random-cluster model is the normalizing constant making $\phi^{\xi}_{G,p,q}$ a probability measure. We denote the random-cluster model with *free* and *wired* boundary conditions by $\phi^{0}_{G,p,q}$ and $\phi^{1}_{G,p,q}$ respectively.

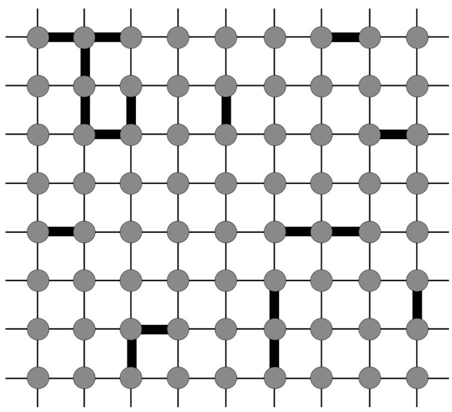

Figure 12: A configuration of the random-cluster model on a subgraph of the square lattice $\mathbb{Z}^2$. Edges and vertices of the graph are drawn in grey. Occupied bonds of the configuration are highlighted in black.

The random-cluster model (or FK-percolation model) with parameter $q = 2$ is intimately connected to the Ising model via the Edwards-Sokal coupling. It is therefore often called *FK-Ising model.*

Definition (Edwards-Sokal coupling)**.** Let G be a finite graph and $\beta \in (0,\infty)$.

- For a spin configuration $\sigma \in \{-1,1\}^{V(G)}$ define random percolation configurations $\boldsymbol{\omega}^{f}_{\beta}(\sigma), \boldsymbol{\omega}^{+}_{\beta}(\sigma) \in \{0,1\}^{E(G)}$ as follows. For $uv \in E(G)$ set

$$\mathbb{P}(\boldsymbol{\omega}^{f}_{\beta}(\sigma)(uv) = 1) = \begin{cases} 1 - e^{-2\beta}, & \sigma(u) = \sigma(v), \\ 0, & \text{otherwise.} \end{cases}$$

[14]Here, we identify the configuration ω with the set of occupied edges $\{e \in E(V) \mid \omega(e) = 1\}$.

$$\mathbb{P}(\boldsymbol{\omega}_\beta^+(\sigma)(uv)=1)=\begin{cases}1, & u,v\in\partial_V G,\\ 1-e^{-2\beta}, & \sigma(u)=\sigma(v),\\ 0, & \text{otherwise.}\end{cases}$$

- Given a percolation configuration $\omega\in\{0,1\}^{E(G)}$, partition $V(G)$ such that two vertices $v,u\in V(G)$ are part of the same subset if and only if they are connected by a path of occupied edges in ω. Now construct a random spin configuration $\boldsymbol{\sigma}^f(\omega)\in\{-1,1\}^{V(G)}$ by assigning to each subset of the partition a constant spin, chosen uniformly at random from $\{-1,1\}$. The random spin configuration $\boldsymbol{\sigma}^+(\omega)\in\{-1,1\}^{V(G)}$ is obtained analogously except that clusters intersecting $\partial_V G$ are assigned $+1$ spin deterministically.

Proposition 3.11 (Edwards-Sokal coupling). *Let G be a finite graph and fix parameters $\beta\in(0,\infty)$, $p\in(0,1)$. Set $p^*=1-e^{-2\beta}$ and $\beta^*=-\frac{1}{2}\ln(1-p)$. Then, the following relations hold.*

- *If $\sigma\sim\mu^f_{G,\beta}$, then $\boldsymbol{\omega}^f_\beta(\sigma)\sim\phi^0_{G,p^*,2}$.*
- *If $\omega\sim\phi^0_{G,p,2}$, then $\boldsymbol{\sigma}^f(\omega)\sim\mu^f_{G,\beta^*}$.*
- *If $\sigma\sim\mu^+_{G,\beta}$, then $\boldsymbol{\omega}^+_\beta(\sigma)\sim\phi^1_{G,p^*,2}$.*
- *If $\omega\sim\phi^1_{G,p,2}$, then $\boldsymbol{\sigma}^+(\omega)\sim\mu^+_{G,\beta^*}$.*

Proof. We will proof the first two statements, the proof of the last two statements is analogous. We begin by showing the first claim. Let us compute the joint law $\gamma_{G,\beta}$ of $\sigma\sim\mu_{G,\beta}$ and $\boldsymbol{\omega}^f(\sigma)$. The measure $\gamma_{G,\beta}$ is supported on pairs of configurations $\sigma\in\{-1,1\}^{V(G)}$, $\omega\in\{0,1\}^{E(G)}$, which are *compatible* in the sense that $\omega(uv)=1$ implies $\sigma(u)=\sigma(v)$. We write $\sigma\perp\omega$ when this is the case. Now, a quick calculation shows that for such a pair of compatible configurations $\sigma\perp\omega$, we get

$$\gamma(\sigma,\omega)\propto(1-e^{-2\beta})^{|\omega|}(e^{-2\beta})^{|E(G)|-|\omega|}.$$

Hence, the marginal distribution of ω under γ is given by

$$\begin{aligned}\sum_{\sigma:\sigma\perp\omega}\gamma(\sigma,\omega)&\propto\sum_{\sigma:\sigma\perp\omega}(1-e^{-2\beta})^{|\omega|}\,(e^{-2\beta})^{|E(G)|-|\omega|}\\&\propto 2^{k^\emptyset(\omega)}(1-e^{-2\beta})^{|\omega|}\,(e^{-2\beta})^{|E(G)|-|\omega|},\end{aligned}$$

which is the law of $\phi^\emptyset_{G,p^*,2}$ as claimed.

We move on to the second claim. We compute the joint law $\lambda_{G,p}$ of $\omega\sim\phi^\emptyset_{G,p,2}$ and $\boldsymbol{\sigma}^f(\omega)\in\{0,1\}^{E(G)}$. Consider two compatible configurations $\sigma\in\{-1,1\}^{V(G)}$, $\omega\in\{0,1\}^{E(G)}$. Since the conditional law $\lambda_{G,p}(\sigma\mid\omega)$ is uniform over all $2^{-k^\emptyset(\omega)}$ spin configurations $\sigma\in\{-1,1\}^{V(G)}$ which are compatible with ω, we obtain

$$\lambda_{G,p}(\sigma,\omega)\propto 2^{-k^\emptyset(\omega)}\,\phi^\emptyset_{G,p,2}(\omega)\propto p^{|\omega|}(1-p)^{|E(G)|-|\omega|}.$$

It follows, that $\lambda_{G,p}=\gamma_{G,\beta^*}$ and hence by the above we conclude that the marginal distribution of σ under $\lambda_{G,p}$ is given by μ_{G,β^*} as desired. □

We state two corollaries of the Edwards-Sokal coupling to highlight how it connects the Ising model to the random-cluster model.

Corollary 3.12. *Let G be a finite graph, $\beta\in(0,\infty)$ and set $p^*=1-e^{-2\beta}$. Then, for any two vertices $u,v\in V(G)$,*

$$\mu^f_{G,\beta}[\sigma_u\sigma_v]=\phi^0_{G,p*,2}(x\leftrightarrow y),$$

where $\{x\leftrightarrow y\}$ denotes the event, that x and y are connected by a path of occupied edges.

Proof. Let $\omega \sim \phi^0_{G,p*,2}$ and consider two vertices $u, v \in V(G)$. If $u \leftrightarrow v$ in ω, then $\boldsymbol{\sigma}^+(\omega)(u) = \boldsymbol{\sigma}^+(\omega)(u)$ by construction. If $u \not\leftrightarrow v$ in ω, then

$$\mathbb{P}(\boldsymbol{\sigma}^+(\omega)(u) = \boldsymbol{\sigma}^+(\omega)(v)) = \frac{1}{2}.$$

A quick calculation shows that,

$$\mathbb{E}[\boldsymbol{\sigma}^f(\omega)(u)\, \boldsymbol{\sigma}^+(\omega)(v)] = \phi^0_{G,p*,2}(u \leftrightarrow v),$$

and the result follows from Proposition 3.11. □

Using the Edwards-Sokal coupling, we can show that existence of a phase transition in the Ising model implies the existence of a phase transition in the FK-Ising model.

Corollary 3.13 (Phase transition of the FK-Ising model)**.** *Let $d \geq 2$ be an integer, $\beta \in (0, \infty)$ and set $p^* = 1 - e^{-2\beta}$. Then, the limit*

$$\phi^1_{p*,2}(0 \leftrightarrow \infty) := \lim_{k \in \mathbb{N}} \phi^1_{\Lambda^d_k,p*,2}(0 \leftrightarrow \partial_V \Lambda^d_k),$$

exists and is given by $\phi^1_{p*,2}(0 \leftrightarrow \infty) = M^d(\beta)$. In particular, the FK-Ising model exhibits the following phase transition. For any $p < p^*$, we have

$$\phi^1_{p*,2}(0 \leftrightarrow \infty) = 0,$$

while for any $p > p^*$, we have

$$\phi^1_{p*,2}(0 \leftrightarrow \infty) > 0.$$

Proof. Consider $\omega \sim \phi^1_{\Lambda^d_k,p*,2}$. By Proposition 3.11, we know that $\boldsymbol{\sigma}^+(\omega) \sim \mu^+_{\Lambda^d_k,\beta}$. We distinguish two cases. If $0 \leftrightarrow \partial_V \Lambda^d_k$ in ω, then $\boldsymbol{\sigma}^+(\omega)(0) = 1$ by construction. If $0 \not\leftrightarrow \partial_V \Lambda^d_k$ in ω, then

$$\mathbb{P}(\boldsymbol{\sigma}^+(\omega)(0) = 1) = \mathbb{P}(\boldsymbol{\sigma}^+(\omega)(0) = -1) = \frac{1}{2}.$$

Hence,

$$\mu^+_{\Lambda^d_k,\beta}[\sigma_0] = \mathbb{E}[\boldsymbol{\sigma}^+(\omega)(0)] = \phi^1_{\Lambda^d_k,p*,2}(0 \leftrightarrow \partial_V G),$$

or put differently

$$M^d_k(\beta) = \phi^1_{\Lambda^d_k,p*,2}(0 \leftrightarrow \partial_V \Lambda^d_k).$$

The claimed convergence and phase transition now immediately follow from Theorem 1.1. □

We will now introduce the Swendsen-Wang algorithm for simulating the Ising model.

The Swendsen-Wang algorithm The Swendsen-Wang algorithm is a direct consequence of the Edwards-Sokal coupling. See Figure 13 for an illustration.

Proposition 3.14 (Swendsen-Wang update)**.** *Let G be a domain of $\mathbb{T}$ and $\beta \in (0, \infty)$. If $\sigma \sim \mu^+_{G,\beta}$, then $\boldsymbol{\sigma}^+(\boldsymbol{\omega}^+_\beta(\sigma)) \sim \mu^+_{G,\beta}$ as well.*

Proof. By Proposition 3.11, we know that if $\sigma \sim \mu^+_{G,\beta}$, then $\boldsymbol{\omega}^+_\beta(\sigma) \sim \phi^1_{G,p^*,2}$, where $p^* = 1 - e^{-2\beta}$ and therefore that $\boldsymbol{\omega}_\beta(\sigma) \sim \mu^+_{G,\beta^*}$, where in this case $\beta^* = -\frac{1}{2}\ln(1 - p^*)$. A quick calculation shows that indeed $\beta^* = \beta$ and the result follows. □

Corollary 3.15 (Convergence of Swendsen-Wang dynamics)**.** *Let G be a domain of $\mathbb{T}$ and let $\beta \in (0, \infty), \sigma_0 \in \{-1, 1\}^{V(\mathbb{T})}$. We construct a Markov chain $\mathsf{SW}(G, \beta, \sigma_0) = (X_k)_{k \in \mathbb{N}}$ as follows. Set $X_0 = \sigma_0$, and for $k \in \mathbb{N}$ inductively define*

$$X_{k+1} := \boldsymbol{\sigma}(\boldsymbol{\omega}^+_\beta(X_k)).$$

Then $\mathsf{SW}(G, \beta)$ converges to the Ising model $\mu^+_{G,\beta}$ with $+1$ boundary conditions.

Proof. By Corollary 3.14 the Swendsen-Wang Markov chain $\mathsf{SW}(G, \beta)$ is stationary with respect to $\mu^+_{G,\beta}$. Moreover, from the definition of the coupling it is evident, that $\mathsf{SW}(G, \beta)$ may reach any configuration in a single step from any other configuration in the state-space. Hence, $\mathsf{SW}(G, \beta)$ is irreducible and aperiodic and the result follows. □

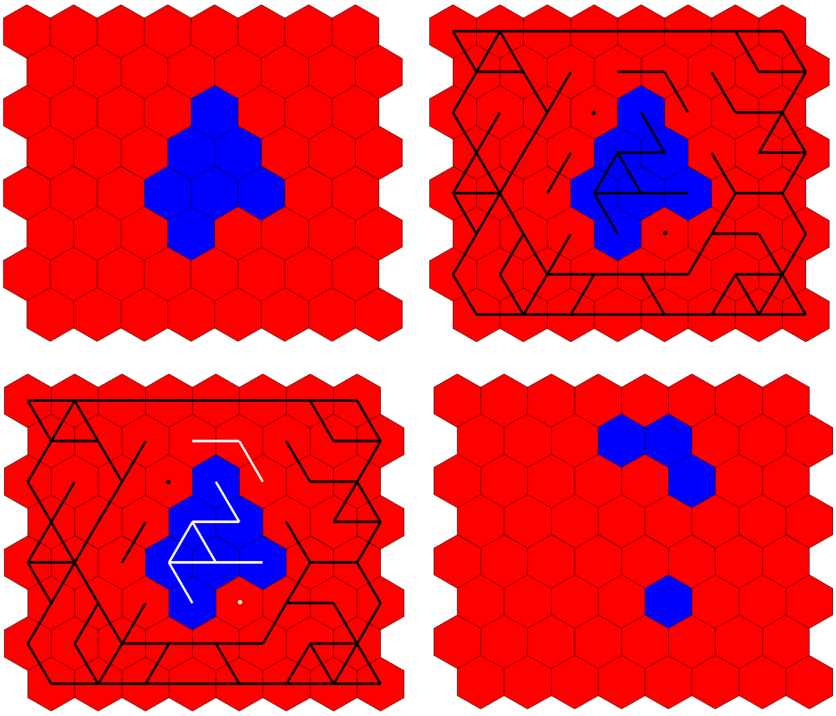

Figure 13: One step of the Swendsen-Wang algorithm illustrated. $+1$ spins are coloured red and -1 spins are coloured blue. *Top left:* The initial spin configuration. *Top right:* A percolation configuration on the dual lattice drawn in black. *Bottom left:* The clusters of the percolation configuration which are selected to be flipped are highlighted in white. *Bottom right:* The spin configuration obtained by flipping the spin values of the selected percolation clusters.

Proposition 3.16. *Let $G \subseteq \mathbb{T}$ be a domain of $\mathbb{T}$ and let $D = D[G]$ be the domain $\mathbb{H}$ consisting of edges and vertices which are part of hexagons in G and let $\beta \in (0, \infty)$. Now, if $\sigma \sim \mu^+_{G,\beta}$, then $\Phi(\sigma) \sim \mathbb{P}^{\emptyset}_{D,1,x}$, where $x = e^{-\beta}$.*

Corollary 3.17. *Let $G \subseteq \mathbb{T}$ be a finite induced subgraph and let $D = D[G]$ be the domain of $\mathbb{H}$ which contains all edges and vertices which are part of a hexagon in G. Let $\beta \in (0, \infty)$, $\sigma_0 \in \{-1, 1\}^{V(\mathbb{T})}$ and consider the Markov chain $X = \mathsf{SW}(G, \beta, \sigma_0)$. Then the Markov chain $(\Phi(X_k))_{k\in\mathbb{N}}$ converges to $\mathbb{P}^{\emptyset}_{D,1,x}$.*

3.4 Chayes-Machta algorithm

An important advance in the development of efficient Markov-chain-Monte-Carlo algorithms in statistical physics was made in 1998 by Chayes and Machta [CM98], who proposed a cluster algorithm for simulating the random-cluster model for any real $q \geq 1$. As the Swendsen-Wang algorithm, the Chayes-Machta algorithm relies on a certain coupling of the random-cluster model to another model on the graph and successively resamples the conditioned marginal distributions.

We now introduce a slightly more general version of the original method [DGMOPS07; DGGBS07], which in particular applies to the loop $O(n)$ model.

Definition. Let $G = (V(G), E(G))$ be a finite graph and let $A \subseteq E(G)$.

- Denote by $\mathcal{P}(G) \subseteq \mathcal{P}(V(G)) \times \mathcal{P}(E(G))$ the set of subgraphs of G.
- Denote by $\mathcal{P}_c(G) \subseteq \mathcal{P}(G)$ the set of *connected*[15] subgraphs of G.
- Denote by $G[A]$ the spanning subgraph $(V(G), A)$ of G.
- Denote by $K_G(A) \subseteq \mathcal{P}_c(G)$ the set of connected components of $G[A]$.

Definition (Generalized cluster model)**.** Let G be a finite graph. The *generalized cluster model* on G with weight function $W : \mathcal{P}_c(G) \to [0, \infty)$ is the probability distribution $\Phi_{G,W} : E(G) \to [0, 1]$ given by the following expression. For $A \subseteq E(G)$, set

$$\phi_{G,W}(A) \coloneqq \frac{1}{Z_{G,W}} \prod_{C \in K_G(A)} W(C),$$

where

$$Z_{G,W} \coloneqq \sum_{A \subseteq E(G)} \prod_{C \in K_G(A)} W(C)$$

is the corresponding partition function.

Remark 8. Let G be a domain of $\mathbb{T}$, let D be a domain of $\mathbb{H}$ and consider parameters $p, q, n, x > 0$.

- Observe that we are indeed dealing with a generalization of the random-cluster model. Consider the weight function given by the following expression. For $C \in \mathcal{P}_c(G)$, set

  $$W^{\mathrm{RCM}}_{p,q}(C) \coloneqq q\, p^{|E(C)|}.$$

  Then,

  $$\phi_{G,W^{\mathrm{RCM}}_{p,q}} \cong \phi^0_{G,p,q}.$$

- Similarly, the loop $O(n)$ model can realized as a generalized cluster model. Indeed, consider the weight function given by the following expression. For $C \in \mathcal{P}_c(D)$, set

  $$W^{\mathrm{loop}}_{n,x}(C) \coloneqq \begin{cases} 1, & \text{if } C \text{ is an isolated vertex,} \\ n\, x^{|E(C)|}, & \text{if } C \text{ is a loop ,} \\ 0, & \text{otherwise.} \end{cases}$$

  Then,

  $$\phi_{D,W^{\mathrm{loop}}_{n,x}} \cong \mathbb{P}^{\emptyset}_{D,n,x}.$$

Definition. Let G be a finite graph, $A \subseteq E(G)$ and $n, k \in \mathbb{N}$.

- A *n-coloring* of G is a map $\alpha : \mathcal{P}_c(G) \to \{1, \dots, n\}$. Denote by $\mathsf{Col}_n(G)$ the set of n-colorings of G.
- Consider some $\alpha \in \mathsf{Col}_n(G)$. Denote by $G^{\alpha,k}$ the subgraph of G containing all $H \in \mathcal{P}_c(G)$ such that $\alpha(H) = i$.
- Consider a pair $A \subseteq E(G)$, $\alpha \in \mathsf{Col}_n(G[A])$. For $i \in \{1, \dots, n\}$, denote by $A^{\alpha,i} \subseteq A$ the set of edges which is assigned color i by α.

[15] We say that a graph $G = (V(G), E(G))$ is connected, if for all vertices $v, u \in V(G)$ we have $\{v, u\} \in E(G)$.

Definition (Chayes-Machta coupling)**.** Let G be a finite graph, $A \subseteq E(G)$ and $n \in \mathbb{N}$. Consider a set of weight functions $\mathcal{W} = \{W_k : \mathcal{P}_c(G) \to [0, \infty)\}_{k=1}^n$ and set $W := \sum_{k=1}^n W_k$. We construct a random coloring $\boldsymbol{\alpha}_{\mathcal{W}}(A) \in \mathsf{Col}_n(G[A])$ as follows. Independently for each cluster $C \in \mathcal{P}_c(G[A])$ choose one of the n colors according its relative weight. That is, for each $i \in \{1, \dots, n\}$ we set

$$\mathbb{P}(\boldsymbol{\alpha}(C) = i) = \frac{W_i(C)}{W(C)}.$$

Denote by $\Phi_{G,\mathcal{W}}$ the joint law of $A \sim \phi_{G,W}$ and $\alpha \sim \boldsymbol{\alpha}_{\mathcal{W}}(A)$.

Proposition 3.18. *Let G be a finite graph, and $n \in \mathbb{N}$. Consider a set of weight functions $\mathcal{W} = \{W_k : \mathcal{P}_c(G) \to [0, \infty)\}_{k=1}^n$ and set $W := \sum_{k=1}^n W_k$. Then, the joint law $\Phi_{G,\mathcal{W}}$ is supported on pairs $A \subseteq E(G)$, $\alpha \in \mathsf{Col}_n(G[A])$ and is given by*

$$\Phi_{G,\mathcal{W}}(A, \alpha) \propto \prod_{i=1}^n \phi_{G^{\alpha,i},W_i}(A^{\alpha,i}).$$

Proof. Let $A \subseteq E(G)$ and $\alpha \in \mathsf{Col}_n(G[A])$. Then,

$$\begin{aligned}
\Phi_{G,\mathcal{W}}(A, \alpha) &\propto \phi_{G,W}(A) \prod_{C \in K(A)} \frac{W_{\alpha(C)}(C)}{W(C)} \\
&\propto \prod_{C \in K(A)} W(C) \prod_{C \in K(A)} \frac{W_{\alpha(C)}(C)}{W(C)} \\
&= \prod_{C \in K(A)} W_{\alpha(C)}(C) \\
&= \prod_{i=1}^n \left(\prod_{\substack{C \in K(A), \\ \alpha(C) = i}} W_i(C) \right) \\
&\propto \prod_{i=1}^n \phi_{G^{\alpha,i},W_i}(A^{\alpha,i}).
\end{aligned}$$

□

Corollary 3.19 (Marginal distribution on subgraph)**.** *Let G be a finite graph, and $n \in \mathbb{N}$. Consider a set of weight functions $\mathcal{W} = \{W_k : \mathcal{P}_c(G) \to [0, \infty)\}_{k=1}^n$ and set $W := \sum_{k=1}^n W_k$. Fix a coloring $\alpha \in \mathsf{Col}_n(G)$. Then, for any $i \in \{1, \dots, n\}$ the marginal distribution of $\Phi_{G,\mathcal{W}}(\cdot \mid \alpha)$ on $G^{\alpha,i}$ is given by $\phi_{G^{\alpha,i},W_i}$.*

Definition (Chayes-Machta dynamics)**.** Let G be a finite graph and $n \in \mathbb{N}$. Consider a set of weight functions $\mathcal{W} = \{W_k : \mathcal{P}_c(G) \to [0, \infty)\}_{k=1}^n$ and set $W := \sum_{k=1}^n W_k$. Moreover, for each $k \in \{1, \dots, n\}$ and $H \in \mathcal{P}(G)$ let $\mathbf{U}_{k,H} : \mathcal{P}(E(G)) \to \mathcal{P}(E(G))$ be a random function[16], such that

$$A \sim \phi_{H,W_k} \implies \mathbf{U}_{k,H}(A) \sim \phi_{H,W_k}.$$

Denote the family of these random functions by $\mathcal{U}$. We inductively construct a Markov chain $\mathsf{CM}(G, \mathcal{W}, \mathcal{U}) = (A_k, \alpha_k)_{k \in \mathbb{N}}$ as follows. Let $k \in \mathbb{N}$ and assume $A_k = A, \alpha_k = \alpha$ for some $A \subseteq E(G)$, $\alpha \in \mathsf{Col}_n(G[A])$. For $i \in \{1, \dots, n\}$, set $B_i := \mathbf{U}_{i,G^{\alpha,i}}(A)$. The next state of the Markov chain is given by

$$A_{k+1} := \bigcup_{i=1}^n B_i, \quad \alpha_{k+1} := \boldsymbol{\alpha}_{\mathcal{W}}(A_k).$$

[16]That is, for each $A \subseteq E(G)$, $\mathbf{U}_{k,H}(A)$ is a random variable taking values in $\mathcal{P}(E(G))$.

Proposition 3.20 (Stationarity of Chayes-Machta dynamics)**.** *Let G be a finite graph and $n \in \mathbb{N}$. Consider a set of weight functions $\mathcal{W} = \{W_k : \mathcal{P}_c(G) \to [0,\infty)\}_{k=1}^n$ and set $W \coloneqq \sum_{k=1}^n W_k$. Moreover, consider a family of random functions $\mathcal{U}$ as defined above. Then, the joint law $\Phi_{G,\mathcal{W}}$ is stationary with respect to the Markov chain $\mathsf{CM}(G,\mathcal{W},\mathcal{U})$.*

Proof. Consider $\mathsf{CM}(G,\mathcal{W},\mathcal{U}) = (A_k,\alpha_k)_{k\in\mathbb{N}}$ as defined above and fix $k \in \mathbb{N}$. It suffices to show that

$$(A_k,\alpha_k) \sim \Phi_{G,\mathcal{W}} \Longrightarrow (A_{k+1},\alpha_k) \sim \Phi_{G,\mathcal{W}},$$

and that

$$(A_k,\alpha_k) \sim \Phi_{G,\mathcal{W}} \Longrightarrow (A_k,\alpha_{k+1}) \sim \Phi_{G,\mathcal{W}}.$$

We begin with the second claim. Indeed, assume that $(A_k,\alpha_k) \sim \Phi_{G,\mathcal{W}}$, then by construction

$$(A_k,\alpha_{k+1}) \sim \phi_{G,W}(A_k) \prod_{C\in K(A_k)} \frac{W_{\alpha_{k+1}(C)}(C)}{W(C)},$$

and the claim follows from the proof of Proposition 3.18. We move on to the second claim. Again, assume that $(A_k,\alpha_k) \sim \Phi_{G,\mathcal{W}}$, and consider a pair $A \subseteq E(G)$, $\alpha \in \mathsf{Col}_n(G[A])$. Then,

$$\begin{aligned}
\mathbb{P}(A_{k+1} = A \mid \alpha_k = \alpha) &= \prod_{i=1}^n \mathbb{P}(A_{k+1}^{\alpha,i} = A^{\alpha,i}) \\
&= \prod_{i=1}^n \mathbb{P}(\mathbf{U}_{i,G_{\alpha,i}}(A_k^{\alpha,i}) = A^{\alpha,i}) \\
&= \prod_{i=1}^n \mathbb{P}(A_k^{\alpha,i} = A^{\alpha,i}) \\
&= \mathbb{P}(A_k = A \mid \alpha_k = \alpha),
\end{aligned}$$

and the result follows. □

The Chayes-Machta algorithm for simulating the loop O(n) model The generalized Chayes-Machta algorithm introduced above allows us to extend Markov-chain-Monte-Carlo algorithms for the loop $O(1)$ model, to the loop $O(n)$ model for any $n \geq 1$.

Corollary 3.21 (Chayes-Machta algorithm for the loop $O(n)$ model)**.** *Let D be a domain of $\mathbb{H}$ and consider real numbers $n > 1$, $x > 0$. For $H \subseteq D$, let $\mathbf{U}_H : \mathcal{P}(E(H)) \to \mathcal{P}(E(H))$ be a random function, satisfying*

$$A \sim \mathbb{P}^{\emptyset}_{H,1,x} \Longrightarrow \mathbf{U}_H(A) \sim \mathbb{P}^{\emptyset}_{H,1,x}.$$

and define

$$\boldsymbol{U}_{k,H} \coloneqq \begin{cases} \mathbf{U}_H, & \text{if } k = 1, \\ \mathbf{Id}_H & \text{otherwise,} \end{cases}$$

where $\mathbf{Id}_H : \mathcal{P}(E(H)) \to \mathcal{P}(E(H))$, $A \mapsto A$. Set $\mathcal{W} \coloneqq \{W^{loop}_{1,x}, W^{loop}_{n-1,x}\}$ and $\mathcal{U} = \{\mathbf{U}_{k,H} \mid k \in \{1,2\},\ H \subseteq E(D)\}$. Then, the Markov chain $\mathsf{CM}(G,\mathcal{W},\mathcal{U})$ is stationary with respect to $\mathbb{P}^{\emptyset}_{D,n,x}$.

In particular using the Chayes-Machta method, we can now use the Swendsen-Wang algorithm to simulate the loop $O(n)$ model for any $n \geq 1$.

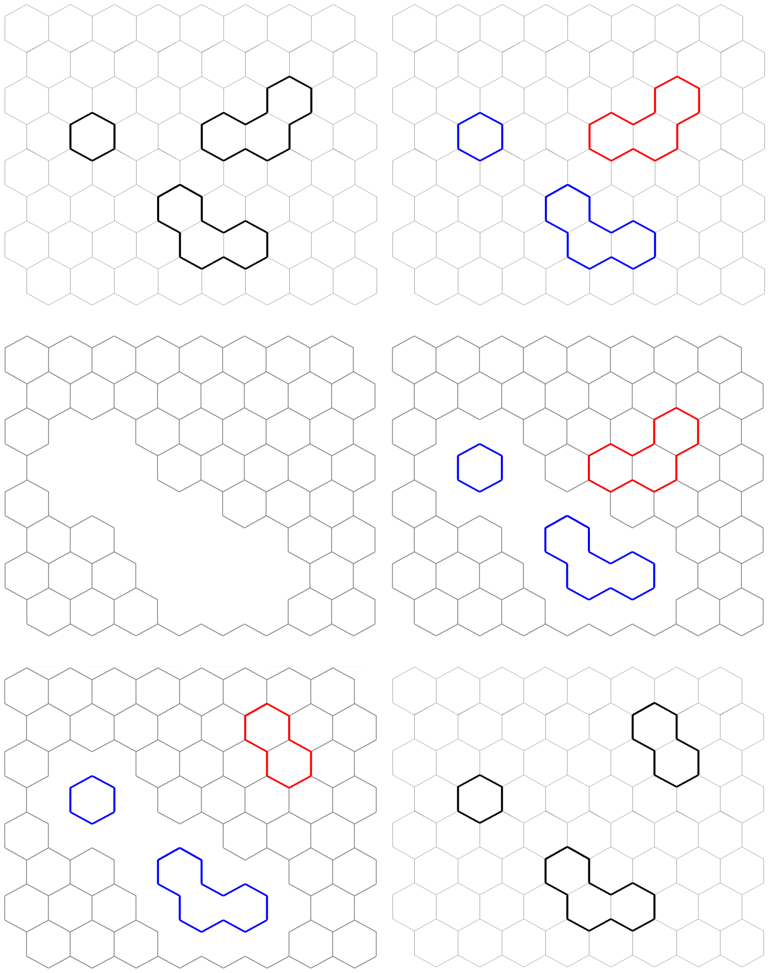

Figure 14: One step of the coloring method for the loop $O(n)$ model illustrated. *Top left:* A loop configuration drawn in black on a domain D of $\mathbb{H}$. *Top right:* A coloring of the loop configuration. Red corresponds to the active color and blue to the inactive color. *Middle row:* The active subgraph corresponding the chosen coloring with and without the loop configuration. *Bottom left:* A Markov chain update is performed on the active subgraph. The inactive loops remain untouched. *Bottom right:* The resulting loop configuration.

3.5 Worm dynamics

In 2001 Prokof'ev and Svistunov [PS01] introduced a family of Markov-chain-Monte-Carlo algorithms aimed at simulating certain loop representations of classical spin models. The name *Worm dynamics*, alludes to the type of state-space on which these Markov chains are defined. Indeed, the Markov chains underlying worm algorithms are defined on spaces of edge-configurations, each of which can be decomposed into a loop configuration and a simple path - the *worm* - connecting two distinguished vertices.

In the original publication [PS01], the authors illustrate how Worm algorithms apply to classical spin models, using so-called *high-temperature expansions*. We demonstrate the method for the Ising model on the hexagonal lattice.

The high-temperature expansion of the Ising model The high-temperature expansion of the Ising model is a graphical representation obtained by rewriting the partition function of the model. The high-temperature allows us to rewrite spin correlations in the Ising model in a convenient way. More specifically, this representation induces a probability distribution on generalized loop configurations with which we can express and study quantities like the two-point function of the Ising model. For more information on this and other graphical represenations of the Ising model see [Dum17b].

Definition. Let D be a domain of $\mathbb{H}$, $\beta \in (0, \infty)$. For a subset $A \subseteq E(D)$ denote by $\partial A \subseteq V(D)$ the set of vertices in the subgraph $(V(D), A)$ which have odd degree. We define

$$\mathcal{E}_D = \{(A, u, v) \mid u, v \in V,\ A \in \mathsf{LoopConf}_D^{u,v}\},$$

where

$$\mathsf{LoopConf}_D^{a,b} = \begin{cases} \{A \subseteq E(D) \mid \partial A = \{a, b\}\}, & \text{if } a \neq b, \\ \mathsf{LoopConf}_D, & \text{otherwise.} \end{cases}$$

Observe, that for any $A \in \mathsf{LoopConf}_D^{a,b}$, we can find a loop configuration $\omega \in \mathsf{LoopConf}_D$ and a simple path $\gamma \subseteq E(D)$ connecting a and b, such that $\omega \cap \lambda = \emptyset$ and $A = \omega \cup \lambda$.[17] See Figure 15 for an illustration.

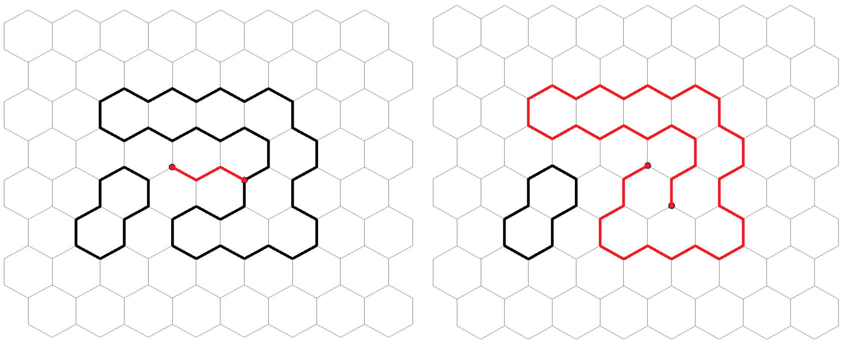

Figure 15: Two edge configurations $\omega_1 \cup \gamma_1 \in \mathsf{LoopConf}_D^{a_1,b_1}$ and $\omega_1 \cup \gamma_1 \in \mathsf{LoopConf}_D^{a_2,b_2}$. The domain D is drawn in grey. The vertices a_1, a_2, b_1, b_2 and the simple paths γ_1, γ_2 are drawn in red. The loop configurations ω_1, ω_2 are drawn in black.

Proposition 3.22 (High-temperature expansion of the Ising model)**.** *Let D be a domain of $\mathbb{H}$ and fix $\beta \in (0, \infty)$. Then,*

1.

$$Z_{D,\beta}^f = \cosh(\beta)^{|E(D)|}\, 2^{|V(D)|} \sum_{A \in \mathsf{LoopConf}_D} (\tanh \beta)^{|A|}.$$

2. For any two distinct fixed vertices $a, b \in V(D)$, the two-point correlation function of the Ising model is given by

$$\mu_{D,\beta}^f[\sigma_a \sigma_b] = \frac{\sum_{A \in \mathsf{LoopConf}_D^{a,b}} (\tanh \beta)^{|A|}}{\sum_{A \in \mathsf{LoopConf}_D} (\tanh \beta)^{|A|}}.$$

Proof. To obtain the high temperature expansion we will rewrite the partition function of the Ising model. In the following calculation, $\sum_\sigma$ denotes a sum running over the set of all spin configurations $\sum_{\sigma \in \{-1,1\}^{V(D)}}$ on our graph. Observe that for $x \in \{-1, 1\}, \beta > 0$, we have

$$e^{\beta x} = \cosh(\beta)\,(1 + \tanh(\beta)\, x).$$

[17] This decomposition is not necessarily unique. This however is irrelevant for what follows.

Hence, we obtain

$$\begin{aligned}
Z^f_{D,\beta} &= \sum_\sigma \prod_{uv\in E(D)} e^{\beta\sigma_u\sigma_v} = \sum_\sigma \prod_{uv\in E(D)} \cosh(\beta)\,(1+\tanh(\beta)\sigma_u\sigma_v) \\
&= \cosh(\beta)^{|E(D)|} \sum_\sigma \prod_{uv\in E(D)} (1+\tanh(\beta)\sigma_u\sigma_v) \\
&= \cosh(\beta)^{|E(D)|} \sum_\sigma \sum_{A\subseteq E(D)} \tanh(\beta)^{|A|} \prod_{uv\in A} \sigma_u\sigma_v \\
&= \cosh(\beta)^{|E(D)|} \sum_{A\subseteq E} \tanh(\beta)^{|A|} \sum_\sigma \prod_{uv\in A} \sigma_u\sigma_v \\
&= \cosh(\beta)^{|E(D)|}\, 2^{|V(D)|} \sum_{A\in\mathcal{E}_D} \tanh(\beta)^{|A|}
\end{aligned}$$

where the last equality follows from

$$\sum_\sigma \prod_{uv\in A} \sigma_u\sigma_v = \begin{cases} 2^{|V(D)|}, & A\in \mathsf{LoopConf}_D \\ 0, & \text{otherwise.}\end{cases}$$

Now for fixed $a,b\in V(D)$, by definition

$$\mu^f_{D,\beta}[\sigma_a\sigma_b] = \frac{\sum_\sigma \sigma_a\sigma_b \prod_{uv\in E(D)} e^{\beta\sigma_u\sigma_v}}{\sum_\sigma \prod_{uv\in E(D)} e^{\beta\sigma_u\sigma_v}} = \frac{\sum_\sigma \sigma_a\sigma_b \prod_{uv\in E(D)} e^{\beta\sigma_u\sigma_v}}{Z_{D,\beta}}$$

so as above we get

$$\begin{aligned}
Z_{D,\beta}\,\mu^f_{D,\beta}[\sigma_a\sigma_b] &= \sum_\sigma \sigma_a\sigma_b \prod_{uv\in E} e^{\beta\sigma_u\sigma_v} \\
&= \sum_\sigma \sigma_a\sigma_b \prod_{uv\in E} \cosh(\beta)\,(1+\tanh(\beta)\sigma_u\sigma_v) \\
&= \cosh(\beta)^{|E|} \sum_\sigma \sigma_a\sigma_b \prod_{uv\in E} (1+\tanh(\beta)\sigma_u\sigma_v) \\
&= \cosh(\beta)^{|E|} \sum_\sigma \sigma_a\sigma_b \sum_{A\subseteq E} \tanh(\beta)^{|A|} \prod_{uv\in A} \sigma_u\sigma_v \\
&= \cosh(\beta)^{|E|} \sum_{A\subseteq E} \tanh(\beta)^{|A|} \sum_\sigma \sigma_a\sigma_b \prod_{uv\in A} \sigma_u\sigma_v \\
&= \cosh(\beta)^{|E|}\, 2^{|V|} \sum_{A\in\mathsf{LoopConf}^{a,b}_D} \tanh(\beta)^{|A|}
\end{aligned}$$

where again the last equality follows from

$$\sum_\sigma \sigma_a\sigma_b \prod_{uv\in A} \sigma_u\sigma_v = \begin{cases} 2^{|V(D)|}, & A\in \mathsf{LoopConf}^{a,b}_D \\ 0, & \text{otherwise}\end{cases}.$$

□

Definition (High-temperature model)**.** Let D be a domain of $\mathbb{H}$, $\beta\in(0,\infty)$. The high-temperature model $\mathsf{HT}_{D,\beta}$ is the probability distribution on $\mathcal{E}_D$ given by

$$\mathsf{HT}_{D,\beta}(A,u,v) = \frac{1}{Z^{\mathsf{HT}}_{D,\beta}} \tanh(\beta)^{|A|},$$

where

$$Z^{\mathsf{HT}}_{D,\beta} = \sum_{(A,u,v)\in\mathcal{E}_D} \tanh(\beta)^{|A|}$$

is the corresponding partition function.

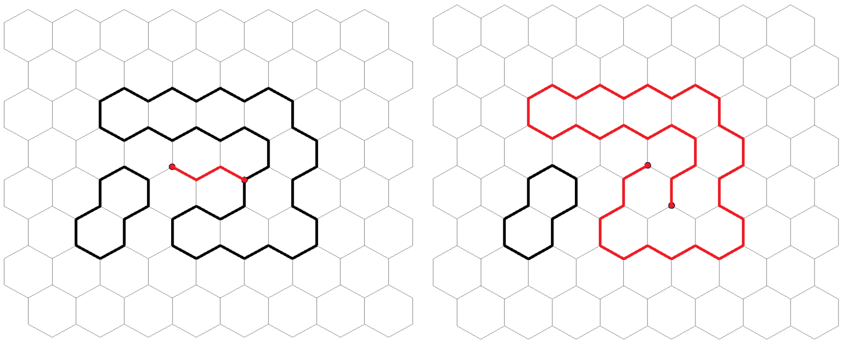

Figure 16: Two edge configurations $\omega_1 \cup \gamma_1 \in \mathsf{LoopConf}_D^{a_1,b_1}$ and $\omega_1 \cup \gamma_1 \in \mathsf{LoopConf}_D^{a_2,b_2}$. The domain D is drawn in grey. The two *worms* γ_1, γ_2 are drawn in red and the corresponding *heads* and *tails* a_1, a_2, b_1, b_2 are indicated by red dots. The loop configurations ω_1, ω_2 are drawn in black.

Let us restate Proposition 3.22 in terms of the high-temperature model just introduced. For any two distinct fixed vertices $a, b \in V(D)$, we have

$$\mu^f_{D,\beta}[\sigma_a \sigma_b] = |V(D)| \cdot \frac{\mathsf{HT}_{D,\beta}(\mathcal{E}_D^{a,b})}{\mathsf{HT}_{D,\beta}(\mathcal{E}_D^{\emptyset})},$$

where

$$\mathcal{E}_D^{a,b} = \{(A, u, v) \in \mathcal{E}_D \mid \{u, v\} = \{a, b\}\},$$
$$\mathcal{E}_D^{\emptyset} = \{(A, u, v) \in \mathcal{E}_D \mid u = v\}.$$

Moreover the partition function $Z^f_{D,\beta}$ of the Ising model can be expressed as follows,

$$Z^f_{D,\beta} = \cosh(\beta)^{|E(D)|}\, 2^{|V(D)|}\, Z^{\mathsf{HT}}_{D,\beta}.$$

In particular, we see that we can express observables of the Ising model like its two-point function in terms of the high-temperature model.

Worm dynamics on $\mathcal{E}_D$ We can simulate models on $\mathcal{E}_D$ like the high-temperature measure using *Worm dynamics*, a family of Markov-chain-Monte-Carlo algorithms based on the Metropolis scheme. At each iteration the proposal chain underlying a Worm algorithm updates the current state by randomly choosing on of the two vertices with odd-degree and moving it to a neighbouring vertex, thus adding or deleting an edge of the previous state.

Definition (Worm update)**.** Let D be a domain of $\mathbb{H}$ and let $(A, u, v) \in \mathcal{E}_D$. Construct a random configuration $\mathbf{w}((A, u, v)) \in \mathcal{E}_D$ as follows. Uniformly at random pick a vertex in $\{u, v\}$. If u was chosen, pick an edge $uu' \in E(D)$ incident with $u \in V(D)$ uniformly at random and set

$$\mathbf{w}((A, u, v)) := (A \triangle uu', u', v),$$

where $\omega \triangle uu'$ denotes the symmetric difference of the sets ω and $\{uu'\}$. Analogously, if v was chosen, then pick an edge $vv' \in E(D)$ incident with v uniformly at random and set

$$\mathbf{w}((A, u, v)) := (A \triangle vv', u, v').$$

Definition (Worm algorithms)**.** Let D be a domain of $\mathbb{H}$ and let μ_D be a strictly positive probability measure on $\mathcal{E}_D$. We construct a Markov chain $X = (X_k)_{k \in \mathbb{N}}$ as follows. For $k \in \mathbb{N}$ inductively define

$$X_{k+1} := \mathbf{w}(X_k),$$

where $\mathbf{w}$ is the random update function given by the preceding Definition. The Worm algorithm for simulating μ_D, is the Markov Chain

$$\mathsf{W}(D, \mu_D, h, \bar{\omega}) := \mathsf{MH}(X, \mu_D, h, \bar{\omega}),$$

where $\bar{\omega} \in \mathcal{E}_D$ is some arbitrarily chosen initial state and $h : [0, \infty) \to [0, 1]$ is some function satisfying the Metropolis condition.

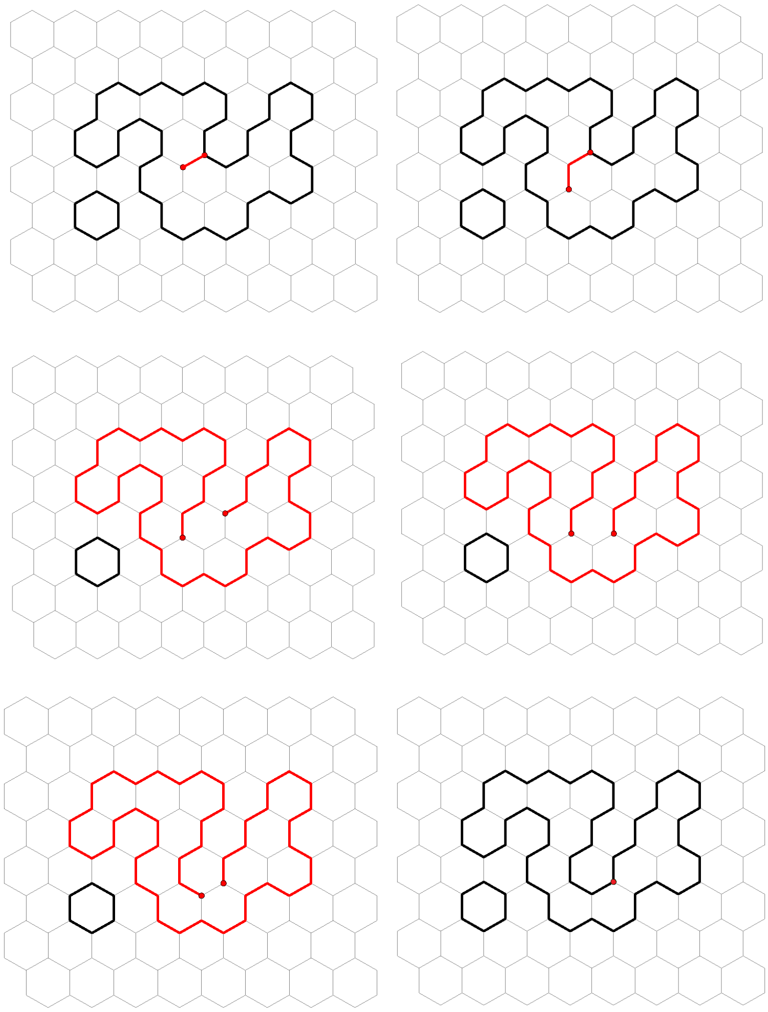

Figure 17: Snapshots of successive states of a worm Markov chain. A worm is drawn in red while closed loops are drawn in black. *Top left:* The first state of the chain after leaving the set of closed loop configurations $\mathsf{LoopConfig}_D$. *Top right, middle row, bottom left:* The worm ends randomly move through the domain until they meet. *Bottom right:* The worm ends have met and the Markov chain returns to the set $\mathsf{LoopConf}_D$.

Proposition 3.23 (Convergence of Worm algorithms)**.** *Let D be a domain of $\mathbb{H}$ and let μ_D be a strictly positive probability measure on $\mathcal{E}_D$. Then, for any initial state $\bar{\omega} \in \mathcal{E}_D$ and any any function $h : [0, \infty) \to [0, 1]$ satisfying the Metropolis condition, the Markov chain $\bar{\mathsf{W}}(D, \mu_D, h, \bar{\omega})$ converges to μ_D.*

Proof. This is a direct consequence of Proposition 3.5, since the worm dynamics are irreducible. $\square$

Thus, we obtain a valid algorithm for simulating the high-temperature measure.

Worm dynamics for the loop $O(n)$ model Enlarging the state-space of loop configurations by permitting two vertices with odd degree, allows us to employ a Worm algorithm for simulating the loop $O(n)$ model.

Definition (Generalized loop $O(n)$ model)**.** Let D be a domain of $\mathbb{H}$ and $n, x > 0$. The loop $O(n)$ model on $\mathcal{E}_D$ with edge-weight x is the probability distribution $\bar{\mathbb{P}}_{D,n,x}$ given by

$$\bar{\mathbb{P}}_{D,n,x}(A, u, v) \propto n^{c_D(A)} x^{|A|},$$

where $c_D(\omega)$ denotes the cyclomatic number[18] of the graph $(V(D), \omega)$.

Proposition 3.24. *Let D be a domain of $\mathbb{H}$ and $n, x > 0$. Then,*

$$(A, u, v) \sim \bar{\mathbb{P}}_{D,n,x}(\cdot \mid \mathcal{E}_D^{\emptyset}) \implies A \sim \mathbb{P}_{D,n,x}.$$

That is, the loop $O(n)$ model can be recovered from the generalized loop $O(n)$ model, by conditioning on the set of proper loop configurations.

Proof. Let $(A, u, u) \in \mathcal{E}_D^{\emptyset}$. Then,

$$\begin{aligned}
\bar{\mathbb{P}}_{D,n,x}((A, u, u) \mid \mathcal{E}_D^{\emptyset}) &= \frac{\bar{\mathbb{P}}_{D,n,x}((A, u, u))}{\bar{\mathbb{P}}_{D,n,x}(\mathcal{E}_D^{\emptyset})} \\
&= n^{c_D(A)} x^{|A|} \left(\sum_{(B,v,v) \in \mathcal{E}_D^{\emptyset}} n^{c_D(B)} x^{|B|} \right)^{-1} \\
&= n^{c_D(A)} x^{|A|} \left(|V(D)| \sum_{B \in \mathsf{LoopConfig_D}} n^{l_D(B)} x^{|B|} \right)^{-1} \\
&= \frac{\mathbb{P}_{D,n,x}(A)}{|V(D)|}.
\end{aligned}$$

Summing this equation over all vertices $u \in V(D)$ yields the result. □

By Proposition 3.23, we know that we can use a worm algorithm for simulating the generalized loop $O(n)$ model defined above. Based on the upcoming Lemma, we will show that the subchain obtained by only observing a Worm algorithm when the two worm ends meet, yields a valid algorithm for simulating the loop $O(n)$ model.

Lemma 3.25. *Let X be an irreducible Markov chain on a finite state-space S with stationary distribution π, and let $T \subseteq S$. Define a new Markov chain Y be only recording only those states visited by the original chain X which lie in T. Then, Y is an irreducible Markov chain on T with stationary distribution π_T given by*

$$\pi_T(s) := \frac{\pi(s)}{\sum_{t \in T} \pi(t)}, \qquad s \in T.$$

Proof. A proof can be found in [KS76, Section 6.1]. □

Corollary 3.26 (Worm dynamics on $\mathsf{LoopConf}_D$)**.** *Let D be a domain of $\mathbb{H}$ and $n, x > 0$. and $\bar{\omega} \in \mathcal{E}_D$. Consider the Worm Markov chain*

$$\mathsf{W}(D, \bar{\mathbb{P}}_{D,n,x}, h, \bar{\omega}) = ((A_k, u_k, v_k))_{k \in \mathbb{N}}$$

on $\mathcal{E}_D$ for some initial state $\bar{\omega} \in \mathcal{E}_D$ and Metropolis map $f : [0, \infty) \to [0, 1]$. Now, let $\mathsf{W}^{\emptyset}(D, n, x, \bar{\omega})$ be the Markov chain on $\mathsf{LoopConf}_D$ obtained by recording A_k for any $k \in \mathbb{N}$ such that $u_k = v_k$. Then, $\mathsf{W}^{\emptyset}(D, n, x, \bar{\omega})$ converges to $\mathbb{P}_{D,n,x}$.

Proof. This follows from Proposition 3.24 and Lemma 3.25. □

[18]That is, the minimum number of edges that must be removed to make the graph cycle-free.

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