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„The Neimark-Sacker Bifurcation“

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Abstract

This master thesis is about the Neimark-Sacker bifurcation, also known as Discrete Hopf bifurcation. At such a bifurcation, an invariant closed curve bifurcates from a fixed point, which changes stability in the process. In this thesis it is described when and how such a bifurcation takes place, as well as what behaviour can be expected on this invariant curve. The theory will be applied in a few examples, concluding this thesis.

Zusammenfassung

Diese Masterarbeit handelt von der Neimark-Sacker Bifurkation, auch diskrete Hopf-Bifurkation genannt. Bei dieser Art der Bifurkation ändert ein Fixpunkt seine Stabilität, während eine invariante Kurve um den Fixpunkt entsteht. In dieser Arbeit wird beschrieben, wann und wie eine solche Bifurkation abläuft und welches Verhalten auf der invarianten Kurve erwartet werden kann. Abschließend wird die Theorie in einigen Beispielen angewendet.

Contents

Introduction

This master's thesis is about the Neimark-Sacker bifurcation, sometimes also referred to as the discrete Hopf bifurcation.
As the second name suggests, the Neimark-Sacker bifurcation occurs in models set in discrete time and describes the changing of a system, in particular the stability of its fixed points, dependent on a parameter μ.

We consider a one-parameter family of maps F_μ of $\mathbb{R}^2$. A Neimark-Sacker bifurcation occurs for a system from this family as a pair of complex valued eigenvalues, $\lambda(\mu)$ and $\overline{\lambda(\mu)}$, of the Jacobian of F at its fixed point x crosses the unit circle as the parameter μ varies in value. The fixed point changes its stability and an invariant closed curve bifurcates from the fixed point. There are two types of Neimark-Sacker bifurcations, just like in the continuous case of the Hopf bifurcation.
Let us assume that x is stable for $\mu < 0$ and unstable for $\mu > 0$ and the Jacobian of the map F at the fixed point x at $\mu = 0$ has a complex conjugate pair of eigenvalues on the unit circle. If the invariant curve exists for $\mu > 0$, one speaks of a supercritical Neimark-Sacker bifurcation. If the curve exists for $\mu < 0$, then the bifurcation is called subcritical.

This thesis begins with the description of the normal form for the Neimark-Sacker bifurcation. A map in this form is not hard to analyse and the bifurcation is easy to see. Then the focus will be on the generic Neimark-Sacker bifurcation and on how one can achieve a form similar to the normal form, which is needed for the stating and the proof of the bifurcation theorem that will follow.
The dynamics of the map on the invariant closed curve will also be discussed. It is of interest to know whether there are dense orbits or periodic points on the curve.
To make the theory more clear and to see where such a bifurcation can take place, this thesis concludes with examples of biological models.

The Theory from this thesis was taken from the books of Kuznetsov [5], Iooss [2] as well as Whitley [7]. The examples were taken from [5], Lauwerier and Metz [6] and Khan [3].

1 Theory

The Neimark-Sacker bifurcation occurs for maps of $\mathbb{R}^2$. For a map to be in this family of maps it needs two complex-conjugate eigenvalues, depending on a parameter, that cross the unit circle. In doing so, a fixed point changes its stability and an invariant closed curve bifurcates, which is of opposite stability as the fixed point. To begin the analysis of this bifurcation we will consider the normal form first. For this form it is easy to see that an invariant closed curve exists. This makes it the form one aspires to transform all other maps, that also undergo a Neimark-Sacker bifurcation, into. After successfully transforming any map of this family into a form like this but with small perturbations, we will commence with the stating and, to some extent, proving of the theorem of existence and uniqueness of the bifurcating invariant closed curve. The dynamics on the invariant curve will also be discussed.

The contents of this chapter regarding the normal form, the reduction to it and the bifurcation theorem can be found in the books of Kuznetsov [5] and Iooss [2]. The discussion on the dynamics on the invariant circle can be found in the writings of Arrowsmith and Place [1] and Whitley [7].

1.1 Normal form of the Neimark-Sacker bifurcation

The following can be found in [5] on pages 125-128.

We consider the following two dimensional discrete-time system, the normal form of the Neimark-Sacker bifurcation

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto (1+\gamma)\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (x_1^2+x_2^2)\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{1.1.1}$$

where γ is the bifurcation parameter of which θ is a function of as well as a and b which are both smooth. This system has a fixed point at $x_1 = x_2 = 0$ for all γ with the Jacobian matrix

$$A(\gamma) = (1+\gamma)\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

which has the eigenvalues $\lambda_{1,2} = (1+\gamma)e^{\pm i\theta}$.

For small γ these eigenvalues are not equal to zero, therefore making the map invertible. At $\gamma = 0$ the fixed point is non-hyperbolic, since both eigenvalues are on the unit circle.

To start the analysis of the bifurcation we introduce a complex valued variable $z = x_1 + ix_2$ and set $d = a + ib$.

The above map for z reads

$$z \mapsto e^{i\theta} z(1 + \gamma + d|z|^2) = \lambda z + cz|z|^2,$$

where $\lambda = \lambda(\gamma) = (1+\gamma)e^{i\theta(\gamma)}$ and $c = c(\gamma) = e^{i\theta(\gamma)} d(\gamma)$, both complex functions of γ. Switching to the polar coordinates of z, $z = re^{i\phi}$, $r = |z|$, gives us the following map for r:

$$r \mapsto r|1 + \gamma + d(\gamma)r^2|.$$

We have

$$\begin{aligned} |1 + \gamma + d(\gamma)r^2| &= \sqrt{(1 + \gamma + d(\gamma)r^2)\overline{(1 + \gamma + d(\gamma)r^2)}} \\ &= (1+\gamma)\left(1 + \frac{2a(\gamma)}{1+\gamma}r^2 + \frac{|d(\gamma)|^2}{(1+\gamma)^2}r^4\right)^{\frac{1}{2}} \\ &= 1 + \gamma + a(\gamma)r^2 + O(r^3) \end{aligned}$$

With this we obtain our map in polar coordinates:

$$\begin{cases} r \mapsto r(1 + \gamma + a(\gamma)r^2) + r^4 R_\gamma(r) \\ \phi \mapsto \phi + \theta(\gamma) r^2 Q_\gamma(r) \end{cases} \tag{1.1.2}$$

where R and Q are smooth functions of (r, γ).
Since the mapping of r is independent of ϕ, it is easier to analyse.

The map for r defines a one-dimensional dynamical system with the fixed points $r = 0$ and $r_0(\gamma) = \sqrt{-\frac{\gamma}{a(\gamma)}} + O(\gamma)$. The fixed pint zero exists for all values of γ, it is stable if $\gamma < 0$ and unstable if $\gamma > 0$. The existence of the fixed point r_0 depends on γ and the sign of $a(\gamma)$, which has to be unequal to zero. If the sign of $a(\gamma)$ has the opposite sign of γ then the fixed point exists and since the map for ϕ is a rotation by an angle depending on r and γ this fixed point r_0 is the invariant circle bifurcating from the fixed point 0. If $a(0) < 0$ then $r_0(\gamma)$ exists for $\gamma > 0$ and the circle will bifurcate as γ passes through zero from negative to positive values, the supercritical Neimark-Sacker bifurcation occurs, see Figure 1.1. If $a(0) > 0$ the second fixed point exists for negative values of γ and the invariant circle bifurcates as γ passes through zero from positive to negative values, the subcritical Neimark-Sacker bifurcation occurs, see Figure 1.2.

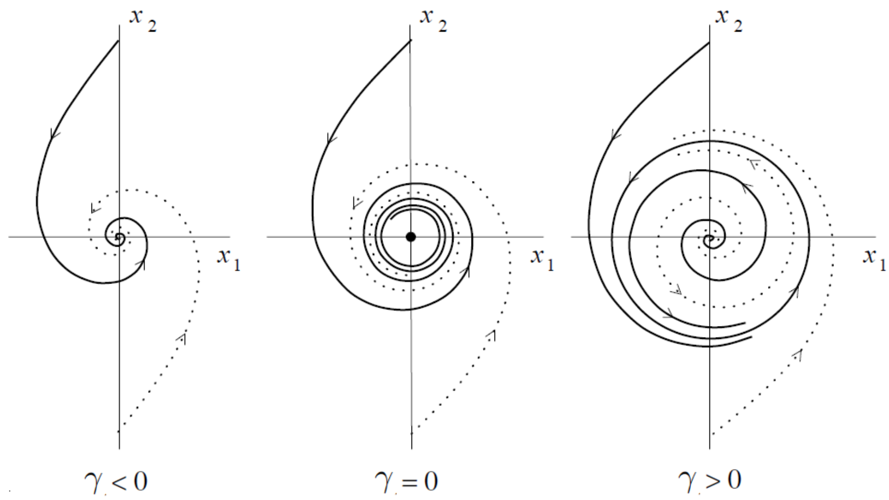

Figure 1.1: Supercritical Neimark-Sacker bifurcation

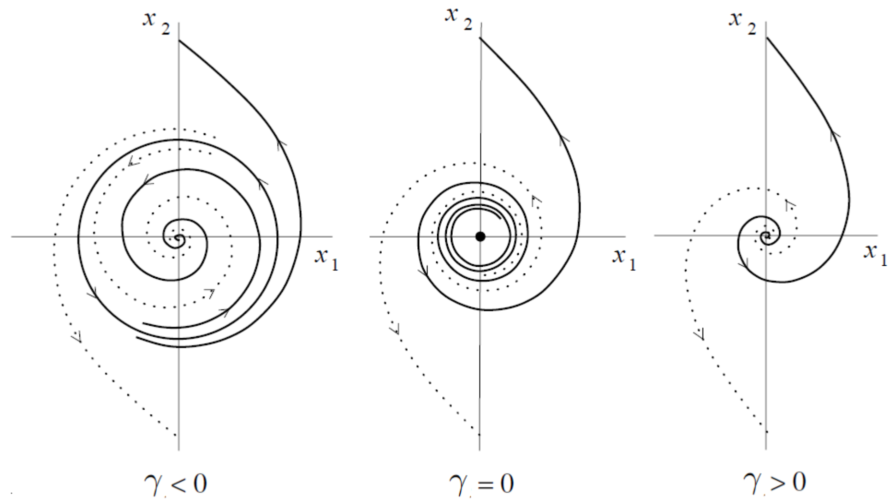

Figure 1.2: Subcritical Neimark-Sacker bifurcation

1.2 Generic Neimark-Sacker bifurcation

The contents of this section can be found in [5], pages 129-135.

We want to show that any two-dimensional system undergoing a Neimark-Sacker bifurcation can be transformed into the form

$$\begin{aligned}\begin{pmatrix}x_1\\x_2\end{pmatrix} &\mapsto (1+\gamma)\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}\\ &\quad+(x_1^2+x_2^2)\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}a & -b\\ b & a\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}+O(||x||^4),\end{aligned} \tag{1.2.1}$$

where γ is the bifurcation parameter of which θ is a function as well as a and b which are both smooth like before. This system has a fixed point at $x_1 = x_2 = 0$ for all γ with the Jacobian matrix

$$A(\gamma) = (1+\gamma)\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}.$$

We rewrite this system, by introduction of a complex valued variable $z = x_1 + ix_2$ and functions α and β, as follows

$$z \mapsto F_\mu(z) = \lambda(\mu)z + \alpha(\mu)z^2\overline{z} + \beta(\mu)\overline{z}^4 + O(|z|^5). \tag{1.2.2}$$

This expression will be used in the proof of the existence of an invariant curve.

1.2.1 Reduction to normal form

We shall commence with the calculations and transformations that will show that any map that undergoes a Neimark-Sacker bifurcation can be written in the form of (1.2.2).

Consider a system $x \mapsto f(x,\gamma)$ with $x \in \mathbb{R}^2$, $\gamma \in \mathbb{R}$ and f smooth, which has the fixed point $x = 0$ at $\gamma = 0$ with simple eigenvalues $\lambda_{1,2} = e^{\pm i\theta_0}$, $0 < \theta_0 < \pi$.

The implicit function theorem tells us, that the system also has a fixed point $x(\gamma)$ for $|\gamma|$ small enough, we may shift this fixed point to the origin.

Now the system can be written as

$$x \mapsto A(\gamma)x + F(x,\gamma), \tag{1.2.3}$$

where F is a smooth vector function whose components $F_{1,2}$ have Taylor expansions in x starting with at least quadratic terms and $F(0,\gamma) = 0$ for all sufficiently small $|\gamma|$. The Jacobian matrix $A(\gamma)$ has the eigenvalues

$$\lambda_{1,2}(\gamma) = r(\gamma)e^{\pm i\phi(\gamma)}, \tag{1.2.4}$$

where $r(0) = 1$, $\phi(0) = \theta_0$. So $r(\gamma) = 1 + \mu(\gamma)$ for some smooth function $\mu(\gamma)$ with $\mu(0) = 0$.
Suppose that $\mu'(0) \neq 0$, then we use μ as our new parameter and rewrite the eigenvalues of A as follows:

$$\lambda_{1,2}(\mu) = (1 + \mu)e^{\pm i\theta(\mu)},$$

with a smooth function $\theta(\mu)$ such that $\theta(0) = \theta_0$.

Lemma 1.2.1. *By the introduction of a complex variable and a new parameter, the system*

$$x \mapsto A(\gamma)x + F(x, \gamma)$$

can be transformed for all sufficiently small $|\gamma|$ into the form

$$z \mapsto \lambda(\mu)z + g(z, \overline{z}, \mu),$$

where $\mu \in \mathbb{R}$, $z \in \mathbb{C}$, $\lambda(\mu) = (1 + \mu)e^{i\theta(\mu)}$ and g is a complex-valued smooth function, whose Taylor expansion with respect to $(z, \overline{z})$ contains quadratic and higher order terms:

$$g(z, \overline{z}, \mu) = \sum_{k+\ell \geq 2} g_{k\ell}(\mu) z^k \overline{z}^\ell,$$

where

$$g_{k\ell}(\mu) = \frac{1}{k!\ell!} \frac{\partial^{k+\ell}}{\partial z^k \partial \overline{z}^\ell} \langle p(\mu), F(zq(\mu) + \overline{zq}(\mu), \mu)\rangle \Big|_{z=0}$$

for $k + \ell \geq 2, k, \ell = 0, 1, \ldots$, where p and q are eigenvectors of A.

It is possible to significantly simplify this expression, which will be done in the next lemmas, all following the same idea.

Lemma 1.2.2. *Elimination of quadratic terms*
The map

$$z \mapsto z' = \lambda z + g_{20}z^2 + g_{11}z\overline{z} + g_{02}\overline{z}^2 + O(|z|^3)$$

with $\lambda = \lambda(\mu) = (1 + \mu)e^{i\theta(\mu)}$
can be transformed into a map without quadratic terms

$$w \mapsto \lambda w + O(|w|^3)$$

for all $|\mu|$ sufficiently small, provided that $e^{i\theta_0} \neq 1$ and $e^{3i\theta_0} \neq 1$.

For this the invertible parameter-dependent change of complex coordinates

$$w = z + h_{20}z^2 + h_{11}z\overline{z} + h_{02}\overline{z}^2$$

is used.

Proof. We can use our mapping

$$z' = \lambda z + g_{20}z^2 + g_{11}z\overline{z} + g_{02}\overline{z}^2 + O(|z|^3)$$

for all z and $\overline{z}$ in the expression for w and get a result

$$\begin{aligned} w \mapsto w' = \lambda z &+ g_{20}z^2 + g_{11}z\overline{z} + g_{02}\overline{z}^2 + O(|z|^3) \\ &+ h_{20}(\lambda^2 z^2 + O(|z|^3)) \\ &+ h_{11}(\lambda z\overline{\lambda z} + O(|z|^3)) \\ &+ h_{02}(\overline{\lambda z}^2 + O(|z|^3)). \end{aligned}$$

With the inverse change of variables

$$z = w - h_{20}w^2 - h_{11}w\overline{w} - h_{02}\overline{w}^2 + O(|w|^3) \tag{1.2.5}$$

we get:

$$\begin{aligned} w' = \lambda w &+ (g_{20} - (\lambda - \lambda^2)h_{20})w^2 \\ &+ (g_{11} - (\lambda - |\lambda|^2)h_{11})w\overline{w} \\ &+ (g_{02} - (\lambda - \overline{\lambda}^2)h_{02})\overline{w}^2 \\ &+ O(|w|^3). \end{aligned}$$

Now we choose the following substitutions:

$$h_{ij} = \frac{g_{ij}}{\lambda - \lambda^i\overline{\lambda}^j}$$

with $i, j \geq 0$ and $i + j = 2$.

They are valid since for $0 < \theta_0 < \pi$:

$$\begin{aligned} \lambda(0) - \lambda^2(0) &= e^{i\theta_0}(1 - e^{i\theta_0}) \neq 0, \\ \lambda(0) - |\lambda(0)|^2 &= e^{i\theta_0} - 1 \neq 0, \\ \lambda(0) - \overline{\lambda}^2(0) &= e^{i\theta_0}(1 - e^{-3i\theta_0}) \neq 0 \end{aligned}$$

This brings us to the desired expression. □

Lemma 1.2.3. *Elimination of all but one cubic terms*
The map

$$z \mapsto \lambda z + g_{30}z^3 + \frac{g_{21}}{2}z^2\overline{z} + g_{12}z\overline{z}^2 + g_{03}\overline{z}^3 + O(|z|^4)$$

with $\lambda = \lambda(\mu) = (1 + \mu)e^{i\theta(\mu)}$
can be transformed into a map with only one cubic term

$$w \mapsto \lambda w + \alpha(\mu)w^2\overline{w} + O(|w|^4)$$

for all $|\mu|$ sufficiently small, provided that $e^{2i\theta_0} \neq 1$ and $e^{4i\theta_0} \neq 1$.
For this the invertible parameter-dependent change of complex coordinates

$$w = z + h_{30}z^3 + h_{21}z^2\overline{z} + h_{12}z\overline{z}^2 + h_{03}\overline{z}^3$$

is used.

Proof. Proceeding as in the last proof with using our mapping for all z and $\overline{z}$ in the expression for w and using the inverse change of variables

$$z = w - h_{30}w^3 - h_{21}w^2\overline{w} - h_{12}w\overline{w}^2 - h_{03}\overline{w}^3 + O(|w|^4).$$

Therefore we get:

$$\begin{aligned} w \mapsto \lambda w &+ (g_{30} - (\lambda - \lambda^3)h_{30})w^3 \\ &+ (g_{21} - (\lambda - \lambda|\lambda|^2)h_{21})w^2\overline{w} \\ &+ (g_{12} - (\lambda - \overline{\lambda}|\lambda|^2)h_{12})w\overline{w}^2 \\ &+ (g_{03} - (\lambda - \overline{\lambda}^3)h_{03})\overline{w}^3 \\ &+ O(|w|^4). \end{aligned}$$

The substitutions are in the same fashion as above

$$h_{ij} = \frac{g_{ij}}{\lambda - \lambda^i\overline{\lambda}^j}$$

with $i, j \geq 0$ and $i + j = 3$ which are all valid for $0 < \theta_0 < \pi$ except for h_{12}:

$$\begin{aligned} \lambda^3(0) - \lambda(0) &= e^{i\theta_0}(e^{2i\theta_0} - 1) \neq 0, \\ \overline{\lambda}(0)|\lambda(0)|^2 - \lambda(0) &= e^{-i\theta_0}(1 - e^{2i\theta_0}) \neq 0, \\ \overline{\lambda}^3(0) - \lambda(0) &= e^{-3i\theta_0}(1 - e^{4i\theta_0}) \neq 0. \end{aligned}$$

The $w^2\overline{w}$-term remains since the substitution $h_{21} = \frac{g_{21}}{\lambda(1-|\lambda|^2)}$ is not valid, considering the denominator vanishes at $\mu = 0$ so we set $h_{12} = 0$.
This brings us to the desired expression. □

The quartic terms can also be simplified.

Lemma 1.2.4. *Elimination of quartic terms*
The map

$$z \mapsto \lambda z + g_{40}z^4 + g_{31}z^3\overline{z} + g_{22}z^2\overline{z}^2 + g_{13}z\overline{z}^3 + g_{04}\overline{z}^4 + O(|z|^5)$$

with $\lambda = \lambda(\mu) = (1 + \mu)e^{i\theta(\mu)}$
can be transformed into the map

$$w \mapsto \lambda w + \beta(\mu)\overline{w}^4 + O(|w|^5)$$

for all $|\mu|$ sufficiently small, provided that $e^{ki\theta_0} \neq 1$ for $k = 1, 2, 3, 4$
or even

$$w \mapsto \lambda w + O(|w|^5)$$

provided that $e^{5i\theta_0} \neq 1$ also.
For this the invertible parameter-dependent change of complex coordinates

$$w = z + h_{40}z^4 + h_{31}z^3\overline{z} + h_{22}z^2\overline{z}^2 + h_{13}z\overline{z}^3 + h_{04}\overline{z}^4$$

is used.

Proof. Proceeding exactly as before with the following substitutions

$$h_{ij} = \frac{g_{40}}{\lambda - \lambda^i\overline{\lambda}^j}$$

with $i, j \geq 0$ and $i + j = 4$ except for h_{04}, for which we have the case distinction

$$h_{04} = \begin{cases} \frac{g_{04}}{\lambda - \overline{\lambda}^4} & \text{if} \quad e^{5i\theta_0} \neq 1 \\ 0 & \text{else} \end{cases}$$

This gives us the desired expressions.

□

Combining the previous lemmas gives us the following compact simplification of any map that undergoes a Neimark-Sacker bifurcation.

Lemma 1.2.5. *By the introduction of a complex variable and a new parameter, the system*

$$x \mapsto A(\gamma)x + F(x, \gamma)$$

can be transformed into a map

$$w \mapsto \lambda w + \alpha(\mu)w^2\overline{w} + \beta(\mu)\overline{w}^4 + O(|w|^5)$$

with $\beta(\mu) = 0$ if $e^{5i\theta_0} \neq 1$
for all $|\mu|$ sufficiently small, with $\lambda = \lambda(\mu) = (1 + \mu)e^{i\theta(\mu)}$ and $\theta_0 = \theta(0)$ such that $e^{ik\theta_0} \neq 1$ for $k = 1, 2, 3, 4$, using an invertible parameter-dependent change of complex coordinates.

1.2.2 Calculating $\alpha(\mu)$

We will see later, that the constant $\mathfrak{a} = -\Re(\alpha(0)\overline{\lambda_0})$ determines what type of Neimark-Sacker bifurcation takes place. Therefore we shall compute $\alpha(0)$.

Since the coordinate change in the first Lemma (1.2.2) regarding the quadratic terms changed all terms of higher order as well, it is necessary to express the inverse change of coordinates more precisely, by computing it up to terms of order 4.

We have

$$w = z + h_{20}z^2 + h_{11}z\overline{z} + h_{02}\overline{z}^2$$
$$z = w + b_{20}w^2 + b_{11}w\overline{w} + b_{02}\overline{w}^2 + b_{30}w^3 + b_{21}w^2\overline{w} + b_{12}w\overline{w}^2 + b_{03}\overline{w}^3 + O(|w|^4).$$

One can see immediately that

$$b_{20} = -h_{20}, \qquad b_{11} = -h_{11}, \qquad b_{02} = -h_{02}$$

with this we get

$$\begin{aligned} w = w &+ w^3(b_{30} - 2h_{20}^2 - h_{11}\overline{h_{02}}) \\ &+ w^2\overline{w}(b_{21} - 2h_{11}h_{20} - h_{20}h_{11} - \overline{h_{11}}h_{11} - 2h_{02}\overline{h_{02}}) \\ &+ w\overline{w}^2(b_{12} - 2h_{02}h_{20} - h_{11}^2 - \overline{h_{20}}h_{11} - 2\overline{h_{11}}h_{02}) \\ &+ \overline{w}^3(b_{03} - h_{02}h_{11} - 2\overline{h_{20}}h_{02}). \end{aligned}$$

This gives us

$$z = w - h_{20}w^2 - h_{11}w\overline{w} - h_{02}\overline{w}^2 + w^2\overline{w}(2h_{11}h_{20} + h_{20}h_{11} + \overline{h_{11}}h_{11} + 2h_{02}\overline{h_{02}}) + H$$

where H contains the other terms of order 3 and all terms of higher order.
Using this expression for z in the coordinate change gives us

$$\alpha(\mu) = |g_{11}|^2 \frac{1}{1-\overline{\lambda}} + g_{11}g_{20}\frac{2\lambda - 3|\lambda| + \overline{\lambda}}{(\lambda-\lambda^2)(1-\overline{\lambda})} + |g_{02}|^2 \frac{2(\overline{\lambda}^2 - \lambda^2\overline{\lambda})}{(\lambda - \overline{\lambda}^2)(\overline{\lambda} - \lambda^2)} + g_{21} \tag{1.2.6}$$

and at $\mu = 0$ this simplifies to

$$\alpha(0) = |g_{11}(0)|^2 \frac{1}{1-\overline{\lambda}} + g_{11}(0)g_{20}(0)\frac{2\lambda - 1}{\lambda - \lambda^2} + |g_{02}(0)|^2 \frac{2}{\lambda^2 - \overline{\lambda}} + g_{21}. \tag{1.2.7}$$

1.3 Bifurcation theorem and sketch of proof

The transformation of the system and the theorem are taken from Iooss [2], pages 27-44.

Theorem 1.3.1. *Let us assume F_μ to be of class C^k near 0, $k \geq 6$, and let the following assumptions be satisfied:*

- *the Jacobian of F_μ at* 0*,* $A(0)$*, has two non-real eigenvalues* λ_0*,* $\overline{\lambda_0}$ *with* $|\lambda_0| = 1$
- $\frac{d|\lambda(\mu)|}{d\mu}|_{\mu=0} > 0$
- $\lambda = \lambda(0)$ *is not an m-th root of unity for* $m = 1, 2, 3, 4$

Then F_μ van be written as

$$F_\mu(z) = \lambda(\mu)z + \alpha(\mu)z^2\overline{z} + \beta(\mu)\overline{z}^4 + O(|z|^5). \tag{1.3.1}$$

Let $\mathfrak{a} = -\Re(\alpha(0)\overline{\lambda_0})$.
If $\mathfrak{a} > 0$ (0 *is attractive for* $\mu = 0$), *there exists a right neighborhood of* $\mu = 0$, *in which there is an invariant* *attractive* *circle, bifurcating from* 0. *In this case the fixed point* 0 *changes stability from stable to unstable when the bifurcation takes place.*
If $\mathfrak{a} < 0$, *there exists a left neighborhood of* $\mu = 0$, *in which there is an invariant* *repelling* *circle, bifurcation from* 0. *For* $\mu > 0$ *the fixed point* 0 *is unstable and for* $\mu < 0$ *it is stable.*

The following is a sketch of the proof, which can be found in Iooss pages 34 ff.

Proof. We start the proof by transforming our system into polar coordinates

$z = re^{i\pi\phi}$, $F_\mu(z) = Re^{i\pi\Phi}$, where

$$\begin{cases} R = |\lambda(\mu)|r[1 + \Re\left(\frac{\alpha(\mu)}{\lambda(\mu)}\right)r^2 + \Re\left(\frac{\beta(\mu)}{\lambda(\mu)}e^{-10i\pi\phi}\right)r^3 + O(r^4)] \\ \Phi = \phi + \theta(\mu) + \frac{1}{2\pi}\Im\left(\frac{\alpha(\mu)}{\lambda(\mu)}\right)r^2 + \frac{1}{2\pi}\Im\left(\frac{\beta(\mu)}{\lambda(\mu)}e^{-10i\pi\phi}\right)r^3 + O(r^4) \ \ (mod\ 1) \end{cases}$$

using the expression for λ and θ below.
Writing λ as $\lambda(\mu) = \lambda_0(1 + \mu\lambda_1 + o(|\mu|))$, $\Re(\lambda_1) > 0$ if F_μ is C^2 near 0 we get

$$\begin{cases} |\lambda(\mu)| = 1 + \mu\Re(\lambda_1) + o(|\mu|) \\ \theta(\mu) = \frac{1}{2\pi}\arg\lambda(\mu) = \theta_0 + \mu\theta_1 + o(|\mu|) \\ \theta_0 = \frac{1}{2\pi}\arg\lambda_0, \qquad \theta_1 = \frac{1}{2\pi}\Im(\lambda_1). \end{cases}$$

We can rewrite the the coordinates as follows

$$\begin{cases} R = (1 + \mu\Re(\lambda_1))r - \mathfrak{a}r^3 + \Re(\beta(0)\overline{\lambda_0}e^{-10i\pi\phi})r^4 + O(|\mu|^2 r + |\mu|r^3 + r^5) \\ \Phi = \phi + \theta_0 + \mu\theta_1 + \omega_1 r^2 + O(|\mu|^2 + |\mu|^2 r^2 + r^3) \ \ (mod\ 1) \end{cases}$$

with

$$\begin{cases} \mathfrak{a} = -\Re(\alpha(0)\overline{\lambda_0}) \\ \omega_1 = \frac{1}{2\pi}\Im(\alpha(0)\overline{\lambda_0}). \end{cases}$$

We used

$$\begin{cases} |\lambda(\mu)|\Re\left(\frac{1}{\lambda(\mu)}\right) = |\lambda(\mu)|\Re\left(\frac{\overline{\lambda(\mu)}}{|\lambda(\mu)|^2}\right) = |\lambda(\mu)|\frac{\Re(\overline{\lambda_0})(1+\mu\Re(\overline{\lambda_1})+o(|\mu|))}{|\lambda(\mu)|^2} = \Re(\overline{\lambda_0}) \\ \Im\left(\frac{1}{\lambda(\mu)}\right) = \Im\left(\frac{\overline{\lambda(\mu)}}{|\lambda(\mu)|^2}\right) = \frac{\Im(\overline{\lambda_0}(1+\mu\Im(\overline{\lambda_1})+o(|\mu|)))}{|\lambda(\mu)|^2} \xrightarrow[\mu=0]{} \Im(\overline{\lambda_0}). \end{cases}$$

If we assume that $\mathfrak{a} \neq 0$, then we have the following map for $\mu = 0$:

$$F_0 : \begin{cases} R = r(1 - \mathfrak{a}r^2) + O(r^4) \\ \Phi = \phi + \theta_0 + O(r^2). \end{cases}$$

Let us assume that $\mathfrak{a} > 0$.
We make another change of coordinates to obtain the principal part of the invariant curve:

$$r = r_0(\mu)(1 + \sqrt{\mu}x) \quad \text{with} \quad r_0(\mu) = \sqrt{\frac{\mu\Re(\lambda_1)}{\mathfrak{a}}} \tag{1.3.2}$$

(any invariant bifurcated closed curve lies in the annulus $|x| \leq 1$).
Then $R = r_0(\mu)(1 + \sqrt{\mu}X)$ where

$$\begin{cases} X = (1 - 2\mu\Re(\lambda_1))x + \mu(\frac{\Re(\lambda_1)}{\mathfrak{a}})^{3/2}\Re(\beta(0)\overline{\lambda_0}e^{-10i\pi\phi}) + \mu^{3/2}X_1(x, \phi, \mu) \\ \Phi = \phi + \theta_0 + \mu\theta_2 + \mu^{3/2}\Phi_1(x, \phi, \mu)(\mod 1), \;\; \theta_2 = \theta_1 + \omega_1\frac{\Re(\lambda_1)}{\mathfrak{a}}. \end{cases} \tag{1.3.3}$$

X_1 and Φ_1 are C^{k-5} with respect to $(x, \phi, \sqrt{\mu})$ in the domain $[-1, 1] \times T^1 \times [0, \delta)$, where $T^1 = \mathbb{R}/\mathbb{Z}$, which means both X_1 and Φ_1 are 1-periodic in ϕ.
If $\mathfrak{a} < 0$ we make a similar change of coordinates

$$r = r_0(\mu)(1 + \sqrt{-\mu}x) \quad \text{with } r_0(\mu) \text{ as above.} \tag{1.3.4}$$

Then $R = r_0(\mu)(1 + \sqrt{-\mu}X)$ and

$$\begin{cases} X = (1 - 2\mu\Re(\lambda_1))x - \mu(\frac{\Re(\lambda_1)}{-\mathfrak{a}})^{3/2}\Re(\beta(0)\overline{\lambda_0}e^{-10i\pi\phi}) + (-\mu)^{3/2}X_1(x, \phi, \mu) \\ \Phi = \phi + \theta_0 + \mu\theta_2 + (-\mu)^{3/2}\Phi_1(x, \phi, \mu)(\mod 1) \end{cases} \tag{1.3.5}$$

where $\mu < 0$.

Now our system is in the right form to start the proof.
The proof consists of five parts:

- the first and second step show the existence of the invariant curve under the assumption that $\lambda_0^5 \neq 1$ and $\mathfrak{a} > 0$
- in the third step stability is discussed
- the fourth step contains the scenario in which $\mathfrak{a} < 0$ but $\lambda_0^5 \neq 1$ is still given
- in step five it is shown that even if $\lambda_0^5 = 1$ one can achieve the desired existence with the same regularity.

Step 1: Existence
Lets assume that $\lambda_0^5 \neq 1$, which means that $\beta = 0$ and $\mathfrak{a} > 0$. With this (1.3.3) becomes

$$\begin{cases} X = (1 - 2\mu\Re(\lambda_1))x + \mu^{3/2}X_1(x, \phi, \mu) \\ \Phi = \phi + \theta_0 + \mu\theta_2 + \mu^{3/2}\Phi_1(x, \phi, \mu) \end{cases} \tag{1.3.6}$$

where $X_1, \Phi_1 \in C^{k-5,1}$.

Consider the metric space

$$U = \{u : T^1 \mapsto \mathbb{R} : |u(\phi)| \leq 1, |u(\phi_1) - u(\phi_2)| \leq |\phi_1 - \phi_2|\} \tag{1.3.7}$$

with $dist(u, u') = ||u - u'|| = \sup_{\phi \in T^1} |u(\phi) - u'(\phi)|$.

We start with $u \in U$ and want to show that we can define a $\hat{u} \in U$ so that if $x = u(\phi)$, $X = \hat{u}(\Phi)$. Then we have a map $\mathfrak{F} : U \mapsto U$.
Next we want to show that this map $\mathfrak{F}$ is a contraction in U, which would give the existence of a unique fixed point u^*

$$u^* : x = u^*(\phi) \to X = u^*(\Phi).$$

First we fix $\mu > 0$ and assume $X_1, \Phi_1 \in C^{0,1}$ with respect to (x, ϕ).

Define $\rho = \sup_{\mu \in (0,\delta), |x| \leq 1, \phi \in T^1} (||X_1||_{0,1}, ||\Phi_1||_{0,1})$
which means

$$\begin{cases} |X_1(x, \phi, \mu)| \leq \rho \\ |X_1(x, \phi, \mu) - X_1(x', \phi, \mu)| \leq \rho |x - x'| \\ |X_1(x, \phi, \mu) - X_1(x, \phi', \mu)| \leq \rho |\phi - \phi'| \end{cases} \tag{1.3.8}$$

the same holds for Φ_1.

Now write

$$\Phi_u(\phi) = \phi + \theta_0 + \mu\theta_2 + \mu^{3/2}\Phi_1(u(\phi), \phi, \mu) (\text{mod } 1) \tag{1.3.9}$$

and show that Φ_u is a Lipschitz-homeomorphism of T^1 onto T^1:
For $\phi \geq \phi'$ if we consider the map Φ in $\mathbb{R}$ instead of T^1 we have

$$\begin{aligned} \Phi_u(\phi) - \Phi_u(\phi') =& \phi + \theta_0 + \mu\theta_2 + \mu^{3/2}\Phi_1(u(\phi), \phi, \mu) \\ & -\phi' - \theta_0 - \mu\theta_2 - \mu^{3/2}\Phi_1(u(\phi'), \phi', \mu) \\ =& \phi - \phi' + \mu^{3/2}(\Phi_1(u(\phi), \phi, \mu) - \Phi_1(u(\phi'), \phi', \mu)) \\ \geq& \left(1 - 2\rho\mu^{3/2}\right)(\phi - \phi'). \end{aligned}$$

We assume $2\rho\mu^{3/2} < 1$.
Then Φ_u is strictly increasing in $\mathbb{R}$, and $\Phi_u(\phi + 1) = \Phi_u(\phi) + 1$ since $\Phi_1(x, \phi, \mu)$ is 1-periodic in ϕ.
This leads to the existence of an inverse function $\Phi_u^{-1} : \mathbb{R} \to \mathbb{R}$ so that $\Phi_u^{-1}(\hat{\phi} + 1) = \Phi_u^{-1}(\hat{\phi}) + 1$ and

$$|\Phi_u^{-1}(\hat{\phi}) - \Phi_u^{-1}(\hat{\phi}')| \leq \left(1 - 2\rho\mu^{3/2}\right)^{-1} |\hat{\phi} - \hat{\phi}'| \tag{1.3.10}$$

Reconsider Φ_u^{-1} on T^1 instead of $\mathbb{R}$ which also satisfies (1.3.10) for $\hat{\phi}, \hat{\phi}' \in T^1$.
With this we can rewrite (1.3.6) as follows

$$\begin{cases} X = (1 - 2\mu\Re(\lambda_1))u(\phi) + \mu^{3/2} X_1(u(\phi), \phi, \mu) \\ \phi = \Phi_u^{-1}(\Phi) \end{cases} \tag{1.3.11}$$

which is just our original system with $x = u(\phi)$ applied and $\phi = \Phi_u^{-1}(\Phi) \Leftrightarrow \Phi_u(\phi) = \Phi$.
(1.3.11) defines function $X = \hat{u}(\Phi)$, where $\hat{u} : T^1 \to \mathbb{R}$ and

$$|\hat{u}(\Phi)| \leq 1 - 2\mu\Re(\lambda_1) + \rho\mu^{3/2} \leq 1.$$

The first inequality holds because $u \in U$ and (1.3.8). The second one holds for μ small enough.
The next step is to check whether $\hat{u}$ is in U or not.
We have

$$\begin{aligned} |\hat{u}(\Phi_1) - \hat{u}(\Phi_2)| =& |(1 - 2\mu\Re(\lambda_1))(u(\Phi_u^{-1}(\Phi_1)) - u(\Phi_u^{-1}(\Phi_2))) \\ &+ \mu^{3/2}(X_1(u(\Phi_u^{-1}(\Phi_1)), \Phi_u^{-1}(\Phi_1), \mu) - X_1(u(\Phi_u^{-1}(\Phi_2)), \Phi_u^{-1}(\Phi_2), \mu))|. \end{aligned}$$

We know

$$|u(\Phi_u^{-1}(\Phi_1)) - u(\Phi_u^{-1}(\Phi_2))| \leq |\Phi_u^{-1}(\Phi_1) - \Phi_u^{-1}(\Phi_2)|$$

and

$$\begin{aligned} &|X_1(u(\phi_1), \phi_1, \mu) - X_1(u(\phi_2), \phi_2, \mu)| \\ \leq& \underbrace{|X_1(u(\phi_1), \phi_1, \mu) - X_1(u(\phi_2), \phi_1, \mu)|}_{\leq \rho|u(\phi_1) - u(\phi_2)| \leq \rho|\phi_1 - \phi_2|} + \underbrace{|X_1(u(\phi_2), \phi_1, \mu) - X_1(u(\phi_2), \phi_2, \mu)|}_{\leq \rho|\phi_1 - \phi_2|} \\ \leq& 2\rho|\phi_1 - \phi_2|. \end{aligned}$$

We have

$$|\hat{u}(\Phi_1) - \hat{u}(\Phi_2)| \leq \left(1 - 2\mu\Re(\lambda_1) + 2\rho\mu^{3/2}\right) |\Phi_u^{-1}(\Phi_1) - \Phi_u^{-1}(\Phi_2)| \leq |\Phi_1 - \Phi_2|$$

where, for the last inequality, we used (1.3.10) and $\frac{1 - 2\mu\Re(\lambda_1) + 2\rho\mu^{3/2}}{1 - 2\rho\mu^{3/2}} \leq 1$, which holds if μ is small enough.
This shows that $\hat{u} \in U$ and we therefore have defined a map $\mathfrak{F} : U \to U$: $u \mapsto \hat{u}$.
Now we show that it is a contraction:

$$\begin{aligned} &|\mathfrak{F}(u_1)](\Phi) - [\mathfrak{F}(u_2)](\Phi)| = |\hat{u}_1(\Phi) - \hat{u}_2(\Phi)| = \\ =& |(1 - 2\mu\Re(\lambda_1))(u_1(\Phi_{u_1}^{-1}(\Phi)) - u_2(\Phi_{u_2}^{-1}(\Phi))) \\ &+ \mu^{3/2}(X_1(u_1(\Phi_{u_1}^{-1}(\Phi)), \Phi_{u_1}^{-1}(\Phi), \mu) - X_1(u_2(\Phi_{u_2}^{-1}(\Phi)), \Phi_{u_2}^{-1}(\Phi), \mu)|. \end{aligned}$$

To simplify we write $\Phi_{u_j}^{-1}(\Phi) = \phi_j$ in the above equation and continue:

$$\begin{aligned}
&|(1-2\mu\Re(\lambda_1))(u_1(\phi_1)-u_2(\phi_2)) + \mu^{3/2}(X_1(u_1(\phi_1),\phi_1,\mu) - X_1(u_2(\phi_2),\phi_2,\mu))| \\
\leq &(1-2\mu\Re(\lambda_1))|(u_1(\phi_1)-u_2(\phi_2))| \\
+&\mu^{3/2}\underbrace{|X_1(u_1(\phi_1),\phi_1,\mu) - X_1(u_1(\phi_1),\phi_2,\mu)|}_{\leq\rho|\phi_1-\phi_2|} + \underbrace{|X_1(u_1(\phi_1),\phi_2,\mu) - X_1(u_2(\phi_2),\phi_2,\mu)|}_{\leq\rho|u_1(\phi_1)-u_2(\phi_2)|} \\
\leq &\left(1-2\mu\Re(\lambda_1)+\rho\mu^{\frac{3}{2}}\right)|u_1(\phi_1)-u_2(\phi_2)| + \rho\mu^{3/2}|\phi_1-\phi_2|.
\end{aligned}$$

For the continuation of the proof we write $\Phi_{u_1}(\phi_1)$ and $\Phi_{u_2}(\phi_2)$ in the form (1.3.9) and subtract the later from the first

$$\begin{aligned}
\Phi_{u_1}(\phi_1) &= \phi_1 + \theta_0 + \mu\theta_2 + \mu^{3/2}\Phi_1(u_1(\phi_1),\phi_1,\mu)(\text{mod } 1) \\
\Phi_{u_2}(\phi_2) &= \phi_2 + \theta_0 + \mu\theta_2 + \mu^{3/2}\Phi_1(u_2(\phi_2),\phi_2,\mu)(\text{mod } 1)
\end{aligned}$$

$$\Phi_{u_1}(\phi_1) - \Phi_{u_2}(\phi_2) = (\phi_1-\phi_2) + \mu^{3/2}(\Phi_1(u_1(\phi_1),\phi_1,\mu) - \Phi_1(u_2(\phi_2),\phi_2,\mu))$$

Since we have $\Phi_{u_j}^{-1}(\Phi) = \phi_j$ which is equivalent to $\Phi_{u_j}(\phi_j) = \Phi$, the left hand side of the above equation is zero, resulting in

$$\begin{aligned}
(\phi_2-\phi_1) =&\mu^{3/2}(\Phi_1(u_1(\phi_1),\phi_1,\mu) - \Phi_1(u_2(\phi_2),\phi_2,\mu)) \\
\Leftrightarrow (\phi_2-\phi_1) =&\mu^{3/2}(\Phi_1(u_1(\phi_1),\phi_1,\mu) - \Phi_1(u_2(\phi_2),\phi_1,\mu) + \Phi_1(u_2(\phi_2),\phi_1,\mu) - \Phi_1(u_2(\phi_2),\phi_2,\mu)) \\
\Leftrightarrow |\phi_1-\phi_2| =&\mu^{3/2}|\Phi_1(u_1(\phi_1),\phi_1,\mu) - \Phi_1(u_2(\phi_2),\phi_1,\mu) + \Phi_1(u_2(\phi_2),\phi_1,\mu) - \Phi_1(u_2(\phi_2),\phi_2,\mu)| \\
\leq&\mu^{3/2}(\underbrace{|\Phi_1(u_1(\phi_1),\phi_1,\mu) - \Phi_1(u_2(\phi_2),\phi_1,\mu)|}_{\leq\rho|u_1(\phi_1)-u_2(\phi_2)|} + \underbrace{|\Phi_1(u_2(\phi_2),\phi_1,\mu) - \Phi_1(u_2(\phi_2),\phi_2,\mu)|}_{\leq\rho|\phi_1-\phi_2|}) \\
\Rightarrow |\phi_1-\phi_2| \leq&\rho\mu^{3/2}(|u_1(\phi_1)-u_2(\phi_2)| + |\phi_1-\phi_2|).
\end{aligned}$$

With this inequality we can deduce the following estimate:

$$\begin{aligned}
|\phi_1-\phi_2| \leq&\rho\mu^{3/2}(|u_1(\phi_1)-u_2(\phi_2)| + |\phi_1-\phi_2|) \\
\Leftrightarrow |\phi_1-\phi_2| \leq&\rho\mu^{3/2}(|u_1(\phi_1)-u_1(\phi_2)+u_1(\phi_2)-u_2(\phi_2)| + |\phi_1-\phi_2|) \\
\leq&\rho\mu^{3/2}(\underbrace{|u_1(\phi_1)-u_1(\phi_2)|}_{\leq|\phi_1-\phi_2|} + \underbrace{|u_1(\phi_2)-u_2(\phi_2)|}_{\leq||u_1-u_2||} + |\phi_1-\phi_2|) \\
\Rightarrow |\phi_1-\phi_2| \leq&\frac{\rho\mu^{3/2}}{(1-2\rho\mu^{3/2})}||u_1-u_2||. \quad (*)
\end{aligned}$$

With this estimate we return to our estimation of $\mathfrak{F}$:

$$
\begin{aligned}
&(1-2\mu\Re(\lambda_1)+\rho\mu^{3/2})|u_1(\phi_1)-u_2(\phi_2)|+\rho\mu^{3/2}|\phi_1-\phi_2| \\
&\leq (1-2\mu\Re(\lambda_1)+\rho\mu^{3/2})(|\phi_1-\phi_2|+||u_1-u_2||)+\rho\mu^{3/2}|\phi_1-\phi_2| \\
&\overset{*}{\leq} (1-2\mu\Re(\lambda_1)+\rho\mu^{3/2})\frac{\rho\mu^{3/2}}{(1-2\rho\mu^{3/2})}||u_1-u_2||+||u_1-u_2||)+\rho\mu^{3/2}\frac{\rho\mu^{3/2}}{(1-2\rho\mu^{3/2})}||u_1-u_2|| \\
&= ||u_1-u_2||\left((1-2\mu\Re(\lambda_1)+\rho\mu^{3/2})\left(1+\frac{\rho\mu^{3/2}}{(1-2\rho\mu^{3/2})}\right)+\rho\mu^{3/2}\frac{\rho\mu^{3/2}}{(1-2\rho\mu^{3/2})}\right) \\
&= ||u_1-u_2||\left(1+\frac{\rho\mu^{3/2}}{(1-2\rho\mu^{3/2})}\right)\left(1-2\mu\Re(\lambda_1)+\rho\mu^{3/2}+\rho\mu^{3/2}\frac{\frac{\rho\mu^{3/2}}{(1-2\rho\mu^{3/2})}}{1+\frac{\rho\mu^{3/2}}{(1-2\rho\mu^{3/2})}}\right).
\end{aligned}
$$

Since

$$
\frac{\frac{\rho\mu^{3/2}}{(1-2\rho\mu^{3/2})}}{1+\frac{\rho\mu^{3/2}}{(1-2\rho\mu^{3/2})}}=\frac{\rho\mu^{3/2}}{1-\rho\mu^{3/2}}<1
$$

(since $\rho\mu^{3/2}<\frac{1}{2}$) the following holds

$$
\begin{aligned}
&||u_1-u_2||\left(1+\frac{\rho\mu^{3/2}}{(1-2\rho\mu^{3/2})}\right)\left(1-2\mu\Re(\lambda_1)+\rho\mu^{3/2}+\rho\mu^{3/2}\frac{\frac{\rho\mu^{3/2}}{(1-2\rho\mu^{3/2})}}{1+\frac{\rho\mu^{3/2}}{(1-2\rho\mu^{3/2})}}\right) \\
&\leq ||u_1-u_2||\underbrace{\left(1+\frac{\rho\mu^{3/2}}{(1-2\rho\mu^{3/2})}\right)\left(1-2\mu\Re(\lambda_1)+2\rho\mu^{3/2}\right)}_{\eta}
\end{aligned}
$$

with $\eta<1$ when μ is small enough.
So we get

$$
|\mathfrak{F}(u_1)](\Phi)-[\mathfrak{F}(u_2)](\Phi)|<\eta||u_1-u_2||.
$$

So $\mathfrak{F}$ is a contraction and therefore there exists a fixed point u^* in U for $\mathfrak{F}$: $u^*=\mathfrak{F}(u^*)$.

This finishes the first part of the proof.

Step 2: Regularity
This step is omitted here and can be read in full in Iooss.
In step 1 our functions were of class $C^{k-5,1}$. In Step 2 we assume $X_1,\Phi_1\in C^{l,1}$ with respect to (x,ϕ) uniformly in $\mu\in(0,\delta)$. The new function space is:

$$
U_l=\{u:T^1\times\mathbb{R}\mapsto\mathbb{R};||u||_{l,1}\leq 1\}
$$

for which a contraction is found like in Step 1.

Additionally, Step 2 also shows that a contraction can be found if $X_1, \Phi_1 \in C^{l,1}$ with respect to (x, ϕ, μ), where the function space must be chosen appropriately.

Step 3: Stability
We have shown, that if $x = u(\phi)$ is in U_0, then iterating the map, we get the manifold $\{x_n = \hat{u}_n(\phi) : \phi \in T^1\}$. This tends uniformly to the graph u^*. Thus for a starting point x_0, ϕ_0 such that $|x_0| \leq 1$, we can consider the constant $x = x_0$ as u in U_0

$$\Rightarrow \inf_{\phi \in T^1} |x_n - u^*(\phi)| \underset{n\to\infty}{\longrightarrow} 0$$

which proves the stability.

Step 4: $\mathfrak{a} < 0$
In the case that $\mathfrak{a} < 0$ we have the system 1.3.5 for F_μ in $\mathbb{R}$ with $\beta(0) = 0$, since we still assume $\lambda_0^5 \neq 1$ and $\mu < 0$.
Consider F_μ^{-1} which is of the form:

$$\begin{cases} x = (1 + 2\mu\Re(\lambda_1))X + o(|\mu|^{3/2}) \\ \phi = \Phi - \theta_0 - \mu\theta_2 + o(|\mu|^{3/2}) \end{cases} . \tag{1.3.12}$$

Now proceed exactly as in the Steps 1 and 2.
This gives the result that F_μ^{-1} has an attractive invariant circle, with all the regularities as F_μ has for $\mathfrak{a} > 0$.
This meas that F_μ for $\mathfrak{a} < 0$ has an invariant circle which is repelling.

Step 5: $\lambda_0^5 = 1$
We now have to start the proof with the system (1.3.3) (or (1.3.5) depending on $\mathfrak{a}$, but since the following changes to the system can be done in the same fashion, we will only discuss the case $\mathfrak{a} > 0$ in this step).
The goal is to end up with a system like (1.3.6).
We rewrite (1.3.3) as

$$\begin{cases} X = (1 - 2\mu\Re(\lambda_1))x + \mu f(\phi) + O(|\mu|^{3/2}) \\ \Phi = \phi + \theta_0 + \mu\theta_2 + O(|\mu|^{3/2}) \end{cases} \tag{1.3.13}$$

where $f(\phi + 1/5) = f(\phi)$ is of class C^∞ and $5\theta_0 \in \mathbb{Z}$.
We make a change of variables:

$$\tilde{x} = x + g(\phi), \quad \tilde{\phi} = \phi \quad \text{with } g \text{ of period } 1/5.$$

Then $\tilde{X} = X + g(\Phi)$ satisfies the equation

$$\tilde{X} = (1 - 2\mu\Re(\lambda_1))[\tilde{x} - g(\tilde{\phi})] + \mu f(\tilde{\phi}) + g(\tilde{\phi} + \theta_0 + \mu\theta_2 + O(|\mu|^{3/2})) + O(|\mu|^{3/2}).$$

Assume g to be C^∞. Then because θ_0 is a multiple of $1/5$ we have

$$\tilde{X} = (1 - 2\mu\Re(\lambda_1))\tilde{x} + O(|\mu|^{3/2}),$$

provided that

$$\begin{aligned}
0 =& -(1-2\mu\Re(\lambda_1))g(\phi)+\mu f(\phi)+g(\phi)+\mu\theta_2 g'(\phi)\\
\Leftrightarrow 0 =& -g(\phi)+2\mu\Re(\lambda_1)g(\phi)+\mu f(\phi)+g(\phi)+\mu\theta_2 g'(\phi)\\
\Leftrightarrow 0 =& 2\Re(\lambda_1)g(\phi)+f(\phi)+\theta_2 g'(\phi).
\end{aligned}$$

(Here g was linearized: $g(\phi+\theta_0+\mu\theta_2)=g(\phi+\mu\theta_2)=g(\phi)+g'(\phi)(\phi+\mu\theta_2-\phi)$.)

For the above it is possible to find a solution $g \in C^\infty$ of period $1/5$, which means that the new map can be transformed into the old one with all the regularities still holding up.

□

Remark. For any given map the value of $\mathfrak{a}$ can be computed using (1.2.7):

$$\mathfrak{a} = \frac{|g_{11}(0)|^2}{2} + Re\left(g_{11}(0)g_{20}(0)\frac{(2\lambda-1)\overline{\lambda}^2}{\lambda-1}\right) + |g_{02}(0)|^2 - Re(\overline{\lambda}g_{21}(0)). \tag{1.3.14}$$

1.4 Dynamics on the invariant circle

The subject of interest in this chapter is the behaviour of the map on the invariant circle. Will there be periodic orbits or dense ones? This is related to the so called "rotation number", which will be introduced in the following section. There also are two cases and theories thereof for where the eigenvalue crosses the unit circle. These are called weak and strong resonance.
The contents of this section can be found in Arrowsmith and Place [1], pages 248 ff. and Whitley [7], pages 201-205.

1.4.1 The rotation number and Arnold's circle map

Definition. Suppose that $f : S^1 \to S^1$ is a homeomorphism of the circle S^1. The function f may be lifted to a map $F : \mathbb{R} \to \mathbb{R}$ via the projection $\exp : \mathbb{R} \to S^1$, $t \to e^{2i\pi t}$.
The *rotation number* is the limit

$$\rho(f) = \lim_{n\to\infty} \frac{F^n(t)-t}{n}$$

which exists and is independent of the choice of F and t.
The rotation number $\rho(f)$ has the following basic properties:

- if $\rho(f) \in \mathbb{Q}$, i.e. $\rho(f) = p/q$, then f has a periodic orbit of period q.
- if $\rho(f) \in \mathbb{R} \setminus \mathbb{Q}$ then f has no periodic orbit and if f is sufficiently smooth then it is conjugate to an irrational rotation and the orbit of each point is dense.

We will make a short excursion to Arnold's circle map. We will see when this map has a rational rotation number and then return to our map to see how the theory can be applied there.

Definition. Arnold's circle map is a two-parameter family of diffeomorphisms
$f_{(\alpha,\epsilon)} : S^1 \mapsto S^1$

$$f_{(\alpha,\epsilon)}(\theta) = \theta + \alpha + \epsilon \sin(\theta)$$

where $\epsilon \in [0, 1)$, θ and $\alpha \in [0, 2\pi]$.

The goal is to determine for which values (α, ϵ) the diffeomorphism $f_{(\alpha,\epsilon)}$ has a rotation number p/q in lowest terms ($q \in \mathbb{Z}^+$, $p \in \{0, 1, .., q-1\}$).

If we measure θ in units of 2π, then the iterate of the map is of the form

$$\theta_{n+1} = f_{(\alpha,\epsilon)}(\theta_n) = \theta_n + \alpha + \epsilon \sin(2\pi\theta_n) \tag{1.4.1}$$

with α and $\theta \in [0, 1]$ and $\epsilon \in [0, \pi/2)$.

The lift of $f_{(\alpha,\epsilon)}$ is

$$\overline{f}_{(\alpha,\epsilon)}(x) = x + \alpha + \epsilon \sin(2\pi x), \quad x \in \mathbb{R}. \tag{1.4.2}$$

We have the following Proposition:

Proposition 1.4.1. *The following holds for the rotation number*

$$\rho(f_{(\alpha,\epsilon)}) = p/q \Leftrightarrow \overline{f}^q_{(\alpha,\epsilon)}(x) - (x + p) = 0 \tag{1.4.3}$$

for some $x \in \mathbb{R}$.

The iterate of (1.4.2) is

$$\overline{f}^q_{(\alpha,\epsilon)}(x) = x + q\alpha + F(\alpha, \epsilon, x)$$

with $F(\alpha, \epsilon, x) = \epsilon \sum_{k=0}^{q-1} \sin(2\pi \overline{f}^k_{(\alpha,\epsilon)}(x))$.
We can rewrite this to make it easier to see when the proposition will hold, using $\alpha = p/q + \beta$:

$$\overline{f}^q_{(\alpha,\epsilon)}(x) = x + p + G_{p/q}(\beta, \epsilon, x) \tag{1.4.4}$$

where $G_{p/q}(\beta, \epsilon, x) = q\beta + F(p/q + \beta, \epsilon, x)$.

So for the proposition to hold $G_{p/q}$ needs to be zero. G is bounded and attains its maximum and minimum for x on the interval $[0, 1]$. So for each ϵ, there is an interval of β on which $G_{p/q}(\beta, \epsilon, x) = 0$ for some $x \in [0, 1]$. The boundaries of these regions can be approximated and β considered as a function of ϵ and x is at least quadratic in ϵ for all $q \geq 2$.

The sets for which $G_{p/q} = 0$ are written as

$$T_{p/q} = \left\{(\alpha, \epsilon) | \rho(f_{(\alpha,\epsilon)}) = p/q\right\} \text{ for } p/q \in \mathbb{Q} \cap [0, 1).$$

These sets are called Arnold tongues. The system of tongues is symmetric around $\alpha = 1/2$. There is a separate tongue for each rational in $[0, 1)$ and the greater q is the thinner is the tongue.
Figure 1.3 gives an example of what these tongues look like.

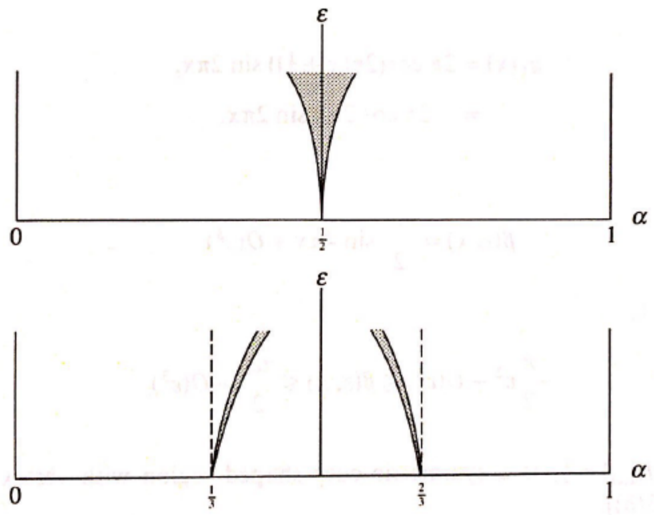

Figure 1.3: Arnold tongues for $T_{1/2}$(above) and $T_{1/3}$ and $T_{2/3}$(below)

1.4.2 Behaviour on the invariant circle and weak resonance

Returning to our map of the normal form (1.1.2), we have already seen that for $r = r_0(\gamma)$ the map reduces to a rotation. In the generic case the map reduces to a diffeomorphism on the circle, which is not necessarily reducible to a rotation. We shall call our map on the circle f_μ^C.

If f_μ^C has a rational rotation number, then it will have an even number of periodic points alternately stable and unstable, which are nodes and saddle points if considered not only on the circle, see Fig 1.4.

To see the dependence of $\rho(f_\mu^C)$ on μ we will reconsider our family as a two-parameter family $F_{(\mu_1,\mu_2)}$ which is again smooth and has a fixed point at $x = 0$ for all (μ_1, μ_2) near $(0, 0)$.

$$F_{(\mu)}(z) = \lambda(\mu)z\left(1 + \sum_{m=1}^{N} \tilde{a}_{2m+1}(\mu)|z|^{2m} + R_N(\mu, z)\right)$$

where $\mu = (\mu_1, \mu_2)$, $\lambda(\mu) = |\lambda(\mu)| \exp(2\pi\beta(\mu))$ and $R_N(\mu, z)$ is $O(|z|^{2N+2})$ and all mentioned functions are smooth. Furthermore $\lambda(0)\tilde{a}_3(0)$ is the parameter α from (1.3.1).

The map taking $(\mu_1, \mu_2) \in \mathbb{R}^2$ to $\lambda(\mu_1, \mu_2) \in \mathbb{C}$ is generically a local diffeomorphism and convenient coordinates for λ are

$$\begin{aligned}(\nu_1, \nu_2): \quad \nu_1 &= |\lambda| - 1\\ 2\pi\nu_2 &= \arg(\lambda) - 2\pi\beta.\end{aligned}$$

A change of variables gives

$$F_{(\nu)}(z) = (1+\nu_1)\exp(2\pi i(\beta+\nu_2))z\left(1+\sum_{m=1}^{N}\tilde{a}_{2m+1}(\nu)|z|^{2m}+R_N(\nu,z).\right)$$

If we take $N = 1$ and neglect R_1 there exists an invariant circle C, which has the following formula for its radius

$$r_0(\nu)^2 = -\frac{\nu_1}{\Re(\tilde{a}_3(0))} + O(|\nu|^2).$$

On this circle we have the map

$$f_\nu^C(\theta) = \theta + \beta + \nu_2 - \frac{\nu_1}{2\pi}\frac{\Im(\tilde{a}_3(0))}{\Re(\tilde{a}_3(0))} + O(|\nu|^2),$$

ν_1 determines the size of the invariant circle and ν_2 gives the rotation of the map for a given ν_1.

When R_1 is included the invariant circle exists if ν_1 and $\Re(\tilde{a}_3(0))$ have opposite sign and are not zero.
The map is then of the form

$$f_\nu^C(\theta) = \theta + \beta + \nu_2 + \Phi(\nu,\theta).$$

This expression can be compared with the Arnold map by identifying $\beta + \nu_2 = \alpha$ and $\nu_1 = \epsilon$.
If we assume that f_ν^C exhibits analogous behaviour to $f_{(\alpha,\epsilon)}$, then we expect to find a tongue $T_{p/q}$ associated with each rational p/q near β.

Theorem 1.4.2. *(Weak resonance)*
Let $F_{(\mu_1,\mu_2)}$ be a smooth 2-parameter family of maps on $\mathbb{R}^2$ such that for all (μ_1,μ_2) in a neighbourhood of 0 in $\mathbb{R}^2$ the following hold:

1. $F_{(\mu_1,\mu_2)}(0) = 0$
2. *$dF_{(\mu_1,\mu_2)}(0)$ has complex-conjugate eigenvalues $\lambda(\mu_1,\mu_2)$, $\overline{\lambda(\mu_1,\mu_2)}$ with $|\lambda(0,0)| = 1$ and $\arg\lambda(0,0) = 2\pi p/q$ (with p/q in lowest terms and $q \geq 5$)*
3. *the map $(\mu_1,\mu_2) \to \lambda(\mu_1,\mu_2)$ is non-singular at $\mu_1 = \mu_2 = 0$.*

Then there is a smooth reparameterization such that near $\mu_1 = \mu_2 = 0$

$$\lambda(\mu_1,\mu_2) = (1+\mu_1)e^{i((2\pi p/q)+\mu_2)}$$

and a smooth, parameter-dependent coordinate change bringing $F_{(\mu_1,\mu_2)}$ into the form

$$F_{(\mu_1,\mu_2)}(z) = \lambda(\mu_1,\mu_2)z + \sum_{m=1}^{[\frac{1}{2}(q-2)]}\alpha_{2m+1}(\mu_1,\mu_2)z^{m+1}\overline{z}^m + \beta(\mu_1,\mu_2)\overline{z}^{q-1} + O(|z|^q).$$

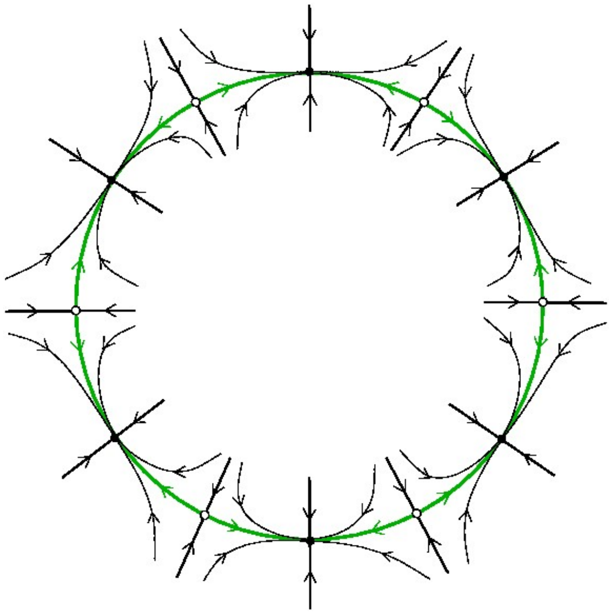

Figure 1.4: Periodic points on invariant curve [4]

In polar coordinates $z = re^{i\phi}$, $F_{(\mu_1,\mu_2)}(z) = Re^{i\Phi}$ *this reads*

$$\begin{cases} R = (1+\mu_1)r + \sum_{m=1}^{[\frac{1}{2}(q-2)]} a_{2m+1}(\mu_1,\mu_2)r^{2m+1} + \tilde{a}_{q-1}(\mu_1,\mu_2,\phi)r^{q-1} \\ \Phi = \frac{2\pi p}{q} + \mu_2 + \sum_{m=1}^{[\frac{1}{2}(q-2)]} b_{2m}(\mu_1,\mu_2)r^{2m} + \tilde{b}_{q-2}(\mu_1,\mu_2,\phi)r^{q-2} - O(r^{q-1}) \end{cases}$$

where $[\frac{1}{2}(q-2)]$ *denotes the integer part.*

The a_n, b_n, α_n *and* β *are smooth functions of* μ *for* (μ_1,μ_2) *near* $(0,0)$ *and* $\tilde{a}_{q-1}$ *and* $\tilde{a}_{q-2}$ *both are of the form* $A(\mu_1,\mu_2)\cos(q\phi) + B(\mu_1,\mu_2)\sin(q\phi)$*. Furthermore* $\alpha_3(\mu_1,\mu_2)$ *is the equivalent to* $\alpha(\mu)$ *in (1.3.1) and* $a_3(0,0) = -\mathfrak{a}$ *from (1.3.14).*

For all sufficiently small (μ_1,μ_2) *with* $|\lambda(\mu_1,\mu_2)| > 1$ *(or* $|\lambda(\mu_1,\mu_2)| < 1$ *respectively) the map* $F_{(\mu_1,\mu_2)}$ *has an attracting (repelling) invariant circle if* $a_3 = a_3(0,0) < 0$ *(or* $a_3 > 0$ *respectively).*
Furthermore, if $\beta = \beta(0,0) \neq 0$ *and* $b_2 = b_2(0,0) \neq 0$*, the map has two orbits of period* q*, one stable and one unstable orbit, for values of* μ_1, μ_2 *lying within a narrow "tongue" in the parameter plane whose boundaries are*

$$\mu_2 \cong \frac{b_2}{a_3}\mu_1 \pm \frac{|\beta|^2}{|a_3|^{\frac{1}{2}(q-2)}}\mu_1^{\frac{1}{2}(q-2)}$$

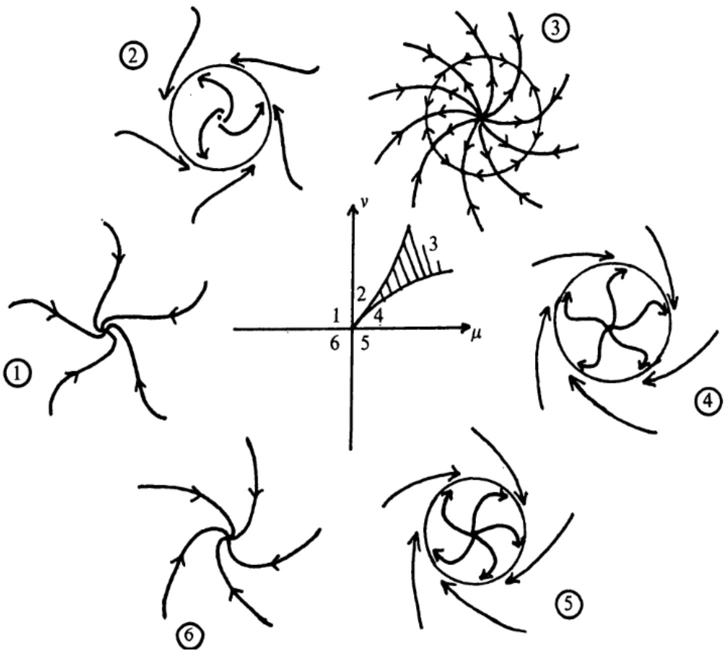

Figure 1.5: Weak resonance

which have a common tangent at $\mu_1 = \mu_2 = 0$.

Figure 1.5 pictures how the map behaves in the μ-plane, where $(\mu_1, \mu_2) = (\mu, \nu)$. If the two values are in region 3 we have periodic orbits on our circle.

We can see in Figure 1.6, where we have identified (μ_1, μ_2) with the real and the imaginary part of our eigenvalue, that as our bifurcation parameter varies and our eigenvalue λ crosses the unit circle, many Arnold tongues are entered and exited very rapidly. This means that periodic orbits appear and vanish as the parameter changes value.

1.4.3 strong resonance

It is important to note that all the above theory is only applicable to the case where $\lambda^k \neq 1$ for $k \in \{1, 2, 3, 4\}$.
When this is not given, one speaks of strong resonance. These cases require separate, more complicated theories and are more difficult to analyse. They will not be discussed in this thesis. For further reading on this topic see Whitley [7] pages 205-209 and Arrowsmith and Place [1] pages 262 ff.

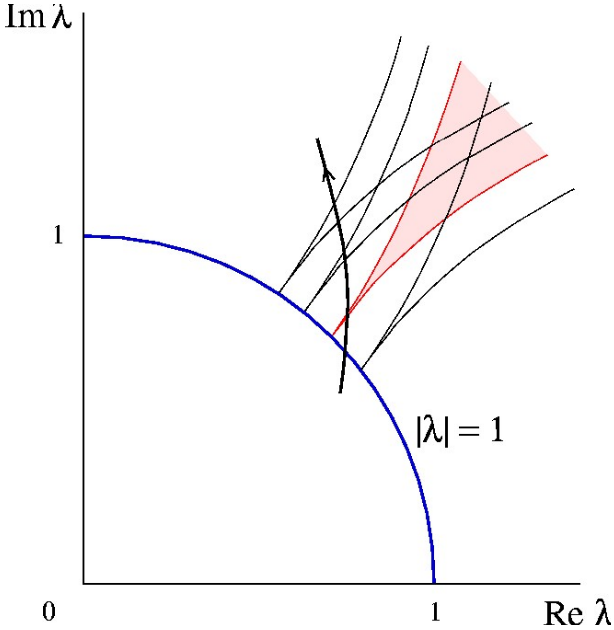

Figure 1.6: Eigenvalue λ passing through Arnold tongues [4]

2 Examples

2.1 Delayed Logistic Equation

This example was taken from [5], page 135.
The starting point is the recurrence equation

$$u_{k+1} = ru_k(1 - u_{k-1}) \tag{2.1.1}$$

where u_k is the density of a population at time k and $r > 0$ is the growth rate.
Now we introduce a new variable $v_k = u_{k-1}$ which makes our system two dimensional:

$$\begin{aligned} u' &= ru(1 - v) \\ v' &= u. \end{aligned} \tag{2.1.2}$$

This system has the fixed points:

- $(0, 0)$
- $(1 - \frac{1}{r}, 1 - \frac{1}{r})$.

Its Jacobian matrix reads

$$A(r) = \begin{pmatrix} 1 & 1-r \\ 1 & 0 \end{pmatrix}$$

which has determinant $\det = r - 1$. For the eigenvalues to be on the unit circle, r needs to be 2.
The eigenvalues of $A(r)$ are

$$\lambda_{1,2}(r) = \frac{1}{2} \pm \sqrt{\frac{5}{4} - r}$$

and at the critical value of $r = 2$ the non-trivial fixed point is (1/2,1/2) and the eigenvalues read

$$\lambda_{1,2}(2) = \frac{1 \pm i\sqrt{3}}{2} = e^{\pm i\frac{\pi}{3}}.$$

One can easily see that all requirements for λ for a Neimark-Sacker bifurcation are satisfied.

From now on we may only consider our system at the critical value of r and start the reduction to the normal form by shifting the non-trivial fixed point to the origin:

$$\begin{aligned} x &= u - 1/2 \\ y &= v - 1/2. \end{aligned}$$

In these new coordinates the map becomes

$$\begin{aligned} x' &= x - 2xy - y \\ y' &= x. \end{aligned} \tag{2.1.3}$$

We now use the ansatz $z = ax + by$ with a and b being complex numbers.
The map for z reads

$$z' = ax' + by' = x(a+b) - ay - 2axy. \tag{2.1.4}$$

This must be of the form (1.3.1) and expressing z and $\overline{z}$ in terms of x and y gives the following

$$\begin{aligned} z' =& \lambda z + g_{20}z^2 + g_{11}z\overline{z} + g_{02}\overline{z}^2 \\ =& \lambda(ax+by) + g_{20}(a^2x^2 + 2abxy + b^2y^2) + g_{11}(a\overline{a}x^2 + xy(a\overline{b} + \overline{a}b) + b\overline{b}y^2) \\ &+ g_{02}(\overline{a}^2x^2 + 2\overline{a}\overline{b}xy + \overline{b}^2y^2) \\ =& x(\lambda a) + y(\lambda b) + xy(2abg_{20} + (a\overline{b} + \overline{a}b)g_{11} + 2\overline{a}\overline{b}g_{02}) \\ &+ x^2(a^2g_{20} + a\overline{a}g_{11} + \overline{a}^2g_{02}) + y^2(b^2g_{20} + b\overline{b}g_{11} + \overline{b}^2g_{02}). \end{aligned} \tag{2.1.5}$$

The need for (2.1.4) and (2.1.5) to be identical leads to the following system of linear equations

$$\begin{aligned} \mathrm{I} &: a + b = \lambda a \\ \mathrm{II} &: -a = \lambda b \\ \mathrm{III} &: -2a = 2abg_{20} + (a\overline{b} + \overline{a}b)g_{11} + 2\overline{a}\overline{b}g_{02} \\ \mathrm{IV} &: 0 = a^2g_{20} + a\overline{a}g_{11} + \overline{a}^2g_{02} \\ \mathrm{V} &: 0 = b^2g_{20} + b\overline{b}g_{11} + \overline{b}^2g_{02}. \end{aligned} \tag{2.1.6}$$

Choosing $b = -1$, which gives $a = \lambda$, leads to

$$g_{20} = -\frac{2i}{\sqrt{3}}, \quad g_{11} = -1 + \frac{i}{\sqrt{3}}, \quad g_{02} = 1 + \frac{i}{\sqrt{3}}. \tag{2.1.7}$$

Calculating $\mathfrak{a}$ from (1.3.14) results in $\mathfrak{a} = 2$ making the bifurcation a supercritical Neimark-Sacker bifurcation.
In the figure below one can see that the fixed point is asymptotically stable for $r = 1.8$ and $r = 1.95$. For $r = 2.1$ there is an attracting invariant curve.

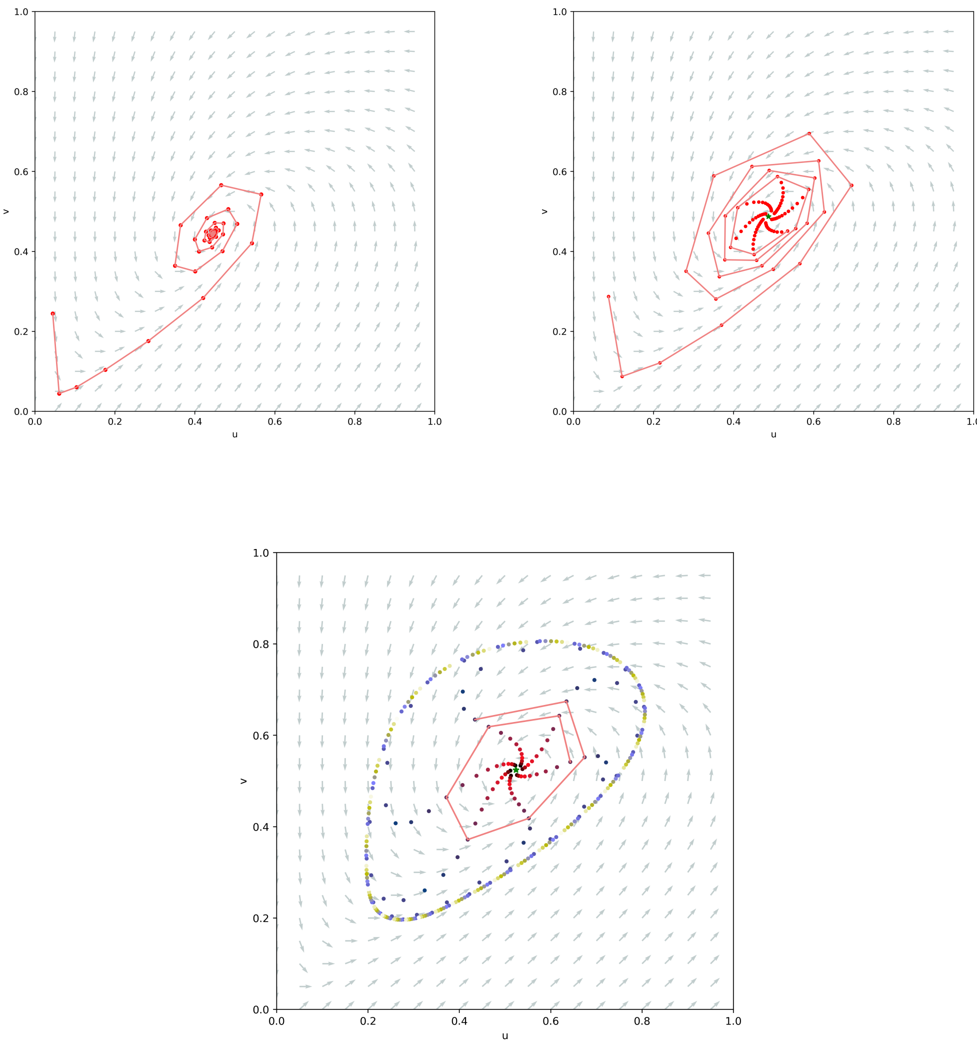

Figure 2.1: r = 1.8, 1.95, 2.1

2.2 Discrete-time Predator-Prey model

This example was taken from Kuznetsov [5], page 139 and the analysis of it can be found in Khan [3].

The model is as follows:

$$\begin{aligned} x_{k+1} &= \alpha x_k(1-x_k) - x_k y_k \\ y_{k+1} &= \frac{1}{\beta} x_k y_k. \end{aligned} \tag{2.2.1}$$

It has the fixed points

- $(0,0)$ for all α and β
- $(1-\frac{1}{\alpha}, 0)$ for all β and $\alpha > 0$
- $(\beta, \alpha(1-\beta)-1)$.

We are interested in the last listed fixed point and shall call it $\mathfrak{P}$ for reference. For $\mathfrak{P}$ to be in the positive quadrant we need $0 < \beta < 1$ and $\alpha > \frac{1}{1-\beta}$. The Jacobi matrix of this system reads

$$A = \begin{pmatrix} \alpha(1-2x) - y & -x \\ y/\beta & x/\beta \end{pmatrix}.$$

At our fixed point $\mathfrak{P}$ it is

$$A(\mathfrak{P}) = \begin{pmatrix} 1-\alpha\beta & -\beta \\ (\alpha(1-\beta)-1)/\beta & 1 \end{pmatrix} \tag{2.2.2}$$

and its characteristic polynomial is

$$P(\lambda) = \lambda^2 + \lambda(\alpha\beta - 2) + \alpha - 2\alpha\beta. \tag{2.2.3}$$

The Jury stability criterion states that $|\lambda_{1,2}| < 1$ if and only if
$0 < \det(A) < 1$ and $|\mathrm{tr}(A)| < 1 + \det(A)$.
Calculating this gives

$$\begin{aligned} 0 < \det(A) < 1 : 0 < \alpha - 2\alpha\beta < 1 \Leftrightarrow 0 < \alpha(1-2\beta) < 1 \\ \Rightarrow \beta < \frac{1}{2} \\ \Rightarrow \alpha < \frac{1}{1-2\beta}. \end{aligned}$$

From this we get that α is our bifurcation parameter and the fixed point $\mathfrak{P}$ loses its stability for $\alpha = 1/(1-2\beta)$.

Now we calculate the eigenvalues:

$$\begin{aligned}\lambda_{1,2} &= \frac{2-\alpha\beta}{2} \pm \sqrt{\frac{(2-\alpha\beta)^2}{4} - \alpha + 2\alpha\beta} \\ &= \frac{2-\alpha\beta}{2} \pm \frac{1}{2}\sqrt{(2+\alpha\beta^2) - 4\alpha} \\ &= \frac{2-\alpha\beta}{2} \pm i\frac{1}{2}\sqrt{4\alpha - (2+\alpha\beta)^2}.\end{aligned}$$

Lets check if all the assumptions for the Neimark-Sacker bifurcation are satisfied.

For a Neimark-Sacker bifurcation to occur we need $4\alpha - (2+\alpha\beta)^2 > 0$ at the critical value of α:

$$4\alpha - (2+\alpha\beta)^2 = 4\frac{1}{1-2\beta} - 4 - 4\frac{\beta}{1-2\beta} - \frac{\beta^2}{(1-2\beta)^2}$$

$$\begin{aligned}& 0 < 4(1-2\beta) - 4(1-2\beta)^2 - 4\beta(1-2\beta) - \beta^2 \\ \Leftrightarrow\ & 0 < 4\beta - 9\beta^2 \\ \Leftrightarrow\ & \beta < \frac{4}{9}.\end{aligned}$$

Next we need to check if $\frac{d|\lambda(\alpha)|}{d\alpha}|_{\alpha=1(/1-2\beta)} > 0$:

$$\begin{aligned}&|\lambda(\alpha)| = \alpha - 2\alpha\beta \\ &\frac{d|\lambda(\alpha)|}{d\alpha}|_{\alpha=1/(1-2\beta)} = 1 - 2\beta \overset{\beta<\frac{1}{2}}{>} 0.\end{aligned}$$

The last condition we need to check is that $\lambda^k \neq 1$ for $k = 1, 2, 3, 4$:

$$\begin{aligned}\lambda = 1 &\Leftrightarrow \beta = 0 \\ \lambda^2 = 1 &\Leftrightarrow \beta \in \{0, 4/9\} \\ \lambda^3 = 1 &\Leftrightarrow \beta \in \{0, 3/7\} \\ \lambda^4 = 1 &\Leftrightarrow \beta \in \{0, 2/5, 4/9\}.\end{aligned}$$

So β must be in the interval $(0, 4/9)$ and is not allowed to be $3/7$ or $2/5$.

The next step is to change our system (2.2.1) such that we can calculate $\mathfrak{a}$ from (1.3.14) easily.
Firstly we shift our fixed point to the origin

$$\begin{aligned}u_k &= x_k - \beta \\ v_k &= y_k - \alpha(1-\beta) + 1\end{aligned}$$

which gives the new system

$$\begin{aligned}u_{k+1} &= -\alpha u_k^2 + u_k(1-\alpha\beta) - \beta v_k - u_k v_k \\ v_{k+1} &= \frac{1}{\beta}(\alpha - \alpha\beta - 1)u_k + v_k + \frac{1}{\beta}u_k v_k.\end{aligned}$$

In this example it is not hard to change the coordinates to achieve the form

$$\begin{pmatrix} X \\ Y \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} f(X,Y) \\ g(X,Y) \end{pmatrix}$$

with the eigenvalue being $\lambda = e^{i\theta}$. In this example this takes the form

$$\begin{pmatrix} X \\ Y \end{pmatrix} \mapsto \begin{pmatrix} \frac{2-\alpha\beta}{2} & -\frac{1}{2}\sqrt{4\beta - (2+\alpha\beta)^2} \\ \frac{1}{2}\sqrt{4\beta - (2+\alpha\beta)^2} & \frac{2-\alpha\beta}{2} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} f(X,Y) \\ g(X,Y) \end{pmatrix}.$$

This can be achieved by the following change of coordinates

$$\begin{pmatrix} u_k \\ v_k \end{pmatrix} = \begin{pmatrix} -\beta & 0 \\ \frac{\alpha\beta}{2} & -\frac{1}{2}\sqrt{4\beta - (2+\alpha\beta^2)} \end{pmatrix} \begin{pmatrix} X_k \\ Y_k \end{pmatrix}$$

and the new system, using $\lambda = \lambda_{\Re} + i\lambda_{\Im}$ is

$$\begin{aligned}X_{k+1} &= \lambda_{\Re} X_k - \lambda_{\Im} Y_k + f(X_k, Y_k) \\ Y_{k+1} &= \lambda_{\Im} X_k + \lambda_{\Re} Y_k + g(X_k, Y_k)\end{aligned} \tag{2.2.4}$$

with f and g being

$$\begin{aligned}f(X,Y) &= \frac{\alpha\beta}{2}X^2 - \lambda_{\Im} XY \\ g(X,Y) &= \frac{\alpha\beta(2+\alpha\beta(2+\beta))}{4\lambda_{\Im}}X^2 - Re(\lambda)XY.\end{aligned} \tag{2.2.5}$$

If we introduce the complex valued variable $z = X + iY$ we get the system

$$\begin{aligned}z \mapsto &\lambda_{\Re} - \lambda_{\Im} + f(X,Y) + i(\lambda_{\Im} + Re(\lambda) + g(X,Y)) \\ = &(\lambda_{\Re} + i\lambda_{\Im})\, X + (\lambda_{\Re} + i\lambda_{\Im})\, iY + f(X,Y) + ig(X,Y) \\ = &\lambda z + f(X,Y) + ig(X,Y).\end{aligned} \tag{2.2.6}$$

Next we rewrite f and g such that the map for z consists only of z and $\overline{z}$ terms. Using

$$\begin{aligned}X^2 &= \frac{1}{4}(z^2 + \overline{z}^2 + 2z\overline{z}) \\ XY &= \frac{1}{4i}(z^2 - \overline{z}^2)\end{aligned}$$

f becomes

$$\begin{aligned} f(X,Y) &= \frac{\alpha\beta}{2}X^2 - \lambda_{\Im}XY \\ &= \frac{1}{2}f_{XX}(0,0)X^2 + f_{XY}(0,0)XY \\ &= \frac{1}{2}f_{XX}(0,0)\frac{1}{4}(z^2+\overline{z}^2+2z\overline{z}) + f_{XY}(0,0)\frac{1}{4i}(z^2-\overline{z}^2) \end{aligned}$$

where $f_{XY} = \partial^2 f(X,Y)/\partial X\partial Y$.
Similarly we can change g:

$$ig(X,Y) = \frac{i}{2}g_{XX}(0,0)\frac{1}{4}(z^2+\overline{z}^2+2z\overline{z}) + \frac{1}{4}g_{XY}(0,0)(z^2-\overline{z}^2).$$

With this we can rewrite (2.2.6) as follows

$$\begin{aligned} z \mapsto \lambda z &+ z^2\left(\frac{1}{8}f_{XX}(0,0) + \frac{1}{4}g_{XY}(0,0) + i\left(-\frac{1}{4}f_{XY}(0,0) + \frac{1}{8}g_{XX}(0,0)\right)\right) \\ &+ z\overline{z}\left(\frac{1}{4}f_{XX}(0,0) + \frac{i}{4}g_{XX}(0,0)\right) \\ &+ \overline{z}^2\left(\frac{1}{8}f_{XX}(0,0) - \frac{1}{4}g_{XY}(0,0) + i\left(\frac{1}{4}f_{XY}(0,0) + \frac{1}{8}g_{XX}(0,0)\right)\right). \end{aligned} \tag{2.2.7}$$

With this $\mathfrak{a}$ from (1.3.14) can easily be computed since we know what our g_{jk} look like:

$$\begin{aligned} g_{20} &= \frac{1}{4}f_{XX}(0,0) + \frac{1}{2}g_{XY}(0,0) + i\left(-\frac{1}{2}f_{XY}(0,0) + \frac{1}{4}g_{XX}(0,0)\right) \\ g_{11} &= \frac{1}{4}f_{XX}(0,0) + \frac{i}{4}g_{XX}(0,0) \\ g_{02} &= \frac{1}{4}f_{XX}(0,0) - \frac{1}{2}g_{XY}(0,0) + i\left(\frac{1}{2}f_{XY}(0,0) + \frac{1}{4}g_{XX}(0,0)\right) \\ g_{21} &= 0. \end{aligned}$$

Now the calculation for $\mathfrak{a}$ results in $\mathfrak{a} > 0$ for $\beta < 4/9$ therefore an attracting invariant closed curve bifurcates, a Supercritical Neimark-Sacker bifurcation takes place.

In the figure below one can see the bifurcation take place at $\beta = 0.27$ which means that the critical value of α is 2.1739. The first two images are for $\alpha = 1.8$ and $\alpha = 2$ and the second two, where the bifurcation has already taken place are at $\alpha = 2.2$ and $\alpha = 2.25$. The blue path always starts very close to the border equilibrium.

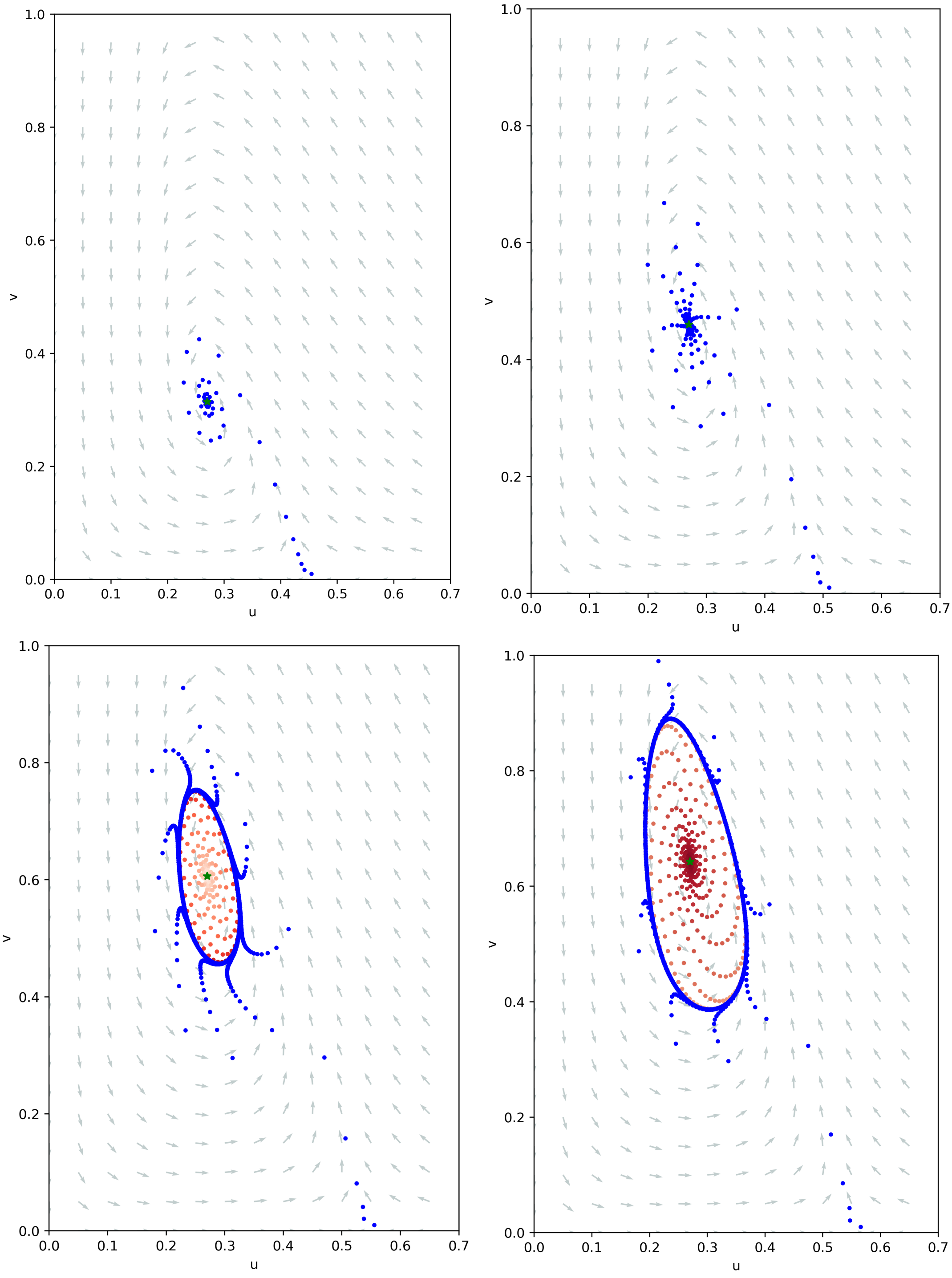

Figure 2.2: $\beta = 0.27,\ \alpha = 1.8, 2, 2.2, 2.25$

2.3 Host-Parasitoid Models

The next three models all have the same type and were taken from Lauwerier and Metz [6]. The models are of the form

$$\begin{aligned} x_{k+1} &= x_n \Phi(y_n) \\ y_{k+1} &= a x_k - x_{k+1} \end{aligned} \tag{2.3.1}$$

where x_k and y_k are the numbers of hosts and parasitoids at time k. The parameter a is greater than 1 and $\Phi(y)$ is assumed to be differentiable with $\Phi'(y) < 0$, $\Phi(0) = a$ and $\Phi(\infty) < 1$ and it contains another parameter b.
The equilibria of this system are

- $(0, 0)$
- $P = (\frac{c}{a-1}, c)$

with c being the unique root of $\Phi(c) = 1$.

The Jacobian matrix of this system reads

$$A = \begin{pmatrix} \Phi(y) & x\Phi'(y) \\ a - \Phi(y) & -x\Phi'(y) \end{pmatrix}. \tag{2.3.2}$$

At the non-trivial fixed point P it is

$$A(P) = \begin{pmatrix} 1 & \frac{c}{a-1}\Phi'(c) \\ a - 1 & -\frac{c}{a-1}\Phi'(c) \end{pmatrix}. \tag{2.3.3}$$

The eigenvalue equation reads

$$\chi(\lambda) = \lambda^2 - \lambda \left(1 - \frac{c\Phi'(c)}{a-1}\right) - \frac{ac\Phi'(c)}{a-1} \tag{2.3.4}$$

and if we substitute $L = -ac\Phi'(c)/(a-1)$ the equation simplifies to

$$\chi(\lambda) = \lambda^2 - \lambda(1 + \frac{L}{a}) + L. \tag{2.3.5}$$

For the eigenvalues to be on the unit circle, L needs to be 1. With this in mind we can write the eigenvalues as

$$\lambda_{1,2} = \frac{1}{2a}\left(a + 1 \pm i\sqrt{3a^2 - 2a - 1}\right). \tag{2.3.6}$$

The next step is to reduce this system to the normal form. For this we first introduce a new variable $w = x\Phi(y)$, changing our original system (2.3.1) into

$$\begin{aligned} x_{k+1} &= w_k \\ w_{k+1} &= w_k \Phi(a x_k - w_k). \end{aligned}$$

Now we use the complex valued variable z determined by

$$cz = a\sigma(\overline{\lambda}x - w) - c_0, \tag{2.3.7}$$

where c and λ are the same as above, c_0 is complex valued and determined by the condition

$$z = 0 \quad \text{for} \quad x = w = \frac{c}{a-1}.$$

This gives us

$$c_0 = \frac{a\sigma c(\overline{\lambda} - 1)}{a-1}$$

where σ is chosen such that

$$ax - w = c(1 + z + \overline{z}) \tag{2.3.8}$$

making it

$$\sigma = \frac{1}{a}\frac{\lambda - a}{\lambda - \overline{\lambda}}.$$

Our map in the new variable z is now

$$\begin{aligned} cz_{k+1} =& a\sigma(\overline{\lambda}x_{k+1} - w_{k+1}) - c_0 \\ =& a\sigma w_k(\overline{\lambda} - \Phi(ax_k - w_k)) - c_0 \\ =& a\sigma w_k(\overline{\lambda} - \Phi(c(1 + z + \overline{z}))) - c_0. \end{aligned} \tag{2.3.9}$$

To express w in terms of z and $\overline{z}$, we first use (2.3.8) to describe x in terms of w, z and $\overline{z}$ and then use this in (2.3.7) to get

$$w = \frac{c}{a-1}(1 + c_1 z + c_2\overline{z}) \tag{2.3.10}$$

where

$$c_1 = \frac{(\sigma\overline{\lambda} - 1)(a-1)}{\sigma(a - \overline{\lambda})}, \qquad c_2 = \frac{\overline{\lambda}(a-1)}{a - \overline{\lambda}}.$$

Now (2.3.7) can be written as

$$z_{k+1} = \frac{a\sigma}{c}\frac{c}{a-1}(1 + c_1 z_k + c_2\overline{z}_k)(\overline{\lambda} - \Phi(c(1 + z_k + \overline{z}_k))) - \frac{c_0}{c}. \tag{2.3.11}$$

If $\Phi(y)$ has, at the equilibrium value, the Taylor expansion

$$\Phi(c(1+t)) = 1 - At + ABt^2 - ACt^3 + \dots, \tag{2.3.12}$$

(2.3.11) becomes

$$z' = \frac{a\sigma}{a-1}(1 + c_1 z + c_2\overline{z})(\overline{\lambda} - 1 + A(z+\overline{z}) - AB(z+\overline{z})^2 + AC(z+\overline{z})^3 + \dots) - \frac{c_0}{c}. \tag{2.3.13}$$

At the bifurcation point the above map for z can be simplified:
Since $-A = \Phi(c(1+t))' = \Phi'(c(1+t))c$ we can use the condition $L = 1$ from the eigenvalue equation to get

$$A = \frac{a-1}{a}.$$

The coefficient c_1 is

$$\begin{aligned} c_1 &= \frac{(\sigma\overline{\lambda}-1)(a-1)}{\sigma(a-\overline{\lambda})} = \frac{((\lambda-a)\overline{\lambda}-a(\lambda-\overline{\lambda}))(a-1)}{(\overline{\lambda}-a)(a-\overline{\lambda})} = \frac{(1-a\lambda)(a-1)}{-a(a-1)} \\ &= \lambda - \frac{1}{a} = \frac{1}{2a}(a-1+i\Im(\lambda)) = 1-\overline{\lambda} \end{aligned}$$

and c_2 is

$$c_2 = \frac{\overline{\lambda}(a-1)}{a-\overline{\lambda}} = \frac{\overline{\lambda}a-\overline{\lambda}}{a-\overline{\lambda}} = \frac{\overline{\lambda}a-a}{a-\overline{\lambda}}+1 = \frac{a\overline{\lambda}-1-a+\lambda}{a-1}+1 = 1-\lambda.$$

Furthermore

$$\frac{a\sigma}{a-1}(\overline{\lambda}-1) - \frac{c_0}{c} = 0$$

and

$$\sigma = \frac{\lambda^2}{1+\lambda}$$

and so the map for z is

$$\begin{aligned} z' &= \frac{a\lambda^2}{(a-1)(1+\lambda)}[1+(1-\overline{\lambda})z+(1-\lambda)\overline{z}][\overline{\lambda}-1+A(z+\overline{z})-\ldots] - \frac{c_0}{c} \\ &= z\left(\frac{a\lambda^2}{(a-1)(1+\lambda)}(1-\overline{\lambda})(\overline{\lambda}-1)\right) + \overline{z}\left(\frac{a\lambda^2}{(a-1)(1+\lambda)}(1-\lambda)(\overline{\lambda}-1)\right) \\ &+ \frac{\lambda^2}{1+\lambda}[1+(1-\overline{\lambda})z+(1-\lambda)\overline{z}][z+\overline{z}-B(z+\overline{z})^2+C(z+\overline{z})^3-\ldots]. \end{aligned} \tag{2.3.14}$$

From the last line one can easily read off the coefficients we need for (1.3.14):

$$\begin{aligned} a_{20} &= \frac{\lambda^2}{1+\lambda}(1-\overline{\lambda}-B), \quad a_{11} = \frac{\lambda^2}{1+\lambda}(2-\lambda-\overline{\lambda}-2B), \\ a_{02} &= \frac{\lambda^2}{1+\lambda}(1-\lambda-B), \quad a_{21} = \frac{\lambda^2}{1+\lambda}[(\lambda+2\overline{\lambda}-3)B+3C]. \end{aligned}$$

Calculating $\mathfrak{a}$ gives the simple result

$$\mathfrak{a} = \frac{1}{2}(A-(A+1)B+4B^2-3C) \tag{2.3.15}$$

which will make the analysis of the following examples remarkably easier.

Even more so if $\Phi(y) = \exp(-f(y))$ because if

$$f(c(1+t)) = At + AB_1t^2 + AC_1t^3 + \dots$$

we can insert this into the Taylor expansion of the exponential function to get the coefficients needed above:

$$B = -B_1 + \frac{A}{2}, \quad C = C_1 - AB_1 + \frac{A^2}{6}$$

resulting in

$$\mathfrak{a} = \frac{1}{2}\left(\frac{1}{2}A + B_1 + 4B_1^2 - 3C_1\right). \tag{2.3.16}$$

2.3.1 The Hassel-Varley Model

The Hassel-Varley Model has the form

$$\begin{aligned} x_{k+1} &= ax_k \exp(-y_k^b) \\ y_{k+1} &= ax_k - x_{k+1} \end{aligned} \tag{2.3.17}$$

with $0 < b \leq 1$.
In this example $\Phi(y) = a\exp(-y^b)$ and $c = \log(a)^{1/b}$ so the non-trivial equilibrium is

$$(x, y) = \left(\frac{\log(a)^{1/b}}{a-1}, \log(a)^{1/b}\right).$$

The eigenvalue equation reads

$$\chi(\lambda) = \lambda^2 - \lambda\left(1 + \frac{b\log(a)}{a-1}\right) + \frac{ab\log(a)}{a-1}.$$

The eigenvalues are on the unit circle if $ab\log(a) = a - 1$ which is equal to
$b = (a-1)/(\log(a)a)$.

Since in this example $\Phi(y) = a\exp(-y^b) = \exp(\log(a) - y^b)$ we can use (2.3.16) to calculate $\mathfrak{a}$ with

$$f(c(1+t)) = -\log(a) + (c(1+t))^b = \log(a)(-1 + (1+t)^b).$$

This gives us

$$A = b\log(a), \quad B_1 = \frac{1}{2}(b-1), \quad C_1 = \frac{1}{6}(b-1)(b-2)$$

making $\mathfrak{a} = 1/4(b^2 + b\log(a) - 1)$. Inserting the critical value of $b = (a-1)/(\log(a)a)$ gives

$$\mathfrak{a} = \frac{\log(a)^2a + (a-1)^2}{4\log(a)^2a^2},$$

which is always positive and bounded since $a > 1$ is assumed. This means that a Subcritical Neimark-Sacker bifurcation takes place.

2.3.2 Parasitoid-Parasitoid Model

Here there are two groups of parasitoids, one with single individuals looking for a host and another with pairs fighting each other. This scenario can be viewed as a chemical reaction

$$u + u \underset{\alpha_2}{\overset{\alpha_1}{\rightleftharpoons}} 2v$$

with u being the number of parasitoids of the first kind and v the number of pairs of the second, such that $P = u + 2v$ is the total number of parasitoids. The above reaction gives the equation

$$\dot{u} = -\alpha_1 u^2 + 2\alpha_2 v.$$

For the equilibrium to exist it is needed that $\alpha_1 u^2 - 2\alpha_2 v = 0$ and if we write v in terms of u and P this equation is

$$\alpha_1 u^2 + \alpha_2 u - \alpha_2 P = 0.$$

Solving this for u results in

$$u = \frac{1}{2\alpha_1}\left(-\alpha_2 + \sqrt{\alpha_2^2 + 4\alpha_1\alpha_2 P}\right).$$

By setting $b = 2\alpha_1/\alpha_2$ and assuming that $4\alpha_1 \approx \alpha_2$ we can plug this into the Nicholson-Bailey model

$$\begin{aligned} x' &= ax\exp(-u) \\ y' &= ax - x', \end{aligned}$$

where x are the hosts and $y = P$ are the parasitoids. This gives the model for this section:

$$\begin{aligned} x' &= ax\exp\left(-\frac{\sqrt{1+y}-1}{b}\right) \\ y' &= ax - x' \end{aligned}$$

with $b > 0$.
The non-trivial equilibrium is

$$(x, y) = \left(\frac{(1 + b\log(a))^2 - 1}{a-1}, (1 + b\log(a))^2 - 1\right)$$

and the eigenvalue equation reads

$$\chi(\lambda) = \lambda^2 - \lambda\left(1 + \frac{a\log(a)(2 + b\log(a))}{2a(a-1)(1 - b\log(a))}\right) + \frac{a\log(a)(2 + b\log(a))}{2(a-1)(1 - b\log(a))}.$$

For the stability of the non-trivial equilibrium, we need

$$\begin{aligned}
1 &> 1 + \frac{a\log(a)(2+b\log(a))}{2a(a-1)(1-b\log(a))} \\
\Leftrightarrow b &> \frac{2a\log(a)-2(a-1)}{\log(a)(2(a-1)-a\log(a))} \\
\Leftrightarrow b &> \frac{a}{2(a-1)-a\log(a)} - \frac{1}{\log(a)}.
\end{aligned}$$

We need to restrict a to $1 < a < 4,9215536$ since $2(a-1) - a\log(a)$ is zero at the upper bound.

In this example $\Phi(y) = a\exp(-\frac{\sqrt{1+y}-1}{b})$ so we can use (2.3.16) again to calculate $\mathfrak{a}$. With

$$f(c(1+t)) = -\log(a) + \frac{\sqrt{1+c(1+t)}-1}{b} = \frac{\sqrt{1+c+ct}-\sqrt{1+c}}{b}$$

the coefficients are:

$$\begin{aligned}
A &= \frac{c}{2b(1+c)^{1/2}} = \frac{(1+b\log(a))^2-1}{2b\sqrt{(1+b\log(a))}} \\
B_1 &= -\frac{c}{4(1+c)} = -\frac{(1+b\log(a))^2-1}{4(1+b\log(a))^2} \\
C_1 &= \frac{c^2}{8(1+c)^2} = \frac{((1+b\log(a))^2-1)^2}{8(1+b\log(a))^4}.
\end{aligned}$$

Now $\mathfrak{a}$ can be calculated:

$$\mathfrak{a} = \frac{c}{8b(1+c)^{1/2}} - \frac{2c+7c^2}{8(1+c)^2}.$$

We can write this as an equation of a by substituting the critical values of b and c, i.e.

$$b = \frac{2a\log(a)-2(a-1)}{\log(a)(2(a-1)-a\log(a))}, \quad c = \left(\frac{a\log(a)}{2(a-1)-a\log(a)}\right)^2 - 1.$$

This results in

$$\mathfrak{a} = \frac{a-1}{4a} - \frac{1}{2}\frac{(a-1)}{a\log(a)} - \frac{1}{2}\left(\frac{a-1}{a\log(a)}\right)^2 + 2\left(\frac{a-1}{a\log(a)}\right)^3 - \left(\frac{a-1}{a\log(a)}\right)^4$$

which is negative for $1 < a < 3,8524$ and positive for $3,8524 < a < 4,92155$. This means that we have a subcritical Neimark-Sacker bifurcation in the first interval and a supercritical bifurcation in the latter one.

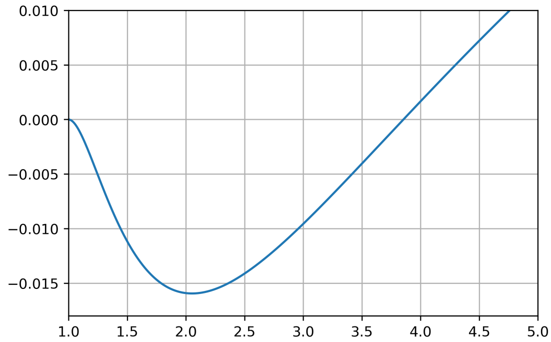

Figure 2.3: $\mathfrak{a}$ as a function of a

Figure 2.4 shows the system at $a = 4.2$, which makes $\mathfrak{a}$ positive, but is very close to where it makes $\mathfrak{a}$ change its sign. In this image there are three orbits shown, where the intensity of the points increases with each step. One can see that the supercritical bifurcation has taken place and the invariant stable curve, here in blue, is attracting from both outside and inside of the curve. There is however another repelling invariant curve outside of the attracting one, making the area of attraction of the inner curve bounded.

2.3.3 Simple model

The Simple Model reads

$$\begin{aligned} x_{k+1} &= \frac{a x_k}{1 + y_k^b} \\ y_{k+1} &= \frac{a x_k y_k^b}{1 + y_k^b} \end{aligned} \tag{2.3.18}$$

with $0 < b$.

Here $\Phi(y) = \frac{a}{1+y^b}$ and $c = (a-1)^{1/b}$ so the non-trivial equilibrium is

$$(x, y) = \Big((a-1)^{1/b-1}), (a-1)^{1/b})\Big)$$

The eigenvalue equation reads

$$\chi(\lambda) = \lambda^2 - \lambda\left(1 + \frac{b}{a}\right) + b$$

so the non-trivial equilibrium is stable for $0 < b < 1$ and unstable for $b > 1$.
At $b = 1$

$$\Phi(c(1+t)) = \frac{a}{a + ct} = 1 - \frac{c}{a}t + \frac{c^2}{a^2}t^2 + \frac{c^3}{a^3}t^3 + \ldots$$

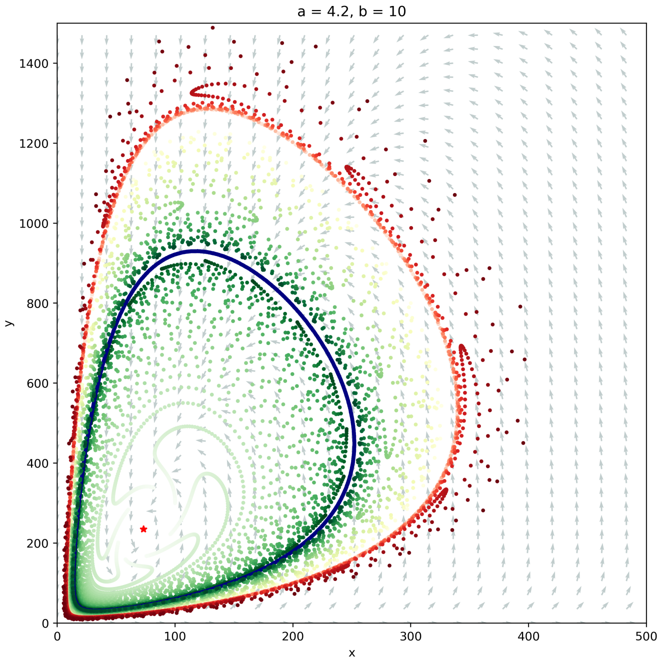

Figure 2.4: PP-model for $a = 4.2$ and $b = 10$

giving us

$$A = \frac{c}{a}, \quad B = \frac{c}{a}, \quad C = \frac{c^2}{a^2}.$$

If we use the formula (2.3.15) we get

$$\mathfrak{a} = \frac{1}{2}\left(\frac{c}{a} - \left(\frac{c}{a} + 1\right)\frac{c}{a} + 4\frac{c^2}{a^2} - 3\frac{c^2}{a^2}\right)$$

which is zero for all values of a so our theory does not apply. Computer experiments show that there are periodic cycles of order 9 and of every greater order, as seen in figure 2.5.

The following gives some insight on why this might be.
If we take the inverse map

$$\begin{aligned} x_k &= \frac{x_{k+1} + y_{k+1}}{a} \\ y_k &= \frac{y_{k+1}}{x_{k+1}} \end{aligned} \tag{2.3.19}$$

and rescale it using

$$u = \log(x), \quad v = \log\left(\frac{y}{a-1}\right) \tag{2.3.20}$$

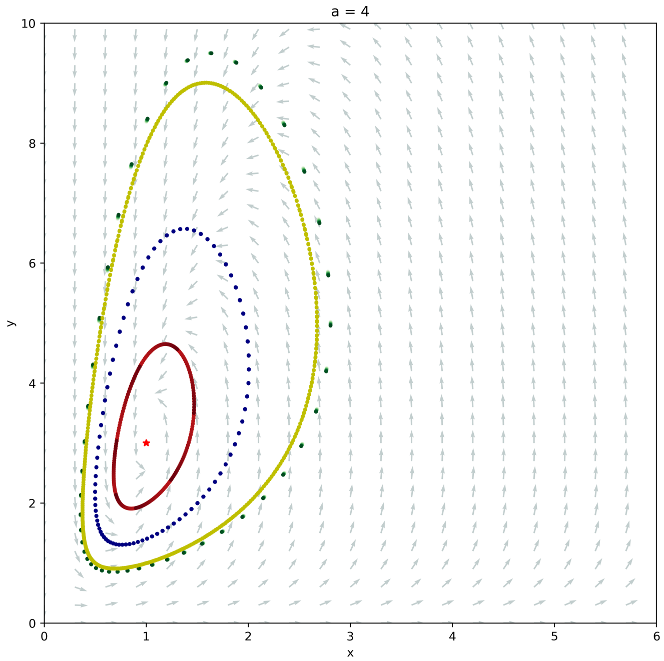

Figure 2.5: Simple model for $a = 4$

we get the system

$$\begin{aligned} u_{k+1} &= \log\left(\frac{e^{u_k} + (a-1)e^{v_k}}{a}\right) \\ v_{k+1} &= v_k - u_k. \end{aligned} \tag{2.3.21}$$

This is in fact a symplectic map since its Jacobian matrix is

$$\begin{pmatrix} \frac{e^u}{e^u+(a-1)e^v} & \frac{(a-1)e^v}{e^u+(a-1)e^v} \\ -1 & 1 \end{pmatrix}$$

and its determinant is 1. So the Simple model is closely related to an area-preserving map. This alone does not guarantee invariant cycles like we have seen in previous examples. In Figure 2.6 one can see that orbits consisting of island chains occur.

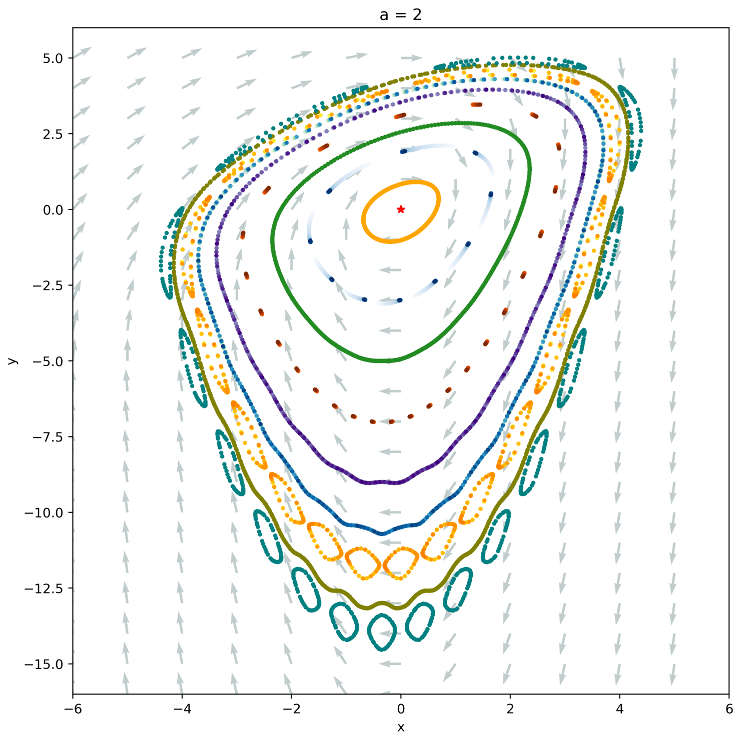

Figure 2.6: Inverse Simple model on a logarithmic scale for $a = 2$

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