# MASTERARBEIT / MASTER'S THESIS

Titel der Masterarbeit / Title of the Master's Thesis

„Nematic Alignment of Cells in Fibers“

verfasst von / submitted by

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angestrebter akademischer Grad / in partial fulfilment of the requirements for the degree of

Master of Science (MSc)

Wien, 2023 / Vienna, 2023

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| Studienkennzahl lt. Studienblatt / degree programme code as it appears on the student record sheet: | UA 066 821 |
| Studienrichtung lt. Studienblatt / degree programme as it appears on the student record sheet: | Masterstudium Mathematik |
| Betreut von / Supervisor: | Assoz. Prof. Sara Merino Aceituno, PhD |

# Acknowledgements

I would like to greatly thank my supervisor Sara Merino Aceituno for providing amazing support throughout the entire process of the thesis.
I would also like to thank my family for always providing advice and, especially my sister, helping me not to worry too much.

# Contents

# 1. Introduction

This Master's Thesis is concerned with analyzing biological models for fibers and cells, using methods of Kinetic Theory. There are many works deriving models for particles, cells, or other biological species, starting from individual based stochastic differential equations and then letting the particle number go to infinity. This procedure is known as the *mean-field limit* and yields a partial differential equation for the density of each species. These derivations are done in [2], [3], [4], [8], and [9]. In this thesis, however, we skip this process and start with the equations for the distributions.

In contrast to many papers, such as the ones mentioned above, this thesis aims to analyze a system of two PDEs rather than a single equation, modelling the behaviour of cells and fibers simultaneously. In Chapter 2 we start with two equations that have only minor coupling. We will see in Chapter 3 that more complex interactions, in this case cells aligning nematically to fibers and vice versa, cause problems in the mathematical treatment. The chosen procedure is similar to [2], which uses well established methods of Kinetic Theory, particularly the use of *Generalized Collision Invariants (GCI)*. This concept was suggested in [4] and has proven to be of great use. The general structure is as follows: We start by transforming the system to dimensionless units as well as scaling it in Section 2.1. We can then rewrite the PDEs with operators $Q^{(1)}$ and $Q^{(2)}$, which we analyze thereafter. In particular, we want to obtain the kernel of these objects, the set of *equilibria*. We further define GCI and prove existence and uniqueness. This is done in Section 2.2. Finally, in Section 2.3 we can compute the *Hydrodynamic Limit* using GCI.

The aforementioned problem for complex two-species models lies in the difficulty of defining GCI appropriately, which is the content of Section 3.1. When naïvely trying to modify the definition of GCI to multi-species models with nematic alignment, one obtains the condition that the species' mean nematic directions are parallel, which should only hold in the limit. Furthermore, we discuss some examples to show the problems arising for two-species models even under the assumption of suitable GCI in Section 3.2. When computing the macroscopic equations via the hydrodynamic limit, we arrive at an overdetermined system. This means we do not obtain a satisfactory result. Therefore, in Section 3.3 we suggest a method for solving the problems occurring in our treatment of two-species biological models with nematic alignment. The core of this approach is letting the difference of the alignment vectors be of order $\varepsilon$ and using a Hilbert expansion. Since we only give an outlook, this procedure can be further researched.

# 2. A Simple Two Species Model with Nematic Alignment

In this chapter we want to introduce and analyze a simplified system of partial differential equations, aiming to describe the interaction between fibers and cells, where in this chapter, the reciprocal influence is small. In Chapter 3 we will modify the system to a more complex one.
Usually one starts with a stochastic differential equation on the particle level and then goes to the kinetic equations via the so-called mean-field limit, however, in this thesis we only discuss the resulting equations. Similar models and their respective derivation from a stochastic differential equation can for example be found in [2], [3], [4], [9], and [8].
We denote by $f = f(x,t,v)$ the cell density, which depends on the space variable $x \in \mathbb{R}^d$, the time variable $t \in \mathbb{R}$, and the orientation $v \in \mathbb{S}^{d-1}$. Analogously, $g = g(x,t,w)$ denotes the distribution of fibers, where one can omit the spacial dependence as can be seen from the equation for $g$. We will usually not denote the variables for better readability. The system of equations is given by

$$\begin{aligned} \partial_t f + s(|v \cdot u_f|) v \cdot \nabla_x f = &- \nabla_v \cdot \left(f \nu \nabla_v ((v \cdot u_f)^2) + \alpha f \nabla_v ((\bar{y}_g - x) \cdot v)\right) \\ &+ \sigma_c \nabla_v \cdot ((v \cdot u_f)^2 \nabla_v f), \end{aligned} \tag{2.1}$$

$$\partial_t g = -\nabla_w \cdot (g(\widetilde{\nu} \nabla_w ((w \cdot \bar{w}_g)^2)) + \sigma \nabla_w \cdot ((w \cdot \bar{w}_g)^2 \nabla_w g)), \tag{2.2}$$

where $\bar{w}_g$ is the leading eigenvector of

$$\mathcal{Q}_g(y) = \int K(|y - z|) \left( w \otimes w - \frac{1}{d} Id \right) g(z,w) \,\mathrm{d}z \,\mathrm{d}w, \tag{2.3}$$

$u_f$ is the leading eigenvector of

$$\mathcal{Q}_f(x) = \int K(|x - z|) \left( v \otimes v - \frac{1}{d} Id \right) f(z,v) \,\mathrm{d}z \,\mathrm{d}v, \tag{2.4}$$

and

$$\bar{y}_g(x,t) = \frac{\int K(|x-y|) y \rho_g(y) \,\mathrm{d}y}{\int K(|x-y|) \rho_g(y) \,\mathrm{d}y}, \tag{2.5}$$

with $\rho_g(y) = \int g(y,t,w) \,\mathrm{d}w$ being the density and $K$ being a sensing kernel, usually chosen as $K(x) = \mathbb{1}_{|x| \le R}$ for an interaction radius $R > 0$. The quantities $\nu$, $\widetilde{\nu}$, $\alpha$, $\sigma$, and $\sigma_c$ are real numbers.

The vectors $\bar{w}_g$ and $u_f$ are the so-called mean nematic directions or alignment vectors of the fibers and cells, respectively, and $\bar{y}_g$ is the center of mass of the fibers.
In comparison to many models for biological phenomena discussed in the literature, this is a system of equations, rather than a singular one. The two given PDEs are only coupled by involving the term $-\nabla_v \cdot (\alpha f \nabla_v(\bar{y}_g - x))$, which describes cells moving towards the center of mass of the fibers, $\bar{y}_g$.
Note, however, that this is already a simplified version of the full model. We would like to change the potentials to $(v \cdot \bar{w}_g)^2$ and $(w \cdot u_f)^2$, respectively, which model cells aligning with fibers and vice versa. Meaning, we want to swap $u_f$ and $\bar{y}_g$, which we will discuss in Chapter 3. Furthermore, note that this system of equations does not include a transport term for the fibers.

*Remark:* We will carry out many computations of this thesis for $d = 2$, as it is done in [2], where we can write integrals over the sphere as integrals with respect to an angle $\theta \in (-\pi, \pi)$. Rephrasing $v \in \mathbb{S}^1$ as $v = (\cos\theta, \sin\theta)^T$ as well as an $\bar{w}_g = (\cos\bar{\theta}_2, \sin\bar{\theta}_2)^T$ and using trigonometric identities, we get

$$v \cdot \bar{w}_g = \cos(\theta - \bar{\theta}_2). \tag{2.6}$$

We will use this quite frequently, so in our case $\bar{\theta}$ will always denote the angle of an alignment vector.

## 2.1. Scaling of the Equations

We first want to transform our system to dimensionless units and then carry out a rescaling, dependent on a parameter $\varepsilon > 0$. For this, we first define reference time and space $t_0 = 1/\nu$ and $x_0 = v_0 t_0$ to get the dimensionless $t' = t/t_0$ and $x' = x/x_0$. Further define $\sigma_c' = \sigma_c/\nu$. Now we additionally want to scale these quantities. To do this, set $\hat{t} = \varepsilon t/t_0$ and $\hat{x} = \varepsilon x/x_0$.
To consider scaled functions, depending on the scaled variables, define $\hat{f}(\hat{t}, \hat{x}, v) = \varepsilon^{-d} f(t', x', v)$ and $\hat{g}(\hat{t}, \hat{x}, w) = \varepsilon^{-d} g(t', x', w)$. We now look at different terms of the equations separately. In [8], the same matrix $\mathcal{Q}_f$ is scaled to obtain a scaled version of the leading eigenvector, which in our case is $u_f$. We give a short overview on how this is done.
First, we define

$$\hat{\mathcal{Q}}_f(\hat{t}, \hat{x}) = \varepsilon^{-d} \mathcal{Q}_f(t', x') = \mathcal{Q}_{\hat{f}}(\hat{t}, \hat{x}), \tag{2.7}$$

with rescaled kernel $\hat{K}(|\hat{x}|) = K(\varepsilon x)$. Then we can write

$$\hat{\mathcal{Q}}_f = \int (v \otimes v - \frac{1}{d} Id) f(x, t, v) \, \mathrm{d}v + O(\varepsilon^2), \tag{2.8}$$

and

$$\hat{u}_f \otimes \hat{u}_f = u_f \otimes u_f + O(\varepsilon^2). \tag{2.9}$$

For the operator $\mathcal{Q}_g$ and its leading eigenvector $\bar{w}_g$, one can use the exact same procedure to obtain

$$\hat{\mathcal{Q}}_g = \int (w \otimes w - \frac{1}{d} Id) g(x,t,w)\, \mathrm{d}w + O(\varepsilon^2), \tag{2.10}$$

and

$$\hat{\bar{w}}_g \otimes \hat{\bar{w}}_g = w_g \otimes w_g + O(\varepsilon^2). \tag{2.11}$$

From now on we will skip the primes for better readability. It remains to be seen how the center of mass of the fibers, $\bar{y}_g$, is scaled. Consider

$$\rho_g(\hat{t}/\varepsilon, \hat{y}/\varepsilon) = \int g(\hat{t}/\varepsilon, \hat{y}/\varepsilon, w)\, \mathrm{d}w = \int g(t,y,w)\, \mathrm{d}w = \varepsilon^d \rho_{\hat{g}}. \tag{2.12}$$

Now we can compute

$$\int \hat{K}(|\hat{x}/\varepsilon - y|) y \rho_g(y)\, \mathrm{d}y = \varepsilon^{-d} \int \hat{K}(|\hat{x} - \hat{y}|/\varepsilon) \hat{y} \varepsilon^{-1} \rho_g(\hat{y}/\varepsilon)\, \mathrm{d}\hat{y} \tag{2.13}$$

$$= \varepsilon^{-1}\varepsilon^{-d} \int \hat{K}(|\hat{x} - \hat{y}|/\varepsilon) \hat{y} \rho_g(\hat{y}/\varepsilon)\, \mathrm{d}\hat{y}. \tag{2.14}$$

So we obtain

$$\bar{y}_g(x,t) = \hat{y}(\hat{t},\hat{x}) = \frac{\varepsilon^{-1}\varepsilon^{-d} \int \hat{K}(|\hat{x} - \hat{y}|/\varepsilon) \hat{y} \rho_g(\hat{y}/\varepsilon)\, \mathrm{d}\hat{y}}{\varepsilon^{-d} \int \hat{K}(|\hat{x} - \hat{y}|/\varepsilon) \rho_g(\hat{y}/\varepsilon)\, \mathrm{d}\hat{y}} \tag{2.15}$$

$$= \varepsilon^{-1} \frac{\int K(|\hat{x} - \hat{y}|) \hat{y} \rho_{\hat{g}}(\hat{y})\, \mathrm{d}\hat{y}}{\int K(|\hat{x} - \hat{y}|) \rho_{\hat{g}}(\hat{y})\, \mathrm{d}\hat{y}}. \tag{2.16}$$

Let $\alpha = O(\varepsilon)$, then $\tilde{\alpha} = \varepsilon^{-1}\alpha = O(1)$. We can consider this as a modelling choice.

Putting everything together, replacing $\hat{f}$ by $f^\varepsilon$ and doing the same for all other quantities as well as dropping the hats and tildes, we arrive at the rescaled system of equations:

$$\begin{aligned} \partial_t f^\varepsilon + s(|v \cdot \bar{u}_{f^\varepsilon}|) v \cdot \nabla_x f^\varepsilon = &- \varepsilon^{-1} \nabla_v \cdot (f^\varepsilon \nabla_v ((v \cdot \bar{u}_{f^\varepsilon})^2)) \\ &- \nabla_v \cdot \alpha f^\varepsilon \nabla_v ((\bar{y}_{g^\varepsilon} - x) \cdot v)) \\ &+ \varepsilon^{-1} \sigma_c \nabla_v \cdot ((v \cdot u_{f^\varepsilon})^2 \nabla_v f^\varepsilon) + O(\varepsilon), \end{aligned} \tag{2.17}$$

$$\partial_t g^\varepsilon = -\varepsilon^{-1} \nabla_w \cdot (g^\varepsilon (\tilde{\nu} \nabla_w ((w \cdot \bar{w}_g)^2)) + \sigma \nabla_w \cdot ((w \cdot \bar{w}_g)^2 \nabla_w g^\varepsilon)) + O(\varepsilon). \tag{2.18}$$

We can also write these equations in terms of operators $Q^{(1)}(f^\varepsilon)$ and $Q^{(2)}(g^\varepsilon)$:

$$\partial_t f^\varepsilon + s(|v \cdot u_{f^\varepsilon}|) v \cdot \nabla_x f^\varepsilon + \alpha f^\varepsilon \nabla_v \cdot (\nabla_v ((y_{g^\varepsilon} - x) \cdot v)) = \varepsilon^{-1} Q^{(1)}(f^\varepsilon) + O(\varepsilon), \tag{2.19}$$

$$\partial_t g^\varepsilon = \varepsilon^{-1} Q^{(2)}(g^\varepsilon) + O(\varepsilon). \tag{2.20}$$

Our goal is to compute the limit $\varepsilon \to 0$, which is done in Section 2.3. For this, however, we first analyze the two operators

$$Q^{(1)}(f^\varepsilon) = \sigma_c \nabla_v \cdot ((v \cdot u_{f^\varepsilon})^2 \nabla_v f^\varepsilon) - \nabla_v \cdot (f^\varepsilon (\nabla_v ((v \cdot u_{f^\varepsilon})^2)), \tag{2.21}$$

and

$$Q^{(2)}(g^\varepsilon) = -\nabla_w \cdot (g^\varepsilon (\widetilde{\nu} \nabla_w ((w \cdot \bar{w}_{g^\varepsilon})^2)) + \sigma \nabla_w \cdot ((w \cdot \bar{w}_{g^\varepsilon})^2 \nabla_w g^\varepsilon)), \tag{2.22}$$

in the following Section 2.2.

## 2.2. Equilibria

This section is concerned with finding the kernels of the operators $Q^{(1)}(f^\varepsilon)$ and $Q^{(2)}(g^\varepsilon)$, which are also called the sets of equilibria. The importance of this is justified, since by letting $\varepsilon \to 0$ in (2.19)-(2.20), the resulting solutions will lie in the kernels of the respective operators.
As done in many papers analyzing similar equations, we will first rewrite the operators in terms of a probability distribution. We will closely follow the structure and methods of [2]. We start with the following definition:

**Definition 2.2.1:** *The* Generalized von Mises (GVM) distribution *on the half-space* $\{w \cdot u > 0\}$ *with direction* $u$ *and concentration* $\kappa$ *is given by*

$$M_u(w) = \frac{1}{Z} \exp\left( -\frac{\kappa}{(w \cdot u)^2} \right), \tag{2.23}$$

*where*

$$Z = \int_{(w \cdot u) > 0} \exp\left( -\frac{\kappa}{(w \cdot u)^2} \right) \, \mathrm{d}w < \infty \tag{2.24}$$

*is a normalizing constant.*

We remark, that this definition varies in some papers, depending on the alignment one wants to model. In our case, as well as in [2], we only want our species' direction to be parallel their respective mean nematic direction. In contrast, often one is interested in the species' direction striving to the exact same vector as the mean nematic direction. In this case, the definition of $M_u(w)$ varies slightly, see for example Chapter 3 or [3].
We now rewrite $Q^{(1)}$ in terms of the GVM distribution. Since $Q^{(2)}$ has the same structure, the same computations apply with only slight modifications. For convenience, we will for now drop all $\varepsilon$'s and write $u$ for $u_{f^\varepsilon}$ as well as $\bar{w}$ for $\bar{w}_{g^\varepsilon}$.

**Lemma 2.2.2:** *The operator* $Q^{(1)}(f)$ *can be written as*

$$Q^{(1)}(f) = \sigma_c \nabla_v \cdot \left[ (v \cdot u)^2 M_u \nabla_v \left( \frac{f}{M_u} \right) \right]. \tag{2.25}$$

*Proof.* First, we do the following computation for $\kappa = 1/\sigma_c$:

$$\begin{aligned} Q^{(1)}(f) =& -\nabla_v f \cdot \nabla_v((v\cdot u)^2) - f\nabla_v \cdot (\nabla_v((v\cdot u)^2)) \\ &+ \sigma_c \nabla_v((v\cdot u)^2)\cdot\nabla_v f + \sigma_c (v\cdot u)^2 \nabla_v \cdot (\nabla_v f) \end{aligned} \tag{2.26}$$

$$=\sigma_c \nabla_v \cdot (-\kappa \nabla_v((v\cdot u)^2) f + (v\cdot u)^2 \nabla_v f) \tag{2.27}$$

$$=\sigma_c \nabla_v \cdot \left[(v\cdot u)^2 \left(\kappa \nabla((v\cdot u)^2)\frac{-f}{(v\cdot u)^2} + \nabla_v f\right)\right]. \tag{2.28}$$

Now, we calculate:

$$M_u \nabla_v \left(\frac{f}{M_u}\right) = \nabla_v f - f\nabla_v((\log(M_u)) \tag{2.29}$$

$$= \nabla_v f - f\nabla\left((\log\left(\frac{1}{Z}\right) - \frac{\kappa}{(v\cdot u)^2})\right) \tag{2.30}$$

$$= \nabla_v f + \kappa f \nabla_v \left(\frac{1}{(v\cdot u)^2}\right) \tag{2.31}$$

$$= \nabla_v f + \kappa f \frac{1}{(v\cdot u)^4}(-\nabla_v((v\cdot u)^2)). \tag{2.32}$$

If we multiply the last equation by $(v\cdot u)^2$ and compare it with the above computation for $Q^{(1)}$, we get our desired result. □

Note that for $Q^{(2)}$ we need to use $\kappa_g = \widetilde{\nu}/\sigma_c$. By abuse of notation, we will not write if we use $\kappa_f$ or $\kappa_g$ in the definition of $M$, since it is clear from the notation of the alignment vector. We continue with the following definition.

**Definition 2.2.3** (From [2, Definition 3.3])**:** *(i) Let $\varphi$ be smooth and let $u \in \mathbb{S}^{d-1}$. We define the linear operator*

$$\mathcal{L}_u \varphi = \sigma \frac{1}{M_u} \nabla_w ((w\cdot u)^2 M_u \nabla_w \varphi). \tag{2.33}$$

*(ii) Let*

$$\mathcal{H}_u = \left\{ g \textit{ measurable} : \int g(w) M_u(w)\, \mathrm{d}w < \infty \right\}. \tag{2.34}$$

*Endowed with the inner product*

$$(g,h)_u = \int g(w)h(w)M_u(w)\, \mathrm{d}w, \tag{2.35}$$

*it is a Hilbert space.*

*(iii) We also need the Hilbert space*

$$\mathcal{V}_u = \{ g \in \mathcal{H}_u : \sqrt{M_u(w)} |w\cdot u| \nabla_w g(w) \in L^2(\mathbb{S}^{d-1})\}, \tag{2.36}$$

*with inner product*

$$(g,h)_{\mathcal{V}_u} = (g,h)_u + \int (\sqrt{M_u(w)}|w\cdot u|\nabla_w g(w))(\sqrt{M_u(w)}|w\cdot u|\nabla_w h(w))\,\mathrm{d}w. \tag{2.37}$$

(iv) *The map $\mathcal{L}_u : \mathcal{V}_u \to \mathcal{V}_u'$ defines a continuous map by*

$$\langle \mathcal{L}_u\varphi, \psi\rangle_{\mathcal{V}_u',\mathcal{V}_u} = -\sigma \int (\sqrt{M_u(w)}|w\cdot u|\nabla_w\varphi(w))(\sqrt{M_u(w)}|w\cdot u|\nabla_w\psi(w))\,\mathrm{d}w, \tag{2.38}$$

*for all $\varphi, \psi \in \mathcal{V}_u$, where $\langle .,. \rangle_{\mathcal{V}_u',\mathcal{V}_u}$ denotes the duality bracket. This extends the $\mathcal{H}_u$-inner product, i.e. for $h \in \mathcal{H}_u \subseteq \mathcal{V}_u'$ and $g \in \mathcal{V}_u$ we have $\langle h, g\rangle_{\mathcal{V}_u',\mathcal{V}_u} = \int hgM_u\,\mathrm{d}w$. Further, the operator $\mathcal{L}_u$ is formally self-adjoint, in the sense that for $\varphi, \psi \in \mathcal{V}_u$ we have*

$$\langle \mathcal{L}_u\varphi, \psi\rangle_{\mathcal{V}_u',\mathcal{V}_u} = \langle \mathcal{L}_u\psi, \varphi\rangle_{\mathcal{V}_u',\mathcal{V}_u}.$$

(v) *The operator $Q^{(1)}$ is defined for $f \in L^1(\mathbb{S}^{d-1})$ and $f/M_u \in \mathcal{V}_u$ by*

$$Q^{(1)}(f) = M_u\mathcal{L}_u(f/M_u). \tag{2.39}$$

*The operator $Q^{(1)}$ is defined similarly.*

We will simply write

$$\langle \mathcal{L}_u\varphi, \psi\rangle_{\mathcal{V}_u',\mathcal{V}_u} = -\sigma \int \nabla_w\varphi \cdot \nabla_w\psi M_u(w\cdot u)^2\,\mathrm{d}w. \tag{2.40}$$

Our aim is to solve $\mathcal{L}_u\varphi = \psi$, where $\psi$ is a given function. The weak formulation of this problem reads

$$\begin{cases} \varphi \in \mathcal{V}_u, \\ -\sigma\int \nabla_w\varphi\cdot\nabla_w\xi M_u(w\cdot u)^2\,\mathrm{d}w = \int \psi\xi M_u\,\mathrm{d}w, \quad \forall \xi \in \mathcal{V}_u. \end{cases} \tag{$*$}$$

**Proposition 2.2.4** (From [2, Proposition 3.4])**:** *(i) Suppose $\psi = 0$. Then the unique solutions of $(*)$ are piecewise constant functions*

$$\varphi(w) = \begin{cases} C_+, & w\cdot u > 0, \\ C_-, & w\cdot u < 0. \end{cases} \tag{2.41}$$

(ii) *Let $d = 2$ and let $\psi \in \mathcal{V}_u$. Then there exists a solution of $(*)$ iff the following two solvability conditions hold*

$$\int_{w\cdot u>0} \psi M_u\,\mathrm{d}w = 0, \quad \textit{and} \quad \int_{w\cdot u<0} \psi M_u\,\mathrm{d}w = 0. \tag{2.42}$$

*We then have a unique solution $\varphi$ to $(*)$, which satisfies (2.42). The general solution in $\mathcal{V}_u$ to $(*)$ is the sum of the solution of (i) and the above unique solution (2.41).*

(iii) *Let $\varphi_0$ be the unique solution to $(*)$ with $\psi_0 \in \mathcal{V}_{(1,0)^T}$ satisfying the solvability conditions (2.42). Let $u \in \mathbb{S}^1$ with $u = (\cos(\theta), \sin(\theta))^T$ and let $\widetilde{\theta} \in [-\pi/2, \pi/2)$. Define $\psi(\theta) = \psi_0(\theta - \widetilde{\theta})$ and $\varphi(\theta) = \varphi_0(\theta - \widetilde{\theta})$. Then $\varphi$ is the solution to $(*)$ satisfying (2.42).*

*Proof.* We can always choose $u = (1, 0, \ldots, 0)^T$ without loss of generality, which translates to $\bar{\theta} = 0$ in $d = 2$. Note that in this case, we get $(w \cdot u) = w_1$ and in particular $(w \cdot u) = \cos(\theta)$ for $d = 2$.

(i) First, we show that $\varphi \in \mathcal{V} := \mathcal{V}_u$ and satisfies $(*)$ with $\psi = 0$. For this, we compute $\nabla_w \varphi$. Note that it is distributionally defined, so

$$(-1)^d \int \xi \nabla_w \varphi = \int \nabla_w \xi \varphi = \int_{w_1>0} \nabla_w \xi C_+ + \int_{w_1<0} \nabla_w \xi C_- \tag{2.43}$$

$$= \int (C_+ \chi_{\{w_1>0\}} + C_- \chi_{\{w_1<0\}}) \nabla_w \xi. \tag{2.44}$$

Since the distributional derivative of an indicator function, is the $\delta$-distribution, we have

$$\nabla_w \varphi = (C_+ - C_-)\delta_{w_1=0}. \tag{2.45}$$

Therefore,

$$\sqrt{M}|w_1|\nabla_w \varphi = 0, \tag{2.46}$$

implying that $\varphi \in \mathcal{V}$. Furthermore,

$$-\sigma \int \nabla_w \varphi \cdot \nabla_w \xi M w_1^2 \, \mathrm{d}w = 0, \tag{2.47}$$

which means that $\varphi$ satisfies $(*)$ with $\psi = 0$.

Now we suppose that $\varphi \in \mathcal{V}$ satisfies $(*)$ with $\psi = 0$. We take $\xi = \varphi$ as a test function, then we get

$$|\nabla_w \varphi|^2 \sqrt{M} w_1 = 0. \tag{2.48}$$

But this shows that $\varphi$ is constant on both $\{w_1 > 0\}$ and $\{w_1 < 0\}$, so it is of the desired form of (2.41).

(ii) Let $\psi \in \mathcal{V}$ with corresponding solution $\varphi$ of $(*)$. Take

$$\xi = \begin{cases} C_+, & w_1 > 0, \\ C_-, & w_1 < 0, \end{cases} \tag{2.49}$$

then we see that $\psi$ must satisfy the solvability conditions (2.42), since

$$-\sigma \int \nabla_w \varphi \cdot \nabla_w \xi M w_1^2 \, \mathrm{d}w = \int \psi \xi M \, \mathrm{d}w. \tag{2.50}$$

This shows the first implication.

For the other direction we define $\widetilde{\mathcal{V}} = \{\varphi \in \mathcal{V} : \int_{w_1>0} \varphi M = \int_{w_1<0} \varphi M = 0\} \subseteq \mathcal{V}$. Now we suppose $\psi$ satisfies the solvability conditions (2.42) and $\varphi$ solves

$$\begin{cases} \varphi \in \widetilde{\mathcal{V}}, \\ -\sigma \int \nabla_w \varphi \cdot \nabla_w \xi M w_1^2 \, \mathrm{d}w = \int \psi \xi M \, \mathrm{d}w, \quad \forall \xi \in \widetilde{\mathcal{V}}. \end{cases} \tag{$\tilde{*}$}$$

We claim that if $\varphi$ solves ($\tilde{*}$), then it also solves ($*$). Indeed, let $\xi \in \mathcal{V}$ and define $\widetilde{\xi} = \xi - \xi_0$ with

$$\xi_0 = \begin{cases} C_+, & w_1 > 0, \\ C_-, & w_1 < 0, \end{cases} \tag{2.51}$$

where $C_+$ and $C_-$ are explicitly given by

$$C_\pm = \int_{\pm w_1 > 0} \xi M \, \mathrm{d}w. \tag{2.52}$$

Because of this choice, $\widetilde{\xi} \in \widetilde{\mathcal{V}}$. Now we apply ($\tilde{*}$) with test function $\widetilde{\xi}$. Again, by the definition of $\xi_0$, all terms corresponding to $\xi_0$ vanish. This shows the claim.

We now want to apply Lax-Milgram to show that ($\tilde{*}$) has a unique solution. The linear forms $\varphi \mapsto \int_{\pm w_1>0} \varphi M \, \mathrm{d}w$ are continuous, so $\widetilde{\mathcal{V}}$ is a closed subspace of the Hilbert space $\mathcal{V}$, making it a Hilbert space itself. We can rewrite ($\tilde{*}$) as

$$\begin{cases} \varphi \in \widetilde{\mathcal{V}}, \\ a(\varphi, \xi) = L(\xi), \quad \forall \xi \in \widetilde{\mathcal{V}}, \end{cases} \tag{2.53}$$

with $a(.,.)$ symmetric, continuous and bilinear and $L$ linear. We need to show coercivity of $a(.,.)$, i.e. $a(\varphi, \varphi) \geq C\|\varphi\|^2$. This is equivalent to

$$\int |\nabla_w \varphi|^2 M w_1^2 \geq C \left( \|\varphi\|_{\mathcal{H}_u} + \int |\nabla_w \varphi|^2 M w_1^2 \right), \tag{2.54}$$

which can be simplified to

$$\int |\nabla_w \varphi|^2 M w_1^2 \, \mathrm{d}w \geq C \int |\varphi|^2 M \, \mathrm{d}w. \tag{2.55}$$

This Poincaré-type inequality has been proven in [2] for $d = 2$.

(iii) This is a simple change of variables, which has also been done in [2].

□

For all following definitions and theorems that allow the dimension being arbitrary, we will keep the current notation of vectors on $\mathbb{S}^{d-1}$ and change to angles, if it is needed. We now introduce the concept equilibria, which will be crucial for analyzing the limit $\varepsilon \to 0$ in our system of PDEs.

**Definition 2.2.5:** *An* equilibrium *of $Q^{(1)}$ is a function $f$ with $f \in L^1(\mathbb{S}^{d-1})$, $f/M_{\bar{w}} \in \mathcal{V}_{\bar{w}}$ and $Q^{(1)}(f) = 0$. We denote the set of equilibria of $Q^{(1)}$ by $\mathcal{E}_1$. For $Q^{(2)}$, we define $\mathcal{E}_2$ analogously.*
*An equilibrium of the system* (2.19)-(2.20) *is a pair of functions $(f, g) \in \mathcal{E}_1 \times \mathcal{E}_2$. We denote the set of equilibria of the system by $\mathcal{E}$.*

As we will see in Lemma 2.2.7 below, the set of equilibria $\mathcal{E}$ can be characterized by the GVM multiplied by a constant, depending on the orientation vector. For this we introduce the following notation:

**Definition 2.2.6:** *Let $\rho_+, \rho_- \in [0, \infty)$ and let $u \in \mathbb{S}^{d-1}$. We define*

$$\bar{g}_{\rho_+,\rho_-,u}(w) = \begin{cases} \rho_+ M_u(w), & w \cdot u > 0, \\ \rho_- M_u(w), & w \cdot u < 0, \end{cases} \tag{2.56}$$

*or equivalently,*

$$\bar{g}_{\rho_+,\rho_-,u}(w) = M_u(w)(\rho_+ \chi_u^+(w) + \rho_- \chi_u^-(w)), \tag{2.57}$$

*where the indicator functions $\chi_u^\pm$ are defined as*

$$\chi_u^\pm(w) = \begin{cases} 1, & \pm(w \cdot u) > 0, \\ 0, & \pm(w \cdot u) < 0. \end{cases} \tag{2.58}$$

*We define $\bar{f}$ analogously.*

With this definition, we have a nice form for $\mathcal{E}$:

**Lemma 2.2.7** (From [2, Lemma 3.7])**:** *The set of equilibria of $Q^{(2)}$ can be written as*

$$\mathcal{E}_2 = \{\bar{g}_{\rho_+,\rho_-,u} : \ \rho_+, \rho_- \in [0, \infty), u \in \mathbb{S}^1\}. \tag{2.59}$$

*The set $\mathcal{E}_1$ can be written analogously.*

*Proof.* Let $u \in \mathbb{S}^1, u = (\cos(\bar{\theta}), \sin(\bar{\theta}))^T$ be arbitrary. By Proposition 2.2.4, we get

$$M_u \mathcal{L}_u \left( \frac{g}{M_u} \right) = 0 \iff \frac{g}{M_u} \text{ is of the form (2.41)} \tag{2.60}$$

$$\iff \exists \rho_+^g, \rho_-^g \in [0, \infty) : g = \bar{g}_{\rho_+^g, \rho_-^g, u}. \tag{2.61}$$

So if $g \in \mathcal{E}_2$, then $g$ is of the desired form. The same holds for $f \in \mathcal{E}_1$.
For the other direction, if $\rho_+^g, \rho_-^g \in [0,\infty)$, we show that $Q^{(2)}(\bar{g}_{\rho_+^g,\rho_-^g,u}) = 0$. But by the equivalence above, it suffices to show that $u = u_{\bar{g}}$. Now, using an equivalent definition of $u$ being an the leading eigenvector of $\mathcal{Q}_g$ in two dimensions, see [5, Appendix A], we obtain the condition

$$\int_{-\pi}^{\pi} \sin(2(\theta - \bar{\theta}))\bar{g}_{\rho_+^g,\rho_-^g,u} \,\mathrm{d}\theta = 0. \tag{2.62}$$

Using the change of variables $\theta' = \theta - \bar{\theta}$, which we also will see multiple times below, we get the condition

$$\int_{-\pi}^{\pi} \sin(2\theta)\bar{g}_{\rho_+^g,\rho_-^g,0} \,\mathrm{d}\theta = 0, \tag{2.63}$$

where we used periodicity. Due to the symmetry of $\bar{g}_{\rho_+^g,\rho_-^g,0}$, the condition holds, which completes the proof. □

We now want to introduce the concept of Generalized Collision Invariants, which has proven to be crucial to compute the hydrodynamic limit.

**Definition 2.2.8:** *Let $u_1, u_2 \in \mathbb{S}^{d-1}$. We say that $(\Psi_1, \Psi_2) \in \mathcal{V}_{u_1} \times \mathcal{V}_{u_2}$ is a* Generalized Collision Invariant *(GCI) associated with $u_1$ and $u_2$ if*

$$\begin{gathered}\langle \mathcal{L}_{u_1} \frac{f}{M_{u_1}}, \Psi_1 \rangle_{\mathcal{V}'_{u_1},\mathcal{V}_{u_1}} = 0, \quad \text{and} \quad \langle \mathcal{L}_{u_2} \frac{g}{M_{u_2}}, \Psi_2 \rangle_{\mathcal{V}'_{u_2},\mathcal{V}_{u_2}} = 0, \\ \forall f \in \mathcal{V}_{u_1} \cap L^1(\mathbb{S}^{d-1}) : u_1 \text{ is an eigenvector of } \mathcal{Q}_f, \\ \forall g \in \mathcal{V}_{u_2} \cap L^1(\mathbb{S}^{d-1}) : u_2 \text{ is an eigenvector of } \mathcal{Q}_g.\end{gathered} \tag{2.64}$$

*If we recall the definition of $\langle .,. \rangle_{\mathcal{V}'_{u_2},\mathcal{V}_{u_2}}$, then we can rewrite the condition for $g$ as*

$$\int \nabla_w (g/M_{u_2}) \cdot \nabla_w \Psi_2 M_{u_2}(w)(w \cdot u_2)^2 \,\mathrm{d}w = 0, \tag{2.65}$$

*and we can do the same for the condition for $f$.*

Our goal is to explicitly write down the set of GCI. For this we will use Proposition 2.2.4, which is again done for the case $d = 2$. First, however, we will find an equivalent condition for a pair of functions being a GCI. For this we use [2, Lemma 3.10] and modify it for two species. We remark that a similar characterization exists in arbitrary dimensions, see [3]. Nevertheless, for simplicity we will only discuss this result for $d = 2$:

**Lemma 2.2.9:** *The pair $(\Psi_1, \Psi_2) \in \mathcal{V}_{u_1} \times \mathcal{V}_{u_2}$ associated with $u_1$ and $u_2$ is a GCI if and only if there exist $\beta_1 \in \mathbb{R}$ and $\beta_2 \in \mathbb{R}$ such that*

$$\begin{aligned}\mathcal{L}_{u_1}(\Psi_1) &= \beta_1 \sin(2(\theta - \bar{\theta}_1)), \\ \mathcal{L}_{u_2}(\Psi_2) &= \beta_2 \sin(2(\theta - \bar{\theta}_2)).\end{aligned} \tag{2.66}$$

*Proof.* As in Lemma 2.2.7, we use that $\bar{\theta}_f = \bar{\theta}_1$ can be rewritten as the condition

$$\int_{-\pi}^{\pi} \sin(2(\theta - \theta_1)) f(\theta) \, \mathrm{d}\theta = 0. \tag{2.67}$$

We can also do the same for $\bar{\theta}_g = \bar{\theta}_2$. By a duality argument already used in [4], the condition (3.6) can be reformulated as there being two constants $\beta_1, \beta_2 \in \mathbb{R}$ such that

$$\begin{cases} \langle \mathcal{L}_{\theta_1} \frac{f}{M_{\theta_1}}, \Psi_1 \rangle_{\mathcal{V}'_{\theta_1}, \mathcal{V}_{\theta_1}} = \beta_1 \int_{-\pi}^{\pi} \sin(2(\theta - \bar{\theta}_1)) f(\theta) \, \mathrm{d}\theta, \ \forall f \in \mathcal{V}_{\bar{\theta}_1} \cap L^1(-\pi, \pi), \\ \langle \mathcal{L}_{\theta_2} \frac{g}{M_{\theta_2}}, \Psi_2 \rangle_{\mathcal{V}'_{\theta_2}, \mathcal{V}_{\theta_2}} = \beta_2 \int_{-\pi}^{\pi} \sin(2(\theta - \bar{\theta}_2)) g(\theta) \, \mathrm{d}\theta, \ \forall g \in \mathcal{V}_{\bar{\theta}_2} \cap L^1(-\pi, \pi). \end{cases} \tag{2.68}$$

Having seen that an operator of the form $\mathcal{L}_u$ is formally self-adjoint, we can rewrite the above as

$$\begin{cases} \langle \mathcal{L}_{\theta_1} \Psi_1, \frac{f}{M_{\theta_1}} \rangle_{\mathcal{V}'_{\theta_1}, \mathcal{V}_{\theta_1}} = \beta_1 (\sin(2(\theta - \theta_1)), \frac{f}{M_{\theta_1}})_{\mathcal{H}_{\theta_1}}, \ \forall f \in \mathcal{V}_{\bar{\theta}_1} \cap L^1(-\pi, \pi), \\ \langle \mathcal{L}_{\theta_2} \Psi_2, \frac{g}{M_{\theta_2}} \rangle_{\mathcal{V}'_{\theta_2}, \mathcal{V}_{\theta_2}} = \beta_2 (\sin(2(\theta - \theta_2)), \frac{g}{M_{\theta_2}})_{\mathcal{H}_{\theta_2}}, \ \forall g \in \mathcal{V}_{\bar{\theta}_2} \cap L^1(-\pi, \pi). \end{cases} \tag{2.69}$$

Therefore, we get (2.66). The converse is clear. □

**Proposition 2.2.10:** *(i) Given $u_1, u_2 \in \mathbb{S}^2$, the set $\mathcal{G}_{u_1,u_2}$ of GCI associated with $u_1$ and $u_2$ is given by*

$$\mathcal{G}_{u_1,u_2} = \operatorname{span}(\{\chi^+_{u_1}, \chi^-_{u_1}, \psi_{u_1}\} \times \{\chi^+_{u_2}, \chi^-_{u_2}, \psi_{u_2}\}), \tag{2.70}$$

*where $\chi^{\pm}_{u_i}$ given by (2.58) and $\psi_{u_i}$ is the unique solution to the problem $(*)$ with right-hand side $\beta_i \sin(2(\theta - \bar{\theta}_i))$.*

*(ii) We can write*

$$\chi^{\pm}_{\bar{\theta}_i}(\theta) = \chi(\theta - \bar{\theta}_i), \quad \text{and} \quad \psi_{\bar{\theta}_i}(\theta) = \psi(\theta - \bar{\theta}_i), \tag{2.71}$$

*where $\chi^{\pm}$ and $\psi$ are defined as $\chi_0$ and $\psi_0$. We also have the following symmetries:*

$$\psi(-\theta) = -\psi(\theta), \quad \text{and} \quad \psi(\pi - \theta) = -\psi(\theta). \tag{2.72}$$

*Proof.* Since this has been proven in [2] for one species, we only give a sketch of the proof.
Using the above Lemma 2.2.9, we get $\mathcal{L}_{u_1} \Psi_1 = \beta_1 \sin(2(\theta - \bar{\theta}_1))$. We want to find the solution space of this problem. But we have already shown that this problem has existence and uniqueness of a solution if the solvability conditions are satisfied. However, one can easily see that they hold by symmetry. Therefore, we get a unique solution $\psi_{u_1}$ and all other solutions differ by a solution of the homogeneous problem (2.41). The same holds for $u_2$ and thus $(i)$ is proven.
For $(ii)$, Proposition 2.2.4 can be applied. □

## 2.3. Hydrodynamic limit

In this section, we consider the (formal) limit $\varepsilon \to 0$. So suppose $f^\varepsilon \to f^0$ and $g^\varepsilon \to f^0$. By looking at the system of scaled equations (2.19)-(2.20), we see that

$$Q^{(1)}(f^0) = 0, \quad \text{and} \quad Q^{(2)}(g^0) = 0. \tag{2.73}$$

But, since we have already computed the kernel, or rather equilibria, of the operators $Q^{(1)}$ and $Q^{(2)}$ in Proposition 2.2.4, we conclude that there exist $\rho^f_\pm(x,t)$ and $\rho^g_\pm(x,t)$ as well as $u_1(x,t)$ and $u_2(x,t)$ such that

$$f^0(x,t,v) = \bar{f}_{\rho^f_+(x,t),\rho^f_-(x,t),u_1(x,t)}, \tag{2.74}$$

$$g^0(x,t,w) = \bar{g}_{\rho^g_+(x,t),\rho^g_-(x,t),u_2(x,t)}. \tag{2.75}$$

Our task is to find equations for $\rho^f_\pm$, $\rho^g_\pm$, $u_1$, and $u_2$. The method for obtaining the defining equations for the functions, we are looking for, is using GCI, as it is done in for example [2], [4], and many other papers. More precisely, we will multiply the scaled equations by their respective GCI and integrate over the sphere. We summarize the results in the following theorem.

**Theorem 2.3.1:** *In the limit $\varepsilon \to 0$, we get*

$$f^\varepsilon(x,t,v) \to \bar{f}_{\rho^f_+(x,t),\rho^f_-(x,t),u_1(x,t)}, \tag{2.76}$$

$$g^\varepsilon(x,t,w) \to \bar{g}_{\rho^g_+(x,t),\rho^g_-(x,t),u_2(x,t)}, \tag{2.77}$$

*where the quantities $\rho^f_\pm(x,t)$, $\rho^g_\pm(x,t)$, and $u_1(x,t) = (\cos(\bar\theta_1), \sin(\bar\theta_1))^T$, $u_2(x,t) = (\cos(\bar\theta_2), \sin(\bar\theta_2))^T$ are given by the equations*

$$\begin{aligned} \partial_t \rho^f_+ &+ A_{11} v(\bar\theta_1) \cdot \nabla_x \rho^f_+ \\ &+ A_{12} \rho^f_+ v^\perp(\bar\theta_1) \cdot \nabla_x \bar\theta_1 \\ &- A_{13} \rho^f_+ (\bar{y}_g - x) \cdot v(\bar\theta_1) = 0, \end{aligned} \tag{2.78}$$

$$\begin{aligned} \partial_t \rho^f_- &- A_{11} v(\bar\theta_1) \cdot \nabla_x \rho^f_- \\ &+ A_{12} \rho^f_- v^\perp(\bar\theta_1) \cdot \nabla_x \bar\theta_1 \\ &+ A_{13} \rho^f_- (\bar{y}_g - x) \cdot v(\bar\theta_1) = 0, \end{aligned} \tag{2.79}$$

$$\begin{aligned} A_{21} v^\perp(\bar\theta_1) &\cdot \nabla_x (\rho^f_+ - \rho^f_-) \\ &+ A_{22} \partial_t \bar\theta_1 (\rho^f_+ + \rho^f_-) \\ &+ A_{23} (\rho^f_+ + \rho^f_-) v(\bar\theta_1) \cdot \nabla_x \bar\theta \\ &+ A_{24} (\rho^f_+ + \rho^f_-)(\bar{y}_g - x) \cdot v^\perp(\bar\theta_1) = 0, \end{aligned} \tag{2.80}$$

*and*

$$\partial_t \rho_+^g = 0, \tag{2.81}$$
$$\partial_t \rho_-^g = 0, \tag{2.82}$$
$$\partial_t \bar{\theta}_2 (\rho_+^g - \rho_-^g) = 0. \tag{2.83}$$

*Introducing the notation*

$$\langle f \rangle := \int_{\cos(\theta)>0} M(\theta) f(\theta) \,\mathrm{d}\theta, \tag{2.84}$$

*and*

$$\langle f \rangle_s := \int_{\cos(\theta)>0} M(\theta) s(|\cos(\theta)|) f(\theta) \,\mathrm{d}\theta, \tag{2.85}$$

*the constants are given by*

$$A_{11} = \langle \cos \rangle_s, \tag{2.86}$$
$$A_{12} = 2\kappa_f \langle \frac{\sin^2}{\cos^3} \rangle_s, \tag{2.87}$$
$$A_{13} = \alpha \left( 2\kappa_f \langle \frac{\sin^2}{\cos^3} \rangle - \langle \cos \rangle \right), \tag{2.88}$$
$$A_{21} = \langle \psi \sin \rangle_s, \tag{2.89}$$
$$A_{22} = 2\kappa_f \langle \psi \frac{\sin}{\cos^3} \rangle_s, \tag{2.90}$$
$$A_{23} = 2\kappa_f \langle \psi \frac{\sin}{\cos^2} \rangle_s, \tag{2.91}$$
$$A_{24} = \alpha \left( -2\kappa_f \langle \psi \frac{\sin}{\cos^2} \rangle + \langle \psi \sin \rangle \right). \tag{2.92}$$

### 2.3.1. The constant $\mathrm{Q}^{(1)}$-GCI

First, we recall that the constant GCI associated to $Q^{(1)}$ are $\chi^{\pm}_{u_{f^\varepsilon}}$.
We take equation (2.19), multiply it by $\chi^{\pm}_{u_{f^\varepsilon}}$, and integrate with respect to $v$ to obtain

$$\int Q^{(1)} f^\varepsilon(v) \chi^{\pm}_{u_{f^\varepsilon}}(v) \,\mathrm{d}v = \langle \mathcal{L}_{u_{f^\varepsilon}} \frac{f^\varepsilon}{M_{u_{f^\varepsilon}}}, \chi^{\pm}_{u_{f^\varepsilon}}(v) \rangle_{\mathcal{V}_{u_{f^\varepsilon}}, \mathcal{V}_{u_{f^\varepsilon}}} = 0, \tag{2.93}$$

by the definition of GCI. Now multiplying by $\varepsilon$, we obtain

$$\int_{\pm v \cdot u_{f^\varepsilon}(x,t)>0} \partial_t f^\varepsilon + s(|v \cdot u_1(x,t)|) v \cdot \nabla_x f^\varepsilon + \alpha \nabla_v \cdot (f^\varepsilon \nabla_v ((\bar{y}_g - x) \cdot v)) \,\mathrm{d}v = 0. \tag{2.94}$$

Let $\varepsilon \to 0$, then

$$\begin{aligned} X = \int_{\pm v \cdot u_1(x,t)>0} & \partial_t \bar{f}_{\rho_+^f, \rho_-^f, u_1} + s(|v \cdot u_1|) v \cdot \nabla_x \bar{f}_{\rho_+^f, \rho_-^f, u_1} \\ & + \alpha \nabla_v \cdot (\bar{f}_{\rho_+^f, \rho_-^f, u_1} \nabla_v ((\bar{y}_g - x) \cdot v))) \,\mathrm{d}v. \end{aligned} \tag{2.95}$$

We will start by computing $\partial_t \bar{f}$ and $\nabla_x \bar{f}$. First we calculate

$$\partial_t M_{u_1(x,t)} = \frac{1}{Z}\partial_t \exp\left(-\frac{\kappa_1}{(w\cdot u_1(x,t))^2}\right) \tag{2.96}$$

$$= M_{u_1(x,t)} \cdot 2\kappa_f \frac{v\cdot \partial_t u_1(x,t)}{(v\cdot u_1(x,t))^3}. \tag{2.97}$$

So now we obtain

$$\partial_t \bar{f} = \partial_t[M_{u_1(x,t)}(\rho_+^f(x,t)\chi^+_{u_1(x,t)} + \rho_-^f(x,t)\chi^-_{u_1(x,t)})] \tag{2.98}$$

$$\begin{aligned} &= \partial_t M_{u_1(x,t)}(\rho_+^f(x,t)\chi^+_{u_1(x,t)} + \rho_-^f(x,t)\chi^-_{u_1(x,t)}) \\ &\quad + M_{u_1(x,t)}(\partial_t\rho_+^f(x,t)\chi^+_{u_1(x,t)} + \partial_t\rho_-^f(x,t)\chi^-_{u_1(x,t)}) \end{aligned} \tag{2.99}$$

$$\begin{aligned} &= M_{u_1(x,t)}\Big[2\kappa_f \frac{v\cdot\partial_t u_1(x,t)}{(v\cdot u_1(x,t))^3}(\rho_+^f(x,t)\chi^+_{u_1(x,t)} + \rho_-^f(x,t)\chi^-_{u_1(x,t)}) \\ &\quad + \partial_t\rho_+^f(x,t)\chi^+_{u_1(x,t)} + \partial_t\rho_-^f(x,t)\chi^-_{u_1(x,t)}\Big]. \end{aligned} \tag{2.100}$$

The computation for $\nabla_x \bar{f}$ is basically the same. We will do all of the following computations in $d=2$ and switch to an integral with respect to an angle. We abbreviate $T = (\partial_t + s(|\cos(\theta-\bar{\theta}_1)|)v(\theta)\cdot\nabla_x)$ and get

$$T\bar{f} = M_{u_1}\Big[2\kappa_f \frac{\sin(\theta-\bar{\theta}_1)}{\cos(\theta-\bar{\theta}_1)^3} T\bar{\theta}_1(\rho_+^f\chi^+_{u_1} + \rho_-^f\chi^-_{u_1}) + \chi^+_{u_1}T\rho_+^f + \chi^-_{u_1}T\rho_-^f\Big]. \tag{2.101}$$

If $\cos(\theta-\bar{\theta}_1)>0$ we obtain

$$\begin{aligned} X = &\int_{\cos(\theta-\bar{\theta}_1)>0} M_{u_1}(\theta)\partial_t\rho_+^f \,\mathrm{d}\theta \\ &+ \int_{\cos(\theta-\bar{\theta}_1)>0} M_{u_1}(\theta)s(|\cos(\theta-\bar{\theta}_1)|)v(\theta)\cdot\nabla_x\rho_+^f \\ &+ 2\kappa_f\rho_+^f \int_{\cos(\theta-\bar{\theta}_1)>0} M_{u_1}(\theta)\partial_t\bar{\theta}_1 \frac{\sin(\theta-\bar{\theta}_1)}{\cos(\theta-\bar{\theta}_1)^3}\,\mathrm{d}\theta \\ &+ 2\kappa_f\rho_+^f \int_{\cos(\theta-\bar{\theta}_1)>0} M_{u_1}(\theta)s(|\cos(\theta-\bar{\theta}_1)|)v(\theta)\cdot\nabla_x\bar{\theta}_1 \frac{\sin(\theta-\bar{\theta}_1)}{\cos(\theta-\bar{\theta}_1)^3}\,\mathrm{d}\theta \\ &+ \alpha\int_{\cos(\theta-\bar{\theta}_1)>0} \partial_\theta(M_{u_1}(\theta)\rho_+^f\partial_\theta((\bar{y}_g - x)\cdot v(\theta)M_{u_1}(\theta)\rho_+^f))\,\mathrm{d}\theta \end{aligned} \tag{2.102}$$

$$= X_1 + X_2 + X_3 + X_4 + X_5. \tag{2.103}$$

Observe that by the normalization property of $M$ we get $X_1 = \partial_t\rho_+^f$. Further

$$X_3 = 2\kappa_f\rho_+^f\partial_t\bar{\theta}_1(x,t)\int_{\cos(\theta)>0} M(\theta)\frac{\sin(\theta)}{\cos(\theta)^3}\,\mathrm{d}\theta = 0, \tag{2.104}$$

by symmetry. Using

$$v(\theta) = \cos(\theta - \bar{\theta}_1) v(\bar{\theta}_1) + \sin(\theta - \bar{\theta}_1) v^\perp(\bar{\theta}_1), \tag{2.105}$$

we further obtain

$$\begin{aligned} X_2 = \int_{\cos(\theta-\bar{\theta}_1)>0} & s(|\cos(\theta - \bar{\theta}_1)|) \cos(\theta - \bar{\theta}_1) v(\bar{\theta}_1) \cdot \nabla_x \rho_+^f \\ & + s(|\cos(\theta - \bar{\theta}_1)|) \sin(\theta - \bar{\theta}_1) v^\perp(\bar{\theta}_1) \cdot \nabla_x \rho_+^f \,\mathrm{d}\theta \end{aligned} \tag{2.106}$$

$$\begin{aligned} =& v(\bar{\theta}_1) \cdot \nabla_x \rho_+^f \int_{\cos(\theta)>0} M(\theta) s(|\cos(\theta)|) \cos(\theta) \,\mathrm{d}\theta \\ & + v^\perp(\bar{\theta}_1) \cdot \nabla_x \rho_+^f \int_{\cos(\theta)>0} M(\theta) s(|\cos(\theta)|) \sin(\theta) \,\mathrm{d}\theta, \end{aligned} \tag{2.107}$$

where the second term vanishes by symmetry. Similarly, one can compute

$$\begin{aligned} X_4 =& 2\kappa_f \rho_+^f v(\bar{\theta}_1) \cdot \nabla_x \bar{\theta}_1(x,t) \int_{\cos(\theta)>0} M(\theta) s(|\cos(\theta)|) \frac{\sin(\theta)}{\cos(\theta)^2} \,\mathrm{d}\theta \\ & + 2\kappa_f \rho_+^f v^\perp(\bar{\theta}_1) \cdot \nabla_x \bar{\theta}_1(x,t) \int_{\cos(\theta)>0} M(\theta) s(|\cos(\theta)|) \frac{\sin(\theta)^2}{\cos(\theta)^3} \,\mathrm{d}\theta, \end{aligned} \tag{2.108}$$

with the first term being zero.

For $X_5$ we start with a side computation. Note that by definition $\bar{y}_g$ does not depend on $\theta$. We have, by rewriting the gradient as a derivative with respect to $\theta$,

$$\begin{aligned} \partial_\theta(M_{\bar{\theta}_1}(\theta) \partial_\theta((\bar{y}_g - x) \cdot v(\theta))) = & \\ =& \partial_\theta M_{\bar{\theta}_1}(\theta) \partial_\theta((\bar{y}_g - x) \cdot v(\theta)) \\ & + M_{\bar{\theta}_1}(\theta) \partial_\theta^2((\bar{y}_g - x) \cdot v(\theta)) \end{aligned} \tag{2.109}$$

$$\begin{aligned} =& - 2\kappa \frac{\sin(\theta - \bar{\theta}_1)}{\cos^3(\theta - \bar{\theta}_1)} M_{\bar{\theta}_1} (\bar{y}_g - x) \cdot v(\bar{\theta}_1) \sin(\theta - \bar{\theta}_1) \\ & + 2\kappa \frac{\sin(\theta - \bar{\theta}_1)}{\cos^3(\theta - \bar{\theta}_1)} M_{\bar{\theta}_1} (\bar{y}_g - x) \cdot v^\perp(\bar{\theta}_1) \cos(\theta - \bar{\theta}_1) \\ & - M_{\bar{\theta}_1} (\bar{y}_g - x) \cdot (v(\bar{\theta}_1) \cos(\theta - \bar{\theta}_1) + v^\perp(\bar{\theta}_1) \sin(\theta - \bar{\theta}_1)). \end{aligned} \tag{2.110}$$

Therefore, we obtain

$$
\begin{aligned}
X_5 = &- \rho_+^f(\bar{y}_g - x) \cdot v(\bar{\theta}_1)\Big[2\kappa \int_{\cos(\theta)>0} M(\theta)\frac{\sin^2(\theta)}{\cos^3(\theta)}\,\mathrm{d}\theta\Big] \\
&+ \rho_+^f(\bar{y}_g - x) \cdot v^\perp(\bar{\theta}_1)\Big[2\kappa \int_{\cos(\theta)>0} M(\theta)\frac{\sin(\theta)}{\cos^2(\theta)}\,\mathrm{d}\theta\Big] \\
&- \rho_+^f(\bar{y}_g - x) \cdot v(\bar{\theta}_1) \int_{\cos(\theta)>0} M(\theta)\cos(\theta)\,\mathrm{d}\theta \\
&- \rho_+^f(\bar{y}_g - x) \cdot v^\perp(\bar{\theta}) \int_{\cos(\theta)>0} M(\theta)\sin(\theta)\,\mathrm{d}\theta
\end{aligned}
\tag{2.111}
$$

$$
= -A_{13}\rho_+^f(\bar{y}_g - x) \cdot v(\bar{\theta}_1), \tag{2.112}
$$

since the constants corresponding to $(\bar{y}_g - x) \cdot v^\perp(\bar{\theta}_1)$ are zero by symmetry.
Putting everything together, we get

$$
\begin{aligned}
\partial_t \rho_+^f &+ A_{11} v(\bar{\theta}_1) \cdot \nabla_x \rho_+^f \\
&+ A_{12}\rho_+^f v^\perp(\bar{\theta}_1) \cdot \nabla_x \bar{\theta}_1 \\
&- A_{13}\rho_+^f(\bar{y}_g - x) \cdot v(\bar{\theta}) = 0.
\end{aligned}
\tag{2.113}
$$

The other case, $v \cdot u_1 < 0$, or equivalently $\cos(\theta - \bar{\theta}_1) < 0$, works similarly with only slight sign changes.

### 2.3.2. The non-constant $Q^{(1)}$-GCI

The (single) non-constant GCI associated to $Q^{(1)}$ is $\psi_{u_f^\varepsilon}$. We proceed analogously to the previous chapter to obtain

$$
\int Q^{(1)} f^\varepsilon(v)\psi_{u_{f^\varepsilon}}\,\mathrm{d}v = \langle \mathcal{L}_{u_{f^\varepsilon}} \frac{f^\varepsilon}{M_{u_{f^\varepsilon}}}, \psi_{u_{f^\varepsilon}}\rangle = 0. \tag{2.114}
$$

Again, by letting $\varepsilon \to 0$, we get

$$
X = \int \psi_{u_1}(\partial_t \bar{f} + s(|v \cdot u_1|)v \cdot \nabla_x \bar{f} + \alpha \bar{f} \nabla_v \cdot (\nabla_v((\bar{y}_g - x) \cdot v)))\,\mathrm{d}v = 0. \tag{2.115}
$$

We can now use the above calculations (2.98)-(2.100) to obtain $X = X_1 + X_2 + X_3 + X_4 + X_5$ with the following expressions, already written in terms of angles:

$$X_1 = \int \psi_{u_1} M_{u_1} (\partial_t \rho_+^f \chi_{u_1}^+ + \partial_t \rho_-^f \chi_{u_1}^-) \,\mathrm{d}\theta, \tag{2.116}$$

$$X_2 = \int \psi_{u_1} M_{u_1} s(|\cos(\theta - \bar\theta_1)|) v(\theta) \cdot (\chi_{u_1}^+ \nabla_x \rho_+^f + \chi_{u_1}^- \nabla_x \rho_-^f) \,\mathrm{d}\theta, \tag{2.117}$$

$$X_3 = \int 2\kappa_f \psi_{u_1} M_{u_1} \partial_t \bar\theta_1 \frac{\sin(\theta - \bar\theta_1)}{\cos(\theta - \bar\theta_1)^3} (\chi_{u_1}^+ \rho_+^f - \chi_{u_1}^- \rho_-^f) \,\mathrm{d}\theta, \tag{2.118}$$

$$X_4 = \int \psi_{u_1} M_{u_1} (s(|\cos(\theta - \bar\theta_1)|) v \cdot \nabla_x) \bar\theta_1 2\kappa_f \frac{\sin(\theta - \bar\theta_1)}{\cos(\theta - \bar\theta_1)^3} (\chi_{u_1}^+ \rho_+^f - \chi_{u_1}^- \rho_-^f) \,\mathrm{d}\theta, \tag{2.119}$$

$$X_5 = \alpha \int \psi_{u_1} \partial_\theta (M_{u_1}(\theta)(\chi_f^+ \rho_+^f + \chi_f^- \rho_-^f) \partial_\theta ((\bar y_g - x) \cdot v(\theta) M_{u_1}(\theta)(\chi_f^+ \rho_+^f + \chi_f^- \rho_-^f))) \,\mathrm{d}\theta. \tag{2.120}$$

By the symmetry relations (2.72), we get that $X_1 = 0$. We again use

$$v(\theta) = \cos(\theta - \bar\theta_1) v(\bar\theta_1) + \sin(\theta - \bar\theta_1) v^\perp(\bar\theta_1) \tag{2.121}$$

to obtain

$$\begin{aligned} X_2 = &\int_{\cos(\theta)>0} \psi M s(|\cos(\theta)|) \cos(\theta) v(\bar\theta_1) \cdot \nabla_x \rho_+^f \,\mathrm{d}\theta \\ &+ \int_{\cos(\theta)>0} \psi M s(|\cos(\theta)|) \sin(\theta) v^\perp(\bar\theta_1) \cdot \nabla_x \rho_+^f \,\mathrm{d}\theta \\ &+ \int_{\cos(\theta)<0} \psi M s(|\cos(\theta)|) \cos(\theta) v(\bar\theta_1) \cdot \nabla_x \rho_-^f \,\mathrm{d}\theta \\ &+ \int_{\cos(\theta)<0} \psi M s(|\cos(\theta)|) \sin(\theta) v^\perp(\bar\theta_1) \cdot \nabla_x \rho_-^f \,\mathrm{d}\theta \end{aligned} \tag{2.122}$$

$$\begin{aligned} = &A_{21} v(\bar\theta_1) \cdot \nabla_x \rho_+^f + A_{22} v^\perp(\bar\theta_1) \cdot \nabla_x \rho_+^f \\ &+ A_{23} v(\bar\theta_1) \cdot \nabla_x \rho_-^f + A_{24} v^\perp(\bar\theta_1) \cdot \nabla_x \rho_-^f. \end{aligned} \tag{2.123}$$

Note that

$$A_{21} = \int_{\cos(\theta)>0} \psi M s(|\cos(\theta)|) \cos(\theta) = \int_{\cos(\theta)<0} \psi M s(|\cos(\theta)|) \cos(\theta) = A_{23}, \tag{2.124}$$

and

$$A_{22} = \int_{\cos(\theta)>0} \psi M s(|\cos(\theta)|) \sin(\theta) = -\int_{\cos(\theta)<0} \psi M s(|\cos(\theta)|) \sin(\theta) = -A_{24}. \tag{2.125}$$

However, by symmetry, $A_{21} = A_{23} = 0$.

For $X_3$ we simply have

$$
\begin{aligned}
X_3 =& 2\kappa_f \partial_t \bar{\theta}_1 \rho_+^f \int_{\cos(\theta)>0} \psi M \frac{\sin(\theta)}{\cos(\theta)^3} \,\mathrm{d}\theta \\
&- 2\kappa_f \partial_t \bar{\theta}_1 \rho_-^f \int_{\cos(\theta)<0} \psi M \frac{\sin(\theta)}{\cos(\theta)^3} \,\mathrm{d}\theta,
\end{aligned} \tag{2.126}
$$

where the integrals have the same absolute value.
Finally, for $X_4$ we can compute

$$
\begin{aligned}
X_4 =& 2\kappa_f \rho_+^f v(\bar{\theta}_1) \cdot \nabla_x \bar{\theta}_1 \int_{\cos(\theta)>0} \psi M s(|\cos(\theta)|) \frac{\sin(\theta)^2}{\cos(\theta)^3} \\
&+ 2\kappa_f \rho_+^f v^\perp(\bar{\theta}_1) \cdot \nabla_x \bar{\theta}_1 \int_{\cos(\theta)>0} \psi M s(|\cos(\theta)|) \frac{\sin(\theta)}{\cos(\theta)^2} \\
&- 2\kappa_f \rho_-^f v(\bar{\theta}_1) \cdot \nabla_x \bar{\theta}_1 \int_{\cos(\theta)<0} \psi M s(|\cos(\theta)|) \frac{\sin(\theta)^2}{\cos(\theta)^3} \\
&- 2\kappa_f \rho_-^f v^\perp(\bar{\theta}_1) \cdot \nabla_x \bar{\theta}_1 \int_{\cos(\theta)<0} \psi M s(|\cos(\theta)|) \frac{\sin(\theta)}{\cos(\theta)^2}
\end{aligned} \tag{2.127}
$$

$$
\begin{aligned}
=& A_{25} \rho_+^f v(\bar{\theta}_1) \cdot \nabla_x \bar{\theta}_1 + A_{26} \rho_+^f v^\perp(\bar{\theta}_1) \cdot \nabla_x \bar{\theta}_1 \\
&- A_{27} \rho_-^f v(\bar{\theta}_1) \cdot \nabla_x \bar{\theta}_1 - A_{28} \rho_-^f v^\perp(\bar{\theta}_1) \cdot \nabla_x \bar{\theta}_1.
\end{aligned} \tag{2.128}
$$

However,

$$
\begin{aligned}
A_{25} =& 2\kappa_f \int_{\cos(\theta)>0} \psi M s(|\cos(\theta)|) \frac{\sin(\theta)^2}{\cos(\theta)^3} \\
=& 2\kappa_f \int_{\cos(\theta)<0} \psi M s(|\cos(\theta)|) \frac{\sin(\theta)^2}{\cos(\theta)^3} = A_{27} = 0,
\end{aligned} \tag{2.129}
$$

and

$$
\begin{aligned}
A_{26} =& 2\kappa_f \int_{\cos(\theta)>0} \psi M s(|\cos(\theta)|) \frac{\sin(\theta)}{\cos(\theta)^2} \\
=& -2\kappa_f \int_{\cos(\theta)<0} \psi M s(|\cos(\theta)|) \frac{\sin(\theta)}{\cos(\theta)^2} = -A_{28}.
\end{aligned} \tag{2.130}
$$

As above we can compute $X_5$, however due to the odd function $\psi_{u_1}$ appearing in the constants, the terms multiplied with $(\bar{y}_g - x) \cdot v(\bar{\theta})$ vanish, while the others do not. Therefore, we get

$$
X_5 = B_1 (\rho_+^f + \rho_-^f)(\bar{y}_g - x) \cdot v^\perp(\bar{\theta}). \tag{2.131}
$$

Collecting all terms and renaming the constants, we get the equation (2.80).

### 2.3.3. The constant $Q^{(2)}$-GCI

For this operator the calculations are very similar. However, we will still give a clear computation of the results.
First, we proceed as in the previous two sections and multiply (2.20) by $\chi^{\pm}_{u^\varepsilon_g}$, which are the constant GCI associated to $Q^{(2)}$, and integrate with respect to $w$. We obtain

$$\int_{w\cdot u_{g^\varepsilon}>0} \partial_t g^\varepsilon \,\mathrm{d}w = 0. \tag{2.132}$$

Taking the limit $\varepsilon \to 0$ and using that $g^0 = \bar{g}_{\rho^g_+(x,t),\rho^g_-(x,t),u_2(x,t)}$ as well as (2.98)-(2.100), we get for $w \cdot u_2 > 0$

$$\begin{aligned}
0 &= \partial_t \rho^g_+ \int_{w\cdot u_2>0} M_{u_2}(w)\,\mathrm{d}w + 2\kappa_g \rho^g_+ \int_{w\cdot u_2>0} M_{u_2}(w) \frac{w\cdot \partial_t u_2(x,t)}{(w\cdot u_2(x,t))^3}\,\mathrm{d}w && (2.133)\\
&= \partial_t \rho^g_+ + 2\kappa_g \rho^g_+ \int_{w\cdot u_2>0} M_{u_2}(w) \frac{w\cdot \partial_t u_2(x,t)}{(w\cdot u_2(x,t))^3}\,\mathrm{d}w && (2.134)\\
&= \partial_t \rho^g_+ + 2\kappa_g \rho^g_+ \int_{\cos(\theta-\bar{\theta}_2)>0} M_{u_2}(\theta)\partial_t\bar{\theta}_2(x,t) \frac{\sin(\theta-\bar{\theta}_2)}{\cos(\theta-\bar{\theta}_2)^3}\,\mathrm{d}\theta && (2.135)\\
&= \partial_t \rho^g_+, && (2.136)
\end{aligned}$$

since the integral term vanishes by symmetry. The analogous statement holds for the other case.

### 2.3.4. The non-constant $Q^{(2)}$-GCI

We follow the same steps with the non-constant $Q^{(2)}$-GCI $\psi_{u_2}$ and take the limit $\varepsilon \to 0$ to obtain

$$X = \int M_{u_2}\psi_{u_2}\partial_t \bar{g}\,\mathrm{d}w = 0. \tag{2.137}$$

We now split this integral into two parts $X = X_1 + X_2$, where

$$\begin{aligned}
X_1 &= \int \psi_{u_2} M_{u_2} (\partial_t \rho^g_+ \chi^+_{u_2} + \partial_t \rho^g_- \chi^-_{u_2})\,\mathrm{d}w, && (2.138)\\
X_2 &= 2\kappa_g \int \psi_{u_2} M_{u_2} \frac{w\cdot \partial_t u_2(x,t)}{(w\cdot u_2(x,t))^3} (\rho^g_+ \chi^+_{u_2} + \rho^g_- \chi^-_{u_2})\,\mathrm{d}w. && (2.139)
\end{aligned}$$

Due to the symmetry relations, we have seen in (2.72), we obtain that $X_1 = 0$. Writing $X_2$ in terms of angles and computing, we get

$$\begin{aligned}
X_2 &= 2\kappa_g \int \psi_{u_2} M_{u_2} \partial_t \bar{\theta}_2(x,t) \frac{\sin(\theta-\bar{\theta}_2)}{\cos(\theta-\bar{\theta}_2)^3} (\rho^g_+ \chi^+_{u_2} + \rho^g_- \chi^-_{u_2})\,\mathrm{d}w && (2.140)\\
&= B\partial_t \bar{\theta}_2 (\rho^g_+ - \rho^g_-). && (2.141)
\end{aligned}$$

# 3. More Complex Two Species Models

In this chapter we will discuss why more complex interactions between species present problems in the mathematical treatment. We take the equations (2.19)-(2.20) and swap $u_f$ with $\bar{w}_g$. This is a model that describes cells aligning with the mean nematic direction of fibers and vice versa. We obtain the following system of equations, which scales just as the system of equations presented in the first chapter:

$$\begin{aligned} \partial_t f + s(|v \cdot \bar{w}_g|) v \cdot \nabla_x f = &- \varepsilon^{-1} \nabla_v \cdot (f \nu \nabla_v ((v \cdot \bar{w}_g)^2) \\ &+ \alpha \nabla_v (f(\bar{y}_g - x) \cdot v)) \\ &+ \varepsilon^{-1} \sigma_c \nabla_v \cdot ((v \cdot \bar{w}_g)^2 \nabla_v f), \end{aligned} \tag{3.1}$$

$$\partial_t g = -\varepsilon^{-1} \nabla_w \cdot (g(\widetilde{\nu} \nabla_w ((w \cdot u_f)^2)) + \varepsilon^{-1} \sigma \nabla_w \cdot ((w \cdot u_f)^2 \nabla_w g)). \tag{3.2}$$

Again, we want to take the limit $\varepsilon \to 0$. Analogously to the procedure in the previous chapter, one can rewrite the equations in terms of operators $Q^1(f^\varepsilon, g^\varepsilon)$ and $Q^2(g^\varepsilon, f^\varepsilon)$, which only differ by switching $u_f$ and $\bar{w}_g$ to the ones introduced in Chapter 2. The computation of the set of equilibria is the same. For the limiting functions $f^0 = \bar{f}_{\rho_+^f, \rho_-^f, u_1}$ and $g^0 = \bar{g}_{\rho_+^g, \rho_-^g, u_2}$, defined as in Definition 2.2.6, we want to have $u_1 = \bar{w}_{g^0}$ and $u_2 = u_{f^0}$, which is the so-called consistency relation. Let $d = 2$ and let $\bar{\theta}_1$ and $\bar{\theta}_2$ be the angles associated to the alignment vectors $u_1$ and $u_2$, respectively. We can then recast the consistency relation into the following system of equations:

$$\begin{cases} \bar{\theta}_1 = \bar{\theta}_{\bar{g}_{\bar{\theta}_2}}, \\ \bar{\theta}_2 = \bar{\theta}_{\bar{f}_{\bar{\theta}_1}}. \end{cases} \tag{3.3}$$

This can be equivalently formulated as, see [5, Appendix A],

$$\begin{cases} \int_{-\pi}^{\pi} \sin(2(\theta - \bar{\theta}_1)) \bar{g}_{\bar{\theta}_2} \, \mathrm{d}\theta = 0, \\ \int_{-\pi}^{\pi} \sin(2(\theta - \bar{\theta}_2)) \bar{f}_{\bar{\theta}_1} \, \mathrm{d}\theta = 0. \end{cases} \tag{3.4}$$

After changing the variables in both integrals, we obtain

$$\begin{cases} \int_{-\pi}^{\pi} \sin(2\theta) \bar{g}_{\bar{\theta}_2 - \bar{\theta}_1} \, \mathrm{d}\theta = 0, \\ \int_{-\pi}^{\pi} \sin(2\theta) \bar{f}_{\bar{\theta}_1 - \bar{\theta}_2} \, \mathrm{d}\theta = 0, \end{cases} \tag{3.5}$$

which is solved for $\bar{\theta} := \bar{\theta}_1 = \bar{\theta}_2$. Therefore, we see that in the limit the consistency relation holds. Now we turn to the question, of how one adjusts the definition of GCI for this system.

## 3.1. GCI for Two Species Models

We modify the definition of GCI as follows.

**Definition 3.1.1:** *Let $u_1, u_2 \in \mathbb{S}^{d-1}$. We say that $(\Psi_1, \Psi_2) \in \mathcal{V}_{u_1} \times \mathcal{V}_{u_2}$ is a* Generalized Collision Invariant *(GCI) associated with $u_1$ and $u_2$ if*

$$\begin{aligned} \langle \mathcal{L}_{u_1} \frac{f}{M_{u_1}}, \Psi_1 \rangle_{\mathcal{V}'_{u_1}, \mathcal{V}_{u_1}} = 0, \quad \textit{and} \quad \langle \mathcal{L}_{u_2} \frac{g}{M_{u_2}}, \Psi_2 \rangle_{\mathcal{V}'_{u_2}, \mathcal{V}_{u_2}} = 0, \\ \forall f \in \mathcal{V}_{u_1} \cap L^1(\mathbb{S}^{d-1}) : u_1 \textit{ is an eigenvector of } \mathcal{Q}_g, \\ \forall g \in \mathcal{V}_{u_2} \cap L^1(\mathbb{S}^{d-1}) : u_2 \textit{ is an eigenvector of } \mathcal{Q}_f. \end{aligned} \tag{3.6}$$

Here we changed the condition of $u_1$ being an eigenvector of $\mathcal{Q}_f$ to being an eigenvector of $\mathcal{Q}_g$. We will use this definition, whenever we have nematic alignment. Since some models using polar alignment are well-suited to show simple expamles, for this case we need to change the definition of GCI. We leave the first part and change the conditions on $f$ and $g$ to

$$P_{u_1^\perp}(\mathcal{Q}_g u_1) = P_{u_2^\perp}(\mathcal{Q}_f u_2) = 0, \tag{3.7}$$

where $P_{u^\perp}$ denotes the orthogonal projection onto the orthogonal space of $\{u\}$. The following example of a classical, coupled model shows that this definition for two species GCI does not lead to a good result.
Consider the operators

$$Q^{(1)}(f,g) = \rho_f M_{\bar{w}_g} - f, \quad \text{and} \quad Q^{(2)}(f,g) = \rho_g M_{u_f} - g. \tag{3.8}$$

In this case, we have polar alignment, meaning we do not have to distinguish between two half-spaces. This means, instead of $\rho^f_\pm$ and $\rho^g_\pm$, we only have single functions $\rho^f$ and $\rho^g$ in the respective equilibria. We can then generally define

$$\mathcal{L}_u(h) = \rho_h M_u - h. \tag{3.9}$$

This operator has been studied many times already, for example in [7]. We now look at characterizations of $(\Psi_1, \Psi_2)$ being a GCI as we have done in Lemma 2.2.9. The equivalences here are shown in [3] and we only give the most important parts. Let $u_1, u_2 \in \mathbb{S}^{d-1}$. Then, requiring $u_1 = \bar{w}_g$ and $u_2 = u_f$, we have

$$(\Psi_1, \Psi_2) \text{ GCI} \Leftrightarrow \begin{cases} \int \mathcal{L}_{u_1} \Psi_1 \,\mathrm{d}v = 0 & \forall f : P_{u_2^\perp}(\mathcal{Q}_f u_2) = 0, \\ \int \mathcal{L}_{u_2} \Psi_2 \,\mathrm{d}w = 0 & \forall g : P_{u_1^\perp}(\mathcal{Q}_g u_1) = 0 \end{cases} \tag{3.10}$$

$$\Leftrightarrow \begin{cases} \int M_{u_1} \Psi_1 \,\mathrm{d}v - \Psi_1 = \beta_2 \cdot v & \forall \beta_2 \in u_2^\perp, \\ \int M_{u_2} \Psi_2 \,\mathrm{d}w - \Psi_2 = \beta_1 \cdot w & \forall \beta_1 \in u_1^\perp. \end{cases} \tag{3.11}$$

Multiplying the equations in (3.11) by $M_{u_1}$ and $M_{u_2}$, respectively, and integrating. We then obtain

$$0 = \int (\beta_2 \cdot v) M_{u_1}.$$

This further implies $\beta_2 \cdot (cu_1) = 0$. We then get that $u_1$ and $u_2$ are parallel. So here we have the condition that $u_f = \bar{w}_g$. This, however, should only be true in the limit $\varepsilon \to 0$, as we have seen above. The problem of finding a correct definition of GCI for multi-species models is yet to be solved and we leave this as an open question for investigation.

## 3.2. Overdetermination due to the Consistency Relation

In this chapter, we assume that the GCI for a two species model corresponding to the respective operators are equal, such that we can try to use this method for more complex systems. Unfortunately, one still encounters problems in the limiting equations. Consider the system (3.1)-(3.2) from above. By assuming that we have suitable GCI, we can analyze this system analogously to the one presented in Chapter 2. However, due to the consistency relation, in the limit $\varepsilon \to 0$ one now has only one mean nematic direction $u$ for both species. Letting $d = 2$ as before, one then obtains the same equations as in Theorem 2.3.1 with $\bar{\theta}_1 = \bar{\theta}_2 =: \bar{\theta}$.
First, note that we have six equations for five unknowns. Looking at the equation

$$\partial_t \bar{\theta}(\rho_+^g - \rho_-^g) = 0, \tag{3.12}$$

which corresponds to (2.83) from Theorem 2.3.1, we then either have $\partial_t \bar{\theta} = 0$ or $\rho_+^g = \rho_-^g$. Both of these cases lead to a non satisfactory result. So even if we had suitable GCI, we would not obtain a meaningful solution for this system. Hence, we try to look at different models that could remedy this.

### 3.2.1. Negative Transport Term

We try to manipulate the sign in front of the transport term of $g$ to achieve cancellations in the limiting equations. Consider the already scaled system

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon} Q^{(1)}(f^\varepsilon, g^\varepsilon), \tag{3.13}$$

$$\partial_t g^\varepsilon - w \cdot \nabla_x g^\varepsilon = \frac{1}{\varepsilon} Q^{(2)}(f^\varepsilon, g^\varepsilon). \tag{3.14}$$

Since the structure of the equations is similar to the one considered for $f$ in the Chapter 2, equation (2.19), we can use the result Theorem 2.3.1 and modify it accordingly. For simplicity we have dropped the term that models cells moving towards the center of mass of the fibers. So in the hydrodynamic limit, we obtain the following equations with some constants yet to be determined to be equal to zero:

$$\begin{aligned}\partial_t\rho^f_+ &+ A_{11}v(\bar\theta)\cdot\nabla_x\rho^f_+\\ &+ A_{12}\rho^f_+v^\perp(\bar\theta)\cdot\nabla_x\bar\theta\\ &- A_{13}\rho^f_+(\bar y_g - x)\cdot v(\bar\theta) = 0,\end{aligned} \tag{3.15}$$

$$\begin{aligned}\partial_t\rho^f_- &- A_{11}v(\bar\theta)\cdot\nabla_x\rho^f_-\\ &+ A_{12}v^\perp(\bar\theta)\rho^f_-\cdot\nabla_x\bar\theta\\ &+ A_{13}\rho^f_-(\bar y_g - x)\cdot v(\bar\theta) = 0,\end{aligned} \tag{3.16}$$

$$\begin{aligned}A_{21}v^\perp(\bar\theta)&\cdot\nabla_x(\rho^f_+ - \rho^f_-)\\ &+ A_{22}\partial_t\bar\theta(\rho^f_+ + \rho^f_-)\\ &+ A_{23}(\rho^f_+ + \rho^f_-)v(\bar\theta)\cdot\nabla_x\bar\theta\\ &+ A_{24}(\rho^f_+ + \rho^f_-)(\bar y_g - x)\cdot v^\perp(\bar\theta) = 0,\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}\partial_t\rho^g_+ &- A_{11}v(\bar\theta)\cdot\nabla_x\rho^g_+\\ &- A_{12}\rho^g_+v^\perp(\bar\theta)\cdot\nabla_x\bar\theta\\ &- A_{13}\rho^g_+(\bar y_g - x)\cdot v(\bar\theta) = 0,\end{aligned} \tag{3.18}$$

$$\begin{aligned}\partial_t\rho^g_- &+ A_{11}v(\bar\theta)\cdot\nabla_x\rho^g_-\\ &- A_{12}v^\perp(\bar\theta)\rho^g_-\cdot\nabla_x\bar\theta\\ &+ A_{13}\rho^g_-(\bar y_g - x)\cdot v(\bar\theta) = 0,\end{aligned} \tag{3.19}$$

$$\begin{aligned}-A_{21}v^\perp(\bar\theta)&\cdot\nabla_x(\rho^g_+ - \rho^g_-)\\ &+ A_{22}\partial_t\bar\theta(\rho^g_+ + \rho^g_-)\\ &- A_{23}(\rho^g_+ + \rho^g_-)v(\bar\theta)\cdot\nabla_x\bar\theta\\ &+ A_{24}(\rho^g_+ + \rho^g_-)(\bar y_g - x)\cdot v^\perp(\bar\theta) = 0,\end{aligned} \tag{3.20}$$

where the constants do not change.
We can see that there is again no hope for a meaningful solution.

### 3.2.2. Simple Coupled Model

We have not yet found a satisfactory result, so we make some changes to the system. Consider

$$\begin{aligned}\partial f^\varepsilon + v\cdot\nabla_x f^\varepsilon =&\nabla_v\cdot(f^\varepsilon\nabla_v((\bar y_{g^\varepsilon} - x)\cdot v))\\ &+ \nabla_v\cdot(f^\varepsilon\nabla_v(v\cdot\bar w_{g^\varepsilon})^2) + \varepsilon^{-1}Q^{(1)}(f^\varepsilon),\end{aligned} \tag{3.21}$$

$$\partial_t g^\varepsilon = Q^{(2)}(g^\varepsilon, f^\varepsilon), \tag{3.22}$$

where

$$Q^{(1)}(f^\varepsilon) = \rho^{f^\varepsilon} M_{u_{f^\varepsilon}} - f^\varepsilon, \tag{3.23}$$

and

$$Q^{(2)}(g^\varepsilon, f^\varepsilon) = \rho^{g^\varepsilon} M_{u_{f^\varepsilon}} - g^\varepsilon, \tag{3.24}$$

with $M_u(v) = \exp(\kappa(w \cdot u))/Z$, $Z$ being a normalizing constant, and $\rho^h = \int_{\mathbb{S}^{d-1}} h$. Similar models are discussed in [1]and [6].

Here we keep the term of cells moving towards the center of mass of the fibers and cells aligning with the mean nematic direction of fibers, however, we scale these terms differently, not multiplying them by $\varepsilon^{-1}$. Instead, we add a simple operator $Q^{(1)}$, that is scaled. We have already encountered this simple operator in Section 3.1. By definition of $Q^{(2)}$, we can see that in the limit the consistency relation has to be satisfied.

In the limit $\varepsilon \to 0$ we get $f^\varepsilon \to f^0 = \rho^f(x,t) M_{u(x,t)}$. Now, as done in Section 2.3, integrate and let $\varepsilon \to 0$ to obtain

$$\begin{aligned} 0 =& \partial_t \left( \int \rho^f M_u \,\mathrm{d}v \right) + \nabla_x \cdot \int v \rho^f M_u \,\mathrm{d}v \\ &- \int \nabla_v \cdot (\rho M_u \nabla_v ((\bar{y}_g - x) \cdot v)) \,\mathrm{d}v - \int \nabla_v \cdot (\rho^f M_u \nabla_v ((v \cdot u(x,t))^2)) \end{aligned} \tag{3.25}$$

$$= \partial_t \rho + c_1 \nabla_x \cdot (\rho^f u) - X_1 - X_2. \tag{3.26}$$

We proceed with the following side computation in two dimensions, where we again use

$$v(\theta) = \cos(\theta - \bar{\theta}) v(\bar{\theta}) + \sin(\theta - \bar{\theta}) v^\perp(\bar{\theta}), \tag{3.27}$$

as in the computation in Section 2.3:

$$\begin{aligned} \partial_\theta (M_u(\theta) \partial_\theta (\bar{y}_g - x) \cdot v(\theta)) =& - \partial_\theta M_u(\theta) (\bar{y}_g - x) \cdot v(\bar{\theta}) \sin(\theta - \bar{\theta}) \\ &+ \partial_\theta M_u(\theta) (\bar{y}_g - x) \cdot v^\perp(\theta) \cos(\theta - \bar{\theta}) \\ &- M_u(\theta) (\bar{y}_g - x) \cdot v(\bar{\theta}) \cos(\theta - \bar{\theta}) \\ &- M_u(\theta) (\bar{y}_g - x) \cdot v^\perp(\bar{\theta}) \sin(\theta - \bar{\theta}). \end{aligned} \tag{3.28}$$

Using

$$\partial_\theta M_u(\theta) = -\kappa \sin(\theta - \bar{\theta}) M_u(\theta), \tag{3.29}$$

we obtain

$$
\begin{aligned}
X_1 =&\rho^f \int \kappa \sin(\theta - \bar{\theta}) M_u(\theta)(\bar{y}_g - x) \cdot v(\bar{\theta}) \sin(\theta - \bar{\theta}) \\
&- \rho^f \int \kappa \sin(\theta - \bar{\theta}) M_u(\theta)(\bar{y}_g - x) \cdot v^{\perp}(\bar{\theta}) \cos(\theta - \bar{\theta}) \\
&- \rho^f \int \cos(\theta - \bar{\theta}) M_u(\theta)(\bar{y}_g - x) \cdot v(\bar{\theta}) \\
&- \rho^f \int \sin(\theta - \bar{\theta}) M_u(\theta)(\bar{y}_g - x) \cdot v^{\perp}(\bar{\theta})
\end{aligned} \tag{3.30}
$$

$$
\begin{aligned}
=&A_{11} \rho^f (\bar{y}_g - x) \cdot v(\bar{\theta}) \\
&- A_{12} \rho^f (\bar{y}_g - x) \cdot v^{\perp}(\bar{\theta}) \\
&- A_{13} \rho^f (\bar{y}_g - x) \cdot v(\bar{\theta}) \\
&- A_{14} \rho^f (\bar{y}_g - x) \cdot v^{\perp}(\bar{\theta}),
\end{aligned} \tag{3.31}
$$

where $A_{12} = A_{13} = 0$.
We further have

$$
\begin{aligned}
X_2 =&\rho^f \int \partial_\theta M_u(\theta)(-2\cos(\theta - \bar{\theta}) \sin(\theta - \bar{\theta}))\,\mathrm{d}\theta \\
&- \rho^f \int -2M_u(\theta)(\sin^2(\theta - \bar{\theta}) - \cos^2(\theta - \bar{\theta}))\,\mathrm{d}\theta
\end{aligned} \tag{3.32}
$$

$$
\begin{aligned}
=&\rho^f \int 2\kappa M_u(\theta) \cos(\theta - \bar{\theta}) \sin^2(\theta - \bar{\theta})\,\mathrm{d}\theta \\
&+ \rho^f \int 2M_u(\theta)(\sin^2(\theta - \bar{\theta}) - \cos^2(\theta - \bar{\theta}))\,\mathrm{d}\theta
\end{aligned} \tag{3.33}
$$

$$
=A_{15}\rho^f. \tag{3.34}
$$

Next we want to use the GCI $\psi_\beta(v) = (\beta \cdot v)$ for $\beta \in u^{\perp}$, which has been computed in [4]. Again, we multiply the equation by the GCI, integrate, and let $\varepsilon \to 0$ to get

$$
\begin{aligned}
0 =& \int \psi_\beta(v)(\partial_t(\rho^f M_u) + v \cdot \nabla_x(\rho^f M_u)) \\
&- \int \psi_\beta(v)(\nabla_v \cdot (\rho^f M_u \nabla_v((\bar{y}_g - x) \cdot v)) - \nabla_v \cdot (f \nabla_v((v \cdot u)^2)))
\end{aligned} \tag{3.35}
$$

$$
=X_1 + X_2 - X_3 - X_4. \tag{3.36}
$$

First, we have

$$
X_1 = \int (\beta \cdot v)(\partial_t \rho^f M_u + \rho^f \partial_t M_u) \tag{3.37}
$$

$$
= \int (\beta \cdot v)(\partial_t \rho^f M_u - \kappa \rho^f \partial_t \bar{\theta} \sin(\theta - \bar{\theta}) M_u) \tag{3.38}
$$

$$
=A_{21} \partial_t \rho^f - A_{22} \rho^f \partial_t \bar{\theta}. \tag{3.39}
$$

Next we get

$$X_2 = \int (\beta \cdot v) v \cdot (\nabla_x \rho^f M_u + \kappa \rho^f \nabla_x \bar{\theta} M_u) \tag{3.40}$$

$$\begin{aligned} &= \int (\beta \cdot v) \cos(\theta - \bar{\theta}) v(\bar{\theta}) \cdot \nabla_x \rho^f M_u \\ &\quad + \int (\beta \cdot v) \sin(\theta - \bar{\theta}) v^\perp(\bar{\theta}) \cdot \nabla_x \rho^f M_u \\ &\quad - \int (\beta \cdot v) \cos(\theta - \bar{\theta}) \sin(\theta - \bar{\theta}) v(\bar{\theta}) \cdot \rho^f \nabla_x \bar{\theta} M_u \\ &\quad - \int (\beta \cdot v) \sin^2(\theta - \bar{\theta}) v^\perp(\bar{\theta}) \cdot \rho^f \nabla_x \bar{\theta} M_u \end{aligned} \tag{3.41}$$

$$= A_{23} v^\perp(\bar{\theta}) \cdot \nabla_x \rho^f - A_{24} \rho^f v(\bar{\theta}) \cdot \nabla_x \bar{\theta}. \tag{3.42}$$

As further above in (3.30)-(3.34), we can obtain $X_3$ and $X_4$, however, the constants that vanish in the above computations , now are nonzero and vice versa. We therefore have

$$X_3 = -A_{25} \rho^f (\bar{y}_g - x) \cdot v^\perp(\bar{\theta}) - A_{26} \rho^f (\bar{y}_g - x) \cdot v(\bar{\theta}), \tag{3.43}$$

and

$$X_4 = A_{27} \rho^f. \tag{3.44}$$

Altogether, we get the following equations for $f$:

$$\begin{aligned} &\partial_t \rho^f + A_{11} \nabla_x \cdot (\rho^f \bar{\theta}) - A_{12} \rho^f (\bar{y}_g - x) \cdot v(\bar{\theta}) \\ &\quad + A_{13} \rho^f (\bar{y}_g - x) \cdot v^\perp(\bar{\theta}) - A_{14} \rho^f = 0, \end{aligned} \tag{3.45}$$

$$\begin{aligned} &A_{21} \partial_t \rho^f - A_{22} \rho^f \partial_t \bar{\theta} \\ &\quad + A_{23} v^\perp(\bar{\theta}) \cdot \nabla_x \rho^f - A_{24} \rho^f v(\bar{\theta}) \cdot \nabla_x \bar{\theta} \\ &\quad + A_{25} \rho^f (\bar{y}_g - x) \cdot v^\perp(\bar{\theta}) + A_{26} \rho^f (\bar{y}_g - x) \cdot v(\bar{\theta}) \\ &\quad - A_{27} \rho^f = 0. \end{aligned} \tag{3.46}$$

Since the equation for $g$ is very simple, we can directly write down the macroscopic equations

$$\partial_t \rho^g = 0, \tag{3.47}$$

$$\rho^g \partial_t \bar{\theta} = 0. \tag{3.48}$$

We now see that due to the consistency relation, we again have overdetermination of the limiting equations. Hence, we do not write down the explicit constants in this case.

## 3.3. Possible Solution for GCI

Finally, we want to discuss a solution to how one could circumvent the problem that occurs for two species GCI. The idea is to let the difference between the alignment vectors of $f$ and $g$ be of order $\varepsilon$ and using a Hilbert expansion. Consider the following model:

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon = \varepsilon^{-1}(\rho^{f^\varepsilon} M_{\bar{w}_{g^\varepsilon}} - f^\varepsilon) =: \varepsilon^{-1} Q^{(1)}(f^\varepsilon, g^\varepsilon), \tag{3.49}$$

$$\partial_t g^\varepsilon + w \cdot \nabla_x g^\varepsilon = \varepsilon^{-1}(\rho^{g^\varepsilon} M_{u_{f^\varepsilon}} - g^\varepsilon) =: \varepsilon^{-1} Q^{(2)}(f^\varepsilon, g^\varepsilon). \tag{3.50}$$

Integrating with respect to $v$ and $w$, respectively, and writing $f, g$ instead of $f^\varepsilon, g^\varepsilon$ for notational convenience yields

$$\rho^f c\bar{w}_g - J_f = O(\varepsilon) \quad \text{implies} \quad \bar{w}_g = \frac{J_f}{\rho^f c} + O(\varepsilon), \tag{3.51}$$

$$\rho^g c u_f - J_g = O(\varepsilon) \quad \text{implies} \quad u_f = \frac{J_g}{\rho^g c} + O(\varepsilon), \tag{3.52}$$

where

$$J_f = \int_{\mathbb{S}^{d-1}} f v \,\mathrm{d}v = \int v\rho^f M_{\bar{w}_g} \,\mathrm{d}v + O(\varepsilon) = \rho^f c\bar{w}_g + O(\varepsilon), \tag{3.53}$$

and the analogous for $J_g$. This yields

$$\bar{w}_g - u_f = \frac{J_f}{\rho^f c} - \frac{J_g}{\rho^g c} + O(\varepsilon) =: R^\varepsilon + O(\varepsilon). \tag{3.54}$$

We will use the expansion

$$f = \rho^f M_{\bar{w}_g} + f_1\varepsilon + O(\varepsilon^2). \tag{3.55}$$

Now

$$\begin{aligned} M_{u_f + R^\varepsilon} &= M_{u_f} \exp(v\varepsilon R^\varepsilon) = M_{u_f} \exp(w(\varepsilon R^\varepsilon f_1 + O(\varepsilon^2))) &(3.56)\\ &\approx M_{u_f}(1 + \varepsilon R^\varepsilon f_1 + O(\varepsilon^2)), &(3.57)\end{aligned}$$

so we can obtain

$$\begin{aligned} Q^{(1)}(f,g) &= \rho_f M_{\bar{w}_g} - f = \rho^f M_{u_f + R^\varepsilon} - f &(3.58)\\ &= \rho^f M_{u_f} - f + \varepsilon\rho^f M_{u_f}(vR^\varepsilon f_1) + O(\varepsilon^2). &(3.59)\end{aligned}$$

With this (3.49) can be rewritten as

$$\partial_t f + v \cdot \nabla_x f = \varepsilon^{-1} \underbrace{(\rho^f M_{u_f} - f)}_{=:\widetilde{Q}^{(1)}} + \rho^g M_{\bar{w}_g}(w f_1 R^\varepsilon) + O(\varepsilon). \tag{3.60}$$

We now use the Hilbert expansion $f = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \cdots$ and get

$$O(\varepsilon^{-1}) : \rho_0 M_{u_{f_0}} - f_0 = 0, \tag{3.61}$$

$$O(\varepsilon^{0}) : \partial_t f_0 + v \cdot \nabla_x f_0 = \rho_{f_1} M_{u_{f_0}} + \rho_0 M_{u_{f_0}}(v f_1 R^\varepsilon) - f_1 \approx -R^\varepsilon. \tag{3.62}$$

One can then use the GCI for $Q^{(1)}$ and proceed as usual. Furthermore, it is possible to have a explicit expression for $R^\varepsilon$. However, our goal was to only give the idea of this method. Computations above this point would exceed the extend of this thesis, so we leave this topic open for further research.

# 4. Conclusion

Altogether we have seen that a simple two-species model with only slight coupling can be analyzed using the method of Generalized Collision Invariants without any problems occurring. However, when considering more complex models that include the two species aligning with the respective other, it is difficult to find a correct definition for GCI. By studying a simple model, we have shown that when using a natural modification of the definitions, the alignment vectors of the species have to be parallel, though this should only hold in the limit. Furthermore, we assumed the GCI to be equal and considered two modified versions of the original system we wanted to study to see that we still encounter problems in the limiting equations, namely overdetermination. Finally, we suggested a method, letting the difference between the alignment vectors be of order $\varepsilon$ and using a Hilbert expansion that could lead to solving the problems we encountered.
In future research, one could further inspect the definition for GCI for multi-species models as well as further develop the method suggested at the end of the thesis.

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# A. Appendix

## A.1. Abstract

This Master's Thesis aims to analyze two-species biological models, describing the interaction between fibers and cells with nematic alignment, using established methods in Kinetic Theory, particularly Generalized Collision Invariants (GCI). First, an analysis of a system with only minor coupling is provided. Furthermore, the problems of defining and using GCI for more complex models, such as overdetermination, are discussed and illustrated with a simple case. Lastly, a method using the Hilbert expansion is suggested, aiming to solve these difficulties.

## A.2. Kurzfassung

Ziel dieser Masterarbeit ist biologische Modelle mit zwei Spezien, Faser sowie Zellen, und *Nematic Alignment* mithilfe etablierter Methoden aus Kinetischer Theorie, vor allem des Konzeptes von *Generalized Collision Invariants* (GCI), zu analysieren. Vorerst wird ein System mit nur geringer Interaktion zwischen den Spezien präsentiert. Weiters werden Probleme der Defintion und Verwendung von GCI für komplexere Modelle, wie das Erscheinen überdeterminierter Systeme, durch einfache Beispiele veranschaulicht und diskutiert. Zum Schluss wird eine Methode mit Verwendung der Hilbert Entwicklung vorgeschlagen, um diese Problematik lösen zu können.