DISSERTATION / DOCTORAL THESIS

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Whittaker periods of isobaric sums over totally real fields and special *L*-values

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Abstract

In the last few years, the arithmetic theory of special values of L-functions has enjoyed enormous progress. In particular, thanks to the introduction of representation-theoretic invariants of cohomological automorphic representations called *Whittaker–Betti periods*, remarkable results have been obtained by Raghuram–Shahidi, Raghuram, Grobner–Harris, Grobner, Grobner–Lin, Li–Liu–Sun and others about the special values of Rankin–Selberg automorphic L-functions for $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$ over arbitrary number fields.

In contrast to this, a lot less seems to be known about special values of Rankin–Selberg automorphic L-functions for $\mathrm{GL}_n \times \mathrm{GL}_m$ with $1 \leq m < n-1$. Among the most prominent results we have, on the one hand, a theorem of Grobner–Sachdeva over CM-fields (with a parity restriction on m and n), and on the other, results of Harder–Raghuram and Raghuram about *quotients* of special L-values (the base field being assumed totally real and totally imaginary, respectively) where such quotients are brought into connection with quantities known as *relative periods*.

In my thesis, I prove a result on special values of Rankin–Selberg L-functions for $\mathrm{GL}_n \times \mathrm{GL}_m$ over a totally real field, with n odd and $m < n$ even, by a method similar to the one used in the work of Grobner–Sachdeva. I then infer a result on quotients of such special values, which, when compared and combined with Harder–Raghuram's results, yields a relation between Whittaker periods, on the one hand, and Harder–Raghuram relative periods on the other. In the special case $m = n - 1$, I compare the latter result with earlier work of Raghuram about relative periods being equal to quotients of Whittaker periods of opposite signatures, up to an explicit power of the imaginary unit i.

Zusammenfassung

Die arithmetische Theorie spezieller Werte von L-Funktionen erfuhr in den letzten Jahren großen Fortschritt. Es seien in dieser Hinsicht Resultate von Raghuram-Shahidi, Raghuram, Grobner-Harris, Grobner, Grobner-Lin, Li-Liu-Sun und anderer genannt, die dank der Einführung gewisser darstellungstheoretischer Invarianten von kohomologischen automorphen Darstellungen (sogenannter *Whittaker-Betti-Perioden*) wichtige Resultate über spezielle Werte von Rankin-Selberg automorphen L-Funktionen für $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$ über beliebigen Zahlkörpern beweisen konnten.

Im Gegensatz zum Fall von $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$, ist über die speziellen Werte von Rankin-Selberg automorphen L-Funktionen für $\mathrm{GL}_n \times \mathrm{GL}_m$, mit $1 \leq m < n-1$, deutlich weniger bekannt. Unter den bedeutendsten Resultaten rangieren in Hinblick auf die vorliegende Arbeit einerseits ein Theorem von Grobner-Sachdeva über CM-Körpern (mit einer Paritätsbedingung an m und n), und andererseits Resultate von Harder-Raghuram, resp. Raghuram, über *Quotienten* von speziellen L-Werten (unter der Annahme, dass der Grundkörper total reell, resp. total imaginär, sei), in denen dergestalte Quotienten mit als „*relative Perioden*“ bekannten Größen in Verbindung gebracht werden.

In meiner Dissertation zeige ich ein Resultat für spezielle Werte von Rankin-Selberg automorphen L-Funktionen für $\mathrm{GL}_n \times \mathrm{GL}_m$ über einem total reellen Körper, so n ungerade und $m < n$ gerade sei. Dabei verwende ich ähnliche Methoden wie in der Arbeit von Grobner-Sachdeva. Aus diesem Resultat schließe ich eine Aussage über Quotienten von solchen speziellen Werten, die, wenn man sie mit den Ergebnissen von Harder-Raghuram vergleicht und kombiniert, eine Verbindung zwischen Whittaker-Perioden, auf der einen Seite, und Harder-Raghuram relativen Perioden, auf der anderen, herstellt. Im Spezialfall $m = n-1$ vergleiche ich letzteres Resultat mit einer früheren Arbeit von Raghuram, in der gezeigt wurde, dass relative Perioden, bis auf eine explizite Potenz der imaginären Einheit i, sich als Quotienten von Whittaker-Perioden mit entgegengesetzten Vorzeichen darstellen lassen.

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"No doubt there was much more wisdom that I failed to recognize."

Stanislaus Grumman, in *The Subtle Knife*, by Philip Pullman

First and foremost, I am deeply grateful to my advisor Harald Grobner, who's had a big hand in shaping this thesis into what you now hold in your hand or on your screen. I thank him for careful reading of early drafts of this work, for many valuable suggestions on both substance and form, and for illuminating many aspects I thought I understood better than I did, as well as for keeping up with the crazy pace I subjected myself and him to in the last few weeks. Looking back on the entirety of my PhD, I'd like to thank him for his help, his time and his patience over the years; for the guidance he has provided me in navigating the world of research; for the pleasant and insightful conversations; and for the opportunities he has given me and helped me in getting.

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On a perhaps unusual note, I'd like to acknowledge that writing this thesis would have been a vastly different experience had it not been for the Internet. I'm indebted to a multitude of people and projects dedicated to making research-level resources widely available for free, including (but not limited to) the arXiv, Stack Exchange, MathOverflow, and everyone's favorite search engine.

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"Come, Mr. Frodo!" he cried. "I can't carry it for you, but I can carry you and it as well."
Samwise Gamgee, in *The Return of the King*,
by J. R. R. Tolkien

This thesis is the culmination and conclusion of a long and important period in my life — the final sprint in a marathon, if you will. I have many people to thank for making these last few years as pleasant as they have been, and helping me, in one way or other, get to the finish line. So many people, in fact, that I'm terrified that I might forget someone. But I'm going to risk it anyways.

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Contents

Introduction

"Campi sconfinati che si arrendono alla sera
Qualche finestra accesa
Mentre il vento arpeggia una ringhiera
Tu vivresti qui per sempre
Dici che dovrei staccare un po' la mente
Ma io. . . "

Colapesce, Dimartino, *Splash*

Ever since their initial introduction in work of Dirichlet on primes in arithmetic progressions, *L-functions* of various types and flavors have become a predominant object of study in analytic number theory. Not only can one extract rich arithmetic information from the analytic behaviour of L-functions (as encapsulated e.g. in the Riemann hypothesis and the Birch–Swinnerton-Dyer conjecture, both of which are as-yet-unsolved Millennium Prize Problems) — it is also understood that they hold the key to deep, and for the most part still conjectural, correspondences between number-theoretic, representation-theoretic, and algebro-geometric objects, in that such correspondences can be expressed as equalities of the L-functions of the objects on both sides.

By the *arithmetic of L-functions*, one means the study of the rationality, or algebraicity, of certain "special" values of L-functions, obtained by evaluating at distinguished points in the complex plane. Given an L-value $L(s_0)$, the goal is typically to independently and intrinsically define a quantity Ω such that the ratio $L(s_0)/\Omega$ can be shown to lie in some concretely defined algebraic number field. For instance, Euler's classic solution to the Basel problem,

$$\zeta(2) = \frac{\pi^2}{6},$$

can be reinterpreted as the statement that two independently defined invariants of the same motive — namely, the special L-value $\zeta(2)$ on the one hand and the *Deligne period* $(2\pi i)^2$ on the other — only differ by a rational factor.

In this thesis, we will focus exclusively on *automorphic L-functions* — more precisely, our research revolves around special values of *Rankin–Selberg automorphic L-functions* for $\mathrm{GL}_n \times \mathrm{GL}_m$ over a totally real number field, with n odd and $m < n$ even. Given such an L-function $L(s, \Pi \times \Pi')$, the goal will be to express its value at a *critical point* s_0 in terms of invariants which, by analogy with the aforementioned Deligne periods, are also called *periods* and are obtained by comparing natural rational structures on isomorphic representation spaces. Several types of periods have been defined in the literature on special values of automorphic L-functions; in this thesis, we will mostly devote our attention to *bottom-degree Whittaker–Betti periods*, or *Whittaker periods* for short, but our results will also entail a comparison with *Harder–Raghuram relative periods*.

The outline of the thesis is as follows.

Chapter 1 is mainly devoted to recalling basic notions and results that will underlie many discussions later on, culminating in the definition of the adèlic group $\mathrm{GL}_n(\mathbb{A}_F)$ and the description of its "factors" $\mathrm{GL}_n(\mathbb{A}_\mathrm{f})$ and $\mathrm{GL}_n(F_\infty)$. In the final section §1.8, we have also included a thorough account of *highest-weight modules* for $\mathrm{GL}_n(F)$, for F totally real, whose level of detail will prove particularly useful at a later juncture. Since the contents of this chapter are well-known, our purpose in writing it has been, for the most part, to fix terminology and notations.

The second chapter, in which we introduce *automorphic representations* of $\mathrm{GL}_n(\mathbb{A}_F)$, is written with a similar philosophy and goal in mind. Our focus will be on giving precise definitions of cuspidal, isobaric, and generic representations, and on reviewing their fundamental properties. We will conclude the chapter with the notion of a *cohomological automorphic representation*, which is the starting point for the results on special values of Rankin–Selberg automorphic L-functions presented in Chapters 5 and 6.

Chapter 3 concludes the purely automorphic preparations; in this short chapter, we recall the definition of Rankin–Selberg automorphic L-functions, define the notion of a *critical point*, and review the explicit description of the critical points of $L(s, \Pi \times \Pi')$ when Π and Π' are both cohomological and their "ranks" differs by one.

In Chapter 4, we introduce three related notions — namely, those of *twists*, *rational structures* and *periods* — which are at the core of the techniques used to establish the main results of Chapters 5 and 6. A salient feature of this chapter is that it is not restricted to the automorphic world — instead, we have tried to capture the bare minimum of structure that is needed to make sense of these notions, and prove as many key properties as possible in this more general and abstract setting.

In Chapter 5, we return to the automorphic setting, and apply the results of the preceding chapter to cohomological automorphic representations. We define *Whittaker periods*, as well as *Harder–Raghuram relative periods*, and state two recent results ([LLS21, Thm. 1.2] and [HR20, Thm. 7.21]) as an example of how these invariants can be used to express special L-values, resp. quotients thereof.

Finally, in Chapter 6 we present original results we have obtained in our research on special L-values. In the first main result, we leverage [LLS21, Thm. 1.2], which is a result on special L-values for $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$, to derive an analogous statement for special L-values for $\mathrm{GL}_n \times \mathrm{GL}_m$, subject to the parity conditions mentioned earlier. The methods used in this proof are akin to those used in [GS20] (cf. also the final remark in [Mah05]), except that, since we work over totally real fields rather than CM fields, we first need to establish an auxiliary result on the existence of cohomological representations of $\mathrm{GL}_2(\mathbb{A}_F)$ with suitable properties (Proposition 6.1.1). Our first main result immediately implies a corollary on quotients of consecutive special L-values, which we then compare with the aforementioned [HR20, Thm. 7.21] to obtain a relation between relative periods and quotients of Whittaker periods, see Corollary 6.4.1.

It is perhaps worth noting that, while our original results are concentrated in the final chapter, we have, at other points in this thesis, included arguments which, to our knowledge, had not yet appeared in this level of completeness of generality in the literature. We hope that this will be helpful to other researchers in the area. Examples include Theorem 2.4.4, whose detailed proof was provided to us by H. Grobner; Theorem 3.2.3, which seems to be well-known to experts but for which we had not previously been able to find a complete argument in recent sources; the *main theorem on periods* (Theorems 4.6.6, 4.6.7, 4.6.9 and 4.6.10), which we have proved under weaker assumptions than had been done thus far; and the "Galoisness" of the rationality field $\mathbb{Q}(E_{\underline{\mu}})$ of a highest-weight module (Proposition 5.1.3), which, to our knowledge, is an entirely novel observation.

Chapter 1

Background

"Unfortunately I must be rather pedantic."
Robert P. Langlands, Letter to André Weil

1.1 Locally compact groups

1.1.1. Recall that a *locally compact group* is a Hausdorff topological group whose topology is locally compact. Similarly, a *compact group* is a Hausdorff topological group whose topology is compact.

1.1.2. Let G be a locally compact group. A nonzero Radon measure[1] on G which is left-invariant (resp., right-invariant) is called a *left* (resp., *right*) *Haar measure* on G. Crucially, every locally compact group G possesses a left Haar measure, and any two left Haar measures on a given G are proportional by a positive real number. The same is true of right Haar measures. (See e.g. [Fol99, §11.1].)

For many important classes of groups, every left Haar measure is also a right Haar measure and viceversa, so we may speak simply of *Haar measure* without further qualifiers. A group for which left and right Haar measures coincide is said to be *unimodular*. All locally compact abelian groups, all discrete groups and all compact groups are unimodular, see e.g. [Bou04, Chap. VII, p. 7] (or, for the case of compact groups specifically, [Fol99, Prop. 11.14]).

1.1.3. Let G be a locally compact group, and μ be a fixed choice of Haar measure on G. Then, for any Borel set $E \subseteq G$, we will denote $\mu(E)$ by $\mathrm{vol}(E)$, and if a function $f\colon E \to \mathbb{C}$ is integrable with respect to μ, we will denote the integral

[1] See e.g. [Fol99, §7.1] for the definition we will be adopting here.

$\int_E f \, \mathrm{d}\mu = \int_E f(g) \, \mathrm{d}\mu(g)$ simply by $\int_E f(g) \, \mathrm{d}g$. In this way, we will not need to introduce specialized notation for Haar measure on each group we consider.

1.1.4. On many groups of interest, there is a unique Haar measure with a certain key property, in which case it is standard to consider the group at hand with this distinguished Haar measure. For instance, discrete groups are always regarded with *counting measure*, absent any mention to the contrary. Similarly, it is understood that Haar measure on a compact group $G = K$ is *normalized* in such a way that the volume of the whole group is 1. (Cf. [Bou04, Chap. VII, p. 7].)

Now let G be a unimodular locally compact group with a distinguished *compact open subgroup* K. Then there exists a distinguished Haar measure on G, namely the unique one whose restriction to K is normalized Haar measure on K.[2] Such groups G will *always* be considered with this distinguished Haar measure unless expressly stated to the contrary.

1.1.5. We will need the fact that, if G be a unimodular locally compact group, and $H \subseteq G$ is a closed subgroup which is also unimodular, then, for each choice of Haar measure of G and H, there is a unique *quotient measure* on $H\backslash G$ such that *Weil's integration formula* (see e.g. [Fol95, eq. (2.50)]) holds. We refer to [Fol95, Thm. 2.49] and [Bou04, Chap. VII, §2, Nos. 5 & 7] for details.

1.2 Locally compact fields

1.2.1. We will freely use the notions of *topological rings*, *topological fields*, and *topological vector spaces* in the sequel, for the definitions and basic properties of which we refer e.g. to [War89]. As in *op. cit.*, such objects are not automatically assumed or required to be Hausdorff by definition, but those we will consider in this thesis will invariably have the Hausdorff property.

1.2.2. In accordance with the literature, by a *locally compact field* we mean a Hausdorff topological field whose topology is locally compact. We further define a *local field* to be a nondiscrete locally compact field.

With our definition, each of $\mathbb{R}$ and $\mathbb{C}$, with the usual "Euclidean" topology, is a local field. An *Archimedean* local field is one which is isomorphic, as a topological field, to either $\mathbb{R}$ or $\mathbb{C}$; a local field which is not Archimedean is called *non-Archimedean*, or *ultrametric*.

[2] Indeed, on an open subgroup, every left (resp., right) Haar measure is obtained by restriction from precisely one left (resp., right) Haar measure on the ambient group, cf. SE1335588 or, for an elaboration, [Cas21, Prop. 8.1].

If L is an ultrametric local field, then it has a largest compact subring, denoted by $\mathcal{O}_L$ and called the *ring of integers* of L. It is a discrete valuation ring with finite residue field (see [RV99, Lemmas 4-15]); as usual, any generator of the maximal ideal of $\mathcal{O}_L$ will be called a *uniformizer* of $\mathcal{O}_L$. Topologically, $\mathcal{O}_L$ is both compact and open of L, so we may and will equip (the additive group of) L with the unique Haar measure for which $\mathrm{vol}(\mathcal{O}_L) = 1$, cf. 1.1.4.

1.3 Algebraic number fields

1.3.1. An *algebraic number field* is, by definition, a finite-degree extension of the rational field $\mathbb{Q}$. In some sources, this is taken to mean

(A) a subfield of $\mathbb{C}$ which is of finite degree over $\mathbb{Q}$

(in which case an algebraic number field is literally a field consisting of algebraic numbers). In this thesis, we opt for a more abstract viewpoint instead: we view an algebraic number field as being

(B) a field F together with a field homomorphism $\mathbb{Q} \to F$ which turns F into a finite-dimensional $\mathbb{Q}$-linear space.

As is well-known, (A) and (B) describe *precisely the same objects* up to isomorphism; nevertheless, since (B) is formally more general, we will stick to (B) throughout the thesis.

1.3.2. *Remarks on* 1.3.1.

(1) Throughout this thesis, algebraic number fields — and in fact most fields — will be denoted by letters such as F, E, ... (We reserve the letter K, and variations thereof, for certain compact subgroups.)

(2) For any algebraic number field F, the (necessarily unique) field homomorphism $\mathbb{Q} \to F$ will be treated as an inclusion and omitted from notation.

1.3.3. The *ring of integers* of an algebraic number field F will be denoted by $\mathcal{O}_F$. A nonzero prime ideal of $\mathcal{O}_F$ will typically be denoted by such a symbol as $\mathfrak{p}$. For a nonzero prime ideal $\mathfrak{p}$, there is a corresponding map $\mathrm{ord}_{\mathfrak{p}} \colon F^* \to \mathbb{Z}$, for which we refer e.g. to [Lan94, Chap. 1, §6]; we will call it the $\mathfrak{p}$-*adic valuation on* F.

1.3.4. For an algebraic number field F, we let $\mathrm{Hom}(F, \mathbb{C})$ denote the set of field embeddings $F \hookrightarrow \mathbb{C}$. The cardinality of $\mathrm{Hom}(F, \mathbb{C})$ is precisely the degree of F over $\mathbb{Q}$. For each $\iota \in \mathrm{Hom}(F, \mathbb{C})$, the embedding *conjugate* to ι — i.e. the map which sends each $x \in F$ to the complex conjugate of $\iota(x) \in \mathbb{C}$ — will be denoted by $\bar{\iota}$.

The set $\mathrm{Hom}(F,\mathbb{R})$ of *real embeddings* will be viewed as a subset of $\mathrm{Hom}(F,\mathbb{C})$ via the natural inclusion $\mathbb{R} \subset \mathbb{C}$. Observe that $\mathrm{Hom}(F,\mathbb{R}) \subseteq \mathrm{Hom}(F,\mathbb{C})$ consists precisely of those $\iota \in \mathrm{Hom}(F,\mathbb{C})$ which satisfy $\iota = \bar{\iota}$, whereas the non-real embeddings of F come in pairs $\{\iota, \bar{\iota}\}$ of conjugate embeddings. (More conceptually, this corresponds to partitioning $\mathrm{Hom}(F,\mathbb{C})$ into $\mathrm{Aut}_{\mathrm{ct}}(\mathbb{C})$-orbits, where $\mathrm{Aut}_{\mathrm{ct}}(\mathbb{C})$ denotes the group of continuous field automorphisms of $\mathbb{C}$ — which has only two elements, namely the identity and complex conjugation — and its action on $\mathrm{Hom}(F,\mathbb{C})$ is on the left and given simply by composition of maps.)

1.4 Completions of number fields

1.4.1. Recall that an *absolute value* on a field F is a real-valued function on F which is positive definite, multiplicative, and sub-additive, i.e. satisfies the triangle inequality. As usual, we classify absolute values on a field into *Archimedean* and *non-Archimedean* (or: *ultrametric*), see [War89, Def. 18.11]. (The *trivial* absolute value is technically an example of a non-Archimedean absolute value, but will usually be excluded from consideration.) Finally, by a *place* on a field we mean an equivalence class of absolute values on that field — we refer to [War89, 18.3, 18.4] for the notion of equivalence.

1.4.2. Now let F be a number field. A place of F is *nontrivial* if it contains some nontrivial absolute value; a nontrivial place v of F is *Archimedean*, or *infinite*, if any (hence: every) absolute value in the equivalence class v is Archimedean, and *non-Archimedean*, or *finite*, otherwise. If v is infinite, we usually write this in symbols as "$v \mid \infty$"; if v is finite, we may write $v < \infty$ or $v \nmid \infty$.

The nontrivial places of F are classified by a result known (at least in the special case $F = \mathbb{Q}$) as *Ostrowski's theorem*, which we will now paraphrase. (Cf. [RV99, Thm. 4-30].) First, for each nonzero prime ideal $\mathfrak{p} \subset \mathcal{O}_F$, one defines the *(normalized)* $\mathfrak{p}$*-adic absolute value* on F as follows:

$$x \mapsto |x|_{\mathfrak{p}} := \begin{cases} 0, & \text{if } x = 0, \\ (\mathrm{N}\mathfrak{p})^{-\mathrm{ord}_{\mathfrak{p}}(x)}, & \text{else,} \end{cases}$$

where $\mathrm{N}\mathfrak{p}$ is shorthand for the cardinality of the finite field $\mathcal{O}_F/\mathfrak{p}$. We recall that $|\cdot|_{\mathfrak{p}}$ is a nontrivial non-Archimedean absolute value on F, and that each finite place of F contains precisely one $|\cdot|_{\mathfrak{p}}$; in other words, the finite places of F are parameterized by the nonzero prime ideals of $\mathcal{O}_F$.

Next, for $\iota \in \mathrm{Hom}(F,\mathbb{C})$, set

$$|x|_{\iota} := |\iota(x)|_{\mathbb{C}} \qquad \text{for all } x \in F.$$

(Here, $|\cdot|_{\mathbb{C}}$ denotes the usual absolute value on $\mathbb{C}$, which extends the standard absolute value on $\mathbb{R}$.) We have that two such absolute values $|\cdot|_\iota$, $|\cdot|_{\iota'}$ are equivalent to each other if and only if the corresponding embeddings ι, ι' lie in the same $\mathrm{Aut}_{\mathrm{ct}}(\mathbb{C})$-orbit (in which case $|\cdot|_\iota = |\cdot|_{\iota'}$). Moreover, every Archimedean absolute value of F is equivalent to $|\cdot|_\iota$ for some $\iota \in \mathrm{Hom}(F, \mathbb{C})$. Thus, the infinite places of F are parameterized by the $\mathrm{Aut}_{\mathrm{ct}}(\mathbb{C})$-orbits in $\mathrm{Hom}(F, \mathbb{C})$.

1.4.3. The *completion* of a number field F at a nontrivial place v is defined in the usual way, cf. e.g. [CF90, Chap. II, §5]. It is a complete valued field F_v together with a field embedding $i_v \colon F \to F_v$ with dense image. One customarily treats i_v as an inclusion map and suppresses it in the notations.

Let v be a finite place of F, and $\mathfrak{p}$ be the unique prime ideal of $\mathcal{O}_F$ corresponding to v as per 1.4.2. Then:

(a) F_v is a local field, which may be called the *field of* $\mathfrak{p}$*-adic numbers*;
(b) its ring of integers $\mathcal{O}_{F_v}$ (cf. 1.2.2) is called the *ring of* $\mathfrak{p}$*-adic integers*;
(c) the residue class field of $\mathcal{O}_{F_v}$ is canonically isomorphic to $\mathcal{O}_F/\mathfrak{p}$ and thus has cardinality $\mathrm{N}\mathfrak{p}$;
(d) the normalized discrete valuation on $\mathcal{O}_{F_v}$ extends the $\mathfrak{p}$-adic valuation $\mathrm{ord}_\mathfrak{p}$ on $\mathcal{O}_F$ introduced in 1.3.3.

We equip F_v with the unique absolute value which extends the normalized $\mathfrak{p}$-adic absolute value on F, and we write $|\cdot|_v$ both for $|\cdot|_\mathfrak{p}$ and for its extension to F_v.

Now let v be an infinite place. Then v corresponds either to a single real embedding ι or to a pair $\{\iota, \bar{\iota}\}$ of complex conjugate embeddings. In the first case, ι extends to a topological isomorphism $F_v \xrightarrow{\sim} \mathbb{R}$, which is necessarily unique since $\mathbb{R}$ has no nontrivial continuous field automorphisms. In this case, we say that v is a *real* place of F. If v is not real, then F_v is isomorphic to $\mathbb{C}$, and v is called a *complex* place of F. Both in the real and complex case, $|\cdot|_v$ will be used to denote both the original absolute value $|\cdot|_\iota$ on F and its unique extension to F_v.

1.5 Adèle rings

1.5.1. Let F be a number field. The *adèle ring* of F, denoted $\mathbb{A}_F$, is the *restricted topological product* of the F_v with respect to the $\mathcal{O}_{F_v}$, where v ranges over all the nontrivial places of F, but $\mathcal{O}_{F_v}$ is only defined when v is finite. (We refer to [CF90, Chap. II, §13] for the definition and basic properties of restricted topological products.) The elements of $\mathbb{A}_F$, called *adèles*, are tuples $(a_v)_v$ where v ranges over the nontrivial places of F and the v-th component a_v lies in F_v; by definition, it is even true that a_v lies in $\mathcal{O}_{F_v}$ for almost all v. Since all completions of number fields are locally compact fields, and $\mathcal{O}_{F_v}$ is compact for $v < \infty$, the results of *loc. cit.* implies that $\mathbb{A}_F$ is a locally compact Hausdorff topological ring.

If we take the restricted product of the F_v with respect to the $\mathcal{O}_{F_v}$ but *only* let v range over the *finite* places of F, we get, by definition, the ring of *finite adèles*, denoted by $\mathbb{A}_{F,\mathrm{f}}$ or simply $\mathbb{A}_\mathrm{f}$ when the ground field F is clear. This is again a locally compact ring. By [CF90, first Lemma on p. 63], $\mathbb{A}_F$ can be regarded as the product $\mathbb{A}_F \cong \mathbb{A}_\mathrm{f} \times F_\infty$, where we let $F_\infty := \prod_{v|\infty} F_v$. We will identify $\mathbb{A}_\mathrm{f}$ and F_∞ with subrings of $\mathbb{A}_F$ in the obvious manner.

There are compatible diagonal embeddings of F in each of the rings $\mathbb{A}_F$, $\mathbb{A}_\mathrm{f}$ and F_∞. We will not distinguish between F and the copy of F sitting diagonally inside $\mathbb{A}_F$, which should more precisely be called the subring of *principal adèles*. It is a fundamental fact (see [CF90, p. 64, Theorem]) that F is discrete in $\mathbb{A}_F$ and that the quotient $F \backslash \mathbb{A}_F$ is compact.

1.5.2. We take a moment to observe, for later reference, that the topological ring $F_\infty = \prod_{v|\infty} F_v$ introduced above is isomorphic to the tensor product $F \otimes_\mathbb{Q} \mathbb{R}$. More precisely, if the latter is given its natural topology as a finite-dimensional real vector space, then $F \otimes_\mathbb{Q} \mathbb{R}$ and F_∞ are isomorphic as topological rings, as proved in [CF90, p. 57, Theorem]. Of course, both are also isomorphic, as topological rings, to $\mathbb{R}^r \times \mathbb{C}^c$, where r, resp. c, stands for the number of real, resp. complex, places of F. If F is *totally real*, i.e. if all infinite places of F are real, then all these isomorphisms are unique (up to permutations of the factors) since $\mathbb{R}$ has no nontrivial continuous field automorphisms.

1.6 Reductive groups

1.6.1. By a *linear algebraic group* over a field k we mean what Getz–Hahn call an *affine algebraic group* over k in [GH23, Def. 1.13]. Thus, a linear algebraic group is in particular a functor from the category of k-algebras to the category of groups, and maps between linear algebraic groups (also called *morphisms*) are natural transformations.

In this thesis, linear algebraic groups will be denoted by symbols such as G, H, etc.; if we want to mention the base field k explicitly, we may write G/k.

1.6.2. By a *connected reductive group* over a field k we mean a smooth connected linear algebraic group over k which is *reductive* as per [GH23, Def. 1.15]. (We refer to *op. cit.* for the notions of connectedness and smoothness — observe that, if k has characteristic 0, the smoothness assumption is automatically satisfied by [GH23, Thm. 1.5.2].)

1.6.3. In this thesis, the main connected reductive group of interest will be the *general linear group*, defined as follows. Let k be a field, and for a finite-dimensional vector space V over field k, let GL_V be the functor which takes a k-algebra A to the group of invertible endomorphisms of the finite-rank free A-module $V \otimes_k A$. In the special case $V = k^n$, we write GL_n for GL_{k^n}. (Clearly, a choice of basis on any V induces an isomorphism $\mathrm{GL}_V \cong \mathrm{GL}_{\dim V}$.) For every $n \geq 1$, the functor GL_n is indeed a connected reductive linear algebraic group over k ([Mil17, 2.8, 2.40 & 19.19]), as claimed.

1.6.4. If G is a connected reductive group over k, and k' is an extension of k, then the *base change* $\mathrm{G}_{k'}$ is again connected reductive by [Mil17, Prop. 19.13]. Conversely, if k'/k is a finite separable extension and G is a connected reductive group over k', then the *Weil restriction of scalars* $\mathrm{Res}_{k'/k}(\mathrm{G})$ is again connected reductive, cf. [Spr09, 16.2.6].

1.6.5. We now recall some elements of the structure theory of GL_n / k, where k is any field of characteristic 0, cf. [GH23, Ex. 1.14, 1.15].

The standard choice of *split maximal torus* in GL_n is the group of diagonal matrices inside GL_n, denoted by T_n. The *characters* of T_n are the maps of the form

$$\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mapsto t_1^{k_1} \cdots t_n^{k_n}$$

for $k_1, \ldots, k_n \in \mathbb{Z}$; thus, the *character group* of T_n, denoted $X^*(\mathrm{T}_n)$, is isomorphic to $\mathbb{Z}^n$. The *roots* of GL_n relative to T_n are the characters $\alpha \in X^*(\mathrm{T}_n)$ of the form

$$\varepsilon_i - \varepsilon_j \colon \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mapsto t_i t_j^{-1}$$

for $1 \leq i, j \leq n$ and $i \neq j$. The standard system of *positive roots* is that consisting of the roots $\varepsilon_i - \varepsilon_j$ for $i < j$; the standard system of *simple roots* is that consisting of the roots $\varepsilon_i - \varepsilon_{i+1}$ for $i = 1, \ldots, n-1$. The *root subgroup* corresponding to the root α is

$$\mathrm{U}_\alpha \coloneqq \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & * & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \subset \mathrm{GL}_n, \tag{1.1}$$

where the $*$ is on the i-th row and j-th column if $\alpha = \varepsilon_i - \varepsilon_j$.

The *standard Borel subgroup* of GL_n is the subgroup of invertible upper-triangular matrices, denoted by B_n. One can write $\mathrm{B}_n = \mathrm{T}_n \mathrm{U}_n$, where U_n is the subgroup of unipotent matrices in B_n. A *standard parabolic subgroup* of GL_n is a subgroup containing B_n. (This includes both B_n and GL_n as special cases.) Standard parabolic subgroups P of GL_n are parameterized by subsets of the set of simple roots. Alternatively, they are parameterized by ordered partitions of n — more precisely, let $(n_1, \ldots, n_d)$ be an ordered tuple of strictly positive integers which sum to n, and define subgroups

$$\mathrm{M} = \left\{ \begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_d \end{pmatrix} : g_i \in \mathrm{GL}_{n_i} \right\}, \qquad \mathrm{N} = \left\{ \begin{pmatrix} I_{n_1} & * & \cdots & * \\ & I_{n_2} & \cdots & * \\ & & \ddots & * \\ & & & I_{n_d} \end{pmatrix} \right\}. \tag{1.2}$$

Then $\mathrm{P} \coloneqq \mathrm{MN}$ is a standard parabolic subgroup of GL_n, and conversely, every standard parabolic subgroup P of GL_n is of this form. One calls $\mathrm{M} \cong \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_d}$ the *standard Levi subgroup* of P; the decomposition $\mathrm{P} = \mathrm{MN}$ is the *standard Levi decomposition* of P, cf. [GH23, Thm. 1.5.4 & Def. 1.17]. If M is the standard Levi subgroup of some standard parabolic subgroup $\mathrm{P} \subseteq \mathrm{GL}_n$, we might simply call M a *Levi subgroup* of G for brevity, even though this is at odds with the standard usage ([GH23, Def. 1.17]).

1.7 Real, $\mathfrak{p}$-adic, and adèlic groups

Throughout, F is an algebraic number field.

1.7.1. Let v be a nontrivial place of F. Then the completion F_v is a local field, and $\mathrm{GL}_n(F_v)$ can be topologized in a natural way; indeed, one can realize $\mathrm{GL}_n(F_v)$ as a closed subset of $F_v^{n^2+1}$ in the standard way, and then endow it with the subspace topology. This turns each $\mathrm{GL}_n(F_v)$ into a locally compact topological group.

If v is infinite, then $\mathrm{GL}_n(F_v)$ is even a real Lie group. It is connected if v is complex, and has precisely two connected components if v is real.

If v is finite, $\mathrm{GL}_n(F_v)$ is a *locally profinite group*, i.e. a locally compact group whose topology is totally disconnected. (In fact, F_v is itself a locally profinite group with respect to addition.) The subgroup $\mathrm{GL}_n(\mathcal{O}_{F_v})$ is compact and open in F_v. (In particular, 1.1.4 applies.)

1.7.2. Now consider the topological rings F_∞, $\mathbb{A}_\mathrm{f}$ and $\mathbb{A}_F = F_\infty \times \mathbb{A}_\mathrm{f}$ from 1.5.1. By the same reasoning as in 1.7.1, each of the groups $\mathrm{GL}_n(F_\infty)$, $\mathrm{GL}_n(\mathbb{A}_\mathrm{f})$ and $\mathrm{GL}_n(\mathbb{A}_F)$ is a locally compact group in a natural way. A more convenient description is as follows: we have equalities (or rather, isomorphisms of topological groups)

$$\mathrm{GL}_n(F_\infty) = \prod_{v|\infty} \mathrm{GL}_n(F_v),$$
$$\mathrm{GL}_n(\mathbb{A}_\mathrm{f}) = \prod_{v<\infty}{}' \mathrm{GL}_n(F_v),$$

where $\prod'$ denotes the restricted product with respect to the compact open subgroups $\mathrm{GL}_n(\mathcal{O}_{F_v})$, and

$$\begin{aligned}\mathrm{GL}_n(\mathbb{A}_F) &= \mathrm{GL}_n(F_\infty) \times \mathrm{GL}_n(\mathbb{A}_\mathrm{f})\\ &= \prod_{v \text{ nontriv.}}{}' \mathrm{GL}_n(F_v),\end{aligned}$$

where $\prod'$ denotes, again, the restricted product relative to the compact open subgroups $\mathrm{GL}_n(\mathcal{O}_{F_v})$, $v < \infty$. (See e.g. [GH23, Prop. 2.3.1] or [PR94, p. 248f.].) The diagonal embedding of F into F_∞ (resp., $\mathbb{A}_\mathrm{f}$, resp. $\mathbb{A}_F$) induces a diagonal embedding of $\mathrm{GL}_n(F)$ into $\mathrm{GL}_n(F_\infty)$ (resp., $\mathrm{GL}_n(\mathbb{A}_\mathrm{f})$, resp. $\mathrm{GL}_n(\mathbb{A}_F)$), which will be treated simply as an inclusion. It is well-known that the image of $\mathrm{GL}_n(F)$ in $\mathrm{GL}_n(F_\infty)$ is dense (this follows from *weak approximation*), whereas the image of $\mathrm{GL}_n(F)$ in $\mathrm{GL}_n(\mathbb{A}_F)$ is discrete.

1.8 Highest-weight modules

Let F be a number field. In this section, we introduce certain representations of the real Lie group $\mathrm{GL}_n(F_\infty)$, and its subgroup $\mathrm{GL}_n(F)$, which will feature prominently in the sequel. We will *restrict to the case where F is totally real*, i.e. where all infinite places of F are real places.

1.8.1. By the assumption on F, we have (cf. 1.4.3, 1.5.2), at each infinite place v of F, a unique continuous isomorphism $F_v \xrightarrow{\sim} \mathbb{R}$. By treating this map as an identification, we obtain the following diagram:

$$\begin{array}{ccccccc} & & F_\infty & & F \otimes_\mathbb{Q} \mathbb{R} & & F \otimes_\mathbb{Q} \mathbb{C} \\ & & \| & & \| & & \| \\ F & \subset & \prod_{v|\infty} F_v & \xrightarrow{\sim} & \prod_{v|\infty} \mathbb{R} & \subset & \prod_{v|\infty} \mathbb{C}, \end{array} \tag{1.3}$$

where the inclusion on the far left is the diagonal embedding $F \hookrightarrow F_\infty$ discussed in 1.5.1. By applying the functor GL_n, we obtain an entirely analogous diagram in the category of groups:

$$\begin{array}{ccccccc} & & \mathrm{GL}_n(F_\infty) & & \mathrm{GL}_n(F\otimes_{\mathbb{Q}}\mathbb{R}) & & \mathrm{GL}_n(F\otimes_{\mathbb{Q}}\mathbb{C}) \\ & & \| & & \| & & \| \\ \mathrm{GL}_n(F) & \subset & \prod_{v\mid\infty} \mathrm{GL}_n(F_v) & \xrightarrow{\simeq} & \prod_{v\mid\infty} \mathrm{GL}_n(\mathbb{R}) & \subset & \prod_{v\mid\infty} \mathrm{GL}_n(\mathbb{C}). \end{array} \tag{1.4}$$

Finally, we recall that the set of infinite places of F is in canonical bijection with $\mathrm{Hom}(F,\mathbb{R})$, so products such as, say, $\prod_{v\mid\infty} \mathrm{GL}_n(\mathbb{C})$ in the above diagrams can equivalently be rewritten as $\prod_{\iota:F\to\mathbb{R}} \mathrm{GL}_n(\mathbb{C})$.

1.8.2. Recall that, for a linear algebraic group G over a field k, one can look at *representations* of G on k-linear spaces, see e.g. [Mil17, Chap. 4]. In particular, if $\mathbb{C}$ is a k-algebra, a representation of G on a k-linear space V induces a representation of $\mathrm{G}(\mathbb{C})$ on the complex vector space $V\otimes_k \mathbb{C}$.

A complex representation of $\mathrm{GL}_n(\mathbb{C})$ which arises from a representation of the underlying algebraic group GL_n will be called a *rational representation* of $\mathrm{GL}_n(\mathbb{C})$. Similarly, a complex representation of

$$\prod_{\iota:F\to\mathbb{R}} \mathrm{GL}_n(\mathbb{C}) = \mathrm{GL}_n(F\otimes_{\mathbb{Q}}\mathbb{C}) = \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_n)(\mathbb{C})$$

is *rational* if it arises from a representation of the underlying algebraic group $\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_n)$.

1.8.3. We now briefly recall the classification of irreducible rational representations of $\mathrm{GL}_n(\mathbb{C})$. To that end, we will freely use the notations and terminology introduced in 1.6.5, and complement them with a few additions.

Firstly, we introduce the term "*weight*" as an equivalent description for a character of T_n. A weight μ can and will be identified with the corresponding tuple $(\mu_1,\dots,\mu_n)\in\mathbb{Z}^n$. We say that μ is *dominant* (relative to the standard choice of positive roots) if $\mu_1\geq\cdots\geq\mu_n$.

Given a dominant weight μ, one can construct an irreducible finite-dimensional complex representation of $\mathrm{GL}_n(\mathbb{C})$ as follows. First, μ defines a group homomorphism $\mathrm{T}_n(\mathbb{C})\to\mathbb{C}^*$, namely

$$\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mapsto t_1^{\mu_1}\cdots t_n^{\mu_n}.$$

(Maps of this form, with $\mu_i\in\mathbb{Z}$ for all i, are called *algebraic characters* of $\mathrm{T}_n(\mathbb{C})$.) Next, let $\overline{\mathrm{B}}_n(\mathbb{C})\subset\mathrm{GL}_n(\mathbb{C})$ denote the subgroup of *lower*-triangular matrices. We have a decomposition $\overline{\mathrm{B}}_n(\mathbb{C}) = \mathrm{T}_n(\mathbb{C}) \ltimes \overline{\mathrm{U}}_n(\mathbb{C})$, so it is meaningful to let $\overline{\mathrm{B}}_n(\mathbb{C})$ acts on the one-dimensional space $\mathbb{C}$ via

$$(t\overline{u})(z) = \mu(t)z \qquad \text{for all } t \in \mathrm{T}_n(\mathbb{C}), \overline{u} \in \overline{\mathrm{U}}_n(\mathbb{C}), z \in \mathbb{C}.$$

Let us denote this one-dimensional representation of $\overline{\mathrm{B}}_n(\mathbb{C})$ by χ_μ. Now, let $\mathcal{O}$ denote the *coordinate ring* of $\mathrm{GL}_n(\mathbb{C})$, i.e. the ring of maps

$$\begin{aligned} f\colon\ \mathrm{GL}_n(\mathbb{C}) &\to \mathbb{C} \\ A &\mapsto f(A) \end{aligned}$$

which are polynomial in the entries of A and in $\det(A)^{-1}$, and set

$$\mathrm{Ind}_{\overline{\mathrm{B}}_n(\mathbb{C})}^{\mathrm{GL}_n(\mathbb{C})}(\chi_\mu) \coloneqq \{f \in \mathcal{O} : f(\bar{b}g) = \chi_\mu(\bar{b})f(g) \text{ for all } \bar{b} \in \overline{\mathrm{B}}_n(\mathbb{C}), g \in \mathrm{GL}_n(\mathbb{C})\}.$$

Then $\mathrm{Ind}_{\overline{\mathrm{B}}_n(\mathbb{C})}^{\mathrm{GL}_n(\mathbb{C})}(\chi_\mu)$ is a finite-dimensional subspace of $\mathcal{O}$, stable under the action of $\mathrm{GL}_n(\mathbb{C})$ on $\mathcal{O}$ by right translation. As a complex representation of $\mathrm{GL}_n(\mathbb{C})$, it turns out to be an irreducible rational representation — it is called the *highest-weight module* attached to μ, and denoted by E_μ. (Cf. [Mil17, 22.30].) The classification result now states that, for every irreducible rational representation π of $\mathrm{GL}_n(\mathbb{C})$, there exists precisely one dominant weight μ such that π is equivalent to E_μ.

1.8.4. The irreducible rational representations of $\mathrm{GL}_n(F \otimes_\mathbb{Q} \mathbb{C})$ admit an entirely analogous classification — in fact, the preceding discussion goes through with only minor changes. The role of the dominant weights is now played by *tuples* $\underline{\mu} = (\mu^\iota)_{\iota:F\to\mathbb{R}}$, where each μ^ι is a dominant weight of T_n; as before, a tuple of weights $\underline{\mu}$ defines a character of $\mathrm{T}_n(F \otimes_\mathbb{Q} \mathbb{C}) = \prod_{\iota:F\to\mathbb{R}} \mathrm{T}_n(\mathbb{C})$, namely

$$\underline{\mu}((t_\iota)_{\iota:F\to\mathbb{R}}) = \prod_{\iota:F\to\mathbb{R}} \mu^\iota(t_\iota).$$

By $\mathcal{O}$ we shall now denote the coordinate ring of $\mathrm{GL}_n(F \otimes_\mathbb{Q} \mathbb{C}) = \prod_{\iota:F\to\mathbb{R}} \mathrm{GL}_n(\mathbb{C})$. Then,

$$\begin{aligned} \mathrm{Ind}_{\overline{\mathrm{B}}_n(F\otimes_\mathbb{Q}\mathbb{C})}^{\mathrm{GL}_n(F\otimes_\mathbb{Q}\mathbb{C})}(\underline{\mu}) \coloneqq \{f \in \mathcal{O} : f(t\overline{u}g) = \underline{\mu}(t)f(g) &\text{ for all } t \in \mathrm{T}_n(F \otimes_\mathbb{Q} \mathbb{C}), \\ &\overline{u} \in \overline{\mathrm{U}}_n(F \otimes_\mathbb{Q} \mathbb{C}), g \in \mathrm{GL}_n(F \otimes_\mathbb{Q} \mathbb{C})\} \end{aligned}$$

is an irreducible rational representation of $\mathrm{GL}_n(F \otimes_\mathbb{Q} \mathbb{C})$, which we call the *highest-weight module with highest weight* $\underline{\mu}$ and denote by $E_{\underline{\mu}}$. It is again the case that every irreducible rational representation of $\mathrm{GL}_n(F \otimes_{\mathbb{Q}} \mathbb{C})$ is of this form up to isomorphism.

It is important to note that, since every object appearing in the preceding discussion factors over the set of embeddings $\iota\colon F \to \mathbb{R}$, an irreducible rational representation of $\mathrm{GL}_n(F \otimes_{\mathbb{Q}} \mathbb{C}) = \prod_{\iota: F \to \mathbb{R}} \mathrm{GL}_n(\mathbb{C})$ necessarily arises as a tensor product of irreducible rational representations of the individual factors — more precisely, $E_{\underline{\mu}} \cong \bigotimes_{\iota: F \to \mathbb{R}} E_{\mu^\iota}$. However, it will ultimately be more convenient to work with the realization given above, cf. §5.1.

1.8.5. The above terminology naturally extends as follows. First, let G denote either $\mathrm{GL}_n(\mathbb{R})$ or $\mathrm{GL}_n(F_v)$ for an infinite place v of F, and view G as a subgroup of $\mathrm{GL}_n(\mathbb{C})$ in the natural way. Then a finite-dimensional complex representation of G will be called a *rational representation*, resp. a *highest-weight module*, of G if it arises as the restriction of a rational representation, resp. a highest-weight module, of $\mathrm{GL}_n(\mathbb{C})$. For a dominant weight μ ot $\mathrm{T}_n(\mathbb{C})$, we denote the corresponding highest-weight module of G again by E_μ.

Now we pass to the global level. Thus, let G denote either $\mathrm{GL}_n(F)$ or $\mathrm{GL}_n(F_\infty)$, viewed as a subgroup of $\mathrm{GL}_n(F \otimes_{\mathbb{Q}} \mathbb{C})$ via (1.4). As before, a finite-dimensional complex representation of G will be called a *rational representation*, resp. a *highest-weight module*, of G if it arises as the restriction of a rational representation, resp. a highest-weight module, of $\mathrm{GL}_n(F \otimes_{\mathbb{Q}} \mathbb{C})$. For a tuple $\underline{\mu} = (\mu^\iota)_{\iota: F \to \mathbb{R}}$, we denote the corresponding highest-weight module of G again by $E_{\underline{\mu}}$.

Chapter 2

Automorphic representations

"To paraphrase my remote ancestor (Humpty Dumpty), words mean whatever you define them to mean. But where does a definition lead?"
Jim Humphreys, comment on MathOverflow question n. 21899

2.1 Notations

Throughout this chapter, F will be a *totally real* number field. We will be interested in automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$, where n is a fixed positive integer. The notations introduced in this section will be used freely in the rest of the chapter.

2.1.1. For each infinite place v of F, we introduce the abbreviation G_v for $\mathrm{GL}_n(F_v)$; similarly, we let $\mathfrak{g}_v \coloneqq \mathrm{Lie}(G_v)$, the *Lie algebra* of G_v, and set $K_v \coloneqq \mathrm{O}_n(F_v)$, the *standard maximal compact subgroup* of G_v. Additionally, we write A_v for the connected central subgroup $\mathbb{R}_{>0} I_n \subset G_v$, where I_n denotes the identity matrix in G_v.

Similarly, we abbreviate

$$\begin{aligned} G_\infty \coloneqq \mathrm{GL}_n(F_\infty) &= \prod_{v|\infty} G_v, \\ \mathfrak{g}_\infty \coloneqq \mathrm{Lie}(G_\infty) &= \bigoplus_{v|\infty} \mathfrak{g}_v, \\ K_\infty \coloneqq \prod_{v|\infty} \mathrm{O}_n(F_v) &= \prod_{v|\infty} K_v. \end{aligned}$$

We also let A_∞ denote the connected central subgroup $\mathbb{R}_{>0}\boldsymbol{I}_n$, where $\boldsymbol{I}_n$ is the element of G_∞ whose component at each $v \mid \infty$ is the identity matrix in G_v. We stress the fact that $A_\infty \neq \prod_{v\mid\infty} A_v$ in general.

2.1.2. For any real Lie group G, the connected component of the neutral element will be denoted by G^0, and the *component group* G/G^0 will be denoted by $\pi_0(G)$. At each $v \mid \infty$, we have canonical isomorphisms

$$\pi_0(G_v) \cong \pi_0(K_v) \cong \pi_0(F_v^*) \cong \pi_0(\mathbb{R}^*) \cong \mathbb{Z}/2\mathbb{Z};$$

similarly,

$$\pi_0(G_\infty) \cong \pi_0(K_\infty) \cong \pi_0(F_\infty^\times) \cong \pi_0((\mathbb{R}^r)^\times) \cong (\mathbb{Z}/2\mathbb{Z})^r,$$

where we have written r for the number of real places of F.

2.2 Cuspidal and isobaric representations

2.2.1. The notion of a cuspidal automorphic representation will be essential in the sequel. Beside the classical definition given in the seminal reference [BJ79, 4.6] (see also [GH23, Defs. 6.6, 6.10]), there are, in the literature, several variations and refinements which we will find it convenient to work with, and which we will now spend a few words about.

For the purposes of this thesis, the fundamental notion will be that of a *unitary cuspidal automorphic representation of* $\mathrm{GL}_n(\mathbb{A}_F)$, by which we mean an irreducible representation of $\mathrm{GL}_n(\mathbb{A}_F)$ which is isomorphic to a subrepresentation of $L^2_{\mathrm{cusp}}(\mathrm{GL}_n(F)A_\infty\backslash \mathrm{GL}_n(\mathbb{A}_F))$. (Cf. [GH23, §6.5] and [Gro23, Chap. 13] for more details.)

If Π is a unitary cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, whose representation space we agree to again denote by the same symbol Π, then the subspace of *(adèlic) smooth vectors* $\Pi^{\infty_\mathbb{A}}$ is a *cuspidal smooth-automorphic representation* of $\mathrm{GL}_n(\mathbb{A}_F)$. We refer the reader to [GŽ21, §4.5] and [Gro23, Def. 15.4] for the definition of a cuspidal smooth-automorphic representation, and to [GŽ21, Thm. 4.31] and [Gro23, Thm. 15.7] for a proof of the claim. We also remark that, by the aforementioned theorems, *every* cuspidal smooth-automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ is obtained as the space of smooth vectors in a unitary cuspidal automorphic representation.

For the sake of completeness, as well as for future reference, we recall how the aforementioned notions relate to the classical one. (See [GH23, Thm. 6.5.1] and

[Gro23, §13.2] for an in-depth discussion.) If Π is a unitary cuspidal representation of $\mathrm{GL}_n(\mathbb{A}_F)$, then the subspace $\Pi^{\infty_{\mathbb{A}}}_{(K_\infty)}$ of K_∞-finite vectors in $\Pi^{\infty_{\mathbb{A}}}$ is (isomorphic to) an irreducible $(\mathfrak{g}_\infty, K_\infty) \times \mathrm{GL}_n(\mathbb{A}_\mathrm{f})$-submodule of the space

$$\mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_n, K_\infty, A_\infty)$$

of cuspidal K_∞-finite adèlic automorphic forms for GL_n which are invariant under A_∞. In particular, $\Pi^{\infty_{\mathbb{A}}}_{(K_\infty)}$ is a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ on which A_∞ acts trivially. (In the terminology of [GH23, §6.5], it is a *cuspidal automorphic representation of* $A_\infty \backslash \mathrm{GL}_n(\mathbb{A}_F)$.) Conversely, every representation of the latter kind is obtained as the subspace of K_∞-finite vectors in a unitary cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$.

It will often be convenient to simultaneously keep in mind the unitary, smooth and K_∞-finite viewpoints, and even to switch back and forth between them depending on the application at hand. To ease the burden on notation, we will usually not distinguish between Π, $\Pi^{\infty_{\mathbb{A}}}$ and $\Pi^{\infty_{\mathbb{A}}}_{(K_\infty)}$.

2.2.2. It follows from the *Multiplicity One Theorem* for the general linear group (see [GH23, Thm. 11.4.3] or the original source [Sha74, Thm. 5.5]), together with the above discussion, that a unitary cuspidal automorphic representation has, in fact, a *unique* realization as a subrepresentation of $L^2(\mathrm{GL}_n(F)A_\infty \backslash \mathrm{GL}_n(\mathbb{A}_F))$.

2.2.3. Next, we discuss isobaric representations, first introduced in [Lan79]. We refer to [Clo90, §1.1] and [GH23, §10.7] for a more complete treatment in the "classical", K_∞-finite language.

Let $n = \sum_{i=1}^d n_i$ be a partition of the positive integer n into strictly positive parts, and for each i, let Π_i be a unitary cuspidal automorphic representation of $\mathrm{GL}_{n_i}(\mathbb{A}_F)$. Then, if P denotes the parabolic subgroup of $\mathrm{GL}_n(\mathbb{A}_F)$ which corresponds to the ordered partition $(n_1, \ldots, n_d)$, cf. 1.6.5, we can construct the unitarily induced representation

$$\mathrm{Ind}_{\mathrm{P}(\mathbb{A}_F)}^{\mathrm{GL}_n(\mathbb{A}_F)}(\Pi_1 \otimes \cdots \otimes \Pi_d) \tag{2.1}$$

(see the proof of [Gro23, Thm. 15.21] for a detailed account of the construction). It is a fundamental result that the resulting representation of $\mathrm{GL}_n(\mathbb{A}_F)$ is *irreducible*. This is proved by combining local considerations at all places; for a uniform and recent treatment, we refer the reader to [LM16] for the non-Archimedean places and [BR10] for the Archimedean ones. One calls (2.1) the *isobaric sum* of the Π_i, and denotes it by $\boxplus_{i=1}^d \Pi_i$.

In this thesis, representations Σ of $\mathrm{GL}_n(\mathbb{A}_F)$ obtained by the above procedure will be called "*unitary isobaric representation*" to emphasize that the summands are unitary. It is a fundamental fact (cf. [JS81, (4.3)], [Lan79]) that, if Σ is a unitary isobaric representation of $\mathrm{GL}_n(\mathbb{A}_F)$, then Σ is, like its summands, again *automorphic*, in the sense that its subspace of K_∞-finite vectors is an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. This justifies calling Σ a "*unitary isobaric automorphic representation*", as we will do in the sequel. No distiction will be made notationally between Σ and its subspace of K_∞-finite vectors.

2.2.4. It is important to note that, if Σ is a unitary isobaric automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, and we write $\Sigma = \boxplus_{i=1}^d \Pi_i$ for unitary cuspidal automorphic representations Π_i, then, in fact, the Π_i are uniquely determined by Σ up to a permutation of the indices. This follows from a celebrated result of Jacquet–Shalika, see [JS81, Thm. 4.4].

We also wish to recall that, if Σ is a unitary isobaric representation, then Σ (or rather, its underlying K_∞-finite automorphic representation) has a *unique* realization as a subrepresentation of the space of K_∞-finite, A_∞-invariant adèlic automorphic forms on GL_n. This is the *Multiplicity One Theorem for isobaric representations*, which is obtained by combining results found in [Sha74], [JL70], and [FS98, §3.3]. The concrete realization is obtained by an application of the theory of *Eisenstein series*, for the definition of which we refer to [Sha10, Def. 6.1.4]; more precisely, the map

$$\Sigma = \mathrm{Ind}_{\mathrm{P}(\mathbb{A}_F)}^{\mathrm{GL}_n(\mathbb{A}_F)}(\Pi_1 \otimes \cdots \otimes \Pi_d) \to \mathcal{A}(\mathrm{GL}_n, K_\infty, A_\infty),$$
$$h \mapsto E_{\mathrm{P}}(\mathrm{id}, h, 0),$$

which sends an element h in the space of Σ to the value of the corresponding Eisenstein series at the identity, defines a nonzero $(\mathfrak{g}_\infty, K_\infty)\times\mathrm{GL}_n(\mathbb{A}_{\mathrm{f}})$-equivariant map between the two spaces. We refer e.g. to [GL21, §1.4.3] for more details.

2.3 Local components

2.3.1. It is well-known that automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$ decompose along the nontrivial places of the ground field F. More precisely, one starts with the easy observation ([GH23, Lemma 6.3.3]) that every automorphic representation Π decomposes as a tensor product

$$\Pi \cong \Pi_{\mathrm{f}} \otimes \Pi_\infty, \tag{2.2}$$

where Π_{f} is an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{A}_{\mathrm{f}})$ and Π_∞ is an irreducible $(\mathfrak{g}_\infty, K_\infty)$-module, and each is uniquely determined by Π up to isomorphism.

Next, we have that Π_∞ decomposes along the infinite places of F,

$$\Pi_\infty \cong \bigotimes_{v|\infty} \Pi_v, \tag{2.3}$$

where each Π_v is an irreducible admissible $(\mathfrak{g}_v, K_v)$-module. Similarly, the factor Π_{f} decomposes as a *restricted tensor product* along the finite places of F,

$$\Pi_{\mathrm{f}} \cong {\bigotimes_{v<\infty}}' \Pi_v, \tag{2.4}$$

where each Π_v is an irreducible admissible $\mathrm{GL}_n(F_v)$-module. The classical reference for this latter result is [Fla79], see also [GH23, Thm. 5.7.1, Thm. 6.3.4]. We emphasize that each *local component* Π_v is (only) determined by Π *up to equivalence*.

2.3.2. For completeness, we also mention the "smooth version" of the above result, derived in [GŽ21, Thm. 3.15], which reads as follows: if Π is a smooth-automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, then Π is isomorphic to a completed inductive tensor product $\Pi_\infty \mathbin{\overline{\otimes}} \Pi_{\mathrm{f}}$, where the second factor is an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{A}_{\mathrm{f}})$, equipped with the finest locally convex topology, and Π_∞ is an irreducible admissible representation of $\mathrm{GL}_n(F_\infty)$. The factor Π_{f} decomposes as a restricted tensor product $\bigotimes'_{v\nmid\infty} \Pi_v$ as per 2.3.1, and the factor Π_∞ decomposes as a completed inductive tensor product of irreducible admissible Casselman–Wallach $\mathrm{GL}_n(F_v)$-representations Π_v, where v ranges over the infinite places of F. Again, the local components at the non-Archimedean places are unique up to equivalence, and the local components at the infinite places are uniquely determined by Π and the requirement of being Casselman–Wallach representations.

2.3.3. Recall the *Strong Multiplicity Theorem for isobaric representations*, which follows from [JS81, Thm. 4.4]: if Σ and Σ' are unitary isobaric automorphic representations whose local components Σ_v and Σ'_v are isomorphic for all but finitely many places v of F, then, in fact, $\Sigma \cong \Sigma'$. (If we view both Σ and Σ' as subrepresentations of the space of automorphic representations, then we can even conclude that $\Sigma = \Sigma'$ by Multiplicity One, see 2.2.4.)

2.3.4. Recall that irreducible admissible representations of the local groups $\mathrm{GL}_n(F_v)$ can be described in terms of essentially square-integrable representations of $\mathrm{M}(F_v)$, as M runs over the Levi subgroups of $\mathrm{GL}_n(F_v)$ — this is the content of the *square-integrable local Langlands classification*, which is summarized in [GH23, §10.6] for the general linear group and in [Gro23, Thm. 5.7] for general reductive groups. For irreducible admissible essentially square-integrable representations σ_i of $\mathrm{GL}_{n_i}(F_v)$, we will denote their *Langlands quotient* (or *local isobaric sum*) by $\boxplus_{i=1}^{d} \sigma_i$; more generally, if π_i are irreducible admissible representations, then we can define $\boxplus_{i=1}^{d} \pi_i$ by writing each π_i as a Langlands quotient and exploiting the (essential) associativity and commutativity of the binary operation $\boxplus$. (Cf. [GH23, eq. (10.30)].) With this convention, we have that, if Σ is a unitary isobaric automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, and hence of the form $\Sigma = \boxplus_{i=1}^{d} \Pi_i$ for unitary cuspidal automorphic representations Π_i of $\mathrm{GL}_{n_i}(\mathbb{A}_F)$, then, at each place v, the local component Σ_v is none other than $\boxplus_{i=1}^{d} (\Pi_i)_v$, see [GH23, Thm. 10.7.5].

2.3.5. We recall that, at the real places, the local Langlands classification can be formulated much more explicitly as follows: if π is an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{R})$, then π is infinitesimally equivalent to a representation of the form

$$(D_{l_1} \otimes |\det|^{t_1}) \boxplus \cdots \boxplus (D_{l_r} \otimes |\det|^{t_r}) \boxplus (\mathrm{sgn}^{\varepsilon_1} |\cdot|^{t'_1}) \boxplus \cdots \boxplus (\mathrm{sgn}^{\varepsilon_{r'}} |\cdot|^{t'_{r'}}). \quad (2.5)$$

Here, the l_i are positive integers; the D_{l_i} denote discrete series representations of $\mathrm{SL}_2^{\pm}(\mathbb{R})$, cf. [Kna94, §2]; and the t_i are complex numbers, so that each summand of the form $D_{l_i} \otimes |\det|^{t_i}$ is an essentially square-integrable representation of $\mathrm{GL}_2(\mathbb{R})$. Similarly, the exponents ε_j are in $\{0, 1\}$, with the convention that sgn^0 is the trivial character, and the t'_j are again complex numbers, so each summand of the form $\mathrm{sgn}^{\varepsilon_j} |\cdot|^{t'_j}$ is an essentially square-integrable representation of $\mathrm{GL}_1(\mathbb{R})$. We further remark that the indices r, r' satisfy the equality $2r + r' = n$, and that π determines the l_i, t_i, ε_j, and t'_j up to permutations. We refer to *loc. cit.* for more details.

2.3.6. For the purposes of this thesis, it will be convenient to review the *local Langlands correspondence* for $\mathrm{GL}_n(\mathbb{R})$. We follow [Kna94, §3].

As a first step, recall that the *Weil group* of $\mathbb{R}$, denoted by $W_{\mathbb{R}}$, is, essentially by definition, a disjoint union

$$W_{\mathbb{R}} = \mathbb{C}^{\times} \sqcup j\mathbb{C}^{\times},$$

where the group operation on $\mathbb{C}^{\times}$ is extended to $W_{\mathbb{R}}$ by the rules $j^2 = -1$ and $jzj^{-1} = \overline{z}$ for all $z \in \mathbb{C}^{\times}$, where $\overline{z}$ denotes the complex conjugate of z. (Thus, $W_{\mathbb{R}}$ fits into a nonsplit short exact sequence $1 \to \mathbb{C}^{\times} \to W_{\mathbb{R}} \to \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to 1$.) The local Langlands correspondence can then be formulated as a one-to-one correspondence between infinitesimal equivalence classes of irreducible admissible representations of $\mathrm{GL}_n(\mathbb{R})$, on the one side, and equivalence classes of n-dimensional *semisimple* representations of $W_{\mathbb{R}}$, on the other. (Here, a finite-dimensional complex representation $\rho\colon W_{\mathbb{R}} \to \mathrm{GL}_n(\mathbb{C})$ is *semisimple* if $\rho(g)$ is semisimple, or in other words diagonalizable, for all $g \in W_{\mathbb{R}}$.) Under the local Langlands correspondence, a typical irreducible admissible representation π of $\mathrm{GL}_n(F_v)$ as in (2.5) corresponds to the n-dimensional semisimple complex representation of $W_{\mathbb{R}}$ which, in the notations of *loc. cit.*, is denoted as

$$(l_1, t_1) \oplus \cdots \oplus (l_r, t_r) \oplus (\tilde{\varepsilon}_1, t'_1) \oplus \cdots \oplus (\tilde{\varepsilon}_{r'}, t'_{r'}). \tag{2.6}$$

(Here, $\tilde{\varepsilon}_j$ is $+$ or $-$ according as $\varepsilon_j = 0$ or 1, respectively.)

2.4 Genericity

2.4.1. Recall the subgroup $\mathrm{U}_n \subset \mathrm{GL}_n$ introduced in 1.6.5, and let ψ be a character of $\mathrm{U}_n(\mathbb{A}_F)$ which is trivial on $\mathrm{U}_n(F)$. Throughout, we assume that ψ is *generic* in the sense of [GH23, Def. 11.1], cf. [Gro23, Def. 7.1, Ex. 7.4].

Let Π be a unitary isobaric automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, realized as a subrepresentation of the space of smooth-automorphic forms as in 2.2.4. Recall that Π is *(globally) ψ-generic* if there exists a function φ in the space of Π such that the *Fourier–Whittaker coefficient* of φ with respect to ψ, i.e. the function

$$\begin{aligned} W_\psi(\varphi)\colon A_\infty \backslash \mathrm{GL}_n(\mathbb{A}_F) &\to \mathbb{C}, \\ g &\mapsto \int_{\mathrm{U}_n(F)\backslash \mathrm{U}_n(\mathbb{A}_F)} \varphi(ug)\overline{\psi}(u)\,\mathrm{d}u \end{aligned} \tag{2.7}$$

is nonzero for some (hence: every) choice of Haar measure on the quotient group $\mathrm{U}_n(F)\backslash\mathrm{U}_n(\mathbb{A}_F)$. If this is the case, then each local component Π_v of Π has a *ψ-Whittaker model*, which is defined as the image of Π_v under any *Whittaker functional* and is hence automatically unique since the space of Whittaker functionals is one-dimensional. (Here, some care is required at at the Archimedean places: there, one needs to restrict to *continuous* Whittaker functionals in order for the assertion to hold true. We refer to [Gro23, §7.1] for a uniform treatment.) As a consequence of the local uniqueness of Whittaker models, cf. *loc. cit.*, we obtain the following result.

2.4.2. Proposition. *Let Π and ψ be as above, and factor $\Pi = \Pi_{\mathrm{f}} \otimes \Pi_\infty$ as well as $\psi = \psi_{\mathrm{f}} \otimes \psi_\infty$ as per 2.3.1. Then,*

(i) *the space*

$$\begin{aligned} \mathrm{Ind}_{\mathrm{U}_n(\mathbb{A}_{\mathrm{f}})}^{\mathrm{GL}_n(\mathbb{A}_{\mathrm{f}})}(\psi_{\mathrm{f}}) = \{W \in C^\infty(\mathrm{GL}_n(\mathbb{A}_{\mathrm{f}})) : W(ug) = \psi_{\mathrm{f}}(u)W(g) \\ \textit{for all } g \in \mathrm{GL}_n(\mathbb{A}_{\mathrm{f}}), u \in \mathrm{U}_n(\mathbb{A}_{\mathrm{f}})\}, \end{aligned}$$

acted on by $\mathrm{GL}_n(\mathbb{A}_\mathrm{f})$ *by right translation, has a* unique *subrepresentation which belongs to the isomorphism type* Π_f*.*

(ii) *The space*

$$\{W \in C^\infty(\mathrm{GL}_n(F_\infty)) : W \textit{ is of moderate growth and } W(ug) = \psi_\infty(u)W(g) \textit{ for all } g \in \mathrm{GL}_n(F_\infty), u \in \mathrm{U}_n(F_\infty)\},$$

acted on by $\mathrm{GL}_n(F_\infty)$ *by right translation, has a* unique *subrepresentation which belongs to the isomorphism type* Π_∞*.*

2.4.3. We will call the representation of $\mathrm{GL}_n(\mathbb{A}_\mathrm{f})$, resp. $\mathrm{GL}_n(F_\infty)$, defined by part (i), resp. (ii), of the above proposition the *Whittaker model* of Π_f, resp. Π_∞. It will be denoted by $\mathcal{W}(\Pi_\mathrm{f}, \psi_\mathrm{f})$, resp. $\mathcal{W}(\Pi_\infty, \psi_\infty)$.

It is a classical result by Shalika (see [Sha74]) that cuspidal automorphic representations are globally ψ-generic for every possible choice of ψ. The following result generalizes this result to unitary isobaric automorphic representations under a key assumption.

2.4.4. Theorem. *Let* Σ *be a unitary isobaric automorphic representation,* $\Sigma = \boxplus_{i=1}^d \Pi_i$ *for unitary cuspidal* Π_i*. Then* Σ *is globally generic if the* Π_i *are pairwise non-isomorphic.*

Proof. As remarked e.g. in [GH16], the claim follows from a combination of results contained in Shahidi's book [Sha10]. A more detailed outline of the argument was communicated to us by H. Grobner and is reproduced below.

First, we introduce some more notation: recall that, by assumption,

$$\Sigma = \mathrm{Ind}_{\mathrm{P}(\mathbb{A}_F)}^{\mathrm{GL}_n(\mathbb{A}_F)}(\Pi_1 \otimes \cdots \otimes \Pi_d),$$

where P is a standard parabolic subgroup of GL_n, with Levi subgroup $\mathrm{M} \cong \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_d}$, and Π_i is a unitary cuspidal automorphic representation of $\mathrm{GL}_{n_i}(\mathbb{A}_F)$, for all $i = 1, \ldots, d$. Moreover, we fix a generic character ψ.

We will now identify Σ with its concrete realization in the space of smooth-automorphic forms of GL_n, as per 2.2.4. Our goal is to prove that $W_\psi(\varphi)$ is nonzero for some φ in the space of Π. Recall from 2.2.4 that φ is of the form $E_\mathrm{P}(\mathrm{id}, h, 0)$ for some $h \in \mathrm{Ind}_{\mathrm{P}(\mathbb{A}_F)}^{\mathrm{GL}_n(\mathbb{A}_F)}(\Pi_1 \otimes \cdots \otimes \Pi_d)$, so we want to find an h such that

$$E_\mathrm{P}(\mathrm{id}, h, 0)_\psi := W_\psi(E_\mathrm{P}(\mathrm{id}, h, 0))$$

is not the zero function. We will achieve this using the concrete computations of *non-constant theorems of Eisenstein series* found in the aforementioned book by Shahidi.

First, we resort to *op. cit.*, Thm. 7.1.2, which gives us a choice of h for which $E_{\mathrm{P}}(\mathrm{id}, h, 0)_\psi$ can be written as a product

$$\prod_{v \in S} W_v(\mathrm{id}_v) \prod_{i=1}^{m} L^S(1 + is, \Pi_1 \otimes \cdots \otimes \Pi_d, r_i^\vee)^{-1} \tag{2.8}$$

(see *loc. cit.* for details), and to the proof of *op. cit.*, Thm. 7.2.1, which explains how one can further refine the choice of h in such a way that the finitely many factors $W_v(\mathrm{id}_v)$, $v \in S$ appearing in the first product on the right-hand side of (2.8) are all nonzero. It thus only remains to prove that the remaining factor $\prod_{i=1}^m L^S(1, \bigotimes_{i=1}^d \Pi_i, r_i^\vee)^{-1}$ is nonzero; here, the r_i are, as in [Sha10, p. 90], the irreducible summands of the adjoint action r of ${}^L M$ on $\mathrm{Lie}({}^L N)$ (see [Sha10, Section 2.4] for these notations and further discussion).

For the final remaining step, we take advantage of the equality

$$\prod_{i=1}^{m} L^S(1 + is, \Pi_1 \otimes \cdots \otimes \Pi_d, r_i^\vee) = \prod_{i<j} L^S(s, \Pi_i \times \Pi_j^\vee), \tag{2.9}$$

where the factors on the right-hand side are *(partial) Rankin–Selberg L-functions*, that will be discussed in more detail in the next chapter. (The equality is not immediate but nevertheless well-known: it can be obtained directly, i.e. by working out the r_i, or inferred from the knowledge of the normalization factors in the standard intertwining operators appearing in the functional equations of Eisenstein series, cf. e.g. [Bad08, Appendix A].) We now recall (see e.g. [GH23, Thm. 11.7.1]) that a Rankin–Selberg automorphic L-function $L(s, \Pi \times \Pi')$, with Π and Π' unitary cuspidal, is holomorphic everywhere unless Π' is isomorphic to the *dual* $\Pi^\vee$ of Π (cf. 3.3.1 for the notion of the dual automorphic representation). Since we assumed all Π_i to be pairwise non-isomorphic, the product (2.9) is holomorphic everywhere, and so its reciprocal, appearing in (2.8) is surely nonzero, as desired. □

2.5 Cohomological representations

In this section, we finally introduce cohomological automorphic representations. Before giving the definition (2.5.2 below), we start with some local preparations.

2.5.1. Recall the notion of *relative Lie algebra cohomology*: if $\mathfrak{g} \supseteq \mathfrak{k}$ are two Lie algebras over, say, the complex field $\mathbb{C}$, and V is a $\mathfrak{g}$-module — meaning: a complex vector space which affords an action of $\mathfrak{g}$ — then one defines complex vector spaces $H^q(\mathfrak{g}, \mathfrak{k}; V)$, where $q \in \mathbb{N}$, as in [BW00, I, §1].

We will compute relative Lie algebra cohomology in a special case. Resume the notations G_∞, $\mathfrak{g}_\infty$ and K_∞ first introduced in §2.1, and set $\boldsymbol{K}_\infty \coloneqq K_\infty A_\infty = (\prod_{v|\infty} K_v) A_\infty$. (Thus, $\boldsymbol{K}_\infty^0 = (\prod_{v|\infty} K_v^0) A_\infty$.) If V is a $(\mathfrak{g}_\infty, K_\infty)$-*module*, see e.g. [BW00, 0, §2.5] for the definition, we set

$$H^\bullet(\mathfrak{g}_\infty, \boldsymbol{K}_\infty^0; V) \coloneqq H^\bullet(\mathfrak{g}_{\infty,\mathbb{C}}, \mathrm{Lie}(\boldsymbol{K}_\infty)_\mathbb{C}; V). \tag{2.10}$$

(Cf. *op. cit.*, I, §5.) Here, the subscript $_\mathbb{C}$ denotes complexification, as usual.

2.5.2. Let $E = E_{\underline{\mu}}$ be a highest-weight module for G_∞. A unitary isobaric automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$ is *cohomological with respect to E* if

$$H^\bullet(\mathfrak{g}_\infty, \boldsymbol{K}_\infty^0; \Pi_\infty \otimes E) \neq 0.$$

(Here, E is also viewed as a $(\mathfrak{g}_\infty, K_\infty)$-module in a natural way.) We stress that this is a condition only on the *infinite* part Π_∞ of Π.

If a representation Π is cohomological, this imposes strong conditions on the shape of Π_∞. To explain this in the required level of detail, we first have to consider each Π_v, with $v \mid \infty$, individually, and then piece the local information back together. We start with the local analogue of the preceding definitions.

2.5.3. Let v be an infinite place of F; let G_v, $\mathfrak{g}_v$, K_v and A_v be as in §2.1; and finally, set $\boldsymbol{K}_v = K_v A_v$. For a $(\mathfrak{g}_v, K_v)$-module V, we proceed analogously to 2.5.1 and set

$$\begin{aligned} H^\bullet(\mathfrak{g}_v, \boldsymbol{K}_v^0; V) &\coloneqq H^\bullet(\mathfrak{g}_{v,\mathbb{C}}, \mathrm{Lie}(\boldsymbol{K}_v)_\mathbb{C}; V) \\ &= H^\bullet(\mathfrak{g}_{v,\mathbb{C}}, (\mathfrak{k}_v \oplus \mathfrak{a}_v)_\mathbb{C}; V), \end{aligned} \tag{2.11}$$

where $\mathfrak{k}_v = \mathrm{Lie}(K_v)$ and $\mathfrak{a}_v = \mathrm{Lie}(A_v)$. Since $\mathfrak{a}_v$ lies in the center of $\mathfrak{g}_v$, it is in particular a direct summand of $\mathfrak{g}_v$ — thus, if $\mathfrak{a}_v$ acts trivially on V, we can apply the *Künneth rule* to obtain

$$H^\bullet(\mathfrak{g}_v, \boldsymbol{K}_v^0; V) = H^\bullet(\tilde{\mathfrak{g}}_{v,\mathbb{C}}, \mathfrak{k}_{v,\mathbb{C}}; V), \tag{2.12}$$

where $\tilde{\mathfrak{g}}_v$ is the direct summand of $\mathfrak{g}_v$ complementary to $\mathfrak{a}_v$.

2.5.4. Let $E = E_\mu$ be a highest-weight module for G_v. An irreducible admissible representation π of G_v is *cohomological with respect to E* if

$$H^\bullet(\mathfrak{g}_v, \boldsymbol{K}_v^0; \pi \otimes E) \neq 0,$$

where the cohomology is actually taken with coefficients in $\pi_{(K_v)} \otimes E$.

2.5.5. The cohomological, *essentially tempered* representations of $G = \mathrm{GL}_n(\mathbb{R})$ admit an elegant classification, which we now recall. (Cf. [Mah05, 3.1.3], [GR14, Sec. 5.5].)

First, let P denote the standard parabolic subgroup of GL_n corresponding to the ordered partition $(2, \dots, 2)$, if n is even, and to $(2, \dots, 2, 1)$, if n is odd. We agree to denote $\mathrm{P}(\mathbb{R})$ simply by P.

We will now look at tuples $(\boldsymbol{l}, w) = (l_1, \dots, l_{\lfloor n/2 \rfloor}, w) \in \mathbb{Z}^{\lfloor n/2 \rfloor + 1}$ such that

(1) $l_1 > \cdots > l_{\lfloor n/2 \rfloor}$ and

(2) $l_1 \equiv l_2 \equiv \cdots \equiv l_{\lfloor n/2 \rfloor} \equiv w - n + 1 \pmod 2$.

If n is even, then for such a tuple $(\boldsymbol{l}, w)$ we set

$$\pi^{\boldsymbol{l},w} := \mathrm{Ind}_P^G((D_{l_1} \otimes |\det|^{-w/2}) \otimes \cdots \otimes (D_{l_{\lfloor n/2 \rfloor}} \otimes |\det|^{-w/2}));$$

on the other hand, if n is odd, then for such a tuple $(\boldsymbol{l}, w)$ and an $\varepsilon \in \{0, 1\}$ we set

$$\pi^{\boldsymbol{l},w,\varepsilon} := \mathrm{Ind}_P^G((D_{l_1} \otimes |\det|^{-w/2}) \otimes \cdots \otimes (D_{l_{\lfloor n/2 \rfloor}} \otimes |\det|^{-w/2}) \otimes (\mathrm{sgn}^{\varepsilon} | \cdot |^{-w/2})).$$

(cf. 2.3.4). The aforementioned classification can now be phrased as follows. Each $\pi^{\boldsymbol{l},w}$, resp. $\pi^{\boldsymbol{l},w,\varepsilon}$, is an irreducible admissible representation of G which is additionally essentially tempered, generic, and cohomological with respect to some highest-weight module E_μ. Conversely, if an irreducible admissible representation π of G is essentially tempered and cohomological with respect to some E_μ, then it is isomorphic to some $\pi^{\boldsymbol{l},w}$, or $\pi^{\boldsymbol{l},w,\varepsilon}$, depending on whether n is even or odd.

If π is cohomological with respect to E_μ, and hence $\pi \cong \pi^{\boldsymbol{l},w}$ or $\pi \cong \pi^{\boldsymbol{l},w,\varepsilon}$, then one can relate μ to the data $(\boldsymbol{l}, w)$ as follows. (Cf. again [Mah05, 3.1.3], [GR14, Sec. 5.5].)

2.5.6. Proposition. *Let* $(\boldsymbol{l}, w) = (l_1, \dots, l_{\lfloor n/2 \rfloor}, w)$ *be as in* 2.5.5*, and let* $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$ *be the unique* n*-tuple of integers satisfying*

(a) $l_i + w - (n-1) = 2(\mu_i - i + 1)$ *for* $i = 1, \dots, \lfloor n/2 \rfloor$*;*

(b) $\mu_i + \mu_{n-i+1} = w$ *for* $i = 1, \dots, n$*.*

Then μ *is a dominant weight, and*

$$\begin{cases} \pi^{\boldsymbol{l},w} \text{ is cohomological with respect to } E_\mu & (n \text{ even}); \\ \pi^{\boldsymbol{l},w,0} \text{ and } \pi^{\boldsymbol{l},w,1} \text{ are both cohomological w.r.t. } E_\mu & (n \text{ odd}). \end{cases} \tag{2.13}$$

Conversely, let μ *be a dominant weight satisfying* (b) *above for some* w*, and define* l_i*, with* $i = 1, \dots, \lfloor n/2 \rfloor$*, by* (a)*. Then* $(\boldsymbol{l}, w) = (l_1, \dots, l_{\lfloor n/2 \rfloor}, w)$ *satisfies* (1)–(2) *from* 2.5.5*, and* (2.13) *is again verified.*

2.5.7. *Remark.* In particular, if π is cohomological with respect to E_μ, then π determines μ uniquely, and μ determines π uniquely up to equivalence (if n is even) resp. uniquely "up to a sign" (if n is odd). $\diamond$

We now go back to the global discussion.

2.5.8. Proposition. *Let Π be a unitary isobaric automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, and, for any infinite place v of F, let Π_v denote the local component of Π at v. Suppose that Π is cohomological with respect to a highest-weight module $E_{\underline{\mu}} = \bigotimes_{v|\infty} E_{\mu^v}$ of G_∞. Then the dominant weights μ^v satisfy*

$$\mu_i^v + \mu_{n-i+1}^v = 0 \qquad \textit{for all } i = 1, \ldots, n \textit{ and all } v \mid \infty, \tag{2.14}$$

and each local component Π_v is cohomological with respect to E_{μ^v}.

Conversely, suppose that each Π_v is cohomological with respect to some E_{μ^v}. Then μ^v satisfies (2.14), and Π is cohomological with respect to $E_{\underline{\mu}} = \bigotimes_{v|\infty} E_{\mu^v}$.

2.5.9. *About the proof.* The ultimate reason for the "purity condition" (2.14) is the fact (see [BC83, Lemma 1.3]) that any irreducible unitary representation of a connected real Lie group is *conjugate self-dual.* It then follows ([BW00, Cor. I.4.2]) that, if the representation Π_v at hand is cohomological, its coefficient system E_{μ^v} must also be conjugate self-dual, i.e., (2.14) must hold true. (The argument goes through even though $\mathrm{GL}_n(\mathbb{R})$ is disconnected since the restriction to $\mathrm{GL}_n(\mathbb{R})^0$ certainly decomposes into a direct sum of irreducible representations, all isomorphic to each other, and and we can then apply the result from the connected case.)

2.5.10. Corollary. *Let Σ be unitary isobaric, $\Sigma = \boxplus_{i=1}^d \Pi_i$ for unitary cuspidal representations Π_i of $\mathrm{GL}_{n_i}(\mathbb{A}_F)$, and assume that Σ is cohomological with respect to some highest-weight module. Then:*

(i) *at most one of the n_i is odd,*
(ii) *each summand Π_i is again cohomological, and*
(iii) *Σ is globally generic.*

Proof. Fix an infinite place v and consider only local components at v. Then, taking into account the above classification results as well as induction in stages, we immediately derive (i)–(ii) as well as the fact that the Π_i must be pairwise distinct. The third claim then follows from this last fact by an application of Theorem 2.4.4. $\square$

As for the explicit computation of the cohomology groups $H^\bullet(\mathfrak{g}_\infty, \boldsymbol{K}_\infty^0; \Pi_\infty \otimes E_{\underline{\mu}})$, we record the following result. We refer to [Clo90, Lemme 3.14], where the general result [BW00, Thm. III.3.3] is made explicit in the case at hand.

2.5.11. Proposition. *Let Π be a unitary isobaric automorphic representation of* $\mathrm{GL}_n(\mathbb{A}_F)$ *which is cohomological with respect to some highest-weight module $E_{\underline{\mu}}$. Then,*

$$H^q(\mathfrak{g}_\infty, \boldsymbol{K}^0_\infty; \Pi_\infty \otimes E) \neq 0 \quad \textit{if and only if} \quad b_n \leq q \leq t_n,$$

where

$$b_n := [F:\mathbb{Q}]\left\lfloor \frac{n^2}{4} \right\rfloor,$$
$$t_n := b_n + [F:\mathbb{Q}]\left\lceil \frac{n}{2} \right\rceil - 1.$$

Moreover, in the bottom degree *$q = b_n$ we have the following description of $H^q(\mathfrak{g}_\infty, \boldsymbol{K}^0_\infty; \Pi_\infty \otimes E)$ as a $\pi_0(G_\infty)$-module:*

(i) *If n is even, then*

$$H^{b_n}(\mathfrak{g}_\infty, \boldsymbol{K}^0_\infty; \Pi_\infty \otimes E) = \bigoplus_{\varepsilon \in \widehat{\pi_0(G_\infty)}} \varepsilon.$$

(ii) *If n is odd, and we write $\Pi_\infty \cong \bigotimes_{v|\infty} \pi^{\boldsymbol{l}^v, 0, \varepsilon_v}$, then*

$$H^{b_n}(\mathfrak{g}_\infty, \boldsymbol{K}^0_\infty; \Pi_\infty \otimes E) = (-1)^{[F:\mathbb{Q}](n-1)/2}\varepsilon,$$

where we have set $\varepsilon := (\varepsilon_v)_{v|\infty}$.

2.5.12. *About the proof.* The claim follows with the help of the Künneth rule from the corresponding "local" assertions at each $v \mid \infty$. It is worth remarking that, since $\boldsymbol{K}_\infty \neq \prod_{v|\infty} \boldsymbol{K}_v$ in general, one has to take into account an additional factor $(\bigoplus_{v|\infty} \mathfrak{a}_v)/\mathfrak{a}_\infty$, as explained in the proof of [HR20, Prop. 3.4]. □

2.5.13. We conclude with a definition. Let Π be a cohomological unitary isobaric representation of $\mathrm{GL}_n(\mathbb{A}_F)$, and let ε be a unitary character of $\pi_0(G_\infty)$. We will say that ε is *permissible* for Π if the ε-isotypic component in the bottom-degree cohomology group $H^{b_n}(\mathfrak{g}_\infty, \boldsymbol{K}^0_\infty; \Pi_\infty \otimes E)$ is nonzero. Thus, every ε is permissible for every Π, if n is even, whereas, if n is odd, a cohomological representation Π uniquely determines a permissible signature $\varepsilon_{\mathrm{perm}}(\Pi)$.

Chapter 3

Rankin–Selberg automorphic L-functions

"The authors must confess that they have not understood the final meaning of this result."

Hervé Jacquet, Ilya I. Piatetskiĭ-Shapiro,
Joseph A. Shalika, *Rankin–Selberg Convolutions* [JPS83]

Throughout this chapter, the ground field F is a totally real number field. Our goal will be to define Rankin–Selberg automorphic L-functions and to discuss critical points of such L-functions.

3.1 Local L-factors

Let v be a nontrivial place of F; let m and n be positive integers; and let π and π' be irreducible admissible representations of $\mathrm{GL}_n(F_v)$ and $\mathrm{GL}_m(F_v)$, respectively. Throughout, we assume $m \leq n$.

3.1.1. If v is a *finite* place of F, we define the *local L-factor* $L(s, \pi \times \pi')$ as in the classical source [JPS83].

In more detail, suppose first that π and π' are *generic*. Then the local L-factor is defined as per *op. cit.*, (2.7). This applies in particular when π and π' are *essentially square-integrable*.

Next, let π and π' be arbitrary irreducible admissible representations. Then, as we saw in 2.3.4, we may write π and π' as Langlands quotients $\pi = \boxplus_{i=1}^{d} \sigma_i$

and $\pi' = \boxplus_{j=1}^{d'} \sigma'_j$ for essentially square-integrable representations σ_i and σ'_j. Following *op. cit.*, (9.4), we set

$$L(s, \pi \times \pi') \coloneqq \prod_{i,j} L(s, \sigma_i \times \sigma'_j). \tag{3.1}$$

As remarked in *loc. cit.*, this yields the previous definition if π and π' happen to be generic, so the construction is well-defined.

3.1.2. Now let v be an *infinite* place of F, and let ρ (resp., ρ') be the n-dimensional (resp., m-dimensional) semisimple complex representation of the Weil group $W_{\mathbb{R}}$ corresponding to Π_v (resp., Π'_v) via the local Langlands correspondence. Then $\rho \otimes \rho'$ decomposes as a direct sum of irreducible representations ρ_j, and one defines

$$L(s, \Pi_v \times \Pi'_v) = \prod_j L(s, \rho_j),$$

where each factor on the right-hand side is defined as in [Kna94, eq. (3.6)], i.e.:

$$L(s, \rho_j) = \begin{cases} 2(2\pi)^{-(s+t+\frac{l}{2})}\Gamma(s+t+\frac{l}{2}) & \text{if } \rho_j \cong (l,t), \\ \pi^{-(s+t)/2}\Gamma(\frac{s+t}{2}) & \text{if } \rho_j \cong (+,t), \\ \pi^{-(s+t+1)/2}\Gamma(\frac{s+t+1}{2}) & \text{if } \rho_j \cong (-,t), \end{cases}$$

where Γ denotes the classical gamma function.

3.2 Global L-functions

Let m and n again be positive integers with $m \leq n$, and let Π, resp. Π', be an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, resp. $\mathrm{GL}_m(\mathbb{A}_F)$. In this section, we define the global L-function $L(s, \Pi \times \Pi')$ *under the assumption that* Π *and* Π' *are unitary isobaric*. Since the standard literature is mostly restricted to the case in which Π and Π' are both cuspidal, we will define global L-functions with some additional care, and in particular expand upon the discussion found e.g. in [Cog, Lecture 4] or [GH23, §11.6, §11.7].

3.2.1. Recall that the representations Π and Π' are *unramified* at almost all places of F. In more details, there exists a finite set S of nontrivial places of F, containing all infinite places, such that, for each place $v \notin S$, the representations Π_v and Π'_v are *unramified* in the sense of [GH23, Def. 7.1].

Whenever Π_v and Π'_v are both unramified, the local L-factor can also be computed in terms of the *Satake parameters* (see e.g. [GH23, §7.5]) of Π_v and Π'_v. This is the so-called *unramified calculation*, and is carried out e.g. in [Cog, §3.1.3].

3.2.2. We now take advantage of the assumption that Π and Π' are *unitary isobaric* automorphic representations. Let S be a finite set of places as in 3.2.1, and consider the infinite product

$$\prod_{v \notin S} L(s, \Pi_v \times \Pi'_v).$$

By [JS81, (3.2)], this product is absolutely convergent for $\mathrm{Re}(s)$ sufficiently large. Thus, it is meaningful to write

$$L(s, \Pi \times \Pi') := \prod_{v} L(s, \Pi_v \times \Pi'_v), \tag{3.2}$$

with the product now ranging over *all* nontrivial places of F, so long as $\mathrm{Re}(s)$ is sufficiently large. It is also customary to write

$$L(s, \Pi_{\mathrm{f}} \times \Pi'_{\mathrm{f}}) := \prod_{v \nmid \infty} L(s, \Pi_v \times \Pi'_v), \tag{3.3}$$

$$L(s, \Pi_{\infty} \times \Pi'_{\infty}) := \prod_{v \mid \infty} L(s, \Pi_v \times \Pi'_v). \tag{3.4}$$

In the statement of the following well-known result, we assume the automorphic representations Π and Π' to be unitary isobaric; it will be convenient to temporarily denote them by Σ and Σ', respectively.

3.2.3. Theorem. *Write $\Sigma = \boxplus_{i=1}^{d} \Pi_i$ and $\Sigma' = \boxplus_{j=1}^{d'} \Pi'_j$, where the Π_i and Π'_j are unitary cuspidal. Then*

$$L(s, \Sigma \times \Sigma') = \prod_{i,j} L(s, \Pi_i \times \Pi'_j) \tag{3.5}$$

for $\mathrm{Re}(s)$ *sufficiently large.*

Proof. Let s be a complex number with sufficiently large real part, and expand each factor on either side of (3.5) into an infinite product as per (3.2). Since the convergence of all these products is absolute, and hence unconditional, it suffices to prove

$$L(s, \Sigma_v \times \Sigma'_v) = \prod_{i,j} L(s, (\Pi_i)_v \times (\Pi'_j)_v) \tag{3.6}$$

for every nontrivial place v. At the finite places, this holds true by definition ((3.1)) since, as we remarked in 2.3.4, we have equalities $\Sigma_v = \boxplus_{i=1}^{d} (\Pi_i)_v$ and $\Sigma'_v = \boxplus_{j=1}^{d'} (\Pi'_j)_v$. At the infinite places, the claim (3.6) follows from the fact that the local Langlands correspondence "sends isobaric sums to direct sums" (see [GH23, §12.4, esp. (12.18)]). $\square$

3.2.4. When Π and Π' are unitary cuspidal, it is known (cf. [GH23, Thm. 11.7.1]) that the product defining $L(s, \Pi \times \Pi')$ is absolutely convergent for $\mathrm{Re}(s) > 1$, and that the global L-function admits a meromorphic continuation to the whole complex plane. It then follows from Theorem 3.2.3 that the same is true of $L(s, \Sigma \times \Sigma')$ when Σ and Σ' are unitary isobaric, and that the equality (3.5) remains true in the field of meromorphic functions on $\mathbb{C}$.

3.3 Critical points

Let Π and Π' be unitary isobaric automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$ and $\mathrm{GL}_m(\mathbb{A}_F)$, respectively, where m and n are again positive integers with $m \leq n$.

3.3.1. We start by recalling the *functional equation* satisfied by Rankin–Selberg L-functions:

$$L(s, \Pi \times \Pi') = \varepsilon(s, \Pi \times \Pi') L(1 - s, \Pi^\vee \times \Pi'^\vee),$$

where the *ε-factor* $\varepsilon(s, \Pi \times \Pi')$ is defined as in [GH23, eq. (11.24)], and $\Pi^\vee$, resp. $\Pi'^\vee$, denotes the *dual representation* (also called *contragredient representation*) to Π, resp. Π', for which we refer e.g. to [Bum98, Sect. 3.3].

3.3.2. A complex number $s_0 \in \frac{n-m}{2} + \mathbb{Z} \subset \mathbb{C}$ is a *critical point* of $L(s, \Pi \times \Pi')$, or alternatively of $L(s, \Pi_\mathrm{f} \times \Pi'_\mathrm{f})$, if neither of the meromorphic functions

$$\begin{aligned} &s \mapsto L(s, \Pi_\infty \times \Pi'_\infty), \\ &s \mapsto L(1 - s, \Pi^\vee_\infty \times \Pi'^\vee_\infty) \end{aligned}$$

(cf. (3.4)) has a pole in $s = s_0$. The set of critical points of $L(s, \Pi \times \Pi')$ will be denoted by $\mathrm{Crit}(\Pi \times \Pi')$.

3.3.3. Suppose that Π is unitary cuspidal and Π' is unitary isobaric. Let us agree to denote Π' as Σ, instead, and let us write $\Sigma = \boxplus_{i=1}^d \Pi_i$, where Π_i is a unitary cuspidal automorphic representation of $\mathrm{GL}_{m_i}(\mathbb{A}_F)$ and $\sum_{i=1}^d m_i = m$. We then know from 3.2.3, 3.2.4 that $L(s, \Pi \times \Sigma) = \prod_{i=1}^d L(s, \Pi \times \Pi_i)$ for all $s \in \mathbb{C}$. We now want to relate the critical points of the L-function on the left-hand side with the critical points of the factors on the right.

Recall from 3.1.2 that the meromorphic function $L(s, \Pi_\infty \times \Pi'_\infty)$ is given, up to a nonzero scalar, by a product of factors of the form $\Gamma(\ell_j(s))$, where each $\ell_j(s)$ is a suitable linear shift of the variable s. Since the gamma function never vanishes, we readily derive the following result, a version of which has appeared in [GS20].

3.3.4. Proposition. *Suppose that the integers $m_1, \dots, m_d$ and $m = \sum_{i=1}^d m_i$ all have the same parity. Then a complex number $s_0 \in \frac{n-m}{2} + \mathbb{Z}$ is a critical point for $L(s, \Pi \times \Sigma)$ if and only if it is a critical point for $L(s, \Pi \times \Pi_i)$ for each $i = 1, \dots, d$. In other words, we have*

$$\mathrm{Crit}(\Pi \times \Sigma) = \bigcap_{i=1}^{d} \mathrm{Crit}(\Pi \times \Pi_i).$$

3.4 Critical points for cohomological representations

Let $n \geq 2$ be an integer, Π be a unitary cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ and $\Sigma = \boxplus_{i=1}^d \Pi_i$ be a unitary isobaric automorphic representation of $\mathrm{GL}_{n-1}(\mathbb{A}_F)$. Furthermore, we assume that both Π and Σ are *cohomological* in the sense of 2.5.2.

3.4.1. We start with some general considerations on highest-weight modules. The inclusion $\mathrm{GL}_{n-1}(\mathbb{C}) \hookrightarrow \mathrm{GL}_n(\mathbb{C})$ given by $g \mapsto \left(\begin{smallmatrix} g & \\ & 1 \end{smallmatrix}\right)$ allows us to regard representations of $\mathrm{GL}_n(\mathbb{C})$ as representations of $\mathrm{GL}_{n-1}(\mathbb{C})$. In general, an irreducible representation of $\mathrm{GL}_n(\mathbb{C})$ does not remain irreducible upon restricting to $\mathrm{GL}_{n-1}(\mathbb{C})$, but decomposes into a sum of irreducibles. We have the following *branching law* ([GW09, Thm. 8.1.1], see also [Rag16, Prop. 2.19]).

3.4.2. Proposition. *Let E_μ be a highest-weight module for $\mathrm{GL}_n(\mathbb{C})$, and E_λ be a highest-weight module for $\mathrm{GL}_{n-1}(\mathbb{C})$. Then the following are equivalent:*

(i) *E_λ occurs in the restriction $E_\mu|_{\mathrm{GL}_{n-1}(\mathbb{C})}$, i.e., $\mathrm{Hom}_{\mathrm{GL}_{n-1}(\mathbb{C})}(E_\lambda, E_\mu) \neq 0$;*

(ii) *the highest weights μ and λ* interlace *in the sense that*

$$\mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_n.$$

Moreover, if E_λ occurs in $E_\mu|_{\mathrm{GL}_{n-1}(\mathbb{C})}$, then it does so with multiplicity one.

3.4.3. Let E_μ, resp. E_λ, be the highest-weight module with respect to which Π, resp. Σ, is cohomological. (Thus, E_μ, resp. E_λ, is a highest-weight module for $\mathrm{GL}_n(F_\infty)$, resp. $\mathrm{GL}_{n-1}(F_\infty)$.) Then there is an explicit description of the critical points of $L(s, \Pi \times \Sigma)$ if the coefficient modules E_μ and E_λ are *compatible* in the sense that, for all $v \mid \infty$,

$$\mathrm{Hom}_{\mathrm{GL}_{n-1}(\mathbb{C})}(E_{\mu^v} \otimes E_{\lambda^v}, \mathbb{C}) \neq 0.$$

(By Proposition 3.4.2, this is equivalent to the assumption that, at each infinite place v, the weights μ^v and λ^v satisfy the *piano condition*

$$\mu_1^v \geq -\lambda_{n-1}^v \geq \mu_2^v \geq -\lambda_{n-2}^v \geq \cdots \geq -\lambda_1^v \geq \mu_n^v,$$

cf. [GL21, Hypothesis 1.44].) Under this assumption, we have ([Rag16, Thm. 2.21]) that a half-integer $s_0 = \frac{1}{2} + m$ is critical for $L(s, \Pi \times \Sigma)$ if and only if

$$\mu_1^v \geq -\lambda_{n-1}^v + m \geq \mu_2^v \geq -\lambda_{n-2}^v + m \geq \cdots \geq -\lambda_1^v + m \geq \mu_n^v \tag{3.7}$$

for all $v \mid \infty$. It follows that there exist integers $m_{\mu,\lambda}^- \leq 0 \leq m_{\mu,\lambda}^+$ such that

$$\left\{ m \in \mathbb{Z} : \frac{1}{2} + m \text{ is critical for } L(s, \Pi \times \Sigma) \right\} = \{ m \in \mathbb{Z} : m_{\mu,\lambda}^- \leq m \leq m_{\mu,\lambda}^+ \},$$

see e.g. *op. cit.*, Cor. 2.35.

3.4.4. *Remark*. In the literature (see e.g. [LLS21], cf. also [JST19]), one encounters the notion of a *balanced place*; in our notation, an integer m is a *balanced place* for the pair (E_μ, E_λ) if (3.7) holds. It follows from [Rag16, Thm. 2.21] that, *if* a balanced place exists, then $m \mapsto \frac{1}{2} + m$ defines a bijection from the set of balanced places for (E_μ, E_λ) to the set of critical points for $L(s, \Pi \times \Sigma)$. However, if the set of balanced places is empty, then it does not follow that the set of critical points is also empty, see *op. cit.*, Section 2.4.5.4.

Chapter 4

Twists, rational structures, and periods

"Some proofs are given in a more general context, but only when they could not be made shorter by specializing the hypothesis to the concrete situation we have in mind. It is not our intention to write a treatise on commutative algebra."

Serge Lang, *Algebraic Number Theory*

For the entirety of this chapter, we leave the world of automorphic representations and look more generally at representations of some fixed group G. Unless expressly stated to the contrary, all representations of G will tactily be over the same ground field $\boldsymbol{C}$, which is fixed throughout, but arbitrary. (Our projected use case will be $\boldsymbol{C} = \mathbb{C}$, but everything applies equally well to, say, $\boldsymbol{C} = \overline{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$ inside $\mathbb{C}$.)

Recall that, to be entirely precise, a $\boldsymbol{C}$-linear representation of G is defined to be a pair (π, V), where V is a $\boldsymbol{C}$-vector space and $\pi \colon G \to \mathrm{GL}(V)$ is a group homomorphism. However, by abuse of notation, we might sometimes refer to π itself as the representation of G.

4.1 Twists

4.1.1. As usual, $\mathrm{Aut}(\boldsymbol{C})$ denotes the group of field automorphisms of $\boldsymbol{C}$. For $\sigma \in \mathrm{Aut}(\boldsymbol{C})$, recall the definition of a *σ-linear map* (between two $\boldsymbol{C}$-vector spaces):

we say that $f\colon V \to W$ is σ-linear if

(a) $f(v+v') = f(v) + f(v')$ for all $v, v' \in V$, and

(b) $f(\lambda v) = \sigma(\lambda) f(v)$ for all $v \in V$ and $\lambda \in \boldsymbol{C}$.

A *semilinear* map is a map which is σ-linear for some $\sigma \in \mathrm{Aut}(\boldsymbol{C})$. We remark that, if f is a semilinear map and f is not identically zero, then there exists a *unique* $\sigma \in \mathrm{Aut}(\boldsymbol{C})$ such that f is σ-linear.

It is easily seen that a semilinear map takes linear subspaces to linear subspaces, and that the composition of semilinear maps is again semilinear — more precisely, if $f\colon V_1 \to V_2$ is σ-linear and $g\colon V_2 \to V_3$ is τ-linear, then $g \circ f$ is $\tau\sigma$-linear.

Notions known from linear algebra can be extended to semilinear maps without much effort — for instance, a *semilinear* (resp., *σ-linear*) *isomorphism* is a bijective semilinear (resp., σ-linear) map, and similarly, we can define semilinear (resp., σ-linear) endomorphisms, automorphisms, and so on. We record the readily-seen fact that the inverse of a σ-linear isomorphism is a σ^{-1}-linear isomorphism.

4.1.2. One defines G-equivariance for semilinear maps in much the same way as for linear maps. That is, let (π, V) and (π', V') be representations of G — then a semilinear map $f\colon V \to V'$ is called *G-equivariant* if it satisfies $\pi'(g) \circ f = f \circ \pi(g)$ for all $g \in G$.

In the next pages, we will frequently encounter *G-equivariant semilinear isomorphisms* between representations of G. We will call such maps *twisting maps*, or *transition maps*, and denote them mostly by the symbol T and variations thereof. Note that a linear transition map is nothing other than an isomorphism of G-representations, i.e. an invertible intertwining operator.

It is easy to deduce from the discussion in 4.1.1 that a transition map takes G-stable subspaces to G-stable subspaces and that compositions and inverses of transition maps are again transition maps.

4.1.3. Let (π, V) be a representation of G, and let $\sigma \in \mathrm{Aut}(\boldsymbol{C})$. If a representation (π', V') of G affords a σ-linear transition map $T\colon V \to V'$, we will say that π' is a *σ-twist* of π. We have the following basic result.

4.1.4. Proposition. *Twists always exist — more precisely, for any representation (π, V) of G and any $\sigma \in \mathrm{Aut}(\boldsymbol{C})$, there exists a representation π' which is a σ-twist of π.*

Proof. We borrow a construction found e.g. in [Wal85, I.1], see also [GR14, §7.1] for an elaboration.

Let (π, V) and σ be as in the statement, and let V' be the $\boldsymbol{C}$-linear space with the same underlying additive group as V, but with the following modified scalar multiplication: for $v \in V'$ and $\lambda \in \boldsymbol{C}$, define $\lambda \star v := \sigma^{-1}(\lambda)v$. (On the right-hand side, juxtaposition denotes the original scalar multiplication on V.) Then the identity map $V \to V'$ is a σ-linear isomorphism, which we denote by t_σ. The action π' of G on V' may be defined via $\pi'(g) := t_\sigma \circ \pi(g) \circ t_\sigma^{-1}$ for all $g \in G$. It is immediate that this indeed yields a linear representation (π', V') of G, and by construction, t_σ is a σ-linear transition map $V \to V'$. ☐

4.1.5. Let (π, V) be a representation of G, and let $\sigma \in \operatorname{Aut}(\boldsymbol{C})$. It is clear from 4.1.1, 4.1.2 that, if a representation (π', V') is a σ-twist of π, then π is a σ^{-1}-twist of π', and that any τ-twist of π', where $\tau \in \operatorname{Aut}(\boldsymbol{C})$, is also a $\tau\sigma$-twist of π. From this, it is an easy exercise to conclude that *twists are unique up to isomorphism*, or more precisely:

4.1.6. Proposition. *Let π be a representation of G, let $\sigma \in \operatorname{Aut}(\boldsymbol{C})$, and let π' be a σ-twist of π. Then any σ-twist of π is isomorphic to π', and conversely, every representation isomorphic to π' is a σ-twist of π.* ☐

4.1.7. For the sake of definedness, we will never speak of "*the*" twist of a representation π by an automorphism σ. However, we may and will do so for isomorphism classes of G-representations. In more detail, let Π be an isomorphism class of G-representations, and let $\sigma \in \operatorname{Aut}(\boldsymbol{C})$. Then it follows from Propositions 4.1.4 and 4.1.6 that there is a unique, well-defined isomorphism class of representations of G consisting of all σ-twists of all elements of Π. This class will be denoted by ${}^\sigma\Pi$ and called *the σ-twist* of Π.

For any isomorphism class Π, the family $\{{}^\sigma\Pi : \sigma \in \operatorname{Aut}(\boldsymbol{C})\}$ of all its twists is a *set*,[1] on which $\operatorname{Aut}(\boldsymbol{C})$ naturally acts from the left via ${}^\sigma({}^\tau\Pi) := {}^{\sigma\tau}\Pi$. The action is transitive but might in general have nontrivial stabilizers, which we may denote by the letter $\mathcal{S}$, as in: $\mathcal{S}(\Pi) := \{\sigma \in \operatorname{Aut}(\boldsymbol{C}) : {}^\sigma\Pi = \Pi\} \subseteq \operatorname{Aut}(\boldsymbol{C})$. By general results on group actions (or by direct verification), we have $\mathcal{S}({}^\sigma\Pi) = \sigma\mathcal{S}(\Pi)\sigma^{-1} \subseteq \operatorname{Aut}(\boldsymbol{C})$ for all σ.

If (π, V) is a representation of G, we will allow ourselves to write $\mathcal{S}(\pi)$ for the stabilizer of the isomorphism class of π. In view of the definition of a twist (4.1.3), it is immediate that

$$\mathcal{S}(\pi) = \left\{\sigma \in \operatorname{Aut}(\boldsymbol{C}) : \begin{array}{c}\text{there exists a } \sigma\text{-linear } G\text{-equivariant} \\ \text{automorphism } V \to V\end{array}\right\}. \tag{4.1}$$

[1]Indeed, by the proof of Proposition 4.1.4, each isomorphism class ${}^\sigma\Pi$, with $\sigma \in \operatorname{Aut}(\boldsymbol{C})$, can be realized on the same additive group.

4.2 Extension of scalars: a review

In this short section, we review the notion of *extension of scalars* in the context of representations of G. Just for this section, the default ground field for representations of G will be denoted by F rather than $\boldsymbol{C}$ — this will harmonize better with the notations used in the following sections.

4.2.1. First, recall the basic concept of extension of scalars from linear algebra: if V is a vector space over a field F, and $i\colon F \hookrightarrow E$ is a field extension, then the tensor product $V \otimes_{F,i} E$ is an E-linear space in a natural way, which may be denoted by V_E (as long as i is clear from context).

Using the functorial properties of tensor products, and the fact that every vector space has a basis, one may conclude that extension of scalars preserves both injective and surjective linear maps. In other words, if V and W are F-linear spaces and $\varphi\colon V \to W$ is an injective (resp., surjective, bijective) F-linear map between them, then $\varphi \otimes \mathrm{id}_E\colon V_E \to W_E$ is an injective (resp., surjective, bijective) E-linear map.

It will be useful to partially generalize the preceding observation to semilinear maps. To wit, let $\varphi\colon V \to W$ be a bijective F-linear map, and let σ be an F-linear[2] field automorphism of E. Then $\varphi \otimes \sigma\colon V_E \to W_E$ is a σ-linear isomorphism of E-linear spaces.

4.2.2. We may now adapt the preceding discussion to G-representations.

Let F, i and E be as above, and let (π, V) be an F-linear representation of the group G. Then we may define an E-linear representation of G on V_E in a natural way: for a pure tensor $v \otimes \lambda \in V_E$ and a group element $g \in G$, we let $g \cdot (v \otimes \lambda) := (g \cdot v) \otimes \lambda$, and then we extend the action by linearity.

Let (ρ, W) be another F-linear representation of G, and W_E be the corresponding E-linear representation obtained by extension of scalars. If an F-linear map $\varphi\colon V \to W$ is G-equivariant, then the same is true for the E-linear extension $\varphi \otimes \mathrm{id}_E\colon V_E \to W_E$ considered in 4.2.1. Essentially the same argument shows that, if φ is bijective and G-equivariant, and σ is an F-linear field automorphism of E, then $\varphi \otimes \sigma\colon V_E \to W_E$ is a σ-linear transition map.

[2] A field automorphism $\sigma \in \mathrm{Aut}(E)$ is F-linear if and only if it satisfies $\sigma \circ i = i$, i.e. it and only if it leaves $i(F) \subseteq E$ fixed pointwise.

4.3 Rational structures

The next topic to which we devote our attention is that of *rational structures*. This is a classical topic, treated in some form already in [Spr09, Chap. 11].

4.3.1. In the sequel, a "representation" of our fixed group G is again, as per the conventions set up at the beginning of the chapter, a $\boldsymbol{C}$-linear representation unless otherwise specified. We will be interested in those representations which arise, by extension of scalars, from representations over some subfield of $\boldsymbol{C}$.

For convenience, we fix a subfield $F \subseteq \boldsymbol{C}$ throughout. We will view $\boldsymbol{C}$ as an extension of F via the inclusion map, and we will freely apply extension of scalars to F-linear representations of G and make free use of notations from §4.2.

4.3.2. Let (π, V) be a representation of G, and let V_0 be a G-stable F-linear subspace of V. (Observe that such a subspace always exists, the trivial choices being $V_0 = \{0\}$ and $V_0 = V$.) Let incl denote the inclusion map $V_0 \hookrightarrow V$ — then the map $\mathrm{incl} \otimes \mathrm{id}_{\boldsymbol{C}}$, i.e. the unique $\boldsymbol{C}$-linear map $V_0 \otimes_F \boldsymbol{C} \to V$ which is given on pure tensors by $v \otimes \lambda \mapsto \lambda v$ (cf. 4.2.2), will be called the *natural map* $V_0 \otimes_F \boldsymbol{C} \to V$, and denoted by nat_{V,V_0}. Since V_0 is assumed G-stable, it follows from 4.2.2 that nat_{V,V_0} is G-equivariant.

If nat_{V,V_0} is bijective, one calls V_0 an *F-rational structure on V* (or *for π*). It is worth noting that this is a restrictive condition since nat_{V,V_0} is, in general, neither injective nor surjective, as is quickly seen in simple examples. If V has an F-rational structure, we also say that V is *defined over F*.

We now record a few elementary facts about rational structures. We start with a simple but useful lemma.

4.3.3. Lemma. *Let (π, V) be a representation of G, and let $V_0 \subseteq V$ be a G-stable F-linear subspace. The following are equivalent:*

(i) *V_0 is an F-rational structure for π;*
(ii) *V_0 has a basis (over F) which is also a basis for V (over $\boldsymbol{C}$);*
(iii) every *basis for V_0 (over F) is also a basis for V (over $\boldsymbol{C}$).*

Proof. Recall that, if $\{v_\alpha\}$ is an F-basis for V_0, then $\{v_\alpha \otimes 1\}$ is a $\boldsymbol{C}$-basis for $V_0 \otimes_F \boldsymbol{C}$, and that by definition, $\mathrm{nat}_{V,V_0}(v_\alpha \otimes 1)$ is simply v_α. Thus, the claim follows from the well-known characterization of bijective maps from basic linear algebra, applied to the linear map nat_{V,V_0}. $\square$

4.3.4. Corollary. *A representation (π, V) of G is defined over F if and only if V has a basis whose F-linear span is G-stable.* $\square$

The next statement says that rational structures can be "transported" along isomorphisms.

4.3.5. Proposition. *Let (π, V) be a representation of G, let $\Phi\colon V \to V'$ be a $\boldsymbol{C}$-linear isomorphism of G-representations, and let $V_0 \subseteq V$ be an F-rational structure on V. Then $\Phi(V_0)$ is an F-rational structure on V'.*

Proof. Clearly, $\Phi(V_0)$ is a G-stable F-linear subspace of V'. Now let $\{v_\alpha\}$ be an F-basis of V_0. Then $\{v_\alpha\}$ is also a $\boldsymbol{C}$-basis of V, by Lemma 4.3.3, hence $\{\Phi(v_\alpha)\}$ is a $\boldsymbol{C}$-basis of V'. Since the F-linear span of this basis is simply $\Phi(V_0)$, we are done by the implication (ii) $\Longrightarrow$ (i) in Lemma 4.3.3. □

4.3.6. Corollary. *Let V_0 be an F-rational structure on a representation (π, V) of G, and $\Phi\colon V \to V$ be a* homothety*, i.e. a map of the form $\Phi = \lambda \mathrm{id}_V$ for some $\lambda \in \boldsymbol{C}^*$. Then $\Phi(V_0)$, equivalently denoted λV_0, is again an F-rational structure on V. If $V \neq \{0\}$, the only homotheties which preserve the subspace V_0 are those of the form $\lambda \mathrm{id}_V$ with $\lambda \in F^*$.*

Proof. The first claim is clear from Proposition 4.3.5 since homotheties are always G-equivariant linear isomorphisms. As for the second claim, choose an F-basis $\{v_\alpha\}$ for V_0 — then, by Lemma 4.3.3, this is a basis for V over $\boldsymbol{C}$, so every element of V can be expressed uniquely as a finite $\boldsymbol{C}$-linear combination of the v_α. Clearly, then, an element of V is in V_0 if and only if all its coefficients with respect to the basis $\{v_\alpha\}$ lie in F. Using this, the second claim becomes clear. □

As a partial converse to Proposition 4.3.5, we have that representations with isomorphic F-rational structures are themselves isomorphic. The precise statement reads as follows.

4.3.7. Proposition. *Let (π, V) and (π', V') be representations of G with F-rational structures V_0 and V_0', respectively, and let $\varphi\colon V_0 \to V_0'$ be a G-equivariant F-linear isomorphism. Then, there exists a unique G-equivariant $\boldsymbol{C}$-linear isomorphism $\Phi\colon V \to V'$ which extends φ.*

Proof. To show uniqueness, choose an F-basis $\{v_\alpha\}$ of V_0, which is then also a $\boldsymbol{C}$-basis for V by Lemma 4.3.3 — then the map Φ must send a finite $\boldsymbol{C}$-linear combination $\sum \lambda_\alpha v_\alpha$ to $\sum \lambda_\alpha \varphi(v_\alpha)$. In other words, Φ is necessarily equal to $\mathrm{nat}_{V',V_0'} \circ (\varphi \otimes \mathrm{id}_{\boldsymbol{C}}) \circ (\mathrm{nat}_{V,V_0})^{-1}$. That this latter map indeed has all the desired properties follows from 4.2.2 together with 4.3.2. □

4.4 Rational structures via transition maps

The goal of this section is to show how rational structures both induce and can be obtained from families of semilinear G-equivariant automorphisms. In order to do that, it will be necessary to fix some notations and introduce some others.

4.4.1. As in the previous section, F is a subfield of the "usual" ground field $\boldsymbol{C}$. We let $\mathrm{Aut}(\boldsymbol{C}/F)$ denote the subgroup of automorphisms $\sigma \in \mathrm{Aut}(\boldsymbol{C})$ which leave F fixed pointwise. On the other hand, for a subgroup $H \subseteq \mathrm{Aut}(\boldsymbol{C})$, we may define the *fixed field* $\mathrm{Fix}(H) := \{x \in \boldsymbol{C} : \sigma(x) = x \text{ for all } \sigma \in H\} \subseteq \boldsymbol{C}$.

It is well-known that the two assignments $F \mapsto \mathrm{Aut}(\boldsymbol{C}/F)$ and $H \mapsto \mathrm{Fix}(H)$ give a *Galois connection* between the subfields of $\boldsymbol{C}$ and the subgroups of $\mathrm{Aut}(\boldsymbol{C})$, see e.g. [Cla11]. In particular, for every subfield $F \subseteq \boldsymbol{C}$ we have $F \subseteq \mathrm{Fix}(\mathrm{Aut}(\boldsymbol{C}/F))$, and for every $H \subseteq \mathrm{Aut}(\boldsymbol{C})$ we have $H \subseteq \mathrm{Aut}(\boldsymbol{C}/\,\mathrm{Fix}(H))$.

We start with a straightforward generalization of Proposition 4.3.7:

4.4.2. Proposition. *Let (π, V) and (π', V') be representations of G with F-rational structures V_0 and V_0', respectively, and let $\varphi\colon V_0 \to V_0'$ be a G-equivariant F-linear isomorphism. Then, for every $\sigma \in \mathrm{Aut}(\boldsymbol{C}/F)$, there exists a unique σ-linear transition map $\Phi\colon V \to V'$ which extends φ.*

Proof. The proof is essentially identical to that of Proposition 4.3.7. To show uniqueness, choose an F-basis $\{v_\alpha\}$ of V_0, which is then also a $\boldsymbol{C}$-basis for V by Lemma 4.3.3 — then the map Φ must send a finite $\boldsymbol{C}$-linear combination $\sum \lambda_\alpha v_\alpha$ to $\sum \sigma(\lambda_\alpha)\varphi(v_\alpha)$. In other words, Φ is necessarily equal to $\mathrm{nat}_{V',V_0'} \circ (\varphi \otimes \sigma) \circ (\mathrm{nat}_{V,V_0})^{-1}$. That this latter map indeed has all the desired properties follows from 4.2.2 together with 4.3.2. □

4.4.3. Corollary. *Let (π, V) be a representation of G with F-rational structure V_0. Then, for every $\sigma \in \mathrm{Aut}(\boldsymbol{C}/F)$, there exists a unique σ-linear transition map $t_\sigma\colon V \to V$ which is the identity on V_0.* □

Notably, there is a partial converse to the corollary.

4.4.4. Proposition. *Let (π, V) be a representation of G. Suppose that there exists a subgroup $H \subseteq \mathrm{Aut}(\boldsymbol{C})$ such that, for each $\sigma \in H$, we have a corresponding σ-linear transition map $t_\sigma\colon V \to V$, and set*

$$V^H := \{v \in V : t_\sigma(v) = v \text{ for all } \sigma \in H\}.$$

Then V^H is an F'-linear, G-stable subspace of V, where $F' = \mathrm{Fix}(H) \subseteq \boldsymbol{C}$, and the natural map $V^H \otimes_{F'} \boldsymbol{C} \to V$ is injective. In particular, V^H is an F'-rational structure for π if it spans V over $\boldsymbol{C}$.

Proof. It is easily seen that V^H is an F'-linear, G-stable subspace. We now present the argument for injectivity, which is well-known, see e.g. [Wal85, proof of Lemme I.1] or [Ber10, proof of Lemma III.8.21]

We argue by contradiction and suppose that there exist elements $v_1, \dots, v_n \in V^H$ which are linearly independent over F' but such that

$$\sum_{i=1}^{n} \lambda_i v_i = 0 \tag{$\star$}$$

for suitable $\lambda_i \in \boldsymbol{C}$, not all zero. We can choose n to be minimal, scale the coefficients so that $\lambda_1 = 1$, and reorder them so that $\lambda_2 \notin F'$. Then there must exist an automorphism $\sigma \in H$ such that $\sigma(\lambda_2) \neq \lambda_2$. But then, applying the σ-linear map t_σ to ($\star$), we get

$$\sum_{i=1}^{n} \sigma(\lambda_i) v_i = 0,$$

and subtracting one equality from the other (recall that $\lambda = 1$!) we obtain

$$\sum_{i=2}^{n} (\sigma(\lambda_i) - \lambda_i) v_i = 0,$$

contradicting the minimality of n. ☐

4.4.5. Corollary. *Keep the notations from the statement of Proposition 4.4.4. Suppose, moreover, that V is* irreducible. *If V^H contains a nonzero vector, then it is an F'-rational structure on V.* ☐

4.4.6. Corollary. *Let V_0 be an F-rational structure for a G-representation (π, V), and for every $\sigma \in \mathrm{Aut}(\boldsymbol{C}/F)$, let t_σ be as in the statement of Corollary 4.4.3. Furthermore, let $F' := \mathrm{Fix}(\mathrm{Aut}(\boldsymbol{C}/F)) \supseteq F$. Then the joint fixed space of the t_σ, denoted $V^{\mathrm{Aut}(\boldsymbol{C}/F)}$, is simply the F'-linear span of V_0 in V. In particular, it is an F'-rational structure for V.*

Proof. The first claim is established easily by working "in coordinates". In more details, fix an F-basis $\{v_\alpha\}$ for V_0 — then t_σ is the map which sends a finite $\boldsymbol{C}$-linear combination $\sum \lambda_\alpha v_\alpha$ to $\sum \sigma(\lambda_\alpha) v_\alpha$. It is then clear that a vector in V is fixed by all t_σ if and only if its coefficients relative to the basis $\{v_\alpha\}$ are all in F', and the claim follows.

As for the second claim, it suffices, by Proposition 4.4.4, to show that $V^{\mathrm{Aut}(\boldsymbol{C}/F)}$ spans V over $\boldsymbol{C}$. But $V^{\mathrm{Aut}(\boldsymbol{C}/F)}$ contains V_0, and the latter spans V over $\boldsymbol{C}$ by assumption, so the same must be true *a fortiori* of $V^{\mathrm{Aut}(\boldsymbol{C}/F)}$. ☐

4.5 Rationality fields

In the previous sections, the subfield $F \subseteq \boldsymbol{C}$ was fixed. Now, we will allow ourselves to consider various possible fields of definition for a given representation. We start with an observation.

4.5.1. It is easy to see that, if a representation is defined over some subfield $F \subseteq \boldsymbol{C}$, then it is also defined over every intermediate field $F \subseteq E \subseteq \boldsymbol{C}$ — in fact, if V_0 is an F-rational structure on (π, V), then it follows from Lemma 4.3.3 that the E-linear span of V_0 inside V is an E-rational structure on π. (Cf. also the final step in the proof of Corollary 4.4.6.)

We are led to ask the following question: what is the "minimal field" over which a given representation π can be defined? Answering this question is only a matter of putting together previous results, but it will be convenient to give another definition first.

4.5.2. It follows from the properties of the *Galois connection* mentioned in 4.4.1 that, for a subfield $F \subseteq \boldsymbol{C}$, the following are equivalent:
(i) $F = \mathrm{Fix}(H)$ for some subgroup $H \subseteq \mathrm{Aut}(\boldsymbol{C})$;
(ii) $F = \mathrm{Fix}(\mathrm{Aut}(\boldsymbol{C}/F_0))$ for some subfield $F_0 \subseteq \boldsymbol{C}$;
(iii) $F = \mathrm{Fix}(\mathrm{Aut}(\boldsymbol{C}/F))$.
(See e.g. [Cla11, Cor. 9 & Prop. 10].) If F satisfies these properties, we will say that it is *closed*. We remind the reader that F is certainly closed in $\boldsymbol{C}$ if either $\boldsymbol{C}/F$ is a Galois extension or $\boldsymbol{C}$ is algebraically closed of characteristic 0 (or both), see *op. cit.*, Thm. 5b) and Cor. 53c), respectively.

4.5.3. Proposition. *Let (π, V) be a representation of G, and let $F \subseteq \boldsymbol{C}$ be a subfield over which π is defined. Suppose that F is closed in the sense of 4.5.2. Then F must contain the field* $\mathrm{Fix}(\mathcal{S}(\pi))$*, where $\mathcal{S}(\pi)$ is defined as in 4.1.7.*

Proof. (cf. [GS18, Lemma A.2.1].) Let V_0 be an F-rational structure for V. For every $\sigma \in \mathrm{Aut}(\boldsymbol{C}/F)$, we have a unique σ-linear transition map $t_\sigma\colon V \to V$ which is the identity on V_0, by Corollary 4.4.3. But then, $\mathrm{Aut}(\boldsymbol{C}/F) \subseteq \mathcal{S}(\pi)$, by definition of the latter (cf. (4.1)). Taking fixed fields on both sides of the inclusion completes the proof. □

4.5.4. For a representation (π, V) of G, the field $\mathrm{Fix}(\mathcal{S}(\pi))$ is called the *rationality field* of π. It is automatically a closed subfield of $\boldsymbol{C}$ (see 4.5.2).

A representation π need *not* be defined over its rationality field in general, even when the group G is finite, see [Rag11, Ex. 2.5] for a concrete example.

Nevertheless, [Wal85, Lemme I.1] shows that π is defined over its rationality field if there exists a subgroup $G' \subseteq G$ such that $\dim_{\boldsymbol{C}} V^{G'} = 1$.

The following easy observation will be useful later.

4.5.5. Lemma. *If a representation π is defined over its rationality field $F = F(\pi)$, then $\mathcal{S}(\pi) = \mathrm{Aut}(\boldsymbol{C}/F(\pi))$.*

Proof. The left-to-right inclusion is automatic by the properties of Galois connections (4.4.1), and the reverse inclusion follows as in the proof of Proposition 4.5.3. □

4.5.6. Just for this paragraph, let $\boldsymbol{C}$ denote either the complex field $\mathbb{C}$ or the subfield $\overline{\mathbb{Q}} \subset \mathbb{C}$ of algebraic numbers. This is the case most often considered in the literature (see e.g. [Wal85], [Har83], etc.), and in this special situation, it is common to denote the rationality field of a $\boldsymbol{C}$-linear representation (π, V) by $\mathbb{Q}(\pi)$, as though it were the field "generated" by π over $\mathbb{Q}$.

To fully justify this notation, one should convince oneself that $\mathbb{Q}$ is precisely the fixed field of $\mathrm{Aut}(\boldsymbol{C})$. We spend a few words on why this is the case. First, by the observations made at the end of 4.5.2, we have $\mathrm{Fix}(\mathrm{Aut}(\boldsymbol{C}/\mathbb{Q})) = \mathbb{Q}$, hence $\mathrm{Fix}(\mathrm{Aut}(\boldsymbol{C})) \subseteq \mathrm{Fix}(\mathrm{Aut}(\boldsymbol{C}/\mathbb{Q})) = \mathbb{Q}$. But since $\mathbb{Q}$ is the smallest subfield of $\boldsymbol{C}$, we must have equality, i.e.: $\mathrm{Fix}(\mathrm{Aut}(\boldsymbol{C})) = \mathbb{Q}$, as claimed.

4.6 Periods

In research on the arithmetic of L-functions, one often relates the L-values of interest to certain invariants, obtained from a comparison of rational structures on isomorphic spaces. These invariants are usually called *periods* with various qualifiers depending on the nature of the spaces whose rational structures are compared — e.g., one may speak of *(bottom-degree) Whittaker–Betti periods*, *(top-degree) Shalika–Betti periods*, *relative periods*, and so on. Another word appearing in earlier work of Borel for a similar type of invariants is "*regulators*", cf. [Bor77].

The use of the term "periods" comes from the motivic world: automorphic periods for the standard L-functions of cohomological cuspidal representations are expected to correspond, via the Langlands conjectures, to the periods defined by Deligne in [Del79], which are indeed periods in the more widespread meaning of the word (namely: they are integrals of algebraic cohomology classes over algebraic cycles; we refer to the standard reference [KZ01] for more details).

In this section, we will use the word "periods" with the meaning of "scalar invariants obtained from comparing rational structures on isomorphic spaces". We now explain the specifics of the construction of such invariants. The essential assumption is that of *Schur-irreducibility*, whose definition we will recall momentarily (s. 4.6.2 below).

4.6.1. Let (π, V) and (ρ, W) be $\boldsymbol{C}$-linear representations of G. We introduce the notation

$$\mathrm{Hom}_G(V, W) \coloneqq \{\Phi \in \mathrm{Hom}_{\boldsymbol{C}}(V, W) : \Phi \circ \pi(g) = \rho(g) \circ \Phi \text{ for all } g \in G\}$$

for the space of $\boldsymbol{C}$-linear intertwining operators from V to W. Furthermore, we set $\mathrm{End}_G(V) \coloneqq \mathrm{Hom}_G(V, V)$.

Clearly, $\mathrm{Hom}_G(V, W)$ is always a $\boldsymbol{C}$-linear space; additionally, $\mathrm{End}_G(V)$ carries a $\boldsymbol{C}$-bilinear multiplication (namely: composition of maps), turning it into a unital associative algebra over $\boldsymbol{C}$.

4.6.2. Recall that a $\boldsymbol{C}$-linear representation (π, V) of G *satisfies Schur's lemma*, or is *Schur-irreducible*, if $\dim_{\boldsymbol{C}} \mathrm{End}_G(V) = 1$, i.e. if and only if

$$\mathrm{End}_G(V) = \{\lambda \cdot \mathrm{id}_V : \lambda \in \boldsymbol{C}\}.$$

4.6.3. *Caveat.* Despite the nomenclature, Schur-irreducibility is neither a stronger nor a weaker property than irreducibility in general, see e.g. [BG16, Ex. 3.3, Ex. 3.5]. Nevertheless, the various versions of *Schur's lemma* guarantee that certain irreducible representations with additional properties (e.g., for the classical statement: finite-dimensionality) are Schur-irreducible.

4.6.4. It is easy to see that, if (π, V) and (ρ, W) are isomorphic representations of G and one of them is Schur-irreducible, then so is the other, and the space $\mathrm{Hom}_G(V, W)$ of intertwining operators $V \to W$ is also one-dimensional over $\boldsymbol{C}$. In fact, these claims follow from part (iii) of the following lemma by setting $\sigma = \mathrm{id}_{\boldsymbol{C}}$.

4.6.5. Lemma. *Let $T \colon V \to V'$ be a σ-linear transition map between representations (π, V), (π', V') of G, where $\sigma \in \mathrm{Aut}(\boldsymbol{C})$. Then:*

(i) *If $T' \colon W \to W'$ is another σ-linear transition map, then $\varphi \mapsto T' \circ \varphi \circ T^{-1}$ is a σ-linear bijection from $\mathrm{Hom}_G(V, W)$ to $\mathrm{Hom}_G(V', W')$.*

(ii) *In particular, $\varphi \mapsto T \circ \varphi \circ T^{-1}$ is a bijective σ-linear ring homomorphism from $\mathrm{End}_G(V)$ to $\mathrm{End}_G(V')$.*

(iii) *In particular: if π is Schur-irreducible, then so is π', and every σ-linear G-equivariant map from V to V' is a scalar multiple of T.*

Proof. The first claim is entirely straightforward. By setting $V = W$, $V' = W'$, and $T = T'$ in (i), one obtains that the map in (ii) is σ-linear; to show that it is a ring homomorphism, one needs to check that $T \circ (\varphi \circ \psi) \circ T^{-1} = (T \circ \varphi \circ T^{-1}) \circ (T \circ \psi \circ T^{-1})$ whenever φ, $\psi \in \mathrm{End}_G(V)$, which is, however, trivial.

As for (iii), assume that π is Schur-irreducible, hence that $\dim_{\boldsymbol{C}} \mathrm{End}_G(V) = 1$. By (ii), there is a σ-linear bijection $\mathrm{End}_G(V) \to \mathrm{End}_G(V')$, and so in particular the two spaces have the same dimension over $\boldsymbol{C}$, proving that π' is also Schur-irreducible. If $T' \colon V \to V'$ is σ-linear and G-equivariant, then $T^{-1} \circ T'$ is a linear G-equivariant map from V to itself, i.e. an element of $\mathrm{End}_G(V)$; thus, $T^{-1} \circ T' = \lambda \cdot \mathrm{id}_V$ for some $\lambda \in \boldsymbol{C}$, or in other words, $T' = \lambda T$. □

We now come to a result we call the *main theorem on periods*. We will state it in a handful of equivalent ways (Theorems 4.6.6, 4.6.7, 4.6.9 and 4.6.10), which we have dubbed versions 0, 1, 2, and 3, respectively. We will give a direct proof for version 1, and explain how to derive the other statements from it.

To simplify the statements, *we henceforth fix a subfield $F \subseteq \boldsymbol{C}$*. Our proofs will rest on the assumption that F is *closed* in the sense of 4.5.2.

We start with the version of the statement which is perhaps the most elegant and intrinsic, if less informative at first glance. In order to state it, we introduce the notation $\mathrm{Aut}_G(V) := \mathrm{End}_G(V)^\times$ for the group of invertible G-equivariant linear automorphisms of a given representation (π, V). Recall from Proposition 4.3.5 that $\mathrm{Aut}_G(V)$ acts on the set of F-rational structures on V by

$$\Phi.V_0 := \Phi(V_0). \tag{4.2}$$

4.6.6. Main Theorem on Periods (version 0)**.** *Let (π, V) be a representation of G which is defined over F. If π is Schur-irreducible, the action* (4.2) *is transitive.*

Now note that, when (π, V) is Schur-irreducible, one has $\mathrm{Aut}_G(V) = \{\lambda \mathrm{id}_V : \lambda \in \boldsymbol{C}^*\}$. Keeping this in mind, we immediately derive the following equivalent reformulation of Theorem 4.6.6. To state it, we make use of the notion of *homotheties* introduced in Corollary 4.3.6.

4.6.7. Main Theorem on Periods (version 1)**.** *Let (π, V) be a representation of G which is defined over F. Suppose that π is* Schur-irreducible*. Then V has a unique F-rational structure* up to homotheties*. In other words, if V_0 and V_1 are F-rational structures on V, then there exists a scalar $\lambda \in \boldsymbol{C}^*$ such that $V_1 = \lambda V_0$.*

Proof. Let V_0 and V_1 be as in the statement, and for each $\sigma \in \mathrm{Aut}(\boldsymbol{C}/F)$, let t_σ (resp., t'_σ) be the unique σ-linear transition map from V to itself which is the identity on V_0 (resp., V_1), as per Corollary 4.4.3. Since F is closed, the joint fixed space of the maps t_σ (resp., t'_σ) is precisely V_0 (resp., V_1), by Corollary 4.4.6.

Since π is Schur-irreducible by assumption, Lemma 4.6.5.(iii) tells us that there is, for each $\sigma \in \mathrm{Aut}(\boldsymbol{C}/F)$, a nonzero scalar $c_\sigma \in \boldsymbol{C}$ such that $t'_\sigma = c_\sigma t_\sigma$. The next step will be to show that there exists a λ such that c_σ is of the form $\frac{\lambda}{\sigma(\lambda)}$ for all $\sigma \in \mathrm{Aut}(\boldsymbol{C}/F)$. To that end, we are going to take a nonzero[3] vector $v \in V_1$ and express it in terms of a basis $\{v_\alpha\}$ for V_0, thus getting $v = \sum \lambda_\alpha v_\alpha$ for uniquely determined $\lambda_\alpha \in \boldsymbol{C}$, only finitely many of which are nonzero. Pick an index α_0 such that $\lambda_{\alpha_0} \neq 0$, and let $\sigma \in \mathrm{Aut}(\boldsymbol{C}/F)$. Then, comparing the expressions for $c_\sigma t_\sigma(v) = t'_\sigma(v) = v$, we obtain that $\frac{\lambda_{\alpha_0}}{\sigma(\lambda_{\alpha_0})} = c_\sigma$. Since $\sigma \in \mathrm{Aut}(\boldsymbol{C}/F)$ was arbitrary, we see that our claim is verified for $\lambda := \lambda_{\alpha_0}$.

For the final step, we show that $\lambda V_0 \subseteq V_1$ and $\lambda^{-1} V_1 \subseteq V_0$. The first inclusion is proved as follows: let $v \in V_0$. Then $t_\sigma(v) = v$ for all $\sigma \in \mathrm{Aut}(\boldsymbol{C}/F)$. Because $t'_\sigma = \frac{\lambda}{\sigma(\lambda)} t_\sigma$ for all σ, we conclude that λv lies in the joint fixed space of the maps t'_σ, which is V_1. The second inclusion is proved by the same argument in the reverse direction. □

4.6.8. *Remark on the proof.* Consider again the step in which we proved that there exists a $\lambda \in \boldsymbol{C}^*$ such that $c_\sigma = \frac{\lambda}{\sigma(\lambda)}$ for all $\sigma \in \mathrm{Aut}(\boldsymbol{C}/F)$. The argument we used in carrying out this step was lifted *verbatim* from [Clo90, p. 102] and requires no assumptions on the extension $\boldsymbol{C}/F$.

An alternative argument, inspired by [Har87, p. 50], is available if $\boldsymbol{C}/F$ is known to be a *Galois* extension. In fact, let t_σ and t'_σ be as in the above proof for every $\sigma \in \mathrm{Gal}(\boldsymbol{C}/F)$. By the explicit shape of these maps (cf. the proof of Proposition 4.4.2), it is easy to see that we have equalities $t^{(\prime)}_{\sigma\tau} = t^{(\prime)}_\sigma \circ t^{(\prime)}_\tau$ for all $\sigma, \tau \in \mathrm{Gal}(\boldsymbol{C}/F)$. It follows that

$$c_{\sigma\tau} = c_\sigma \cdot \sigma(c_\tau) \qquad \text{for all } \sigma, \tau \in \mathrm{Gal}(\boldsymbol{C}/F),$$

i.e.: the map $\sigma \mapsto c_\sigma$ is (algebraically) a 1-cocycle of $\mathrm{Gal}(\boldsymbol{C}/F)$ with values in $\boldsymbol{C}^*$. We claim that this cocycle is *continuous* (where $\mathrm{Gal}(\boldsymbol{C}/F)$ and $\boldsymbol{C}^*$ are given the Krull topology and the discrete topology, respectively). The existence of a suitable λ can then be deduced from (a suitably general version of) Hilbert's Theorem 90, cf. e.g. [Ber10, Prop. III.8.24].

To prove continuity of c_σ, it suffices to show that the subgroup $H := \{\sigma \in \mathrm{Gal}(\boldsymbol{C}/F) : c_\sigma = 1\}$ is open in $\mathrm{Gal}(\boldsymbol{C}/F)$. (Recall that $\boldsymbol{C}^*$ is equipped with

[3] V_1 must contain a nonzero vector: for suppose that V_1 were the zero space. Then V would also have to be zero (because V_1 spans V over $\boldsymbol{C}$), contradicting Schur-irreducibility.

the discrete topology.) Since $\mathrm{Gal}(\boldsymbol{C}/F)$ is compact, we have that H is open in $\mathrm{Gal}(\boldsymbol{C}/F)$ if and only if it is closed and of finite index.

We will now exhibit an intermediate field $F \subseteq E \subseteq \boldsymbol{C}$ such that E is finite over F and $H \supseteq \mathrm{Gal}(\boldsymbol{C}/E)$. This will imply that H is of finite index in $\mathrm{Gal}(\boldsymbol{C}/F)$; moreover, by definition of the Krull topology, $\mathrm{Gal}(\boldsymbol{C}/E)$ will be a closed subgroup of $\mathrm{Gal}(\boldsymbol{C}/F)$, and hence H, being a finite union of $\mathrm{Gal}(\boldsymbol{C}/E)$-cosets, will itself be closed in $\mathrm{Gal}(\boldsymbol{C}/F)$, as required.

To construct E, we proceed as follows. Pick F-bases $\{v_\alpha\}$ and $\{v'_\beta\}$ of V_0 and V_1, respectively, and fix a nonzero $v \in V$. Then $v = \sum \lambda_\alpha v_\alpha = \sum \mu_\beta v'_\beta$ for suitable coefficients λ_α, μ_β, which are uniquely determined by v. Let E be the extension of F generated by the finitely many nonzero coefficients λ_α, μ_β. Then, for all $\sigma \in \mathrm{Gal}(\boldsymbol{C}/E)$, we have $t_\sigma(v) = v = t'_\sigma(v)$ and hence $c_\sigma = 1$, as claimed. ◇

We proceed to give two more versions of the main theorem. Recall that we assume F to be *closed* in the sense of 4.5.2.

4.6.9. Main Theorem on Periods (version 2). *Let (π, V) and (ρ, W) be isomorphic representations of G, and assume that either of them is Schur-irreducible and defined over F. (Hence, both are.) Consider V and W as equipped with F-rational structures V_0 and W_0, respectively. Then there exists a G-equivariant linear isomorphism $\Phi_0 \colon V \to W$ which maps V_0 onto W_0.*

4.6.10. Main Theorem on Periods (version 3). *Let (π, V) and (ρ, W) be isomorphic representations of G, and assume that either of them is Schur-irreducible and defined over F. (Hence, both are.) Let V_0 and W_0 be F-rational structures on V and W, respectively, and let $\Phi \colon V \to W$ be a G-equivariant linear isomorphism. Then there exists a nonzero scalar $\lambda \in \boldsymbol{C}^*$ such that $\lambda^{-1}\Phi$ maps V_0 onto W_0.*

Proof of 4.6.10. By Proposition 4.3.5, $\Phi(V_0)$ is an F-rational structure on W. Moreover, W satisfies the assumptions of Theorem 4.6.7, so, by that result, there exists a $\lambda \in \boldsymbol{C}^*$ such that $\Phi(V_0) = \lambda W_0$. This λ has the required property. □

Proof of 4.6.9. By assumption, $\mathrm{Hom}_G(V, W)$ contains an isomorphism $\Phi \colon V \to W$. The assumptions of Theorem 4.6.10 are verified, so there is a $\lambda \in \boldsymbol{C}^*$ such that $\lambda^{-1}\Phi$ sends V_0 onto W_0. But then we may take $\Phi_0 := \lambda^{-1}\Phi$. □

4.6.11. Let (π, V) and (ρ, W) be as in the statement of Theorem 4.6.10, where F is, again, a closed subfield of $\boldsymbol{C}$. Suppose given the following datum: an F-rational structure V_0 on V; an F-rational structure W_0 on W; and a G-equivariant linear isomorphism $\Phi \colon V \to W$. Then Theorem 4.6.10 guarantees the existence of a nonzero scalar $\lambda \in \boldsymbol{C}^*$ such that $\lambda^{-1}\Phi(V_0) = W_0$. The scalar itself is not unique, but it is an easy consequence of Corollary 4.3.6 that its class in $\boldsymbol{C}^*/F^*$ is uniquely determined by the initial datum (V_0, W_0, Φ). We call this class the *period* attached to the triple (V_0, W_0, Φ). We remark in passing that a class in $\boldsymbol{C}^*/F^*$ yields, for every intermediate field $F \subseteq E \subseteq \boldsymbol{C}$, a well-defined class in $\boldsymbol{C}^*/E^*$.

Let (V_1, W_1, Φ_1) be another datum as above. Then, by Theorem 4.6.7 and Lemma 4.6.5, there exist nonzero scalars α, β, $\gamma \in \boldsymbol{C}^*$ such that $V_1 = \alpha V_0$, $W_1 = \beta W_0$ and $\Phi_1 = \gamma\Phi$. An easy computation then shows that the period attached to (V_1, W_1, Φ_1) is equal to $\alpha\beta^{-1}\gamma$ times the period attached to (V_0, W_0, Φ).

4.6.12. *Remark.* To conclude the section, we discuss the attribution of the main theorem on periods.

The arguments in our proof of Theorem 4.6.7 are in part borrowed from the proof of [Wal85, Lemme I.1], as well as the elaboration given in [Clo90, p. 102]. We remark, however, that Waldspurger's lemma was only proved over the ground field $\boldsymbol{C} = \mathbb{C}$ and with a stronger assumption on the representation (π, V), and we have reason to believe that Theorem 4.6.7 has not yet appeared in the literature in the level of generality in which we have phrased it. We also invite the reader to compare our Theorem 4.6.7 with [JST19, Lemma 4.5], which has a similar statement (but again with stronger assumptions) and whose proof is of a different flavor.

The implication (version 1)$\Rightarrow$(version 3) is well-known, see e.g. [Mah05, §3.4] or the proof of [RS10, Def./Prop. 3.3]. As for version 2, one may compare its statement with [LLS21, Lemma 6.4], which is proved by an application of [Spr09, Prop. 11.1.6]. (This latter result is, in turn, similar to our Proposition 4.4.4, except that it only applies to Galois extensions.) $\diamond$

4.7 Compatible families of transition maps

In this final section, we want to lay down an abstract framework for studying concrete situations which we will be confronted with in the next chapter. The main goal is to extend the central result Proposition 4.4.4 so as to allow us to simultaneously define rational structures on all twists of a given representation π. Of course, since twists are only well-defined at the level of isomorphism classes (see 4.1.7), this is only meaningful if we *choose* a σ-twist of π at each σ in a reasonable way. We make this precise in the discussion below. Our treatment owes a lot to [Har87, 1.3].

4.7.1. Let $\mathcal{F} = \{\Pi_\lambda\}_{\lambda\in\mathcal{L}}$ be a set of isomorphism classes of representations of G, where $\mathcal{L}$ is an arbitrary index set. (It is implicit in this notation that $\Pi_\lambda \neq \Pi_\mu$ when $\lambda \neq \mu$.) We will assume that the collection $\{\Pi_\lambda\}$ is *closed under twists* in the sense that, for every $\lambda \in \mathcal{L}$ and every $\sigma \in \mathrm{Aut}(\boldsymbol{C})$, there exists a (necessarily unique) $\sigma\lambda \in \mathcal{L}$ such that $\Pi_{\sigma\lambda} = {}^\sigma\Pi_\lambda$. (Cf. 4.1.7.) Note that this effectively defines a left action of $\mathrm{Aut}(\boldsymbol{C})$ on the index set $\mathcal{L}$; for any $\lambda \in \mathcal{L}$, the stabilizer $\mathcal{S}(\lambda) = \{\sigma : \sigma\lambda = \lambda\}$ is none other than the stabilizer $\mathcal{S}(\Pi_\lambda)$ of Π_λ, defined as in 4.1.7.

For each $\lambda \in \mathcal{L}$, let (π_λ, V_λ) be a realization of Π_λ, i.e. a representation of G which lies in the isomorphism class Π_λ. (Implicit in this formulation is the requirement that $(\pi_{\sigma\lambda}, V_{\sigma\lambda}) = (\pi_\lambda, V_\lambda)$ whenever $\sigma \in \mathcal{S}(\lambda)$.) Moreover, for each $\lambda \in \mathcal{L}$ and each $\sigma \in \mathrm{Aut}(\boldsymbol{C})$, let $t_{\sigma;\lambda} \colon V_\lambda \to V_{\sigma\lambda}$ be a σ-linear transition map. We *assume* that the diagram

$$\begin{tikzcd} V_\lambda \arrow[r, "t_{\sigma;\lambda}"] \arrow[dr, "t_{\tau\sigma;\lambda}"'] & V_{\sigma\lambda} \arrow[d, "t_{\tau;\sigma\lambda}"] \\ & V_{\tau\sigma\lambda} \end{tikzcd} \tag{4.3}$$

commutes for all $\sigma, \tau \in \mathrm{Aut}(\boldsymbol{C})$ and all $\lambda \in \mathcal{L}$. Then we can draw the following conclusions.

4.7.2. Lemma.

(i) $t_{\mathrm{id}_{\boldsymbol{C}};\lambda} = \mathrm{id}_{V_\lambda}$ *for all* $\lambda \in \mathcal{L}$*;*

(ii) $t_{\tau^{-1};\tau\lambda} = t_{\tau;\lambda}^{-1}$ *for all* $\lambda \in \mathcal{L}$ *and all* $\tau \in \mathrm{Aut}(\boldsymbol{C})$*.*

Proof. The first claim follows easily by (4.3) taking into account the idempotence of $\mathrm{id}_{\boldsymbol{C}}$ and the fact that transition maps are bijective. The second claim is an easy consequence of (4.3) and (i). □

4.7.3. For each $\lambda \in \mathcal{L}$, we set

$$V_{\lambda,0} \coloneqq \{v \in V_\lambda : t_{\sigma;\lambda}(v) = v \text{ for all } \sigma \in \mathcal{S}(\lambda)\}. \tag{4.4}$$

Recall from (the proof of) Proposition 4.4.4 that $V_{\lambda,0} \subseteq V_\lambda$ is a G-stable $F(\lambda)$-linear subspace, where $F(\lambda) \coloneqq \mathrm{Fix}(\mathcal{S}(\lambda)) = \mathrm{Fix}(\mathcal{S}(\Pi_\lambda))$ is shorthand for the rationality field of Π_λ.

4.7.4. Proposition. *For every* $\tau \in \mathrm{Aut}(\boldsymbol{C})$ *and every* $\lambda \in \mathcal{L}$*, we have*

$$t_{\tau;\lambda}(V_{\lambda,0}) = V_{\tau\lambda,0}.$$

Proof. Let $v' \in V_{\tau\lambda}$ and $v = t_{\tau;\lambda}^{-1}(v') \in V_\lambda$. We want to show that $v' \in V_{\tau\lambda,0}$ if and only if $v \in V_{\lambda,0}$.

Recall from 4.1.7 that the map $\sigma \mapsto \tau\sigma\tau^{-1}$ defines a group isomorphism $\mathcal{S}(\Pi) \xrightarrow{\sim} \mathcal{S}({}^\tau\Pi)$. Applying (4.3) and Lemma 4.7.2.(ii), we have

$$\begin{aligned} v' \in V_{\tau\lambda,0} &\iff t_{\tau\sigma\tau^{-1};\lambda}(v') = v' && \text{for all } \sigma \in \mathcal{S}(\lambda) \\ &\iff t_{\tau;\lambda}(t_{\sigma;\lambda}(t_{\tau;\lambda}^{-1}(v'))) = v' && \text{for all } \sigma \in \mathcal{S}(\lambda) \\ &\iff t_{\sigma;\lambda}(v) = v && \text{for all } \sigma \in \mathcal{S}(\lambda) \\ &\iff v \in V_{\lambda,0}, \end{aligned}$$

as claimed. □

4.7.5. Proposition. *Suppose that $V_{\lambda,0}$ is an $F(\lambda)$-rational structure on V_λ for* some $\lambda \in \mathcal{L}$. *Then $V_{\tau\lambda,0}$ is an $F(\tau\lambda)$-rational structure on $V_{\tau\lambda}$ for all $\tau \in \mathrm{Aut}(\boldsymbol{C})$.*

Proof. We know from Proposition 4.7.4 that $V_{\tau\lambda,0} = t_{\tau;\lambda}(V_{\lambda 0})$, and it is easy to see (cf. 4.1.7) that $F(\tau\lambda) = \tau(F(\lambda))$. Thus, we will be done if we can prove that a τ-linear transition map sends an F-rational structure on its domain to a $\tau(F)$-rational structure on its target space. But this can be shown by the same argument used in the proof of Proposition 4.3.5. □

4.7.6. The preceding proposition finally achieves our goal to simultaneously pin down rational structures on all twists of a given representation: more precisely, if we can show that $V_{\lambda,0}$ is an $F(\lambda)$-rational structure on V_λ for *some* $\lambda \in \mathcal{L}$, then it follows that $V_{\mu,0}$ is an $F(\mu)$-rational structure on V_μ whenever $\mu = \tau\lambda$ for some $\tau \in \mathrm{Aut}(\boldsymbol{C})$. The rational structures $V_{\mu,0}$ are chosen compatibly with each other, and with the transition maps $t_{\sigma;\lambda}$ fixed in 4.7.1, in the sense that, by construction,

(1) the rational structures are mapped to each other by the transition maps (this is precisely Proposition 4.7.4), and
(2) for each $\lambda \in \mathcal{L}$ and each $\sigma \in \mathcal{S}(\lambda)$, the map $t_{\sigma;\lambda}$ is the unique σ-linear automorphism of V_λ which is the identity on $V_{\lambda,0}$, cf. Corollary 4.4.3.

4.7.7. To conclude the section, we go back to the topic of periods from the previous section. The goal is to show (Proposition 4.7.8 below) how periods "behave under twisting", in a sense that we will now make precise using the preceding discussion. We remark that Proposition 4.7.8 is sometimes even used as the definition of periods, cf. e.g. [RS10, Def./Prop. 3.3].

Recall that, in 4.7.1, we fixed a family $\mathcal{F} = \{\Pi_\lambda\}_{\lambda\in\mathcal{L}}$ of isomorphism classes of representations of G, closed under twists. We will now *assume that each* $\Pi \in$

$\mathcal{F}$ *is Schur-irreducible*, or more precisely, that each representation of G whose isomorphism class lies in $\mathcal{F}$ is Schur-irreducible. (See 4.6.2.) We also remind the reader that, for each λ, the rationality field $F(\lambda)$ is closed in the sense of 4.5.2, since it is, by definition, the fixed field of a subgroup of $\mathrm{Aut}(\boldsymbol{C})$.

Now let $\{(\pi_\lambda, V_\lambda)\}_{\lambda\in\mathcal{L}}$ be a family of realizations of the Π_λ, as in 4.7.1, equipped with maps $t_{\sigma;\lambda}$ satisfying (4.3). We assume that $V_{\lambda,0}$ is an $F(\lambda)$-rational structure on V_λ for all $\lambda \in \mathcal{L}$, cf. 4.7.6. Similarly, let $\{(\rho_\lambda, W_\lambda)\}_{\lambda\in\mathcal{L}}$ be another family of representatives, equipped with maps $u_{\sigma;\lambda}$, subject to the same conditions on both the transition maps and the subspaces $W_{\lambda,0}$. We may now state the final result.

4.7.8. Proposition. *For each $\lambda \in \mathcal{L}$, let $\Phi_\lambda\colon V_\lambda \to W_\lambda$ be an isomorphism of G-representations. Assume that, for all $\lambda \in \mathcal{L}$ and all $\sigma \in \mathrm{Aut}(\boldsymbol{C})$, the diagram*

$$\begin{array}{ccc} V_\lambda & \xrightarrow{\Phi_\lambda} & W_\lambda \\ \downarrow{\scriptstyle t_{\sigma;\lambda}} & & \downarrow{\scriptstyle u_{\sigma;\lambda}} \\ V_{\sigma\lambda} & \xrightarrow{\Phi_{\sigma\lambda}} & W_{\sigma\lambda} \end{array} \tag{4.5}$$

commutes. If $\Omega \in \boldsymbol{C}^*/F(\lambda)^*$ and $\Omega' \in \boldsymbol{C}^*/F(\sigma\lambda)^*$ denote the periods attached (see 4.6.11) to the triples $(V_{\lambda,0}, W_{\lambda,0}, \Phi_\lambda)$ and $(V_{\sigma\lambda,0}, W_{\sigma\lambda,0}, \Phi_{\sigma\lambda})$, respectively, then $\Omega' = \sigma(\Omega)$.

(In the final line of the statement, it is understood that $\sigma\colon \boldsymbol{C} \to \boldsymbol{C}$ descends to a well-defined map $\boldsymbol{C}^*/F(\lambda)^* \to \boldsymbol{C}^*/F(\sigma\lambda)^*$ since $F(\sigma\lambda) = \sigma(F(\lambda))$.)

Proof. Fix a $\lambda \in \mathcal{L}$ and a $\sigma \in \mathrm{Aut}(\boldsymbol{C})$. To reduce notational clutter, we replace the commutative diagram in the statement with

$$\begin{array}{ccc} V & \xrightarrow{\Phi} & W \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle u} \\ V' & \xrightarrow{\Phi'} & W'. \end{array} \tag{4.6}$$

Moreover, we let $V_0 \coloneqq V_{\lambda,0} \subseteq V$, and define W_0, V_0', W_0' in a similar way as subspaces of W, V', W', respectively. Recall from Proposition 4.7.4 that $t(V_0) = V_0'$ and similarly $u(W_0) = W_0'$.

Pick a representative $\tilde\Omega \in \boldsymbol{C}^*$ of the equivalence class $\Omega \in \boldsymbol{C}^*/F(\lambda)^*$, and set $\Phi'' \coloneqq u \circ (\tilde\Omega^{-1}\Phi) \circ t^{-1}$. We have

$$\begin{aligned} \Phi'' &= u \circ (\tilde\Omega^{-1}\Phi) \circ t^{-1} \\ &= \sigma(\tilde\Omega)^{-1} \cdot (u \circ \Phi \circ t^{-1}) \\ &= \sigma(\tilde\Omega)^{-1}\Phi', \end{aligned}$$

where in the middle step we have used Lemma 4.6.5.(i), and in the last step we have used the commutativity of the diagram (4.6). Thus, Φ'' is a G-equivariant linear isomorphism $W \to W'$. By the definition of the period Ω, it satisfies $\Phi''(V_0') = W_0'$. Since the period Ω' for the triple (V_0', W_0', Φ') is unique as an element of $\boldsymbol{C}^*/F(\sigma\lambda)^*$, it follows that Ω' is the class of $\sigma(\tilde{\Omega})$, as claimed. $\square$

4.7.9. *Remark.* We observe in passing that, if the condition (4.5) is satisfied, the isomorphism Φ_λ uniquely determines all isomorphisms $\Phi_{\tau\lambda}$ as τ ranges over the automorphisms of $\boldsymbol{C}$. (Indeed, $\Phi_{\tau\lambda} = u_{\tau;\lambda} \circ \Phi_\lambda \circ (t_{\tau;\lambda})^{-1}$.) $\diamond$

Chapter 5

Special L-values via periods: state of the art

“... jusques à ce que j’ai trouvé la somme de la série réciproque des quarrés, & ensuite de toutes les autres puissances paires: ayant demontré que les sommes de toutes ces séries dépendent du rapport de la circonférence d’un cercle π *à son diametre* 1.”

Leonhard Euler, *Remarques sur un beau rapport entre les séries des puissances tant directes que réciproques*

As in Chapters 2 and 3, we let F be a totally real number field and restrict our attention to the connected reductive group GL_n/F, where n is a positive integer. We make free use of the notations introduced in §2.1, as well as the notation $\boldsymbol{K}_\infty := K_\infty A_\infty$ introduced in §2.5.

5.1 Transition maps between highest-weight modules

5.1.1. Let $E = E_{\underline{\mu}}$ be a highest-weight module for $\mathrm{GL}_n(F)$. It is well-known ([Mil17, Thm. 22.18]) that the subspace of $\mathrm{B}_n(F)$-invariant vectors inside the space of $E_{\underline{\mu}}$ is one-dimensional. Thus, it follows from [Wal85, Lemme I.1] that $E_{\underline{\mu}}$ is defined over its rationality field $\mathbb{Q}(E_{\underline{\mu}})$.

We will now derive this fact again by exploiting the machinery set up in §4.7.

Firstly, recall from §1.8 that $\{E_{\underline{\mu}}\}_{\underline{\mu}\in\mathcal{L}}$, where

$$\mathcal{L} \coloneqq \{\underline{\mu} = (\mu^{\iota})_{\iota: F\to\mathbb{R}} : \text{each } \mu^{\iota} = (\mu_1^{\iota}, \dots, \mu_n^{\iota}) \text{ is dominant}\},$$

contains precisely one representative out of each isomorphism class of irreducible rational representations of $\mathrm{GL}_n(F)$. As in [GL21, §1.3.1], we observe that twists of irreducible rational representations by automorphisms $\sigma \in \mathrm{Aut}(\mathbb{C})$ are necessarily again irreducible rational representations. Thus, we get a left action of $\mathrm{Aut}(\mathbb{C})$ on $\mathcal{L}$, as in 4.7.1. For $\underline{\mu} \in \mathcal{L}$, let $\mathcal{S}(\underline{\mu})$ denote the stabilizer $\{\sigma \in \mathrm{Aut}(\mathbb{C}) : \sigma\underline{\mu} = \underline{\mu}\}$, and set $\mathbb{Q}(\underline{\mu})$ equal to the fixed field of $\mathcal{S}(\underline{\mu})$ in $\mathbb{C}$. As argued in 4.7.1, $\mathcal{S}(\underline{\mu}) = \mathcal{S}(E_{\underline{\mu}})$, therefore $\mathbb{Q}(\underline{\mu}) = \mathbb{Q}(E_{\underline{\mu}})$.

We now proceed to construct distinguished transition maps $t_{\sigma;\underline{\mu}} \colon E_{\underline{\mu}} \to E_{\sigma\underline{\mu}}$ for all σ and all $\underline{\mu}$. Recall the explicit description of $E_{\underline{\mu}}$ as an induced representation, and thus as a subspace of

$$\mathcal{O} = \left\{ f \colon \prod_{\iota: F\to\mathbb{R}} \mathrm{GL}_n(\mathbb{C}) \to \mathbb{C} : f \text{ is a regular map} \right\}$$

(cf. 1.8.4). We have an action of $\mathrm{Aut}(\mathbb{C})$ on $F \otimes_{\mathbb{Q}} \mathbb{C}$:

$$\sigma.(x \otimes z) \coloneqq x \otimes \sigma(z),$$

which in turn induces an action of $\mathrm{Aut}(\mathbb{C})$ on $\mathrm{GL}_n(F \otimes_{\mathbb{Q}} \mathbb{C})$, again denoted by a dot. Armed with this, we may define, for each $\sigma \in \mathrm{Aut}(\mathbb{C})$, a map $T_\sigma \colon \mathcal{O} \to \mathcal{O}$ as follows:

$$T_\sigma(f)(x) = \sigma(f(\sigma^{-1}.x)) \qquad \text{for } x = (x^{\iota})_{\iota: F\to\mathbb{R}} \in \prod_{\iota} \mathrm{GL}_n(\mathbb{C}).$$

The following properties of the T_σ are easily checked:

(1) each T_σ is a σ-linear map $\mathcal{O} \to \mathcal{O}$;
(2) each T_σ is equivariant for the action of $\prod_\iota \mathrm{GL}_n(\mathbb{C})$ on $\mathcal{O}$ by right translation;
(3) $T_\sigma \circ T_\tau = T_{\sigma\tau}$ for all σ, $\tau \in \mathrm{Aut}(\mathbb{C})$;
(4) T_σ sends the irreducible subspace $E_{\underline{\mu}}$ onto the irreducible subspace $E_{\underline{\mu}^\sigma}$, where $\underline{\mu}^\sigma \coloneqq (\mu^{\sigma^{-1}\circ\iota})_{\iota: F\to\mathbb{R}}$.

Thus, we obtain the sought-after transition maps $t_{\sigma;\underline{\mu}}$ by restricting the domain and codomain of T_σ to $E_{\underline{\mu}}$ and $E_{\underline{\mu}^\sigma}$, respectively.

Finally, we want to describe the subspace $E_{\underline{\mu},0}$, defined following (4.4). The primary goal is to exhibit a nonzero element $v_{\underline{\mu}}$ of $E_{\underline{\mu},0}$. Having done that, we will know, by irreducibility of $E_{\underline{\mu}}$, that $E_{\underline{\mu},0}$ is a $\mathbb{Q}(E_{\underline{\mu}})$-rational structure on $E_{\underline{\mu}}$

(recall Corollary 4.4.5), and that, in fact, $E_{\underline{\mu},0}$ is the $\mathbb{Q}(E_{\underline{\mu}})$-span of the $\mathrm{GL}_n(F)$-translates of $v_{\underline{\mu}}$ (cf. the proof of [Wal85, Lemme I.1].)

Following [LLS21, §2.1], we will take $v_{\underline{\mu}}$ to be a distinguished element of the one-dimensional subspace of $\mathrm{B}_n(F)$-invariant vectors in $E_{\underline{\mu}}$. To that end, recall first that, for an irreducible rational representation $E_{\underline{\nu}}$, its dual representation $E_{\underline{\nu}}^{\vee}$ is again an irreducible rational representation, and hence isomorphic to $E_{\underline{\nu}^{\vee}}$ for some highest weight $\underline{\nu}^{\vee}$. More concretely, we can identify $E_{\underline{\mu}}$ with the space of linear functionals on

$$\mathrm{Ind}_{\mathrm{B}_n(F\otimes_{\mathbb{Q}}\mathbb{C})}^{\mathrm{GL}_n(F\otimes_{\mathbb{Q}}\mathbb{C})}(\underline{\mu}^{-1}) \coloneqq \{f \in \mathcal{O} : f(tug) = \underline{\mu}(t)^{-1} f(g) \text{ for all } t \in \mathrm{T}_n(F \otimes_{\mathbb{Q}} \mathbb{C}),$$
$$u \in \mathrm{U}_n(F \otimes_{\mathbb{Q}} \mathbb{C}), g \in \mathrm{GL}_n(F \otimes_{\mathbb{Q}} \mathbb{C})\}. \quad (5.1)$$

Consider, then, the linear functional on the space (5.1) which sends an element f to $f(\mathrm{id})$, where by id we mean the identity element in $\mathrm{GL}_n(F \otimes_{\mathbb{Q}} \mathbb{C})$. A direct computation shows that this linear functional, when viewed as an element of $E_{\underline{\mu}}$, is $\mathrm{B}_n(F)$-invariant and fixed by all maps $t_{\sigma;\underline{\mu}}$, $\sigma \in \mathcal{S}(\mu)$. (Cf. also [JST19, Lemma 4.6].) We take this element of $E_{\underline{\mu}}$ to be the sought-after $v_{\underline{\mu}}$.

5.1.2. The above argument shows that the abstractly defined weight $\sigma\underline{\mu}$ is in fact none other than $\underline{\mu}^{\sigma} = (\mu^{\sigma^{-1}\circ\iota})_{\iota:F\to\mathbb{R}}$ for all $\underline{\mu} \in \mathcal{L}$ and all $\sigma \in \mathrm{Aut}(\mathbb{C})$. Our choice to distinguish notationally between $\sigma\underline{\mu}$ and $\underline{\mu}^{\sigma}$ is due to the fact that $\sigma.\underline{\mu} \coloneqq \underline{\mu}^{\sigma}$ is, in fact, a *right* action of $\mathrm{Aut}(\mathbb{C})$ on $\mathcal{L}$, whereas the action $\sigma.\underline{\mu} \coloneqq \sigma\underline{\mu}$ coming from abstract considerations on irreducible rational representations is a *left* action. The fact that these actions agree allows us to prove the following statement about the rationality field $\mathbb{Q}(E_{\underline{\mu}})$.

5.1.3. Proposition. *$\mathbb{Q}(E_{\underline{\mu}}) = \mathbb{Q}(\underline{\mu})$ is a finite Galois extension of $\mathbb{Q}$.*

Proof. The finiteness of $\mathbb{Q}(E_{\underline{\mu}})$ over $\mathbb{Q}$ is well-known, cf. e.g. [GR14, Lemma 7.1]. To see why it holds, it suffices (see e.g. [Cla11, Thm. 30]) to check that $\mathcal{S}(\underline{\mu})$ is a finite-index subgroup of $\mathrm{Aut}(\mathbb{C})$. But this is clear since the $\mathrm{Aut}(\mathbb{C})$-orbit of $\underline{\mu}$ can only have finitely many elements.

It follows that $\mathbb{Q}(\underline{\mu}) \subset \overline{\mathbb{Q}}$, where $\overline{\mathbb{Q}} \subset \mathbb{C}$ denotes the subfield of algebraic complex numbers, and so it is meaningful to look at $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\underline{\mu})) \subset \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. It remains to show that $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\underline{\mu}))$ is normal in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Because

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) = \mathrm{Aut}(\mathbb{C})/\,\mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}}),$$
$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\underline{\mu})) = \mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\underline{\mu}))/\,\mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}}),$$

and passing to quotients preserves normality of subgroups, it suffices to show that $\mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\underline{\mu}))$ is normal in $\mathrm{Aut}(\mathbb{C})$. Observe that we have $\mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\underline{\mu})) = \mathrm{Aut}(\mathbb{C}/\mathbb{Q}(E_{\underline{\mu}})) = \mathcal{S}(E_{\underline{\mu}}) = \mathcal{S}(\underline{\mu})$, where the equality in the middle follows from Lemma 4.5.5. Accordingly, the goal is to show that $\mathcal{S}(\underline{\mu})$ is normal in $\mathrm{Aut}(\mathbb{C})$.

To that end, let σ, $\tau \in \mathrm{Aut}(\mathbb{C})$ be arbitrary. Then from

$$(\sigma\tau)\underline{\mu} = \sigma(\tau\underline{\mu}) = \sigma(\underline{\mu}^{\tau}) = (\underline{\mu}^{\tau})^{\sigma} = \underline{\mu}^{\tau\sigma} = (\tau\sigma)\underline{\mu}$$

we deduce $[\sigma,\tau]\underline{\mu} = \underline{\mu}$, where as usual $[\sigma,\tau]$ stands for the *commutator* $\sigma\tau\sigma^{-1}\tau^{-1}$. In other words, $\mathcal{S}(\underline{\mu})$ contains the commutator subgroup of $\mathrm{Aut}(\mathbb{C})$. Since any subgroup containing the commutator subgroup is automatically normal, this completes the proof. $\square$

5.2 Twists of cohomological representations

We recall some fundamental and well-known facts.

5.2.1. Proposition. *Let Π be a unitary isobaric automorphic representation which is cohomological with respect to some highest-weight module $E_{\underline{\mu}}$ of $\mathrm{GL}_n(F)$. Then:*

(i) *Π_{f} is Schur-irreducible as a representation of $\mathrm{GL}_n(\mathbb{A}_{\mathrm{f}})$;*
(ii) *Π_{f} is defined over its rationality field $\mathbb{Q}(\Pi_{\mathrm{f}})$, which is a number field;*
(iii) *for every $\sigma \in \mathrm{Aut}(\mathbb{C})$, there is a unitary isobaric automorphic representation whose finite part is the σ-twist of Π_{f}. This representation is uniquely determined by the data σ and Π and is cohomological with respect to $E_{\underline{\mu}^{\sigma}}$.*

5.2.2. *About the proof.* The first claim is immediate from the fact that Π_{f} is an irreducible admissible representation and Schur's lemma holds for such representations, see e.g. [Ren09, III.1.8]. Moreover, it is shown in [Clo90, Prop. 3.1] that every irreducible admissible complex representation of $\mathrm{GL}_n(\mathbb{A}_{\mathrm{f}})$ is defined over its rationality field, which proves the first half of (ii). The remaining claims follow from [Clo90, Thm. 3.13] for Π cuspidal; the extension to the isobaric case can be found in [Gro18, §1.2.5] for CM fields and in [LLS21, Remark 6.1] for general number fields.

The representation determined by (iii) of the preceding proposition will henceforth be denoted by ${}^{\sigma}\Pi$. From the last assertion in (iii), we immediately obtain:

5.2.3. Corollary. *Let Π and $E_{\underline{\mu}}$ be as in the statement of Proposition 5.2.1. Then $\mathbb{Q}(\Pi_{\mathrm{f}}) \supseteq \mathbb{Q}(E_{\underline{\mu}})$.*

5.3 Cohomology of locally symmetric spaces

5.3.1. Consider the space

$$X := (\mathrm{GL}_n(F_\infty)/K_\infty^0 A_\infty) \times \mathrm{GL}_n(\mathbb{A}_\mathrm{f}).$$

It affords an action of $\mathrm{GL}_n(F)$, diagonally embedded in $\mathrm{GL}_n(F_\infty)$, coming from the first factor, and an action of $\mathrm{GL}_n(\mathbb{A}_\mathrm{f})$ coming from the second. Set

$$S^{\mathrm{GL}_n} := \mathrm{GL}_n(F)\backslash X.$$

This space is a projective limit of adèlic locally symmetric spaces, see [Roh96]; in more detail, we have

$$S^{\mathrm{GL}_n} = \varprojlim_{K_\mathrm{f}} S^{\mathrm{GL}_n}_{K_\mathrm{f}},$$

where the limit is over all compact open subgroups of $\mathrm{GL}_n(\mathbb{A}_\mathrm{f})$, and, for such a subgroup K_f,

$$\begin{aligned} S^{\mathrm{GL}_n}_{K_\mathrm{f}} &:= \mathrm{GL}_n(F)\backslash X/K_\mathrm{f} \\ &= A_\infty \,\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F)/K_\infty^0 K_\mathrm{f} \\ &= \qquad \mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F)/\boldsymbol{K}_\infty^0 K_\mathrm{f} \end{aligned}$$

is the *adèlic locally symmetric space with level structure* K_f.

5.3.2. For any highest-weight module $E_{\underline{\mu}}$ of G_∞, we will denote the corresponding *local system* on S^{GL_n} by $\mathcal{E}_{\underline{\mu}}$. Recall that this is a sheaf of complex vector spaces on S^{GL_n}, cf. *op. cit.*, §2. We may then look at the cohomology groups

$$H^q(S^{\mathrm{GL}_n}, \mathcal{E}_{\underline{\mu}}),$$

which are naturally $\mathrm{GL}_n(\mathbb{A}_\mathrm{f}) \times \pi_0(G_\infty)$-modules. We are interested in the isotypic components for the action of these two groups, especially in the *bottom degree* $q = b_n$ (cf. Proposition 2.5.11).

The proposition below establishes a fundamental relation to $(\mathfrak{g}_\infty, \boldsymbol{K}_\infty^0)$-cohomology.

5.3.3. Proposition. *Let* Π *be a unitary isobaric automorphic representation of* $\mathrm{GL}_n(\mathbb{A}_F)$ *which is cohomological with respect to the highest-weight module* $E_{\underline{\mu}}$*, and let* $\varepsilon \in \widehat{\pi_0(G_\infty)}$ *be permissible for* Π*. (Cf. 2.5.13.) Then* $H^{b_n}(S^{\mathrm{GL}_n}, \mathcal{E}_{\underline{\mu}})$ *has a unique* $\mathrm{GL}_n(\mathbb{A}_\mathrm{f}) \times \pi_0(G_\infty)$*-stable subspace belonging to the isomorphism class*

$\Pi_{\mathrm{f}} \times \varepsilon$. *More precisely, let* $\mathcal{M}(\Pi_{\mathrm{f}})$ *and* $\mathcal{M}(\Pi_\infty)$ *be realizations of* Π_{f} *and* Π_∞, *respectively — then there is a* $\mathrm{GL}_n(\mathbb{A}_{\mathrm{f}}) \times \pi_0(G_\infty)$*-equivariant linear injection*

$$\mathcal{M}(\Pi_{\mathrm{f}}) \otimes H^{b_n}(\mathfrak{g}_\infty, \boldsymbol{K}_\infty^0; \mathcal{M}(\Pi_\infty) \otimes E_{\underline{\mu}})(\varepsilon) \hookrightarrow H^{b_n}(S^{\mathrm{GL}_n}, \mathcal{E}_{\underline{\mu}}),$$

which is furthermore unique up to a nonzero scalar.

We refer the reader to [Gro18, Prop. 1.6], [GL21, Props. 1.19, 1.21], and [LLS21, Prop. 6.2].

5.4 Rational structure on Betti cohomology

5.4.1. Let $E_{\underline{\mu}}$ be a highest-weight module for $\mathrm{GL}_n(F)$, and consider the sheaf cohomology groups $H^\bullet(S^{\mathrm{GL}_n}, \mathcal{E}_{\underline{\mu}})$ introduced in 5.3.2. By the functorial properties of sheaf cohomology, we can leverage the discussion from §5.1 to obtain analogous results about the cohomology groups $H^\bullet(S^{\mathrm{GL}_n}, \mathcal{E}_{\underline{\mu}})$. In more detail,

(1) the σ-linear transition maps $t_{\sigma;\underline{\mu}}$ induce σ-linear transition maps

$$\tilde{t}_{\sigma;\underline{\mu}} \colon H^\bullet(S^{\mathrm{GL}_n}, \mathcal{E}_{\underline{\mu}}) \to H^\bullet(S^{\mathrm{GL}_n}, \mathcal{E}_{\underline{\mu}^\sigma});$$

(2) the subspaces $E_{\underline{\mu},0} \subset E_{\underline{\mu}}$ induce local systems $\mathcal{E}_{\underline{\mu},0}$, which are sheaves of $\mathbb{Q}(E_{\underline{\mu}})$-vector spaces, and we have a natural inclusion

$$H^\bullet(S^{\mathrm{GL}_n}, \mathcal{E}_{\underline{\mu},0}) \subset H^\bullet(S^{\mathrm{GL}_n}, \mathcal{E}_{\underline{\mu}});$$

(3) $H^\bullet(S^{\mathrm{GL}_n}, \mathcal{E}_{\underline{\mu},0})$ is precisely the joint fixed space of the $\tilde{t}_{\sigma;\underline{\mu}}$ with $\sigma \in \mathcal{S}(\underline{\mu})$;

(4) $H^\bullet(S^{\mathrm{GL}_n}, \mathcal{E}_{\underline{\mu},0})$ is a $\mathbb{Q}(E_{\underline{\mu}})$-rational structure on $H^\bullet(S^{\mathrm{GL}_n}, \mathcal{E}_{\underline{\mu}})$.

5.4.2. Now let Π be a unitary isobaric automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ which is cohomological with respect to $E_{\underline{\mu}}$, and let $\varepsilon \in \widehat{\pi_0(G_\infty)}$ be permissible for Π. Then ε is also permissible for each twist ${}^\sigma\Pi$ of Π. (If n is even, this is clear since then every ε is permissible; if n is odd, this can be read off [LLS21, Remark 6.3].)

Let $\mathcal{H}(\Pi_{\mathrm{f}} \times \varepsilon) \subset H^{b_n}(S^{\mathrm{GL}_n}, \mathcal{E}_{\underline{\mu}})$ denote the unique subspace which is in the isomorphism class $\Pi_{\mathrm{f}} \times \varepsilon$ (cf. Proposition 5.3.3). We will now again apply the observations we made in §4.7 to simultaneously fix a $\mathbb{Q}({}^\sigma\Pi_{\mathrm{f}})$-rational structure on $\mathcal{H}({}^\sigma\Pi_{\mathrm{f}} \times \varepsilon)$ for each automorphism $\sigma \in \mathrm{Aut}(\mathbb{C})$.

5.4.3. Proposition. *For every* $\sigma \in \mathrm{Aut}(\mathbb{C})$, *the map* $\tilde{t}_{\sigma;\underline{\mu}}$ *sends the subspace* $\mathcal{H}(\Pi_{\mathrm{f}} \times \varepsilon)$ *to the subspace* $\mathcal{H}({}^\sigma\Pi_{\mathrm{f}} \times \varepsilon)$. *In particular,* $\mathcal{H}(\Pi_{\mathrm{f}} \times \varepsilon)$ *is stable under the maps* $\tilde{t}_{\sigma;\underline{\mu}}$ *with* $\sigma \in \mathcal{S}(\Pi_{\mathrm{f}})$.

The claim follows swiftly from the uniqueness part of Proposition 5.3.3. Cf. [Gro18, Prop. 1.7], [LLS21, Prop. 6.2].

5.4.4. Proposition. *Let* $H^{b_n}(S^{\mathrm{GL}_n}, \mathcal{E}_{\underline{\mu}})_{\mathbb{Q}(\Pi_{\mathrm{f}})} \subset H^{b_n}(S^{\mathrm{GL}_n}, \mathcal{E}_{\underline{\mu}})$ *denote the* $\mathbb{Q}(\Pi_{\mathrm{f}})$*-linear span of* $H^{b_n}(S^{\mathrm{GL}_n}, \mathcal{E}_{\underline{\mu},0})$*. (See 5.4.1.) Then*

$$\mathcal{H}(\Pi_{\mathrm{f}} \times \varepsilon) \cap H^{b_n}(S^{\mathrm{GL}_n}, \mathcal{E}_{\underline{\mu}})_{\mathbb{Q}(\Pi_{\mathrm{f}})}$$

defines a $\mathbb{Q}(\Pi_{\mathrm{f}})$*-rational structure on* $\mathcal{H}(\Pi_{\mathrm{f}} \times \varepsilon)$*.*

Proof. First, recall from 5.4.1 that $H^{b_n}(S^{\mathrm{GL}_n}, \mathcal{E}_{\underline{\mu},0})$ is a $\mathbb{Q}(E_{\underline{\mu}})$-rational structure on $H^{b_n}(S^{\mathrm{GL}_n}, \mathcal{E}_{\underline{\mu},0})$. By Corollary 5.2.3, $\mathbb{Q}(\Pi_{\mathrm{f}}) \supseteq \mathbb{Q}(E_{\underline{\mu}})$, so it follows from 4.5.1 that $H^{b_n}(S^{\mathrm{GL}_n}, \mathcal{E}_{\underline{\mu}})_{\mathbb{Q}(\Pi_{\mathrm{f}})}$ is a $\mathbb{Q}(\Pi_{\mathrm{f}})$-rational structure on $H^{b_n}(S^{\mathrm{GL}_n}, \mathcal{E}_{\underline{\mu}})$. The claim is then an immediate consequence of [Spr09, Prop. 11.1.4] (cf. also [Clo90, Lemme 3.2.1]) in light of Proposition 5.4.3. □

5.5 Harder–Raghuram relative periods

Throughout this section, n will be an even positive integer, and Π will be fixed *unitary cuspidal* automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, which is cohomological with respect to a highest-weight module $E_{\underline{\mu}}$ of $\mathrm{GL}_n(F)$. Furthermore, let $\varepsilon \in \widehat{\pi_0(G_\infty)}$ — since n is even, ε is permissible for Π. We will define the so-called *relative period* $\Omega^{\varepsilon}_{\mathrm{rel}}(\Pi_{\mathrm{f}})$ attached to Π and ε, following [HR20, §5.2].

5.5.1. For each $\varepsilon \in \widehat{\pi_0(G_\infty)}$, let w^{ε} be a generator of the one-dimensional space $H^{b_n}(\mathfrak{g}_\infty, \boldsymbol{K}^0_\infty; \mathcal{W}(\Pi_\infty) \otimes \mathcal{E}_{\underline{\mu}})(\varepsilon)$, where we abbreviate $\mathcal{W}(\Pi_\infty) := \mathcal{W}(\Pi_\infty, \psi_\infty)$. A compatible choice of the w^{ε} as ε ranges in the set $\widehat{\pi_0(G_\infty)}$ will be called an *entangled system* of generators. We now explain how to obtain such a system $\underline{w} = \{w^{\varepsilon}\}_{\varepsilon}$.

Recall from the explicit description of $(\mathfrak{g}_\infty, \boldsymbol{K}^0_\infty)$-cohomology that a generator $w^{\mathbf{1}}$ of $H^{b_n}(\mathfrak{g}_\infty, \boldsymbol{K}^0_\infty; \mathcal{W}(\Pi_\infty) \otimes \mathcal{E}_{\underline{\mu}})(\mathbf{1})$, where $\mathbf{1}$ denotes the identity element of $\widehat{\pi_0(G_\infty)}$, can be written as

$$w^{\mathbf{1}} = \sum_{\underline{i}=(i_1,\dots,i_{b_n})} \sum_{\alpha=1}^{\dim E_{\underline{\mu}}} X^*_{\underline{i}} \otimes \xi_{\infty,\underline{i},\alpha} \otimes e_\alpha.$$

(We refer to [GL21, §1.5.2, esp. eq. (1.17)] for the unexplained notations and the assumptions on the objects appearing in the expression; in particular, we assume that $\xi_{\infty,\underline{i},\alpha}$ factors as

$$\xi_{\infty,\underline{i},\alpha} = \otimes_{v|\infty} \xi_{v,\underline{i},\alpha},$$

where $\xi_{v,\underline{i},\alpha}$ lies in the local Whittaker model $\mathcal{W}(\Pi_v, \psi_v)$.) We say that the w^ε are entangled if, for all ε,

$$w^\varepsilon = \sum_{\underline{i}=(i_1,\dots,i_{b_n})} \sum_{\alpha=1}^{\dim E_{\underline{\mu}}} X_{\underline{i}}^* \otimes \xi^\varepsilon_{\infty,\underline{i},\alpha} \otimes e_\alpha$$

where

$$\xi^\varepsilon_{\infty,\underline{i},\alpha} = \otimes_{v|\infty} \xi^{\varepsilon_v}_{v,\underline{i},\alpha}$$

and the action of the superscript ε_v on each factor is trivial if $\varepsilon_v = 1$, and amounts to twisting by a sign if $\varepsilon_v = -1$. Since each isotypic subspace $H^{b_n}(\mathfrak{g}_\infty, \boldsymbol{K}^0_\infty; \mathcal{W}(\Pi_\infty) \otimes \mathcal{E}_{\underline{\mu}})(\varepsilon)$ is one-dimensional, it follows that, if $\{w_\varepsilon\}^\varepsilon$ and $\{\tilde{w}^\varepsilon\}_\varepsilon$ are two entangled systems, then there exists a nonzero scalar c such that $\tilde{w}^\varepsilon = cw^\varepsilon$ for all ε.

5.5.2. Let $\mathcal{M}(\Pi_{\mathrm{f}})$ be any realization of Π_{f}. Following [HR20, p. 55], we fix a $\mathrm{GL}_n(\mathbb{A}_{\mathrm{f}})$-equivariant linear isomorphism

$$\phi\colon \mathcal{M}(\Pi_{\mathrm{f}}) \otimes H^{b_n}(\mathfrak{g}_\infty, \boldsymbol{K}^0_\infty; \mathcal{W}(\Pi_\infty) \otimes E_{\underline{\mu}}) \to H^{b_n}(S^{\mathrm{GL}_n}, \mathcal{E}_{\underline{\mu}})(\Pi_{\mathrm{f}}),$$

where $H^{b_n}(S^{\mathrm{GL}_n}, \mathcal{E}_{\underline{\mu}})(\Pi_{\mathrm{f}})$ denotes the Π_{f}-*isotypic component* in $H^{b_n}(S^{\mathrm{GL}_n}, \mathcal{E}_{\underline{\mu}})$, and we remark that the choice of ϕ is unique up to nonzero scalars.

Given a ϕ as above, and an entangled system $\underline{w} = \{w^\varepsilon\}_\varepsilon$ as in 5.5.1, we may define, for each $\varepsilon \in \widehat{\pi_0(G_\infty)}$, a $\mathrm{GL}_n(\mathbb{A}_{\mathrm{f}}) \times \pi_0(G_\infty)$-equivariant isomorphism $\psi_\varepsilon = \psi_\varepsilon^{\phi,\underline{w}}$,

$$\psi_\varepsilon\colon \mathcal{M}(\Pi_{\mathrm{f}}) \xrightarrow{\sim} \mathcal{H}(\Pi_{\mathrm{f}} \times \varepsilon), \tag{5.2}$$

by the rule $\psi_\varepsilon(v) = \phi(v \otimes w^\varepsilon)$ for all $\varepsilon \in \widehat{\pi_0(G_\infty)}$. In other words, ψ_ε arises as the composition of the maps

$$\begin{array}{ccccc} \mathcal{M}(\Pi_{\mathrm{f}}) & \to & \mathcal{M}(\Pi_{\mathrm{f}}) \otimes H^{b_n}(\mathfrak{g}_\infty, \boldsymbol{K}^0_\infty; \mathcal{W}(\Pi_\infty) \otimes E_{\underline{\mu}})(\varepsilon) & \to & \mathcal{H}(\Pi \times \varepsilon), \\ v & \mapsto & v \otimes w^\varepsilon & \mapsto & \phi(v \otimes w^\varepsilon). \end{array}$$

The central result is now as follows.

5.5.3. Proposition. *There exists a unique isomorphism*

$$\Phi_{\mathrm{can}}\colon \mathcal{H}(\Pi_f \times \varepsilon) \to \mathcal{H}(\Pi_{\mathrm{f}} \times -\varepsilon)$$

such that, for all realizations $\mathcal{M}(\Pi_{\mathrm{f}})$ of Π_{f} and all collections of maps $\{\psi_\varepsilon\}_\varepsilon$ as in 5.5.2, the diagram

$$\begin{tikzcd} & \mathcal{M}(\Pi_{\mathrm{f}}) \arrow[dl, "\psi_\varepsilon"'] \arrow[dr, "\psi_{-\varepsilon}"] & \\ \mathcal{H}(\Pi_{\mathrm{f}} \times \varepsilon) \arrow[rr, "\Phi_{\mathrm{can}}"] & & \mathcal{H}(\Pi_{\mathrm{f}} \times -\varepsilon), \end{tikzcd}$$

commutes.

Proof. We first define Φ_{can} by choosing a realization $\mathcal{M}(\Pi_{\mathrm{f}})$ and an "entangled system" $\{\psi_\varepsilon\}_\varepsilon$ and setting $\Phi_{\mathrm{can}} := \psi_{-\varepsilon} \circ \psi_\varepsilon^{-1}$. Since the data ϕ and $\underline{w}$ on which the ψ_ε depend are each unique up to a nonzero scalar independent of ε, we see that Φ_{can} does not depend on the choice of $\{\psi_\varepsilon\}_\varepsilon$; in other words,

$$\Phi_{\mathrm{can}} = \psi_{-\varepsilon} \circ \psi_\varepsilon^{-1}$$

for *any* entangled system of maps $\psi_\varepsilon \colon \mathcal{M}(\Pi_{\mathrm{f}}) \to \mathcal{H}(\Pi_{\mathrm{f}} \times \varepsilon)$. It remains to show that this property is not contingent on the choice of realization $\mathcal{M}(\Pi_{\mathrm{f}})$.

Thus, let $\mathcal{M}'(\Pi_{\mathrm{f}})$ be another realization of Π_{f}. Let $\psi'_\varepsilon \colon \mathcal{M}'(\Pi_{\mathrm{f}}) \to \mathcal{H}(\Pi_{\mathrm{f}} \times \varepsilon)$ be entangled maps, $\varepsilon \in \widehat{\pi_0(G_\infty)}$, and let $\Theta \colon \mathcal{M}(\Pi_{\mathrm{f}}) \to \mathcal{M}'(\Pi_{\mathrm{f}})$ be an isomorphism. Then $\{\psi'_\varepsilon \circ \Theta\}_\varepsilon$ is again an entangled system of maps from $\mathcal{M}(\Pi_{\mathrm{f}})$ to $\mathcal{H}(\Pi_{\mathrm{f}} \times \varepsilon)$, so, by the arguments of the preceding paragraph,

$$\begin{aligned}
\Phi_{\mathrm{can}} &= (\psi'_{-\varepsilon} \circ \Theta) \circ (\psi'_\varepsilon \circ \Theta)^{-1} \\
&= \psi'_{-\varepsilon} \circ \Theta \circ \Theta^{-1} \circ (\psi'_\varepsilon)^{-1} \\
&= \psi'_{-\varepsilon} \circ (\psi'_\varepsilon)^{-1},
\end{aligned}$$

independently of the choice of Θ and ψ'_ε. $\square$

5.5.4. Consider the equivariant isomorphism

$$\Phi \colon \mathcal{H}(\Pi_{\mathrm{f}} \times \varepsilon) \to \mathcal{H}(\Pi_{\mathrm{f}} \times -\varepsilon)$$

given by $\Phi := \mathrm{i}^{[F:\mathbb{Q}]n/2}\Phi_{\mathrm{can}}$, where Φ_{can} is the isomorphism defined in Proposition 5.5.3. Let $\mathcal{H}(\Pi_{\mathrm{f}} \times \varepsilon)_0 \subset \mathcal{H}(\Pi_{\mathrm{f}} \times \varepsilon)$ and $\mathcal{H}(\Pi_{\mathrm{f}} \times -\varepsilon)_0 \subset \mathcal{H}(\Pi_{\mathrm{f}} \times -\varepsilon)$ be the natural $\mathbb{Q}(\Pi_{\mathrm{f}})$-rational structures defined as in Proposition 5.4.4. To this triple, we can attach, by 4.6.11, a period, which is an element of $\mathbb{C}^*/\mathbb{Q}(\Pi_{\mathrm{f}})^*$. We will denote this period by $\Omega^\varepsilon(\Pi_{\mathrm{f}})$ and call it the *relative period* attached to the representation Π and the signature ε.

5.6 Rational structure on Whittaker models

Now let n again be an arbitrary positive integer. The contents of this section are by now standard, see [Har83, III], [Mah05, §3.3], [RS10, §3.2], [Gro18, §1.4], [LLS21, §6.2] among others.

5.6.1. Let Π be a unitary isobaric automorphic representation which is cohomological with respect to a highest-weight module $E_{\underline{\mu}}$, and fix a generic character ψ of $\mathrm{U}_n(\mathbb{A}_F)$ which is trivial on $\mathrm{U}_n(F)$. We know from Corollary 2.5.10.(iii) that Π is globally ψ-generic, which, by definition, means that the map W_ψ from (2.7) is not identically zero. By Proposition 2.4.2.(i), the finite part Π_{f} of Π has a unique ψ-Whittaker model

$$\mathcal{W}(\Pi_{\mathrm{f}}, \psi_{\mathrm{f}}) \subset \{W \in C^\infty(\mathrm{GL}_n(\mathbb{A}_{\mathrm{f}})) : W(ug) = \psi(u)W(g) \text{ for all } g \in \mathrm{GL}_n(\mathbb{A}_{\mathrm{f}}), u \in \mathrm{U}_n(\mathbb{A}_{\mathrm{f}})\}. \quad (5.3)$$

As in the references cited at the beginning of this section, one defines, for each $\sigma \in \mathrm{Aut}(\mathbb{C})$, a σ-linear $\mathrm{GL}_n(\mathbb{A}_{\mathrm{f}})$-equivariant automorphism of the space on the right-hand side of (5.3), which we will denote by T_σ. By the uniqueness of Whittaker models, T_σ sends the irreducible subspace $\mathcal{W}(\Pi_{\mathrm{f}}, \psi_{\mathrm{f}})$ to the irreducible subspace $\mathcal{W}({}^\sigma\Pi_{\mathrm{f}}, \psi_{\mathrm{f}})$. (Note the similarity to the discussion in §5.1.) Thus, we obtain transition maps $t_\sigma : \mathcal{W}(\Pi_{\mathrm{f}}, \psi_{\mathrm{f}}) \to \mathcal{W}({}^\sigma\Pi_{\mathrm{f}}, \psi_{\mathrm{f}})$, for each $\sigma \in \mathrm{Aut}(\mathbb{C})$, by restricting the domain and codomain of T_σ.

We now again apply the observations gathered in §4.7. Let $\mathcal{W}(\Pi_{\mathrm{f}})_0$ be the $\mathrm{GL}_n(\mathbb{A}_{\mathrm{f}})$-stable $\mathbb{Q}(\Pi_{\mathrm{f}})$-linear subspace of $\mathcal{W}(\Pi_{\mathrm{f}}, \psi_{\mathrm{f}})$ obtained as the joint fixed space of the maps t_σ with $\sigma \in \mathcal{S}(\Pi_{\mathrm{f}})$. We have the following result.

5.6.2. Proposition. *$\mathcal{W}(\Pi_{\mathrm{f}})_0$ is a $\mathbb{Q}(\Pi_{\mathrm{f}})$-rational structure on $\mathcal{W}(\Pi_{\mathrm{f}}, \psi_{\mathrm{f}})$.*

Proof. Since Π_{f}, and hence $\mathcal{W}(\Pi_{\mathrm{f}}, \psi_{\mathrm{f}})$, is irreducible, $\mathcal{W}(\Pi_{\mathrm{f}})_0$ is a rational structure on $\mathcal{W}(\Pi_{\mathrm{f}}, \psi_{\mathrm{f}})$ if it contains a nonzero vector, see Corollary 4.4.5. This latter assertion can be showed as follows (cf. [GM23, §3.1]). Firstly, let v be a finite place of F — then, one defines the notion of a *new vector* in the local ψ_v-Whittaker model $\mathcal{W}(\Pi_v, \psi_v)$ as in [JPS81, Def. (4.4)]. One shows (see *op. cit.*, Thm. 5.1.(ii) and [Jac12]) that new vectors form a one-dimensional subspace of $\mathcal{W}(\Pi_v, \psi_v)$. Finally, one shows ([Mat13], or independently, [Miy14, Cor. 4.4]) that nonzero new vectors do not vanish at the identity, so that one may normalize the Whittaker new vector W_v, at each v, so that its value at the identity is 1. Then $W = \bigotimes_{v \nmid \infty} W_v$ is an element of $\mathcal{W}(\Pi_{\mathrm{f}}, \psi_{\mathrm{f}})$ with the sought-after properties. (Note the similarity with the argument given for highest-weight modules in 5.1.1.) □

5.7 Whittaker–Betti periods

5.7.1. Let Π be a cohomological unitary isobaric automorphic representation, and ε be permissible for Π. Recall that ε yields a representation of $\pi_0(G_\infty)$ — for the sake of definedness, we specify that the space of this representation is $\mathbb{C}$ and the action is given by $g.v = \varepsilon(g)v$ for all $g \in \pi_0(G_\infty)$. As is standard, we will not distinguish notationally between the character ε and the corresponding representation.

With this convention, consider now the representation $\mathcal{W}(\Pi_{\mathrm{f}}) \times \varepsilon$ of $\mathrm{GL}_n(\mathbb{A}_{\mathrm{f}}) \times \pi_0(G_\infty)$. It follows from Proposition 5.6.2 that this representation is defined over $\mathbb{Q}(\Pi_{\mathrm{f}})$, the natural rational structure being given, of course, by $\mathcal{W}(\Pi_{\mathrm{f}})_0 \otimes \varepsilon_0$, where the space of ε_0 is simply $\mathbb{Q}(\Pi_{\mathrm{f}}) \subset \mathbb{C}$.

5.7.2. We are now ready to define Whittaker–Betti periods, following [GL21, §1.5] and [LLS21, §6.3]. Let $\mathcal{W}(\Pi_\infty)$ be the Whittaker model of the infinite part Π_∞ of Π. Then the first cited source, see also [Gro18], defines a chain of isomorphisms

$$\begin{aligned}
\mathcal{W}(\Pi_{\mathrm{f}}) \otimes \varepsilon &\xrightarrow{\sim} H^{b_n}(\mathfrak{g}_\infty, \boldsymbol{K}_\infty^0; \mathcal{W}(\Pi_\infty) \otimes E_{\underline{\mu}})(\varepsilon) \otimes \mathcal{W}(\Pi_{\mathrm{f}}) \\
&\xrightarrow{\sim} H^{b_n}(\mathfrak{g}_\infty, \boldsymbol{K}_\infty^0; \mathcal{W}(\Pi) \otimes E_{\underline{\mu}})(\varepsilon) \\
&\xrightarrow{\sim} H^{b_n}(\mathfrak{g}_\infty, \boldsymbol{K}_\infty^0; \Pi \otimes E_{\underline{\mu}})(\varepsilon) \\
&\xrightarrow{\sim} H^{b_n}(\mathfrak{g}_\infty, \boldsymbol{K}_\infty^0; \mathcal{A}_{\mathcal{J},\{\mathrm{P}\},\varphi_{\mathrm{P}}}(\mathrm{GL}_n) \otimes E_{\underline{\mu}})(\varepsilon) \\
&\xrightarrow{\sim} \mathcal{H}(\Pi_{\mathrm{f}} \times \varepsilon).
\end{aligned} \tag{5.4}$$

(We refer to [GL21] for the meaning of the unexplained notations; let us simply remark that, in the middle row, Π denotes a concrete realization of Π as a subrepresentation of the space of automorphic forms on GL_n, and that, for the second row from above, we fix the isomorphism to be the natural one.)

Let Φ denote the composition of the isomorphisms in (5.4). Recall the $\mathbb{Q}(\Pi_{\mathrm{f}})$-rational structures $\mathcal{W}(\Pi_{\mathrm{f}})_0 \times \varepsilon_0$ and $\mathcal{H}(\Pi_{\mathrm{f}} \times \varepsilon)_0$ on the domain and target space of Φ, respectively. Then, by 4.6.11, we obtain a period attached to these rational structures and our choice of Π. This period will be called the *bottom-degree Whittaker–Betti period*, or simply *Whittaker period*, attached to Π_{f} and ε, and will be denoted by $\Omega^{\varepsilon}_{\mathrm{Wh}}(\Pi_{\mathrm{f}})$.

5.8 Relationship to L-values

In this section, we recall a couple of recent results which establish a relation between the values of Rankin–Selberg L-functions at critical points, on the one hand, and the periods introduced in the previous section, on the other.

5.8.1. Recall that the ground field F is a totally real number field, fixed throughout. Let $n > 1$ be an odd integer, and let Π, resp. Σ, be a unitary cuspidal, resp.

unitary isobaric, automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, resp. $\mathrm{GL}_{n-1}(\mathbb{A}_F)$. Furthermore, we assume that Π and Σ are cohomological with respect to highest-weight modules $E_{\underline{\mu}}$ and $E_{\underline{\lambda}}$, respectively. We identify both $\pi_0(\mathrm{GL}_n(F_\infty))$ and $\pi_0(\mathrm{GL}_{n-1}(F_\infty))$ with $\pi_0(F_\infty^\times)$ in the canonical way as per §2.1. The following is a special case of [LLS21, Thm. 1.2].

5.8.2. Theorem (Li–Liu–Sun, 2021). *Suppose that the weights $\underline{\mu} = (\mu^\iota)_{\iota:F\to\mathbb{R}}$ and $\underline{\lambda} = (\lambda^\iota)_{\iota:F\to\mathbb{R}}$ satisfy the piano condition*

$$\mu_1^\iota \geq -\lambda_{n-1}^\iota \geq \cdots \geq -\lambda_1^\iota \geq \mu_n^\iota \qquad \text{for all } \iota$$

(Cf. §3.4.) Let $\varepsilon_{\mathrm{perm}}(\Pi)$ denote the unique signature permissible for Π. Then, for every critical point $s_0 = \frac{1}{2} + j$ of $L(s, \Pi \times \Sigma)$,

$$L(s_0, \Pi \times \Sigma) \sim \mathrm{i}^e\, \Omega_{\mathrm{Wh}}^{\eta}(\Pi)\, \Omega_{\mathrm{Wh}}^{\eta'}(\Sigma)\, \mathcal{G}(\chi_\Sigma),$$

where:

$$\begin{aligned} e &:= j\frac{n(n-1)}{2}[F:\mathbb{Q}] - \sum_{\iota:F\to\mathbb{R}} \sum_{i=1}^{n-1} i(\mu_{i+1}^\iota + \lambda_i^\iota); \\ \eta &:= (-1)^{(n-1)(n-2)/2} \varepsilon_{\mathrm{perm}}(\Pi); \\ \eta' &:= (-1)^j \eta; \end{aligned}$$

$\mathcal{G}(\chi_\Sigma)$ is the Gauß sum of the central character of Σ; and $\sim$ means "up to a nonzero element of the compositum of the rationality fields of Π and Σ".

5.8.3. Let m be an even positive integer and n be an odd integer greater than m, and let Π, resp. Π', be a unitary cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, resp. $\mathrm{GL}_m(\mathbb{A}_F)$ which is cohomological with respect to a coefficient module $E_{\underline{\mu}}$, resp. $E_{\underline{\lambda}}$. We again identify both $\pi_0(\mathrm{GL}_n(F_\infty))$ and $\pi_0(\mathrm{GL}_m(F_\infty))$ with $\pi_0(F_\infty^\times)$ in the canonical way. As a special case of [HR20, Thm. 7.21], we have the following result.

5.8.4. Theorem (Harder–Raghuram, 2020). *Assume that $L(s, \Pi \times \Pi')$ has a critical point s_0 such that s_0+1 is also critical and such that $L(s_0+1, \Pi\times\Pi') \neq 0$. Then,*

$$\frac{L(s_0, \Pi \times \Pi')}{L(s_0+1, \Pi \times \Pi')} \sim \Omega_{\mathrm{rel}}^{(-1)^k\cdot\varepsilon}(\Pi'_{\mathrm{f}}),$$

where the tuple $\varepsilon = (\varepsilon_v)_{v|\infty}$ is uniquely determined by the isomorphism

$$\Pi_\infty \cong \bigotimes_{v|\infty} \pi^{\boldsymbol{l}_v, 0, \varepsilon_v};$$

the integer k equals $s_0 + (n+m)/2$, and $\sim$ means "up to a nonzero element of the Galois closure of the compositum of $\mathbb{Q}(\Pi_{\mathrm{f}})$ and $\mathbb{Q}(\Pi'_{\mathrm{f}})$".

5.8.5. We spend a few words on how Theorem 5.8.4 is obtained from the aforementioned [HR20, Thm. 7.21]. All references in this paragraph are to [HR20].

Firstly, since we assume the existence of at least two critical points, it is surely the case that Π and Π' have *disjoint cuspidal parameters* (cf. Def. 7.4, Prop. 7.7). Next, recall that s_0 is critical for $L(s, \Pi\times\Pi')$ if and only if $s_0 - k = -\frac{n+m}{2}$ is critical for $L(s, \Pi \times \Pi'(k))$, where $\Pi'(k))$ denotes a *Tate twist* (cf. §5.2.4). Keeping in mind that the *cohomological L-function* appearing in the statement of Thm. 7.21 is related to the Rankin–Selberg L-function by an explicit shift by $\mathsf{m}_0 + \frac{n+m}{2}$, where m_0 is defined as in §7.2.3.1, we now have that $-\frac{n+m}{2}$ and $1 - \frac{n+m}{2}$ are both critical for $L(s, \Pi \times \Pi'(k))$. In other words, the assumptions of Thm. 7.21 are met, so we can apply Thm. 7.21 to the quotient of special values

$$\frac{L(-\frac{n+m}{2}, \Pi \times \Pi'(k))}{L(1 - \frac{n+m}{2}, \Pi \times \Pi'(k))}.$$

The above theorem now follows by retracing the argument in the opposite direction and recalling the transformation rule for relative periods under Tate twists (§5.2.4).

Chapter 6

Original results

"Εὕρηκᾰ! Εὕρηκᾰ!"
attributed to Archimedes of Syracuse

"I shall call it... blellow!"
Reese, *Malcolm in the middle*

As was the case in the previous chapters, we fix a totally real number field F. Throughout this chapter, n and m are positive integers subject to the following conditions: *n is odd, m is even, and $m < n$.* We will freely use the notation $r := (n - m - 1)/2$. Finally, Π and Π' are cohomological unitary cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$ and $\mathrm{GL}_m(\mathbb{A}_F)$, respectively.

6.1 An auxiliary result

6.1.1. Proposition. *There exist unitary cuspidal cohomological representations $\Pi_1, \ldots, \Pi_r$ of $\mathrm{GL}_2(\mathbb{A}_F)$ such that the isobaric sum*

$$\Sigma := \Pi' \boxplus \Pi_1 \boxplus \cdots \boxplus \Pi_r$$

is again globally generic and cohomological.

Proof. We will freely use the notations and results from §2.5. Recall that, for each infinite place v of F, the local component Π_v is equivalent to $\pi^{\boldsymbol{l}_v,0}$ for some tuple $l_1^v, \ldots, l_{m/2}^v$ of pairwise distinct positive integers of the same parity. For each v, pick positive integers $l'^v_1, \ldots, l'^v_r$ such that

$$l_1^v, \ldots, l_v^{m/2}, l'^v_1, \ldots, l'^v_r$$

are, again, pairwise distinct with the same parity.

For each $j = 1, \dots, r$, let

$$\pi_j \coloneqq \bigotimes_{v|\infty} \pi^{l'^v_j, 0}.$$

We will now need the well-known fact that every discrete series representation π of $\mathrm{GL}_2(F_\infty)$ occurs as the infinite component Π_∞ of a unitary cuspidal automorphic representation Π of $\mathrm{GL}_2(\mathbb{A}_F)$. This can be proved by exploiting the correspondence between cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_F)$ and *primitive holomorphic Hilbert modular forms*, a modern and complete account of which can be found in [RT11, Thm. 1.4, §4]. In more detail, the parameters l'^v_j determine a tuple of weights $k = (k_1, \dots, k_d)$, where d is the degree of F over $\mathbb{Q}$; the central character of π determines a nebentypus character ω; and by passing to a sufficiently deep level $\mathfrak{n}$, it follows from well-known dimension formulae that there exists a primitive $\mathbf{f} \in S_k(\mathfrak{n}, \omega)_{\mathrm{prim}}$, in which case $\Pi = \Pi(\mathbf{f})$ will have the desired property.

Thus, let Π_j be a unitary cuspidal automorphic representation whose component at infinity $(\Pi_j)_\infty$ is isomorphic to π_j, for $j = 1, \dots, r$, and consider the isobaric sum $\Sigma = \Pi' \boxplus \Pi_1 \boxplus \cdots \boxplus \Pi_r$. By induction in stages, we have, at each infinite place v,

$$\begin{aligned} \Sigma_v &= \pi^{\boldsymbol{l}_v, 0} \boxplus \pi^{l'^v_1, 0} \boxplus \cdots \boxplus \pi^{l'^v_r, 0} \\ &= \pi^{\hat{\boldsymbol{l}}_v, 0}, \end{aligned}$$

where we have denoted by $\hat{\boldsymbol{l}}_v$ the tuple of distinct positive integers consisting of $l^v_1, \dots, l^v_{m/2}, l'^v_1, \dots, l'^v_r$, suitably reordered so as to form a decreasing sequence. In particular, Σ is cohomological. Finally, since all l'^v_j are pairwise distinct, we also have that the Π_j are pairwise non-isomorphic, and so Σ is globally generic by Theorem 2.4.4. □

6.2 An expression for the special values

6.2.1. Theorem. *Let Π_i, with $i = 1, \dots, r$, be as in the statement of Proposition 6.1.1, and set $\Sigma \coloneqq \Pi' \boxplus \Pi_1 \boxplus \cdots \boxplus \Pi_r$. Let $\underline{\mu} = (\mu^\iota)_{\iota: F \to \mathbb{R}}$, resp. $\underline{\lambda} = (\lambda^\iota)_{\iota: F \to \mathbb{R}}$, denote the highest weight of the coefficient module $E_{\underline{\mu}}$, resp. $E_{\underline{\lambda}}$, with respect to which Π, resp. Σ is cohomological. Assume that the piano condition*

$$\mu^\iota_1 \geq -\lambda^\iota_{n-1} \geq \cdots \geq -\lambda^\iota_1 \geq \mu^\iota_n \qquad \textit{for all } \iota.$$

is satisfied. Let $s_0 = \frac{1}{2} + j$ *be a critical point of* $L(s, \Pi \times \Pi')$ *which is also critical for* $L(s, \Pi \times \Pi_i)$ *for* $i = 1, \dots, r$*, and assume that* $L(s_0, \Pi \times \Pi_i) \neq 0$ *for all* i*. Let* $\varepsilon_{\mathrm{perm}}(\Pi)$ *denote the unique signature permissible for* Π*. Then*

$$L(s_0, \Pi \times \Pi') \sim \mathrm{i}^e\, \Omega^{\eta}_{\mathrm{Wh}}(\Pi)\, \Omega^{\eta'}_{\mathrm{Wh}}(\Sigma)\, \mathcal{G}(\chi_\Sigma) \prod_{i=1}^{r} L(s_0, \Pi \times \Pi_i)^{-1},$$

where, as in the statement of Theorem 5.8.2,

$$\begin{aligned} e &:= j\frac{n(n-1)}{2}[F : \mathbb{Q}] - \sum_{\iota : F \to \mathbb{R}} \sum_{i=1}^{n-1} i(\mu^{\iota}_{i+1} + \lambda^{\iota}_i); \\ \eta &:= (-1)^{(n-1)(n-2)/2} \varepsilon_{\mathrm{perm}}(\Pi); \\ \eta' &:= (-1)^j \eta; \end{aligned}$$

$\mathcal{G}(\chi_\Sigma)$ *is the Gauß sum of the central character of* Σ*; and* $\sim$ *means "up to a nonzero element of the compositum of the rationality fields of* Π *and* Σ*".*

Proof. By our assumptions on s_0, together with Proposition 3.3.4, we have that s_0 is critical for $L(s, \Pi \times \Sigma)$, and hence $j = s_0 - \frac{1}{2}$ is a balanced place of the pair $(E_{\underline{\mu}}, E_{\underline{\lambda}})$. By Proposition 6.1.1, Π and Σ satisfy the assumptions of Theorem 5.8.2, so we have

$$L(s_0, \Pi \times \Sigma) \sim \mathrm{i}^e\, \Omega^{\eta}_{\mathrm{Wh}}(\Pi)\, \Omega^{\eta'}_{\mathrm{Wh}}(\Sigma)\, \mathcal{G}(\chi_\Sigma)$$

with notations as in the statement of Theorem 5.8.2. By Theorem 3.2.3, we can write the left-hand side as

$$L(s_0, \Pi \times \Pi') \prod_{i=1}^{r} L(s_0, \Pi \times \Pi_i),$$

and the claim follows immediately. □

6.2.2. *Remark.* The preceding statement leaves an important question open, namely that of when, and in how many ways, one can choose the Π_i in such a way that the highest weights of Π and Σ satisfy the piano condition. We observe that this is a purely combinatorial question, and hope to obtain more progress on the matter in future work. ◇

6.3 Quotients of special values

6.3.1. Corollary. *Resume the notations and assumptions from the statement of Theorem 6.2.1. Suppose that* $L(s, \Pi \times \Sigma)$ *has at least two consecutive critical points*

s_0, s_0+1 and that they both fulfill the assumptions of Theorem 6.2.1. Then, if $L(s_0+1,\Pi\times\Sigma)\neq 0$, we have

$$\frac{L(s_0,\Pi\times\Pi')}{L(s_0+1,\Pi\times\Pi')}\sim \mathrm{i}^{[F:\mathbb{Q}](n-1)/2}\frac{\Omega^{\eta'}_{\mathrm{Wh}}(\Sigma)}{\Omega^{-\eta'}_{\mathrm{Wh}}(\Sigma)}\prod_{i=1}^{r}\left(\frac{L(s_0,\Pi\times\Pi_i)}{L(s_0+1,\Pi\times\Pi_i)}\right)^{-1}.$$

Proof. Under the assumptions stated before the corollary, we can apply Theorem 6.2.1 both at $s=s_0$ and at $s=s_0+1$. Resuming the notation of Theorem 5.8.2, and working out the dependence on $j=s_0-1/2$ in the second relation, we obtain

$$L(s_0,\Pi\times\Pi')\sim \mathrm{i}^{e}\,\Omega^{\eta}_{\mathrm{Wh}}(\Pi)\,\Omega^{\eta'}_{\mathrm{Wh}}(\Sigma)\,\mathcal{G}(\chi_\Sigma)\prod_{i=1}^{r}L(s_0,\Pi\times\Pi_i)^{-1},$$
$$L(s_0+1,\Pi\times\Pi')\sim \mathrm{i}^{e'}\,\Omega^{\eta}_{\mathrm{Wh}}(\Pi)\,\Omega^{\eta'}_{\mathrm{Wh}}(\Sigma)\,\mathcal{G}(\chi_\Sigma)\prod_{i=1}^{r}L(s_0+1,\Pi\times\Pi_i)^{-1},$$

where $e'=e+[F:\mathbb{Q}]n(n-1)/2$. By our assumptions on s_0, we may divide the second relation by the first, and obtain

$$\frac{L(s_0,\Pi\times\Pi')}{L(s_0+1,\Pi\times\Pi')}\sim \mathrm{i}^{[F:\mathbb{Q}]n(n-1)/2}\frac{\Omega^{\eta'}_{\mathrm{Wh}}(\Sigma)}{\Omega^{-\eta'}_{\mathrm{Wh}}(\Sigma)}\prod_{i=1}^{r}\left(\frac{L(s_0,\Pi\times\Pi_i)}{L(s_0+1,\Pi\times\Pi_i)}\right)^{-1}.$$

Finally, since n is odd, we have

$$\mathrm{i}^{\frac{n(n-1)}{2}[F:\mathbb{Q}]}=\left(\mathrm{i}^{[F:\mathbb{Q}](n-1)/2}\right)^{n}=\pm\mathrm{i}^{[F:\mathbb{Q}](n-1)/2},$$

which completes the proof. □

6.4 Relating different periods

6.4.1. Corollary. *Resume the notations and assumptions from the statement of Corollary 6.3.1. Furthermore, let ε be the signature determined by Π_∞ in virtue of the isomorphism*

$$\Pi_\infty\cong\bigotimes_{v|\infty}\pi^{\boldsymbol{l}_v,0,\varepsilon_v},$$

(cf. Proposition 2.5.8), and, for an even integer b, let $\varepsilon_b:=(-1)^{s_0+(n+b)/2}\varepsilon$. Then

$$\frac{\Omega^{\eta'}_{\mathrm{Wh}}(\Sigma)}{\Omega^{-\eta'}_{\mathrm{Wh}}(\Sigma)}\sim \mathrm{i}^{[F:\mathbb{Q}](n-1)/2}\Omega^{\varepsilon_m}_{\mathrm{rel}}(\Pi'_{\mathrm{f}})\cdot\prod_{i=1}^{r}\Omega^{\varepsilon_2}_{\mathrm{rel}}(\Pi_{i,\mathrm{f}}),\tag{6.1}$$

where $\sim$ means "up to a nonzero element of the Galois closure of the compositum of $\mathbb{Q}(\Pi_{\mathrm{f}})$ and $\mathbb{Q}(\Sigma_{\mathrm{f}})$". In particular, if $m = n-1$, i.e. $\Sigma = \Pi'$ and $r = 0$, then

$$\Omega_{\mathrm{rel}}^{\varepsilon_{n-1}}(\Pi'_{\mathrm{f}}) \sim \mathrm{i}^{[F:\mathbb{Q}](n-1)/2} \frac{\Omega_{\mathrm{Wh}}^{\eta'}(\Pi')}{\Omega_{\mathrm{Wh}}^{-\eta'}(\Pi')},$$

where $\sim$ means "up to a nonzero element of the Galois closure of the compositum of $\mathbb{Q}(\Pi_{\mathrm{f}})$ and $\mathbb{Q}(\Pi'_{\mathrm{f}})$".

Proof. By our assumptions, together with

$$\mathrm{Crit}(\Pi \times \Sigma) = \mathrm{Crit}(\Pi \times \Pi') \cap \bigcap_{i=1}^{r} \mathrm{Crit}(\Pi \times \Pi_i)$$

(cf. Proposition 3.3.4), we have that both s_0 and $s_0 + 1$ are critical for each of the L-functions $L(s, \Pi \times \Pi')$ and $L(s, \Pi \times \Pi_i)$ for $i = 1, \ldots, r$. Since each of Π', $\Pi_1, \ldots, \Pi_r$ is unitary cuspidal and cohomological, we can apply Theorem 5.8.4 to each quotient of consecutive special values appearing in (6.3.1), obtaining

$$\frac{L(s_0, \Pi \times \Pi')}{L(s_0+1, \Pi \times \Pi')} \sim \Omega_{\mathrm{rel}}^{(-1)^{k_1} \cdot \varepsilon}(\Pi'_{\mathrm{f}}),$$

with $k_1 = s_0 + (n+m)/2$, as well as

$$\frac{L(s_0, \Pi \times \Pi_i)}{L(s_0+1, \Pi \times \Pi_i)} \sim \Omega_{\mathrm{rel}}^{(-1)^{k_2} \cdot \varepsilon}(\Pi_{i,\mathrm{f}}),$$

with $k_2 = s_0 + (n+2)/2$. (In both relations, ε has the same meaning as in the statement of Theorem 5.8.4.) These observations, together with the fact that a power of i and its inverse are either equal or differ only by a sign, complete the proof. $\square$

6.4.2. From the definition of η' in the statement of Theorem 6.2.1 and the relation between $\varepsilon_{\mathrm{perm}}(\Pi)$ and ε, see Proposition 2.5.11.(ii), we obtain that the signs η' and ε_{n-1} are related as follows (throughout, $j = s_0 - \frac{1}{2}$):

$$\begin{aligned}
\eta' &= (-1)^j \eta \\
&= (-1)^{j + \frac{(n-1)(n-2)}{2}} \varepsilon_{\Pi_\infty} \\
&= (-1)^{j + \frac{(n-1)(n-2)}{2}} (-1)^{[F:\mathbb{Q}](n-1)/2} \varepsilon \\
&= (-1)^{j + \frac{(n-1)(n-2)}{2}} (-1)^{[F:\mathbb{Q}](n-1)/2} (-1)^{j+n} \varepsilon_{n-1} \\
&= (-1)^{\frac{n-1}{2}([F:\mathbb{Q}]+1)+1} \varepsilon_{n-1},
\end{aligned}$$

where in the last step we have applied repeatedly the fact that n is odd. Thus, η' and ε_{n-1} agree except possibly for a sign.

In view of Proposition 5.5.3, it is to be expected that, in fact, $\eta' = \varepsilon_{n-1}$, regardless of the parity of $\frac{n-1}{2}$ and $[F : \mathbb{Q}]$. We invite the reader to compare this assertion with [Rag13, Thm. 3.1.(1)], which in our notation, and taking into account the differing normalizations of the relative periods in *op. cit.* and [HR20], reads precisely

$$\Omega^{\varepsilon}_{\mathrm{rel}}(\Pi'_{\mathrm{f}}) \sim \mathrm{i}^{[F:\mathbb{Q}](n-1)/2} \cdot \frac{\Omega^{\varepsilon}_{\mathrm{Wh}}(\Pi'_{\mathrm{f}})}{\Omega^{-\varepsilon}_{\mathrm{Wh}}(\Pi'_{\mathrm{f}})}$$

for *all* signatures $\varepsilon \in \widehat{\pi_0(F^{\times}_{\infty})}$. (Here, $\sim$ means "up to a nonzero element in $\mathbb{Q}(\Pi'_{\mathrm{f}})$".) Ascertaining the reason for this apparent discrepancy with the expected formulae will be a top priority in follow-up work.

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"I šta ćemo sad?"

Konstrakta, *In corpore sano*