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”a quadrat + a quadrat = zwa quadrat”

– *Unknown*

During three years of doctoral studies, I was often asked what my research is about. Sometimes, I might have been successful at explaining something, but more often than not, I failed miserably and told a math joke instead. If you had the honor of being told one by me – thank you for enduring it!

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Abstract

This thesis is concerned with the quantitative description of the spectrum of two-dimensional canonical systems of differential equations. For arbitrary Hamiltonians, we determine the imaginary part of the Weyl coefficient and the norm of the monodromy matrix along the imaginary axis up to universal multiplicative constants.

In the limit point case, this yields a criterion for a canonical system to have resolvents belonging to a Schatten–von Neumann class with small index. Most notably, we obtain a characterisation of membership in trace class.

Our second focus is the problem of describing the growth of the monodromy matrix. We present an algorithmic method that lets us determine this growth up to a logarithmic multiplicative factor, and which is applicable to every canonical system in limit circle case. Particular emphasis is put on Hamiltonians associated with indeterminate Hamburger moment problems. Under mild well-behavedness assumptions, we compute the growth of the monodromy matrix explicitly in terms of the parameters of the Hamiltonian. The formulae we obtain lead to a profound understanding of intrinsically appearing case distinctions between large and small orders.

Zusammenfassung

Diese Arbeit beschäftigt sich mit der quantitativen Beschreibung des Spektrums von zweidimensionalen Kanonischen Systemen von Differentialgleichungen. Für beliebige Hamiltonians bestimmen wir den Imaginärteil des Weyl-Koeffizienten und die Norm der Monodromiematrix entlang der imaginären Achse bis auf universelle multiplikative Konstanten.

Im Grenzpunktfall ergibt dies ein Kriterium dafür, wann die Resolventen eines Kanonischen Systems zu einer Schatten–von-Neumann-Klasse mit kleinem Index gehören. Insbesondere erhalten wir eine Charakterisierung für Zugehörigkeit zur Spurklasse.

Unser zweiter Fokus ist das Problem, das Wachstum der Monodromiematrix zu beschreiben. Wir präsentieren eine algorithmische Methode, die uns dieses Wachstum bis auf einen logarithmischen multiplikativen Faktor bestimmen lässt, und die auf jedes Kanonische System im Grenzkreisfall anwendbar ist. Ein besonderer Schwerpunkt sind Hamiltonians, die aus nichtdeterminierten Hamburgerschen Momentenproblemen resultieren. Unter leichten Regularitätsvoraussetzungen berechnen wir das Wachstum der Monodromiematrix explizit in Abhängigkeit von den Parametern des Hamiltonians. Die so erhaltenen Formeln führen zu einem profunden Verständnis von intrinsisch auftretenden Fallunterscheidungen zwischen großen und kleinen Ordnungen.

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Introduction

Second-order linear differential operators are among the most studied objects in spectral theory. Whether it be a Schrödinger operator, Dirac operator, or Krein string, there are different physical motivations in the background. The theories behind these operators, however, are to a certain extent analogous. This is where *canonical systems* of differential equations enter. They formally include all of the operators mentioned above, and have the unique property that every self-adjoint operator with simple spectrum is unitarily equivalent to the model operator of a canonical system.

Let us recall the definition of what will be in the center of our attention. A (two-dimensional) canonical system is an equation of the form

$$y'(t) = zJH(t)y(t), \qquad t \in (a,b) \text{ a.e.}, \tag{0.0.1}$$

with $-\infty < a < b \leq \infty$, $z \in \mathbb{C}$ is the spectral parameter, and $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is a signature matrix. The $\mathbb{R}^{2\times 2}$-valued function H is called the *Hamiltonian*. It should be locally integrable on $[a,b)$ and such that $H(t) \geq 0$ and $H(t) \neq 0$ for a.e. $t \in (a,b)$.

Basic notions

Operator theoretic aspects. A Hamiltonian gives rise to a Hilbert space $L^2(H)$, defined as the set of (equivalence classes $H(t)\,\mathrm{d}t$-a.e. of) measurable functions $f : (a,b) \to \mathbb{C}^2$ satisfying

(i) $\int_a^b f(t)^* H(t) f(t)\,\mathrm{d}t < \infty$ and

(ii) If $I \subseteq (a,b)$ is an interval and there exists ϕ such that $(\cos\phi, \sin\phi)JH \equiv 0$ on I, then $(\cos\phi, \sin\phi)f$ is constant on I.

An interval I with the property described in (ii) is called an *H-indivisible interval.*

Consider now the linear relation

$$T_H := \left\{ (f,g) \in L^2(H) \times L^2(H) \;\middle|\; \begin{array}{l} f \text{ has an absolutely continu-} \\ \text{ous representative } f_0 \text{ with} \\ f_0' = JHg \text{ and } (1,0)f_0(a) = 0 \end{array} \right\}. \tag{0.0.2}$$

T_H^* is symmetric, and one of the following two alternatives takes place.

(a) If $\int_a^b \operatorname{tr} H(t)\,\mathrm{d}t < \infty$, then T_H^* has deficiency indices $(1,1)$. The endpoint b is said to be in *limit circle case*. All self-adjoint extensions of T_H^* in $L^2(H)$ are given by

$$A_{H,\beta} := \left\{ (f,g) \in T_H \;\middle|\; \big(\cos(\beta), \sin(\beta)\big) f_0(b) = 0 \right\}, \qquad (\beta \in [0,\pi)).$$

The spectrum of $A_{H,\beta}$ is purely discrete.

(b) If $\int_a^b \operatorname{tr} H(t)\,\mathrm{d}t = \infty$, then T_H is self-adjoint. We say that *limit point case* holds at b.

We will now define our central objects of study: the monodromy matrix (in limit circle case) and the Weyl coefficient (in limit point case).

The monodromy matrix. Consider the fundamental solution $W = (w_{ij})_{i,j=1}^2 \colon [a,b) \times \mathbb{C} \to \mathbb{C}^{2\times 2}$ that solves the initial value problem

$$\begin{cases} \frac{d}{dt} W(t;z)J = zW(t;z)H(t), & t \in (a,b), \\ W(a,z) = I. \end{cases}$$

If limit circle case holds at b, the fundamental solution exists up to $t = b$ and gives rise to the *monodromy matrix*

$$W_H(z) := \begin{pmatrix} w_{H,11} & w_{H,12} \\ w_{H,21} & w_{H,22} \end{pmatrix} := W(b,z). \tag{0.0.3}$$

Since $\big(w_{21}(\cdot,z), w_{22}(\cdot,z)\big)^\top$ satisfies the boundary condition at a, the eigenvalues of $A_{H,\beta}$ are precisely the zeros of $\cos(\beta) w_{H,21} + \sin(\beta) w_{H,22}$.

The Weyl coefficient. If b is in limit point case, a replacement for the monodromy matrix is needed. Consider the disks

$$D_{t,z} := \left\{ \frac{w_{11}(t,z)\tau + w_{12}(t,z)}{w_{21}(t,z)\tau + w_{22}(t,z)} \,\middle|\, \tau \in \overline{\mathbb{C}_+} \right\} \subseteq \overline{\mathbb{C}_+}. \tag{0.0.4}$$

If $z \in \mathbb{C}_+$ is fixed, the disks are nested and have a single point in common:

$$\bigcap_{t\in(a,b)} D_{t,z} = \{q_H(z)\}.$$

The function $z \mapsto q_H(z)$ is called the *Weyl coefficient* of H. It is a Herglotz function, i.e., analytic on $\mathbb{C}_+$ with $\operatorname{Im} q_H(z) \geq 0$. As such, it admits an integral representation of the form

$$q_H(z) = \alpha + \beta z + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \mathrm{d}\mu_H(t), \qquad z \in \mathbb{C}_+. \tag{0.0.5}$$

The measure μ_H in this representation is the spectral measure of (the operator part of) the self-adjoint relation T_H.

Questions related to growth

Following the structure of this thesis, we talk first about the growth of the Weyl coefficient.

High-energy behaviour of the Weyl coefficient. Under *high-energy behaviour* we understand the behaviour of $q_H(ir)$ for $r \to \infty$, which depends principally on the values of H near the left endpoint a. It plays an important role in inverse spectral theory [Lev87, Tes09] and enables the quantitative description of (the distribution function of) μ_H through classical Abelian-Tauberian theorems. For example, provided $\beta = 0$ in (0.0.5), it holds for every $\alpha \in (0,2)$ that

$$\int_{\mathbb{R}} \frac{\mathrm{d}\mu_H(t)}{1+|t|^\alpha} < \infty \qquad \Leftrightarrow \qquad \int_1^\infty \frac{\operatorname{Im} q_H(ir)}{r^\alpha} \,\mathrm{d}r < \infty, \tag{0.0.6}$$

cf. [Kac56]. This leads to the following important question: How can we describe the high-energy behaviour of q_H (and in particular its imaginary part which is the Poisson integral of μ_H) in terms of H?

Growth of the monodromy matrix. W_H is an entire function of finite exponential type given by the *Krein–de Branges formula*

$$\text{e.t.}(W_H) = \int_a^b \sqrt{\det H(t)} \,\mathrm{d}t. \tag{0.0.7}$$

Consider a self-adjoint realisation $A_{H,\beta}$ and let $(\lambda_n^\pm)_{n=1}^\infty$ be the sequences of positive and negative eigenvalues of $A_{H,\beta}$, respectively, arranged by increasing modulus. Using classical theory of entire functions, the Krein–de Branges formula implies

$$\lim_{n\to\infty} \frac{n}{|\lambda_n^\pm|} = \frac{1}{\pi}\text{e.t.}(W_H) = \frac{1}{\pi}\int_a^b \sqrt{\det H(t)} \,\mathrm{d}t. \tag{0.0.8}$$

If the right side does not vanish, this is as precise a description of the spectrum as we could hope for. In case that $\det H \equiv 0$, relation (0.0.8) only tells us that the eigenvalues are asymptotically less dense than the integers. A viable strategy of describing the density of eigenvalues nonetheless is determining the growth of W_H. On a rough scale, one could ask for its order which coincides with the convergence exponent of the sequence of all (positive and negative) eigenvalues. Since the order is obtained from comparing $\log \|W_H(z)\|$ to powers only, it is even more desirable to describe $\log \|W_H(z)\|$ itself and capture, for example, logarithmic terms.

Remarks on literature

Former work on the growth of the monodromy matrix. The growth of a monodromy matrix W is often measured in terms of its order as an entire function:

$$\rho := \limsup_{r\to\infty} \frac{\log\log \max_{|z|=r} \|W(z)\|}{\log r} \quad \in [0,1].$$

Many results on the order are written not in the "language" of canonical systems, but in terms of Hamburger moment problems, Jacobi matrices, and Krein strings. For Schrödinger operators, there is not much to study, since the order is always equal to $\frac{1}{2}$ (e.g., [Tes09, Lemma 9.18]). From the perspective of moment problems, the task of estimating the order was studied already in [Liv39], where a lower bound was obtained. Berezanskii [Ber56] and Berg-Szwarc [BS14] calculated the order of Jacobi matrices under quite restrictive assumptions.
In [Val99], G. Valent made a conjecture about the order and type of certain Jacobi matrices associated to birth and death processes with polynomial rates. The conjecture for the order was proved independently in [BS17] and in [Rom17], and the conjecture about type was confirmed in [Boc19].
In terms of the Hamiltonian of a canonical system, the growth of W_H was studied intensively in recent years [Rom17, PW19, PW23, Pru20]. For example, a generalised version of Berezanskii's Theorem was obtained in the language of canonical systems in [PW19]. A very different approach at calculating the order was found by R. Romanov [Rom17], for Hamiltonians assuming only two possible values.

Former work on the growth of the Weyl coefficient. A very early example of an asymptotic formula for the Weyl coefficient is given in [Mar52]. Specifically, it is shown that the spectral function of a Schrödinger operator behaves asymptotically like the square root, regardless of the potential; using Abelian-Tauberian theorems, one obtains a similar asymptotic formula for the Weyl coefficient. Since then, Weyl coefficient asymptotics have been found, e.g., for Sturm-Liouville problems, Dirac operators, or Krein strings [Eve72, Kas75, EHS83, Ben89].
There are also some results that estimate, under weaker assumptions, the modulus of the Weyl coefficient, or even determine it up to multiplicative constants [Hil63, Atk88, Ben89, JL99, HRdS00]. We know of only one paper [LPW] where estimates specifically for the imaginary part are obtained from direct estimates of Weyl disks.

Summary of results

We give a short account of our results.

In our very first result, Theorem I.1.1, we use a new approach to study the estimate for the imaginary part of the Weyl coefficient q_H from [LPW]. What we obtain is a formula that determines $\operatorname{Im} q_H(ir)$ up to universal multiplicative constants.

Writing out the differential equation, we relate the norm of the fundamental solution to an integral over the imaginary parts of Weyl coefficients of cut-offs of H. For Hamiltonians with discrete spectrum, this results in Theorem II.3.1 that gives a formula for the Stieltjes transform of the eigenvalue counting function, again up to universal multiplicative constants.

Theorem II.3.1 gives rise to two major theorems. In Theorem II.5.4, we present a characterisation of membership in p-Schatten–von Neumann classes of resolvents of the model operator for $p \in [0, 2]$. Theorem II.6.2 introduces, for Hamiltonians in limit circle case, an algorithmic method of determining the growth of the monodromy matrix. This yields a formula for the order of any Hamiltonian in limit circle case, cf. Corollary II.6.3.

In the third and fourth paper, we consider *Hamburger Hamiltonians* in limit circle case (which arise from rewriting indeterminate Hamburger moment problems to the canonical systems setting). We explicitly determine the growth of the monodromy matrix in a large number of situations. The results (and methods) provide a clear picture of the intrinsically different behaviour between orders less than $\frac{1}{2}$ (cf. Theorem III.5.3) and larger than $\frac{1}{2}$ (cf. Theorem IV.1.1).

Description of manuscripts

1 A quantitative formula for the imaginary part of a Weyl coefficient

Jakob Reiffenstein

to appear in Journal of Spectral Theory (accepted on Feb 15, 2023)

arXiv: https://arxiv.org/abs/2207.04768

Let H be a Hamiltonian in limit point case with Weyl coefficient q_H. We present a function that is explicitly given in terms of H, which coincides with $\operatorname{Im} q_H(ir)$ up to multiplicative constants independent of H. The formula reflects the fact that the behaviour of the Weyl coefficient near $+i\infty$ is controlled by the values of H near the left endpoint a.

Using Abelian-Tauberian theorems, we obtain quantitative results on the growth of the spectral measure, e.g., criteria for integrability of

given comparison functions with respect to the spectral measure.
Comparing formulae for absolute value and imaginary part of q_H, restrictions on the growth of $|q_H(ir)|$ can be observed whenever the angle of $q_H(ir)$ approaches 0 or π. Conversely, if $|q_H(ir)|$ is positively increasing, the angle of $q_H(ir)$ has to stay away from 0 and π. Using the de Branges correspondence, these results can be viewed as statements on abstract Herglotz functions.
We also answer a number of questions from [LPW] related to the position of $q_H(ir)$ within certain Weyl disks.

2 Eigenvalue distribution of canonical systems: Trace class and sparse spectrum

Matthias Langer, Jakob Reiffenstein, Harald Woracek
in preparation

Unlike the Weyl coefficient, the growth at infinity of the fundamental solution is not governed only by the local behaviour near a of H. By globalising the formula for $\operatorname{Im} q_H(ir)$ from the first paper, we arrive at a formula in terms of H for $\log|w_{22}(t, ir)|$ up to multiplicative constants. The formula is a key tool that we make extensive use of in this and in the last paper.
Our first main result treats canonical systems with sparse discrete spectrum. We present an equivalent criterion for resolvents of a canonical system to belong to Schatten–von Neumann classes with exponent p between 0 and 2. This complements recent work by R. Romanov and H. Woracek [RW20], who gave criteria for every p larger than 1. Note that the method used in [RW20] depends on symmetrically normed operator ideals, while we use the theory of entire functions to treat this problem. There is also an intrinsic reason why different strategies are needed: The inclusion of resolvents in a Schatten–von Neumann class with $p > 1$ is independent of the off-diagonal entry of H, but this is (almost trivially) not true for $p \leq 1$.
As a second important consequence of the formula for $\log|w_{22}|$, we describe an algorithmic approach for computing the growth of W_H for systems in limit circle case.

This manuscript was written in close collaboration.

3 An upper bound for the Nevanlinna matrix of an indeterminate moment sequence

Raphael Pruckner, Jakob Reiffenstein, Harald Woracek
submitted; preprint published on arXiv
arXiv: https://arxiv.org/abs/2307.10748

This paper deals with limit circle Hamburger Hamiltonians specifically. Such Hamiltonians are parametrised by two sequences, $(l_j)_{j=1}^{\infty}$ and $(\phi_j)_{j=1}^{\infty}$, which we refer to as the lengths and angles of the Hamiltonian. We develop an upper bound for the norm of the monodromy matrix in terms of these sequences of parameters. Due to our use of well-behaved comparison functions, any Hamburger Hamiltonian may be treated.

The upper bound can be evaluated very explicitly if one uses comparison functions that are regularly varying. The computations involve a number of (intrinsic) case distinctions that we lay out in detail. In the case of fast decay, corresponding to orders less than $\frac{1}{2}$, the upper bound is attained (thus determining the growth) whenever the lengths and increments of angles are close to being regularly varying.

This manuscript was written in close collaboration.

4 Lower bounds for the order of a Nevanlinna matrix
Jakob Reiffenstein
in preparation

We stay in the setting of limit circle Hamburger Hamiltonians. From literature on the order problem (frequently written in the language of moment problems and Jacobi matrices), several upper bounds are known. However, apart from notational differences there is but one lower bound for the order – the convergence exponent of the off-diagonal sequence of Jacobi parameters – which holds under more or less restrictive assumptions.

Using the algorithmic approach from Theorem II.6.2, we obtain a fundamentally different lower bound, corresponding to orders larger than $\frac{1}{2}$. This result establishes sharpness of the upper bound in Theorem III.4.6 also for large order.

As a second main result, we revisit the lower bound previously known in some special cases and show that it holds, in fact, for *any* Jacobi matrix in limit circle case.

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Individual papers

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Chapter I

A quantitative formula for the imaginary part of a Weyl coefficient

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A quantitative formula for the imaginary part of a Weyl coefficient

Jakob Reiffenstein [‡]

Abstract. We investigate two-dimensional canonical systems $y' = zJHy$ on an interval, with positive semi-definite Hamiltonian H. Let q_H be the Weyl coefficient of the system. We prove a formula that determines the imaginary part of q_H along the imaginary axis up to multiplicative constants, which are independent of H. We also provide versions of this result for Sturm-Liouville operators and Krein strings.
Using classical Abelian-Tauberian theorems, we deduce characterizations of spectral properties such as integrability of a given comparison function w.r.t. the spectral measure μ_H, and boundedness of the distribution function of μ_H relative to a given comparison function.
We study in depth Hamiltonians for which $\arg q_H(ir)$ approaches 0 or π (at least on a subsequence). It turns out that this behavior of $q_H(ir)$ imposes a substantial restriction on the growth of $|q_H(ir)|$. Our results in this context are interesting also from a function theoretic point of view.

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I.1 Introduction

We study two-dimensional *canonical systems*

$$y'(t) = zJH(t)y(t), \qquad t \in [a,b) \text{ a.e.,} \tag{I.1.1}$$

where $-\infty < a < b \leq \infty$, $z \in \mathbb{C}$ is a spectral parameter and $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The *Hamiltonian* H is assumed to be a locally integrable, $\mathbb{R}^{2\times 2}$-valued function on $[a,b)$ that further satisfies

- ▷ $H(t) \geq 0$ and $H(t) \neq 0, \qquad t \in [a,b)$ a.e.;
- ▷ H is definite, i.e., if $v \in \mathbb{C}^2$ is s.t. $H(t)v \equiv 0$ on $[a,b)$, then $v = 0$;

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▷ $\int_a^b \operatorname{tr} H(t)\,\mathrm{d}t = \infty$ (limit point case at b).

Together with a boundary condition at a, the equation (I.1.1) becomes the eigenvalue equation of a self-adjoint (possibly multi-valued) operator A_H in a Hilbert space $L^2(H)$ associated with H. Throughout this paper, we fix the boundary condition $(1,0)y(a) = 0$, which is no loss of generality.
Many classical second-order differential operators such as Schrödinger and Sturm-Liouville operators, Krein strings, and Jacobi operators can be transformed to the form (I.1.1), see, e.g., [Rem18, Tes09, BHdS20, KWW07, Kac99]. Canonical systems thus form a unifying framework.
All of the above operators have in common that their spectral theory is centered around the Weyl coefficient q of the operator (also referred to as Titchmarsh-Weyl m-function). This function is constructed by Weyl's nested disk method and is a Herglotz function, i.e., it is holomorphic on $\mathbb{C}\setminus\mathbb{R}$ and satisfies there $\frac{\operatorname{Im} q(z)}{\operatorname{Im} z} \geq 0$ as well as $q(\overline{z}) = \overline{q(z)}$. It can thus be represented as

$$q(z) = \alpha + \beta z + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \mathrm{d}\mu(t), \qquad z \in \mathbb{C} \setminus \mathbb{R} \tag{I.1.2}$$

with $\alpha \in \mathbb{R}$, $\beta \geq 0$, and μ a positive Borel measure on $\mathbb{R}$ satisfying $\int_{\mathbb{R}} \frac{d\mu(t)}{1+t^2} < \infty$. The measure μ in the integral representation (I.1.2) of the Weyl coefficient is a spectral measure of the underlying operator model if $\beta = 0$ (if $\beta > 0$, a one-dimensional component has to be added). The importance of canonical systems in this context lies in the Inverse Spectral Theorem of L. de Branges, stating that each Herglotz function q is the Weyl coefficient of a unique (suitably normalized) canonical system.

Given a Hamiltonian H, we are ultimately interested in the description of properties of its spectral measure μ_H in terms of H. The correspondence between H and μ_H can be best understood using the Weyl coefficient q_H, whose imaginary part $\operatorname{Im} q_H$ determines μ_H via the Stieltjes inversion formula.
In their recent paper [LPW], Langer, Pruckner, and Woracek gave a two-sided estimate for $\operatorname{Im} q_H(ir)$ in terms of the coefficients of H:

$$L(r) \lesssim \operatorname{Im} q_H(ir) \lesssim A(r), \qquad r > 0, \tag{I.1.3}$$

where L, A are explicit in terms of H, and we used the notation $f(r) \lesssim g(r)$ to state that $f(r) \leq Cg(r)$ for a constant $C > 0$. Moreover, in (I.1.3) the constants implicit in $\lesssim$ are independent of H. The exact formulation of this result will be recalled in Theorem I.2.1.

It may happen that $L(r) = \mathrm{o}(A(r))$, and $\operatorname{Im} q_H(ir)$ is not determined by (I.1.3). A toy example for this is the Hamiltonian

$$H(t) = t \begin{pmatrix} |\log t| & |\log t|^2 \\ |\log t|^2 & |\log t|^3 \end{pmatrix}, \qquad t \in [0, \infty).$$

For $r \to \infty$, a calculation shows that

$$L(r) \asymp (\log r)^{-3}, \qquad A(r) \asymp (\log r)^{-1},$$

where $f(r) \asymp g(r)$ means that both $f(r) \lesssim g(r)$ and $g(r) \lesssim f(r)$.

The following theorem, which is our main result, improves the estimate (I.1.3) by giving a formula for $\operatorname{Im} q_H(ir)$ up to universal multiplicative constants.

I.1.1 Theorem. *Let H be a Hamiltonian on $[a, b)$, and denote*[1]

$$H(t) = \begin{pmatrix} h_1(t) & h_3(t) \\ h_3(t) & h_2(t) \end{pmatrix}, \quad \Omega_H(t) = \begin{pmatrix} \omega_1^{(H)}(t) & \omega_3^{(H)}(t) \\ \omega_3^{(H)}(t) & \omega_2^{(H)}(t) \end{pmatrix} := \int_a^t H(s)\,\mathrm{d}s. \tag{I.1.4}$$

Let $\hat{t} : (0, \infty) \to (a, b)$ be a function satisfying[2]

$$\det \Omega_H(\hat{t}(r)) \asymp \frac{1}{r^2}, \qquad r \in (0, \infty). \tag{I.1.5}$$

Then

$$\operatorname{Im} q_H(ir) \asymp \left| q_H(ir) - \frac{\omega_3^{(H)}(\hat{t}(r))}{\omega_2^{(H)}(\hat{t}(r))} \right| \asymp \frac{1}{r\omega_2^{(H)}(\hat{t}(r))}, \tag{I.1.6}$$

$$\frac{\operatorname{Im} q_H(ir)}{|q_H(ir)|^2} \asymp \frac{1}{r\omega_1^{(H)}(\hat{t}(r))}, \tag{I.1.7}$$

for $r \in (0, \infty)$. The constants implicit in $\asymp$ in (I.1.6) and (I.1.7) depend on the constants hidden in $\asymp$ in (I.1.5), but not on H.
If, in addition, $\operatorname{Im} q_H(ir) = \mathrm{o}(|q_H(ir)|)$ for $r \to \infty$ (or $r \to 0$), then[3]

$$q_H(ir) \sim \frac{\omega_3^{(H)}(\hat{t}(r))}{\omega_2^{(H)}(\hat{t}(r))}, \qquad r \to \infty \quad (r \to 0). \tag{I.1.8}$$

[1] When there is no risk of ambiguity, we write Ω and ω_j instead of Ω_H and $\omega_j^{(H)}$ for short.

[2] We will see later that the equation $\det \Omega_H(t) = \frac{1}{r^2}$ has a unique solution for every $r > 0$. A possible choice of $\hat{t}$ is thus the function that maps $r > 0$ to this solution.

[3] With $f(r) \sim g(r)$ meaning $\lim \frac{f(r)}{g(r)} = 1$.

The two-sided estimate (I.1.6) has some useful features: its pointwise nature, its applicability for $r \to \infty$ and $r \to 0$, and the universality of the constants hidden in $\asymp$. However, it is rather different from an asymptotic formula: it does not capture small oscillations of $\operatorname{Im} q_H(ir)$ around $\frac{1}{r\omega_2^{(H)}(\hat{t}(r))}$.
Note also that the first relation in (I.1.6) can be seen as a statement about the real part of $q_H(ir)$. In fact, $\operatorname{Im} q_H(ir)$ is also obtained if we subtract $\operatorname{Re} q_H(ir)$ from $q_H(ir)$, then take absolute values. It is an open question whether $\operatorname{Re} q_H(ir)$ can be described more directly in terms of H.

A most important class of operators is that of Sturm-Liouville (in particular, Schrödinger) operators. Let us provide a reformulation of Theorem I.1.1 for these operators right away.

Sturm-Liouville operators

We provide a version of Theorem I.1.1 for Sturm-Liouville equations

$$-(py')' + qy = zwy \tag{I.1.9}$$

on (a,b), where $1/p, q, w \in L^1_{loc}(a,b)$, $w > 0$ and p, q are real-valued. Suppose that a is in limit circle case and b is in limit point case. Impose a Dirichlet boundary condition at a, i.e., $y(a) = 0$. The Weyl coefficient for this problem is the unique number $m(z)$ with

$$c(z,\cdot) + m(z)s(z,\cdot) \in L^2((a,b), w(x)\,\mathrm{d}x)$$

where $c(z,\cdot)$ and $s(z,\cdot)$ are solutions of (I.1.9) with initial values

$$\begin{pmatrix} p(a)c'(z,a) \\ c(z,a) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} p(a)s'(z,a) \\ s(z,a) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

I.1.2 Theorem. *For each $t \in (a,b)$, let $(.,.)_t$ and $\|.\|_t$ denote the scalar product and norm on $L^2((a,t), w(x)\,\mathrm{d}x)$, i.e.,*

$$(f,g)_t = \int_a^t f(x)\overline{g(x)}w(x)\,\mathrm{d}x.$$

For $\xi \in \mathbb{R}$, let $\hat{t}_\xi : (0,\infty) \to (a,b)$ be a function satisfying

$$\|c(\xi,\cdot)\|^2_{\hat{t}_\xi(r)}\|s(\xi,\cdot)\|^2_{\hat{t}_\xi(r)} - (c(\xi,\cdot), s(\xi,\cdot))^2_{\hat{t}_\xi(r)} \asymp \frac{1}{r^2}, \qquad r \in (0,\infty). \tag{I.1.10}$$

Then

$$\operatorname{Im} m(\xi + ir) \asymp \frac{1}{r\|s(\xi,\cdot)\|^2_{\hat{t}_\xi(r)}}, \tag{I.1.11}$$

$$\frac{\operatorname{Im} m(\xi + ir)}{|m(\xi+ir)|^2} \asymp \frac{1}{r\|c(\xi,\cdot)\|^2_{\hat{t}_\xi(r)}}, \tag{I.1.12}$$

for $r \in (0,\infty)$*. The constants implicit in* $\asymp$ *are independent of* p, q, w *as well as* ξ*, but do depend on the constants pertaining to* $\asymp$ *in (I.1.10).*

In fact, Theorem I.1.2 is a direct consequence of Theorem I.1.1 upon employing a transformation (cf. [Rem18] for $p = w = 1$ and $\xi = 0$) that maps solutions of (I.1.9) to solutions of the canonical system $y' = (z - \xi)JH_\xi y$, where

$$H_\xi(t) = w(t) \cdot \begin{pmatrix} c(\xi,t)^2 & -s(\xi,t)c(\xi,t) \\ -s(\xi,t)c(\xi,t) & s(\xi,t)^2 \end{pmatrix}, \qquad t \in [a,b).$$

The Weyl coefficients then satisfy $m(z) = q_{H_\xi}(z - \xi)$.

Historical remarks

The origins of the Weyl coefficient in the theory of the Sturm-Liouville differential equation are well summarized in Everitt's paper [Eve04]. We give a short account specifically on the history of estimates for the growth of the Weyl coefficient, which date back at least to the 1950s. Particular attention was often given to the deduction of asymptotic formulae for the Weyl coefficient [Mar52, Kac73, Eve72, Kas75, Atk81, Ben89]. However, asymptotic results usually depend on rather strong assumptions on the data. When weakening these assumptions, one can still ask for explicit estimates for $q(z)$ as $z \to \infty$ nontangentially in the upper half-plane. There is a number of rather early results that determine $|q(z)|$ up to $\asymp$, e.g., [Hil63, Atk88, Ben89], although these still depend on data subject to additional restrictions. Fundamental progress has been made by Jitomirskaya and Last [JL99], who considered Schrödinger operators with arbitrary (real-valued and locally integrable) potentials. They found a formula up to $\asymp$ for $|q(z)|$, which also covers the case $z \to 0$. An analog of this formula for canonical systems was given in [HRdS00].

When it comes to $\operatorname{Im} q(z)$, however, no such formula was available. Only the very recent estimate (I.1.3) from [LPW, Theorem 1.1] made it possible to obtain our main result that determines $\operatorname{Im} q(z)$ up to $\asymp$.

Structure of the paper

The proof of Theorem I.1.1, together with some immediate corollaries, makes up Section I.2. In Section I.3, we continue with a first application, a criterion for integrability of a given comparison function with respect to μ_H. We also characterize boundedness of the distribution function of μ_H relative to a given comparison function.

Section I.4 is dedicated to the boundary behavior of Herglotz functions. Cauchy integrals and the relative behavior of its imaginary and real part have been intensively studied. For example, for a Herglotz function q it is known [Pol03] that the set of $\xi \in \mathbb{R}$ for which

$$\lim_{r \to 0} \frac{\operatorname{Im} q(\xi + ir)}{|q(\xi + ir)|} = 0 \tag{I.1.13}$$

is a zero set w.r.t. μ. In contrast to measure theoretic results like this, we use the de Branges correspondence $H \leftrightarrow q_H$ to investigate this behavior pointwise w.r.t. ξ. In Theorem I.4.3a we show that if ξ is such that (I.1.13) holds, then $|q(\xi + ir)|$ is slowly varying (cf. Definition I.4.2). Theorem I.4.3b is a partial converse of this statement.

In Section I.5 we turn to a finer study of $\operatorname{Im} q_H(ir)$ in the context of the geometric origins of (I.1.3) and (I.1.6). Namely, the functions L and A describe the imaginary parts of bottom and top of certain Weyl disks containing $q_H(ir)$. We show that there are restrictions on the possible location of $q_H(ir)$ within the disks, and construct a Hamiltonian H for which $q_H(ir)$ oscillates back and forth between the bottoms and tops of the disks. This construction allows us to answer several open problems that were posed in [LPW].

We conclude our work with a reformulation of Theorem I.1.1 for the principal Titchmarsh-Weyl coefficient q_S of a Krein string. This reformulation is the content of Section I.6.

Notation associated to Hamiltonians

Let H be a Hamiltonian on $[a, b)$.

An interval $(c, d) \subseteq [a, b)$ is called *H-indivisible* if $H(t)$ takes the form $h(t)\binom{\cos\varphi}{\sin\varphi}\binom{\cos\varphi}{\sin\varphi}^*$ a.e. on (c, d), with scalar-valued h and fixed $\varphi \in [0, \pi)$. The angle φ is then called the *type* of the interval.

I.1.3 Definition. Let

$$\mathring{a}(H) := \inf\Big\{t > a \,\Big|\, (a,t) \text{ is not } H\text{-indivisible of type } 0 \text{ or } \frac{\pi}{2}\Big\}, \tag{I.1.14}$$

$$\hat{a}(H) := \inf\Big\{t > a \,\Big|\, (a,t) \text{ is not } H\text{-indivisible}\Big\}. \tag{I.1.15}$$

Usually, we write $\mathring{a}$ and $\hat{a}$ for short. Since H is assumed to be definite, both of these numbers are smaller than b.

Note that $(\omega_1\omega_2)(t) > 0$ if and only if (a,t) is not H-indivisible of type 0 or $\frac{\pi}{2}$, i.e., $t > \mathring{a}$. Using the assumption $\int_a^b \operatorname{tr} H(t)\,\mathrm{d}t = \infty$, we infer that $\omega_1\omega_2$ is an increasing bijection from $(\mathring{a}, b)$ to $(0,\infty)$.
Similarly, $\det\Omega(t) > 0$ is equivalent to $t > \hat{a}$. We have

$$\frac{d}{dt}\Big(\frac{\det\Omega(t)}{\omega_1(t)}\Big) = \omega_1(t)^{-2}\begin{pmatrix} -\omega_3(t) \\ \omega_1(t)\end{pmatrix}^* H(t)\begin{pmatrix} -\omega_3(t) \\ \omega_1(t)\end{pmatrix} \geq 0$$

and (by symmetry) $\frac{d}{dt}\big(\frac{\det\Omega}{\omega_2}\big) \geq 0$. Since at least one of ω_1 and ω_2 is unbounded, $\det\Omega$ is an increasing bijection from $(\hat{a}, b)$ to $(0,\infty)$.

I.1.4 Definition. For a Hamiltonian H and a number $\eta > 0$, set

$$\mathring{r}_{\eta,H} : \left\{\begin{array}{ccc} (\mathring{a}, b) & \to & (0,\infty) \\ t & \mapsto & \frac{\eta}{2\sqrt{(\omega_1\omega_2)(t)}}, \end{array}\right. \qquad \hat{r}_{\eta,H} : \left\{\begin{array}{ccc} (\hat{a}, b) & \to & (0,\infty) \\ t & \mapsto & \frac{\eta}{2\sqrt{\det\Omega(t)}}. \end{array}\right.$$

Both of these functions are decreasing and bijective. We define their inverse functions,

$$\mathring{t}_{\eta,H} := \mathring{r}_{\eta,H}^{-1} \,:\, (0,\infty) \to (\mathring{a}, b), \qquad \hat{t}_{\eta,H} := \hat{r}_{\eta,H}^{-1} \,:\, (0,\infty) \to (\hat{a}, b). \tag{I.1.16}$$

Note that the functions $\hat{t}_{\eta,H}$, for any $\eta > 0$, satisfy (I.1.5). Functions of this form will be the default choice of $\hat{t}$ for the sake of Theorem I.1.1. We will often fix η and H and write $\mathring{r}$, $\mathring{t}$, $\hat{r}$, $\hat{t}$ for short. If η is fixed but the Hamiltonian is ambiguous, we may write $\mathring{r}_H$, $\mathring{t}_H$, $\hat{r}_H$, $\hat{t}_H$ to indicate dependence on H.

I.2 On the imaginary part of the Weyl coefficient

We start by providing the details of the estimate (I.1.3), which is the central result in [LPW].

I.2.1 Theorem ([LPW, Theorem 1.1]). *Let H be a Hamiltonian on $[a,b)$, and let $\eta \in (0, 1-\frac{1}{\sqrt{2}})$ be fixed. For $r>0$, let $\mathring{t}(r)$ be the unique number satisfying*

$$(\omega_1^{(H)}\omega_2^{(H)})(\mathring{t}(r)) = \frac{\eta^2}{4r^2}, \tag{I.2.1}$$

cf. Definition I.1.4. Set[4]

$$A_{\eta,H}(r) := \frac{\eta}{2r\omega_2^{(H)}(\mathring{t}(r))}, \qquad L_{\eta,H}(r) := \frac{\det \Omega_H(\mathring{t}(r))}{(\omega_1^{(H)}\omega_2^{(H)})(\mathring{t}(r))} \cdot A_{\eta,H}(r).$$

Then the Weyl coefficient q_H associated with the Hamiltonian H satisfies

$$|q_H(ir)| \asymp A_{\eta,H}(r), \tag{I.2.2}$$

$$L_{\eta,H}(r) \lesssim \operatorname{Im} q_H(ir) \lesssim A_{\eta,H}(r) \tag{I.2.3}$$

for $r \in (0,\infty)$. The constants implicit in these relations are independent of H. Their dependence on η is continuous.

In the following proof of Theorem I.1.1, we will also show that Theorem I.2.1 still holds if $\mathring{t} : (0,\infty) \to (a,b)$ is a function satisfying $(\omega_1\omega_2)(\mathring{t}(r)) \asymp \frac{1}{r^2}$, and

$$A(r) := \frac{1}{r\omega_2^{(H)}(\mathring{t}(r))}, \qquad L(r) := \frac{\det \Omega_H(\mathring{t}(r))}{(\omega_1^{(H)}\omega_2^{(H)})(\mathring{t}(r))} \cdot A(r).$$

In particular, we can choose any $\eta > 0$ in (I.2.1).

Proof of Theorem I.1.1. Let $\hat{t}_{\eta,H}$ be defined as in Definition I.1.4. We show that for any $\eta > 0$, Theorem I.1.1 holds for $\hat{t}_{\eta,H}$ in place of $\hat{t}$, and that the dependence on η of the constants hidden in $\asymp$ in (I.1.6) and (I.1.7) is continuous. This then implies that Theorem I.1.1 holds for any function $\hat{t}$ satisfying (I.1.5).

The proof is divided into steps.

Step 1. We introduce a family of transformations of H that leave the imaginary part of the Weyl coefficient unchanged. If $p \in \mathbb{R}$ and

$$H_p(t) := \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} H(t) \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix} = \begin{pmatrix} h_1(t) + 2ph_3(t) + p^2h_2(t) & h_3(t) + ph_2(t) \\ h_3(t) + ph_2(t) & h_2(t) \end{pmatrix},$$

an easy calculation shows that the Weyl coefficient q_p of H_p is given by $q_p(z) = q_0(z) + p = q_H(z) + p$.

[4] If η and H are clear from the context, we may write A and L for short.

Step 2. We prove (I.1.6)-(I.1.8) for fixed $\eta \in (0, 1 - \frac{1}{\sqrt{2}})$. The following abbreviations are used only in Step 2:

short form	meaning	short form	meaning	short form	meaning
$\mathring{t}$	$\mathring{t}_{\eta,H}$	$\mathring{t}_p$	$\mathring{t}_{\eta,H_p}$	Ω_p	Ω_{H_p}
$\hat{t}$	$\hat{t}_{\eta,H}$	$\hat{t}_p$	$\hat{t}_{\eta,H_p}$	$\omega_j^{(p)}$	$\omega_j^{(H_p)}$
L_p	L_{η,H_p}	A_p	A_{η,H_p}	Ω	Ω_H

Let $r > 0$ be fixed (this is important). Our first observation is that $\hat{t}_p(r) = \hat{t}(r)$ for any p since $\det \Omega_p(t) = \det \Omega(t)$ does not depend on p. If we can find p such that $\mathring{t}_p(r) = \hat{t}_p(r) = \hat{t}(r)$, then clearly

$$\frac{L_p(r)}{A_p(r)} = \frac{\det \Omega_p(\mathring{t}_p(r))}{(\omega_1^{(p)}\omega_2^{(p)})(\mathring{t}_p(r))} = \frac{\det \Omega_p(\hat{t}_p(r))}{(\omega_1^{(p)}\omega_2^{(p)})(\mathring{t}_p(r))} = 1.$$

We apply Theorem I.2.1 with η and H_p. The estimate (I.2.3) then takes the form

$$A_p(r) = L_p(r) \lesssim \operatorname{Im} q_H(ir) \lesssim A_p(r) \tag{I.2.4}$$

while (I.2.2) turns into

$$|q_H(ir) + p| \asymp A_p(r), \tag{I.2.5}$$

where

$$A_p(r) = \frac{\eta}{2r\omega_2^{(p)}(\mathring{t}_p(r))} = \frac{\eta}{2r\omega_2(\hat{t}(r))}.$$

The right choice of p is

$$p = -\frac{\omega_3(\hat{t}(r))}{\omega_2(\hat{t}(r))},$$

leading to $\omega_3^{(p)}(\hat{t}(r)) = 0$ and thus

$$(\omega_1^{(p)}\omega_2^{(p)})(\hat{t}(r)) = \det \Omega_p(\hat{t}(r)) = \det \Omega(\hat{t}(r)) = \frac{\eta^2}{4r^2}.$$

Consequently, $\mathring{t}_p(r) = \hat{t}(r)$. Observe that the implicit constants in (I.2.2) and (I.2.3) are independent of H and r and depend continuously on η. This shows that (I.1.6) holds, with constants depending continuously on η.

Step 3. (I.1.7) follows from an application of (I.1.6) to $\tilde{H} := J^\top H J = \begin{pmatrix} h_2 & -h_3 \\ -h_3 & h_1 \end{pmatrix}$ and note that $\hat{t}_{\eta,\tilde{H}} = \hat{t}_{\eta,H}$. Thus

$$\frac{\operatorname{Im} q_H(ir)}{|q_H(ir)|^2} = \operatorname{Im}\Big(-\frac{1}{q_H(ir)}\Big) = \operatorname{Im} q_{\tilde{H}}(ir) \asymp \frac{1}{r\omega_2^{(\tilde{H})}(\hat{t}_{\eta,\tilde{H}}(r))} = \frac{1}{r\omega_1^{(H)}(\hat{t}_{\eta,H}(r))}.$$

Formula (I.1.8) follows if we divide (I.1.6) by $|q_H(ir)|$. Hence, we proved the assertion for $\eta \in (0, 1 - \frac{1}{\sqrt{2}})$.

In the remaining steps we treat the missing case $\eta \geq 1 - \frac{1}{\sqrt{2}}$.

Step 4. Let $k > 0$. For use in Step 5, we show that

$$\operatorname{Im} q_H(ir) \asymp \operatorname{Im} q_H(ikr), \qquad |q_H(ir)| \asymp |q_H(ikr)| \tag{I.2.6}$$

for $r \in (0, \infty)$, where the constants in $\asymp$ depend continuously on k and are independent of H.

For the imaginary part, the statement is easy to see from the integral representation (I.1.2). For the absolute value, we use the Hamiltonian $\tilde{H}$ from Step 3 to obtain

$$\frac{\operatorname{Im} q_H(ir)}{|q_H(ir)|^2} = \operatorname{Im} q_{\tilde{H}}(ir) \asymp \operatorname{Im} q_{\tilde{H}}(ikr) = \frac{\operatorname{Im} q_H(ikr)}{|q_H(ikr)|^2}.$$

This shows that $|q_H(ir)| \asymp |q_H(ikr)|$ as well.

Step 5. Fix a Hamiltonian H, and let $\eta_0 \geq 1 - \frac{1}{\sqrt{2}}$. Then

$$\mathring{t}_{\eta_0,H}(r) = \mathring{t}_{\frac{1}{4},\frac{1}{4\eta_0}H}(r), \qquad \hat{t}_{\eta_0,H}(r) = \hat{t}_{\frac{1}{4},\frac{1}{4\eta_0}H}(r)$$

and

$$A_{\eta_0,H}(r) = A_{\frac{1}{4},\frac{1}{4\eta_0}H}(r), \qquad L_{\eta_0,H}(r) = L_{\frac{1}{4},\frac{1}{4\eta_0}H}(r).$$

Since $\frac{1}{4}$ is less than $1 - \frac{1}{\sqrt{2}}$, we can use Theorem I.2.1 with $\eta := \frac{1}{4}$ to obtain

$$\begin{aligned} L_{\eta_0,H}(r) = L_{\frac{1}{4},\frac{1}{4\eta_0}H}(r) &\lesssim \operatorname{Im} q_{\frac{1}{4\eta_0}H}(ir) \\ &\leq |q_{\frac{1}{4\eta_0}H}(ir)| \asymp A_{\frac{1}{4},\frac{1}{4\eta_0}H}(r) = A_{\eta_0,H}(r) \end{aligned} \tag{I.2.7}$$

for $r \in (0, \infty)$. Since $q_{\frac{1}{4\eta_0}H}(z) = q_H\big(\frac{z}{4\eta_0}\big)$ and by Step 4, we see that Theorem I.2.1 holds for arbitrary $\eta > 0$. It is easy to check that continuous dependence of constants on η is retained. Repeating Steps $1 - 3$ now shows that also Theorem I.1.1 holds for $\hat{t}_{\eta,H}$ for any $\eta > 0$. Moreover, it is not hard to see that everything still works if $\hat{t}$ is a function satisfying (I.1.5). $\square$

I.2.2 Remark. Theorem I.2.1 and Theorem I.1.1, in the form we stated them, give information about $q_H(z)$ for $z = ir$. However, if $\vartheta \in (0, \pi)$ is fixed, these theorems also hold

- ▷ for $z = re^{i\vartheta}$ uniformly for $r \in (0, \infty)$ and
- ▷ for $z = re^{i\varphi}$ uniformly for $r \in (0, \infty)$ and $|\frac{\pi}{2} - \varphi| \leq |\frac{\pi}{2} - \vartheta|$.

We restate the explicit constants coming from [LPW]. Fix $\eta \in (0, 1 - \frac{1}{\sqrt{2}})$ and set $\sigma := (1 - \eta)^{-2} - 1 \in (0, 1)$. With

$$c_-(\eta, \vartheta) = \frac{\eta \sin \vartheta}{2(1 + |\cos \vartheta|)} \cdot \frac{1 - \sigma}{1 + \sigma}, \qquad c_+(\eta, \vartheta) = \frac{\sigma + \frac{2}{\eta \sin \vartheta}}{1 - \sigma},$$

we have[5]

$$c_-(\eta, \vartheta) \cdot \frac{\eta}{2} \cdot \frac{1}{r\omega_2(\hat{t}_{\eta,H}(r))} \leq \operatorname{Im} q_H(re^{i\vartheta}) \leq c_+(\eta, \vartheta) \cdot \frac{\eta}{2} \cdot \frac{1}{r\omega_2(\hat{t}_{\eta,H}(r))}, \tag{I.2.8}$$

$$c_-(\eta, \vartheta) \cdot \frac{\eta}{2} \cdot \frac{1}{r\omega_1(\hat{t}_{\eta,H}(r))} \leq \frac{\operatorname{Im} q_H(re^{i\vartheta})}{|q_H(re^{i\vartheta})|^2} \leq c_+(\eta, \vartheta) \cdot \frac{\eta}{2} \cdot \frac{1}{r\omega_1(\hat{t}_{\eta,H}(r))}. \tag{I.2.9}$$

In order to show (I.2.8), we need to slightly adapt the proof of Theorem I.1.1 by replacing ir with $re^{i\vartheta}$ in (I.2.4) and taking into account the constants provided in [LPW, Theorem 1.1]. Then (I.2.9) follows as in Step 3 of the proof.
For $\vartheta = \frac{\pi}{2}$, the optimal choice of η is around 0.13833 which gives

$$c_+(0.13833, \frac{\pi}{2}) \approx 1.568, \quad c_-(0.13833, \frac{\pi}{2}) \approx 0.002, \quad \frac{c_+(0.13833, \frac{\pi}{2})}{c_-(0.13833, \frac{\pi}{2})} \approx 675.772.$$

While it is possible to derive explicit constants also for $\eta \geq 1 - \frac{1}{\sqrt{2}}$, doing so does not result in an improvement of the quotient c_+/c_-.

Immediate consequences of Theorem I.1.1

In order to simplify calculations, unless specified otherwise, we will always assume that $\mathring{t}(r)$ and $\hat{t}(r)$ are defined implicitly by

$$(\omega_1\omega_2)(\mathring{t}(r)) = \frac{1}{r^2}, \quad \det \Omega(\hat{t}(r)) = \frac{1}{r^2}, \tag{I.2.10}$$

[5] Since c_- and c_+ are clearly monotonic in ϑ, (I.2.8) and (I.2.9) still hold when $q_H(re^{i\vartheta})$ is replaced by $q_H(re^{i\varphi})$, where $|\frac{\pi}{2} - \varphi| \leq |\frac{\pi}{2} - \vartheta|$.

and similarly for $\mathring{r}$ and $\hat{r}$ (cf. Definition I.1.4 with $\eta = 2$).
We revisit the example from the introduction in more generality. The following example was communicated by Matthias Langer. The calculations can be found in the appendix.

I.2.3 Example. Let $\alpha > 0$ and $\beta_1, \beta_2 \in \mathbb{R}$ where $\beta_1 \neq \beta_2$. Set $\beta_3 := \frac{\beta_1+\beta_2}{2}$ and define, for $t \in (0, \infty)$,

$$H(t) = t^{\alpha-1} \begin{pmatrix} |\log t|^{\beta_1} & |\log t|^{\beta_3} \\ |\log t|^{\beta_3} & |\log t|^{\beta_2} \end{pmatrix}.$$

Then for $r \to \infty$, we have

$$L(r) \asymp (\log r)^{\frac{\beta_1-\beta_2}{2}-2} \text{ and}$$

$$A(r) \asymp |q_H(ir)| \asymp (\log r)^{\frac{\beta_1-\beta_2}{2}},$$

i.e., $L(r) = \mathrm{o}(A(r))$. Using Theorem I.1.1, we can now continue the calculations, leading to

$$\operatorname{Im} q_H(ir) \asymp (\log r)^{\frac{\beta_1-\beta_2}{2}-1} \asymp \sqrt{L(r)A(r)}.$$

It is an immediate consequence of Theorem I.1.1 that $\operatorname{Im} q_H$ depends monotonically on the off-diagonal of H.

I.2.4 Corollary. *Let $H = \begin{pmatrix} h_1 & h_3 \\ h_3 & h_2 \end{pmatrix}$ and $\tilde{H} = \begin{pmatrix} h_1 & \tilde{h}_3 \\ \tilde{h}_3 & h_2 \end{pmatrix}$ be two Hamiltonians on $[a, b)$. If $t > \hat{a}(H)$ such that*

$$\Big| \int_a^t h_3(s)\, \mathrm{d}s \Big| \geq \Big| \int_a^t \tilde{h}_3(s)\, \mathrm{d}s \Big|,$$

then

$$\operatorname{Im} q_H(i\hat{r}_H(t)) \lesssim \operatorname{Im} q_{\tilde{H}}(i\hat{r}_H(t))$$

with a constant independent of t, H, and $\tilde{H}$.

Proof. Our condition states that $|\omega_3(t)| \geq |\tilde{\omega}_3(t)|$. Taking into account that $t > \hat{a}(H)$, this means that $0 < \det \Omega(t) \leq \det \tilde{\Omega}(t)$. Hence $\hat{r}_H(t) \geq \hat{r}_{\tilde{H}}(t)$, and further $\hat{t}_{\tilde{H}}(\hat{r}_H(t)) \leq t$. Now, by (I.1.6),

$$\operatorname{Im} q_H(i\hat{r}_H(t)) \asymp \frac{1}{\hat{r}_H(t)\omega_2(t)} \leq \frac{1}{\hat{r}_H(t)\omega_2(\hat{t}_{\tilde{H}}(\hat{r}_H(t)))} \asymp \operatorname{Im} q_{\tilde{H}}(i\hat{r}_H(t)).$$

□

The following result elaborates on the relative behavior of $\operatorname{Im} q_H$ and $|q_H|$. We obtain a quantitative and pointwise relation between $\frac{\operatorname{Im} q_H}{|q_H|}$ and $\frac{\det \Omega}{\omega_1\omega_2}$, leading to the equivalence

$$\lim_{r\to\infty} \frac{\operatorname{Im} q_H(ir)}{|q_H(ir)|} = 0 \iff \lim_{t\to\hat{a}} \frac{\det \Omega(t)}{(\omega_1\omega_2)(t)} = 0. \tag{I.2.11}$$

The relation between $\frac{\det \Omega}{\omega_1\omega_2}$ and $\frac{\operatorname{Im} q_H(ir)}{|q_H(ir)|}$ has been investigated also in [LPW21]. Their proof of (I.2.11)[6] is based on compactness arguments.
Note that our result shows that (I.2.11) holds true for $r \to 0$ and $t \to b$ as well.

I.2.5 Proposition. *Let H be a Hamiltonian on $[a,b)$. Then*[7]

$$\frac{\operatorname{Im} q_H(ir)}{|q_H(ir)|} \asymp \frac{\mathring{r}(\hat{t}(r))}{r} = \sqrt{\frac{\det \Omega(\hat{t}(r))}{(\omega_1\omega_2)(\hat{t}(r))}} \tag{I.2.12}$$

for $r \in (0,\infty)$. Moreover,

$$\big|q_H\big(i\mathring{r}(\hat{t}(r))\big)\big| \asymp |q_H(ir)|, \qquad r \in (0,\infty). \tag{I.2.13}$$

All constants implicit in $\asymp$ do not depend on H.

Proof. By definition of $\mathring{r}$ and using (I.1.6) and (I.1.7),

$$\mathring{r}(\hat{t}(r)) = \frac{1}{\sqrt{(\omega_1\omega_2)(\hat{t}(r))}} \asymp r\frac{\operatorname{Im} q_H(ir)}{|q_H(ir)|}.$$

We also have

$$\sqrt{\frac{\det \Omega(\hat{t}(r))}{(\omega_1\omega_2)(\hat{t}(r))}} = \frac{1}{\sqrt{r^2(\omega_1\omega_2)(\hat{t}(r))}} = \frac{\mathring{r}(\hat{t}(r))}{r},$$

and (I.2.12) follows.
For the proof of (I.2.13), we need the formula

$$\omega_1(\mathring{t}(r)) \asymp \frac{|q_H(ir)|}{r}$$

[6] In [LPW21], $\lim_{t\to a}$ was considered instead of $\lim_{t\to\hat{a}}$.
[7] $\mathring{r}(\hat{t}(r))$ is well-defined because of $\hat{t}(r) \in (\hat{a},b) \subseteq (\mathring{a},b)$.

which we get from Theorem I.2.1 applied to $J^\top HJ$. Combine this with (I.2.2) to get

$$|q_H(ir)|^2 \asymp \frac{\omega_1(\mathring{t}(r))}{\omega_2(\mathring{t}(r))}.$$

On the other hand, (I.1.6) and (I.1.7) give

$$|q_H(ir)|^2 \asymp \frac{\omega_1(\hat{t}(r))}{\omega_2(\hat{t}(r))} = \frac{\omega_1\Big(\mathring{t}\big(\mathring{r}(\hat{t}(r))\big)\Big)}{\omega_2\Big(\mathring{t}\big(\mathring{r}(\hat{t}(r))\big)\Big)} \asymp \big|q_H\big(i\mathring{r}(\hat{t}(r))\big)\big|^2.$$

$\square$

The freedom in the choice of η leads to the following formula that we will refer to later on.

I.2.6 Corollary. *Let H be a Hamiltonian on $[a,b)$. Then, for any $k>0$,*

$$\operatorname{Im} q_H(ikr) \asymp \left|q_H(ikr) - \frac{\omega_3(\hat{t}(r))}{\omega_2(\hat{t}(r))}\right| \asymp \left|q_H(ir) - \frac{\omega_3(\hat{t}(r))}{\omega_2(\hat{t}(r))}\right| \tag{I.2.14}$$

with constants depending on k, but not on H.
If $\operatorname{Im} q_H(ir) = \mathrm{o}(|q_H(ir)|)$ for $r\to\infty$ $[r\to 0]$, then

$$q_H(ikr) \sim \frac{\omega_3(\hat{t}(r))}{\omega_2(\hat{t}(r))}, \qquad r\to\infty \quad [r\to 0]. \tag{I.2.15}$$

Proof. Apply Theorem I.1.1 to H using $\hat{t}_{1,H}$, and to kH using $\hat{t}_{k,kH}$. Then $\hat{t}_{1,H}(r) = \hat{t}_{k,kH}(r)$, and we write $\hat{t}(r)$ for short. Keeping in mind that $q_{kH}(z) = q_H(kz)$, this leads to

$$\operatorname{Im} q_H(ir) \asymp \left|q_H(ir) - \frac{\omega_3^{(H)}(\hat{t}(r))}{\omega_2^{(H)}(\hat{t}(r))}\right| \asymp \frac{1}{r\omega_2^{(H)}(\hat{t}(r))}$$

as well as

$$\operatorname{Im} q_H(ikr) \asymp \left|q_H(ikr) - \frac{k\omega_3^{(H)}(\hat{t}(r))}{k\omega_2^{(H)}(\hat{t}(r))}\right| \asymp \frac{1}{kr\cdot\omega_2^{(H)}(\hat{t}(r))}.$$

(I.2.14) follows. Now (I.2.15) is obtained by dividing (I.2.14) by $|q_H(ikr)|$. $\square$

I.3 Behavior of tails of the spectral measure

Theorem I.1.1 that approximately determines the imaginary part of $q_H(ir)$ allows us to determine the growth of the spectral measure μ_H relative to suitable comparison functions. Let us introduce the measure $\tilde{\mu}_H$ on $[0,\infty)$ by

$$\tilde{\mu}_H([0,r)) := \tilde{\mu}_H(r) := \mu_H((-r,r)), \qquad r > 0. \tag{I.3.1}$$

In Section I.3.1, equivalent conditions are given for when the function $r \mapsto \tilde{\mu}_H$ is integrable w.r.t. a given weight function, and also when the measure $\tilde{\mu}_H$ is finite w.r.t. to a rescaling function.

On the other hand, we can view $\tilde{\mu}_H$ as a function of the positive real parameter r, and compare this to a given function $\mathfrak{g}$. This is what we do in Section I.3.2.

We note that the content of this section is analogous to [LPW, Section 4]. The availability of formula (I.1.6) leads to improved results in the present article, however we provide less detail as was given in [LPW].

The proofs in this section are based on standard theorems of Abelian-Tauberian type, relating μ_H to its Poisson integral

$$\mathcal{P}[\mu_H](z) := \int_{\mathbb{R}} \operatorname{Im}\left(\frac{1}{t-z}\right) \mathrm{d}\mu_H(t). \tag{I.3.2}$$

By (I.1.2), we have $\mathcal{P}[\mu_H](z) = \operatorname{Im} q_H(z) - \beta \operatorname{Im} z$. If $\beta = 0$, we can proceed with the application of Abelian-Tauberian theorems without problems. The case $\beta > 0$ is equivalent to a being the left endpoint of an H-indivisible interval of type $\frac{\pi}{2}$, i.e., $\mathring{a}(H) > a$ and h_2 vanishes a.e. on $[a, \mathring{a}(H))$. The restricted Hamiltonian $H_- := H\big|_{[\mathring{a}(H),b)}$ then has the Weyl coefficient $q_{H_-}(z) = q_H(z) - \beta z$ and thus $\operatorname{Im} q_{H_-}(z) = \mathcal{P}[\mu_H](z)$. Hence, we can investigate μ_H by applying the theorems from this section to H_-.

I.3.1 Finiteness of the spectral measure w.r.t. given weight functions

I.3.1 Theorem. *Let H be a Hamiltonian defined on $[a,b)$, and assume that h_2 does not vanish identically in a neighborhood of a. Let $\mathfrak{f}$ be a continuous, non-decreasing function, and denote by μ_H the spectral measure of H.*
Then the following statements are equivalent:

(i)
$$\int_1^\infty \tilde{\mu}_H(r) \frac{\mathfrak{f}(r)}{r^3} \, \mathrm{d}r < \infty; \tag{I.3.3}$$

(ii) *There is $a' \in (\hat{a}, b)$ such that*

$$\int_{\hat{a}}^{a'} \frac{1}{\omega_2(t)^2} \begin{pmatrix} \omega_2(t) \\ -\omega_3(t) \end{pmatrix}^* H(t) \begin{pmatrix} \omega_2(t) \\ -\omega_3(t) \end{pmatrix} \cdot \not{\ell}\big(\det \Omega(t)^{-\frac{1}{2}}\big) \,\mathrm{d}t < \infty.$$

If, in addition, $\not{\ell}$ is differentiable, then the above conditions hold if and only if there is $a' \in (\hat{a}, b)$ such that

$$\int_{\hat{a}}^{a'} \frac{(\det \Omega)'(t)}{\omega_2(t) \det \Omega(t)^{\frac{1}{2}}} \not{\ell}'\big(\det \Omega(t)^{-\frac{1}{2}}\big) \,\mathrm{d}t < \infty.$$

Proof. First note that finiteness of the integrals in the proposition clearly does not depend on $a' \in (\hat{a}, b)$.
Let ξ be the measure on $[1, \infty)$ such that $\not{\ell}(r) = \xi([1, r))$, $r \geq 1$. It follows from [Kac82, Lemma 4] that

$$\int_{[1,\infty)} \frac{\mathcal{P}[\mu_H](ir)}{r} \,\mathrm{d}\xi(r) < \infty \quad \Longleftrightarrow \quad \int_1^\infty \frac{\tilde{\mu}_H(r)\not{\ell}(r)}{r^3} \,\mathrm{d}r < \infty.$$

Since h_2 does not vanish identically in a neighborhood of a, we have $\mathcal{P}[\mu_H] = \operatorname{Im} q_H$. By Theorem I.1.1, we have

$$\frac{\mathcal{P}[\mu_H](ir)}{r} \asymp \frac{1}{r^2 \omega_2(\hat{t}(r))} \asymp \frac{\det \Omega(\hat{t}(r))}{\omega_2(\hat{t}(r))}.$$

Hence

$$\int_1^\infty \tilde{\mu}_H(r) \frac{\not{\ell}(r)}{r^3} \,\mathrm{d}r < \infty \quad \Longleftrightarrow \quad \int_{[1,\infty)} \frac{\det \Omega\big(\hat{t}(r)\big)}{\omega_2\big(\hat{t}(r)\big)} \,\mathrm{d}\xi(r) < \infty. \tag{I.3.4}$$

We define a measure ν on $(0, \infty)$ via $\nu((r, \infty)) = \frac{\det \Omega(\hat{t}(r))}{\omega_2(\hat{t}(r))}$, $r > 0$. Let $\hat{\nu}$ be the measure on $(\hat{a}, b)$ satisfying $\hat{\nu}((\hat{a}, t)) = \nu((\hat{r}(t), \infty)) = \frac{\det \Omega(t)}{\omega_2(t)}$, $t > \hat{a}$. Integrating by parts (see, e.g., [Kac65, Lemma 2]), we can rewrite the first integral in (I.3.4) as follows:

$$\begin{aligned}
&\int_{[1,\infty)} \frac{\det \Omega(\hat{t}(r))}{\omega_2(\hat{t}(r))} \,\mathrm{d}\xi(r) = \int_{[1,\infty)} \nu\big((r, \infty)\big) \,\mathrm{d}\xi(r) \\
&= \int_{[1,\infty)} \not{\ell}(r) \,\mathrm{d}\nu(r) = \int_{(\hat{a}, \hat{t}(1)]} \not{\ell}(\hat{r}(t)) \,\mathrm{d}\hat{\nu}(t) = \int_{(\hat{a}, \hat{t}(1)]} \not{\ell}(\hat{r}(t)) \,\mathrm{d}\Big(\frac{\det \Omega}{\omega_2}\Big)(t) \\
&= \int_{\hat{a}}^{\hat{t}(1)} \not{\ell}\big(\hat{r}(t)\big) \cdot \frac{1}{\omega_2(t)^2} \begin{pmatrix} \omega_2(t) \\ -\omega_3(t) \end{pmatrix}^* H(t) \begin{pmatrix} \omega_2(t) \\ -\omega_3(t) \end{pmatrix} \mathrm{d}t.
\end{aligned}$$

To prove the additional statement, let us assume that $\mathscr{f}$ is differentiable. Using a substitution we can rewrite the second integral in (I.3.4) differently:

$$\int_{[1,\infty)} \frac{\det\Omega\big(\hat{t}(r)\big)}{\omega_2\big(\hat{t}(r)\big)}\, \mathrm{d}\xi(r) = \int_1^\infty \frac{\det\Omega\big(\hat{t}(r)\big)}{\omega_2\big(\hat{t}(r)\big)} \mathscr{f}'(r)\,\mathrm{d}r$$

$$= \int_{\hat{t}(r)}^{\hat{a}} \frac{\det\Omega(t)}{\omega_2(t)} \mathscr{f}'(\hat{r}(t))\hat{r}'(t)\,\mathrm{d}t = \frac{1}{2}\int_{\hat{a}}^{\hat{t}(r)} \frac{\det\Omega(t)}{\omega_2(t)} \mathscr{f}'(\hat{r}(t)) \frac{(\det\Omega)'(t)}{\det\Omega(t)^{\frac{3}{2}}}\,\mathrm{d}t.$$

□

The following result provides, in particular, information on when the measure $\tilde{\mu}_H$ is finite w.r.t. a regularly varying rescaling function $\mathscr{g}$.

I.3.2 Corollary. *Let H be a Hamiltonian on $[a,b)$, and assume that h_2 does not vanish identically in a neighborhood of a. Let $\mathscr{g}$ be a continuous function that is regularly varying with index $\alpha \in [0,2]$, and denote by μ_H the spectral measure of H as in (I.1.2). Then, for $\alpha \in (0,2)$ and every $a' \in (\hat{a}, b)$, the following statements are equivalent:*

(i) $$\int_{[1,\infty)} \frac{\mathrm{d}\tilde{\mu}_H(r)}{\mathscr{g}(r)} < \infty;$$

(ii) $$\int_{\hat{a}}^{a'} \frac{1}{\omega_2(t)^2} \begin{pmatrix} \omega_2(t) \\ -\omega_3(t) \end{pmatrix}^* H(t) \begin{pmatrix} \omega_2(t) \\ -\omega_3(t) \end{pmatrix} \frac{\mathrm{d}t}{\det\Omega(t)\mathscr{g}\big(\det\Omega(t)^{-\frac{1}{2}}\big)} < \infty.$$

(iii) $$\int_{\hat{a}}^{a'} \frac{(\det\Omega)'(t)}{\omega_2(t)\det\Omega(t)\mathscr{g}\big(\det\Omega(t)^{-\frac{1}{2}}\big)}\,\mathrm{d}t < \infty;$$

If $\alpha = 0$, then (iii) $\Rightarrow$ (i) and (iii) $\Leftrightarrow$ (ii), while for $\alpha = 2$ we have (iii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (ii).

Proof. The increasing function $\mathscr{f}(r) := \int_1^r \frac{t}{\mathscr{g}(t)}\,\mathrm{d}t$ is regularly varying by Karamata's Theorem ([BGT89, Propositions 1.5.8 and 1.5.9a]. Moreover,

$$\mathscr{f}(r) \begin{cases} \asymp \frac{r^2}{\mathscr{g}(r)}, & 0 \le \alpha < 2, \\ \gg \frac{r^2}{\mathscr{g}(r)}, & \alpha = 2. \end{cases} \tag{I.3.5}$$

Clearly (*iii*) is equivalent to

$$\int_{\hat{a}}^{a'} \frac{(\det\Omega)'(t)}{\omega_2(t)\det\Omega(t)^{\frac{1}{2}}} \mathscr{f}'\big(\det\Omega(t)^{-\frac{1}{2}}\big)\,\mathrm{d}t < \infty$$

which is the term appearing in the additional statement of Theorem I.3.1. Applying Theorem I.3.1 and using (I.3.5), this is equivalent to (for $\alpha \in [0,2)$) or implies (for $\alpha = 2$) both *(ii)* and

$$\int_1^\infty \tilde{\mu}_H(r) \frac{dr}{r\mathfrak{g}(r)} < \infty.$$

By [LPW, Proposition 4.5], this is further equivalent to (for $\alpha \in (0,2]$) or implies (for $\alpha = 0$) the first item. $\square$

I.3.2 Comparative growth of the distribution function

In this section we investigate lim sup-conditions for the quotient $\frac{\tilde{\mu}_H(r)}{\mathfrak{g}(r)}$ instead of integrability conditions. Let us introduce the corresponding classes of measures.

I.3.3 Definition. Let $\mathfrak{g}(r)$ be a regularly varying function with index $\alpha \in [0,2]$ and $\lim_{r\to\infty} \mathfrak{g}(r) = \infty$. Then we set

$$\mathcal{F}_{\mathfrak{g}} := \big\{\mu \mid \tilde{\mu}(r) \lesssim \mathfrak{g}(r),\, r \to \infty\big\}, \qquad \mathcal{F}^0_{\mathfrak{g}} := \big\{\mu \mid \tilde{\mu}(r) = \mathrm{o}(\mathfrak{g}(r)),\, r \to \infty\big\},$$

where again $\tilde{\mu}(r) := \mu((-r,r))$.

It should be mentioned that, for non-decreasing $\mathfrak{g}$, if

$$\int_{[1,\infty)} \frac{\mathrm{d}\tilde{\mu}(r)}{\mathfrak{g}(r)} < \infty,$$

then $\mu \in \mathcal{F}^0_{\mathfrak{g}} \subseteq \mathcal{F}_{\mathfrak{g}}$. For further discussion of this relation, the reader is referred to [LPW].

I.3.4 Theorem. *Let H be a Hamiltonian on $[a,b)$, and assume that h_2 does not vanish identically in a neighborhood of a. Let $\mathfrak{g}(r)$ be a regularly varying function with index $\alpha \in [0,2]$ and $\lim_{r\to\infty} \mathfrak{g}(r) = \infty$. Denote by μ_H the spectral measure of H. For $\alpha < 2$, the following statements hold:*

$$\text{(i)} \qquad \mu_H \in \mathcal{F}_{\mathfrak{g}} \quad \Leftrightarrow \quad \limsup_{t\to\hat{a}} \frac{1}{\omega_2(t)\mathfrak{g}\big(\det\Omega(t)^{-\frac{1}{2}}\big)} < \infty;$$

$$\text{(ii)} \qquad \mu_H \in \mathcal{F}^0_{\mathfrak{g}} \quad \Leftrightarrow \quad \lim_{t\to\hat{a}} \frac{1}{\omega_2(t)\mathfrak{g}\big(\det\Omega(t)^{-\frac{1}{2}}\big)} = 0.$$

If $\alpha = 2$, then the right hand side of (i), (ii) implies the left hand side, respectively.

Proof. We use [LPW, Lemma 4.16] which, adapted to our situation, reads as

$$c_\alpha \limsup_{r\to\infty}\Big(\frac{r}{\mathcal{g}(r)}\mathcal{P}[\mu_H](ir)\Big) \leq \limsup_{r\to\infty}\frac{\tilde{\mu}_H(r)}{\mathcal{g}(r)} \leq c'_\alpha \limsup_{r\to\infty}\Big(\frac{r}{\mathcal{g}(r)}\mathcal{P}[\mu_H](ir)\Big),$$

and the second inequality holds even for $\alpha = 2$. Since h_2 does not vanish identically in a neighborhood of a, we have $\mathcal{P}[\mu_H] = \operatorname{Im} q_H$. Therefore, the assertion follows from Theorem I.1.1 and a substitution $r = \hat{r}(t)$. $\square$

I.4 Weyl coefficients with tangential behavior

In this section, we investigate the scenario

$$\lim_{r\to\infty}\frac{\operatorname{Im} q_H(ir)}{|q_H(ir)|} = 0 \qquad \text{or} \qquad \liminf_{r\to\infty}\frac{\operatorname{Im} q_H(ir)}{|q_H(ir)|} = 0. \tag{I.4.1}$$

This is equivalent to tangential behavior of $q_H(ir)$, i.e.,

$$\lim_{r\to\infty}\arg q_H(ir) \in \{0,\pi\} \quad \text{or} \quad \liminf_{r\to\infty}\min\big\{\arg q_H(ir),\, \pi - \arg q_H(ir)\big\} = 0.$$

From Proposition I.2.5 we get that

$$\lim_{n\to\infty}\frac{\operatorname{Im} q_H(ir_n)}{|q_H(ir_n)|} = 0 \iff \lim_{n\to\infty}\frac{\det\Omega(\hat{t}(r_n))}{(\omega_1\omega_2)(\hat{t}(r_n))} = 0. \tag{I.4.2}$$

for every sequence $r_n \to \infty$. All results in this section can be seen from the canonical systems perspective as well as from the Herglotz functions perspective.

To start with, we observe that the second assertion in (I.4.1) implies the first unless the limit inferior is assumed only along very sparse sequences. We formulate this fact in the language of Herglotz functions, and prove it within the canonical systems setting. However, we do not know a purely function theoretic proof (which may very well exist in the literature).

I.4.1 Lemma. *Let q be a Herglotz function. Suppose there is a sequence $(r_n)_{n\in\mathbb{N}}$ with $r_n \to \infty$, $\sup_{n\in\mathbb{N}}\frac{r_{n+1}}{r_n} < \infty$, and*

$$\lim_{n\to\infty}\frac{\operatorname{Im} q(ir_n)}{|q(ir_n)|} = 0.$$

Then $\lim_{r\to\infty}\frac{\operatorname{Im} q(ir)}{|q(ir)|} = 0$.

Proof. Let H be a Hamiltonian (on $[0,\infty)$), such that $q = q_H$.
Let $d(t) := \frac{\det\Omega(t)}{(\omega_1\omega_2)(t)}$. Set $t_n := \hat{t}(r_n)$, then by (I.2.12),

$$d(t_n) \asymp \left(\frac{\operatorname{Im} q(ir_n)}{|q(ir_n)|}\right)^2 \xrightarrow{n\to\infty} 0.$$

Suppose that the assertion was not true, i.e., there is a sequence $\xi_1 > \xi_2 > ...$ converging to 0, such that $d(\xi_k) \geq C > 0$ for all k. For $k \in \mathbb{N}$, set $n(k) := \max\{n \in \mathbb{N} \mid t_n > \xi_k\}$. We obtain

$$\begin{aligned}\left(\frac{r_{n(k)+1}}{r_{n(k)}}\right)^2 &= \frac{\det\Omega(t_{n(k)})}{\det\Omega(t_{n(k)+1})} \geq \frac{\det\Omega(\xi_k)}{\det\Omega(t_{n(k)+1})}\\ &= \frac{d(\xi_k)}{d(t_{n(k)+1})}\cdot\frac{(\omega_1\omega_2)(\xi_k)}{(\omega_1\omega_2)(t_{n(k)+1})} \geq \frac{C}{d(t_{n(k)+1})} \xrightarrow{k\to\infty} \infty\end{aligned}$$

which contradicts our assumption. $\square$

Recall formulae (I.2.13) and (I.2.12). On an intuitive level, they tell us that in the case that $\operatorname{Im} q_H(ir) \not\asymp |q_H(ir)|$, the growth of $|q_H(ir)|$ is restricted since $\mathring{r}(\hat{t}(r))$ is then far away from r. If read in the other direction, this means that if $|q_H(ir)|$ grows quickly and without oscillating too much, then $\mathring{r}(\hat{t}(r))$ and r should be close to each other, and hence the quotient $\frac{\operatorname{Im} q_H(ir)}{|q_H(ir)|}$ should not decay.
The following definition introduces the notions needed in Theorems I.4.3a and I.4.3b, which confirm this intuition.

I.4.2 Definition.

- ▷ A measurable function $f : (0,\infty) \to (0,\infty)$ is called *regularly varying (at infinity) with index* $\alpha \in \mathbb{R}$ if, for any $\lambda > 0$,

$$\lim_{r\to\infty} \frac{f(\lambda r)}{f(r)} = \lambda^\alpha. \tag{I.4.3}$$

If $\alpha = 0$, then f is also called *slowly varying (at infinity)*.

- ▷ A measurable function $f : (0,\infty) \to (0,\infty)$ is *positively increasing (at infinity)* if there is $\lambda \in (0,1)$ such that

$$\limsup_{r\to\infty} \frac{f(\lambda r)}{f(r)} < 1. \tag{I.4.4}$$

Let us say explicitly that we do not require f to be monotone.

I.4.3a Theorem. *Let $q \neq 0$ be a Herglotz function. If $|q(ir)|$ or $\frac{1}{|q(ir)|}$ is positively increasing at infinity (in particular, if $|q(ir)|$ is regularly varying with index $\alpha \neq 0$), then $\operatorname{Im} q(ir) \asymp |q(ir)|$ as $r \to \infty$.*

I.4.3b Theorem. *Let $q \neq 0$ be a Herglotz function. If $\operatorname{Im} q(ir) = \mathrm{o}(|q(ir)|)$ as $r \to \infty$, then, for every $\delta \in [0,1)$,*

$$\lim_{r\to\infty} \frac{q\Big(ir\Big[\frac{\operatorname{Im} q(ir)}{|q(ir)|}\Big]^{\delta}\Big)}{q(ir)} = 1. \tag{I.4.5}$$

For $k > 0$, we also have $\lim_{r\to\infty} \frac{q(ikr)}{q(ir)} = 1$, in particular, $|q(ir)|$ is slowly varying at infinity.

I.4.4 Remark. In Theorem I.4.3a, the requirement that $|q(ir)|$ should be positively increasing is meaningful. It is not enough that $|q(ir)|$ grows sufficiently fast, say, $|q(ir)| \gtrsim r^\delta$ for $r \to \infty$ and some $\delta > 0$.
In fact, for any given $\delta \in (0,1)$, we construct in Definition I.5.2 a Hamiltonian[8] H whose Weyl coefficient q_H satisfies (see Lemma I.5.6) $|q_H(ir)| \gtrsim r^\delta$ as $r \to \infty$, but

$$\liminf_{r\to\infty} \frac{\operatorname{Im} q_H(ir)}{|q_H(ir)|} = 0.$$

In other words, $\operatorname{Im} q_H(ir) \not\asymp |q_H(ir)|$.
Note also that for the above-mentioned H, certainly $|q_H(ir)|$ is not slowly varying [BGT89, Proposition 1.3.6]. Hence, in Theorem I.4.3b it is not enough to require $\operatorname{Im} q(ir_n) = \mathrm{o}(|q(ir_n)|)$ on some sequence $r_n \to \infty$.

I.4.5 Example. Let $q(z) = \log z$, satisfying $|q(ir)| = \big[(\log r)^2 + \frac{\pi^2}{4}\big]^{1/2}$ which is increasing. However, $\operatorname{Im} q(ir)$ is constant and hence $\operatorname{Im} q(ir) = \mathrm{o}(|q(ir)|)$ as $r \to \infty$. Theorem I.4.3a fails because $|q(ir)|$ is not positively increasing.

Proof of Theorem I.4.3a. Assume first that $|q(ir)|$ is positively increasing. Then there are $\lambda, \sigma \in (0,1)$ and $R > 0$ such that

$$\frac{|q(i\lambda r)|}{|q(ir)|} \leq \sigma, \qquad r \geq R. \tag{I.4.6}$$

Let H be a Hamiltonian with Weyl coefficient $q_H = q$, allowing us to use (I.2.13).

[8] Choose suitable parameters $p, l \in (0,1)$, such that $\delta = \frac{\log l}{\log(pl)}$, i.e., $p = l^{\delta^{-1}-1}$.

Suppose that the assertion was not true. Then there is a (w.l.o.g., monotone) sequence $r_n \to \infty$ with $\lim_{n\to\infty} \frac{\operatorname{Im} q(ir_n)}{|q(ir_n)|} = 0$. Let $m(n)$ be such that

$$\lambda^{m(n)+1} \leq \frac{\mathring{r}(\hat{t}(r_n))}{r_n} < \lambda^{m(n)}.$$

Note that $m(n) \to \infty$ because of (I.2.12).
Furthermore, (I.2.13) ensures that there is $\beta > 0$ with

$$\beta \leq \frac{|q(i\mathring{r}(\hat{t}(r)))|}{|q(ir)|}, \quad r \in (0, \infty).$$

We will also need that for $0 < r < r'$,

$$\frac{|q(ir)|}{|q(ir')|} \asymp \frac{r'\omega_2(\mathring{t}(r'))}{r\omega_2(\mathring{t}(r))} \leq \frac{r'}{r}$$

because ω_2 is nondecreasing.
Choosing n so big that $\mathring{r}(\hat{t}(r_n)) \geq R$, we get the contradiction

$$\begin{aligned}\beta &\leq \frac{\left|q\big(i\mathring{r}(\hat{t}(r_n))\big)\right|}{\left|q\big(ir_n\big)\right|} = \frac{\left|q\big(i\mathring{r}(\hat{t}(r_n))\big)\right|}{\left|q\big(i\lambda^{m(n)}r_n\big)\right|} \cdot \prod_{j=0}^{m(n)-1} \frac{\left|q\big(i\lambda^{j+1}r_n\big)\right|}{\left|q\big(i\lambda^j r_n\big)\right|} \\ &\lesssim \frac{\lambda^{m(n)}r_n}{\mathring{r}(\hat{t}(r_n))}\sigma^{m(n)} \leq \frac{\sigma^{m(n)}}{\lambda} \xrightarrow{n\to\infty} 0.\end{aligned}$$

This proves the theorem in the case that $|q(ir)|$ is positively increasing.
If, on the other hand, $\frac{1}{|q(ir)|}$ is positively increasing, we may set $\tilde{q} := -\frac{1}{q}$, for which $|\tilde{q}(ir)|$ is positively increasing. We obtain

$$\frac{\operatorname{Im} q(ir)}{|q(ir)|} = \frac{\operatorname{Im} \tilde{q}(ir)}{|\tilde{q}(ir)|} \asymp 1.$$

Finally, we note that if $|q(ir)|$ is regularly varying with index $\alpha > 0$, then it is also positively increasing. If $|q(ir)|$ is regularly varying with index $\alpha < 0$, then $\frac{1}{|q(ir)|}$ is regularly varying with index $-\alpha > 0$ and thus positively increasing. $\square$

Our proof of Theorem I.4.3b is elementary - only folklore facts that follow from the Herglotz integral representation (I.1.2) are needed. We would be interested in an elementary proof of Theorem I.4.3a as well, which so far we have not found.

One fact needed in the following proof is the following: For any Herglotz function q and any $z \in \mathbb{C}_+$, we have

$$|q'(z)| \leq \frac{\operatorname{Im} q(z)}{\operatorname{Im} z}. \tag{I.4.7}$$

This can be seen using the representation (I.1.2): We write

$$q'(z) = b + \int_{\mathbb{R}} \frac{d\sigma(t)}{(t-z)^2}$$

and obtain

$$|q'(z)| \leq b + \int_{\mathbb{R}} \frac{d\sigma(t)}{|t-z|^2} = \frac{\operatorname{Im} q(z)}{\operatorname{Im} z}.$$

Proof of Theorem I.4.3b. Let $k \in (0,1)$. Then

$$|\log q\big(ikr\big) - \log q(ir)| = \Big| \int_{kr}^{r} i(\log q)'(is)\, \mathrm{d}s \Big| \leq \int_{kr}^{r} \big|(\log q)'(is)\big|\, \mathrm{d}s. \tag{I.4.8}$$

Apply (I.4.7) to $\log q$ and to $i\pi - \log q$ to obtain

$$\begin{aligned} |(\log q)'(is)| &\leq \frac{1}{s} \min\big\{ \operatorname{Im}[\log q(is)], \pi - \operatorname{Im}[\log q(is)] \big\} \\ &= \frac{1}{s} \min\big\{ \arg q(is), \pi - \arg q(is) \big\} \asymp \frac{\operatorname{Im} q(is)}{s|q(is)|} \end{aligned}$$

for all $s > 0$. We will also need monotonicity in s of $s\frac{\operatorname{Im} q(is)}{|q(is)|}$. In fact, it is easy to see from (I.1.2) that $s \operatorname{Im} q(is)$ is nondecreasing in s. Now we can write

$$s\frac{\operatorname{Im} q(is)}{|q(is)|} = \sqrt{s \operatorname{Im} q(is) \cdot s \operatorname{Im}\Big(-\frac{1}{q(is)}\Big)}$$

and hence $s\frac{\operatorname{Im} q(is)}{|q(is)|}$ is nondecreasing in s. Putting together and continuing the estimation in (I.4.8), we obtain

$$\begin{aligned} |\log q\big(ikr\big) - \log q(ir)| &\lesssim \int_{kr}^{r} \frac{\operatorname{Im} q(is)}{s|q(is)|}\, \mathrm{d}s \leq r\frac{\operatorname{Im} q(ir)}{|q(ir)|} \cdot \int_{kr}^{r} \frac{ds}{s^2} \\ &= r\frac{\operatorname{Im} q(ir)}{|q(ir)|}\Big(\frac{1}{kr} - \frac{1}{r}\Big) \asymp \frac{\operatorname{Im} q(ir)}{|q(ir)|} \xrightarrow{r\to\infty} 0. \end{aligned} \tag{I.4.9}$$

This shows $\lim_{r\to\infty} \frac{q(ikr)}{q(ir)} = 1$. To prove (I.4.5), set $k(r) := \frac{\operatorname{Im} q(ir)}{|q(ir)|}$ and repeat the calculations up to the second to last term in (I.4.9), but with k replaced by $k(r)^\delta$, where $\delta \in [0,1)$. Since

$$rk(r)\Big(\frac{1}{rk(r)^\delta} - \frac{1}{r}\Big) \asymp k(r)^{1-\delta} \xrightarrow{r\to\infty} 0,$$

we arrive at (I.4.5). $\square$

Note that $\lim_{r\to\infty}\frac{q(ikr)}{q(ir)} = 1$ is also a consequence of (I.2.15). The preceding proof, in addition to being elementary, is needed to show (I.4.5) which, upon taking absolute values, can be seen as slow variation with a rate.

I.5 Maximal oscillation within Weyl disks

In order to explain the aim of this section, let us first recall the notion of Weyl disks. Let $W(t,z) \in \mathbb{C}^{2\times 2}$ be the fundamental solution of

$$\frac{d}{dt}W(t,z)J = zW(t,z)H(t), \tag{I.5.1}$$

with initial condition $W(a,z) = I$, solving the transpose of equation (I.1.1). We define the *Weyl disks*

$$D_{t,z} := \left\{ \frac{w_{11}(t,z)\tau + w_{12}(t,z)}{w_{21}(t,z)\tau + w_{22}(t,z)} \middle| \tau \in \overline{\mathbb{C}_+} \right\} \subseteq \overline{\mathbb{C}_+}, \tag{I.5.2}$$

where $\mathbb{C}_+ = \{z \in \mathbb{C} | \operatorname{Im} z > 0\}$, and the closure is taken in the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. For fixed $z \in \mathbb{C}_+$ and $t_1 \leq t_2$, we have $D_{t_1,z} \supseteq D_{t_2,z}$, and the disks shrink down to a single point which is $q_H(z)$:

$$\bigcap_{t\in[a,b)} D_{t,z} = \{q_H(z)\}.$$

Now we review the estimate (I.1.3) which has a geometric interpretation. Namely, the functions $L(r)$ and $A(r)$ give, up to $\asymp$, the imaginary part of the bottom and top point of $D_{\mathring{t}(r),ir}$, respectively. The size of $\operatorname{Im} q_H(ir)$ relative to $L(r)$ and $A(r)$ thus corresponds to the vertical position of $q_H(ir)$ within the disk $D_{\mathring{t}(r),ir}$.
In this section we give answers to several questions from [LPW]. For instance, the question was raised whether there is a Hamiltonian H for which $L(r) \asymp \operatorname{Im} q_H(ir) \not\asymp A(r)$ for $r \to \infty$. The answer to this particular question is no, cf. Proposition I.5.1. However[9], $L(r_n) \asymp \operatorname{Im} q_H(ir_n) \ll A(r_n)$ on a subsequence $r_n \to \infty$ is possible, and we provide examples for this in Definition I.5.2 and in Example I.5.7. The Weyl coefficient of the Hamiltonian constructed in Definition I.5.2 exhibits "maximal" oscillatory behaviour in the sense that it goes back and forth between the bottoms and tops of the disks $D_{\mathring{t}(r),ir}$.

I.5.1 Proposition. *Let H be a Hamiltonian on (a,b). The following statements hold:*

[9] In this section, we use the more transparent notation $f(r) \ll g(r)$ instead of $f(r) = \mathrm{o}(g(r))$.

(*i*) *Suppose that* $L(r) \not\asymp A(r)$ *as* $r \to \infty$. *Then there exists a sequence* $(r_n)_{n\in\mathbb{N}}$ *such that* $r_n \to \infty$, $L(r_n) \ll A(r_n)$, *and*

$$\operatorname{Im} q_H(ir_n) \gtrsim \sqrt{L(r_n)A(r_n)}.$$

(*ii*) *Suppose that* $L(r) \not\asymp A(r)$, *but not* $L(r) \ll A(r)$ *as* $r \to \infty$. *Then there is also* $(r'_n)_{n\in\mathbb{N}}$ *with* $r'_n \to \infty$, $L(r'_n) \ll A(r'_n)$, *and*

$$\operatorname{Im} q_H(ir'_n) \asymp \sqrt{L(r'_n)A(r'_n)}. \tag{I.5.3}$$

Proof. We shorten notation by setting $d(t) := \frac{\det \Omega(t)}{(\omega_1\omega_2)(t)}$. By assumption, $\liminf_{t\to\hat{a}} d(t) = 0$. Let $c \in (\hat{a}, b)$ be fixed and set $t_n := \max\{t \le c \mid d(t) \le \frac{1}{n}\}$. With $t_n^+ := \hat{t}(\mathring{r}(t_n)) \ge t_n$, we have $d(t_n^+) \ge \frac{1}{n} = d(t_n)$ if n is large enough for $t_n^+ \le c$ to hold. Using (I.2.12), we obtain

$$\left(\frac{\operatorname{Im} q_H(i\mathring{r}(t_n))}{A(\mathring{r}(t_n))}\right)^2 \asymp d(t_n^+) \ge d(t_n) = \frac{L(\mathring{r}(t_n))}{A(\mathring{r}(t_n))}.$$

Note that $L(\mathring{r}(t_n)) \ll A(\mathring{r}(t_n))$ because of $d(t_n) \to 0$.
Suppose now that $s := \limsup_{r\to\infty} \frac{L(r)}{A(r)} > 0$. Set $\xi_n := \max\{t \le t_n \mid d(\xi_n) = \frac{s}{2}\}$ and find τ_n between ξ_n and t_n such that $d(\tau_n) = \min\{d(t) \mid t \in [\xi_n, t_n]\}$. Certainly, $d(\tau_n) \le d(t_n) = \frac{1}{n}$ and $d(\tau_n) \le d(t)$ for all $t \in [\xi_n, c]$. Also note that by the same arguments as above,

$$\operatorname{Im} q_H(i\mathring{r}(\tau_n)) \gtrsim \sqrt{L(\mathring{r}(\tau_n)A(\mathring{r}(\tau_n))}. \tag{I.5.4}$$

We prove next that $\hat{r}(\tau_n) \ll \mathring{r}(\xi_n)$. Note that by passing to a subsequence and possibly switching signs of ω_3 by looking at $J^\top HJ$ instead of H, we can assume that

$$\lim_{n\to\infty} \frac{\omega_3(\tau_n)}{\sqrt{(\omega_1\omega_2)(\tau_n)}} = 1.$$

A calculation shows that $\sqrt{(\omega_1\omega_2)(t)} - \omega_3(t)$ is increasing. Hence

$$\begin{aligned}
\left(\frac{\hat{r}(\tau_n)}{\mathring{r}(\xi_n)}\right)^2 &= \frac{(\omega_1\omega_2)(\xi_n)}{\det\Omega(\tau_n)} \qquad \text{(I.5.5)}\\
&= \frac{(\omega_1\omega_2)(\xi_n)\Big(1-\frac{\omega_3(\tau_n)}{\sqrt{(\omega_1\omega_2)(\tau_n)}}\Big)}{\Big(\sqrt{(\omega_1\omega_2)(\tau_n)}-\omega_3(\tau_n)\Big)^2\Big(1+\frac{\omega_3(\tau_n)}{\sqrt{(\omega_1\omega_2)(\tau_n)}}\Big)}\\
&\leq \frac{(\omega_1\omega_2)(\xi_n)\Big(1-\frac{\omega_3(\tau_n)}{\sqrt{(\omega_1\omega_2)(\tau_n)}}\Big)}{\Big(\sqrt{(\omega_1\omega_2)(\xi_n)}-\omega_3(\xi_n)\Big)^2\Big(1+\frac{\omega_3(\tau_n)}{\sqrt{(\omega_1\omega_2)(\tau_n)}}\Big)}\\
&= \frac{\Big(1-\frac{\omega_3(\tau_n)}{\sqrt{(\omega_1\omega_2)(\tau_n)}}\Big)}{\Big(1-\frac{\omega_3(\xi_n)}{\sqrt{(\omega_1\omega_2)(\xi_n)}}\Big)^2\Big(1+\frac{\omega_3(\tau_n)}{\sqrt{(\omega_1\omega_2)(\tau_n)}}\Big)} \lesssim 1-\frac{\omega_3(\tau_n)}{\sqrt{(\omega_1\omega_2)(\tau_n)}} \to 0.
\end{aligned}$$

Let $\tau_n^- := \mathring{t}(\hat{r}(\tau_n))$. By the calculation above, $\mathring{r}(\tau_n^-) = \hat{r}(\tau_n) < \mathring{r}(\xi_n)$ for large enough n, implying $\tau_n^- > \xi_n$ and hence $d(\tau_n^-) \geq d(\tau_n)$. Consequently,

$$\frac{L(\hat{r}(\tau_n))}{A(\hat{r}(\tau_n))} = d(\tau_n^-) \geq d(\tau_n) \asymp \left(\frac{\operatorname{Im} q_H(i\hat{r}(\tau_n))}{A(\hat{r}(\tau_n))}\right)^2.$$

This means that

$$\operatorname{Im} q_H(i\hat{r}(\tau_n)) \leq C\sqrt{L(\hat{r}(\tau_n))A(\hat{r}(\tau_n))}$$

for some $C > 0$ and all large n. Recall (I.5.4) and choose $C' > 0$, w.l.o.g. $C' < C$, such that

$$\operatorname{Im} q_H(i\mathring{r}(\tau_n)) \geq C'\sqrt{L(\mathring{r}(\tau_n))A(\mathring{r}(\tau_n))}$$

for large n. By continuity, we find, for each large n, an $r_n' \in [\mathring{r}(\tau_n), \hat{r}(\tau_n)]$ with

$$\frac{\operatorname{Im} q_H(ir_n')}{\sqrt{L(r_n')A(r_n')}} \in [C', C],$$

such that $(r_n')_{n\in\mathbb{N}}$ satisfies (I.5.3). The only thing left to prove is that $L(r_n') \ll A(r_n')$.

Suppose not, then on a subsequence we would have $L(r_n') \asymp A(r_n')$. Consider $\xi_n' := \mathring{t}(r_n') \leq \tau_n$ which would then satisfy $d(\xi_n') \gtrsim 1$ and hence

$$1 - \frac{\omega_3(\xi_n')}{\sqrt{(\omega_1\omega_2)(\xi_n')}} \gtrsim 1.$$

Now look at (I.5.5), but with ξ_n replaced by ξ_n'. It follows that, for large n, $\hat{r}(\tau_n) < r_n'$, contradicting the choice of r_n'. □

In the following definition, we construct a Hamiltonian by prescribing $f := \frac{\omega_3}{\sqrt{\omega_1\omega_2}}$ and choosing f to be a highly oscillating function. It should be mentioned that the method we use for prescription works on a general basis: Any locally absolutely continuous function with values in $(-1,1)$ occurs as $\frac{\omega_3}{\sqrt{\omega_1\omega_2}}$ for some Hamiltonian. Details can be found in the appendix.

I.5.2 Definition. Let $(t_n)_{n\in\mathbb{N}}$, $(\xi_n)_{n\in\mathbb{N}}$ be sequences of positive numbers converging to zero, where $\xi_{n+1} < t_n < \xi_n$ for all $n \in \mathbb{N}$. Choose $p, l \in (0,1)$ and set

$$f(t_n) = 1 - p^n, \qquad f(\xi_n) = l^n$$

and interpolate between those points using monotone and absolutely continuous functions (e.g., linear interpolation). Set

$$\alpha_1(t) := \begin{cases} \frac{f'(t)}{1-f(t)}, & t \in (\xi_{n+1}, t_n), \\ 0, & t \in (t_n, \xi_n) \end{cases}$$

and

$$\alpha_2(t) := \begin{cases} \frac{f'(t)}{1-f(t)}, & t \in (\xi_{n+1}, t_n), \\ -2\frac{f'(t)}{f(t)}, & t \in (t_n, \xi_n) \end{cases}$$

For $t \in [0, t_1]$, let $\omega_i(t) := \exp\Big(-\int_t^{t_1} \alpha_i(s)\,\mathrm{d}s\Big)$, $i = 1,2$, and $\omega_3(t) := \sqrt{(\omega_1\omega_2)(t)} \cdot f(t)$. Set $h_i(t) = \omega_i'(t)$, $i = 1,2,3$, $t \in [0,t_1]$. For $t \in (t_1, \infty)$, let $h_1(t) := 1$ and $h_2(t) := h_3(t) := 0$. Finally, define

$$H_{p,l} := \begin{pmatrix} h_1 & h_3 \\ h_3 & h_2 \end{pmatrix}.$$

I.5.3 Lemma. *$H_{p,l}$ is a Hamiltonian on $[0,\infty)$, and $\omega_i(t) = \int_0^t h_i(s)\,\mathrm{d}s$ for $i = 1,2,3$ and $t \in [0,t_1]$. Moreover, 0 is not the left endpoint of an $H_{p,l}$-indivisible interval.*

Proof. We write H instead of $H_{p,l}$ for short. First we show that $H(t) \geq 0$ for all $t \in [0,t_1]$. Start by noting that, for $i = 1,2$,

$$\frac{h_i(t)}{\omega_i(t)} = (\log \omega_i)'(t) = \alpha_i(t),$$

and calculate

$$\frac{h_3(t)^2}{(\omega_1\omega_2)(t)} = \frac{\big[(\sqrt{\omega_1\omega_2}f)'(t)\big]^2}{(\omega_1\omega_2)(t)} = \Big(f'(t) + \frac{1}{2}\Big[\frac{h_1(t)}{\omega_1(t)} + \frac{h_2(t)}{\omega_2(t)}\Big]f(t)\Big)^2$$
$$= \Big(f'(t) + \frac{\alpha_1(t)+\alpha_2(t)}{2}f(t)\Big)^2.$$

If $t \in (t_n, \xi_n)$, then this equates to 0, as does

$$\frac{(h_1 h_2)(t)}{(\omega_1 \omega_2)(t)} = \alpha_1(t)\alpha_2(t) = 0.$$

For $t \in (\xi_{n+1}, t_n)$,

$$\Big(f'(t) + \frac{\alpha_1(t) + \alpha_2(t)}{2} f(t)\Big)^2 = \Big(\frac{f'(t)}{1 - f(t)}\Big)^2 = \alpha_1(t)\alpha_2(t) = \frac{(h_1 h_2)(t)}{(\omega_1 \omega_2)(t)}.$$

In both cases, $\det H(t) = 0$. For $i = 1, 2$, as $\alpha_i(t) \geq 0$, $t \in [0, t_1]$, certainly $\omega_i(t)$ is increasing and thus $h_i(t) \geq 0$. This suffices to show that $H(t) \geq 0$. H is in limit point case since, for $t > t_1$, the trace of $H(t)$ equals 1. To show that $\omega_i(t) = \int_0^t h_i(s)\, \mathrm{d}s$, $i = 1, 2, 3$, $t \in [0, t_1]$, we need to check that $\lim_{t \to 0} \omega_i(t) = 0$. For $i = 1$, this follows from

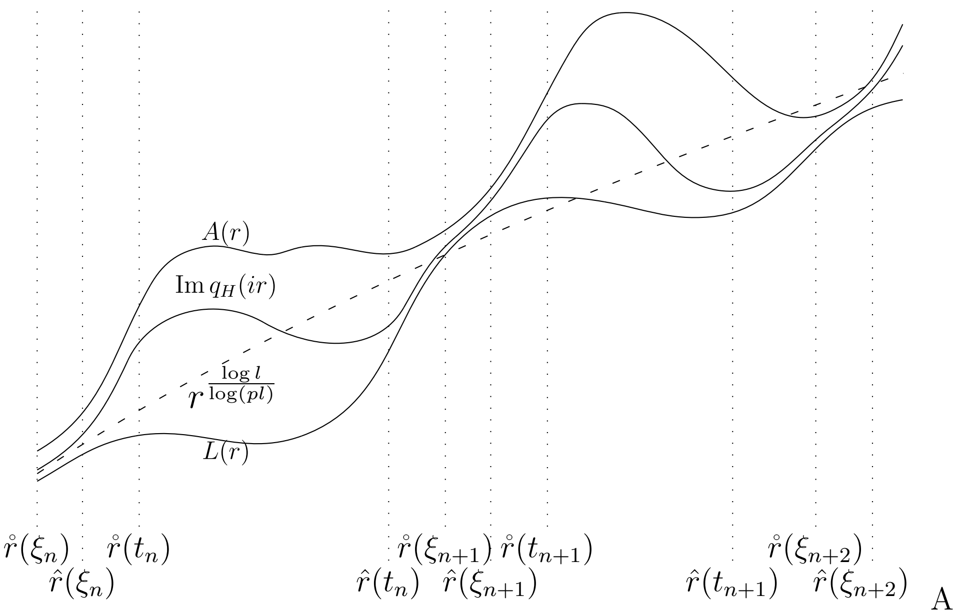
$$\int_0^{t_1} \alpha_1(s)\, \mathrm{d}s = \sum_{n=1}^{\infty} \int_{\xi_{n+1}}^{t_n} \frac{f'(s)}{1 - f(s)}\, \mathrm{d}s = \sum_{n=1}^{\infty} \big[\log(1 - l^{n+1}) - \log(p^n)\big] = \infty. \tag{I.5.6}$$

For $i = 2$, it follows from the fact that $\alpha_2(t) \geq \alpha_1(t)$ for all $t \in [0, t_1]$, and for $i = 3$ it follows from the definition of ω_3 and the fact that $f(t) < 1$, $t \in [0, t_1]$. Finally, 0 is not the left endpoint of an H-indivisible interval because

$$\det \Omega(t) = (\omega_1 \omega_2)(t)\big(1 - f(t)^2\big) > 0$$

for all $t \in (0, t_1]$. $\square$

We investigate the behaviour for $r \to \infty$ of $\operatorname{Im} q_{H_{p,l}}(ir)$ as well as $L(r)$ and $A(r)$. A rough description of the situation is:



sketch of the behaviour of $q_{H_{p,l}}$

Formal details are given in the following theorem as well as in Lemma I.5.6.

I.5.4 Theorem. *Let $p, l \in (0,1)$. For the Hamiltonian $H = H_{p,l}$ from Definition I.5.2 and for all sufficiently large $n \in \mathbb{N}$, we have*

$$\mathring{r}(\xi_n) < \hat{r}(\xi_n) < \mathring{r}(t_n) < \hat{r}(t_n) < \mathring{r}(\xi_{n+1}). \tag{I.5.7}$$

On the intervals delimited by the terms in (I.5.7), the functions $L(r)$, $\operatorname{Im} q_H(ir)$, and $A(r)$ behave in the following way:

(*i*) $\operatorname{Im} q_H(ir) \asymp A(r)$ *uniformly for* $r \in [\mathring{r}(\xi_n), \hat{r}(\xi_n)]$, $n \in \mathbb{N}$.

(*ii*) $\operatorname{Im} q_H(ir) \asymp A(r)$ *uniformly for* $r \in [\hat{r}(\xi_n), \mathring{r}(t_n)]$, $n \in \mathbb{N}$.
Moreover, $L(\mathring{r}(t_n)) \ll A(\mathring{r}(t_n))$.

(*iii*) $L(r) \ll A(r)$ *uniformly for* $r \in [\mathring{r}(t_n), \hat{r}(t_n)]$, $n \in \mathbb{N}$.
In addition, $L(\mathring{r}(t_n)) \ll \operatorname{Im} q_H(i\mathring{r}(t_n)) \asymp A(\mathring{r}(t_n))$ *as well as* $L(\hat{r}(t_n)) \asymp \operatorname{Im} q_H(i\hat{r}(t_n)) \ll A(\hat{r}(t_n))$.

(*iv*) $L(r) \asymp \operatorname{Im} q_H(ir) \ll A(r)$ *uniformly for* $r \in [\hat{r}(t_n), \mathring{r}(\xi_{n+1})]$, $n \in \mathbb{N}$.

The proof of this theorem involves some (partly tedious) computations that are partly contained in the forthcoming lemma.
The symbol $\approx$ should mean equality up to an additive term that is bounded in n and t.

I.5.5 Lemma. *For the Hamiltonian $H_{p,l}$, the following formulae hold.*

$\log \mathring{r}(t_n) \approx -n^2 \frac{\log(pl)}{2} - n \frac{\log\left(\frac{l}{p}\right)}{2}$	$\log \hat{r}(t_n) \approx -n^2 \frac{\log(pl)}{2} - n \frac{\log l}{2}$
$\log \mathring{r}(\xi_n) \approx -n^2 \frac{\log(pl)}{2} + n \frac{\log(pl)}{2}$	$\log \hat{r}(\xi_n) \approx -n^2 \frac{\log(pl)}{2} + n \frac{\log(pl)}{2}$

$\log \mathring{r}(t) \approx -n^2 \frac{\log(pl)}{2} - n \frac{\log\left(\frac{l}{p}\right)}{2} + \log f(t),$	$t \in [t_n, \xi_n]$.
$\log \mathring{r}(t) \approx -n^2 \frac{\log(pl)}{2} - n \frac{\log(pl)}{2} + \log(1 - f(t)),$	$t \in [\xi_{n+1}, t_n]$.
$\log \hat{r}(t) \approx -n^2 \frac{\log(pl)}{2} - n \frac{\log\left(\frac{l}{p}\right)}{2} + \log f(t) - \frac{\log(1-f(t))}{2},$	$t \in [t_n, \xi_n]$.
$\log \hat{r}(t) \approx -n^2 \frac{\log(pl)}{2} - n \frac{\log(pl)}{2} + \frac{\log(1-f(t))}{2},$	$t \in [\xi_{n+1}, t_n]$.

Proof. First we calculate

$$\begin{aligned}
\log(\mathring{r}(t_n)) &= -\frac{1}{2}\log[(\omega_1\omega_2)(t_n)] = \frac{1}{2}\int_{t_n}^{t_1}(\alpha_1(s)+\alpha_2(s))\,\mathrm{d}s \\
&= \sum_{k=1}^{n-1}\Big(\int_{t_{k+1}}^{\xi_{k+1}}\frac{-f'(s)}{f(s)}\,\mathrm{d}s + \int_{\xi_{k+1}}^{t_k}\frac{f'(s)}{1-f(s)}\,\mathrm{d}s\Big) \\
&= \sum_{k=1}^{n-1}\Big(\log(1-p^{k+1}) - (k+1)\log l + \log(1-l^{k+1}) - k\log p\Big) \\
&\approx -n^2\frac{\log(pl)}{2} - n\frac{\log\left(\frac{l}{p}\right)}{2}. \qquad \text{(I.5.8)}
\end{aligned}$$

This also leads to

$$\begin{aligned}
\log\hat{r}(t_n) &= -\frac{1}{2}\log(1-f(t_n)^2) + \log\mathring{r}(t_n) \approx -\frac{1}{2}\log(1-f(t_n)) + \log\mathring{r}(t_n) \\
&\approx -n^2\frac{\log(pl)}{2} - n\frac{\log l}{2}.
\end{aligned}$$

If $t \in [t_n, \xi_n]$, then

$$\log\mathring{r}(t) = \log\mathring{r}(t_n) - \int_{t_n}^{t}\frac{-f'(s)}{f(s)}\,\mathrm{d}s \approx -n^2\frac{\log(pl)}{2} - n\frac{\log\left(\frac{l}{p}\right)}{2} + \log f(t).$$

If $t \in [\xi_{n+1}, t_n]$, then

$$\begin{aligned}
\log\mathring{r}(t) &= \log\mathring{r}(t_n) + \int_{t}^{t_n}\frac{f'(s)}{1-f(s)}\,\mathrm{d}s \\
&\approx -n^2\frac{\log(pl)}{2} - n\frac{\log(pl)}{2} + \log(1-f(t)).
\end{aligned}$$

By adding $-\frac{1}{2}\log(1-f(t)^2) \approx -\frac{1}{2}\log(1-f(t))$, the analogous formula for $\hat{r}(t)$ follows. Lastly,

$$\begin{aligned}
\log\mathring{r}(\xi_n) &\approx -n^2\frac{\log(pl)}{2} - n\frac{\log\left(\frac{l}{p}\right)}{2} + \log f(\xi_n) \\
&\approx -n^2\frac{\log(pl)}{2} + n\frac{\log(pl)}{2}.
\end{aligned}$$

and

$$\log\hat{r}(\xi_n) = -\frac{1}{2}\log(1-f(\xi_n)^2) + \log\mathring{r}(\xi_n) \approx \log\mathring{r}(\xi_n).$$

□

Proof of Theorem I.5.4. It follows from Lemma I.5.5 that $\hat{r}(\xi_n) < \mathring{r}(t_n)$ and $\hat{r}(t_n) < \mathring{r}(\xi_{n+1})$ for large enough n. The remaining two inequalities in (I.5.7) follow from the basic fact that $\mathring{r}(t) < \hat{r}(t)$ for all $t \in (0, \infty)$.
We will now prove $(i) - (iv)$ in reverse order.

<u>(iv)</u>: $\xi_{n+1} \leq \mathring{t}(r) \leq t_n$ and $\xi_{n+1} \leq \hat{t}(r) \leq t_n$. By Lemma I.5.5,

$$
\begin{aligned}
&- n^2 \frac{\log(pl)}{2} - n\frac{\log(pl)}{2} + \frac{1}{2}\log\left(1 - f(\hat{t}(r))\right) \approx \log \hat{r}(\hat{t}(r)) = \log r \\
&= \log \mathring{r}(\mathring{t}(r)) \approx -n^2 \frac{\log(pl)}{2} - n\frac{\log(pl)}{2} + \log\left(1 - f(\mathring{t}(r))\right).
\end{aligned}
$$

Hence,

$$
\frac{\operatorname{Im} q_H(ir)}{A(r)} \asymp \sqrt{1 - f\left(\hat{t}(r)\right)^2} \asymp 1 - f\left(\mathring{t}(r)\right)^2 = \frac{L(r)}{A(r)}.
$$

In addition,

$$
\frac{L\left(\mathring{r}(\xi_{n+1})\right)}{A\left(\mathring{r}(\xi_{n+1})\right)} \asymp 1 - f(\xi_{n+1})^2 \asymp 1,
$$

while

$$
\frac{L\left(\hat{r}(t_n)\right)}{A\left(\hat{r}(t_n)\right)} \asymp 1 - f\left(\mathring{t}(\hat{r}(t_n))\right)^2 \asymp \sqrt{1 - f(t_n)^2} = p^{\frac{n}{2}} \ll 1.
$$

<u>(iii)</u>: $\xi_{n+1} \leq \mathring{t}(r) \leq t_n$ and $t_n \leq \hat{t}(r) \leq \xi_n$. Thus

$$
\begin{aligned}
&- n^2 \frac{\log(pl)}{2} - n\frac{\log\left(\frac{l}{p}\right)}{2} + \log f(\hat{t}(r)) - \frac{1}{2}\log\left(1 - f(\hat{t}(r))\right) \approx \log \hat{r}(\hat{t}(r)) \\
&= \log \mathring{r}(\mathring{t}(r)) \approx -n^2 \frac{\log(pl)}{2} - n\frac{\log(pl)}{2} + \log\left(1 - f(\mathring{t}(r))\right).
\end{aligned}
$$

Consequently,

$$
\frac{1}{2}\log\left(1 - f(\hat{t}(r))\right) \approx n \log p + \log f(\hat{t}(r)) - \log\left(1 - f(\mathring{t}(r))\right),
$$

which implies

$$
\sqrt{1 - f(\hat{t}(r))} \asymp p^n \frac{f(\hat{t}(r))}{1 - f(\mathring{t}(r))}.
$$

Let us check that the term $f(\hat{t}(r))$ can be neglected. Using that $f(\mathring{t}(r)) \leq 1 - p^n$, we get

$$
\sqrt{1 - f(\hat{t}(r))} \lesssim f(\hat{t}(r))
$$

which is only possible if $f(\hat{t}(r))$ stays away from 0. As $f(\hat{t}(r)) < 1$, this means that $f(\hat{t}(r)) \asymp 1$, leading to

$$\frac{\operatorname{Im} q_H(ir)}{A(r)} \asymp \sqrt{1 - f(\hat{t}(r))} \asymp \frac{p^n}{1 - f(\mathring{t}(r))}.$$

Hence, $\operatorname{Im} q_H(i\mathring{r}(t_n)) \asymp A(\mathring{r}(t_n))$. Looking back at case (iv), we know that $\operatorname{Im} q_H(i\hat{r}(t_n)) \asymp L(\hat{r}(t_n)) \ll A(\hat{r}(t_n))$. In particular, since $\frac{L(r)}{A(r)} = 1 - f(\mathring{t}(r))^2$ is increasing for r in $[\mathring{r}(t_n), \hat{r}(t_n)]$, we have $L(r) \ll A(r)$ uniformly on this interval.

<u>(ii):</u> $t_n \leq \mathring{t}(r) \leq \xi_n)$ and $t_n \leq \hat{t}(r) \leq \xi_n$, leading to

$$\begin{aligned}
&-n^2\frac{\log(pl)}{2} - n\frac{\log\left(\frac{l}{p}\right)}{2} + \log f(\hat{t}(r)) - \frac{1}{2}\log\left(1 - f(\hat{t}(r))\right) \approx \log \mathring{r}(\hat{t}(r)) \\
&= \log \mathring{r}(\mathring{t}(r)) \approx -n^2\frac{\log(pl)}{2} - n\frac{\log\left(\frac{l}{p}\right)}{2} + \log f(\mathring{t}(r)).
\end{aligned}$$

Hence

$$\sqrt{1 - f(\hat{t}(r))} \asymp \frac{f(\hat{t}(r))}{f(\mathring{t}(r))} > f(\hat{t}(r)).$$

In particular, $1 - f(\hat{t}(r))$ stays away from 0, which means that

$$\frac{\operatorname{Im} q_H(ir)}{A(r)} \asymp \sqrt{1 - f(\hat{t}(r))} \asymp 1.$$

In other words, $\operatorname{Im} q_H(ir) \asymp A(r)$ uniformly for $r \in [\hat{r}(\xi_n), \mathring{r}(t_n)]$. As we already know, $L(\mathring{r}(t_n)) \ll \operatorname{Im} q_H(i\mathring{r}(t_n)) \asymp A(\mathring{r}(t_n))$.

<u>(i):</u> $t_n \leq \mathring{t}(r) \leq \xi_n$ and $\xi_n \leq \hat{t}(r) \leq t_{n-1}$. In this case

$$\begin{aligned}
&-n^2\frac{\log(pl)}{2} + n\frac{\log(pl)}{2} + \frac{1}{2}\log\left(1 - f(\hat{t}(r))\right) \approx \log \hat{r}(\hat{t}(r)) = \log \mathring{r}(\mathring{t}(r)) \\
&\approx -n^2\frac{\log(pl)}{2} - n\frac{\log\left(\frac{l}{p}\right)}{2} + \log f(\mathring{t}(r)).
\end{aligned}$$

Taking into account that $f(\mathring{t}(r)) \geq l^n$ by definition, it follows that

$$\frac{\operatorname{Im} q_H(ir)}{A(r)} \asymp \sqrt{1 - f(\hat{t}(r))} \asymp \frac{f(\mathring{t}(r))}{l^n} \asymp 1.$$

Therefore, $\operatorname{Im} q_H(ir) \asymp A(r)$ uniformly for $r \in [\mathring{r}(\xi_n), \hat{r}(\xi_n)]$. At the left end of this interval, we even have $L(\mathring{r}(\xi_n)) \asymp A(\mathring{r}(\xi_n))$ by case (iv). □

Before we state our next result, we note that by definition of $H_{p,l}$,

$$\liminf_{t\to 0}\frac{\det\Omega(t)}{(\omega_1\omega_2)(t)}=\liminf_{t\to 0}\big(1-f(t)^2\big)=0. \tag{I.5.9}$$

In view of (I.2.12), we have $\liminf_{r\to\infty}\frac{\operatorname{Im} q_{H_{p,l}}(ir)}{|q_{H_{p,l}}(ir)|}=0$ and hence $\operatorname{Im} q_{H_{p,l}}(ir)\not\asymp |q_{H_{p,l}}(ir)|$.

Nevertheless, the following lemma shows that $|q_{H_{p,l}}(ir)|$ grows faster than a power. Recalling Theorem I.4.3a, this means that $|q(ir)|\gtrsim r^\delta$ for $r\to\infty$ is not a sufficient condition for $\operatorname{Im} q(ir)\asymp|q(ir)|$ as $r\to\infty$. Instead, we see that $|q(ir)|$ being positively increasing really means that not only does $|q(ir)|$ grow sufficiently fast, but also without oscillating too much.

I.5.6 Lemma. *Let* $\delta:=\frac{\log l}{\log(pl)}\in(0,1)$. *Then*

- ▷ $|q_{H_{p,l}}(ir)|\gtrsim r^\delta$, $r\to\infty$,
- ▷ $|q_{H_{p,l}}(i\mathring{r}(\xi_n))|\asymp\mathring{r}(\xi_n)^\delta$.

Proof. We start the proof with calculating, for $t\in[t_n,\xi_n]$,

$$\begin{aligned}\log\sqrt{\frac{\omega_1(t)}{\omega_2(t)}}&=\frac{1}{2}\log\Big(\frac{\omega_1(t)}{\omega_2(t)}\Big)=\sum_{k=1}^{n-1}\int_{t_{k+1}}^{\xi_{k+1}}\frac{-f'(s)}{f(s)}\,\mathrm{d}s+\int_{t_n}^{t}\frac{f'(s)}{f(s)}\,\mathrm{d}s\\&=\sum_{k=1}^{n-1}\Big(\log(1-p^{k+1})-(k+1)\log l\Big)+\log f(t)-\log(1-p^n)\\&\approx-(n^2+n)\frac{\log l}{2}+\log f(t).\end{aligned}$$

Now we use our formula for $\log\mathring{r}(t)$:

$$\begin{aligned}&\log\sqrt{\frac{\omega_1(t)}{\omega_2(t)}}\\&\approx\frac{\log l}{\log(pl)}\log\mathring{r}(t)+\frac{1}{2}\Big(\frac{\log(l)\log(\frac{l}{p})}{\log(pl)}-\log l\Big)n+\Big(1-\frac{\log l}{\log(pl)}\Big)\log f(t)\\&=\frac{\log l}{\log(pl)}\log\mathring{r}(t)+\frac{\log p}{\log(pl)}\big(\log f(t)-n\log l\big),\qquad t\in[t_n,\xi_n].\end{aligned}\tag{I.5.10}$$

Since f was assumed to be monotone decreasing on $[t_n,\xi_n]$, and $\log f(\xi_n)=n\log l$,

$$\log\sqrt{\frac{\omega_1(t)}{\omega_2(t)}}\gtrsim\frac{\log l}{\log(pl)}\log\mathring{r}(t)=\delta\log\mathring{r}(t),$$

where $\gtrapprox$ indicates that the inequality holds up to an additive term that is bounded in n and t. Therefore

$$|q_{H_{p,l}}(i\mathring{r}(t))| \asymp \sqrt{\frac{\omega_1(t)}{\omega_2(t)}} \gtrapprox \mathring{r}(t)^\delta, \quad t \in [t_n, \xi_n].$$

Observing that $\frac{\omega_1}{\omega_2}$ is constant on $[\xi_{n+1}, t_n]$ (since $\alpha_1 - \alpha_2 = 0$ there), we obtain this estimate also for $t \in [\xi_{n+1}, t_n]$:

$$|q_{H_{p,l}}(i\mathring{r}(t))| \asymp \sqrt{\frac{\omega_1(t)}{\omega_2(t)}} = \sqrt{\frac{\omega_1(\xi_{n+1})}{\omega_2(\xi_{n+1})}} \gtrapprox \mathring{r}(\xi_{n+1})^\delta \geq \mathring{r}(t)^\delta.$$

Finally, setting $t = \xi_n$ in (I.5.10) yields $|q_{H_{p,l}}(i\mathring{r}(\xi_n))| \asymp \mathring{r}(\xi_n)^\delta$. $\square$

I.5.7 Example. Let H be as in Definition I.5.2, but $f(\xi_n) = 1 - l^{n-1}$ instead, where $l > \sqrt{p}$. Similarly to Theorem I.5.4, one can show that

$$L(\hat{r}(t_n)) \asymp \operatorname{Im} q_H(i\hat{r}(t_n)) \ll A(\hat{r}(t_n)).$$

However, for our new Hamiltonian,

$$\lim_{t\to 0} \frac{\det \Omega(t)}{(\omega_1\omega_2)(t)} = \limsup_{t\to 0} \frac{\det \Omega(t)}{(\omega_1\omega_2)(t)} = \limsup_{t\to 0} \left(1 - f(t)^2\right) = 0$$

as opposed to (I.5.9).

I.6 Reformulation for Krein strings

Recall that a *Krein string* is a pair $S[L, \mathfrak{m}]$ consisting of a number $L \in (0, \infty]$ and a nonnegative Borel measure $\mathfrak{m}$ on $[0, L]$, such that $\mathfrak{m}([0, t])$ is finite for every $t \in [0, L)$, and $\mathfrak{m}(\{L\}) = 0$. To this pair we associate the equation

$$y'_+(x) + z \int_{[0,x]} y(t)\, \mathrm{d}\mathfrak{m}(t) = 0, \qquad x \in [0, L), \tag{I.6.1}$$

where y'_+ denotes the right-hand derivative of y, and z is a complex spectral parameter.

For each string, we can construct a function q_S called the *principal Titchmarsh-Weyl coefficient* of the string ([LW98] following [KK68]). This function belongs to the Stieltjes class, i.e., it is analytic on $\mathbb{C} \setminus [0, \infty)$, its imaginary part is nonnegative on $\mathbb{C}_+$, and its values on $(-\infty, 0)$ are positive.

The correspondence between Krein strings and functions of Stieltjes class is bijective, as was shown by M.G.Krein.

Theorem I.6.1 below is the reformulation of Theorem I.1.1 for the Krein string case.

I.6.1 Theorem. *Let $S[L, \mathfrak{m}]$ be a Krein string and set*

$$\delta(t) := \Big(\int_{[0,t)} \xi^2 \, \mathrm{d}\mathfrak{m}(\xi)\Big) \cdot \Big(\int_{[0,t)} \mathrm{d}\mathfrak{m}(\xi)\Big) - \Big(\int_{[0,t)} \xi \, \mathrm{d}\mathfrak{m}(\xi)\Big)^2. \tag{I.6.2}$$

for $t \in [0, L)$. Let

$$\hat{\tau}(r) := \inf\big\{t > 0 \,\big|\, \frac{1}{r^2} \leq \delta(t)\big\}, \qquad r \in (0, \infty).$$

We set

$$f(r) := \mathfrak{m}([0, \hat{\tau}(r))) + \mathfrak{m}(\{\hat{\tau}(r)\}) \frac{\frac{1}{r^2} - \delta(\hat{\tau}(r))}{\delta(\hat{\tau}(r)+) - \delta(\hat{\tau}(r))} \tag{I.6.3}$$

if δ is discontinuous at $\hat{\tau}(r)$, and $f(r) := \mathfrak{m}([0, \hat{\tau}(r)))$ otherwise. Then

$$\operatorname{Im} q_S(ir) \asymp \frac{1}{r f(r)}, \qquad r \in (0, \infty), \tag{I.6.4}$$

with constants independent of the string.

Before proving Theorem I.6.1, we need to introduce the concept of dual strings as well as a Hamiltonian associated to a string. Writing

$$m(t) := \mathfrak{m}([0, t)), \qquad t \in [0, L)$$

we can define the dual string $S[\hat{L}, \hat{\mathfrak{m}}]$ of $S[L, \mathfrak{m}]$ by setting

$$\hat{L} := \begin{cases} m(L) & \text{if } L + m(L) = \infty, \\ \infty & \text{else} \end{cases}$$

and

$$\hat{m}(\xi) := \inf\{t > 0 \,|\, \xi \leq m(t)\}.$$

The function $\hat{m}$ is increasing and left-continuous and thus gives rise to a nonnegative Borel measure $\hat{\mathfrak{m}}$.

The Hamiltonian defined by

$$H(t) := \begin{cases} \begin{pmatrix} \hat{m}(t)^2 & \hat{m}(t) \\ \hat{m}(t) & 1 \end{pmatrix} & \text{if } t \in [0, \hat{L}], \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \text{if } \hat{L} + \int_0^{\hat{L}} \hat{m}(t)^2 \, \mathrm{d}t < \infty, \ \hat{L} < t < \infty \end{cases} \tag{I.6.5}$$

then satisfies $q_S = q_H$, see e.g. [KWW07].

Proof of Theorem I.6.1. In view of Theorem I.1.1 and the fact that $q_S = q_H$ for the Hamiltonian H defined in (I.6.5), our task is to express $\hat{t}_H(r)$ in terms of the string. If $\delta(\hat{\tau}(r)) = \frac{1}{r^2}$, this is easy because of [KWW07, Corollary 3.4] giving

$$\det \Omega_H(m(\hat{\tau}(r))) = \delta(\hat{\tau}(r)) = \frac{1}{r^2}$$

and hence $\hat{t}_H(r) = m(\hat{\tau}(r))$.
Otherwise, we have $\delta(\hat{\tau}(r)) < \frac{1}{r^2}$ and $\delta(\hat{\tau}(r)+) \geq \frac{1}{r^2}$. Using again [KWW07, Corollary 3.4], we have

$$\det \Omega_H(m(\hat{\tau}(r))) = \delta(\hat{\tau}(r)) < \frac{1}{r^2}, \qquad \det \Omega_H(m(\hat{\tau}(r)+)) = \delta(\hat{\tau}(r)+) \geq \frac{1}{r^2} \tag{I.6.6}$$

which tells us that $\hat{t}_H(r) \in \big(m(\hat{\tau}(r)), m(\hat{\tau}(r)+)\big]$. By [KWW07, Lemma 3.1], $\hat{m}$ is constant on this interval. Therefore, for $t \in \big(m(\hat{\tau}(r)), m(\hat{\tau}(r)+)\big]$,

$$\begin{aligned} \det \Omega_H(t) &= \bigg(\int_0^{m(\hat{\tau}(r))} \hat{m}(x)^2 \, \mathrm{d}x + \big(t - m(\hat{\tau}(r))\big)\hat{m}(t)^2 \bigg) \cdot t \\ &\quad - \bigg(\int_0^{m(\hat{\tau}(r))} \hat{m}(x) \, \mathrm{d}x + \big(t - m(\hat{\tau}(r))\big)\hat{m}(t) \bigg)^2 = c_1(r)t + c_2(r) \end{aligned}$$

for some constants $c_1(r), c_2(r)$. Using (I.6.6), this leads to

$$\det \Omega_H(t) = \delta(\hat{\tau}(r)) + \frac{t - m(\hat{\tau}(r))}{m(\hat{\tau}(r)+) - m(\hat{\tau}(r))} \big(\delta(\hat{\tau}(r)+) - \delta(\hat{\tau}(r))\big).$$

If we equate this to $\frac{1}{r^2}$, we find that

$$\hat{t}_H(r) = m(\hat{\tau}(r)) + \big(m(\hat{\tau}(r)+) - m(\hat{\tau}(r))\big) \frac{\frac{1}{r^2} - \delta(\hat{\tau}(r))}{\delta(\hat{\tau}(r)+) - \delta(\hat{\tau}(r))} = f(r).$$

Now we have $\omega_{H;2}(t) = \int_0^t h_2(s) \, \mathrm{d}s = t$, and Theorem I.1.1 now shows

$$\operatorname{Im} q_S(ir) = \operatorname{Im} q_H(ir) \asymp \frac{1}{r\hat{t}_H(r)} = \frac{1}{rf(r)}.$$

$\square$

Appendix I.A A construction method for Hamiltonians with prescribed angle of the Weyl coefficient

Let H be a Hamiltonian on (a, b). Assume for simplicity that $\hat{a} = a$, i.e., a is not the left endpoint of an H-indivisible interval. As discussed at the beginning of Section I.4, the behavior of

$$\frac{\det \Omega(t)}{(\omega_1\omega_2)(t)} = 1 - \frac{\omega_3(t)^2}{(\omega_1\omega_2)(t)} > 0$$

towards the left endpoint a corresponds to the angle of $q_H(ir)$ for $r \to \infty$. It is thus desirable to be able to construct examples of Hamiltonians with prescribed $\frac{\omega_3(t)}{\sqrt{(\omega_1\omega_2)(t)}}$, which is what we did in Definition I.5.2. We give now a general version of this idea.
The following result is formulated for Hamiltonians in limit circle case, making the statement cleaner. When we made use of this construction method in Definition I.5.2, we obtained a Hamiltonian in limit point case by simply appending an infinitely long indivisible interval.

I.A.1 Proposition. *Let f be locally absolutely continuous on $(a, b]$ and such that $f(t) \in (-1, 1)$ for all $t \in (a, b]$. Then there is a Hamiltonian, in limit circle case at b, with the properties*

(i) $a = \hat{a}$ is not the left endpoint of an H-indivisible interval, and

(ii) $f(t) = \frac{\omega_3(t)}{\sqrt{(\omega_1\omega_2)(t)}}$ for all $t \in (a, b]$.

In addition, let

$$\Delta(f) := \frac{2|f'|}{1 - \operatorname{sgn}(f')f}$$

which is in $L^1_{loc}((a, b])$. Then all possible choices for $(\omega_1\omega_2)(t)$ are given by functions of the form

$$\exp\Big(c - \int_t^b g(s)\,\mathrm{d}s\Big)$$

where $c \in \mathbb{R}$, $g \in L^1_{loc}((a, b]) \setminus L^1((a, b])$ with $g(t) \geq \Delta(f)(t)$ and $g(t) > 0$ for $t \in (a, b]$ a.e.

Proof. If H is given and such that $(i),(ii)$ hold, then clearly f is locally absolutely continuous and takes values in $(-1,1)$.
Let f be as in the statement. Then clearly $f' \in L^1_{loc}((a,b])$. Also, the denominator of $\Delta(f)$ is locally bounded below by a positive number, and hence $\Delta(f) \in L^1_{loc}((a,b])$. We check the conditions that $\omega_1(t), \omega_2(t)$ must satisfy in order that they, together with $\omega_3(t) := \sqrt{(\omega_1\omega_2)(t)}f(t)$, give rise to a Hamiltonian through $h_i(t) := \omega_i'(t)$, $i = 1,2,3$. Clearly, ω_1, ω_2 have to be increasing, absolutely continuous on $[a,b]$ and satisfy $\omega_1(0) = \omega_2(0) = 0$. Moreover, we want

$$(h_1h_2)(t) \geq h_3(t)^2 = \Big(\sqrt{(\omega_1\omega_2)(t)}f'(t) + \frac{h_1(t)\omega_2(t) + \omega_1(t)h_2(t)}{2\sqrt{(\omega_1\omega_2)(t)}}f(t)\Big)^2.$$

This is equivalent to

$$\frac{(h_1h_2)(t)}{(\omega_1\omega_2)(t)} \geq \bigg(f'(t) + \frac{1}{2}\Big(\frac{h_1(t)}{\omega_1(t)} + \frac{h_2(t)}{\omega_2(t)}\Big)f(t)\bigg)^2$$

Setting

$$\alpha_i(t) := \frac{h_i(t)}{\omega_i(t)}, \quad i = 1,2, \qquad g := \alpha_1 + \alpha_2 \in L^1_{loc}((a,b]),$$

the inequality takes the form

$$\alpha_i(g - \alpha_i) \geq \Big(f' + \frac{1}{2}gf\Big)^2$$

which is equivalent to

$$\alpha_i \in \left[\frac{g}{2} - \sqrt{\frac{g^2}{4} - \Big(f' + \frac{1}{2}gf\Big)^2}, \frac{g}{2} + \sqrt{\frac{g^2}{4} - \Big(f' + \frac{1}{2}gf\Big)^2}\right]. \tag{I.A.1}$$

In particular,

$$\begin{aligned}\frac{g^2}{4} - \Big(f' + \frac{1}{2}gf\Big)^2 \geq 0 &\iff \frac{g}{2} \geq \Big|f' + \frac{1}{2}gf\Big| \\ &\iff g \geq \frac{2f'}{1-f} \text{ and } g \geq \frac{-2f'}{1+f} \\ &\iff g \geq \frac{2|f'|}{1 - \operatorname{sgn}(f')f} = \Delta(f).\end{aligned}$$

Since $\Delta(f) \in L^1_{loc}((a,b])$, we can find $g \in L^1_{loc}((a,b])$, $g(t) \geq \Delta(f)(t)$ a.e. on $(a,b]$, and additionally, $g \notin L^1((a,b])$. Choose measurable functions α_1, α_2 such that

- ▷ $\alpha_1 + \alpha_2 = g$,
- ▷ (I.A.1) holds for α_1 at almost all $t \in (a,b)$ (and hence for α_2), and
- ▷ $\alpha_1, \alpha_2 \notin L^1((a,b])$.

Note that α_1 and α_2 belong to $L^1_{loc}((a,b])$ since g does, and that such a choice is possible because one can always take $\alpha_1 = \alpha_2 = \frac{g}{2}$.
From the construction it is clear that for a Hamiltonian H with $\frac{d}{dt}[\log \omega_i(t)] = \alpha_i(t)$, $i = 1,2$, there is $c \in \mathbb{R}$ such that

$$\omega_i(t) := \exp\Big(c - \int_t^b \alpha_i(s)\,\mathrm{d}s\Big), \qquad i = 1,2.$$

□

Appendix I.B Calculations for Example I.2.3

Let H be the Hamiltonian from Example I.2.3,

$$H(t) = t^{\alpha-1} \begin{pmatrix} |\log t|^{\beta_1} & |\log t|^{\beta_3} \\ |\log t|^{\beta_3} & |\log t|^{\beta_2} \end{pmatrix}, \qquad t \in (0,\infty),$$

where $\alpha > 0$, $\beta_1, \beta_2 \in \mathbb{R}$ such that $\beta_1 \neq \beta_2$, and $\beta_3 := \frac{\beta_1+\beta_2}{2}$. We carry out the calculations to justify the claimed asymptotics from the example. They were communicated by Matthias Langer.

In order to calculate $\mathring{t}(r)$ and $\hat{t}(r)$, two lemmas are needed.

I.B.1 Lemma. *Let $f : (0,\varepsilon) \to (0,\infty)$ be increasing and $f(t) \sim ct^a|\log t|^b$ as $t \to 0$, for $a > 0$, $c > 0$, $b \in \mathbb{R}$. Then*

$$f^{-1}(s) \sim \Big(ca^{-b}s|\log s|^{-b}\Big)^{\frac{1}{a}}, \quad s \to 0.$$

Proof. We have

$$\lim_{t\to 0} \frac{f(t)}{t^a|\log t|^b} = c. \tag{I.B.1}$$

Therefore,

$$\lim_{t\to 0} \Big[\log f(t) - a\log t - b\log|\log t|\Big] = \log c$$

and further

$$\lim_{t\to 0}\left[\frac{\log f(t)}{\log t}-a\right]=\lim_{t\to 0}\left[\frac{\log f(t)}{\log t}-a-b\frac{\log|\log t|}{\log t}\right]=0.$$

In other words, $|\log t|\sim\frac{1}{a}|\log f(t)|$. At the same time, by (I.B.1),

$$t\sim\left(c^{-1}f(t)|\log t|^{-b}\right)^{\frac{1}{a}}\sim\left(c^{-1}f(t)[\frac{1}{a}|\log f(t)|]^{-b}\right)^{\frac{1}{a}}$$

which implies the assertion. $\square$

I.B.2 Lemma. *Let $a>-1$ and $b\in\mathbb{R}$. Then*

$$\int_0^t s^a(-\log s)^b\,\mathrm{d}s=\frac{1}{a+1}t^{a+1}(-\log t)^b \cdot\left[1-\frac{b}{a+1}(-\log t)^{-1}+\frac{b(b-1)}{(a+1)^2}(-\log t)^2+\mathrm{O}\big((-\log t)^{-3}\big)\right]$$

Proof.

$$\int_0^t s^a(-\log s)^b\,\mathrm{d}s=\frac{1}{a+1}s^{a+1}(-\log s)^b\Big|_0^t-\frac{1}{a+1}\int_0^t s^a b(-\log s)^{b-1}\,\mathrm{d}s$$
$$=\frac{1}{a+1}t^{a+1}(-\log t)^b-\frac{b}{a+1}\int_0^t s^a(-\log s)^{b-1}\,\mathrm{d}s \tag{I.B.2}$$

Using (I.B.2) two more times:

$$\begin{aligned}&=\frac{1}{a+1}t^{a+1}(-\log t)^b\\&\quad-\frac{b}{a+1}\left[\frac{1}{a+1}t^{a+1}(-\log t)^{b-1}-\frac{b-1}{a+1}\int_0^t s^a(-\log s)^{b-2}\,\mathrm{d}s\right]\\&=\frac{1}{a+1}t^{a+1}(-\log t)^b\left[1-\frac{b}{a+1}(-\log t)^{-1}+\frac{b(b-1)}{(a+1)^2}(-\log t)^{-2}\right]\\&\quad+c(a,b)\int_0^t s^a(-\log s)^{b-3}\,\mathrm{d}s.\end{aligned}$$

The assertion follows using Karamata's Theorem [BGT89, Prop. 1.5.8 and 1.5.9a]. $\square$

We are now in position to determine $L(r)$, $\operatorname{Im} q_H(ir)$, and $A(r)$.

Calculation of $\mathring{t}(r)$: By Karamata's Theorem we have

$$\omega_i(t)\sim\frac{1}{\alpha}t^\alpha(-\log t)^{\beta_i},\qquad i=1,2,3.$$

Hence

$$(\omega_1\omega_2)(t) \sim \frac{1}{\alpha^2} t^{2\alpha}(-\log t)^{\beta_1+\beta_2}. \tag{I.B.3}$$

Applying Lemma I.B.1 yields

$$(\omega_1\omega_2)^{-1}(s) \sim c \cdot s^{\frac{1}{2\alpha}}(-\log s)^{-\frac{\beta_3}{\alpha}}.$$

We arrive at

$$\mathring{t}(r) = (\omega_1\omega_2)^{-1}(r^{-2}) \sim c' \cdot r^{-\frac{1}{\alpha}}(\log r)^{-\frac{\beta_3}{\alpha}}. \tag{I.B.4}$$

Calculation of $A(r)$:

$$\begin{aligned} A(r) &= \sqrt{\frac{\omega_1(\mathring{t}(r))}{\omega_2(\mathring{t}(r))}} \sim (-\log \mathring{t}(r))^{\frac{\beta_1-\beta_2}{2}} \sim \left[-\log\left(r^{-\frac{1}{\alpha}}(\log r)^{-\frac{\beta_3}{\alpha}}\right)\right]^{\frac{\beta_1-\beta_2}{2}} \\ &\sim (\alpha \log r)^{\frac{\beta_1-\beta_2}{2}}. \end{aligned}$$

Calculation of $\hat{t}(r)$: We use Lemma I.B.2 to calculate $\omega_i(t)$ with more precision:

$$\begin{aligned} \omega_i(t) &= \int_0^t s^{\alpha-1}(-\log s)^{\beta_i}\, \mathrm{d}s = \frac{1}{\alpha} t^{\alpha}(-\log t)^{\beta_i} \\ &\quad \cdot \left[1 - \frac{\beta_i}{\alpha}(-\log t)^{-1} + \frac{\beta_i(\beta_i-1)}{\alpha^2}(-\log t)^{-2} + \mathrm{O}\big((-\log t)^{-3}\big)\right] \end{aligned}$$

We get

$$\begin{aligned} &\det \Omega(t) = \frac{1}{\alpha^2} t^{2\alpha}(-\log t)^{\beta_1+\beta_2} \\ &\cdot \Bigg[1 - \frac{\beta_1+\beta_2}{\alpha}(-\log t)^{-1} + \frac{\beta_1(\beta_1-1)+\beta_2(\beta_2-1)+\beta_1\beta_2}{\alpha^2}(-\log t)^{-2} \\ &- \left(1 - \frac{2\beta_3}{\alpha}(-\log t)^{-1} + \frac{2\beta_3(\beta_3-1)+\beta_3^2}{\alpha^2}(-\log t)^{-2} + \mathrm{O}\big((-\log t)^{-3}\big)\right)\Bigg] \\ &= \frac{1}{\alpha^2} t^{2\alpha}(-\log t)^{\beta_1+\beta_2}\left[\left(\frac{\beta_1-\beta_2}{2\alpha}\right)^2(-\log t)^{-2} + \mathrm{O}\big((-\log t)^{-3}\big)\right] \\ &\sim c \cdot t^{2\alpha}(-\log t)^{2(\beta_3-1)}. \end{aligned} \tag{I.B.5}$$

By Lemma I.B.1,

$$\big(\det \Omega\big)^{-1}(s) \sim c' \cdot s^{\frac{1}{2\alpha}}(-\log s)^{-\frac{\beta_3-1}{\alpha}}$$

and further

$$\hat{t}(r) = \big(\det\Omega\big)^{-1}(r^{-2}) \sim c'' \cdot r^{-\frac{1}{\alpha}}(\log r)^{-\frac{\beta_3-1}{\alpha}}.$$

Calculation of $\operatorname{Im} q_H(ir)$:

$$\begin{aligned}\operatorname{Im} q_H(ir) &\asymp \frac{1}{r\omega_2(\hat{t}(r))} \sim \frac{\alpha}{r}\big(\hat{t}(r)\big)^{-\alpha}\big(-\log\hat{t}(r)\big)^{-\beta_2}\\ &\sim c''' \cdot \frac{\alpha}{r} r(\log r)^{\beta_3-1}(\log r)^{-\beta_2} = c''' \cdot (\log r)^{\frac{\beta_1-\beta_2}{2}-1}.\end{aligned}$$

Calculation of $L(r)$: Using (I.B.5), (I.B.3) and (I.B.4), we have

$$\frac{\det\Omega(\mathring{t}(r))}{(\omega_1\omega_2)(\mathring{t}(r))} \sim \Big(\frac{\beta_1-\beta_2}{2\alpha}\Big)^2(\alpha\log r)^{-2}.$$

Multiplying by $A(r)$, we obtain

$$L(r) \sim c(\log r)^{\frac{\beta_1-\beta_2}{2}-2}.$$

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Chapter II

Eigenvalue distribution of canonical systems: trace class and sparse spectrum

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Eigenvalue distribution of canonical systems: trace class and sparse spectrum

Matthias Langer * Jakob Reiffenstein * Harald Woracek ‡

Abstract: In this paper we consider two-dimensional canonical systems with discrete spectrum and study their eigenvalue densities. We develop a formula that determines the Stieltjes transform of the eigenvalue counting function up to universal multiplicative constants. An explicit criterion is given for the resolvents of the model operator to belong to a Schatten-von Neumann class with index $p \in (0, 2)$, thus giving an answer to the long standing question of which canonical systems have trace class resolvents.
For canonical systems with two limit circle endpoints, an algorithmic approach is available for determining the growth of the monodromy matrix. We discuss connections to previous work by R. Romanov and present examples that demonstrate how to apply this method even for complicated data.

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Keywords: canonical system, discrete spectrum, density of eigenvalues, regularly varying function

II.1 Introduction

A two-dimensional *canonical system* is a differential equation of the form

$$y'(t) = zJH(t)y(t) \tag{II.1.1}$$

on some interval $[a, b)$, where H is a locally integrable 2×2-matrix-valued function, $z \in \mathbb{C}$, and $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Equations of this form originate from Hamiltonian mechanics (with one-dimensional phase space). The function H is called the *Hamiltonian* of the system, and is supposed to take nonnegative matrices as values. Canonical systems provide a unifying treatment for various equations of mathematical physics and pure mathematics, such as Dirac systems, Schrödinger equations, Krein strings, or Jacobi matrices. In fact, every self-adjoint operator with simple spectrum can be realised as the differential operator A_H induced by some canonical system (and an appropriate boundary condition). The theory of two-dimensional canonical systems

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was developed by M.G. Krein [GK70] and L. de Branges [dB68]; recent references are [Rem18, BHdS20].

In this paper we consider the situation when A_H has discrete spectrum, and address the question how to determine the asymptotic density of the sequence of eigenvalues of A_H. Here we think of the density of a sequence having no finite accumulation point in ways familiar from complex analysis.

Let us start with explaining a simplified version of our results. Denote by σ_H the spectrum of A_H, assume that σ_H is discrete, and consider the following question: *for which $p > 0$ does the series*

$$\sum_{\lambda\in\sigma_H\setminus\{0\}} \frac{1}{|\lambda|^p} \tag{II.1.2}$$

converge?

In [RW20] explicit conditions on H were given which characterise discreteness of σ_H and convergence of the series (II.1.2) when $p > 1$. The approach was operator-theoretic: the series converges if and only if resolvents of A_H belong to the Schatten–von Neumann class $\mathfrak{S}_p$. The proof of the convergence criterion relies heavily on the theory of symmetrically normed operator ideals, in particular Matsaev's Theorem, and breaks down when $p \leq 1$. As examples show, the results indeed fail to hold for such p. In particular, it remained an open question when resolvents of A_H belong to the trace class.

In the present paper we develop a very different approach which allows to deal with small values of p. The method is analytic, and it works for $p < 2$. Our first main theorem is Theorem II.3.1, where we consider the counting function

$$n_H(r) := \#\big\{\lambda\in\sigma_H \mid |\lambda| < r\big\},$$

and determine the Stieltjes transform of the function $\frac{n_H(\sqrt{t})}{t}$ up to universal constants $c_-, c_+ > 0$:

$$c_-\cdot I(r) \leq \int_0^\infty \frac{1}{t+r^2}\cdot\frac{n_H(\sqrt{t})}{t}\,\mathrm{d}t \leq c_+\cdot I(r), \tag{II.1.3}$$

where the function $I(r)$ is given explicitly in terms of the Hamiltonian H. An answer to the above stated question is then deduced by applying a variant of Karamata's Tauberian theorem. The proof of (II.1.3) relies on recent work from [LPW, Reiar] about the high-energy behaviour of the Weyl coefficient of a canonical system. The connection with this seemingly unrelated topic is made with a globalisation trick, then the usual relations between growth and zero distribution of a canonical product are exploited, and a limiting process has to be carried out. The reason that the present approach is bound to values $p < 2$ is that for larger p higher order regularising factors occur in canonical products.

It is now apparent that our result (II.1.3) goes far beyond merely characterising convergence of (II.1.2). Using the language of complex analysis, we can characterise convergence class, as well as minimal/finite/maximal type, with respect to regularly varying comparison functions. This is achieved by applying refined variants of the mentioned Tauberian theorem. What we cannot do is to characterise actual eigenvalue asymptotics. This is an intrinsic limitation of the method, and happens because of the constants in (II.1.3) whose presence allows for uncontrollable oscillations of the Stieltjes transform.

The function $I(r)$ in (II.1.3) is an explicit but cumbersome expression involving the entries of H, integrals, and inverse functions. We believe that for small growth, meaning $p \leq 1$, there cannot exist a plain and simple formula characterising convergence of (II.1.2) or finer growth behaviour. This believe is supported by earlier results in the literature about special types of systems, e.g., from [BS14] for Jacobi matrices or [Kac86, Kac90] for Krein strings. Hence, we inevitably face the challenge to evaluate the complicated expression $I(r)$ at least asymptotically.

In our second main result, Theorem II.6.2, we address this question of applicability. In that theorem we consider the case when the Hamiltonian H is not only locally integrable on the half-open interval $[a, b)$, but even integrable on the whole interval $[a, b]$. This case, called *Weyl's limit circle case*, is often considered as being simpler than the general case when it comes to spectral theory. Assuming that limit circle case takes place, it is well-known that σ_H is discrete and that (this is the *Krein–de Branges formula*)

$$\lim_{r\to\infty} \frac{n_H(r)}{r} = \frac{2}{\pi} \int_a^b \sqrt{\det H(t)}\, \mathrm{d}t. \tag{II.1.4}$$

However, in the present context this does not help much: the formula does not tell anything about the true speed of growth of n_H as soon as $\det H$ vanishes identically. And the situation when $\det H = 0$ is not at all obscure: for example it occurs for every Hamiltonian associated with a Jacobi matrix and most realisations of Schrödinger equations.

In Theorem II.6.2 we give an algorithm which produces a function $\kappa_H(r)$ such that

$$\begin{aligned} c'_-\kappa_H(r) - \mathrm{O}(\log r) \leq r^2 \int_0^\infty \frac{1}{t+r^2} \cdot \frac{n_H(\sqrt{t})}{t}\, \mathrm{d}t \\ \leq c'_+\kappa_H(r) \log r \big(1 + \mathrm{o}(1)\big) + \mathrm{O}(\log r) \end{aligned} \tag{II.1.5}$$

with universal constants $c'_-, c'_+ > 0$. This theorem is a trade-off: it enhances applicability tremendously, but we pay for it with accepting less precision quantified by the $\log r$-factor on the right-hand side.

The algorithm itself, and the deduction of (II.1.5) from (II.1.3), are elementary and based on computing an appropriate partitioning of the interval $[a, b]$. The theorem as it stands is sharp. We give examples which show that both bounds in (II.1.5) can be attained. It is bound to the limit circle case because the functions hidden in O-relations will explode as $\int_a^b \|H(t)\| \, \mathrm{d}t$ increases.

The closest relative to the present results in the literature is the paper [Kac86] by I.S. Kac, where he gives a criterion for a Krein string to have discrete spectrum of convergence class w.r.t. a proximate order. Rewriting a string to a canonical system leads to a diagonal Hamiltonian, and we can reobtain his results. It is noteworthy that the methods of proof are essentially different: while our approach relies on a direct estimate of Weyl discs via [LPW], Kac's method can be seen as using differential inequalities for an associated Riccati equation.

II.2 Preliminaries

II.2.1 Nevanlinna functions

In the spectral theory of self-adjoint operators, a particular class of analytic functions plays a prominent role.

II.2.1 Definition. We call q a *Nevanlinna function*, if q is an analytic function on the open upper half-plane $\mathbb{C}^+$ and maps this half-plane into its closure $\mathbb{C}^+ \cup \mathbb{R}$. The set of all Nevanlinna functions is denoted by $\mathcal{N}$.

◀

Sometimes, also the function constant equal to ∞ is included in $\mathcal{N}$. We will not do this, and write $\mathcal{N} \cup \{\infty\}$ whenever necessary.

In the literature such functions are also often called *Herglotz functions*. Indeed, G. Herglotz proved that they admit an integral representation.

II.2.2 Theorem. *A function $q\colon \mathbb{C}^+ \to \mathbb{C}$ is a Nevanlinna function, if and only if there exist $\alpha \in \mathbb{R}$, $\beta \geq 0$, and a positive Borel measure μ on $\mathbb{R}$ with $\int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{1+t^2} < \infty$ such that*

$$q(z) = \alpha + \beta z + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \mathrm{d}\mu(t), \qquad z \in \mathbb{C}^+.$$

The data in this integral representation can be obtained from q by explicit formulae. Namely, $\alpha = \operatorname{Re} q(i)$, $\beta = \lim_{y\to\infty} \frac{1}{iy} q(iy)$, and μ by the Stieltjes inversion formula, e.g., [RR94, 5.4].

II.2.2 Canonical systems

As already said in the introduction, we study systems (II.1.1) whose Hamiltonian H has some integrability and positivity properties.

II.2.3 Definition. Let $-\infty < a < b \leq \infty$. We denote by $\mathbb{H}_{a,b}$ the set of all measurable functions $H\colon (a,b) \to \mathbb{R}^{2\times 2}$, such that

- ▷ $H \in L^1\big((a,c), \mathbb{R}^{2\times 2}\big)$ for all $c \in (a,b)$,
- ▷ $H(t) \geq 0$ for $t \in (a,b)$ a.e.,
- ▷ $\{t \in (a,b) \mid H(t) = 0\}$ has measure zero.

Functions which coincide a.e. will tacitly be identified.

We say that H is in *limit circle case* if $H \in L^1\big((a,b), \mathbb{R}^{2\times 2}\big)$, and that H is in *limit point case* otherwise.

◀

If there is need to refer to the entries of H, we always use the generic notation

$$H(t) = \begin{pmatrix} h_1(t) & h_3(t) \\ h_3(t) & h_2(t) \end{pmatrix}. \tag{II.2.1}$$

Due to positivity, H is in limit circle case if and only if

$$\int_a^b \operatorname{tr} H(t)\,\mathrm{d}t < \infty.$$

Intervals where H is a scalar multiple of a constant rank 1 matrix play an intrinsically exceptional role. In the following we denote

$$\xi_\phi := \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}, \qquad \phi \in \mathbb{R}.$$

II.2.4 Definition. Let $H \in \mathbb{H}_{a,b}$. A nonempty interval $(c,d) \subseteq (a,b)$ is called *indivisible* for H, if there exists $\phi \in \mathbb{R}$ such that

$$H(t) = \operatorname{tr} H(t) \cdot \xi_\phi \xi_\phi^T \qquad \text{for } t \in (c,d) \text{ a.e.}$$

The number ϕ, which is determined up to integer multiples of π is called the *type* of the indivisible interval (c,d).

We denote by I_{reg} the set of all $t \in (a,b)$ which are not inner point of an indivisible interval. We call H *definite*, if (a,b) is not indivisible.

◀

Next, let us very briefly recall the operator model of $H \in \mathbb{H}_{a,b}$. A Hilbert space $L^2(H)$ is defined as a certain closed subspace of the usual L^2-space of the matrix measure $H(t)\,\mathrm{d}t$. Namely, as the subspace containing all those (equivalence classes $H(t)\,\mathrm{d}t$–a.e. of) functions f, which have the following property: for every indivisible interval $(c,d) \subseteq (a,b)$ the function $\xi_\phi^T f(t)$ is constant a.e. on (c,d) where ϕ is the type of (c,d).

In order to define the model operator $A(H)$, we have to distinguish between limit circle and limit point case. Sometimes, $A(H)$ is a multi-valued operator (i.e., a linear relation), and in order to include these cases in the definition we specify $A(H)$ by its graph. Also, we assume that H is definite, since this implies that absolutely continuous representatives are unique and hence boundary values are well-defined.

▷ Assume H is in limit circle case. Then

$$A(H) := \left\{ (f,g) \in L^2(H) \times L^2(H) \;\middle|\; \begin{array}{l} f \text{ has an absolutely continuous} \\ \text{representative with } f' = JHg \text{ a.e.} \\ \text{and } (1,0)f(a) = (0,1)f(b) = 0 \end{array} \right\}.$$

▷ Assume H is in limit point case. Then

$$A(H) := \left\{ (f,g) \in L^2(H) \times L^2(H) \mid \begin{array}{l} f \text{ has an absolutely continuous} \\ \text{representative with } f' = JHg \text{ a.e.} \\ \text{and } (1,0)f(a) = 0 \end{array} \right\}.$$

In both cases $A(H)$ is self-adjoint, and we denote the spectrum of (the operator part of) $A(H)$ as σ_H. The spectrum is always simple. If H is in limit circle case, $A(H)$ is invertible and $A(H)^{-1}$ is compact.

Discreteness of the spectrum, and membership in Schatten–von Neumann ideals can be characterised. We recall the following two facts, which are shown in [RW20, KW07]:

▷ $A(H)$ has compact resolvents, i.e. σ_H is discrete, if and only if there exists $\phi \in \mathbb{R}$ such that

$$\int_a^b \xi_\phi^T H(t) \xi_\phi < \infty, \quad \lim_{t \to b} \left(\int_a^t \xi_{\phi+\frac{\pi}{2}}^T H(s) \xi_{\phi+\frac{\pi}{2}} \, \mathrm{d}s \cdot \int_t^b \xi_\phi^T H(s) \xi_\phi \, \mathrm{d}s \right) = 0.$$

▷ $A(H)$ has resolvent belonging to the Hilbert–Schmidt class, i.e. $\sum_{\lambda \in \sigma_H \setminus \{0\}} \frac{1}{\lambda^2} < \infty$, if and only if there exists $\phi \in \mathbb{R}$ such that

$$\int_a^b \xi_\phi^T H(t) \xi_\phi < \infty, \quad \int_a^b \left(\int_a^t \xi_{\phi+\frac{\pi}{2}}^T H(s) \xi_{\phi+\frac{\pi}{2}} \, \mathrm{d}s \right) \cdot \xi_\phi^T H(s) \xi_\phi \, \mathrm{d}t < \infty.$$

II.2.5 Remark. Applying rotation isomorphism always allows to reduce to the case that $\phi = 0$. This is a common procedure; for details see e.g. [RW20, §6.1]. Assuming that $\int_a^b \binom{1}{0}^T H(t) \binom{1}{0} < \infty$, the above two statements rewrite as

$$\sigma_H \text{ discrete} \quad \Longleftrightarrow \quad \lim_{t \to b} \left(\int_t^b h_2(s) \, \mathrm{d}s \cdot \int_a^t h_1(s) \, \mathrm{d}s \right) = 0,$$

$$\sum_{\lambda \in \sigma_H \setminus \{0\}} \frac{1}{\lambda^2} < \infty \quad \Longleftrightarrow \quad \int_a^b \left(\int_a^t h_2(s) \, \mathrm{d}s \right) \cdot h_1(s) \, \mathrm{d}t < \infty.$$

◁

A scalar spectral measure for $A(H)$ can be constructed from the solutions of the equation (II.1.1). Let us briefly explain this procedure. Given $H \in \mathbb{H}_{a,b}$, the initial value problem

$$\begin{cases} \frac{\partial}{\partial t} W_H(t;z)J = zW_H(t;z)H(t), & t \in (a,b), \\ W_H(a,z) = I. \end{cases} \tag{II.2.2}$$

has a unique solution $W_H(t;z)$ on $[a,b)$. We refer to $W_H(t;z)$ as the *fundamental solution* of H. If H is in limit circle case, this solution can be extended continuously to b, and we call $W_H(b,z)$ the *monodromy matrix* of H. For each fixed t the function $W_H(t;z)$ is an entire function in z. Its entries $w_{H,ij}(t;z)$, $i,j = 1,2$, are real along the real axis, have only real and simple zeroes, and are of bounded type in $\mathbb{C}^+$.

▷ Assume H is in limit circle case. Then the function

$$q_H(z) := \frac{w_{H,12}(b,z)}{w_{H,22}(b,z)}$$

is a Nevanlinna function. The measure in its Herglotz integral representation is a scalar spectral measure for the operator part of $A(H)$. In particular, σ_H is discrete and coincides with the set of zeroes of the function $w_{H,22}(b,z)$.

▷ Assume H is in limit point case. Then the limit

$$q_H(z) := \lim_{t \to b} \frac{w_{H,12}(t;z)}{w_{H,22}(t;z)}$$

exists locally uniformly on $\mathbb{C} \setminus \mathbb{R}$ and is a Nevanlinna function. The measure in its Herglotz integral representation is a scalar spectral measure for the operator part of $A(H)$.

We shall refer to the function q_H above as the *Weyl coefficient* of H. Let us also mention in this place that $A(H)$ is multi-valued if and only if in the integral representation of q_H the linear term bz is present.

II.2.3 The function $\det \Omega_H$

Given a Hamiltonian $H \in \mathbb{H}_{a,b}$, we set

$$\Omega_H(s,t) := \int_s^t H(u)\,\mathrm{d}u,$$

where $s,t \in [a,b)$ with $s \le t$. If H is in limit circle case, we can extend this definition to the range $s,t \in [a,b]$, $s \le t$.

If there is need to refer to the entries of Ω_H, we shall use the notation (recall (II.2.1))

$$\omega_{H,j}(s,t) := \int_s^t h_j(u)\,\mathrm{d}u, \quad j = 1,2,3, \tag{II.2.3}$$

so that

$$\Omega_H(s,t) = \begin{pmatrix} \omega_{H,1}(s,t) & \omega_{H,3}(s,t) \\ \omega_{H,3}(s,t) & \omega_{H,2}(s,t) \end{pmatrix},$$

Since $H(t) \geq 0$ and $\operatorname{tr} H(t) > 0$ a.e., we have $\Omega_H(s,t) \geq 0$ for all $s \leq t$ and $\Omega_H(s,t) = 0$ if and only $s = t$.

If there is no risk of ambiguity, we drop explicit notation of H and write $\Omega(s,t)$ and $\omega_j(s,t)$ instead of $\Omega_H(s,t)$ and $\omega_{H,j}(s,t)$, respectively.

We make extensive use of the function

$$\det\Omega\colon \begin{cases} \{(s,t) \in [a,b) \mid s \leq t\} & \to & [0,\infty) \\ (s,t) & \mapsto & \det\Omega(s,t) \end{cases}$$

and thus collect some of its properties.

II.2.6 Lemma. *Let $H \in \mathbb{H}_{a,b}$.*

(i) *The function $\det\Omega$ is continuous.*

(ii) *We have $\det\Omega(s,t) = 0$ if and only if (s,t) is indivisible.*

(iii) *Let $s < t$ and assume that $\det\Omega(s,t) > 0$. Then, for all $s' \in [a,s]$ and $t' \in [t,b]$ we have*

$$\frac{\det\Omega(s,t)}{\omega_1(s,t)} \leq \frac{\det\Omega(s',t')}{\omega_1(s',t')}, \qquad \frac{\det\Omega(s,t)}{\omega_2(s,t)} \leq \frac{\det\Omega(s',t')}{\omega_2(s',t')}.$$

(iv) *Fix $s \in [a,b)$, and set*

$$c(s) := \sup\{t \in [s,b] \mid (s,t) \text{ indivisible}\}.$$

Then the function $t \mapsto \det\Omega(s,t)$ is strictly increasing on $[c(s),b)$.

(v) *Fix $t \in (a,b)$, and set*

$$d(t) := \inf\{s \in [a,t] \mid (s,t) \text{ indivisible}\}.$$

Then the function $s \mapsto \det\Omega(s,t)$ is strictly decreasing on $[a,d(s)]$.

Proof. Item (i) is clear since $\Omega(s,t)$ is an integral over an L^1_{loc}-function. For the proof of (ii) note that, for each $x \in \mathbb{C}^2$,

$$\big(\Omega(s,t)x,x\big) = \int_s^t \big(H(u)x,x\big)\,\mathrm{d}u,$$

Using that $H(u)$ and $\Omega(s,t)$ are positive semidefinite, we see that $\Omega(s,t)x = 0$ if and only if $H(u)x = 0$ for a.a. $u \in (s,t)$.

Item (iii) follows by differentiating, see [LPW, (2.17)]. Items (iv) and (v) are immediate consequences of (iii), since either $\omega_1(s,t)$ or $\omega_2(s,t)$ is strictly increasing. $\square$

By monotonicity we can extend $\det\Omega$ to the right endpoint b regardless whether H is limit circle or limit point. Namely, set

$$\det\Omega(s,b) := \lim_{t\to b} \det\Omega(s,t), \quad s \in [a,b).$$

This extension, now defined on $\{(s,t) \in [a,b) \times [a,b] \mid s \le t\}$ again has all properties listed in Lemma II.2.6. In addition we have the following fact.

II.2.7 Lemma. *Let $H \in \mathbb{H}_{a,b}$ be definite. Then H is in limit point case, if and only if $\det\Omega(s,b) = \infty$ for some (and hence for every) $s \in [a,b)$.*

Proof. We use the same notation as in the proof of Lemma II.2.6. Since

$$\int_a^t \operatorname{tr} H(u)\,\mathrm{d}u = \lambda_1\big(\Omega(a,t)\big) + \lambda_2\big(\Omega(a,t)\big),$$

limit point case takes place if and only if $\lim_{t\to b} \lambda_1(\Omega(a,t)) = \infty$. Since H is definite, we find $c \in (a,b)$ such that $\lambda_2(\Omega(a,c)) > 0$, and hence

$$\lim_{t\to b} \lambda_1(\Omega(a,t)) = \infty \;\Leftrightarrow\; \lim_{t\to b} \Big(\lambda_1\big(\Omega(a,t)\big) \cdot \lambda_2\big(\Omega(a,t)\big)\Big) = \infty.$$

$\square$

The following notion plays a central role in our present investigations.

II.2.8 Definition. Let $H \in \mathbb{H}_{a,b}$, $r_0 \ge 0$, and $c_-, c_+ > 0$.

(i) We call $\hat{t}$ a *compatible function* for H, r_0 with constants c_-, c_+ if

$$\hat{t}\colon (r_0,\infty) \to (a,b),$$
$$\forall r \in (r_0,\infty).\; \frac{c_-}{r^2} \le \det\Omega\big(a,\hat{t}(r)\big) \le \frac{c_+}{r^2}.$$

(ii) If $\hat{t}$ is a compatible function for H, r_0, we denote

$$\Gamma(\hat{t}) := \big\{(t,r) \in (a,b) \times (r_0,\infty) \mid \hat{t}(r) \leq t\big\}.$$

(iii) We call $(\hat{t},\hat{s})$ a *compatible pair* for H, r_0 with constants c_-, c_+, if $\hat{t}$ is a compatible function for H, r_0 with constants c_-, c_+, and

$$\begin{aligned}&\hat{s}\colon \Gamma(\hat{t}) \to [a,b),\\ &\forall (t,r) \in \Gamma(\hat{t}).\ \hat{s}(t;r) \leq t,\ \frac{c_-}{r^2} \leq \det\Omega\big(\hat{s}(t;r),t\big) \leq \frac{c_+}{r^2}.\end{aligned}$$

◀

The properties of $\det\Omega$ established in Lemma II.2.6 imply that compatible pairs exist.

II.2.9 Corollary. *Let $H \in \mathbb{H}_{a,b}$ be definite and let $c > 0$. Then there exists a compatible pair $(\hat{t},\hat{s})$ for $H, \big(\frac{c}{\det\Omega(a,b)}\big)^{\frac{1}{2}}$ with constants c, c.*

Proof. Let $r > \big(\frac{c}{\det\Omega(a,b)}\big)^{\frac{1}{2}}$. Then $0 < \frac{c}{r^2} < \det\Omega(a,b)$. Since the function $t \mapsto \det\Omega(a,t)$ is a continuous and increasing bijection from $[c(a),b)$ onto $[0,\det\Omega(a,b))$, we find a unique point $\hat{t}(r) \in (a,b)$ such that

$$\det\Omega(a,\hat{t}(r)) = \frac{c}{r^2}.$$

Let $(t,r) \in \Gamma(\hat{t})$. Then

$$0 < \frac{c}{r^2} = \det\Omega(a,\hat{t}(r)) \leq \det\Omega(a,t),$$

and since $s \mapsto \det\Omega(s,t)$ is a continuous and decreasing bijection from $[a,d(t)]$ onto $[0,\det\Omega(a,t)]$, we find a unique point $\hat{s}(t;r) \in [a,t)$ such that

$$\det\Omega(\hat{s}(t;r),t) = \frac{c}{r^2}.$$

□

II.2.10 Remark. For many purposes it would be enough to work with the compatible pairs constructed in Corollary II.2.9, even using only "$c = 1$". Our reasons to introduce the terminology of a "compatible pair" at all, and to use the additional parameters r_0, c_-, c_+, are that this makes it easier to trace constants in estimates and to actually apply the abstract results in concrete situations. Both features are essential for our present work.

II.2.4 Behaviour of q_H towards $i\infty$

We use the following notation to compare functions up to multiplicative constants.

II.2.11 Notation. Let X be a set and let $f, g\colon X \to (0,\infty)$ be functions. Then we write $f \lesssim g$ (or $f(x) \lesssim g(x)$) to say that

$$\exists c > 0\ \forall x \in X.\ f(x) \leq cg(x).$$

Moreover, we write

$$f \gtrsim g :\Leftrightarrow g \lesssim f, \quad f \asymp g :\Leftrightarrow f \lesssim g \wedge f \gtrsim g.$$

◀

The following result is shown in [Reiar, Theorem I.1.1]. It is a crucial tool for the present paper.

Recall the notation $\omega_{H,j}$ (abbreviated to ω_j) from (II.2.3).

II.2.12 Theorem. *Let $H \in \mathbb{H}_{a,b}$ be definite and in limit point case. Further, let $r_0 \geq 0$, $c_-, c_+ > 0$, and $\hat{t}$ a compatible function for H, r_0 with constants c_-, c_+. Then*

$$|q_H(ir)| \asymp \sqrt{\frac{\omega_1(a,\hat{t}(r))}{\omega_2(a,\hat{t}(r))}}, \qquad r \in (r_0,\infty),$$

$$\operatorname{Im} q_H(ir) \asymp \frac{1}{r\omega_2(a,\hat{t}(r))}, \qquad r \in (r_0,\infty).$$

The constants implicit in the two relations "$\asymp$" depend on c_-, c_+, but not on $H, r_0, \hat{t}$.

II.3 Growth of the eigenvalue counting function

In the below theorem we give a formula which determines the growth of (the Stieltjes transform of) the counting function of the spectrum of H provided σ_H is discrete and $\sum_{\lambda\in\sigma_H\setminus\{0\}}\frac{1}{\lambda^2}<\infty$. This formula is complicated, but it is explicit in the Hamiltonian.

Recall the notion of a compatible pair from Definition II.2.8, and that such pairs always exist, cf. Corollary II.2.9. Moreover, recall Remark II.2.5.

II.3.1 Theorem. *Let $H\in\mathbb{H}_{a,b}$ be a Hamiltonian with discrete spectrum and $\int_a^b\binom{1}{0}H(t)\binom{1}{0}\,\mathrm{d}t<\infty$, and denote*

$$n_H(r):=\#\big\{\lambda\in\sigma_H \mid |\lambda|<r\big\}.$$

Let $r_0\geq 0$, $c_-,c_+>0$, assume we have a compatible pair $(\hat t,\hat s)$ for H,r_0 with constants c_-,c_+, and denote

$$K(t;r):=\mathbb{1}_{[a,\hat t(r))}(t)\frac{\omega_{H,2}(a,t)h_1(t)}{r^{-2}+\omega_{H,3}(a,t)^2}+\mathbb{1}_{[\hat t(r),b)}(t)\frac{h_1(t)}{\omega_{H,1}\big(\hat s(t;r),t\big)}. \qquad \text{(II.3.1)}$$

Set $r_0':=\max\{r_0,\inf\{r>r_0 \mid \hat t(r)<\sup I_{\mathrm{reg}}\}\}$ (note here: if H does not end with an indivisible interval towards b, then $r_0'=r_0$). Then

$$\int_0^\infty\frac{1}{t+r^2}\cdot\frac{n_H(\sqrt t)}{t}\,\mathrm{d}t\asymp\frac{1}{r^2}\int_a^b K(t;r)\,\mathrm{d}t,\qquad r>r_0', \qquad \text{(II.3.2)}$$

where this relation includes that one side is finite if and only if the other side is. The constants implicit in "$\asymp$" depend on c_-,c_+, but not on $H,r_0,\hat t,\hat s$.

The proof of this result will be given below in three subsections. Before that, let us add a couple of remarks about the theorem and its formulation.

II.3.2 Remark.

(i) If we are willing to accept that the constants in (II.3.2) depend also on $H,r_0,\hat t,\hat s$, then the first summand of $K(t;r)$ can be dropped. Details are given in Proposition II.4.4.

(ii) Knowing the behaviour of the Stieltjes transform of the measure $\frac{n_H(\sqrt t)}{t}\,\mathrm{d}t$, allows to employ classical Abelian- and Tauberian arguments to extract knowledge about n_H itself. For details see Theorems II.5.4 and II.5.7.

(iii) The formula (II.3.2) gives a meaningful result about n_H only if the integrals on either side are finite. The integral on the left side of (II.3.2) is finite for some (equivalently, for every) $r > 0$, if and only if

$$\int_0^\infty \frac{n_H(\sqrt{t})}{t^2}\,\mathrm{d}t < \infty,$$

which in turn is equivalent to $\sum_{\lambda\in\sigma_H} \frac{1}{\lambda^2} < \infty$.

(iv) Applying Theorem II.3.1 in practice is often challenging. We will return to this topic with an algorithmic approach in Theorem II.6.2, and in particular will see examples where the integral over $K(t;r)$ can be evaluated explicitly, cf. Theorem II.7.1 and proposition II.7.7. Those examples will also show that both bounds in Theorem II.6.2 are sharp.

$\lhd$

Let us now turn to the proof of Theorem II.3.1. It proceeds in three stages. First, we use a recently obtained result about the high-energy behaviour of Weyl coefficients to determine the growth of the monodromy matrix $W_H(b,z)$ for a Hamiltonian in limit circle case. This is the essence of the argument. Second, we rewrite the so obtained knowledge to the counting function $n_H(r)$, and finally we make a limit argument to include the limit point case.

II.3.1 Growth of the monodromy matrix

In this subsection we consider the limit circle case, and prove a formula for the growth of (one entry of) the monodromy matrix. Recall the notation $W_H(t;z) = (w_{H,ij}(t;z))_{i,j=1}^2$.

II.3.3 Theorem. *Let $H \in \mathbb{H}_{a,b}$ be in limit circle case. Let $r_0 \geq 0$, $c_-, c_+ > 0$, and assume we have a compatible pair $(\hat{t},\hat{s})$ for H, r_0 with constants c_-, c_+. Further, let $K(t;r)$ be as in (II.3.1). Then*

$$\log|w_{H,22}(b;ir)| \asymp \int_a^b K(t;r)\,\mathrm{d}t, \qquad r > r_0 \tag{II.3.3}$$

The constants implicit in the relation "$\asymp$" depend on c_-, c_+, but not on $H, r_0, \hat{t}, \hat{s}$.

The first step is to observe that $\log|w_{H,22}(b;ir)|$ can be written as an integral over (a,b), where the integrand is built from the imaginary part of Weyl coefficients of Hamiltonians depending on t.

II.3.4 Lemma. *Let $H \in \mathbb{H}_{a,b}$, $a \leq t_1 \leq t_2 < b$, and $r > 0$. Then*

$$\log |w_{H,22}(t_2; ir)| - \log |w_{H,22}(t_1; ir)| = r \int_{t_1}^{t_2} \operatorname{Im}\Big(-\frac{w_{H,21}(t; ir)}{w_{H,22}(t; ir)}\Big) h_1(t)\, \mathrm{d}t.$$

Proof. The right lower entry of the matrix equation (II.2.2) reads as

$$\frac{\partial}{\partial t} w_{H,22}(t; z) = z\Big(w_{H,22}(t; z)h_3(t) + w_{H,21}(t; z)h_1(t)\Big).$$

The function $w_{H,22}(t; \cdot)$ is zero-free on $\mathbb{C}^+$, and $w_{H,22}(t; 0) = 1$. Let $\log w_{H,22}(t; \cdot)$ be the branch of the logarithm analytic in a domain containing $\mathbb{C}^+ \cup \{0\}$ with $\log w_{H,22}(t; 0) = 0$. Then the function $t \mapsto \log w_{H,22}(t; z)$ is differentiable a.e., and

$$\frac{\partial}{\partial t} \log w_{H,22}(t; z) = \frac{\frac{\partial}{\partial t} w_{H,22}(t; z)}{w_{H,22}(t; z)}, \qquad z \in \mathbb{C}^+ \cup \{0\}.$$

Hence,

$$\begin{aligned} \frac{\partial}{\partial t} \log |w_{H,22}(t; z)| &= \frac{\partial}{\partial t}\Big(\operatorname{Re} \log w_{H,22}(t; z)\Big) = \operatorname{Re} \frac{\frac{\partial}{\partial t} w_{H,22}(t; z)}{w_{H,22}(t; z)} \\ &= (\operatorname{Re} z) h_3(t) + \operatorname{Re}\Big(z \frac{w_{H,21}(t; z)}{w_{H,22}(t; z)}\Big) h_1(t). \end{aligned}$$

For $z = ir$ this yields

$$\frac{\partial}{\partial t} \log |w_{H,22}(t; ir)| = r \operatorname{Im}\Big(-\frac{w_{H,21}(t; ir)}{w_{H,22}(t; ir)}\Big) h_1(t).$$

Integrating both sides of this relation we obtain the stated formula. $\square$

II.3.5 Lemma. *Let $H \in \mathbb{H}_{a,b}$ and $t \in (a, b)$. Denote*

$$H_{(t)}(s) := \begin{cases} RH(a + t - s)R, & s \in [a, t], \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, & s \in (t, \infty), \end{cases} \tag{II.3.4}$$

with $R := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $H_{(t)} \in \mathbb{H}_{a,\infty}$, is in limit point case, and

$$q_{H_{(t)}}(z) = -\frac{w_{H,21}(t; z)}{w_{H,22}(t; z)}. \tag{II.3.5}$$

Proof. An elementary calculation shows that, on the interval $[a,t]$, the fundamental solution of $H_{(t)}$ is given by

$$W_{H_{(t)}}(s;z)=RW_H(t;z)^{-1}W_H(a+t-s;z)R,\qquad s\in[a,t],$$

and hence

$$W_{H_{(t)}}(t;z)=RW_H(t;z)^{-1}R=\begin{pmatrix} w_{H,11}(t;z) & -w_{H,21}(t;z)\\ -w_{H,12}(t;z) & w_{H,22}(t;z)\end{pmatrix}.$$

Since the interval $[t,\infty)$ is $H_{(t)}$-indivisible of type $\frac{\pi}{2}$, relation (II.3.5) follows. $\square$

The above lemmata allow to invoke Theorem II.2.12.

II.3.6 Lemma. *Let $H\in\mathbb{H}_{a,b}$, $r_0\geq 0$, $c_-,c_+>0$, and let $(\hat{t},\hat{s})$ be a compatible pair for H,r_0 with constants c_-,c_+. Then*

$$-\operatorname{Im}\frac{w_{H,21}(t;ir)}{w_{H,22}(t;ir)}\asymp \mathbb{1}_{[a,\hat{t}(r))}(t)\frac{\omega_{H,2}(a,t)}{r\big(\frac{1}{r^2}+\omega_{H,3}(a,t)^2\big)}+\frac{\mathbb{1}_{[\hat{t}(r),b)}(t)}{r\omega_{H,1}(\hat{s}(t;r),t)},$$
$$r\in(r_0,\infty),\ t\in[a,b). \qquad \text{(II.3.6)}$$

The constants implicit in the relation "$\asymp$" depend on c_-,c_+, but not on $H,r_0,\hat{t},\hat{s}$.

Proof. If (a,t) is indivisible of type 0, then both sides of (II.3.6) are equal to zero, and hence (II.3.6) holds with any constants in "$\asymp$" for such t. For the rest of the proof we may thus assume that (a,t) is not indivisible of type 0.

For $r>r_0$ and $t\in(a,b)$ we define

$$\hat{t}_{(t)}(r):=\begin{cases} t+\frac{1}{\omega_{H,2}(a,t)}\Big(\frac{c_+}{r^2}-\det\Omega_H(a,t)\Big) & \text{if } t\in(a,\hat{t}(r)),\\ a+t-\hat{s}(t;r) & \text{if } t\in(\hat{t}(r),b).\end{cases}$$

Note here that $\omega_{H,2}(a,t)\neq 0$ since (a,t) is not indivisible of type 0.

The definition of $H_{(t)}$ in (II.3.4) gives

$$\Omega_{H_{(t)}}(a,s)=\begin{cases} R\Omega(a+t-s,t)R & \text{if } s\in[a,t],\\ R\Omega(a,t)R+(s-t)\begin{pmatrix}0&0\\0&1\end{pmatrix} & \text{if } s\in(t,\infty),\end{cases}$$

and, consequently,

$$\det\Omega_{H_{(t)}}(a,s)=\begin{cases}\det\Omega(a+t-s,t) & \text{if } s\in[a,t],\\ \det\Omega(a,t)+(s-t)\omega_2(a,t) & \text{if } s\in(t,\infty).\end{cases}$$

Plugging the definition of $\hat{t}_{(t)}(r)$ yields

$$\det \Omega_{H_{(t)}}\big(a, \hat{t}_{(t)}(r)\big) = \begin{cases} \det \Omega_H(\hat{s}(t;r), t) & \text{if } \hat{t}(r) \in [a,t], \\ \frac{c_+}{r^2} & \text{if } \hat{t}(r) \in (t,\infty). \end{cases}$$

We see that $\hat{t}$ is a compatible function for $H_{(t)}, r_0$ with constants c_-, c_+.

Now we apply Theorem II.2.12 which yields that

$$-\operatorname{Im} \frac{w_{H,21}(t;ir)}{w_{H,22}(t;ir)} = \operatorname{Im} q_{H_{(t)}}(ir) \asymp \frac{1}{r\omega_{H_{(t)}}(a, \hat{t}_{(t)}(r))},$$

where the constants in "$\asymp$" depend on c_-, c_+, but not on $H, t, r_0, \hat{t}, \hat{s}$.

The remaining task is to express $\omega_{(t),2}\big(a, \hat{t}_{(t)}(r)\big)$ in terms of H. The definition of $H_{(t)}$ gives

$$\omega_{H_{(t)},2}(a,s) = \begin{cases} \omega_{H,1}(a+t-s, t) & \text{if } s \in [a,t], \\ \omega_{H,1}(a,t) + (s-t) & \text{if } s \in (t,\infty), \end{cases}$$

and plugging the definition of $\hat{t}_{(t)}(r)$ we obtain

$$\omega_{H_{(t)},2}\big(a, \hat{t}_{(t)}(r)\big) = \begin{cases} \omega_{H,1}\big(\hat{s}(t;r), t\big) & \text{if } \hat{t}(r) \in [a,t], \\ \frac{1}{\omega_{H,2}(a,t)}\big(\frac{1}{r^2} + \omega_{H,3}(a,t)^2\big) & \text{if } \hat{t}(r) \in (t,\infty). \end{cases}$$

Putting together, this yields (II.3.6). □

Rewriting the formulae resulting from the above lemma leads to (II.3.3).

Proof of Theorem II.3.3. Combining Lemma II.3.4 with Lemma II.3.6, yields

$$\begin{aligned} \log |w_{H,22}(b;ir)| &= r \int_a^b \operatorname{Im}\Big(-\frac{w_{H,21}(t;ir)}{w_{H,22}(t;ir)}\Big) h_1(t)\, \mathrm{d}t \\ &\asymp r \int_a^b \Bigg[\mathbb{1}_{[a,\hat{t}(r))}(t) \frac{\omega_{H,2}(a,t)}{r\big(\frac{1}{r^2} + \omega_{H,3}(a,t)^2\big)} + \frac{\mathbb{1}_{[\hat{t}(r),b)}(t)}{r\omega_{H,1}(\hat{s}(t;r),t)}\Bigg] h_1(t)\, \mathrm{d}t \\ &= \int_a^b K(t;r)\, \mathrm{d}t. \end{aligned}$$

□

II.3.2 Rewriting in terms of n_H

We settle the limit circle case in Theorem II.3.1 by reformulating the statement of Theorem II.3.3 in terms of $n_H(r)$. To achieve this we use symmetrisation and common tools from complex analysis.

II.3.7 Lemma. *Let f be an entire function with $f(0) = 1$, which is real along the real axis, has only real zeroes, and is of bounded type in $\mathbb{C}^+$. Denote by $n_f(r)$ the number of zeroes of f in the interval $(-r, r)$ (counted according to their multiplicities). Then*

$$\log|f(ir)| = \frac{r^2}{2}\int_0^\infty \frac{1}{t+r^2}\cdot\frac{n_f(\sqrt{t})}{t}\,\mathrm{d}t.$$

Proof. Let $\lambda_1, \lambda_2, \dots$ be the sequence of zeroes of f arranged according to nondecreasing modulus. Since f is of bounded type, the genus of f is 0 or 1 and

$$f(z) = \lim_{R\to\infty}\prod_{|\lambda_n|<R}\Big(1-\frac{z}{\lambda_n}\Big), \tag{II.3.7}$$

e.g. [Lev80, V.Theorem 11].

Set $\kappa_n := \lambda_n^2$. Then $\sum\limits_n \frac{1}{\kappa_n} < \infty$, and we may consider the canonical product

$$g(z) := \prod_n \Big(1+\frac{z}{\kappa_n}\Big).$$

Clearly, the counting functions n_g of the zeroes of g is related to n_f by

$$n_g(r) = \#\big\{n\in\mathbb{N} \mid \kappa_n < r\big\} = n_f(\sqrt{r}).$$

Using (II.3.7) and that f is symmetric w.r.t. the real axis, we see that

$$g(r) = f(i\sqrt{r})f(-i\sqrt{r}) = |f(i\sqrt{r})|^2.$$

On the other hand, we have

$$\log g(r) = r\int_0^\infty \frac{1}{t+r}\cdot\frac{n_g(t)}{t}\,\mathrm{d}t,$$

e.g. [Boa54, (4.1.4)]; note here that convergence of the series $\sum\limits_n \frac{1}{\kappa_n}$ implies that $n_g(r) = \mathrm{o}(r)$. □

Proof of Theorem II.3.1 (limit circle case). The spectrum σ_H is nothing but the zero set of $w_{H,22}(b;\cdot)$, and $w_{H,22}(b;\cdot)$ is real along the real axis, has only real and simple zeroes, and is of bounded type in $\mathbb{C}^+$. The above lemma thus shows that that the statement (II.3.3) of Theorem II.3.3 is equivalent to

$$r^2 \int_0^\infty \frac{1}{t+r^2} \cdot \frac{n_H(\sqrt{t})}{t}\,\mathrm{d}t \asymp \int_a^b K(t;r)\,\mathrm{d}t, \qquad r > r_0$$

$\square$

II.3.3 Making a limit argument

It remains to make the passage to the limit point case. The basis is to thoroughly understand the relation between the operator models of a Hamiltonian and its truncations.

Let $H \in \mathbb{H}_{a,b}$ and $c \in I_{\mathrm{reg}}$. Then we have the maps

$$\iota_c\colon \begin{cases} L^2(H|_{(a,c)}) & \to \quad L^2(H) \\ [f]_{H|_{(a,c)}} & \mapsto \quad \left[t \mapsto \begin{cases} f(t) & \text{if } t \in (a,c) \\ 0 & \text{if } t \in [c,b) \end{cases}\right]_H \end{cases}$$

$$\rho_c\colon \begin{cases} L^2(H) & \to \quad L^2(H|_{(a,c)}) \\ [f]_H & \mapsto \quad [f|_{(a,c)}]_{H|_{(a,c)}} \end{cases}$$

Clearly, ι_c is an isometry, ρ_c is a partial isometry with $\ker \rho_c = (\operatorname{ran} \iota_c)^\perp$ and $\rho_c \circ \iota_c = \mathrm{id}$, and $P_c := \iota_c \circ \rho_c$ is the orthogonal projection acting as $P_c f = \mathbb{1}_{(a,c)} f$.

The following result, relating the operators $A(H)$ and $A(H|_{(a,c)})$, is known from the literature. In fact, a more general version is shown in [PW23, Theorem 3.4]. The proof in the presently considered situation is much simpler than in the general case; and for the convenience of the reader we provide the argument.

II.3.8 Lemma. *Let $H \in \mathbb{H}_{a,b}$ be definite with $0 \in \rho(A(H))$ and $\binom{1}{0} \in L^2(H)$. Moreover, let $c \in I_{\mathrm{reg}}$. Then*

$$P_c A(H)^{-1} P_c \circ \iota_c = \iota_c \circ A(H|_{(a,c)})^{-1}.$$

Proof. Let $g \in L^2(H|_{(a,c)})$ and set $\hat{g} := \iota_c g$. Let f be the unique absolutely continuous representative of $A(H|_{(a,c)})^{-1} g$ such that

$$f' = JHg \text{ a.e.}, \quad (1,0)f(a) = (0,1)f(c) = 0.$$

Set

$$\hat{f}(t) := \begin{cases} f(t) & \text{if } t \in [a,c], \\ f(c) & \text{if } t \in (c,b]. \end{cases}$$

Then $\hat{f}$ is absolutely continuous and satisfies

$$\begin{aligned}
&\hat{f}'(t) = JH(t)\hat{g}(t), \qquad t \in (a,b) \text{ a.e.},\\
&(1,0)\hat{f}(a) = 0, \quad (0,1)\hat{f}(t) = 0,\ t \in [c,b].
\end{aligned}$$

We see that $\hat{f} \in L^2(H)$ and $A(H)^{-1}\hat{g} = \hat{f}$.

It remains to note that $P_c \circ \iota_c = \iota_c$ and $P_c\hat{f} = \iota_c f$. □

We need a simple (and well-known) fact about compact operators, which we state explicitly for completeness but otherwise skip details. For a compact operator T denote by $s_n(T)$ its n-th s-number, and

$$n(T,\delta) := \#\big\{n \geq 1 \mid s_n(T) > \delta\big\}.$$

II.3.9 Lemma. *Let T be a compact operator, and let B_l, B, C_l, C be bounded operators such that $\lim_{l\to\infty} B_l = B$ and $\lim_{l\to\infty} C_l^* = C^*$ strongly. Then*

$$\forall \delta \in (0,\infty) \setminus \big\{s_n(BTC) \mid n \geq 1\big\}.\ \lim_{l\to\infty} n(B_lTC_l,\delta) = n(BTC,\delta).$$

II.3.10 Lemma. *Let $H \in \mathbb{H}_{a,b}$ be definite with σ_H discrete and $\binom{1}{0} \in L^2(H)$.*

(i) *For each $r > 0$ and $c, c' \in I_{\text{reg}}$ with $a < c \leq c' < b$ we have*

$$n_{H|_{(a,c)}}(r) \leq n_{H|_{(a,c')}}(r).$$

(ii) *Assume additionally that $\sup I_{\text{reg}} = b$ or that limit point case takes place. Then, for each $r > 0$ which is a point of continuity of n_H, it holds that*

$$\lim_{\substack{c\to\sup I_{\text{reg}}\\ c\in I_{\text{reg}}}} n_{H|_{(a,c)}}(r) = n_H(r).$$

Proof. Observe that $n_H(r) = n(A(H)^{-1}, \frac{1}{r})$, and the same for $H|_{(a,c)}$. Now we can apply the above lemmata. First, by Lemma II.3.8,

$$n\big(A(H|_{(a,c)})^{-1},\delta\big) = n\big(P_cA(H)^{-1}P_c,\delta\big),$$

and the assertion in (i) follows since $P_c = P_cP_{c'}$. For the assertion in (ii) note that the stated additional assumption ensures that the union of the ranges of P_c is dense and hence

$$\lim_{\substack{c\to\sup I_{\text{reg}}\\ c\in I_{\text{reg}}}} P_c = I$$

strongly. Now apply Lemma II.3.9. □

We can now pass to the limit in (II.3.2).

Proof of Theorem II.3.1 (limit point case). By the monotone convergence theorem we have

$$\lim_{\substack{c\to\sup I_{\mathrm{reg}}\\ c\in I_{\mathrm{reg}}}} \int_0^\infty \frac{1}{t+r^2}\cdot\frac{n_{H|_{(a,c)}}(\sqrt{t})}{t}\,\mathrm{d}t = \int_0^\infty \frac{1}{t+r^2}\cdot\frac{n_H(\sqrt{t})}{t}\,\mathrm{d}t$$

for all $r>0$.

Consider the case that $\sup I_{\mathrm{reg}} = b$. For $c\in I_{\mathrm{reg}}$ set

$$r_{0,c} := \inf\{r>r_0 \mid \hat{t}(r)<c\},$$
$$\hat{t}_c\colon \begin{cases} (r_{0,c},\infty) & \to & (a,c)\\ r & \mapsto & \hat{t}(r)\end{cases}$$

Then, clearly, $\hat{t}_c$ is a compatible function for $H|_{(a,c)}, r_{0,c}$ with constants c_+, c_-. Now set

$$\hat{s}_c\colon \begin{cases} \Gamma(\hat{t}_c) & \to & [a,c)\\ (t,r) & \mapsto & \hat{s}(t;r)\end{cases}$$

then $(\hat{t}_c,\hat{s}_c)$ is a compatible pair for $H|_{(a,c)}, r_{0,c}$ with constants c_+, c_-. Let K_c be the function (II.3.1) constructed from $(\hat{t}_c,\hat{s}_c)$. Then

$$K_c = K|_{[a,c)\times(r_{0,c},\infty)}.$$

Thus we can apply the monotone convergence theorem to conclude that

$$\lim_{\substack{c\to b\\ c\in I_{\mathrm{reg}}}} \int_a^c K_c(t,r)\,\mathrm{d}t = \int_a^b K(t;r)\,\mathrm{d}t,$$

and the assertion of the theorem follows.

Assume now that H ends with an indivisible interval towards b. Then $h_1(t)=0$ for $t\in(\max I_{\mathrm{reg}}, b)$, and hence

$$\int_a^b K(t;r)\,\mathrm{d}t = \int_a^{\sup I_{\mathrm{reg}}} K(t;r)\,\mathrm{d}t.$$

The assertion of the theorem follows by applying that we already know the result for the limit circle case. ☐

II.4 Additions to the Theorem

II.4.1 Two monotonicity results

In this subsection we show two results which establish that the function $K(t;r)$, and with it the quantities on the respective left sides of (II.3.2) and (II.3.3), depend monotonically on H in different ways.

The first result is a consequence of [Reiar, Corollary I.2.4].

II.4.1 Corollary. *Let $H,\tilde{H}\in\mathbb{H}_{a,b}$, and assume that $h_1=\tilde{h}_1$, $h_2=\tilde{h}_2$, and*

$$\forall a\leq c<d<b.\ \Big|\int_c^d h_3(t)\,\mathrm{d}t\Big|\leq\Big|\int_c^d \tilde{h}_3(t)\,\mathrm{d}t\Big|.$$

Let $r_0\geq 0$, $c_-,c_+>0$, assume we have compatible pairs for H,r_0 and $\tilde{H},r_0$ with constants c_-,c_+, and denote by $K(t;r)$ and $\tilde{K}(t;r)$ be the corresponding functions (II.3.1). Then

$$\tilde{K}(t;r)\lesssim K(t;r).$$

The constant implicit in the relation "$\lesssim$" depends on c_-,c_+, but not on $H,\tilde{H}$.

Proof. Our assumption on the off-diagonal entries of H and $\tilde{H}$ ensures that we can apply [Reiar, Corollary I.2.4] to each of the pairs of Hamiltonians $H_{(t)}$ and $\tilde{H}_{(t)}$ as defined in (II.3.4). The constants occurring from this application are independent of $H,\tilde{H},t$. Hence, Lemma II.3.6 yields the assertion. □

Our second result will be monotonicity in the style of [LPW, Proposition 1.4]. To establish it, we need the analogue of this result in the setting of [Reiar].

II.4.2 Proposition. *Let $H,\widetilde{H}\in\mathbb{H}_{a,b}$ be in limit point case, and assume that (a,b) is not indivisible of type 0 for H. Further, let $c_-,c_+>0$ and assume that*

$$\begin{aligned}&\det\Omega_{\widetilde{H}}(a,t)\leq c_+^2\cdot\det\Omega(a,t), && t\in[a,b),\\ &\omega_{\widetilde{H},2}(a,t)\geq c_-\cdot\omega_2(a,t), && t\in[a,b).\end{aligned}$$

Then

$$\operatorname{Im}q_{\widetilde{H}}(ir)\lesssim\operatorname{Im}q_H(ir),\qquad r\in(0,\infty). \tag{II.4.1}$$

The constant implicit in the relation "$\lesssim$" depends on c_-,c_+, but not on $H,\widetilde{H}$.

Proof. If $\widetilde{H}$ is not definite, then $q_{\widetilde{H}}$ is a real constant, and (II.4.1) holds with any constant in "$\lesssim$". If H is not definite, then

$$\det\Omega_{\widetilde{H}}(a,b)\leq c_+^2\cdot\det\Omega(a,b)=0,$$

and hence $\widetilde{H}$ is also not definite. Again (II.4.1) holds with any constant in "$\lesssim$".

Assume now that $H, \widetilde{H}$ are both definite. Let $\hat{t}_H(r)$ and $\hat{t}_{\widetilde{H}}(r)$ be compatible functions for $H, 0$ and $\widetilde{H}, 0$, respectively, with constants $1, 1$. Our assumption on determinants implies that

$$\det \Omega\big(a, \hat{t}_{\widetilde{H}}(r)\big) \geq \frac{1}{c_+^2} \det \Omega_{\widetilde{H}}\big(a, \hat{t}_{\widetilde{H}}(r)\big) = \frac{1}{(c_+ r)^2} = \det \Omega\big(a, \hat{t}_H(c_+ r)\big).$$

It follows that $\hat{t}_{\widetilde{H}}(r) \geq \hat{t}_H(c_+ r)$. We obtain

$$\begin{aligned} \operatorname{Im} q_{\widetilde{H}}(ir) &\asymp \frac{1}{r\omega_{\widetilde{H},2}\big(a, \hat{t}_{\widetilde{H}}(r)\big)} \leq \frac{1}{c_- \cdot r\omega_2\big(a, \hat{t}_{\widetilde{H}}(r)\big)} \\ &\leq \frac{1}{c_- \cdot r\omega_2\big(a, \hat{t}_H(c_+ r)\big)} \asymp \operatorname{Im} q_H(ic_+ r) \asymp \operatorname{Im} q_H(ir), \end{aligned}$$

where the last asymptotic relation follows from the inequalities

$$\min\Big\{c_+, \frac{1}{c_+}\Big\} \leq \frac{\operatorname{Im} q_H(ic_+ r)}{\operatorname{Im} q_H(ir)} \leq \max\Big\{c_+, \frac{1}{c_+}\Big\},$$

which can easily be shown with the help of the Herglotz integral representation of q_H. $\square$

II.4.3 Corollary. *Let $H, \tilde{H} \in \mathbb{H}_{a,b}$, $c_-, c_+ > 0$, and assume that*

$$\begin{aligned} \tilde{H}(t) &\leq c_+ H(t), \qquad t \in (a,b) \text{ a.e.}, \\ \tilde{h}_1(t) &\geq c_- h_1(t), \qquad t \in (a,b) \text{ a.e.} \end{aligned}$$

Let $r_0 \geq 0$, assume we have compatible pairs for H, r_0 and $\tilde{H}, r_0$ with constants c_-, c_+, and denote by $K(t;r)$ and $\tilde{K}(t;r)$ be the corresponding functions (II.3.1). Then

$$\tilde{K}(t;r) \lesssim K(t;r).$$

The constants implicit in the relation "$\lesssim$" depend on c_-, c_+, but not on $H, \tilde{H}$.

Proof. For $t \in (a,b)$, let $H_{(t)}$ and $\widetilde{H}_{(t)}$ be the Hamiltonians as in (II.3.4) corresponding to H and $\widetilde{H}$. Our present assumption implies that for all $t \in (a,b)$

$$\widetilde{H}_{(t)}(s) \leq \max\{1, c_+\} \cdot H_{(t)}(s),\ \tilde{h}_{(t),2}(s) \geq \min\{1, c_-\} \cdot h_{(t),2}(s), \quad s \in (a, \infty) \text{ a.e.}$$

Applying Proposition II.4.2 with $H_{(t)}$ and $\widetilde{H}_{(t)}$, yields

$$\operatorname{Im} q_{\widetilde{H}_{(t)}}(ir) \lesssim \operatorname{Im} q_{H_{(t)}}(ir), \qquad r > 0,\ t \in (a,b),$$

where the constant implicit in "$\lesssim$" depends on c_+, c_-, but not on $H, \widetilde{H}, t$. Again, referring to Lemma II.3.6 finishes the proof. $\square$

II.4.2 The role of the first summand of $K(t;r)$

We now shoe that, for each fixed Hamiltonian H, the integral over the first summand of $K(t;r)$ gives only a contribution of logarithmic size. Thus, it can be neglected if one is not interested in uniformity of constants w.r.t. H.

II.4.4 Proposition. *Consider the setting of Theorem II.3.1, and set* $c' := \max\{c_+, 1\}$. *Then*

$$\int_a^{\hat{t}(r)} \frac{\omega_2(a,t)h_1(t)}{r^{-2}+\omega_3(a,t)^2}\,\mathrm{d}t \leq c_- + 2c'\Big[\log r + \log \frac{\operatorname{tr}\Omega(a,\hat{t}(r_0))}{2\cdot\sqrt{c_-}}\Big].$$

Proof. We estimate the integrand by distinguishing small and large values of t. Let $\mathring{t}(r) \in (a, \hat{t}(r)]$ be the unique number such that

$$\omega_1(a,\mathring{t}(r))\omega_2(a,\mathring{t}(r)) = \frac{c_-}{r^2}.$$

For $t \in [a, \mathring{t}(r)]$ we estimate

$$\frac{\omega_2(a,t)h_1(t)}{r^{-2}+\omega_3(a,t)^2} \leq r^2\omega_2(a,\mathring{t}(r))h_1(t),$$

and hence

$$\int_a^{\mathring{t}(r)} \frac{\omega_2(a,t)h_1(t)}{r^{-2}+\omega_3(a,t)^2}\,\mathrm{d}t \leq r^2\omega_2(a,\mathring{t}(r))\omega_1(a,\mathring{t}(r)) = c_-.$$

For $t \in [\mathring{t}(r), \hat{t}(r)]$ we estimate

$$\begin{aligned}
\frac{1}{r^2} + \omega_3(a,t)^2 &\geq \frac{1}{c'}\Big(\frac{c_+}{r^2} + \omega_3(a,t)^2\Big) \\
&\geq \frac{1}{c'}\Big(\det\Omega(a,\hat{t}(r)) + \omega_3(a,t)^2\Big) \\
&\geq \frac{1}{c'}\Big(\det\Omega(a,t) + \omega_3(a,t)^2\Big) \\
&= \frac{1}{c'}\omega_1(a,t)\omega_2(a,t).
\end{aligned}$$

This yields

$$\begin{aligned}
\int_{\mathring{t}(r)}^{\hat{t}(r)} \frac{\omega_2(a,t)h_1(t)}{r^{-2}+\omega_3(a,t)^2}\,\mathrm{d}t &\leq c' \int_{\mathring{t}(r)}^{\hat{t}(r)} \frac{h_1(t)}{\omega_1(a,t)}\,\mathrm{d}t = c' \log \frac{\omega_1(a,\hat{t}(r))}{\omega_1(a,\mathring{t}(r))} \\
&\leq c' \log \left[\frac{\omega_1(a,\hat{t}(r))}{\omega_1(a,\mathring{t}(r))} \cdot \frac{\omega_2(a,\hat{t}(r))}{\omega_2(a,\mathring{t}(r))}\right] \\
&= c' \log \left[\frac{r^2}{c_-} \cdot \omega_1(a,\hat{t}(r))\omega_2(a,\hat{t}(r))\right] \\
&\leq 2c' \left[\log r + \log \frac{\operatorname{tr}\Omega(a,\hat{t}(r))}{2\cdot\sqrt{c_-}}\right] \leq 2c' \left[\log r + \log \frac{\operatorname{tr}\Omega(a,\hat{t}(r_0))}{2\cdot\sqrt{c_-}}\right].
\end{aligned}$$

□

II.5 Growth relative to a regularly varying comparison function

In this section we make the transition from the knowledge about the Stieltjes transform obtained in Theorem II.3.1 to the function n_H itself. There are two types of results, cf. Theorem II.5.4 and Theorem II.5.7. Using terminology as in complex analysis these are convergence class properties and finite or minimal type properties. Naturally, our results are bound to the situation that $\sum_{\lambda\in\sigma_H}\frac{1}{\lambda^2}<\infty$. This, however, is no essential restriction since the case of dense spectrum (even with any convergence exponent strictly larger than 1) has been settled in [RW20].

To measure the growth of n_H, we use regularly varying functions in Karamata's sense as comparison functions. This is a very fine scale, and includes all proximate orders in the sense of Valiron. In particular, of course, the classical case of comparing growth with powers is covered.

II.5.1 Functions of regular variation

The growth of a function is often compared with functions of the form $\exp(\mathfrak{f}(r))$. The most classical comparison functions are powers $\mathfrak{f}(r)=r^\rho$. A refined comparison scale is established by regularly varying functions in Karamata sense.

II.5.1 Definition. A function $\mathfrak{f}\colon[1,\infty)\to(0,\infty)$ is called *regularly varying* (at ∞) with *index* $\alpha\in\mathbb{R}$, if it is measurable and

$$\forall\lambda\in(0,\infty).\ \lim_{r\to\infty}\frac{\mathfrak{f}(\lambda r)}{\mathfrak{f}(r)}=\lambda^\alpha. \tag{II.5.1}$$

We write $\operatorname{ind}\mathfrak{f}$ for the index of regular variation of $\mathfrak{f}$. A regularly varying function with index 0 is also called *slowly varying*.

◀

Our standard reference for the theory of such functions is [BGT89].

Standard examples of regularly varying functions are power-log functions, and by this we mean functions that behave for $r\to\infty$ like

$$r^\alpha\cdot\big(\log r\big)^{\beta_1}\cdot\big(\log\log r\big)^{\beta_2}\cdot\ldots\cdot\big(\underbrace{\log\cdots\log}_{m^{\text{th}}\text{ iterate}}r\big)^{\beta_m}, \tag{II.5.2}$$

where $\alpha\in\mathbb{R}$ and $\beta_1,\ldots,\beta_m\in\mathbb{R}$. Such functions were introduced in the context of complex analysis already at a very early stage by E. Lindelöf.

Regularly varying functions ℓ are used to quantify growth for $r \to \infty$, and hence the values of $\ell(r)$ for small r are irrelevant. This allows to change ℓ on any finite interval without changing the essence of results, which is an often practical freedom.

We cite a number of fundamental theorems on regularly varying functions. Proofs can be found in [BGT89].

II.5.2 Theorem (Karamata's Theorem)**.** *Let ℓ be regularly varying with index $\alpha \in \mathbb{R}$.*

(i) *Assume that $\alpha \geq 0$. Then the function $t \mapsto \int_1^t \ell(s)\frac{ds}{s}$ is regularly varying with index α, and*

$$\lim_{t\to\infty}\left(\ell(t) \Big/ \int\limits_1^t \ell(s)\tfrac{ds}{s}\right) = \alpha.$$

(ii) *Assume that $\alpha \leq 0$ and $\int_1^\infty \ell(s)\frac{ds}{s} < \infty$. Then the function $t \mapsto \int_t^\infty \ell(s)\frac{ds}{s}$ is regularly varying with index α, and*

$$\lim_{t\to\infty}\left(\ell(t) \Big/ \int\limits_t^\infty \ell(s)\tfrac{ds}{s}\right) = -\alpha.$$

A regularly varying function ℓ with positive index is – at least asymptotically – invertible. In fact, with

$$\ell^-(x) := \sup\left\{t \in [1,\infty) \mid \ell(t) < x\right\},$$

we have the following result, cf. [BGT89, Theorem 1.5.12].

II.5.3 Theorem. *Let ℓ be regularly varying with index $\alpha > 0$. Then ℓ^- is regularly varying with index $\frac{1}{\alpha}$, and*

$$(\ell \circ \ell^-)(x) \sim (\ell^- \circ \ell)(x) \sim x. \tag{II.5.3}$$

Any regularly varying function ℓ^- with the property (II.5.3) is called an *asymptotic inverse* of ℓ, and asymptotic inverses are determined uniquely up to $\sim$.

II.5.2 Characterising growth of n_H

In the context of convergence class we obtain the following characterisation. Note the particular case of the theorem when $g(r) = r^\rho$ for some $\rho \in (0,2)$. It should be emphasised that this result applies also when the convergence exponent of the eigenvalues is 0. In particular, we can characterise occurrence of very sparse spectrum on the level of (again terminology from complex analysis) logarithmic order or similar measures for growth of order zero.

Recall the discussion concerning Hilbert–Schmidt class in Remarks II.2.5 and II.3.2, and the notion of compatible pairs from Definition II.2.8.

II.5.4 Theorem. *Let $H \in \mathbb{H}_{a,b}$ be definite, and assume that resolvents of $A(H)$ belong to the Hilbert–Schmidt class. Let $r_0 \geq 0$, $c_-, c_+ > 0$, assume we have a compatible pair $(\hat{t}, \hat{s})$ for H, r_0 with constants c_-, c_+, and let $K(t;r)$ be as in (II.3.1). Let $\mathfrak{g}$ be a regularly varying function with $\operatorname{ind}\mathfrak{g} \in [0,2]$, $\mathfrak{g}(t) = \mathrm{o}(t^2)$, and[1] $\mathfrak{g}|_Y \asymp 1$ on every compact $Y \subseteq [1,\infty)$.*

- ▷ *If $\operatorname{ind}\mathfrak{g} \in (0,2)$, set $\mathfrak{g}^* := \mathfrak{g}$.*
- ▷ *If $\operatorname{ind}\mathfrak{g} = 0$, assume we have a regularly varying function $\mathfrak{g}^*$ with $\frac{1}{\mathfrak{g}^*}$ locally integrable and*

$$\int_t^\infty \int_u^\infty \frac{1}{\mathfrak{g}^*(s)} \frac{\mathrm{d}s}{s} \frac{\mathrm{d}u}{u} \asymp \frac{1}{\mathfrak{g}(t)}. \tag{II.5.4}$$

- ▷ *If $\operatorname{ind}\mathfrak{g} = 2$, assume we have a regularly varying function $\mathfrak{g}^*$ with*

$$\int_1^t \frac{s^2}{\mathfrak{g}^*(s)} \frac{\mathrm{d}s}{s} \asymp \frac{t^2}{\mathfrak{g}(t)}. \tag{II.5.5}$$

Then it holds that

$$\sum_{\substack{\lambda\in\sigma_H\\ |\lambda|\geq 1}} \frac{1}{\mathfrak{g}(|\lambda|)} < \infty \quad\Longleftrightarrow\quad \int_1^\infty \frac{1}{r\mathfrak{g}^*(r)} \int_a^b K(t;r)\,\mathrm{d}t\,\mathrm{d}r < \infty.$$

If $\operatorname{ind}\mathfrak{g} \in \{0,2\}$ it seems to be not known whether (II.5.4) or (II.5.5), respectively, can always be satisfied, cf. [Peš23, Remark 3.7]. If $\mathfrak{g}^*$ exists, then $\operatorname{ind}\mathfrak{g}^* = \operatorname{ind}\mathfrak{g}$ but necessarily $\mathfrak{g} = \mathrm{o}(\mathfrak{g}^*)$.

In the below example we show that (II.5.4) and (II.5.5) can be satisfied for all $\mathfrak{g}$ in the class of Lindelöf comparison functions (II.5.2). In fact, we explicitly specify a suitable function $\mathfrak{g}^*$, cf. (II.5.7) and (II.5.8). It is noteworthy, and somewhat unexpected, that the cases $\operatorname{ind}\mathfrak{g} = 0$ and $\operatorname{ind}\mathfrak{g} = 2$ behave differently when it comes to the gap between $\mathfrak{g}$ and $\mathfrak{g}^*$.

[1] Note that modifying $\mathfrak{g}$ on a finite interval does not affect the validity of (II.5.1).

II.5.5 Example. Let $N \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_N \in \mathbb{R}$, and consider the slowly varying function

$$\ell(r) := \prod_{k=1}^{N} \big(\log^{[k]} r\big)^{\alpha_k},$$

where $\log^{[k]}$ is the k-th iterate of the function log. We assume in the following that

$$\alpha_k = 0, \quad k < n; \qquad \alpha_n \neq 0 \tag{II.5.6}$$

with some $n \in \{1, \dots, N\}$. Further, denote by D the operator

$$(Df)(r) := rf'(r).$$

A computation using induction with respect to m shows that, for each $m \in \mathbb{N}$,

$$(D^m \ell)(r) \in \frac{\ell(r)}{(\log r)^m \prod_{j=2}^{n} \log^{[j]} r} \Bigg[\alpha_n \prod_{j=1}^{m-1} (\alpha_1 - j) \\ + \operatorname{span}\Bigg\{\prod_{l=2}^{N} (\log^{[l]} r)^{\nu_l} \mid \nu_l \leq 0,\, \nu_l \in \mathbb{Z},\, (\nu_2, \dots, \nu_N) \neq (0, \dots, 0)\Bigg\}\Bigg].$$

▷ Assume that $\alpha_1, \dots, \alpha_N$ are such that

$$\int_x^\infty \frac{\ell(t)}{\log t} \frac{\mathrm{d}t}{t} < \infty,$$

and let $n \in \{1, \dots, N\}$ such that (II.5.6) holds. Then $\alpha_1 \leq 0$ and hence the function $\ell^*(r) := \frac{\ell(r)}{(\log r)^2 \prod_{j=2}^{n} \log^{[j]} r}$ satisfies $D^2 \ell \asymp \ell^*$, which implies that

$$\int_t^\infty \int_u^\infty \ell^*(s) \frac{\mathrm{d}s}{s} \frac{\mathrm{d}u}{u} \asymp \int_t^\infty \int_u^\infty (D^2 \ell)(s) \frac{\mathrm{d}s}{s} \frac{\mathrm{d}u}{u} \asymp \ell(t).$$

If, in the situation of Theorem II.5.4, the function $\mathfrak{g}$ is a Lindelöf comparison function with $\operatorname{ind} \mathfrak{g} = 0$ such that (II.5.6) holds, then we can satisfy (II.5.4) with the choice

$$\mathfrak{g}^*(r) := \mathfrak{g}(r) \cdot (\log r)^2 \prod_{j=2}^{n} \log^{[j]} r. \tag{II.5.7}$$

▷ Assume that $\alpha_1, \dots, \alpha_N$ are such that

$$\int_x^\infty \frac{\ell(t)}{\log t} \frac{\mathrm{d}t}{t} = \infty,$$

and let $n \in \{1, \dots, N\}$ such that (II.5.6) holds. Then the function $\ell^*(r) := \frac{\ell(r)}{\log r \cdot \prod_{j=2}^{n} \log^{[j]} r}$ satisfies $D\ell \asymp \ell^*$, and hence (t_0 sufficiently large)

$$\int_{t_0}^{t} \ell^*(s) \frac{\mathrm{d}s}{s} \asymp \int_{t_0}^{t} (D\ell)(s) \frac{\mathrm{d}s}{s} \asymp \ell(t).$$

If, in the situation of Theorem II.5.4, the function g is a Lindelöf comparison function with $\operatorname{ind} g = 2$ such that (II.5.6) holds, then we can satisfy (II.5.5) with the choice

$$g^*(r) := g(r) \cdot \log r \cdot \prod_{j=2}^{n} \log^{[j]} r. \tag{II.5.8}$$

$\lhd$

We come to the proof of Theorem II.5.4. To deduce this result from Theorem II.3.1 it is enough to apply the following simple Abelian-Tauberian lemma. It is shown by standard methods; for completeness we provide details.

II.5.6 Lemma. *Let f be regularly varying and locally integrable with*

$$\int_1^\infty f(r)\, \mathrm{d}r = \infty, \qquad \int_1^\infty \frac{f(r)}{1+r}\, \mathrm{d}r < \infty.$$

Let ν be a positive Borel measure on $[1,\infty)$, and set $n(t) := \nu([1,t))$. Then

$$\int_1^\infty f(r) \Big(\int_1^\infty \frac{1}{t+r} \cdot \frac{n(\sqrt{t})}{t}\, \mathrm{d}t \Big)\, \mathrm{d}r < \infty \quad \Longleftrightarrow \quad \int_1^\infty \hat{f}(t)\, \mathrm{d}\nu(t) < \infty$$

where

$$\hat{f}(t) := \int_t^\infty \frac{1}{u} \int_u^\infty \frac{1}{s^3} \int_1^s f(r^2) r\, \mathrm{d}r\, \mathrm{d}s\, \mathrm{d}u.$$

If $-1 < \operatorname{ind} f < 0$, then $\hat{f}(t) \asymp f(t^2)$. If $\operatorname{ind} f \in \{-1, 0\}$, then $f(t^2) = \mathrm{o}(\hat{f}(t))$.

Proof. Consider the measure $\mathrm{d}\mu(t) := f(t)\, \mathrm{d}t$. Using Karamata's theorems (in the form stated in [LW23, Appendix A]), we obtain that its distribution function

$$\mu([1,r)) = \int_1^r f(t)\, \mathrm{d}t$$

is regularly varying, and its Stieltjes transform satisfies

$$\int_1^\infty \frac{1}{t+r} \cdot f(r)\, \mathrm{d}r \asymp \int_t^\infty \frac{1}{s^2} \int_1^s f(r)\, \mathrm{d}r\, \mathrm{d}s.$$

We use Fubini's theorem, a change of variables, and integration by parts to compute (here we write "$\diamond$" to say that two expressions are together finite, infinite, or zero)

$$\begin{aligned}
\int\limits_1^\infty \ell(r)\Big(\int\limits_1^\infty \frac{1}{t+r}\cdot\frac{n(\sqrt{t})}{t}\,\mathrm{d}t\Big)\,\mathrm{d}r &= \int\limits_1^\infty\Big(\int\limits_1^\infty \frac{1}{t+r}\cdot \ell(r)\,\mathrm{d}r\Big)\cdot\frac{n(\sqrt{t})}{t}\,\mathrm{d}t \\
&\diamond \int\limits_1^\infty\Big(\int\limits_t^\infty \frac{1}{s^2}\int\limits_1^s \ell(r)\,\mathrm{d}r\,\mathrm{d}s\Big)\cdot\frac{n(\sqrt{t})}{t}\,\mathrm{d}t \\
&\diamond \int\limits_1^\infty \frac{1}{u}\int\limits_{u^2}^\infty \frac{1}{s^2}\int\limits_1^s \ell(r)\,\mathrm{d}r\,\mathrm{d}s\cdot n(u)\,\mathrm{d}u \\
&= \int\limits_1^\infty\Big(\int\limits_t^\infty \frac{1}{u}\int\limits_{u^2}^\infty \frac{1}{s^2}\int\limits_1^s \ell(r)\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}u\Big)\,\mathrm{d}\nu(t).
\end{aligned}$$

It remains to compute

$$\int\limits_{u^2}^\infty \frac{1}{s^2}\int\limits_1^s \ell(r)\,\mathrm{d}r\,\mathrm{d}s = 2\int\limits_{u^2}^\infty \frac{1}{s^2}\int\limits_1^{\sqrt{s}} \ell(p^2)p\,\mathrm{d}p\,\mathrm{d}s = 4\int\limits_u^\infty \frac{1}{v^3}\int\limits_1^v \ell(p^2)p\,\mathrm{d}p\,\mathrm{d}v.$$

To see the additional statements, note that

$$\operatorname{ind}\ell > -1 \quad\Rightarrow\quad \frac{1}{s^2}\int\limits_1^s \ell(r^2)r\,\mathrm{d}r \asymp \ell(s^2)$$

$$\operatorname{ind}\ell < 0 \quad\Rightarrow\quad \int\limits_u^\infty \frac{\ell(s^2)}{s}\,\mathrm{d}s \asymp \ell(s^2)$$

while $\ell(s^2)$ is a $\mathrm{o}(\cdot)$ of the respective integral if $\operatorname{ind}\ell = -1$ and $\operatorname{ind}\ell = 0$. $\square$

Proof of Theorem II.5.4. We check that Lemma II.5.6 is applicable with the function

$$\ell(r) := \frac{1}{g^*(\sqrt{r})}.$$

Note that $\operatorname{ind}\ell = -\frac{1}{2}\operatorname{ind} g$. If $\operatorname{ind} g \in [0,2)$, it is clear that $\int_1^\infty \ell(t)\,\mathrm{d}t = \infty$. If $\operatorname{ind} g = 2$, we have

$$\int_1^t \ell(r)\,\mathrm{d}r = 2\int_1^{t^2} \frac{s^2}{g^*(s)}\frac{\mathrm{d}s}{s} \asymp \frac{t^4}{g(t^2)} \to \infty.$$

If $\operatorname{ind} g \in (0,2]$, it is clear that $\int_1^\infty f(t)\frac{\mathrm{d}t}{t} < \infty$. If $\operatorname{ind} g = 0$, we have

$$\int_1^\infty \int_{\sqrt{u}}^\infty f(s)\frac{\mathrm{d}s}{s}\frac{\mathrm{d}u}{u} = 2\int_1^\infty \int_u^\infty \frac{1}{g^*(s)}\frac{\mathrm{d}s}{s}\frac{\mathrm{d}u}{u} < \infty,$$

and thus $\int_1^\infty f(s)\frac{\mathrm{d}s}{s} < \infty$.

Next, we identify $\hat{f}$ from Lemma II.5.6. If $\operatorname{ind} g \in (0,2)$, it is already stated in the lemma that $\hat{f}(t) \asymp f(t^2) \asymp \frac{1}{g(t)}$. If $\operatorname{ind} g = 0$ or $\operatorname{ind} g = 2$, we use Karamata's Theorem to compute

$$\hat{f}(t) \asymp \int_t^\infty \int_u^\infty f(s^2)\frac{\mathrm{d}s}{s}\frac{\mathrm{d}u}{u} \asymp \frac{1}{g(t)}.$$

The proof of the theorem is completed by computing

$$\begin{aligned}
\sum_{\lambda\in\sigma_H} \frac{1}{g(|\lambda|)} &\diamond \int_1^\infty \hat{f}(t)\,\mathrm{d}n_H(t) \diamond \int_1^\infty \frac{1}{g^*(\sqrt{r})} \int_1^\infty \frac{1}{t+r}\cdot\frac{n_H(\sqrt{t})}{t}\,\mathrm{d}t\,\mathrm{d}r \\
&\diamond \int_1^\infty \frac{r}{g^*(r)} \int_1^\infty \frac{1}{t+r^2}\cdot\frac{n_H(\sqrt{t})}{t}\,\mathrm{d}t\,\mathrm{d}r \\
&\diamond \int_1^\infty \frac{1}{r g^*(r)} \int_a^b K(t;r)\,\mathrm{d}t\,\mathrm{d}r.
\end{aligned}$$

$\square$

In the context of finite or minimal type we obtain the following result. Contrasting Theorem II.5.4, it amounts to a characterisation only if the comparison function g satisfies $\operatorname{ind} g \in (0,2)$: note in the formulation of the theorem that $g = \mathrm{o}(g_*)$ if $\operatorname{ind} g \in \{0,2\}$.

II.5.7 Theorem. *Let $H \in \mathbb{H}_{a,b}$ be definite, and assume that resolvents of $A(H)$ belong to the Hilbert–Schmidt class. Let $r_0 \geq 0$, $c_-, c_+ > 0$, assume we have a compatible pair $(\hat{t}, \hat{s})$ for H, r_0 with constants c_-, c_+, and let $K(t;r)$ be as in (II.3.1). Let g be a regularly varying function with $\operatorname{ind} g \geq 0$ and $\int_1^\infty \frac{g(t)}{t^3}\,\mathrm{d}t < \infty$, and set*

$$g_*(r) := \begin{cases} \int_1^r \frac{g(t)}{t}\,\mathrm{d}t & \text{if } \operatorname{ind} g = 0, \\ g(r) & \text{if } \operatorname{ind} g \in (0,2), \\ r^2 \int_r^\infty \frac{g(t)}{t^3}\,\mathrm{d}t & \text{if } \operatorname{ind} g = 2. \end{cases}$$

Then it holds that

$$\limsup_{r\to\infty}\frac{\delta}{\mathfrak{g}_*(r)}\int_a^b K(t;r)\,\mathrm{d}t \le \limsup_{r\to\infty}\frac{n_H(r)}{\mathfrak{g}(r)} \le \limsup_{r\to\infty}\frac{\Delta}{\mathfrak{g}(r)}\int_a^b K(t;r)\,\mathrm{d}t.$$

The constants δ, Δ *depend on* $c_-, c_+, \operatorname{ind}\mathfrak{g}$*, but not on* $H, r_0, \hat{t}, \hat{s}, \mathfrak{g}$.

As an example let us again consider a Lindelöf comparison function $\mathfrak{g}$ of the form

$$\mathfrak{g}(r) = r^\alpha \prod_{k=1}^{N} (\log^{[k]} r)^{\beta_k}$$

with $\alpha = \operatorname{ind}\mathfrak{g} \in \{0, 2\}$ and assume that $\beta_k = -1$ for $k < n$ and $\beta_n \neq -1$. We can use the computation from Example II.5.5 to obtain

$$\mathfrak{g}_*(r) \asymp \mathfrak{g}(r) \cdot \prod_{j=1}^{n} \log^{[j]} r.$$

Contrasting the situation of Theorem II.5.4, here the boundary cases produce the same gap between $\mathfrak{g}$ and $\mathfrak{g}_*$. In the case $\operatorname{ind}\mathfrak{g} = 0$ and if $\mathfrak{g}(r) \to \infty$ as $r \to \infty$, we have $\beta_1 \ge 0$ and hence $n = 1$.

Proof of Theorem II.5.7. Using monotonicity as a Tauberian condition, we get a rough estimate: for each $r > 0$ it holds that

$$\begin{aligned}\int_0^\infty \frac{r^2}{t+r^2}\cdot\frac{n_H(\sqrt{t})}{t}\,\mathrm{d}t &\ge \int_{r^2}^\infty \frac{r^2}{t+r^2}\cdot\frac{n_H(\sqrt{t})}{t}\,\mathrm{d}t \\ &\ge n_H(r)\int_{r^2}^\infty \Big(\frac{1}{t} - \frac{1}{t+r^2}\Big)\,\mathrm{d}t \\ &= n_H(r)\cdot \log\frac{t}{t+r^2}\Big|_{r^2}^{\infty} = \log 2\cdot n_H(r).\end{aligned}$$

This shows the second inequality in the assertion.

The argument for the proof of the first inequality is more involved. Set $\alpha := \sup\{t \ge 0 \mid n_H(t) = 0\}$; then $\alpha > 0$. Assume that $\limsup_{r\to\infty}\frac{n_H(r)}{\mathfrak{g}(r)} < \infty$, fix $\gamma > \limsup_{r\to\infty}\frac{n_H(r)}{\mathfrak{g}(r)}$, and choose $x > \max\{1, \sqrt{\alpha}\}$ such that

$$\forall t \ge x.\ n_H(t) \le \gamma\mathfrak{g}(t).$$

We have

$$\begin{aligned}&\int_\alpha^{x^2} \frac{r^2}{t+r^2}\cdot\frac{n_H(\sqrt{t})}{t}\,\mathrm{d}t \le \frac{x^2 n_H(x)}{\alpha}, \\ &\int_1^\infty \frac{r^2}{t+r^2}\cdot\frac{\gamma\mathfrak{g}(\sqrt{t})}{t}\,\mathrm{d}t \ge n_H(x)\int_{x^2}^\infty \frac{r^2}{t+r^2}\cdot\frac{1}{t}\,\mathrm{d}t,\end{aligned}$$

and hence

$$\begin{aligned}
\int_0^\infty \frac{r^2}{t+r^2}\frac{n_H(\sqrt{t})}{t}\,\mathrm{d}t &= \int_{x^2}^\infty \frac{r^2}{t+r^2}\frac{n_H(\sqrt{t})}{t}\,\mathrm{d}t + \int_\alpha^{x^2} \frac{r^2}{t+r^2}\frac{n_H(\sqrt{t})}{t}\,\mathrm{d}t \\
&\leq \gamma\cdot\int_{x^2}^\infty \frac{r^2}{t+r^2}\frac{\mathfrak{g}(\sqrt{t})}{t}\,\mathrm{d}t + \frac{x^2 n_H(x)}{\alpha} \\
&\leq \gamma\cdot\int_1^\infty \frac{r^2}{t+r^2}\frac{\mathfrak{g}(\sqrt{t})}{t}\,\mathrm{d}t\cdot\Big(1+\frac{x^2}{\alpha}\Big[\int_{x^2}^\infty \frac{r^2}{t+r^2}\frac{1}{t}\,\mathrm{d}t\Big]^{-1}\Big).
\end{aligned}$$

By monotone convergence the second term in the rounded bracket tends to 0 when $r\to\infty$, and we are left with the task to give a bound for

$$\limsup_{r\to\infty}\Big[\frac{1}{\mathfrak{g}^*(r)}\cdot\int_1^\infty \frac{r^2}{t+r^2}\frac{\mathfrak{g}(\sqrt{t})}{t}\,\mathrm{d}t\Big].$$

We integrate by parts, split up the integral to estimate separately, and make some changes of variables. This leads to

$$\begin{aligned}
\int_1^\infty \frac{r^2}{t+r^2}\frac{\mathfrak{g}(\sqrt{t})}{t}\,\mathrm{d}t &= \int_1^\infty\Big(\int_1^t \frac{\mathfrak{g}(\sqrt{s})}{s}\,\mathrm{d}s\Big)\frac{r^2}{(t+r^2)^2}\,\mathrm{d}t \\
&\leq \int_1^{r^2}\Big(\int_1^t \frac{\mathfrak{g}(\sqrt{s})}{s}\,\mathrm{d}s\Big)\frac{1}{r^2}\,\mathrm{d}t + \int_{r^2}^\infty\Big(\int_1^t \frac{\mathfrak{g}(\sqrt{s})}{s}\,\mathrm{d}s\Big)\frac{r^2}{t^2}\,\mathrm{d}t \\
&= \frac{4}{r^2}\int_1^r t^2\Big(\int_1^t \mathfrak{g}(s)\frac{\mathrm{d}s}{s}\Big)\frac{\mathrm{d}t}{t} + 4r^2\int_r^\infty \frac{1}{t^2}\Big(\int_1^t \mathfrak{g}(s)\frac{\mathrm{d}s}{s}\Big)\frac{\mathrm{d}t}{t}.
\end{aligned}$$

Applying Karamata's theorem in each of the cases $\operatorname{ind}\mathfrak{g}=0$, $\operatorname{ind}\mathfrak{g}\in(0,2)$, $\operatorname{ind}\mathfrak{g}=2$, separately, we obtain

$$\lim_{r\to\infty}\Big[\frac{1}{\mathfrak{g}^*(r)}\cdot\frac{1}{r^2}\int\limits_1^r t^2\Big(\int\limits_1^t \mathfrak{g}(s)\frac{\mathrm{d}s}{s}\Big)\frac{\mathrm{d}t}{t}\Big] = \begin{cases} \frac{1}{2} & \text{if } \operatorname{ind}\mathfrak{g}=0, \\ \frac{1}{(2+\operatorname{ind}\mathfrak{g})\operatorname{ind}\mathfrak{g}} & \text{if } \operatorname{ind}\mathfrak{g}\in(0,2), \\ 0 & \text{if } \operatorname{ind}\mathfrak{g}=2, \end{cases}$$

$$\lim_{r\to\infty}\Big[\frac{1}{\mathfrak{g}^*(r)}\cdot r^2\int\limits_r^\infty \frac{1}{t^2}\Big(\int\limits_1^t \mathfrak{g}(s)\frac{\mathrm{d}s}{s}\Big)\frac{\mathrm{d}t}{t}\Big] = \begin{cases} \frac{1}{2} & \text{if } \operatorname{ind}\mathfrak{g}=0, \\ \frac{1}{(2-\operatorname{ind}\mathfrak{g})\operatorname{ind}\mathfrak{g}} & \text{if } \operatorname{ind}\mathfrak{g}\in(0,2), \\ \frac{1}{2} & \text{if } \operatorname{ind}\mathfrak{g}=2. \end{cases}$$

We see that

$$\limsup_{r\to\infty}\Big[\frac{1}{\mathfrak{g}^*(r)}\cdot\int_0^\infty \frac{r^2}{t+r^2}\frac{n_H(\sqrt{t})}{t}\,\mathrm{d}t\Big]\leq c\gamma$$

with a constant c that depends only on $\operatorname{ind}\mathfrak{g}$. Since γ was arbitrary, the required inequality follows. □

Using Theorem II.5.7 we obtain a corollary for the limit circle case. In Theorem II.3.3 we saw that the integral over $K(t;r)$ governs the growth of the right lower entry $w_{22}(b;z)$ of the monodromy matrix along the imaginary axis. Using machinery from complex analysis, we can pass to the growth of $\max_{|z|=r}\|W(b;z)\|$.

II.5.8 Proposition. *Let $H \in \mathbb{H}_{a,b}$ be in limit circle case, and let $\mathfrak{g}$ be a regularly varying function with $\int_1^\infty \frac{\mathfrak{g}(t)}{t^2}\,\mathrm{d}t < \infty$ and $\mathfrak{g}(r) = \mathrm{o}(r)$. Set*

$$\mathfrak{g}^{**}(r) := \begin{cases} \int_1^r \frac{\mathfrak{g}(t)}{t}\,\mathrm{d}t & \text{if } \operatorname{ind}\mathfrak{g} = 0, \\ \frac{1}{\operatorname{ind}\mathfrak{g}(1-\operatorname{ind}\mathfrak{g})}\cdot \mathfrak{g}(r) & \text{if } \operatorname{ind}\mathfrak{g} \in (0,1), \\ r\int_r^\infty \frac{\mathfrak{g}(t)}{t^2}\,\mathrm{d}t & \text{if } \operatorname{ind}\mathfrak{g} = 1. \end{cases}$$

Then

$$\limsup_{r\to\infty} \frac{1}{\mathfrak{g}^{**}(r)} \log\Big(\max_{|z|=r}\|W(b;z)\|\Big) \leq \Delta \limsup_{r\to\infty}\frac{1}{\mathfrak{g}(r)}\int_a^b K(t;r)\,\mathrm{d}t$$

where Δ depends on $\operatorname{ind}\mathfrak{g}$ only.

Proof. Set $\gamma := \limsup_{r\to\infty}\frac{1}{\mathfrak{g}(r)}\int_a^b K(t;r)\,\mathrm{d}t$. If $\gamma = \infty$ there is nothing to prove; hence assume that $\gamma < \infty$.

Set $f(z) := w_{22}(b;z)$. Our first aim is to show that f is a canonical product with genus 0. Theorem II.5.7 implies that $\limsup_{r\to\infty}\frac{n_H(r)}{\mathfrak{g}(r)} \leq \Delta\gamma < \infty$. Hence, the sequence of zeros of f has genus 0. Let F be the canonical product with the same zeros as f; then F is of minimal exponential type. The function f is of bounded type in both half-planes, and by Theorem II.3.3 its mean type is equal to 0. Thus f is also of minimal exponential type. In the exponential factor $e^{\alpha+\beta z}$ of the Hadamard factorisation of f, we must thus have $\beta = 0$. Since $f(0) = 1$, also $\alpha = 0$, and we see that $f = F$. Now we refer to [Lev80, Lemma 3], make a "limit superior" argument as in the proof of the last theorem, and apply Karamata's theorem to evaluate integrals. This leads to

$$\limsup_{r\to\infty}\frac{1}{\mathfrak{g}^{**}(r)}\log\Big(\max_{|z|=r}\|f(z)\|\Big) \leq \Delta\gamma.$$

To finish the proof it remains to refer to [BW06, Proposition 2.3]. □

Note in this context that one can always choose a regularly varying function $\mathfrak{g}$ such that

$$\limsup_{r\to\infty}\frac{1}{\mathfrak{g}(r)}\int_a^b K(t;r)\,\mathrm{d}t = 1,$$

cf. [BGT89, Theorem 2.3.11].

II.6 Algorithmic approach to order

We saw in Theorems II.3.1 and II.3.3 and proposition II.5.8 that the growth of the zero counting function in general, and of the monodromy matrix in the limit circle case, is governed by the integral over the function $K(t;r)$. The challenge thus is to handle that integral, and this is often difficult.

In this section we consider the limit circle case, and give an algorithm that leads to an evaluation of $\int_a^b K(t;r)\,dt$ up to a possible logarithmic error. This algorithm enhances applicability tremendously, and has interesting consequences, e.g. Proposition II.6.9. It is methodologically related to the partitioning theorems given in [Rom17]; we will discuss details in Section II.6.2.

II.6.1 Evaluating $\int_a^b K(t;r)\,dt$ by partitioning

The size of $\int_a^b K(t;r)\,dt$ can be determined by using a clever partitioning of the interval $[a,b]$.

II.6.1 Definition. Let $H \in \mathbb{H}_{a,b}$ be in limit circle case and let $c > 0$. For each $r > 0$ we define points $\sigma_j^{(r)}$ and a number $\kappa_H(r)$ by the following procedure:

- ▷ Set $\sigma_0^{(r)} := a$.
- ▷ If $\det\Omega\big(\sigma_{j-1}^{(r)}, b\big) > \frac{c}{r^2}$, let $\sigma_j^{(r)} \in (\sigma_{j-1}^{(r)}, b)$ be the unique point such that

$$\det\Omega\big(\sigma_{j-1}^{(r)}, \sigma_j^{(r)}\big) = \frac{c}{r^2}.$$

  Otherwise, set $\sigma_j^{(r)} := b$ and $\kappa_H(r) := j$, and terminate.

◀

By Minkowski's determinant inequality, the algorithm terminates after at most

$$\Big\lfloor r \cdot \frac{\sqrt{\det\Omega(a,b)}}{\sqrt{c}} \Big\rfloor$$

many steps.

II.6.2 Theorem. *Let $H \in \mathbb{H}_{a,b}$ be in limit circle case and let $c > 0$. For $r > \big(\frac{c}{\det\Omega(a,b)}\big)^{\frac{1}{2}}$ let $\hat{t}(r) \in [a,b]$ be the unique point with $\det\Omega(a,\hat{t}(r)) = \frac{c}{r^2}$, and for $t \in [\hat{t}(r), b]$ let $\hat{s}(t;r) \in [a,t]$ be the unique point with $\det\Omega(\hat{s}(t;r),t) = \frac{c}{r^2}$.*

Further, let $K(t;r)$ *be the function defined in (II.3.1) using this choice of* $\hat{t}$ *and* $\hat{s}$*. Then*

$$\kappa_H(r)\cdot\log 2-\mathrm{O}(\log r)\leq\int_a^b K(t;r)\,\mathrm{d}t\leq\kappa_H(r)\log r\cdot\big(2e+\mathrm{o}(1)\big)+\mathrm{O}(\log r).$$

The functions $\mathrm{O}(\log r)$ *and* $\mathrm{o}(1)$ *depend only on* c *and* $\operatorname{tr}\Omega(a,b)$*; explicit formulae are (II.6.3), (II.6.6), and (II.6.7).*

Recalling Theorem II.3.3, we immediately obtain a formula for the order.

II.6.3 Corollary. *Let* $H\in\mathbb{H}_{a,b}$ *be in limit circle case, and let* ρ_H *be the order of the monodromy matrix* $W_H(b,z)$*. Let* $c>0$ *and let* $\kappa_H(r)$ *be the numbers produced by the algorithm in Definition II.6.1. Then*

$$\rho_H=\limsup_{n\to\infty}\frac{\log\kappa_H(n)}{\log r}.$$

The upper and lower bounds in Theorem II.6.2 are proved independently, and we start with the lower bound.

II.6.4 Lemma. *Assume we have points* $s_1,\ldots,s_k$ *such that (we set* $s_0:=a$*)*

$$a<s_1<\ldots<s_k\leq b,\qquad\det\Omega(s_{j-1},s_j)\geq\frac{c}{r^2},\ j\in\{1,\ldots,k\}.\tag{II.6.1}$$

Then

$$\int_{s_1}^{s_k}\frac{h_1(t)}{\omega_1(\hat{s}(t;r),t)}\,\mathrm{d}t\geq k\log 2-\Big[\log r+\log\frac{\operatorname{tr}\Omega(a,b)}{\sqrt{c}}\Big].$$

Proof. Our assumption (II.6.1) implies that

$$\hat{s}(t;r)\in[s_{j-2},t]\quad\text{for all }t\in[s_{j-1},s_j],\ j=2,\ldots,k.$$

Using that ω_1 is monotone and additive, we obtain

$$\begin{aligned}\int\limits_{s_1}^{s_k}\frac{h_1(t)}{\omega_1(\hat{s}(t;r),t)}\,\mathrm{d}t&=\sum_{j=2}^{k}\int\limits_{s_{j-1}}^{s_j}\frac{h_1(t)}{\omega_1(\hat{s}(t;r),t)}\,\mathrm{d}t\geq\sum_{j=2}^{k}\int\limits_{s_{j-1}}^{s_j}\frac{h_1(t)}{\omega_1(s_{j-2},t)}\,\mathrm{d}t\\&=\sum_{j=2}^{k}\Big(\log\omega_1(s_{j-2},s_j)-\log\omega_1(s_{j-2},s_{j-1})\Big)\\&=\sum_{j=2}^{k}\log\Big(1+\frac{\omega_1(s_{j-1},s_j)}{\omega_1(s_{j-2},s_{j-1})}\Big)\geq\sum_{j=2}^{k}\log\Big(2\Big[\frac{\omega_1(s_{j-1},s_j)}{\omega_1(s_{j-2},s_{j-1})}\Big]^{\frac{1}{2}}\Big)\\&=(k-1)\log 2+\frac{1}{2}\log\frac{\omega_1(s_{k-1},s_k)}{\omega_1(a,s_1)}.\end{aligned}\tag{II.6.2}$$

We have

$$\omega_1(s_{k-1}, s_k) \geq \frac{\det \Omega(s_{k-1}, s_k)}{\omega_2(s_{k-1}, s_k)} \geq \frac{c}{r^2} \cdot \frac{1}{\omega_2(s_{k-1}, s_k)},$$

and therefore

$$\frac{\omega_1(s_{k-1}, s_k)}{\omega_1(a, s_1)} \geq \frac{c}{r^2} \cdot \frac{1}{\omega_1(a,b)\omega_2(a,b)} \geq \frac{4c}{r^2(\operatorname{tr}\Omega(a,b))^2}.$$

We can thus further estimate the expression from (II.6.2) as

$$\geq k\log 2 - \log 2 + \frac{1}{2}\log\frac{4c}{r^2(\operatorname{tr}\Omega(a,b))^2} = k\log 2 - \Big[\log r + \log\frac{\operatorname{tr}\Omega(a,b)}{\sqrt{c}}\Big].$$

□

Proof of Theorem II.6.2; lower bound. First of all note that $\hat{t}(r) = \sigma_1^{(r)}$. Using Lemma II.6.4 we obtain

$$\begin{aligned}
\int_a^b K(t;r)\,\mathrm{d}t &\geq \int_{\sigma_1^{(r)}}^{\sigma^{(r)}_{\kappa_H(r)-1}} \frac{h_1(t)}{\omega_1(\hat{s}(t;r),t)}\,\mathrm{d}t \\
&\geq (\kappa_H(r)-1)\log 2 - \Big[\log r + \log\frac{\operatorname{tr}\Omega(a,b)}{\sqrt{c}}\Big] \\
&= \kappa_H(r)\log 2 - \Big[\log r + \log\frac{2\operatorname{tr}\Omega(a,b)}{\sqrt{c}}\Big]. \qquad \text{(II.6.3)}
\end{aligned}$$

□

The argument which yields the upper bound from the theorem relies on the monotonicity property Lemma II.2.6 (iii) in the form of the next lemma.

II.6.5 Lemma. *Assume we have points s_0, s_1, s_2 such that*

$$a \leq s_0 < s_1 < s_2 \leq b, \qquad \det\Omega(s_0,s_1) \geq \frac{c}{r^2} \geq \det\Omega(s_1,s_2).$$

Then

$$\int_{s_1}^{s_2} \frac{h_1(t)}{\omega_1(\hat{s}(t;r),t)}\,\mathrm{d}t \leq e\cdot\log\left(\frac{r^2\det\Omega(s_0,s_2)}{c}\right).$$

Proof. For each $t \in [s_1, s_2]$ we have $\hat{s}(t;r) \in [s_0, s_1]$. Let $\gamma > 0$. Using

Lemma II.2.6 (iii) we estimate

$$\begin{aligned}
\int\limits_{s_1}^{s_2} \frac{h_1(t)}{\omega_1(\hat{s}(t;r),t)}\,\mathrm{d}t &= \int\limits_{s_1}^{s_2} \Big(\frac{r^2}{c}\Big)^{\gamma} \Big(\frac{\det\Omega(\hat{s}(t;r),t)}{\omega_1(\hat{s}(t;r),t)}\Big)^{\gamma} \frac{h_1(t)}{\omega_1(\hat{s}(t;r),t)^{1-\gamma}}\,\mathrm{d}t \\
&\le \Big(\frac{r^2}{c}\Big)^{\gamma} \Big(\frac{\det\Omega(s_0,s_2)}{\omega_1(s_0,s_2)}\Big)^{\gamma} \cdot \begin{cases} \int\limits_{s_1}^{s_2} \frac{h_1(t)}{\omega_1(s_1,t)^{1-\gamma}}\,\mathrm{d}t & \text{if } \gamma \in (0,1] \\ \int\limits_{s_1}^{s_2} \frac{h_1(t)}{\omega_1(s_0,t)^{1-\gamma}}\,\mathrm{d}t & \text{if } \gamma > 1 \end{cases} \\
&= \Big(\frac{r^2 \det\Omega(s_0,s_2)}{c}\Big)^{\gamma} \Big(\frac{1}{\omega_1(s_0,s_2)}\Big)^{\gamma} \\
&\quad \cdot \frac{1}{\gamma} \begin{cases} \omega_1(s_1,s_2)^{\gamma} & \text{if } \gamma \in (0,1] \\ \omega_1(s_0,s_2)^{\gamma} - \omega_1(s_0,s_1)^{\gamma} & \text{if } \gamma > 1 \end{cases} \\
&\le \frac{1}{\gamma} \Big(\frac{r^2 \det\Omega(s_0,s_2)}{c}\Big)^{\gamma}.
\end{aligned}$$

The last expression becomes minimal for

$$\gamma := \Big[\log\Big(\frac{r^2\det\Omega(s_0,s_2)}{c}\Big)\Big]^{-1}.$$

When we use this value for γ, the asserted inequality follows. $\square$

II.6.6 Lemma. *Let $l, l' \in \mathbb{N}$ with $1 \le l < l' \le \kappa_H(r)$. Then*

$$\int\limits_{\sigma_l^{(r)}}^{\sigma_{l'}^{(r)}} \frac{h_1(t)}{\omega_1(\hat{s}(t;r),t)}\,\mathrm{d}t \le \sum_{j=l+1}^{l'} e \log\Big(\frac{r^2 \det\Omega(\sigma_{j-2}^{(r)},\sigma_j^{(r)})}{c}\Big). \tag{II.6.4}$$

Proof. Lemma II.6.5 can be applied with each of the triples $\sigma_{j-2}^{(r)}, \sigma_{j-1}^{(r)}, \sigma_j^{(r)}$. This yields that

$$\begin{aligned}
\int\limits_{\sigma_l^{(r)}}^{\sigma_{l'}^{(r)}} \frac{h_1(t)}{\omega_1(\hat{s}(t;r),t)}\,\mathrm{d}t &= \sum_{j=l+1}^{l'} \int\limits_{\sigma_{j-1}^{(r)}}^{\sigma_j^{(r)}} \frac{h_1(t)}{\omega_1(\hat{s}(t;r),t)}\,\mathrm{d}t \\
&\le \sum_{j=l+1}^{l'} e \log\Big(\frac{r^2\det\Omega(\sigma_{j-2}^{(r)},\sigma_j^{(r)})}{c}\Big).
\end{aligned}$$

$\square$

Proof of Theorem II.6.2; upper bound. To find a bound for the integral of $K(t;r)$ over the interval $(\sigma_1^{(r)}, b)$, we have to estimate the sum (II.6.4). We do this using that

$$\sqrt{\det \Omega(\sigma_{j-2}^{(r)}, \sigma_j^{(r)})} \leq \frac{\operatorname{tr} \Omega(\sigma_{j-2}^{(r)}, \sigma_j^{(r)})}{2} \leq \frac{\operatorname{tr} \Omega(a,b)}{2}. \tag{II.6.5}$$

This leads to

$$\begin{aligned} \int_{\sigma_1^{(r)}}^{b} K(t;r)\,\mathrm{d}t &\leq \sum_{j=2}^{\kappa_H(r)} e \log\Big(\frac{r^2 \det \Omega(\sigma_{j-2}^{(r)}, \sigma_j^{(r)})}{c}\Big) \\ &\leq 2e \cdot (\kappa_H(r) - 1) \cdot \Big(\log r + \log \frac{\operatorname{tr} \Omega(a,b)}{2\sqrt{c}}\Big). \end{aligned} \tag{II.6.6}$$

The integral of $K(t;r)$ over the remaining part $(a, \sigma_1^{(r)})$ is bounded by

$$\int_a^{\sigma_1^{(r)}} K(t;r)\,\mathrm{d}t \leq c + 2\max\{c,1\}\Big(\log r + \log \frac{\operatorname{tr}\Omega(a,b)}{2\sqrt{c}}\Big) \tag{II.6.7}$$

by Proposition II.4.4. Putting together, the assertion of the theorem follows. □

It is sometimes practical to notice that one can get estimates for $\kappa_H(r)$ without carrying out the algorithm which produces the points $\sigma_j^{(r)}$ precisely.

II.6.7 Remark. Assume we have points $s_0, \ldots, s_k$ with $a = s_0 < s_1 < \cdots < s_{k-1} < s_k = b$.

(i) If $\det \Omega(s_{l-1}, s_l) \leq \frac{c}{r^2}$ for all $l \in \{1, \ldots, k\}$, then $\kappa_H(r) \leq k$.

(ii) If $\det \Omega(s_{l-1}, s_l) \geq \frac{c}{r^2}$ for all $l \in \{1, \ldots, k\}$, then $\kappa_H(r) \geq k$.

For the proof of (i) consider the pairwise disjoint intervals $(\sigma_{j-1}^{(r)}, \sigma_j^{(r)}]$, $j \in \{1, \ldots, \kappa_H(r)\}$. By Lemma II.2.6 (iv),(v) each of these intervals must contain at least one of the points s_l. The proof of (ii) is analogous: each of the pairwise disjoint intervals $(s_{l-1}, s_l]$, $l \in \{1, \ldots, k\}$ must contain at least of the points $\sigma_j^{(r)}$.

◁

A counting argument like the above gives an analogue to the statement in Lemma II.6.4 for the upper bound.

II.6.8 Proposition. *Assume we have points $s_1, \dots, s_k$ such that $a < s_1 < \dots < s_k \leq b$ and*

$$\det \Omega(a, s_1) \geq \frac{c}{r^2}, \quad \det \Omega(s_{j-1}, s_j) \leq \frac{c}{r^2}, \; j \in \{2, \dots, k\}. \tag{II.6.8}$$

Then

$$\int_{s_1}^{s_k} K(t; r)\, \mathrm{d}t \leq e(k-1) \cdot \Big[2 \log r + \log \frac{\det \Omega(a, b)}{c} \Big]. \tag{II.6.9}$$

Proof. Set

$$l := \max\{j \mid \sigma_j^{(r)} \leq s_1\}, \quad l' := \min\{j \mid s_k \leq \sigma_j^{(r)}\}.$$

Due to (II.6.8) and the fact that $\det \Omega$ is strictly monotone, cf. Lemma II.2.6, each of the pairwise disjoint intervals

$$\big[\sigma_{j-1}^{(r)}, \sigma_j^{(r)}\big], \; j = l+1, \dots, l',$$

must contain at least one of $s_1, \dots, s_k$. Thus $k \geq l - l'$.

Using $\det \Omega(\sigma_{j-2}^{(r)}, \sigma_j^{(r)}) \leq \det \Omega(a, b)$, we obtain from Lemma II.6.6 applied with the points $\sigma_l^{(r)}, \dots, \sigma_{l'}^{(r)}$ that

$$\begin{aligned} \int\limits_{s_1}^{s_k} \frac{h_1(t)}{\omega_1(\hat{s}(t; r), t)}\, \mathrm{d}t &\leq \int\limits_{\sigma_l^{(r)}}^{\sigma_{l'}^{(r)}} \frac{h_1(t)}{\omega_1(\hat{s}(t; r), t)}\, \mathrm{d}t \\ &\leq (k-1) \cdot e \Big(2 \log r + \log \frac{\det \Omega(a, b)}{c} \Big). \end{aligned}$$

$\square$

It is a very intuitive fact that the growth of the monodromy matrix should not increase when one cuts out pieces of the Hamiltonian. A result giving meaning to this intuition is shown in [PW23, Theorem 3.4 and Corollary 3.5]. There a certain assumption is placed on the piece which is cut out, and it is shown with an operator theoretic argument that the zero counting function does not increase. Using the above Theorem II.6.2 we can show a result which does not require any assumptions and implies that growth might increase at most by a logarithmic factor.

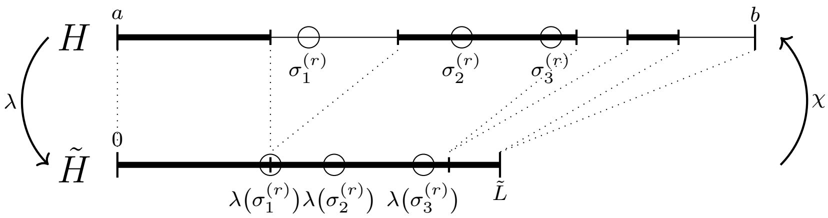
II.6.9 Proposition. *Let $H \in \mathbb{H}_{a,b}$ be in limit circle case and let $\Delta \subseteq [a, b]$ be measurable. Set*

$$\begin{aligned}
&\lambda(t) := \int_a^t \mathbb{1}_\Delta(u)\,\mathrm{d}u,\ t \in [a, b], \qquad \tilde{a} := 0,\ \tilde{b} := \lambda(b), \\
&\chi(s) := \min\big\{t \in [a, b] \mid \lambda(t) \geq s\big\},\ s \in [\tilde{a}, \tilde{b}], \\
&\tilde{H}(s) := H(\chi(s)),\ s \in [\tilde{a}, \tilde{b}].
\end{aligned}$$

Then $\tilde{H} \in \mathbb{H}_{\tilde{a},\tilde{b}}$, and we have

$$\kappa_H(r) \geq \kappa_{\tilde{H}}(r) \quad \textit{for all } r > 0.$$

The intuition behind the proof is simple and is expressed in the picture



Proof of Proposition II.6.9. The fact that $\tilde{H} \in \mathbb{H}_{\tilde{a},\tilde{b}}$ is shown in the proof of [PW23, Theorem 3.4], where it is also seen that

$$H(t)\mathbb{1}_\Delta(t) = (\tilde{H} \circ \lambda)(t)\mathbb{1}_\Delta(t), \quad t \in [a, b] \text{ a.e.}$$

Using this we make a change of variable with the absolutely continuous function λ, and obtain

$$\int_s^t H(u)\,\mathrm{d}u \geq \int_s^t H(u)\mathbb{1}_\Delta(u)\,\mathrm{d}u = \int_s^t (\tilde{H} \circ \lambda)(u)\lambda'(u)\,\mathrm{d}u = \int_{\lambda(s)}^{\lambda(t)} \tilde{H}(v)\,\mathrm{d}v.$$

This implies that (quantities $\tilde{\Omega}, \tilde{\sigma}_j^{(r)}$ refer to $\tilde{H}$)

$$\det \Omega(s, t) \geq \det \tilde{\Omega}(\lambda(s), \lambda(t)), \quad a \leq s \leq t \leq b.$$

Hence, each of the pairwise disjoint intervals

$$[\tilde{\sigma}_{j-1}^{(r)}, \tilde{\sigma}_j^{(r)}), \quad j \in \{1, \ldots, \kappa_{\tilde{H}}(r) - 1\}$$

contains at least one of the point $\lambda(\sigma_j^{(r)})$, and it follows that $\kappa_H(r) \geq \kappa_{\tilde{H}}(r)$. $\square$

II.6.2 Connection with Romanov's Theorems I and II

We revisit two results due to R. Romanov that provide knowledge about the growth of the monodromy matrix starting from partitions or coverings of the interval $[a, b]$, and make the connection to our present theorems.

The first of the two is the approximation theorem [Rom17, Theorem 1], or more precisely its improved version [PW23, Theorem 4.1].

II.6.10 Theorem (Romanov's Theorem I – improved)**.** *Let $H \in \mathbb{H}_{a,b}$ be in limit circle case such that $\det H = 0$ a.e., and write*[2] $H(t) = \operatorname{tr} H(t) \cdot \xi_{\varphi(t)}\xi_{\varphi(t)}^T$ *with a measurable function $\varphi\colon [a, b] \to \mathbb{R}$. Assume we are given*

- ▷ *a partition $a = s_0 < s_1 < \cdots < s_k = b$ of $[a, b]$,*
- ▷ *rotation parameters $\psi_1, \ldots, \psi_k \in \mathbb{R}$,*
- ▷ *distortion parameters $a_1, \ldots, a_k \in (0, 1]$,*

and set

$$A_1 := \sum_{j=1}^{k} a_j^2 \int_{s_{j-1}}^{s_j} \cos^2\big(\varphi(t) - \psi_j\big) \cdot \operatorname{tr} H(t)\, dt,$$

$$A_2 := \sum_{j=1}^{k} \frac{1}{a_j^2} \int_{s_{j-1}}^{s_j} \sin^2\big(\varphi(t) - \psi_j\big) \cdot \operatorname{tr} H(t)\, dt,$$

$$A_3 := \sum_{j=1}^{k-1} \log\Bigg(\max\Big\{\frac{a_j}{a_{j+1}}, \frac{a_{j+1}}{a_j}\Big\} \cdot \big|\cos\big(\psi_j - \psi_{j+1}\big)\big| + \frac{|\sin(\psi_j - \psi_{j+1})|}{a_j a_{j+1}}\Bigg),$$

$$A_4 := -\log a_1 - \log a_N.$$

Then

$$\forall r > 0.\ \log\Big(\max_{|z|=r} \|W_H(z)\|\Big) \leq r \cdot (A_1 + A_2) + A_3 + A_4.$$

Here we use the spectral norm on $\mathbb{C}^{2\times 2}$.

Note that the integrals appearing in A_1 and A_2 can be written in a more geometric way emphasising the relation with the function Ω: since $\xi_\alpha^T \xi_\beta = \cos(\beta - \alpha)$, we have

$$\int_{s_{j-1}}^{s_j} \cos^2\big(\varphi(t) - \psi_j\big) \cdot \operatorname{tr} H(t)\, dt = \xi_{\psi_j}^T \Omega(s_{j-1}, s_j) \xi_{\psi_j},$$

[2] Recall the definition $\xi_\phi := \binom{\cos\phi}{\sin\phi}$ from the preliminaries.

$$\int_{s_{j-1}}^{s_j} \sin^2\big(\varphi(t)-\psi_j\big)\cdot \operatorname{tr} H(t)\,\mathrm{d}t = \xi^T_{\psi_j+\frac{\pi}{2}}\Omega(s_{j-1},s_j)\xi_{\psi_j+\frac{\pi}{2}}.$$

The proof of Theorem II.6.10 is carried out by approximating H with a Hamiltonian consisting of a finite number of indivisible intervals, and estimating the monodromy matrix of that approximation (which is a polynomial). Applying the theorem is probably more challenging than proving it: in order to find a good bound, one has to come up with a clever choice of the input data.

Romanov's Theorem I provides for each data set consisting of a partition, rotation parameters, and distortion parameters, a pointwise bound for $\log\big(\max_{|z|=r}\|W(z)\|\big)$. The best one can get from that theorem is thus obtained by taking the infimum of the bounds over all such data sets. Call that infimum $J(r)$. It is not known how to produce a sequence of data sets (or one single data set) which approximates (or evaluates) $J(r)$. It is not known whether $J(r)$ always gives the correct growth of $\log\max_{|z|=r}\|W(z)\|$, even in a very weak sense like order.

With a somewhat tricky application of Theorem II.6.10, we can prove an upper bound which uses only a partition as input, and yet comes close to $J(r)$ even with a deterministic choice of the partition. As we see further below, in the discussion after its proof, it also implies that $J(r)$ always coincides with $\log\max_{|z|=r}\|W(z)\|$ up to a possible gap of size $\log r$. In particular we establish that $J(r)$ always gives the correct order.

II.6.11 Theorem. *Let $H\in\mathbb{H}_{a,b}$ be in limit circle case with* $\det H = 0$*. Assume we have $r>0$, $\gamma>0$, and a partition*

$$a = s_0 < s_1 < \cdots < s_k = b \tag{II.6.10}$$

such that

$$\sum_{j=1}^{k}\sqrt{\det\Omega(s_{j-1},s_j)} \le \gamma\cdot\frac{k}{r}. \tag{II.6.11}$$

Then

$$\log\Big(\max_{|z|=r}\|W_H(z)\|\Big) \le k\log^+ r + \gamma'\cdot k + \log^+ r + \gamma'',$$

where

$$\gamma' := \log^+ \operatorname{tr}\Omega(a,b) + 3(1+\gamma) + 2\log 2,\qquad \gamma'' := \log^+\operatorname{tr}\Omega(a,b) + \log 2.$$

Proof. Our aim is to apply Theorem II.6.10 with the partition given in the present assumption. To do this, we still need to find suitable parameters ψ_j and a_j.

▷ The rotation parameters ψ_j are chosen such that the integral in A_2 is minimised. Let λ_j be the smallest eigenvalue of $\Omega(s_{j-1}, s_j)$, and let ψ_j be such that $\xi_{\psi_j+\frac{\pi}{2}}$ is an eigenvector for λ_j.

▷ The distortion parameters are chosen such that A_1 and A_2 are balanced.

$$a_j^2 := \begin{cases} \frac{\max\left\{\frac{1}{r}, \sqrt{\det\Omega(s_{j-1},s_j)}\right\}}{\operatorname{tr}\Omega(s_{j-1},s_j)} & \text{if } \operatorname{tr}\Omega(s_{j-1}, s_j) \geq \frac{1}{r} \\ 1 & \text{if } \operatorname{tr}\Omega(s_{j-1}, s_j) < \frac{1}{r}. \end{cases}$$

Note that $a_j \leq 1$ for all j since

$$\sqrt{\det\Omega(s_{j-1}, s_j)} \leq \sqrt{(\omega_1\omega_2)(s_{j-1}, s_j)} \leq \frac{1}{2}\operatorname{tr}\Omega(s_{j-1}, s_j).$$

It remains to estimate A_1, A_2, A_3, A_4:

$$\begin{aligned} A_1 &= \sum_{j=1}^{k} a_j^2 \int_{s_{j-1}}^{s_j} \cos^2\left(\varphi(t) - \psi_j\right) \cdot \operatorname{tr} H(t)\,\mathrm{d}t \\ &\leq \sum_{j=1}^{k} a_j^2 \int_{s_{j-1}}^{s_j} \operatorname{tr} H(t)\,\mathrm{d}t = \sum_{j=1}^{k} a_j^2 \operatorname{tr}\Omega(s_{j-1}, s_j) \\ &\leq \sum_{\substack{j=1 \\ \operatorname{tr}\Omega(s_{j-1},s_j)<\frac{1}{r}}}^{k} \operatorname{tr}\Omega(s_{j-1}, s_j) \;\; + \sum_{\substack{j=1 \\ \operatorname{tr}\Omega(s_{j-1},s_j)\geq\frac{1}{r}}}^{k} \left(\frac{1}{r} + \sqrt{\det\Omega(s_{j-1}, s_j)}\right) \\ &\leq (2+\gamma) \cdot \frac{k}{r}, \end{aligned}$$

$$\begin{aligned} A_2 &= \sum_{j=1}^{k} \frac{1}{a_j^2} \cdot \xi^T_{\psi_j+\frac{\pi}{2}}\Omega(s_{j-1}, s_j)\xi_{\psi_j+\frac{\pi}{2}} = \sum_{j=1}^{k} \frac{1}{a_j^2} \cdot \lambda_j \\ &\leq \sum_{\substack{j=1 \\ \operatorname{tr}\Omega(s_{j-1},s_j)<\frac{1}{r}}}^{k} \operatorname{tr}\Omega(s_{j-1}, s_j) \;\; + \sum_{\substack{j=1 \\ \operatorname{tr}\Omega(s_{j-1},s_j)\geq\frac{1}{r}}}^{k} \frac{\operatorname{tr}\Omega(s_{j-1}, s_j)}{\sqrt{\det\Omega(s_{j-1}, s_j)}} \cdot 2\frac{\det\Omega(s_{j-1}, s_j)}{\operatorname{tr}\Omega(s_{j-1}, s_j)} \\ &\leq (1+2\gamma) \cdot \frac{k}{r}. \end{aligned}$$

In order to estimate A_3 and A_4, simply observe that

$$\frac{1}{a_j^2} \leq \max\left\{1, r\operatorname{tr}\Omega(a,b)\right\} \leq 1 + r\operatorname{tr}\Omega(a,b),$$

and remember $a_j \leq 1$. It follows that

$$\begin{aligned} A_3 &\leq \sum_{j=1}^{k-1} \log\Big[\sqrt{1+r\operatorname{tr}\Omega(a,b)} + (1+r\operatorname{tr}\Omega(a,b))\Big] \\ &\leq k\cdot\Big[\log^+ r + \log^+\operatorname{tr}\Omega(a,b) + 2\log 2\Big], \end{aligned}$$

$$A_4 \leq 2\log^+\sqrt{1+r\operatorname{tr}\Omega(a,b)} \leq \log^+ r + \log^+\operatorname{tr}\Omega(a,b) + \log 2.$$

Putting together the assertion of the theorem follows. □

Let us establish what was claimed in the last paragraph before the theorem. This is based on the following simple observation:

II.6.12 Remark. Theorem II.6.11 yields a one-line proof of the upper bound from Theorem II.6.2 (with a different constant): for the partition $\sigma_0^{(r)},\dots,\sigma_{\kappa_H(r)}^{(r)}$ produced by the algorithm from Definition II.6.1 it holds that

$$\sum_{j=1}^{\kappa_H(r)} \sqrt{\det\Omega(\sigma_{j-1}^{(r)},\sigma_j^{(r)})} \leq \kappa_H(r)\cdot\frac{\sqrt{c}}{r};$$

now remember Theorem II.3.3.

◁

Let $\tilde{J}(r)$ (keeping $\gamma > 0$ in the above theorem fixed) be the infimum of the bounds from Theorem II.6.11 taken over all partitions (II.6.10) subject to (II.6.11).

▷ Since the proof of Theorem II.6.11 is carried out by an application of Theorem II.6.10, we have $J(r) \leq \tilde{J}(r)$.

▷ Since Theorem II.6.11 applied as in Remark II.6.12 produces an upper bound $\lesssim \kappa_H(r)\log r$, we have $\tilde{J}(r) \lesssim \kappa_H(r)\log r$.

Invoking the lower bound from Theorem II.6.2 we thus obtain the following result.

II.6.13 Corollary. *With the above notation it holds that*

$$\begin{aligned} 1 \leq J(r)\Big/\log\Big(\max_{|z|=r}\|W_H(z)\|\Big) &\leq \tilde{J}(r)\Big/\log\Big(\max_{|z|=r}\|W_H(z)\|\Big) \\ &\lesssim \big(\kappa_H(r)\log r\big)\Big/\log\Big(\max_{|z|=r}\|W_H(z)\|\Big) \lesssim \log r. \end{aligned}$$

We turn to the second of Romanov's theorems, which is [Rom17, Theorem 2]. It deals with diagonal (meaning $h_3(t) = 0$ a.e.) and trace-normalised (meaning $\operatorname{tr} H(t) = 1$ a.e.) Hamiltonians with $\det H = 0$, and expresses the order of the monodromy matrix as a certain infimum.

A diagonal positive semidefinite 2×2-matrix with trace equal to 1 and zero determinant can only be either $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Hence, a diagonal and trace-normalised Hamiltonian with zero determinant is determined by the two sets

$$I_1 := \Big\{ t \in [a,b] \mid H(t) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \Big\}, \quad I_2 := \Big\{ t \in [a,b] \mid H(t) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \Big\}.$$

II.6.14 Theorem (Romanov's Theorem II)**.** *Let $H \in \mathbb{H}_{a,b}$ be in limit circle case with $\det H = 0$, and assume that H is diagonal and trace-normalised. Then the order of the monodromy matrix is the infimum of all numbers $\alpha > 0$ that possess the following property:*

($\star$) *For every sufficiently large $N \in \mathbb{N}$ there exists a covering $\{\omega_j \mid j = 1, \ldots, K(N)\}$ of $[a,b]$ by intervals ω_j, such that (here λ denotes the Lebesgue measure)*

$$K(N) \leq N, \qquad \sum_{j=1}^{K(N)} \sqrt{\lambda(\omega_j \cap I_1)\lambda(\omega_j \cap I_2)} \lesssim N^{1-\frac{1}{\alpha}}.$$

Note that the number $\alpha := 1$ always has the property ($\star$); just take the covering $\{[a,b]\}$. The upper bound in this theorem is proven by an application of Romanov's Theorem I, and the lower bound by using the differential inequality [Rom17, Lemma 8]. The assumption in Theorem II.6.14 that H is trace-normalised is no loss in generality; it can always be achieved by a change of the independent variable. Contrasting this, the assumption that H is diagonal is a severe restriction.

Contrasting Romanov's Theorem II, Theorem II.6.11 does not request any assumptions on H, except that $\det H = 0$ which is no essential restriction due to (II.1.4). Moreover, according to Corollary II.6.13, it gives the growth of W almost precisely, and not only on the rough scale of order. It can – and should – be viewed as a generalisation and refinement of Romanov's Theorem II, for the reason that we can deduce from it the following result. Here we use the notation

$$\det \Omega(\omega) := \det \Omega\big(\inf \omega, \sup \omega\big) \quad \text{for } \omega \subseteq [a,b] \text{ interval}.$$

II.6.15 Theorem. *Let $H \in \mathbb{H}_{a,b}$ be in limit circle case. Let $\mathfrak{f}$ be a regularly varying function with $\operatorname{ind} \mathfrak{f} > 0$ which possess the following property:*

($\star$) *For every sufficiently large $N \in \mathbb{N}$ there exists a covering $\{\omega_j \mid j = 1, \dots, K(N)\}$ of $[a,b]$ by intervals ω_j, such that*

$$K(N) \lesssim N, \qquad \sum_{j=1}^{K(N)} \sqrt{\det \Omega(\omega_j)} \lesssim \frac{N}{\mathfrak{f}^-(N)}.$$

Here $\mathfrak{f}^-$ denotes any asymptotic inverse of $\mathfrak{f}$.

Then (for, say, $r \geq 2$)

$$\log\Big(\max_{|z|=r} \|W_H(z)\|\Big) \lesssim \mathfrak{f}(r) \log r.$$

Proof. For each sufficiently large r, so that we can apply the assumption ($\star$) with $N(r) := \lfloor \mathfrak{f}(r) \rfloor$, let $\omega_1, \dots, \omega_{K(N(r))}$ be intervals having the stated properties. Choose a partition $s'_0, \dots, s'_{k'(r)}$ such that

$$\forall l \in \{1, \dots, k'(r)\}.\ (s'_{l-1}, s'_l) \subseteq \omega_{\iota(l)}$$

with some injective function $\iota\colon \{1, \dots, k'(r)\} \to \{1, \dots, K(N(r))\}$. Further, choose a refinement $s_0, \dots, s_{k(r)}$ of $s'_0, \dots, s_{k'(r)}$ with $k(r) \asymp \mathfrak{f}(r)$. Then we have, remembering Minkowski's determinant theorem,

$$\begin{aligned}
\sum_{j=1}^{k(r)} \sqrt{\det \Omega(s_{j-1}, s_j)} &\leq \sum_{l=1}^{k'(r)} \sqrt{\det \Omega(s'_{l-1}, s'_l)} \leq \sum_{l=1}^{k'(r)} \sqrt{\det \Omega(\omega_{\iota(l)})} \\
&\leq \sum_{l=1}^{K(N(r))} \sqrt{\det \Omega(\omega_j)} \lesssim \frac{N(r)}{\mathfrak{f}^-(N(r))} \asymp \frac{k(r)}{r}.
\end{aligned}$$

Theorem II.6.11 implies that

$$\log\Big(\max_{|z|=r} \|W_H(z)\|\Big) \lesssim k(r) \log r \asymp \mathfrak{f}(r) \log r.$$

□

To make the connection with Theorem II.6.14 observe that for a diagonal and trace-normalised Hamiltonian

$$\det \Omega(\omega) = \lambda(\omega \cap I_1)\lambda(\omega \cap I_2),$$

use $\mathfrak{f}(r) := r^\alpha$, and pass to the infimum in α.

II.6.3 Recovering finite type results with universal constants

We revisit some upper bounds for order and type known from the literature, and give an alternative approach by using the algorithm developed above. This leads to estimates that do not depend on the Hamiltonian in any other way than in the constants used in the algorithm. The important feature is that these bounds hold pointwise with universal constants, and not what they give as asymptotic bounds for $r \to \infty$. This feature will also be exploited in the proof of Theorem II.7.1 below.

The logic of the arguments is appealingly simple. To explain it, let us analyse the proof of the upper bound in Theorem II.6.2. The task was to estimate the sum (II.6.4) where $l = 1$ and $l' = \kappa_H(r)$. In other words, one has to estimate the determinant of $\Omega(.,.)$ and sum up. This can be done in different ways, and one of them led to the bound from Theorem II.6.2. Namely, we used the crude estimate (II.6.5) for the determinant and left the logarithm. The sum then is the number of summands times that value. Now we proceed differently: we use the crude estimate "$e \log x \leq \frac{x^\alpha}{\alpha}$" for the logarithm, but estimate $\det\Omega(.,.)$ in such a way that summing up is possible.

As a first illustration of the previous results, we discuss the Krein–de Branges formula (II.1.4) in the language of growth (rather than eigenvalue asymptotics). The following statement is of course much weaker, but it is pointwise and with explicit constants.

II.6.16 Proposition. *Let $H \in \mathbb{H}_{a,b}$ be in limit circle case and let the notation be as in Theorem II.6.2 with $c := \det\Omega(a,b)$. Then*

$$\frac{\int_a^b \sqrt{\det H(t)}\,\mathrm{d}t}{\sqrt{\det\Omega(a,b)}}\log 2\cdot r - \big[\log r + 2\log 2\big] \leq \int_{\hat{t}(r)}^b K(t;r)\,\mathrm{d}t \leq \frac{2\operatorname{tr}\Omega(a,b)}{\sqrt{c}}\cdot r.$$

Proof. For the upper bound we estimate

$$\sum_{j=2}^{\kappa_H(r)} e\log\left(\frac{r^2\det\Omega(\sigma_{j-2}^{(r)},\sigma_j^{(r)})}{c}\right) \leq \frac{2}{\sqrt{c}}\cdot r\cdot\sum_{j=2}^{\kappa_H(r)}\sqrt{\det\Omega(\sigma_{j-2}^{(r)},\sigma_j^{(r)})}$$
$$\leq \frac{1}{\sqrt{c}}\cdot r\cdot\sum_{j=2}^{\kappa_H(r)}\operatorname{tr}\Omega(\sigma_{j-2}^{(r)},\sigma_j^{(r)}) \leq \frac{2}{\sqrt{c}}\cdot r\cdot\operatorname{tr}\Omega(a,b).$$

The lower bound is shown by an application of Lemma II.6.4. If $\det H = 0$ a.e. there is nothing to prove. Otherwise, choose $s_j \in [a,b]$ such that

$$\int_a^{s_j} \sqrt{\det H(t)}\,\mathrm{d}t = j \cdot \frac{\sqrt{c}}{r}, \quad j = 1, \dots, \left\lfloor \frac{r}{\sqrt{c}} \int_a^b \sqrt{\det H(t)}\,\mathrm{d}t \right\rfloor.$$

Then

$$\sqrt{\det \Omega(s_{j-1}, s_j)} = \Big(\det \int_{s_{j-1}}^{s_j} H(t)\,\mathrm{d}t\Big)^{\frac{1}{2}} \geq \int_{s_{j-1}}^{s_j} \sqrt{\det H(t)}\,\mathrm{d}t = \frac{\sqrt{c}}{r}.$$

$\square$

Next we consider a definite Hamiltonian H with zero determinant. W.l.o.g. assume that $H(t) = \xi_{\varphi(t)}\xi_{\varphi(t)}^T$ where $\varphi\colon [a,b] \to \mathbb{R}$ is a bounded measurable and nonconstant function. Recall that the *oscillation* of a bounded function φ is

$$\operatorname{osc}_\varphi(s,t) := \operatorname{ess\,sup}\big(\varphi|_{[s,t]}\big) - \operatorname{ess\,inf}\big(\varphi|_{[s,t]}\big).$$

II.6.17 Lemma. *Let $\varphi \in L^\infty([a,b],\mathbb{R})$. Then, for all $a \leq s < t \leq b$,*

$$\det \Omega(s,t) \leq \Big[\frac{t-s}{2} \cdot \operatorname{osc}_\varphi(s,t)\Big]^2.$$

Proof. Set

$$\kappa := \frac{1}{2}\big[\operatorname{ess\,sup}(\varphi|_{[s,t]}) + \operatorname{ess\,inf}(\varphi|_{[s,t]})\big],$$

so that

$$\|\varphi|_{[s,t]} - \kappa\|_\infty \leq \frac{1}{2}\operatorname{osc}_\varphi(s,t).$$

Since

$$\xi_{\varphi(x)-\kappa}\xi_{\varphi(x)-\kappa}^T = \begin{pmatrix} \cos\kappa & \sin\kappa \\ -\sin\kappa & \cos\kappa \end{pmatrix} \xi_{\varphi(x)}\xi_{\varphi(x)}^T \begin{pmatrix} \cos\kappa & -\sin\kappa \\ \sin\kappa & \cos\kappa \end{pmatrix},$$

we obtain

$$\begin{aligned}
\det \Omega(s,t) &= \det \int_s^t \xi_{\varphi(u)}\xi_{\varphi(u)}^T\,\mathrm{d}u = \det \int_s^t \xi_{\varphi(u)-\kappa}\xi_{\varphi(u)-\kappa}^T\,\mathrm{d}u \\
&\leq \Big(\int_s^t \cos^2\big(\varphi(u)-\kappa\big)\,\mathrm{d}u\Big)\Big(\int_s^t \sin^2\big(\varphi(u)-\kappa\big)\,\mathrm{d}u\Big) \\
&\leq (t-s)\cdot(t-s)\Big(\frac{\operatorname{osc}_\varphi(s,t)}{2}\Big)^2.
\end{aligned}$$

$\square$

The above lemma leads to the following statements; again we emphasise that these bounds hold pointwise, and one should not think of them asymptotically.

II.6.18 Proposition. *Let $\varphi\colon [a,b] \to \mathbb{R}$ be measurable and nonconstant, and consider the Hamiltonian $H(t) := \xi_{\varphi(t)}\xi_{\varphi(t)}^T$. Further notation is as in Theorem II.6.2.*

(i) *If $\varphi \in L^\infty(a,b)$, then*

$$\int_{\hat{t}(r)}^b K(t;r)\,\mathrm{d}t \le \frac{2}{\sqrt{c}}(b-a)\operatorname{osc}_\varphi(a,b)\cdot r.$$

(ii) *If φ is Hölder continuous on $[a,b]$ with exponent ν and constant σ, then*

$$\int_{\hat{t}(r)}^b K(t;r)\,\mathrm{d}t \le 4(1+\nu)(b-a)\Big(\frac{\sigma}{2\sqrt{c}}\Big)^{\frac{1}{1+\nu}}\cdot r^{\frac{1}{1+\nu}}.$$

(iii) *If φ is of bounded variation on $[a,b]$, and $V_a^b(\varphi)$ denotes its total variation, then*

$$\int_{\hat{t}(r)}^b K(t;r)\,\mathrm{d}t \le \frac{2\sqrt{2}}{\sqrt[4]{c}}\sqrt{(b-a)V_a^b(\varphi)}\cdot r^{\frac{1}{2}}.$$

Proof. In the proof we will repeatedly use the inequality $e\log x \le \frac{x^\alpha}{\alpha}$. For the proof of (i) we note that $\operatorname{osc}_\varphi(s,t)$ is monotone in s and t, and thus

$$\begin{aligned}\sum_{j=2}^{\kappa_H(r)} e\log\left(\frac{r^2\det\Omega(\sigma_{j-2}^{(r)},\sigma_j^{(r)})}{c}\right) &\le \frac{2}{\sqrt{c}}\cdot r\cdot\sum_{j=2}^{\kappa_H(r)}\frac{\sigma_j^{(r)}-\sigma_{j-2}^{(r)}}{2}\operatorname{osc}_\varphi(a,b)\\ &\le \frac{2}{\sqrt{c}}\cdot r\cdot(b-a)\operatorname{osc}_\varphi(a,b).\end{aligned}$$

Under the assumption of (ii) we have $\operatorname{osc}_\varphi(s,t)\le\sigma(t-s)^\nu$, and hence

$$\begin{aligned}&\sum_{j=2}^{\kappa_H(r)} e\log\left(\frac{r^2\det\Omega(\sigma_{j-2}^{(r)},\sigma_j^{(r)})}{c}\right)\\ &\qquad\le 2(1+\nu)\cdot r^{\frac{1}{1+\nu}}\cdot\sum_{j=2}^{\kappa_H(r)}\Big[\frac{1}{\sqrt{c}}\frac{\sigma_j^{(r)}-\sigma_{j-2}^{(r)}}{2}\operatorname{osc}_\varphi(\sigma_{j-2}^{(r)},\sigma_j^{(r)})\Big]^{\frac{1}{1+\nu}}\\ &\qquad\le 4(1+\nu)\Big(\frac{\sigma}{2\sqrt{c}}\Big)^{\frac{1}{1+\nu}}\cdot r^{\frac{1}{1+\nu}}\cdot(b-a).\end{aligned}$$

Finally, if φ is of bounded variation, we note that $\operatorname{osc}_\varphi(s,t) \leq V_s^t(\varphi)$, and estimate as

$$\sum_{j=2}^{\kappa_H(r)} e \log\left(\frac{r^2 \det \Omega(\sigma_{j-2}^{(r)}, \sigma_j^{(r)})}{c}\right)$$

$$\leq \frac{4}{\sqrt[4]{c}} \cdot r^{\frac{1}{2}} \cdot \sum_{j=2}^{\kappa_H(r)} \left(\frac{\sigma_j^{(r)} - \sigma_{j-2}^{(r)}}{2} \cdot V_{\sigma_{j-2}^{(r)}}^{\sigma_j^{(r)}}(\varphi)\right)^{\frac{1}{2}}$$

$$\leq \frac{2}{\sqrt[4]{c}} \cdot r^{\frac{1}{2}} \cdot \left(\sum_{j=2}^{\kappa_H(r)} \frac{\sigma_j^{(r)} - \sigma_{j-2}^{(r)}}{2}\right)^{\frac{1}{2}} \left(\sum_{j=2}^{\kappa_H(r)} V_{\sigma_{j-2}^{(r)}}^{\sigma_j^{(r)}}(\varphi)\right)^{\frac{1}{2}}$$

$$\leq \frac{2\sqrt{2}}{\sqrt[4]{c}} \cdot r^{\frac{1}{2}} \cdot \sqrt{(b-a)V_a^b(\varphi)}.$$

$\square$

II.7 Sharpness of bounds

II.7.1 An example with oscillating rotation

We give an example where the integral of $K(t;r)$ can be evaluated precisely (and which does not fall into a "simple" category). It requires quite complex computations because the Hamiltonian oscillates, still is accessible, since oscillation accumulates only at one endpoint.

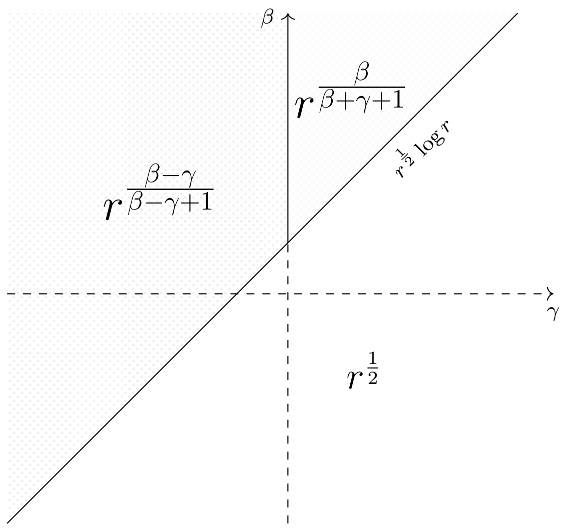
II.7.1 Theorem. *Let $(\gamma,\beta)\in\mathbb{R}^2\setminus(0,0)$, and let $\varphi\colon[0,1]\to\mathbb{R}$ be the chirp signal*

$$\varphi(t):=t^\gamma\sin\Big(\frac{1}{t^\beta}\Big).$$

Consider the Hamiltonian $H(t):=\xi_{\varphi(t)}\xi_{\varphi(t)}^T$, and let $W(z)$ be its monodromy matrix. Then

$$\log|w_{22}(ir)|\asymp\begin{cases} r^{\frac{\beta}{\beta+\gamma+1}} & \text{if } \beta>\gamma+1\wedge\gamma\geq 0,\\ r^{\frac{\beta-\gamma}{\beta-\gamma+1}} & \text{if } \beta>\gamma+1\wedge\gamma<0,\\ r^{\frac12}\log r & \text{if } \beta=\gamma+1,\\ r^{\frac12} & \text{if } \beta<\gamma+1.\end{cases}$$

for $r\to\infty$.



In the situation of the following auxiliary lemmata, we use the given function φ to define the Hamiltonian $H(t):=\xi_{\varphi(t)}\xi_{\varphi(t)}^T$ and abbreviate $\Omega=\Omega_H$.

The first step is to express $\det\Omega$ in terms of φ.

II.7.2 Lemma. *Let $\varphi : (a,b) \to \mathbb{R}$ be measurable, and $a \le s < t \le b$. Then*

$$\det \Omega(s,t) = \frac{1}{2} \int_s^t \int_s^t \sin^2\bigl(\varphi(x) - \varphi(y)\bigr) \, \mathrm{d}x \, \mathrm{d}y. \tag{II.7.1}$$

Proof. We have

$$\begin{aligned}
&\det \Omega(s,t) \\
&= \int_s^t \int_s^t \bigl[\sin^2 \varphi(x) \cos^2 \varphi(y) - \sin \varphi(x) \cos \varphi(x) \sin \varphi(y) \cos \varphi(y) \, \mathrm{d}x \, \mathrm{d}y \\
&= \int_s^t \int_s^t \sin \varphi(x) \cos \varphi(y) \sin\bigl(\varphi(x) - \varphi(y)\bigr) \, \mathrm{d}x \, \mathrm{d}y.
\end{aligned}$$

Swapping x and y we obtain

$$\det \Omega(s,t) = -\int_s^t \int_s^t \sin \varphi(y) \cos \varphi(x) \sin\bigl(\varphi(x) - \varphi(y)\bigr) \, \mathrm{d}x \, \mathrm{d}y.$$

Adding both sides of these relations we arrive at

$$2 \det \Omega(s,t) = \int_s^t \int_s^t \sin^2\bigl(\varphi(x) - \varphi(y)\bigr) \, \mathrm{d}x \, \mathrm{d}y.$$

□

It is usually difficult to estimate the right hand side in (II.7.1) from below. One possibility is to exploit smoothness of φ. We also include an estimate from above, even though we will not need it.

II.7.3 Lemma. *Let $\varphi : (a,b) \to \mathbb{R}$ be monotone, absolutely continuous and not constant. The following statements hold for $a \le s < t \le b$:*

(i) *If $\varphi' \in L^\infty(a,b)$, then*

$$\det \Omega(s,t) \ge \frac{\bigl(\varphi(t) - \varphi(s)\bigr)^2 - \sin^2\bigl(\varphi(t) - \varphi(s)\bigr)}{4\|\varphi'\|_\infty^2}.$$

(ii) *If $\frac{1}{\varphi'} \in L^\infty(a,b)$, then*

$$\det \Omega(s,t) \le \Bigl\|\frac{1}{\varphi'}\Bigr\|_\infty^2 \frac{\bigl(\varphi(t) - \varphi(s)\bigr)^2 - \sin^2\bigl(\varphi(t) - \varphi(s)\bigr)}{4}.$$

Proof. Since, by assumption, φ is monotone, we have $\varphi'(x)\varphi'(y) \geq 0$ for a.e. $x, y \in (a,b)$.

If $\varphi' \in L^\infty(a,b)$, then, by Lemma II.7.2,

$$\begin{aligned}\det \Omega(s,t) &= \frac{1}{2}\int_s^t \int_s^t \sin^2\big(\varphi(x)-\varphi(y)\big)\,\mathrm{d}x\,\mathrm{d}y \\ &\geq \frac{1}{2\|\varphi'\|_\infty^2}\int_s^t \int_s^t \sin^2\big(\varphi(x)-\varphi(y)\big)\varphi'(x)\varphi'(y)\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{II.7.2}$$

If $\frac{1}{\varphi'} \in L^\infty(a,b)$, then

$$\begin{aligned}\det \Omega(s,t) &= \frac{1}{2}\int_s^t \int_s^t \sin^2\big(\varphi(x)-\varphi(y)\big)\,\mathrm{d}x\,\mathrm{d}y \\ &\leq \frac{1}{2}\Big\|\frac{1}{\varphi'}\Big\|_\infty^2\int_s^t \int_s^t \sin^2\big(\varphi(x)-\varphi(y)\big)\varphi'(x)\varphi'(y)\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{II.7.3}$$

With a straightforward calculation we obtain

$$\begin{aligned}&\int_s^t \int_s^t \sin^2\big(\varphi(x)-\varphi(y)\big)\varphi'(x)\varphi'(y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{\big(\varphi(t)-\varphi(s)\big)^2 - \sin^2\big(\varphi(t)-\varphi(s)\big)}{2}\end{aligned}$$

which, together with (II.7.2) and (II.7.3), proves the inequalities in (i) and (ii).

□

We present a lemma that should be seen as a tool for enabling Lemma II.6.4.

II.7.4 Lemma. *Let $\varphi : (a,b) \to \mathbb{R}$ be monotone, absolutely continuous and not constant. Then there are points $s_j \in [a,b]$ with*

$$a = s_0 < s_1 < \ldots < s_{k-1} < s_k \leq b \tag{II.7.4}$$

with the following properties.

$$\text{(i)} \quad r \geq \|\varphi'\|_\infty \quad \Rightarrow \quad k = \left\lfloor |\varphi(a) - \varphi(b)| \Big(\frac{r}{\|\varphi'\|_\infty} \Big)^{\frac{1}{2}} \right\rfloor \\ \wedge \ \forall j \in \{1, \dots, k\}. \quad \det \Omega(s_{j-1}, s_j) > \frac{1}{16r^2}. \tag{II.7.5}$$

$$\text{(ii)} \quad r \leq \|\varphi'\|_\infty \quad \Rightarrow \quad k = \left\lfloor |\varphi(a) - \varphi(b)| \frac{r}{\|\varphi'\|_\infty} \right\rfloor \\ \wedge \ \forall j \in \{1, \dots, k\}. \quad \det \Omega(s_{j-1}, s_j) > \frac{1}{16r^2}. \tag{II.7.6}$$

Proof. We prove (i) first. Let $f(x) := x^2 - \sin^2(x)$ and observe that $\frac{f(x)}{x^4}$ is decreasing. Hence

$$x \leq 1 \quad \Rightarrow \quad f(x) \geq \frac{f(1)}{1^4} \cdot x^4 > \frac{x^4}{4}. \tag{II.7.7}$$

Since φ is monotone, $\varphi([a, b])$ is an interval. Assume for simplicity that φ is nondecreasing, i.e., $\varphi([a, b]) = [\varphi(a), \varphi(b)]$. Let

$$h := \Big(\frac{\|\varphi'\|_\infty}{r} \Big)^{\frac{1}{2}} \leq 1$$

and split $[\varphi(a), \varphi(b)]$ into smaller intervals $[d_{j-1}, d_j]$ of length h each, starting with $d_0 := 0$. Clearly, we can choose up to

$$k := \left\lfloor \frac{|\varphi(b) - \varphi(a)|}{h} \right\rfloor = \left\lfloor |\varphi(a) - \varphi(b)| \Big(\frac{r}{\|\varphi'\|_\infty} \Big)^{\frac{1}{2}} \right\rfloor$$

many such intervals. Since $\varphi\colon [a, b] \to \varphi([a, b])$ is surjective, there are points s_j such that $s_0 = a$ and $\varphi(s_j) = d_j$ for $j = 0, 1, \dots, k$. By Lemma II.7.3 and (II.7.7)

$$\det \Omega(s_{j-1}, s_j) \geq \frac{f\big(|\varphi(s_{j-1}) - \varphi(s_j)|\big)}{4\|\varphi'\|_\infty^2} = \frac{f(h)}{4\|\varphi'\|_\infty^2} > \frac{h^4}{16\|\varphi'\|_\infty^2} = \frac{1}{16r^2}.$$

In the proof of (ii), we need to estimate $f(x)$ for $x \geq 1$. Note that

$$x \geq 1 \quad \Rightarrow \quad f(x) \geq \frac{f(1)}{1^2} \cdot x^2 > \frac{x^2}{4}. \tag{II.7.8}$$

Choosing points s_j analogously, but for $h := \frac{\|\varphi'\|_\infty}{r} \geq 1$, we obtain

$$k = \left\lfloor \frac{|\varphi(b) - \varphi(a)|}{h} \right\rfloor = \left\lfloor |\varphi(a) - \varphi(b)| \frac{r}{\|\varphi'\|_\infty} \right\rfloor$$

and, for $j = 1, 2, \dots, k$,

$$\det \Omega(s_{j-1}, s_j) \geq \frac{f\big(|\varphi(s_{j-1}) - \varphi(s_j)|\big)}{4\|\varphi'\|_\infty^2} = \frac{f(h)}{4\|\varphi'\|_\infty^2} > \frac{h^2}{16\|\varphi'\|_\infty^2} = \frac{1}{16r^2}.$$

□

Now we are only two auxiliary lemmata away from being able to prove Theorem II.7.1. Lemma II.7.5 contains a lower bound while Lemma II.7.6 gives a bound from above, and both results are very specifically adapted to the situation in Theorem II.7.1.

II.7.5 Lemma. *Let $\varphi : (a, b) \to \mathbb{R}$ be locally absolutely continuous and let $H(t) := \xi_{\varphi(t)}\xi_{\varphi(t)}^T$. Suppose that $\|\varphi'|_K\|_\infty < \infty$ for each compact $K \subseteq (a, b]$, and that we are given*

- ▷ *a decreasing sequence $(\xi_n)_{n=1}^\infty$ with $\xi_1 = b$, such that φ is monotone on each interval $[\xi_n, \xi_{n-1}]$;*
- ▷ *a function $T : (X, \infty) \to \mathbb{N}$ with the property that, for any $r \in (X, \infty)$ and $n \in \{2, \dots, T(r)\}$,*

$$r \geq \big\|\varphi'|_{[\xi_n,\xi_{n-1}]}\big\|_\infty \;\wedge\; |\varphi(\xi_n) - \varphi(\xi_{n-1})|\Big(\frac{r}{\|\varphi'|_{[\xi_n,\xi_{n-1}]}\|_\infty}\Big)^{\frac{1}{2}} \geq 1. \qquad \text{(II.7.9)}$$

Then, if $(\hat{t}, \hat{s})$ is a compatible pair for H, $\big(\frac{1}{16\det\Omega(a,b)}\big)^{\frac{1}{2}}, \frac{1}{16}, \frac{1}{16}$, we have

$$\int_{\xi_{T(r)}}^b K(t; r)\,\mathrm{d}t$$
$$\geq \frac{\log 2}{2} r^{\frac{1}{2}} \sum_{n=2}^{T(r)} |\varphi(\xi_n) - \varphi(\xi_{n-1})|\big\|\varphi'|_{[\xi_n,\xi_{n-1}]}\big\|_\infty^{-\frac{1}{2}} - \log\big(4(b-a)r\big).$$

Proof. For each $n \in \{2, \dots, T(r)\}$, Lemma II.7.4 provides

$$k(n, r) = \Big\lfloor |\varphi(\xi_n) - \varphi(\xi_{n-1})|\Big(\frac{r}{\|\varphi'|_{[\xi_n,\xi_{n-1}]}\|_\infty}\Big)^{\frac{1}{2}}\Big\rfloor \geq 1$$

many disjoint subintervals $[s_{j-1}^{(n,r)}, s_j^{(n,r)}]$ of $[\xi_n, \xi_{n-1}]$ with the property that $\det \Omega\big(s_{j-1}^{(n,r)}, s_j^{(n,r)}\big) > \frac{1}{16r^2}$ each. Taking the sum over all $n \in 2, \dots, T(r)$, at least

$$\sum_{n=2}^{T(r)} k(n, r) \geq \frac{1}{2} r^{\frac{1}{2}} \sum_{n=2}^{T(r)} |\varphi(\xi_n) - \varphi(\xi_{n-1})|\big\|\varphi'|_{[\xi_n,\xi_{n-1}]}\big\|_\infty^{-\frac{1}{2}}$$

many disjoint intervals with this property are contained in $[\xi_{T(r)}, 1]$. Lemma II.6.4 now gives the desired lower bound. □

II.7.6 Lemma. *Let $\varphi : (a,b) \to \mathbb{R}$ be locally absolutely continuous and let $H(t) := \xi_{\varphi(t)}\xi_{\varphi(t)}^T$. Suppose that $\|\varphi'|_K\|_\infty < \infty$ for each compact $K \subseteq (a,b]$, and that we are given*

- ▷ *a decreasing sequence $(\xi_n)_{n=1}^\infty$ with $\xi_1 = b$;*
- ▷ *a function $T : (X,\infty) \to \mathbb{N}$ with*

$$\forall n \in \{2,\ldots,T(r)\}. \quad \det\Omega(\xi_n,\xi_{n-1}) > \frac{1}{16r^2}.$$

Then, if $(\hat{t},\hat{s})$ is a compatible pair for H, $\big(\frac{1}{16\det\Omega(a,b)}\big)^{\frac{1}{2}}, \frac{1}{16}, \frac{1}{16}$, the estimate

$$\int_{\xi_{T(r)-1}}^{b} K(t;r)\,\mathrm{d}t \leq 8\sqrt{2}\cdot r^{\frac{1}{2}} \sum_{j=2}^{T(r)-1} (\xi_{j-1}-\xi_{j+1})\big\|\varphi'|_{[\xi_{j+1},\xi_{j-1}]}\big\|_\infty^{\frac{1}{2}}$$

holds.

Proof. Fix $r > X$ and recall that, for $t \in [\xi_{T(r)-1}, b]$ we have

$$K(t;r) = \frac{h_1(t)}{\omega_1(\hat{s}(t;r),t)}$$

because of

$$\det\Omega(a,\xi_{T(r)-1}) \geq \det\Omega(\xi_{T(r)},\xi_{T(r)-1}) \geq \frac{1}{16r^2}.$$

For each $n \in \{2,\ldots,\leq T(r)-1\}$, set $I_n := [\xi_{n+1},\xi_{n-1}]$ and define the Hamiltonian $H_n := H|_{I_n}$. We choose a compatible pair $(\hat{t}_n,\hat{s}_n)$ for H_n, $\big(\frac{1}{16\det\Omega(\xi_{n+1},\xi_{n-1})}\big)^{\frac{1}{2}}, \frac{1}{16}, \frac{1}{16}$, cf. Corollary II.2.9. Since

$$\det\Omega(\xi_{n+1},\xi_n) > \frac{1}{16r^2}$$

it follows that $\hat{t}_n(r) < \xi_n$. Hence $\hat{s}_n(t;r)$ is well-defined for $t \in I_n$ and satisfies

$$\det\Omega(\hat{s}_n(t;r),t) = \frac{1}{16r^2} = \det\Omega(\hat{s}_n(t;r),t), \qquad t \in I_n.$$

This implies $\hat{s}_n(t;r) = \hat{s}(t;r)$ for $t \in I_n$. Note that for $s,t \in I_n$ it holds that $|\varphi(s)-\varphi(t)| \leq \big\|\varphi'|_{I_n}\big\|_\infty |s-t|$, i.e., $\varphi|_{I_n}$ is Hölder continuous with exponent 1 and constant $\big\|\varphi'|_{I_n}\big\|_\infty$. Applying Proposition II.6.18, (ii), to H_n, we obtain

$$\int_{\xi_n}^{\xi_{n-1}} \frac{h_1(t)}{\omega_1(\hat{s}(t;r),t)}\,\mathrm{d}t = \int_{\xi_n}^{\xi_{n-1}} \frac{h_1(t)}{\omega_1(\hat{s}_n(t;r),t)}\,\mathrm{d}t \leq \int_{\hat{t}_n(r)}^{\xi_{n-1}} \frac{h_1(t)}{\omega_1(\hat{s}_n(t;r),t)}\,\mathrm{d}t$$
$$\leq 8\sqrt{2}(\xi_{j-1}-\xi_{j+1})\big\|\varphi'|_{I_n}\big\|_\infty^{\frac{1}{2}} r^{\frac{1}{2}}.$$

Now take the sum over $n \in \{2,\ldots,T(r)-1\}$ to complete the proof.

□

Proof of Theorem II.7.1 for $\beta \leq 0$. We clearly have $\varphi(t) \sim t^{\gamma-\beta}$ for $t \to 0$. There are three cases to distinguish, depending on which of the intervals $[0,\infty), [-1,0)$, and $(-\infty,-1)$ the exponent $\gamma-\beta$ belongs to.

① Case $\gamma - \beta \geq 0$. In this situation, φ is of bounded variation, and Proposition II.6.18, (iii), leads to $\log|w_{22}(ir)| \lesssim r^{\frac{1}{2}}$. Choosing a subinterval (a',b') of (a,b) such that $0 < \|\varphi'|_{(a',b')}\|_\infty < \infty$, Lemma II.7.4, (i) gives (for each large enough r) a total of $k(r) \gtrsim r^{\frac{1}{2}}$ many points $s_j^{(r)} \in [a',b'] \subseteq [a,b]$ such that

$$\det \Omega\big(s_{j-1}^{(r)}, s_j^{(r)}\big) \geq \frac{1}{16r^2}$$

for large r and $j \in \{1,\dots,k(r)\}$. Now Lemma II.6.4 shows that $\log|w_{22}(ir)| \gtrsim r^{\frac{1}{2}}$.

② Case $-1 \leq \gamma - \beta < 0$. We use the abbreviation $\alpha := \gamma - \beta$. Setting $\xi_n := n^{\frac{1}{\alpha}}$, we have

$$\begin{aligned} &\varphi(\xi_n) \sim n, && |\xi_{n-1} - \xi_n| \sim \Big(-\frac{1}{\alpha}\Big) n^{\frac{1-\alpha}{\alpha}}, \\ &\varphi'(\xi_n) \sim \alpha n^{\frac{\alpha-1}{\alpha}}, && |\varphi(\xi_{n-1}) - \varphi(\xi_n)| \sim 1 \end{aligned} \tag{II.7.10}$$

for $n \to \infty$. We would like to identify numbers n such that Lemma II.7.4, (i) applied to $\varphi|_{[\xi_n,\xi_{n-1}]}$ gives $k \geq 1$ subintervals of $[\xi_n, \xi_{n-1}]$ (which, in particular, means that $\det\Omega(\xi_n,\xi_{n-1}) > \frac{1}{16r^2}$). In other words, we want

$$r \geq \|\varphi'|_{[\xi_n,\xi_{n-1}]}\|_\infty \quad \wedge \quad |\varphi(\xi_n) - \varphi(\xi_{n-1})|\Big(\frac{r}{\|\varphi'|_{[\xi_n,\xi_{n-1}]}\|_\infty}\Big)^{\frac{1}{2}} \geq 1.$$

A calculation shows that for a small enough $\eta > 0$ and $n \leq T(r) := \lfloor \eta r^{\frac{\alpha}{\alpha-1}} \rfloor$, the above inequalities are valid. By Lemma II.7.6 and (II.7.10),[3]

$$\int_{\xi_{T(r)-1}}^1 K(t;r)\,\mathrm{d}t \lesssim r^{\frac{1}{2}} \sum_{j=2}^{T(r)-1} n^{\frac{1-\alpha}{2\alpha}} \asymp r^{\frac{1}{2}} \int_1^{r^{\frac{\alpha}{\alpha-1}}} x^{\frac{1+\alpha}{2\alpha}-1}\,\mathrm{d}x.$$

If $\alpha > -1$, the integral on the right hand side is bounded; if $\alpha = -1$, it is equal to $\frac{1}{2}\log r$. Hence

$$\int_{\xi_{T(r)-1}}^1 K(t;r)\,\mathrm{d}t \lesssim \begin{cases} r^{\frac{1}{2}}, & \alpha > -1; \\ r^{\frac{1}{2}}\log r, & \alpha = -1. \end{cases}$$

[3] $K(t;r)$ is understood w.r.t. the compatible pair for H, $\big(\frac{1}{16\det(0,1)}\big)^{\frac{1}{2}}, \frac{1}{16}, \frac{1}{16}$.

By Proposition II.6.16, we also have

$$\int_{\hat{t}(r)}^{\xi_{T(r)-1}} K(t;r)\,\mathrm{d}t \lesssim r\xi_{T(r)-1} \lesssim r^{\frac{\alpha}{\alpha-1}} \le r^{\frac{1}{2}},$$

and as we know from Proposition II.4.4, the contribution of $[0, \hat{t}(r)]$ is neglectable. Combining the estimates, we obtain

$$\log|w_{22}(ir)| \asymp \int_0^1 K(t;r)\,\mathrm{d}t \lesssim \begin{cases} r^{\frac{1}{2}}, & \alpha > -1; \\ r^{\frac{1}{2}}\log r, & \alpha = -1. \end{cases}$$

Since (II.7.9) holds for $2 \le n \le T(r)$, Lemma II.7.5 yields

$$\begin{aligned}\int_0^1 K(t;r)\,\mathrm{d}t &\gtrsim r^{\frac{1}{2}} \sum_{n=2}^{T(r)} |\varphi(\xi_n) - \varphi(\xi_{n-1})| \big\|\varphi'|_{[\xi_n,\xi_{n-1}]}\big\|_\infty^{-\frac{1}{2}} \\ &\asymp r^{\frac{1}{2}} \sum_{n=2}^{T(r)} n^{\frac{1-\alpha}{2\alpha}} \asymp \begin{cases} r^{\frac{1}{2}}, & \alpha > -1; \\ r^{\frac{1}{2}}\log r, & \alpha = -1. \end{cases}\end{aligned}$$

③ Case $\gamma - \beta < -1$. We write again $\alpha := \gamma - \beta$ for short. With $\xi_n := n^{-1}$, we have

$$\begin{aligned} &\varphi(\xi_n) \sim n^{-\alpha}, && |\xi_{n-1} - \xi_n| \sim n^{-2}, \\ &\varphi'(\xi_n) \sim \alpha n^{1-\alpha}, && |\varphi(\xi_{n-1}) - \varphi(\xi_n)| \sim -\alpha n^{-\alpha-1}. \end{aligned} \tag{II.7.11}$$

We see that (II.7.9) is satisfied for all $n \le T(r) := \lfloor \epsilon r^{\frac{1}{1-\alpha}} \rfloor$ if $\epsilon > 0$ is small enough. For n satisfying (II.7.9), Lemma II.7.4, (i), shows that $\det \Omega(\xi_n, \xi_{n-1}) > \frac{1}{16r^2}$. Using Lemma II.7.6 and (II.7.11),

$$\begin{aligned}\int_{\xi_{T(r)-1}}^1 K(t;r)\,\mathrm{d}t &\lesssim r^{\frac{1}{2}} \sum_{j=2}^{T(r)-1} n^{-\frac{1+\alpha}{2}-1} \asymp r^{\frac{1}{2}} \int_1^{r^{\frac{1}{1-\alpha}}} x^{-\frac{1+\alpha}{2}-1}\,\mathrm{d}x. \\ &\asymp r^{\frac{1}{2}} \cdot r^{\frac{1+\alpha}{2(\alpha-1)}} = r^{\frac{\alpha}{\alpha-1}}.\end{aligned}$$

Proposition II.6.16 gives

$$\int_{\hat{t}(r)}^{\xi_{T(r)-1}} K(t;r)\,\mathrm{d}t \lesssim r\xi_{T(r)-1} \asymp r^{\frac{\alpha}{\alpha-1}},$$

leading to $\log|w_{22}(ir)| \lesssim r^{\frac{\alpha}{\alpha-1}}$. In analogy to the previous case, a lower bound for $\log|w_{22}(ir)|$ is given by

$$r^{\frac{1}{2}} \sum_{n=2}^{T(r)} |\varphi(\xi_n) - \varphi(\xi_{n-1})| \big\|\varphi'|_{[\xi_n,\xi_{n-1}]}\big\|_\infty^{-\frac{1}{2}} \gtrsim r^{\frac{1}{2}} \sum_{j=2}^{T(r)-1} n^{-\frac{1+\alpha}{2}-1} \asymp r^{\frac{\alpha}{\alpha-1}}.$$

□

Proof of Theorem II.7.1 for $\beta > 0$. This is the case where φ oscillates. A few preparations are needed before we can treat the different cases.

▷ The derivative of φ is given by

$$\varphi'(t) = t^{\gamma-1}\big[\gamma\sin(t^{-\beta}) - \beta t^{-\beta}\cos(t^{-\beta})\big]. \tag{II.7.12}$$

For $n = 2, 3, \ldots$, we have $\varphi\big((\pi n)^{-\frac{1}{\beta}}\big) = 0$ and there is a unique stationary point

$$\xi_n \in \big((\pi n)^{-\frac{1}{\beta}}, (\pi(n-1))^{-\frac{1}{\beta}}\big).$$

Setting $\xi_1 := 1$, the function φ is monotone on each interval $[\xi_n, \xi_{n-1}]$.

▷ Let

$$\xi_n^{-\beta} = \pi(n-1) + x_n, \qquad 0 < x_n < \pi.$$

From $\varphi'(\xi_n) = 0$ we infer

$$\pi(n-1) + x_n = \frac{\gamma}{\beta}\tan(x_n),$$

i.e. $\lim_{n\to\infty}\tan(x_n) = \infty$ and therefore $\lim_{n\to\infty} x_n = \frac{\pi}{2}$. We see that

$$\varphi(\xi_n) = \xi_n^\gamma \sin(\pi(n-1) + x_n) = \xi_n^\gamma \sin(x_n)(-1)^{n-1} \sim (-1)^{n-1}(\pi n)^{-\frac{\gamma}{\beta}}.$$

and hence

$$|\varphi(\xi_n) - \varphi(\xi_{n-1})| \sim 2\pi^{-\frac{\gamma}{\beta}} n^{-\frac{\gamma}{\beta}},$$
$$\xi_{n-1} - \xi_n \sim \frac{1}{\beta\pi^{\frac{1}{\beta}}} n^{-\frac{\beta+1}{\beta}}.$$

▷ From (II.7.12) we obtain

$$\beta n^{\frac{\beta-\gamma+1}{\beta}} \le \|\varphi'|_{[\xi_n,\xi_{n-1}]}\|_\infty \le (\beta+\gamma) n^{\frac{\beta-\gamma+1}{\beta}}.$$

① Case $\gamma \ge 0$. Let $r > 0$ be fixed. A computation shows that there is $\eta > 0$ such that (II.7.9) holds for all $n \le T(r) := \lfloor \eta r^{\frac{\beta}{\beta+\gamma+1}} \rfloor$. For these n, Lemma II.7.4 guarantees $\det\Omega(\xi_n, \xi_{n-1}) > \frac{1}{16r^2}$. Hence Lemma II.7.6 is applicable and gives

$$\int_{\xi_{T(r)-1}}^1 K(t;r)\,\mathrm{d}t \lesssim r^{\frac{1}{2}} \sum_{j=2}^{T(r)-1} (\xi_{j-1} - \xi_{j+1}) \big\|\varphi'|_{[\xi_{j+1},\xi_{j-1}]}\big\|_\infty^{\frac{1}{2}}$$
$$\asymp r^{\frac{1}{2}} \int_1^{r^{\frac{\beta}{\beta+\gamma+1}}} x^{\frac{\beta-\gamma-1}{2\beta}-1}\,\mathrm{d}x \asymp \begin{cases} r^{\frac{\beta}{\beta+\gamma+1}}, & \beta > \gamma+1; \\ r^{\frac{1}{2}}\log r, & \beta = \gamma+1; \\ r^{\frac{1}{2}}, & \beta < \gamma+1. \end{cases}$$

Proposition II.6.18, (ii), applied to $H|_{[0,\xi_{T(r)-1}]}$, gives

$$\int_{\hat{t}(r)}^{\xi_{T(r)-1}} \frac{h_1(t)}{\omega_1(\hat{s}(t;r),t)}\,\mathrm{d}t \lesssim \xi_{T(r)-1}\cdot r\,\big|\varphi(\xi_{T(r)-1})-\varphi(\xi_{T(r)})\big|$$
$$\asymp r^{-\frac{1}{\beta+\gamma+1}}\cdot r\cdot r^{-\frac{\gamma}{\beta+\gamma+1}} \asymp r^{\frac{\beta}{\beta+\gamma+1}}.$$

If $\beta\le\gamma+1$, then $\frac{\beta}{\beta+\gamma+1}\le\frac{1}{2}$. Together with the bound we obtained for the contribution of $[\xi_{T(r)-1},1]$, we have

$$\log|w_{22}(ir)| \asymp \int_0^1 K(t;r)\,\mathrm{d}t \lesssim \begin{cases} r^{\frac{\beta}{\beta+\gamma+1}}, & \beta>\gamma+1;\\ r^{\frac{1}{2}}\log r, & \beta=\gamma+1;\\ r^{\frac{1}{2}}, & \beta<\gamma+1.\end{cases}$$

Since (II.7.9) is satisfied for $n\le T(r)$, Lemma II.7.5 tells us that

$$\int_0^1 K(t;r)\,\mathrm{d}t \gtrsim \int_{\xi_{T(r)}}^1 K(t;r)\,\mathrm{d}t$$
$$\gtrsim r^{\frac{1}{2}}\sum_{n=2}^{T(r)}|\varphi(\xi_n)-\varphi(\xi_{n-1})|\big\|\varphi'|_{[\xi_n,\xi_{n-1}]}\big\|_\infty^{-\frac{1}{2}} \asymp r^{\frac{1}{2}}\int_1^{r^{\frac{\beta}{\beta+\gamma+1}}} x^{\frac{\beta-\gamma-1}{2\beta}-1}\,\mathrm{d}x$$

which coincides (up to a multiplicative constant) with the upper bound we obtained.

② Case $\gamma<0$. We fix $r>0$. Then there is $\eta>0$ such that (II.7.9) holds for all $n\le T(r):=\lfloor\eta r^{\frac{\beta}{\beta-\gamma+1}}\rfloor$. For these n, Lemma II.7.4 guarantees $\det\Omega(\xi_n,\xi_{n-1})>\frac{1}{16r^2}$. By Lemma II.7.6,

$$\int_{\xi_{T(r)-1}}^1 K(t;r)\,\mathrm{d}t \lesssim r^{\frac{1}{2}}\sum_{j=2}^{T(r)-1}(\xi_{j-1}-\xi_{j+1})\big\|\varphi'|_{[\xi_{j+1},\xi_{j-1}]}\big\|_\infty^{\frac{1}{2}}$$
$$\asymp r^{\frac{1}{2}}\int_1^{r^{\frac{\beta}{\beta-\gamma+1}}} x^{\frac{\beta-\gamma-1}{2\beta}-1}\,\mathrm{d}x \asymp \begin{cases} r^{\frac{\beta-\gamma}{\beta-\gamma+1}}, & \beta>\gamma+1;\\ r^{\frac{1}{2}}\log r, & \beta=\gamma+1;\\ r^{\frac{1}{2}}, & \beta<\gamma+1.\end{cases}$$

Using Proposition II.6.16 with $H|_{[0,\xi_{T(r)-1}]}$, we obtain

$$\int_{\hat{t}(r)}^{\xi_{T(r)-1}} K(t;r)\,\mathrm{d}t \lesssim \xi_{T(r)-1}\cdot r \asymp r^{\frac{\beta-\gamma}{\beta-\gamma+1}}.$$

Since $T(r)$ was chosen such that (II.7.5) holds for $n \leq T(r)$, we obtain from Lemma II.7.5 the lower estimate

$$\begin{aligned}\int_0^1 K(t;r)\,\mathrm{d}t \geq \int_{\xi_{T(r)}}^1 K(t;r)\,\mathrm{d}t &\gtrsim r^{\frac{1}{2}} \sum_{n=2}^{T(r)} |\varphi(\xi_n) - \varphi(\xi_{n-1})| \left\| \varphi'|_{[\xi_n,\xi_{n-1}]} \right\|_\infty^{-\frac{1}{2}} \\ &\asymp r^{\frac{1}{2}} \int_1^{r^{\frac{\beta}{\beta-\gamma+1}}} x^{\frac{\beta-\gamma-1}{2\beta}-1}\,\mathrm{d}x\end{aligned}$$

and this is $\asymp$ to the upper bound.

$\square$

II.7.2 Sharpness of the upper bound

We give a specific example that the upper bound in Theorem II.6.2 can be attained. Let H be the Hamiltonian defined on the interval $[0,1]$ a.e. by

$$H(t) := \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \text{if } e^{-(2n+1)} \leq t < e^{-2n}, \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } e^{-(2n+2)} \leq t < e^{-(2n+1)}, \end{cases} \qquad \text{for } n \geq 0.$$

II.7.7 Proposition. *Let $\kappa_H(r)$ and $K(t;r)$ be as in Theorem II.6.2 where, for definiteness, we use $c := 1$. Then*

$$\kappa_H(r) \asymp \log r, \qquad \int_a^b K(t;r)\,\mathrm{d}t \asymp (\log r)^2.$$

Proof. As a first step we compute $\det \Omega$ on an initial section of the interval $[0,1]$. Let $n \in \mathbb{N}$, then

$$\begin{aligned} \det \Omega(0, e^{-2n}) &= \sum_{l=n}^{\infty} \left[e^{-2l} - e^{-(2l+1)}\right] \cdot \sum_{l=n}^{\infty} \left[e^{-(2l+1)} - e^{-(2l+2)}\right] \\ &= \frac{1}{e}\big(1 - \frac{1}{e}\big)^2 \Big(\sum_{l=n}^{\infty} e^{-2l}\Big)^2 = \alpha e^{-4n}, \end{aligned}$$

where $\alpha := \frac{e}{(e+1)^2}$. This formula yields that

$$\det \Omega(0, e^{-2n}) \leq \frac{1}{r^2} \quad \Leftrightarrow \quad n \geq \frac{1}{2}\log r + \frac{1}{4}\log \alpha.$$

Now note that

$$\det \Omega\big(e^{-(2n+1)}, e^{-2n}\big) = \det \Omega\big(e^{-(2n+2)}, e^{-(2n+1)}\big) = 0.$$

For sufficiently large r the interval

$$\Big[\frac{1}{2}\log r + \frac{1}{4}\log \alpha, \log r\Big]$$

contains at least one integer. Let $n(r)$ be one such, and set

$$s_0 := 0,\ s_1 := e^{-2n(r)}, \quad s_l := e^{-2n(r)+l-1} \text{ for } l = 2, \dots, 2n(r)+1.$$

Then we can apply Remark II.6.7 (i), and obtain that $\kappa_H(r) \leq 2\log r + 1$. In turn it follows that $\int_a^b K(t;r)\,\mathrm{d}t \lesssim (\log r)^2$.

The reverse inequalities are obtained by explicit computation of the integral of $K(t;r)$ over certain subintervals. Let $t \in [e^{-(2n+1)}, e^{-2n}]$. Then we clearly have $\hat{s}(t;r) \le e^{-(2n+1)}$. Furthermore,

$$\det \Omega(e^{-(2n+2)}, t) = (e-1)e^{-(2n+2)}\big(t - e^{-(2n+1)}\big),$$

and hence

$$\hat{s}(t;r) \ge e^{-(2n+2)} \quad \Leftrightarrow \quad t \ge e^{-(2n+1)} + \frac{e^{2n+2}}{r^2(e-1)}.$$

The interval

$$\Big[e^{-(2n+1)} + \frac{e^{2n+2}}{r^2(e-1)}, e^{-2n}\Big]$$

is nonempty, if and only if

$$n \le \frac{1}{2}\log r + \frac{1}{4}\log\beta,$$

where $\beta := \frac{(e-1)^2}{e^3}$. For such n we can estimate

$$\begin{aligned}\int\limits_{e^{-(2n+1)}}^{e^{-2n}} K(t;r)\,\mathrm{d}t &\ge \int\limits_{e^{-(2n+1)}+\frac{e^{2n+2}}{r^2(e-1)}}^{e^{-2n}} \frac{h_1(t)}{\omega_1(\hat{s}(t;r),t)}\,\mathrm{d}t \\ &= \int\limits_{e^{-(2n+1)}+\frac{e^{2n+2}}{r^2(e-1)}}^{e^{-2n}} \frac{1}{t - e^{-(2n+1)}}\,\mathrm{d}t = 2\log r - 4n + \log\beta.\end{aligned}$$

Set $N := \lfloor \frac{1}{4}\log r \rfloor$. Then we have

$$\begin{aligned}\int_a^b K(t;r)\,\mathrm{d}t &\ge \sum_{n=0}^{N} \int_{e^{-(2n+1)}}^{e^{-2n}} K(t;r)\,\mathrm{d}t \ge \sum_{n=0}^{N}\big(2\log r - 4n + \log\beta\big) \\ &\ge (N+1)\cdot 2\log r - N\cdot 4N + (N+1)\cdot\log\beta \\ &\ge \frac{1}{4}(\log r)^2 - \mathrm{O}(\log r).\end{aligned}$$

Having this lower bound for the integral, clearly implies that $\kappa_H(r) \gtrsim \log r$. $\square$

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Chapter III

An upper bound for the Nevanlinna matrix of an indeterminate moment sequence

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An upper bound for the Nevanlinna matrix of an indeterminate moment sequence

Raphael Pruckner ∗ Jakob Reiffenstein ∗ Harald Woracek ‡

Abstract: The solutions of an indeterminate Hamburger moment problem can be parameterised using the Nevanlinna matrix of the problem. The entries of this matrix are entire functions of minimal exponential type, and any growth less than that can occur.
An indeterminate moment problem can be considered as a canonical system in limit circle case by rewriting the three-term recurrence of the problem to a first order vector-valued recurrence. We give a bound for the growth of the Nevanlinna matrix in terms of the parameters of this canonical system. In most situations this bound can be evaluated explicitly. It is sharp in the sense that for well-behaved parameters it coincides with the actual growth of the Nevanlinna matrix up to multiplicative constants.

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III.1 Introduction

Let $(s_n)_{n=0}^\infty$ be a sequence of real numbers. The *Hamburger moment problem* is the task of describing the set

$$\mathcal{M}\big((s_n)_{n=0}^\infty\big) := \left\{ \mu \;\middle|\; \begin{array}{l} \mu \text{ positive Borel measure on } \mathbb{R} \\ s_n = \int_{\mathbb{R}} t^n \, d\mu(t) \text{ for } n = 0,1,2,\dots \end{array} \right\}.$$

This is a classical problem of analysis and was treated extensively in work of H. Hamburger, M. Riesz, R. Nevanlinna, and many others. Standard references are, e.g., [ST43, Akh61, Sch17]. It is known that one of the following alternatives takes place for the set $\mathcal{M}((s_n)_{n=0}^\infty)$: it is either empty, or contains exactly one element, or contains infinitely many elements. In the last case $(s_n)_{n=0}^\infty$ is called an *indeterminate moment sequence*, and this is the case we are concerned with in the present paper.

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For an indeterminate moment sequence $(s_n)_{n=0}^\infty$ the set $\mathcal{M}\big((s_n)_{n=0}^\infty\big)$ can be parameterised: There exist four entire functions A, B, C, D, such that the formula

$$\int_{\mathbb{R}} \frac{1}{t-z}\,\mathrm{d}\mu(t) = \frac{A(z)\tau(z)+B(z)}{C(z)\tau(z)+D(z)} \tag{III.1.1}$$

establishes a bijection between $\mathcal{M}((s_n)_{n=0}^\infty)$ and the set of all functions τ that are analytic in the open upper half-plane $\mathbb{C}^+$ and have nonnegative imaginary part (formally including the function constant equal to ∞). The matrix

$$W(z) := \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}$$

is called the *Nevanlinna matrix* of the sequence $(s_n)_{n=0}^\infty$.

The Nevanlinna matrix admits an operator theoretic interpretation (and this viewpoint could be used to prove (III.1.1)). To the moment sequence there is an associated sequence $(p_n)_{n=0}^\infty$ of orthogonal polynomials that satisfies a three-term recurrence of the form

$$zp_n(z) = b_n p_{n+1}(z) + a_n p_n(z) + b_{n-1}p_{n-1}(z), \qquad n = 0, 1, 2, \dots \tag{III.1.2}$$

with certain parameters $a_n \in \mathbb{R}$ and $b_n > 0$ for $n = 0, 1, 2, \dots$ (and formally $b_{-1} = 0$). The *Jacobi operator* is the closure of the operator

$$(Ju)_n = \begin{cases} b_0u_1 + a_0u_0 & \text{if } n = 0, \\ b_nu_{n+1} + a_nu_n + b_{n-1}u_{n-1} & \text{if } n \geq 1 \end{cases}$$

defined on $\mathcal{D} := \{u \in \ell^2 \mid u_n = 0 \text{ for almost all } n\}$, and (III.1.2) is the formal eigenvalue equation for J. The Jacobi operator is closed and symmetric, and has deficiency index $(1, 1)$. Hence, the self-adjoint extensions of the Jacobi operator (we write again J for simplicity) are described by M.G. Krein's resolvent formula. The Nevanlinna matrix of $(s_n)_{n=0}^\infty$ is precisely the u-resolvent matrix of J for a certain generating element u.

The entries A, B, C, D of the Nevanlinna matrix all have the same growth [BP94], and a classical theorem of M. Riesz in [Rie23] states that they are of minimal exponential type. Moreover, it is known that any growth smaller than that may occur, e.g. [BS98]. Revealing more refined information about the growth of the Nevanlinna matrix is an intricate problem. It is of great interest also for spectral theoretic reasons: due to the above interpretation of W as a resolvent matrix, the set of zeros of D coincides with the spectrum of a particular self-adjoint extension of J. Hence, if the growth of W is known, information about the distribution of eigenvalues can be obtained using standard tools from complex analysis.

In the present paper we give an upper bound for

$$M(r) := \max_{|z|=r} \|W(z)\|$$

of which there are two versions: First, a general formulation where any (possibly rough) data is admitted, and second, a much more explicit bound for data satisfying mild regularity conditions. If, in addition, the data decays sufficiently fast, then the upper bound coincides with $M(r)$ up to multiplicative constants.

So far, we talked about two different (equivalent) objects, i.e., the moment sequence and the Jacobi operator. Our method actually relies on a third object – a *canonical system* of differential equations. It is obtained from rewriting the three-term recurrence (III.1.2) as a first order vector difference equation and interpreting this as a discrete differential equation. From the fundamental solution of the canonical system, we obtain its *monodromy matrix* which again coincides with the Nevanlinna matrix W.

The canonical systems occurring in the context of moment problems are represented by a *Hamiltonian* which reflects the discrete nature of the difference equation. It is determined by two sequences, its *lengths* $l_j > 0$ and *angles* $\phi_j \in \mathbb{R}$, which we call the *Hamiltonian parameters* (details are given below, cf. Definition III.1.1). This model for an indeterminate moment sequence is well suited for the study of various properties. For example, the moment sequence is indeterminate if and only if $(l_j)_{j=1}^{\infty}$ is summable. Our results will almost exclusively be formulated in terms of the Hamiltonian parameters.

Let us briefly review some earlier results on the growth of Nevanlinna matrices. The history of the subject starts probably with a theorem of M.S. Livšic in [Liv39] that gives a lower bound for $M(r)$ in terms of the moment sequence itself. This bound is easy to handle, but will usually not give the correct size. In recent work of C. Berg and R. Szwarc [BS14] it is shown that the order and type of W

$$\rho := \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r}, \qquad \tau := \limsup_{r\to\infty} \frac{\log M(r)}{r^{\rho}},$$

coincide with those of a certain entire function built in a complicated way from the coefficients of the orthonormal polynomials. A theorem which takes the Jacobi parameters as input is due to Yu.M. Berezanskii [Ber56] and was generalised in [BS14]. It states that the order of W coincides with the convergence exponent of the sequence $(b_n)_{n=0}^{\infty}$, but the assumptions are very restrictive, involving regularity of b_n and smallness of a_n. Bounds for the order of W in terms of the Hamiltonian parameters, which under certain conditions give the correct value, are obtained in [PRW17].

In our present theorems we start with the Hamiltonian parameters, and give bounds for the type of W with respect to a general comparison function (e.g., a proximate order in the sense of Valiron). This improves significantly upon [PRW17] in several ways: we work on a much more refined scale of measuring growth, we obtain type estimates, and in some situations our bound improves the earlier results even on the rough scale of order. Several cases occur, which are presented in Theorem III.4.6. The path that leads to this result is divided into two main sections: First, we prove a very general, albeit complicated, upper bound in Theorem III.2.2. Second, in Theorem III.4.1 we use J. Karamata's theory of regularly varying functions to evaluate this general bound. For the convenience of the reader, the significantly simpler case of usual order and type is covered separately in Corollary III.2.5. We show in Theorem III.5.3 that the upper bound is attained if the lengths and angle differences are themselves close to regularly varying and decay sufficiently fast (corresponding to order less than $\frac{1}{2}$). In this case the growth of $\log M(r)$ is fully determined up to multiplicative constants.

The proof of our foundational theorem is based on a somewhat tricky application of a recent result from [PW23], approximating the target Hamiltonian with a finite dimensional one. Building upon that, a detailed (and partly tedious) analysis of functions follows. We recommend the reader to get an overall picture before diving into the actual estimates.

Canonical systems with Hamburger Hamiltonian

A two-dimensional canonical system is an equation of the form

$$y'(t) = zJH(t)y(t), \qquad t \in (a,b) \text{ a.e.},$$

where

- $-\infty \le a < b \le \infty$,
- $H \in L^1_{\mathrm{loc}}\big((a,b), \mathbb{R}^{2\times 2}\big)$, and $H(t) \ge 0$ and $H(t) \ne 0$ for $t \in (a,b)$ a.e.,
- $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $z \in \mathbb{C}$,

and the solution $y\colon (a,b) \to \mathbb{C}^2$ is required to be locally absolutely continuous. The function H is called the *Hamiltonian* of the system.

Canonical systems that occur from indeterminate moment sequences are those whose Hamiltonian has the following particular – discrete – form. Here we denote

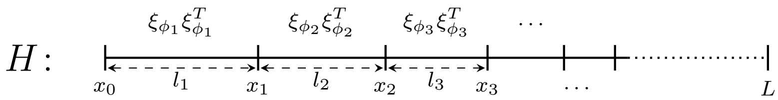
$$\xi_\phi := \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}, \qquad \phi \in \mathbb{R}.$$

III.1.1 Definition. Let $(l_j)_{j=1}^\infty$ be a summable sequence of positive numbers and $(\phi_j)_{j=1}^\infty$ be a sequence of real numbers. Set $x_0 := 0$, $x_j := \sum_{i=1}^j l_i$ for $j \geq 1$, and $L := \sum_{j=1}^\infty l_j < \infty$. The *Hamburger Hamiltonian* with *lengths* l_j and *angles* ϕ_j is the function H defined on the interval $(0, L)$ as

$$H(t) := \xi_{\phi_j}\xi_{\phi_j}^T \quad \text{for } j \in \mathbb{N} \text{ and } x_{j-1} \leq t < x_j.$$

◀

A Hamburger Hamiltonian H thus can be pictured as



Since a Hamburger Hamiltonian in the sense of the above definition is even integrable on the whole interval $(0, L)$, there exists a unique 2×2 matrix-valued solution $W\colon [0, L] \times \mathbb{C} \to \mathbb{C}^{2\times 2}$ of the initial value problem

$$\begin{cases} \frac{\partial}{\partial t} W(t; z)J = zW(t; z)H(t), & t \in (0, L) \text{ a.e.}, \\ W(0; z) = I. \end{cases}$$

We refer to W as the *fundamental solution* of H, and to the matrix $W_H(z) := W(L; z)$ as its *monodromy matrix*.

A notational convention

We frequently compare functions up to multiplicative constants or asymptotically, and throughout the paper use the following notation.

III.1.2 Notation. Let X be a set and $f, g\colon X \to (0, \infty)$.

(i) We write "$f \lesssim g$" (or "$f(x) \lesssim g(x)$") to say that there exists a constant $C > 0$ such that $f(x) \leq C \cdot g(x)$ for all $x \in X$. We write "$f \gtrsim g$" if $g \lesssim f$, and "$f \asymp g$" if $f \lesssim g$ and $f \gtrsim g$.

(ii) Assume that X is directed. We say that "$f \lesssim g$ *for sufficiently large* x", if there exists $x_0 \in X$ such that $f|_Y \lesssim g|_Y$ where $Y := \{x \in X \mid x \succeq x_0\}$.

(iii) Assume again that X is directed. We write "$f \sim g$" if $\lim_{x\in X} \frac{f(x)}{g(x)} = 1$, and "$f \ll g$" if $\lim_{x\in X} \frac{f(x)}{g(x)} = 0$, and "$f \approx g$" if $\lim_{x\in X} \frac{f(x)}{g(x)}$ exists in $(0,\infty)$.

(iv) Assume that X is a subset of a topological space. We say that "$f \lesssim g$ *locally*", if $f|_K \lesssim g|_K$ for every compact subset K of X. Analogous wording applies to "$\asymp$".

◀

For the convenience of the reader, we include an appendix where the definition and some basic results about regularly varying functions in Karamata sense are recalled.

III.2 An upper bound for the monodromy matrix

In this section we give a generic upper bound for $\log \|W_H(z)\|$. The functions $\text{đ}_l, \text{đ}_\phi$ used below play the role of well-behaved comparison functions for the lengths and for the differences of consecutive angles of the Hamiltonian.

III.2.1 Definition. Let $\text{đ}_l\colon [1,\infty) \to (0,\infty)$ and $\text{đ}_\phi\colon [1,\infty) \to (0,1]$ be measurable with $\text{đ}_l \asymp 1 \asymp \text{đ}_\phi$ locally. Then we denote

$$\text{ꝁ}(R) := \sup\Big\{t \in [1,\infty) \mid \sup_{1\leq s\leq t} \frac{2}{R(\text{đ}_l\text{đ}_\phi)(s)} \leq 1\Big\}, \tag{III.2.1}$$

$$\hbar(R) := \sup\Big\{t \in [1,\infty) \mid \sup_{1\leq s\leq t} \frac{\text{đ}_\phi(s)}{R\text{đ}_l(s)} \leq 1\Big\}, \tag{III.2.2}$$

where $R \in [\frac{2}{(\text{đ}_l\text{đ}_\phi)(1)},\infty)$ and $R \in [\frac{\text{đ}_\phi(1)}{\text{đ}_l(1)},\infty)$, respectively. Further, we set

$$g(t,R) := \begin{cases} \log\big(R(\text{đ}_l\text{đ}_\phi)(t)\big) & \text{if } 1 \leq t < \text{ꝁ}(R),\\ R^{\frac{1}{2}}(\text{đ}_l\text{đ}_\phi)^{\frac{1}{2}}(t) & \text{if } \text{ꝁ}(R) \leq t < \hbar(R),\\ R\text{đ}_l(t) & \text{if } \hbar(R) \leq t, \end{cases}$$

where $(t,R) \in [1,\infty) \times [\frac{2}{(\text{đ}_l\text{đ}_\phi)(1)},\infty)$, and

$$\text{ǥ}(t,R) = \int_1^t g(s,R)\,\mathrm{d}s. \tag{III.2.3}$$

◀

Note here that $\mathscr{d}_\phi(t) \leq 1$, and hence $\mathscr{k}(R) \leq \hbar(R)$ for all $R \geq \frac{2}{(d_l \mathscr{d}_\phi)(1)}$.

III.2.2 Theorem. *Let $(l_j)_{j=1}^\infty$ be a summable sequence of positive numbers, and $(\phi_j)_{j=1}^\infty$ a sequence of real numbers. Denote by H the Hamburger Hamiltonian with these lengths and angles, and let W_H be its monodromy matrix.*

Let $\psi \in \mathbb{R}$ and let $\mathscr{d}_l, \mathscr{d}_\phi, \mathscr{c}_l, \mathscr{c}_\phi \colon [1, \infty) \to (0, \infty)$ be measurable and nonincreasing, with $\mathscr{d}_\phi \leq 1$ and $\mathscr{c}_\phi \leq \mathscr{c}_l$. Assume that

$$\forall j \in \mathbb{N}.\ l_j \leq \mathscr{d}_l(j) \ \wedge \ |\sin(\phi_{j+1} - \phi_j)| \leq \mathscr{d}_\phi(j), \tag{III.2.4}$$

$$\forall N \in \mathbb{N}.\ \sum_{j=N+1}^\infty l_j \leq \mathscr{c}_l(N) \ \wedge \ \sum_{j=N+1}^\infty l_j \sin^2(\phi_j - \psi) \leq \mathscr{c}_\phi(N). \tag{III.2.5}$$

Let $\mathscr{g}(t, R)$ be as in (III.2.3), and set

$$\begin{aligned} L(t, R) &:= 1 + \log^+ R + \log^+ \frac{\mathscr{c}_l(\lceil t \rceil)}{\mathscr{c}_\phi(\lceil t \rceil)} + \log^+ \frac{\mathscr{d}_l(1)}{\mathscr{d}_\phi(1)} \\ &+ \log^+ \frac{\mathscr{d}_l(\min\{\lceil t \rceil, \lfloor \hbar(R) \rfloor\})}{\mathscr{d}_\phi(\min\{\lceil t \rceil, \lfloor \hbar(R) \rfloor\})} + \sum_{j=1}^{\min\{\lceil t \rceil, \lfloor \hbar(R) \rfloor\} - 1} \left| \log \left(\frac{\mathscr{d}_\phi(j)}{\mathscr{d}_l(j)} \Big/ \frac{\mathscr{d}_\phi(j+1)}{\mathscr{d}_l(j+1)} \right) \right|. \end{aligned} \tag{III.2.6}$$

Then we have, for all $R \geq \frac{2}{(d_l \mathscr{d}_\phi)(1)}$,

$$\log \Big(\max_{|z|=R} \|W_H(z)\| \Big) \leq 9 \cdot \inf_{t \geq 1} \Big(\max \Big\{ \mathscr{g}(t, R), R(\mathscr{c}_l \mathscr{c}_\phi)^{\frac{1}{2}}(t) \Big\} + L(t, R) \Big). \tag{III.2.7}$$

III.2.3 Remark. The terms appearing in the upper bound in (III.2.7) have the following meaning:

(i) $\mathscr{g}(t, R)$ estimates the contribution of the first $\lceil t \rceil$ intervals of H;

(ii) $R(\mathscr{c}_l \mathscr{c}_\phi)^{\frac{1}{2}}(t)$ estimates the contribution of the remaining intervals of H;

(iii) $L(t, R)$ is usually a remainder term.

$\triangleleft$

We mention two possible scenarios. If lengths and angle differences are nonincreasing but decay slowly, choosing $\mathscr{d}_l(j), \mathscr{d}_\phi(j)$ such that equality holds in (III.2.4) makes $\mathscr{g}(t, R)$ a rather precise bound for the contribution of the first $\lceil t \rceil$ intervals. On the other hand, if lengths and angle differences behave irregularly, then $\mathscr{g}(t, R)$ likely overestimates the contribution of the first $\lceil t \rceil$ intervals. However, choosing $\mathscr{c}_l, \mathscr{c}_\phi$ such that equality holds in (III.2.5), the bound (III.2.7) is still a good estimate for $\log \|W_H(z)\|$ as long as the decay is sufficiently fast.

If $\lim_{t\to\infty}(\mathfrak{c}_l\mathfrak{c}_\phi)(t) > 0$, the bound (III.2.7) is $\gtrsim R$ and thus trivial: we know that W_H is of minimal exponential type by the classical Krein-de Branges formula. Hence we may safely assume that $\lim_{t\to\infty}(\mathfrak{c}_l\mathfrak{c}_\phi)(t) = 0$ whenever this is convenient.

The following lemma hints at a way to evaluate the upper bound in Theorem III.2.2.

III.2.4 Lemma. *Let $\mathfrak{d}_l\colon [1,\infty) \to (0,\infty)$ and $\mathfrak{d}_\phi\colon [1,\infty) \to (0,1]$ be measurable with $\mathfrak{d}_l \asymp 1 \asymp \mathfrak{d}_\phi$ locally, and let $\mathfrak{c}_l, \mathfrak{c}_\phi$ be continuous and nonincreasing with $\lim\limits_{t\to\infty}(\mathfrak{c}_l\mathfrak{c}_\phi)(t) = 0$. Then the equation*

$$\mathfrak{g}(t,R) = R(\mathfrak{c}_l\mathfrak{c}_\phi)^{\frac{1}{2}}(t), \tag{III.2.8}$$

has a unique solution $T(R)$, and

$$\min_{t\geq 1} \max\left\{\mathfrak{g}(t,R), R(\mathfrak{c}_l\mathfrak{c}_\phi)^{\frac{1}{2}}(t)\right\} = \mathfrak{g}(T(R),R) = R(\mathfrak{c}_l\mathfrak{c}_\phi)^{\frac{1}{2}}(T(R)).$$

In addition, $\lim_{R\to\infty} T(R) = \infty$.

Proof. For each fixed R the function $t \mapsto \mathfrak{g}(t,R)$ is continuous and increasing while $t \mapsto R(\mathfrak{c}_l\mathfrak{c}_\phi)^{\frac{1}{2}}(t)$ is continuous and nonincreasing. Moreover, $\mathfrak{g}(1,R) = 0 < R(\mathfrak{c}_l\mathfrak{c}_\phi)^{\frac{1}{2}}(1)$ and $\lim_{t\to\infty} R(\mathfrak{c}_l\mathfrak{c}_\phi)^{\frac{1}{2}}(t) = 0$. By the intermediate value theorem, (III.2.8) has a unique solution $T(R)$.

If $\liminf_{R\to\infty} T(R) < \infty$, choose a sequence $(R_n)_{n\in\mathbb{N}}$ such that $(T(R_n))_{n\in\mathbb{N}}$ is bounded. Using the crude estimate

$$\mathfrak{g}(t,R) \leq t\log\big(R(\mathfrak{d}_l\mathfrak{d}_\phi)(1)\big) + \sqrt{2}\,t + t \lesssim t\log R,$$

this leads to the contradiction

$$R(\mathfrak{c}_l\mathfrak{c}_\phi)^{\frac{1}{2}}(T(R_n)) = \mathfrak{g}(T(R_n),R) \lesssim T(R_n)\cdot\log R \ll R \asymp R(\mathfrak{c}_l\mathfrak{c}_\phi)^{\frac{1}{2}}(T(R_n)).$$

□

In many situations, we are going to first determine $T(R)$, and then show that $L(T(R),R)$ is small. The bound (III.2.7) is then asymptotically equal to $R(\mathfrak{c}_l\mathfrak{c}_\phi)^{\frac{1}{2}}(T(R))$.

Before we go into the proof of Theorem III.2.2, let us discuss a particular situation.

III.2.5 Corollary. *Let $(l_j)_{j=1}^{\infty}$ be a summable sequence of positive numbers, and $(\phi_j)_{j=1}^{\infty}$ a sequence of real numbers. Denote by H the Hamburger Hamiltonian with these lengths and angles, and let W_H be its monodromy matrix.*

Assume that we have $\delta_l, \delta_\phi, \gamma_l, \gamma_\phi \geq 0$, such that

$$\forall j \in \mathbb{N}.\ l_j \lesssim j^{-\delta_l} \ \wedge\ |\sin(\phi_{j+1} - \phi_j)| \lesssim j^{-\delta_\phi},$$
$$\forall N \in \mathbb{N}.\ \sum_{j=N+1}^{\infty} l_j \lesssim N^{-\gamma_l} \ \wedge\ \sum_{j=N+1}^{\infty} l_j \sin^2(\phi_j - \psi) \lesssim N^{-\gamma_\phi},$$

i.e., we are in the situation of Theorem III.2.2 with

$$d_l(t) = ct^{-\delta_l},\ d_\phi(t) = ct^{-\delta_\phi},\ c_l(t) = ct^{-\gamma_l},\ c_\phi(t) = ct^{-\gamma_\phi},$$

where c is some positive constant. Set $\delta := \delta_l + \delta_\phi$, $\gamma := \frac{1}{2}(\gamma_l + \gamma_\phi)$, and assume that $\delta > 0$ and $\gamma > 0$. Then we have, for sufficiently large R, the following bounds for $T(R)$ (cf. Lemma III.2.4), $\log \max_{|z|=R} \|W_H(z)\|$ and the order ρ_H of W_H.

Data satisfies			$T(R) \asymp$	$\log \max\limits_{\lvert z\rvert=R} \|W_H(z)\| \lesssim$	$\rho_H \leq$
$\delta < 1+\gamma$			$\left(\frac{R}{\log R}\right)^{\frac{1}{1+\gamma}}$	$R^{\frac{1}{1+\gamma}}(\log R)^{\frac{\gamma}{1+\gamma}}$	$\frac{1}{1+\gamma}$
$\delta = 1+\gamma$			$R^{\frac{1}{\delta}}$	$R^{\frac{1}{\delta}}$	$\frac{1}{\delta}$
$1+\gamma < \delta,$	$\delta > 2,$		$R^{\frac{\delta-1}{\gamma\delta}}$	$R^{\frac{1}{\delta}}$	$\frac{1}{\delta}$
	$\delta = 2,$		$\left(\frac{R^{\frac{1}{2}}}{\log R}\right)^{\frac{1}{\gamma}}$	$R^{\frac{1}{2}} \log R$	$\frac{1}{2}$
	$\delta < 2,$	$\delta_l \leq 1+\gamma$	$R^{\frac{1}{2-\delta+2\gamma}}$	$R^{\frac{2-\delta+\gamma}{2-\delta+2\gamma}}$	$\frac{2-\delta+\gamma}{2-\delta+2\gamma}$
		$1+\gamma < \delta_l$	$R^{\frac{\delta_l-1}{\gamma(\delta_l-\delta_\phi)}}$	$R^{\frac{1-\delta_\phi}{\delta_l-\delta_\phi}}$	$\frac{1-\delta_\phi}{\delta_l-\delta_\phi}$

Proof. To start with we observe

$$k(R) = \left(\frac{c^2 R}{2}\right)^{\frac{1}{\delta}}, \quad h(R) = \begin{cases} R^{\frac{1}{\delta_l - \delta_\phi}} & \text{if } \delta_l > \delta_\phi, \\ \infty & \text{otherwise} \end{cases},$$
$$R(c_l c_\phi)^{\frac{1}{2}}(t) = Rt^{-\gamma}.$$

The main effort is to evaluate $g(t, R)$, and this is where the case distinctions come from.

▷ Range $t \in [1, \not{k}(R)]$:

$$\begin{aligned} \not{g}(t,R) &= \int_1^t \log\left(c^2 R s^{-\delta}\right) \mathrm{d}s = t\log(c^2R) - \delta \int_1^t \log s \,\mathrm{d}s \\ &= t\log(c^2R) - \delta\big(t\log t - t + 1\big) = t\log\left(c^2 R t^{-\delta}\right) + \delta t - \delta. \end{aligned} \tag{III.2.9}$$

In particular,

$$\begin{aligned} \not{g}(\not{k}(R), R) &= \not{k}(R)\log\big(\underbrace{R\not{k}(R)^{-\delta}}_{=2}\big) + \delta\not{k}(R) - \delta \\ &= (\log 2 + \delta)\not{k}(R) - \delta \asymp \not{k}(R). \end{aligned}$$

▷ Range $t \in [\not{k}(R), \hbar(R)]$:

$$\int_{\not{k}(R)}^t s^{-\frac{\delta}{2}}\,\mathrm{d}s = \begin{cases} \frac{1}{1-\frac{\delta}{2}}\left(t^{1-\frac{\delta}{2}} - \not{k}(R)^{1-\frac{\delta}{2}}\right) & \text{if } \delta < 2, \\ \log\frac{t}{\not{k}(R)} & \text{if } \delta = 2, \\ \frac{1}{\frac{\delta}{2}-1}\left(\not{k}(R)^{1-\frac{\delta}{2}} - t^{1-\frac{\delta}{2}}\right) & \text{if } \delta > 2. \end{cases}$$

Let $\sigma > \frac{1}{\delta}$ and $\sigma \leq \frac{1}{\delta_l - \delta_\phi}$ in case $\delta_l > \delta_\phi$. Then

$$\not{g}(R^\sigma, R) \asymp R^{\frac{1}{\delta}} + R^{\frac{1}{2}} \cdot \begin{cases} R^{\sigma(1-\frac{\delta}{2})} & \text{if } \delta < 2, \\ \log R & \text{if } \delta = 2, \\ R^{\frac{1}{\delta}(1-\frac{\delta}{2})} & \text{if } \delta > 2. \end{cases}$$

Observe that for $\delta < 2$

$$\frac{1}{\delta} = \frac{1}{2} + \frac{1}{\delta}\big(1 - \frac{\delta}{2}\big) < \frac{1}{2} + \sigma\big(1 - \frac{\delta}{2}\big).$$

Hence,

$$\not{g}(R^\sigma, R) \asymp \begin{cases} R^{\frac{1}{2}+\sigma(1-\frac{\delta}{2})} & \text{if } \delta < 2, \\ R^{\frac{1}{2}}\log R & \text{if } \delta = 2, \\ R^{\frac{1}{\delta}} & \text{if } \delta > 2. \end{cases} \tag{III.2.10}$$

▷ Range $t \in [\hbar(R), \infty)$:
This case can occur only if $\delta_l > \delta_\phi$ (since otherwise $\hbar(R) = \infty$).

$$\int_{\hbar(R)}^t s^{-\delta_l}\,\mathrm{d}s = \begin{cases} \frac{1}{1-\delta_l}\left(t^{1-\delta_l} - \hbar(R)^{1-\delta_l}\right) & \text{if } \delta_l < 1, \\ \log\frac{t}{\hbar(R)} & \text{if } \delta_l = 1, \\ \frac{1}{\delta_l - 1}\left(\hbar(R)^{1-\delta_l} - t^{1-\delta_l}\right) & \text{if } \delta_l > 1. \end{cases}$$

Let $\sigma > \frac{1}{\delta_l - \delta_\phi}$. Then

$$\mathcal{g}(R^\sigma, R) \asymp \begin{cases} R^{\frac{1-\delta_\phi}{\delta_l-\delta_\phi}} + R^{1+\sigma(1-\delta_l)} & \text{if } \delta < 2, \delta_l < 1, \\ R^{\frac{1-\delta_\phi}{\delta_l-\delta_\phi}} + R\log R & \text{if } \delta < 2, \delta_l = 1, \\ R^{\frac{1-\delta_\phi}{\delta_l-\delta_\phi}} + R^{1+\frac{1-\delta_l}{\delta_l-\delta_\phi}} & \text{if } \delta < 2, \delta_l > 1, \\ R^{\frac{1}{2}}\log R + R^{1+\frac{1-\delta_l}{\delta_l-\delta_\phi}} & \text{if } \delta = 2 \quad (\Rightarrow \delta_l > 1), \\ R^{\frac{1}{\delta}} + R^{1+\frac{1-\delta_l}{\delta_l-\delta_\phi}} & \text{if } \delta > 2 \quad (\Rightarrow \delta_l > 1). \end{cases}$$

Observe that

$$\delta_l < 1, \sigma > \frac{1}{\delta_l - \delta_\phi} \;\Rightarrow\; \frac{1-\delta_\phi}{\delta_l-\delta_\phi} < 1 + \sigma(1-\delta_l)$$

since

$$\begin{aligned} &\frac{1-\delta_\phi}{\delta_l-\delta_\phi} < 1 + \sigma(1-\delta_l) \\ &\qquad \Leftrightarrow 1 - \delta_\phi < \delta_l - \delta_\phi + \sigma(1-\delta_l)(\delta_l - \delta_\phi) \\ &\qquad \Leftrightarrow 1 - \delta_l < \sigma(1-\delta_l)(\delta_l - \delta_\phi) \\ &\qquad \Leftrightarrow 1 < \sigma(\delta_l - \delta_\phi) \end{aligned}$$

Next,

$$\delta \geq 2 \;\Rightarrow\; \frac{1-\delta_\phi}{\delta_l-\delta_\phi} \leq \frac{1}{\delta}$$

since

$$\begin{aligned} &\frac{1-\delta_\phi}{\delta_l-\delta_\phi} \leq \frac{1}{\delta} \\ &\qquad \Leftrightarrow (1-\delta_\phi)(\delta_l + \delta_\phi) \leq \delta_l - \delta_\phi \\ &\qquad \Leftrightarrow 0 \leq \delta_\phi(\delta - 2) \end{aligned}$$

It follows that

$$\mathcal{g}(R^\sigma, R) \asymp \begin{cases} R^{1+\sigma(1-\delta_l)} & \text{if } \delta < 2, \delta_l < 1, \\ R\log R & \text{if } \delta < 2, \delta_l = 1, \\ R^{\frac{1-\delta_\phi}{\delta_l-\delta_\phi}} & \text{if } \delta < 2, \delta_l > 1, \\ R^{\frac{1}{2}}\log R & \text{if } \delta = 2, \\ R^{\frac{1}{\delta}} & \text{if } \delta > 2. \end{cases} \tag{III.2.11}$$

Denote by $\tau(R)$ the function (whose form depends on the considered case) given in the second column of the table in Corollary III.2.5. Using (III.2.9), (III.2.10), and (III.2.11), we see that

$$\mathscr{g}(\tau(R),R) \asymp R\tau(R)^{-\gamma} = R(\mathscr{c}_l\mathscr{c}_\phi)^{\frac{1}{2}}(\tau(R)) \tag{III.2.12}$$

and hence $T(R) \asymp \tau(R)$. Moreover, the upper bound (III.2.7) for $\log\max_{|z|=R}\|W_H(z)\|$ is $\asymp$ to all terms in (III.2.12).

It remains to check that $L(T(R),R)$ is small. First, observe that the function $\frac{d_\phi}{d_l}$ is just a power and thus monotone. Hence, the last summand of $L(t,R)$ is a telescoping sum. Using the facts that also $\frac{c_l}{c_\phi}$ is a power and that $T(R)$ is, in each case, bounded from below and above by some positive power, we have $L(T(R),R) \lesssim \log R$. ☐

Corollary III.2.5 already shows that our present results may improve drastically upon our previous work [PRW17]. The main feature that enables this is the use of $\mathscr{c}_\phi$: the present results exploit convergence of angles more efficiently.

III.2.6 Example. Let $\alpha > 1$ and $\beta \geq 0$, and let H be the Hamburger Hamiltonian with lengths and angles

$$l_j := j^{-\alpha}, \quad \phi_j := (-1)^j j^{-\beta}.$$

Considering the expressions (III.2.4) and (III.2.5), we see that Corollary III.2.5 can be applied with

$$\delta_l := \alpha,\ \delta_\phi := \beta,\ \gamma_l := \alpha - 1,\ \gamma_\phi := \alpha + 2\beta - 1.$$

Since $\delta = \alpha + \beta$ and $\gamma = \alpha + \beta - 1$, this yields

$$\log\Big(\max|z| = R\|W_H(z)\|\Big) \lesssim R^{\frac{1}{\alpha+\beta}},$$

and hence $\rho_H \leq \frac{1}{\alpha+\beta}$. Now recall [PRW17, Example 2.23], where we saw that the order ρ_H is at least $\frac{1}{\alpha+\beta}$. Thus, $\rho_H = \frac{1}{\alpha+\beta}$.

Let us compare this with what we can obtain from [PRW17]. For $\alpha + \beta \geq 2$, the formula $\rho_H = \frac{1}{\alpha+\beta}$ was already established in [PRW17, Example 2.23]. For $\alpha+\beta < 2$, we had obtained the upper bound $\frac{1-\beta}{\alpha-\beta}$ in that place. However, this did not take into account that angles converge. If we want to take convergence into account, we should use [PRW17, Corollary 2.9]. The quantities used there identify as

$$\Delta_l^+ = \alpha, \quad \Delta_\phi^* = \Lambda^* = \beta,$$

and the bound for order so obtained thus is $\frac{1-\frac{\beta}{2}}{\alpha}$. If $\beta > 0$, this lies strictly between $\frac{1}{\alpha+\beta}$ and $\frac{1-\beta}{\alpha-\beta}$, since

$$\frac{1-\beta}{\alpha-\beta} - \frac{1-\frac{\beta}{2}}{\alpha} = \frac{\frac{\beta}{2}(2-\alpha-\beta)}{(\alpha-\beta)\alpha} > 0,$$
$$\frac{1-\frac{\beta}{2}}{\alpha} - \frac{1}{\alpha+\beta} = \frac{\frac{\beta}{2}(2-\alpha-\beta)}{\alpha(\alpha+\beta)} > 0.$$

III.2.1 Proof of Theorem III.2.2

Denote by $W(s,t;z)$ the unique solution of the initial value problem

$$\begin{cases} \frac{\partial}{\partial t} W(s,t;z)J = zW(s,t;z)H(t), & t \in (0,L) \text{ a.e.,} \\ W(s,s;z) = I. \end{cases}$$

and observe the multiplicativity property

$$W(s,t;z)W(t,u;z) = W(s,u;z).$$

Our method for the proof of Theorem III.2.2 is to fix $t \geq 1$ and estimate $\|W(0,x_{\lceil t \rceil};z)\|$ and $\|W(x_{\lceil t \rceil},L;z)\|$ separately. The contribution of the first $\lceil t \rceil$ intervals is estimated using the explicit form of the fundamental solution on indivisible intervals. On the remaining part, we use Grönwall's Lemma. This method is a refined version of an idea from [Rom17] and its improved variant from [PW23].

Set $W_j(z) := W(x_{j-1},x_j;z)$, then by a direct computation

$$W_j(z) = I + zl_j\xi_{\phi_j}\xi_{\phi_j}^T J.$$

Moreover, multiplicativity yields

$$W(0,x_N;z) = W_1(z) \cdot \ldots \cdot W_N(z), \qquad N \in \mathbb{N}. \tag{III.2.13}$$

We often use the matrices introduced in [PW23, Definition 4.4]: for $a > 0$ and $\psi \in \mathbb{R}$, denote

$$\Omega(a,\psi) := \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \exp(-\psi J) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix}.$$

Recall the following properties from [PW23, Lemma 4.6].

III.2.7 Lemma. *Let $a, b > 0$ and $\phi, \psi \in \mathbb{R}$.*

(i) $\|\Omega(a,\psi)\| = \|\Omega(a,\psi)^{-1}\| = \max\{a, a^{-1}\}$;

(ii) $\|\Omega(a,\psi)\xi_\phi\xi_\phi^T J\Omega(a,\psi)^{-1}\| = a^2\cos^2(\phi-\psi) + \frac{1}{a^2}\sin^2(\phi-\psi)$;

(iii) $\|\Omega(a,\psi)\Omega(b,\phi)^{-1}\| \leq \max\left\{\frac{a}{b}, \frac{b}{a}\right\}|\cos(\phi-\psi)| + \max\left\{ab, \frac{1}{ab}\right\}|\sin(\phi-\psi)|$.

We use $\Omega(a_j,\psi_j)$ to rotate and dilate $W_j(z)$ such that the norm becomes small. In the following lemma, we make the obvious choice $\psi_j = \phi_j$, but retain the freedom of choosing suitable parameters a_j.

III.2.8 Lemma. *Let $N \in \mathbb{N}$ and let numbers $a_j \in (0,1]$, $j = 1,\dots,N$, be given. Then, for each $z \in \mathbb{C}$, we have the estimate*

$$\|W(0,x_N;z)\| \leq \frac{1}{a_1 a_N} \cdot \prod_{j=1}^{N}\Big(1 + |z| l_j a_j^2\Big) \tag{III.2.14}$$
$$\cdot \prod_{j=1}^{N-1}\left(\max\left\{\frac{a_j}{a_{j+1}}, \frac{a_{j+1}}{a_j}\right\} \cdot \big|\cos\big(\phi_j - \phi_{j+1}\big)\big| + \frac{|\sin(\phi_j - \phi_{j+1})|}{a_j a_{j+1}}\right).$$

Proof. With $\Omega_j := \Omega(a_j,\phi_j)$ we have

$$\Omega_j W_j(z)\Omega_j^{-1} = I + z l_j \Omega_j \xi_{\phi_j}\xi_{\phi_j}^T J\Omega_j^{-1},$$

and we can use Lemma III.2.7, (ii), to estimate

$$\|\Omega_j W_j(z)\Omega_j^{-1}\| \leq 1 + |z| l_j a_j^2.$$

Expanding on (III.2.13),

$$W(0,x_N;z) = \Omega_1^{-1} \cdot (\Omega_1 W_1(z)\Omega_1^{-1}) \cdot \Omega_1\Omega_2^{-1} \cdot \ldots \cdot (\Omega_N W_N(z)\Omega_N^{-1}) \cdot \Omega_N.$$

Using submultiplicativity of the norm and Lemma III.2.7, (i), (iii), we arrive at the desired estimate. □

The next lemma is the ingredient of Theorem III.2.2 that accounts for the term $\mathfrak{g}(t,R)$ in (III.2.7).

III.2.9 Lemma. *Let $(l_j)_{j=1}^\infty$ be a summable sequence of positive numbers, and $(\phi_j)_{j=1}^\infty$ a sequence of real numbers. Let H be the Hamburger Hamiltonian with these lengths and angles. Let $\mathfrak{d}_l\colon [1,\infty) \to (0,\infty)$ and $\mathfrak{d}_\phi\colon [1,\infty) \to (0,1]$ be measurable and nonincreasing, and assume that*

$$\forall j \in \mathbb{N}.\ l_j \leq \mathfrak{d}_l(j) \ \wedge\ |\sin(\phi_{j+1} - \phi_j)| \leq \mathfrak{d}_\phi(j).$$

Then, for every $(N, R) \in \mathbb{N} \times \left[\frac{2}{(d_l d_\phi)(1)}, \infty\right)$,

$$\log\Big(\max_{|z|=R} \|W(0, x_N; z)\|\Big) \leq \tfrac{7}{2} \cdot \Big[g(N, R) + \lambda\big(\min\{N, \hbar(R)\}, R\big)\Big]$$

where

$$\lambda(t, R) := 1+\log^+ R+\log^+ \frac{d_l(1)}{d_\phi(1)}+\log^+ \frac{d_l(\lfloor t\rfloor)}{d_\phi(\lfloor t\rfloor)}+\sum_{j=1}^{\lfloor t\rfloor-1} \left|\log \frac{d_\phi(j)d_l(j+1)}{d_l(j)d_\phi(j+1)}\right|.$$

Proof. Fix z with $|z| = R$. In the first step we estimate $\log\|W(0, x_N; z)\|$ when $N \leq \hbar(R)$. This is done by an application of Lemma III.2.8 with

$$a_j := \left(\frac{d_\phi(j)}{R d_l(j)}\right)^{\frac{1}{4}}, \qquad j = 1, \dots, N.$$

Since $N \leq \hbar(R)$ we have $a_j \in (0, 1]$. The factors in (III.2.14) are treated separately:

$$\frac{1}{a_1 a_N} = R^{\frac{1}{2}} \cdot \left(\frac{d_l(1)}{d_\phi(1)}\right)^{\frac{1}{4}} \cdot \left(\frac{d_l(N)}{d_\phi(N)}\right)^{\frac{1}{4}},$$

$$1 + |z| l_j a_j^2 \leq 1 + R d_l(j) a_j^2 = 1 + \big(R(d_l d_\phi)(j)\big)^{\frac{1}{2}},$$

$$\begin{aligned}
&\max\left\{\frac{a_j}{a_{j+1}}, \frac{a_{j+1}}{a_j}\right\} \big|\cos\big(\psi_j - \psi_{j+1}\big)\big| + \frac{|\sin(\psi_j - \psi_{j+1})|}{a_j a_{j+1}} \\
&\quad= \max\left\{\frac{a_j}{a_{j+1}}, \frac{a_{j+1}}{a_j}\right\} \cdot \left(|\cos(\phi_{j+1} - \phi_j)| + \frac{|\sin(\phi_{j+1} - \phi_j)|}{\max\{a_j^2, a_{j+1}^2\}}\right) \\
&\quad\leq \max\left\{\frac{a_j}{a_{j+1}}, \frac{a_{j+1}}{a_j}\right\} \cdot \left(1 + \frac{d_\phi(j)}{a_j^2}\right) \\
&\quad= \left[\max\left\{\frac{d_\phi(j)d_l(j+1)}{d_l(j)d_\phi(j+1)}, \frac{d_l(j)d_\phi(j+1)}{d_\phi(j)d_l(j+1)}\right\}\right]^{\frac{1}{4}} \cdot \Big(1 + \big(R(d_l d_\phi)(j)\big)^{\frac{1}{2}}\Big).
\end{aligned}$$

Now (III.2.14) yields

$$\begin{aligned}
\log\|W(0, x_N; z)\| &\leq \frac{1}{2}\log^+ R + \frac{1}{4}\log^+ \frac{d_l(1)}{d_\phi(1)} + \frac{1}{4}\log^+ \frac{d_l(N)}{d_\phi(N)} \\
&\quad+ \sum_{j=1}^{N} \log\Big(1 + \big(R(d_l d_\phi)(j)\big)^{\frac{1}{2}}\Big) \\
&\quad+ \frac{1}{4}\sum_{j=1}^{N-1} \left|\log \frac{d_\phi(j)d_l(j+1)}{d_l(j)d_\phi(j+1)}\right| + \sum_{j=1}^{N-1} \log\Big(1 + \big(R(d_l d_\phi)(j)\big)^{\frac{1}{2}}\Big).
\end{aligned}$$

Since $\not d_l, \not d_\phi$ are nonincreasing, we can estimate sums by integrals as

$$\sum_{j=2}^{N} \log\left(1+\big(R(\not d_l \not d_\phi)(j)\big)^{\frac12}\right) \le \int_1^N \log\left(1+\big(R(\not d_l \not d_\phi)(s)\big)^{\frac12}\right) \mathrm{d}s$$
$$\le \tfrac{3}{2} \cdot \int_1^{\min\{N,\hbar(R)\}} \log\big(R(\not d_l \not d_\phi)(s)\big)\, \mathrm{d}s + \int_{\min\{N,\hbar(R)\}}^{N} \big(R(\not d_l \not d_\phi)(s)\big)^{\frac12} \mathrm{d}s.$$

Noting that

$$\log\left(1+\big(R(\not d_l \not d_\phi)(1)\big)^{\frac12}\right) \le \frac{3}{2}\Big(\log^+ R + \log^+(\not d_l \not d_\phi)(1)\Big)$$
$$\le \frac{3}{2}\Big(\log^+ R + \log^+ \frac{\not d_l(1)}{\not d_\phi(1)}\Big),$$

leads to

$$\log\|W(0,x_N;z)\| \le \frac{7}{2}\log^+ R + \frac{13}{4}\log^+ \frac{\not d_l(1)}{\not d_\phi(1)} + \frac{1}{4}\log^+ \frac{\not d_l(N)}{\not d_\phi(N)}$$
$$+ \frac{1}{4}\sum_{j=1}^{N-1} \left|\log \frac{\not d_\phi(j) \not d_l(j+1)}{\not d_l(j) \not d_\phi(j+1)}\right|$$
$$+ 3 \cdot \int_1^{\min\{N,\hbar(R)\}} \log\big(R(\not d_l \not d_\phi)(s)\big)\, \mathrm{d}s + 2 \cdot \int_{\min\{N,\hbar(R)\}}^{N} \big(R(\not d_l \not d_\phi)(s)\big)^{\frac12} \mathrm{d}s. \qquad \text{(III.2.15)}$$

We see that

$$\log\|W(0,x_N;z)\| \le \tfrac{7}{2}\cdot\big[\not g(N,R) + \lambda(N,R)\big].$$

Consider now the case that $N > \hbar(R)$. Since $\|\xi_{\phi_j}\xi_{\phi_j}^T J\| = 1$, we see that

$$\|W(x_{\lfloor \hbar(R)\rfloor}, x_N; z)\| = \left\| \prod_{j=\lfloor \hbar(R)\rfloor+1}^{N} W(x_{j-1}, x_j; z)\right\|$$
$$\le \prod_{j=\lfloor \hbar(R)\rfloor+1}^{N} \left\| I + z l_j \xi_{\phi_j}\xi_{\phi_j}^T J\right\| \le \prod_{j=\lfloor \hbar(R)\rfloor+1}^{N} \big(1 + R \not d_l(j)\big).$$

Noting that

$$R \not d_l\big(\lfloor \hbar(R)\rfloor + 1\big) < 1,$$

and estimating the sum by an integral we arrive at

$$\begin{aligned}\log\|W(x_{\lfloor \hbar(R)\rfloor},x_N;z)\| &\leq \sum_{j=\lfloor \hbar(R)\rfloor+1}^{N}\log\Big(1+R\mathcal{d}_l(j)\Big)\\ &< \log 2 \;+\int\limits_{\lfloor \hbar(R)\rfloor+1}^{N} R\mathcal{d}_l(s)\,\mathrm{d}s.\end{aligned}$$

We combine this with (III.2.15) to obtain

$$\begin{aligned}\log\|W(0,x_N;z)\| &\leq \log\|W(0,x_{\lfloor \hbar(R)\rfloor};z)\|+\log\|W(x_{\lfloor \hbar(R)\rfloor},x_N;z)\|\\ &\leq \tfrac{7}{2}\cdot\big[\mathcal{g}(N,R)+\lambda(\hbar(R),R)\big].\end{aligned}$$

□

The second lemma accounts for the contribution of the remaining intervals.

III.2.10 Lemma. *Let $(l_j)_{j=1}^{\infty}$ be a summable sequence of positive numbers, and $(\phi_j)_{j=1}^{\infty}$ a sequence of real numbers. Denote by H the Hamburger Hamiltonian with these lengths and angles, and let W_H be its monodromy matrix.*

Let $\mathcal{c}_l,\mathcal{c}_\phi\colon[1,\infty)\to(0,\infty)$ be two functions with $\mathcal{c}_\phi\leq\mathcal{c}_l$. Choose $\psi\in\mathbb{R}$ and assume that

$$\forall N\in\mathbb{N}.\;\sum_{j=N+1}^{\infty}l_j\leq\mathcal{c}_l(N)\;\wedge\;\sum_{j=N+1}^{\infty}l_j\sin^2(\phi_j-\psi)\leq\mathcal{c}_\phi(N).$$

Then, for any $N\in\mathbb{N}$ and $z\in\mathbb{C}$,

$$\|W(x_N,L;z)\|\leq\Big(\frac{\mathcal{c}_l(N)}{\mathcal{c}_\phi(N)}\Big)^{\frac12}\exp\bigg(2|z|\big(\mathcal{c}_l\mathcal{c}_\phi\big)^{\frac12}(N)\bigg).$$

Proof. Let $\Omega\in\mathrm{GL}(2,\mathbb{R})$ and consider the function $\tilde{W}(t):=\Omega W(x_N,t;z)\Omega^{-1}$ defined on $[x_N,L]$. This function satisfies $\tilde{W}(x_N)=I$ and

$$\tilde{W}'(t)=-z\tilde{W}(t)\Omega H(t)J\Omega^{-1},\qquad t\in[x_N,L].$$

By Grönwall's Lemma,

$$\|\tilde{W}(t)\|\leq\exp\bigg(|z|\int_{x_N}^{t}\|\Omega H(s)J\Omega^{-1}\|\,\mathrm{d}s\bigg).$$

Set $a := \left(\frac{\not{c}_\phi(N)}{\not{c}_l(N)}\right)^{\frac{1}{4}}$ and $\Omega = \Omega(a,\psi)$. Using Lemma III.2.7, (i), (ii), we obtain

$$\begin{aligned}
\|W(x_N,L;z)\| &\leq \|\Omega^{-1}\|\|\tilde{W}(L)\|\|\Omega\| \leq \frac{1}{a^2}\exp\left(|z|\int_{x_N}^{L}\|\Omega H(s)J\Omega^{-1}\|\,\mathrm{d}s\right)\\
&= \left(\frac{\not{c}_l(N)}{\not{c}_\phi(N)}\right)^{\frac{1}{2}}\exp\left(|z|\sum_{j=N+1}^{\infty} l_j\cdot\|\Omega\xi_{\phi_j}\xi_{\phi_j}^T J\Omega^{-1}\|\right)\\
&= \left(\frac{\not{c}_l(N)}{\not{c}_\phi(N)}\right)^{\frac{1}{2}}\exp\left(|z|\sum_{j=N+1}^{\infty} l_j\Big[a^2\cos^2(\phi_j-\psi)+\frac{1}{a^2}\sin^2(\phi_j-\psi)\Big]\right)\\
&\leq \left(\frac{\not{c}_l(N)}{\not{c}_\phi(N)}\right)^{\frac{1}{2}}\exp\left(|z|\Big[a^2\sum_{j=N+1}^{\infty} l_j+\frac{1}{a^2}\sum_{j=N+1}^{\infty} l_j\sin^2(\phi_j-\psi)\Big]\right)\\
&\leq \left(\frac{\not{c}_l(N)}{\not{c}_\phi(N)}\right)^{\frac{1}{2}}\exp\left(2|z|\big(\not{c}_l\not{c}_\phi\big)^{\frac{1}{2}}(N)\right)
\end{aligned}$$

□

The proof of Theorem III.2.2 is now easily completed.

Proof of Theorem III.2.2. Let $(t,R)\in[1,\infty)\times\Big[\frac{2}{(d_l \not{d}_\phi)(1)}\Big)$ and set $N := \lceil t\rceil$. We use Lemma III.2.9 and Lemma III.2.10 to estimate, for $|z| = R$,

$$\begin{aligned}
\log\|W_H(z)\| &\leq \log\|W(0,x_N;z)\| + \log\|W(x_N,L;z)\|\\
&\leq \tfrac{7}{2}\cdot\Big[\not{g}(N,R)+\lambda\big(\min\{N,\hbar(R)\},R\big)\Big]\\
&\quad+\Big[\tfrac{1}{2}\log\frac{\not{c}_l(N)}{\not{c}_\phi(N)}+2R\big(\not{c}_l\not{c}_\phi\big)^{\frac{1}{2}}(N)\Big]\\
&\leq \tfrac{7}{2}\cdot\Big[\not{g}(t,R)+R\big(\not{c}_l\not{c}_\phi\big)^{\frac{1}{2}}(t)\Big]+\tfrac{7}{2}\cdot\big|\not{g}(N,R)-\not{g}(t,R)\big|\\
&\quad+\tfrac{7}{2}\cdot\lambda\big(\min\{N,\hbar(R)\},R\big)+\tfrac{1}{2}\log\frac{\not{c}_l(N)}{\not{c}_\phi(N)}.
\end{aligned}$$

Since $0\leq N-t\leq 1$ and

$$\begin{aligned}
g(s,R) &\leq \begin{cases}\frac{3}{2}\big(\log^+ R+\log^+\frac{d_l(1)}{d_\phi(1)}\big) & \text{if } 1\leq s<\hslash(R),\\ \sqrt{2} & \text{if } \hslash(R)\leq s<\hbar(R),\\ 1 & \text{if } \hbar(R)\leq s,\end{cases}\\
&\leq \frac{3}{2}\cdot\lambda\big(\min\{N,\hbar(R)\},R\big),
\end{aligned}$$

we have

$$\big|\mathfrak{g}(N,R)-\mathfrak{g}(t,R)\big|\leq\frac{3}{2}\cdot\lambda\big(\min\{N,\hbar(R)\},R\big),$$

and obtain

$$\log\|W_H(z)\|\leq 9\cdot\Big[\max\Big\{\mathfrak{g}\big(t,R\big),R(\mathfrak{c}_l\mathfrak{c}_\phi)^{\frac{1}{2}}(t)\Big\}+L(t,R)\Big].$$

Since t was arbitrary, we can pass to the infimum on the right side. □

III.3 Properties of the upper bound

In this section we study the expression on the right side of (III.2.7) independently of its meaning in the context of Theorem III.2.2.

III.3.1 Definition. Let $\mathfrak{d}_l\colon[1,\infty)\to(0,\infty)$ and $\mathfrak{d}_\phi\colon[1,\infty)\to(0,1]$ be measurable with $\mathfrak{d}_l\asymp 1\asymp\mathfrak{d}_\phi$ locally. Then we denote

$$B(R):=\inf_{t\geq 1}\Big(\max\Big\{\mathfrak{g}\big(t,R\big),R(\mathfrak{c}_l\mathfrak{c}_\phi)^{\frac{1}{2}}(t)\Big\}+L(t,R)\Big).\tag{III.3.1}$$

◀

We consider the questions

- ▷ Is $L(t,R)$ small compared to $\max\big\{\mathfrak{g}\big(t,R\big),R(\mathfrak{c}_l\mathfrak{c}_\phi)^{\frac{1}{2}}(t)\big\}$?
- ▷ Does $\inf_{t\geq 1}\max\big\{\mathfrak{g}\big(t,R\big),R(\mathfrak{c}_l\mathfrak{c}_\phi)^{\frac{1}{2}}(t)\big\}$ depend monotonically on the data $\mathfrak{d}_l,\mathfrak{d}_\phi,\mathfrak{c}_l,\mathfrak{c}_\phi$?

It turns out that already very weak assumptions on the data, concerning monotonicity, continuity, variation, and power boundedness, are sufficient to ensure that the answers are affirmative. This tells us that usually $L(t,R)$ can be dropped from (III.3.1), and hence $B(R)$ can be evaluated much more easily, and that usually the natural intuition which stems from the context of Theorem III.2.2, that faster decay means smaller growth, is indeed reflected in $B(R)$.

III.3.1 About monotonicity in the data

The essence is the following result.

III.3.2 Proposition. *Let* $d_l, \widehat{d}_l\colon [1,\infty) \to (0,\infty)$ *and* $d_\phi, \widehat{d}_\phi\colon [1,\infty) \to (0,1]$ *be measurable with* $d_l \asymp d_\phi \asymp \widehat{d}_l \asymp \widehat{d}_\phi \asymp 1$ *locally, and assume that* $\widehat{d}_l\widehat{d}_\phi$ *is* $\asymp$ *to some nonincreasing function. If* $\widehat{d}_l \lesssim d_l$ *and* $\widehat{d}_\phi \lesssim d_\phi$ *on* $[1,\infty)$*, then (quantities* $\widehat{g}(t,R)$ *etc. are defined correspondingly for* $\widehat{d}_l, \widehat{d}_\phi$*)*

$$\widehat{g}(t,R) \lesssim g(t,R) \tag{III.3.2}$$

for $R \geq \frac{2}{(d_l d_\phi)(1)}$ *and a.e.* (t,R) *with* $t < \widehat{\hbar}(R)$ *or* $t > \hbar(R)$*. If additionally* $\frac{\widehat{d}_\phi}{\widehat{d}_l}$ *is bounded or* $\asymp$ *on* $[1,\infty)$ *to a nondecreasing function, then (III.3.2) holds for* R *sufficiently large and a.e.* $t \in [1,\infty)$*.*

The constant that is implicit in the relation (III.3.2) depends on the constants implicit in the assumptions, but not on the actual functions $d_l, d_\phi, \widehat{d}_l, \widehat{d}_\phi$*.*

We start with a preparatory lemma.

III.3.3 Lemma. *Let* $\Phi, \Psi\colon [1,\infty) \to (0,\infty)$ *with* Φ *nondecreasing, and assume that* $\alpha, \alpha' > 0$ *are such that*

$$\forall t \in [1,\infty).\ \alpha \cdot \Phi(t) \leq \Psi(t) \leq \alpha' \cdot \Phi(t).$$

Set

$$t_0 := \sup\Big(\big\{t \in [1,\infty) \mid \sup_{1\leq s\leq t} \Psi(s) \leq 1\big\} \cup \{1\}\Big) \in [1,\infty].$$

Then, if $t_0 < \infty$*,*

$$\forall t \in (t_0,\infty).\ \Psi(t) > \frac{\alpha}{\alpha'}.$$

Proof. Since Φ is nondecreasing, we have

$$\begin{aligned}\big\{t \in [1,\infty) \mid \alpha'\Phi(t) \leq 1\big\} \cup \{1\} &= \big\{t \in [1,\infty) \mid \sup_{1\leq s\leq t} \alpha'\Phi(s) \leq 1\big\} \cup \{1\} \\ &\subseteq \big\{t \in [1,\infty) \mid \sup_{1\leq s\leq t} \Psi(s) \leq 1\big\} \cup \{1\}.\end{aligned}$$

Set

$$t_1 := \sup\Big(\big\{t \in [1,\infty) \mid \alpha'\Phi(t) \leq 1\big\} \cup \{1\}\Big) \in [1,\infty],$$

then $t_1 \leq t_0$. Again by monotonicity, $\alpha'\Phi(t) > 1$ for all $t > t_1$. and this yields that for $t > t_1$

$$\Psi(t) \geq \alpha\Phi(t) > \frac{\alpha}{\alpha'}.$$

$\square$

Proof of Proposition III.3.2. We have to trace constants, and thus make the constants from the assumptions explicit: choose a nondecreasing function $\boldsymbol{u}$ and $\alpha, \alpha' > 0$, such that

$$\alpha \boldsymbol{u}(t) \leq \frac{2}{(\widehat{\boldsymbol{d}}_l \widehat{\boldsymbol{d}}_\phi)(t)} \leq \alpha' \boldsymbol{u}(t), \qquad t \in [1, \infty),$$

and choose $\kappa_l, \kappa_\phi > 0$, such that

$$\widehat{\boldsymbol{d}}_l(t) \leq \kappa_l \boldsymbol{d}_l(t), \quad \widehat{\boldsymbol{d}}_\phi(t) \leq \kappa_\phi \boldsymbol{d}_\phi(t), \qquad t \in [1, \infty).$$

Moreover, set

$$\lambda := \max_{t \geq 2} \frac{\log t}{\sqrt{t}},$$

and note that

$$x \leq \gamma y \ \Rightarrow \ \log x \leq \Big(1 + \frac{\log^+ \gamma}{\log 2}\Big) \cdot \log y \quad \text{for } x > 0, y \geq 2, \gamma > 0.$$

Next, observe that

$$\big[R(\widehat{\boldsymbol{d}}_l \widehat{\boldsymbol{d}}_\phi)(t)\big]^{\frac{1}{2}} < \Big(\frac{2\alpha'}{\alpha}\Big)^{\frac{1}{2}}, \qquad t > \widehat{\bar{k}}(R). \tag{III.3.3}$$

In order to see this, let R be fixed and apply Lemma III.3.3 with $\Phi(t) = \frac{u(t)}{R}$ and $\Psi(t) = \frac{2}{R(\widehat{d}_l \widehat{d}_\phi)(t)}$. Then $t_0 = \widehat{\bar{k}}(R)$, and hence for $t > \widehat{\bar{k}}(R)$ we have $\Psi(t) > \frac{\alpha}{\alpha'}$, which implies (III.3.3).

In order to establish that $\widehat{g}(t, R) \lesssim g(t, R)$ we first assume $t < \widehat{\hbar}(R)$ and distinguish the following six cases according to the definitions of $\widehat{g}(t, R)$ and $g(t, R)$.

▷▷ $1 \leq t < \widehat{\bar{k}}(R) \ \wedge \ 1 \leq t < \bar{k}(R)$:

$$\log\big[R(\widehat{\boldsymbol{d}}_l \widehat{\boldsymbol{d}}_\phi)(t)\big] \leq \Big(1 + \frac{\log^+(\kappa_l \kappa_\phi)}{\log 2}\Big) \log\big[R(\boldsymbol{d}_l \boldsymbol{d}_\phi)(t)\big].$$

▷▷ $1 \leq t < \widehat{\bar{k}}(R) \ \wedge \ \bar{k}(R) < t < \hbar(R)$:

$$\log\big[R(\widehat{\boldsymbol{d}}_l \widehat{\boldsymbol{d}}_\phi)(t)\big] \leq \lambda\big[R(\widehat{\boldsymbol{d}}_l \widehat{\boldsymbol{d}}_\phi)(t)\big]^{\frac{1}{2}} \leq \lambda(\kappa_l \kappa_\phi)^{\frac{1}{2}}\big[R(\boldsymbol{d}_l \boldsymbol{d}_\phi)(t)\big]^{\frac{1}{2}}.$$

▷▷ $1 \leq t < \widehat{\bar{k}}(R) \ \wedge \ \hbar(R) < t$:

$$\begin{aligned}\log\big[R(\widehat{\boldsymbol{d}}_l \widehat{\boldsymbol{d}}_\phi)(t)\big] &\leq \lambda\big[R(\widehat{\boldsymbol{d}}_l \widehat{\boldsymbol{d}}_\phi)(t)\big]^{\frac{1}{2}} \leq \lambda\big[R(\widehat{\boldsymbol{d}}_l \widehat{\boldsymbol{d}}_\phi)(t)\big] \\ &\leq \lambda\big[R\widehat{\boldsymbol{d}}_l(t)\big] \leq \lambda\kappa_l\big[R\boldsymbol{d}_l(t)\big].\end{aligned}$$

$\triangleright\triangleright$ $\widehat{\mathscr{k}}(R) < t < \widehat{\hbar}(R) \ \wedge\ 1 \leq t < \mathscr{k}(R)$. By (III.3.3),

$$\big[R(\widehat{d}_l \widehat{d}_\phi)(t)\big]^{\frac{1}{2}} < \Big(\frac{2\alpha'}{\alpha}\Big)^{\frac{1}{2}} \leq \frac{1}{\log 2}\Big(\frac{2\alpha'}{\alpha}\Big)^{\frac{1}{2}} \log\big[R(d_l d_\phi)(t)\big].$$

$\triangleright\triangleright$ $\widehat{\mathscr{k}}(R) < t < \widehat{\hbar}(R) \ \wedge\ \mathscr{k}(R) < t < \hbar(R)$:

$$\big[R(\widehat{d}_l \widehat{d}_\phi)(t)\big]^{\frac{1}{2}} \leq (\kappa_l \kappa_\phi)^{\frac{1}{2}}\big[R(d_l d_\phi)(t)\big]^{\frac{1}{2}}.$$

$\triangleright\triangleright$ $\widehat{\mathscr{k}}(R) < t < \widehat{\hbar}(R) \ \wedge\ \hbar(R) < t$:

$$\big[R(\widehat{d}_l \widehat{d}_\phi)(t)\big]^{\frac{1}{2}} \leq R\widehat{d}_l(t) \leq \kappa_l R d_l(t).$$

Since (t, R) is assumed to satisfy $t < \widehat{\hbar}(R) \vee\ \hbar(R) < t$, and $t < \widehat{\hbar}(R)$ was already covered, the only case left to treat is

$\triangleright\triangleright$ $\widehat{\hbar}(R) < t \ \wedge\ \hbar(R) < t$:

$$R\widehat{d}_l(t) \leq \kappa_l R d_l(t).$$

Now we assume the additional condition on $\frac{\widehat{d}_\phi}{\widehat{d}_l}$. If this quotient is bounded, then $\widehat{\hbar}(R) = \infty$ for sufficiently large R, and there is nothing to show. Otherwise, choose a nondecreasing function $\mathfrak{v}$ and $\beta, \beta' > 0$ such that

$$\beta\mathfrak{v}(t) \leq \frac{\widehat{d}_\phi}{\widehat{d}_l} \leq \beta'\mathfrak{v}(t), \qquad t \in [1, \infty).$$

We claim that

$$R\widehat{d}_l(t) \leq \frac{\beta'}{\beta}\widehat{d}_\phi(t), \qquad t > \widehat{\hbar}(R). \tag{III.3.4}$$

In fact, if we fix R and apply Lemma III.3.3 with $\Phi(t) = \frac{\mathfrak{v}(t)}{R}$ and $\Psi(t) = \frac{\widehat{d}_\phi(t)}{R\widehat{d}_l(t)}$, we find that $t_0 = \widehat{\hbar}(R)$ and $\Psi(t) > \frac{\beta}{\beta'}$ for $t > \widehat{\hbar}(R)$. This is equivalent to (III.3.4).

With these preparations, the assertion follows from the following simple inequalities:

$\triangleright\triangleright$ $\widehat{\hbar}(R) < t \ \wedge\ 1 \leq t < \mathscr{k}(R)$:

$$R\widehat{d}_l(t) \leq \frac{\beta'}{\beta}\widehat{d}_\phi(t) \leq \frac{\beta'}{\beta} \leq \frac{\beta'}{\beta \log 2} \log\big[R(d_l d_\phi)(t)\big].$$

▷▷ $\widehat{\hbar}(R) < t \ \wedge\ \hslash(R) < t < \hbar(R)$:

$$R\widehat{d}_l(t) \le \frac{\beta'}{\beta}\widehat{d}_\phi(t) \le \frac{\beta'}{\beta}\kappa_\phi d_\phi(t) \le \frac{\beta'}{\beta}\kappa_\phi\big[R(d_l d_\phi)(t)\big]^{\frac{1}{2}}.$$

□

III.3.4 Corollary. *Let $d_l, d_\phi, \widehat{d}_l, \widehat{d}_\phi$ be as in Proposition III.3.2 and subject to the additional condition of this proposition. Further, let $c_l, c_\phi, \widehat{c}_l, \widehat{c}_\phi\colon [1,\infty) \to (0,\infty)$. If*

$$\widehat{d}_l \lesssim d_l,\ \ \widehat{d}_\phi \lesssim d_\phi,\ \ \widehat{c}_l \lesssim c_l,\ \ \widehat{c}_\phi \lesssim c_\phi$$

on $[1,\infty)$, then

$$\inf_{t\ge 1}\max\big\{\widehat{g}\big(t,R\big), R(\widehat{c}_l\widehat{c}_\phi)^{\frac{1}{2}}(t)\big\} \lesssim \inf_{t\ge 1}\max\big\{g\big(t,R\big), R(c_l c_\phi)^{\frac{1}{2}}(t)\big\}$$

for sufficiently large R. The constant that is implicit in the assertion depends on the constants implicit in the assumptions, but not on the actual functions $d_l, \widehat{d}_l$ etc.

III.3.2 About smallness of $L(t,R)$

To start with, let us observe that $g(t,R)$ must grow at least logarithmically.

III.3.5 Lemma. *Let $d_l\colon [1,\infty) \to (0,\infty)$ and $d_\phi\colon [1,\infty) \to (0,1]$ be measurable with $d_l \asymp 1 \asymp d_\phi$ locally. Then, for each $\varepsilon > 0$, there exists a positive constant $c > 0$ such that, for $t \ge 1+\varepsilon$ and $R \ge \sup_{1\le s\le 1+\varepsilon}\frac{2}{(d_l d_\phi)(s)}$ we have*

$$g(t,R) \ge c\log R.$$

Proof. Let $\eta := \sup_{1\le s\le 1+\varepsilon}\frac{2}{(d_l d_\phi)(s)}$. Then $R \ge \eta$ is equivalent to $1+\varepsilon \le \hslash(R)$. Hence

$$\begin{aligned} g(t,R) &\ge \int_1^{1+\varepsilon} \log\big(R(d_l d_\phi)(s)\big)\,\mathrm{d}s \\ &= \Big(\varepsilon + \frac{1}{\log R}\int_1^{1+\varepsilon}\log(d_l d_\phi)(s)\,\mathrm{d}s\Big)\cdot\log R. \end{aligned}$$

The term in the bracket tends to ε for $R \to \infty$, and hence we find $R_0 \ge \eta$ such that $g(t,R) \ge \frac{\varepsilon}{2}\cdot\log R$ for all $t \ge 1+\varepsilon$ and $R \ge R_0$. The function g is nonzero, and hence bounded away from zero, on the compact set $\{1+\varepsilon\} \times [\eta, R_0]$. By monotonicity, it is therefore also bounded away from zero on $[1+\varepsilon,\infty) \times [\eta, R_0]$. Together, we see that a constant $c > 0$ can be chosen as required. □

The most cumbersome part of $L(t,R)$ is the sum written as the last of the six summands in (III.2.6). Assuming monotonicity of the quotient $\frac{d_\phi}{d_l}$, this sum turns into a telescoping sum and can be estimated.

III.3.6 Lemma. *Let $d_l, d_\phi\colon [1,\infty) \to (0,\infty)$ and let $\hbar(R)$ be as in (III.2.2). Assume that the quotient $\frac{d_\phi}{d_l}$ is eventually monotone but not eventually constant. Then*

$$\sum_{j=1}^{\min\{\lceil t\rceil, \lfloor \hbar(R)\rfloor\}-1} \left| \log\left(\frac{d_\phi(j)}{d_l(j)} \Big/ \frac{d_\phi(j+1)}{d_l(j+1)} \right) \right| \asymp 1 + \left| \log \frac{d_\phi(\min\{\lceil t\rceil, \lfloor \hbar(R)\rfloor\})}{d_l(\min\{\lceil t\rceil, \lfloor \hbar(R)\rfloor\})} \right|$$

for $(t,R) \in [1,\infty) \times [\frac{d_\phi(1)}{d_l(1)}, \infty)$.

Proof. We abbreviate $T := \min\{\lceil t\rceil, \lfloor \hbar(R)\rfloor\}$. Choose $M \in \mathbb{N}$ such that $\frac{d_\phi}{d_l}$ is monotone on $[M,\infty)$ and $\frac{d_\phi(M)}{d_l(M)} \neq \frac{d_\phi(M+1)}{d_l(M+1)}$. For all (t,R) such that $T \geq M+1$ we have

$$\begin{aligned}\sum_{j=M}^{T-1} \left| \log\left(\frac{d_\phi(j)}{d_l(j)} \Big/ \frac{d_\phi(j+1)}{d_l(j+1)} \right) \right| &= \left| \sum_{j=M}^{T-1} \log\left(\frac{d_\phi(j)}{d_l(j)} \Big/ \frac{d_\phi(j+1)}{d_l(j+1)} \right) \right| \\ &= \left| \log \frac{d_\phi(M)}{d_l(M)} - \log \frac{d_\phi(T)}{d_l(T)} \right|.\end{aligned}$$

Clearly, this value is nondecreasing in T and positive for $T \geq M+1$. If $\left|\log \frac{d_\phi(T)}{d_l(T)}\right|$ is bounded, it is $\asymp 1$. If $\left|\log \frac{d_\phi(T)}{d_l(T)}\right|$ is unbounded, it is $\asymp \left|\log \frac{d_\phi(T)}{d_l(T)}\right|$.

The beginning part of the sum which we cut off, i.e.,

$$\sum_{j=1}^{M-1} \left| \log\left(\frac{d_\phi(j)}{d_l(j)} \Big/ \frac{d_\phi(j+1)}{d_l(j+1)} \right) \right|$$

is some nonnegative number independent of t and R. □

III.3.7 Corollary. *Let $d_l, d_\phi\colon [1,\infty) \to (0,\infty)$ and let $\hbar(R)$ be as in (III.2.2). Assume that one of the following two assumptions holds.*

(i) *The quotient $\frac{d_\phi}{d_l}$ is eventually nondecreasing and $\frac{d_\phi}{d_l}(t) \lesssim \sup_{1\leq s<t} \frac{d_\phi}{d_l}(s)$ for sufficiently large t.*

(ii) *The quotient $\frac{d_\phi}{d_l}$ is eventually nonincreasing and power bounded from below (i.e. there exists $\alpha > 0$ such that $\frac{d_\phi(t)}{d_l(t)} \gtrsim t^{-\alpha}$ for sufficiently large t).*

Then

$$\sum_{j=1}^{\min\{\lceil t\rceil,\lfloor\hbar(R)\rfloor\}-1}\left|\log\left(\frac{d_\phi(j)}{d_l(j)}\Big/\frac{d_\phi(j+1)}{d_l(j+1)}\right)\right| \lesssim \begin{cases}1+\log^+ R & \text{if (i) holds,}\\ 1+\log t & \text{if (ii) holds,}\end{cases}$$

for $(t,R)\in[1,\infty)\times[\frac{d_\phi(1)}{d_l(1)},\infty)$.

Proof. Again abbreviate $T:=\min\{\lceil t\rceil,\lfloor\hbar(R)\rfloor\}$, and let $M\in\mathbb{N}$ be such that $\frac{d_\phi}{d_l}$ is monotone on $[M,\infty)$. Under the assumption (i) we have

$$\frac{d_\phi(T)}{d_l(T)} \leq \frac{d_\phi(\hbar(R))}{d_l(\hbar(R))} \lesssim R,$$

and hence

$$\left|\log\frac{d_\phi(T)}{d_l(T)}\right| \lesssim 1+\log R.$$

Under the assumption (ii) we have

$$t^{-\alpha}\lesssim T^{-\alpha}\lesssim\frac{d_\phi(T)}{d_l(T)}\lesssim 1,$$

and hence

$$\left|\log\frac{d_\phi(T)}{d_l(T)}\right| \lesssim 1+\log t.$$

□

Another type of condition on the quotient $\frac{d_\phi}{d_l}$ that ensures that the sum can be estimated, is that its variation is not too fast.

III.3.8 Lemma. *Let* $d_l, d_\phi\colon[1,\infty)\to(0,\infty)$ *and let* $\hbar(R)$ *be as in (III.2.2). Assume that the quotient* $\frac{d_\phi}{d_l}$ *can be represented as*

$$\frac{d_\phi(t)}{d_l(t)} = c(t)\cdot\exp\left(\int_1^t \epsilon(u)\frac{\mathrm{d}u}{u}\right),\qquad t\in[1,\infty),$$

where $\epsilon\colon[1,\infty)\to\mathbb{R}$ *is locally integrable and eventually bounded, and where* $c\colon[1,\infty)\to(0,\infty)$ *is eventually constant. Then*

$$\sum_{j=1}^{\min\{\lceil t\rceil,\lfloor\hbar(R)\rfloor\}-1}\left|\log\left(\frac{d_\phi(j)}{d_l(j)}\Big/\frac{d_\phi(j+1)}{d_l(j+1)}\right)\right| \lesssim 1+\log t$$

for $(t,R)\in[1,\infty)\times[\frac{d_\phi(1)}{d_l(1)},\infty)$.

Proof. We again abbreviate $T := \min\{\lceil t\rceil, \lfloor \hbar(R)\rfloor\}$. For $M \in \mathbb{N}$ such that $\epsilon(t)$ is bounded and $c(t)$ is constant on $[M,\infty)$, we can estimate, for all (t,R) such that $T > M$,

$$\begin{aligned}\sum_{j=M}^{T-1} \left| \log\left(\frac{d_\phi(j)}{d_l(j)} \Big/ \frac{d_\phi(j+1)}{d_l(j+1)} \right) \right| &= \sum_{j=M}^{T-1} \left| \int_j^{j+1} \epsilon(u) \frac{\mathrm{d}u}{u} \right| \\ &\leq \sum_{j=M}^{T-1} \int_j^{j+1} |\epsilon(u)| \frac{\mathrm{d}u}{u} = \int_M^T |\epsilon(u)| \frac{\mathrm{d}u}{u} \lesssim \log t.\end{aligned}$$

□

Now we give three sets of conditions, each of which ensures that $L(t,R)$ can be neglected.

III.3.9 Proposition. *Let $d_l\colon [1,\infty) \to (0,\infty)$ and $d_\phi\colon [1,\infty) \to (0,1]$ be measurable with $d_l \asymp 1 \asymp d_\phi$ locally, let c_l, c_ϕ be continuous and nonincreasing with $\lim_{t\to\infty} (c_l c_\phi)(t) = 0$, and let $T(R)$ be the unique solution of (III.2.8). Assume that one of the following three sets of assumptions holds.*

(i) *The quotient $\frac{d_\phi}{d_l}$ is eventually nondecreasing and $\frac{d_\phi}{d_l}(t) \lesssim \sup_{1\leq s<t} \frac{d_\phi}{d_l}(s)$ for sufficiently large t. We have $c_\phi(t) \lesssim c_\phi(t+1)$ for sufficiently large t.*

(ii) *The quotient $\frac{d_\phi}{d_l}$ is eventually nonincreasing and power bounded from below. There exists $\alpha > 0$ such that $(c_l c_\phi)(t) \lesssim t^{-\alpha}$ for sufficiently large t. The quotient $\frac{c_l}{c_\phi}$ is power bounded from above or $c_\phi(t) \lesssim c_\phi(t+1)$ for sufficiently large t.*

(iii) *The quotient $\frac{d_\phi}{d_l}$ can be represented as*

$$\frac{d_\phi(t)}{d_l(t)} = c(t) \cdot \exp\left(\int_1^t \epsilon(u) \frac{\mathrm{d}u}{u} \right), \qquad t \in [1,\infty),$$

where $\epsilon\colon [1,\infty) \to \mathbb{R}$ is locally integrable and eventually bounded, and where $c\colon [1,\infty) \to (0,\infty)$ is eventually constant. There exists $\alpha > 0$ such that $(c_l c_\phi)(t) \lesssim t^{-\alpha}$ for sufficiently large t. The quotient $\frac{c_l}{c_\phi}$ is power bounded from above or $c_\phi(t) \lesssim c_\phi(t+1)$ for sufficiently large t.

Let $B(R)$ be as in (III.3.1). Then $L(T(R),R) \lesssim 1 + \log^+ R$, and

$$B(R) \asymp g(T(R),R) = R(c_l c_\phi)^{\frac{1}{2}}(T(R)).$$

Proof. Assume (i). The first sentence of (i) is nothing but the assumption (i) in Corollary III.3.7, and hence the sum in the definition of $L(t,R)$ is $\lesssim 1+\log^+ R$ independently of t. The last but one summand in (III.2.6) is $\lesssim 1$ by monotonicity of $\frac{d_\phi}{d_l}$. It remains to estimate the third summand in (III.2.6), and here we use the second sentence in the present assumption: using $\lim_{R\to\infty} T(R)=\infty$ and Lemma III.3.5,

$$\begin{aligned}\log^+ \frac{c_l(\lceil T(R)\rceil)}{c_\phi(\lceil T(R)\rceil)} &\lesssim 1+\log^+ \frac{c_l(T(R))}{c_\phi(T(R))} \lesssim 1+\log^+ \frac{1}{(c_l c_\phi)(T(R))} \\ &= 1+\log^+ \frac{R^2}{g(T(R),R)^2} \lesssim 1+\log^+ R.\end{aligned} \tag{III.3.5}$$

Assume (ii). Let $\alpha>0$ be as in the second sentence of (ii), then $R(c_l c_\phi)^{\frac12}(R^{\frac{2}{\alpha}}) \lesssim 1 \ll g(R^{\frac{2}{\alpha}},R)$, and hence $T(R)\lesssim R^{\frac{2}{\alpha}}$. The first sentence of (ii) is nothing but assumption (ii) in Corollary III.3.7. This yields that the sum in the definition of $L(t,R)$ is $\lesssim 1+\log^+ t$, and hence the sum in $L(T(R),R)$ is $\lesssim 1+\log^+ R$. The same holds for the last but one summand in (III.2.6) by power boundedness of $\frac{d_\phi}{d_l}$, and for the third summand (III.2.6) in case that $\frac{c_l}{c_\phi}$ is power bounded. If $c_\phi(t)\lesssim c_\phi(t+1)$ for sufficiently large t, then the third summand in (III.2.6) is $\lesssim 1+\log^+ R$ by (III.3.5).

Assume (iii). The second sentence again ensures that $T(R)$ is power bounded from above. The first sentence is the assumption of Lemma III.3.8, and it follows that the sum in $L(T(R),R)$ is $\lesssim 1+\log^+ R$. The form of $\frac{d_\phi}{d_l}$ implies that this quotient is power bounded from below, and hence the last but one summand (III.2.6) is $\lesssim 1+\log^+ R$. Concerning the third summand, just argue as above.

To finish the proof it remains to refer to Lemma III.3.5. □

III.4 Regularly varying decay

In the preceding section, we studied properties of the function $B(R)$, in particular, we saw that the term $L(t,R)$ is usually neglectable. Our next goal is to explicitly evaluate

$$\min_{t\geq 1}\left(\max\left\{g(t,R), R(c_l c_\phi)^{\frac12}(t)\right\}\right)$$

in the situation that d_l, d_ϕ, c_l, c_ϕ are all regularly varying, cf. Theorem III.4.1. As in the preceding section, this is a pure analysis of functions, and independent of the meaning in the context of Theorem III.2.2. However, after having completed the proof of Theorem III.4.1, we will return to the study of Hamburger Hamiltonians and combine Theorem III.4.1 with Theorem III.2.2. This establishes an explicit upper bound for the growth of the monodromy matrix, cf. Theorem III.4.6.

The setup for Theorem III.4.1 is as follows:

(i) We are given regularly varying functions $\mathscr{d}_l, \mathscr{d}_\phi, \mathscr{c}_l, \mathscr{c}_\phi \colon [1,\infty) \to (0,\infty)$ with nonpositive indices, $\mathscr{d}_\phi \leq 1$, and $\mathscr{d}_l \asymp 1 \asymp \mathscr{d}_\phi$ locally.

(ii) We denote

$$\delta_l := -\operatorname{ind}\mathscr{d}_l, \quad \delta_\phi := -\operatorname{ind}\mathscr{d}_\phi, \quad \gamma_l := -\operatorname{ind}\mathscr{c}_l, \quad \gamma_\phi := -\operatorname{ind}\mathscr{c}_\phi,$$

and further

$$\mathscr{D}(t) := \frac{1}{(\mathscr{d}_l\mathscr{d}_\phi)(t)}, \quad \delta := \operatorname{ind}\mathscr{D} = \delta_l + \delta_\phi, \tag{III.4.1}$$

$$\mathscr{C}(t) := \frac{1}{(\mathscr{c}_l\mathscr{c}_\phi)^{\frac{1}{2}}(t)}, \quad \gamma := \operatorname{ind}\mathscr{C} = \frac{\gamma_l + \gamma_\phi}{2}. \tag{III.4.2}$$

(iii) The quantities of interest are

$$\begin{aligned} \mathscr{B}(t,R) &:= \max\Big\{\mathscr{g}(t,R), \frac{R}{\mathscr{C}(t)}\Big\}, \\ \mathscr{B}(R) &:= \min_{t\geq 1}\mathscr{B}(t,R). \end{aligned} \tag{III.4.3}$$

Four fundamentally different cases occur. They depend on the relative size of $\mathscr{D}(t)$ vs. $\mathscr{C}(t)$, and on the absolute size of $\mathscr{D}(t)$. Moreover, there are a few exceptional cases that have to be ruled out.

III.4.1 Theorem. *Assume we are given data as in* (i) *above, and let notation be as in* (ii) *and* (iii). *Assume further that $\delta > 0$, that $\mathscr{C}$ is $\asymp$ on $[1,\infty)$ to a nondecreasing function, and that $\lim_{t\to\infty}\mathscr{C}(t) = \infty$.*

Then the following estimates for $\mathscr{B}(R)$ hold for sufficiently large R.

$\boxed{\text{A}}$ Case $\mathscr{D}(t) \lesssim t\mathscr{C}(t)$:

Choose $\alpha \geq 4 \cdot \sup_{t\geq 1}\frac{\mathscr{D}(t)}{t\mathscr{C}(t)}$, and set $\mathscr{f}(t) := t\mathscr{C}(t)\log\big[\alpha\frac{t\mathscr{C}(t)}{\mathscr{D}(t)}\big]$. Then an asymptotic inverse of $\mathscr{f}$ exists, and with $\tau(R) := \mathscr{f}^-\big(\frac{1}{\alpha}R\big)$ we have

$$\mathscr{B}(R) \asymp \mathscr{B}\big(\tau(R),R\big) \asymp \frac{R}{\mathscr{C}(\mathscr{f}^-(R))}. \tag{III.4.4}$$

The function on the right side is regularly varying with index $\frac{1}{1+\gamma}$.

If $\delta < 1+\gamma$ we have $\mathscr{f}(t) \asymp t\mathscr{C}(t)\log t$, and $\mathscr{B}(R)$ is (up to $\asymp$) independent of $\mathscr{d}_l$ and $\mathscr{d}_\phi$.

$\boxed{\text{B}}$ Case $t\mathscr{C}(t) \lesssim \mathscr{D}(t)$, $\int\limits_1^\infty \mathscr{D}(s)^{-\frac{1}{2}}\,\mathrm{d}s < \infty$, and if $(\delta_l, \delta_\phi, \gamma) = (1,1,0)$ then $\frac{d_\phi}{d_l}$ is bounded or $\asymp$ to a monotone function:

Then

$$\hbar(R) \lesssim \mathscr{B}(R) \lesssim R^{\frac{1}{2}} \int\limits_{\hbar(R)}^{\infty} \mathscr{D}(s)^{-\frac{1}{2}}\,\mathrm{d}s. \tag{III.4.5}$$

Both bounds in (III.4.5) are regularly varying with index $\frac{1}{\delta}$.

If $\delta > 2$, *then* $\asymp$ *holds throughout (III.4.5) and* $\mathscr{B}(R)$ *is (up to* $\asymp$*) independent of* c_l *and* c_ϕ.

$\boxed{\text{C}}$ Case $\frac{1}{d_l(t)} \lesssim t\mathscr{C}(t) \lesssim \mathscr{D}(t)$, $\int\limits_1^\infty \mathscr{D}(s)^{-\frac{1}{2}}\,\mathrm{d}s = \infty$, and $(\delta, \gamma) \neq (2, 0)$:

Set $f_0(t) := \frac{t^2\mathscr{C}(t)^2}{\mathscr{D}(t)}$ *and* $f_1(t) := \Big(\mathscr{C}(t) \int\limits_1^t \mathscr{D}(s)^{-\frac{1}{2}}\,\mathrm{d}s\Big)^2$. *Then asymptotic inverses* f_0^-, f_1^- *exist, and with*

$$\tau(R) := \min\big\{\max\big\{f_1^-(R), \hbar(R)\big\}, \hbar(R)\big\}$$

we have

$$\frac{R}{\mathscr{C}(f_0^-(R))} \lesssim \mathscr{B}(R) \lesssim \mathscr{B}(\tau(R), R) \lesssim \frac{R}{\mathscr{C}(f_1^-(R))}. \tag{III.4.6}$$

Both bounds in (III.4.6) are regularly varying with index $\frac{2-\delta+\gamma}{2-\delta+2\gamma}$.

If $\delta < 2$, *then* $\asymp$ *holds throughout (III.4.6).*

$\boxed{\text{D}}$ Case $t\mathscr{C}(t) \lesssim \frac{1}{d_l(t)}$, $\int\limits_1^\infty \mathscr{D}(s)^{-\frac{1}{2}}\,\mathrm{d}s = \infty$, and $\int_1^\infty d_l(s)\,\mathrm{d}s < \infty$:

Then $\frac{d_\phi}{d_l}$ *is unbounded and*

$$R\hbar(R)d_l(\hbar(R)) \lesssim \mathscr{B}(R) \lesssim R^{\frac{1}{2}} \int\limits_1^{\hbar(R)} \mathscr{D}(s)^{-\frac{1}{2}}\,\mathrm{d}s + R\int\limits_{\hbar(R)}^{\infty} d_l(s)\,\mathrm{d}s. \tag{III.4.7}$$

If $\delta_l > \delta_\phi$, *then both bounds in (III.4.7) are regularly varying with index* $\frac{1-\delta_\phi}{\delta_l-\delta_\phi}$.

If $\delta < 2$ *and* $\delta_l > 1$, *then* $\asymp$ *holds throughout (III.4.7).*

III.4.1 Proof of Theorem III.4.1

We start with two lemmata that determine the function $\mathfrak{g}(t,R)$ on the intervals $[1,\mathfrak{k}(R)]$ and $[\mathfrak{k}(R),\mathfrak{h}(R)]$, respectively.

III.4.2 Lemma. *Let $\mathfrak{D}\colon[1,\infty)\to(0,\infty)$ be regularly varying with $\operatorname{ind}\mathfrak{D}>0$ and $\mathfrak{D}\asymp 1$ locally. Set*

$$\mathfrak{k}(R):=\sup\Big\{t\in[1,\infty)\mid \sup_{1\le s\le t}\frac{2\mathfrak{D}(s)}{R}\le 1\Big\},\qquad R\ge 2\mathfrak{D}(1).\tag{III.4.8}$$

Then we have

$$\int_1^t \log\frac{R}{\mathfrak{D}(s)}\,\mathrm{d}s\asymp t\log\frac{R}{\mathfrak{D}(t)}\tag{III.4.9}$$

for $R\in[2\mathfrak{D}(1),\infty)$ and $t\in\big[2\log\frac{R}{\mathfrak{D}(1)},\mathfrak{k}(R)\big]$.

Proof. In the first part of the proof we establish the assertion under the additional assumption that $\mathfrak{D}$ is continuously differentiable with $\mathfrak{D}'(t)>0$ for all $t\ge 1$. Let $R\in[2\mathfrak{D}(1),\infty)$ and $t\in\big[2\log\frac{R}{\mathfrak{D}(1)},\mathfrak{k}(R)\big]$. Integration by parts yields that

$$\int_1^t \log\frac{R}{\mathfrak{D}(s)}\,\mathrm{d}s=t\log\frac{R}{\mathfrak{D}(t)}-\log\frac{R}{\mathfrak{D}(1)}+\int_1^t s\frac{\mathfrak{D}'(s)}{\mathfrak{D}(s)}\,\mathrm{d}s.\tag{III.4.10}$$

The relation "$\gtrsim$" in (III.4.9) readily follows: The integral on the right side is positive, and by the definition of $\mathfrak{k}(R)$ we have $\frac{R}{\mathfrak{D}(t)}\ge 2$. For $t\ge 2\log\frac{R}{\mathfrak{D}(1)}$ it follows that

$$\begin{aligned}\int_1^t \log\frac{R}{\mathfrak{D}(s)}\,\mathrm{d}s&\ge\Big(\frac12\cdot t\log\frac{R}{\mathfrak{D}(t)}+\frac12\cdot 2\log\frac{R}{\mathfrak{D}(1)}\cdot\log 2\Big)-\log\frac{R}{\mathfrak{D}(1)}\\&\ge\frac12\cdot t\log\frac{R}{\mathfrak{D}(t)}.\end{aligned}$$

To show that "$\lesssim$" holds in (III.4.9) we make a change of variable and use Karamata's theorem. The inverse function $\mathfrak{D}^{-1}$ exists and is continuously differentiable with positive derivative, and is regularly varying with positive index. We obtain that, for $v\to\infty$,

$$\int_1^v s\frac{\mathfrak{D}'(s)}{\mathfrak{D}(s)}\,\mathrm{d}s=\int_{\mathfrak{D}(1)}^{\mathfrak{D}(v)}\mathfrak{D}^{-1}(u)\frac{\mathrm{d}u}{u}\sim(\operatorname{ind}\mathfrak{D})\cdot v.$$

As a function of $v\in[2\log 2,\infty)$, both v and the integral above are continuous and nonzero. Hence

$$\int_1^v s\frac{\mathfrak{D}'(s)}{\mathfrak{D}(s)}\,\mathrm{d}s\asymp v,\qquad v\in[2\log 2,\infty).$$

Referring to (III.4.10) and again using that $\frac{R}{\mathscr{D}(t)} \geq 2$, we obtain

$$\int_1^t \log \frac{R}{\mathscr{D}(s)} \, \mathrm{d}s \leq t \log \frac{R}{\mathscr{D}(t)} + \int_1^t s \frac{\mathscr{D}'(s)}{\mathscr{D}(s)} \, \mathrm{d}s \lesssim t \log \frac{R}{\mathscr{D}(t)}.$$

The second part of the proof is to reduce to the above settled special case. Hence, assume that $\mathscr{D}$ is given as in the statement of the lemma. Lemma III.A.9 gives a regularly varying function $\mathscr{D}_1$ which is continuously differentiable with $\mathscr{D}_1' > 0$, such that $\mathscr{D} \asymp \mathscr{D}_1$ on $[1, \infty)$. Dividing $\mathscr{D}_1$ by a sufficiently large constant, we obtain a function $\tilde{\mathscr{D}}$ with the properties listed above and

$$\frac{1}{\alpha}\mathscr{D}(t) \leq \tilde{\mathscr{D}}(t) \leq \mathscr{D}(t) \quad \text{for } t \geq 1$$

for some $\alpha \geq 1$.

Clearly, $\mathscr{k}(R) \leq \tilde{\mathscr{k}}(R)$ (where $\tilde{\mathscr{k}}(R)$ is defined by (III.4.8) with $\tilde{\mathscr{D}}$ in place of $\mathscr{D}$). It follows that for $t < \mathscr{k}(R)$

$$\log \frac{R}{\mathscr{D}(t)} \leq \log \frac{R}{\tilde{\mathscr{D}}(t)} \leq \log \Big(\alpha \frac{R}{\mathscr{D}(t)}\Big) \leq \log \frac{R}{\mathscr{D}(t)} + \log \alpha \leq \Big(1 + \frac{\log \alpha}{\log 2}\Big) \log \frac{R}{\mathscr{D}(t)}.$$

We see that

$$\int_1^t \log \frac{R}{\mathscr{D}(s)} \, \mathrm{d}s \asymp \int_1^t \log \frac{R}{\tilde{\mathscr{D}}(s)} \, \mathrm{d}s \asymp t \log \frac{R}{\tilde{\mathscr{D}}(t)} \asymp t \log \frac{R}{\mathscr{D}(t)}.$$

□

III.4.3 Lemma. *Let* $\mathscr{D} \colon [1, \infty) \to (0, \infty)$ *be regularly varying with* $\operatorname{ind} \mathscr{D} > 0$ *and* $\mathscr{D} \asymp 1$ *locally. Then we have*

$$R^{\frac{1}{2}} t \mathscr{D}(t)^{-\frac{1}{2}} \lesssim \mathscr{k}(R) + R^{\frac{1}{2}} \int_{\mathscr{k}(R)}^t \mathscr{D}(s)^{-\frac{1}{2}}(s) \, \mathrm{d}s \lesssim R^{\frac{1}{2}} \int_1^t \mathscr{D}(s)^{-\frac{1}{2}} \, \mathrm{d}s$$

for $R \in [2\mathscr{D}(1), \infty)$ *and* $t \in [\mathscr{k}(R), \infty)$.

Proof. The asserted estimate from above is easy to see: by Karamata's theorem we have

$$\mathscr{k}(R) \asymp R^{\frac{1}{2}} \mathscr{k}(R) \mathscr{D}(\mathscr{k}(R))^{-\frac{1}{2}} \lesssim R^{\frac{1}{2}} \int_1^{\mathscr{k}(R)} \mathscr{D}(s)^{-\frac{1}{2}} \, \mathrm{d}s.$$

To obtain an estimate from below, we use Lemma III.A.9. This gives a regularly varying function $\mathscr{D}_1$ which is continuously differentiable with $\mathscr{D}_1' >$

0 and satisfies $\mathscr{D} \asymp \mathscr{D}_1$. By multiplying $\mathscr{D}_1$ with a sufficiently large constant, we obtain a function $\tilde{\mathscr{D}}$ with the properties listed above and

$$\mathscr{D}(t) \leq \tilde{\mathscr{D}}(t) \leq \alpha \mathscr{D}(t)$$

with some $\alpha \geq 1$.

Consider the functions

$$f_1(t) := R^{\frac{1}{2}} t \tilde{\mathscr{D}}(t)^{-\frac{1}{2}}, \quad f_2(t) := \sqrt{2}\, \hbar(R) + R^{\frac{1}{2}} \int_{\hbar(R)}^{t} \tilde{\mathscr{D}}(s)^{-\frac{1}{2}}\, \mathrm{d}s.$$

Then

$$\begin{aligned} f_1(\hbar(R)) &= R^{\frac{1}{2}} \hbar(R) \tilde{\mathscr{D}}(\hbar(R))^{-\frac{1}{2}} \\ &\leq R^{\frac{1}{2}} \hbar(R) \mathscr{D}(\hbar(R))^{-\frac{1}{2}} = \sqrt{2}\, \hbar(R) = f_2(\hbar(R)), \end{aligned}$$

and, for $t > \hbar(R)$,

$$f_1'(t) = R^{\frac{1}{2}} \tilde{\mathscr{D}}(t)^{-\frac{1}{2}} + R^{\frac{1}{2}} t \underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\left(\tilde{\mathscr{D}}(t)^{-\frac{1}{2}}\right)}_{<0} \leq R^{\frac{1}{2}} \tilde{\mathscr{D}}(t)^{-\frac{1}{2}} = f_2'(t).$$

We see that

$$f_1(t) \leq f_2(t) \quad \text{for } t \geq \hbar(R).$$

It remains to note that

$$f_1(t) \gtrsim R^{\frac{1}{2}} t \mathscr{D}(t)^{-\frac{1}{2}}, \quad f_2(t) \lesssim \hbar(R) + R^{\frac{1}{2}} \int_{\hbar(R)}^{t} \mathscr{D}(s)^{-\frac{1}{2}}\, \mathrm{d}s.$$

□

III.4.4 Lemma. *Consider regularly varying functions $d_l, d_\phi, \mathscr{D}$ as introduced in the discussion preceding Theorem III.4.1. Suppose that $\int_1^\infty \mathscr{D}(s)^{-\frac{1}{2}}\, \mathrm{d}s < \infty$. If $\delta_\phi = \delta_l = 1$, we also assume that $\frac{d_\phi}{d_l}$ is bounded or $\asymp$ to a nondecreasing function. Then for sufficiently large R we have, independently of t, that*

$$g(t, R) \lesssim R^{\frac{1}{2}} \int_{\hbar(R)}^{\infty} \mathscr{D}(s)^{-\frac{1}{2}}\, \mathrm{d}s.$$

Proof. We distinguish cases.

▷ Assume that $\frac{d_\phi}{d_l}$ is bounded, i.e., $\hbar(R) = \infty$ for all sufficiently large R. This is surely the case if $\delta_l < \delta_\phi$.

By Karamata's theorem

$$k(R) = \sqrt{2}\, R^{\frac{1}{2}} k(R) \mathcal{D}(k(R))^{-\frac{1}{2}} \lesssim R^{\frac{1}{2}} \int_{k(R)}^{\infty} \mathcal{D}(s)^{-\frac{1}{2}} \, \mathrm{d}s.$$

Now it follows that, for all $t \geq k(R)$,

$$g(t,R) \asymp k(R) + R^{\frac{1}{2}} \int\limits_{k(R)}^{t} \mathcal{D}(s)^{-\frac{1}{2}} \, \mathrm{d}s \lesssim R^{\frac{1}{2}} \int\limits_{k(R)}^{\infty} \mathcal{D}(s)^{-\frac{1}{2}} \, \mathrm{d}s.$$

▷ Assume that $\frac{d_\phi}{d_l}$ is $\asymp$ to a nondecreasing function. This is surely the case if $\delta_l > \delta_\phi$.

If R is large enough and $t > \hbar(R)$,

$$R d_l(t) = \left(\frac{R d_l(t)}{d_\phi(t)} \right)^{\frac{1}{2}} \cdot \left[R(d_l d_\phi)(t) \right]^{\frac{1}{2}} \lesssim \left[R(d_l d_\phi)(t) \right]^{\frac{1}{2}}.$$

We see that also in this case,

$$\begin{aligned} g(t,R) &\asymp k(R) + R^{\frac{1}{2}} \int\limits_{k(R)}^{\min\{t,\hbar(R)\}} \mathcal{D}(s)^{-\frac{1}{2}} \, \mathrm{d}s + R \int\limits_{\min\{t,\hbar(R)\}}^{t} d_l(s) \, \mathrm{d}s \\ &\lesssim R^{\frac{1}{2}} \int\limits_{k(R)}^{\infty} \mathcal{D}(s)^{-\frac{1}{2}} \, \mathrm{d}s. \end{aligned}$$

▷ Assume that $\delta_l = \delta_\phi > 1$.

By Karamata's theorem,

$$R \int_{\hbar(R)}^{\infty} d_l(s) \, \mathrm{d}s \asymp R \hbar(R) d_l(\hbar(R)) \asymp \hbar(R) d_\phi(\hbar(R)).$$

Since $\delta_\phi > 1$, the function $t \mapsto t d_\phi(t)$ is eventually decreasing. Since it is locally bounded, it is even bounded on $[1, \infty)$. Again we see that $g(t,R) \lesssim R^{\frac{1}{2}} \int\limits_{k(R)}^{\infty} \mathcal{D}(s)^{-\frac{1}{2}} \, \mathrm{d}s$ independently of t.

□

III.4.5 Lemma. *In the situation of Theorem III.4.1, the inequality*

$$\min\Big\{\mathscr{g}(s,R),\frac{R}{\mathscr{C}(s)}\Big\}\lesssim\mathscr{B}(R)\le\mathscr{B}(t,R)=\max\Big\{\mathscr{g}(t,R),\frac{R}{\mathscr{C}(t)}\Big\}\quad\text{(III.4.11)}$$

holds for $(s,t,R)\in[1,\infty)^2\times[\frac{2}{(d_l d_\phi)(1)},\infty)$. *In particular, if* $\tau(R)$ *is a function satisfying* $\mathscr{g}(\tau(R),R)\asymp\frac{R}{\mathscr{C}(\tau(R))}$, *then*

$$\mathscr{B}(R)\asymp\mathscr{g}(\tau(R),R)\asymp\frac{R}{\mathscr{C}(\tau(R))}.$$

Proof. The upper bound in (III.4.11) is obvious from the definition of $\mathscr{B}(t,R)$ and $\mathscr{B}(R)$. For the lower bound, we observe that

$$\begin{aligned} s\le t &\quad\Rightarrow\quad \mathscr{g}(s,R)\le\mathscr{g}(t,R),\\ s>t &\quad\Rightarrow\quad \frac{R}{\mathscr{C}(s)}\lesssim\frac{R}{\mathscr{C}(t)} \end{aligned}$$

and hence

$$\min\Big\{\mathscr{g}(s,R),\frac{R}{\mathscr{C}(s)}\Big\}\lesssim\mathscr{B}(t,R)\qquad(s,t)\in[1,\infty)^2.$$

Taking the infimum over t proves the lower bound in (III.4.11). □

Now we are ready for the proof of Theorem III.4.1.

Proof of Case $\boxed{A}$. We have $\operatorname{ind}\mathscr{f}=1+\gamma>0$, and hence $\mathscr{f}^-$ exists. Set

$$\tau(R):=\mathscr{f}^-\big(\frac{1}{\alpha}R\big).$$

Then, for $R\to\infty$,

$$\frac{1}{\alpha}R\sim\mathscr{f}(\tau(R))=\tau(R)\mathscr{C}(\tau(R))\log\Big[\alpha\frac{\tau(R)\mathscr{C}(\tau(R))}{\mathscr{D}(\tau(R))}\Big],$$

and hence

$$\frac{R}{\mathscr{D}(\tau(R))}\sim\Big(\alpha\frac{\tau(R)\mathscr{C}(\tau(R))}{\mathscr{D}(\tau(R))}\Big)\log\Big[\alpha\frac{\tau(R)\mathscr{C}(\tau(R))}{\mathscr{D}(\tau(R))}\Big].$$

Our choice of α implies that $2\mathscr{D}(\tau(R))\le R$ for all sufficiently large values of R, and therefore that $\tau(R)\le\mathscr{k}(R)$ for such R.

It holds that

$$\frac{t}{\log t}\circ t\log t\asymp t\quad\text{for } t\ge 4,$$

and we further obtain

$$\frac{\frac{R}{\mathscr{D}(\tau(R))}}{\log\left(\frac{R}{\mathscr{D}(\tau(R))}\right)} \asymp \frac{\tau(R)\mathscr{C}(\tau(R))}{\mathscr{D}(\tau(R))}.$$

Using this and Lemma III.4.2, thus

$$\frac{R}{\mathscr{C}(\tau(R))} \asymp \tau(R)\log\left(\frac{R}{\mathscr{D}(\tau(R))}\right) \asymp \mathscr{g}(\tau(R),R).$$

In view of Lemma III.4.5, this completes the proof of (III.4.4).

To see the additional statements, note first that

$$\operatorname{ind}\frac{R}{\mathscr{C}(\mathscr{f}^-(R))} = 1-\gamma\cdot\frac{1}{1+\gamma} = \frac{1}{1+\gamma}.$$

Further $\delta < 1+\gamma$ implies $\operatorname{ind}\frac{t\mathscr{C}(t)}{\mathscr{D}(t)} > 0$, and thus $\log\left(\alpha\frac{t\mathscr{C}(t)}{\mathscr{D}(t)}\right) \asymp \log t$. □

Proof of Case $\boxed{B}$. To obtain the bound from below, we use Lemma III.4.2 and our assumption that $t\mathscr{C}(t) \lesssim \mathscr{D}(t)$. Setting $t = \mathscr{k}(R) \asymp \mathscr{D}^-(R)$, we find that

$$\mathscr{g}(\mathscr{k}(R),R) \asymp \mathscr{k}(R), \qquad \frac{R}{\mathscr{C}(\mathscr{k}(R))} \asymp \frac{R}{\mathscr{C}(\mathscr{D}^-(R))} \gtrsim \mathscr{D}^-(R) \asymp \mathscr{k}(R).$$

Now we prove the upper bound for $\mathscr{B}(R)$. If Lemma III.4.4 is applicable, then

$$\mathscr{g}(t,R) \lesssim R^{\frac{1}{2}} \int\limits_{\mathscr{k}(R)}^{\infty} \mathscr{D}(s)^{-\frac{1}{2}}\,\mathrm{d}s.$$

Choosing $\tau(R)$ so large that $\frac{R}{\mathscr{C}(\tau(R))}$ is less than the right hand side of this relation, we see that

$$\mathscr{B}(R) \le \mathscr{B}(\tau(R),R) \lesssim R^{\frac{1}{2}} \int\limits_{\mathscr{k}(R)}^{\infty} \mathscr{D}(s)^{-\frac{1}{2}}\,\mathrm{d}s.$$

If Lemma III.4.4 is not applicable, we must have $\delta_l = \delta_\phi = 1$ and $\gamma > 0$. Choose ρ so large that

$$\operatorname{ind}\mathscr{k} = \frac{1}{\delta} > 1-\gamma\rho = \operatorname{ind}\left(\frac{R}{\mathscr{C}(R^\rho)}\right).$$

Since $\frac{\mathscr{d}_\phi}{\mathscr{d}_l}$ is slowly varying, we have $R^\rho \lesssim \hbar(R)$, cf. Lemma III.A.8. Therefore,

$$\mathscr{g}(\hbar(R), R) \gtrsim \bar{k}(R) \gg \frac{R}{\mathscr{C}(R^\rho)} \geq \frac{R}{\mathscr{C}(\hbar(R))}.$$

We thus obtain

$$\mathscr{B}(R) \lesssim \mathscr{B}(\hbar(R), R) \asymp \mathscr{g}(\hbar(R), R) \lesssim R^{\frac{1}{2}} \int\limits_{\bar{k}(R)}^{\infty} \mathscr{D}(s)^{-\frac{1}{2}}\, \mathrm{d}s.$$

To see the additional statements, note first that $\operatorname{ind} \bar{k} = \frac{1}{\delta}$ and also

$$\operatorname{ind}\Big(R^{\frac{1}{2}} \int_{\bar{k}(R)}^{\infty} \mathscr{D}(s)^{-\frac{1}{2}}\, \mathrm{d}s\Big) = \frac{1}{2} + \frac{1}{\delta} \cdot \Big(1 - \frac{\delta}{2}\Big) = \frac{1}{\delta}.$$

Further, $\delta > 2$ implies that

$$R^{\frac{1}{2}} \int_{\bar{k}(R)}^{\infty} \mathscr{D}(s)^{-\frac{1}{2}}\, \mathrm{d}s \asymp R^{\frac{1}{2}} \bar{k}(R) \mathscr{D}(\bar{k}(R))^{-\frac{1}{2}} \asymp \bar{k}(R).$$

$\square$

Proof of Case $\boxed{C}$. We have

$$\operatorname{ind} \mathscr{f}_0 = 2 - \delta + 2\gamma, \quad \operatorname{ind} \mathscr{f}_1 = 2\Big(\gamma + \big(1 - \frac{\delta}{2}\big)\Big) = 2 - \delta + 2\gamma.$$

Since $\mathscr{D}^{-\frac{1}{2}}$ is not integrable, we must have $\delta \leq 2$. The case that $\delta = 2$ and $\gamma = 0$ is ruled out by assumption, and it follows that $2 - \delta + 2\gamma > 0$. Thus, asymptotic inverses $\mathscr{f}_0^-$ and $\mathscr{f}_1^-$ exist.

To prove the bound from below, we use $\mathscr{f}_0^-(R)$ pushed into the interval $[\bar{k}(R), \hbar(R)]$, namely we set

$$\tau(R) := \min\big\{\max\big\{\mathscr{f}_0^-(R), \bar{k}(R)\big\}, \hbar(R)\big\}.$$

Our assumption that $\frac{1}{\mathscr{d}_l(t)} \lesssim t\mathscr{C}(t) \lesssim \mathscr{D}(t)$ gives

$$\frac{\mathscr{d}_\phi(t)}{\mathscr{d}_l(t)} \lesssim \mathscr{f}_0(t) \lesssim \mathscr{D}(t),$$

and hence $\bar{k}(R) \lesssim \mathscr{f}_0^-(R) \lesssim \hbar(R)$. Thus, $\tau(R) \asymp \mathscr{f}_0^-(R)$.

Using Lemma III.4.3 we obtain

$$\mathscr{g}(\tau(R), R) \gtrsim R^{\frac{1}{2}} \tau(R) \mathscr{D}(\tau(R))^{-\frac{1}{2}} \asymp R^{\frac{1}{2}} \mathscr{f}_0^-(R) \mathscr{D}(\mathscr{f}_0^-(R))^{-\frac{1}{2}}.$$

On the other hand, the definition of $\mathscr{f}_0$ yields for $R \to \infty$ that

$$R \sim \mathscr{f}_0(\mathscr{f}_0^-(R)) = \frac{\mathscr{f}_0^-(R)^2 \mathscr{C}(\mathscr{f}_0^-(R))^2}{\mathscr{D}(\mathscr{f}_0^-(R))},$$

and hence

$$\frac{R^{\frac{1}{2}}}{\mathscr{C}(\mathscr{f}_0^-(R))} \sim \mathscr{f}_0^-(R)\mathscr{D}(\mathscr{f}_0^-(R))^{-\frac{1}{2}}.$$

Together, thus,

$$\min\Big\{\mathscr{g}(\tau(R),R), \frac{R}{\mathscr{C}(\tau(R))}\Big\} \gtrsim R^{\frac{1}{2}}\mathscr{f}_0^-(R)\mathscr{D}(\mathscr{f}_0^-(R))^{-\frac{1}{2}} \asymp \frac{R}{\mathscr{C}(\mathscr{f}_0^-(R))}.$$

For the proof of the upper bound we use

$$\tau(R) := \min\big\{\max\big\{\mathscr{f}_1^-(R), \mathscr{k}(R)\big\}, \mathscr{h}(R)\big\}.$$

Since $\mathscr{f}_0 \lesssim \mathscr{f}_1$ we have $\mathscr{f}_1^- \lesssim \mathscr{f}_0^- \lesssim \mathscr{h}$, and therefore $\mathscr{f}_1^-(R) \lesssim \tau(R)$. This readily implies that

$$\frac{R}{\mathscr{C}(\tau(R))} \lesssim \frac{R}{\mathscr{C}(\mathscr{f}_1^-(R))}.$$

To estimate $\mathscr{g}(\tau(R),R)$ we distinguish two cases. First assume that $\mathscr{f}_1^-(R) \le \mathscr{k}(R)$. Then $\tau(R) = \mathscr{k}(R)$, and hence

$$\mathscr{g}(\tau(R),R) \asymp \mathscr{k}(R) \lesssim \frac{R}{\mathscr{C}(\mathscr{k}(R))} \le \frac{R}{\mathscr{C}(\mathscr{f}_1^-(R))}.$$

Second assume that $\mathscr{f}_1^-(R) \ge \mathscr{k}(R)$. Then $\tau(R) \le \mathscr{f}_1^-(R)$. We use Lemma III.4.3 to obtain

$$\mathscr{g}(\tau(R),R) \lesssim R^{\frac{1}{2}} \int_1^{\tau(R)} \mathscr{D}(s)^{-\frac{1}{2}}\,\mathrm{d}s \lesssim R^{\frac{1}{2}} \int_1^{\mathscr{f}_1^-(R)} \mathscr{D}(s)^{-\frac{1}{2}}\,\mathrm{d}s.$$

The last integral can be evaluated from the definition of $\mathscr{f}_1$: for $R \to \infty$

$$R \sim \mathscr{f}_1(\mathscr{f}_1^-(R)) = \Big(\mathscr{C}(\mathscr{f}_1^-(R)) \int_1^{\mathscr{f}_1^-(R)} \mathscr{D}(s)^{-\frac{1}{2}}\,\mathrm{d}s\Big)^2,$$

and hence

$$\int_1^{\mathscr{f}_1^-(R)} \mathscr{D}(s)^{-\frac{1}{2}}\,\mathrm{d}s \sim \frac{R^{\frac{1}{2}}}{\mathscr{C}(\mathscr{f}_1^-(R))}.$$

Altogether we see that always

$$\max\Big\{\mathscr{g}(\tau(R),R), \frac{R}{\mathscr{C}(\tau(R))}\Big\} \lesssim \frac{R}{\mathscr{C}(\mathscr{f}_1^-(R))}.$$

This completes the proof of (III.4.6).

To see the additional statements, recall that $\operatorname{ind} f_0 = \operatorname{ind} f_1 = 2 - \delta + 2\gamma$. This implies that

$$\operatorname{ind} \frac{R}{\mathscr{C}(f_0^-(R))} = \operatorname{ind} \frac{R}{\mathscr{C}(f_1^-(R))} = 1 - \gamma \cdot \frac{1}{2 - \delta + 2\gamma} = \frac{2 - \delta + \gamma}{2 - \delta + 2\gamma}.$$

Further, $\delta < 2$ implies that $\int_1^t \mathscr{D}(s)^{-\frac{1}{2}}\,\mathrm{d}s \asymp t\mathscr{D}(t)^{-\frac{1}{2}}$, and hence $f_0 \asymp f_1$. $\square$

Proof of Case $\boxed{D}$. The first thing to show is that $\frac{d_\phi}{d_l}$ is unbounded: if we had $d_\phi(t) \lesssim d_l(t)$, it would follow that

$$\mathscr{D}(t)^{-\frac{1}{2}} = \sqrt{d_l(t) d_\phi(t)} \lesssim d_l(t),$$

and this contradicts the fact that $d_l(t)$ is integrable while $\mathscr{D}^{-\frac{1}{2}}(t)$ is not.

To show the bound from below asserted in (III.4.7), we note that

$$g(\hbar(R), R) \gtrsim R^{\frac{1}{2}} \hbar(R) \mathscr{D}(\hbar(R))^{-\frac{1}{2}}$$

by Lemma III.4.3, and that

$$\frac{R}{\mathscr{C}(\hbar(R))} \gtrsim R\hbar(R) d_l(\hbar(R))$$

by our assumption that $t\mathscr{C}(t) \lesssim \frac{1}{d_l(t)}$. Since $R \asymp \frac{d_\phi}{d_l}(\hbar(R))$ for $R \to \infty$, we obtain

$$R\hbar(R) d_l(\hbar(R)) \asymp R^{\frac{1}{2}} \hbar(R) \Big(\frac{d_\phi(\hbar(R))}{d_l(\hbar(R))}\Big)^{\frac{1}{2}} d_l(\hbar(R)) = R^{\frac{1}{2}} \hbar(R) \mathscr{D}(\hbar(R))^{-\frac{1}{2}},$$

and see that

$$\min\Big\{ g(\hbar(R), R), \frac{R}{\mathscr{C}(\hbar(R))} \Big\} \gtrsim R\hbar(R) d_l(\hbar(R)).$$

For the proof of the bound from above, we show that $g(t, R)$ is bounded independently of $t \geq \hbar(R)$. Using again Lemma III.4.3, we find

$$\begin{aligned} g(t, R) &= g(\hbar(R), R) + R \int_{\hbar(R)}^t d_l(s)\,\mathrm{d}s \\ &\lesssim R^{\frac{1}{2}} \int_1^{\hbar(R)} \mathscr{D}(s)^{-\frac{1}{2}}\,\mathrm{d}s + R \int_{\hbar(R)}^\infty d_l(s)\,\mathrm{d}s. \end{aligned}$$

It remains to choose $\tau(R)$ sufficiently large, so that $\frac{R}{\mathscr{C}(\tau(R))} \leq g(\tau(R)), R)$, which is clearly possible. This completes the proof of (III.4.7).

To see the additional statements, assume first that $\delta_l > \delta_\phi$. Then $\hbar(R)$ is regularly varying with index $\frac{1}{\delta_l - \delta_\phi}$. We obtain

$$\operatorname{ind}\Big(R\hbar(R)d_l(\hbar(R))\Big) = 1 + \frac{1}{\delta_l - \delta_\phi} - \frac{\delta_l}{\delta_l - \delta_\phi} = \frac{1 - \delta_\phi}{\delta_l - \delta_\phi}.$$

By Karamata's theorem

$$\operatorname{ind}\Big(R^{\frac{1}{2}} \int_1^{\hbar(R)} \mathscr{D}(s)^{-\frac{1}{2}}\,\mathrm{d}s\Big) = \frac{1}{2} + \big(1 - \frac{\delta}{2}\big)\frac{1}{\delta_l - \delta_\phi} = \frac{1 - \delta_\phi}{\delta_l - \delta_\phi},$$

$$\operatorname{ind}\Big(R \int_{\hbar(R)}^{\infty} d_l(s)\,\mathrm{d}s\Big) = 1 + (1 - \delta_l)\frac{1}{\delta_l - \delta_\phi} = \frac{1 - \delta_\phi}{\delta_l - \delta_\phi}.$$

Further, again referring to Karamata's theorem, $\delta < 2$ implies

$$R^{\frac{1}{2}} \int_1^{\hbar(R)} \mathscr{D}(s)^{-\frac{1}{2}}\,\mathrm{d}s \asymp R^{\frac{1}{2}}\hbar(R)\mathscr{D}(\hbar(R))^{-\frac{1}{2}} \asymp R\hbar(R)d_l(\hbar(R)),$$

and $\delta_l > 1$ implies

$$R \int_{\hbar(R)}^{\infty} d_l(s)\,\mathrm{d}s \asymp R\hbar(R)d_l(\hbar(R)).$$

□

III.4.2 Bound for the monodromy matrix

We combine Theorem III.2.2 with Theorem III.4.1 to obtain a bound for the growth of W_H when the lengths and angle differences of H are bounded by regularly varying functions. This yields a far reaching generalisation of Corollary III.2.5.

III.4.6 Theorem. *Let $(l_j)_{j=1}^\infty$ be a summable sequence of positive numbers, and $(\phi_j)_{j=1}^\infty$ a sequence of real numbers. Denote by H the Hamburger Hamiltonian with these lengths and angles, and let W_H be its monodromy matrix.*

Let $\psi \in \mathbb{R}$ and let d_l, d_ϕ, c_l, c_ϕ be regularly varying functions that are $\approx$ to some nonincreasing functions, such that $d_l \asymp 1 \asymp d_\phi$ locally, $c_\phi(t) \lesssim$

$c_l(t)$ for sufficiently large t, $\operatorname{ind} d_l + \operatorname{ind} d_\phi < 0$, and $\lim_{t\to\infty}\big(c_l c_\phi\big)(t) = 0$. Assume that

$$l_j \lesssim d_l(j), \qquad |\sin(\phi_{j+1}-\phi_j)| \lesssim d_\phi(j), \quad j \text{ sufficiently large},$$
$$\sum_{j=N+1}^{\infty} l_j \lesssim c_l(N), \quad \sum_{j=N+1}^{\infty} l_j \sin^2(\phi_j-\psi) \lesssim c_\phi(N), \quad N \text{ sufficiently large}.$$

Denote

$$\mathscr{D}(t) := \frac{1}{(d_l d_\phi)(t)}, \ \delta := \operatorname{ind}\mathscr{D}, \quad \mathscr{C}(t) := \frac{1}{(c_l c_\phi)^{\frac{1}{2}}(t)}, \ \gamma := \operatorname{ind}\mathscr{C}.$$

Then we have the following bounds for W_H and its order ρ_H.

<table>
<tr><th colspan="3">Data satisfies</th><th>$\log\big(\max_{|z|=R}\|W_H(z)\|\big)$ is $\lesssim$</th><th>$\rho_H \leq$</th></tr>
<tr><td colspan="3">$\mathscr{D}(t) \lesssim t\mathscr{C}(t)$</td><td>$\frac{R}{\mathscr{C}(f^-(R))}$ where
$f(t) := t\mathscr{C}(t)\log\big[\alpha\frac{t\mathscr{C}(t)}{\mathscr{D}(t)}\big]$
$\alpha := 4\sup_{t\geq 1}\frac{\mathscr{D}(t)}{t\mathscr{C}(t)}$</td><td>$\frac{1}{1+\gamma}$</td></tr>
<tr><td rowspan="3">$t\mathscr{C}(t) \lesssim \mathscr{D}(t)$,</td><td colspan="2">$\int_1^\infty \mathscr{D}(s)^{-\frac{1}{2}}\,\mathrm{d}s < \infty$,
($\gamma>0$ or $\frac{d_\phi}{d_l} \approx$ to nondecreasing)</td><td>$R^{\frac{1}{2}}\int_{h(R)}^{\infty} \mathscr{D}(s)^{-\frac{1}{2}}\,\mathrm{d}s$</td><td>$\frac{1}{\delta}$</td></tr>
<tr><td rowspan="2">$\int_1^\infty \mathscr{D}(s)^{-\frac{1}{2}}\,\mathrm{d}s = \infty$,</td><td>$\frac{1}{d_l(t)} \lesssim t\mathscr{C}(t)$,
$(\delta,\gamma) \neq (2,0)$</td><td>$\frac{R}{\mathscr{C}(f_1^-(R))}$ where
$f_1(t) := [\mathscr{C}(t)\int_1^t \mathscr{D}(s)^{-\frac{1}{2}}\,\mathrm{d}s]^2$</td><td>$\frac{2-\delta+\gamma}{2-\delta+2\gamma}$</td></tr>
<tr><td>$t\mathscr{C}(t) \lesssim \frac{1}{d_l(t)}$,
$\int_1^\infty d_l(s)\,\mathrm{d}s < \infty$,
$\delta_l > \delta_\phi$</td><td>$R^{\frac{1}{2}}\int_1^{h(R)} \mathscr{D}(s)^{-\frac{1}{2}}\,\mathrm{d}s$
$+R\int_{h(R)}^{\infty} d_l(s)\,\mathrm{d}s$</td><td>$\frac{1-\delta_\phi}{\delta_l-\delta_\phi}$</td></tr>
</table>

In each case the relation $\lesssim$ holds for R sufficiently large.

The main assumptions on the data d_l, d_ϕ, c_l, c_ϕ put in this theorem are that those functions are regularly varying and that $\operatorname{ind} d_l + \operatorname{ind} d_\phi < 0$ (i.e., $\neq 0$). Monotonicity assumptions are only a minor restriction: for example they are automatically fulfilled whenever the function under consideration is not slowly varying. The assumption that $c_\phi \lesssim c_l$ is no loss in generality, since replacing c_ϕ by $\min\{c_\phi, c_l\}$ does not affect validity of any of the other assumptions.

Proof of Theorem III.4.6.

① Observe that neither the assumptions of the theorem nor the case distinction in the assertion of the theorem depends on the equivalence class modulo $\approx$ of the functions d_l, d_ϕ, c_l, c_ϕ. Further, using Remark III.A.7, we see that the functions written in the second column of the table change only up to $\approx$ when we pass to other data equivalent modulo $\approx$ to d_l, d_ϕ, c_l, c_ϕ.

Hence, it is enough to prove the theorem for suitable modifications $\widehat{d}_l, \widehat{d}_\phi, \widehat{c}_l, \widehat{c}_\phi$ of d_l, d_ϕ, c_l, c_ϕ which differ only up to $\approx$.

② The task is to define $\widehat{d}_l, \widehat{d}_\phi, \widehat{c}_l, \widehat{c}_\phi$ in such a way that Theorem III.2.2, Proposition III.3.9, and Theorem III.4.1 become applicable. To this end we use Lemma III.A.10 and the freedom of choice of nonincreasing smoothenings mentioned in Remark III.A.11.

We choose $\mathcal{S}[c_\phi]$ such that

$$\forall N \in \mathbb{N}. \ \sum_{j=N+1}^{\infty} l_j \sin^2(\phi_j - \psi) \le \mathcal{S}[c_\phi](N),$$

which is clearly possible by first choosing an arbitrary nonincreasing smoothening of c_ϕ and then multiplying it with a sufficiently large positive constant. Next, we choose $\mathcal{S}[c_l]$ such that

$$\mathcal{S}[c_\phi] \le \mathcal{S}[c_l], \qquad \forall N \in \mathbb{N}. \ \sum_{j=N+1}^{\infty} l_j \le \mathcal{S}[c_l](N).$$

A suitable modification of d_ϕ is found as follows. If $\liminf_{t\to\infty} d_\phi(t) > 0$, we choose $\mathcal{S}[d_\phi] := 1$ (which corresponds to the choices $\epsilon[d_\phi] := 0$ and $\kappa[d_\phi] := 1$). If $\lim_{t\to\infty} d_\phi(t) = 0$, we choose $\mathcal{S}[d_\phi]$ such that

$$\mathcal{S}[d_\phi] \le 1, \qquad \forall j \in \mathbb{N}. \ |\sin(\phi_{j+1} - \phi_j)| \le \mathcal{S}[d_\phi](j),$$

which is possible by choosing an arbitrary nonincreasing smoothening of d_ϕ, then multiplying it with a sufficiently large positive constant, and then cutting it off at 1.

It remains to define a modification of d_l. In the generic case that $\gamma > 0$ we proceed just the same as above and choose $\mathcal{S}[c_l]$ such that

$$\forall j \in \mathbb{N}. \ l_j \le \mathcal{S}[d_l](j). \tag{III.4.12}$$

In the boundary case that $\gamma = 0$ we make a further case distinction. If we are in the situation of the 1st or the 3rd row of the table in the theorem, we do just the same as above. If we are in the situation of the 2nd or the 4th row, the additional assumption ensures that $\frac{d_l}{d_\phi}$ is $\approx$ to some nonincreasing function, and we choose

$$\mathcal{S}[d_l] := \mathcal{S}[d_\phi] \cdot \mathcal{S}\Big[\frac{d_l}{d_\phi}\Big],$$

where $\mathcal{S}\big[\frac{d_l}{d_\phi}\big]$ is sufficiently large to ensure that (III.4.12) holds.

Now we set

$$\widehat{d}_l := \mathcal{S}[d_l],\ \widehat{d}_\phi := \mathcal{S}[d_\phi],\ \widehat{c}_l := \mathcal{S}[c_l],\ \widehat{c}_\phi := \mathcal{S}[c_\phi].$$

③ We apply our previous results with the data $\widehat{d}_l, \widehat{d}_\phi, \widehat{c}_l, \widehat{c}_\phi$. Here we denote by $\widehat{B}(R)$, $\widehat{L}(t,R)$, $\widehat{\mathscr{B}}(R)$, etc. the correspondingly defined functions.

Theorem III.2.2 implies that

$$\log\Big(\max_{|z|=R} \|W_H(z)\|\Big) \lesssim \widehat{B}(R),$$

and we face the task to control $\widehat{L}(t,R)$. In almost all cases Proposition III.3.9 takes care of this:

- ▷ Proposition III.3.9 (i) applies if $\delta_\phi > \delta_l$, or if $\gamma = 0$ and we are in the situation of the 2nd row of the table.
- ▷ Proposition III.3.9 (ii) applies if $\gamma > 0$ and $\delta_\phi < \delta_l$.
- ▷ Proposition III.3.9 (iii) applies if $\gamma > 0$ and $\delta_\phi = \delta_l$.

Thus, in all of these cases, $\widehat{B}(R) \asymp \widehat{\mathscr{B}}(R)$, and the bounds asserted in the second column of the table follows from Theorem III.4.1.

It remains to study the situation that $\gamma = 0$, $\delta_\phi \leq \delta_l$, and we are in the 1st or 3rd row of the table. Applying Corollary III.3.7 (ii) if $\delta_\phi < \delta_l$ and Lemma III.3.8 if $\delta_\phi = \delta_l$, yields

$$\widehat{L}(t,R) \lesssim 1 + \log t.$$

Let $\tau(R)$ be the power bounded function used in Theorem III.4.1 $\boxed{\text{A}}\boxed{\text{C}}$ to estimate $\widehat{\mathscr{B}}(R)$. Then

$$\widehat{B}(R) \leq \widehat{\mathscr{B}}(\tau(R),R) + \widehat{L}(\tau(R),R) \lesssim \widehat{\mathscr{B}}(\tau(R),R).$$

The bounds asserted in the second column of the table thus follow from Theorem III.4.1.

④ The bounds for ρ_H arise simply by taking the indices of the regularly varying functions in the second column.

$\square$

In the following corollary we revisit the setting of Theorem III.4.6, except we are not given functions $\mathpzc{c}_l, \mathpzc{c}_\phi$.

III.4.7 Corollary. *Let H be a Hamburger Hamiltonian and let $\mathpzc{d}_l, \mathpzc{d}_\phi$ be regularly varying. Assume that $\mathpzc{d}_\phi(t)$ is $\sim$ to a nonincreasing function as $t \to \infty$ and that $\mathpzc{d}_l \asymp 1 \asymp \mathpzc{d}_\phi$ locally.*

Assume that the lenghts and angles of H are bounded as

$$l_j \lesssim \mathpzc{d}_l(j), \ |\sin(\phi_{j+1} - \phi_j)| \lesssim \mathpzc{d}_\phi(j) \text{ for sufficiently large } j,$$

and that $\mathpzc{d}_l \in L^1([1, \infty))$. Then the following statements hold.

- ▷ *If $\delta > 2$, then $\log\big(\max_{|z|=R} \|W_H(z)\|\big) \lesssim \mathpzc{k}(R)$.*
- ▷ *If $0 < \delta < 2$, then $\log\big(\max_{|z|=R} \|W_H(z)\|\big) \lesssim R \int\limits_{\mathpzc{k}(R)}^{\infty} \mathpzc{d}_l(x)\,\mathrm{d}x$.*
- ▷ *If $0 < \delta \leq 2$, $\delta_l > 1$, and there exists $\psi \in \mathbb{R}$ such that $|\sin(\phi_j - \psi)| \lesssim |\sin(\phi_{j+1} - \phi_j)|$, then again $\log\big(\max_{|z|=R} \|W_H(z)\|\big) \lesssim \mathpzc{k}(R)$.*
- ▷ *If $\delta = 2$ and $(\delta_l, \delta_\phi) \neq (1, 1)$, then $\rho_H \leq \frac{1}{2}$.*

Proof. Our goal is to apply Theorem III.4.6, and in order to do that we need to construct suitable functions $\mathpzc{c}_l, \mathpzc{c}_\phi$. Note that the assumption $\mathpzc{d}_l \in L^1([1, \infty))$ implies $\delta_l \geq 1$.

① Without any a priori assumption, we can set

$$\mathpzc{c}_l(t) := \mathpzc{c}_\phi(t) := \int_t^\infty \mathpzc{d}_l(x)\,\mathrm{d}x \gtrsim t\mathpzc{d}_l(t).$$

Then $\mathpzc{d}_l, \mathpzc{d}_\phi$ together with $\mathpzc{c}_l, \mathpzc{c}_\phi$ satisfy the general assumptions of Theorem III.4.6. We have

$$t\mathscr{C}(t) \lesssim \frac{1}{\mathpzc{d}_l(t)} \lesssim \mathscr{D}(t)$$

and $\gamma = \delta_l - 1 \geq 0$. This choice of $\mathpzc{c}_l, \mathpzc{c}_\phi$ is sufficient to prove the asserted bounds in the following cases:

- ▷ $\delta > 2 \wedge \delta_l > 1$. Since $\gamma > 0$, Theorem III.4.6 gives the upper bound $R^{\frac{1}{2}} \int\limits_{\mathpzc{k}(R)}^{\infty} \mathscr{D}(s)^{-\frac{1}{2}}\,\mathrm{d}s \asymp \mathpzc{k}(R)$.

▷ $\delta < 2$. Then $\delta_\phi = \delta - \delta_l < 1 \leq \delta_l$, hence $\frac{d_\phi}{d_l}$ has positive index and is eventually nondecreasing. Since

$$R^{\frac{1}{2}} \int_1^{\hbar(R)} \mathfrak{D}(s)^{-\frac{1}{2}}\,\mathrm{d}s \asymp R^{\frac{1}{2}}\hbar(R)\mathfrak{D}(\hbar(R))^{-\frac{1}{2}} \lesssim R\int_{\hbar(R)}^\infty d_l(s)\,\mathrm{d}s,$$

Theorem III.4.6 gives the asserted upper bound.

▷ $\delta = 2 \wedge \delta_l > 1$. Then we are either in the second or fourth row of Theorem III.4.6, but in both cases the upper bound for the order is equal to $\frac{1}{2}$.

② Assume $\delta > 2$ and $\delta_\phi > 1$. This implies that $\sum_{j=1}^\infty |\sin(\phi_{j+1} - \phi_j)| < \infty$. We want to choose $\psi := \lim_{j\to\infty}\phi_j$, so we need to prove that this limit actually exists (at least if the angles ϕ_j are all modified by adding integer multiples of π which leaves H unchanged). Start by choosing n_0 so large that $\sum_{j=n_0}^\infty |\sin(\phi_{j+1} - \phi_j)| \leq \frac{\pi}{8}$. Adding to ϕ_j an integer multiple of π, we may assume $|\phi_j - \phi_{n_0}| \leq \frac{\pi}{2}$ for all $j \in \mathbb{N}$. Since $|x| \leq 2|\sin(x)|$ for $|x| \leq \frac{\pi}{2}$, and $|\sin(x+y)| \leq |\sin(x)| + |\sin(y)|$, we have for $j > k \geq n_0$

$$\begin{aligned}|\phi_j - \phi_k| &\leq |\phi_j - \phi_{n_0}| + |\phi_{n_0} - \phi_k| \leq 2\big(|\sin(\phi_j - \phi_{n_0})| + |\sin(\phi_{n_0} - \phi_k)|\big)\\ &\leq 4\sum_{n=n_0}^\infty |\sin(\phi_{n+1} - \phi_n)| \leq \frac{\pi}{2}.\end{aligned}$$

Therefore,

$$|\phi_j - \phi_k| \leq 2|\sin(\phi_j - \phi_k)| \leq 2\sum_{n=k}^{j-1} |\sin(\phi_{n+1} - \phi_n)|$$

and thus $(\phi_j)_{j=1}^\infty$ is a Cauchy sequence. Let ψ be its limit. We set

$$c_l(t) := \int_t^\infty d_l(x)\,\mathrm{d}x, \qquad c_\phi(t) := c_l(t)\cdot\Big(\int_t^\infty d_\phi(x)\,\mathrm{d}x\Big)^2$$

and observe that d_l, d_ϕ together with c_l, c_ϕ satisfy the assumptions of Theorem III.4.6. A calculation shows that $\gamma = \delta - 2 > 0$. Hence $\mathrm{ind}[t\mathscr{C}(t)| = \delta - 1 < \delta$ and in particular $t\mathscr{C}(t) \ll \mathfrak{D}(t)$. Again Theorem III.4.6 provides the desired upper bound.

③ Assume that we have $\psi \in \mathbb{R}$ such that $|\phi_j - \psi| \lesssim |\phi_{j+1} - \phi_j|$, and $\delta_l > 1$. Set

$$\begin{aligned} c_l(t) &:= \int_t^\infty d_l(x)\,\mathrm{d}x \asymp t d_l(t),\\ c_\phi(t) &:= \int_t^\infty d_l(x) d_\phi(x)^2\,\mathrm{d}x \asymp t d_l(t) d_\phi(t)^2.\end{aligned}$$

Then $t\mathscr{C}(t) \asymp \mathscr{D}(t)$ and we are in the first row of Theorem III.4.6. Now we note that the bound given in the theorem is $\asymp$ to $\mathscr{k}(R)$.

□

III.5 Additions and examples

III.5.1 Combining with a bound from below

In the same way that upper bounds for lengths and angle differences lead to upper bounds for the growth of the monodromy matrix, lower bounds lead to lower bounds. We recall a result obtained in [PW23, Corollary 2.5].

III.5.1 Proposition. *Let H be a Hamburger Hamiltonian with lengths $(l_j)_{j=1}^\infty$ and angles $(\phi_j)_{j=1}^\infty$, and let $\mathscr{f}$ be regularly varying with positive index. If*

$$l_{j+1}l_j \sin^2(\phi_{j+1} - \phi_j) \gtrsim \frac{1}{\mathscr{f}(j)}, \qquad j \in \mathbb{N}, \tag{III.5.1}$$

then

$$\log\Big(\max_{|z|=R} \|W_H(z)\|\Big) \gtrsim \mathscr{f}^-(R^2)$$

for sufficiently large R.

Let us translate Proposition III.5.1 to the setting where we compare the lengths and the angle differences to regularly varying functions $\mathscr{d}_l, \mathscr{d}_\phi$.

III.5.2 Corollary. *Let $(l_j)_{j=1}^\infty$ be a summable sequence of positive numbers, and let $(\phi_j)_{j=1}^\infty$ be a sequence of real numbers. Denote by H the Hamburger Hamiltonian with these lengths and angles, and let W_H be its monodromy matrix. Assume that $\mathscr{d}_l, \mathscr{d}_\phi$ are regularly varying and satisfy*

$$\forall j \in \mathbb{N}. \quad l_j \gtrsim \mathscr{d}_l(j) \qquad |\sin(\phi_{j+1} - \phi_j)| \gtrsim \mathscr{d}_\phi(j).$$

Then

$$\log\Big(\max_{|z|=R} \|W_H(z)\|\Big) \gtrsim \Big[\frac{1}{\mathscr{d}_l\mathscr{d}_\phi}\Big]^-(R)$$

for sufficiently large R.

In particular, the order of W_H is at least $\frac{1}{\delta}$, where $\delta := -(\operatorname{ind} \mathscr{d}_l + \operatorname{ind} \mathscr{d}_\phi)$.

Proof. Since $\mathscr{d}_l$ is regularly varying, we have $\mathscr{d}_l(t+1) \sim \mathscr{d}_l(t)$. Setting $\mathscr{D}(t) := \frac{1}{(\mathscr{d}_l \mathscr{d}_\phi)(t)}$, we see that (III.5.1) is satisfied for $\mathscr{f}(t) := \mathscr{D}(t)^2$. Since $\mathscr{f}^-(t) = \mathscr{D}^-(t^{\frac{1}{2}})$, we obtain

$$\log\Big(\max_{|z|=R} \|W_H(z)\|\Big) \gtrsim \mathscr{f}^-(R^2) = \mathscr{D}^-(R) = \Big[\frac{1}{\mathscr{d}_l \mathscr{d}_\phi}\Big]^-(R).$$

□

If the lengths and angle differences are well-behaved and summable, the growth of W_H can be determined up to $\asymp$. Note that no functions $\mathscr{c}_l, \mathscr{c}_\phi$ appear in the formulation of the following theorem.

III.5.3 Theorem. *Let $(l_j)_{j=1}^\infty$ be a summable sequence of positive numbers, and let $(\phi_j)_{j=1}^\infty$ be a sequence of real numbers. Denote by H the Hamburger Hamiltonian with these lengths and angles. Consider regularly varying functions $\mathscr{d}_l, \mathscr{d}_\phi$ with $\mathscr{d}_l \asymp 1 \asymp \mathscr{d}_\phi$ locally. If*

(i) *$l_j \asymp \mathscr{d}_l(j)$ and $|\sin(\phi_{j+1} - \phi_j)| \asymp \mathscr{d}_\phi(j)$ for sufficiently large j,*

(ii) *$\delta := -\big(\operatorname{ind} \mathscr{d}_l + \operatorname{ind} \mathscr{d}_\phi\big) > 2$,*

then

$$\log\Big(\max_{|z|=R} \|W_H(z)\|\Big) \asymp \Big[\frac{1}{\mathscr{d}_l \mathscr{d}_\phi}\Big]^-(R) \qquad \textit{for sufficiently large } R,$$

and

$$\rho_H = \frac{1}{\delta}.$$

Proof. Due to Corollary III.5.2, we only need to show that

$$\log\big(\max_{|z|=R} \|W_H(z)\|\big) \lesssim \big[\frac{1}{\mathscr{d}_l \mathscr{d}_\phi}\big]^-(R).$$

We notice that $\operatorname{ind} \mathscr{d}_l \leq -1$ since $(l_j)_{j=1}^\infty$ is summable. Hence $\mathscr{d}_l$ is $\sim$ to an eventually monotone function, and thus

$$\int_M^\infty \mathscr{d}_l(x)\,\mathrm{d}x \lesssim \sum_{j=M}^\infty \mathscr{d}_l(j) \lesssim \sum_{j=M}^\infty l_j < \infty.$$

Since $\mathscr{d}_l \asymp 1$ locally, this shows that $\mathscr{d}_l \in L^1([1,\infty))$. By Corollary III.4.7,

$$\log\Big(\max_{|z|=R} \|W_H(z)\|\Big) \lesssim \mathscr{k}(R) \asymp \Big[\frac{1}{\mathscr{d}_l \mathscr{d}_\phi}\Big]^-(R).$$

□

III.5.2 Power-log-majorisations and exceptional cases

By considering $\mathscr{d}_l, \mathscr{d}_\phi, \mathscr{c}_l, \mathscr{c}_\phi$ consisting of a power times a power of a logarithm, we can gain some insight into the exceptional cases of our results. For data of this form all functions occurring in our results can in principle be computed explicitly. We do not aim at giving a complete picture, but rather give a couple of illustrative examples. It should also be added that not all phenomena (counterexamples, exceptional cases, or similar) can be illustrated with functions of this form; one would have to admit an additional double-logarithmic factor.

The facts presented below are shown by straightforward, yet tedious and elaborate, computations. It is practical to use the lexicographic order on $\mathbb{R}^2$, and we denote it by $\preceq$. Explicitly, thus

$$(\alpha, \beta) \preceq (\alpha', \beta') :\Leftrightarrow \alpha < \alpha' \vee \big(\alpha = \alpha' \wedge \beta \leq \beta'\big)$$

and, as ususal, $\prec$ stands for "$\preceq$ but not $=$".

III.5.4 Setting and Notation. Assume we are given parameters

$$(\delta_l, \alpha_l), (\delta_\phi, \alpha_\phi), (\gamma_l, \beta_l), (\gamma_\phi, \beta_\phi) \in [0, \infty) \times \mathbb{R},$$

and denote

$$\delta := \delta_l + \delta_\phi, \;\; \alpha := \alpha_l + \alpha_\phi, \gamma := \frac{1}{2}(\gamma_l + \gamma_\phi), \;\; \beta := \frac{1}{2}(\beta_l + \beta_\phi).$$

Assume that these parameters satisfy

- ▷ $(\delta_l, \alpha_l), (\delta_\phi, \alpha_\phi), (\gamma_l, \beta_l), (\gamma_\phi, \beta_\phi) \succeq (0, 0)$,
- ▷ $\delta > 0, (\gamma, \beta) \succ (0, 0)$,
- ▷ $(\gamma_l, \beta_l) \preceq (\gamma_\phi, \beta_\phi)$.

Let $\mathscr{d}_l, \mathscr{d}_\phi, \mathscr{c}_l, \mathscr{c}_\phi$ be continuous and nonincreasing functions, such that $\mathscr{d}_\phi \leq 1$, $\mathscr{c}_\phi \leq \mathscr{c}_l$, and that (for sufficiently large t)

$$\begin{aligned} \mathscr{d}_l(t) &= t^{-\delta_l}(\log t)^{-\alpha_l}, \quad & \mathscr{d}_\phi(t) &= t^{-\delta_\phi}(\log t)^{-\alpha_\phi}, \\ \mathscr{c}_l(t) &= t^{-\gamma_l}(\log t)^{-\beta_l}, \quad & \mathscr{c}_\phi(t) &= t^{-\gamma_\phi}(\log t)^{-\beta_\phi}. \end{aligned}$$

◀

We compute some of the basic ingredients. Here, and throughout the following, all formulas are understood to hold for sufficiently large t or R.

III.5.5 Lemma. *We have*

(i) $\mathscr{D}(t) = t^{\delta}(\log t)^{\alpha}, \quad \mathscr{C}(t) = t^{\gamma}(\log t)^{\beta},$

$$\frac{d_\phi(t)}{d_l(t)} = t^{\delta_l-\delta_\phi}(\log t)^{\alpha_l-\alpha_\phi}, \quad \frac{c_l(t)}{c_\phi(t)} = t^{\gamma_\phi-\gamma_l}(\log t)^{\beta_\phi-\beta_l}.$$

(ii) $\mathscr{k}(R) \approx R^{\frac{1}{\delta}}(\log R)^{-\frac{\alpha}{\delta}},$

$$\hbar(R) = \infty \;\Leftrightarrow\; (\delta_l, \alpha_l) \preceq (\delta_\phi, \alpha_\phi),$$

$$\hbar(R) \approx \begin{cases} R^{\frac{1}{\delta_l-\delta_\phi}}(\log R)^{-\frac{\alpha_l-\alpha_\phi}{\delta_l-\delta_\phi}} & \text{if } \delta_l > \delta_\phi, \\ \exp\left(R^{\frac{1}{\alpha_l-\alpha_\phi}}\right) & \text{if } \delta_l = \delta_\phi \wedge \alpha_l > \alpha_\phi. \end{cases}$$

Proof. The formulas stated in (i) follow directly from the definitions. To see (ii) we first use Remark III.A.6: the function $\mathscr{k}(R)$ is $\approx$ to an asymptotic inverse of $\mathscr{D}(t)$, and if $\delta_l > \delta_\phi$ then $\hbar(R)$ is an asymptotic inverse of $\frac{d_\phi(t)}{d_l(t)}$.

If $(\delta_l, \alpha_l) \preceq (\delta_\phi, \alpha_\phi)$, then the quotient $\frac{d_\phi}{d_l}$ is bounded and hence $\hbar(R) = \infty$. If $\delta_l = \delta_\phi$ and $\alpha_l > \alpha_\phi$, we solve the equation $\left(\frac{d_\phi}{d_l} \circ \hbar\right)(R) = R$. □

III.5.6 Lemma. *We have*

$$\int_1^\infty \mathscr{D}(s)^{-\frac{1}{2}}\,\mathrm{d}s < \infty \;\Leftrightarrow\; (\delta, \alpha) \succ (2, 2)$$

$$\int_t^\infty \mathscr{D}(s)^{-\frac{1}{2}}\,\mathrm{d}s = \begin{cases} \frac{1}{\frac{\delta}{2}-1} \cdot t^{1-\frac{\delta}{2}}(\log t)^{-\frac{\alpha}{2}} & \text{if } \delta > 2 \\ \frac{1}{\frac{\alpha}{2}-1} \cdot (\log t)^{1-\frac{\alpha}{2}} & \text{if } \delta = 2 \wedge \alpha > 2 \end{cases}$$

$$\int_1^t \mathscr{D}(s)^{-\frac{1}{2}}\,\mathrm{d}s \sim \begin{cases} \frac{1}{1-\frac{\delta}{2}} \cdot t^{1-\frac{\delta}{2}}(\log t)^{-\frac{\alpha}{2}} & \text{if } \delta < 2, \\ \frac{1}{1-\frac{\alpha}{2}} \cdot (\log t)^{1-\frac{\alpha}{2}} & \text{if } \delta = 2 \wedge \alpha < 2, \\ \log\log t & \text{if } \delta = \alpha = 2. \end{cases}$$

Proof. The equivalence stated in the first line is clear, and the stated formulas for $\delta > 2$ and $\delta < 2$ follow for example from Karamata's theorem (or explicit calculation). Assume that $\delta = 2$. A primitive of $\frac{1}{t}(\log t)^{-\frac{\alpha}{2}}$ is given by

$$\begin{cases} \frac{1}{1-\frac{\alpha}{2}} \cdot (\log t)^{1-\frac{\alpha}{2}} & \text{if } \alpha \neq 2, \\ \log\log t & \text{if } \alpha = 2, \end{cases}$$

and also in this case the assertion follows. □

In our first example we discuss the role of the term $L(t,R)$; this fits the context of Proposition III.3.9. We show that there are situations where $L(t,R)$ cannot be neglected, but also that the assumptions in Proposition III.3.9 are only sufficient for $B(R) \asymp \mathscr{B}(R)$.

III.5.7 Example. Assume that

$$(\delta_l, \alpha_l) \prec (\delta_\phi, \alpha_\phi), \quad \gamma = 0, \quad (\delta, \alpha) \succ (2,2). \tag{III.5.2}$$

The first assumption is there to rule out applicability of Proposition III.3.9 (i), the second to rule out applicability of Proposition III.3.9 (ii),(iii), and the third to reduce computational effort (the facts we want to illustrate occur already under this additional assumption).

For parameters subject to (III.5.2) it holds that

$$\mathscr{B}(R) \asymp \begin{cases} R^{\frac{1}{2}}(\log R)^{1-\frac{\alpha}{2}} & \text{if } \delta = 2, \\ R^{\frac{1}{\delta}}(\log R)^{-\frac{\alpha}{\delta}} & \text{if } \delta > 2. \end{cases} \tag{III.5.3}$$

$$B(R) \begin{cases} \asymp \mathscr{B}(R) & \text{if } \delta_l = \delta_\phi \vee \beta > 1 \vee (\beta = \delta - 1 > 1 \wedge \alpha < 0), \\ \asymp R^{\frac{1}{1+\beta}} \gg \mathscr{B}(R) & \text{otherwise .} \end{cases} \tag{III.5.4}$$

Proof. We start with showing (III.5.3). If $\delta > 2$, the stated formula follows from Theorem III.4.1 $\boxed{\text{B}}$.

Assume that $\delta = 2$, then $\alpha > 2$. By Lemma III.4.4 we have

$$\mathscr{B}(R) \leq \sup_{t \geq 1} \mathscr{g}(t,R) \lesssim R^{\frac{1}{2}} \int_{\mathscr{k}(R)}^{\infty} \mathscr{D}(s)^{-\frac{1}{2}} \, \mathrm{d}s \asymp R^{\frac{1}{2}} (\log R)^{1-\frac{\alpha}{2}}.$$

Set $\tau(R) := \exp\left(R^{\frac{1}{2\beta}}\right)$. Since

$$\frac{R}{\mathscr{C}(\tau(R))} = R^{\frac{1}{2}} \gg R^{\frac{1}{2}}(\log R)^{1-\frac{\alpha}{2}} \gtrsim \mathscr{g}(\tau(R), R),$$

and

$$\int_{\tau(R)}^{\infty} \mathscr{D}(s)^{-\frac{1}{2}} \, \mathrm{d}s \asymp R^{\frac{1-\frac{\alpha}{2}}{2\beta}},$$

and $1 - \frac{\alpha}{2} < 0$, we have

$$\begin{aligned} \mathscr{B}(R) \gtrsim \mathscr{g}(\tau(R), R) &\gtrsim R^{\frac{1}{2}} \int_{\mathscr{k}(R)}^{\tau(R)} \mathscr{D}(s)^{-\frac{1}{2}} \, \mathrm{d}s \\ &\gtrsim R^{\frac{1}{2}}(\log R)^{1-\frac{\alpha}{2}} - R^{\frac{1}{2}} R^{\frac{1-\frac{\alpha}{2}}{2\beta}} \asymp R^{\frac{1}{2}}(\log R)^{1-\frac{\alpha}{2}}. \end{aligned}$$

For the proof of (III.5.4) we have to include the term $L(t,R)$ into the discussion. We have

$$B(R) \asymp \inf_{t\geq 1} \max\Big\{ \mathscr{g}(t,R), \frac{R}{\mathscr{C}(t))}, L(t,R) \Big\},$$

and hence

$$\max\Big\{ \mathscr{B}(R), \inf_{t\geq 1}\Big\{\frac{R}{\mathscr{C}(t))}, L(t,R)\Big\}\Big\} \lesssim B(R)$$
$$\lesssim \max\Big\{ \sup_{t\geq 1} \mathscr{g}(t,R), \inf_{t\geq 1}\Big\{\frac{R}{\mathscr{C}(t))}, L(t,R)\Big\}\Big\}.$$

By what we already showed $\mathscr{B}(R) \asymp \sup_{t\geq 1} \mathscr{g}(t,R)$, and it remains to evaluate the written infimum. By Lemma III.3.6 and the fact that

$$\log^+ \frac{\mathscr{c}_l(t)}{\mathscr{c}_\phi(t)} \lesssim \log\log t,$$

we obtain that

$$L(t,R) \asymp 1 + \log^+ R + \begin{cases} \log t & \text{if } \delta_l < \delta_\phi, \\ \log\log t & \text{if } \delta_l = \delta_\phi. \end{cases}$$

Note that therefore $t \mapsto L(t,R)$ is $\asymp$ to a nondecreasing continuous function.

Consider first the case that $\delta_l < \delta_\phi$. Then we set $\tau(R) := \exp\big(R^{\frac{1}{1+\beta}}\big)$, and obtain

$$\frac{R}{\mathscr{C}(\tau(R))} = R^{1-\frac{\beta}{1+\beta}} = R^{\frac{1}{1+\beta}} \asymp L(\tau(R), R).$$

and hence

$$\min_{t\geq 1} \max\Big\{\frac{R}{\mathscr{C}(t)}, L(t,R)\Big\} \asymp R^{\frac{1}{1+\beta}}.$$

Assume now that $\delta_l = \delta_\phi$. Then we set $\tau(R) := \exp\Big(\big(\frac{R}{\log R}\big)^{\frac{1}{\beta}}\Big)$, and obtain

$$\frac{R}{\mathscr{C}(\tau(R))} = \log R \asymp L(\tau(R), R),$$

which leads to

$$\min_{t\geq 1} \max\Big\{\frac{R}{\mathscr{C}(t)}, L(t,R)\Big\} \asymp \log R.$$

Putting together, the relation (III.5.4) follows. $\square$

In our second example we discuss the bounds from Theorem III.4.1 $\boxed{\text{C}}$. We show that they are sharp but need not necessarily be attained.

III.5.8 Example. Assume that

$$(\delta_l, \alpha_l) \preceq (1+\gamma, \beta) \preceq (\delta, \alpha), \quad \delta = 2, \alpha \leq 2, \quad \gamma > 0. \tag{III.5.5}$$

These assumptions ensure that we are in the situation of Theorem III.4.1 $\boxed{\text{C}}$ and that in (III.4.6) we do not automatically have equality.

For parameters subject to (III.5.5) it holds that

$$\frac{R}{\mathscr{C}(\mathscr{f}_0^-(R))} \asymp R^{\frac{1}{2}}(\log R)^{-\frac{\alpha}{2}} \asymp \mathscr{k}(R), \tag{III.5.6}$$

$$\frac{R}{\mathscr{C}(\mathscr{f}_1^-(R))} \asymp \begin{cases} R^{\frac{1}{2}}(\log R)^{1-\frac{\alpha}{2}} & \text{if } \alpha < 2, \\ R^{\frac{1}{2}} \log\log R & \text{if } \alpha = 2. \end{cases} \tag{III.5.7}$$

$$\mathscr{B}(R) \asymp \begin{cases} R^{\frac{1}{2}}(\log R)^{1-\frac{\alpha}{2}} & \text{if } \gamma < 1, \\ R^{\frac{1}{2}}(\log R)^{-\frac{\alpha}{2}} \log\log R & \text{if } \gamma = 1 \wedge \alpha > \beta, \\ R^{\frac{1}{2}}(\log R)^{-\frac{\alpha}{2}} & \text{if } \gamma = 1 \wedge \alpha = \beta. \end{cases} \tag{III.5.8}$$

We see that

- ▷ $\mathscr{B}(R) \asymp \frac{R}{\mathscr{C}(\mathscr{f}_1^-(R))}$ if $\gamma < 1 \wedge \alpha < 2$,
- ▷ $\frac{R}{\mathscr{C}(\mathscr{f}_0^-(R))} \ll \mathscr{B}(R) \asymp \frac{R}{\mathscr{C}(\mathscr{f}_1^-(R))}$ if $\gamma < 1 \wedge \alpha = 2$ or $\gamma = 1 \wedge \alpha > \beta$,
- ▷ $\mathscr{B}(R) \asymp \frac{R}{\mathscr{C}(\mathscr{f}_0^-(R))}$ if $\gamma = 1 \wedge \alpha = \beta$.

Proof. Plugging the definitions and using Remark III.A.6 yields

$$\mathscr{f}_0(t) = t^{2\gamma}(\log t)^{2\beta-\alpha}, \quad \mathscr{f}_0^-(R) \asymp \left(\frac{R}{(\log R)^{2\beta-\alpha}}\right)^{\frac{1}{2\gamma}},$$

$$\mathscr{f}_1(t) \sim \begin{cases} \frac{1}{2-\alpha} t^{2\gamma}(\log t)^{2-\alpha+2\beta} & \text{if } \alpha < 2, \\ t^{2\gamma}(\log t)^{2\beta}(\log\log t)^2 & \text{if } \alpha = 2, \end{cases}$$

$$\mathscr{f}_1^-(R) \approx \begin{cases} \left(\frac{R}{(\log R)^{2-\alpha+2\beta}}\right)^{\frac{1}{2\gamma}} & \text{if } \alpha < 2, \\ \left(\frac{R}{(\log R)^{2\beta}(\log\log R)^2}\right)^{\frac{1}{2\gamma}} & \text{if } \alpha = 2. \end{cases}$$

From this (III.5.6) and (III.5.7) follow immediately.

Consider the case that $\gamma < 1$. Set

$$\tau(R) := \left(\frac{R}{(\log R)^{2-\alpha+2\beta}}\right)^{\frac{1}{2\gamma}}.$$

Then

$$\mathscr{k}(R) \ll \tau(R) \ll \mathscr{f}_0^-(R) \lesssim \mathscr{h}(R),$$

and hence

$$\mathscr{g}(\tau(R), R) \asymp \mathscr{k}(R) + R^{\frac{1}{2}} \int_{\mathscr{k}(R)}^{\tau(R)} \mathscr{D}(s)^{-\frac{1}{2}}\, \mathrm{d}s.$$

Using Lemma III.5.6, we obtain

$$\int_{\mathscr{k}(R)}^{\tau(R)} \mathscr{D}(s)^{-\frac{1}{2}}\, \mathrm{d}s \approx \begin{cases} (\log \tau(R))^{1-\frac{\alpha}{2}} - (\log \mathscr{k}(R))^{1-\frac{\alpha}{2}} & \text{if } \alpha < 2 \\ \log\log \tau(R) - \log\log \mathscr{k}(R) & \text{if } \alpha = 2 \end{cases}$$

$$\asymp \begin{cases} (\log R)^{1-\frac{\alpha}{2}} & \text{if } \alpha < 2 \\ 1 & \text{if } \alpha = 2 \end{cases} \asymp (\log R)^{1-\frac{\alpha}{2}}.$$

Hence,

$$\mathscr{g}(\tau(R), R) \asymp R^{\frac{1}{2}} (\log R)^{1-\frac{\alpha}{2}}.$$

Plugging the definitions shows that also

$$\frac{R}{\mathscr{C}(\tau(R))} \asymp R^{\frac{1}{2}} (\log R)^{1-\frac{\alpha}{2}},$$

and it follows that $\mathscr{B}(R) \asymp R^{\frac{1}{2}} (\log R)^{1-\frac{\alpha}{2}}$.

Next, consider the case that $\gamma = 1$ and $\alpha > \beta$. Set

$$\tau(R) := \frac{R^{\frac{1}{2}} (\log R)^{\frac{\alpha}{2}-\beta}}{\log\log R}.$$

Again

$$\mathscr{k}(R) \ll \tau(R) \ll \mathscr{f}_0^-(R) \ll \mathscr{h}(R),$$

and therefore

$$\mathscr{g}(\tau(R), R) \asymp \mathscr{k}(R) + R^{\frac{1}{2}} \int_{\mathscr{k}(R)}^{\tau(R)} \mathscr{D}(s)^{-\frac{1}{2}}\, \mathrm{d}s.$$

The 1$^{\text{st}}$ mean value theorem provides us with $\xi(R) \in [\mathscr{k}(R), \tau(R)]$ such that

$$\int_{\mathscr{k}(R)}^{\tau(R)} \mathscr{D}(s)^{-\frac{1}{2}}\, \mathrm{d}s = \int_{\mathscr{k}(R)}^{\tau(R)} s^{-1} (\log s)^{-\frac{\alpha}{2}}\, \mathrm{d}s$$

$$= (\log \xi(R))^{-\frac{\alpha}{2}} \int_{\mathscr{k}(R)}^{\tau(R)} s^{-1}\, \mathrm{d}s = (\log \xi(R))^{-\frac{\alpha}{2}} \log \frac{\tau(R)}{\mathscr{k}(R)}.$$

We have $\log \tau(R) \sim \log \mathscr{k}(R) \approx \log R$, and hence $\log \xi(R) \asymp \log R$. Moreover,

$$\frac{\tau(R)}{\mathscr{k}(R)} = \frac{(\log R)^{\alpha-\beta}}{\log\log R}.$$

It follows that

$$\int_{\mathscr{k}(R)}^{\tau(R)} \mathscr{D}(s)^{-\frac{1}{2}}\,\mathrm{d}s \approx (\log R)^{-\frac{\alpha}{2}} \log\log R,$$

and in turn

$$\mathscr{g}(\tau(R), R) \asymp R^{\frac{1}{2}}(\log R)^{-\frac{\alpha}{2}} \log\log R.$$

Again simply plugging the definitions shows that also

$$\frac{R}{\mathscr{C}(\tau(R))} \asymp R^{\frac{1}{2}}(\log R)^{-\frac{\alpha}{2}} \log\log R,$$

and thus $\mathscr{B}(R) \asymp R^{\frac{1}{2}}(\log R)^{-\frac{\alpha}{2}} \log\log R$.

It remains to settle the case that $\gamma = 1$ and $\alpha = \beta$, but this is easily done. Simply plug the definitions to obtain

$$\frac{R}{\mathscr{C}(\mathscr{k}(R))} \asymp \mathscr{k}(R) \asymp \mathscr{g}(\mathscr{k}(R), R),$$

and therefore $\mathscr{B}(R) \asymp \mathscr{k}(R) \asymp R^{\frac{1}{2}}(\log R)^{-\frac{\alpha}{2}}$. □

In our third example we discuss the exceptional case "$(\delta, \gamma) = (2, 0)$" in the third row of the table in Theorem III.4.6. We show that in some cases (interpreting $\mathscr{f}_1^-$ appropriately) the written bound still holds and is even attained by $B(R)$, while in others $B(R)$ is strictly larger due to domination of $L(t, R)$.

III.5.9 Example. Assume that

$$(\delta_l, \alpha_l) \preceq (1, 1+\beta), \quad (\delta, \gamma) = (2, 0), \quad \alpha \leq 2. \tag{III.5.9}$$

These assumptions ensure that the exceptional case from the third row of the table in Theorem III.4.6 (and also of Theorem III.4.1 $\boxed{\mathrm{C}}$) takes place. Note that (III.5.9) implies that $\delta_l \leq \delta_\phi$. Moreover, $\beta > 0$ since $(\gamma, \beta) \succ (0, 0)$.

For parameters subject to (III.5.9) it holds that

$$\mathscr{B}(R) \asymp \begin{cases} R^{\frac{2-\alpha+\beta}{2-\alpha+2\beta}} & \text{if } \alpha < 2, \\ R^{\frac{1}{2}} \log R & \text{if } \alpha = 2, \end{cases}$$

and

$$B(R) \begin{cases} \asymp \mathscr{B}(R) & \text{if } \delta_l = \delta_\phi \vee (\delta_l < \delta_\phi \wedge \alpha \leq 1+\beta), \\ \asymp R^{\frac{1}{1+\beta}} \gg \mathscr{B}(R) & \text{if } \delta_l < \delta_\phi \wedge \alpha > 1+\beta. \end{cases}$$

Observe, moreover, that the bound for order in the third row of the table in Theorem III.4.6 has no continuous extension to $(2,0)$; its directional limits vary from $\frac{1}{2}$ to 1. The above formula shows that the actual order of the bound $B(R)$ has nothing to do with (δ,γ) (being equal to $(2,0)$). Yet, it is sometimes given by the same formula, only with the "logarithmic exponents" α,β instead of δ,γ. Also note that the exponent $\frac{1}{1+\beta}$ also occurred in Example III.5.7.

Proof. We have

$$\mathscr{f}_1(t) \sim \begin{cases} \frac{1}{2-\alpha}(\log t)^{2-\alpha+2\beta} & \text{if } \alpha < 2, \\ (\log t)^{2\beta}(\log\log t)^2 & \text{if } \alpha = 2. \end{cases}$$

The function $\mathscr{f}_1 \circ \exp$ is regularly varying with positive index, and has approximate inverse

$$(\mathscr{f}_1 \circ \exp)^{-}(R) = \begin{cases} R^{\frac{1}{2-\alpha+2\beta}} & \text{if } \alpha < 2, \\ \left(\frac{R^{\frac{1}{2}}}{\log R}\right)^{\frac{1}{\beta}} & \text{if } \alpha = 2. \end{cases}$$

We define

$$\mathscr{f}_1^{-}(R) := (\exp\circ(\mathscr{f}_1 \circ \exp)^{-})(R) = \begin{cases} \exp\left(R^{\frac{1}{2-\alpha+2\beta}}\right) & \text{if } \alpha < 2, \\ \exp\left(\left(\frac{R^{\frac{1}{2}}}{\log R}\right)^{\frac{1}{\beta}}\right) & \text{if } \alpha = 2. \end{cases},$$

so that

$$(\mathscr{f}_1 \circ \mathscr{f}_1^{-})(R) \asymp R, \quad \log\left[(\mathscr{f}_1^{-} \circ \mathscr{f}_1)(R)\right] \asymp \log R.$$

For the sake of consistency we use also here the notation of a function $\tau(R)$, and set $\tau(R) := \mathscr{f}_1^{-}(R)$. Then

$$\mathscr{k}(R) \ll \tau(R) \ll \hbar(R),$$

where the second relation is seen as follows: If $\hbar(R) = \infty$, there is nothing to prove. Otherwise, we must have $\delta_l = \delta_\phi$ and $\alpha_l > \alpha_\phi$. Since $\delta = 2$, thus $\delta_l = \delta_\phi = 1$, and it follows that $\alpha_l \leq 1+\beta$. This implies that

$$\alpha_l - \alpha_\phi \leq 2 - \alpha + 2\beta.$$

We conclude that

$$\mathscr{g}(\tau(R), R) \asymp \mathscr{k}(R) + R^{\frac{1}{2}} \int_{\mathscr{k}(R)}^{\tau(R)} \mathscr{D}(s)^{-\frac{1}{2}}\, \mathrm{d}s.$$

Using Lemma III.5.6, we obtain

$$\begin{aligned} \int_{\mathscr{k}(R)}^{\tau(R)} \mathscr{D}(s)^{-\frac{1}{2}}\, \mathrm{d}s &\approx \begin{cases} (\log \tau(R))^{1-\frac{\alpha}{2}} - (\log \mathscr{k}(R))^{1-\frac{\alpha}{2}} & \text{if } \alpha < 2, \\ \log\log \tau(R) - \log\log \mathscr{k}(R) & \text{if } \alpha = 2, \end{cases} \\ &\asymp \begin{cases} R^{\frac{1-\frac{\alpha}{2}}{2-\alpha+2\beta}} & \text{if } \alpha < 2, \\ \log R & \text{if } \alpha = 2. \end{cases} \end{aligned}$$

Hence,

$$\mathscr{g}(\tau(R), R) \asymp \begin{cases} R^{\frac{2-\alpha+\beta}{2-\alpha+2\beta}} & \text{if } \alpha < 2, \\ R^{\frac{1}{2}} \log R & \text{if } \alpha = 2. \end{cases}$$

Plugging the definitions yields

$$\frac{R}{\mathscr{C}(\tau(R))} \asymp \begin{cases} R^{\frac{2-\alpha+\beta}{2-\alpha+2\beta}} & \text{if } \alpha < 2, \\ R^{\frac{1}{2}} \log R & \text{if } \alpha = 2, \end{cases}$$

and the formula asserted for $\mathscr{B}(R)$ follows.

In order to show the asserted formulas for $B(R)$, we have to include the term $L(t, R)$ into the discussion. We distinguish several cases.

Assume that $\delta_l = \delta_\phi$ and $\alpha_l \geq \alpha_\phi$. The Proposition III.3.9 (i) shows that $B(R) \asymp \mathscr{B}(R)$. Assume that $\delta_l = \delta_\phi$ and $\alpha_l < \alpha_\phi$. We have

$$\log^+ \frac{\mathscr{c}_l(\tau(R))}{\mathscr{c}_\phi(\tau(R))} = \log^+ \Big[(\log \tau(R))^{\beta_\phi - \beta_l}\Big] \lesssim \log R,$$

$$\log^+ \frac{\mathscr{d}_l(\tau(R))}{\mathscr{d}_\phi(\tau(R))} = \log^+ \Big[(\log \tau(R))^{\alpha_\phi - \alpha_l}\Big] \approx \log R.$$

Now Lemma III.3.6 implies that $L(\tau(R), R) \asymp \log R$, and we see that $B(R) \asymp \mathscr{B}(R)$. Assume that $\delta_l < \delta_\phi$ and $\alpha \leq 1 + \beta$. Then

$$\log^+ \frac{\mathscr{d}_l(\tau(R))}{\mathscr{d}_\phi(\tau(R))} \asymp \log \tau(R) \asymp \begin{cases} R^{\frac{1}{2-\alpha+2\beta}} & \text{if } \alpha < 2, \\ \left(\frac{R^{\frac{1}{2}}}{\log R}\right)^{\frac{1}{\beta}} & \text{if } \alpha = 2, \end{cases}$$

and hence, again referring to Lemma III.3.6, also

$$L(\tau(R), R) \asymp \begin{cases} R^{\frac{1}{2-\alpha+2\beta}} & \text{if } \alpha < 2, \\ \left(\frac{R^{\frac{1}{2}}}{\log R}\right)^{\frac{1}{\beta}} & \text{if } \alpha = 2. \end{cases}$$

This shows that $L(\tau(R), R) \lesssim \mathscr{B}(R)$, and therefore again $B(R) \asymp \mathscr{B}(R)$.

Assume now that $\delta_l < \delta_\phi$ and $\alpha > 1+\beta$. This case is different, and we also use a different function $\tau(R)$, namely, we now set

$$\tau(R) := \exp\Big(R^{\frac{1}{1+\beta}}\Big).$$

We have $\log \frac{d_l(t)}{d_\phi(t)} \asymp \log t$, and hence obtain

$$L(\tau(R), R) \asymp R^{\frac{1}{1+\beta}}.$$

Further,

$$\frac{R}{\mathscr{C}(\tau(R))} = R^{\frac{1}{1+\beta}}, \quad \mathscr{g}(\tau(R), R) \asymp \begin{cases} R^{\frac{1}{2}+\frac{1-\frac{\alpha}{2}}{1+\beta}} & \text{if } \alpha < 2, \\ R^{\frac{1}{2}} \log R & \text{if } \alpha = 2. \end{cases}$$

Since $\alpha > 1+\beta$, we have

$$\frac{1}{2} + \frac{1-\frac{\alpha}{2}}{1+\beta} < \frac{1}{1+\beta},$$

and see that

$$B(R) \asymp \min_{t\geq 1} \max\Big\{\mathscr{g}(t,R), \frac{R}{\mathscr{C}(t)}, L(t,R)\Big\} \asymp R^{\frac{1}{1+\beta}}.$$

□

III.5.3 Two corollaries given in terms of Jacobi parameters

We present two applications of Theorem III.5.3 in which we return to the regime of power moment problems. The first is a supplement to a result from [Pru20], and in the second we give examples where the Nevanlinna matrix has prescribed growth.

At this point we need the concrete formulae relating Jacobi parameters with Hamiltonian parameters. They read as

$$\begin{aligned} \frac{1}{b_n} &= \sin(\phi_{n+1} - \phi_n)\sqrt{l_{n+1} l_n}, \\ a_n &= -\frac{1}{l_n}\big[\cot(\phi_{n+1} - \phi_n) + \cot(\phi_n - \phi_{n-1})\big], \end{aligned} \tag{III.5.10}$$

where the angles are chosen such that $\phi_{n+1} - \phi_n \in [0, \pi)$, cf. [Kac99]. Given the Jacobi parameters, it is in general hard to solve the equations (III.5.10) for the Hamiltonian parameters. Under the assumption that b_n, a_n have a certain power-like asymptotic, the approximate size of l_n and $|\sin(\phi_{n+1}-\phi_n)|$ can be determined.

III.5.10 Corollary. *Let $a_n \in \mathbb{R}$ and $b_n > 0$ be sequences which have asymptotics*

$$b_n = n^\sigma\left(\frac{|y_0|}{2} + \frac{x_1}{n} + \frac{x_2}{n^2} + \mathrm{O}\Big(\frac{1}{n^{2+\epsilon}}\Big)\right),$$
$$a_n = n^\sigma\left(y_0 + \frac{y_1}{n} + \frac{y_2}{n^2} + \mathrm{O}\Big(\frac{1}{n^{2+\epsilon}}\Big)\right),$$

where

$$\sigma > 2,\ y_0 \neq 0, \quad x_1, x_2, y_1, y_2 \in \mathbb{R}, \quad \epsilon > 0.$$

Assume that the moment problem with these Jacobi parameters is indeterminate, and let $W(z)$ be its Nevanlinna matrix. Then

$$\log\left(\max_{|z|=R}\|W(z)\|\right) \asymp R^{\frac{1}{\sigma}}.$$

Before we come to the proof, let us put this statement in the right context. It deals with the critical situation that off-diagonal and diagonal of the Jacobi matrix are comparable with ratio 2. This setting was considered in [Pru20, Theorem 2]. In that theorem occurrence of limit circle case was characterised in terms of the data of the expansions, and it was shown that $W(z)$ is of order $\frac{1}{\sigma}$ with positive type. The significance of Corollary III.5.10 is that now we know that $W(z)$ is also of finite type.

Proof. In the proof of [Pru20, Theorem 2] it was shown that

$$l_n \asymp \lambda(n)^2, \quad |\sin(\phi_{n+1}-\phi_n)| \asymp \frac{1}{n^\sigma \lambda(n)^2},$$

where λ is a function of the form $\lambda(t) = n^\tau$ or $\lambda(t) = n^\tau \log t$ with some $\tau \in [-\frac{\sigma}{2}, -\frac{1}{2})$. The assumptions of Theorem III.5.3 are thus satisfied with $d_l(t) := \lambda(t)^2$ and $d_\phi(t) := \frac{1}{n^\sigma \lambda(n)^2}$. Applying this theorem yields

$$\log\left(\max_{|z|=R}\|W(z)\|\right) \asymp \left[\frac{1}{d_l d_\phi}\right]^-(R) \asymp R^{\frac{1}{\sigma}}.$$

$\square$

We come to our second corollary, where we produce a variety of examples with prescribed growth of the Nevanlinna matrix (slower than the threshold $R^{\frac{1}{2}}$). Thereby, the speed of growth is always determined by the off-diagonal, and the diagonal can be as large or as small as we please.

III.5.11 Corollary. *Let g be regularly varying with* $\operatorname{ind} g \in (0, \frac{1}{2})$.

(i) *Let $\omega \in [-2, 2]$. Then there exist Jacobi parameters b_n and a_n, such that*

$$b_n \sim g^-(n), \ \frac{a_n}{b_n} \to \omega, \quad \log\Big(\max_{|z|=R} \|W(z)\|\Big) \asymp g(R).$$

(ii) *Let $\omega_n \neq 0$, $\omega_n \to 0$, be such that $\lim_{n\to\infty} \frac{\omega_{n-1}}{\omega_n}$ exists in $(-1, \infty)$. Then there exist Jacobi parameters b_n and a_n, such that*

$$b_n \sim g^-(n), \ \frac{a_n}{b_n} \sim \omega_n, \quad \log\Big(\max_{|z|=R} \|W(z)\|\Big) \asymp g(R).$$

The condition in (i) that $|\omega| \leq 2$ is no restriction, since otherwise we could not have limit circle case by Wouk's theorem.

Proof. We specify lengths and angles, which is done differently in different cases.

▷ Assume that $\omega \in (-2, 2)$. Let $\psi \in (0, \pi)$ be such that $\cos\psi = -\frac{\omega}{2}$, and set

$$l_n := \frac{1}{\sin\psi \cdot g^-(n)}, \qquad \phi_{n+1} := n\psi, \ n \in \mathbb{N},$$

so that $\phi_{n+1} - \phi_n = \psi$.

▷ Assume that $\omega = -2$. Set

$$l_n := \frac{n}{g^-(n)}, \qquad \phi_{n+1} := \sum_{k=1}^{n} \frac{1}{k}, \ n \in \mathbb{N},$$

so that $\phi_{n+1} - \phi_n = \frac{1}{n}$.

▷ Assume that $\omega = 2$. Set

$$l_n := \frac{n}{g^-(n)}, \qquad \phi_{n+1} := n\pi - \sum_{k=1}^{n} \frac{1}{k}, \ n \in \mathbb{N},$$

so that $\phi_{n+1} - \phi_n = \pi - \frac{1}{n}$.

▷ Assume that $\omega_n \neq 0$, $\omega_n \to 0$, and $\gamma := \lim_{n\to\infty} \frac{\omega_{n-1}}{\omega_n}$ exists in $(-1, \infty)$. Set

$$l_n := \frac{1}{\mathfrak{g}^-(n)}, \qquad \phi_{n+1} := n\frac{\pi}{2} + \frac{1}{2(1+\gamma)} \sum_{\substack{k=1 \\ |\omega_n|<\pi}}^{n} \omega_n, \ n \in \mathbb{N},$$

so that $\phi_{n+1} - \phi_n = \frac{\pi}{2} + \frac{\omega_n}{2(1+\gamma)}$ for all sufficiently large n.

Let b_n and a_n be the Jacobi parameters given by (III.5.10). Then, in all cases, $b_n \sim \mathfrak{g}^-(n)$. Multiplying the two equations from (III.5.10), shows that

$$\frac{a_n}{b_n} = -\sqrt{\frac{l_{n+1}}{l_n}} \cdot \left(\cos(\phi_{n+1} - \phi_n) + \cos(\phi_n - \phi_{n-1}) \frac{\sin(\phi_{n+1} - \phi_n)}{\sin(\phi_n - \phi_{n-1})} \right),$$

and this implies the asserted property of $\frac{a_n}{b_n}$. Finally, we apply Theorem III.5.3 with the obvious choices for $\mathfrak{d}_l, \mathfrak{d}_\phi$ to obtain that

$$\log \left(\max_{|z|=R} \|W(z)\| \right) \asymp \mathfrak{g}(R).$$

□

Appendix III.A Regularly varying functions

In complex analysis the growth of an analytic function is compared with functions of the form $\exp(\mathfrak{a}(r))$. The most classical comparison functions are powers $\mathfrak{a}(r) = r^\rho$, and this leads to the notions of order and type. A refined comparison scale was introduced already at a very early stage by E.Lindelöf [Lin05] who considered comparison functions behaving for $r \to \infty$ like

$$r^\alpha \cdot \big(\log r\big)^{\beta_1} \cdot \big(\log\log r\big)^{\beta_2} \cdot \ldots \cdot \big(\underbrace{\log \cdots \log}_{m^{\text{th}} \text{ iterate}} r\big)^{\beta_m},$$

where $\alpha > 0$ and $\beta_1, \ldots, \beta_m \in \mathbb{R}$. Functions which are nowadays commonly used as comparison functions are regularly varying functions in Karamata sense, cf. [BGT89, Chapter 7] (for other levels of generality see also [Lev80, Rub96]). Lindelöf's comparison functions are examples of functions of that kind.

Let us now recall Karamata's definition of regular variation.

III.A.1 Definition. A function $a\colon [1,\infty) \to (0,\infty)$ is called *regularly varying* at ∞ with *index* $\alpha \in \mathbb{R}$, if it is measurable and

$$\forall \lambda \in (0,\infty).\ \lim_{r\to\infty} \frac{a(\lambda r)}{a(r)} = \lambda^\alpha.$$

We write $\operatorname{ind} a$ for the index of regular variation of function a. A regularly varying function with index 0 is also called *slowly varying*.

◀

Regularly varying function a are used to quantify growth for $r \to \infty$, and hence the values of $a(r)$ for small r are irrelevant. This allows to change a on any finite interval without changing the essence of results, and this freedom can often be used to assume a has some additional practical properties.

We cite a number of fundamental theorems on regularly varying functions. Proofs can be found, e.g., in [BGT89] or [Sen76]. We start with an *representation theorem.*

III.A.2 Theorem (Representation theorem)**.** *Let $\alpha \in \mathbb{R}$. A function $a\colon [1,\infty) \to (0,\infty)$ is regularly varying with index α if and only if it has a representation of the form*

$$a(r) = r^\alpha \cdot c(r) \exp\left(\int_1^r \epsilon(u)\frac{\mathrm{d}u}{u}\right), \qquad r \in [1,\infty),$$

where c, ϵ are measurable, $\lim_{r\to\infty} c(r) = c \in (0,\infty)$, and $\lim_{r\to\infty} \epsilon(r) = 0$.

If a is slowly varying (i.e., $\alpha = 0$) and eventually nondecreasing (nonincreasing), then ϵ may be taken eventually nonnegative (nonpositive).

It is a legitimate intuition that regularly varying functions fill in the scale of powers, and that a regularly varying function with index α behaves roughly like the power r^α. The following results, which we will use frequently, express this intuition very clearly. The first is a variant of the *Potter bounds*, and the second is the classical *Karamata Theorem* about asymptotic integration.

III.A.3 Theorem (Potter bounds (variant))**.** *Let a be regularly varying with index $\alpha \in \mathbb{R}$.*

(i) $\forall \epsilon > 0.\ r^{\alpha-\epsilon} \ll a(r) \ll r^{\alpha+\epsilon}$,

(ii) $\lim_{r\to\infty} \frac{\log a(r)}{\log r} = \alpha$,

(iii) *For all $\epsilon > 0$ the quotients $\frac{a(r)}{r^{\alpha-\epsilon}}$ and $\frac{r^{\alpha+\epsilon}}{a(r)}$ are $\sim$ to an eventually increasing function.*

III.A.4 Theorem (Karamata's Theorem)**.** *Let $\mathcal{a}$ be regularly varying with index $\alpha \in \mathbb{R}$.*

(i) *Assume that $\alpha \geq -1$. Then the function $x \mapsto \int_1^x \mathcal{a}(t)\,\mathrm{d}t$ is regularly varying with index $\alpha + 1$, and*

$$\lim_{x\to\infty} \left(x\mathcal{a}(x) \Big/ \int_1^x \mathcal{a}(t)\,\mathrm{d}t \right) = \alpha + 1.$$

(ii) *Assume that $\alpha \leq -1$ and $\int_1^\infty \mathcal{a}(t)\,\mathrm{d}t < \infty$. Then the function $x \mapsto \int_x^\infty \mathcal{a}(t)\,\mathrm{d}t$ is regularly varying with index $\alpha + 1$, and*

$$\lim_{x\to\infty} \left(x\mathcal{a}(x) \Big/ \int_x^\infty \mathcal{a}(t)\,\mathrm{d}t \right) = -(\alpha + 1).$$

A regularly varying function $\mathcal{a}$ with positive index is – at least asymptotically – invertible. In fact, if

$$\mathcal{a}^-(x) := \sup \left\{ t \in [1, \infty) \mid \mathcal{a}(t) < x \right\},$$

we have the following result, cf. [BGT89, Theorem 1.5.12].

III.A.5 Theorem. *Let $\mathcal{a}$ be regularly varying with index $\alpha > 0$. Then $\mathcal{a}^-$ is regularly varying with index $\frac{1}{\alpha}$, and*

$$(\mathcal{a} \circ \mathcal{a}^-)(x) \sim (\mathcal{a}^- \circ \mathcal{a})(x) \sim x. \tag{III.A.1}$$

Any regularly varying function $\mathcal{a}^-$ with the property (III.A.1) is called an *asymptotic inverse* of $\mathcal{a}$, and asymptotic inverses are determined uniquely up to $\sim$. We recall a useful formula for computing asymptotic inverses for functions of a particular form.

III.A.6 Remark. Assume that $\rho > 0$ and that $\mathcal{f}$ is regularly varying. Set $\mathcal{g} := \mathcal{f} \circ \log$ (for sufficiently large t). Then

$$\mathcal{a}(t) := t^\rho \mathcal{g}(t), \quad \mathcal{a}^-(t) := \rho^{\frac{\mathrm{ind}\,\mathcal{f}}{\rho}} \cdot \left(\frac{t}{\mathcal{g}(t)} \right)^{\frac{1}{\rho}}$$

are asymptotic inverses of each other.

$\lhd$

Another practical observation is the following.

III.A.7 Remark. Let $\mathcal{f}$ be regularly varying with $\mathrm{ind}\,\mathcal{f} > 0$, and assume we have a function $\mathcal{g}$ with $\mathcal{g} \approx \mathcal{f}$. Then $\mathcal{g}$ is regularly varying with $\mathrm{ind}\,\mathcal{g} = \mathrm{ind}\,\mathcal{f}$ and $\mathcal{g}^- \approx \mathcal{f}^-$.

$\lhd$

By the Potter bounds every regularly varying function $\mathcal{a}$ is bounded and bounded away from zero on every interval $[r_1, r_2]$ sufficiently far to the right. Sometimes it is needed for technical reasons to assume this property for all compact intervals in the domain of $\mathcal{a}$. Of course, this is no loss in generality; remember that modification on a finite interval does not change the essence of the function $\mathcal{a}$.

III.A.8 Lemma. *Let $\mathcal{a} \colon [1, \infty) \to (0, \infty)$ be slowly varying and assume that $\mathcal{a} \lesssim 1$ locally. For $R \geq \mathcal{a}(1)$ set*

$$\mathcal{b}(R) := \sup \Big\{ t \in [1, \infty) \mid \sup_{1 \leq s \leq t} \frac{\mathcal{a}(s)}{R} \leq 1 \Big\} \in [1, \infty].$$

Then $\mathcal{b}$ grows faster than any power, i.e., $R^\rho \leq \mathcal{b}(R)$ for every $\rho > 0$ and R sufficiently large.

Proof. Let $\rho > 0$ and set $\epsilon := \frac{1}{\rho}$. By Theorem III.A.3, there exists $M > 0$ such that $\mathcal{a}(r) \leq r^\epsilon$ for all $r \geq M$. If $R \geq \sup_{r \in [1,M]} \mathcal{a}(r)$, this means that

$$\Big\{ t \in [1, \infty) \mid \sup_{1 \leq s \leq t} \frac{s^\epsilon}{R} \leq 1 \Big\} \subseteq \Big\{ t \in [1, \infty) \mid \sup_{1 \leq s \leq t} \frac{\mathcal{a}(s)}{R} \leq 1 \Big\}.$$

The assertion follows if we take suprema of both sets. □

III.A.9 Lemma. *Let $\mathcal{a}$ be regularly varying with index $\alpha > 0$ and assume that $\mathcal{a} \asymp 1$ locally. Then there exists a continuously differentiable and regularly varying function $\mathcal{s}$, where $\mathcal{s}'(t) > 0$ for $t \in [1, \infty)$,*

$$\begin{aligned} \mathcal{a}(t) &\asymp \mathcal{s}(t), \qquad && t \in [1, \infty), \\ \mathcal{a}(t) &\sim s(t), && t \to \infty. \end{aligned}$$

Proof. We use the smooth variation theorem [BGT89, Theorem 1.8.2]. This gives a function $\mathcal{s}$ that is continuously differentiable, whose derivative is positive for all sufficiently large t, and such that $\mathcal{a} \sim \mathcal{s}$ for $t \to \infty$. W.l.o.g. we assume $\mathcal{s}'(t) > 0$ for all $t \in [1, \infty)$.

Choose $t_0 \geq 1$ such that $\frac{1}{2}\mathcal{a}(t) \leq \mathcal{s}(t) \leq \frac{3}{2}\mathcal{a}(t)$ for all $t \geq t_0$. By our assumption, $\mathcal{a} \asymp 1$ on $[1, t_0]$. Since $\mathcal{s}$ is continuous, $\mathcal{s} \asymp 1 \asymp \mathcal{a}$ on $[1, t_0]$. Summing up, we have $\mathcal{s} \asymp \mathcal{a}$ on $[1, \infty)$. □

Another lemma in a similar direction is the following. The proof is immediate from the representation theorem and we do not go into details.

III.A.10 Lemma. *Let $\mathfrak{a}$ be regularly varying, and assume that $\mathfrak{a}$ is $\approx$ to some nonincreasing function. Then there exist*

$$\epsilon[\mathfrak{a}], \kappa[\mathfrak{a}] \colon [1, \infty) \to (0, \infty)$$

such that

- ▷ $\epsilon[\mathfrak{a}]$ *is locally integrable,* $\lim_{t\to\infty} \epsilon[\mathfrak{a}](t) = 0$*, and* $\epsilon[\mathfrak{a}] \leq 0$ *if* $\operatorname{ind} \mathfrak{a} = 0$*,*
- ▷ $\kappa[\mathfrak{a}]$ *is eventually constant,*
- ▷ *The function*

$$\mathcal{S}[\mathfrak{a}](t) := \kappa[\mathfrak{a}](t) \cdot t^{\operatorname{ind} \mathfrak{a}} \cdot \exp\left(\int_1^r \epsilon[\mathfrak{a}](u) \frac{\mathrm{d}u}{u} \right)$$

is nonincreasing, continuous, and $\approx \mathfrak{a}$*.*

We speak of any function $\mathcal{S}[\mathfrak{a}]$ as in the lemma as a *nonincreasing smoothening* of $\mathfrak{a}$.

Of course, $\kappa[\mathfrak{a}]$ and $\epsilon[\mathfrak{a}]$ are far from unique. We mention two particular instances of the freedom of choice in $\mathcal{S}[\mathfrak{a}]$.

III.A.11 Remark.

(i) If $\mathcal{S}[\mathfrak{a}]$ is some nonincreasing smoothening of $\mathfrak{a}$ and $\alpha > 0$, then also $\alpha \cdot \mathcal{S}[\mathfrak{a}]$ is a nonincreasing smoothening of $\mathfrak{a}$. This corresponds to multiplying $\kappa[\mathfrak{a}]$ by α.

(ii) If $\mathcal{S}[\mathfrak{a}]$ is some nonincreasing smoothening of $\mathfrak{a}$ and $\alpha > \lim_{t\to\infty} \mathcal{S}[\mathfrak{a}](t)$, then also $\min\{\mathcal{S}[\mathfrak{a}](t), \alpha\}$ is a nonincreasing smoothening of $\mathfrak{a}$. This is seen by modifying $\kappa[\mathfrak{a}]$ on a finite interval.

◁

One more property of this construction is as follows. Assume we have two functions $\mathfrak{a}_1, \mathfrak{a}_2$ that are both regularly varying and $\approx$ to some nonincreasing function. If $\mathcal{S}[\mathfrak{a}_1]$ and $\mathcal{S}[\mathfrak{a}_2]$ are nonincreasing smoothenings of $\mathfrak{a}_1$ and $\mathfrak{a}_2$, respectively, then $\mathcal{S}[\mathfrak{a}_1] \cdot \mathcal{S}[\mathfrak{a}_2]$ is a nonincreasing smoothening of $\mathfrak{a}_1 \cdot \mathfrak{a}_2$. This corresponds to taking

$$\kappa[\mathfrak{a}_1\mathfrak{a}_2] := \kappa[\mathfrak{a}_1] \cdot \kappa[\mathfrak{a}_2], \quad \epsilon[\mathfrak{a}_1\mathfrak{a}_2] := \epsilon[\mathfrak{a}_1] + \epsilon[\mathfrak{a}_2].$$

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Chapter IV

Lower bounds for the growth of a Nevanlinna matrix

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Lower bounds for the growth of a Nevanlinna matrix

Jakob Reiffenstein [‡]

Abstract. Our objects of study are two-dimensional canonical systems that arise from indeterminate Hamburger moment problems as well as from half-line Jacobi operators in limit circle case. The monodromy matrix of such a system coincides with the Nevanlinna matrix of the associated moment problem, and its growth corresponds to the density of eigenvalues of self-adjoint realisations of the system. We present new estimates for the growth of the monodromy matrix, emphasising two goals. On the one hand, we investigate orders larger than $\frac{1}{2}$, assuming some regularity of the data. By explicitly determining the growth of the monodromy matrix for large classes of canonical system, we show that the growth depends much more sensibly on the data than for orders less than $\frac{1}{2}$. On the other hand, we remove all assumptions on the data and ask what can still be said. One of the results we obtain states that the order of a Nevanlinna matrix is always greater than or equal to the convergence exponent of the off-diagonal sequence of Jacobi parameters.

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IV.1 Introduction

In the present paper, we consider two-dimensional *canonical systems*

$$y'(t) = zJH(t)y(t), \qquad t \in (0, L) \text{ a.e.,} \tag{IV.1.1}$$

where $L \in (0, \infty)$, $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $z \in \mathbb{C}$, and the *Hamiltonian* H is a 2×2 matrix-valued function on $(0, L)$ that satisfies

$$H(t) \geq 0, \quad \operatorname{tr} H(t) = 1, \qquad t \in (0, L) \text{ a.e.,}$$

and is of a particular discrete form (the precise definition is given further below). Hamiltonians of this kind are called *Hamburger Hamiltonians*, and this term is reminiscent of the fact that they form an equivalent model of Hamburger moment problems. In fact, the orthogonal polynomials p_n of

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an indeterminate moment sequence $(s_n)_{n=0}^\infty$ satisfy a three-term recurrence relation

$$b_n p_{n+1}(z) + a_n p_n(z) + b_{n-1} p_{n-1}(z) = z p_n(z) \tag{IV.1.2}$$

with certain parameters $b_n > 0$ and $a_n \in \mathbb{R}$. The recurrence can be brought to the form (IV.1.1), with H being a Hamburger Hamiltonian (and this process can be reversed).

Properties of the moment sequence $(s_n)_{n=0}^\infty$ can be reformulated in terms of the corresponding Hamburger Hamiltonian H. Let us mention three prominent instances that are important for the present work:

- ▷ $(s_n)_{n=0}^\infty$ is indeterminate if and only if $L < \infty$;
- ▷ If $(s_n)_{n=0}^\infty$ is indeterminate, its Nevanlinna matrix $W = (w_{ij})_{i,j=1}^2$ is the monodromy matrix of H;
- ▷ The minimal operator of the Jacobi matrix with parameters a_n, b_n is unitarily equivalent to the minimal operator of H.

We deal with the indeterminate case, and investigate specifically growth and zero distribution of the entries w_{ij} of the Nevanlinna matrix. Due to the interpretation of W as the monodromy matrix of the system (IV.1.1), the zero distribution of w_{ij} has operator theoretic meaning: the zeros of, say, w_{22} coincide with the spectrum of the self-adjoint operator arising from (IV.1.1) and the boundary conditions $(1,0)y(0) = (0,1)y(L) = 0$. Hence, obtaining more detailed information about w_{ij} will lead to knowledge about the spectrum of the minimal Jacobi operator. As a simplest example: the (common) order of w_{ij} coincides with the (common) convergence exponent of the spectrum of self-adjoint extensions of the minimal Jacobi operator.

A classical theorem of M. Riesz [Rie23], or, alternatively, the Krein-de Branges formula [Kre51, dB61], tells us that w_{ij} is of minimal exponential type. There is a line of research, starting already in the early days of the study of moment problems, that deals with finding upper and lower bounds for the growth of the Nevanlinna matrix; a few examples are [Liv39, Ber56, BS14, Rom17, Pru20, PRW23]. We would like to emphasise two points. First, while there are many methods of obtaining upper bounds, all known lower bounds come from comparison of entries of W to entire functions derived from $(s_n)_{n=0}^\infty$ or coefficients of p_n [Liv39, BS14]. Estimates in terms of the Jacobi or Hamiltonian parameters can be obtained from these bounds only under regularity assumptions [Ber56, BS14, PW19, Pru20, PW23]. Second, the case of fast growth, meaning growth of order in $(\frac{1}{2}, 1]$, is much more intricate than the case of slow growth.

Hamburger Hamiltonians

Let $(l_j)_{j=1}^{\infty}$ be a summable sequence of positive numbers and $(\phi_j)_{j=1}^{\infty}$ be a sequence of real numbers. Using the notation

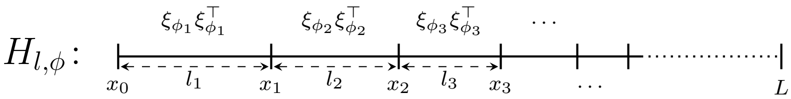
$$x_0 := 0, \qquad x_n := \sum_{j=1}^{n} l_j, \qquad n \in \mathbb{N},$$
$$L := \sum_{j=1}^{\infty} l_j < \infty$$

and

$$\xi_\phi := \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix},$$

we define a Hamiltonian on $(0, L)$ by setting

$$H_{l,\phi}(t) := \xi_{\phi_j}\xi_{\phi_j}^{\top} \quad \text{for } j \in \mathbb{N} \text{ and } x_{j-1} \le t < x_j. \tag{IV.1.3}$$



The numbers l_j and ϕ_j are referred to as *lengths* and *angles* of the Hamiltonian. Note that the sequences of lengths is unique for each Hamburger Hamiltonian, while the angles are not: adding an integer multiple of π to any ϕ_j leaves the Hamiltonian invariant. Hence, whenever we make an assumption on "the sequence of angles of a Hamburger Hamiltonian H", what we mean is that *there exists* $(\phi_j)_{j=1}^{\infty}$ such that $H = H_{l,\phi}$.

Let $W(t; z)$ be the fundamental solution of (the transpose of) (IV.1.1):

$$\begin{cases} \frac{d}{dt}W(t;z)J = zW(t;z)H(t), & t \in [0, L], \\ W(0, z) = I. \end{cases}$$

Since $L < \infty$, i.e., limit circle case holds at L, we can define

$$W_H(z) := W(L; z), \qquad \rho_H := \limsup_{r\to\infty} \frac{\log\log \max_{|z|=r} \|W_H(z)\|}{\log r}.$$

The function W_H is called the *monodromy matrix*. It is an entire function of exponential type 0, and its order ρ_H can be any number in $[0, 1]$.

Auxiliary notions

1. A function $\mathpzc{a}\colon [1,\infty) \to (0,\infty)$ is called *regularly varying* at ∞, if it is measurable and satisfies

$$\forall \lambda \in (0,\infty).\ \lim_{r\to\infty} \frac{\mathpzc{a}(\lambda r)}{\mathpzc{a}(r)} = \lambda^\alpha$$

for some $\alpha \in \mathbb{R}$ called the *index* of $\mathpzc{a}$. We write $\operatorname{ind}\mathpzc{a} := \alpha$.

2. Set

$$\mathpzc{a}^-(x) := \sup\big\{t \in [1,\infty) \mid \mathpzc{a}(t) < x\big\}. \tag{IV.1.4}$$

If $\mathpzc{a}$ is regularly varying with nonzero index, then $\mathpzc{a}^-$ is an asymptotic inverse of $\mathpzc{a}$, i.e.,

$$(\mathpzc{a} \circ \mathpzc{a}^-)(x) \sim (\mathpzc{a}^- \circ \mathpzc{a})(x) \sim x. \tag{IV.1.5}$$

3. We refer to [BGT89] for further information on regularly varying functions.

4. Consider functions $f, g : X \to (0,\infty)$, where X is any set. We use the following notation to compare f and g.

   ▷ We write $f \lesssim g$ if there exists $C > 0$ such that $f(x) \leq Cg(x)$ for all $x \in X$. By $f \gtrsim g$ we mean $g \lesssim f$, and $f \asymp g$ is used for "$f \lesssim g$ and $f \gtrsim g$".

   ▷ If $(X, \preceq)$ is directed, we write "$f(x) \lesssim g(x)$ for sufficiently large x" if there exists $x_0 \in X$ such that $f|_Y \lesssim g|_Y$, where $Y := \{x \in X\colon\ x \succeq x_0\}$. Similar notation is used for $\gtrsim$ and $\asymp$.

   ▷ Let again $(X, \preceq)$ be directed. We use the symbols $\sim$ and $\mathrm{O}(\cdot)$ in a standard way.

Main results

In Section IV.3, we compare the lengths and the increments of angles of H to regularly varying functions. If the decay of these functions is slow enough, the order is greater than $\frac{1}{2}$. Our first main theorem, Theorem IV.3.1, establishes a new lower bound for the growth of the monodromy matrix in this case. Let us present an overview of the situation that combines Theorem IV.3.1 with [PRW23, Corollaries III.4.7 and III.5.2].

IV.1.1 Theorem. *Let H be a Hamburger Hamiltonian, and suppose that*

$$l_j \asymp \mathscr{d}_l(j), \quad |\sin(\phi_{j+1} - \phi_j)| \asymp \mathscr{d}_\phi(j), \tag{IV.1.6}$$

where $\mathscr{d}_l$ is regularly varying with index $-\delta_l$, and $\mathscr{d}_\phi$ is regularly varying with index $-\delta_\phi$. Assume that $\mathscr{d}_\phi(t)$ is $\sim$ to a nonincreasing function as $t \to \infty$ and that $\mathscr{d}_l \asymp 1 \asymp \mathscr{d}_\phi$ on each compact subset of $[1, \infty)$.

Set

$$\mathscr{k}(r) := \Big[\frac{1}{\mathscr{d}_l \mathscr{d}_\phi}\Big]^-(r), \qquad \mathscr{m}(r) := r \int_{\left[\frac{\mathscr{d}_\phi}{\mathscr{d}_l}\right]^-(r)}^\infty \mathscr{d}_l(x)\,\mathrm{d}x. \tag{IV.1.7}$$

Then, if

$$0 < \delta_l + \delta_\phi < 2,$$

the following statements hold.

(i) *In general, we have*

$$\mathscr{k}(r) \lesssim \max_{|z|=r} \log \|W_H(z)\| \lesssim \mathscr{m}(r)$$

for sufficiently large r. In particular, $\frac{1}{\delta_l+\delta_\phi} \leq \rho_H \leq \frac{1-\delta_\phi}{\delta_l-\delta_\phi}$.

(ii) *If there is $\psi \in \mathbb{R}$ such that $|\sin(\phi_j - \psi)| \lesssim |\sin(\phi_{j+1} - \phi_j)|$, and if $\delta_l > 1$, then for sufficiently large r it holds that*

$$\max_{|z|=r} \log \|W_H(z)\| \asymp \mathscr{k}(r).$$

We have $\rho_H = \frac{1}{\delta_l+\delta_\phi}$.

(iii) *If ϕ_j is monotone with $|\phi_{j+1} - \phi_j| \asymp |\sin(\phi_{j+1} - \phi_j)| \asymp \mathscr{d}_\phi(j)$ and $\delta_\phi > 0$, the growth of W_H is given, for large enough r, by*

$$\max_{|z|=r} \log \|W_H(z)\| \asymp \mathscr{m}(r)$$

and its order is $\rho_H = \frac{1-\delta_\phi}{\delta_l-\delta_\phi}$.

The contribution of Theorem IV.3.1 to this overview is the lower bound in (iii).

Section IV.4 (which is independent of Section IV.3) is dedicated to estimates that do not depend on well-behavedness of the data. Unlike previous results, we obtain lower bounds directly in terms of the Jacobi parameters. For example, Theorem IV.4.4 states that the order of a Jacobi matrix is *always* greater than or equal to the convergence exponent of $(b_n)_{n=0}^{\infty}$.

Finally, in Theorem IV.4.7 we give an abstract upper estimate. In follow-up results, we obtain a number of bounds that are strikingly explicit in view of the weak assumptions on the data.

IV.2 Preliminaries

We collect auxiliary notions and properties of H.

A method of estimating W_H

Let us recall some notions from [LRW23]. First, we introduce the notation

$$\Omega_H(s,t) = \int_s^t H(x)\,dx. \tag{IV.2.1}$$

If the reference to H is unambiguous, we will write Ω instead of Ω_H.

In [LRW23, Theorem II.3.3] a function K_H is defined, which is derived explicitly from H and such that

$$\log|w_{H,22}(ir)| \asymp \int_0^L K_H(t;r)\,\mathrm{d}t. \qquad r>0, \tag{IV.2.2}$$

The precise form of K_H is not relevant for our purposes, as we will only need the following, more explicit, estimate.

IV.2.1 Lemma ([LRW23, Lemma II.6.5])**.** *Assume we have points s_0, s_1, s_2 such that*

$$0 \le s_0 < s_1 < s_2 \le L, \qquad \det\Omega_H(s_0,s_1) \ge \frac{1}{r^2} \ge \det\Omega_H(s_1,s_2).$$

Then

$$\int_{s_1}^{s_2} K_H(t;r)\,\mathrm{d}t \le e\log\big(r^2\det\Omega_H(s_0,s_2)\big). \tag{IV.2.3}$$

Further properties of $\det \Omega_H$

Let H be a Hamburger Hamiltonian with lengths $(l_j)_{j=1}^{\infty}$ and angles $(\phi_j)_{j=1}^{\infty}$. We will frequently use [LRW23, Lemma II.7.2], expressing $\det \Omega_H$ in terms of these parameters:

$$\det \Omega_H(x_m, x_n) = \frac{1}{2} \sum_{j,k=m+1}^{n} l_j l_k \sin^2(\phi_j - \phi_k). \tag{IV.2.4}$$

There are explicit relations between the lengths and angles of H and the corresponding Jacobi parameters a_j, b_j.

Let the angles of the Hamiltonian be normalised by setting $\phi_0 := \frac{\pi}{2}$ and $\phi_{j+1} - \phi_j \in [0, \pi)$. With $l_0 := 1$, we have

$$a_0 = \tan \phi_1, \tag{IV.2.5}$$

$$a_j = -\frac{\sin(\phi_{j+1} - \phi_{j-1})}{l_j \sin(\phi_{j+1} - \phi_j) \sin(\phi_j - \phi_{j-1})}, \qquad j = 1, 2, \ldots, \tag{IV.2.6}$$

$$b_j = \frac{1}{\sqrt{l_j l_{j+1}} \sin(\phi_{j+1} - \phi_j)}, \quad j = 0, 1, 2, \ldots, \tag{IV.2.7}$$

cf. relations (3.16) and (3.17) in [Kac99] .

This also leads to the following interpretation of $\det \Omega_H$ in terms of a_j, b_j:

$$\det \Omega_H(x_{n-1}, x_{n+1}) = l_{n+1} l_n \sin^2(\phi_{n+1} - \phi_n) = \frac{1}{b_n^2}, \tag{IV.2.8}$$

$$\det \Omega_H(x_{n-1}, x_{n+2}) = \frac{a_{j+1}^2 + b_j^2 + b_{j+1}^2}{b_j^2 b_{j+1}^2}.$$

IV.3 Regularly varying decay of parameters of H

Our first main result contributes to Theorem IV.1.1 the lower bound in item (iii). It treats sequences of angles that are monotone and unbounded, and shows that in this case the upper bound in Theorem IV.1.1, (i), is attained. Intuitively, this means that there is no cancellation, as opposed to item (ii) of Theorem IV.1.1.

IV.3.1 Theorem. *Consider a Hamburger Hamiltonian H. Let $\mathfrak{d}_l, \mathfrak{d}_\phi : [1, \infty) \to (0, \infty)$ be regularly varying. Assume that $\delta_\phi := -\operatorname{ind} \mathfrak{d}_\phi \in (0, 1)$. Suppose that angles $(\phi_j)_{j=1}^{\infty}$ of H can be chosen such that*

- $\triangleright$ $l_j \gtrsim \mathfrak{d}_l(j), \quad |\phi_{j+1} - \phi_j| \asymp \mathfrak{d}_\phi(j)$ *for sufficiently large j;*
- $\triangleright$ $(\phi_j)_{j=1}^{\infty}$ *is eventually monotone.*

Then

$$\log|w_{H,22}(ir)| \gtrsim r\int_{\left[\frac{d_\phi}{d_l}\right]^{-}(r)}^{\infty} \mathpzc{d}_l(x)\,\mathrm{d}x. \tag{IV.3.1}$$

for r sufficiently large.

The proof needs two auxiliary lemmata. The first one is folklore, but due to lack of reference we provide a proof.

IV.3.2 Lemma. *Let $\mathpzc{a} : [1,\infty) \to (0,\infty)$ be regularly varying and let $k > 0$. Then there exists $M > 0$ such that*

$$\mathpzc{a}(x+h) \asymp \mathpzc{a}(x), \qquad x \in [M,\infty), \quad h \in [-k,k]. \tag{IV.3.2}$$

In particular,

$$\int_x^y \mathpzc{a}(s)\,\mathrm{d}s \asymp (y-x)\mathpzc{a}(x) \asymp \frac{y-x}{y'-x'}\int_{x'}^{y'} \mathpzc{a}(s)\,\mathrm{d}s. \tag{IV.3.3}$$

on $\Big\{(x,y,x',y') \colon M \le x < y < x+k \wedge M \le x' < y' \wedge [x',y'] \subseteq [x-k,x+k]\Big\}.$

Proof. Let $K := \left[\frac{1}{k+1}, k+1\right]$. Then for $x \ge k+1$ and $h \in [-k,k]$ we have $\frac{x+h}{x} \in K$. By [BGT89, Theorem 1.2.1], we have $\lim_{x\to\infty} \frac{\mathpzc{a}(\lambda x)}{\mathpzc{a}(x)} = \lambda^\alpha$ uniformly for $\lambda \in K$. Consequently, there exists $M \ge k+1$ such that

$$\left|\frac{\mathpzc{a}(x+h)}{\mathpzc{a}(x)} - \left(\frac{x+h}{x}\right)^\alpha\right| < \frac{1}{2(k+1)^\alpha}, \qquad x \ge M.$$

Since $\left(\frac{x+h}{x}\right)^\alpha \ge \frac{1}{(k+1)^\alpha}$, (IV.3.2) follows. Now (IV.3.3) holds because of

$$\int_x^y \mathpzc{a}(s)\,\mathrm{d}s \asymp (y-x)\mathpzc{a}(x) = \frac{y-x}{y'-x'}\cdot(y'-x')\mathpzc{a}(x) \asymp \frac{y-x}{y'-x'}\int_{x'}^{y'} \mathpzc{a}(s)\,\mathrm{d}s.$$

□

Our main intuition in the proof of Theorem IV.3.1 is that for monotone angles, $|\sin(\phi_j - \phi_k)|$ stays away from zero for sufficiently many indices j,k – which is important because of (IV.2.4). In the following lemma, we develop the necessary tool for quantifying this intuition.

IV.3.3 Lemma. *Let H be a Hamburger Hamiltonian. Then, for $m,n \in \mathbb{N}$ with $m < n$ and a permutation σ of $\{m+1,\dots,n\}$,*

$$\det\Omega(x_m,x_n) \ge \frac{1}{8}\bigg(\sum_{j=m+1}^{n} \min\{l_j, l_{\sigma(j)}\}|\sin(\phi_j - \phi_{\sigma(j)})|\bigg)^2.$$

Proof. We use the elementary fact $\sin^2(a+b) \le 2[\sin^2(a)+\sin^2(b)]$. By (IV.2.4),

$$
\begin{aligned}
\det\Omega(x_m,x_n) &= \frac{1}{2}\sum_{j,k=m+1}^{n} l_j l_k \sin^2(\phi_j-\phi_k)\\
&= \frac{1}{4}\sum_{j,k=m+1}^{n}\Big[l_j l_k \sin^2(\phi_j-\phi_k) + l_k l_{\sigma(j)}\sin^2(\phi_k-\phi_{\sigma(j)})\Big]\\
&\ge \frac{1}{4}\sum_{j,k=m+1}^{n} l_k \min\{l_j,l_{\sigma(j)}\}\big[\sin^2(\phi_j-\phi_k)+\sin^2(\phi_k-\phi_{\sigma(j)})\big]\\
&\ge \frac{1}{8}\Big(\sum_{k=m+1}^{n} l_k\Big)\Big(\sum_{j=m+1}^{n}\min\{l_j,l_{\sigma(j)}\}\sin^2(\phi_j-\phi_{\sigma(j)})\Big)\\
&\ge \frac{1}{8}\Big(\sum_{j=m+1}^{n}\sqrt{l_j\min\{l_j,l_{\sigma(j)}\}}|\sin(\phi_j-\phi_{\sigma(j)})|\Big)^2\\
&\ge \frac{1}{8}\Big(\sum_{j=m+1}^{n}\min\{l_j,l_{\sigma(j)}\}|\sin(\phi_j-\phi_{\sigma(j)})|\Big)^2.
\end{aligned}
$$

$\square$

Proof of Theorem IV.3.1. Our strategy is to find indices m,n such that $\det\Omega(x_m,x_n)$ can be estimated from below without much loss, then use [LRW23, Lemma II.6.4]. We do that by using Lemma IV.3.3 and finding permutations σ of subsets of $\mathbb{N}$ such that $|\sin(\phi_k-\phi_{\sigma(k)})| \asymp 1$ on a sufficiently large k-set.

We may assume that d_ϕ is continuous: If not, replace it by the function obtained from linearly interpolating the integer values of d_ϕ. Let

$$\Phi(t) := \int_1^t d_\phi(s)\,\mathrm{d}s, \qquad t \ge 1.$$

The function $\Phi : [1,\infty) \to [\phi_1,\infty)$ is increasing and regularly varying with index $1-\delta_\phi > 0$. In particular, it is bijective.

Step 1. For fixed $h>0$, the mean value theorem yields

$$\Phi^{-1}(x+h) - \Phi^{-1}(x) = h\big(\Phi^{-1}\big)'(\xi_x) = \frac{h}{\Phi'\big(\Phi^{-1}(\xi_x)\big)}$$

for some $\xi_x \in [x, x+h]$. Since $\lim_{x\to\infty} \frac{\xi_x}{x} = 1$ and because Φ^{-1} and Φ' are regularly varying, we have

$$\Phi^{-1}(x+h) - \Phi^{-1}(x) \sim \frac{h}{\Phi'\big(\Phi^{-1}(x)\big)} = \frac{h}{d_\phi\big(\Phi^{-1}(x)\big)}.$$

By Karamata's Theorem [BGT89, Theorem 1.5.11], $\Phi(t) \sim \frac{t d_\phi(t)}{1-\delta_\phi}$, and hence

$$\Phi^{-1}(x+h) - \Phi^{-1}(x) \sim \frac{h}{1-\delta_\phi} \cdot \frac{\Phi^{-1}(x)}{x}. \tag{IV.3.4}$$

Step 2. The right hand side of (IV.3.4) is regularly varying with index $\frac{\delta_\phi}{1-\delta_\phi} > 0$, and thus tends to ∞. Fix numbers $u, w, u', w' \in (0,1)$ with $u < w$, $u' < w'$, and $w - u < w' - u'$. Consider, for $j \in \mathbb{N}$, the discrete intervals

$$I_j^+ := \Big\{k \in \mathbb{N} : \Big\lceil \Phi^{-1}\big(j+u\big)\Big\rceil \le k \le \Big\lfloor \Phi^{-1}\big(j+w\big)\Big\rfloor \Big\}$$

and

$$I_j^- := \Big\{l \in \mathbb{N} : \Big\lceil \Phi^{-1}\big(j-w'\big)\Big\rceil \le l \le \Big\lfloor \Phi^{-1}\big(j-u'\big)\Big\rfloor \Big\}.$$

By (IV.3.4), the cardinalities of I_j^+ and I_j^- satisfy

$$\lim_{j\to\infty} \frac{|I_j^+|}{|I_j^-|} = \frac{w-u}{w'-u'} < 1.$$

In particular, $|I_j^+| < |I_j^-|$ for all sufficiently large j.

Step 3. For the following estimates, we need a (finite) number of properties that only hold for sufficiently large indices. We will frequently assume implicitly that each variable that occurs is large enough.

Given sufficiently large $n, m \in \mathbb{N}$ with $n \ge m+2$, there exists a permutation σ of $\big\{\big\lceil \Phi^{-1}\big(m\big)\big\rceil, \ldots, \big\lceil \Phi^{-1}\big(n\big)\big\rceil\big\}$ with the property $\sigma(I_j^+) \subseteq I_j^-$ for all $m+1 \le j \le n$. Our goal is to choose u, w, u', w' such that

$$|\sin(\phi_k - \phi_{\sigma(k)})| \asymp 1, \quad m, n \text{ suff. large}, \quad m+1 \le j \le n, \quad k \in I_j^+. \tag{IV.3.5}$$

Let $k \in I_j^+$. Then

$$\begin{aligned} u+u' &\le \Phi\Big(\Big\lceil \Phi^{-1}\big(j+u\big)\Big\rceil\Big) - \Phi\Big(\Big\lfloor \Phi^{-1}\big(j-u'\big)\Big\rfloor\Big) \le \Phi(k) - \Phi(\sigma(k)) \\ &\le \Phi\Big(\Big\lfloor \Phi^{-1}\big(j+w\big)\Big\rfloor\Big) - \Phi\Big(\Big\lceil \Phi^{-1}\big(j-w'\big)\Big\rceil\Big) \le w+w'. \end{aligned} \tag{IV.3.6}$$

Our next goal is to obtain similar bounds for $|\phi_k - \phi_{\sigma(k)}|$. By Lemma IV.3.2,

$$\Phi(k) - \Phi(\sigma(k)) = \int_{\sigma(k)}^{k} d_\phi(s)\, \mathrm{d}s = \sum_{i=\sigma(k)}^{k-1} \int_i^{i+1} d_\phi(x)\, \mathrm{d}x \asymp \sum_{i=\sigma(k)}^{k-1} d_\phi(i)$$
$$\asymp \sum_{i=\sigma(k)}^{k-1} \left|\phi_{i+1} - \phi_i\right| = \left|\phi_k - \phi_{\sigma(k)}\right|$$

and hence there exist constants c, c' such that

$$c\big(\Phi(k) - \Phi(\sigma(k))\big) \le |\phi_k - \phi_{\sigma(k)}| \le c'\big(\Phi(k) - \Phi(\sigma(k))\big)$$

for m, n large, $m+1 \le j \le n$ and $k \in I_j^+$. Remembering (IV.3.6), we see that

$$c(u + u') \le \phi_k - \phi_{\sigma(k)} \le c'(w + w') \tag{IV.3.7}$$

for m, n large, $m+1 \le j \le n$ and $k \in I_j^+$.

Let $w := w' := \frac{\pi}{4} \min\{1, \frac{1}{c'}\}$ and $u = \frac{w}{2}$, $u' = \frac{w}{4}$. Due to (IV.3.7), there is $\epsilon > 0$ such that $\phi_k - \phi_{\sigma(k)} \in [\epsilon, \frac{\pi}{2}]$. Hence (IV.3.5) holds for this choice of u, w, u', w'.

Step 4. Recalling (IV.3.5), Lemma IV.3.3 leads to

$$\begin{aligned}
&\sqrt{\det \Omega\big(x_{\lceil \Phi^{-1}(m)\rceil}, x_{\lceil \Phi^{-1}(n)\rceil}\big)} \\
&\gtrsim \sum_{k=\lceil \Phi^{-1}(m)\rceil+1}^{\lceil \Phi^{-1}(n)\rceil} \min\{l_k, l_{\sigma(k)}\} |\sin(\phi_k - \phi_{\sigma(k)})| \\
&\gtrsim \sum_{j=m+1}^{n} \sum_{k \in I_j^+} \min\{l_k, l_{\sigma(k)}\} \gtrsim \sum_{j=m+1}^{n} \sum_{k \in I_j^+} \min\{d_l(k), d_l(\sigma(k))\} \\
&= \sum_{j=m+1}^{n} \sum_{k \in I_j^+} d_l(k).
\end{aligned} \tag{IV.3.8}$$

By Lemma IV.3.2,

$$\sum_{j=m+1}^{n} \sum_{k \in I_j^+} d_l(k) \asymp \sum_{j=m+1}^{n} \sum_{k \in I_j^+} \int_k^{k+1} d_l(x)\, \mathrm{d}x$$
$$= \sum_{j=m+1}^{n} \int_{\lceil \Phi^{-1}(j+u)\rceil}^{1+\lfloor \Phi^{-1}(j+w)\rfloor} d_l(x)\, \mathrm{d}x \asymp \sum_{j=m+1}^{n} \int_{\Phi^{-1}(j+u)}^{\Phi^{-1}(j+w)} d_l(x)\, \mathrm{d}x$$

for j sufficiently large. Combining this with (IV.3.8),

$$\sqrt{\det \Omega\big(x_{\lceil \Phi^{-1}(m)\rceil}, x_{\lceil \Phi^{-1}(n)\rceil}\big)} \gtrsim \sum_{j=m+1}^{n} \int_{\Phi^{-1}(j+u)}^{\Phi^{-1}(j+w)} d_l(x)\, \mathrm{d}x \tag{IV.3.9}$$

Step 5. We further refine (IV.3.9). For all $\xi < \zeta$, substituting $x = \Phi^{-1}(t)$ yields

$$\int_{\Phi^{-1}(\xi)}^{\Phi^{-1}(\zeta)} d_l(x)\, \mathrm{d}x = \int_{\xi}^{\zeta} \frac{d_l\big(\Phi^{-1}(t)\big)}{\Phi'\big(\Phi^{-1}(t)\big)}\, \mathrm{d}t = \int_{\xi}^{\zeta} \frac{d_l\big(\Phi^{-1}(t)\big)}{d_\phi\big(\Phi^{-1}(t)\big)}\, \mathrm{d}t. \tag{IV.3.10}$$

Using Lemma IV.3.2, this leads to

$$\begin{aligned}
&\sqrt{\det \Omega\big(x_{\lceil \Phi^{-1}(m)\rceil}, x_{\lceil \Phi^{-1}(n)\rceil}\big)} \gtrsim \sum_{j=m+1}^{n} \int_{j+u}^{j+w} \frac{d_l\big(\Phi^{-1}(t)\big)}{d_\phi\big(\Phi^{-1}(t)\big)}\, \mathrm{d}t \\
&\asymp \sum_{j=m+1}^{n} \frac{(j+w)-(j+u)}{(j+1)-j} \int_{j}^{j+1} \frac{d_l\big(\Phi^{-1}(t)\big)}{d_\phi\big(\Phi^{-1}(t)\big)}\, \mathrm{d}t \\
&\asymp \int_{m+1}^{n+1} \frac{d_l\big(\Phi^{-1}(t)\big)}{d_\phi\big(\Phi^{-1}(t)\big)}\, \mathrm{d}t \asymp \int_{m}^{n} \frac{d_l\big(\Phi^{-1}(t)\big)}{d_\phi\big(\Phi^{-1}(t)\big)}\, \mathrm{d}t.
\end{aligned} \tag{IV.3.11}$$

Step 6. For $r > 0$, define

$$m_0(r) := 1 + \max\left\{ m \in \mathbb{N} \colon \int_{m}^{m+1} \frac{d_l\big(\Phi^{-1}(t)\big)}{d_\phi\big(\Phi^{-1}(t)\big)}\, \mathrm{d}t \geq \frac{1}{r} \right\}$$

and

$$m_n(r) := \min\left\{ m \in \mathbb{N} \colon m > m_{n-1}(r) \wedge \int_{m_{n-1}(r)}^{m} \frac{d_l\big(\Phi^{-1}(t)\big)}{d_\phi\big(\Phi^{-1}(t)\big)}\, \mathrm{d}t \geq \frac{1}{r} \right\}$$

where $\min \emptyset := \infty$. For $n \geq 1$ with $m_n(r) < \infty$ we have, by (IV.3.11),

$$\det \Omega\big(x_{\lceil \Phi^{-1}(m_{n-1}(r))\rceil}, x_{\lceil \Phi^{-1}(m_n(r))\rceil}\big) \gtrsim \frac{1}{r^2}. \tag{IV.3.12}$$

Let $k(r) := \min\{n \in \mathbb{N} \colon m_n(r) = \infty\}$. For large enough $r > 0$, the ($k(r)$ many) points

$$0,\ x_2,\ x_{\lceil \Phi^{-1}(m_1(r))\rceil},\ x_{\lceil \Phi^{-1}(m_2(r))\rceil}, \ldots, x_{\lceil \Phi^{-1}(m_{k(r)-2}(r))\rceil},\ L$$

satisfy the condition of [LRW23, Lemma II.6.4], and this yields that

$$\log|w_{H,22}(ir)| \gtrsim \int_{x_2}^{L} K_H(t;r)\,\mathrm{d}t \gtrsim k(r) + \mathrm{O}(\log r). \tag{IV.3.13}$$

Step 7. We shall estimate $k(r)$ from below. By definition of $m_0(r)$, for $1 \leq n < k(r)$ it holds that

$$\frac{1}{r} \leq \int_{m_{n-1}(r)}^{m_n(r)} \frac{d_l\big(\Phi^{-1}(t)\big)}{d_\phi\big(\Phi^{-1}(t)\big)}\,\mathrm{d}t \leq \frac{1}{r} + \int_{m_n(r)-1}^{m_n(r)} \frac{d_l\big(\Phi^{-1}(t)\big)}{d_\phi\big(\Phi^{-1}(t)\big)}\,\mathrm{d}t \leq \frac{2}{r}. \tag{IV.3.14}$$

In addition we have

$$\int_{m_{k(r)-1}(r)}^{\infty} \frac{d_l\big(\Phi^{-1}(t)\big)}{d_\phi\big(\Phi^{-1}(t)\big)}\,\mathrm{d}t \leq \frac{1}{r}.$$

This leads to

$$\frac{k(r)}{r} \gtrsim \sum_{n=1}^{k(r)} \int_{m_{n-1}(r)}^{m_n(r)} \frac{d_l\big(\Phi^{-1}(t)\big)}{d_\phi\big(\Phi^{-1}(t)\big)}\,\mathrm{d}t = \int_{m_0(r)}^{\infty} \frac{d_l\big(\Phi^{-1}(t)\big)}{d_\phi\big(\Phi^{-1}(t)\big)}\,\mathrm{d}t. \tag{IV.3.15}$$

By Lemma IV.3.2,

$$\frac{d_l\big(\Phi^{-1}(m_0(r))\big)}{d_\phi\big(\Phi^{-1}(m_0(r))\big)} \asymp \int_{m_0(r)}^{m_0(r)+1} \frac{d_l\big(\Phi^{-1}(t)\big)}{d_\phi\big(\Phi^{-1}(t)\big)}\,\mathrm{d}t \asymp \frac{1}{r}$$

implying

$$m_0(r) \asymp \Phi\Big(\Big[\frac{d_\phi}{d_l}\Big]^{-}(r)\Big). \tag{IV.3.16}$$

Substituting back using (IV.3.10), the estimate (IV.3.15) becomes

$$k(r) \gtrsim r \int_{m_0(r)}^{\infty} \frac{d_l\big(\Phi^{-1}(t)\big)}{d_\phi\big(\Phi^{-1}(t)\big)}\,\mathrm{d}t \asymp r \int_{\left[\frac{d_\phi}{d_l}\right]^{-}(r)}^{\infty} d_l(x)\,\mathrm{d}x. \tag{IV.3.17}$$

Using (IV.3.13), the theorem is proved. $\square$

We return for a moment to the situation of Theorem IV.1.1. If (IV.1.6) is satisfied with $\delta_l + \delta_\phi = 2$ but not $(\delta_l, \delta_\phi) = (1,1)$, then [PRW23, Corollaries III.4.7 and III.5.2] state that $\rho_H = \frac{1}{2}$. Our next result shows that the point $(\delta_l, \delta_\phi) = (1,1)$ is intrinsically exceptional.

IV.3.4 Proposition. *Let H be a Hamburger Hamiltonian satisfying*

$$l_j \asymp j^{-1}(\log j)^{-\nu}, \qquad \phi_{j+1} - \phi_j = |\phi_{j+1} - \phi_j| \asymp j^{-1}$$

for j sufficiently large, where $\nu \in (1,2)$. Then

$$\log \max_{|z|=r} \|W_H(z)\| \asymp r^{\frac{1}{\nu}}$$

for r sufficiently large. In particular, $\rho_H = \frac{1}{\nu}$.

Proof of Proposition IV.3.4. Using the notation from the beginning of the section, we have

$$d_l(t) = t^{-1}(\log t)^{-\nu}, \qquad d_\phi(t) = t^{-1}.$$

Further, let

$$c_l(t) := (\nu - 1)\int_t^\infty d_l(s)\,\mathrm{d}s = (\log t)^{1-\nu}, \qquad c_\phi := c_l.$$

A calculation shows that we are in the situation of [PRW23, Example III.5.9], and we obtain the upper bound

$$\log \max_{|z|=r} \|W_H(z)\| \lesssim r^{\frac{1}{\nu}}.$$

The proof of the lower bound is the same as that of Theorem IV.3.1, except for a few modifications. We have

$$\Phi(t) = \int_1^t d_\phi(s)\,\mathrm{d}s = \log t$$

and thus $\Phi^{-1}(x) = e^x$. Defining I_j^+ and I_j^- as in the proof of Theorem IV.3.1, we have

$$|I_j^+| \sim \Phi^{-1}(j+w) - \Phi^{-1}(j+u) = e^j\big(e^w - e^u\big),$$
$$|I_j^-| \sim \Phi^{-1}(j-u') - \Phi^{-1}(j-w') = e^j\big(e^{-u'} - e^{-w'}\big)$$

We will later choose $u, w, u', w' \in (0,1)$ such that $|I_j^+| < |I_j^-|$ for sufficiently large j. Proceeding as in Step 3, we arrive at (IV.3.7) and hence w, w' need to be chosen such that $c'(w + w') \leq \frac{\pi}{2}$. One possibility is making the ansatz $u = \frac{w}{2}$, $u' = \frac{w'}{2}$, $w' := \frac{\pi}{4}\min\{1, \frac{1}{c'}\}$, then choosing w very small. Then, by our construction, (IV.3.5) holds.

We proceed with Step 4, where no adjustments are necessary. Hence (IV.3.9) holds, leading to

$$\begin{aligned}\sqrt{\det \Omega\big(x_{\lceil \Phi^{-1}(m)\rceil}, x_{\lceil \Phi^{-1}(n)\rceil}\big)} &\gtrsim \sum_{j=m+1}^{n} \int_{\Phi^{-1}(j+u)}^{\Phi^{-1}(j+w)} d_l(x)\,\mathrm{d}x \\ &= \sum_{j=m+1}^{n} \int_{e^{j+u}}^{e^{j+w}} x^{-1}(\log x)^{-\nu}\,\mathrm{d}x = \sum_{j=m+1}^{n} \int_{j+u}^{j+w} t^{-\nu}\,\mathrm{d}t \\ &\asymp \sum_{j=m+1}^{n} \int_{j}^{j+1} t^{-\nu}\,\mathrm{d}t \asymp \int_{m}^{n} t^{-\nu}\,\mathrm{d}t.\end{aligned}$$

Steps 6 and 7 of the proof of Theorem IV.3.1 can again be copied without changes, since

$$\frac{d_l\big(\Phi^{-1}(t)\big)}{d_\phi\big(\Phi^{-1}(t)\big)} = t^{-\nu}$$

is regularly varying. By (IV.3.16), we have $m_0(r) \asymp r^{\frac{1}{\nu}}$, and thus the asserted lower bound is given by (IV.3.17). □

IV.3.5 Remark.

▷ If $\nu = 2$ in Proposition IV.3.4, then $\rho_H = \frac{1}{2}$, and

$$r^{\frac{1}{2}} \lesssim \log \max_{|z|=r} \|W_H(z)\| \lesssim r^{\frac{1}{2}} \log r.$$

The proof of this is the same as that of Proposition IV.3.4, except the upper bound taken from [PRW23, Example III.5.9] has a different form.

▷ If $\nu > 2$, we also have $\rho_H = \frac{1}{2}$. All we can say about the precise growth is

$$r^{\frac{1}{2}}(\log r)^{-\frac{\nu}{2}} \lesssim \log \max_{|z|=r} \|W_H(z)\| \lesssim r^{\frac{1}{2}}(\log r)^{1-\frac{\nu}{2}}.$$

In order to see this, use [PRW23, Corollary III.5.2] for the lower bound and the second row of [PRW23, Theorem III.4.6] for the upper bound.

IV.3.1 Jacobi matrices with large diagonal

In [Pru20, Theorem 2], Jacobi matrices J with parameters satisfying

$$\begin{aligned} a_n &= n^{\beta}\Big(y_0 + \frac{y_1}{n} + \frac{y_2}{n^2} + \mathrm{O}\Big(\frac{1}{n^{2+\epsilon}}\Big)\Big), \qquad n \to \infty, \\ b_n &= n^{\beta}\Big(\frac{|y_0|}{2} + \frac{x_1}{n} + \frac{x_2}{n^2} + \mathrm{O}\Big(\frac{1}{n^{2+\epsilon}}\Big)\Big), \qquad n \to \infty \end{aligned} \tag{IV.3.18}$$

with $y_0 \neq 0$ are considered. It is shown that J is in the limit circle case if $\kappa := \frac{4x_1}{|y_0|} - \frac{2y_1}{y_0} > \frac{3}{2}$ and $\beta \in (\frac{3}{2}, \kappa)$. If even $\kappa > 2$ and $\beta \in [2, \kappa)$, the order is equal to $\frac{1}{\beta}$.

In contrast to this, if $\frac{3}{2} < \beta < \min\{2, \kappa\}$, Pruckner's result only shows that the order belongs to $[\frac{1}{\beta}, \frac{1}{2(\beta-1)}]$. Using Theorem IV.1.1, we can now construct examples of Jacobi matrices of this form, for which the order is in fact equal to $\frac{1}{2(\beta-1)}$.

IV.3.6 Corollary. *For each $\beta \in (\frac{3}{2}, 2)$, there exist sequences $(a_n), (b_n)$ with asymptotics of the form*

$$a_n = n^\beta \Big(-2 + \frac{7}{6n} + \frac{y_2}{n^2} + \mathrm{O}\Big(\frac{1}{n^3}\Big)\Big), \qquad n \to \infty,$$
$$b_n = n^\beta \Big(1 + \frac{\frac{\beta}{2} - \frac{1}{12}}{n} + \frac{x_2}{n^2} + \mathrm{O}\Big(\frac{1}{n^3}\Big)\Big), \qquad n \to \infty,$$

such that the Jacobi matrix J with these parameters is in limit circle case, and its Nevanlinna matrix satisfies

$$\log \max_{|z|=r} \|W(z)\| \asymp r^{\frac{1}{2(\beta-1)}}, \qquad r \textit{ sufficiently large.} \tag{IV.3.19}$$

Proof. Define a Hamburger Hamiltonian by

$$l_n := n^{\frac{1}{2}-\beta}, \qquad \phi_{n+1} - \phi_n = n^{-\frac{1}{2}},$$

whose monodromy matrix satisfies (IV.3.19) due to Theorem IV.1.1. Now use (IV.2.7),(IV.2.6) to compute the asymptotics of the corresponding Jacobi matrix. ☐

IV.3.7 Open Problem. We suspect, but are unable to prove, that asymptotics of the form (IV.3.18), with $\frac{3}{2} < \beta < \min\{2, \kappa\}$, will always lead to

$$\log \max_{|z|=r} \|W(z)\| \asymp r^{\frac{1}{2(\beta-1)}}.$$

◁

IV.4 One-sided estimates for the growth of $w_{H,22}$

The overarching theme of this section is to find explicit bounds without imposing many restrictions on the data. The key challenge is to estimate $\det \Omega(x_m, x_n)$ with as little loss as possible.

IV.4.1 Lower bounds: Hamiltonian parameters

In the following two results, we establish lower bounds directly in terms of $\det\Omega$. Theorem IV.4.1 gives a bound for the order only, but is generally easier to evaluate than Theorem IV.4.2 which estimates the growth of $\log|w_{H,22}(ir)|$. In the case of (almost) regularly varying data, the bounds for order obtained from these theorems coincide, and Theorem IV.4.2 also gives an estimate for the type with respect to the bound for the order.

IV.4.1 Theorem. *Let H be a Hamburger Hamiltonian on $(0,L)$ and let $s\in\mathbb{N}$. For $j\in\mathbb{N}$ set*

$$\beta_j := \frac{1}{\sqrt{\det\Omega(x_j,x_{j+s})}}.$$

Then the order of W_H is greater than or equal to the convergence exponent of $(\beta_j)_{j=1}^\infty$.

Proof. Let α be the convergence exponent of $(\beta_j)_{j=1}^\infty$, then for $\epsilon>0$

$$\sum_{n=0}^\infty \sqrt{\det\Omega(x_j,x_{j+s})}^{\,\alpha-\epsilon} = \infty.$$

For

$$N_r := \Big|\Big\{j\colon\ \sqrt{\det\Omega(x_j,x_{j+s})} \geq \frac{1}{r}\Big\}\Big|,$$

we obtain $\neg\big(N_r \lesssim r^{\alpha-2\epsilon}\big)$ from [BS14, Lemma 5.5]. Hence, there exists an increasing sequence $(r_m)_{m=1}^\infty$ such that $N_{r_m} \geq r_m^{\alpha-2\epsilon}$ for all $m\in\mathbb{N}$.

We find that for each $m\in\mathbb{N}$, there are at least $k(r_m) := \lfloor\frac{N_{r_m}}{s}\rfloor$ non-overlapping intervals $[x_j,x_{j+s}]$ with $\det\Omega(x_j,x_{j+s}) \geq \frac{1}{r_m^2}$. By (IV.2.2) and [LRW23, Lemma II.6.4]

$$\log|w_{H,22}(ir_m)| \asymp \int_0^L K_H(t,r_m)\,\mathrm{d}t \gtrsim k(r_m) \asymp N_{r_m} \geq r_m^{\alpha-2\epsilon}.$$

This shows that the order of W is at least $\alpha-2\epsilon$. Since ϵ was arbitrary, this completes the proof. ☐

The following theorem can be seen as a generalisation of [PW23, Corollary 2.5], which only admits (close to) regularly varying data and $s=2$.

IV.4.2 Theorem. *Let H be a Hamburger Hamiltonian on $(0,L)$ and let $s\in\mathbb{N}$. Suppose that we are given a sequence $(f_j)_{j=1}^\infty$ of nonnegative numbers such that*

$$\sqrt{\det\Omega(x_j,x_{j+s})} \geq cf_j, \qquad j\in\mathbb{N}$$

for some $c>0$. Then

(i) *We always have*

$$\log|w_{H,22}(ir)| \gtrsim \frac{r}{s} \sum_{j:\ \forall k \geq j.\, f_k < \frac{1}{r}} f_j + \mathrm{O}(\log r), \qquad r \text{ sufficiently large.}$$

The constant implicit in this estimate depends only on c and on L.

(ii) *If $f_j = \frac{1}{g(j)}$ for some regularly varying function g, then*

$$\log|w_{H,22}(ir)| \gtrsim g^{-}(r)$$

and the order of W_H is greater than or equal to $\frac{1}{\operatorname{ind} g}$.

IV.4.3 Remark. The number $\frac{1}{\operatorname{ind} g}$ appearing in item (ii) coincides with the convergence exponent of $(f_j^{-1})_{j=1}^{\infty}$.

Proof. We start by proving (i). Consider an interval $[x_m, x_n]$. Using the Minkowski Determinant Theorem,

$$\sqrt{\det \Omega(x_m, x_n)} \geq \sum_{j=1}^{\lfloor \frac{n-m}{s} \rfloor} \sqrt{\det \Omega(x_{m+(j-1)s}, x_{m+js})} \geq c \sum_{j=1}^{\lfloor \frac{n-m}{s} \rfloor} f_{m+(j-1)s}.$$

Applying this inequality also to the intervals $[x_{m+k}, x_n]$, we find that

$$\sqrt{\det \Omega(x_{m+k}, x_n)} \geq c \sum_{j=1}^{\lfloor \frac{n-m-k}{s} \rfloor} f_{m+k+(j-1)s}.$$

Summing up these inequalities for $k = 0, \dots, s-1$ leads to

$$s\sqrt{\det \Omega(x_m, x_n)} \geq \sum_{k=0}^{s-1} \sqrt{\det \Omega(x_{m+k}, x_n)}$$
$$\geq c \sum_{k=0}^{s-1} \sum_{j=1}^{\lfloor \frac{n-m-k}{s} \rfloor} f_{m+k+(j-1)s} = c \sum_{j=m}^{n-s} f_j.$$

Let $m_0(r) := 1 + \max\left\{j :\ f_j \geq \frac{1}{r}\right\}$ and define inductively

$$m_{n+1}(r) := \min\left\{m > m_n(r) \ :\ \sum_{j=m_n(r)}^{m} f_j \geq \frac{2s}{r}\right\}$$

where $\min \emptyset := \infty$.

Recalling $f_j < \frac{1}{r}$ for $j \geq m_0(r)$, we obtain for each n with $m_{n+1}(r) < \infty$

$$\sqrt{\det \Omega(x_{m_n(r)}, x_{m_{n+1}(r)})} \geq \frac{c}{s} \sum_{j=m_n(r)}^{m_{n+1}(r)-s} f_j$$

$$\geq \frac{c}{s}\left(\sum_{j=m_n(r)}^{m_{n+1}(r)} f_j - \frac{s}{r}\right) \geq \frac{c}{r}.$$

Set $k(r) := \min\{n : m_n(r) = \infty\}$, then for sufficiently large $r > 0$, the ($k(r)$ many) points

$$0,\ x_{m_0(r)},\ x_{m_1(r)}, \ldots x_{m_{k(r)-3}(r)},\ x_{m_{k(r)-2}(r)},\ L$$

meet the condition of [LRW23, Lemma II.6.4]. Hence $\log |w_{H,22}(ir)\| \gtrsim k(r) + \mathrm{O}(\log r)$, which implies (i) by estimating

$$\frac{2s}{r} k(r) \geq \sum_{n=0}^{k(r)-2} \sum_{j=m_n(r)}^{m_{n+1}(r)-1} f_j + \sum_{j=m_{a(r)-1}(r)}^{\infty} f_j = \sum_{j:\ \forall k \geq j.\, f_k < \frac{1}{r}} f_j.$$

It remains to prove (ii). Suppose that $f_j = \frac{1}{\mathfrak{g}(j)}$ for regularly varying $\mathfrak{g}$. By Karamata's Theorem,

$$r \sum_{j:\ \forall k \geq j.\, f_k < \frac{1}{r}} f_j = r \sum_{j:\ \forall k \geq j.\, \mathfrak{g}(k) > r} \frac{1}{\mathfrak{g}(j)} \asymp r \int_{\mathfrak{g}^-(r)}^{\infty} \frac{1}{\mathfrak{g}(x)}\, \mathrm{d}x$$

$$\gtrsim r \frac{\mathfrak{g}^-(r)}{\mathfrak{g}(\mathfrak{g}^-(r))} \asymp \mathfrak{g}^-(r).$$

$\square$

IV.4.2 Lower bounds: Jacobi parameters

Berezanskii's Theorem [Ber65, VII, Theorem 1.5] gives a certain (rather narrow) family of Jacobi matrices having order equal to the convergence exponent of $(b_n)_{n=0}^{\infty}$. This number also appears in other specific situations as a lower bound for the order, e.g., in Corollary IV.3.6 and the discussion preceding it.

We generalise this observation and show that the convergence exponent of $(b_n)_{n=0}^{\infty}$ is *always* a lower bound for the order.

IV.4.4 Theorem. *Let J be a Jacobi matrix in limit circle case with corresponding Nevanlinna matrix W. Then the order of W is greater than or equal to the convergence exponent of $(b_n)_{n=0}^{\infty}$.*

Proof. Use (IV.2.8) and Theorem IV.4.1 for $s = 2$. □

Using Theorem IV.4.2, we also obtain a lower bound for $\log|w_{22}(ir)|$ in terms of the Jacobi parameters.

IV.4.5 Corollary. *Let J be a Jacobi matrix in limit circle case with corresponding Nevanlinna matrix W. Then*

$$\log|w_{22}(ir)| \gtrsim r \sum_{n:\ \forall k \geq n.\, b_k > r} \frac{1}{b_n} + \mathrm{O}(\log r), \qquad r \text{ sufficiently large.}$$

If $b_n \asymp \mathfrak{b}(n)$ for some regularly varying function $\mathfrak{b}$, then

$$\log|w_{22}(ir)| \gtrsim \mathfrak{b}^-(r).$$

Proof. Consider the Hamburger Hamiltonian H corresponding to J, which satisfies $W = W_H$. By (IV.2.8), Theorem IV.4.2 is applicable with $s = 2$, $c = 1$, and $f_j = \frac{1}{b_{j+1}}$. If $b_n \asymp \mathfrak{b}(n)$ where $\mathfrak{b}$ is regularly varying, we instead take $f_j = \frac{1}{\mathfrak{b}(j+1)}$ and c sufficiently small. □

IV.4.6 Remark. If the additional assumption is satisfied in Corollary IV.4.5, the index of regular variation of $\mathfrak{b}^-(r)$ coincides with the convergent exponent of $(b_n)_{n=1}^\infty$.

IV.4.3 A flexible upper bound

When estimating the growth of $w_{H,22}$ from above, one faces the task of tackling the different behaviours near 0 and near L. First, we give a general upper bound for the contribution of subintervals $[x_N, L]$.

IV.4.7 Theorem. *Let H be a Hamburger Hamiltonian on $(0, L)$. Suppose that there exists $\nu > 0$ and sequences $(f_n)_{n=0}^\infty, (g_n)_{n=0}^\infty$ of positive numbers, that are nondecreasing, bounded, and such that*

$$\det \Omega(x_m, x_n)^\nu \leq c^2 \cdot \big(f_n - f_m\big)\big(g_n - g_m\big), \qquad m, n \in \mathbb{N}_0,\ m < n. \quad \text{(IV.4.1)}$$

Let $f := \lim_{n\to\infty} f_n$ and $g := \lim_{n\to\infty} g_n$. Then there exists $r_0 \geq 0$ such that, for any $(N, r) \in \mathbb{N}_0 \times (r_0, \infty)$, the estimate

$$\int_{x_N}^L K_H(t; r)\,\mathrm{d}t \leq \frac{6c}{\nu} r^\nu (f - f_N)^{\frac{1}{2}} (g - g_N)^{\frac{1}{2}} + \mathrm{O}(\log(Lr)). \qquad \text{(IV.4.2)}$$

holds. The constant implicit in $\mathrm{O}(\log(Lr))$ is universal.

Proof. Fix $(N,r) \in \mathbb{N}_0 \times (0,\infty)$. We define points σ_k and a number $d \in \mathbb{N}$ according to the following rules.

- ▷ Set $\sigma_0 := x_N$;
- ▷ If $\det \Omega(\sigma_{k-1}, L) > \frac{1}{r^2}$, let σ_k be the unique point satisfying $\det \Omega(\sigma_{k-1}, \sigma_k) = \frac{1}{r^2}$;
- ▷ If $\det \Omega(\sigma_{k-1}, L) \leq \frac{1}{r^2}$, set $\sigma_k := L$, $d := k$, and terminate.

By Lemma IV.2.1 and using that $e \log x \leq \frac{x^\alpha}{\alpha}$ for $x \geq 1$ and $\alpha > 0$,

$$\int_{\sigma_{k-1}}^{\sigma_k} K_H(t;r)\,\mathrm{d}t \leq e \log\big(r^2 \det \Omega(\sigma_{k-2}, \sigma_k)\big) \tag{IV.4.3}$$

$$\leq \frac{2}{\nu} r^\nu \big(\det \Omega(\sigma_{k-2}, \sigma_k)\big)^{\frac{\nu}{2}}, \quad j = 2, \ldots, d. \tag{IV.4.4}$$

We need to pass from σ_k to the nearest node x_n. Let

$$n(k) := \max\{n \colon x_n \leq \sigma_k\}.$$

For $k = 2, \ldots, d-1$, (IV.4.1) leads to

$$\begin{aligned} \int_{\sigma_{k-1}}^{\sigma_k} K_H(t;r)\,\mathrm{d}t &\leq \frac{2}{\nu} r^\nu \big(\det \Omega(x_{n(k-2)}, x_{n(k)+1})\big)^{\frac{\nu}{2}} \\ &\leq \frac{2c}{\nu} r^\nu \Big[\big(f_{n(k)+1} - f_{n(k-2)}\big)\big(g_{n(k)+1} - g_{n(k-2)}\big)\Big]^{\frac{1}{2}}. \end{aligned} \tag{IV.4.5}$$

We take the sum to obtain

$$\begin{aligned} &\int_{\sigma_1}^{\sigma_{d-1}} K_H(t;r)\,\mathrm{d}t \\ &\leq \frac{2c}{\nu} r^\nu \sum_{k=2}^{d-1} \Big[\big(f_{n(k)+1} - f_{n(k-2)}\big)\big(g_{n(k)+1} - g_{n(k-2)}\big)\Big]^{\frac{1}{2}} \\ &\leq \frac{2c}{\nu} r^\nu \Big[\sum_{k=2}^{d-1} f_{n(k)+1} - f_{n(k-2)}\Big]^{\frac{1}{2}} \Big[\sum_{k=2}^{d-1} g_{n(k)+1} - g_{n(k-2)}\Big]^{\frac{1}{2}}. \end{aligned}$$

Note that $n(k) + 1 \leq n(k+1)$ for $k = 2, \ldots, d-1$. Both of these sums are thus estimated from above by a telescoping sum:

$$\sum_{k=2}^{d-1} f_{n(k)+1} - f_{n(k-2)} \leq \sum_{i=1}^{3} \sum_{\substack{k=2 \\ k \equiv i \bmod 3}}^{d-1} \big(f_{n(k+1)} - f_{n(k-2)}\big) \leq 3\big(f - f_{n(0)}\big)$$

and analogously for (g_n) instead of (f_n). In total, we get

$$\int_{\sigma_1}^{\sigma_{d-1}} K_H(t;r)\,\mathrm{d}t \le \frac{6c}{\nu} r^\nu \big(f - f_N\big)^{\frac{1}{2}} \big(g - g_N\big)^{\frac{1}{2}}.$$

It remains to estimate the contributions of $[\sigma_0, \sigma_1]$ and $[\sigma_{d-1}, L]$. Since $\det \Omega(\sigma_0, \sigma_1) = \frac{1}{r^2}$ and $\det \Omega(\sigma_{d-1}, L) \le \frac{1}{r^2}$, Lemma IV.2.1 and (possibly) [LRW23, Lemma II.6.5] show that both intervals contribute not more than $c' \log(Lr)$ each. □

Theorem IV.4.7 can be used to easily reobtain [BS14, Theorem 1.2] (with a slightly different estimate for the type). For the translation from the language of canonical systems to the language of moment problems see [Kac99].

IV.4.8 Corollary. *If the sequence of lengths $(l_j)_{j=1}^\infty$ of a Hamburger Hamiltonian H belongs to $\ell^\alpha(\mathbb{N})$ for some $\alpha \in (0,1]$, then*

$$\log |w_{H,22}(ir)| \le r^\alpha \cdot M \sum_{j=1}^\infty l_j^\alpha, \qquad r \text{ sufficiently large}, \tag{IV.4.6}$$

where M is a universal constant.

Proof. Since

$$\det \Omega(x_m, x_n)^\alpha \le \Big(\sum_{j=m+1}^n l_j \Big)^{2\alpha} \le \Big(\sum_{j=m+1}^n l_j^\alpha \Big)^2,$$

(IV.4.1) is satisfied with $\nu = \alpha$ and $f_n = g_n = \sum_{j=1}^n l_j^\alpha$. Hence (IV.4.6) follows from Theorem IV.4.7 (with $N = 0$) and (IV.2.2). □

In [PW23, Corollary 5.3] Hamiltonians with continuously rotating angle $\phi(t)$ are considered. It is shown that for Hölder continuous ϕ with exponent $\alpha \in (0,1]$, the monodromy matrix is of order at most $\frac{1}{1+\alpha}$ with finite type. The following result shows that a similar bound holds for sequences of angles that satisfy a discrete Hölder-type condition.

IV.4.9 Corollary. *Let H be a Hamburger Hamiltonian on $(0,L)$. If there is $\alpha > 0$ such that*

$$|\phi_{m+1} - \phi_n| \lesssim |x_m - x_n|^\alpha, \qquad m,n \in \mathbb{N},$$

then

$$\log |w_{H,22}(ir)| \le ML \cdot r^{\frac{1}{1+\alpha}}, \qquad r \text{ sufficiently large},$$

where M is a universal constant.

Proof. Let $m < n$. Then

$$\begin{aligned}\det \Omega(x_m, x_n) &= \frac{1}{2} \sum_{j,k=m+1}^{n} l_j l_k \sin^2(\phi_j - \phi_k) \\ &\lesssim (x_n - x_m)^{2\alpha} \sum_{j,k=m+1}^{n} l_j l_k = (x_n - x_m)^{2\alpha+2}.\end{aligned} \tag{IV.4.7}$$

Hence (IV.4.1) is satisfied with $f_n := g_n := x_n$ and $\nu = \frac{1}{1+\alpha}$. The assertion follows from Theorem IV.4.7 and (IV.2.2). ☐

IV.4.4 Bounds using balancing of contributions

Here we present two results that use the freedom of choice of N in Theorem IV.4.7. The first result, Theorem IV.4.10, is an upper bound that needs no assumption on H whatsoever. The second one, Theorem IV.4.12, only needs the increments of angles of H to be summable.

IV.4.10 Theorem. *Let H be a Hamburger Hamiltonian on $(0, L)$ and choose $\psi \in \mathbb{R}$. Consider the decreasing function*

$$F\colon \begin{cases} \mathbb{N} & \to \quad (0,\infty) \\ N & \mapsto \quad \frac{1}{N}\big(\sum_{j=N+1}^{\infty} l_j\big)^{\frac{1}{2}}\big(\sum_{j=N+1}^{\infty} l_j \sin^2(\phi_j - \psi)\big)^{\frac{1}{2}} \end{cases}$$

Then

$$\log|w_{H,22}(ir)| \lesssim F^{-}\Big(\frac{\log r}{r}\Big) \cdot \log r, \qquad r \text{ sufficiently large},$$

where $F^{-}(x) = \min\{N \in \mathbb{N}\colon\ F(N) \leq x\}$.

In order to estimate the contribution of $[x_N, L]$, we could use [PRW23, Lemma III.2.10]. Here we present a similar result, which we prove directly using Theorem IV.4.7.

IV.4.11 Lemma. *Let H be a Hamburger Hamiltonian on $(0, L)$ and choose $\psi \in \mathbb{R}$. Then, for $(N, r) \in \mathbb{N}_0 \times (r_0, \infty)$,*

$$\int_{x_N}^{L} K_H(t; r)\,\mathrm{d}t \leq 6r\Big(\sum_{j=N+1}^{\infty} l_j\Big)^{\frac{1}{2}}\Big(\sum_{j=N+1}^{\infty} l_j \sin^2(\phi_j - \psi)\Big)^{\frac{1}{2}} + \mathrm{O}(\log(Lr)).$$

The constant implicit in $\mathrm{O}(\log(Lr))$ *is universal.*

Proof. We estimate $\det \Omega$ by

$$\begin{aligned}
\det \Omega(x_m, x_n) &= \det\Big(\sum_{j=m+1}^{n} \xi_{\phi_j}\xi_{\phi_j}^{\top}\Big) \\
&= \det\Bigg[\begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix} \sum_{j=m+1}^{n} \xi_{\phi_j}\xi_{\phi_j}^{\top} \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix}\Bigg] \\
&= \det\Big(\sum_{j=m+1}^{n} \xi_{\phi_j-\psi}\,\xi_{\phi_j-\psi}^{\top}\Big) \leq \Big(\sum_{j=m+1}^{n} l_j\Big)\Big(\sum_{j=m+1}^{n} l_j \sin^2(\phi_j-\psi)\Big)
\end{aligned}$$

This shows that (IV.4.1) is satisfied with $\nu = 1$, $f_n := x_n$ and

$$g_n := \sum_{j=1}^{n} l_j \sin^2(\phi_j - \psi).$$

The assertion thus follows from Theorem IV.4.7. □

Proof of Theorem IV.4.10. We fix N and apply [LRW23, Lemma II.6.5] using $s_j := x_{j+1}$, $j = 1, \ldots, N-1$. For sufficiently large r, this yields

$$\int_{x_2}^{x_N} K_H(t;r)\,\mathrm{d}t \lesssim N\log r + \mathrm{O}(1).$$

The contribution of $[0, x_2]$ is $\lesssim \log r + \mathrm{O}(1)$ by [LRW23, Proposition II.4.4] and Lemma IV.2.1. Applying Lemma IV.4.11 for $N = F^{-}\big(\frac{\log r}{r}\big)$ leads to

$$\int_{x_N}^{L} K_H(t;r)\,\mathrm{d}t \leq 6r \cdot NF(N) + \mathrm{O}(\log(Lr)) \leq 6N\log r + \mathrm{O}(\log(Lr)).$$

Combining these estimates, the theorem follows from (IV.2.2). □

If the increments of consecutive angles of H are summable, another bound is available. For regularly varying data, this bound is worse (by at most a logarithmic term) than [PRW23, Corollary III.4.7], but gives the same estimate for the order of W_H. In return, we get a much more general upper bound that only needs summability of angle increments.

IV.4.12 Theorem. *Let H be a Hamburger Hamiltonian on $(0, L)$ and suppose that $\sum_{j=1}^{\infty} |\sin(\phi_{j+1} - \phi_j)| < \infty$. Let G be the decreasing function*

$$G\colon \begin{cases} \mathbb{N} & \to & (0,\infty) \\ N & \mapsto & \frac{1}{N}\big(\sum_{j=N+1}^{\infty} l_j\big)^{\frac{1}{2}}\big(\sum_{j=N+1}^{\infty} |\sin(\phi_{j+1}-\phi_j)|\big)^{\frac{1}{2}} \end{cases}$$

Then

$$\log|w_{H,22}(ir)| \lesssim G^{-}\Big(\frac{\log r}{r^{\frac{1}{2}}}\Big)\cdot \log r, \qquad r \textit{ sufficiently large.}$$

IV.4.13 Lemma. *Let H be a Hamburger Hamiltonian on $(0, L)$ and suppose that $\sum_{j=1}^{\infty} |\sin(\phi_{j+1} - \phi_j)| < \infty$. Then, for $(N, r) \in \mathbb{N}_0 \times (r_0, \infty)$,*

$$\int_{x_N}^{L} K_H(t; r)\,\mathrm{d}t \leq 12 r^{\frac{1}{2}} \Big(\sum_{j=N+1}^{\infty} l_j \Big)^{\frac{1}{2}} \Big(\sum_{j=N+1}^{\infty} |\sin(\phi_j - \phi_{j-1})| \Big)^{\frac{1}{2}} + \mathrm{O}(\log(Lr)). \tag{IV.4.8}$$

The constant implicit in $\mathrm{O}(\log(Lr))$ *is universal.*

Proof. We repeatedly use the inequality $|\sin(a+b)| \leq |\sin a| + |\sin b|$ for $a, b \in \mathbb{R}$, which is easy to check. It holds that

$$\begin{aligned} \det \Omega(x_m, x_n) &= \frac{1}{2} \sum_{j,k=m+1}^{n} l_j l_k \sin^2(\phi_j - \phi_k) = \sum_{\substack{j,k=m+1 \\ j<k}}^{n} l_j l_k \sin^2(\phi_j - \phi_k) \\ &\leq \sum_{\substack{j,k=m+1 \\ j<k}}^{n} l_j l_k \Big(\sum_{s=j+1}^{k-1} |\sin(\phi_{s+1} - \phi_s)| \Big)^2 \\ &\leq \Big(\sum_{s=m+1}^{n} |\sin(\phi_{s+1} - \phi_s)| \Big)^2 \Big(\sum_{j=m+1}^{n} l_j \Big)^2. \end{aligned}$$

In other words, (IV.4.1) is satisfied with $\nu = \frac{1}{2}$, $f_n := x_n$, and $g_n := \sum_{s=1}^{n} |\sin(\phi_{s+1} - \phi_s)|$. Thus (IV.4.8) follows from Theorem IV.4.7. □

Proof of Theorem IV.4.12. As in the proof of Theorem IV.4.10, we have

$$\int_0^{x_N} K_H(t; r)\,\mathrm{d}t \lesssim N \log r + \mathrm{O}(1).$$

From Lemma IV.4.11 with $N = G^{-}\big(\frac{\log r}{r^{\frac{1}{2}}}\big)$, we obtain

$$\int_{x_N}^{L} K_H(t; r)\,\mathrm{d}t \leq 12 r^{\frac{1}{2}} \cdot N G(N) + \mathrm{O}(\log(Lr)) \leq 12 N \log r + \mathrm{O}(\log(Lr))$$

which proves the theorem. □

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