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„Continuous time approach of a perturbed second order dynamical system combining a vanishing damping term with a Hessian-analogue damping term"

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Abstract

Für Dynamiken in stetiger Zeit des Problems $V(z) = 0$ eines monotonen und stetigen elementwertigen Operators V via Betrachtung eines dynamischen Systems zweiter Ordnung haben Boţ, Csetnek und Nguyen in ihrer Arbeit [3] verbesserte Konvergenzraten für ein System zweiter Ordnung gefunden, das einen asymptotisch verschwindenden Dämpfungsterm mit einem Hesse-analogen Dämpfungterm kombiniert. Die vorliegende Masterarbeit betrachtet dieses System mit Störungsterm g und zeigt dieselben Konvergenzresultate, die im ungestörten Fall in [3] bewiesen wurden, wobei die einzige zusätzliche Voraussetzung durch die Integrierbarkeit von $t \mapsto t\|g(t)\|$ gegeben ist.
Das erste Kapitel führt einige notwendige Konzepte und Resultate aus der monotonen Operatortheorie ein, die alle in [2] gefunden werden können. Ebenso enthält dieses Kapitel eine Aussage aus [4], die in einigen Beweisen der Hauptaussagen im zweiten Kapitel benötigt wird. Dieses Kapitel verfolgt einen analogen Aufbau zu [3], wobei die entsprechenden Resultate für den Fall mit Störung g eingeführt und bewiesen werden. Die Beweise wurden von der Autorin an diesen Fall angepasst. Außerdem werden einige notwendige Aussagen wie das stetige Opial-Lemma und ein Lemma über die Existenz und den Wert des Grenzwertes einer stetig differenzierbaren Funktion aus [1] angeführt. Zum Abschluss findet sich ein Kommentar zu den Beobachtungen, die im Laufe der Masterarbeit gemacht werden und ein Ausblick auf interessante Fragestellungen, die sich natürlicherweise aus den diskutierten Resultaten und Konzepten ergeben.

Abstract

Considering continuous time dynamics for the problem $V(z) = 0$ for a monotone and continuous single-valued operator V via a second-order dynamical system, Boț, Csetnek and Nguyen established improved convergence rates for a second-order dynamical system employing an asymptotic vanishing damping term with a Hessian-analogue damping term in their paper [3]. This master's thesis studies the system with perturbation term g and establishes the same convergence results as those discovered for the unperturbed case in [3] for the perturbed case, with the sole additional assumption that $t \mapsto t\|g(t)\|$ be integrable.
The first chapter introduces some necessary notions and results from monotone operator theory, all of which can be found in [2], as well as a result from [4] that is employed in some of the proofs of the main theorems in the second chapter. This chapter follows a course analogous to [3], introducing the corresponding theorems for the perturbed case and providing their proofs which have been adapted to the perturbed case by the author, as well as introducing or recalling a few necessary results such as the continuous Opial's Lemma and a lemma on the existence and value of the limit of a continuously differentiable function from [1]. Finally, the conclusion comments on the observations made over the course of the thesis and provides an outlook on interesting questions naturally arising from the results and notions discussed.

Contents

1 Preliminaries

Throughout this thesis, $\mathcal{H}$ denotes a real Hilbert space with scalar product $\langle \cdot, \cdot \rangle$, where $\langle x, x \rangle = \|x\|^2$ for any $x \in \mathcal{H}$.
Recall the definition of a (maximally) monotone set-valued operator:

Definition 1.0.1. Let $V : \mathcal{H} \rightrightarrows \mathcal{H}$ be a set-valued operator.

(i) It is called *monotone* if for all $x, y \in \mathcal{H}$ and $x^* \in V(x)$, $y^* \in V(y)$, it holds

$$\langle y^* - x^*, y - x \rangle \geq 0.$$

(ii) The *graph* of V is defined to be

$$G(V) := \{(x, x^*) : x \in \mathcal{H}, x^* \in V(x)\}.$$

(iii) A monotone operator $V : \mathcal{H} \rightrightarrows \mathcal{H}$ is called *maximally monotone* if there exists no monotone operator V' such that $G(V) \subsetneq G(V')$.

Over the course of this thesis, we will use the following result concerning the maximal monotonicity of a continuous operator:

Proposition 1.0.2. *Let $V : \mathcal{H} \to \mathcal{H}$ be a single-valued continuous monotone operator. Then, it is maximally monotone.*

Proof. Let $(x, u) \in \mathcal{H} \times \mathcal{H}$ such that

$$\langle V(y) - u, y - x \rangle \geq 0 \quad \text{for all } y \in \mathcal{H}. \tag{1.1}$$

For all $n \in \mathbb{N}$ and $y_n := x + \frac{1}{n}(u - V(x))$, we have

$$\langle V(y_n) - u, u - V(x) \rangle \geq 0.$$

Because $V(y_n) \to V(y)$ as $n \to +\infty$ from the continuity of V, this means

$$0 \leq \langle V(x) - u, u - V(x) \rangle = -\|V(x) - u\|^2,$$

and hence $V(x) = u$. Thus, for any (x, u) satisfying (1.1), it follows $(x, u) \in G(V)$. This implies that V is maximally monotone: If this was not the case, there would exist a monotone operator $V' : \mathcal{H} \to \mathcal{H}$ such that $G(V) \subsetneq G(V')$. Let $(x, V'(x)) \in G(V') \backslash G(V)$. Then, by montonicity of V', for any $(y, V(y)) \in G(V)$, it holds

$$\langle V(y) - V'(x), y - x \rangle \geq 0,$$

which means that $(x, V'(x)) \in G(V)$ by (1.1), a contradiction. Thus, V must be maximally monotone. $\square$

Finally, we will also need the following result on the sequential closedness of the graph of a maximally monotone operator:

Proposition 1.0.3. *Let V be a maximally monotone operator. Then, its graph is sequentially closed in* $(\mathcal{H}, \tau_{weak}) \times (\mathcal{H}, \tau_{strong})$.

Proof. Let $(z_n, V(z_n)) \to (z, u)$ with respect to $\tau_{\text{weak}} \times \tau_{\text{strong}}$. Then, consider $y \in \mathcal{H}$ and

$$\langle y - z, V(y) - u\rangle = \lim_{n\to+\infty} \langle y - z_n, V(y) - u\rangle = \lim_{n\to+\infty} \langle y - z_n, V(y) - V(z_n)\rangle \geq 0,$$

since for $a_n := y - z_n$, $b_n := V(y) - V(z_m)$, $a := y - z$ and $b := V(y) - u$, it holds that a_n converges weakly to a, denoted as $a_n \rightharpoonup a$, as well as $b_n \to b$ and

$$\begin{aligned} |\langle a_n, b_n\rangle - \langle a, b\rangle| &\leq |\langle a_n, b_n\rangle - \langle a_n, b\rangle| + |\langle a_n, b\rangle - \langle a, b\rangle| \\ &\leq \underbrace{\|a_n\|}_{\text{bounded}} \underbrace{\|b_n - b\|}_{\to 0} + \underbrace{|\langle a_n - a, b\rangle|}_{\to 0} \\ &\to 0 \quad \text{as } n \to +\infty, \end{aligned} \tag{1.2}$$

using that any weakly convergent sequence must be bounded. Now, (1.2) implies by the maximal monotonicity of V that $(z, u) \in G(V)$, i.e. $V(z) = u$. $\square$

Let $V : \mathcal{H} \to \mathcal{H}$ denote a monotone and continuous operator and consider the monotone equation

$$V(z) = 0,$$

with solution set $\mathcal{Z}$. The paper [3] associates the following dynamical system to this equation:

$$\begin{cases} \ddot{z}(t) + \frac{\alpha}{t}\dot{z}(t) + \beta(t)\frac{d}{dt}V(z(t)) + \frac{1}{2}\left(\dot{\beta}(t) + \frac{\alpha}{t}\beta(t)\right)V(z(t)) = 0 \\ z(t_0) = z^0 \text{ and } \dot{z}(t_0) = \dot{z}^0, \end{cases}$$

and proves convergence results along with convergence rates for expressions such as $\langle z(t) - z_*, V(z(t))\rangle$ or $\|V(z(t))\|$, where $z_* \in \mathcal{Z}$. It seems to be a natural question to ask about corresponding results for a *perturbed* version of this system,

$$\begin{cases} \ddot{z} + \frac{\alpha}{t}\dot{z} + \beta(t)\frac{d}{dt}V(z(t)) + \frac{1}{2}\left(\dot{\beta}(t) + \frac{\alpha}{t}\beta(t)\right)V(z(t)) = g(t) \\ z(t_0) = z^0 \text{ and } \dot{z}(t_0) = \dot{z}^0, \end{cases}$$

and the conditions needed to impose on g to obtain similar convergence results. This master's thesis presents such results: The convergence rates remain the same as in the unperturbed case, with the only additional assumption being that g is an L^1-function. The techniques needed are very similar to those used for the unperturbed case in [3].

2 Analysis of the Perturbed System

Consider the perturbed dynamical system

$$\begin{cases} \ddot{z}(t) + \frac{\alpha}{t}\dot{z}(t) + \beta(t)\frac{d}{dt}V(z(t)) + \frac{1}{2}\left(\dot{\beta}(t) + \frac{\alpha}{t}\beta(t)\right)V(z(t)) = g(t) \\ z(t_0) = z^0 \text{ and } \dot{z}(t_0) = \dot{z}^0, \end{cases} \tag{2.1}$$

where $\alpha \geq 2, (z^0, \dot{z}^0) \in \mathcal{H} \times \mathcal{H}$ and $\beta : [t_0, +\infty) \to (0, +\infty)$ is a continuously differentiable and nondecreasing function which satsfies the so-called *growth condition*

$$0 \leq \sup_{t \geq t_0} \frac{t\dot{\beta}(t)}{\beta(t)} \leq \alpha - 2. \tag{2.2}$$

Also, assume that $t \mapsto V(z(t))$ is differentiable on $[t_0, +\infty)$ and let $t \mapsto t\|g(t)\|$ be integrable. Analogously to the analysis of the unperturbed case in [3], it is possible to prove the desired results concerning weak convergence of the trajectory as well as convergence rates for expressions such as $\langle z(t) - z_*, V(z(t))\rangle$ or $\|V(z(t))\|$, where $z_* \in \mathcal{Z}$: First, we define a suitable energy function. Then, we will require an additional step compared to the unperturbed case because the value of the energy function at $t \in [t_0, +\infty)$ for the perturbed case contains the integral

$$\int_t^{+\infty} \langle 4s\lambda(z(s) - z_*) + 2s^2(2\dot{z}(s) + \beta(s)V(z(s))), g(s)\rangle ds,$$

whose well-definedness needs to be proven. Next, we show an estimate on the derivative of the energy function to establish the convergence (rates) of $\langle z(t) - z_*, V(z(t))\rangle$ and $\|V(z(t))\|$ as $t \to +\infty$. Then, we will prove existence and uniqueness of a solution of (2.1) as well as a result on the weak convergence of this solution. As a final result in the analysis of the perturbed system, we show that the convergence rates can be improved under the *strict* growth condition

$$0 \leq \sup_{t \geq t_0} \frac{t\dot{\beta}(t)}{\beta(t)} < \alpha - 2. \tag{2.3}$$

2.1 The Energy Function

Let $z_* \in \mathcal{H}$ such that $V(z_*) = 0$ and $b + 1 \leq \lambda \leq \alpha - 1$, and let $T \geq t_0$. Set

$$\begin{aligned} E_{\lambda,T}(t) := \frac{1}{2}\|2\lambda(z(t) - z_*) + t(2\dot{z}(t) + \beta(t)V(z(t)))\|^2 + 2\lambda(\alpha - 1 - \lambda)\|z(t) - z_*\|^2 \\ + 2\lambda t\beta(t)\langle z(t) - z_*, V(z(t))\rangle + \frac{1}{2}t^2\beta(t)^2\|V(z(t))\|^2 \\ + \int_t^T \langle r(s), g(s)\rangle ds, \end{aligned}$$

where

$$r(t) := 4t\lambda(z(t) - z_*) + 2t^2(2\dot{z}(t) + \beta(t)V(z(t))).$$

First, we consider an estimate of the derivative of $E_{\lambda,T}$.

Lemma 1. *Let $z : [t_0, +\infty) \to \mathcal{H}$ be a solution of (2.1), $z_* \in \mathcal{H}$ such that $V(z_*) = 0$ and $0 \leq \lambda \leq \alpha - 1$. Then for every $t \geq t_0$, it holds*

$$\begin{aligned}\frac{d}{dt}E_{\lambda,T}(t) \leq &- 2\lambda w(t)\langle z(t) - z_*, V(z(t))\rangle + t\beta(t)\left((\alpha - 1 - \lambda)\beta(t) - 2tw(t)\right)\|V(z(t))\|^2 \\ &- (\alpha - 1 - \lambda)t\|2\dot{z}(t) + \beta(t)V(z(t))\|^2,\end{aligned}$$

where

$$w : [t_0, +\infty) \to \mathbb{R}, \quad w(t) = \frac{1}{2}\left((\alpha - 2)\frac{\beta(t)}{t} - \dot{\beta}(t)\right). \tag{2.4}$$

Proof. First, observe from (2.1) that

$$2t\ddot{z}(t) + t\beta(t)\frac{d}{dt}V(z(t)) = -2\alpha\dot{z}(t) - t\beta(t)\frac{d}{dt}V(z(t)) - (t\dot{\beta}(t) + \alpha\beta(t))V(z(t)) + 2tg(t). \tag{2.5}$$

Consider

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\|2\lambda(z(t) - z_*) + t(2\dot{z}(t) + \beta(t)V(z(t)))\|^2\right) \\ &= \Big\langle 2\lambda(z(t) - z_*) + t(2\dot{z}(t) + \beta(t)V(z(t))), \\ &\qquad\qquad 2(\lambda + 1)\dot{z}(t) + (t\dot{\beta}(t) + \beta(t))V(z(t)) + 2t\ddot{z}(t) + t\beta(t)\frac{d}{dt}V(z(t))\Big\rangle \\ &\overset{(2.5)}{=} 4\lambda(\lambda + 1 - \alpha)\langle z(t) - z_*, \dot{z}(t)\rangle + 2\lambda(1 - \alpha)\beta(t)\langle z(t) - z_*, V(z(t))\rangle \\ &\quad - 2\lambda t\beta(t)\left\langle z(t) - z_*, \frac{d}{dt}V(z(t))\right\rangle + 4(\lambda + 1 - \alpha)t\|\dot{z}(t)\|^2 \\ &\quad + 2t(\lambda + 2 - 2\alpha)\beta(t)\langle\dot{z}(t), V(z(t))\rangle - 2t^2\beta(t)\left\langle\dot{z}(t), \frac{d}{dt}V(z(t))\right\rangle \\ &\quad + (1 - \alpha)t\beta(t)^2\|V(z(t))\|^2 - t^2\beta(t)^2\left\langle V(z(t)), \frac{d}{dt}V(z(t))\right\rangle \\ &\quad + \langle\underbrace{4t\lambda(z(t) - z_*) + 2t^2(2\dot{z}(t) + \beta(t)V(z(t)))}_{=r(t)}, g(t)\rangle.\end{aligned} \tag{2.6}$$

Then, it follows from the definition of $E_{\lambda,T}$ that the terms containing g cancel out in the derivative, and thus the proof follows exactly as in [3] for the derivative of the energy function: Differentiating the remaining terms of $E_{\lambda,T}$ gives

$$\begin{aligned}&\frac{d}{dt}\left(2\lambda(\alpha - 1 - \lambda)\|z(t) - z_*\|^2 + 2\lambda t\beta(t)\langle z(t) - z_*, V(z(t))\rangle + \frac{1}{2}t^2\beta(t)^2\|V(z(t))\|^2\right) - \langle r(t), g(t)\rangle \\ &= 4\lambda(\alpha - 1 - \lambda)\langle z(t) - z_*, \dot{z}(t)\rangle + 2\lambda\left(\beta(t) + t\dot{\beta}(t)\right)\langle z(t) - z_*, V(z(t))\rangle \\ &\quad + 2\lambda t\beta(t)\langle\dot{z}(t), V(z(t))\rangle + 2\lambda t\beta(t)\left\langle z(t) - z_*, \frac{d}{dt}V(z(t))\right\rangle \\ &\quad + t\beta(t)\left(\beta(t) + t\dot{\beta}(t)\right)\|V(z(t))\|^2 + t^2\beta(t)^2\left\langle V(z(t)), \frac{d}{dt}V(z(t))\right\rangle - \langle r(t), g(t)\rangle.\end{aligned} \tag{2.7}$$

Now, summing up (2.6) and (2.7) and using (2.4), the definition of w, obtain

$$\begin{aligned}\frac{d}{dt}E_{\lambda,T}(t) = &-2\lambda t w(t)\langle z(t)-z_*, V(z(t))\rangle + 4(\lambda+1-\alpha)t\|\dot{z}(t)\|^2 \\ &+4(\lambda+1-\alpha)t\beta(t)\langle \dot{z}(t), V(z(t))\rangle - 2t^2\beta(t)\left\langle \dot{z}(t), \frac{d}{dt}V(z(t))\right\rangle \\ &-2t^2\beta(t)w(t)\|V(z(t))\|^2.\end{aligned}$$

Now observe the following equality:

$$\begin{aligned}&4(\lambda+1-\alpha)t\|\dot{z}(t)\|^2 + 4(\lambda+1-\alpha)t\beta(t)\langle \dot{z}(t), V(z(t))\rangle - 2t^2\beta(t)w(t)\|V(z(t))\|^2 \\ &\quad = t(\alpha-1-\lambda)\beta(t)^2\|V(z(t))\|^2 - 2t^2\beta(t)w(t)\|V(z(t))\|^2 \\ &\qquad -(\alpha-1-\lambda)t(4\|\dot{z}(t)\|^2 + 4\langle \dot{z}(t), \beta(t)V(z(t))\rangle + \beta(t)^2\|V(z(t))\|^2) \\ &\quad = t\beta(t)((\alpha-1-\lambda)\beta(t) - 2tw(t))\|V(z(t))\|^2 - (\alpha-1-\lambda)t\|2\dot{z}(t)+\beta(t)V(z(t))\|^2\end{aligned}$$

Further, for every $t \geq t_0$, the monotonicity of V implies that

$$\left\langle \dot{z}(t), \frac{d}{dt}V(z(t))\right\rangle \geq 0,$$

which can be seen as follows: Let $(t_n)_{n\geq 0}$ such that $t_n \to t$. Then, we can write

$$\begin{aligned}\left\langle \dot{z}(t), \frac{d}{dt}V(z(t))\right\rangle &= \lim_{n\to+\infty}\left\langle \frac{1}{t_n-t}(z(t_n)-z(t)), \frac{1}{t_n-t}(V(z(t_n))-V(z(t)))\right\rangle \\ &=: \lim_{n\to+\infty}\langle a_n, b_n\rangle.\end{aligned}$$

Since $a_n \to \dot{z}(t) =: a$ as $n \to +\infty$ and $b_n \to \frac{d}{dt}V(z(t)) =: b$ as $n \to +\infty$, use relation (1.2) to obtain $\langle a, b\rangle = \lim_{n\to+\infty}\langle a_n, b_n\rangle$, which means applied to our case

$$\left\langle \dot{z}(t), \frac{d}{dt}V(z(t))\right\rangle = \lim_{n\to+\infty}\frac{1}{(t_n-t)^2}\underbrace{\langle z(t_n)-z(t), V(z(t_n))-V(z(t))\rangle}_{\geq 0 \text{ by monotonicity of } V} \geq 0.$$

Hence, the desired inequality holds. □

Remark 2.1.1. Note that using the growth condition (2.2), for $\lambda \in \left[\sup_{t\geq t_0}\frac{t\dot{\beta}(t)}{\beta(t)}+1, \alpha-1\right]$, this estimate yields that $E_{\lambda,T}$ is nonincreasing: It follows immediately that $w(t) \geq 0$ for all $t \in [t_0, +\infty)$. Thus, it only remains to consider

$$\begin{aligned}0 \overset{!}{\geq} (\alpha-1-\lambda)\beta(t) - 2tw(t) &= (\alpha-1-\lambda)\beta(t) - 2t\frac{1}{2}\left((\alpha-2)\frac{\beta(t)}{t} - \dot{\beta}(t)\right) \\ &= (\alpha-1-\lambda)\beta(t) - (\alpha-2)\beta(t) + t\dot{\beta}(t) = (-\lambda+1)\beta(t) + t\dot{\beta}(t),\end{aligned}$$

which is equivalent to

$$(\lambda-1)\beta(t) \overset{!}{\geq} t\dot{\beta}(t) \Leftrightarrow \lambda \geq 1 + \frac{t\dot{\beta}(t)}{\beta(t)} \quad \forall t \in [t_0, +\infty),$$

and this condition is satisfied for $\lambda \geq b+1$, as claimed.

The second result aims to extend the integral to the interval $[t_0, +\infty)$. This will allow us to define an energy function E_λ independent of the parameter T by performing the extension that is shown to be well-defined in the lemma below. This extended integral version of the energy function will be used for the remainder of this thesis. For the proof of this result, we will use the Gronwall-Bellman Lemma in the following formulation that can also be found in [4]:

Lemma 2 (Gronwall-Bellman Lemma). *Let $m \in L^1([t_0, T])$ such that $m \geq 0$ almost everywhere on (t_0, T) and let c be a nonnegative constant. Suppose that $\phi : [t_0, T] \to \mathbb{R}$ is a continuous function such that for all $t \in [t_0, T]$,*

$$\frac{1}{2}\phi(t)^2 \leq \frac{1}{2}c^2 + \int_{t_0}^{t} m(\tau)\phi(\tau)d\tau$$

holds. Then, for all $t \in [t_0, T]$,

$$|\phi(t)| \leq c + \int_{t_0}^{t} m(\tau)d\tau.$$

Proof. Let $\varepsilon > 0$ and set

$$\psi_\varepsilon(t) := \frac{1}{2}(c+\varepsilon)^2 + \int_{t_0}^{t} m(s)\phi(s)ds = \frac{1}{2}\varepsilon^2 + \underbrace{c\varepsilon}_{\geq 0} + \underbrace{\frac{1}{2}c^2 + \int_{t_0}^{t} m(s)\phi(s)ds}_{\geq \frac{1}{2}\phi(t)^2 \geq 0}. \tag{2.8}$$

Observe that ψ_ε is absolutely continuous. Now,

$$\frac{d}{dt}\psi_\varepsilon(t) = m(t)\phi(t) \quad \text{almost everywhere on } (t_0, T)$$

and

$$\frac{1}{2}\phi(t)^2 \leq \psi_0(t) \leq \psi_\varepsilon(t)$$

for all $t \in [0, T]$. Combining these inequalities yields

$$\frac{d}{dt}\psi_\varepsilon(t) \leq m(t)\sqrt{2}\sqrt{\psi_\varepsilon(t)}.$$

By (2.8), we have

$$\psi_\varepsilon(t) \geq \frac{1}{2}\varepsilon^2 \quad \text{for all } t \in [0, T].$$

Thus, $\sqrt{\psi_\varepsilon}$ must be absolutely continuous almost everywhere since ψ_ε is absolutely continuous and bounded from below by the constant $\frac{1}{2}\varepsilon^2$. Hence, it is differentiable almost everywhere and

$$\frac{d}{dt}\sqrt{\psi_\varepsilon(t)} = \frac{1}{2\sqrt{\psi_\varepsilon}}\frac{d}{dt}\psi_\varepsilon(t) \quad \text{almost everywhere on } (t_0, T).$$

Hence,

$$\frac{d}{dt}\sqrt{\psi_\varepsilon(t)} \leq \frac{1}{\sqrt{2}}m(t) \quad \text{almost everywhere on } (t_0, T),$$

and therefore

$$\sqrt{\psi_\varepsilon(t)} \leq \sqrt{\psi_\varepsilon(t_0)} + \frac{1}{\sqrt{2}} \int_{t_0}^{t} m(s) ds.$$

From this, it follows that

$$|\phi(t)| \leq \sqrt{2}\sqrt{\psi_\varepsilon(t)} \leq \sqrt{2\psi_\varepsilon(t_0)} + \int_{t_0}^{t} m(s) ds \leq c + \varepsilon + \int_{t_0}^{t} m(s) ds \quad \text{for all } t \in [t_0, T], \varepsilon > 0.$$

Letting $\varepsilon \to 0$ in this inequality yields the desired result. □

Lemma 3. *If $t \mapsto t\|g(t)\|$ is integrable and $\lambda \in \left[\sup_{t \geq t_0} \frac{t\dot{\beta}(t)}{\beta(t)} + 1, \alpha - 1\right]$, it holds*

$$\int_{t_0}^{\infty} \langle r(s), g(s) \rangle ds < +\infty,$$

and hence,

$$\begin{aligned} E_\lambda(t) := \frac{1}{2}\|2\lambda(z(t) - z_*) + t(2\dot{z}(t) + \beta(t)V(z(t)))\|^2 + 2\lambda(\alpha - 1 - \lambda)\|z(t) - z_*\|^2 \\ + 2\lambda t\beta(t)\langle z(t) - z_*, V(z(t))\rangle + \frac{1}{2}t^2\beta(t)^2\|V(z(t))\|^2 \\ + \int_t^{\infty} \langle r(s), g(s) \rangle ds \end{aligned} \tag{2.9}$$

is well-defined.

Proof. Fix a real number $T > t_0$ and a real number $t \in [t_0, T]$ and recall that by Remark 2.1.1, the energy function is nonincreasing, i.e.

$$\begin{aligned} &\frac{1}{2}\|2\lambda(z(t) - z_*) + t(2\dot{z}(t) + \beta(t)V(z(t)))\|^2 + 2\lambda t\beta(t)\langle z(t) - z_*, V(z(t))\rangle \\ &\quad + \frac{1}{2}t^2\beta(t)^2\|V(z(t))\|^2 + \int_t^T \langle r(s), g(s)\rangle ds \\ &\quad \leq \frac{1}{2}\|2\lambda(z(t_0) - z_*) + t_0(2\dot{z}(t_0) + \beta(t_0)V(z(t_0)))\|^2 \\ &\qquad + 2\lambda t_0\beta(t_0)\langle z(t_0) - z_*, V(z(t_0))\rangle \\ &\qquad + \frac{1}{2}t_0^2\beta(t_0)^2\|V(z(t_0))\|^2 + \int_{t_0}^T \langle r(s), g(s)\rangle ds. \end{aligned}$$

Each summand of the left-hand side is ≥ 0, which implies

$$\frac{1}{8t^2}\|r(t)\|^2 \leq C + \int_{t_0}^{t} \langle r(s), g(s)\rangle ds \leq C + \int_{t_0}^{t} \left\|\frac{1}{2s}r(s)\right\| 2s\|g(s)\| ds, \tag{2.10}$$

where the constant C is given by

$$\begin{aligned} C := \frac{1}{2}\|2\lambda(z(t_0) - z_*) + t_0(2\dot{z}(t_0) + \beta(t_0)V(z(t_0)))\|^2 + 2\lambda t_0\beta(t_0)\langle z(t_0) - z_*, V(z(t_0))\rangle \\ + \frac{1}{2}t_0^2\beta(t_0)^2\|V(z(t_0))\|^2, \end{aligned}$$

meaning that there exists a bound on $\frac{1}{2t}\|r(t)\|$ that holds for all $t \in [t_0, T]$. Now let $t \geq t_0$, choose $T > t \geq 0$ and perform the same calculation as above to obtain the estimate for all $s \in [t_0, T]$, hence in particular for t. Thus, the estimate does not depend on T and holds for all $t \in [t_0, +\infty)$. Using the Gronwall-Bellman Lemma 2, we obtain

$$\|r(t)\| \leq \sqrt{2C} + \int_{t_0}^{t} 2s\|g(s)\|ds \leq \sqrt{2C} + 2\int_{t_0}^{\infty} s\|g(s)\|ds.$$

If g is integrable, this bound is finite and does not depend on t, and thus it follows

$$\begin{aligned}\left|\int_t^T \langle r(s), g(s)\rangle ds\right| &\leq \int_t^T |\langle r(s), g(s)\rangle| ds \leq \int_t^T \|r(s)\|\|g(s)\|ds \\ &\leq \int_{t_0}^{\infty} \underbrace{\|r(s)\|}_{\leq\sqrt{2C}+2\int_{t_0}^{\infty} s\|g(s)\|ds} \|g(s)\|ds \leq \left(\sqrt{2C} + 2\int_{t_0}^{\infty} s\|g(s)\|ds\right)\int_{t_0}^{\infty} \|g(s)\|ds,\end{aligned}$$

which is a constant estimate that depends neither on t nor on T. Thus, for any $t \geq t_0$, the integral

$$\int_t^{\infty} |\langle r(s), g(s)\rangle| ds$$

is well-defined. Consider $t_0 \leq t \leq T_1 < T_2 < \infty$ and observe

$$\begin{aligned}\left|\int_t^{T_2} \langle r(s), g(s)\rangle ds - \int_t^{T_1} \langle r(s), g(s)\rangle ds\right| &= \left|\int_{T_1}^{T_2} \langle r(s), g(s)\rangle ds\right| \leq \int_{T_1}^{T_2} |\langle r(s), g(s)\rangle| ds \\ &\leq \int_{T_1}^{T_2} \|r(s)\|\|g(s)\|ds \leq \left(\sqrt{2C} + 2\int_{t_0}^{\infty} s\|g(s)\|ds\right)\int_{T_1}^{T_2} \|g(s)\|ds,\end{aligned}$$

so by integrability of $t \mapsto t\|g(t)\|$, we know that $\int_t^{T_n} \langle r(s), g(s)\rangle ds$ is a Cauchy sequence for any $t \geq t_0$, $(T_n)_{n\in\mathbb{N}}$ such that $T_n \to +\infty$ as $n \to +\infty$ and $T_n \geq t$ for all $n \in \mathbb{N}$. Hence, it is convergent by completeness of $\mathbb{R}$. Thus, the integral

$$\int_t^{\infty} \langle r(s), g(s)\rangle ds$$

is well-defined and hence

$$\begin{aligned}E_\lambda(t) := \frac{1}{2}\|2\lambda(z(t) - z_*) + t(2\dot{z}(t) + \beta(t)V(z(t)))\|^2 + 2\lambda(\alpha - 1 - \lambda)\|z(t) - z_*\|^2 \\ + 2\lambda t\beta(t)\langle z(t) - z_*, V(z(t))\rangle + \frac{1}{2}t^2\beta(t)^2\|V(z(t))\|^2 \\ + \int_t^{\infty} \langle r(s), g(s)\rangle ds\end{aligned}$$

is well-defined. □

Remark 2.1.2. Contrary to the unperturbed case, the well-definedness of the energy function uses the growth condition (2.2): Since this proof only establishes well-definedness of E_λ in the case $\lambda \in \left[\sup_{t\geq t_0} \frac{t\dot{\beta}(t)}{\beta(t)} + 1, \alpha - 1\right]$, this interval must be assumed to be nonempty, which is true if and only if the growth condition holds. This also means that going forward, for every result involving the energy function, we will implicitly use the growth condition.

Remark 2.1.3. Note that knowing that E_λ is well-defined for any $\lambda \in \left[\sup_{t\geq t_0} \frac{t\dot\beta(t)}{\beta(t)} + 1, \alpha - 1\right]$, we can perform exactly the same proof as in Lemma 1 to obtain the estimate of the derivative of the energy function formulated with E_λ instead of $E_{\lambda,T}$ with the only additional assumption being that $\lambda \in \left[\sup_{t\geq t_0} \frac{t\dot\beta(t)}{\beta(t)} + 1, \alpha - 1\right]$. This interval is nonempty as a result of the growth condition (2.2) which is assumed to hold true throughout this thesis, in contrast to the strict growth condition (2.3) that is only required in some theorems, for example when establishing the improved o-convergence rates for $\|V(z(t))\|$ and $\langle z(t) - z_*, V(z(t))\rangle$.

Now that we have established the well-definedness of E_λ as well as the analogue of Lemma 1 for E_λ, we move to the (asymptotic) analysis of the trajectory $z(t)$.

2.2 Analysis of the Trajectory

2.2.1 Existence of the Trajectory

This subsection is devoted to proving the existence and uniqueness of solutions of the dynamical system (2.1) in a very general setting.

Theorem 1. *Let $\alpha > 2$ and assume that $V : \mathcal{H} \to \mathcal{H}$ is continuously differentiable, $\beta : [t_0, \infty) \to (0, +\infty)$ is a continuously differentiable and nondecreasing function which satisfies (2.3), g such that $t \mapsto tg(t)$ is a Lipschitz continuous function with $t \mapsto t\|g(t)\|$ being integrable, and that V and $\dot\beta$ are Lipschitz continuous on bounded sets. Then for every initial condition $z(t_0) = z^0 \in \mathcal{H}$ and $\dot z(t_0) = \dot z^0 \in \mathcal{H}$ the dynamical system (2.1) has a unique global twice continuously differentiable solution $z : [t_0, \infty) \to \mathcal{H}$.*

Proof. First, rewrite the system as a first-order ODE:

$$\begin{cases} \dot z(t) = \frac{1}{2t}u(t) - \frac{1}{t}(\alpha - 1)z(t) - \beta(t)V(z(t)) \\ \dot u(t) = (t\dot\beta(t) + (2-\alpha)\beta(t))V(z(t)) + 2tg(t) \\ (z(t_0), u(t_0)) = (z^0, 2(\alpha-1)z^0 + 2t_0\dot z^0 + 2t_0\beta(t_0)V(z^0) + 2t_0 g(t_0)), \end{cases} \tag{2.11}$$

where we set $u(t) := 2(\alpha - 1)z(t) + 2t\dot z(t) + 2t\beta(t)V(z(t))$. Hence,

$$\begin{aligned} \dot u(t) &= 2\alpha \dot z(t) + 2t\ddot z(t) + 2(t\dot\beta(t) + \beta(t))V(z(t)) + 2t\beta(t)\frac{d}{dt}V(z(t)) \\ &= (t\dot\beta(t) + (2-\alpha)\beta(t))V(z(t)) + 2tg(t). \end{aligned}$$

Next, define

$$\begin{aligned} &G : [t_0, +\infty) \times \mathcal{H} \times \mathcal{H} \to \mathcal{H} \times \mathcal{H}, \\ &G(t, \zeta, \mu) := \left((t\dot\beta(t) + (2-\alpha)\beta(t))V(\zeta) + 2tg(t), \frac{1}{2t}\mu - \frac{1}{t}(\alpha-1)\zeta + \beta(t)V(\zeta) \right) \end{aligned}$$

and rewrite (2.11) to read

$$\begin{cases} (\dot u(t), \dot z(t)) = G(t, z(t), u(t)) \\ (z(t_0), u(t_0)) = (z^0, 2(\alpha-1)z^0 + 2t_0\dot z^0 + 2t_0\beta(t_0)V(z^0)) \end{cases}$$

In the next step, we show that G is Lipschitz continuous on bounded sets. To this end, define $t_1 < t_2 \in [t_0, \infty)$ and $\delta > 0$. Because β is continuously differentiable, it is in particular Lipschitz continuous on $[t_1, t_2]$. Denote its Lipschitz constant by $\ell_\beta > 0$. Further, V and $\dot\beta$ are Lipschitz continuous on $B(0,\delta)$ and $[t_1, t_2]$, respectively. Denote their Lipschitz constants by L and ℓ_d. Finally, $t \mapsto tg(t)$ is also Lipschitz continuous by assumption. Denote its Lipschitz constant by L_g. Define constants

$$\begin{aligned}
L_t &:= \left(\sup_{s\in[t_1,t_2]} |\dot\beta(s)| + t_2\ell_d + (\alpha-2)\ell_\beta \right)(L\delta + \|V(0)\|) \\
&\qquad\qquad + \left(\frac{\delta(2\alpha-1)}{2t_1^2} + \ell_\beta(L\delta + \|V(0)\|) \right) + 2L_g > 0 \\
L_\zeta &:= (\alpha-2)\beta(t_2)L + \left(\frac{\alpha-1}{t_1} + L\beta(t_2) \right) > 0 \\
L_\mu &:= \frac{1}{2t_1} > 0 \\
L_G &:= \sqrt{L_t^2 + L_\zeta^2 + L_\mu^2}.
\end{aligned}$$

Calculations completely analogous to the unperturbed case in [3] yield Lipschitz continuity of G with Lipschitz constant L_G: First, observe that for every $t \in [t_1, t_2]$, the strict growth condition (2.3) along with the fact that β is nonincreasing implies that

$$(2-\alpha)\beta(t_2) \le (2-\alpha)\beta(t) \le t\dot\beta(t) + (2-\alpha)\beta(t) \le 0.$$

Let $(t,\zeta,\mu), (\tilde t, \tilde\zeta, \tilde\mu) \in [t_1,t_2] \times B(0,\delta) \times B(0,\delta)$. By the triangle inequality,

$$\begin{aligned}
&\left\| \left(t\dot\beta(t) + (2-\alpha)\beta(t) \right) V(\zeta) - \left(\tilde t\dot\beta(\tilde t) + (2-\alpha)\beta(\tilde t) \right) V\left(\tilde\zeta\right) \right\| \\
&\le \left| t\dot\beta(t) + (2-\alpha)\beta(t) \right| \left\| V(\zeta) - V\left(\tilde\zeta\right) \right\| \\
&\qquad + \left| \left(t\dot\beta(t) + (2-\alpha)\beta(t) \right) - \left(\tilde t\dot\beta(\tilde t) + (2-\alpha)\beta(\tilde t) \right) \right| \left\| V\left(\tilde\zeta\right) \right\| \\
&\le (\alpha-2)\beta(t_2)L \left\| \zeta - \tilde\zeta \right\| + \left(\left|\dot\beta(t)\right| \left|t - \tilde t\right| + \tilde t \left| \dot\beta(t) - \dot\beta\left(\tilde t\right) \right| + (\alpha-2)\left|\beta(t) - \beta\left(\tilde t\right)\right| \right) \left\| V\left(\tilde\zeta\right) \right\| \\
&\le (\alpha-2)\beta(t_2)L \left\| \zeta - \tilde\zeta \right\| + \left(\sup_{\tau\in[t_1,t_2]} \left|\dot\beta(\tau)\right| + t_2\ell_d + (\alpha-2)\ell_\beta \right) (L\delta + \|V(0)\|) \left| t - \tilde t \right|.
\end{aligned} \tag{2.12}$$

For the second component of the value of G, it holds

$$\begin{aligned}
&\left\| \frac{1}{2t}\mu - \frac{1}{t}(\alpha-1)\zeta - \beta(t)V(\zeta) - \frac{1}{2\tilde t}\tilde\mu + \frac{1}{\tilde t}(\alpha-1)\tilde\zeta + \beta\left(\tilde t\right) V\left(\tilde\zeta\right) \right\| \\
&\le \frac{1}{2}\left\| \frac{1}{t}\mu - \frac{1}{\tilde t}\tilde\mu \right\| + (\alpha-1)\left\| \frac{1}{t}\zeta - \frac{1}{\tilde t}\tilde\zeta \right\| + \left\| \beta(t)V(\zeta) - \beta\left(\tilde t\right)V\left(\tilde\zeta\right) \right\| \\
&\le \frac{1}{2}\left(\|\tilde\mu\| + 2(\alpha-1)\left\|\tilde\zeta\right\| \right) \left| \frac{1}{t} - \frac{1}{\tilde t} \right| + \frac{1}{2t}\|\mu - \tilde\mu\| + \frac{1}{t}(\alpha-1)\left\|\zeta - \tilde\zeta\right\| \\
&\qquad + \beta(t)\left\| V(\zeta) - V\left(\tilde\zeta\right) \right\| + \|V(\zeta)\| \left|\beta(t) - \beta\left(\tilde t\right)\right| \\
&\le \left(\frac{1}{2t_1^2}\delta(2\alpha-1) + \ell_\beta(L\cdot\delta + \|V(0)\|) \right) \left|t - \tilde t\right| + \frac{1}{2t_1}\|\mu - \tilde\mu\| + \left(\frac{1}{t_1}(\alpha-1) + L\beta(t_2) \right) \left\| \zeta - \tilde\zeta \right\|.
\end{aligned} \tag{2.13}$$

Combining the equations (2.12) and (2.13) yields the claimed Lipschitz continuity of G with Lipschitz constant L_G.
The argument for existence and uniqueness only uses the derivative of E_λ and thus can be copied from [3]: Using the local existence and uniqueness theorem, we know that there exists a unique continuous differentiable solution $(z, u) \in \mathcal{H} \times \mathcal{H}$ of (2.11) defined on a maximal interval $[t_0, T_{\max})$ where $0 < t_0 < T_{\max} \leq +\infty$. Also, either

$$T_{\max} = +\infty \qquad \text{or} \qquad \lim_{t \to T_{\max}} \|(z(t), u(t))\| = +\infty.$$

We will show that indeed, it holds $T_{\max} = +\infty$. From

$$\begin{aligned}\frac{d}{dt} E_{\alpha-1,T}(t) &\leq \underbrace{-2(\alpha-1)w(t)\langle z(t) - z_*, V(z(t))\rangle}_{\leq 0} - 2t^2\beta(t)w(t)\|V(z(t))\|^2 \\ &\leq -2t^2\beta(t)w(t)\|V(z(t))\|^2,\end{aligned}$$

deduce using integration that for $t_0 \leq t < T_{\max}$,

$$E_{\alpha-1}(t) + 2\int_{t_0}^{t} \tau^2\beta(\tau)w(\tau)\|V(z(t))\|^2 d\tau \leq E_{\alpha-1}(t_0) < +\infty.$$

From the definition of the energy function (2.9), observe that the functions

$$\begin{aligned}&t \mapsto \frac{1}{2t} r(t) + 2(\alpha-1)z_* = u(t) - t\beta(t)V(z(t)) \quad \text{and} \\ &t \mapsto t\beta(t)V(z(t))\end{aligned}$$

are bounded on $[t_0, T_{\max})$, which implies boundedness of $u(t)$ on $[t_0, T_{\max})$. By the strict growth condition, there exists $0 < \varepsilon \leq \alpha - 2$ such that

$$\sup_{t \geq t_0} \frac{t\dot{\beta}(t)}{\beta(t)} = \alpha - 2 - \varepsilon < \alpha - 2. \tag{2.14}$$

Thus,

$$w(t) = -\frac{1}{2}\dot{\beta}(t) + \frac{1}{2}(\alpha-2)\frac{\beta(t)}{t} \geq \frac{\varepsilon}{2}\frac{\beta(t)}{t} > 0, \tag{2.15}$$

which implies

$$\begin{aligned}+\infty > E_{\alpha-1}(t_0) &\geq E_{\alpha-1}(t) + 2\int_{t_0}^{t} \tau^2\beta(\tau)w(\tau)\|V(z(t))\|^2 d\tau \\ &\geq \lim_{t \to T_{\max}} E_{\alpha-1}(t) + 2\int_{t_0}^{T_{\max}} \tau^2\beta(\tau)\frac{\varepsilon}{2}\frac{\beta(\tau)}{\tau}\|V(z(\tau))\|^2 d\tau \\ &\geq \lim_{t \to T_{\max}} E_{\alpha-1}(t) + \varepsilon\int_{t_0}^{T_{\max}} \tau\beta(\tau)^2\|V(z(\tau))\|^2 d\tau.\end{aligned}$$

This can be rewritten to read

$$\int_{t_0}^{T_{\max}} \tau\beta(\tau)^2\|V(z(\tau))\|^2 d\tau \leq \frac{1}{\varepsilon}\left(E_{\alpha-1}(t_0) - \lim_{t \to T_{\max}} E_{\alpha-1}(t)\right) < +\infty,$$

for suitable $\varepsilon > 0$. Now for $0 < \lambda < \alpha - 1$, it holds

$$\frac{d}{dt}E_\lambda(t) \leq 2(\alpha - 1 - \lambda)t\beta(t)^2\|V(z(t))\|^2,$$

which follows from Lemma 1: First, recall the inequality $\|a + b\|^2 \leq 2\|a\|^2 + 2\|b\|^2$ which holds for any elements a, b of the Hilbert space, and apply it to obtain

$$4\|\dot{z}(t)\|^2 = \|2\dot{z}(t)\|^2 \leq 2\|2\dot{z}(t) + \beta(t)V(z(t))\|^2 + 2\|\beta(t)V(z(t))\|^2,$$

which can equivalently be rewritten to read

$$-\|2\dot{z}(t) + \beta(t)V(z(t))\|^2 \leq -2\|\dot{z}(t)\|^2 + \beta(t)^2\|V(z(t))\|^2.$$

From this, using the estimate from Lemma 1, for every $\lambda \in \left[\sup_{t\geq t_0} \frac{t\dot{\beta}(t)}{\beta(t)} + 1, \alpha - 1\right]$ and every $t \geq t_0$, it follows

$$\begin{aligned}
\frac{d}{dt}E_\lambda(t) &\leq (\alpha - 1 - \lambda)t\beta(t)^2\|V(z(t))\|^2 - (\alpha - 1 - \lambda)t\|2\dot{z}(t) + \beta(t)V(z(t))\|^2 \\
&\leq (\alpha - 1 - \lambda)t\beta(t)^2\|V(z(t))\|^2 + (\alpha - 1 - \lambda)t(-2\|\dot{z}(t)\|^2 + \|\beta(t)V(z(t))\|^2) \\
&\leq 2(\alpha - 1 - \lambda)t\beta(t)^2\|V(z(t))\|^2 - 2(\alpha - 1 - \lambda)t\|\dot{z}(t)\|^2 \\
&\leq 2(\alpha - 1 - \lambda)t\beta(t)^2\|V(z(t))\|^2.
\end{aligned} \tag{2.16}$$

Integrating this inequality yields for $t_0 \leq t < T_{\max}$

$$\begin{aligned}
2\lambda(\alpha - 1 - \lambda)\|z(t) - z_*\|^2 &\leq E_\lambda(t) - \int_t^{+\infty} \langle r(s), g(s)\rangle ds \\
&\leq E_\lambda(t_0) + 2(\alpha - 1 - \lambda)\int_{t_0}^t \tau\beta(\tau)^2\|V(z(\tau))\|^2 d\tau + \underbrace{\int_{t_0}^{+\infty} \|r(s)\|\|g(s)\|ds}_{=:C<+\infty \text{ by Lemma 3}} \\
&\leq E_\lambda(t_0) + 2(\alpha - 1 - \lambda)\int_{t_0}^{T_{\max}} \tau\beta(\tau)^2\|V(z(\tau))\|^2 d\tau + C \\
&\leq E_\lambda(t_0) + \frac{2}{\varepsilon}(\alpha - 1 - \lambda)\left(E_{\alpha-1}(t_0) - \lim_{t\to T_{\max}} E_{\alpha-1}(t)\right) + C \\
&< +\infty.
\end{aligned}$$

Combining the fact that $t \mapsto u(t)$ is bounded with the above inequality yields that

$$\lim_{t\to T_{\max}} \|(z(t), u(t))\| < +\infty,$$

hence, $T_{\max} = +\infty$, as desired. This finishes the proof. □

2.2.2 Asymptotic Analysis of the Trajectory

As the first result for this section, we show a lemma that will prove to be essential to the convergence of $E_\lambda(t)$ for $t \to +\infty$.

Lemma 4. *Suppose $F : [0, +\infty) \to \mathbb{R}$ is locally absolutely continuous, bounded from below, and there exists $G \in L^1([0, +\infty))$ such that for almost all t,*

$$\frac{d}{dt}F(t) \leq G(t). \tag{2.17}$$

Then there exists $\lim_{t\to+\infty} F(t) \in \mathbb{R}$.

Proof. Integrating the inequality (2.17) yields for all $0 < s < t < +\infty$

$$F(t) - F(s) \leq \int_s^t G(s)ds,$$

which can equivalently be rewritten as

$$F(t) - \int_0^t G(\tau)d\tau \leq F(s) - \int_0^s G(\tau)d\tau,$$

i.e., the function $h : t \mapsto F(t) - \int_0^t G(\tau)d\tau$ is nonincreasing. Because $G \in L^1([0, +\infty))$ and F is bounded from below, h is bounded from below, as well. Hence, the limit $\lim_{t\to+\infty} F(t) - \int_0^t G(\tau)d\tau$ exists in $\mathbb{R}$ and using again that G is an L^1-function, it follows that there exists $\lim_{t\to+\infty} F(t)$, as claimed. □

The next result yields not only convergence of $V(z(t))$ and $\langle z(t) - z_*, V(z(t))\rangle$ as $t \to +\infty$, but also gives convergence rates with the only assumption on g being integrability of $t \mapsto t\|g(t)\|$.

Theorem 2. *Let $z : [t_0, +\infty) \to \mathcal{H}$ be a solution of* (2.1)*, $t \mapsto t\|g(t)\|$ be integrable and z_* such that $V(z_*) = 0$. For every $t \geq t_0$ it holds*

$$\begin{aligned} 0 \leq \|V(z(t))\| &\leq \sqrt{2E_{\alpha-1}(t_0)}\frac{1}{t\beta(t)}, \\ 0 \leq \langle z(t) - z_*, V(z(t))\rangle &\leq \frac{E_{\alpha-1}(t_0)}{2(\alpha-1)}\frac{1}{t\beta(t)}, \end{aligned} \tag{2.18}$$

and the following results are true

$$\begin{aligned} \int_{t_0}^{+\infty} tw(t)\langle z(t) - z_*, V(z(t))\rangle dt < +\infty, \\ \int_{t_0}^{+\infty} t^2\beta(t)w(t)\|V(z(t))\|^2 dt < +\infty. \end{aligned} \tag{2.19}$$

If we assume in addition that the strict growth condition (2.3) *holds, the trajectory $t \mapsto z(t)$ is bounded and it holds*

$$\int_{t_0}^{+\infty} t\|\dot{z}(t)\|^2 dt < +\infty, \tag{2.20}$$

and $\lim_{t\to+\infty} E_\lambda(t) \in \mathbb{R}$ for every $\lambda \in \left[\sup_{t\geq t_0} \frac{t\dot{\beta}(t)}{\beta(t)} + 1, \alpha - 1\right]$.

Proof. First, choose $\lambda := \alpha - 1$ in E_λ. Then, the estimate from Lemma 1 gives that $E_{\alpha-1}$ is nonincreasing. Thus,

$$\begin{aligned} \|V(z(t))\|^2 &\leq 2\frac{1}{t^2\beta(t)^2}\underbrace{E_{\alpha-1}(t)}_{\leq E_{\alpha-1}(t_0)} - 2\frac{1}{t^2\beta(t)^2}\int_t^\infty \langle r(s), g(s)\rangle ds \\ &\leq 2\frac{1}{t^2\beta(t)^2}\left(E_{\alpha-1}(t_0) + \underbrace{\left|\int_{t_0}^\infty \langle r(s), g(s)\rangle ds\right|}_{\text{stays bounded as in the proof of Lemma 3}}\right), \end{aligned}$$

which means that the right-hand side actually consists of a constant divided by $t^2\beta(t)^2$. Hence,

$$\|V(z(t))\| \leq \frac{D}{t\beta(t)}$$

for some suitable constant D. Similarly,

$$\begin{aligned} 0 \leq \langle z(t) - z_*, V(z(t))\rangle &\leq \frac{E_{\alpha-1}(t_0)}{2(\alpha-1)} \frac{1}{t\beta(t)} - \frac{1}{2(\alpha-1)t\beta(t)} \int_t^{\infty} \langle r(s), g(s)\rangle ds \\ &\leq \frac{1}{t\beta(t)} \left(\frac{1}{2(\alpha-1)} \left(E_{\alpha-1}(t_0) + \left| \int_{t_0}^{\infty} \langle r(s), g(s)\rangle ds \right| \right) \right), \end{aligned}$$

which implies that for a suitable constant $\tilde{D}$, we have

$$0 \leq \langle z(t) - z_*, V(z(t))\rangle \leq \frac{1}{t\beta(t)} \tilde{D},$$

which finishes the proof of the inequalities (2.18). The proof of the statements (2.19) only relies on the derivative of $E_{\alpha-1}(t)$, which means that the proof for these inequalities is identical to the one in [3] since the derivatives of the energy functions for the perturbed and unperturbed cases coincide: Consider the inequality

$$\frac{d}{dt} E_{\alpha-1}(t) \leq \underbrace{-2(\alpha-1)w(t)\langle z(t) - z_*, V(z(t))\rangle}_{(*)} \underbrace{-2t^2\beta(t)w(t)\|V(z(t))\|^2}_{(**)} \leq 0.$$

Then, observe that the expressions $(*)$ and $(**)$ are both ≤ 0, hence bounded from below by $\frac{d}{dt}E_{\alpha-1}(t)$, and integrate those inequalities to obtain

$$\begin{aligned} &\int_{t_0}^{+\infty} w(t)\langle z(t) - z_*, V(z(t))\rangle \leq \lim_{t\to+\infty} E_{\alpha-1}(t) - E_{\alpha-1}(t_0) < +\infty \quad \text{and} \\ &\int_{t_0}^{+\infty} t^2\beta(t)w(t)\|V(z(t))\|^2 \leq \lim_{t\to+\infty} E_{\alpha-1}(t) - E_{\alpha-1}(t_0) < +\infty, \end{aligned}$$

proving the inequalities (2.19).
For the boundedness of $z(t)$, observe that

$$\begin{aligned} \|z(t) - z_*\| &\leq \frac{1}{2\lambda(\alpha-1-\lambda)} \left(E_\lambda(t) - \int_t^{\infty} \langle r(s), g(s)\rangle ds \right) \\ &\leq \frac{1}{2\lambda(\alpha-1-\lambda)} \left(E_\lambda(t_0) + \int_{t_0}^{\infty} \|r(s)\|\|g(s)\| ds \right), \end{aligned}$$

which implies the boundedness of the trajectory. Statement (2.20) and the existence of $\lim_{t\to+\infty} E_\lambda(t) \in \mathbb{R}$ for $\lambda \in \left(\sup_{t\geq t_0} \frac{t\dot{\beta}(t)}{\beta(t)} + 1, \alpha - 1 \right)$ only use the derivative of E_λ and can hence be proven exactly as in [3]: Assume that the strict growth condition (2.3) holds. Then, as in Theorem 1, there exists $0 < \varepsilon \leq \alpha - 2$ such that the inequality (2.14) holds. From this, it follows using (2.15) that

$$t^2\beta(t)w(t) \geq \frac{\varepsilon}{2} t\beta(t)^2 \quad \forall t \geq t_0$$

holds. Using the second line of (2.19), it follows

$$\int_{t_0}^{+\infty} t\beta(t)^2\|V(z(t))\|^2 dt < +\infty. \tag{2.21}$$

Recall the estimate (2.16) from the proof of Theorem 1:

$$\begin{aligned}
\frac{d}{dt}E_\lambda(t) &\leq (\alpha-1-\lambda)t\beta(t)^2\|V(z(t))\|^2 - (\alpha-1-\lambda)t\|2\dot{z}(t)+\beta(t)V(z(t))\|^2 \\
&\leq (\alpha-1-\lambda)t\beta(t)^2\|V(z(t))\|^2 + (\alpha-1-\lambda)t(-2\|\dot{z}(t)\|^2 + \|\beta(t)V(z(t))\|^2) \\
&\leq 2(\alpha-1-\lambda)t\beta(t)^2\|V(z(t))\|^2 - 2(\alpha-1-\lambda)t\|\dot{z}(t)\|^2 \\
&\leq 2(\alpha-1-\lambda)t\beta(t)^2\|V(z(t))\|^2.
\end{aligned}$$

Then, integrating the penultimate line of (2.16) for $\lambda \in \left[\sup_{t\geq t_0}\frac{t\dot{\beta}(t)}{\beta(t)}+1, \alpha-1\right)$ fixed yields

$$\lim_{t\to+\infty} E_\lambda(t) - E_\lambda(t_0) \leq 2(\alpha-1-\lambda)\int_{t_0}^{+\infty} t\beta(t)^2\|V(z(t))\|^2 dt - 2(\alpha-1-\lambda)\int_{t_0}^{+\infty} t\|\dot{z}(t)\|^2 dt,$$

which is equivalent to

$$\begin{aligned}
2(\alpha-1-\lambda)\int_{t_0}^{+\infty} t\|\dot{z}(t)\|^2 dt \leq 2(\alpha-1-\lambda)\int_{t_0}^{+\infty} & t\beta(t)^2\|V(z(t))\|^2 dt \\
&+ E_\lambda(t_0) - \lim_{t\to+\infty} E_\lambda(t) < +\infty,
\end{aligned}$$

thus proving the claim (2.20). It remains to consider the existence of the limit $\lim_{t\to+\infty} E_\lambda(t)$. To accomplish this for any $\lambda \in \left[\sup_{t\geq t_0}\frac{t\dot{\beta}(t)}{\beta(t)}+1, \alpha-1\right]$, use Lemma 4 with $F(t) := E_\lambda(t)$ and $G(t) := 2(\alpha-1-\lambda)t\beta(t)^2\|V(z(t))\|^2$. □

The next theorem provides a result concerning the convergence of the trajectories of (2.1). Its proof will use Opial's Lemma in continuous form which we recall for convenience:

Lemma 5 (Opial's Lemma)**.** *Let $\mathcal{S}$ be a nonempty subset of $\mathcal{H}$ and $z : [t_0, \infty) \to \mathcal{H}$. Assume that*

(i) for every $z_* \in \mathcal{S}$, the limit $\lim_{t\to+\infty}\|z(t)-z_*\|$ exists;

(ii) every weak sequential cluster point of the trajectory $z(t)$ as $t \to +\infty$ belongs to $\mathcal{S}$.

Then, $z(t)$ converges weakly to a point in $\mathcal{S}$ as $t \to +\infty$.

We will also use the following result that can be found in [1]:

Lemma 6. *Let $a > 0$ and $q : [t_0, +\infty) \to \mathcal{H}$ be a continuously differentiable function such that*

$$\lim_{t\to+\infty}\left(q(t) + \frac{t}{a}\dot{q}(t)\right) = \ell \in \mathcal{H}.$$

Then it holds

$$\lim_{t\to+\infty} q(t) = \ell.$$

Proof. Let $v := q - \ell$ and $\varepsilon > 0$. There exists $T \geq t_0$ such that for $t \geq T$,

$$\left\|v(t) + \frac{t}{\alpha}\dot{v}(t)\right\| < \varepsilon.$$

Multiplying by $\alpha t^{\alpha-1}$ yields

$$\left\|\alpha t^{\alpha-1}v(t)+t^{\alpha}\dot{v}(t)\right\|=\left\|\frac{d}{dt}\left(t^{\alpha}v(t)\right)\right\|<\varepsilon\alpha t^{\alpha-1}.$$

Next, we integrate between T and $t\geq T$ to obtain

$$\|t^{\alpha}v(t)-T^{\alpha}v(T)\|=\left\|\int_T^t\frac{d}{ds}(s^{\alpha}v(s))ds\right\|\leq\int_T^t\left\|\frac{d}{ds}(s^{\alpha}v(s))\right\|ds<\varepsilon(t^{\alpha}-T^{\alpha}).$$

Thus,

$$\|v(t)\|\leq\left(\frac{T}{t}\right)^{\alpha}\|v(T)\|+\varepsilon\left(1-\left(\frac{T}{t}\right)^{\alpha}\right),$$

which implies that

$$\limsup_{t\to+\infty}\|v(t)\|\leq\varepsilon.$$

Since $\varepsilon>0$ was arbitrary and $\|v(t)\|\geq 0$ for all $t\in[t_0,+\infty)$, it follows that

$$\lim_{t\to+\infty}\|q(t)-\ell\|=\lim_{t\to+\infty}\|v(t)\|=0,$$

which concludes the proof. □

Now, having established all the necessary results, we can show

Theorem 3. *Let $\alpha>2$ and $z:[t_0,+\infty)\to\mathcal{H}$ be a solution of (2.1) and assume that β : $[t_0,+\infty)$ satisfies the strict growth condition (2.3) and that $t\mapsto t\|g(t)\|$ is integrable. Then $z(t)$ converges weakly to a solution of the equation $V(z)=0$ as $t\to+\infty$.*

Proof. Consider $\sup_{t\geq t_0}\frac{t\dot{\beta}(t)}{\beta(t)}+1\leq\lambda_1<\lambda_2\leq\alpha-1$ and $z_*\in\mathcal{Z}$. Then, by definition of E_λ, it follows

$$\begin{aligned}E_{\lambda_2}(t)-E_{\lambda_1}(t)=4(\lambda_2-\lambda_1)&\left(t\langle z(t)-z_*,\dot{z}(t)+\beta(t)V(z(t))\rangle+\frac{1}{2}(\alpha-1)\|z(t)-z_*\|^2\right)\\&+\int_t^{+\infty}2(\lambda_2-\lambda_1)\langle z(s)-z_*,g(s)\rangle ds.\end{aligned}$$

First, we need to prove that

$$\int_t^{+\infty}\langle z(s)-z_*,g(s)\rangle ds$$

is well-defined for all $t\geq t_0$. To achieve this, proceed as in the proof of Lemma 3: Observe

$$\begin{aligned}&\left|\int_t^{T_2}\langle z(s)-z_*,g(s)\rangle ds-\int_t^{T_1}\langle z(s)-z_*,g(s)\rangle ds\right|=\left|\int_{T_1}^{T_2}\langle z(s)-z_*,g(s)\rangle ds\right|\\&\qquad\leq\int_{T_1}^{T_2}\|z(s)\|\|g(s)\|ds+\int_{T_1}^{T_2}\|z_*\|\|g(s)\|ds\overset{z\text{ is bounded by Theorem 2}}{\leq}(D+\|z_*\|)\int_{T_1}^{T_2}\|g(s)\|ds,\end{aligned}$$

which implies that $\int_t^{T_n} \langle z(s) - z_*, g(s)\rangle ds$ is a Cauchy sequence in $\mathbb{R}$, hence convergent for $T_n \geq t \quad \forall t \geq t_0$ and $T_n \to +\infty$. Thus, we can set

$$p(t) := t\langle z(t) - z_*, \dot{z}(t) + \beta(t)V(z(t))\rangle + \frac{1}{2}(\alpha - 1)\|z(t) - z_*\|^2 + \frac{1}{2}\int_t^{+\infty} \langle z(s) - z_*, g(s)\rangle ds$$

$$q(t) := \frac{1}{2}\|z(t) - z_*\|^2 + \int_{t_0}^t \beta(s)\langle z(s) - z_*, V(z(s))\rangle ds,$$

which yields

$$\dot{q}(t) = \langle z(t) - z_*, \dot{z}(t)\rangle + \beta(t)\langle z(t) - z_*, V(z(t))\rangle = \langle z(t) - z_*, \dot{z}(t) + \beta(t)V(z(t))\rangle,$$

and thus

$$(\alpha - 1)q(t) + t\dot{q}(t) = p(t) + (\alpha - 1)\int_{t_0}^t \beta(s)\langle z(s) - z_*, V(z(s))\rangle ds - \frac{1}{2}\int_t^{\infty} \langle z(s) - z_*, g(s)\rangle ds. \tag{2.22}$$

Since by Theorem 2, $\lim_{t\to+\infty} E_\lambda(t)$ exists for all $\lambda \in \left[\sup_{t\geq t_0} \frac{t\dot{\beta}(t)}{\beta(t)} + 1, \alpha - 1\right]$, it follows that $\lim_{t\to+\infty} p(t)$ exists. The remaining arguments to check the conditions for Opial's Lemma proceed exactly as in the paper [3]: Note that

$$\int_{t_0}^t \beta(s)\langle z(s) - z_*, V(z(s))\rangle ds$$

is nondecreasing with respect to t, and by (2.15), it holds for every $t \geq t_0$ that

$$\frac{\varepsilon}{2}\int_{t_0}^t \beta(s)\langle z(s) - z_*, V(z(s))\rangle ds \leq \int_{t_0}^t sw(s)\langle z(s) - z_*, V(z(s))\rangle ds.$$

Therefore, it follows from the first inequality of (2.19) that

$$\lim_{t\to+\infty} \int_{t_0}^t \beta(s)\langle z(s) - z_*, V(z(s))\rangle ds \in \mathbb{R}. \tag{2.23}$$

Hence, combining this with the fact that $\lim_{t\to+\infty} p(t)$ exists and (2.22) guarantees that

$$\lim_{t\to+\infty} (\alpha - 1)q(t) + t\dot{q}(t) \in \mathbb{R}$$

exists. Now use Lemma 6 to obtain the existence of $\lim_{t\to+\infty} q(t) \in \mathbb{R}$. Combined with the definition of q and (2.23), this implies that

$$\lim_{t\to+\infty} \|z(t) - z_*\| \in \mathbb{R},$$

which means that the first assumption of Opial's Lemma holds. For the second assumption, let $\bar{z}$ be a weak sequential cluster point of $z(t)$ as $t \to +\infty$. By definition, there must exist a sequence $(z(t_n))_{n\geq 0}$ such that

$$z(t_n) \rightharpoonup \bar{z} \text{ as } n \to +\infty,$$

where $\rightharpoonup$ denotes weak convergence. Theorem 2 guarantees that

$$V(z(t_n)) \to 0 \text{ as } n \to +\infty.$$

Since V is monotone and continuous, it must be maximally monotone as established in Proposition 1.0.2. Hence, the graph of V must be sequentially closed in $(\mathcal{H}, \tau_{\text{weak}}) \times (\mathcal{H}, \tau_{\text{strong}})$, and thus, $V(\bar{z}) = 0$. This means that the second hypothesis of Opial's Lemma holds, which completes the proof. □

Finally, observe that the convergence rates from Theorem 2 did not require the *strict* growth condition (2.3), it was only necessary to assume the growth condition (2.2) in order to ensure well-definedness of the energy function. In the unperturbed case, where well-definedness of the energy function is not tied to the growth condition, assuming it to be true can improve the convergence rates. This is carried into the perturbed case, as is shown in the following

Theorem 4. *Let $\alpha > 2$ and $z : [t_0, +\infty) \to \mathcal{H}$ be a solution of* (2.1)*, $t \mapsto t\|g(t)\|$ integrable, z_* such that $V(z_*) = 0$, and assume that $\beta : [t_0, +\infty) \to (0, +\infty)$ satisfies the strict growth conditon* (2.3)*. Then it holds*

$$\|\dot{z}(t)\| = o\left(\frac{1}{t}\right),$$

$$\langle z(t) - z_*, V(z(t))\rangle = o\left(\frac{1}{t\beta(t)}\right) \text{ and } \|V(z(t))\| = o\left(\frac{1}{t\beta(t)}\right) \text{ as } t \to +\infty.$$

Proof. Analogously to [3], observe that

$$\begin{aligned}
E_\lambda(t) &= \frac{1}{2}\left\|2\lambda(z(t) - z_*) + t(2\dot{z}(t) + \beta(t)V(z(t)))\right\|^2 + 2\lambda(\alpha - 1 - \lambda)\|z(t) - z_*\|^2 \\
&\quad + 2\lambda\beta(t)\langle z(t) - z_*, V(z(t))\rangle + \frac{1}{2}t^2\beta(t)^2\|V(z(t))\|^2 + \int_t^{+\infty}\langle r(s), g(s)\rangle ds \\
&= 2\lambda(\alpha - 1)\|z(t) - z_*\|^2 + 4\lambda t\langle z(t) - z_*, \dot{z}(t) + \beta(t)V(z(t))\rangle \\
&\quad + \frac{1}{2}t^2\|2\dot{z}(t) + \beta(t)V(z(t))\|^2 + \frac{1}{2}t^2\beta(t)^2\|V(z(t))\|^2 + \int_t^{+\infty}\langle r(s), g(s)\rangle ds \\
&= 4\lambda p(t) + t^2\|\dot{z}(t) + \beta(t)V(z(t))\|^2 + t^2\|\dot{z}(t)\|^2,
\end{aligned}$$

where we use the definition of p and the equality

$$\|x\|^2 + \|y\|^2 = \frac{1}{2}\left(\|x + y\|^2 + \|x - y\|^2\right) \geq \frac{1}{2}\|x + y\|^2 \qquad \forall x, y \in \mathcal{H}. \tag{2.24}$$

The remainder of the proof proceeds exactly as in [3]: Recall that both limits $\lim_{t\to+\infty} E_\lambda(t) \in \mathbb{R}$ and $\lim_{t\to+\infty} p(t) \in \mathbb{R}$ exist, which means that the limit of

$$h : [t_0, +\infty) \to \mathbb{R}, \quad h(t) := t^2\|\dot{z}(t) + \beta(t)V(z(t))\|^2 + t^2\|\dot{z}(t)\|^2$$

as $t \to +\infty$ must exist as well, and

$$\lim_{t\to+\infty} h(t) \in [0, +\infty).$$

From the statements (2.20) and (2.21), we obtain

$$\begin{aligned}
\int_{t_0}^{+\infty}\frac{1}{t}h(t)dt &= \int_{t_0}^{+\infty} t\|\dot{z}(t) + \beta(t)V(z(t))\|^2 + t\|\dot{z}(t)\|^2 dt \\
&\overset{(2.24)}{\leq} 3\int_{t_0}^{+\infty} t\|\dot{z}(t)\|^2 dt + 2\int_{t_0}^{+\infty} t\beta(t)^2\|V(z(t))\|^2 dt < +\infty,
\end{aligned}$$

which implies that since $\lim_{t\to+\infty} h(t)$ exists, it must be

$$\lim_{t\to+\infty} h(t) = 0.$$

Therefore,

$$\lim_{t\to+\infty} t\|\dot{z}(t)+\beta(t)V(z(t))\| = \lim_{t\to+\infty} t\|\dot{z}(t)\|^2 = 0,$$

and thus,

$$\lim_{t\to+\infty} t\beta(t)\|V(z(t))\| = 0,$$

which proves the convergence rates for $\|\dot{z}(t)\|$ and $\|V(z(t))\|$. Finally, using the Cauchy-Schwarz inequality and the fact that the trajectory $t \mapsto z(t)$ is bounded, it follows

$$0 \le t\beta(t)\langle z(t)-z_*, V(z(t))\rangle \le t\beta(t)\|z(t)-z_*\|\|V(z(t))\|,$$

finishing the proof. $\square$

Example 2.2.1. These results are illustrated for $V(x,y) = \begin{pmatrix} A^T y \\ -Ax \end{pmatrix}$, where $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in the plots below: Figure 2.1 uses the perturbation $g(t) = \left(\frac{1}{t^3}, 0, 0, 0\right)$, and thus, all the necessary conditions for the convergence results as $t \to +\infty$ are satisfied. Hence, the convergence theorems hold. However, if the perturbation g does not satisfy the integrability of $t \mapsto t\|g(t)\|$, the results can break down, as evidenced by Figure 2.2 using the perturbation $g(t) = (t, 0, 0, 0)$.

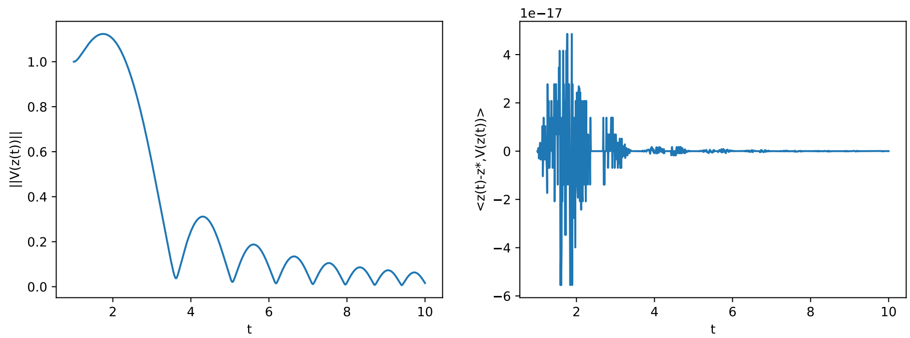

Figure 2.1: Perturbation $g(t) = \left(\frac{1}{t^3}, 0, 0, 0\right)$: The convergence results hold.

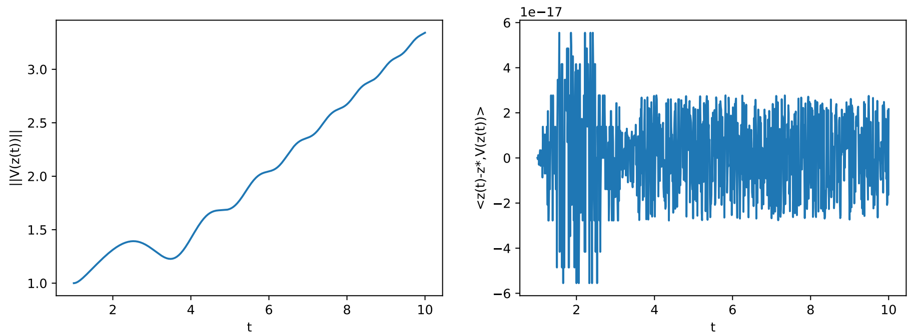

Figure 2.2: Perturbation $g(t) = (t, 0, 0, 0)$ which does not satisfy the integrability of $t \mapsto t\|g(t)\|$: The convergence results do not hold.

3 Conclusion

Having established the analogous results to the unperturbed case from [3] with the only requirement on the perturbation g being integrability, it is a very interesting question whether the same can be said for the implicit or explicit time discretizations, and what conditions need to be imposed on the discrete g_k: If this was the case under moderate requirements on the perturbation, as in the continuous case, these discretizations would yield fast converging algorithms for $V(z) = 0$ while allowing for perturbations, which translates to small errors made in the numerical computation of the next iterate.
As a generalization, the fast continuous time dynamics of monotone inclusions governed by maximally monotone set-valued operators arise for example when investigating the convex minimization of a given *nonsmooth* function f with constraint $Ax = b$, where A operates on the Hilbert space $\mathcal{H}$ and $b \in \mathcal{H}$: Since the gradient of f can no longer be defined, the problem requires a more general notion of derivative. Consider the *subdifferential* of f, a set-valued maximally monotone operator that can be visualized as containing all the possible directions of tangents to the graph of f:

$$\partial f(x) := \{x^* \in \mathcal{H} : \langle x^*, x - y\rangle \leq f(x) - f(y) \quad \forall x, y \in \mathcal{H}\}.$$

Now, the analogous procedure to the one illustrated above yields a set-valued operator instead of the operator $V : \mathcal{H} \to \mathcal{H}$ that governed the system considered in this thesis, which motivates the investigation of monotone inclusions and the continuous time dynamics of their associated systems in order to generalize the results obtained.

References

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