DISSERTATION / DOCTORAL THESIS

Titel der Dissertation /Title of the Doctoral Thesis

„(Asymptotic) Analysis of the Pauli-Poisswell equation in semi-relativistic Quantum Physics“

verfasst von / submitted by

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angestrebter akademischer Grad / in partial fulfilment of the requirements for the degree of

Doktor der Naturwissenschaften (Dr. rer. nat.)

Wien, 2023 / Vienna 2023

Studienkennzahl lt. Studienblatt / degree programme code as it appears on the student record sheet:	A 796 605 405
Dissertationsgebiet lt. Studienblatt / field of study as it appears on the student record sheet:	Mathematik
Betreut von / Supervisor:	Univ.-Prof. Dr. Norbert Mauser

Abstract

The self-consistent Pauli-Poisswell equation for 2-spinors is the first order in $1/c$ semi-relativistic approximation of the Dirac-Maxwell equation for 4-spinors. It consists of a vector-valued magnetic Schrödinger equation with an additional term coupling spin and magnetic field via the Pauli matrices, which is coupled to 1+3 Poisson type equations representing the magnetostatic approximation of Maxwell's equations. The Pauli-Poisswell equation is a consistent $O(1/c)$ model which keeps both relativistic effects of magnetism and of spin which are both absent in the non-relativistic Schrödinger-Poisson equation and inconsistent in the Schrödinger-Maxwell equation.

The semiclassical ($\hbar \to 0$) limit of the non-relativistic Schrödinger-Poisson equation towards the Vlasov-Poisson equation was established using Wigner measures by P.L. Lions & T. Paul and P. Markowich & N.J. Mauser in a mixed state formulation in $3d$.

We extend these results to the related Pauli-Poisson equation where a strong external magnetic field is applied which is much stronger than the magnetic self-interaction while the electric field is coupled self-consistently via the Poisson equation for the electric potential. Here, magnetic Lieb-Thirring estimates are crucial. The Pauli-Poisson equation converges towards the Vlasov-Poisson equation with Lorentz force.

We prove local wellposedness and existence of finite energy weak solutions of the Pauli-Poisswell equation.

We present the global-in-time semiclassical limit to the Vlasov-Poisswell equation with Lorentz force by the Wigner method: Substantial progress has been achieved using an additional assumption on the initial mixed state that yields the boundedness of the Husimi function. A technical problem of missing regularity due to the nonlinear coupling of the current density $J^{\hbar,c}$ and the presence of the magnetic field $B^{\hbar,c}$ remains an open problem.

We prove the local-in-time semiclassical limit of the Pauli-Poisswell equation to the Euler-Poisswell equation by the WKB method. The Euler equation corresponds to the Vlasov equation with monokinetic initial data. A key step is to obtain a priori energy estimates for which we have to take into account the Poisson equations for the electromagnetic potentials. We also obtain weak convergence of the monokinetic Wigner transform and strong convergence of the density and the current density.

This PhD thesis is based on the following publications:

[72] J. Möller and N. J. Mauser. Nonlinear PDE models in semi-relativistic quantum physics. *Submitted*, 2023.

[71] J. Möller. The Pauli-Poisson equation and its semiclassical limit. *Submitted*, 2023.

[38] P. Germain, N. J. Mauser, and J. Möller. Weak and strong solutions of the Pauli-Poiswell equation. *Manuscript*, 2023.

[73] J. Möller and N. J. Mauser. The semiclassical limit from the Pauli-Poisswell equation to the Vlasov-Poisswell equation by the Wigner method. *Manuscript*, 2023.

[97] C. Yang, J. Möller, and N. Mauser. The semiclassical limit from the Pauli-Poisswell to the Euler-Poisswell system by the WKB method. *Submitted*, 2023.

Zusammenfassung

Die selbstkonsistente Pauli-Poisswell-Gleichung für 2-Spinoren ist die semirelativistische Näherung erster Ordnung in $1/c$ der Dirac-Maxwell-Gleichung für 4-Spinoren. Sie besteht aus einer vektorwertigen magnetischen Schrödingergleichung mit zusätzlichen Kopplungsterm für Spin und Magnetfeld und ist an 1+3 Poisson-Gleichungen als magnetostatische Näherung der Maxwell-Gleichungen gekoppelt. Die Pauli-Poisswell-Gleichung ist ein konsistentes $O(1/c)$-Modell, das die relativistischen Effekte Magnetismus und Spin enthält, die in der nichtrelativistischen Schrödinger-Poisson-Gleichung nicht vorhanden und in der Schrödinger-Maxwell-Gleichung inkonsistent sind.

Der semiklassische ($\hbar \to 0$) Limes der nicht-relativistischen Schrödinger-Poisson-Gleichung zur Vlasov-Poisson-Gleichung wurde von P.L. Lions & T. Paul und P. Markowich & N.J. Mauser unter Verwendung von Wigner-Maßen und einer Dichtematrixformulierung mit gemischten Zuständen in $3d$ bewiesen.

Wir erweitern diese Resultate auf die Pauli-Poisson-Gleichung mit starkem externen Magnetfeld, das viel stärker als die magnetische Selbstwechselwirkung ist, während das elektrische Feld selbstkonsistent über die Poisson-Gleichung für das elektrische Potential gekoppelt ist. Hier sind die magnetischen Lieb-Thirring-Abschätzungen wichtig. Die Pauli-Poisson-Gleichung konvergiert gegen die Vlasov-Poisson-Gleichung mit Lorentz-Kraft.

Wir beweisen die lokale Wohlgestelltheit und die Existenz von schwachen Lösungen der Pauli-Poisswell-Gleichung.

Wir präsentieren den semiklassischen Limes der Pauli-Poisswell-Gleichung zur Vlasov - Poisswell - Gleichung mit Lorentz-Kraft. Wesentliche Fortschritte wurden unter Verwendung einer zusätzlichen Annahme über den gemischten Anfangszustand gemacht, aus der die gleichmäßige Beschränktheit der Husimi-Funktion folgt. Ein offenes technisches Problem ist die fehlende Regularität aufgrund der nichtlinearen Kopplung der Stromdichte $J^{\hbar,c}$ und des Magnetfeldes $B^{\hbar,c}$.

Wir beweisen den lokalen semiklassischen Grenzwert der Pauli-Poisswell-Gleichung zur magnetischen Euler-Poisswell-Gleichung mit der WKB-Methode. Die Euler - Gleichung entspricht der Vlasov-Gleichung mit monokinetischen Anfangsdaten. Ein wichtiger Schritt besteht darin, A-priori-Abschätzungen zu erhalten. Wir erhalten auch schwache Konvergenz der monokinetischen Wigner-Transformation und starke Konvergenz der Dichte und der Stromdichte.

Diese Dissertation basiert auf folgenden Publikationen:

[72] J. Möller and N. J. Mauser. Nonlinear PDE models in semi-relativistic quantum physics. *Submitted*, 2023.

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[97] C. Yang, J. Möller, and N. Mauser. The semiclassical limit from the Pauli-Poisswell to the Euler-Poisswell system by the WKB method. *Submitted*, 2023.

Acknowledgement - Remerciements – Danksagung

First and foremost I want to thank my supervisor Norbert J. Mauser for his continuous support in all aspects and the productive scientific cooperation.

I want to thank Patrick Gérard, Pierre Germain and Changhe Yang for many instructive discussions and cooperations.

I want to thank François Golse and Robert Seiringer for the good discussions and for accepting to referee this thesis; and I want to thank Claude Bardos, Nicolas Besse, Yann Brenier, Pierre Germain and Peter Markowich for participating as jury members.

I also want to thank Jérémie Joudioux and Stefan Fredenhagen for supervising my Bachelor and Master thesis in physics and again Norbert J. Mauser for supervising my Master thesis in mathematics.

Financial support from the Austrian Science Fund (FWF) via the SFB project 'Taming Complexity in Partial Differential Equations' No. F 65 and via the doctoral school DK "Dissipation and Dispersion in Nonlinear PDE" project No. W1450 is acknowledged, as well as from the Vienna Science and Technology Fund (WWTF) project No. MA16-066 "SEQUEX", and from the Wolfgang Pauli Institute.

I also thank the Laboratoire de Mathématiques d'Orsay (U. Paris-Saclay) for their hospitality, in particular Patrick Gérard, Frédéric Rousset and Jean-Claude Saut.

Ich möchte mich auch bei meinen Kollegen Peter Allmer, Daniel Bäumer, Arkadij Bojko und Timon Gutleb sowie bei Lukas Exl, Jan-Frederik Mennemann, Hans Peter Stimming, und auch herzlich bei MMM-WPI-Direktorin Stefanie Preuss für die guten Jahre in Wien bedanken und bei Georg Grützner für die gemeinsame Zeit in Paris.

Zuletzt bedanke ich mich bei meiner Familie, insbesondere meinen Eltern, denn ohne sie wäre es nicht möglich gewesen.

Contents

1 Introduction

This section is based on [72].

Ever since the discovery of quantum physics in the 1920s the question of its asymptotics has been important for multiple reasons: On the one hand a ”semiclassical” approximation yields models more suited to specific settings by simplifying the equations and making calculations easier (and in many cases actually feasible) and on the other hand it manifests the *correspondence principle* which states that quantum systems should behave classically for large quantum numbers, i.e. when the Planck constant $\hbar \approx 1.455 \times 10^{-34}$ Js is small compared to the typical action W of the system. The limit $\hbar \to 0$ is called *semiclassical limit* and $\hbar$ is called *semiclassical parameter*. The Planck constant is a small parameter in quantum evolution equations and the asymptotics prove to be highly non-trivial and require different sophisticated techniques like *Wigner* and *WKB methods*.

In relativistic physics the small parameter is the inverse of the speed of light, i.e. $1/c$ where $c = 299{,}792{,}458$ m/s. Similarly to the quantum-classical correspondence principle one expects classical behavior when $1/c$ is small compared to the typical speed v_0 of a system. The limit $1/c \to 0$ is called *non-relativistic limit*, also called *post-Newtonian limit*. $1/c$ arises as a small parameter in (semi-)relativistic quantum equations together with the Planck constant. Thus, relativistic quantum mechanics depends on two small parameters $\hbar$ and $1/c$.

These two asymptotic regimes in relativistic quantum mechanics imply that there are multiple hierarchies of equations in varying order of $\hbar$ and $1/c$ and that there are intermediate regimes where only terms up to a certain order in one of the two small parameters $\hbar$ and $1/c$ are kept. These are the *semiclassical* (for $\hbar$) and *semi-relativistic* (for $1/c$) regimes.

A third asymptotic is the *mean field limit* $1/N \to \infty$ of linear N-particle equations. A system of N interacting particles is described by a system of N coupled linear PDE. In the limit $1/N \to 0$, which is also called *weak coupling limit* (where a factor $1/N$ weakens the binary interaction potential of the N-body Hamiltonian), the N-body dynamics can be approximated by nonlinear 1-body dynamics, e.g. by the Schrödinger-Poisson equation in $3d$. Thus there are many equations of interest in relativistic, semi-relativistic, (semi-)classical and fully quantum physics. In this thesis we will concentrate on the *Pauli-Poisswell equation* which is a *self-consistent semi-relativistic* nonlinear PDE and on the *Pauli-Poisson equation*, a *semi-self-consistent semi-relativistic* model.

1.1 Self-consistent models in relativistic quantum mechanics

Relativistic quantum mechanics is an immensely successful theory giving a correct description of the behavior of particles on the atomic scale moving at high velocities compared to the speed of light and interacting with the electromagnetic field. On the other hand, non-relativistic quantum mechanics is centered around the Schrödinger equation which is insufficient when relativistic effects such as spin and the magnetic field arise. In relativistic quantum mechanics the description of spin emerges naturally in the Dirac equation which is the correct equation for particles with spin 1/2, i.e. fermions, describing both spin states of the particle and the antiparticle, resulting in a four-component wave function, i.e. a *4-spinor* or *Dirac spinor*. Semi-relativistic quantum mechanics is the theory that keeps relativistic corrections up to order $O(1/c)$ or $O(1/c^2)$ and it has been discovered by Wolfgang Pauli in 1927 that the correct semi-relativistic equation describing charged spin-1/2-particles in an electromagnetic field is the Pauli equation which includes the description of spin and magnetic field which are coupled via the Stern-Gerlach term. In the semi-relativistic regime one keeps the description of magnetism and spin but gives up the description of the antiparticle, resulting in a two-component wave function, a *2-spinor* or *Pauli spinor*.

Since (fast) moving charged particles emit radiation the effect of self-interaction of a charged particle with the electromagnetic field it generates has to be taken into account. Models which describe self-interaction are called *self-consistent models*. In the fully relativistic regime the correct self-consistent models are the *Dirac-Maxwell equation* (cf. [14, 19, 64, 69, 90]) for spin-1/2-particles (fermions) and by the *Klein-Gordon-Maxwell equation* (cf. [57, 64, 65, 86]) for spin-0-particles (bosons). Since Maxwell's equations are relativistic and Lorentz invariant it is natural to couple them self-consistently to the Dirac and Klein-Gordon equations. In the non-relativistic regime the *Schrödinger-Poisson equation* offers a description of the self-interaction with the electric field which is given by a Poisson equation for the electric potential $V^\hbar$ with the particle density $\rho^\hbar$ as a source term. The magnetic field, being a relativistic effect, does not self-interact at the non-relativistic level and neither does spin which is naturally coupled to the magnetic field.

In the first order semi-relativistic regime of $O(1/c)$ the correct self-consistent description is given by the *Pauli-Poisswell equation* (a portmanteau of *Poiss*on and Max*well* coined by Masmoudi and Mauser in [63]). Here, Maxwell's equations for the magnetic vector potential $A^{\hbar,c}$ and the electric scalar potential $V^{\hbar,c}$ are replaced by a magnetostatic approximation given by Poisson equations for $A^{\hbar,c}$ and $V^{\hbar,c}$ (cf. [18, 63]) with the current density $J^{\hbar,c}$ and the particle density $\rho^{\hbar,c}$ as source terms. This is appropriate when the typical velocity of the system is small in the intermediate regime compared to the speed of light.

In the second order semi-relativistic regime of $O(1/c^2)$ the correct self-consistent description is given by the *Pauli-Darwin equation* [55, 66, 67] where the Pauli equation with corrections of order $O(1/c^2)$ (i.e. Darwin term and Zitterbewegung) is coupled to the $O(1/c^2)$ Darwin approximation of Maxwell's equations [18] (as opposed to the $O(1/c)$ Poisswell equations).

The derivation and existence and uniqueness analysis of the Pauli-Poisswell equation is discussed in Section 4, the semiclassical limit of the Pauli-Poisswell equation to the Vlasov-Poisswell equation by the Wigner method is discussed in Section 5 and the semiclassical limit to the Euler-Poisswell equation by the WKB method is discussed in Section 6.

The Pauli-Poisson (cf. [71]), magnetic Schrödinger-Maxwell (cf. [15, 49, 74, 76]) and magnetic Schrödinger-Poisson (cf. [7, 61, 70]) equations are all inconsistent models in the small parameter $1/c$. In fact these models omit terms of order $O(1/c)$ and are therefore only $O(1)$. They are introduced in Section 1.2. The Pauli-Poisson equation including its semiclassical limit to the magnetic Vlasov-Poisson equation and global wellposedness in the energy space is discussed in Section 3.

There are two main methods for the semiclassical limit $\hbar \to 0$:

a) *Wigner methods* yield global-in-time weak limits towards Vlasov equations. A general survey on Wigner transforms and their application to homogenization limits was given by Gérard, Markowich, Mauser and Poupaud in [36] .

b) *WKB methods* (after Wentzel, Kramers and Brillouin) yield local-in-time limits to Euler equations which correspond to monokinetic Vlasov equations for pure states. For a unified presentation of these approaches and an extension to multi-valued WKB see e.g. [91].

The Wigner and WKB methods are introduced in Section 2.

We have the following diagram for the hierarchies of self-consistent models in relativistic quantum mechanics. The horizontal resp. vertical arrows indicate the semiclassical ($\hbar \to 0$), resp. non-relativistic ($c \to \infty$) limits.

Dirac-Maxwell		
Klein Gordon-Maxwell	$\xrightarrow{\hbar\to 0}$	**rel. Vlasov w/ Lorentz force - Maxwell** **Euler w/ Lorentz force - Maxwell**
$\downarrow$		$\downarrow$
Pauli-Darwin $O(1/c^2)$	$\xrightarrow{\hbar\to 0}$	**rel. Vlasov w/ Lorentz force - Darwin** **Euler w/ Lorentz force - Darwin** $O(1/c^2)$
Pauli-Poisswell $O(1/c)$	$\xrightarrow{\hbar\to 0}$	**Vlasov w/ Lorentz force - Poisswell** **Euler w/ Lorentz force - Poisswell** $O(1/c)$
$\downarrow$		$\downarrow$
Pauli-Poisson	$\xrightarrow{\hbar\to 0}$	**Vlasov w/ Lorentz force - Poisson**
magn. Schrödinger-Maxwell	$\xrightarrow{\hbar\to 0}$	**non-rel. Vlasov w/ Lorentz force - Maxwell**
magn. Schrödinger-Poisson	$\xrightarrow{\hbar\to 0}$	**Vlasov w/ Lorentz force - Poisson**
$\downarrow$		$\downarrow$
$\xrightarrow{c\to\infty}$		$\xrightarrow{c\to\infty}$
$\downarrow$		$\downarrow$
Schrödinger-Poisson	$\xrightarrow{\hbar\to 0}$	**Vlasov-Poisson** **Euler-Poisson**

In the following q is the electric charge, m is the mass, $A_k^{\hbar,c}\colon \mathbb{R}^3 \to \mathbb{R}$, $k = 1, 2, 3$ are the components of the magnetic vector potential $A^{\hbar,c}$ and $B^{\hbar,c} = \nabla \times A^{\hbar,c}$ is the magnetic field. The electric scalar potential is denoted by $V^{\hbar,c}\colon \mathbb{R}^3 \to \mathbb{R}$ and the particle or charge density by $\rho^{\hbar,c}$. Depending on the asymptotics of interest we will omit some of the superscripts $\hbar$ and c.

1.1.1 Fully-relativistic: Dirac-Maxwell and Klein-Gordon-Maxwell

The upper left hand corner is the regime of relativistic quantum mechanics. The Dirac-Maxwell and Klein-Gordon-Maxwell equations describe:

(a) The *magnetic field*, which is intrinsically relativistic in nature, and described by Maxwell's equations,

(b) the *spin* of the particle which is coupled to magnetism,

(c) the *antiparticle* (for the Dirac-Maxwell equation).

The Dirac-Maxwell equation models a charged spin-1/2-particle and its antiparticles. A typical example is the electron (e^-) and the positron (e^+). The Klein-Gordon-Maxwell equation models a charged spin-0-particle, for instance a pion (π^+ or π^-)). Since Maxwell's equations are intrinsically relativistic, it is natural to couple them to the Dirac and Klein-Gordon equations. The **Dirac-Maxwell equation** for a 4-spinor $\mathbf{\Psi}^{\hbar,c} = (\psi_1^{\hbar,c}, \psi_2^{\hbar,c}, \psi_3^{\hbar,c}, \psi_4^{\hbar,c}) \in (L^2(\mathbb{R}^3, \mathbb{C}))^4$ in Lorenz gauge[1]

$$\nabla \cdot A^{\hbar,c} + \frac{1}{c}\partial_t V^{\hbar,c} = 0, \tag{1.1}$$

reads

$$i\hbar\partial_t \mathbf{\Psi}^{\hbar,c} = -i\hbar c \sum_{k=1}^{3} \gamma^0\gamma^k \partial_k \mathbf{\Psi}^{\hbar,c} + mc^2\gamma^0\mathbf{\Psi}^{\hbar,c} - q\sum_{k=1}^{3} A_k^{\hbar,c}\gamma^0\gamma^k\mathbf{\Psi}^{\hbar,c} - qV^{\hbar,c}\mathbf{\Psi}^{\hbar,c} \tag{1.2}$$

$$\frac{1}{c^2}\partial_t^2 V^{\hbar,c} - \Delta V^{\hbar,c} = \mathbf{\Psi}^{\hbar,c} \cdot \overline{\mathbf{\Psi}^{\hbar,c}} =: \rho^{\hbar,c} \tag{1.3}$$

$$\frac{1}{c^2}\partial_t^2 A_k^{\hbar,c} - \Delta A_k^{\hbar,c} = c\gamma^0\gamma^k\mathbf{\Psi}^{\hbar,c} \cdot \overline{\mathbf{\Psi}^{\hbar,c}} =: J_k^{\hbar,c} \tag{1.4}$$

The upper two components represent the spin-up and -down states of the particle and the lower two components represent the antiparticle. $J_k^{\hbar,c}$ are the components of the *Dirac current density* and γ^μ, $\mu = 0, 1, 2, 3$ are the *Gamma matrices* defined by:

$$\gamma^0 = \begin{pmatrix} I_{2\times 2} & \\ & -I_{2\times 2} \end{pmatrix}, \qquad \gamma^k = \begin{pmatrix} & \sigma_k \\ -\sigma_k & \end{pmatrix}, \tag{1.5}$$

where the σ_k are the *Pauli matrices*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.6}$$

[1] Named after Ludvig Lorenz (1829-1891), who is not be confused with Hendrik Lorentz (1853-1928) after whom the Lorentz force and the Lorentz transformation are named.

The **Klein-Gordon-Maxwell equation** for a scalar wave function $\psi^{\hbar,c} \in L^2(\mathbb{R}^3, \mathbb{C})$ in Coulomb gauge

$$\nabla \cdot A^{\hbar,c} = 0 \tag{1.7}$$

reads

$$-(\frac{\hbar}{c}\partial_t + iqV^{\hbar,c})^2\psi^{\hbar,c} + (\hbar\nabla - i\frac{q}{c}A^{\hbar,c})^2\psi^{\hbar,c} - m^2c^2\psi^{\hbar,c} = 0, \tag{1.8}$$

$$\frac{1}{c^2}\partial_t^2 A^{\hbar,c} - \Delta A^{\hbar,c} = \frac{1}{c}\mathbb{P}J^{\hbar,c}, \tag{1.9}$$

$$-\Delta V^{\hbar,c} = \rho^{\hbar,c}, \tag{1.10}$$

where

$$J^{\hbar,c} = \mathrm{Im}(\overline{\psi^{\hbar,c}}(\hbar\nabla - i\frac{q}{c}A^{\hbar,c})\psi^{\hbar,c}). \tag{1.11}$$

is the current density and $\mathbb{P}$ is the Leray projection on divergence-free fields, i.e. $\mathbb{P}J^{\hbar,c} = J^{\hbar,c} - \nabla\Delta^{-1}(\nabla \cdot J^{\hbar,c})$.

Global wellposedness of the three dimensional Dirac-Maxwell system with data of arbitrary size is still open, the negative spectrum of the Dirac operator hindering energy estimate techniques (cf. [19, 23, 24, 30, 33]). To our knowledge, the best result so far for the Dirac-Maxwell equation in $3+1$ dimensions was obtained in [65] by Masmoudi and Nakanishi where local wellposedness is shown in $H^{1/2} \times \dot{H}^1 \times L^2$ for $(\mathbf{\Psi}^{\hbar,c}, A, \partial_t A)$.

The rigorous non-relativistic limit of the Dirac equation to the Schrödinger equation with external magnetic field was studied in [12] and of the Dirac-Maxwell equation to the Schrödinger-Poisson equation in [13, 14, 64]. Basic results of the full classical limit of the Dirac-Maxwell equation to the Vlasov-Poisson equation were obtained in [69] by Mauser and Selberg by combining the non-relativistic limit from the Dirac-Maxwell equation to the Schrödinger-Poisson equation [13, 64] with the semiclassical limit from the Schrödinger-Poisson equation to the Vlasov-Poisson equation [59, 62]. Semi-relativistic approximations to the Dirac equation up to second order in $1/c$ were studied in [67].

Now consider first the upper horizontal arrow, i.e. the semiclassical limit $\hbar \to 0$ of Dirac-Maxwell and Klein-Gordon-Maxwell:

$$\begin{array}{ccc} \boxed{\textbf{Dirac-Maxwell}} & & \\ \boxed{\textbf{Klein Gordon-Maxwell}} & \xrightarrow{\hbar\to 0} & \boxed{\begin{array}{c}\textbf{rel. Vlasov w/ Lorentz force - Maxwell} \\ \textbf{Euler w/ Lorentz force - Maxwell}\end{array}} \end{array}$$

We expect (cf. [63], [69]) that the Wigner equation (cf. Section 2) associated to the Dirac-Maxwell equation converges to the relativistic Vlasov equation with Lorentz force coupled to Maxwell's equations, i.e. the **(relativistic) Vlasov-Maxwell equation**

$$\partial_t f^c_\pm + v(p)\cdot\nabla_x f^c_\pm \mp q(\nabla_x V^c + \frac{1}{c}v(p)\times B^c)\cdot\nabla_p f^c_\pm = 0 \tag{1.12}$$

$$\frac{1}{c^2}\partial_t^2 V^c - \Delta V^c = \rho^c \tag{1.13}$$

$$\frac{1}{c^2}\partial_t^2 A^c_k - \Delta A^c_k = cJ^c_k \tag{1.14}$$

with initial data

$$f^c_\pm(x,p,0) = f^c_{\pm,I}(x,p) \quad \text{in } \mathbb{R}^3_x\times\mathbb{R}^3_p, \tag{1.15}$$

where $f^c_\pm$ represents the phase-space density of the positron and electron component, respectively. Here,

$$v(p) = \frac{p}{\sqrt{\frac{|p|^2}{c^2}+1}} \tag{1.16}$$

is the relativistic velocity with the Lorentz factor defined by

$$\gamma = \frac{1}{\sqrt{\frac{|p|^2}{c^2}+1}} \tag{1.17}$$

Moreover,

$$\rho^c(x,t) = \int_{\mathbb{R}^3_p}(f^c_+ + f^c_-)(x,p,t)\mathrm{d}p \tag{1.18}$$

$$J^c(x,t), = \int_{\mathbb{R}^3_p} v(p)(f^c_+ + f^c_-)(x,p,t)\mathrm{d}p. \tag{1.19}$$

are the charge and current density, respectively. Global weak solutions of the Vlasov-Maxwell equation, i.e. Vlasov with Lorentz force coupled to Maxwell, were established in the seminal paper by DiPerna and Lions in [26]. Recently, Bardos, Besse and Nguyen proved an Onsager type conjecture for the Vlasov-Maxwell equation in [8] and Besse and Bechouche sharpened the regularity for the DiPerna-Lions weak solutions in [17]. The non-relativistic limit of the Vlasov-Maxwell equation to the Vlasov-Poisson equation was shown in [3, 25]. For the linear Dirac equation the semiclassical limit to the relativistic Vlasov equation was shown by Spohn [93] based on [36]. The semiclassical limit of the Dirac-Maxwell equation to the Vlasov-Maxwell equation is a very hard open problem and our results on the Pauli-Poisson and Pauli-Poisswell equations are first steps in this direction.

In the WKB ansatz (cf. Section 2) one replaces the wave function by a special ansatz and obtains a coupled system of equations for the density $\rho^{\hbar,c} = |a^{\hbar,c}|^2$ (where $a^{\hbar,c}$ is amplitude) and the velocity $u^{\hbar,c}$. Denote by ρ^c and u^c the respective limits as $\hbar\to 0$ of $\rho^{\hbar,c}$ and $u^{\hbar,c}$. Then ρ^c and u^c satisfy the pressureless **Euler-Maxwell equation**, studied in [37, 48] which is the fully relativistic self-consistent model for a system (plasma) of charged particles of the same sign self-interacting with the electromagnetic field. It reads

$$\begin{aligned}
&\partial_t\rho^c + \nabla\cdot(\rho^c u^c) = 0\\
&\partial_t(\rho^c u^c) + \nabla\cdot(\rho^c u^c \otimes u^c) = \rho^c(E^c + \frac{1}{c}u^c \times B^c),\\
&\partial_t E^c - c\nabla\times B^c = \rho^c u^c, \quad \partial_t B^c + c\nabla\times E^c = 0,\\
&\nabla\cdot E^c = -\rho^c, \quad \nabla\cdot B^c = 0.\\
&(\rho^c, u^c, E^c, B^c)(x,0) = (\rho^c_I(x), u^c_I(x), E^c_I(x), B^c_I(x))
\end{aligned} \tag{1.20}$$

In the Euler-Maxwell case there is no distinction between repulsive or attractive interaction as for the Euler-Poisson equation (which describes either the attractive gravitational or the repulsive electrostatic self-interaction depending on the sign) since the force described by Maxwell's equations is given by the Lorentz force $F_{\mathrm{L}} = q(E+u\times B)$ (in the non-relativistic Euler-Poisson equation it reduces to the repulsive electrostatic force $F_{\mathrm{el}} = qE$). The WKB analysis of the Dirac-Maxwell equation was studied in [90]. The non-relativistic limit $c\to\infty$ of the Euler-Maxwell equation (1.20) to the Euler-Poisson equation was shown in [78, 98]. Similar to the Euler-Poisson case the Euler-Maxwell equation is in fact more stable than the pure Euler system without self-interaction and smooth solutions can exist globally without shock formation [48].

1.1.2 First order semi-relativistic: Pauli-Poisswell

The self-consistent **Pauli-Poisswell equation** arises as the semi-relativistic approximation at first order $O(1/c)$ of the fully relativistic Dirac-Maxwell equation for a 4-spinor (cf. [63]) and is the correct semi-relativistic $O(1/c)$ self-consistent model for charged spin-1/2 particles with self-interaction with the electromagnetic field where the electromagnetic potentials $V^{\hbar,c}$ and $A^{\hbar,c}$ are given by a magnetostatic (i.e. slow in the intermediate regime with respect to c) description via $1+3$ Poisson equations with density and current density as source terms.

The Pauli-Poisswell equation describes
(a) The *magnetic field* in a semi-relativistic, magnetostatic approximation by the Poisswell equations,
(b) the *spin* of the particle by the Pauli equation.
Unlike the Dirac-Maxwell equation, the Pauli-Poisswell equation does not describe the antiparticle.

The Pauli-Poisswell equation for a 2-spinor $\Psi^{\hbar,c} = (\psi_1^{\hbar,c}, \psi_2^{\hbar,c})^T \in (L^2(\mathbb{R}^3,\mathbb{C}))^2$ corresponds to the "electron component" of the Dirac equation for a 4-spinor obtained by a Foldy - Wouthuysen transform and projection on the "upper" ("large") component of the 4 spinor (the "lower" ("small") component corresponds to the positron, cf. [13]), is given by:

$$i\hbar\partial_t\Psi^{\hbar,c} = -\frac{1}{2m}(\hbar\nabla - i\frac{q}{c}A^{\hbar,c})^2\Psi^{\hbar,c} + qV^{\hbar,c}\Psi^{\hbar,c} - \frac{q\hbar}{2mc}(\sigma\cdot B^{\hbar,c})\Psi^{\hbar,c}, \tag{1.21}$$

$$-\Delta V^{\hbar,c} = |\Psi^{\hbar,c}|^2 =: \rho^{\hbar,c}, \tag{1.22}$$

$$-\Delta A^{\hbar,c} = \frac{1}{c}J^{\hbar,c}. \tag{1.23}$$

where the Pauli current density is given by

$$J^{\hbar,c}(\Psi^{\hbar,c}, A^{\hbar,c}) = \mathrm{Im}(\overline{\Psi^{\hbar,c}}(\hbar\nabla - i\frac{q}{c}A^{\hbar,c})\Psi^{\hbar,c}) - \hbar\nabla\times(\overline{\Psi^{\hbar,c}}\sigma\Psi^{\hbar,c}), \tag{1.24}$$

with initial data

$$\Psi^{\hbar,c}(x,0) = \Psi_I^{\hbar,c}(x) \in (L^2(\mathbb{R}^3))^2. \tag{1.25}$$

Here, $|\Psi^{\hbar,c}|^2 := |\psi_1^{\hbar,c}|^2 + |\psi_2^{\hbar,c}|^2$ is the scalar charge density. The Stern-Gerlach term coupling magnetism and spin is denoted by $\sigma\cdot B^{\hbar,c} := \sum_{k=1}^3 \sigma_k B_k^{\hbar,c}$ where the σ_k are the Pauli matrices defined in (1.5). The expressions $\overline{\Psi^{\hbar,c}}\nabla\Psi^{\hbar,c}$ and $\overline{\Psi^{\hbar,c}}\sigma\Psi^{\hbar,c}$ are to be understood as the vectors with components

$$(\overline{\psi_1^{\hbar,c}}, \overline{\psi_2^{\hbar,c}})\partial_k \begin{pmatrix}\psi_1^{\hbar,c}\\ \psi_2^{\hbar,c}\end{pmatrix}, \qquad (\overline{\psi_1^{\hbar,c}}, \overline{\psi_2^{\hbar,c}})\sigma_k \begin{pmatrix}\psi_1^{\hbar,c}\\ \psi_2^{\hbar,c}\end{pmatrix},$$

for $k = 1, 2, 3$. The two components of the Pauli equation describe the two spin states of a fermion, whereas the Poisson equations describe the electrodynamic self-interaction of a fast moving particle with the electromagnetic field that it generates itself due to the finite speed of light. The key semi-relativistic features of the Pauli equation are the **magnetic Laplacian**

$$\Delta_A := \frac{1}{2m}(\hbar\nabla - i\frac{q}{c}A^{\hbar,c})^2, \tag{1.26}$$

and the **Stern-Gerlach term**

$$\frac{\hbar q}{2mc}(\sigma\cdot B^{\hbar,c})\Psi^{\hbar,c}, \tag{1.27}$$

coupling spin and magnetic field. Note that in (1.26) we keep a term of order $O(1/c^2)$ which is necessary to keep covariance of the equation. Compare (1.24) to (1.49) where we have additional terms due to the presence of magnetic fields and spin.

The semiclassical limit $\hbar\to 0$ of the Pauli-Poisswell equation (1.21)-(1.25) is represented by the following arrow in the diagram

$$\boxed{\textbf{Pauli-Poisswell } (O(1/c))} \xrightarrow{\hbar\to 0} \boxed{\begin{array}{c}\textbf{Vlasov w/ Lorentz force - Poisswell}\\ \textbf{Euler w/ Lorentz force - Poisswell}\\ (O(1/c))\end{array}}$$

The **Vlasov-Poisswell equation** is the semi-relativistic $O(1/c)$ approximation of the relativistic Vlasov-Maxwell equation (1.12)-(1.14), given by the Vlasov equation with Lorentz force coupled to the Poisswell equations:

$$\partial_t f^c + p \cdot \nabla_x f^c - (\nabla_x V^c + \frac{1}{c} p \times B^c) \cdot \nabla_p f^c = 0 \tag{1.28}$$

$$-\Delta V^c(x,t) = \rho^c(x,t), \qquad \rho(x,t) = \int_{\mathbb{R}^3_p} f^c(x,p,t)\mathrm{d}p, \tag{1.29}$$

$$-\Delta A^c(x,t) = \frac{1}{c} J^c(x,t), \qquad J^c(x,t) = \int_{\mathbb{R}^3_p} p f^c(x,p,t)\mathrm{d}p, \tag{1.30}$$

with the initial condition

$$f^c(x,p,0) = f^c_I(x,p) \quad \text{in } \mathbb{R}^3_x \times \mathbb{R}^3_p. \tag{1.31}$$

The wellposedness of (1.28)-(1.31) has been studied in [85] (see also Section 5) and numerical analysis in [18]. Compare it to the Vlasov-Maxwell equation where one has to phase space densities f^c_+ and f^c_- for the electron and positron components, respectively. Also note that the velocity is no longer given by the relativistic velocity $v(p)$ defined by (1.16). We discuss the semiclassical limit of the Pauli-Poisswell equation to the Vlasov-Poisswell equation in Section 5.

The WKB ansatz leads to the pressureless **Euler-Poisswell equation**

$$\begin{aligned}
\partial_t a^c + (A^c + u^c) \cdot \nabla a^c + \frac{1}{2} a^c \mathrm{div}(u^c + A^c) &= \frac{i}{2}(\sigma \cdot B^c) a^c, \\
\partial_t u^c + (A^c + u^c) \cdot \nabla u^c + u^c \cdot \nabla \otimes A^c + \nabla(\frac{|A^c|^2}{2} + V^c) &= 0, \\
a^c(x,0) &= a^c_I(x), \\
u^c(x,0) &= u^c_I(x).
\end{aligned} \tag{1.32}$$

where

$$-\Delta V^c = \rho^c, \tag{1.33}$$

$$-\Delta A^c = -\rho^c u^c - \rho^c A^c, \tag{1.34}$$

and

$$\rho^c = |a^c|^2.$$

Multiplying the first equation in (1.32) by $\overline{a}$ and taking the real part yields

$$\begin{aligned}
\partial_t \rho^c + \nabla \cdot (\rho^c (u^c + A^c)) &= 0, \\
\partial_t u^c + (A^c + u^c) \cdot \nabla u^c + u^c \cdot \nabla \otimes A^c + \nabla(\frac{|A^c|^2}{2} + V^c) &= 0, \\
\rho^c(x,0) &= \rho^c_I(x), \\
u^c(x,0) &= u^c_I(x)(x).
\end{aligned} \tag{1.35}$$

and

$$-\Delta V = \rho, \tag{1.36}$$
$$-\Delta A = -\rho u - \rho A, \tag{1.37}$$

since $\mathrm{Re}(i\overline{a}(\sigma \cdot B)a) = 0$. The Euler-Poisswell equation (1.35)-(1.37) is the $O(1/c)$ semi-relativistic approximation of the Euler-Maxwell equation which is natural since the Pauli-Poisswell equation is the semi-relativistic $O(1/c)$ approximation of the Dirac-Maxwell equation and the Euler-Maxwell equation is the semiclassical limit of the Dirac-Maxwell equation. Therefore the Euler-Poisswell equation is the correct semi-relativistic self-consistent model of a system (plasma) of charged particles with self-interaction with the electromagnetic field.

The semiclassical limit of the Pauli-Poisswell equation to the Euler-Poisswell equation using WKB methods as well as existence and uniqueness analysis will be discussed in Section 6.

Remark 1. The numerical analysis of the Pauli equation is not well developed yet. For the linear Pauli equation with given external potentials V and A using an operator splitting scheme was studied in [51] where the linear Pauli Hamiltonian

$$-\Delta + 2iA \cdot \nabla + |A|^2 + V - (\sigma \cdot B) \tag{1.38}$$

is split into the "kinetic step" $-\Delta$, the "advection step" $2iA \cdot \nabla$, the "potential step" $|A|^2 + V + \mathrm{diag}(B_3, -B_3)$ and the "coupling step" which consists of the off-diagonal elements of $\sigma \cdot B$. The most computationally demanding step is the advection step which is solved in [51] by the method of characteristics combined with Fourier interpolation.

The numerical analysis for the Pauli-Poisswell equation is much harder due to the nonlinear coupling of the potentials via the 3+1 Poisson equations. Since already in the linear Pauli equation the advection step is the most complicated one, the fact that in the Pauli-Poisswell equation A is given by the Poisson equation with the current density as source term will require more sophisticated methods for the numerical analysis.

For the non-magnetic case without spin, i.e. the Schrödinger-Poisson equation, Lubich proved stability of an operator splitting in [60]. The first step for the numerical analysis of the Pauli-Poisswell equation could be to generalize this result to the magnetic Schrödinger-Poisson equation and then to the Pauli-Poisson equation (1.61)-(1.63) by adding the spin term.

1.1.3 Second order semi-relativistic: Pauli-Darwin

Keeping terms of order $O(1/c^2)$ in the approximation of the Dirac-Maxwell equation leads to the **Pauli-Darwin equation** which reads (with the same notation as for the Pauli-

Poisswell equation)

$$
\begin{aligned}
i\hbar\partial_t\Psi^{\hbar,c} =& -\frac{1}{2m}(\hbar\nabla - i\frac{q}{c}A^{\hbar,c})^2\Psi^{\hbar,c} + qV^{\hbar,c}\Psi^{\hbar,c} - \frac{q\hbar}{2mc}(\sigma\cdot B^{\hbar,c})\Psi^{\hbar,c},\\
&-\frac{\hbar^4}{8m^3c^2}\Delta^2\Psi^{\hbar,c} - \frac{iq\hbar}{8m^2c^2}\sigma\cdot(\nabla\times E^{\hbar,c})\Psi^{\hbar,c} - \frac{\hbar q}{4m^2c^2}(\sigma\cdot(E^{\hbar,c}\times\nabla))\Psi^{\hbar,c}\\
&-\frac{\hbar q}{8m^2c^2}(\nabla\cdot E^{\hbar,c})\Psi^{\hbar,c},
\end{aligned}
\tag{1.39}
$$

$$-\Delta V^{\hbar,c} = |\Psi^{\hbar,c}|^2, \tag{1.40}$$

$$-\Delta A^{\hbar,c} = \frac{1}{c}\mathbb{P}J^{\hbar,c}. \tag{1.41}$$

where the Pauli current density is given by (1.24), with initial data

$$\Psi^{\hbar,c}(x,0) = \Psi_I^{\hbar,c}(x) \in (L^2(\mathbb{R}^3))^2. \tag{1.42}$$

The additional terms corresponding to Δ^2 is the *mass term*. The other two terms in the second line of (1.39) represent the *spin-orbit coupling* and the last term corresponds to the *Zitterbewegung* and is also known as *Darwin term*, cf. [55, 66, 67].

The semiclassical limit of the Pauli-Darwin is represented by the following arrow in the diagram:

$$\boxed{\textbf{Pauli-Darwin } (O(1/c^2))} \xrightarrow{\hbar\to 0} \boxed{\begin{array}{c}\textbf{Vlasov w/ Lorentz force - Darwin}\\ \textbf{Euler w/ Lorentz force - Darwin}\\ (O(1/c))\end{array}}$$

The **Vlasov-Darwin equation** (cf. [18, 77, 84, 85]) equation is given by

$$\partial_t f^c + v(p)\cdot\nabla_x f + (-\nabla_x V^c + \frac{1}{c}v(p)\times B^c)\cdot\nabla_p f^c = 0 \tag{1.43}$$

$$-\Delta V^c(x,t) = \rho(x,t), \qquad \rho^c(x) = \int_{\mathbb{R}^3_p} f^c(x,p,t)\mathrm{d}p, \tag{1.44}$$

$$-\Delta A^c(x,t) = \frac{1}{c}\mathbb{P}J^c(x,t), \qquad J^c(x,t) = \int_{\mathbb{R}^3_p} pf^c(x,p,t)\mathrm{d}p, \tag{1.45}$$

where $v(p)$ is the relativistic velocity defined by (1.16), with the initial condition

$$f^c(x,p,0) = f_I^c(x,p) \quad \text{in } \mathbb{R}^3_x\times\mathbb{R}^3_p. \tag{1.46}$$

Again, compare the Vlasov-Darwin equation to the Vlasov-Maxwell and Vlasov-Poisswell equations. The main difference to the Vlasov-Maxwell equation is the replacement of Maxwell's equations by the *Darwin approximation* (not to be confused with the Darwin term) of Maxwell's equations which is the magnetostatic approximation in Coulomb gauge (whence the projection on divergence-free fields $\mathbb{P}$). The main difference compared to the Vlasov-Poisswell equation is the relativistic Vlasov equation which in the Vlasov-Poisswell equation is replaced by a semi-relativistic Vlasov equation.

1.1.4 Non-relativistic: Schrödinger-Poisson

In the non-relativistic limit $c \to \infty$ the Dirac-Maxwell and Klein-Gordon-Maxwell equations converge to the **Schrödinger-Poisson equation** (cf. [13, 14, 64]) where spin and magnetic field no longer appear since both are relativistic effects. The magnetic field is absent and the magnetic Laplacian (1.26) reduces to the regular Laplacian. The self-interaction is fully electrostatic and described by the Poisson equation for $V^\hbar$. The corresponding semiclassical limit is represented by

$$\boxed{\textbf{Schrödinger-Poisson}} \xrightarrow{\hbar \to 0} \boxed{\begin{array}{c}\textbf{Vlasov-Poisson}\\ \textbf{Euler-Poisson}\end{array}}$$

The Schrödinger-Poisson equation for a scalar wave function $\psi^\hbar$ is given by

$$i\hbar\partial_t\psi^\hbar = -\frac{\hbar^2}{2}\Delta\psi^\hbar + V^\hbar\psi^\hbar, \tag{1.47}$$

$$-\Delta V^\hbar = |\psi^\hbar|^2, \tag{1.48}$$

where the Schrödinger current density is defined by

$$J^\hbar = \hbar\,\mathrm{Im}(\overline{\psi}^\hbar\nabla\psi^\hbar). \tag{1.49}$$

with initial data

$$\psi^\hbar(x,0) = \psi_I^\hbar(x) \in L^2(\mathbb{R}^3), \tag{1.50}$$

In three space dimensions the Poisson equation (1.48) is equivalent to a Hartree nonlinearity with convolution kernel $|x|^{-1}$ which is the fundamental solution of $-\Delta$ in *three dimensions only*:

$$V^\hbar(\psi^\hbar) = (-\Delta)^{-1}(|\psi^\hbar|^2) = \frac{1}{4\pi|x|} * |\psi^\hbar|^2 = \int_{\mathbb{R}^3}\frac{|\psi^\hbar(y)|^2}{|x-y|}\mathrm{d}y. \tag{1.51}$$

The wellposedness theory of (1.47)-(1.50) at the H^2-level in $d = 3$ in a mixed state formulation was given by Brezzi and Markowich in [20] and at the L^2-level by Castella in [22]. The Schrödinger-Poisson equation also arises as the mean field limit of the linear N-body Schrödinger equation with Coulomb interaction, cf. [11, 9, 29] and Section 1.3 for details.

The lower right hand corner represents the fully classical limit of the equations mentioned so far, i.e. the **Vlasov-Poisson equation** for a phase-space density f:

$$\partial_t f + \xi\cdot\nabla_x f - \nabla_x V\cdot\nabla_\xi f = 0, \tag{1.52}$$

$$-\Delta V(x,t) = \rho(x,t), \qquad\qquad \rho(x,t) := \int_{\mathbb{R}^3} f(x,\xi,t)\mathrm{d}\xi. \tag{1.53}$$

with the initial condition

$$f(x,\xi,0) = f_I(x,\xi) \quad \text{in } \mathbb{R}^3_x \times \mathbb{R}^3_\xi. \tag{1.54}$$

When a magnetic field is present, the electric force $-\nabla V$ has to be replaced by the Lorentz force $-\nabla V + p \times B$ where $p = \xi - A$ and $B = \nabla \times A$. Then one obtains the **Vlasov-Poisson equation with Lorentz force** or **magnetic Vlasov-Poisson equation**

$$\partial_t f + p \cdot \nabla_x f + (-\nabla_x V + p \times B) \cdot \nabla_p f = 0, \tag{1.55}$$

$$-\Delta V(x,t) = \rho(x,t), \qquad \rho(x,t) := \int_{\mathbb{R}^3} f(x,\xi,t) \mathrm{d}\xi. \tag{1.56}$$

with the initial condition

$$f(x,p,0) = f_I(x,p) \quad \text{in } \mathbb{R}^3_x \times \mathbb{R}^3_p. \tag{1.57}$$

The semiclassical limit of (1.47)-(1.49) to the Vlasov-Poisson equation using Wigner transforms was shown by Lions and Paul in [59] and by Markowich and Mauser in [62] (in $d = 1$ the pure state case was proved by Zhang, Zheng and Mauser in [102] for appropriate non-unique measure valued weak solutions of the Vlasov-Poisson equation). The wellposedness of the Vlasov-Poisson equation was discussed in [81]. The limit of strong magnetic fields of the magnetic Vlasov-Poisson equation was established by Golse and Saint-Raymond in [44] and the homogenization limit by Frénod and Sonnendrücker in [31].

The WKB ansatz leads to the (repulsive) pressureless **Euler-Poisson equation**. It is given by

$$\begin{aligned} \partial_t \rho + \nabla \cdot (\rho u) &= 0 \\ \partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) + \rho \nabla V &= 0, \\ \rho(x,0) &= \rho_I(x), \\ u(x,0) &= u_I(x). \end{aligned} \tag{1.58}$$

where

$$-\Delta V = \rho, \qquad \rho = |a|^2. \tag{1.59}$$

The Euler-Poisson equation arises as the non-relativistic limit $c \to \infty$ of the Euler-Maxwell equation (1.20) (cf.[78, 98]) and as the semiclassical limit using WKB analysis of the Schrödinger-Poisson equation to the Euler-Poisson equation (cf. Zhang [100, 101] and Alazard and Carles [1]. Grenier [45] proved the semiclassical limit of the cubic NLS and recently, Gui and Zhang [46] proved the semiclassical limit of the Gross-Pitaevskii equation. Recently, Golse and Paul [42] showed that the bosonic N-particle Schrödinger dynamics with Coulomb interaction converge to the Euler-Poisson equation in the combined semiclassical and mean field limit $\hbar + 1/N \to 0$ if the first marginal of the initial data has monokinetic Wigner measure (for the definition cf. Section 2.2) which is equivalent to the WKB formulation.

Depending on the sign in front of the term $\rho\nabla V$ on the LHS of the second equation in (1.58) the Euler-Poisson equation is called *repulsive* (with a negative sign) or *attractive* (with a positive sign). In other words, the Euler-Poisson equation is repulsive if the sign on the LHS of the second equation in (1.58) and the sign on the RHS of the Poisson equation (1.59) are the same and attractive if they are opposite.

In the repulsive case the Euler-Poisson equation is a non-relativistic self-consistent model (for the self-interaction with the electric field, coupled via the Poisson equation (1.59)) for a system (plasma) of charged particles of charge of the same sign (e.g. electrons). This is because the Coulomb interaction is repulsive for charges of the same sign.

In the attractive case the Euler-Poisson equation is a self-consistent model for non-relativistic gravitational interaction, e.g. gaseous stars, which is attractive.

The analytical properties of the Euler-Poisson equation depend heavily on the sign. The repulsive case is less delicate than the attractive case: The dispersive nature of the electric field may prevent the formation of singularities [50]. In $3d$ Guo and Pausader [47, 50] proved the global existence of small smooth irrotational flows. Recently, Hadžić and Jang constructed global solutions for both the gravitational and electrostatic Euler-Poisson equation in [52].

However it is also possible to have blow-up even for the repulsive case, e.g. [99] or even non-existence, e.g. [79] for the $1d$ case. Further references can also be found in [28].

1.2 Inconsistent self-consistent models

There are other nonlinear models in relativistic quantum mechanics, that are not consistent approximations in an order of $\hbar$ or c throughout the equations. The following three equations are all inconsistent in $1/c$ since they lack terms of order $O(1/c)$ and therefore are in fact $O(1)$.

Note that only the $3d$ case of a coupling to the Poisson equation is physically sound. The Poisson equation in 1 resp. $2d$ models infinite planes, resp. wires and not point charges and the case of 4d or higher is physically irrelevant.

1.2.1 Pauli-Poisson

The **Pauli-Poisson equation**, introduced in [72] along the lines of the magnetic Schrödinger-Poisson equation (e.g. Barbaroux and Vougalter [6, 7]), for a 2-spinor $\Psi^{\hbar,c}(x,t) \in (L^2(\mathbb{R}^3,\mathbb{C}))^2$ reads

$$i\hbar\partial_t\Psi^{\hbar,c} = -\frac{1}{2m}(\hbar\nabla - i\frac{q}{c}A)^2\Psi^{\hbar,c} + qV^{\hbar,c}\Psi^{\hbar,c} + \frac{\hbar q}{2cm}(\sigma\cdot B)\Psi^{\hbar,c}, \tag{1.60}$$

$$\Delta V^{\hbar,c} = -\rho^{\hbar,c} := -|\Psi^{\hbar,c}|^2. \tag{1.61}$$

with initial data

$$\Psi^{\hbar,c}(x,0) = \Psi_I^{\hbar,c}(x) \in (L^2(\mathbb{R}^3))^2, \tag{1.62}$$

and Pauli current density $J^{\hbar,c}$ given by

$$J^{\hbar,c} = \mathrm{Im}(\overline{\Psi^{\hbar,c}}(\hbar\nabla - i\frac{q}{c}A)\Psi^{\hbar,c}) + \hbar\nabla \times (\overline{\Psi^{\hbar,c}}\sigma\Psi^{\hbar,c}). \tag{1.63}$$

More generally we may consider the **Pauli-Hartree equation**

$$i\hbar\partial_t\Psi^{\hbar,c} = -\frac{1}{2m}(\hbar\nabla - i\frac{q}{c}A)^2\Psi^{\hbar,c} + V^{\mathrm{ext}}\Psi^{\hbar,c} - \frac{\hbar q}{2cm}(\sigma\cdot B)\Psi^{\hbar,c} + (W * |\Psi^{\hbar,c}|^2)\Psi^{\hbar,c}, \tag{1.64}$$

where V^{ext} is an external potential and W is an interaction kernel depending on $x \in \mathbb{R}^3$. In $3d$ only the Pauli-Poisson equation is equivalent to the Pauli-Hartree equation with

$$W(x) = -\frac{\lambda}{|x|}, \tag{1.65}$$

where λ is a constant. The Pauli-Poisson equation is the appropriate model when an external magnetic field is applied which is much stronger than the self-consistent magnetic field such that self-interaction with the magnetic field can be neglected, but self-interaction with the electric field is still being taken into account. The Pauli-Poisson equation also arises as the mean field limit of the N-body Pauli equation similar to the Schrödinger-Poisson equation, cf. Section 1.3 and [72]. We will discuss global wellposedness and the semiclassical limit of (1.60)-(1.63) in section 3. We have the diagram

$$\boxed{\textbf{Pauli-Poisson}} \quad \xrightarrow{\hbar\to 0} \quad \boxed{\textbf{Vlasov w/ Lorentz force - Poisson}}$$

The semiclassical limit of the Pauli-Poisson equation to the Vlasov-Poisson equation with Lorentz force will be shown in Section 3 based on [71].

1.2.2 Magnetic Schrödinger-Poisson

If we neglect spin, i.e. the Stern-Gerlach term is omitted, and $\Psi^{\hbar,c}$ is replaced by a scalar wave function $\psi^\hbar$ in (1.60)-(1.63) we obtain the **magnetic Schrödinger-Poisson equation**, cf. [7]:

$$i\hbar\partial_t\psi^\hbar = -\frac{1}{2m}(\hbar\nabla - i\frac{q}{c}A)^2\psi^\hbar + qV^\hbar\psi^\hbar, \tag{1.66}$$

$$\Delta V^\hbar = -|\psi^\hbar|^2, \tag{1.67}$$

with initial data

$$\psi^\hbar(x,0) = \psi_I^\hbar(x) \in L^2(\mathbb{R}^3), \tag{1.68}$$

where the magnetic Schrödinger current density is defined by

$$J^\hbar = \mathrm{Im}(\overline{\psi^\hbar}(\hbar\nabla - i\frac{q}{c}A)\psi^\hbar). \tag{1.69}$$

Like (1.47)-(1.49) the magnetic Schrödinger-Poisson equation is a scalar equation. Compare (1.69) to (1.63) where we have an additional divergence-free term due to the spin coupling which is not present in (1.69). Wellposedness of the magnetic Schrödinger-Poisson equation in $\mathbb{R}^3$ for bounded magnetic potential has been studied in [7].

We can replace the coupling to the Poisson equation by more general Hartree nonlinearities, i.e. $W(x) * |\psi^\hbar(x)|^2$ where W is an interaction kernel. We have the **magnetic Schrödinger-Hartree equation**, introduced in [61, 70]:

$$i\hbar\partial_t\psi^\hbar = -\frac{1}{2m}(\hbar\nabla - i\frac{q}{c}A)^2\psi^\hbar + V^{\text{ext}}\psi^\hbar + (W * |\psi^\hbar|^2)\psi^\hbar, \tag{1.70}$$

with initial data

$$\psi^\hbar(x,0) = \psi_I^\hbar(x) \in L^2(\mathbb{R}^3), \tag{1.71}$$

here, V^{ext} is an external potential. In $\mathbb{R}^3$ and for $W(x) = |x|^{-1}$ the magnetic Schrödinger-Hartree equation and the magnetic Schrödinger-Poisson equation are equivalent.

Note that the magnetic Schrödinger-Hartree and in particular the magnetic-Schrödinger-Poisson equation in $3d$ arise as the mean field limit of the linear N-body magnetic Schrödinger equation with interaction potential, cf. [61].

1.2.3 Schrödinger-Maxwell

Another model extensively studied in the mathematical literature is the **magnetic Schrödinger-Maxwell equation** for a scalar wave function $\psi^\hbar$ given by

$$i\hbar\partial_t\psi^{\hbar,c} = -\frac{1}{2m}(\hbar\nabla - i\frac{q}{c}A^{\hbar,c})^2\psi^{\hbar,c} + qV^{\hbar,c}\psi^{\hbar,c}, \tag{1.72}$$

$$\Box V^\hbar = 4\pi|\psi^\hbar|^2, \qquad \Box A^\hbar = \frac{4\pi}{c}J^\hbar, \tag{1.73}$$

where

$$J^{\hbar,c} = \operatorname{Im}(\overline{\psi^{\hbar,c}}(\hbar\nabla - i\frac{q}{c}A^{\hbar,c})\psi^{\hbar,c}) \tag{1.74}$$

is the current density of the magnetic Schrödinger equation and initial data

$$\psi^{\hbar,c}(x,0) = \psi_I^{\hbar,c} \qquad A^{\hbar,c}(x,0) = a_0^{\hbar,c} \qquad \partial_t A^{\hbar,c}(x,0) = a_1^{\hbar,c}. \tag{1.75}$$

The Schrödinger-Maxwell equation is inconsistent in $1/c$ since Maxwell's equations are fully relativistic and the magnetic Schrödinger equation is non-relativistic since it lacks the spin term which is $O(1/c)$. Moreover, the Schrödinger equation is not Lorentz invariant. The first result on global wellposedness was obtained in [74] using energy estimates. In [49] the authors obtain global weak solutions in the energy space without uniqueness by parabolic regularization. Local and global wellposedness was above the energy level were obtained in [75, 76] using refined Koch-Tzetkov estimates. Scattering was discussed in [39]. Global wellposedness in the energy space was obtained by construction of a parametrix in [15]. Moreover a nonlinear extension by addition of a power nonlinearity can be found in [2].

1.3 N-particle models and mean field limits

Nonlinear PDE like the Schrödinger-Poisson equation also arise as the mean field limit of linear N-body equations with interaction between the particles like the N-body Schrödinger equation with Coulomb interaction. A quantum system consisting of a large number N of interacting particles is described by an N-body wave function, normalized in $L^2(\mathbb{R}^{3N})$,

$$\psi_N^\hbar = \psi_N^\hbar(x_1,\dots,x_N,t), \qquad \int_{\mathbb{R}^{3N}} |\psi_N^\hbar(x_1,\dots,x_N,t)|^2 \mathrm{d}x_1\cdots\mathrm{d}x_N = 1 \tag{1.76}$$

which satisfies the **N-body Schrödinger equation**

$$i\hbar\partial_t\psi_N^\hbar = H_N\psi_N^\hbar \tag{1.77}$$

where H_N is the N-body Hamiltonian given by

$$H_N = -\frac{\hbar^2}{2m}\sum_{j=1}^N \Delta_{x_j} + \frac{1}{N}\sum_{j<k}^N V(|x_j-x_k|) \tag{1.78}$$

where V is a binary interaction potential and the factor $1/N$ is the *weak coupling* scaling. For the Coulomb interaction one has $V(x)\simeq|x|^{-1}$ (which yields the Schrödinger-Hartree equation and in particular in $3d$ the Schrödinger-Poisson equation) and for the "contact interaction" $V(x)=\delta(x)$ (which yields the Gross-Pitaevskii equation). For large N, equation (1.77) becomes impossible to solve numerically. Therefore it is imperative to approximate linear N-body equations by (systems of) nonlinear 1-body equations. The following diagram represents the asymptotic links between N-body linear and 1-body nonlinear equations.

$$\begin{array}{ccc}
\boxed{\textbf{linear } N\textbf{-body Schrödinger}} & \xrightarrow{\hbar\to 0} & \boxed{\textbf{linear } N\textbf{-body Liouville}} \\
\downarrow & & \downarrow \\
N\to\infty & & N\to\infty \\
\downarrow & & \downarrow \\
\boxed{1\textbf{-body nonlinear Schrödinger}} & \xrightarrow{\hbar\to 0} & \boxed{1\textbf{-body nonlinear Vlasov}}
\end{array}$$

The Hartree ansatz for condensate of bosons, i.e. particles with symmetric N-body wave function, is to assume that the inital data are factorized with the same wave function for all bosons,

$$\psi_N^\hbar(x_1,\dots,x_N,t=0) = \prod_{j=1}^N \psi_I^\hbar(x_j) \tag{1.79}$$

which trivially produces a symmetric wave function. This is valid for a boson ensemble (i.e. if the system of bosons is in a condensed state). Note that the general Hartree ansatz for bosons would use different orbitals $\psi^\hbar_{I,j}$. It was shown in [11, 92] that for factorized initial data and bounded V, the N-body linear Schrödinger equation converges to the self-consistent 1-body nonlinear Schrödinger equation or *Schrödinger-Hartree equation*

$$i\hbar\partial_t\psi^\hbar = -\frac{\hbar^2}{2}\psi^\hbar + (V * |\psi^\hbar|^2)\psi^\hbar \tag{1.80}$$

This was extended in [9, 29] to the Coulomb potential

$$V(x) = \frac{\lambda}{|x|} \tag{1.81}$$

which implies the convergence of the three-dimensional N-body Schrödinger equation with Coulomb interaction to the Schrödinger-Poisson equation.

The combined limit $\hbar + 1/N \to 0$ of (1.77) with Coulomb interaction to the Euler-Poisson equation (1.58) was shown in [42] and to the Vlasov-Poisson equation (1.52)-(1.53) in [41] by defining a quantum Wasserstein distance in order to measure the distance between quantum density operators and classical phase space densities. Recently, Golse and Paul extended the results to define a quantum Wasserstein distance between two density operators instead of a density operator and a classical phase space density in [43].

For fermions (i.e. for antisymmetric wave functions) a different ansatz has to be chosen. The *Hartree-Fock ansatz* consists of taking an initial fermionic, i.e. antisymmetric N-body wave function $\psi^\hbar_{N,I} \in L^2_{\text{as}}(\mathbb{R}^{3N})$ (the subspace of L^2 consisting of antisymmetric wave functions) for the N-body Schrödinger equation (1.77). In particular, one chooses the *Slater determinant*

$$\psi^\hbar_{\text{S}}(x_1,\ldots,x_N,t) = \frac{1}{\sqrt{N!}}\det\Big(\psi^\hbar_i(x_j,t)\Big), \tag{1.82}$$

where $\{\psi^\hbar_j\}_{j=1}^N$ is an orthonormal system in $L^2(\mathbb{R}^3,\mathbb{C})$. The associated initial (pure state) one particle reduced density matrix $\rho^{\hbar,(1)}_{N,I}$ should be close in the trace norm to the initial one particle reduced density matrix $\rho^\hbar_{\text{S},I} = \rho^\hbar_{\text{S}}(0)$ of $\psi^\hbar_{\text{S},I}$ in the sense that

$$\operatorname{tr}\Big(\rho^{\hbar,(1)}_{N,I} - \rho^\hbar_{\text{S},I}\Big) \le C, \tag{1.83}$$

uniformly in N. Let $\rho^\hbar_{\text{S}}(t)$ denote the (pure state) time evolution of $\rho^\hbar_{\text{S},I}$. It satisfies the **time dependent Hartree-Fock (TDHF) equation**

$$i\hbar\partial_t\rho^\hbar_{\text{S}}(t) = \left[-\frac{\hbar^2}{2}\Delta + \frac{1}{|x|} * \rho^\hbar_{\text{S,diag}}(t) - X, \rho^\hbar_{\text{S}}(t)\right], \tag{1.84}$$

where

$$\rho^\hbar_{\text{S,diag}}(x) = \frac{1}{N}\rho^\hbar_{\text{S}}(x,x) \tag{1.85}$$

is the density of $\rho^\hbar_{\mathrm{S}}$ and X denotes the exchange term with integral kernel

$$X(x,y) = \frac{1}{N}\frac{1}{|x-y|}\rho^\hbar_{\mathrm{S}}(x,y). \tag{1.86}$$

It is then expected that the time evolution $\rho^{\hbar,(1)}_N(t)$ of the initial one particle reduced density matrix $\rho^{\hbar,(1)}_{N,I}$ should remain close to $\rho^\hbar_{\mathrm{S}}(t)$ and their distance in some (refined) trace norm should vanish in the limit $N \to \infty$. The convergence of the fermionic N-body Schrödinger equation to the TDHF equation was shown for bounded, symmetric binary interaction potentials V (boundedness excludes the Coulomb potential) in [10]. The problem for the Coulomb interaction is hard due to the singular nature of the Coulomb potential. In [82], the case of a simultaneous limit where $\hbar = N^{-1/3}$ was proved, in [80] for a different scaling linking potential and kinetic energy and in [32] for the same scaling as in [11] where it is shown that the time evolution of the fermionic N-body Schrödinger equation with Coulomb potential stays close to the Hartree-Fock equation in some trace norm. The result in [82] holds for $\rho^\hbar_N$ representing a pure state, i.e. $\rho^\hbar_N$ is given by an orthogonal projection on the N-dimensional subspace spanned by antisymmetric wave functions.

Notice that this is at odds with the semiclassical limit of the Schrödinger-Poisson equation in $\mathbb{R}^3$ to the Vlasov-Poisson equation [59, 62] and the limit of the Pauli-Poisson equation to the Vlasov-Poisson equation with Lorentz force in [71] where only mixed states are allowed since the occupation probabilities have to satisfy conditon (2.37) in order for the Wigner transform to be bounded uniformly in L^2. A recent result for the Hartree-Fock dynamics of fermionic mixed states is [16]. However this result does not include Coulomb interaction. Moreover, in [82] the assumptions on the initial data for $\rho^\hbar_N$ are restrictive in the sense that one needs control over the commutator $[x, \rho^\hbar_N]$ for which the authors of [82] did not find non-trivial sufficient conditions.

The N-body magnetic Schrödinger Hamiltonian with Coulomb interaction is given by

$$H^{\mathrm{m}}_N = -\frac{1}{2m}\sum_{j=1}^{N}(\hbar\nabla_{x_j} - i\frac{q}{c}A(x_j))^2 + \frac{1}{N}\sum_{j<k}^{N}\frac{1}{|x_j - x_k|} \tag{1.87}$$

The bosonic N-body wave function $\psi^\hbar_N$ satisfies the **N-body magnetic Schrödinger equation**

$$i\hbar\partial_t\psi^\hbar_N = H^{\mathrm{m}}_N\psi^\hbar_N \tag{1.88}$$

with initial data

$$\psi^\hbar_{N,I} = (\psi^\hbar_I)^{\otimes N} \in L^2(\mathbb{R}^{3N}) \tag{1.89}$$

It was shown by Lührmann [61] that the linear bosonic N-body magnetic Schrödinger equation with Coulomb interaction converges to the magnetic Schrödinger-Hartree equation in the limit $N \to \infty$ for *pure states* and for $\hbar$ *fixed.*

The Pauli-Poisson equation should arise as the mean field limit of the N**-body Pauli equation** given by

$$i\hbar\partial_t \Psi_N^\hbar = H_N^{\mathrm{P}} \Psi_N^\hbar, \tag{1.90}$$

where H_N^{P} is the linear N-body Pauli Hamiltonian with Coulomb interaction given by

$$H_N^{\mathrm{P}} = -\frac{1}{2m}\sum_{j=1}^N (\hbar\nabla_{x_j} - i\frac{q}{c}A(x_j))^2 - \frac{\hbar q}{2cm}(\sigma\cdot B(x_j)) + \frac{1}{N}\sum_{j<k}^N \frac{1}{|x_j - x_k|}, \tag{1.91}$$

with initial data

$$\Psi_{N,I}^\hbar = (\Psi_I^\hbar)^{\otimes N} \in ((L^2(\mathbb{R}^3))^2)^{\otimes N} \cong (L^2(\mathbb{R}^{3N}))^2. \tag{1.92}$$

The following diagram should hold for the Pauli equation:

$$\begin{array}{ccc}
\boxed{\textbf{linear } N\textbf{-body Pauli}} & \xrightarrow{\hbar\to 0} & \boxed{\textbf{linear } N\textbf{-body Liouville w/ Lorentz force}} \\
\downarrow & & \downarrow \\
N\to\infty & & N\to\infty \\
\downarrow & & \downarrow \\
\boxed{1\textbf{-body Pauli-Hartree}} & \xrightarrow{\hbar\to 0} & \boxed{1\textbf{-body Vlasov w/ Lorentz force - Hartree}}
\end{array}$$

Since the Pauli equation holds for fermions the Pauli exclusion principle implies that the Hartree ansatz as for the bosonic N-body magnetic Schrödinger equation is in fact not correct. The correct ansatz is the Hartree-Fock ansatz. However, based on a an approximation of the exchange term by a power nonlinearity $|\psi^\hbar|^{2/3}\psi^\hbar$ due to Slater [89] one may approximate the fermionic model. The corresponding equation is the *Schrödinger-Poisson-Xα equation*, introduced in [68] and studied numerically in [5].

1.4 Outline

The thesis is organized as follows. In section 2 we introduce the *WKB ansatz* and the *Wigner method* and we give precise definitions for the density matrix and the Wigner transform. In section 3 we prove global wellposedness of the Pauli-Poisson equation in the energy space and the semiclassical limit to the Vlasov-Poisson equation with Lorentz force using Wigner transforms. In section 4 we introduce the Pauli-Poisswell equation and prove the existence of global finite energy weak solutions and the existence and uniqueness of local strong solutions. Sections 5 is devoted to the semiclassical limit of the Pauli-Poisswell equation to the Vlasov-Poisswell by the Wigner method and in Section 6 we prove the semiclassial limit of the Pauli-Poisswell equation to the Euler-Poisswell equation using the WKB ansatz where we also show local wellposedness of the Pauli-Poisswell and Euler-Poisswell equations. The main results of this thesis are:

- **Global-in-time semiclassical limit of the Pauli-Poisson equation by the Wigner method:** Theorem 1, cf. [71].
- **Global wellposedness of the Pauli-Poisson equation:** Theorem 2, cf. [71].
- **Global weak solutions in the energy space of the Pauli-Poisswell equation:** Theorem 3, cf. [38].
- **Local strong solutions of the Pauli-Poisswell equation:** Theorem 4, cf. [38].
- **Global-in-time semiclassical limit of the Pauli-Poisswell equation to the Vlasov-Poisswell equation by the Wigner method:** Section 5, cf. [73].
- **Local wellposedness of the Pauli-Poisswell-WKB equation and the Euler-Poisswell equation:** Theorem 5, cf. [97].
- **Semiclassical limit of the Pauli-Poisswell equation to the Euler-Poisswell equation by the WKB method:** Theorem 6, cf. [97].

2 Semiclassical analysis: Preliminaries

We use a scaling where $m = c = q = 1$ and the small parameter is a scaled dimensionless Planck constant, still denoted by $\hbar$. In Sections 3.2, 4.2 and 4.3 we will use a scaling where $\hbar = 1$ and omit superscripts altogether since we do not consider asymptotics there.

2.1 WKB vs Wigner

The limit $\hbar \to 0$ is not straightforward because of the nonlinearities in self-consistent models. For instance it is not obvious how the limit $\hbar \to 0$ of the nonlinearity $V^\hbar \Psi^\hbar$, where $V^\hbar$ is given by the Poisson equation $-\Delta V^\hbar = |\Psi^\hbar|^2$, should be defined since the convergence of $\Psi^\hbar$ will be weak in general and the product of weak limits is a priori not defined (and not equal to the weak limit of the product) and additional compactness is needed. Moreover, physical quantities like the density $\rho^\hbar$, the current density $J^\hbar$ or the energy $E^\hbar$ are quadratic in the wave function and for the same reason the weak limit of these quantities has to be rigorously justified. There are 2 main techniques for such semiclassical limits:

The first approach to this problem is the *WKB ansatz* (Wentzel–Kramers–Brillouin) [101], also called *geometrical optics ansatz* [91], which consists of looking for solutions of the form

$$\Psi^\hbar(x,t) = a^\hbar(x,t) e^{iS^\hbar(x,t)/\hbar} \tag{2.1}$$

where $a^\hbar$ is the amplitude and $S^\hbar$ the phase. In the case of the Dirac-Maxwell equation or the Pauli-Poisswell equation $a^\hbar$ is vector-valued. We always choose an ansatz where the components of $\Psi^{\hbar,0}$ have the same phase $S^{\hbar,0}$. If one would choose more complicated initial data with two different phases $S_j^{\hbar,0}$, $j = 1,2$, then also the solution $\Psi^\hbar$ would have two phases and oscillatory cross terms like

$$\exp\left(\frac{i}{\hbar}(S_j^\hbar - S_k^\hbar)\right),$$

would appear. Note that when dealing with matrix-valued Hamiltonians one usually employs this *multicomponent WKB ansatz* by taking a vector-valued amplitude and a scalar phase which is a sound mathematical choice that is also usually chosen by physicists (see e.g. [56], [96]).

In general one assumes that $A^\hbar$ admits an asymptotic expansion of the form

$$A^\hbar = A^0 + \hbar A^1 + \hbar^2 A^2 + O(\hbar^3)$$

Then one plugs this ansatz into the equation, e.g. the Pauli-Poisswell or Schrödinger-Poisson equation and makes the substitution $u^\hbar = \nabla S^\hbar$. For the Schrödinger-Poisson equation (1.47)-(1.50) the corresponding WKB equation is given by

$$\partial_t a^\hbar + u^\hbar \cdot \nabla a^\hbar + \frac{1}{2} a^\hbar \nabla \cdot u^\hbar = \frac{i\hbar}{2} \Delta a^\hbar, \tag{2.2}$$

$$\partial_t u^\hbar + u^\hbar \cdot \nabla u^\hbar + \nabla V^\hbar = 0, \tag{2.3}$$

$$a^\hbar(0,x) = a^{\hbar,0}(x), \tag{2.4}$$

$$u^\hbar(0,x) = u^0(x). \tag{2.5}$$

where $V^\hbar$ is given by the Poisson equation $-\Delta V^\hbar = |a^\hbar|^2$, cf. [21]. For the Pauli-Poisswell equation we derive the corresponding WKB equation in Section 6 and prove an a priori estimate which will allows to show local wellposedness and the semiclassical limit. The WKB equation (2.3)-(2.5) converges to the *Euler-Poisson equation*, compare (1.58),

$$\partial_t a + u \cdot \nabla a + \frac{1}{2} a \nabla \cdot u = 0, \tag{2.6}$$

$$\partial_t u + u \cdot \nabla u + \nabla V = 0, \tag{2.7}$$

$$a(0,x) = a^{\hbar,0}(x), \tag{2.8}$$

$$u(0,x) = u^0(x). \tag{2.9}$$

There are limitations to this ansatz, the most important one that it is usually only valid locally in time. This is because the WKB equation (2.2)-(2.5) does not admit global solutions in time due to singularities, called *focal points* or *caustics*. There are generalizations of the WKB ansatz to somewhat overcome these drawbacks such as *multivalued phases*, cf. [91] and the references therein.

The second approach is the method of *Wigner transforms*, first mentioned by Wigner in [95]. One introduces the *Wigner transform* f of the wave function $\psi^\hbar$ for the Schrödinger equation or the 2-spinor $\Psi^\hbar$ for the Pauli equation. It is defined by doubling the number of coordinates (”density matrix”) and a change of coordinates to center of mass and relative coordinates on the scale $\hbar$ and by a subsequent Fourier transform to a ”dual” (kinetic) variable:

$$f^\hbar(x,\xi,t) := \frac{1}{(2\pi\hbar)^3} \int_{\mathbb{R}^3} e^{-i\xi\cdot y} \psi^\hbar(x + \frac{\hbar y}{2}) \overline{\psi^\hbar(x - \frac{\hbar y}{2}, t)} \mathrm{d}y. \tag{2.10}$$

The Wigner transform is only a *quasi probability density* since it attains negative values in general. It was shown in [53]) that in the case of a pure state, i.e. a single wave function $\psi^\hbar$ as in (2.10), it is positive precisely when the state $\psi^\hbar$ is a *coherent state*, defined by

$$\psi^\hbar_{x,\xi}(y) := \frac{1}{(\pi\hbar)^{3/2}} e^{(y-x)^2/2\hbar} e^{i\xi y/\hbar}. \tag{2.11}$$

The Wigner transform of a coherent state is the *Husimi function* $\tilde{f}^\hbar(x,\xi)$ which is equivalent to a Wigner transform ”smoothed” by a convolution in phase space in the variables x and ξ with a Gaussian kernel $G(x,\xi)$, i.e.

$$\tilde{f}^\hbar(x,\xi) := f^\hbar(x,\xi) *_x G^\hbar(x) *_\xi G^\hbar(\xi), \quad G^\hbar(z) := \frac{1}{(\pi\hbar)^{3/2}} e^{-|z|^2/\hbar}. \tag{2.12}$$

It is easy to see that $\tilde{f}^\hbar$ defines a pointwise nonnegative function since the variance of the Gaussian is chosen to be $\hbar$. This is related to the uncertainty principle which states that the product of the variances σ_x and σ_ξ of x and ξ is always greater than $\hbar/2$, i.e. $\sigma_x\sigma_\xi \geq \hbar/2$. Note that coherent states are precisely the states with minimal uncertainty, i.e. $\sigma_x\sigma_\xi = \hbar/2$. For more details on the relationship between the Husimi function and the uncertainty principle see [94].

For the Schrödinger-Poisson equation it can be shown by straightforward computation that $f^\hbar$ obeys the *Wigner-Poisson equation* aka *quantum Vlasov-Poisson equation* [62]:

$$\partial_t f^\hbar + \xi \cdot \nabla_x f^\hbar + \theta[V^\hbar] f^\hbar = 0 \tag{2.13}$$

$$-\Delta V^\hbar = |\psi^\hbar|^2, \tag{2.14}$$

$$f^\hbar(x, \xi, 0) = f_I^\hbar \tag{2.15}$$

where $\theta[V^\hbar]$ is the pseudo-differential operator defined by

$$(\theta[V^\hbar] f^\hbar)(x, \xi, t) := \frac{1}{(2\pi)^3} \int_{\mathbb{R}^6} \delta[V^\hbar](x, y, t) f^\hbar(x, \eta, t) e^{-i(\xi-\eta)\cdot y} \mathrm{d}\eta \mathrm{d}y, \tag{2.16}$$

where the symbol is

$$\delta[V^\hbar](x, y, t) = \frac{i}{\hbar}(V(x + \frac{\hbar y}{2}) - V(x - \frac{\hbar y}{2})). \tag{2.17}$$

Although it is not a proper phase space density one can verify easily that the moments of $f^\hbar$ define the densities of the conserved quantities of the Schrödinger-Poisson equation, namely the position (or charge) density and the current density

$$\rho^\hbar(x, t) = |\psi^\hbar(x, t)|^2 = \int f^\hbar(x, \xi, t) \mathrm{d}\xi, \tag{2.18}$$

$$J^\hbar(x, t) = \hbar \,\mathrm{Im}(\overline{\psi^\hbar(x, t)} \nabla \psi^\hbar(x, t)) = \int \xi f^\hbar(x, \xi, t) \mathrm{d}\xi \tag{2.19}$$

and the kinetic energy density

$$e^\hbar_{\mathrm{kin}}(x, t) = \hbar^2 |\nabla \psi^\hbar(x, t)|^2 = \int |\xi|^2 f^\hbar(x, \xi, t) \mathrm{d}\xi \tag{2.20}$$

which give rise to conservation of mass (or charge), momentum and energy, respectively. In particular, the L^2-norm of the Wigner transform is conserved. Note that the (positive) Husimi function does not yield the correct moments, but rather smoothed moments. It can be shown by several methods that the Wigner transform converges weakly to a non-negative Radon measure f which is called *Wigner measure* [36, 59, 62]. It is related to the semiclassical measures in [34],[36] and measures the loss of compactness in L^2 in [35]. One way of showing the positivity of the Wigner measure is by noting that the Husimi function and the Wigner transform converge to the same limit [59] or by the Bochner-Schwartz theorem [36]. As discussed in the introduction on can show under appropriate assumptions on $f^\hbar$ and $V^\hbar$ ([59], [62]) that (2.13)- (2.15) converges to the Vlasov-Poisson equation

$$\partial_t f + \xi \cdot \nabla_x f - \nabla_x V \nabla_\xi f = 0 \tag{2.21}$$

$$-\Delta V = \rho := \int f \mathrm{d}\xi, \tag{2.22}$$

$$f(x, \xi, 0) = f_I \tag{2.23}$$

The relationship between the WKB method and the Wigner method is apparent when we define the *monokinetic Wigner measure*

$$f(x, \xi, t) = \rho(x, t)\delta(\xi - u(x, t)). \tag{2.24}$$

and the monokinetic initial Wigner measure

$$f_I(x, \xi, t) = \rho_I(x, t)\delta(\xi - u_I(x, t)), \tag{2.25}$$

If we set $\rho = |a|^2$ and use (2.25) as initial data for (2.21)-(2.23) we can expect that for short times, $f(x, \xi, t)$ remains monokinetic and obeys the Euler-Poisson equation (2.6)-(2.9). We have the following heuristic principle:

The WKB method leads to local-in-time strong convergence to Euler equations (which correspond to Vlasov equations with monokinetic data).
The Wigner method leads to global-in-time weak convergence to Vlasov equations.

2.2 Wigner transforms of mixed states

Mixed states in quantum mechanics represent a statistical ensemble of possibles states and are the fundamental object of quantum mechanics since a pure state is a special case of a mixed state. It expresses that, in general, we cannot prepare a quantum system in one precise (pure) state, but rather with a certain probability λ_j in one of many possible states ψ_j. The mixed state formulation is necessary from a technical point of view when dealing with the semiclassical limit of the Schrödinger-Poisson and Pauli-Poisson equations since uniform L^2 estimates for the Wigner transform are only possible in a mixed state formulation. A mixed state is represented by the density matrix which is defined as follows:

Let $\{\Psi_j^\hbar\}_{j\in\mathbb{N}}$, $\Psi_j^\hbar = (\psi_{j,1}^\hbar, \psi_{j,2}^\hbar)^T$ be an orthonormal system in $(L^2(\mathbb{R}^3, \mathbb{C}))^2$. We define the density matrix $\rho^\hbar$

$$\begin{aligned}\rho^\hbar(x, y, t) &:= \sum_{j=1}^\infty \lambda_j^\hbar \Psi_j^\hbar(x, t)\overline{\Psi_j^\hbar(y, t)} \\ &= \sum_{j=1}^\infty \lambda_j^\hbar \left(\psi_{j,1}^\hbar(x, t)\psi_{j,1}^\hbar(y, t)^* + \psi_{j,2}^\hbar(x, t)\psi_{j,2}^\hbar(y, t)^*\right),\end{aligned} \tag{2.26}$$

where $\lambda = \{\lambda_j^\hbar\}_{j\in\mathbb{N}}$ is a normally convergent series such that $\lambda_j^\hbar \geq 0$ and $\sum_j \lambda_j^\hbar = 1$. The $\lambda_j^\hbar$'s are the *occupation probabilities* of the states $\Psi_j^\hbar$. If there is a k such that $\lambda_j^\hbar = 1$ for

$j = k$ and $\lambda_j^\hbar = 0$ otherwise then $\rho^\hbar$ represents a *pure state*. Otherwise it represents a *mixed state*. The reason why we write a superscript $\hbar$ denoting a dependence on $\hbar$ will become apparent in Remark 2.

The density matrix $\rho^\hbar$ can be considered as the kernel of a Hilbert-Schmidt, hermitian, positive and trace class operator $\varrho^\hbar$ on $(L^2(\mathbb{R}^3))^2$, called *density operator*. We will often identify $\rho^\hbar$ and $\varrho^\hbar$. The diagonal of $\rho^\hbar(x, y)$ corresponds to the particle density and is defined by

$$\rho^\hbar_{\text{diag}}(x) := \rho^\hbar(x, x) = \sum_{j=1}^{\infty} \lambda_j^\hbar |\Psi_j^\hbar(x)|^2 \in L^1(\mathbb{R}^3_x). \tag{2.27}$$

Note that since $\Psi_j^\hbar \in (L^2(\mathbb{R}^3))^2$ and since $\sum_j \lambda_j^\hbar$ converges normally this expression is well-defined. Indeed, consider

$$\rho^\hbar(x+z, x) = \sum_{j=1}^{\infty} \lambda_j^\hbar \Psi_j^\hbar(x+z)\overline{\Psi_j^\hbar(x)}.$$

We can then let $z \to 0$ since $\rho^\hbar(x+z, x) \in C(\mathbb{R}^3_z, L^1(\mathbb{R}^3_x))$. The time evolution of the density matrix $\rho^\hbar$ is given by the von Neumann equation:

$$i\hbar \frac{\partial \rho^\hbar}{\partial t} = [H, \rho^\hbar]. \tag{2.28}$$

The Wigner transform $f^\hbar(x, \xi, t)$ of $\rho^\hbar$ follows from definition (2.10) as

$$f^\hbar(x, \xi, t) := \frac{1}{(2\pi\hbar)^3} \int_{\mathbb{R}^3} e^{-i\xi\cdot y} \rho^\hbar(x + \frac{\hbar y}{2}, x - \frac{\hbar y}{2}, t)\mathrm{d}y. \tag{2.29}$$

More generally, we define the *Wigner matrix* $F^\hbar$ (cf. [36]) as the Wigner transform of a matrix valued density matrix $R^\hbar$, i.e.

$$F^\hbar(x, \xi, t) = \frac{1}{(2\pi\hbar)^3} \int_{\mathbb{R}^3_y} e^{-i\xi\cdot y} R^\hbar(x + \frac{\hbar y}{2}, x - \frac{\hbar y}{2}, t)\mathrm{d}y, \tag{2.30}$$

where

$$R^\hbar(x, y, t) := \sum_{j=1}^{\infty} \lambda_j^\hbar \Psi_j^\hbar(x, t) \otimes \overline{\Psi_j^\hbar(y, t)},$$

where $\otimes$ denotes the tensor product of vectors. Then we can define the scalar Wigner transform as $f^\hbar = \mathrm{Tr}\big(F^\hbar\big)$ and the scalar density matrix as $\rho^\hbar = \mathrm{Tr}\big(R^\hbar\big)$ where Tr denotes the 2×2 matrix trace. Similarly we can define $R^\hbar_{\text{diag}}$ as

$$R^\hbar_{\text{diag}}(x) = R^\hbar(x, x). \tag{2.31}$$

A simple calculation shows that

$$\rho^\hbar_{\text{diag}}(x) = \int_{\mathbb{R}^3_\xi} f^\hbar(x, \xi)\mathrm{d}\xi. \tag{2.32}$$

and

$$R^{\hbar}_{\mathrm{diag}}(x) = \int_{\mathbb{R}^3_\xi} F^{\hbar}(x,\xi)\mathrm{d}\xi. \tag{2.33}$$

This also implies that $f^{\hbar} \in L^1(\mathbb{R}^3_x \times \mathbb{R}^3_\xi)$ uniformly in L^1. We will also use the notation

$$f^{\hbar} = W^{\hbar}[\rho^{\hbar}] \qquad\qquad F^{\hbar} = W^{\hbar}[R^{\hbar}] \tag{2.34}$$

and

$$\tilde{f}^{\hbar} = \tilde{W}^{\hbar}[\rho^{\hbar}] \qquad\qquad \tilde{F}^{\hbar} = \tilde{W}^{\hbar}[R^{\hbar}] \tag{2.35}$$

Since $\{\Psi^{\hbar}_j\}_{j\in\mathbb{N}}$ is a bounded family in $(L^2(\mathbb{R}^3))^2$, $f^{\hbar}$, resp. $F^{\hbar}$ is a bounded family in $\mathcal{S}'(\mathbb{R}^3_x \times \mathbb{R}^3_\xi)$, resp. $(\mathcal{S}'(\mathbb{R}^3_x \times \mathbb{R}^3_\xi))^{2\times 2}$ (cf. [36], Proposition 1.1). Thus there is a sequence $\{\hbar_k\}$ going to 0 such that $f^{\hbar_k}$, resp. $F^{\hbar_k}$ converges to a limit f, resp. F. Then f, resp. F is a nonnegative (matrix valued) Radon measure, i.e. for all $z \in \mathbb{C}^2$,

$$\sum_{i,j} F_{ij} z_i \overline{z_j} \geq 0.$$

Note that one can also define the following test function space (in fact an algebra) $\mathcal{A}$ which is more adapted to the Wigner transform $f^{\hbar}$,

$$\mathcal{A} := \{\phi \in C_0(\mathbb{R}^3_x \times \mathbb{R}^3_\xi) \colon \mathcal{F}_\xi[\phi](x,\eta) \in L^1(\mathbb{R}^3_\eta, C_0(\mathbb{R}^3_x))\} \tag{2.36}$$

Under the same assumptions one can show that $f^{\hbar}$ and $F^{\hbar}$ have a weak limit in $\mathcal{A}'$. However, this technicality will make no difference to our analysis and we will either use $\mathcal{S}'$ or $\mathcal{A}'$. The algebra $\mathcal{A}$ was introduced in [59] and used e.g. in [102].

The measure f is called *Wigner measure* and F is called *Wigner matrix measure*, (cf. [59, 36]).

Now we prove some crucial properties of the Wigner transform in the limit $\hbar \to 0$. First we prove that $f^{\hbar}$ is bounded in L^2 uniformly in $\hbar$ if a certain condition on the occupation probabilities $\lambda^{\hbar}_j$ is satisfied. Moreover, the Husimi function $\tilde{f}^{\hbar}$ is bounded in L^∞ uniformly in $\hbar$ if another condition is satisfied. The results hold analogously for the matrix versions.

Remark 2. This also implies that a mixed state formulation is necessary since the conditions imply that an infinite number of the $\lambda^{\hbar}_j$ is different from zero. Since (2.37) and (2.38) imply that the sequence $\{\lambda^{\hbar}\}$ depends on $\hbar$ the reason for the superscript $\hbar$ becomes apparent.

Lemma 2.1. *Let $\{\lambda^{\hbar}\} \in \ell^1$ be a normally convergent sequence of occupation probabilities for a density matrix $\rho^{\hbar}$ such that $\sum_j \lambda^{\hbar}_j = 1$ and $\lambda^{\hbar}_j \geq 0$. If*

$$\|\lambda^{\hbar}\|_2^2 = \sum_{j=1}^{\infty} (\lambda^{\hbar}_j)^2 \leq (2\pi\hbar)^3. \tag{2.37}$$

then the Wigner transform $f^\hbar$ is uniformly bounded in $\hbar$ in $L^2(\mathbb{R}^3_x \times \mathbb{R}^3_\xi)$.

If the occupation probabilities $\{\lambda^\hbar\}$ satisfy the condition

$$\|\lambda^\hbar\|_\infty = \sup_{j\in\mathbb{N}} \lambda_j^\hbar \leq (2\pi\hbar)^3. \tag{2.38}$$

then the Husimi function $\tilde{f}^\hbar$ is uniformly bounded in $\hbar$ in $L^\infty(\mathbb{R}^3_x \times \mathbb{R}^3_\xi)$.

Moreover, both conditions (2.37) and (2.38) together with $\lambda_j^\hbar \geq 0$ and $\{\lambda^\hbar\} \in \ell^1$ imply that $\lambda_j^\hbar \neq 0$ for infinitely many j.

Proof. It is easy to see that the L^2 norm of $f^\hbar$ is conserved. Then $\|f^\hbar\|_2 \leq C$ implies that $\hbar^{-3}\operatorname{tr}(\rho_\hbar^2) = \hbar^{-3}\|\lambda^\hbar\|_2^2 \leq C$ (use (2.29), Plancherel and a change of variables). Together with the requirements $\sum_j \lambda_j^\hbar = 1$ and $\lambda_j^\hbar \geq 0$ this implies that an infinite number of $\lambda_j^\hbar$ are different from zero. In fact, assume that there is a $N \in \mathbb{N}$ such that $\lambda_j^\hbar = 0$ for all $j > N$. Then

$$1 = \sum_{j=1}^{\infty} \lambda_j^\hbar = \sum_{j=1}^{N} \lambda_j^\hbar \leq N(2\pi\hbar)^3.$$

Letting $\hbar$ go to zero leads to a contradiction.

For the second part let $\psi^\hbar_{x,\xi} \in L^2(\mathbb{R}^3)$ be a coherent state with unit norm, defined by

$$\psi^\hbar_{x,\xi}(y) := (\pi\hbar)^{-3/2} e^{(y-x)^2/2\hbar} e^{i\xi y/\hbar}.$$

The Wigner transform of $\psi^\hbar_{x,\xi}$ is given by

$$W^\hbar[\psi^\hbar_{x,\xi}](y,\eta) = (\pi\hbar)^{-3} e^{-(y-x)^2} e^{-(\eta-\xi)^2}.$$

Then the Husimi function can be written as

$$\tilde{W}^\hbar[\Psi^\hbar](x,\xi) = \iint W^\hbar[\Psi^\hbar] W^\hbar[\psi^\hbar_{x,\xi}] \mathrm{d}y\mathrm{d}\eta.$$

where the dot emphasizes that the density operator is applied to the coherent state $\psi^\hbar_{x,\xi}$. By Plancherel we have that

$$\begin{aligned}\tilde{W}^\hbar[\Psi^\hbar](x,\xi) &= \left\langle \Psi^\hbar(y+\frac{\hbar v}{2})\overline{\Psi^\hbar(y-\frac{\hbar v}{2})}, \psi^\hbar_{x,\xi}(y+\frac{\hbar v}{2})\overline{\psi^\hbar_{x,\xi}(y-\frac{\hbar v}{2})}\right\rangle \\ &= |\langle \Psi^\hbar, \psi^\hbar_{x,\xi}\rangle|^2.\end{aligned}$$

This shows that the Husimi function can be written as

$$\tilde{W}^\hbar[\rho^\hbar] = \frac{1}{(2\pi\hbar)^3}\langle \rho^\hbar \cdot \psi^\hbar_{x,\xi}, \psi^\hbar_{x,\xi}\rangle_{L^2} \tag{2.39}$$

We have that

$$\tilde{W}^\hbar[\rho^\hbar] = \frac{1}{(2\pi\hbar)^3}\langle \rho^\hbar \cdot \psi^\hbar_{x,\xi}, \psi^\hbar_{x,\xi}\rangle_{L^2} = \frac{1}{(2\pi\hbar)^3}\sum_j \lambda_j^\hbar |\langle \psi^\hbar_{x,\xi}, \Psi^\hbar_j\rangle|^2$$
$$\leq \sum_j |\langle \psi^\hbar_{x,\xi}, \Psi^\hbar_j\rangle|^2 = 1.$$

By spectral theory,

$$\rho^\hbar = \sum_j \lambda_j^\hbar \langle \cdot, \Psi_j^\hbar\rangle_{L^2}\Psi_j^\hbar.$$

The first equality is the definition of the Husimi function via coherent states. The second equality uses the spectral decomposition of $\rho^\hbar$. The inequality follows from $\lambda_j^\hbar \leq (2\pi\hbar)^3$ and the last equality follows from the fact that $\{\Psi_j^\hbar\}$ is an orthonormal system in L^2 and that $\|\psi^\hbar_{x,\xi}\|_2 = 1$. Thus, we have shown that $\tilde{F}^\hbar \in L^\infty(\mathbb{R}^6)^{2\times 2}$. $\square$

Remark 3. The first condition (2.37) on the occupation probabilities of the initial mixed state is known since 1993, when it was found independently in the work of Lions and Paul [59], resp. Markowich and Mauser [62] on the limit of the Schrödinger-Poisson equation to the Vlasov-Poisson equation.

The second condition (2.38) is new and was introduced in [73] in order to obtain more regularity for the Husimi function and hence relax the required regularity for the time derivative of the current density $J^\hbar$ and the magnetic potential $A^\hbar$, which is a major difficulty in the Pauli-Poisswell case.

Now we state some results on the convergence of $f^\hbar$ and $\tilde{f}^\hbar$. Let

$$\begin{aligned}\hat{\rho}^\hbar(\xi,\eta) &:= \mathcal{F}_x[\overline{\mathcal{F}_y}[\rho^\hbar(x,y)]]\\ &= \sum_{j=1}^\infty \lambda_j^\hbar \widehat{\Psi_j^\hbar}(\xi)\overline{\widehat{\Psi_j^\hbar}(\eta)}\end{aligned}$$

and

$$\hat{\rho}^\hbar_{\text{diag}}(\xi) := \hat{\rho}^\hbar(\xi,\xi) = \sum_{j=1}^\infty \lambda_j^\hbar |\widehat{\Psi_j^\hbar}(\xi)|^2$$

A sequence of density matrices $\rho^\hbar$ is said to be $\hbar$-*oscillatory* if for all φ continuous with compact support,

$$\limsup_{\hbar\to 0}\int_{|\xi|\geq R/\hbar} \widehat{\varphi\rho^\hbar_{\text{diag}}}(\xi)\mathrm{d}\xi \to 0 \tag{2.40}$$

as $R\to\infty$. By $\widehat{\varphi\rho^\hbar_{\text{diag}}}(\xi)$ we mean

$$\sum_{j=1}^\infty \lambda_j^\hbar |\widehat{\varphi\Psi_j^\hbar}(\xi)|^2$$

It is *compact at infinity* if

$$\limsup_{\hbar \to 0} \int_{|x| \geq R} \rho^{\hbar}_{\mathrm{diag}}(x) \mathrm{d}x \to 0 \tag{2.41}$$

as $R \to \infty$. For reference of these two definitions consider [36].

Proposition 1. *Let $f^\hbar$ and $\tilde{f}^\hbar$ be the Wigner and Husimi functions of a density matrix $\rho^\hbar$. Denote by f and $\tilde{f}$ their respective limits as $\hbar \to 0$. Then*

$$f \equiv \tilde{f}.$$

In particular, f is a positive, bounded measure on $\mathbb{R}^d_x \times \mathbb{R}^d_\xi$. Moreover, if $\rho^\hbar$ is $\hbar$-oscillatory, then

$$\rho_{\mathrm{diag}}(x) = \int_{\mathbb{R}^3_\xi} f(x, \xi) \mathrm{d}\xi$$

Proof. Theorem III.1 and Theorem III.2 in [59] and Proposition 1.7 in [36]. □

A sufficient condition for (2.40) is provided by the following

Lemma 2.2. *Let $s > 0$ such that*

$$\hbar \operatorname{tr}\left(-\Delta \rho^\hbar\right) \leq C, \tag{2.42}$$

or equivalently,

$$\sum_{j=1}^{\infty} \lambda_j^\hbar \hbar \int |\nabla \Psi_j^\hbar|^2 \mathrm{d}x \leq C.$$

Then $\rho^\hbar$ is $\hbar$-oscillatory.

Proof. By Plancherel, (2.42) is equivalent to

$$\sum_{j=1}^{\infty} \lambda_j^\hbar \int \hbar^2 |\xi|^{2s} |\widehat{\Psi_j^\hbar}(\xi)|^2 \mathrm{d}\xi \leq C$$

It follows that

$$\int_{|\xi| > R/\hbar} \hat{\rho}^\hbar_{\mathrm{diag}}(\xi) \mathrm{d}\xi \leq C R^{-2},$$

Letting $R \to \infty$ proves the claim. □

3 The Pauli-Poisson equation: Wellposedness and semiclassical limit

This section is based on [71].

As shown in Lemma 2.1 the density matrix formulation is necessary because uniform L^2-estimates for the Wigner transform are only possible in this setting. Therefore we rewrite the Pauli-Poisson equation (1.60)-(1.63) in the mixed state formulation. The electric potential $V^\hbar$ depends on $\hbar$ since it is coupled to $\Psi^\hbar$ via (3.2). The magnetic potential A is externally given and does not depend on $\hbar$ which is why we write it without a superscript.

Let $\{\Psi_j\}_{j\in\mathbb{N}}$ be an orthonormal system in $(L^2(\mathbb{R}^3))^2$. Then we have the **mixed state Pauli-Poisson equation**

$$i\hbar\partial_t\Psi_j^\hbar = -\frac{1}{2}(\hbar\nabla - iA)^2\Psi_j^\hbar + V^\hbar\Psi_j^\hbar - \frac{1}{2}\hbar(\sigma\cdot B)\Psi_j^\hbar, \quad j = 1,\dots,\infty \tag{3.1}$$

$$-\Delta V^\hbar = \sum_{j=1}^\infty \lambda_j^\hbar|\Psi_j^\hbar|^2 = \rho_{\text{diag}}^\hbar, \tag{3.2}$$

$$\Psi_j^\hbar(x,0) = \Psi_{j,I}^\hbar(x) \in (L^2(\mathbb{R}^3))^2. \tag{3.3}$$

where the mixed state Pauli current density is given by

$$J^\hbar(\Psi^\hbar, A) = \sum_{j=1}^\infty \lambda_j^\hbar\left[\operatorname{Im}(\overline{\Psi_j^\hbar}(\hbar\nabla - iA)\Psi_j^\hbar) - \hbar\nabla\times(\overline{\Psi_j^\hbar}\sigma\Psi_j^\hbar)\right], \tag{3.4}$$

or equivalently

$$J^\hbar = \sum_{j=1}^\infty \lambda_j^\hbar \operatorname{Re}\left(\overline{\Psi_j^\hbar}\sigma(\sigma\cdot(-i\hbar\nabla - A))\Psi_j^\hbar\right), \tag{3.5}$$

It can be written as a first order moment of the Wigner matrix,

$$J^\hbar = \int_{\mathbb{R}^3_\xi} \operatorname{Tr}\Big(\sigma(\sigma\cdot(\xi - A)F^\hbar)\Big)\mathrm{d}\xi. \tag{3.6}$$

Similarly,

$$\rho_{\text{diag}}^\hbar = \int f^\hbar \mathrm{d}\xi. \tag{3.7}$$

The continuity equation holds for the mixed densities:

$$\partial_t\rho_{\text{diag}}^\hbar + \operatorname{div}_x J^\hbar = 0.$$

Let H be the Pauli Hamiltonian and H_0 the free Pauli Hamiltonian, i.e.

$$H = -\frac{1}{2}((\sigma\cdot(\hbar\nabla - iA))^2 + V^\hbar, \tag{3.8}$$

$$H_0 = -\frac{1}{2}((\sigma\cdot(\hbar\nabla - iA))^2. \tag{3.9}$$

Recall that in three space dimensions the Poisson equation for $V^\hbar$ is equivalent to

$$V^\hbar[\Psi^\hbar] = -\Delta^{-1}\rho^\hbar = \frac{1}{4\pi|x|} * \rho^\hbar = \frac{1}{4\pi|x|} * |\Psi^\hbar|^2. \tag{3.10}$$

Let us derive the Pauli-Wigner equation as described in Section 2.2. From the von Neumann-Liouville equation (2.28) we obtain that the matrix valued density matrix $R^\hbar$ obeys the following equation with $r = x + (\hbar y)/2$ and $s = x - (\hbar y)/2$:

$$\partial_t R^\hbar(r,s,t) = \frac{1}{i\hbar}(H_r - H_s^\dagger)R^\hbar(r,s,t), \tag{3.11}$$

where the subscript indicates that the Hamiltonian is taken with respect to the respective variable and $\dagger$ indicates the conjugate transpose. This means (note that $(\sigma\cdot B)^\dagger = (\sigma\cdot B)$),

$$\begin{aligned}\partial_t R^\hbar(r,s,t) =& \frac{i}{2}[\hbar(\Delta_r - \Delta_s) + 2(A(r)\nabla_r + A(s)\nabla_s) - \frac{i}{2\hbar}(A(r)^2 - A(s)^2) \\ &- \frac{i}{2}(\sigma\cdot B(r) - \sigma\cdot B(s)) - \frac{i}{\hbar}(V^\hbar(r,t) - V^\hbar(s,t))]R^\hbar(r,s,t).\end{aligned} \tag{3.12}$$

Now we set $T^\hbar(x,y,t) := R^\hbar(r,s,t)$. Then, using

$$\mathrm{div}_y(\nabla_x T^\hbar)(x,y,t) = \frac{1}{2}(\Delta_r - \Delta_s)R^\hbar(r,s,t),$$

and

$$\nabla_r = \frac{1}{2}\nabla_x + \frac{1}{\hbar}\nabla_y, \qquad \nabla_s = \frac{1}{2}\nabla_x - \frac{1}{\hbar}\nabla_y,$$

we obtain

$$\begin{aligned}\partial_t T^\hbar &= i\hbar\mathrm{div}_y(\nabla_x T^\hbar) + \beta[A]\cdot\nabla_x T^\hbar - i\delta[A]\cdot\nabla_y T^\hbar \\ &- \frac{i}{2}\delta[A^2]T^\hbar - \frac{i\hbar}{2}\delta[\sigma\cdot B]T^\hbar - i\delta[V^\hbar]T^\hbar.\end{aligned} \tag{3.13}$$

where β and δ are defined as

$$\beta[g] := \frac{1}{2}(g(x + \frac{\hbar y}{2}) + g(x - \frac{\hbar y}{2})), \tag{3.14}$$

and

$$\delta[g] := \frac{i}{\hbar}(g(x + \frac{\hbar y}{2}) - g(x - \frac{\hbar y}{2})). \tag{3.15}$$

Taking the Fourier transform in y for the first term on the RHS, using partial integration and setting $F^\hbar = \mathcal{F}[u^\hbar]$ yields:

$$i\hbar\mathcal{F}_y(\mathrm{div}_y(\nabla_x T^\hbar))(x,\xi,t) = -\xi\cdot\nabla_x F^\hbar(x,\xi,t).$$

Moreover,

$$\mathcal{F}_y[\beta[A]\cdot\nabla_x T^\hbar] = \mathcal{F}_y[\beta[A]] *_\xi \nabla_x F^\hbar,$$

and

$$-i\mathcal{F}_y[\delta[A]\cdot\nabla_y T^\hbar] = \mathcal{F}_y[\delta[A]] *_\xi (\xi F^\hbar).$$

Moreover,

$$\mathcal{F}_y[\delta[\cdot]T^\hbar] = -(\theta[\cdot]F^\hbar),$$

where

$$(\theta[\cdot]F^\hbar)(x,\xi,t) := \frac{1}{(2\pi)^3}\int_{\mathbb{R}^6} \delta[\cdot](x,y,t)F^\hbar(x,\eta,t)e^{-i(\xi-\eta)\cdot y}\mathrm{d}\eta\mathrm{d}y. \tag{3.16}$$

Then we obtain the *Pauli-Wigner-Poisson equation* for $F^\hbar$,

$$\begin{aligned}\partial_t F^\hbar + \xi\cdot\nabla_x F^\hbar - \mathcal{F}_y[\beta[A]] *_\xi \nabla_x F^\hbar - \theta[A](\xi F^\hbar) + \frac{1}{2}\theta[|A|^2]F^\hbar \\ -\frac{\hbar}{2}\theta[\sigma\cdot B]F^\hbar + \theta[V^\hbar]F^\hbar = 0,\end{aligned} \tag{3.17}$$

$$-\Delta V^\hbar = \rho^\hbar_{\text{diag}} = \int_{\mathbb{R}^3_\xi} f^\hbar \mathrm{d}\xi, \tag{3.18}$$

$$F^\hbar(t=0) = F^\hbar_I, \tag{3.19}$$

The initial Wigner matrix $F^\hbar_I$ is the Wigner transform of the initial matrix valued density matrix $R^\hbar_I$, i.e.

$$F^\hbar_I = \frac{1}{(2\pi\hbar)^3}\int_{\mathbb{R}^3} e^{-i\xi\cdot y}R^\hbar_I(x+\frac{\hbar y}{2}, x-\frac{\hbar y}{2})\mathrm{d}y, \tag{3.20}$$

$$R^\hbar_I(x,y) = \sum_{j=1}^\infty \lambda^\hbar_j \Psi^\hbar_{j,I}(x)\otimes\overline{\Psi^\hbar_{j,I}(y)}. \tag{3.21}$$

We will always assume that $F^\hbar_I$ converges (after extraction of a subsequence) weakly* in $(\mathcal{S}'(\mathbb{R}^6))^2$ to a positive (or null) and bounded measure f_I. The energy of the Pauli-Wigner equation is given by

$$E(t) := \operatorname{tr}\Big(H_0 R^\hbar(t)\Big) + \iint_{\mathbb{R}^3_x\times\mathbb{R}^3_y} \frac{\rho^\hbar_{\text{diag}}(x,t)\rho^\hbar_{\text{diag}}(y,t)}{|x-y|}\mathrm{d}x\mathrm{d}y. \tag{3.22}$$

In fact the second term on the RHS is equivalent to

$$\int |\nabla V^\hbar|^2 \mathrm{d}x.$$

3.1 Main result

In the following we will give a rigorous proof by a nontrivial extension of the Wigner measure analysis in [36],[59],[62] from the scalar case of a simple Schrödinger equation to the 2-spinor case of the Pauli equation.

As outlined in Section 2.2 and Lemma 2.1 the uniform boundedness in L^2 of the Wigner transform follows from a bound on the ℓ^2 norm of the sequence $\{\lambda^\hbar\}$ of occupation probabilities. Therefore we make the following assumption.

Assumption 1. *Let $R^\hbar$ or $\rho^\hbar$ be a matrix valued density matrix or density matrix defined by an orthonormal system $\{\Psi_j^\hbar\} \subset (L^2(\mathbb{R}^3))^2$ and occupation probabilities $\lambda_j^\hbar \in [0,1]$. We assume that*

$$\lambda_j^\hbar \geq 0, \quad \sum_{j=1}^{\infty} \lambda_j^\hbar = 1, \tag{3.23}$$

$$\|\lambda^\hbar\|_2^2 = \sum_{j=1}^{\infty} (\lambda_j^\hbar)^2 \leq (2\pi\hbar)^3. \tag{3.24}$$

Theorem 1. *Let $\{\Psi_j^\hbar\}_{j\in\mathbb{N}} \in C(\mathbb{R}_t, (L^2(\mathbb{R}_x^3))^2)$ be a solution of the mixed state Pauli-Poisson equation (3.1)-(3.3) with associated matrix valued density matrix $R^\hbar$ such that the occupation probabilities satisfy Assumption 1. Let $F^\hbar$ be the associated Wigner matrix solving the Pauli-Wigner equation (3.17) with initial data $F_I^\hbar(x,p) = F^\hbar(x,p,0)$. Assume that $F_I^\hbar$ converges up to a subsequence in $\mathcal{S}'(\mathbb{R}_x^3 \times \mathbb{R}_p^3)^{2\times 2}$ to a nonnegative matrix-valued Radon measure F_I. Let $p = \xi - A$.*

(i) Let $A, V \in C^1(\mathbb{R}^3)$ such that $B := \nabla \times A \in C(\mathbb{R}^3)$. Then $F^\hbar$ converges weakly up to a subsequence in $(\mathcal{S}')^{2\times 2}$ to $F \in C_b(\mathbb{R}_t, \mathcal{M}_{w*}^{2\times 2})$ such that F solves the Vlasov-Poisson equation with Lorentz force*

$$\partial_t F + p \cdot \nabla_x F + (-\nabla_x V + p \times B) \cdot \nabla_p F = 0, \tag{3.25}$$

in $(\mathcal{D}')^{2\times 2}$ verifying the initial condition

$$F(x,p,0) = F_I(x,p) \quad \text{in } \mathbb{R}_x^3 \times \mathbb{R}_p^3. \tag{3.26}$$

(ii) Let $V^\hbar$ be given by $-\Delta V^\hbar = \rho_{\text{diag}}^\hbar$ and suppose $A \in W^{1,\frac{7}{2}}(\mathbb{R}^3)$. Moreover, suppose that $\{F_I^\hbar\}$ is a bounded sequence in $(L^2(\mathbb{R}^6))^{2\times 2}$ and that the initial energy $E(0)$ is bounded independently of $\hbar$. Then $F^\hbar$ converges weakly up to a subsequence in $L^\infty(I, L^2(\mathbb{R}_x^3 \times \mathbb{R}_p^3)^{2\times 2})$ to*

$$F \in C_b(\mathbb{R}_t, \mathcal{M}_{w*}^{2\times 2}) \cap L^\infty(I, L^1 \cap L^2(\mathbb{R}_x^3 \times \mathbb{R}_p^3)^{2\times 2}),$$

such that F solves

$$\partial_t F + p \cdot \nabla_x F + (-\nabla_x V + p \times B) \cdot \nabla_p F = 0, \tag{3.27}$$

in $(\mathcal{D}')^{2\times 2}$ and

$$-\Delta V = \rho_{\text{diag}}(x), \quad \rho_{\text{diag}}(x) = \int_{\mathbb{R}_\xi^3} f(x,p)\mathrm{d}p, \tag{3.28}$$

where $f = \mathrm{Tr}(F)$, verifying the initial condition

$$F(x,p,0) = F_I(x,p) \quad \text{in } \mathbb{R}_x^3 \times \mathbb{R}_p^3. \tag{3.29}$$

(iii) Let $A \in W^{1,\frac{7}{2}}(\mathbb{R}^3)$. The mixed state Pauli current density $J^\hbar$ defined by

$$J^\hbar = \sum_{j=1}^{\infty} \lambda_j^\hbar \left[\mathrm{Im}(\overline{\Psi_j^\hbar}(\hbar\nabla - iA)\Psi_j^\hbar) - \hbar\nabla \times (\overline{\Psi_j^\hbar}\sigma\Psi_j^\hbar) \right], \tag{3.30}$$

converges in $\mathcal{D}'$ to

$$J = \int_{\mathbb{R}^3_p} pf \mathrm{d}p. \tag{3.31}$$

We will prove Theorem 1 in Section 3.3. We also prove the following global wellposedness result for the mixed state Pauli-Poisson equation (3.1)-(3.3)in Section 3.2. Here, $\underline{\Psi} := \{\Psi_j\}_{j\in\mathbb{N}}$ and $\underline{\Psi}_I = \{\Psi_{j,I}\}_{j\in\mathbb{N}}$. The energy space $\mathcal{H}^1(\mathbb{R}^3)$ will be defined in Section 3.2.

Theorem 2. *Let $A \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$, $|B| \in L^2(\mathbb{R}^3)$. For any $\Psi_I \in \mathcal{H}^1(\mathbb{R}^3)$ there exists a unique solution to the initial value problem (3.1)-(3.3) in $C(\mathbb{R}, \mathcal{H}^1(\mathbb{R}^3)) \cap C^1(\mathbb{R}, \mathcal{H}^1(\mathbb{R}^3)^*)$. In particular, the solution is in $(L^2(\mathbb{R}^3))^2$ for all times. If $\underline{\Psi}_{n,I}, \underline{\Psi}_I \in \mathcal{H}^1(\mathbb{R}^3)$ are initial data satisfying $\underline{\Psi}_{n,I} \to \underline{\Psi}_I$ in $\mathcal{H}^1(\mathbb{R}^3)$ with corresponding unique solutions $\underline{\Psi}_n \in C(\mathbb{R}, \mathcal{H}^1) \cap C^1(\mathbb{R}, \mathcal{H}^{1*})$ and $\underline{\Psi} \in C(\mathbb{R}, \mathcal{H}^1) \cap C^1(\mathbb{R}, \mathcal{H}^{1*})$ then $\underline{\Psi}_n \to \underline{\Psi}$ in $L^\infty(\mathbb{R}, \mathcal{H}^1(\mathbb{R}^3))$.*

One can generalize the global wellposedness of the Pauli-Poisson equation to the Pauli-Hartree equation (1.64) with slight changes in the proof (cf. [70]).

Corollary 1. *Under the assumptions of Theorem 2 and assuming that W is even, $W \in L^{r_1}(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$ for $3/2 \le r_1 \le \infty$ and $\nabla W \in L^{r_2} + L^\infty$ for $1 \le r_2 \le \infty$ the Pauli-Hartree equation (1.64) equation is globally wellposed in $\mathcal{H}^1(\mathbb{R}^3)$.*

Remark 4. The question arises whether the Pauli-Hartree equation can be posed in arbitrary space dimensions. The three dimensional magnetic field $B = \nabla \times A$ has to be replaced by its d-dimensional generalization $\nabla \wedge A$. In this case, following the result for the M-SH equation [70], the conditions for W would be: W even, $W \in L^{r_1}(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$ for $\max\{1, d/2\} \le r_1 \le \infty$ ($r_1 > 1$ if $d = 2$) and $\nabla W \in L^{r_2} + L^\infty$ for $\max\{1, d/3\} \le r_2 \le \infty$.

In the case of the linear Pauli equation with sufficiently smooth external electromagnetic potentials we can use Theorem 6.1 from [36] in order to heuristically determine the semiclassical limit equation of the Pauli-Wigner-Poisson equation (3.17)-(3.19). Write the Pauli equation as

$$\hbar\partial_t \Psi^\hbar + P(x, \hbar D)\Psi^\hbar = 0, \quad x \in \mathbb{R}^3_x, t \in \mathbb{R} \tag{3.32}$$

$$\Psi^\hbar(x, 0) = \Psi_I^\hbar(x) \in (L^2(\mathbb{R}^3))^2. \tag{3.33}$$

where $P(x, \hbar D)$ is the pseudo-differential operator with matrix-valued symbol given by

$$P(x, \xi) := \frac{i}{2}(\sigma \cdot (\xi - A(x)))^2 + iV\mathrm{Id}.$$

The 2×2-matrix $-iP(x,\xi)$ has one eigenvalue with multiplicity 2, given by

$$\lambda(x,\xi) = \frac{1}{2}\|\xi - A(x)\|^2 + V(x), \tag{3.34}$$

where $\|\xi - A\|^2 = \sum_{j=1}^{3}(\xi_j - A_j)^2$ and we can deduce that $F^\hbar$ defined in (2.29) converges to F solving

$$\partial_t F + \{\lambda, F\} = 0. \tag{3.35}$$

where $\{h,g\} = \nabla_\xi h \cdot \nabla_x g - \nabla_x h \cdot \nabla_\xi g$ denotes the Poisson bracket. Using (3.34) we have

$$\partial_t F + \xi \cdot \nabla_x F + (\nabla_x A)\xi \cdot \nabla_\xi F - (\nabla_x A)A \cdot \nabla_\xi F - A \cdot \nabla_x F - \nabla_x V(x) \cdot \nabla_\xi F = 0 \tag{3.36}$$

We easily obtain after a change of variables $p = \xi - A$ and by using $B = \nabla \times A$,

$$\partial_t F + p \cdot \nabla_x F + (-\nabla_x V + p \times B)\nabla_p F = 0, \tag{3.37}$$

which is the Vlasov equation with Lorentz force for an electron (1.55). Note that for an electron with negative energy (a positron) there would be a minus sign in front of the Lorentz force term.

Remark 5. For the linear Dirac equation it was shown by Spohn in [93] and in [36] that in the semiclassical limit $\hbar \to 0$ the Wigner matrix measure F for the electron component obeys

$$\partial_t F + \{\lambda_-, F\} + i[H_s^{(-)}, F] = 0$$

where

$$H_s^{(-)} := -i[\Pi_-, \{\lambda_-, \Pi_-\}] - \frac{i}{2}\Pi_- \left(\lambda_-\{\Pi_-, \Pi_-\} - \lambda_+\{\Pi_+, \Pi_+\}\right)\Pi_- \tag{3.38}$$

where $\lambda_\pm$ are the two degenerate eigenvalues of the symbol matrix of the Dirac operator and $\Pi_\pm$ the respective projections on the eigenspaces. This is in fact a special case of a theorem from [36] where more general pseudodifferential operators are considered. In [93], $H_s^{(-)}$ is called *spinor Hamiltonian* and it depends on the positron component as well which can be seen from the last term in (3.38). For the linear Pauli equation the situation is different: There is one degenerate eigenvalue (3.34) whose eigenspace coincides with the whole space and the projection on the eigenspace is the identity Id. In this case, H_s vanishes since

$$H_s = -i[\mathrm{Id}, \{\lambda, \mathrm{Id}\}] = 0.$$

Therefore the Wigner matrix equation reduces to

$$\partial_t F + (\nabla_\xi \lambda \cdot \nabla_x F - \nabla_x \lambda \cdot \nabla_\xi F) = 0$$

Also, $R^\hbar_{\mathrm{diag}}$ converges to

$$R_{\mathrm{diag}} = \int_{\mathbb{R}^3_\xi} F \mathrm{d}\xi.$$

The spinor u decouples into two independent Vlasov equations represented by the diagonal elements of the Wigner matrix F. This is consistent with the fact that the Stern-Gerlach term (1.27) which couples the spin components ψ_1 and ψ_2 scales with $\hbar$ and vanishes in the semiclassical limit.

3.2 Global wellposedness of the Pauli-Poisson equation

In this chapter we will prove a global wellposedness result for the mixed state Pauli-Poisson equation, given by

$$i\partial_t\Psi_j = -\frac{1}{2}(\nabla - iA)^2\Psi_j + V\Psi_j + \frac{1}{2}(\sigma\cdot B)\Psi_j, \quad j=1,\dots,\infty \tag{3.39}$$

$$-\Delta V = \rho_{\text{diag}} := -\sum_{j=1}^{\infty}\lambda_j|\Psi_j|^2 \tag{3.40}$$

$$\Psi_j(x,0) = \Psi_{j,I}(x) \in (L^2(\mathbb{R}^3))^2, \tag{3.41}$$

where $\{\Psi_j\}_{j\in\mathbb{N}}$ is an orthonormal system in $(L^2(\mathbb{R}^3))^2$ and $\{\lambda\}_{j\in\mathbb{N}}$ is a sequence satisfying $\lambda_j \geq 0$ and

$$\|\lambda\|_1 := \sum_{j=1}^{\infty}\lambda_j = 1. \tag{3.42}$$

Note that in this section we omit the superscripts. The global wellposedness in the energy space for the magnetic Schrödinger equation with Hartree nonlinearity $W * |\Psi^\hbar|^2$ for the pure state case (1.70)-(1.71), was proven in [70]. The method is to establish an appropriate energy space where the magnetic Laplacian defines a self-adjoint operator. Then one shows that the Hartree nonlinearity is Lipschitz in the energy space and finally one uses energy conservation to extend the solution globally. In [70], the magnetic potential A is assumed to be in $L^2_{\text{loc}}(\mathbb{R}^3)$ which is sufficient for the magnetic Laplacian Δ_A to be self-adjoint on $L^2(\mathbb{R}^3)$ due to a theorem by Leinfelder and Simader, cf. [58]. For the Pauli-Poisson equation we will treat the Stern-Gerlach term as a perturbation of the magnetic Laplacian as in [4].

The result in [70] is established for pure states. In [7] wellposedness is proved for mixed states but only for bounded magnetic fields. Global wellposedness in H^2 of the Schrödinger-Poisson equation without magnetic field for mixed states was obtained in [20],[54] and in L^2 in [22]. We will prove global wellposedness in the energy space for the Pauli-Poisson equation with rougher magnetic potential than in [7] for mixed states, thereby extending the results from [7], [70]. We will use the notation

$$\underline{\Psi} := \{\Psi_j\}_{j\in\mathbb{N}}, \quad \underline{\Psi}_I = \{\Psi_{j,I}\}_{j\in\mathbb{N}}$$

Moreover, we have

$$|\underline{\Psi}| = \sum_{j=1}^{\infty}\lambda_j|\Psi_j| \qquad\qquad |\underline{\Psi}|^2 \equiv \rho_{\text{diag}} = \sum_{j=1}^{\infty}\lambda_j|\Psi_j|^2 \tag{3.43}$$

and denote the mixed state analogues to $(L^p(\mathbb{R}^3))^2$, $(H^s(\mathbb{R}^3))^2$ and $(H^1_A(\mathbb{R}^3))^2$ (similary to [20],[22],[54]) by

$$\begin{aligned}\mathbf{L}^p &:= \{\underline{\Psi}\colon \Psi_j \in (L^p(\mathbb{R}^3))^2\,\forall j \in \mathbb{N};\ \|\underline{\Psi}\|_p^2 := \sum \lambda_j \|\Psi_j\|_p^2 < \infty\},\\ \mathbf{H}^s &:= \{\underline{\Psi}\colon \Psi_j \in (H^s(\mathbb{R}^3))^2\,\forall j \in \mathbb{N};\ \|\underline{\Psi}\|_{H^s}^2 := \sum \lambda_j \|\Psi_j\|_{H^s}^2 < \infty\},\\ \mathbf{H}^1_A &:= \{\underline{\Psi}\colon \Psi_j \in (H^1_A(\mathbb{R}^3))^2\,\forall j \in \mathbb{N};\ \|\underline{\Psi}\|_{H^1_A}^2 := \sum \lambda_j \|\Psi_j\|_{H^1_A}^2 < \infty\}.\end{aligned}$$

where H^1_A is defined as

$$H^1_A = \{f\colon \|f\|_{H^1_A}^2 := \|(\nabla - iA)f\|_2^2 + \|f\|_2^2 < \infty\} \tag{3.44}$$

for given A. The usual inequalities also hold for mixed state spaces, e.g. the Hölder inequality, cf. [22]:

$$\|\underline{\Psi\Phi}\|_r \le \|\underline{\Psi}\|_p \|\underline{\Phi}\|_q,$$

for $1/r = 1/p + 1/q$ and $\underline{\Psi} \in \mathbf{L}^p, \underline{\Phi} \in \mathbf{L}^q$. But also the Sobolev, Young and Hardy-Littlewood-Sobolev inequalities can be easily generalized to mixed state spaces. When we write $\underline{\Psi} \in C_0^\infty$ we mean that $\Psi_j \in (C_0^\infty)^2$ for all $j \in \mathbb{N}$. Operators act component-wise, e.g.

$$\nabla\underline{\Psi} = \{\nabla\Psi_j\}_{j\in\mathbb{N}}. \tag{3.45}$$

We now adapt the proof from [70] for the Pauli-Poisson equation. We define the energy space $\mathcal{H}^1$ for the Pauli equation and show that $|x|^{-1} * |\underline{\Psi}|^2$ is Lipschitz in $\mathcal{H}^1$. After having obtained a local solution by a fixed point argument we extend the local solution globally using energy conservation. The following lemma is based on [4].

Lemma 3.1. *Let $A \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$, $|B| \in L^2(\mathbb{R}^3)$. Then the sesquilinear form*

$$h_0(f,g) := \langle H_0 f, g\rangle, \quad f,g \in (C_0^\infty(\mathbb{R}^3))^2, \tag{3.46}$$

where H_0 is the linear Pauli Hamiltonian defined by (3.9) is symmetric, closable and non-negative in $(L^2(\mathbb{R}^3))^2$. The associated self-adjoint operator H_0 has form domain $\mathfrak{Q}(H_0)$ equal to the completion of $(C_0^\infty(\mathbb{R}^3))^2$ with respect to the norm $\|f\|_{H^1_A}^2 := \|(\nabla - iA)f\|^2 + \|f\|^2$.

Proof. Let $\epsilon > 0$ and write $|B| = B_1 + B_2$ with $\|B_1\|_{L^2} < \epsilon$ and $\|B_2\|_{L^\infty} < C_\epsilon$ where C_ϵ is a constant depending on ϵ. Then

$$\langle H_0 f, f\rangle = \langle -(\nabla - iA)^2 f, f\rangle + \langle (\sigma\cdot B)f, f\rangle.$$

Moreover,

$$\begin{aligned}|\langle(\sigma\cdot B)f,f\rangle| &\le \langle B_1 f, f\rangle + \langle B_1 f, f\rangle\\ &\le \|B_1\|_2\|f\|_4^2 + C_\epsilon\|f\|_2^2\\ &\le \tilde{C}\epsilon\|\nabla|f|\|_2^2 + C_\epsilon\|f\|_2^2\\ &\le \tilde{C}\epsilon\|(\nabla - iA)f\|_2^2 + C_\epsilon\|f\|_2^2\end{aligned}$$

The third inequality follows from the Sobolev inequality and the last inequality follows from the diamagnetic inequality. It is known (cf. [58]) that for $A \in L^2_{\text{loc}}$, $-(\nabla - iA)^2$ defines a non-negative self-adoint operator with form domain $\mathfrak{Q}(H_0)$. Hence $\sigma \cdot B$ defines a relatively form-bounded perturbation of the form associated to $-(\nabla - iA)^2$ and by the KLMN theorem on relatively form bounded perturbations of closed forms, h_0 defines a closable non-negative form with form domain $\mathfrak{Q}(H_0)$. □

Definition 1 (Energy space)**.** Let $A \in L^2_{\text{loc}}(\mathbb{R}^3)$, $|B| \in L^2(\mathbb{R}^3)$. The energy space for the Pauli equation with magnetic potential A and magnetic field B is given by

$$\mathcal{H}^1(\mathbb{R}^3) := \{\underline{\Psi} \in \mathbf{L}^2 \colon (\nabla - iA)\underline{\Psi} \in \mathbf{L}^2, (\sigma \cdot B)_+^{1/2}\underline{\Psi} \in \mathbf{L}^2\}$$

with associated norm

$$\|\underline{\Psi}\|^2_{\mathcal{H}^1} := \|(\nabla - iA)\underline{\Psi}\|_2^2 + \|(\sigma \cdot B)_+^{1/2}\underline{\Psi}\|_2^2 + \|\underline{\Psi}\|_2^2.$$

Lemma 3.2 (Properties of $\mathcal{H}^1(\mathbb{R}^3)$ and H_0)**.** *Let $A \in L^2_{\text{loc}}(\mathbb{R}^3)$, $|B| \in L^2(\mathbb{R}^3)$. Then $\mathcal{H}^1(\mathbb{R}^3)$ is a Hilbert space, complete with respect to $\|\cdot\|_{\mathcal{H}^1}$. The norm $\|\cdot\|_{\mathcal{H}^1}$ is equivalent to $\|\cdot\|_{\mathbf{H}^1_A}$. Moreover, H_0 is unitary on $\mathcal{H}^1(\mathbb{R}^3)$, i.e.*

$$\|e^{-itH_0}\underline{\Psi}\|_{\mathcal{H}^1} = \|\underline{\Psi}\|_{\mathcal{H}^1},$$

for all $f \in \mathcal{H}^1$ and for all $t \in \mathbb{R}$.

Proof. The norms are equivalent because

$$\begin{aligned}
\|\underline{\Psi}\|^2_{\mathcal{H}^1} &\leq \|\nabla_A\underline{\Psi}\|_2^2 + \|(\sigma \cdot B)^{1/2}\underline{\Psi}\|_2^2 + \|\underline{\Psi}\|_2^2 \\
&\lesssim (1+a)\|\nabla_A\underline{\Psi}\|_2^2 + (1+b)\|\underline{\Psi}\|_2^2
\end{aligned}$$

where $a \in (0,1)$ and $b \geq 0$ are the constants from Lemma 3.1 and

$$\begin{aligned}
\|\underline{\Psi}\|^2_{H^1_A} &\leq \|\nabla_A\underline{\Psi}\|_2^2 + \|\underline{\Psi}\|_2^2 \\
&\leq \|\nabla_A\underline{\Psi}\|_2^2 + \|(\sigma \cdot B)^{1/2}\underline{\Psi}\|_2^2 + \|\underline{\Psi}\|_2^2.
\end{aligned}$$

Unitarity follows from the fact that H_0 is self-adjoint on $\mathcal{H}^1(\mathbb{R}^3)$. □

Lemma 3.3 (Local Lipschitz property)**.** *Let $A \in L^2_{\text{loc}}(\mathbb{R}^3)$, $|B| \in L^2(\mathbb{R}^3)$. Moreover, let $\mathcal{H}^1(\mathbb{R}^3)$ be the energy space from Definition 1. Then for any $\underline{\Psi}, \underline{\Phi} \in \mathcal{H}^1(\mathbb{R}^3)$, the nonlinearity $(|x|^{-1} * |\underline{\Psi}|^2)\underline{\Psi}$ is locally Lipschitz, i.e.*

$$\|(\frac{1}{|x|} * |\underline{\Psi}|^2)\underline{\Psi} - (\frac{1}{|x|} * |\underline{\Phi}|^2)\underline{\Phi}\|_{\mathcal{H}^1} \leq C(\|\underline{\Psi}\|^2_{\mathcal{H}^1} + \|\underline{\Phi}\|^2_{\mathcal{H}^1})\|\underline{\Psi} - \underline{\Phi}\|_{\mathcal{H}^1}, \tag{3.47}$$

where C is independent of $\underline{\Psi}$ and $\underline{\Phi}$.

Proof. Assume that $\underline{\Psi}, \underline{\Phi} \in C_0^\infty$ first. Rewrite the magnetic derivative part of the $\mathcal{H}^1$ norm as follows:

$$\begin{aligned}\|\nabla_A(\frac{1}{|x|} * |\underline{\Psi}|^2)\underline{\Psi} - (\frac{1}{|x|} * |\underline{\Phi}|^2)\underline{\Phi}\|_2 \leq \|(\frac{1}{|x|} &* \nabla(|\underline{\Psi}|^2 - |\underline{\Phi}|^2))\underline{\Psi}\|_2 \\ + \|(\frac{1}{|x|} &* \nabla|\underline{\Phi}|^2)(\underline{\Psi} - \underline{\Phi})\|_2 \\ + \|(\frac{1}{|x|} &* (|\underline{\Psi}|^2 - |\underline{\Phi}|^2))\nabla_A\underline{\Psi}\|_2 \\ + \|(\frac{1}{|x|} &* |\underline{\Phi}|^2)\nabla_A(\underline{\Psi} - \underline{\Phi})\|_2,\end{aligned} \tag{3.48}$$

since $\nabla_A(fg) = f\nabla_A g + (\nabla f)g$. Notice that

$$\left\|(\frac{1}{|x|} * (fh))g\right\|_2 \lesssim \|fh\|_p^2\|g\|_q, \tag{3.49}$$

where $2/p + 1/q = 7/6$ due to the Hölder and Hardy-Littlewood-Sobolev inequalities. Choosing $p = 2$ and $q = 6$ and using the Sobolev and diamagnetic inequalities yields

$$\left\|(\frac{1}{|x|} * (fh))g\right\|_2 \lesssim \|f\|_2\|h\|_2\|\nabla_A g\|_2. \tag{3.50}$$

It follows that:

$$\left\|(\frac{1}{|x|} * \nabla|f|^2)g\right\|_2 \lesssim \||f|\|_2\|\nabla|f|\|_2\|g\|_6 \lesssim \|f\|_2\|\nabla_A f\|_2\|\nabla_A g\|_2 \tag{3.51}$$

We now apply theses inequality to the terms in (3.48). The first term yields

$$\begin{aligned}\|(\frac{1}{|x|} * \nabla(|\underline{\Psi}|^2 - |\underline{\Phi}|^2))\underline{\Psi}\|_2 \lesssim \|\underline{\Psi}\|_2\|\nabla_A(\underline{\Psi} - \underline{\Phi})\|_2\|\nabla_A\underline{\Psi}\|_2 \\ + \|\underline{\Psi} - \underline{\Phi}\|_2\|\nabla_A\underline{\Phi}\|_2\|\nabla_A\underline{\Psi}\|_2.\end{aligned} \tag{3.52}$$

For the second term we obtain similarly,

$$\|(\frac{1}{|x|} * \nabla|\underline{\Phi}|^2)(\underline{\Psi} - \underline{\Phi})\|_2 \leq \|\underline{\Phi}\|_2\|\nabla_A\underline{\Phi}\|_2\|\nabla_A(\underline{\Psi} - \underline{\Phi})\|_2. \tag{3.53}$$

For the third term we have using Young's inequality and the fact that $|x|^{-1} \in L^{3/2} + L^\infty$

$$\begin{aligned}\|(\frac{1}{|x|} * (|\underline{\Psi}|^2 - |\underline{\Phi}|^2))\nabla_A\underline{\Psi}\|_2 \lesssim & \||\underline{\Psi}|^2 - |\underline{\Phi}|^2\|_3\|\nabla_A\underline{\Psi}\|_2 + \||\underline{\Psi}|^2 - |\underline{\Phi}|^2\|_1\|\nabla_A\underline{\Psi}\|_2 \\ \leq & \|\underline{\Psi} - \underline{\Phi}\|_6(\|\underline{\Psi}\|_6 + \|\underline{\Phi}\|_6)\|\nabla_A\underline{\Psi}\|_2 \\ & + \|\underline{\Psi} - \underline{\Phi}\|_2(\|\underline{\Psi}\|_2 + \|\underline{\Phi}\|_2)\|\nabla_A\underline{\Psi}\|_2 \\ \leq & \|\nabla_A(\underline{\Psi} - \underline{\Phi})\|_2(\|\nabla_A\underline{\Psi}\|_2 + \|\nabla_A\underline{\Phi}\|_2)\|\nabla_A\underline{\Psi}\|_2 \\ & + \|\underline{\Psi} - \underline{\Phi}\|_2(\|\underline{\Psi}\|_2 + \|\underline{\Phi}\|_2)\|\nabla_A\underline{\Psi}\|_2.\end{aligned} \tag{3.54}$$

For the fourth term we argue similarly and obtain

$$\|(\frac{1}{|x|} * |\underline{\Phi}|^2)\nabla_A(\underline{\Psi} - \underline{\Phi})\|_2 \lesssim \|\nabla_A \underline{\Phi}\|_2 \|\nabla_A(\underline{\Psi} - \underline{\Phi})\|_2 + \|\underline{\Phi}\|_2 \|\nabla_A(\underline{\Psi} - \underline{\Phi})\|_2. \tag{3.55}$$

Finally, we have the spin term which is estimated in the same way as the third term above.

$$\begin{aligned}\|(\frac{1}{|x|} * (|\underline{\Psi}|^2 - |\underline{\Phi}|^2))(\sigma \cdot B)\underline{\Psi}\|_2 \lesssim &\|\nabla_A(\underline{\Psi} - \underline{\Phi})\|_2 (\|\nabla_A \underline{\Psi}\|_2 + \|\nabla_A \underline{\Phi}\|_2)\|(\sigma \cdot B)\underline{\Psi}\|_2 \\ &+ \|\underline{\Psi} - \underline{\Phi}\|_2 (\|\underline{\Psi}\|_2 + \|\underline{\Phi}\|_2)\|(\sigma \cdot B)\underline{\Psi}\|_2.\end{aligned} \tag{3.56}$$

From (3.52)-(3.56) we deduce (3.47) for $\underline{\Psi}, \underline{\Phi} \in C_0^\infty$. It remains to lift the restriction. Consider two sequences $\{\underline{\Psi}_n\}, \{\underline{\Phi}_n\} \subset C_0^\infty$ converging to $\underline{\Psi}, \underline{\Phi}$ in $\mathcal{H}^1$ respectively. It is then easy to see that due to the Lipschitz property of the nonlinearity, $(|x|^{-1} * |\underline{\Psi}_n|^2)\underline{\Psi}_n$ and $(|x|^{-1} * |\underline{\Phi}_n|^2)\underline{\Phi}_n$ are Cauchy sequences in $\mathcal{H}^1$ with respective limits $\mathbf{U}$ and $\mathbf{V}$. Since $\|f\|_2 \leq C\|f\|_{\mathcal{H}^1}$ the convergence is also in L^2 and we can deduce that the convergence to $\mathbf{U}$ and $\mathbf{V}$ is pointwise a.e. Then one concludes that $\mathbf{U} = (|x|^{-1} * |\underline{\Psi}|^2)\underline{\Psi}$ and similarly for $\mathbf{V}$. Taking limits allows to conclude that the Lipschitz property holds for all $\underline{\Psi}, \underline{\Phi} \in \mathcal{H}^1$. □

Define the Banach space

$$X_T = \{\underline{\Psi} \in C(I, \mathcal{H}^1(\mathbb{R}^3)) \colon \|\underline{\Psi}\|_{L^\infty(I, \mathcal{H}^1(\mathbb{R}^3))} \leq M;\ \underline{\Psi}(0, x) = \underline{\Psi}_I(x)\} \tag{3.57}$$

with the norm

$$\|\underline{\Psi}\|_{X_T} := \|\underline{\Psi}\|_{L^\infty(I, \mathcal{H}^1(\mathbb{R}^3))} = \sup_{|t| \leq T} (\|\underline{\Psi}\|_2 + \|H_0 \underline{\Psi}\|_2). \tag{3.58}$$

For every $\underline{\Psi} \in X_T$ and every $t \in I$ we introduce the map $\underline{\Psi} \mapsto \Upsilon(\underline{\Psi})$ defined by

$$\Upsilon(\underline{\Psi}) := e^{itH_0}\underline{\Psi}_0 - i\int_0^t e^{i(t-s)H_0}(\frac{1}{|x|} * |\underline{\Psi}(s)|^2)\underline{\Psi}(s)\mathrm{d}s. \tag{3.59}$$

We will show that Υ is a contraction in X_T and conclude by a fixed point argument.

Lemma 3.4 (Local wellposedness)**.** *Let $A \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$, $|B| \in L^2(\mathbb{R}^3)$. Let $\underline{\Psi}_I \in \mathcal{H}^1(\mathbb{R}^3)$. For every $M > 0$ there is a time $T(M) > 0$ such that for all $\underline{\Psi}_I$ with $\|\underline{\Psi}_I\|_{\mathcal{H}^1} \leq M$ there is a unique $\underline{\Psi} \in C([-T(M), T(M)], \mathcal{H}^1(\mathbb{R}^3))$ solving (3.39)-(3.41). In fact, there exist $T_{\min}, T_{\max} > 0$ such that there is a unique solution $\underline{\Psi} \in C([-T_{\min}, T_{\max}], \mathcal{H}^1(\mathbb{R}^3)) \cap C^1([-T_{\min}, T_{\max}], \mathcal{H}^1(\mathbb{R}^3)^*)$. Moreover, the blow-up alternative holds: If $T_{\max} < \infty$ then $\|\underline{\Psi}(t)\|_{\mathcal{H}^1} \to \infty$ as $t \uparrow T_{\max}$ (resp. as $t \downarrow -T_{\min}$ if $T_{\min} < \infty$).*

Proof. Let us show that Υ maps X_T into X_T for the right choices of $T = T(\overline{M})$ and $\overline{M}$.

Observe that for $t \in [0, T(\overline{M}))$

$$\begin{aligned}
\|\Upsilon(\underline{\Psi})\|_{\mathcal{H}^1} &\leq \|e^{itH_0}\underline{\Psi}_I\|_{\mathcal{H}^1} + \int_0^t \|e^{i(t-s)H_0}(\frac{1}{|x|} * |\underline{\Psi}(s)|^2)\underline{\Psi}(s)\|_{\mathcal{H}^1}\mathrm{d}s \\
&= \|\underline{\Psi}_I\|_{\mathcal{H}^1} + \int_0^t \|(\frac{1}{|x|} * |\underline{\Psi}(s)|^2)\underline{\Psi}(s)\|_{\mathcal{H}^1}\mathrm{d}s \\
&\lesssim \|\underline{\Psi}_I\|_{\mathcal{H}^1} + \|\underline{\Psi}\|_{X_T}^3 T(\overline{M}) \\
&\leq \overline{M} + (2\overline{M})^3 T(\overline{M}).
\end{aligned}$$

where we have used unitarity of e^{itH_0} on $\mathcal{H}^1$ and the local Lipschitz property. Finally, choosing $T(\overline{M}) \simeq 1/(16\overline{M}^2)$ yields that

$$\|\Upsilon(\underline{\Psi})\|_{\mathcal{H}^1} \lesssim 2\overline{M},$$

and thus Υ maps X_T into X_T for $M = 2\overline{M}$. Now we show that $\Upsilon(\underline{\Psi})$ is a contraction on X_T. Given $\underline{\Psi}, \underline{\Phi} \in X_T$ and $t \in [0, T(M))$,

$$\begin{aligned}
\|\Upsilon(\underline{\Psi}) - \Upsilon(\underline{\Phi})\|_{\mathcal{H}^1} &\lesssim \int_0^t \|(\frac{1}{|x|} * |\underline{\Psi}(s)|^2)\underline{\Psi}(s) - (\frac{1}{|x|} * |\underline{\Phi}(s)|^2)\underline{\Phi}(s)\|_{\mathcal{H}^1}\mathrm{d}s \\
&\lesssim \int_0^t (\|\underline{\Psi}(s)\|_{\mathcal{H}^1}^2 + \|\underline{\Phi}(s)\|_{\mathcal{H}^1}^2)\|\underline{\Psi}(s) - \underline{\Phi}(s)\|_{\mathcal{H}^1}\mathrm{d}s \\
&\leq T(\overline{M})(\|\underline{\Psi}\|_{X_T}^2 + \|\underline{\Phi}\|_{X_T}^2)\|\underline{\Psi} - \underline{\Phi}\|_{X_T} \\
&\leq 8\overline{M}^2 T(\overline{M})\|\underline{\Psi} - \underline{\Phi}\|_{X_T} \\
&= \frac{1}{2}\|\underline{\Psi} - \underline{\Phi}\|_{X_T}.
\end{aligned}$$

A symmetric argument for $t \in (-T(M), 0]$ yields the claim. Finally, we have to check the maximality of the time interval. Define

$$\begin{aligned}
T_{\max} &:= \sup\{T > 0 \colon \exists \text{ solution } \underline{\Psi} \in C([0, T_{\max}), \mathcal{H}^1(\mathbb{R}^3)) \\
T_{\min} &:= \sup\{T > 0 \colon \exists \text{ solution } \underline{\Psi} \in C((-T_{\min}, 0], \mathcal{H}^1(\mathbb{R}^3)).
\end{aligned}$$

By definition, for any interval $J = [-T', T] \subseteq (-T_{\min}, T_{\max})$ there is a unique solution $\underline{\Psi} \in C(J, \mathcal{H}^1) \cap C^1(J, \mathcal{H}^{1*})$. The uniqueness follows since for any two $u, v \in C(J, \mathcal{H}^1)$ we have by the Duhamel formula, the unitarity of e^{itH_0} and the local Lipschitz property of the nonlinearity,

$$\begin{aligned}
\|\underline{\Psi}(t) - \underline{\Phi}(t)\|_{\mathcal{H}^1} &\leq \int_0^t \|(\frac{1}{|x|} * |\underline{\Psi}(s)|^2)\underline{\Psi}(s) - (\frac{1}{|x|} * |\underline{\Phi}(s)|^2)\underline{\Phi}(s)\|_{\mathcal{H}^1}\mathrm{d}s \\
&\lesssim (\|\underline{\Psi}\|_{L^\infty(I,\mathcal{H})}^2 + \|\underline{\Phi}\|_{L^\infty(I,\mathcal{H})}^2) \int_0^t \|\underline{\Psi}(s) - \underline{\Phi}(s)\|_{\mathcal{H}^1}\mathrm{d}s.
\end{aligned}$$

Applying Gronwall's inequality we can deduce that $\underline{\Psi}(t) = \underline{\Phi}(t)$ on $[0, T]$ and by an analoguous argument on $[-T', 0]$ as well. Finally we present the argument for the blow-up alternative for $T_{\max}$ (the case for $T_{\min}$ follows analoguously). If $T_{\max} < \infty$ assume there is a $M > 0$ and a sequence $\{t_n\} \subset \mathbb{R}$ such that $t_n \uparrow T_{\max}$ but $\|\underline{\Psi}\|_{\mathcal{H}^1} \leq M$ for all n. We can deduce from the wellposedness theory that there is a solution whose initial datum has $\mathcal{H}^1$-norm below M with local existence time $T(M)$. Then we can construct a solution $\underline{\Phi} \in C((-T(M), T(M)), \mathcal{H}^1)$ with initial datum $\underline{\Psi}(t_k)$ with t_k such that $t_k + T(M) > T_{\max}$. Setting $\tilde{\underline{\Phi}}(t) = \underline{\Phi}(t - t_k)$ we see that $\tilde{\underline{\Phi}} \in C((t_k - T(M), t_k + T(M)), \mathcal{H}^1)$ is a solution. Since it coincides with $\underline{\Psi}$ at $t = t_k$ we can construct a solution with initial datum $\underline{\Psi}_I$ in the space $C((-T_{\min}, t_k + T(M)), \mathcal{H}^1)$ which is a contradiction to the maximality of $T_{\max}$. The claim follows. $\square$

Lemma 3.5 (Continuous dependence on the initial data)**.** *Let $A \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$, $|B| \in L^2(\mathbb{R}^3)$. Let $\underline{\Psi}_{n,I}, \underline{\Psi}_I \in \mathcal{H}^1(\mathbb{R}^3)$ be initial data satisfying $\underline{\Psi}_{n,I} \to \underline{\Psi}_I$ in $\mathcal{H}^1(\mathbb{R}^3)$ with corresponding unique maximal solutions $\underline{\Psi}_n \in C(J_n, \mathcal{H}^1) \cap C^1(J_n, \mathcal{H}^{1*})$ and $\underline{\Psi} \in C(J, \mathcal{H}^1) \cap C^1(J, \mathcal{H}^{1*})$. If $J' \subset J$ is a closed interval then $J' \subset J_n$ for n large enough and $\underline{\Psi}_n \to \underline{\Psi}$ in $C(J', \mathcal{H}^1)$.*

Proof. W.l.o.g. assume that $0 \in J'$. Set $M := 2\sup_{t \in J'}\{\|\underline{\Psi}(t)\|_{\mathcal{H}^1}\}$. According to the previous lemma, $\underline{\Psi} \in C((-T(M), T(M)), \mathcal{H}^1)$ is a solution with local existence time $T(M) \simeq 1/16M^2$ and initial datum satisfying $\|\underline{\Psi}_I\|_{\mathcal{H}^1} \leq M$ (since $J' \subset J$ by assumption). By assumption, $\|\underline{\Psi}_{n,I}\|_{\mathcal{H}^1} \leq M$ for n large enough, thus $\underline{\Psi}_n$ is a solution at least on $[-T(M), T(M)]$, whence $[-T(M), T(M)] \subset J \cap J_n$ for n large enough. Evoking Duhamel's formula we see that for $t \in [0, T(M)]$ (the other case follows analoguously),

$$\|\underline{\Psi}_n(t) - \underline{\Psi}(t)\|_{\mathcal{H}^1} \lesssim \|\underline{\Psi}_{n,I} - \underline{\Psi}_I\|_{\mathcal{H}^1} + \frac{1}{2}\|\underline{\Psi}_n - \underline{\Psi}\|_{L^\infty((-T(M),T(M)),\mathcal{H}^1)}$$

This implies that $\underline{\Psi}_n \to \underline{\Psi}$ in $C((-T(M), T(M)), \mathcal{H}^1)$. Since $T(M)$ depends only on M an iteration of this argument covers the whole interval J'. We conclude that $J' \subset J_n$ for n large enough and $\underline{\Psi}_n \to \underline{\Psi}$ in $C(J', \mathcal{H}^1)$. $\square$

Lemma 3.6 (Energy and charge conservation)**.** *Let $A \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$, $|B| \in L^2(\mathbb{R}^3)$ and let $\underline{\Psi} \in C((-T_{\min}, T_{\max}), \mathcal{H}^1) \cap C^1((-T_{\min}, T_{\max}), \mathcal{H}^{1*})$ be the unique maximal solution to the IVP (3.39)-(3.41). Then the charge*

$$Q(\underline{\Psi}) := \|\underline{\Psi}\|_2^2 = \sum_{j=1}^{\infty} \lambda_j \|\Psi_j\|_2^2 = \int \rho_{\mathrm{diag}} \mathrm{d}x \tag{3.60}$$

and the energy

$$E(\underline{\Psi}) := \frac{1}{2}\int_{\mathbb{R}^3} \sum_{j=1}^{\infty} \lambda_j |(\sigma \cdot (\nabla - iA))\Psi_j|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} |\nabla V|^2 \mathrm{d}x \tag{3.61}$$

are conserved, i.e. $Q(\underline{\Psi}(t)) = Q(\underline{\Psi}_I)$ and $E(\underline{\Psi}(t)) = E(\underline{\Psi}_I)$ for all $t \in (-T_{\min}, T_{\max})$.

Proof. By the continuous dependence on the initial data and since $(C_0^\infty(\mathbb{R}^3))^2$ is dense in $\mathcal{H}^1(\mathbb{R}^3)$ we can assume without loss of generality that $\underline{\Psi}$ is smooth. Charge conservation follows from multiplying the mixed state Pauli equation by $\overline{u}_j$, multiplying by λ_j, integrating and taking the imaginary part. For the energy observe we multiply by $\partial_t\overline{\Psi_j}$, multiply by λ_j, integrate and take the real part to obtain

$$\operatorname{Re}\frac{1}{2}\sum_j\lambda_j\int(\partial_t\overline{\Psi_j})(\sigma\cdot(\nabla-iA))^2\Psi_j\mathrm{d}x+\operatorname{Re}\sum_j\lambda_j\int(\partial_t\overline{\Psi_j})V\Psi_j=0$$

This is equivalent to

$$\partial_t\frac{1}{2}\sum_j\lambda_j\int(\partial_t\overline{\Psi_j})(\sigma\cdot(\nabla-iA))^2\Psi_j\mathrm{d}x+\partial_t\frac{1}{2}\operatorname{Re}\sum_j\lambda_j\int|\nabla V|^2=0.$$

Indeed,

$$\begin{aligned}\operatorname{Re}\int(\partial_t\overline{u}_j)V\Psi_j\mathrm{d}x&=\frac{1}{2}\partial_t\int V|\Psi_j|^2\mathrm{d}x-\int(\partial_tV)|\Psi_j|^2\mathrm{d}x\\&=\frac{1}{2}\partial_t\int|\nabla V|^2\mathrm{d}x-\operatorname{Re}\iint\frac{1}{|x-y|}(\partial_t\overline{\Psi_j}(y))\Psi_j(y)|\Psi_j(x)|^2\mathrm{d}y\mathrm{d}x\\&=\frac{1}{2}\partial_t\int|\nabla V|^2\mathrm{d}x-\operatorname{Re}\int(\partial_t\overline{u}_j)V\Psi_j\mathrm{d}x.\end{aligned}$$

Thus, $\partial_tE(\underline{\Psi}(t))=0$. □

Proof of Theorem 2. By Lemma 3.6, $\sup_{t\in\mathbb{R}}\|\underline{\Psi}\|_{\mathcal{H}^1}<\infty$. Hence $T_{\min}=T_{\max}=\infty$ by the blow-up alternative. □

3.3 Semiclassical limit

In this section we prove Theorem 1.

External electric and magnetic fields

Proof of Theorem 1 (i). We identify the limits. Let us start with the spin term. We need to show for all test functions $\varphi\in\mathcal{S}$ that $\langle\hbar\theta[\sigma\cdot B]F^\hbar,\varphi\rangle$ is bounded independently of $t\in\mathbb{R}$ and that it converges to zero when $\hbar\to0$. Indeed we have

$$\begin{aligned}\langle\hbar\theta[\sigma\cdot B]F^\hbar,\varphi\rangle=&\frac{i}{2}\iint_{(\mathbb{R}^3)^2}F^\hbar(x,\eta)\int_{\mathbb{R}^3}\mathcal{F}_\xi[\varphi](x,y)\\&\cdot e^{i\eta\cdot y}\sigma\cdot(B(x+\frac{\hbar y}{2})-B(x-\frac{\hbar y}{2}))\mathrm{d}y\mathrm{d}x\mathrm{d}\eta\\=&\frac{i}{2}\langle F^\hbar,\psi_1^\hbar\rangle_{\mathcal{S}\times\mathcal{S}'}.\end{aligned}$$

Here,

$$\psi_1^\hbar(x,\eta):=\int_{\mathbb{R}^3}\mathcal{F}_\xi[\varphi](x,y)e^{i\eta\cdot y}\sigma\cdot(B(x+\frac{\hbar y}{2})-B(x-\frac{\hbar y}{2}))\mathrm{d}y,$$

which is in $\mathcal{S}$ by the choice of φ and the assumption that $B\in C(\mathbb{R}^3)$. It is now clear that $\mathcal{F}_\eta[\psi_1^\hbar]$ converges to zero in $L^1(\mathbb{R}^3_z,C(\mathbb{R}^3_x))$ if $\hbar\to0$.

For the $\theta[A^2]F^\hbar$ term we use the same reasoning with $\psi_1^\hbar(x,\eta)$ replaced by

$$\psi_2^\hbar(x,\eta) := \int_{\mathbb{R}^3} \mathcal{F}_\xi[\varphi](x,y)e^{i\eta\cdot y}\frac{1}{\hbar}(A(x+\frac{\hbar y}{2})^2 - A(x-\frac{\hbar y}{2})^2)\mathrm{d}y.$$

Since $A \in C^1(\mathbb{R}^3)$, also $A^2 \in C^1(\mathbb{R}^3)$ and we can take the limit in $\hbar$. However we have to be careful since we take the directional derivative with respect to a vector valued function. In fact,

$$\lim_{\hbar\to 0}\frac{1}{\hbar}A(x+\frac{\hbar y}{2})^2 - A(x-\frac{\hbar y}{2})^2 = \sum_{i,k=1}^{3} A_i y_k \partial_k A_i.$$

Thus,

$$\frac{1}{2}\frac{i}{(2\pi)^3}\int_{\mathbb{R}^3}\mathcal{F}_\xi[\varphi](x,y)A_i y_k \partial_k A_i e^{i\eta_k y_k}\mathrm{d}y = \frac{1}{2}(A_i\partial_k A_i)\nabla_\eta\varphi(x,\eta).$$

where we use Einstein summation convention for clarity. We conclude that $\langle\theta[A^2]F^\hbar,\varphi\rangle$ converges to

$$\int_{(\mathbb{R}^3)^2}(A_i\partial_k A_i)\nabla_\xi\varphi(x,\xi)\mathrm{d}f(t).$$

Next, we consider the $i\mathcal{F}_y[\delta[A]] *_\xi (\xi F^\hbar)$ term. We have

$$\begin{aligned}\langle i\mathcal{F}_y[\delta[A]] *_\xi (\xi F^\hbar),\varphi\rangle &= \frac{1}{(2\pi)^3}\iint F^\hbar(x,\eta)\int(\nabla_y e^{i\eta\cdot y})\delta[A](x,y)\mathcal{F}_\xi[\varphi](x,y)\mathrm{d}y\mathrm{d}x\mathrm{d}\eta \\ &= \frac{1}{(2\pi)^3}\langle F^\hbar,\psi_3^\hbar\rangle_{\mathcal{S}'\times\mathcal{S}},\end{aligned}$$

where

$$\psi_3^\hbar := \int_{\mathbb{R}^3}\mathcal{F}_\xi[\varphi](x,y)(\nabla_y e^{i\eta\cdot y})\delta[A]\mathrm{d}y.$$

Since $A \in C^1(\mathbb{R}^3)$ we can deduce that $\psi_3^\hbar$ converges in $\mathcal{S}$ to

$$\int \mathcal{F}_\xi[\varphi](\nabla_y e^{i\eta\cdot y})y\cdot\nabla_x A\mathrm{d}y.$$

Furthermore,

$$\begin{aligned}\frac{1}{(2\pi)^3}\int(\nabla_y e^{i\eta\cdot y})y\cdot\nabla_x A(\mathcal{F}_\xi\varphi)\mathrm{d}y &= -i\nabla_x A\cdot\nabla_\eta\int\frac{1}{(2\pi)^3}\mathcal{F}_\xi[\varphi]\nabla_y e^{i\eta\cdot y}\mathrm{d}y \\ &= i\nabla_x A\cdot\nabla_\eta\mathcal{F}_y^{-1}[\nabla_y(\mathcal{F}_\xi\varphi)(x,y)] \\ &= \nabla_x A\xi\cdot\nabla_\eta\varphi(x,\eta).\end{aligned}$$

We conclude that $\langle i\mathcal{F}_y[\delta[A]] *_\xi (\xi F^\hbar),\varphi\rangle$ converges to

$$\int_{(\mathbb{R}^3)^2}\nabla_x A\xi\cdot\nabla_\xi\varphi(x,\xi)\mathrm{d}f(t).$$

Consider now $\mathcal{F}_y[\beta[A]] *_\xi \nabla_x F^\hbar$. We have

$$\langle \mathcal{F}_y[\beta[A]] *_\xi \nabla_x F^\hbar, \varphi\rangle = \frac{i}{(2\pi)^3} \iint \nabla_x F^\hbar(x,\eta) \int e^{i\eta\cdot y}\beta[A]\mathcal{F}_\xi[\varphi](x,y)\mathrm{d}y\mathrm{d}x\mathrm{d}\eta$$
$$= \frac{i}{(2\pi)^3}\langle \nabla_x F^\hbar, \psi_4^\hbar\rangle$$

where

$$\psi_4^\hbar := \int_{\mathbb{R}^3} \mathcal{F}_\xi[\varphi](x,y)e^{i\eta\cdot y}\beta[A]\mathrm{d}y.$$

We easily see similarly to the reasoning before that $\psi_4^\hbar$ converges in $\mathcal{S}$ to

$$\int_{\mathbb{R}^3} \mathcal{F}_\xi[\varphi](x,y)A(x)e^{i\eta\cdot y}\mathrm{d}y,$$

which is equivalent to

$$A(x)\varphi(x,\eta).$$

Finally we consider the V-term we proceed as in the case for A^2 and observe that under the assumption $V \in C^1$

$$\langle \theta[V]F^\hbar, \varphi\rangle = i\langle F^\hbar, \psi_5^\hbar\rangle_{\mathcal{S}\times\mathcal{S}'},$$

where

$$\psi_5^\hbar(x,\eta) := \int_{\mathbb{R}^3} \mathcal{F}_\xi[\varphi](x,y)e^{i\eta\cdot y}\frac{1}{\hbar}(V(x+\frac{\hbar y}{2}) - V(x-\frac{\hbar y}{2}))\mathrm{d}y,$$

converges to

$$\int \nabla_x V \nabla_\xi \varphi(x,\xi)\mathrm{d}f(t).$$

After a change of variables $p = \xi - A$ we conclude $F^\hbar$ converges to F in the sense of distributions and F solves the Vlasov equation (3.36). $\square$

Self-consistent electric field, external magnetic field

We now turn to the nonlinear case where $V^\hbar$ is given by the Poisson equation (1.61). We need the following observations.

Lemma 3.7. *Let H_0 be the linear Pauli Hamiltonian from (3.9). If*

$$E(0) := \mathrm{tr}\Big(H_0 R^\hbar\Big) + \frac{1}{2}\iint_{\mathbb{R}^3_x\times\mathbb{R}^3_y} \frac{\rho^\hbar_{\mathrm{diag},I}(x)\rho^\hbar_{\mathrm{diag},I}(y)}{|x-y|}\mathrm{d}x\mathrm{d}y,$$

is bounded independently of $\hbar$ then the total energy

$$E(t) := \mathrm{tr}\Big(H_0 R^\hbar\Big) + \frac{1}{2}\iint_{\mathbb{R}^3_x\times\mathbb{R}^3_y} \frac{\rho^\hbar_{\mathrm{diag}}(x)\rho^\hbar_{\mathrm{diag}}(y)}{|x-y|}\mathrm{d}x\mathrm{d}y,$$

is bounded independently of $\hbar$.

Proof. This follows from Lemma 3.6 by rewriting the energy in the density matrix formulation. In fact,

$$\operatorname{tr}\Big(H_0 R^\hbar\Big) = \frac{1}{2}\sum_{j=1}^{\infty} \lambda_j^\hbar \|\sigma\cdot(\hbar\nabla - iA)\Psi_j^\hbar\|_2^2,$$

and

$$\frac{1}{2}\iint_{\mathbb{R}^3_x\times\mathbb{R}^3_y} \frac{\rho^\hbar_{\mathrm{diag}}(x)\rho^\hbar_{\mathrm{diag}}(y)}{|x-y|}\mathrm{d}x\mathrm{d}y = \frac{1}{2}\int_{\mathbb{R}^3_x} |\nabla V^\hbar|^2 \mathrm{d}x$$

□

Now define

$$R(x,y) := \sum_j \lambda_j \Psi_j(x)\otimes\overline{\Psi_j(y)} \qquad\qquad \rho_{\mathrm{diag}}(x) := \sum \lambda_j |\Psi_j(x)|^2 \tag{3.62}$$

where $\lambda = \{\lambda_j\}_{j\in\mathbb{N}}$ is a decreasing sequence with $\lambda_j \geq 0$ for all $j \in \mathbb{N}$ and $\{\Psi_j\}$ is an orthonormal basis of $L^2(\mathbb{R}^3)^2$. Let

$$\overline{H}_0 := \frac{1}{2}(\sigma\cdot(\nabla - iA))^2 \tag{3.63}$$

Lemma 3.8. *Let $B \in L^{7/2}(\mathbb{R}^3)$ and let*

$$\operatorname{tr}\big(-\overline{H}_0 R\big) := \sum_j \lambda_j^\hbar \int_{\mathbb{R}^3} |(\sigma\cdot(\nabla - iA))\Psi_j|^2 \mathrm{d}x, \qquad\qquad \|\lambda^\hbar\|_2 = \sum_j (\lambda_j^\hbar)^2,$$

Then $R_{\mathrm{diag}}, \rho^\hbar_{\mathrm{diag}} \in L^{7/5}(\mathbb{R}^3)$ with

$$\|R_{\mathrm{diag}}\|_{7/5} \leq \|\lambda^\hbar\|_2^{4/7} \operatorname{tr}\big(-\overline{H}_0 R\big)^{3/7} \tag{3.64}$$

$$\|\rho_{\mathrm{diag}}\|_{7/5} \leq \|\lambda^\hbar\|_2^{4/7} \operatorname{tr}\big(-\overline{H}_0 R\big)^{3/7} \tag{3.65}$$

In particular, if $\lambda^\hbar$ satisfies Assumption 1 it holds that $R_{\mathrm{diag}}, \rho^\hbar_{\mathrm{diag}} \in L^{7/5}(\mathbb{R}^3)$ independently of $\hbar$.

Proof. The main ingredient for the proof is the following magnetic Lieb-Thirring inequality from [87]. Let $\overline{H}_0$ be the Pauli operator from (3.63) and denote by μ_j its negative eigenvalues. Let $\gamma \geq 1$ and $q > 3/2$. Then

$$\sum_j |\mu_j|^\gamma \leq C_1(\gamma,q)\int V_-^{3/2+\gamma}\mathrm{d}x + C_2(\gamma,q)\|B\|_{3q/2}^{3/2}\left(\int V_-^{\gamma q'}\mathrm{d}x\right)^{1/q'} \tag{3.66}$$

We will use an argument from [59] to prove the claim. Define the operator $\overline{H}_0 - tR^\alpha$ where $t > 0$ will be determined later. Then

$$\operatorname{tr}\big((\overline{H}_0 - tR^\alpha)R\big) = \sum_j \lambda_j^\hbar \int |(\sigma\cdot(\nabla - iA))\Psi_j|^2 - tR^\alpha|\Psi_j|^2\mathrm{d}x \geq \sum_j \lambda_j^\hbar |\mu_j|.$$

By the Hölder inequality,

$$t\int R^{\alpha+1}\mathrm{d}x \leq \operatorname{tr}\big(\overline{H}_0 R\big) + \|\lambda^\hbar\|_p \left(\sum |\mu_j|^{p'}\right)^{1/p'}$$

Now we use the magnetic Lieb-Thirring inequality (3.66) and obtain

$$\begin{aligned} t\int R^{\alpha+1}\mathrm{d}x \leq \operatorname{tr}\big(\overline{H}_0 R\big) &+ \|\lambda^\hbar\|_p C_1 \left(\int (tR^\alpha)^{3/2+p'}\right)^{1/p'} \\ &+ \|\lambda^\hbar\|_p C_2 \|B\|_{3q/2}^{3/2} \left(\int (tR^\alpha)^{p'q'}\right)^{1/(q'p')} \end{aligned}$$

where C_1 and C_2 depend on q and p'. Choosing $p = p' = 2$ and setting $\alpha + 1 = 3\alpha/2 + 2\alpha$ yields $\alpha = 2/5$. Therefore,

$$\begin{aligned} t\int R^{7/5}\mathrm{d}x \leq \operatorname{tr}\big(\overline{H}_0 R\big) &+ tt^{3/4}\|\lambda^\hbar\|_2 C_1 \left(\int R^{7/5}\right)^{1/2} \\ &+ t\|\lambda^\hbar\|_2 C_2 \|B\|_{3q/2}^{3/2} \left(\int R^{4q'/5}\right)^{1/(2q')} \end{aligned}$$

By assumption, $B \in L^{7/2}$, so we can choose $q = 7/3 > 3/2$ and $q' = 7/4$. Together with the choice

$$t^{3/4} = \frac{1}{2C_1\|\lambda^\hbar\|_2}\left(\int R^{7/5}\right)^{1/2}$$

we obtain

$$\frac{1}{2}t\int R^{7/5}\mathrm{d}x \leq \operatorname{tr}\big(\overline{H}_0 R\big) + t\|\lambda^\hbar\|_2 C_3 \left(\int R^{7/5}\right)^{2/7}.$$

where C_3 depends on $\|B\|_{7/2}$. It follows that

$$\|R\|_{7/5}^{5/3} \lesssim_{C_1,C_3} \|\lambda^\hbar\|_2^{4/3} \operatorname{tr}\big(\overline{H}_0 R\big) + \|\lambda_2\|R\|_{7/5}^{4/3}. \tag{3.67}$$

Now since the power of the second term on the RHS is smaller than on the LHS we can absorb it into the LHS and obtain

$$\|R\|_{7/5}^{5/3} \lesssim_{C_1,C_3} \|\lambda^\hbar\|_2^{4/3} \operatorname{tr}\big(\overline{H}_0 R\big).$$

For the last statement of the lemma we use (3.65) and Assumption 1 to obtain

$$\|R_{\mathrm{diag}}^\hbar\|_{7/5} \leq C\hbar^{6/7} \operatorname{tr}\big(\overline{H}_0 R\big)^{4/7}$$

Now

$$
\begin{aligned}
\operatorname{tr}\big(\overline{H}_0 R\big)^{4/7} &\leq \left(\frac{1}{2\hbar^2}\sum_j \lambda_j^\hbar \int |(\sigma\cdot\hbar\nabla)\Psi_j|^2 \mathrm{d}x + \frac{1}{2}\sum_j \int \lambda_j^\hbar |(\sigma\cdot A)\Psi_j|^2\mathrm{d}x\right)^{4/7}\\
&\lesssim \frac{1}{\hbar^{6/7}}\left(\sum_j \lambda_j^\hbar \int |(\sigma\cdot(\hbar\nabla - iA))\Psi_j|^2\mathrm{d}x\right)^{4/7} + \left(\sum_j \int \lambda_j^\hbar |(\sigma\cdot A)\Psi_j|^2\mathrm{d}x\right)^{4/7}\\
&\leq \frac{1}{\hbar^{6/7}} K^{4/7} + \|A^2\|_6 \|\rho^\hbar_{\mathrm{diag}}\|_{6/5})^{3/7}\\
&\leq \frac{1}{\hbar^{6/7}} K^{4/7} + C\|\rho^\hbar_{\mathrm{diag}}\|^{1/4}_{7/5}
\end{aligned}
$$

where C does not depend on $\hbar$ and where K denotes the kinetic energy of the Pauli-Poisson equation which is bounded independently of $\hbar$ due to Lemma 3.7. By combining the last two estimates and observing that the powers of $\hbar$ cancel yields the claim. Obviously all calculations hold for $\rho^\hbar_{\mathrm{diag}} = \operatorname{Tr}\big(R^\hbar\big)$ as well. □

Proof of Theorem 1 (ii). By assumption, $\|F^\hbar_I\|_2$ is bounded independently of $\hbar$ (this is due to (2.37)). By conservation of $\|F^\hbar\|_2$ we have that $F^\hbar \in L^2$ independently of $\hbar$. Thus we can extract a subsequence such that $F^\hbar$ converges weakly* to some

$$
F \in C_b(\mathbb{R}, \mathcal{M}^{2\times 2}_{w*}) \cap L^\infty(\mathbb{R}, L^1 \cap L^2(\mathbb{R}^3_x \times \mathbb{R}^3_\xi)^{2\times 2}).
$$

Now let $\psi \in \mathcal{S}(\mathbb{R}^3_x \times \mathbb{R}^3_\xi)$. With the notation of Proposition (i) we have to show that

$$
\begin{aligned}
\psi(x,z)\frac{\hbar}{2}\delta[\sigma\cdot B] &\longrightarrow 0, && (3.68)\\
\psi(x,z)\frac{1}{2}\delta[A^2] &\longrightarrow \psi(x,z) z\cdot(\nabla_x A)A(x), && (3.69)\\
iz\psi(x,z)\delta[A] &\longrightarrow iz\psi(x,z)z\cdot\nabla_x A, && (3.70)\\
\psi(x,z)\beta[A] &\longrightarrow \psi(x,z)A(x) && (3.71)\\
\psi(x,z)\delta[V^\hbar] &\longrightarrow \psi(x,z)z\cdot\nabla_x V. && (3.72)
\end{aligned}
$$

in $L^2(\mathbb{R}^6)$ as $\hbar \to 0$. The first one is immediately clear since $B \in L^2_{\mathrm{loc}}$. For the second one we have to ensure that

$$
(\nabla_x A)A \in L^2_{\mathrm{loc}}(\mathbb{R}^3).
$$

The assumption on A implies

$$
A \in L^{12}_{\mathrm{loc}}(\mathbb{R}^3), \quad \nabla_x A \in L^{12/5}_{\mathrm{loc}}(\mathbb{R}^3),
$$

by Sobolev's inequality. The claim follows from Hölder's inequality. The third and fourth convergences are clear since in particular $A \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$.

It remains to show that $V^\hbar$ converges in $H^1_{\mathrm{loc}}(\mathbb{R}^3)$ which is equivalent to showing that $\nabla V^\hbar = (\nabla|x|^{-1}) * \rho^\hbar_{\mathrm{diag}}$ converges in $L^2_{\mathrm{loc}}(\mathbb{R}^3)$. Note that

$$\nabla|x|^{-1} \in L^{3/2,\infty}(\mathbb{R}^3)$$

so in particular it is in $L^{14/11}(\mathbb{R}^3) + L^q(\mathbb{R}^3)$ for $14/11 < q < \infty$ since $3/2 > 14/11$. By Lemma 3.8, $\rho^\hbar_{\mathrm{diag}}$ is bounded independently of $\hbar$ in $L^{7/5}(\mathbb{R}^3)$ and by Young's inequality we conclude that $\nabla V^\hbar \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$. In order to obtain convergence in $L^\infty((0,T), L^2_{\mathrm{loc}}(\mathbb{R}^3_x))$ we use a compactness argument. By the continuity equation we have

$$\partial_t \nabla V^\hbar = \nabla(-\Delta)^{-1}(\partial_t \rho^\hbar_{\mathrm{diag}}) \simeq J^\hbar.$$

A similar argument for $J^\hbar$ as for $\rho^\hbar_{\mathrm{diag}}$ in Lemma 3.8 implies that $J^\hbar$ is bounded in $L^{7/6}(\mathbb{R}^3)$ independently of $\hbar$. Therefore $\partial_t \nabla V^\hbar \in L^{7/6}(\mathbb{R}^3)$ and we conclude by the Aubin-Lions lemma that $V^\hbar$ converges strongly in $C((0,T), H^1_{\mathrm{loc}}(\mathbb{R}^3_x))$. The last part is to show that $\rho^\hbar$ is $\hbar$-oscillatory. Observe that $\mathrm{tr}\big(H\rho^\hbar\big) \leq C$ implies that

$$\iint f^\hbar(x,\xi)|\xi - A(x)|^2 \mathrm{d}x\mathrm{d}\xi \leq C$$

On the other hand,

$$\begin{aligned}\iint f^\hbar |\xi|^2 \mathrm{d}x\mathrm{d}\xi &\leq \iint f^\hbar |\xi - A|^2 \mathrm{d}x\mathrm{d}\xi + \iint f^\hbar A^2 \mathrm{d}x\mathrm{d}\xi \\ &\leq \mathrm{tr}\Big(H_0 R^\hbar\Big) + \|f^\hbar\|_{p'} \|A^2\|_p\end{aligned}$$

where p' is dual to p. Now take for instance $p = 6, p' = 6/5$ and observe that $f^\hbar \in L^r$ for $r \in [1,2]$ and $A^2 \in L^6$. Therefore we can evoke Lemma 2.2 and the last statement of the Theorem follows from this and the last statement in Theorem 1. $\square$

Current density

Proof of Theorem 1 (iii). One easily checks (using $\nabla \times (\overline{u}\sigma u) = \mathrm{tr}((\nabla \times \sigma\overline{u}) \otimes u) + \mathrm{tr}(\overline{u} \otimes \nabla \times \sigma u)$) that

$$J^\hbar = \int_{\mathbb{R}^3_\xi} (\xi - A(x)) f^\hbar(x,\xi)\mathrm{d}\xi - \hbar\nabla_x \times \int_{\mathbb{R}^3_\xi} \mathrm{tr}\Big(\sigma F^\hbar\Big)(x,\xi)\mathrm{d}\xi,$$

Let $\varphi \in C_0^\infty$ and set

$$j^\hbar = \int_{\mathbb{R}^3_\xi} \xi f^\hbar \mathrm{d}\xi, \quad j := \int \xi f \mathrm{d}\xi.$$

Again, $j^\hbar$ is bounded in $L^{7/6}(\mathbb{R}^3)$ independently of $\hbar$ and

$$\iint j^\hbar \varphi \mathrm{d}x\mathrm{d}t \to \iint j\varphi \mathrm{d}x\mathrm{d}t,$$

as $\hbar \to 0$. Since $\rho^\hbar_{\mathrm{diag}} \in L^{7/5}(\mathbb{R}^3)$ independently of $\hbar$ and $A \in L^{12}(\mathbb{R}^3)$ (by the Sobolev inequality) we have that

$$\iint A\rho^\hbar_{\mathrm{diag}}\varphi \mathrm{d}x\mathrm{d}t \to \iint A\rho_{\mathrm{diag}}\varphi \mathrm{d}x\mathrm{d}t.$$

since $A\varphi \in L^{7/2}$. By the same reasoning we obtain that the spin term is $o(1)$ as $\hbar \to 0$.

□

4 The Pauli-Poisswell equation: Derivation and Wellposedness

This section is based on [38].

Consider the Pauli-Poisswell equation:

$$i\hbar\partial_t\Psi^\hbar = -\frac{1}{2}(\hbar\nabla - iA^\hbar)^2\Psi^\hbar + V^\hbar\Psi^\hbar - \frac{1}{2}\hbar(\sigma\cdot B^\hbar)\Psi^\hbar, \tag{4.1}$$

$$-\Delta V^\hbar = |\Psi^\hbar|^2 =: \rho^\hbar, \tag{4.2}$$

$$-\Delta A^\hbar = J^\hbar. \tag{4.3}$$

$$\Psi^{\hbar,c}(x,0) = \Psi_I^{\hbar,c}(x) \in (L^2(\mathbb{R}^3))^2. \tag{4.4}$$

where the Pauli current density is given by

$$J^\hbar(\Psi^\hbar, A^\hbar) = \mathrm{Im}(\overline{\Psi^\hbar}(\hbar\nabla - iA^\hbar)\Psi^\hbar) - \hbar\nabla\times(\overline{\Psi^\hbar}\sigma\Psi^\hbar), \tag{4.5}$$

We can rewrite $J^\hbar$ as

$$J^\hbar = \mathrm{Re}\left(\overline{\Psi^\hbar}\sigma(\sigma\cdot(-i\hbar\nabla - A^\hbar))\Psi^\hbar\right). \tag{4.6}$$

The analysis of the fully self-consistent Pauli-Poisswell equation is much harder than for the Pauli-Poisson equation since the magnetic field is no longer external but now coupled via the Poisson equation (4.3): For the semiclassical limit the regularity obtained from this is not enough in order to use uniform L^2 bounds on the Wigner function (recall that we needed $A^\hbar\cdot(\nabla\otimes A^\hbar)\in L^2$ for the Pauli-Poisson equation). Using the Husimi function $\tilde{F}^\hbar$, the uniform boundedness of $\tilde{F}^\hbar$ due to Assumption 2 in combination with Lemma 2.1 and the fact that $F^\hbar$ and $\tilde{F}^\hbar$ converge to the same Wigner matrix measure F in the limit $\hbar\to 0$, we can use $\tilde{F}^\hbar$ and the corresponding *Husimi equation*, derived in Lemma 5.1, in order to pass to the limit.

The uniform boundedness of $\tilde{F}^\hbar$ due to Lemma 2.1 relaxes the required regularity for $A^\hbar$ since strong convergence no longer has to occur in H^1_{loc} but $W^{1,p}_{\mathrm{loc}}$.

However, there are still technical problems when proving the convergence of the magnetic potential $A^\hbar$ since a straightforward argument using the Aubin-Lions lemma is not sufficient since the required full time derivative $\partial_t A^\hbar$ of $A^\hbar$ cannot be defined in $L^p(\mathbb{R}^3)$, $p\geq 1$ uniformly in $\hbar$ when $A^\hbar$ is given by (4.3). One can relax the requirement on the time derivative using refined compactness arguments (cf. [88]).

On the other hand we do not know how to obtain enough regularity for $B^\hbar$ in order for magnetic Lieb-Thirring estimates to hold. The magnetic Lieb-Thirring estimates are necessary in order to prove uniform L^p estimates for the density $\rho^\hbar$ and the current density $J^\hbar$.

We give details on these issues in Section 5.

First we discuss the derivation and gauge of (4.1)-(4.4). Then, we discuss the existence of global weak solutions and the existence and uniqueness of local strong solutions.

4.1 Derivation and gauge

The Pauli-Poisswell equation is derived from the Dirac-Maxwell equation (1.2)-(1.4) as follows. Write $\varepsilon := 1/c$. First, the $O(\varepsilon)$ approximation of the Dirac equation is the Pauli equation, cf. [66]. Therefore we have to calculate the $O(\varepsilon)$ approximation of Maxwell's equations. We present the derivation from [63]. The electromagnetic fields are given by

$$E^\varepsilon := -\nabla V^\varepsilon - \varepsilon \partial_t A^c \qquad\qquad B^\varepsilon := \nabla \times A^\varepsilon \tag{4.7}$$

Consider a formal expansion in ε:

$$A^\varepsilon = A^0 + \varepsilon A^1 + O(\varepsilon^2), \qquad\qquad V^\varepsilon = V^0 + \varepsilon V^1 + O(\varepsilon^2), \tag{4.8}$$

$$B^\varepsilon = B^0 + \varepsilon B^1 + O(\varepsilon^2), \qquad\qquad E^\varepsilon = E^0 + \varepsilon E^1 + O(\varepsilon^2). \tag{4.9}$$

Maxwell's equations for the fields E^ε and B^ε read

$$\begin{aligned} &\varepsilon\partial_t E^\varepsilon - \nabla \times B^\varepsilon = -\varepsilon J^\varepsilon, &\qquad& \nabla\cdot E^\varepsilon = \rho^\varepsilon, \\ &\varepsilon\partial_t B^\varepsilon + \nabla \times E^\varepsilon = 0, && \nabla\cdot B^\varepsilon = 0. \end{aligned}$$

For the terms of order $O(1)$ this yields

$$\begin{aligned} &\nabla \times E^0 = 0, &\qquad& \nabla\cdot E^0 = \rho^0, &\qquad& E^0 = -\nabla V^0, \\ &\nabla \times B^0 = 0, && \nabla\cdot B^0 = 0. \end{aligned}$$

From this we can deduce

$$-\Delta V^0 = \rho^0, \qquad\qquad B^0 = 0.$$

For $O(\varepsilon)$ we have

$$\begin{aligned} \partial_t E^0 - \nabla\times B^1 &= -J^0, &\qquad& \nabla\cdot B^1 = 0, &\qquad& \partial_t V^0 = \nabla\times A^1, \\ \nabla\times E^1 &= 0, && \nabla\cdot E^1 = 0. \end{aligned}$$

From this we obtain (using $B^1 = \nabla \times A^1$ and $\nabla \times (\nabla \times A^1) = \nabla(\nabla \cdot A^1) - \Delta A^1$)

$$-\Delta A^1 = J^0, \qquad\qquad E^1 = 0.$$

From this discussion and the definition of the fields we obtain that up to first order in ε we have $\varepsilon A^1 = A^\varepsilon + O(\varepsilon^2)$ and $V^0 = V^\varepsilon + O(\varepsilon^2)$. This gives the $O(\varepsilon)$ approximation of Maxwell's equations, i.e. the **Poisswell equations**.

$$-\Delta V^\varepsilon(x,t) = \rho(x,t), \qquad\qquad -\Delta A^\varepsilon(x,t) = \varepsilon J(x,t). \tag{4.10}$$

Since the equation for the magnetic potential A^ε up to order $O(\varepsilon)$ resembles the electrostatic (Poisson) equation for V^ε one can consider it as a *magnetostatic* approximation. We also remark that since $B^0 = 0$, the magnetic field B^ε is a relativistic object. Since the Pauli equation is $O(\varepsilon)$ as well it is natural to couple it to the Poisswell equations (4.10) to arrive at the Pauli-Poisswell equation.

Another important question is the choice of gauge for the Pauli-Poisswell system. In the derivation above we used Lorenz gauge for Maxwell's equations. As it turns out this fixes the gauge for the Pauli-Poisswell system, namely the Lorenz gauge (1.1). When expanding the magnetic Laplacian

$$-\frac{1}{2}(\hbar\nabla - i\varepsilon A)^2,$$

we are therefore not allowed to assume $\mathrm{div}A = 0$. On the other hand when we choose Coulomb gauge

$$\mathrm{div}A = 0,$$

Maxwell's equations become

$$\Box A = -\varepsilon J + \varepsilon\partial_t\nabla V, \qquad \Delta V = -\rho.$$

Since $\nabla \times (\nabla V) = 0$, the second term in the wave equation for A is rotation-free. Since any vector field can be decomposed into a divergence-free and a rotation-free part we can introduce the Helmholtz projection $\mathbb{P}$ onto divergence free fields and write

$$\Box A = -\varepsilon\mathbb{P}J, \qquad \Delta V = -\rho.$$

Now we can introduce the same Hilbert expansion (4.8)-(4.9) as above and plug it into Maxwell's equations, the Coulomb gauge and the field equations:

$$B^0 = 0, \qquad E^1 = 0,$$
$$V^1 = 0, \qquad A^0 = 0.$$

and

$$\Delta V^0 = -\rho, \qquad \Delta A^1 = -J + \partial_t\nabla V^0,$$

as well as

$$\mathrm{div}A^1 = 0.$$

Again, $-J + \partial_t\nabla V^0$ is the divergence-free part of J, so

$$-\Delta V^\hbar = \rho, \qquad \Delta A^\hbar = -\varepsilon\mathbb{P}J.$$

So we obtain a magnetostatic approximation of Maxwell's equations with a divergence free current as source term. Note that the electric field E in this approximation is given by

$$E^\varepsilon = E^0 + O(\varepsilon^2) = -\nabla V^0 + O(\varepsilon^2).$$

because there is no $\partial_t A$ term which means that E^ε is rotation-free. This is precisely the difference between the Darwin and Poisswell approximation of Maxwell's equations.

4.2 Global finite energy weak solutions

In this section we will prove the global existence of weak solutions to the Pauli-Poisswell equation (4.1)-(4.4).

Theorem 3. *Let $\Psi_0 \in (H^1(\mathbb{R}^3))^2$. Then there is at least one solution to* (1.21)-(1.25) *such that $\Psi \in BC_w(\mathbb{R}, (H^1(\mathbb{R}^3))^2)$.*

We will follow a compactness argument based on *parabolic regularization*. This means adding a "dissipation" (or "viscosity") term $\epsilon\Delta$ which regularizes the equation, yielding strong solutions to the regularized system by a contraction argument. Taking the limit $\epsilon \to 0$ will be justified by obtaining suitable uniform bounds to the regularized solution.

We note that there is no natural scaling for the Pauli-Poiswell equation.

Lemma 4.1. *The Pauli-Poisswell equation* (4.1)-(4.4) *does not admit a scaling that leaves the equation invariant.*

Proof. Define

$$\Psi_\lambda := \lambda^\alpha \Psi(\lambda^2 t, \lambda x) \qquad A_\lambda := \lambda^\beta A(\lambda^2 t, \lambda x) \qquad V_\lambda := \lambda^\gamma V(\lambda^2 t, \lambda x),$$

and plug this into the Pauli-Poisswell equation (4.1)-(4.3). We obtain

$$i\lambda^{2+\alpha}\partial_t \Psi = -\frac{1}{2}\lambda^{2+\alpha}\Delta\Psi + i\lambda^{\alpha+\beta+1}A\cdot\nabla\Psi + \lambda^{\alpha+2\beta}\frac{|A|^2}{2}\Psi + \lambda^{\gamma+\alpha}V\Psi - \frac{i}{2}\lambda^{\alpha+\beta+1}(\sigma\cdot B)\Psi,$$

yielding

$$\beta = 1, \gamma = 2.$$

On the other hand,

$$-\Delta V_\lambda = -\lambda^{\gamma+2}\Delta V = \lambda^{2\alpha}|\Psi|^2,$$

yielding

$$2 + \gamma = 2\alpha \quad \Rightarrow \quad \alpha = 2.$$

But,

$$-\Delta A_\lambda = -\lambda^{\beta+2}\Delta A = \lambda^{2\alpha+1}(\overline{\Psi}\nabla\Psi - \nabla\times(\overline{\Psi}\sigma\Psi)) - \lambda^{2\alpha+\beta}A|\Psi|^2,$$

yielding

$$\beta + 2 = 2\alpha + 1 = 2\alpha + \beta$$

which has no solution. It is clear that there can be no other scaling $(\lambda^\delta x, \lambda^\kappa t)$ since we always have $\delta = 2\kappa$ from the linear Schrödinger part $i\partial_t\Psi + \Delta\Psi$ of the Pauli equation and doing the calculation for this case also yields a contradiction analogous to the one above. Hence there is no scaling of Ψ that leaves the equation invariant. $\square$

In the spirit of [49] the regularized system can be written as

$$\partial_t \Psi - \frac{i+\epsilon}{2}\Delta\Psi = K_\epsilon \tag{4.11}$$

where

$$K_\epsilon = i(\Delta^{-1}|\Psi|^2)\Psi + \frac{i}{2}(\sigma\cdot B)\Psi - \frac{i+\epsilon}{2}(2iA\cdot\nabla\Psi + |A|^2\Psi). \tag{4.12}$$

together with the Poisson equation for A,

$$-\Delta A = J, \tag{4.13}$$

$$J(\Psi, A) = \mathrm{Im}(\overline{\Psi}(\nabla - iA)\Psi) - \nabla\times(\overline{\Psi}\sigma\Psi), \tag{4.14}$$

and the initial data

$$\Psi(x,0) = \Psi_0(x). \tag{4.15}$$

Here we represent the potential V as $\Delta^{-1}|\Psi|^2$, i.e. the convolution of $|x|^{-1}$ with $|\Psi|^2$ since we work in three space dimensions. The strategy is to solve the regularized system in a suitable space and to use uniform bounds on A and Ψ in order to extract a subsequence Ψ_{ϵ_n} converging in the weak sense.

Global solutions of the regularized Pauli-Poisswell equation

Formally (4.11) is solved by the integral equation

$$\Psi = G_{\Psi_0}N(\Psi), \tag{4.16}$$

where $N(\Psi) = K_\epsilon$ and G_{Ψ_0} maps Ψ to Ψ^G and where Ψ^G is given by the integral equation

$$\Psi^G(t) = S(t)\Psi_0 + \int_0^t S(t-s)\Psi(s)\mathrm{d}s, \tag{4.17}$$

where $S(t) := \exp((i+\epsilon)t\Delta)$.

Lemma 4.2. *Let $\Psi \in (H^1(\mathbb{R}^3))^2$. Let A be a solution to $\Delta A = J$. Then A satisfies the bound*

$$\|\nabla A\|_2 \le C\|\Psi\|_{\dot{H}^1}. \tag{4.18}$$

Proof. By multiplication of both sides of the equation with A, followed by integration we obtain

$$\int_{\mathbb{R}^3} |A|^2|\Psi|^2 + |\nabla A|^2 = -i\int_{\mathbb{R}^3} A(\overline{\Psi}\nabla\Psi - (\nabla\overline{\Psi})\Psi).$$

The RHS can be estimated by $\|A\Psi\|_2\|\nabla\Psi\|_2$ due to Hölder's inequality. Then $\|A\Psi\|_2^2 \le \|A\Psi\|_2\|\nabla\Psi\|_2$ so $\|A\Psi\|_2 \le \|\nabla\Psi\|_2$ which implies $\|\nabla A\|_2 \le \|\nabla\Psi\|_2$ and the claim follows. $\square$

Lemma 4.3. *For all $u \in H^1(\mathbb{R}^3)^2$ the operator K_P defined by*

$$K_P u = 2iA \cdot \nabla\Psi + |A|^2 u + (\sigma \cdot B)\Psi, \tag{4.19}$$

satisfies the estimate

$$\|K_P \Psi\|_{L^{3/2}} \lesssim \|\Psi\|_{\dot{H}^1}^2 \tag{4.20}$$

Proof. This is a simple application of Lemma 4.2 and Sobolev embedding:

$$\begin{aligned}\|K^P\Psi\|_{W^{k-1,3/2}} &\lesssim \|A\|_6\|\nabla\Psi\|_2 + \|A\|_6\|A\Psi\|_2 + \|B\|_2\|\Psi\|_6 \\ &\lesssim \|\Psi\|_{H^1}^2 + \|\Psi\|_{H^1}^3,\end{aligned}$$

since $\|B\|_2 = \|\nabla A\|_2 \lesssim \|\Psi\|_{H^1}$. □

Next we show a priori bounds on the solution Ψ. Define X^k as the space of all functions in $C(I, H^k)$ such that $\nabla\Psi \in L^2(I, H^k)$ with the norm

$$\|\Psi\|_{X^k} := \|\Psi\|_{L^\infty(I,H^k)} + \|\nabla\Psi\|_{L^2(I,H^k)}. \tag{4.21}$$

The following lemma is inspired by the analogous results for the Schrödinger-Maxwell equation in [49].

Lemma 4.4. *Let $\epsilon > 0$ and $k \geq 1$. Then G_{Ψ_0} (cf. (4.17)) maps $L^2(I, W^{k,3/2})$ to X^k with*

$$\|G_{\Psi_0}\Psi\|_{X^k} \lesssim_\epsilon \|\Psi_0\|_{H^k} + T^{1/4}\|\Psi\|_{L^2(I,W^{k,3/2})}. \tag{4.22}$$

Proof. Without loss of generality assume $k = 0$. Let $\tilde{\Psi} = S(t)\Psi_0$ be the solution of the linear initial value problem

$$\partial_t\tilde{\Psi} - (i + \epsilon)\Delta\tilde{\Psi} = 0, \quad \tilde{\Psi}(t = 0) = \Psi_0.$$

We have the standard energy estimate of the heat-Schrödinger flow

$$\|S(t)\Psi_0\|_2^2 + 2\epsilon \int_0^t \|\nabla(s)\Psi_0\|_2^2 \mathrm{d}s = \|\Psi_0\|_2^2, \tag{4.23}$$

as well as the $L^p - L^q$ estimates for the heat evolution operator $\exp(\epsilon t\Delta)$

$$\|S(t)f\|_2 \lesssim_\epsilon t^{-1/4}\|f\|_{3/2}, \tag{4.24}$$

$$\|\nabla S(t)f\|_2 \lesssim_\epsilon t^{-3/4}\|f\|_{3/2}. \tag{4.25}$$

Using (4.24) and Hölder's inequality we deduce

$$\begin{aligned}\left\|\int_0^t S(t-s)\Psi(s)\mathrm{d}s\right\| &\lesssim_\epsilon \int_0^t (t-s)^{-1/4}\|\Psi(s)\|_{3/2}\mathrm{d}s \\ &\lesssim t^{1/4}\left(\int_0^t \|\Psi(s)\|_{3/2}^2\mathrm{d}s\right)^{1/2}.\end{aligned}$$

from which we obtain $\Psi^G \in C(I, L^2)$. Moreover, by (4.25), the Hardy-Littlewood-Sobolev inequality and Hölder's inequality we have

$$\begin{aligned}\int_0^t \|\nabla \int_0^\tau S(\tau - s)\Psi(s)\mathrm{d}s\|_2^2 \mathrm{d}\tau &\lesssim \int_0^t \left(\int_0^\tau (\tau - s)^{-3/4}\|\Psi(s)\|_{3/2}\mathrm{d}s\right)^2 \mathrm{d}\tau \\ &\leq \left(\int_0^t \|\Psi(\tau)\|_{3/2}^{4/3}\mathrm{d}\tau\right)^{3/2} \\ &\leq t^{1/2}\int_0^t \|\Psi(\tau)\|_{3/2}^2 \mathrm{d}\tau.\end{aligned}$$

From the last two inequalities and (4.23) we deduce

$$\|\Psi^G\|_{L^\infty(I,H^k)} + \|\nabla\Psi^G\|_{L^2(I,H^k)} \lesssim_\epsilon (\|\Psi_0\|_{H^k} + T^{1/4}\|\Psi\|_{L^2(I,W^{k,3/2})}),$$

which implies (4.22) for $k = 0$ and the lemma is proved. □

In order to set up a contraction argument in X^k we need to establish local Lipschitz properties of the nonlinearity K_ϵ.

Lemma 4.5. *The nonlinearity $N(\Psi) = K_\epsilon$ maps X^1 to $L^2(I, W^{1,3/2})$ and we have the estimates*

$$\|N(\Psi)\|_{L^2(I,W^{1,3/2})} \lesssim_\epsilon (1 + T^{1/2})\beta(\|\Psi\|_{X^1})\|\Psi\|_{X^1}, \tag{4.26}$$

$$\|N(\Psi) - N(\Phi)\|_{L^2(I,W^{1,3/2})} \lesssim_\epsilon (1 + T^{1/2})\beta(\|\Psi\|_{X^1} + \|\Phi\|_{X^1})\|\Psi\|_{X^1}, \tag{4.27}$$

where $\beta(r) = r + r^2 + r^3$.

Proof. We abbreviate $g_1 = (\Delta^{-1}|\Psi|^2)\Psi$, $g_2 = A\cdot\nabla\Psi$, $g_3 = |A|^2\Psi$ and $g_4 = (\sigma\cdot B)\Psi$ such that $K_\epsilon = ig_1 + ig_4 - (i+\epsilon)(2ig_2 + g_3)$. Then

$$\begin{aligned}\|g_1\|_{3/2} &\leq \|\Delta^{-1}|\Psi|^2\|_6\|\Psi\|_2 \lesssim \||\Psi|^2\|_{6/5}\|\Psi\|_2^2 \\ &\leq \|\Psi\|_3\|\Psi\|_2^2 \lesssim \|\Psi\|_{H^1}^3,\end{aligned}$$

where we used Hardy-Littlewood-Sobolev and Hölder. Moreover,

$$\begin{aligned}\|\nabla^k g_1\|_{3/2} &\leq \sum_{\alpha+\beta=k}\|\nabla^\alpha(\Delta^{-1}|\Psi|^2)\|_6\|\nabla^\beta\Psi\|_2 \\ &\lesssim \||\Psi|^2\|_{6/5}\|\nabla^k\Psi\|_2 + \sum_{\substack{\alpha+\beta=k\\ \alpha\neq 0}}\|\nabla^\alpha|\Psi|^2\|_{6/5}\|\nabla^\beta\Psi\|_2 \\ &\lesssim \|\Psi\|_{H^1}^2\|\Psi\|_{H^k} + \sum_{\substack{\alpha+\beta=k\\ \alpha\neq 0}}\left[\sum_{s+r=\alpha}\|\nabla^s\Psi\|_{\frac{6\alpha}{(2\alpha+s)}}\|\nabla^r\Psi\|_{\frac{6\alpha}{(2\alpha+r)}}\right]\|\nabla^\beta\Psi\|_2 \\ &\lesssim \|\Psi\|_{H^1}^2\|\Psi\|_{H^k} + \sum_{\substack{\alpha+\beta=k\\ \alpha\neq 0}}\|\nabla^\alpha\Psi\|_2\|\Psi\|_3\|\nabla^\beta\Psi\|_2 \\ &\lesssim \|\Psi\|_{H^1}^2\|\Psi\|_{H^k}.\end{aligned}$$

Taking the L^2-norm in time yields

$$\|\nabla^k g_1\|_{L^2(I,L^{3/2})} = \left(\int_0^T |\|\Psi\|_{H^1}^2 \|\Psi\|_{H^k}|\right)^{\frac{1}{2}} \lesssim T^{1/2}\|\Psi\|_{L^\infty(I,H^1)}^2 \|\Psi\|_{L^\infty(I,H^k)}.$$

For g_2, observe

$$\|g_2\|_{3/2} \leq \|A\|_6 \|\nabla\Psi\|_2 \lesssim \|\Psi\|_{H^1}^2,$$

by Lemma 4.2. We have the following estimates for the derivatives of g_2:

$$\begin{aligned}\|\nabla g_2\|_{3/2} &\lesssim \sum_{\alpha+\beta=1} \|\nabla^\alpha A\|_{\frac{6}{(1+2\alpha)}} \|\nabla^\beta \nabla\Psi\|_{\frac{6}{(1+2\beta)}} \\ &= \|\nabla A\|_2 \|\nabla\Psi\|_6 + \|A\|_6 \|\nabla^2\Psi\|_2 \\ &\lesssim \|\Psi\|_{H^1} \|\nabla\Psi\|_{H^1}.\end{aligned}$$

It follows that

$$\|\nabla g_2\|_{L^2(I,L^{3/2})} \lesssim \|\Psi\|_{L^\infty(I,H^1)} \|\nabla\Psi\|_{L^2(I,H^1)}.$$

Finally for g_3 we obtain

$$\|g_3\|_{3/2} \leq \|A\|_6^2 \|\nabla\Psi\|_3 \lesssim \|\Psi\|_{H^1}^3,$$

and

$$\begin{aligned}\|\nabla g_3\|_{3/2} &\lesssim \|A\|_6^2 \|\nabla\Psi\|_3 + \|A\|_6 \|\nabla A\|_2 \|\Psi\|_\infty \\ &\lesssim \|\Psi\|_{H^1}^2 \|\nabla\Psi\|_{H^1}.\end{aligned}$$

Thus,

$$\|\nabla g_3\|_{L^2(I,L^{3/2})} \lesssim \|\Psi\|_{L^\infty(I,H^1)}^2 \|\nabla\Psi\|_{L^2(I,H^1)}.$$

Finally for g_4 we have

$$\|g_4\|_{3/2} \lesssim \|\nabla A\|_2 \|\Psi\|_6 \lesssim \|\Psi\|_{H^1}^2,$$

and

$$\begin{aligned}\|\nabla g_4\|_{3/2} &\lesssim \|\nabla^2 A\Psi\|_{3/2} + \|\nabla A \cdot \nabla\Psi\|_{3/2} \\ &\lesssim \|J\|_2 \|\Psi\|_{H^1} + \|\Psi\|_{H^1} \|\nabla\Psi\|_{H^1} \\ &\lesssim \|\Psi\|_{H^1}^2 \|\nabla\Psi\|_2 + \|\Psi\|_{H^1}^4.\end{aligned}$$

From the estimates for g_1, g_2, g_3 and g_4 we obtain

$$\|K_\epsilon\|_{L^2(I,L^{3/2})} \lesssim T^{1/2}(\|\Psi\|_{L^\infty(I,H^1)}^3 + \|\Psi\|_{L^\infty(I,H^1)}^2),$$

and from the estimate on the derivatives

$$\|K_\epsilon\|_{L^2(I,W^{1,3/2})} \lesssim (1+\epsilon)(1+T^{1/2})(\|\Psi\|_{X^1} + \|\Psi\|_{X^1}^2 + \|\Psi\|_{X^1}^3)\|\Psi\|_{X^1},$$

proving the claim. $\square$

Next we establish local wellposedness for the regularized system using a contraction argument. We will also show that the the blow-up alternative holds in the smaller norm H^1 and that we can approximate solutions in X^1.

Lemma 4.6. *Let $\Psi_0 \in (H^1(\mathbb{R}^3))^2$. Then there is a $T_{\max} > 0$ such that the initial value problem*

$$\Psi = G_{\Psi_0} N(\Psi)$$

has a unique solution in $X^1(T)$ for all $0 < T < T_{\max}$ and such that for $T_{\max} < \infty$ we have

$$\lim_{t \uparrow T_{\max}} \|\Psi(t)\|_{H^1} = \infty. \tag{4.28}$$

Moreover, if $\Psi_{0,n} \in H^1$ with $\Psi_{0,n} \to \Psi_0$ in H^1 and if $0 < T < T_{\max}$ then the solution Ψ_n of $\Psi_n = G_{\Psi_{0,n}} N(\Psi_n)$ is in $X^1(T)$ for n large enough and $\Psi_n \to \Psi$ in $X^1(T)$.

Proof. Combining Lemmas 4.4 and 4.5 we obtain

$$\begin{aligned} \|G_{\Psi_0} N(\Psi)\|_{X^1} &\leq \alpha \|\Psi_0\|_{H^1} + T^{1/4} \|N(\Psi)\|_{L^2(I, W^{1,3/2})} \\ &\leq \alpha \|\Psi_0\|_{H^1} + (T^{1/4} + T^{3/4}) \beta(\|\Psi\|_{X^1}) \|\Psi\|_{X^1}, \end{aligned} \tag{4.29}$$

and

$$\|G_{\Psi_0} N(\Psi) - G_{\Psi_0}(N(\Phi)\|_{X^1} \lesssim (T^{1/4} + T^{3/4}) \beta(\|\Psi\|_{X^1} + \|\Phi\|_{X^1}) \|\Psi - \Phi\|_{X^1}.$$

Now choose $R > 0$ and $T_1 = T_1(R) > 0$ such that $\|\Psi_0\|_{H^1} < R$ and $(T_1^{1/4} + T_1^{1/2}) \beta(4\alpha R) < 1/2$. Define $B_1 = B_{T_1}(0, 2\alpha R)$ as the closed ball of radius $2\alpha R$ with center zero. With these choices, $G_{\Psi_0} N(\cdot)$ is a contraction mapping from B_1 to itself and therefore admits a unique solution Ψ in B_1. Moreover T_1 depends only on R and thus there is a $T_{\max}$ such that Ψ extends to a unique solution in $X^k(T)$ for all $0 < T < T_{\max}$. Moreover,

$$\lim_{t \uparrow T_{max}} \|\Psi(t)\|_{X^1(T)} = \infty.$$

In order to show (4.28) assume the contrary. Assume there exists a constant $L > 0$ such that $\|\Psi(t\|_{H^1} < L$ for $0 \leq t < T_{\max}$. By the theory developed above there is a $T_1 > 0$ depending on L only such that the initial value problem $\Phi = G_{\Phi_0} N(\Phi)$ has a unique solution $\Phi \in B_1$ for all $\Phi_0 \in H^1$ with $\|\Phi_0\|_H^1 < L$. Let $0 < \delta < T_1$ and $T_* := T_{\max} - \delta$ and let Ψ_* be the unique solution to $\Psi_* = G_{\Psi(T*)} N(\Psi_*)$. It is easy to see that $\Psi_*(t) = \Psi(t + T_*)$ for $0 \leq t < \delta$ whence $\Psi_* \in X^1(T)$ and (4.29) holds for Ψ_* for $0 < T < \delta$. It follows that

$$\begin{aligned} \|\Psi_*\|_X^1(T) &= \|G_{\Psi(T*)} N(\Psi_*)\|_{X^1(T)} \\ &\leq \alpha \|\Psi(T_*)\|_{H^1} + (T^{1/4} + T^{3/4}) \beta(\|\Psi_*\|_{X^1(T)}) \|\Psi_*\|_{X^1(T)} \\ &\leq \alpha \|\Psi(T_*)\|_{H^1} + (\delta^{1/4} + \delta^{3/4}) \beta(2\alpha L) \|\Psi_*\|_{X^1(T)}. \end{aligned}$$

Choosing δ small enough such that $(\delta^{1/4} + \delta^{3/4}) \beta(2\alpha L) < 1/2$ we obtain

$$\|\Psi_*\|_{X^1(T)} < 2\alpha \|\Psi(T_*)\|_{H^1}$$

with $0 < T < \delta$. Letting $t \uparrow \delta$ in the equation above is equivalent to letting $t \uparrow T_{\max}$ for $\|\Psi(t)\|_{X^k}(T)$ since Ψ coincides with Ψ_* on $[0, \delta)$. But then

$$\lim_{t \uparrow T_{max}} \|\Psi(t)\|_{X^1(T)} \leq 2\alpha \|\Psi(T_*)\|_{H^1} \tag{4.30}$$

which is a contradiction to (4.6). Hence (4.28) is proved.

For the final part of the lemma we can restrict ourselves to showing $\Psi_n \to \Psi$ in $X^1(T_1)$ with T_1 as before. Since $\Psi_{0,n} \to \Psi_0$ in H^1 there is a n such that $\|\Psi_{0,n}\|_{H^1} < R$. It follows that $u, \Psi_n \in B_1$. Using lemmas 4.4 and 4.5 we conclude

$$\|\Psi - \Psi_n\|_{X^1(T_1)} \leq \alpha \|\Psi_0 - \Psi_{0,n}\|_{H^1} + (T_k^{1/4} + T_k^{3/4})\beta(4\alpha R)\|\Psi - \Psi_n\|_{X^1(T_1)}$$

Since $\beta(T_1, 4\alpha R)$ is chosen to be smaller than $1/2$ we obtain convergence of Ψ_n to Ψ in $X^1(T_1)$. $\square$

We are now able to state the existence and uniqueness result as well as the a priori uniform bounds on the solution of the regularized system.

Lemma 4.7. *Let $\epsilon > 0$. Assume that the initial data Ψ_0 is in $(H^1(\mathbb{R}^3))^2$. Then there is a unique solution Ψ to the regularized Pauli-Poisswell system (4.11)-(4.15) such that $\Psi \in C(\mathbb{R}, H^1)$ and $\nabla\Psi \in L^2(\mathbb{R}, H^1)$. Moreover it holds that $\|\Psi\|_{H^1} < C(\|\Psi_0\|_{H^1})$ where C is independent of t and ϵ.*

Proof. By Lemma 4.6 there is a unique solution $\Psi \in C([0,T), H^1)$ with $\nabla\Psi \in L^2([0,T), H^1)$ for some $T > 0$. By (4.28) we deduce that $T = \infty$ if the H^1-norm of Ψ does not blow up in finite time. Next we show that the energy is bounded a priori at time $t = 0$. First, observe that $\|(\nabla - iA)\Psi_0\|_2^2 \leq (\|\nabla\Psi_0\|_2 + \|A\|_6\|\Psi_0\|_3)^2$. All terms involved can be estimated by $\|\Psi_0\|_{H^1}$. Furthermore we have the estimate $\|\nabla A\|_2^2 \lesssim \|\Psi_0\|_{H^1}^4$. Finally,

$$\begin{aligned}
\|(\Delta^{-1}|\Psi_0|^2)^{1/2}\Psi_0\|_2 &\leq \|(\Delta^{-1}|\Psi_0|^2)^{1/2}\|_{12}\|\Psi_0\|_{12/5} \\
&\leq \|(\Delta^{-1}|\Psi_0|^2)\|_6^{1/2}\|\Psi_0\|_{12/5} \\
&\leq \|\Psi_0\|_{12/5}^3
\end{aligned}$$

Now $2 < 12/5 < 6$, so the last expression can be estimated by $\|\Psi_0\|_{H^1}^3$ by Sobolev's embedding. We conclude that

$$E(0) < C(\|\Psi_0\|_{H^1}). \tag{4.31}$$

One can achieve an a priori bound on $\|\nabla\Psi\|_2$ together without smallness assumption:

$$\begin{aligned}
\|\nabla\Psi\|_2 &\leq \|(\nabla - iA)\Psi\|_2 + \|A\Psi\|_2 \\
&\leq E(0)^{1/2} + \|A\|_6\|\Psi\|_3 \\
&\leq E(0)^{1/2} + \|\nabla A\|_2\|\Psi\|_2^{1/2}\|\Psi\|_6^{1/2} \\
&\leq E(0)^{1/2} + CQ(0)^{1/4}\|\nabla\Psi\|_2^{3/2},
\end{aligned}$$

where we have used Hölder, Gagliardo-Nirenberg-Sobolev and Sobolev embedding. From the definition of the energy we know that $\|\nabla A\|_2^2 \leq E(0)$ so eventually we obtain

$$\|\nabla\Psi\|_2 \leq E(0)^{1/2} + CQ(0)^{1/4}E(0)^{1/2} = C(E(0), Q(0)) \tag{4.32}$$

Together with $\|\Psi\|_2^2 \leq Q(0) = \|\Psi_0\|^2$ and (4.31) one obtains the a priori bound $\|\Psi\|_{H^1} \leq C$. The last part of Lemma 4.6 states that the smoothness assumption can be made without loss of generality. □

Global solutions of the Pauli-Poisswell equation

Proof of Theorem 3. Consider a sequence $\{\epsilon_n\}$ of parameters such that $\epsilon_n \to 0$ as $n \to \infty$. Due to Lemma 4.7 there is a solution Ψ_n for each n to the system

$$\begin{aligned}
\partial_t\Psi_n - (i+\epsilon_n)\Delta\Psi_n &= K_{\epsilon_n}(\Psi_n, A_n)\\
\Delta A_n &= -J(\Psi_n, A_n),
\end{aligned}$$

such that $\Psi_n \in C(\mathbb{R}, H^1)$ and initial value Ψ_0. By the Hölder inequality and Sobolev embedding we can estimate

$$\|J(\Psi_n, A_n)\|_{3/2} \leq \|\Psi_n\|_6\|\nabla\Psi_n\|_2 + \|A_n\|_6\|\Psi_n\|_4^2 < \|\Psi_n\|_{H^1}^2(1 + \|\Psi_n\|_{H^1}) < C.$$

Similary,

$$\begin{aligned}
\|K_{\epsilon_n}(\Psi_n, A_n)\|_{3/2} &\leq \|\Psi_n^2\|_{6/5}\|\Psi_n\|_2 + \|A_n\|_6\|\nabla\Psi_n\|_2 + \|A_n\|_6^2\|\Psi_n\|_3\\
&\lesssim \|\Psi_n\|_{H^1}^3 < C.
\end{aligned}$$

The sequence $\{\Psi_n\}$ is bounded in H^1 by virtue of the uniform a priori estimate from Lemma 4.7. The boundedness of these sequences allows us to infer weak*-convergence of the terms involved (by the Banach-Alaoglu theorem), i.e. there is a subsequence which we also denote by Ψ_n such that $\Psi_n \rightharpoonup^* \Psi$ in $L^\infty(\mathbb{R}, H^1)$. Moreover there exist $\alpha, \beta \in L^\infty(\mathbb{R}, L^{3/2})$ such that

$$J(\Psi_n, A_n) \rightharpoonup^* \beta \qquad\qquad K_{\epsilon_n}(\Psi_n, A_n) \rightharpoonup^* \alpha$$

in $L^\infty(\mathbb{R}, L^{3/2})$. Thus we can pass to the limit in the equation and obtain

$$\partial_t\Psi - i\Delta\Psi = \alpha \tag{4.33}$$

$$\Delta A = -\beta, \tag{4.34}$$

in $\mathcal{D}'(\mathbb{R}, H^{-1})$. It remains to establish that the pair (Ψ, A) is the desired solution by showing that the initial condition is satisfied and that $\alpha = K_0(\Psi, A)$ and $\beta = J(\Psi, A)$. From (4.33) and the fact that $\Delta\Psi \in L^\infty(\mathbb{R}, H^{-1})$ since $\Delta\colon H^1(\mathbb{R}^3) \to H^{-1}(\mathbb{R}^3$ is a bounded operator we deduce that $\partial_t\Psi \in L^\infty$. Together with the weak*-convergence of Ψ_n to Ψ in $L^\infty(\mathbb{R}, H^1)$ we deduce that Ψ is weakly continuous from $\mathbb{R}$ to H^1.

To see that Ψ satisfies the initial condition $\Psi_{t=0} = \Psi_0$ we denote by $\langle\cdot,\cdot\rangle$ the duality pairing. Then

$$\int_0^t \langle \Psi_n(s)\partial_t f(s) + (i\Delta\Psi_n + K_{\epsilon_n})f(s), \chi\rangle_{H^1,H^{-1}} = -\langle \Psi_0, \chi\rangle_{H^1,H^{-1}},$$

for all $\chi \in L^2$ and all $f \in C^\infty(\mathbb{R})$ with $f(0) = 1$ and $f(t) = 0$ (this follows from integration by parts). Then taking the limit $n \to \infty$ and using (4.33) and the fact that $\Psi_n \rightharpoonup \Psi$ weak* in $L^\infty(I, H^1)$ we obtain

$$\int_0^t \langle \Psi(s)\partial_t f(s) + \partial_t u(s)f(s), \chi\rangle_{H^1,H^{-1}} = -\Psi_0,$$

implying that $\Psi(0) = \Psi_0$.

Next, we show that we have convergence to the right limits by using a standard compactness argument. First, observe that by Rellich-Kondrachov the embedding $H^1(\Omega) \Subset L^4(\Omega)$ is compact for $\Omega \subset \mathbb{R}^3$ open and bounded. Moreover the embedding $L^4(\Omega) \subset H^{-1}(\Omega)$ is continuous. As Ψ_n is bounded in $L^\infty(I, H^1)$ and $\partial_t\Psi_n$ is bounded in $L^\infty(I, H^{-1})$ the same holds for Ψ_n in $L^4(I, H^1(\Omega))$ and $\partial_t\Psi_n$ in $L^4(I, H^{-1}(\Omega))$. By the Aubin-Lions compactness lemma (cf. [83, Section 7.3]) the embedding $W^{1,4,4}(I, H^1(\Omega), H^{-1}(\Omega)) \Subset L^4(I \times \Omega)$ is compact and Ψ_n converges to Ψ strongly in $L^4(I \times \Omega)$ up to a subsequence. Moreover since $\|\Psi_n(t)\|_{H^1} < C$ we have

$$\||\Psi_n|^2\|_{L^2(I,L^{6/5})} \le \|\Psi_n\|^2_{L^2(I,L^{12/5})} \le \|\Psi_n\|^2_{L^2(I,H^1)} < \infty$$

by Sobolev embedding. Next, $\Delta^{-1}\colon L^{6/5} \to L^6$ is a bounded operator, hence $\Delta^{-1}|\Psi_n|^2$ is bounded in $L^2(I, L^6)$. Since Ψ_n converges strongly in $L^4(I\times\Omega)$ and thus pointwise a.e. in $I\times\Omega$ up to a subsequence we deduce that $|\Psi_n|^2$ converges weakly to $|\Psi|^2$ in $L^2(I, L^{6/5})$ and thus $\Delta^{-1}|\Psi_n|^2$ converges weakly to $\Delta^{-1}|\Psi|^2$ in $L^2(I, L^6)$. By Lemma 4.8, A_n converges strongly in $L^6(I \times \Omega)$. Thus we can conclude that K_{ϵ_n} converges weakly in $L^{3/2}(I \times \Omega)$ and $J(\Psi_n, A_n)$ converges weakls in $L^{6/5}(I \times \Omega)$ completing the proof. $\square$

Lemma 4.8. *Let $\{\Psi_n\}$ be a bounded sequence in $L^\infty(I\times H^1(\Omega))$ solving the Pauli equation. Then the sequence $\{A_n\}$ defined by $\Delta A_n = -2\,\mathrm{Im}(\overline{\Psi_n}(\nabla - iA_n)\Psi_n)$ converges strongly in $L^6(I \times \Omega)$ to A defined by $\Delta A = -2\,\mathrm{Im}(\overline{\Psi}(\nabla - iA)\Psi)$.*

Proof. Subtract the equations for A and A_n, multiply by $A - A_n$ and integrate by parts:

$$-\int(\nabla(A - A_n))^2 \mathrm{d}x - \int (A - A_n)^2|\Psi_n|^2 \mathrm{d}x \simeq \int (A - A_n)(\overline{\Psi}\nabla\Psi - \overline{\Psi}_n\nabla\Psi_n)\mathrm{d}x$$
$$+ \int (A - A_n)A_n(|\Psi|^2 - |\Psi_n|^2)\mathrm{d}x.$$

The second term on the RHS can be estimated by $\|A - A_n\|_6\|A_n\|_6\||\Psi|^2 - |\Psi|_n^2\|_{3/2}$. For the first term observe

$$\int (A - A_n)(\overline{\Psi}\nabla\Psi - \overline{\Psi}_n\nabla\Psi_n)\mathrm{d}x = \int (A - A_n)(\overline{\Psi} - \overline{\Psi}_n)\nabla\Psi \mathrm{d}x$$
$$+ \int (A - A_n)\overline{\Psi}_n(\nabla\Psi - \nabla\Psi_n)\mathrm{d}x$$

The first term on the RHS can be estimated by $\|A-A_n\|_6\|\overline{\Psi}-\overline{\Psi}_n\|_3\|\nabla\Psi\|_2$. Now integrate the remaining term by parts to obtain

$$\begin{aligned}\int (A-A_n)\overline{\Psi}_n(\nabla\Psi-\nabla\Psi_n)\mathrm{d}x = &-\int \nabla(A-A_n)\overline{\Psi}_n(\Psi-\Psi_n)\mathrm{d}x\\ &-\int (A-A_n)\nabla\overline{\Psi}_n(\Psi-\Psi_n)\mathrm{d}x\end{aligned}$$

The first term can be controlled by $\|\nabla(A-A_n)\|_2\|\Psi_n\|_4\|\Psi-\Psi_n\|_4$ whereas the second one can be controlled by $\|A-A_n\|_6\|\nabla\Psi_n\|_2\|\Psi-\Psi_n\|_3$. By Sobolev embedding, $\|A_n\|_{L^6(\mathbb{R}^3)} \lesssim \|A_n\|_{\dot{H}^1(\mathbb{R}^3)}$ which concludes the proof. □

4.3 Local strong solutions

In this section we prove the unique existence of local solutions to the Pauli-Poisswell equation (4.1)-(4.4). We use standard energy estimate techniques combined which lead to an a priori bound on the H^s norm of Ψ. Local existence follows straightforward from a standard fixed point argument.

Theorem 4. *Let $s>5/2$, $T>0$ and $\Psi_0\in H^s(\mathbb{R}^3)$. Then the Pauli-Poisswell equation (4.1)-(4.4) has a unique solution $\Psi\in C([0,T),H^s(\mathbb{R}^3)$ with initial data Ψ_0.*

Proof. We apply ∇^s to (4.1), multiply by $\nabla^s\overline{\Psi}$ and integrate over $\mathbb{R}^3$:

$$\begin{aligned}\int \nabla^s\overline{\Psi}\cdot\partial_t\nabla^s\Psi - i\int\nabla^s\overline{\Psi}\cdot\Delta\nabla^s\Psi = &\int\nabla^s\overline{\Psi}\cdot\nabla^s(A\cdot\nabla\Psi)-\frac{i}{2}\int\nabla^s\overline{\Psi}\cdot\nabla^s(|A|^2\Psi)\\ &-i\int\nabla^s\overline{\Psi}\cdot\nabla^s(V\Psi)+\frac{i}{2}\int\nabla^s\overline{\Psi}\cdot\nabla^s(\sigma\cdot B)\end{aligned}$$

Taking the real part and using $2\,\mathrm{Re}(\overline{f}\partial_t f)=\partial_t|f|^2$ yields

$$\begin{aligned}\frac{1}{2}\partial_t\int|\nabla^s\Psi|^2 = &\,\mathrm{Re}\int\nabla^s\overline{\Psi}\cdot\nabla^s(A\cdot\nabla\Psi)-\mathrm{Re}\,\frac{i}{2}\int\nabla^s\overline{\Psi}\cdot\nabla^s(|A|^2\Psi)\\ &-\mathrm{Re}\,i\int\nabla^s\overline{\Psi}\cdot\nabla^s(V\Psi)+\mathrm{Re}\,\frac{i}{2}\int\nabla^s\overline{\Psi}\cdot\nabla^s(\sigma\cdot B)\end{aligned}$$

Now we can write

$$\mathrm{Re}\int\nabla^s\overline{\Psi}\cdot\nabla^s(A\cdot\nabla\Psi)=\mathrm{Re}\int(\nabla^s(A\cdot\nabla\Psi)-A\cdot\nabla^s\nabla\Psi)\nabla^s\overline{\Psi}+\mathrm{Re}\int(A\cdot\nabla^s\nabla\Psi)\nabla^s\overline{\Psi}.$$

By partial integration the last term on the RHS is equal to

$$-\frac{1}{2}\int(\nabla\cdot A)|\nabla^s\Psi|^2=0,$$

due to $\nabla \cdot A = 0$. The first term on the RHS can be estimated due to the Kato-Ponce inequality,

$$\|\nabla^s(A \cdot \nabla\Psi) - A \cdot \nabla^s\nabla\Psi\|_2\|\nabla^s\Psi\|_2 \lesssim (\|\nabla A\|_\infty\|\nabla\Psi\|_{\dot{H}^{s-1}} + \|A\|_{\dot{H}^s}\|\nabla\Psi\|_\infty)\|\Psi\|_{\dot{H}^s} \tag{4.35}$$

This means we have to control $\nabla^s A$ in L^2 or ∇A in $\dot{H}^{s-1}$. We can generalize Lemma 4.2 to H^s as follows.

Lemma 4.9. *Let A be defined by (1.23) and $s > 0$. Then*

$$\|\nabla^s A\|_2 \leq C(s, \|\Psi\|_{H^1}, \|\Psi\|_{H^2}, \|\Psi\|_\infty)\|u\|_{\dot{H}^s}.$$

Proof. Apply ∇^s to (1.23) and multiply by $\nabla^s A$ (recall that A is real-valued). By the same reasoning as before we obtain

$$\|\nabla^{s+1}A\|_2^2 \lesssim \|\nabla^s(\Psi\nabla_A\Psi)\|_2\|\nabla^s A\|_2$$

By the trivial estimate $\|\nabla^s A\|_2 \leq \|\nabla^{s+1}A\|_2$ we can absorb this term into the LHS. By virtue of the Kato-Ponce inequality the RHS can be bounded by

$$\begin{aligned}\|\nabla^s(\Psi\nabla_A\Psi)\|_2 &\lesssim \|\nabla^s\Psi\|_3\|\nabla_A\Psi\|_6 + \|\Psi\|_\infty\|\nabla^s\nabla_A\Psi\|_2. \\ &\lesssim \|\Psi\|_{\dot{H}^{s+1/2}}(\|\nabla\Psi\|_6 + \|A\Psi\|_6) + \|\Psi\|_\infty(\|\Psi\|_{\dot{H}^{s+1}} + \|\nabla^s(A\Psi)\|_2) \quad \lesssim\end{aligned}$$

By the Kato-Ponce inequality we obtain $\|\nabla^s(A\Psi)\|_2 \lesssim \|\nabla^s A\|_2\|\Psi\|_\infty + \|A\|_\infty\|\Psi\|_{\dot{H}^s}$ which implies together with the inequality above

$$\begin{aligned}\|\nabla^{s+1}A\|_2 \lesssim &\|\Psi\|_{\dot{H}^{s+1/2}}(\|\Psi\|_{\dot{H}^2} + \|A\|_\infty\|\Psi\|_{\dot{H}^1}) + \\ &\|\Psi\|_\infty\|\Psi\|_{\dot{H}^{s+1}} + \|\nabla^s A\|_2\|\Psi\|_\infty^2 + \|A\|_\infty\|\Psi\|_{\dot{H}^s}.\end{aligned} \tag{4.36}$$

By the Gagliardo-Nirenberg interpolation inequality,

$$\begin{aligned}\|A\|_\infty &\lesssim \|A\|_6 + \|A\|_{\dot{W}^{2,6}} \leq \|\nabla A\|_2 + \|J\|_6 \\ &\lesssim \|\Psi\|_{\dot{H}^1} + \|\Psi\|_3\|\nabla\Psi\|_2 + \|\Psi\|_{H^1}\|\Psi\|_\infty^2 \\ &\lesssim \|\Psi\|_{H^1} + \|\Psi\|_{H^1}^2 + \|\Psi\|_{H^1}\|\Psi\|_\infty^2\end{aligned}$$

Now Gagliardo-Nirenberg interpolation for non-integer s (see Brezis-Mironescu) gives

$$\|\nabla^s A\|_2 \lesssim \|\nabla A_2\|^{1/s}\|\nabla^{s+1}A\|_2^{(s-1)/s}.$$

We combine this with Young's inequality for products to obtain

$$\|\nabla^s A\|_2\|\Psi\|_\infty^2 \leq \frac{1}{p}(\|\nabla A\|_2^{1/s}\|\Psi\|_\infty^2)^p + \frac{1}{q}\|\nabla^{s+1}A\|_2^{q(s-1)/s}.$$

Now we can choose q such that $q(s-1)/s = 1$ which implies $q = s/(s-1) > 1$. Therefore we can absorb the second term on the RHS above into the left hand side. Then $p = s$ and the first term on the RHS becomes $(1/s)\|\nabla A\|_2\|\Psi\|_\infty^{2s}$. Combining this with (4.36) and the trivial embeddings $\dot{H}^s \hookrightarrow \dot{H}^{s+1/2} \hookrightarrow \dot{H}^{s+1}$ yields

$$\|\nabla^{s+1} A\|_2 \leq C(s, \|\Psi\|_{H^1}, \|\Psi\|_{H^2}, \|\Psi\|_\infty)\|\Psi\|_{\dot{H}^{s+1}}.$$

proving the claim. $\square$

Returning to (4.35) we obtain

$$\begin{aligned}\|\nabla^s(A\cdot\nabla\Psi) - A\cdot\nabla^s\nabla\Psi\|_2 &\lesssim \|A\|_{W^{1,\infty}}\|\Psi\|_{H^s} + \|\nabla A\|_{\dot{H}^{s-1}}\|\Psi\|_{W^{1,\infty}} \\ &\lesssim C(s, \|\Psi\|_{H^1}, \|\Psi\|_{H^2}, \|\Psi\|_{W^{1,\infty}})\|\Psi\|_{H^s}\end{aligned}$$

We also need estimates for the other terms, starting with

$$\operatorname{Re} i \int \nabla^s\overline{\Psi}\cdot\nabla^s(V\Psi).$$

Here, we have to estimate

$$\|\nabla^s(V\Psi)\|_2$$

where $V = (-\Delta)^{-1}|\Psi|^2$.

Lemma 4.10. *Let V be defined by (4.2). Then,*

$$\|((-\Delta)^{-1}|\Psi|^2)\Psi\|_{H^s} \lesssim \max\{\|\Psi\|_{H^1}^2, \|\Psi\|_\infty)\}\|\Psi\|_{H^s}. \tag{4.37}$$

Proof. We use Kato-Ponce to estimate

$$\|((-\Delta)^{-1}|\Psi|^2)\Psi\|_{H^s} \lesssim \|(-\Delta)^{-1}|\Psi|^2\|_{p_1}\|\Psi\|_{W^{s,p_2}} + \|(-\Delta)^{-1}|\Psi|^2\|_{W^{s,p_3}}\|\Psi\|_{p_4},$$

where $1/2 = 1/p_1 + 1/p_2 = 1/p_3 + 1/p_4$. For the first term on the RHS we have $\Delta^{-1}\colon L^r \to L^{p_1}$ for $0 < 1/r = 1/p_1 + 2/3 < 1$. We choose $p_1 = \infty$, $p_2 = 2$ and $1/r = 2/3$. Then we can use Sobolev's embedding $H^1 \hookrightarrow L^3$ to estimate the first term by $\|\Psi\|_{H^1}^2\|\Psi\|_{H^s}$. For the second term we choose $p_3 = 2$, $p_4 = \infty$, yielding $\|(-\Delta)^{-1}|\Psi|^2\|_{\dot{H}^s}\|\Psi\|_\infty$. Now we can write

$$\|(-\Delta)^{-1}|\Psi|^2\|_{\dot{H}^s} = \|\omega^{s-2}|\Psi|^2\|_2 \lesssim \|\Psi\|_{\dot{H}^{s-2}}\|\Psi\|_\infty.$$

Together with the trivial estimate $\|\Psi\|_{\dot{H}^{s-2}} \leq \|\Psi\|_{H^s}$ we obtain

$$\|((-\Delta)^{-1}|\Psi|^2)\Psi\|_{H^s} \lesssim (\|\Psi\|_{H^1}^2 + \|\Psi\|_\infty)\|\Psi\|_{H^s}.$$

which is what we wanted to show. $\square$

For the other terms we have to estimate

$$\|\nabla^s(|A|^2\Psi)\|_2 \qquad \text{and} \qquad \|\nabla^s((\sigma\cdot B)\Psi)\|_2.$$

For the first one note

$$\begin{aligned}\|\nabla^s(|A|^2\Psi)\|_2 &\lesssim \|A\|_\infty(\|A\|_{\dot{H}^s}\|\Psi\|_\infty + \|A\|_\infty\|\Psi\|_{\dot{H}^s}), \\ &\lesssim C(s, \|\Psi\|_{H^1}, \|\Psi\|_{H^2}, \|\Psi\|_{W^{1,\infty}})\|\Psi\|_{H^s}\end{aligned}$$

For the second one we have

$$\begin{aligned}\|\nabla^s((\sigma\cdot B)\Psi)\|_2 &\lesssim \|\nabla A\|_{\dot{H}^s}\|\Psi\|_\infty + \|\nabla A\|_\infty\|\Psi\|_{\dot{H}^s},\\ &\lesssim C(s, \|\Psi\|_{H^1}, \|\Psi\|_{H^2}, \|\Psi\|_{W^{1,\infty}})\|\Psi\|_{H^s}\end{aligned}$$

Now we can state the a priori estimate

Lemma 4.11. *Let Ψ be a solution to (4.1)-(4.4) and let $s > 2$. Then,*

$$\|\Psi(t)\|_{H^s}^2 \leq 2\|\Psi(0)\|_{H^s}^2 \exp\left(\int_0^t C(s, \|\Psi\|_{H^1}, \|\Psi\|_{H^2}, \|\Psi\|_{W^{1,\infty}})\|\Psi\|_{H^s}\mathrm{d}\tau\right) \tag{4.38}$$

Proof. This follows from the estimates so far and Grönwall's inequality. □

The theorem then follows from a standard fixed point argument using the a priori estimate from Lemma 4.11. □

5 The semiclassical limit of Pauli-Poisswell to Vlasov-Poisswell: Wigner

This section is based on [73].

In this section we discuss the semiclassical limit of the Pauli-Poisswell equation using Wigner transforms. Similar to the Pauli-Poisson equation we first formulate the mixed state Pauli-Poisswell equation. Note that now, also the magnetic potential $A^\hbar$ depends on $\hbar$ which is the reason why we write it with a superscript, like $V^\hbar$.

Let $\{\Psi_j^\hbar\}_{j\in\mathbb{N}}$ be an $(L^2(\mathbb{R}^3))^2$-orthonormal system solving the **mixed state Pauli-Poisswell equation**

$$i\hbar\partial_t\Psi_j^\hbar = -\frac{1}{2}(\hbar\nabla - iA^\hbar)^2\Psi_j^\hbar + V^\hbar\Psi_j^\hbar - \frac{1}{2}\hbar(\sigma\cdot B^\hbar)\Psi_j^\hbar, \quad j=1,\dots,\infty \tag{5.1}$$

$$-\Delta V^\hbar = \sum_{j=1}^\infty \lambda_j^\hbar|\Psi_j^\hbar|^2 = \rho_{\text{diag}}^\hbar, \tag{5.2}$$

$$-\Delta A^\hbar = J^\hbar, \tag{5.3}$$

$$\Psi_j^\hbar(x,0) = \Psi_{j,I}^\hbar(x) \in (L^2(\mathbb{R}^3))^2. \tag{5.4}$$

where the mixed state Pauli current density is given by

$$J^\hbar(\Psi^\hbar, A^\hbar) = \sum_{j=1}^\infty \lambda_j^\hbar\left[\mathrm{Im}(\overline{\Psi_j^\hbar}(\hbar\nabla - iA^\hbar)\Psi_j^\hbar) - \hbar\nabla\times(\overline{\Psi_j^\hbar}\sigma\Psi_j^\hbar)\right], \tag{5.5}$$

or equivalently

$$J^\hbar = \sum_{j=1}^\infty \lambda_j^\hbar\,\mathrm{Re}\left(\overline{\Psi_j^\hbar}\sigma(\sigma\cdot(-i\hbar\nabla - A^\hbar))\Psi_j^\hbar\right), \tag{5.6}$$

It can be written as a first order moment of the Wigner matrix,

$$J^\hbar = \int_{\mathbb{R}^3_\xi}\mathrm{Tr}\Big(\sigma(\sigma\cdot(\xi - A^\hbar)F^\hbar)\Big)\mathrm{d}\xi \tag{5.7}$$

The *Pauli-Wigner-Poisswell equation* reads

$$\begin{aligned}\partial_t F^\hbar + \xi\cdot\nabla_x F^\hbar - \mathcal{F}_y[\beta[A^\hbar]] *_\xi \nabla_x F^\hbar - i\mathcal{F}_y[\delta[A^\hbar]] *_\xi (\xi F^\hbar)&\\ +\frac{1}{2}\theta[(A^\hbar)^2]F^\hbar - \frac{\hbar}{2}\theta[\sigma\cdot B^\hbar]F^\hbar + \theta[V^\hbar]F^\hbar = 0,&\end{aligned} \tag{5.8}$$

$$-\Delta V^\hbar = \rho_{\text{diag}}^\hbar, \tag{5.9}$$

$$-\Delta A^\hbar = J^\hbar \tag{5.10}$$

$$F^\hbar(t=0) = F_I^\hbar, \tag{5.11}$$

where $\theta[\cdot]$ is the pseudo-differential operator defined by (3.16). The energy of the Pauli-Wigner equation is given by

$$E(t) := \operatorname{tr}\Big(H_0 R^\hbar(t)\Big) + \iint_{\mathbb{R}^3_x\times\mathbb{R}^3_y} \frac{\rho^\hbar_{\mathrm{diag}}(x,t)\rho^\hbar_{\mathrm{diag}}(y,t)}{|x-y|}\mathrm{d}x\mathrm{d}y + \int |\nabla A^\hbar(x,t)|^2\mathrm{d}x. \quad (5.12)$$

where the second term is equivalent to

$$\int_{\mathbb{R}^3_x} |\nabla V^\hbar(x,t)|^2\mathrm{d}x. \quad (5.13)$$

We will show below (Lemma 5.1) that the matrix Husimi function $\tilde{F}^\hbar$ solves the *Husimi equation*

$$\begin{aligned}\partial_t\tilde{F}^\hbar + \frac{\hbar}{2}\nabla_x\cdot\nabla_\xi\tilde{F}^\hbar + \xi\cdot\nabla_x\tilde{F}^\hbar - i\frac{\hbar}{2}\mathcal{F}_y[\delta[A^\hbar]] *_\xi (\nabla_\xi\tilde{F}^\hbar) - i\mathcal{F}_y[\delta[A^\hbar]] *_\xi (\xi\tilde{F}^\hbar)\\ +\mathcal{F}_y[\beta[A^\hbar]] *_\xi \nabla_x\tilde{F}^\hbar + \frac{1}{2}\theta[(A^\hbar)^2]\tilde{F}^\hbar - \frac{\hbar}{2}\theta[\sigma\cdot B]\tilde{F}^\hbar + \theta[V^\hbar]\tilde{F}^\hbar = 0.\end{aligned} \quad (5.14)$$

We will use the Husimi equation to pass to the limit because the regularity obtained from the Poisson equation for $A^\hbar$ is not enough for an argument using the mere Wigner equation.

As outlined in Section 2.2 and Lemma 2.1 the uniform boundedness of the Husimi function follows from a bound on the ℓ^∞ norm of the sequence $\{\lambda^\hbar\}$ of occupation probabilities. Therefore we make the following assumption.

Assumption 2. *Let $R^\hbar$ or $\rho^\hbar$ be a matrix valued density matrix or density matrix defined by an orthonormal system $\{\Psi^\hbar_j\}\subset (L^2(\mathbb{R}^3))^2$ and occupation probabilities $\lambda^\hbar_j\in[0,1]$. We assume that*

$$\lambda^\hbar_j \geq 0, \quad \sum_{j=1}^\infty \lambda^\hbar_j = 1, \quad (5.15)$$

$$\|\lambda^\hbar\|_\infty = \sup_{j\in\mathbb{N}}\lambda^\hbar_j \leq (2\pi\hbar)^3. \quad (5.16)$$

Remark 6. The reason why uniform boundedness of $F^\hbar$ in L^2 is not enough is because it implies that $\delta[(A^\hbar)^2]$ should converge to $A\cdot(\nabla\otimes A)$ in L^2_{loc}. By Sobolev's inequality we would need that $A^\hbar$ converges strongly in $W^{1,12/5}(\mathbb{R}^3_x)$. If $A^\hbar$ has less regularity than this we would need $F^\hbar\in L^p$ uniformly in $\hbar$ for $p>2$ but this is not possible in general.

We have the following conjecture (which is not a theorem yet).

Conjecture. *Let $\{\Psi^\hbar_j\}_{j\in\mathbb{N}}\in C(\mathbb{R}_t,(L^2(\mathbb{R}^3_x))^2)$ be a solution of the mixed state Pauli-Poisswell equation (5.1)-(5.4) with associated matrix valued density matrix $R^\hbar$ such that the occuptation probabilities satisfy Assumption 2. Let $\tilde{F}^\hbar$ be the associated matrix Husimi* *function and suppose it solves the matrix Husimi equation (5.14) with initial data $\{\tilde{F}_I^\hbar\}$ bounded uniformly in $(L^1\cap L^\infty(\mathbb{R}^6))^{2\times 2}$ where $A^\hbar$ and $V^\hbar$ are given by the Poisswell equations*

$$-\Delta V^\hbar(x,t)=\rho^\hbar_{\mathrm{diag}}(x,t), \qquad -\Delta A^\hbar(x,t)=J^\hbar(x,t).$$

with $\rho^\hbar_{\mathrm{diag}}$ and $J^\hbar$ as defined in (2.32) and (5.7), respectively. Moreover, suppose that the initial energy $E(0)$ in (5.12) is bounded independently of $\hbar$. Assume that $F_I^\hbar$ converges weakly to F_I in $(L^1\cap L^\infty(\mathbb{R}^6))^{2\times 2}$. Then $\tilde{F}^\hbar$ converges weakly up to a subsequence in $L^\infty(I,(L^q(\mathbb{R}^3_x\times\mathbb{R}^3_p))^{2\times 2})$ (equipped with the weak topology) to*

$$F\in C_b(\mathbb{R}_t,\mathcal{M}^{2\times 2}_{w*})\cap L^\infty(I,(L^q(\mathbb{R}^3_x\times\mathbb{R}^3_p)^{2\times 2}),$$

for all $1\le q\le\infty$ such that F solves the **Vlasov-Poisswell equation**

$$\partial_t F+p\cdot\nabla_x F-(\nabla_x V+p\times B)\cdot\nabla_p F=0,\quad B=\nabla\times A \tag{5.17}$$

$$-\Delta V(x,t)=\rho_{\mathrm{diag}}(x,t), \qquad \rho_{\mathrm{diag}}(x,t)=\int_{\mathbb{R}^3_p} f(x,p,t)\mathrm{d}p, \tag{5.18}$$

$$-\Delta A(x,t)=J(x,t), \qquad J(x,t)=\int_{\mathbb{R}^3_p} pf(x,p,t)\mathrm{d}p, \tag{5.19}$$

where $f=\mathrm{Tr}(F)$ in the distributional sense verifying the initial condition

$$F(x,p,0)=F_I(x,p),\quad \text{in } \mathbb{R}^3_x\times\mathbb{R}^3_p. \tag{5.20}$$

Classical solutions for the Vlasov-Poisswell equation where studied by Seehafer in [85]:

Theorem (Seehafer)**.** *For every non-negative $f_I\in C^2_c(\mathbb{R}^3_x\times\mathbb{R}^3_\xi)$ there exists a $T^*>0$ and a classical solution $f\in C^1([0,T^*)\times\mathbb{R}^3_x\times\mathbb{R}^3_\xi)$ of the Vlasov-Poisswell equation (5.17)-(5.20) satisfying $f(x,\xi,0)=f_I(x,\xi)$.*

Now we discuss the ansatz for the proof of the semiclassical limit of the Pauli-Poisswell equation. The uniform boundedness of $\tilde{F}^\hbar$ under Assumption 2 was proved in Lemma 2.1. Conservation of energy is discussed in Lemma 5.2.

Lemma 5.1. *If $F^\hbar$ satisfies the Wigner equation (5.8), then the Husimi function $\tilde{F}^\hbar\in L^1(\mathbb{R}^6)$ satisfies the* Husimi equation

$$\begin{aligned}\partial_t\tilde{F}^\hbar+\frac{\hbar}{2}\nabla_x\cdot\nabla_\xi\tilde{F}^\hbar+\xi\cdot\nabla_x\tilde{F}^\hbar-i\frac{\hbar}{2}\mathcal{F}_y[\delta[A^\hbar]]*_\xi(\nabla_\xi\tilde{F}^\hbar)-i\mathcal{F}_y[\delta[A^\hbar]]*_\xi(\xi\tilde{F}^\hbar)\\+\mathcal{F}_y[\beta[A^\hbar]]*_\xi\nabla_x\tilde{F}^\hbar+\frac{1}{2}\theta[(A^\hbar)^2]\tilde{F}^\hbar-\frac{\hbar}{2}\theta[\sigma\cdot B]\tilde{F}^\hbar+\theta[V^\hbar]\tilde{F}^\hbar=0.\end{aligned} \tag{5.21}$$

Proof. Consider the Pauli-Wigner equation

$$\begin{aligned}\partial_t F^\hbar + \xi\cdot\nabla_x F^\hbar - \mathcal{F}_y[\beta[A^\hbar]] *_\xi \nabla_x F^\hbar - i\mathcal{F}_y[\delta[A^\hbar]] *_\xi (\xi F^\hbar)\\ + \frac{1}{2}\theta[(A^\hbar)^2]F^\hbar - \frac{\hbar}{2}\theta[\sigma\cdot B]F^\hbar + \theta[V^\hbar]F^\hbar = 0.\end{aligned}\tag{5.22}$$

Now take the convolution in x and ξ with the function $G^\hbar(z) = (\pi\hbar)^{-3/2}\exp\big(|z|^2/\hbar\big)$ (i.e. the Husimi transform). This operation commutes with the time derivative and the potential terms, but one obtains an additional term when exchanging the ξ-convolution with the multiplication by ξ. More precisely we have to calculate the commutator of the operators $C\colon \psi \mapsto G^\hbar *_\xi \psi$ acting on $L^1(\mathbb{R}^6)$ and $M\colon \psi\mapsto \xi\psi$. We claim that $[C,M](\nabla_x\psi) = (\hbar/2)\nabla_x\cdot\nabla_\xi(G^\hbar *_\xi \psi)$. Indeed,

$$\begin{aligned}G^\hbar *_\xi (\xi\nabla_x\psi) - \xi G^\hbar * \nabla_x\psi &= \frac{1}{(\pi\hbar)^{\frac{3}{2}}}\int_{\mathbb{R}^3_\eta}((\xi-\eta)\nabla_x\psi(\xi-\eta) - \xi\cdot\nabla_x\psi(\xi-\eta))e^{-|\eta|^2/\hbar}\mathrm{d}\eta,\\ &= -\frac{1}{(\pi\hbar)^{\frac{3}{2}}}\nabla_x\int \eta\psi(\xi-\eta))e^{-|\eta|^2/\hbar}\mathrm{d}\eta,\\ &= \frac{\hbar}{2}\nabla_x\int\nabla_\eta G^\hbar(\eta)\psi(\xi-\eta)\mathrm{d}\eta,\\ &= \frac{\hbar}{2}\nabla_x\cdot\nabla_\xi(G^\hbar *_\xi \psi).\end{aligned}$$

Now replace ψ by $G^\hbar *_x F^\hbar$ and we obtain, using $CM = [C,M] + MC$,

$$G^\hbar *_{x,\xi} (\xi\cdot\nabla_x f^\hbar) = \frac{\hbar}{2}\nabla_x\cdot\nabla_\xi\tilde{F}^\hbar + \xi\cdot\nabla_x\tilde{F}^\hbar,$$

which proves the claim. Following the same reasoning we apply $G^\hbar *_{x,\xi}$ to the term $-i\widehat{\delta[A^\hbar]} *_\xi (\xi F^\hbar)$ and obtain

$$G^\hbar *_{x,\xi} (-i\widehat{\delta[A^\hbar]} *_\xi (\xi F^\hbar)) = -i\frac{\hbar}{2}\widehat{\delta[A^\hbar]} *_\xi (\nabla_\xi\tilde{F}^\hbar) - i\widehat{\delta[A^\hbar]} *_\xi (\xi\tilde{F}^\hbar).$$

The other terms commute with $G^\hbar *_{x,\xi}$ so that the first part of the lemma follows. □

Remark 7. Note that the Husimi equation is not well-posed in any $L^p(\mathbb{R}^3)$ since the resolvent set of the operator $(-\hbar/2)\nabla_x\cdot\nabla_\xi$ contains no real numbers, as can be easily seen by taking the Fourier transform (if $((-\hbar/2)\nabla_x\cdot\nabla_\xi - \lambda)\Psi(x,\xi) = g(x,\xi)$, then $\widehat{\Psi}(\eta,y) = ((\hbar/2)\eta\cdot y - \lambda)^{-1}\widehat{g}(\eta,y)$ for $\eta, y\in\mathbb{R}^3$. Since $\eta\cdot y$ takes values in all of $\mathbb{R}^3$, the expression $((\hbar/2)\eta\cdot y-\lambda)^{-1}$ is never surjective). However, if the potentials and the initial data are in $H^s(\mathbb{R}^3)$ for $s > 3/2$ one can define a C-existence family for the Husimi operator which produces a strong solution for the Husimi equation. See [27].

Lemma 5.2. *Suppose $\{\Psi_j^\hbar\}$ is a L^2-bounded family solving (5.1)-(5.4). Then the energy*

$$E(t) = \frac{1}{2}\int\sum_{j=1}^\infty \lambda_j^\hbar|\sigma\cdot(\hbar\nabla - iA^\hbar)\Psi_j^\hbar|^2\mathrm{d}x + \frac{1}{2}\int|\nabla A^\hbar(t)|^2 + |\nabla V^\hbar(t)|^2\mathrm{d}x,$$

is conserved, i.e. $\mathrm{d}E(t)/\mathrm{d}t = 0$.

Proof. In the following we omit the calculations involving the $V^\hbar$ term since it is anaologuous to the energy of the Schrödinger-Poisson equation and would make calculations unnecessarily cumbersome. Let us take the time derivative of the first term

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int\sum_{j=1}^{\infty}\lambda_j^\hbar|\sigma\cdot(\hbar\nabla-iA^\hbar)\Psi_j^\hbar|^2\mathrm{d}x,$$

and show that it is equal to $-(1/2)\partial_t\int|\nabla A^\hbar|^2\mathrm{d}x$. We have

$$\begin{aligned}
&=\sum_{j=1}^{\infty}\lambda_j^\hbar\operatorname{Re}\int\overline{(\sigma\cdot(\hbar\nabla-iA^\hbar))\Psi_j^\hbar}\partial_t(\sigma\cdot(\hbar\nabla-iA^\hbar))\Psi_j^\hbar\mathrm{d}x,\\
&=\sum_{j=1}^{\infty}\lambda_j^\hbar\operatorname{Re}\int\overline{(\sigma\cdot(\hbar\nabla-iA^\hbar))\Psi_j^\hbar}((\sigma\cdot(\hbar\nabla-iA^\hbar))\partial_t\Psi_j^\hbar-i(\sigma\cdot(\partial_tA^\hbar))\Psi_j^\hbar)\mathrm{d}x,\\
&=-\frac{1}{2}\sum_{j=1}^{\infty}\lambda_j^\hbar\operatorname{Re}i\int|(\sigma\cdot(\hbar\nabla-iA^\hbar))^2\Psi_j^\hbar|^2\mathrm{d}x-\operatorname{Re}i\int\overline{(\sigma\cdot(\hbar\nabla-iA^\hbar))\Psi_j^\hbar}(\sigma\cdot(\partial_tA^\hbar))\Psi_j^\hbar\mathrm{d}x,\\
&=-\sum_{j=1}^{\infty}\lambda_j^\hbar\operatorname{Re}\int i\overline{(\sigma\cdot\hbar\nabla)\Psi_j^\hbar}(\sigma\cdot(\partial_tA^\hbar))\Psi_j^\hbar\mathrm{d}x+\int A^\hbar\partial_tA^\hbar|\Psi_j^\hbar|^2\mathrm{d}x,
\end{aligned}\tag{5.23}$$

where we used the equation and partial integration in the third equality and the Pauli vector identity in the fourth step by observing that $A\times\partial_tA=0$. Observe now that

$$\operatorname{Re}i\Psi_j^\hbar\overline{(\sigma\cdot\hbar\nabla)\Psi_j^\hbar}=-\operatorname{Re}i\overline{\Psi_j^\hbar}(\sigma\cdot\hbar\nabla)\Psi_j^\hbar,$$

and thus

$$-\operatorname{Re}i\Psi_j^\hbar\overline{(\sigma\cdot\hbar\nabla)\Psi_j^\hbar}(\sigma\cdot(\partial_tA^\hbar))=\frac{1}{2}\operatorname{Re}i(\overline{\Psi_j^\hbar}(\sigma\cdot\hbar\nabla)\Psi_j^\hbar-\Psi_j^\hbar\overline{(\sigma\cdot\hbar\nabla)\Psi_j^\hbar})(\sigma\cdot(\partial_tA^\hbar)).\tag{5.24}$$

Plugging (5.24) in (5.23) yields

$$\begin{aligned}
\cdots&=\frac{1}{2}\operatorname{Re}\int i(\overline{\Psi_j^\hbar}(\sigma\cdot\hbar\nabla)\Psi_j^\hbar-\Psi_j^\hbar\overline{(\sigma\cdot\hbar\nabla)\Psi_j^\hbar})(\sigma\cdot(\partial_tA^\hbar)))\mathrm{d}x+\int A^\hbar\partial_tA^\hbar|\Psi_j^\hbar|^2\mathrm{d}x,\\
&=\operatorname{Re}\int(\Delta A^\hbar-A^\hbar|\Psi_j^\hbar|^2)\partial_tA^\hbar\mathrm{d}x+\int A^\hbar\partial_tA^\hbar|\Psi_j^\hbar|^2\mathrm{d}x,\\
&=-\frac{1}{2}\int\partial_t|\nabla A^\hbar|^2-\int A^\hbar(\partial_tA^\hbar)|\Psi_j^\hbar|^2\mathrm{d}x+\int A^\hbar(\partial_tA^\hbar)|\Psi_j^\hbar|^2\mathrm{d}x,\\
&=-\frac{1}{2}\int\partial_t|\nabla A^\hbar|^2,
\end{aligned}$$

where in the second step we have used

$$\Delta A^\hbar-A^\hbar|\Psi_j^\hbar|^2=i\hbar(\overline{\Psi_j^\hbar}\nabla\Psi_j^\hbar-\overline{\nabla\Psi_j^\hbar}\Psi_j^\hbar+\hbar\nabla\times(\overline{\Psi_j^\hbar}\sigma\Psi_j^\hbar),$$

and in the third step we have used partial integration on the $\Delta A^\hbar\partial_tA^\hbar$ term. $\square$

Lemma 5.3. *Let $J^\hbar$ be the Pauli current density defined by (1.24). Then*

$$\partial_t J^\hbar \simeq -\nabla \mathbb{T}^\hbar + (\nabla V^\hbar - \partial_t A^\hbar - \nabla(\sigma \cdot B^\hbar))\rho^\hbar_{\text{diag}} + 2\mathbb{F} J^\hbar \tag{5.25}$$

where $\mathbb{T}^\hbar$ is defined by

$$(\mathbb{T}^\hbar)_{\ell m} := \frac{2}{\hbar}\sum_{j=1}^\infty \lambda_j^\hbar \operatorname{Re}(\nabla_{A,\ell}\Psi_j^\hbar \otimes \overline{\nabla_{A,m}\Psi_j^\hbar}) - \frac{1}{2}\Delta\rho^\hbar_{\text{diag}}\delta_{\ell m}, \tag{5.26}$$

and $\mathbb{F}^\hbar$ is defined by

$$(\mathbb{F}^\hbar)_{\ell m} := \partial_\ell A_m^\hbar - \partial_m A_\ell^\hbar. \tag{5.27}$$

Proof. The term $\nabla \times (\overline{\Psi_j^\hbar}\sigma\Psi_j^\hbar)$ is of the same type as $\operatorname{Im}(\overline{\Psi_j^\hbar}\nabla\Psi_j^\hbar)$ and we therefore omit it for the sake of simplicity. Now,

$$\begin{aligned}
\partial_t J^\hbar &= \sum_{j=1}^\infty \lambda_j^\hbar \operatorname{Im}(\partial_t\overline{\Psi_j^\hbar}\nabla_A\Psi_j^\hbar + \overline{\Psi_j^\hbar}\partial_t\nabla_A\Psi_j^\hbar)\\
&= \sum_{j=1}^\infty \lambda_j^\hbar \operatorname{Im}(\partial_t\overline{\Psi_j^\hbar}\nabla_A\Psi_j^\hbar + \overline{\Psi_j^\hbar}\nabla_A\partial_t\Psi_j^\hbar) - (\partial_t A^\hbar)\sum_{j=1}^\infty \lambda_j^\hbar|\Psi_j^\hbar|^2
\end{aligned}$$

Using the Pauli equation and its complex conjugate we obtain

$$\begin{aligned}
\partial_t J^\hbar =& \frac{1}{\hbar}\sum_{j=1}^\infty \lambda_j^\hbar \operatorname{Im}(-i\overline{\Delta_A\Psi_j^\hbar}\nabla_A\Psi_j^\hbar + i\overline{\Psi_j^\hbar}\nabla_A\Delta_A\Psi_j^\hbar)\\
&+ \frac{1}{\hbar}\sum_{j=1}^\infty \lambda_j^\hbar \operatorname{Im}(iV^\hbar\overline{\Psi_j^\hbar}\nabla_A\Psi_j^\hbar - i\overline{\Psi_j^\hbar}\nabla_A V^\hbar\Psi_j^\hbar)\\
&+ \sum_{j=1}^\infty \lambda_j^\hbar \operatorname{Im}\left\{-i\overline{(\sigma\cdot B^\hbar)\Psi_j^\hbar}\nabla_A\Psi_j^\hbar + i\overline{\Psi_j^\hbar}\nabla_A(\sigma\cdot B^\hbar)\Psi_j^\hbar\right\} - (\partial_t A^\hbar)\rho^\hbar_{\text{diag}}
\end{aligned} \tag{5.28}$$

For the second term in (5.28) we have

$$-\frac{1}{\hbar}\sum_{j=1}^\infty \lambda_j^\hbar \operatorname{Re}(V^\hbar\overline{\Psi_j^\hbar}\nabla_A\Psi_j^\hbar - \overline{\Psi_j^\hbar}V^\hbar\nabla_A\Psi_j^\hbar - \hbar\nabla V^\hbar|\Psi_j^\hbar|^2) = \nabla V^\hbar\rho^\hbar_{\text{diag}}.$$

and for the third term in (5.28) we have

$$\begin{aligned}
&\sum_{j=1}^\infty \lambda_j^\hbar \operatorname{Re}(\overline{(\sigma\cdot B^\hbar)\Psi_j^\hbar}\nabla_A\Psi_j^\hbar - \overline{\Psi_j^\hbar}\nabla_A(\sigma\cdot B^\hbar)\Psi_j^\hbar)\\
=&\sum_{j=1}^\infty \lambda_j^\hbar \operatorname{Re}\left\{(\overline{(\sigma\cdot B^\hbar)\Psi_j^\hbar} - \overline{\Psi_j^\hbar}(\sigma\cdot B^\hbar))\nabla_A\Psi_j^\hbar - \hbar\overline{\Psi_j^\hbar}(\nabla(\sigma\cdot B^\hbar))\Psi_j^\hbar\right\}\\
=&-\hbar\sum_{j=1}^\infty \lambda_j^\hbar \operatorname{Re}(\overline{\Psi_j^\hbar}(\nabla(\sigma\cdot B^\hbar))\Psi_j^\hbar)\\
\simeq&-\hbar\nabla(\sigma\cdot B^\hbar)\rho^\hbar_{\text{diag}},
\end{aligned}$$

since $\overline{(\sigma\cdot B)\Psi_j^\hbar} - \overline{\Psi_j^\hbar}(\sigma\cdot B) = 0$. We calculate $(\nabla\mathbb{T}^\hbar)_\ell = \nabla_m\mathbb{T}^\hbar_{\ell m}$,

$$\begin{aligned}
\nabla_m\mathbb{T}^\hbar_{\ell m} =& 2\sum_{j=1}^\infty \lambda_j^\hbar \nabla_m \operatorname{Re}\left\{\nabla_{A,\ell}\Psi_j^\hbar \otimes \overline{\nabla_{A,m}\Psi_j^\hbar}\right\} - \frac{1}{2}\nabla_\ell\Delta\rho^\hbar_{\text{diag}}\\
=& 2\sum_{j=1}^\infty \lambda_j^\hbar \operatorname{Re}\left\{\frac{1}{\hbar}(\nabla_{A_m}\nabla_{A,\ell}\Psi_j^\hbar)\otimes\overline{\nabla_{A,m}\Psi_j^\hbar} + \frac{1}{\hbar}\nabla_{A,\ell}\Psi_j^\hbar\otimes\overline{\Delta_A\Psi_j^\hbar}\right\}\\
&-\sum_{j=1}^\infty \lambda_j^\hbar\nabla_\ell|\nabla_A\Psi_j^\hbar|^2 - \nabla_\ell\operatorname{Re}(\overline{\Psi_j^\hbar}\Delta_A\Psi_j^\hbar)\\
=& 2\sum_{j=1}^\infty \lambda_j^\hbar \operatorname{Re}\left\{i\mathbb{F}^\hbar_{\ell m}\Psi_j^\hbar\overline{\nabla_{A,m}\Psi_j^\hbar} + \frac{1}{\hbar}\nabla_{A,\ell}\Psi_j^\hbar\overline{\Delta_A\Psi_j^\hbar} - \frac{1}{2\hbar}\hbar\nabla_\ell(\overline{\Psi_j^\hbar}\Delta_A\Psi_j^\hbar)\right\}
\end{aligned}$$

where we used

$$\frac{1}{\hbar}[\nabla_{A,\ell},\nabla_{A,m}] = i\mathbb{F}^\hbar_{m\ell}.$$

Now we combine the expressions for $\partial_t J^\hbar$ and $\nabla\mathbb{T}^\hbar$,

$$\begin{aligned}
\partial_t J^\hbar_\ell + \nabla_m\mathbb{T}^\hbar_{\ell m} =& (\nabla V^\hbar - \hbar\nabla(\sigma\cdot B^\hbar - \partial A^\hbar)\rho^\hbar_{\text{diag}} + 2\sum_{j=1}^\infty \lambda_j^\hbar\operatorname{Re}\left\{i\mathbb{F}^\hbar_{\ell m}\Psi_j^\hbar\overline{\nabla_{A,m}\Psi_j^\hbar}\right\}\\
&-\frac{1}{\hbar}\sum_{j=1}^\infty\lambda_j^\hbar\operatorname{Re}\left\{\overline{\Delta_A\Psi_j^\hbar}\nabla_A\Psi_j^\hbar + \overline{\Psi_j^\hbar}\nabla_A\Delta_A\Psi_j^\hbar\right.\\
&\left.-2\,\nabla_{A,\ell}\Psi_j^\hbar\overline{\Delta_A\Psi_j^\hbar} + \hbar\nabla_\ell(\overline{\Psi_j^\hbar}\Delta_A\Psi_j^\hbar)\right\}\\
=& (\nabla V^\hbar - \hbar\nabla(\sigma\cdot B^\hbar - \partial A^\hbar)\rho^\hbar_{\text{diag}} + 2\mathbb{F}^\hbar_{\ell m}\sum_{j=1}^\infty\lambda_j^\hbar\operatorname{Im}\left\{\overline{\Psi_j^\hbar}\nabla_{A,m}\Psi_j^\hbar\right\}.
\end{aligned}$$

which proves the claim. □

We now present an approach for a proof of the semiclassical limit. First we show that if $A^\hbar$ is regular enough we can pass to the limit. Then we show where and why some regularity is still missing for the the full proof:

- Uniform estimates for the $\rho^\hbar_{\text{diag}}$ and $J^\hbar$ seem unavaible using magnetic Lieb-Thirring estimates due to missing regularity for $B^\hbar$
- Uniform estimates for $\partial_t J^\hbar$ needed for compactness seem unavailable which is related to the first item

Attempt of Proof of the Conjecture. Let now $\tilde{f}^\hbar = \operatorname{Tr}\tilde{F}^\hbar \in L^q$, $1\le q\le\infty$ where $\tilde{F}^\hbar$ is a solution of the Husimi equation (5.14). From the boundedness of the kinetic energy, i.e. $\operatorname{tr}\big(H_0R^\hbar\big)\le C$, we infer

$$\iint \tilde{f}^\hbar(x,\xi)|\xi - A^\hbar(x)|^2\mathrm{d}x\mathrm{d}\xi \le C.$$

Then,

$$\begin{aligned}\iint \tilde f^\hbar |\xi|^2 \mathrm{d}x\mathrm{d}\xi &\le \iint \tilde f^\hbar |\xi - A^\hbar|^2 \mathrm{d}x\mathrm{d}\xi + \iint \tilde f^\hbar (A^\hbar)^2 \mathrm{d}x\mathrm{d}\xi \\ &\le C + \|\tilde f^\hbar\|_{p'} \|(A^\hbar)\|_{2p}^2 \\ &\le C \langle \|A^\hbar\|_{2p}^2\rangle^2.\end{aligned}$$

where p' is dual to p. Choosing $p = 3$ we infer from the fact that $A^\hbar \in L^6$ uniformly in $\hbar$ (by Lemma 5.2 and Sobolev's inequality) that the RHS is bounded uniformly in $\hbar$. By Lemma 2.2, $\rho^\hbar_{\mathrm{diag}}$ is $\hbar$-oscillatory. Thus it converges weakly to $\rho_{\mathrm{diag}} \in C_b(\mathbb{R}_t, \mathcal{M}_{w^*})$ and

$$\rho_{\mathrm{diag}}(x) = \int_{\mathbb{R}^3_\xi} f(x,\xi)\mathrm{d}\xi.$$

by Proposition 1.

The proof for the convergence of the term involving $V^\hbar$ is analogous to the Pauli-Poisson case. The terms

$$\frac{\hbar}{2}\nabla_x \cdot \nabla_\xi \tilde F^\hbar, \qquad\qquad i\frac{\hbar}{2}\mathcal{F}_y[\delta[A^\hbar]] *_\xi (\nabla_\xi \tilde F^\hbar),$$

converge to zero in any L^q since by partial integration the derivatives act on $\varphi \in \mathcal{S}$ and thus the terms are of order $\mathcal{O}(\hbar)$ as $\hbar \to 0$. We also need to show convergence of

$$\begin{aligned}\varphi(x,z)\frac{\hbar}{2}\delta[\sigma\cdot B^\hbar] &\longrightarrow 0, & iz\varphi(x,z)\delta[A^\hbar] &\longrightarrow iz\varphi(x,z)z\cdot\nabla_x \otimes A,\\ \varphi(x,z)\beta[A^\hbar] &\longrightarrow \varphi(x,z)A(x), & \varphi(x,z)\delta[V^\hbar] &\longrightarrow \varphi(x,z)z\cdot\nabla_x V^\hbar.\end{aligned}$$

for any $\varphi \in \mathcal{S}$ in some $L^q(\mathbb{R}^6)$ as $\hbar \to 0$ for $q \in [1,\infty]$. All these convergences follow once the strong convergence of $A^\hbar$ in $W^{1,r}_{\mathrm{loc}}(\mathbb{R}^3)$ is shown. For the term $\theta[(A^\hbar)^2]\tilde F^\hbar$ we need to show that

$$\frac{1}{2}\delta[(A^\hbar)^2](x,y) \longrightarrow y\cdot A^\hbar(x)(\nabla\otimes A^\hbar(x))$$

strongly in $C([0,T), L^p_{\mathrm{loc}}(\mathbb{R}^3))$ for some $1 \le p \le \infty$ since $\tilde F^\hbar$ converges weakly in $L^1 \cap L^\infty(\mathbb{R}^3_x \times \mathbb{R}^3_\xi)$ to F. Then

$$\frac{1}{2}\langle \theta[(A^\hbar)^2]\tilde F^\hbar, \varphi\rangle_{\mathcal{S}'\times\mathcal{S}}$$

will converge to

$$\begin{aligned}\langle F, ((\nabla\otimes A)A)\cdot\nabla_\xi\varphi\rangle &= \iint ((\nabla\otimes A)A)(x)\cdot\nabla_\xi\varphi(x,\xi)F(x,\xi)\mathrm{d}x\mathrm{d}\xi \\ &= \langle -((\nabla\otimes A)A)\cdot\nabla_\xi F, \varphi\rangle_{\mathcal{S}'\times\mathcal{S}}, \end{aligned} \tag{5.29}$$

by partial integration with respect to ξ.

If we can show this then we can conclude that F solves

$$\partial_t F + \xi \cdot \nabla_x F + (\nabla_x \otimes A)\xi \cdot \nabla_\xi F - (\nabla_x \otimes A)A \cdot \nabla_\xi F - A \cdot \nabla_x F - \nabla_x V \cdot \nabla_\xi F = 0. \quad (5.30)$$

in $\mathcal{S}'$. After a change of variables $p = \xi - A$ and by using $B = \nabla \times A$ we easily obtain the following equation for F:

$$\partial_t F + p \cdot \nabla_x F - (\nabla_x V + p \times B) \cdot \nabla_p F = 0. \quad (5.31)$$

The convergence of the current density $J^\hbar$ to

$$J = \int p f \mathrm{d}p$$

in the sense of distributions is also similar to the Pauli-Poisson equation. The spin term $\hbar\mathbf{S}$ of $J^\hbar$ vanishes in the distributional limit since the curl acts on the test function by partial integration and the integral with respect to ξ over $\mathrm{Tr}\big(\sigma F^\hbar\big)$ is bounded in L^2. It remains to show that

$$j^\hbar = \int \xi f^\hbar \mathrm{d}\xi \longrightarrow \int \xi f \mathrm{d}\xi =: j, \quad (5.32)$$

and

$$A^\hbar \rho^\hbar_{\mathrm{diag}} \longrightarrow A \int f \mathrm{d}\xi = A\rho_{\mathrm{diag}}. \quad (5.33)$$

in the sense of distributions. Then we can conclude by the change of variables $p = \xi - A$. Now let $M > 0$ and write

$$\begin{aligned}\int (j^\hbar - j)\varphi \mathrm{d}x\mathrm{d}t &= \iint_{\mathbb{R}^3_\xi} (f^\hbar - f)\xi \mathrm{d}\xi \varphi \mathrm{d}x\mathrm{d}t \\ &= \iint_{|\xi|>M} (f^\hbar - f)\xi \mathrm{d}\xi\varphi\mathrm{d}x\mathrm{d}t + \iint_{|\xi|\le M} (f^\hbar - f)\xi\mathrm{d}\xi\varphi\mathrm{d}x\mathrm{d}t.\end{aligned}$$

for $\varphi \in \mathcal{S}$. The second integral vanishes for every fixed $M > 0$ since f is the weak* limit of $f^\hbar$ in $L^\infty(I, L^2(\mathbb{R}^3_x \times \mathbb{R}^3_\xi))$. For the first one we write

$$\iint_{|\xi|>M} (f^\hbar - f)\xi\mathrm{d}\xi\varphi\mathrm{d}x\mathrm{d}t \lesssim \frac{1}{M} \iint_{|\xi|>M} (f^\hbar - f)|\xi|^2 \mathrm{d}\xi\varphi\mathrm{d}x\mathrm{d}t.$$

By conservation of energy the integral is bounded independently of M. We conclude by letting $M \to \infty$. Now write

$$\begin{aligned}\int (A^\hbar \rho^\hbar_{\mathrm{diag}} - A\rho_{\mathrm{diag}})\varphi\mathrm{d}x\mathrm{d}t &= \int A^\hbar(\rho^\hbar_{\mathrm{diag}} - \rho_{\mathrm{diag}})\varphi\mathrm{d}x\mathrm{d}t + \int (A^\hbar - A)\rho_{\mathrm{diag}}\varphi\mathrm{d}x\mathrm{d}t \\ &= \iint A^\hbar(f^\hbar - f)\varphi\mathrm{d}\xi\mathrm{d}x\mathrm{d}t + \iint (A^\hbar - A)f\varphi\mathrm{d}\xi\mathrm{d}x\mathrm{d}t\end{aligned}$$

Since $A \in L^6$ uniformly both in $\hbar$ and t we can conclude by the same reasoning as for $j^\hbar$.

The proof is not yet complete when trying to show strong convergence of $A^\hbar$ to A in some Sobolev space $W^{1,r}_{\text{loc}}$ with $r \geq 3/2$. The latter condition on r is necessary due to Sobolev's embedding and Hölder's inequality.

Our first attempt is the Aubin-Lions lemma. Therefore we need

- $\partial_t A^\hbar \in L^\infty([0,T), X_2)$ (L^∞ in time could be replaced by L^r with $r > 1$) such that $W^{1,r}(B_R)$ is continuously embedded into X_2,
- $A^\hbar \in L^\infty([0,T), X_1)$ such that X_1 is compactly embedded into $W^{1,r}(B_R)$.

Then we obtain a compact embedding of $V = \{A^\hbar \in L^\infty([0,T), X_1), \partial_t A^\hbar \in L^\infty([0,T), X_2)\}$ into $C([0,T), W^{1,r}(B_R))$ and thus strong convergence of $\delta[(A^\hbar)^2]$ in $C([0,T), L^r(B_R))$ with $r = 3r/(6-r)$ by Sobolev's embedding.

Problem 1: Magnetic Lieb-Thirring estimates

When we try to prove a uniform estimate for $R^\hbar$ in $L^{5/3}$ using $B = \nabla \times A$ and $-\Delta A = J$ we run into several problems. Indeed, with the notation of Lemma 3.8,

$$t\int R^{\alpha+1}\mathrm{d}x \leq \operatorname{tr}\big(\overline{H}_0 R\big) + \|\lambda^\hbar\|_p C_1 \left(\int (tR^\alpha)^{3/2+p'}\right)^{1/p'} + \|\lambda^\hbar\|_p C_2 \|B\|^{3/2}_{3q/2} \left(\int (tR^\alpha)^{p'q'}\right)^{1/(q'p')}$$

Choosing $\alpha = 2/3$, $p = \infty$, $p' = 1$, $q = 5/3$, $q' = 5/2$ we have

$$t\|R\|^{5/3}_{5/3} \leq \operatorname{tr}\big(\overline{H_0} R\big) + C_1 t^{5/2}\|\lambda^\hbar\|_\infty \|R\|^{5/3}_{5/3} + C_2 t\|\lambda^\hbar\|_\infty \|B\|^{3/2}_{5/2}\|R\|^{2/3}_{5/3}. \tag{5.34}$$

Using $B \simeq \nabla A$ and $A \simeq (-\Delta)^{-1}J$ we have

$$B \simeq |\nabla|^{-1}J = |\nabla|^{-1}((\operatorname{tr} R)^{1/2} k_P^{1/2}) \tag{5.35}$$

where k_P is such that

$$\|k_P^{1/2}\|_2^2 = \int k_P \mathrm{d}x = K_P := \operatorname{tr}\big(\overline{H}_0 R\big) \tag{5.36}$$

is the kinetic energy (without $\hbar$) of the Pauli equation. By Hardy-Littlewood-Sobolev and Hölder,

$$\begin{aligned} \|B\|^{3/2}_{5/2} &\leq \|(\operatorname{tr} R)^{1/2} k_P^{1/2}\|^{3/2}_{15/11} \leq \|R\|^{3/4}_{15/7} \|k_P^{1/2}\|^{3/2}_2 \\ &\leq \|R\|^{3/4}_{15/7} \operatorname{tr}\big(\overline{H}_0 R\big)^{3/4}. \end{aligned}$$

Set

$$t^{3/2} = \frac{1}{2C_1\|\lambda^\hbar\|_\infty} \tag{5.37}$$

Then (5.34) becomes

$$\frac{1}{2}t\|R\|_{5/3}^{5/3} \leq \operatorname{tr}\left(\overline{H_0}R\right) + C_3\|\lambda^\hbar\|_\infty^{1/3}\|R\|_{15/7}^{3/4}\operatorname{tr}\left(\overline{H}_0R\right)^{3/4}\|R\|_{5/3}^{2/3} \tag{5.38}$$

$$\leq \operatorname{tr}\left(\overline{H_0}R\right) + C_4\hbar\|R\|_{15/7}^{3/4}\operatorname{tr}\left(\overline{H}_0R\right)^{3/4}\|R\|_{5/3}^{2/3}. \tag{5.39}$$

where in the second step we used Assumption 2, i.e. $\|\lambda^\hbar\|_\infty \lesssim \hbar^3$. By Sobolev's inequality,

$$\|R\|_3 \leq \operatorname{tr}\left(\overline{H}_0R\right) \tag{5.40}$$

Then we interpolate $\|R\|_{15/7}^{1/2}$ between $L^{5/3}$ and L^3 to obtain

$$\begin{aligned}\frac{1}{2}t\|R\|_{5/3}^{5/3} &\leq \operatorname{tr}\left(\overline{H_0}R\right) + C_4\hbar\|R\|_{5/3}^{27/56}\|R\|_{5/3}^{2/3}\operatorname{tr}\left(\overline{H}_0R\right)^{3/4}\operatorname{tr}\left(\overline{H}_0R\right)^{14/56}\\ &\leq \operatorname{tr}\left(\overline{H_0}R\right) + C_4\hbar\|R\|_{5/3}^{193/168}\operatorname{tr}\left(\overline{H}_0R\right)\end{aligned}$$

It does not seem possible to obtain a power smaller than one for $\|R\|_{5/3}$ in the second term above.

Problem 2: Bound for $\hbar(-\Delta)^{-1}(J^\hbar\rho^\hbar_{\mathrm{diag}})$ in L^p.

Supposing we were able to prove $J^\hbar \in L^\infty([0,T), L^{5/4}_{\mathrm{loc}}(\mathbb{R}^3))$ uniformly in $\hbar$. Then we can deduce that $A^\hbar \in L^\infty([0,T), W^{2,5/4}_{\mathrm{loc}}(\mathbb{R}^3))$. By the Rellich-Kondrachov theorem, $W^{2,5/4}_{\mathrm{loc}}(\mathbb{R}^3)$ is compactly embedded in $W^{1,r}(B_R)$ for $1 \leq r < 15/7$. Now we have to address the choice of X_2. We take the time derivative of $-\Delta A^\hbar = J^\hbar$ and obtain

$$\partial_t A^\hbar = (-\Delta)^{-1}\partial_t J^\hbar$$

That means we have to investigate $\partial_t J^\hbar$. By Lemma 5.3 we have that $\partial_t A^\hbar = (-\Delta)^{-1}\partial_t J^\hbar$ has the following structure:

$$\begin{aligned}\partial_t A^\hbar \simeq& |\nabla|^{-1}\mathbb{T}^\hbar + \nabla\rho^\hbar_{\mathrm{diag}} + (-\Delta)^{-1}\left((\nabla V^\hbar + \hbar\nabla(\sigma\cdot B^\hbar) + \partial_t A^\hbar)\rho^\hbar_{\mathrm{diag}}\right) + (-\Delta)^{-1}(\mathbb{F}^\hbar J^\hbar)\\ \simeq& \sum_{j=1}^\infty \lambda_j^\hbar|\nabla|^{-1}(\nabla_A\Psi_j^\hbar \otimes \overline{\nabla_A\Psi_j^\hbar}) + (-\Delta)^{-1}\left((|\nabla|^{-1}\rho^\hbar_{\mathrm{diag}} + \hbar J^\hbar + \partial_t A^\hbar)\rho^\hbar_{\mathrm{diag}}\right)\\ &+ (-\Delta)^{-1}(\mathbb{F}^\hbar J^\hbar)\end{aligned}$$

Even if we obtain the uniform in $\hbar$ bound for $J^\hbar$ is $L^{5/4}$ and $L^{5/3}$ for $\rho^\hbar$ this can't produce a bound for $J^\hbar\rho^\hbar_{\mathrm{diag}}$ in L^p, $p \geq 1$.

Observe that we can "lose" one power of $\hbar$, i.e. it suffices that $J^\hbar \rho^\hbar_{\text{diag}} = O(\hbar^\alpha)$ in some L^p for $\alpha > -1$. However in order for $J^\hbar \rho^\hbar_{\text{diag}}$ to be in L^1 we need to put $\rho^\hbar_{\text{diag}} \in L^3$ and $J^\hbar \in L^{3/2}$. Indeed,

$$\|\rho^\hbar_{\text{diag}}\|_3 \leq \frac{K}{\hbar^2} \tag{5.41}$$

since

$$\frac{K}{\hbar^2} = \sum_j \lambda^\hbar_j \int |\nabla u^\hbar_j|^2 \mathrm{d}x \geq \sum_j \lambda^\hbar_j \int \|(u^\hbar_j)^2\|_3 \mathrm{d}x$$

by Sobolev's embedding. Similarly,

$$\|J^\hbar\|_{3/2} \leq \frac{K}{\hbar}. \tag{5.42}$$

But then

$$\hbar \int |J^\hbar \rho^\hbar_{\text{diag}}| \mathrm{d}x \leq \frac{K^2}{\hbar^2}. \tag{5.43}$$

But even L^1 would not be enough since we need at least $J^\hbar \rho^\hbar_{\text{diag}} \in L^{3/2+\epsilon}$ in order to use Hardy-Littlewood-Sobolev.

Note that Problem 2 could be circumvented by using a relaxed version of the Aubin-Lions lemma where no longer a full derivative in time of $A^\hbar$ is needed. However, the problem of the magnetic Lieb-Thirring bound remains.

A possible way to complete the proof by gaining the missing regularity could be the use of a quantum kinetic (Wigner) version of the averaging lemma of classical kinetic theory, cf. [40]. □

6 The semiclassical limit of Pauli-Poisswell to Euler-Poisswell: WKB

This section is based on [97].

In this section we prove the semiclassical limit of the Pauli-Poisswell equation to the Euler-Poisswell equation (1.32)-(1.34) via the WKB method which was discussed in section 2. We assume that the initial data $\Psi^{0,\hbar} = (\psi_1^{0,\hbar}, \psi_2^{0,\hbar})^T$ and the solution $\Psi^\hbar = (\psi_1^\hbar, \psi_2^\hbar)^T$ of the Pauli-Poisswell equation are of the form

$$\psi_j^{0,\hbar}(x) = a_j^{0,\hbar}(x) e^{\frac{i}{\hbar} S^0(x)}, \quad j = 1, 2, \tag{6.1}$$

and

$$\psi_j^\hbar(t,x) = a_j^\hbar(t,x) e^{\frac{i}{\hbar} S^\hbar(t,x)}. \tag{6.2}$$

The density $\rho^\hbar$ is then given by $\rho^\hbar = |a^\hbar|^2$. Substituting $\Psi^\hbar$ from (6.2) into (4.1) yields

$$\begin{aligned} i\hbar(\partial_t a_j^\hbar + \frac{i}{\hbar} a_j^\hbar \partial_t S^\hbar) e^{\frac{i}{\hbar} S^\hbar} = &- \frac{\hbar^2}{2}(\Delta a_j^\hbar + \frac{2i}{\hbar} \nabla a_j^\hbar \cdot \nabla S^\hbar + \frac{i}{\hbar} a_j^\hbar \Delta S^\hbar - \frac{1}{\hbar^2} |\nabla S^\hbar|^2 a_j^\hbar) e^{\frac{i}{\hbar} S^\hbar} \\ &- i\hbar A^\hbar \cdot (\nabla a_j^\hbar + \frac{i}{\hbar} a_j^\hbar \nabla S^\hbar) e^{\frac{i}{\hbar} S^\hbar} - \frac{i\hbar}{2} \mathrm{div} A^\hbar a_j^\hbar e^{\frac{i}{\hbar} S^\hbar} \\ &+ (\frac{|A^\hbar|^2}{2} + V^\hbar) a_j^\hbar e^{\frac{i}{\hbar} S^\hbar} - \frac{\hbar}{2} \overset{\hbar}{\mathbf{B}}{}_j^i a_i^\hbar e^{\frac{i}{\hbar} S^\hbar}. \end{aligned}$$

Here, $\mathbf{B}^\hbar = \sigma \cdot B^\hbar = \sum_{k=1}^3 \sigma_k B_k^\hbar$ is a 2×2 matrix and we use summation convention for $\overset{\hbar}{\mathbf{B}}{}_j^i a_i$. Rewrite the equation above as

$$\begin{aligned} &a_j \left(\partial_t S^\hbar + \frac{1}{2} |\nabla S^\hbar|^2 + (\frac{|A^\hbar|^2}{2} + V^\hbar) + A^\hbar \cdot \nabla S^\hbar \right) \\ &-i\hbar \left(\partial_t a_j^\hbar - \frac{i}{2} \overset{\hbar}{\mathbf{B}}{}_j^i a_i^\hbar - \frac{i}{2}(\hbar \Delta a_j^\hbar + 2i \nabla a_j^\hbar \cdot \nabla S^\hbar + i a_j^\hbar \Delta S^\hbar) + \frac{1}{2} \mathrm{div} A^\hbar a_j^\hbar + A^\hbar \cdot \nabla a_j^\hbar \right) = 0. \end{aligned}$$

Then we have the following system:

$$\begin{aligned} \partial_t a_j^\hbar + A^\hbar \cdot \nabla a_j^\hbar + \nabla a_j^\hbar \cdot \nabla S^\hbar + \frac{1}{2} a_j^\hbar (\Delta S^\hbar + \mathrm{div} A^\hbar) &= \frac{i\hbar}{2} \Delta a_j^\hbar + \frac{i}{2} \overset{\hbar}{\mathbf{B}}{}_j^i a_i, \\ \partial_t S^\hbar + \frac{1}{2} |\nabla S^\hbar|^2 + A^\hbar \cdot \nabla S^\hbar + (\frac{|A^\hbar|^2}{2} + V^\hbar) &= 0, \\ a_j^\hbar(x,0) &= a_j^{\hbar,0}(x), \\ S^\hbar(x,0) &= S^{\hbar,0}(x). \end{aligned} \tag{6.3}$$

Let now $u^{\hbar,0} := \nabla S^{\hbar,0}$ and $u^\hbar := \nabla S^\hbar$. By differentiating the second equation in (6.3) we obtain the **Pauli-Poisswell-WKB equation**

$$
\begin{aligned}
\partial_t a_j^\hbar + (A^\hbar + u^\hbar)\cdot\nabla a_j^\hbar + \frac{1}{2}a_j^\hbar \mathrm{div}(u^\hbar + A^\hbar) &= \frac{i\hbar}{2}\Delta a_j^\hbar + \frac{i}{2}\mathbf{B}_j^{\hbar\, i} a_i^\hbar \\
\partial_t u^\hbar + (A^\hbar + u^\hbar)\cdot\nabla u^\hbar + u^\hbar\cdot\nabla\otimes A^\hbar + \nabla(\frac{|A^\hbar|^2}{2} + V^\hbar) &= 0 \\
a_j^\hbar(x,0) &= a_j^{\hbar,0}(x) \\
u^\hbar(x,0) &= u^{\hbar,0}(x).
\end{aligned}
\tag{6.4}
$$

where

$$-\Delta V^\hbar = \rho^\hbar, \tag{6.5}$$

$$-\Delta A^\hbar = \hbar w^\hbar - \hbar v^\hbar - \rho^\hbar u^\hbar - \rho^\hbar A^\hbar, \tag{6.6}$$

and

$$v^\hbar := \frac{1}{2}\nabla\times(\overline{a^\hbar}\sigma_1 a^\hbar, \overline{a^\hbar}\sigma_2 a^\hbar, \overline{a^\hbar}\sigma_3 a^\hbar)^T, \tag{6.7}$$

$$w^\hbar := \frac{i}{2}\sum_{j=1}^{2}(\overline{a_j^\hbar}\nabla a_j^\hbar - a_j^\hbar\nabla\overline{a_j^\hbar}). \tag{6.8}$$

Then J is given by

$$J^\hbar = \hbar w^\hbar - \hbar v^\hbar - \rho^\hbar(u^\hbar + A^\hbar). \tag{6.9}$$

There is a one-to-one match between each solution $\Psi^\hbar$ of the original Pauli-Poisswell equation (4.1) (up to a constant $e^{i\theta}$) and each solution $(a^\hbar, u^\hbar)$ of (6.4). Given $(a^\hbar, u^\hbar)$ solving (6.4), we can construct $\Psi^\hbar$ solving (4.1) by (6.2). Inversely, if one has $\Psi^\hbar$ solving (4.1) one needs to calculate $V^\hbar$ and $A^\hbar$. Then $u^\hbar$ is obtained through the second equation in (6.4) and then $a^\hbar$ through the first equation in (6.4). This is where existence and uniqueness of solutions to (6.4) are needed. However this approach is only local in time since (6.4) has the following characteristics

$$\dot{x} = u^\hbar + A^\hbar, \quad x(0) = x_0.$$

The characteristics will intersect in finite time which means that caustics appear (cf. [91]).

6.1 Main result

Define the normed space X^s by

$$X^s := H^{s-1}(\Omega, (\mathbb{C})^2) \times H^s(\Omega, \qquad \|(a,u)\|_{X^s} := \|a\|_{H^{s-1}} + \|u\|_{H^s}\mathbb{R}^3)$$

We also frequently write $\|(a,u)\|_{X^s}$ instead of $\|(a,u)(t)\|_{X^s}$ for simplicity.

The following two theorems are the main results of this section. The first theorem is about the wellposedness and blow up of the Euler-Poisswell equation (1.32)-(1.34) and the Pauli-Poisswell-WKB equation (6.4)-(6.6). Theorem 5.a will be proved in Proposition 3, Theorem 5.b will be proved in Proposition 4.

Theorem 5.a. Local wellposedness of Euler-Poisswell and Pauli-Poisswell
Let $\Psi^\hbar$ satisfy (6.2) and $\Psi^{\hbar,0}$ satisfy (6.1). Let $s > 7/2$ and let

$$\|u^{\hbar,0}\|_{H^s} + \|a^{\hbar,0}\|_{H^s} + \|u^0\|_{H^s} + \|a^0\|_{H^s} \leq Q,$$

where $Q > 0$ is independent of $\hbar$. Assume that $(a^{\hbar,0}, u^{\hbar,0})$ converges to (a^0, u^0) in X^{s-2}. Then:

(i) There exists a unique local solution $(a, u) \in C([0, T^0), H^{s-1}) \times C([0, T^0), H^s)$ to the Euler-Poisswell equation (1.32)-(1.34) with initial data $(a^0, u^0) \in H^s \times H^{s-1}$. Here $T^0 > 0$ is the maximal (possibly infinite) time of existence.

(ii) There exists a unique local solution $(a^\hbar, u^\hbar) \in C([0, T^\hbar), H^s) \times C([0, T^\hbar), H^s)$ to the Pauli-Poisswell-WKB equation (6.4)-(6.6) with initial data $(a^{\hbar,0}, u^{\hbar,0}) \in H^s \times H^s$. Here $T^\hbar > 0$ is the maximal (possibly infinite) time of existence.

(iii) There exists $\hbar_0 > 0$ such that $T^\hbar$ has a uniform positive lower bound T for $0 < \hbar < \hbar_0$. In other words, there exist $\hbar_0 > 0$ and $T > 0$, such that

$$\inf_{0<\hbar<\hbar_0} T^\hbar > T. \tag{6.10}$$

Theorem 5.b. Blow up alternative for Euler-Poisswell and Pauli-Poisswell
Under the assumptions of Theorem 5.a, either the time of existence is global, or finite time blow up occurs. In the latter case it holds that:

(i) If T^0 is finite, then the solution (a, u) of (1.32) blows up at T^0 such that

$$\lim_{t \to T^0-} (\|a(t)\|_{L^\infty} + \|u(t)\|_{W^{1,\infty}}) = \infty.$$

(ii) If $T^\hbar$ is finite, then the solution $(a^\hbar, u^\hbar)$ of (6.4) blows up at $T^\hbar$ such that

$$\lim_{t \to T^\hbar-} (\|a^\hbar(t)\|_{H^1} + \|a^\hbar(t)\|_{W^{1,\infty}} + \|a^\hbar(t)\|_{W^{2,3}} + \|u^\hbar(t)\|_{W^{1,\infty}}) = \infty.$$

Furthermore, although $\|u^\hbar(t)\|_{W^{1,\infty}} + \|a^\hbar(t)\|_{L^\infty}$ may not blow up at $T^\hbar$ like in the Euler-Poisswell equation, we can still show the following nearly singular behavior: For $\hbar$ small enough, there exists a $K = K(\hbar)$, such that

$$\limsup_{t \to T^\hbar-} \|u^\hbar(t)\|_{W^{1,\infty}} + \|a^\hbar(t)\|_{L^\infty} \geq K.$$

Here, K goes to infinity as $\hbar$ goes to zero.

The second result is about the semiclassical limit.

Theorem 6. Semiclassical limit of Pauli-Poisswell-WKB
Let T^0 and $T^\hbar$ be the maximal times of existence of the Euler-Poisswell equation (1.32)-(1.34) and the Pauli-Poisswell-WKB equation (6.4)-(6.6). Under the assumptions of Theorem 5.a we have:

(i) Suppose that $T < T^0$ is a lower bound of $T^\hbar$ satisfying (6.10). We have the following semiclassical limit

$$\|(a^\hbar - a, u^\hbar - u)\|_{L^\infty([0,T],X^{s-2})} \underset{\hbar\to 0}{\longrightarrow} 0. \tag{6.11}$$

where X^s is defined by (6.1).

(ii) For all $t < T$ the Wigner transform $f^\hbar$ satisfies

$$f^\hbar(x,\xi,t) \underset{\hbar\to 0}{\rightharpoonup} f(x,p,t) = \rho(x,t)\delta(p - u(x,t)) \text{ in } \mathcal{A}'$$

where $\mathcal{A}'$ is the dual of $\mathcal{A}$ defined by (2.36). In particular, f is the solution of (5.17) with initial data

$$f(0,x,\xi) = \rho^0(x,t)\delta(\xi - u^0(x,t)). \tag{6.12}$$

(iii) For the density $\rho^\hbar$ and the current density $J^\hbar$ it holds that

$$\begin{aligned} \rho^\hbar = |a^\hbar|^2 &\underset{\hbar\to 0}{\longrightarrow} \rho = |a|^2 && \text{in } L^\infty([0,T], H^{s-3}(\mathbb{R}^3)), \\ J^\hbar = \hbar w^\hbar - \hbar v^\hbar - \rho^\hbar(u^\hbar + A^\hbar) &\underset{\hbar\to 0}{\longrightarrow} J = -(\rho u + \rho A) && \text{in } L^\infty([0,T], H^{s-3}(\mathbb{R}^3)), \end{aligned}$$

where A is given by $-\Delta A = -\rho u - \rho A$.

(iv) If $s \geq 6$, then the asymptotic behavior of the maximal time of existence $T^\hbar$ satisfies

$$\liminf_{t\to T^\hbar -} T^\hbar \geq T^0. \tag{6.13}$$

Theorem 6 will be proved in Section 6.6.

Remark 8. We can generalize our results to cases in higher dimensions. In $d \geq 4$ we just need to replace the assumption $s > 7/2$ by $s > d/2 + 2$ (due to Sobolev's embedding). However note that the Poisson equation for $V^\hbar$ is then no longer equivalent to the Hartree term.

Remark 9. We draw our inspiration from three different papers in which the semiclassical limit of different nonlinear Schrödinger equations was discussed. In [45], Grenier showed the semiclassical limit for the nonlinear Schrödinger equation with $A^\hbar \equiv 0$ and the nonlinearity given by $f(|\psi|^2)\psi$ with $f \in C^\infty(\mathbb{R}_+, \mathbb{R})$. In [100], Zhang established the semiclassical limit of the Wigner-Poisson equation and in [1], Alazard and Carles showed the semiclassical limit of the Schrödinger-Poisson equation with doping profile in $d \geq 3$ dimensions.

6.2 A priori estimates

In this section we will prove an a priori estimate for (6.4). We will omit the $\hbar$-superscript in this section in order to maintain readability. For

$$a \in L^\infty([0,T], H^s(\Omega, \mathbb{C})), \quad u \in W^{1,\infty}([0,T], H^s(\Omega, \mathbb{R}^3))$$

we define the following functionals. Let $\mu, \mu_1, \mu_2 > 0$.

$$E_s(t) := \|(a,u)(t)\|_{X^s}, \tag{6.14}$$

$$E_s^\mu(t) := E_s(t) + \mu\hbar\|a(t)\|_{H^s}, \tag{6.15}$$

$$E_s^{\mu_1,\mu_2}(t) := E_s^{\mu_1}(t) + \mu_2\|\partial_t u(t)\|_{H^{s-1}}, \tag{6.16}$$

$$M(t) := 1 + \|u(t)\|_{W^{1,\infty}} + \|a(t)\|_{L^\infty} + \hbar\|a(t)\|_{H^1} + \hbar\|a(t)\|_{W^{1,\infty}} + \hbar\|a(t)\|_{W^{2,3}}, \tag{6.17}$$

$$N(t) := \sup_{0\le s\le t} M(s). \tag{6.18}$$

Note that by Sobolev's inequality, when $s > 7/2$, we have

$$M(t) \lesssim E_s^\mu(t). \tag{6.19}$$

Proposition 2. *Suppose u and a solve (6.4). Then we have the following estimate*

$$E_s^1(t) \le CN(t)^{2s+3}E_s^1(0)e^{CN(t)^{2s+3}t}. \tag{6.20}$$

$E_s^1(t)$ is defined by (6.15) with $\mu = 1$. C is some constant that only depends on s.

Remark 10. This proposition is very important for the following two reasons. First, it allows us to prove local wellposedness as well as establish the semiclassical limit $\hbar \to 0$ of (6.4). In fact, the proof of Theorem 5.a is following the spirit of the proof of Proposition 2. Second, this proposition leads to the second part of Theorem 5.b simply by a bootstrap argument. (See Section 6.5)

The remainder of this section is devoted to the proof of this proposition which we split into four steps. The first step (Lemma 6.2) is based on charge conservation (i.e. the conservation of $\|a\|_2$) and is similar to the proof of Lemma 2.2 in [100]. It yields the following estimate

$$\frac{d}{dt}E_s(t) \lesssim (M(t) + \|A\|_{W^{1,\infty}})(E_s(t) + \|\nabla A\|_{H^s}) + \|\nabla V\|_{H^s}.$$

If $\|A\|_{W^{1,\infty}}, \|\nabla A\|_{H^s}$ and $\|\nabla V\|_{H^s}$ would not appear on the RHS one could use Gronwall's inequality directly. Therefore we have to bound these terms: In the second step (Lemma 6.3) we will bound $\|A\|_{W^{1,\infty}}, \|\nabla A\|_{H^s}$ and $\|\nabla V\|_{H^s}$ by $M(t), E_s(t)$ and $\hbar\|a\|_{H^s}$. This is achieved by some standard elliptic estimates. We obtain

$$\frac{d}{dt}E_s(t) \lesssim M(t)^{2s+1}E_s^1(t).$$

in Lemma 6.4. Since $E_s^1(t) = E_s(t) + \hbar\|a(t)\|_{H^s}$ contains $\|a(t)\|_{H^s}$ we will prove a bound for $\hbar\|a\|_{H^s}$ (Lemma 6.5). Combining it with Lemma 6.4, we can close the argument and prove Proposition 2.

Estimate of $E_s(t)$

In this part we will use charge conservation to obtain a first estimate of $E_s(t)$. By multiplying the first equation in (6.4) by $\overline{a_j}$ it is easy to show that

$$\partial_t |a_j|^2 + (u+A)\cdot\nabla|a_j|^2 + |a_j|^2 \mathrm{div}(u+A) = \frac{i\hbar}{2}(\Delta a_j \overline{a}_j - \Delta\overline{a}_j a_j) + \frac{1}{2}\mathrm{Im}(\mathbf{B}_i^j a_i \overline{a}_i).$$

Sum with respect to j and obtain

$$\partial_t \rho + (u+A)\cdot\nabla\rho + \rho\mathrm{div}(u+A) = \frac{i\hbar}{2}\sum_i(\Delta a_i\overline{a}_i - \Delta\overline{a}_i a_i). \tag{6.21}$$

Integrating over Ω and by parts yields

$$\frac{d}{dt}\|a\|_{L^2}^2 = 0, \tag{6.22}$$

which is the conservation of charge. In general we have $\|a\|_{L^2} = 1$. Combining (6.22) with (6.17), interpolation and Hölder's inequality yields the estimate

$$\|a\|_{L^p} \lesssim \|a\|_{L^2} + \|a\|_{L^\infty} \lesssim M(t), \quad 2 \le p \le +\infty, \tag{6.23}$$

where $M(t)$ is defined by (6.17).

Remark 11. There are two key observations concerning the estimate above. The first point is that although we are not able to obtain good estimates for $\nabla\rho$ we can use integration by parts to transfer the derivative from ρ to $(u+A)$. The second point is that the term $(i\hbar/2)\Delta a$ cancels due to Green's second identity since Δa has constant coefficients.

Since we need higher regularity for a later we need to extend the argument above to $\partial_x^\alpha a_j$ and $\partial_x^\alpha u$ which can be done by similar reasoning. To clarify the analogy between the charge conservation and our estimate for $\partial_x^\alpha a_j$, we first show the following lemma, which is inspired by the two observations in Remark 11.

Lemma 6.1. *Consider the following equation,*

$$\partial_t a + (A+u)\cdot\nabla a = \frac{i\hbar}{2}\Delta a + F,$$

with a complex-valued, u real vector-valued and F is a source term. Then,

$$\frac{d}{dt}\|a\|_{L^2}^2 = \int_\Omega div(u+A)|a|^2 + 2\int_\Omega \mathrm{Re}(F\cdot\overline{a}), \tag{6.24}$$

and

$$\frac{d}{dt}\|a\|_{L^2} \lesssim (M(t) + \|A\|_{W^{1,\infty}})\|a\|_{L^2} + \|F\|_{L^2}. \tag{6.25}$$

hold. If the term $(i\hbar/2)\Delta a$ is absent, the two conclusions still hold true.

Proof. Similar to (6.21), it is easy to see that

$$\partial_t|a|^2 + (u+A)\cdot\nabla|a|^2 = \frac{i\hbar}{2}(\Delta a\overline{a} - \Delta\overline{a}a) + 2\,\mathrm{Re}(F\overline{a}).$$

Integrating on Ω and by parts, we obtain (6.24). Again, by integrating by parts, we get rid of the terms ∇a and Δa, for which we do not have good estimates. Noticing that

$$\begin{aligned}
\frac{d}{dt}\|a\|_{L^2}^2 &= 2\|a\|_{L^2}\frac{d}{dt}\|a\|_{L^2}, \\
\left|\int_\Omega \mathrm{Re}(F\cdot\overline{a})\right| &\le \|a\|_{L^2}\|F\|_{L^2}, \\
\left|\int_\Omega \mathrm{div}(u+A)|a|^2\right| &\le (\|\nabla A\|_{L^\infty} + \|u\|_{W^{1,\infty}})\|a\|_{L^2}^2 \\
&\le (M(t) + \|A\|_{W^{1,\infty}})\|a\|_{L^2}^2.
\end{aligned}$$

(6.25) immediately follows. ∎

Applying Lemma 6.1 to system (6.4), we have the following estimate.

Lemma 6.2. *Suppose u and a solve (6.4). Then we have the following estimate*

$$\frac{d}{dt}E_s(t) \lesssim (M(t) + \|A\|_{W^{1,\infty}})(E_s(t) + \|\nabla A\|_{H^s}) + \|\nabla V\|_{H^s}. \tag{6.26}$$

Proof. Apply ∂_x^α to the first equation of (6.4) for $1 \le n(\alpha) \le s-1$ and find

$$\partial_t\partial_x^\alpha a_i + (u+A)\cdot\nabla\partial_x^\alpha a_i = \frac{i\hbar}{2}\Delta\partial_x^\alpha a_i - \mathbf{R}_i^c, \tag{6.27}$$

with

$$\begin{aligned}
\mathbf{R}_i^c = (\mathrm{div}\partial_x^\alpha((u+A)a_i) - (u+A)\cdot\nabla\partial_x^\alpha a_i) - \frac{1}{2}\partial_x^\alpha(a_i\mathrm{div}(u+A)) - \frac{i}{2}\partial_x^\alpha(\mathbf{B}_i^j a_j) \\
:= \sum_{j=1}^3 F_j
\end{aligned} \tag{6.28}$$

for $i = 1,2$. It is easy to see that $\mathbf{R}^c$ does not contain s-th order derivative of a_i. Then due to the Kato-Ponce inequality, we have the following estimate for F_1:

$$\begin{aligned}
\|F_1\|_2 &= (\mathrm{div}\partial_x^\alpha((u+A)a_i) - (u+A)\cdot\nabla\partial_x^\alpha a_i)\|_2 \\
&\lesssim \|a\|_{\dot{H}^{s-1}}\|\nabla(u+A)\|_{L^\infty} + \|a\|_{L^\infty}\|u+A\|_{\dot{H}^s}, \\
&\lesssim (M(t) + \|A\|_{W^{1,\infty}})(E_s(t) + \|\nabla A\|_{H^s}).
\end{aligned}$$

For F_2 we have

$$\begin{aligned}
\|F_2\|_2 &= \|\partial_x^\alpha(a_i\mathrm{div}(u+A))\|_2 \\
&\lesssim \|a\|_{\dot{H}^{s-1}}\|\nabla(u+A)\|_{L^\infty} + \|a\|_{L^\infty}\|u+A\|_{\dot{H}^s} \\
&\lesssim (M(t) + \|A\|_{W^{1,\infty}})(E_s(t) + \|\nabla A\|_{H^s}).
\end{aligned}$$

For F_2 we have

$$\begin{aligned}\|F_3\|_2 &= \|\partial_x^\alpha(\mathbf{B}_i^j a_j)\|_2\\ &\lesssim \|\mathbf{B}\|_{L^\infty}\|a\|_{H^{s-1}} + \|a\|_{L^\infty}\|\mathbf{B}\|_{H^{s-1}}\\ &\lesssim (M(t) + \|A\|_{W^{1,\infty}})(E_s(t) + \|\nabla A\|_{H^s}).\end{aligned}$$

Here, we used the fact $1 \le n(\alpha) \le s-1$. In the estimate for the last term, we also used the following fact,

$$\|\mathbf{B}\|_{H^{s-1}} \lesssim \|\nabla A\|_{H^{s-1}}, \quad \|\mathbf{B}\|_{L^\infty} \lesssim \|\nabla A\|_{L^\infty}. \tag{6.29}$$

It follows that for $\mathbf{R}_i^c$, we have

$$\|\mathbf{R}_i^c\|_{L^2} \lesssim (M(t) + \|A\|_{W^{1,\infty}})(E_s(t) + \|\nabla A\|_{H^s}). \tag{6.30}$$

For (6.27), we can apply estimate (6.25) in Lemma 6.1 to obtain

$$\frac{d}{dt}\|\partial_x^\alpha a_i\|_{L^2} \lesssim (M(t) + \|A\|_{W^{1,\infty}})\|\partial_x^\alpha a_i\|_{L^2} + \|\mathbf{R}_i^c\|_{L^2},$$

Adding up with respect to $i = 1, 2$ and plugging in (6.30), it follows that

$$\frac{d}{dt}\|\partial_x^\alpha a\|_{L^2} \lesssim (M(t) + \|A\|_{W^{1,\infty}})(E_s(t) + \|\nabla A\|_{H^s}).$$

Summing for all $1 \le |\alpha| \le s-1$ yields

$$\frac{d}{dt}\|a\|_{H^{s-1}} \lesssim (M(t) + \|A\|_{W^{1,\infty}})(E_s(t) + \|\nabla A\|_{H^s}). \tag{6.31}$$

The estimate for $\|u\|_{H^s}$ is similar and we obtain

$$\frac{d}{dt}\|u\|_{H^s} \lesssim (M(t) + \|A\|_{W^{1,\infty}})(E_s(t) + \|\nabla A\|_{H^s}) + \|\nabla V\|_{H^s}. \tag{6.32}$$

Combining (6.31) and (6.32) finishes the proof. □

Estimates for the Poisson equations

In (6.26) we need to eliminate $\|\nabla A\|_{H^s}$ and $\|\nabla V\|_{H^s}$ in order to use Gronwall's inequality. Thus in this part, we will estimate $\|\nabla A\|_{H^s}$ and $\|\nabla V\|_{H^s}$. Recall that V and A are given by (6.5) and (6.6). The result is following.

Lemma 6.3. *Let $s \ge 0$. For V and A given by (6.5) and (6.6), we have the following estimates,*

$$\|\nabla V\|_{H^s} \lesssim M(t)\|a\|_{H^{s-1}}, \tag{6.33}$$

$$\|\nabla A\|_{H^s} \lesssim M(t)^{2s}E_s^1(t), \tag{6.34}$$

$$\|A\|_{W^{1,\infty}} \lesssim M(t)^5. \tag{6.35}$$

where $E_s^1(t)$ is given by (6.15) with $\mu = 1$ and $M(t)$ is given by (6.17).

Remark 12. Unfortunately, we have no estimate for $\|A\|_{L^2}$, which means we have to avoid using $\|A\|_{L^2}$ to bound other terms in the analysis.

Proof. We divide the proof into three steps.

Step 1: Estimate of $\|\nabla V\|_{H^s}$. By definition, we have

$$\|\nabla V\|_{H^s} = \|\nabla V\|_{L^2} + \|\nabla^2 V\|_{H^{s-1}}$$

For $\|\nabla V\|_{L^2}$, we need the help of Sobolev inequality and then use the $W^{2,p}$ estimate for the Poisson equation. For $\|\nabla^2 V\|_{H^{s-1}}$, we use the $W^{2,p}$ estimate directly. Sobolev's inequality gives

$$\|\nabla V\|_{L^2} \lesssim \|\nabla^2 V\|_{L^{6/5}}.$$

By Hölder's inequality, we have

$$\|\nabla^2 V\|_{L^{6/5}} \quad \lesssim \||a|^2\|_{L^{6/5}} \lesssim (\|a\|_{L^2}^{5/6}\|a\|_{L^\infty}^{1/6})^2 \lesssim M(t)\|a\|_{L^2}. \tag{6.36}$$

Moreover,

$$\|\nabla^2 V\|_{H^{s-1}} \quad \lesssim \||a|^2\|_{H^{s-1}} \lesssim \|a\|_{H^{s-1}}\|a\|_{L^\infty} \lesssim M(t)\|a\|_{H^{s-1}}. \tag{6.37}$$

In the second step we used the Kato-Ponce inequality. From (6.2) and (6.36), we get

$$\|\nabla V\|_{L^2} \lesssim M(t)\|a\|_{L^2}.$$

Combining the equation above and (6.37), we get (6.33).

Step 2: Estimate of $\|A\|_{W^{1,\infty}}$. By the Gagliardo-Nirenberg inequality,

$$\|A\|_{L^\infty} \lesssim \|A\|_{L^6}^{\alpha_1}\|A\|_{\dot W^{2,6}}^{1-\alpha_1}, \qquad \|\nabla A\|_{L^\infty} \lesssim \|A\|_{L^6}^{\alpha_2}\|A\|_{\dot W^{2,6}}^{1-\alpha_2}.$$

For some $\alpha_1, \alpha_2 \in [0,1]$. Then $\|A\|_{W^{1,\infty}}$ is bounded by

$$\|A\|_{W^{1,\infty}} \lesssim \|A\|_{L^6} + \|A\|_{\dot W^{2,6}}$$

From (4.3), (6.9) we know that

$$\begin{aligned}\|A\|_{\dot W^{2,6}} &\lesssim \|J\|_{L^6} \\ &\lesssim \hbar\|w\|_{L^6} + \hbar\|v\|_{L^6} + \|\rho u\|_{L^6} + \|\rho A\|_{L^6}.\end{aligned}$$

Recalling (6.7) and (6.8), with the help of Hölder inequality, it follows that

$$\begin{aligned}\|A\|_{\dot W^{2,6}} &\lesssim \hbar\|a\|_{L^6}\|a\|_{W^{1,\infty}} + \|\rho\|_{L^6}\|u\|_{L^\infty} + \|\rho\|_{L^\infty}\|A\|_{L^6} \\ &= \hbar\|a\|_{L^6}\|a\|_{W^{1,\infty}} + \|a\|_{L^{12}}^2\|u\|_{L^\infty} + \|a\|_{L^\infty}^2\|A\|_{L^6} \\ &\lesssim M(t)^2(1 + \|A\|_{L^6}).\end{aligned}$$

In the last step, we use (6.23) to bound the norms of a and u. By the Sobolev inequality,

$$\|A\|_{L^6} \lesssim \|\nabla A\|_{L^2}, \tag{6.38}$$

which leads to

$$\|\nabla A\|_{L^2} \lesssim \|\hbar(v-w)+\rho u\|_{L^{6/5}}.$$

Thus $\|A\|_{W^{1,\infty}}$ is bounded by

$$\|A\|_{W^{1,\infty}} \lesssim M(t)^2(1+\|\hbar(v-w)+\rho u\|_{L^{6/5}}).$$

Since

$$\begin{aligned}\|\hbar(v-w)+\rho u\|_{L^{6/5}} &\le \|\rho u\|_{L^{6/5}} + \hbar\|v-w\|_{L^{6/5}}\\ &\lesssim \||a|^2\|_{L^{6/5}}\|u\|_{L^\infty} + \hbar\|a\|_{L^3}\|\nabla a\|_{L^2}\\ &\lesssim M(t)^3.\end{aligned}$$

we can conclude.

Step 3: Estimate of $\|\nabla A\|_{H^s}$. Multiplying (4.3) by A and integrating by parts yields

$$\begin{aligned}\|\nabla A\|_{L^2}^2 &= \int_\Omega (\hbar(v-w)+\rho(u-A))A dx \le \int_\Omega (\hbar(v-w)+\rho u)A dx\\ &\lesssim \|\hbar(v-w)+\rho u\|_{L^{6/5}}\|A\|_{L^6}.\end{aligned}$$

Recalling the Sobolev inequality (6.38) and the definition of v and w via (6.7) and (6.8), by Hölder inequality, we can get

$$\begin{aligned}\|\hbar(v-w)+\rho u\|_{L^{6/5}} &\le \|\rho u\|_{L^{6/5}} + \hbar\|v-w\|_{L^{6/5}}\\ &\lesssim \||a|^2\|_{L^3}\|u\|_{L^2} + \hbar\|a\|_{L^3}\|\nabla a\|_{L^2}\\ &\lesssim M(t)^2 E_s^1(t).\end{aligned}$$

In the last step we used (6.23). It follows that

$$\|\nabla A\|_{L^2} \lesssim M(t)^2 E_s^1(t). \tag{6.39}$$

We obtain the estimate of $\|\nabla A\|_{L^2}$. The next step is to estimate $\|\nabla A\|_{H^s}$. For $s \ge 1$ we have

$$\begin{aligned}\|\nabla A\|_{H^s} &\lesssim \|\nabla^2 A\|_{H^{s-1}} + \|\nabla A\|_{L^2}\\ &\lesssim \hbar\|v-w\|_{H^{s-1}} + \|\rho u\|_{H^{s-1}} + \|\rho A\|_{H^{s-1}} + \|\nabla A\|_{L^2}.\end{aligned}$$

$\|\nabla A\|_{L^2}$ is bounded by (6.39). $\hbar\|v-w\|_{H^{s-1}}$ and $\|\rho u\|_{H^{s-1}}$ are bounded by

$$\begin{aligned}\hbar\|v-w\|_{H^{s-1}} &\lesssim \hbar\|a\|_{L^\infty}\|\nabla a\|_{H^{s-1}}\\ &\lesssim M(t)E_s^1(t),\end{aligned} \tag{6.40}$$

and

$$\begin{aligned}\|\rho u\|_{H^{s-1}} &\lesssim \|\rho\|_{L^\infty}\|u\|_{H^{s-1}} + \|u\|_{L^\infty}\|\rho\|_{H^{s-1}}\\ &\lesssim \|a\|_{L^\infty}^2\|u\|_{H^{s-1}} + \|u\|_{L^\infty}\|a\|_{L^\infty}\|a\|_{H^{s-1}}\\ &\lesssim M(t)^2E_s^1(t). \end{aligned}\tag{6.41}$$

We still need to estimate $\|\rho A\|_{H^{s-1}}$ and the key point is that we should avoid $\|A\|_{L^2}$, for which we have no upper bounds. Since

$$\begin{aligned}\|\rho A\|_{H^{s-1}} &\lesssim \sum_{n(\alpha)+n(\beta)\le s-1, n(\beta)\ge 1}\|\partial^\alpha\rho\partial^\beta A\|_{L^2} + \sum_{n(\alpha)=s-1}\|\partial^\alpha\rho A\|_{L^2}\\ &\lesssim \|\rho\nabla A\|_{H^{s-2}} + \sum_{n(\alpha)=s-1}\|\partial^\alpha\rho\|_{L^2}\|A\|_{L^\infty},\end{aligned}$$

by the Kato-Ponce inequality, $\|\rho A\|_{H^{s-1}}$ is given by

$$\begin{aligned}\|\rho A\|_{H^{s-1}} &\lesssim \|\nabla A\|_{H^{s-2}}\|\rho\|_{L^\infty} + \|\nabla A\|_{L^\infty}\|\rho\|_{H^{s-2}} + \|\rho\|_{H^{s-1}}\|A\|_{L^\infty}\\ &\lesssim \|\nabla A\|_{H^{s-2}}\|a\|_{L^\infty}^2 + \|A\|_{W^{1,\infty}}\|a\|_{L^\infty}\|a\|_{H^{s-1}}.\end{aligned}$$

By the definition of the H^s norm via the Fourier transform and Hölder's inequality, we obtain

$$\|\nabla A\|_{H^{s-2}} \le \|\langle\xi\rangle^{s-2}\widehat{\nabla A}^{\frac{s-2}{s-1}}\|_{\frac{2(s-1)}{s-2}}\|\widehat{\nabla A}^{\frac{1}{s-1}}\|_{2(s-1)} \lesssim \|\nabla A\|_{H^{s-1}}^{\frac{s-2}{s-1}}\|\nabla A\|_2^{\frac{1}{s-1}}.$$

Then we get

$$\|\rho A\|_{H^{s-1}} \lesssim \|\nabla A\|_{H^{s-1}}^{\frac{s-2}{s-1}}\|\nabla A\|_{L^2}^{\frac{1}{s-1}}\|a\|_{L^\infty}^2 + \|A\|_{W^{1,\infty}}\|a\|_{L^\infty}\|a\|_{H^{s-1}}. \tag{6.42}$$

Plugging (6.40),(6.41) and (6.42) into (6.2) yields

$$\begin{aligned}\|\nabla A\|_{H^s} &\lesssim \hbar\|v-w\|_{H^{s-1}} + \|\rho u\|_{H^{s-1}} + \|\nabla A\|_{L^2}\\ &+ \|\nabla A\|_{H^{s-1}}^{\frac{s-2}{s-1}}\|\nabla A\|_{L^2}^{\frac{1}{s-1}}\|a\|_{L^\infty}^2 + \|A\|_{W^{1,\infty}}\|a\|_{L^\infty}\|a\|_{H^{s-1}}\\ &\lesssim \hbar\|v-w\|_{H^{s-1}} + \|\rho u\|_{H^{s-1}} + (1+\|a\|_{L^\infty}^{2(s-1)})\|\nabla A\|_{L^2} + \|A\|_{W^{1,\infty}}\|a\|_{L^\infty}\|a\|_{H^{s-1}}\\ &\lesssim M(t)(\|A\|_{W^{1,\infty}} + M(t))E_s^1(t) + M(t)^{2s-2}\|\nabla A\|_{L^2}.\end{aligned}$$

In the second step we used the following fact (which is just Young's inequality for products),

$$C_1\|\nabla A\|_{H^{s-1}}^{\frac{s-2}{s-1}}(\|\nabla A\|_{L^2}\|a\|_{L^\infty}^{2(s-1)})^{\frac{1}{s-1}} \le \frac{1}{2}\|\nabla A\|_{H^{s-1}} + C_2\|\nabla A\|_{L^2}\|a\|_{L^\infty}^{2(s-1)}.$$

Moreover, we absorbed $(1/2)\|\nabla A\|_{H^{s-1}}$ into the LHS by the trivial estimate $\|\nabla A\|_{H^{s-1}} \le \|\nabla A\|_{H^s}$. Then we can apply (6.39) and get

$$\|\nabla A\|_{H^s} \lesssim M(t)(\|A\|_{W^{1,\infty}} + M(t))E_s^1(t) + M(t)^{2s}E_s^1(t). \tag{6.43}$$

To finish the proof of (6.34), we only need (6.35). $\square$

Combining Lemma 6.2 and Lemma 6.3 leads to

Lemma 6.4. *Suppose u and a solve (6.4). Then we have the following estimate*

$$\frac{d}{dt}E_s(t) \lesssim M(t)^{2s+1}E_s^1(t), \tag{6.44}$$

where $E_s(t)$ is defined by (6.14).

Estimate of $\varepsilon\|a\|_{H^s}$ in $E_s^1(t)$

To apply Gronwall's inequality to (6.44) we need to estimate the term $\hbar\|a\|_{H^s}$ in $E_s^1(t)$. We have the following lemma.

Lemma 6.5. *Suppose S and a solve (6.3). Then we have the following estimate. For any vector α with $n(\alpha) = s$,*

$$\hbar^2\frac{d}{dt}\|\partial_x^\alpha a\|_{L^2}^2 - 2\hbar\sum_{i=1}^{2}\int_\Omega \partial_x^\alpha S\partial_t \operatorname{Im}(\overline{a}_i\partial_x^\alpha a_i)dx \lesssim M(t)^{2s+2}E_s^1(t). \tag{6.45}$$

where $E_s^1(t)$ is defined by (6.15).

Lemma 6.5 in combination with Lemma 6.4 will yield Proposition 2 which will be shown in the last part of this section. In the following section we will motivate Lemma 6.5 by an analogous result for the Schrödinger-Poisson equation.

Consider the case $A \equiv 0$. Then (4.1) becomes

$$i\hbar\partial_t\Psi = -\frac{\hbar^2}{2}\Delta\Psi + V\Psi, \tag{6.46}$$

where $\Psi = (\psi_1, \psi_2)$. By abuse of notation we use Ψ and all other original notations for the respective quantities of the Schrödinger-Poisson equation. For the Schrödinger-Poisson equation the WKB equation (6.4) becomes

$$\begin{aligned}
\partial_t a_j + u\cdot\nabla a_j + \frac{1}{2}a_j\mathrm{div}u &= \frac{i\hbar}{2}\Delta a_j \\
\partial_t u + u\cdot\nabla u + \nabla V &= 0 \\
a_j(x,0) &= a_j^{\hbar,0}(x), \\
u(x,0) &= u^{\hbar,0}(x).
\end{aligned} \tag{6.47}$$

While ρ and w are still defined by (4.2) and (6.8), the current density J becomes

$$J = -\operatorname{Re}(\overline{\Psi}(-i\hbar\nabla)\Psi),$$

or equivalently

$$J = \hbar w - \rho u. \tag{6.48}$$

Then (6.21) becomes

$$\partial_t \rho + \operatorname{div}(\rho u) = \frac{i\hbar}{2} \sum_i (\Delta a_i \overline{a}_i - \Delta \overline{a}_i a_i). \tag{6.49}$$

Then with the help of energy conservation, we will show the following lemma.

Lemma 6.6. *Suppose u and a solve (6.47). Then we have the following equation*

$$-2\hbar \int_\Omega u \cdot \partial_t w dx + \hbar^2 \frac{d}{dt} \|\nabla a\|_{L^2}^2 = 0. \tag{6.50}$$

Remark 13. To see why Lemma 6.6 motivates Lemma 6.5, replace $u = \nabla S$ by $\partial^\alpha S$ and ∇a by $\partial^\alpha a$ on the LHS of (6.50) where α is some vector satisfying $n(\alpha) = s$. Recall that w is defined by (6.8). Then w should be replaced by $\frac{i}{2} \sum_{j=1}^2 (\overline{a}_j \partial^\alpha a_j - a_j \partial^\alpha \overline{a}_j)$. The LHS of (6.50) becomes

$$\hbar^2 \frac{d}{dt} \|\partial^\alpha a\|_{L^2}^2 - i\hbar \int_\Omega \partial^\alpha S \cdot \partial_t \sum_{j=1}^2 (\overline{a}_j \partial^\alpha a_j - a_j \partial^\alpha \overline{a}_j) dx. \tag{6.51}$$

This is exactly the LHS of (6.45) in Lemma 6.5. In fact, inspired by Lemma 6.6, we tried to estimate (6.51) and finally obtained Lemma 6.5. The difficulty is now that for the Pauli-Poisswell equation, $A \neq 0$ and we have to modify our calcuations in the proof of Lemma 6.6.

Proof. The proof is split into three steps. First, we show that the energy of the Schrödinger-Poisson equation (6.46) is conserved and obtain (6.52). Then we will use the WKB system (6.4) to obtain a similar conclusion (6.53). Finally we combine (6.52) and (6.53) and finish the proof.

Step 1. First let us show energy conservation. Let

$$E(t) = \| - i\hbar \nabla \Psi\|_{L^2}^2 + \|\nabla V\|_{L^2}^2,$$

denote the energy of the Schrödinger-Poisson equation (6.46). Multiplying (6.46) with $\partial_t \overline{\Psi}$ and taking the real part, we will get

$$2 \operatorname{Re} \left(\partial_t \overline{\Psi} \left[\frac{1}{2} (-i\hbar \nabla)^2 + V \right] \Psi \right) = 0.$$

Using $2 \operatorname{Re}(V \partial_t \overline{\Psi} \Psi) = V \partial_t |\Psi|^2$ yields

$$0 = \operatorname{Re}(\partial_t \overline{\Psi} (-i\hbar \nabla)^2 \Psi) + V \partial_t |\Psi|^2.$$

Now, $-\Delta V = |\Psi|^2$ and integration by parts yields

$$
\begin{aligned}
0 &= \int_\Omega \mathrm{Re}(\partial_t \overline{\Psi}(-i\hbar\nabla)^2\Psi)dx + \int_\Omega V\partial_t|\Psi|^2 dx \\
&= \mathrm{Re}\int_\Omega \partial_t(-i\hbar\nabla\Psi)^*(-i\hbar\nabla\Psi)dx + \int_\Omega \partial_t(\nabla V)\cdot\nabla \mathrm{Vdx} \\
&= \frac{1}{2}\partial_t\left(\int_\Omega |\hbar\nabla\Psi|^2 dx + \int_\Omega |\nabla V|^2 dx\right) \\
&= \frac{1}{2}\partial_t E(t).
\end{aligned}
$$

We can plug our ansatz (6.2) into $E(t)$ and it follows that

$$
E(t) = \int_\Omega \rho|u|^2 dx + \|\nabla V\|_{L^2}^2 + 2\hbar\,\mathrm{Im}\int_\Omega \overline{a}(u\cdot\nabla)a dx + \hbar^2\|\nabla a\|_{L^2}^2.
$$

Recall w defined by (6.8). Then $(d/dt)E(t) = 0$ becomes

$$
\frac{d}{dt}\int_\Omega \rho|u|^2 dx + \frac{d}{dt}\|\nabla V\|_{L^2}^2 - 2\hbar\frac{d}{dt}\int_\Omega u\cdot w dx + \hbar^2\frac{d}{dt}\|\nabla a\|_{L^2}^2 = 0. \tag{6.52}
$$

Step 2. Now we consider the WKB system (6.47) and try to get another form of energy conservation. From (6.47), we know that

$$
\partial_t|u|^2 + u\cdot\nabla|u|^2 + 2u\cdot\nabla V = 0.
$$

Together with (6.49), we get

$$
\partial_t\rho|u|^2 + \mathrm{div}(|u|^2\rho u) + 2\rho u\cdot\nabla V = \frac{i\hbar}{2}\sum_i(\Delta a_i\overline{a}_i - \Delta\overline{a}_i a_i)|u|^2.
$$

Recalling the expression of J (6.48), we can rewrite the equation above as

$$
\partial_t\rho|u|^2 + \mathrm{div}(|u|^2\rho u) - 2J\cdot\nabla V + 2\hbar w\cdot\nabla V = \frac{i\hbar}{2}\sum_i(\Delta a_i\overline{a}_i - \Delta\overline{a}_i a_i)|u|^2.
$$

Integrating on Ω, it follows that

$$
\partial_t\int_\Omega \rho|u|^2 dx - 2\int_\Omega J\cdot\nabla V dx = \int_\Omega\left(-2\hbar w\cdot\nabla V + \frac{i\hbar}{2}\sum_i(\Delta a_i\overline{a}_i - \Delta\overline{a}_i a_i)|u|^2\right)dx.
$$

By the continuity equation we obtain

$$
\int_\Omega J\cdot\nabla V dx = -\int_\Omega \mathrm{div}(J)V dx = -\int_\Omega \partial_t\rho V dx = \int_\Omega \partial_t(\Delta V)V dx = -\frac{d}{dt}\|\nabla V\|_{L^2}^2.
$$

Then (6.53) becomes

$$\frac{d}{dt}\int_{\Omega}\rho|u|^2dx+\frac{d}{dt}\|\nabla V\|_{L^2}^2=\hbar\int_{\Omega}\left(-2w\cdot\nabla V+\frac{i}{2}\sum_i(\Delta a_i\overline{a}_i-\Delta\overline{a}_i a_i)|u|^2\right)dx. \quad (6.53)$$

This equation could be seen as an analogy of the energy conservation.

Step 3. Combining (6.53) with (6.52), we can easily see that

$$\hbar\int_{\Omega}\left(-2w\cdot\nabla V+\frac{i}{2}\sum_i(\Delta a_i\overline{a}_i-\Delta\overline{a}_i a_i)|u|^2\right)dx-2\hbar\frac{d}{dt}\int_{\Omega}u\cdot w dx+\hbar^2\frac{d}{dt}\|\nabla a\|_{L^2}^2=0.$$

Integrating by parts, we will get

$$-2\hbar\int_{\Omega}w\cdot\left(\nabla V+\frac{1}{2}\nabla|u|^2\right)dx-2\hbar\frac{d}{dt}\int_{\Omega}u\cdot w dx+\hbar^2\frac{d}{dt}\|\nabla a\|_{L^2}^2=0.$$

Since

$$\nabla V+\frac{1}{2}\nabla|u+A|^2=-\partial_t u,$$

it follows that

$$2\hbar\int_{\Omega}w\cdot\partial_t u dx-2\hbar\frac{d}{dt}\int_{\Omega}u\cdot w dx+\hbar^2\frac{d}{dt}\|\nabla a\|_{L^2}^2=0.$$

This immediately leads to (6.50) and finishes the proof. □

Proof of Lemma 6.5

Now we return to the original system (4.1), which means that $A\neq 0$.

Proof of Lemma 6.5. We need to estimate two terms in Lemma 6.5 which we denote by I_1 and I_2:

$$I_1=-i\hbar\int_{\Omega}\partial^{\alpha}S\cdot\partial_t\sum_{j=1}^{2}(\overline{a}_j\partial^{\alpha}a_j-a_j\partial^{\alpha}\overline{a}_j)dx,$$

and

$$I_2=\hbar^2\frac{d}{dt}\|\partial^{\alpha}a\|_{L^2}^2.$$

It turns out that when estimating I_1 and I_2 separately, two terms will cancel each other when adding I_1 and I_2. This leads to an estimate for $\|\partial^{\alpha}a\|_{L^2}^2$.

Estimate of I_2. First, we rewrite (6.27) as

$$\partial_t\partial_x^{\alpha}a_i+(u+A)\cdot\nabla\partial_x^{\alpha}a_i+\frac{1}{2}a_i\partial_x^{\alpha}\mathrm{div}(u+A)+\mathbf{R}_a=\frac{i\hbar}{2}\Delta\partial_x^{\alpha}a_i, \quad (6.54)$$

with

$$\begin{aligned}\mathbf{R}_a =&(\partial_x^\alpha((u+A)\cdot\nabla a_i) - (u+A)\cdot\nabla\partial_x^\alpha a_j) - \frac{i}{2}\partial_x^\alpha(\mathbf{B}_i^j a_j)\\ &+\frac{1}{2}(\partial_x^\alpha(a_i\mathrm{div}(u+A)) - a_i\partial_x^\alpha\mathrm{div}(u+A)).\end{aligned}\tag{6.55}$$

Applying (6.24) in Lemma 6.1 leads to

$$\begin{aligned}&\frac{d}{dt}\|\partial_x^\alpha a\|_{L^2}^2 + \sum_{i=1}^2\int_\Omega \mathrm{Re}(a_i\partial_x^\alpha \mathrm{div}(u+A)\partial_x^\alpha\overline{a}_i)dx\\ =&\sum_{i=1}^2\left(\int_\Omega |\partial_x^\alpha a_i|^2\mathrm{div}(u+A)dx - \int_\Omega \mathrm{Re}(\mathbf{R}_a\partial_x^\alpha\overline{a}_i)dx\right).\end{aligned}$$

Using the Kato-Ponce inequality and (6.29), (6.34) gives the following estimates. For the second term on the LHS of the equation above we have

$$\begin{aligned}\sum_{i=1}^2\int_\Omega \mathrm{Re}(a_i\partial_x^\alpha\mathrm{div}(A)\partial_x^\alpha\overline{a}_i)dx &\lesssim \|a\|_{L^\infty}\|\partial_x^\alpha a\|_{L^2}\|\partial_x^\alpha\mathrm{div}(u+A)\|_{L^2}\\ &\lesssim M(t)\|a\|_{H^s}\|\nabla A\|_{H^s},\\ &\lesssim \frac{M(t)^{2s+1}}{\hbar}E_s^1(t)^2.\end{aligned}$$

For the first term on the RHS we have

$$\begin{aligned}\left|\int_\Omega |\partial_x^\alpha a_i|^2\mathrm{div}(u+A)dx\right| &\lesssim \|\partial_x^\alpha a\|_{L^2}^2\|u+A\|_{W^{1,\infty}}\\ &\lesssim \frac{M(t)^5}{\hbar^2}E_s^1(t)^2.\end{aligned}$$

For the second term on the RHS we have to consider $\mathbf{R}_a$, cf. (6.55). We can write $\int_\Omega \mathrm{Re}(\mathbf{R}_a\partial_x^\alpha\overline{a_i})dx$ as

$$\begin{aligned}\int_\Omega \mathrm{Re}(\mathbf{R}_a\partial_x^\alpha\overline{a_i})dx =& \int_\Omega \mathrm{Re}((\partial_x^\alpha((u+A)\cdot\nabla a_i) - (u+A)\cdot\nabla\partial_x^\alpha a_j)\partial_x^\alpha\overline{a}_i)dx\\ &+\frac{1}{2}\int_\Omega \mathrm{Im}(\partial_x^\alpha(\mathbf{B}_i^j a_j)\partial_x^\alpha\overline{a}_i))dx\\ &+\int_\Omega \mathrm{Re}((\partial_x^\alpha(a_i\mathrm{div}(u+A)) - a_i\partial_x^\alpha\mathrm{div}(u+A))\partial_x^\alpha\overline{a}_i)dx\\ :=& \sum_{j=1}^3 F_j\end{aligned}$$

The estimate for F_1 is given by

$$\begin{aligned}
|F_1| &= \left|\int_\Omega \mathrm{Re}((\partial_x^\alpha((u+A)\cdot\nabla a_i) - (u+A)\cdot\nabla\partial_x^\alpha a_j)\partial_x^\alpha \overline{a}_i)dx\right| \\
&\lesssim (\|a\|_{H^s}\|u+A\|_{W^{1,\infty}} + \|u+A\|_{H^s}\|\nabla a\|_{L^\infty})\|\partial_x^\alpha a\|_{L^2} \\
&\lesssim \frac{M(t)^{2s+1}}{\hbar^2}E_s^1(t)^2.
\end{aligned}$$

For F_2 we have

$$\begin{aligned}
|F_2| &= \left|\int_\Omega \mathrm{Im}(\partial_x^\alpha(\mathbf{B}_i^j a_j)\partial_x^\alpha \overline{a}_i))dx\right| \\
&\lesssim (\|a\|_{H^s}\|\mathbf{B}\|_{L^\infty} + \|\mathbf{B}\|_{H^s}\|a\|_{L^\infty})\|\partial_x^\alpha a\|_{L^2} \\
&\lesssim (\|a\|_{H^s}\|\nabla A\|_{L^\infty} + \|\nabla A\|_{H^s}\|a\|_{L^\infty})\|\partial_x^\alpha a\|_{L^2} \\
&\lesssim \frac{M(t)^{2s+1}}{\hbar}E_s^1(t)^2.
\end{aligned}$$

And for F_3 we have

$$\begin{aligned}
|F_3| &= \left|\int_\Omega \mathrm{Re}((\partial_x^\alpha(a_i \mathrm{div}(u+A)) - a_i\partial_x^\alpha \mathrm{div}(u+A))\partial_x^\alpha \overline{a}_i)dx\right| \\
&\lesssim (\|a\|_{H^s}\|u+A\|_{W^{1,\infty}} + \|u+A\|_{H^s}\|\nabla a\|_{L^\infty})\|\partial_x^\alpha a\|_{L^2} \\
&\lesssim \frac{M(t)^{2s+1}}{\hbar^2}E_s^1(t)^2,
\end{aligned}$$

Applying the estimates above and multiplying by $\hbar^2$ yields

$$\hbar^2\frac{d}{dt}\|\partial_x^\alpha a\|_{L^2}^2 + \hbar^2\sum_{i=1}^2\int_\Omega \mathrm{Re}(a_i\partial_x^\alpha \mathrm{div}(u)\partial_x^\alpha \overline{a}_i)dx \lesssim M(t)^{2s+1}E_s^1(t)^2. \tag{6.56}$$

Since the term

$$\sum_{i=1}^2\int_\Omega \mathrm{Re}(a_i\partial_x^\alpha \mathrm{div}(u)\partial_x^\alpha \overline{a}_i)dx$$

contains a $(s+1)$-th order derivative of u, we cannot apply the Kato-Ponce inequality. Otherwise, we will get $\|u\|_{H^{s+1}}$. But if we add (6.56) to the estimate of I_1, which we will show in the next step, this tricky term will cancel with another term in the estimate of I_1.

Estimate of I_1. Multiplying (6.54) by $\overline{a}_i$, we get

$$\overline{a}_i\partial_t\partial_x^\alpha a_i + \overline{a}_i(u+A)\cdot\nabla\partial_x^\alpha a_i + \frac{1}{2}|a_i|^2\partial_x^\alpha \mathrm{div}(u+A) + \overline{a}_i\mathbf{R}_a = \frac{i\hbar}{2}\overline{a}_i\Delta\partial_x^\alpha a_i.$$

Multiplying the first equation in (6.4) by $\partial_x^\alpha\overline{a}_i$ and complex conjugation yield

$$\partial_x^\alpha a_i\partial_t\overline{a}_i + \partial_x^\alpha a_i(A+u)\cdot\nabla\overline{a}_i + \frac{1}{2}\overline{a}_i\partial_x^\alpha a_i \mathrm{div}(u+A) = -\frac{i\hbar}{2}\Delta\overline{a}_i\partial_x^\alpha a_i - \frac{i}{2}\mathbf{B}_i^j\overline{a}_j\partial_x^\alpha a_i.$$

Adding the two equations above together yields

$$\partial_t(\overline{a}_i\partial_x^\alpha a_i) + (u+A)\cdot\nabla(\overline{a}_i\partial_x^\alpha a_i) + \frac{1}{2}|a_i|^2\partial_x^\alpha \mathrm{div}(u+A) + \mathbf{R}_b = \frac{i\hbar}{2}\overline{a}_i\Delta\partial_x^\alpha a_i - \frac{i\hbar}{2}\Delta\overline{a}_i\partial_x^\alpha a_i,$$

where $\mathbf{R}_b$ is defined as

$$\mathbf{R}_b := \overline{a}_i\mathbf{R}_a + \frac{1}{2}\overline{a}_i\partial_x^\alpha a_i \mathrm{div}(u+A) + \frac{i}{2}\mathbf{B}_i^j\overline{a}_j\partial_x^\alpha a_i.$$

Recall that $\mathbf{R}_a$ was defined as

$$\begin{aligned}\mathbf{R}_a =& (\partial_x^\alpha((u+A)\cdot\nabla a_i) - (u+A)\cdot\nabla\partial_x^\alpha a_j) - \frac{i}{2}\partial_x^\alpha(\mathbf{B}_i^j a_j)\\ &+ \frac{1}{2}(\partial_x^\alpha(a_i\mathrm{div}(u+A)) - a_i\partial_x^\alpha\mathrm{div}(u+A)).\end{aligned}$$

Now take the imaginary part and multiply with $\partial_x^\alpha S$:

$$\begin{aligned}&\partial_x^\alpha S\partial_t\,\mathrm{Im}(\overline{a}_i\partial_x^\alpha a_i) + \partial_x^\alpha S(u+A)\cdot\nabla\,\mathrm{Im}(\overline{a}_i\partial_x^\alpha a_i) + \partial_x^\alpha S\,\mathrm{Im}(\mathbf{R}_b)\\ =& -\frac{\hbar}{2}\partial_x^\alpha S\,\mathrm{Re}(\overline{a}_i\Delta\partial_x^\alpha a_i - \Delta\overline{a}_i\partial_x^\alpha a_i).\end{aligned}$$

Integrate by parts:

$$\begin{aligned}&\int_\Omega \partial_x^\alpha S\partial_t\,\mathrm{Im}(\overline{a}_i\partial_x^\alpha a_i)dx - \int_\Omega \mathrm{div}(\partial_x^\alpha S(u+A))\,\mathrm{Im}(\overline{a}_i\partial_x^\alpha a_i)dx + \int_\Omega \partial_x^\alpha S\,\mathrm{Im}(\mathbf{R}_b)dx\\ =& -\frac{\hbar}{2}\int_\Omega \partial_x^\alpha S\,\mathrm{Re}(\overline{a}_i\Delta\partial_x^\alpha a_i - \Delta\overline{a}_i\partial_x^\alpha a_i)dx\\ =& -\frac{\hbar}{2}\int_\Omega \Delta\partial_x^\alpha S\,\mathrm{Re}(\overline{a}_i\partial_x^\alpha a_i)dx - \hbar\int_\Omega \nabla\partial_x^\alpha S\cdot\mathrm{Re}(\nabla\overline{a}_i\partial_x^\alpha a_i)dx\\ =& -\frac{\hbar}{2}\int_\Omega \partial_x^\alpha\mathrm{div}(u)\,\mathrm{Re}(\overline{a}_i\partial_x^\alpha a_i)dx - \hbar\int_\Omega \partial_x^\alpha u\cdot\mathrm{Re}(\nabla\overline{a}_i\partial_x^\alpha a_i)dx.\end{aligned}$$

The first term on the LHS is the term we want to estimate. The first term on the RHS is the term that will cancel with the tricky term in I_2. That means we have to estimate the remaining terms. For the second term on the LHS we have

$$\begin{aligned}\left|\int_\Omega \mathrm{div}(\partial_x^\alpha S(u+A))\,\mathrm{Im}(\overline{a}_i\partial_x^\alpha a_i)dx\right| &\lesssim \|\partial_x^\alpha a_i\|_2\|a_i\|_\infty\|\mathrm{div}(\partial_x^\alpha S(u+A))\|_2\\ &\lesssim \|\partial_x^\alpha a_i\|_2\|a_i\|_\infty\|\partial_x^\alpha S\|_{H^1}\|u+A\|_{W^{1,\infty}}\\ &\lesssim \frac{M(t)^6}{\hbar}E_s^1(t)^2.\end{aligned}$$

The next five inequalities give the estimate for the third term on the LHS, i.e. $\int_\Omega \partial_x^\alpha S\,\mathrm{Im}(\mathbf{R}_b)dx$.

$$
\begin{aligned}
\left|\int_\Omega \partial_x^\alpha S\,\mathrm{Re}(\overline{a}_i\partial_x^\alpha(\mathbf{B}_i^k a_k)-\mathbf{B}_i^k\overline{a}_k\partial_x^\alpha a_i)dx\right| &\lesssim \|\partial_x^\alpha S\|_2(\|a_i\|_\infty\|\partial_x^\alpha(\mathbf{B}_i^k a_k)\|_2+\|\partial_x^\alpha a_i\|_2\|\mathbf{B}_i^k a_k\|_\infty)\\
&\lesssim \|u\|_{H^s}\|a_i\|_\infty(\|\mathbf{B}\|_{H^s}\|a_i\|_\infty+\|a_i\|_{H^s}\|\mathbf{B}\|_\infty)\\
&\lesssim \|u\|_{H^s}\|a_i\|_\infty(\|\nabla A\|_{H^s}\|a_i\|_\infty+\|a_i\|_{H^s}\|\nabla A\|_\infty)\\
&\lesssim M(t)^{2s+2}\|u\|_{H^s}(\|u\|_{H^s}+\|a_i\|_{H^s})\\
&\lesssim \frac{M(t)^{2s+2}}{\hbar}E_s^1(t)^2.
\end{aligned}
$$

$$
\begin{aligned}
\left|\int_\Omega \partial_x^\alpha S\,\mathrm{Re}(\Delta\overline{a}_i\partial_x^\alpha a_i)dx\right| &\lesssim \|\Delta a_i\|_{L^3}\|\partial_x^\alpha S\|_{L^6}\|\partial_x^\alpha a_i\|_2\\
&\lesssim \|\Delta a_i\|_{L^3}\|\nabla\partial_x^\alpha S\|_2\|\partial_x^\alpha a_i\|_2\\
&\lesssim \frac{M(t)}{\hbar^2}E_s^1(t)^2.
\end{aligned}
$$

$$
\begin{aligned}
\left|\int_\Omega \partial_x^\alpha S\,\mathrm{Im}(\overline{a}_i\partial_x^\alpha a_i)\mathrm{div}(u+A)dx\right| &\lesssim \|\mathrm{div}(u+A)\|_\infty\|a_i\|_\infty\|\partial_x^\alpha a_i\|_2\|\partial_x^\alpha S\|_2\\
&\lesssim \frac{M(t)^6}{\hbar}E_s^1(t)^2,
\end{aligned}
$$

$$
\begin{aligned}
&\left|\int_\Omega \partial_x^\alpha S\,\mathrm{Im}(\overline{a}_i(\partial_x^\alpha(a_i\mathrm{div}(u+A))-a_i\partial_x^\alpha\mathrm{div}(u+A)))dx\right|\\
\lesssim&\|\partial_x^\alpha S\|_2\|a_i\|_\infty\|\partial_x^\alpha(a_i\mathrm{div}(u+A))-a_i\partial_x^\alpha\mathrm{div}(u+A)\|_2\\
\lesssim&\|u\|_{H^s}(\|a_i\|_{W^{1,\infty}}\|\mathrm{div}(u+A)\|_{H^{s-1}}+\|\mathrm{div}(u+A)\|_\infty\|a_i\|_{H^s})\\
\lesssim&\frac{M(t)}{\hbar}\|u\|_{H^s}((\|u\|_{H^s}+\|\nabla A\|_{H^{s-1}})+\hbar\|a_i\|_{H^s})\\
\lesssim&\frac{M(t)^{2s+1}}{\hbar}E_s^1(t)^2.
\end{aligned}
$$

$$
\begin{aligned}
&\left|\int_\Omega \partial_x^\alpha S\,\mathrm{Im}(\overline{a}_i(\partial_x^\alpha((u+A)\cdot\nabla a_i)-(u+A)\cdot\nabla\partial_x^\alpha a_j))dx\right|\\
\lesssim&\|\partial_x^\alpha S\|_2\|a_i\|_\infty\|\partial_x^\alpha((u+A)\cdot\nabla a_i)-(u+A)\cdot\nabla\partial_x^\alpha a_j\|_2\\
\lesssim&\|\partial_x^\alpha S\|_2(\|\nabla(u+A)\|_\infty\|\nabla a_i\|_{\dot{H}^{s-1}}+\|\nabla a_i\|_\infty\|u+A\|_{\dot{H}^s})\\
\lesssim&\frac{M(t)^{2s+1}}{\hbar}E_s^1(t)^2.
\end{aligned}
$$

For the last term on the RHS we have

$$\begin{aligned}\left|\int_\Omega \partial_x^\alpha u\cdot \mathrm{Re}(\nabla \overline{a}_i \partial_x^\alpha a_i)dx\right| &\lesssim \|u\|_{H^s}\|\nabla a_i\|_\infty \|a_i\|_{H^s}\\ &\lesssim \frac{M(t)}{\hbar}\|u\|_{H^s}\|a\|_{H^s}\\ &\lesssim \frac{M(t)}{\hbar^2}E_s^1(t)^2.\end{aligned}$$

Combining all estimates yields

$$-2\hbar\sum_{i=1}^2\int_\Omega \partial_x^\alpha S\partial_t\,\mathrm{Im}(\overline{a}_i\partial_x^\alpha a_i)dx - \hbar^2\sum_{i=1}^2\int_\Omega \partial_x^\alpha \mathrm{div}(u)\,\mathrm{Re}(\overline{a}_i\partial_x^\alpha a_i)dx \lesssim M(t)^{2s+2}E_s^1(t)^2. \tag{6.57}$$

Notice that the second term of the equation above will cancel with the second term of (6.56). We can use this observation to finish the estimate of $\|a\|_{H^s}^2$.

Combining I_1 and I_2. Adding (6.56) and (6.57) immediately yields (6.45). □

6.3 Proof of Proposition 2

Now we will use Lemma 6.4 and Lemma 6.5 to derive the a priori estimate in Proposition 2.

Proof of Proposition 2. Notice that

$$\begin{aligned}&\hbar^2\frac{d}{dt}\|\partial_x^\alpha a\|_{L^2}^2 - 2\hbar\sum_{i=1}^2\int_\Omega \partial_x^\alpha S\partial_t\,\mathrm{Im}(\overline{a}_i\partial_x^\alpha a_i)dx\\ =&\hbar^2\frac{d}{dt}\|\partial_x^\alpha a\|_{L^2}^2 - 2\hbar\frac{d}{dt}\sum_{i=1}^2\int_\Omega \partial_x^\alpha S\,\mathrm{Im}(\overline{a}_i\partial_x^\alpha a_i)dx + 2\hbar\sum_{i=1}^2\int_\Omega \partial_t\partial_x^\alpha S\,\mathrm{Im}(\overline{a}_i\partial_x^\alpha a_i)dx.\end{aligned}$$

First we estimate the third term. By the second equation in (6.3), we have

$$\begin{aligned}2\hbar\sum_{i=1}^2\int_\Omega \partial_t\partial_x^\alpha S\,\mathrm{Im}(\overline{a}_i\partial_x^\alpha a_i)dx &\lesssim \hbar\|\partial_t\partial_x^\alpha S\|_{L^2}\|\partial_x^\alpha a\|_{L^2}\|a\|_{L^\infty}\\ &\lesssim \hbar\|\partial_t u\|_{H^{s-1}}\|a\|_{H^s}\|a\|_{L^\infty}.\end{aligned}$$

Using

$$\begin{aligned}\|\partial_t u\|_{H^{s-1}} &\lesssim \|(A+u)\cdot\nabla u\|_{H^{s-1}} + \|u\cdot\nabla A\|_{H^{s-1}} + \|\nabla|A|^2\|_{H^{s-1}} + \|\nabla V\|_{H^{s-1}}\\ &\lesssim (\|u\|_{L^\infty}+\|A\|_{L^\infty})(\|u\|_{H^s}+\|A\|_{\dot{H}^s}) + \|\nabla V\|_{H^{s-1}}\\ &\lesssim M(t)^{2s+1}E_s^1(t).\end{aligned} \tag{6.58}$$

yields

$$2\hbar\sum_{i=1}^{2}\int_{\Omega}\partial_t\partial_x^\alpha S\,\mathrm{Im}(\overline{a}_i\partial_x^\alpha a_i)dx \lesssim M(t)^{2s+2}E_s^1(t)^2.$$

Now we use Lemma 6.5 and rewrite (6.45) as

$$\hbar^2\frac{d}{dt}\|\partial_x^\alpha a\|_2^2 - 2\hbar\frac{d}{dt}\sum_{i=1}^{2}\int_{\Omega}\partial_x^\alpha S\,\mathrm{Im}(\overline{a}_i\partial_x^\alpha a_i)dx \lesssim M(t)^{2s+2}E_s^1(t)^2.$$

Integrating from 0 to t, we get

$$\begin{aligned}\hbar^2\|\partial_x^\alpha a\|_{L^2}^2 - 2\hbar\sum_{i=1}^{2}\int_{\Omega}\partial_x^\alpha S\,\mathrm{Im}(\overline{a}_i\partial_x^\alpha a_i)dx&\\ -\hbar^2\|\partial_x^\alpha a^{\hbar,0}\|_{L^2}^2 + 2\hbar\sum_{i=1}^{2}\int_{\Omega}\partial_x^\alpha S^{\hbar,0}\,\mathrm{Im}(\overline{a}_i^{\hbar,0}\partial_x^\alpha a_i^{\hbar,0})dx &\lesssim \int_0^t M(r)^{2s+2}E_s^1(r)^2dr.\end{aligned} \tag{6.59}$$

It is easy to see that

$$\begin{aligned}\left|\int_{\Omega}\partial_x^\alpha S\,\mathrm{Im}(\overline{a}_i\partial_x^\alpha a_i)dx\right| &\lesssim \|a\|_{L^\infty}\|\partial_x^\alpha S\|_{L^2}\|\partial_x^\alpha a\|_{L^2}\\ &\lesssim M(t)\|u\|_{H^{s-1}}\|a\|_{H^s}\\ &\lesssim M(t)(E_s(t)^2 + E_s(t)\|a\|_{\dot{H}^s}),\end{aligned}$$

and similarly that

$$\begin{aligned}\hbar^2\|\partial_x^\alpha a^{\hbar,0}\|_{L^2}^2 + \hbar\left|\sum_{i=1}^{2}\int_{\Omega}\partial_x^\alpha S^{\hbar,0}\,\mathrm{Im}(\overline{a}_i^{\hbar,0}\partial_x^\alpha a_i^{\hbar,0})dx\right| &\lesssim \hbar\|u^{\hbar,0}\|_{H^{s-1}}\|a^{\hbar,0}\|_{H^s}\|a^{\hbar,0}\|_{L^\infty}\\ &\quad+\hbar^2\|a^{\hbar,0}\|_{H^s}^2\\ &\lesssim M(0)E_s^1(0)^2.\end{aligned}$$

Plugging the two estimate above into (6.59), it follows that

$$\hbar^2\|\partial_x^\alpha a\|_{L^2}^2 \lesssim M(0)E_s^1(0)^2 + \hbar M(t)E_s(t)\|a\|_{\dot{H}^s} + M(t)E_s(t)^2 + \int_0^t M(r)^{2s+2}E_s^1(r)^2dr.$$

Summing over all α such that $n(\alpha) = s$, we get

$$\hbar^2\|a\|_{\dot{H}^s}^2 \lesssim M(0)E_s^1(0)^2 + \hbar M(t)E_s(t)\|a\|_{\dot{H}^s} + M(t)E_s(t)^2 + \int_0^t M(r)^{2s+2}E_s^1(r)^2dr. \tag{6.60}$$

Noticing that for any constant $C > 0$, we have

$$C\hbar M(t)E_s(t)\|a\|_{\dot{H}^s} \le \frac{1}{2}\hbar^2\|a\|_{\dot{H}^s}^2 + \frac{1}{2}C^2M(t)^2E_s(t)^2,$$

it follows that

$$\hbar^2\|a\|_{\dot{H}^s}^2 \lesssim M(0)E_s^1(0)^2 + M(t)^2E_s(t)^2 + \int_0^t M(r)^{2s+2}E_s^1(r)^2dr. \tag{6.61}$$

From Lemma 6.4, i.e. (6.44), it follows that

$$\frac{d}{dt}E_s(t)^2 \lesssim M(t)^{2s+1}E_s^1(t)^2.$$

The integral form of this inequality is

$$E_s(t)^2 \lesssim M(0)^{2s+1}E_s^1(0)^2 + \int_0^t M(r)^{2s+1}E_s^1(r)^2dr. \tag{6.62}$$

Plugging (6.62) into (6.61), we get

$$\hbar^2\|a\|_{\dot{H}^s}^2 \lesssim M(t)^2M(0)^{2s+1}E_s^1(0)^2 + M(t)^2\int_0^t M(r)^{2s+1}E_s^1(r)^2dr. \tag{6.63}$$

Adding (6.62) with (6.63) and recalling the definition of $N(t)$ given by (6.18), we get

$$E_s^1(t)^2 \lesssim N(t)^{2s+3}\left(E_s^1(0)^2 + \int_0^t E_s^1(r)^2dr\right).$$

By the Gronwall inequality we can conclude that (6.20) holds. This finishes the proof. □

6.4 Existence and Uniqueness of Solutions

In this section we discuss existence and uniqueness of solutions for the Pauli-Poisswell-WKB system (6.4), i.e. Theorem 5.a. Proposition 3 is equivalent to the last two parts in Theorem 5.a. By the same argument, we can easily prove the first part in Theorem 5.a, i.e. the local wellposedness of the Euler-Poisswell equation (1.32). The proof of this part is even simpler because we don't need to deal with the H^s norm of a.

We use a fixed point argument to show existence and uniqueness in the space $Y^{T,s}_{\mu_1,\mu_2}$ defined by (6.64) for some subtly selected μ_1, μ_2. For the fixed point argument we need that the solution map Υ is an involution on a suitable subset of $Y^{T,s}_{\mu_1,\mu_2}$ and that it is a contraction on that set. The first part is shown in Lemma 6.7, the second part in Lemma 6.8.

Proposition 3. *Under the assumptions of Theorem 5 there exists a unique solution $(a^\hbar, u^\hbar)$ of (6.4), such that $a^\hbar \in L^\infty([0,T], H^s), u^\hbar \in W^{1,\infty}([0,T], H^s)$ for some $T > 0$ that only depends on Q, s in Theorem 5. In particular, T is independent of $\hbar$.*

We will close the argument the space $Y^{T,s}_{\mu_1,\mu_2}$:

$$Y^{T,s}_{\mu_1,\mu_2} := L^\infty([0,T], H^s(\Omega,\mathbb{C})) \times W^{1,\infty}([0,T], H^s(\Omega,\mathbb{R}^3)), \tag{6.64}$$

equipped with the following norm:

$$\begin{aligned}\|(a,u)\|_{Y^{T,s}_{\mu_1,\mu_2}} &:= \sup_{t\in[0,T]} E_s^{\mu_1,\mu_2}(t)\\ &= \sup_{t\in[0,T]} (\|(a,u)(t)\|_{X^s} + \mu_1\hbar\|a(t)\|_{H^s} + \mu_2\|\partial_t u(t)\|_{H^{s-1}}),\end{aligned}$$

where $0 < \mu_1, \mu_2 < 1$. In order to improve readability we will simply write $\|(a,u)\|_Y$. Define the operator Υ as:

$$\Upsilon^{\hbar}_{a^{\hbar,0},u^{\hbar,0}} : Y^{T,s}_{\mu_1,\mu_2} \to Y^{T,s}_{\mu_1,\mu_2}, \quad (\tilde{a},\tilde{u}) \mapsto (a,u),$$

where (a,u) is the solution at time t of the following linear system

$$\begin{aligned}\partial_t a_j + (\tilde{A}+\tilde{u})\cdot\nabla a_j + \frac{1}{2}a_j \mathrm{div}(\tilde{u}+\tilde{A}) &= \frac{i\hbar}{2}\Delta a_j + \frac{i}{2}\widetilde{\mathbf{B}}^i_j a_i,\\ \partial_t u + (\tilde{A}+\tilde{u})\cdot\nabla u + u\cdot\nabla\tilde{A} + \nabla(\frac{|\tilde{A}|^2}{2}+\tilde{V}) &= 0,\\ a_j(x,0) &= a_j^{\hbar,0}(x)\\ u(x,0) &= u^{\hbar,0}(x).\end{aligned} \tag{6.65}$$

Here, $\tilde{A}$ and $\tilde{V}$ are given by

$$\tilde{\rho} = |\tilde{a}_1|^2 + |\tilde{a}_2|^2, \qquad -\Delta\tilde{V} = \tilde{\rho}, \qquad -\Delta\tilde{A} = \hbar\tilde{w} - \hbar\tilde{v} - \tilde{\rho}\tilde{u} - \tilde{\rho}\tilde{A},$$

and

$$\begin{aligned}\tilde{v} &= \frac{1}{2}\nabla\times(\tilde{a}^*\sigma_1\tilde{a}, \tilde{a}^*\sigma_2\tilde{a}, \tilde{a}^*\sigma_3\tilde{a})^T,\\ \tilde{w} &= \frac{i}{2}\sum_{j=1}^{2}(\overline{\tilde{a}}_j\nabla\tilde{a}_j - \tilde{a}_j\nabla\overline{\tilde{a}}_j).\end{aligned}$$

with $\widetilde{\mathbf{B}}$ defined similarly. Let $\mathcal{B} = \mathcal{B}_{s,\mu_1,\mu_2,T,L}$ denote the ball of radius L in Y:

$$\mathcal{B}_{s,\mu_1,\mu_2,T,L} := \left\{(a,u) : \|(a,u)\|_{Y^{T,s}_{\mu_1,\mu_2}} \le L\right\}.$$

Proof of Proposition 3. Lemma 6.7 shows that Υ maps $\mathcal{B}$ into itself and Lemma 6.8 shows that it is indeed a contraction. Then we can conclude by Banach's fixed point theorem. □

Lemma 6.7. *Under the assumptions of Theorem 5 there exists $L > 0$ only depending on Q and s in and $\mu_0 > 0$ such that for all $\mu_1, \mu_2 \in (0,\mu_0)$ there is a $T_0 = T_0(\mu_1,\mu_2,L,s) > 0$ independent of $\hbar$ such that for all $T \in [0,T_0)$*

$$\Upsilon^{\hbar}_{a^{\hbar,0},u^{\hbar,0}}(\mathcal{B}_{s,\mu_1,\mu_2,T,L}) \subset \mathcal{B}_{s,\mu_1,\mu_2,T,L}.$$

In other words, $\Upsilon^{\hbar}_{a^{\hbar,0},u^{\hbar,0}}$ maps $\mathcal{B}_{s,\mu_1,\mu_2,T,L}$ into itself.

Lemma 6.8. *Under the assumptions of Theorem 5 and for any $L > 0$, there exist $\mu_{10} = \mu_{10}(L, s) > 0$ and $\mu_{20} = \mu_{20}(\mu_1, L, s) > 0$ such that for all $0 < \mu_1 < \mu_{10}$ and $0 < \mu_2 < \mu_{20}$ there exists $T_0 = T_0(\mu_1, \mu_2, L, s) > 0$ independent of $\hbar$ such that for all $T \in [0, T_0)$ and $(\tilde{a}^1, \tilde{u}^1), (\tilde{a}^2, \tilde{u}^2) \in \mathcal{B}_{s,\mu_1,\mu_2,T,L}$ we have*

$$\|\Upsilon^{\hbar}_{a^{\hbar,0},u^{\hbar,0}}(\tilde{a}^1, \tilde{u}^1) - \Upsilon^{\hbar}_{a^{\hbar,0},u^{\hbar,0}}(\tilde{a}^2, \tilde{u}^2)\|_{Y^{T,s}_{\mu_1,\mu_2}} \leq \tau \|(\tilde{a}^1, \tilde{u}^1) - (\tilde{a}^2, \tilde{u}^2)\|_{Y^{T,s}_{\mu_1,\mu_2}}. \tag{6.66}$$

Here, $\tau \in (0, 1)$ is a constant.

Proof of Lemma 6.7. We need to show that for $(\tilde{a}, \tilde{u})$ such that $\|(\tilde{a}, \tilde{u})\|_Y \leq L$ the image $\Upsilon^{\hbar}_{a^{\hbar,0},u^{\hbar,0}}(\tilde{a}, \tilde{u})$ also satisfies $\|\Upsilon^{\hbar}_{a^{\hbar,0},u^{\hbar,0}}(\tilde{a}, \tilde{u})\|_Y \leq L$. We have

$$\partial_t \partial_x^\alpha a_i + (\tilde{u} + \tilde{A}) \cdot \nabla \partial_x^\alpha a_i = \frac{i\hbar}{2} \Delta \partial_x^\alpha a_i - \tilde{\mathbf{R}}_i^c,$$

with

$$\tilde{\mathbf{R}}_i^c = (\mathrm{div} \partial_x^\alpha ((\tilde{u} + \tilde{A}) a_i) - (\tilde{u} + \tilde{A}) \cdot \nabla \partial_x^\alpha a_i) - \frac{1}{2} \partial_x^\alpha (a_i \mathrm{div}(\tilde{u} + \tilde{A})) - \frac{i}{2} \partial_x^\alpha \left(\tilde{\mathbf{B}}_i^j a_j \right).$$

Step 1: Estimate for $\|(a, u)\|_{X^s}$. Let us first look at the X^s-norm of (a, u) which we have to estimate in terms of the Y-norm of $(\tilde{a}, \tilde{u})$. Similarly to (6.31), we can apply Lemma 6.1 and prove that

$$\begin{aligned}
\frac{d}{dt} \|a\|_{H^{s-1}} &\lesssim (M(\tilde{a}, \tilde{u})(t) + \|\tilde{A}\|_{W^{1,\infty}}) \|(a, u)\|_{X^s} + M(a, u)(t)(\|(\tilde{a}, \tilde{u})\|_{X^s} + \|\nabla \tilde{A}\|_{H^s}) \\
&\lesssim (E_s^1(\tilde{a}, \tilde{u})(t) + \|\nabla \tilde{A}\|_{H^s}) \|(a, u)\|_{X^s}.
\end{aligned}$$

The second inequality follows from (6.19) and Sobolev embedding. Similarly to (6.32), we can prove that

$$\begin{aligned}
\frac{d}{dt} \|u\|_{H^s} \lesssim & (\|\nabla \tilde{A}\|_{H^s} + \|\tilde{u}\|_{H^s}) \|u\|_{L^\infty} + \|u\|_{H^s} (\|\tilde{u}\|_{W^{1,\infty}} + \|\tilde{A}\|_{W^{1,\infty}}) \\
& + \|\nabla \tilde{A}\|_{H^s}^2 + \|\nabla \tilde{V}\|_{H^s} \\
\lesssim & (\|\nabla \tilde{A}\|_{H^s} + E_s^1(\tilde{a}, \tilde{u})(t)) \|(a, u)\|_{X^s} + \|\nabla \tilde{A}\|_{H^s}^2 + \|\nabla \tilde{V}\|_{H^s}
\end{aligned}$$

Now we have to estimate the potential terms. Due to Lemma 6.3, we know that

$$\begin{aligned}
\|\nabla \tilde{V}\|_{H^s} &\lesssim M(\tilde{a}, \tilde{u})(t) \|\tilde{a}\|_{H^{s-1}} \lesssim \|(\tilde{a}, \tilde{u})\|_Y^2, \\
\|\nabla \tilde{A}\|_{H^s} &\lesssim M(\tilde{a}, \tilde{u})(t)^{2s} E_s^1(\tilde{a}, \tilde{u})(t) \lesssim \frac{\|(\tilde{a}, \tilde{u})\|_Y^{2s+1}}{\mu_1}, \\
\|\tilde{A}\|_{W^{1,\infty}} &\lesssim M(\tilde{a}, \tilde{u})(t)^5 \lesssim \|(\tilde{a}, \tilde{u})\|_Y^5
\end{aligned}$$

Here we used (6.19) and the fact that

$$E_s^1(\tilde{a}, \tilde{u})(t) \lesssim \frac{\|(\tilde{a}, \tilde{u})\|_Y}{\mu_1}, \quad \forall 0 < t < T.$$

Now we get a first estimate for $\|(a,u)\|_{X^s}$:

$$\frac{d}{dt}\|(a,u)\|_{X^s} \lesssim \left(\frac{\|(\tilde{a},\tilde{u})\|_Y^{2s+1}+1}{\mu_1}\right)\|(a,u)\|_{X^s} + \left(\frac{\|(\tilde{a},\tilde{u})\|_Y^{4s+2}+1}{\mu_1^2}\right).$$

Multiply both sides with $\|(a,u)\|_{X^s}$ and get

$$\begin{aligned}\frac{d}{dt}\|(a,u)\|_{X^s}^2 &\lesssim \left(\frac{\|(\tilde{a},\tilde{u})\|_Y^{2s+1}+1}{\mu_1}\right)\|(a,u)\|_{X^s}^2 + \|(a,u)\|_{X^s}\left(\frac{\|(\tilde{a},\tilde{u})\|_Y^{4s+2}+1}{\mu_1^2}\right)\\ &\lesssim \left(\frac{\|(\tilde{a},\tilde{u})\|_Y^{2s+1}+1}{\mu_1}\right)\|(a,u)\|_{X^s}^2 + \left(\frac{\|(\tilde{a},\tilde{u})\|_Y^{8s+4}+1}{\mu_1^4}\right).\end{aligned}$$

Writing the estimate in integral form and under the assumptions of Theorem 5 we obtain

$$\|(a,u)\|_{X^s}^2 \lesssim Q^2 + \left(\frac{\|(\tilde{a},\tilde{u})\|_Y^{2s+1}+1}{\mu_1}\right)\int_0^t \|(a,u)(r)\|_{X^s}^2 dr + t\left(\frac{\|(\tilde{a},\tilde{u})\|_Y^{8s+4}+1}{\mu_1^4}\right). \quad (6.67)$$

Step 2: Estimate for $\|a\|_{H^s}$. Applying the proof of (6.56) and (6.57) to system (6.65) yields

$$\begin{aligned}&\hbar^2\frac{d}{dt}\|\partial_x^\alpha a\|_{L^2}^2 + \hbar^2\sum_{i=1}^2\int_\Omega \mathrm{Re}(a_i\partial_x^\alpha \mathrm{div}(\tilde{u})\partial_x^\alpha \overline{a}_i)dx\\ &\lesssim \left(\|(\tilde{a},\tilde{u})\|_Y^{2s+1}+1\right)\left(\hbar\|a\|_{H^s}+\|(a,u)\|_{X^s}\right)^2\\ &\lesssim \left(\frac{\|(\tilde{a},\tilde{u})\|_Y^{2s+1}+1}{\mu_1^2}\right)E_s^{\mu_1}(a,u)(t)^2,\end{aligned}$$

and

$$\begin{aligned}&-2\hbar\sum_{i=1}^2\int_\Omega \partial_x^\alpha\tilde{S}\partial_t\,\mathrm{Im}(\overline{a}_i\partial_x^\alpha a_i)dx - \hbar^2\sum_{i=1}^2\int_\Omega \partial_x^\alpha \mathrm{div}(\tilde{u})\,\mathrm{Re}(\overline{a}_i\partial_x^\alpha a_i)dx\\ &\lesssim \left(\|(\tilde{a},\tilde{u})\|_Y^{2s+2}+1\right)\left(\hbar\|a\|_{H^s}+\|(a,u)\|_{X^s}\right)^2\\ &\lesssim \left(\frac{\|(\tilde{a},\tilde{u})\|_Y^{2s+1}+1}{\mu_1^2}\right)E_s^{\mu_1}(a,u)(t)^2.\end{aligned}$$

In the last step of both estimates, we used

$$E_s^1(a,u)(t) \lesssim \frac{E_s^{\mu_1}(a,u)(t)}{\mu_1}.$$

Adding both estimates leads to the following estimate

$$\hbar^2\frac{d}{dt}\|\partial_x^\alpha a\|_{L^2}^2 - 2\hbar\sum_{i=1}^2\int_\Omega \partial_x^\alpha\tilde{S}\partial_t\,\mathrm{Im}(\overline{a}_i\partial_x^\alpha a_i)dx \lesssim \left(\frac{\|(\tilde{a},\tilde{u})\|_Y^{2s+2}+1}{\mu_1^2}\right)E_s^{\mu_1}(a,u)(t)^2,$$

which is the analogue of (6.45). Since

$$\frac{d}{dt}\sum_{i=1}^{2}\int_{\Omega}\partial_x^{\alpha}\tilde{S}\,\mathrm{Im}(\overline{a}_i\partial_x^{\alpha}a_i)dx=\sum_{i=1}^{2}\int_{\Omega}\partial_x^{\alpha}\tilde{S}\partial_t\,\mathrm{Im}(\overline{a}_i\partial_x^{\alpha}a_i)dx+\sum_{i=1}^{2}\int_{\Omega}\partial_t\partial_x^{\alpha}\tilde{S}\,\mathrm{Im}(\overline{a}_i\partial_x^{\alpha}a_i)dx$$

and

$$\begin{aligned}\hbar\left|\sum_{i=1}^{2}\int_{\Omega}\partial_t\partial_x^{\alpha}\tilde{S}\,\mathrm{Im}(\overline{a}_i\partial_x^{\alpha}a_i)dx\right| &\lesssim \hbar\|\partial_t\partial_x^{\alpha}\tilde{S}\|_{L^2}\|\partial_x^{\alpha}a\|_{L^2}\|a\|_{L^\infty}\\ &\lesssim \hbar\|\partial_t\tilde{u}\|_{H^{s-1}}\|a\|_{H^s}\|a\|_{L^\infty}\\ &\lesssim \frac{1}{\mu_1\mu_2}\|(\tilde{a},\tilde{u})\|_Y E_s^{\mu_1}(a,u)(t)^2,\end{aligned}$$

we have

$$\hbar^2\frac{d}{dt}\|\partial_x^{\alpha}a\|_{L^2}^2-2\hbar\frac{d}{dt}\sum_{i=1}^{2}\int_{\Omega}\partial_x^{\alpha}\tilde{S}\,\mathrm{Im}(\overline{a}_i\partial_x^{\alpha}a_i)dx\lesssim\frac{(\|(\tilde{a},\tilde{u})\|_Y^{2s+2}+1)E_s^{\mu_1}(a,u)(t)^2}{(\mu_1\wedge\mu_2)\mu_1}.$$

Integrating from 0 to t yields

$$\begin{aligned}&\hbar^2\|\partial_x^{\alpha}a\|_{L^2}^2-2\hbar\sum_{i=1}^{2}\int_{\Omega}\partial_x^{\alpha}\tilde{S}\,\mathrm{Im}(\overline{a}_i\partial_x^{\alpha}a_i)dx-\hbar^2\|\partial_x^{\alpha}a^{\hbar,0}\|_{L^2}^2\\ &+2\hbar\sum_{i=1}^{2}\int_{\Omega}\partial_x^{\alpha}\tilde{S}(0,t)\,\mathrm{Im}(\overline{a}_i^{\hbar,0}\partial_x^{\alpha}a_i^{\hbar,0})dx\lesssim\frac{1}{(\mu_1\wedge\mu_2)\mu_1}(\|(\tilde{a},\tilde{u})\|_Y^{2s+2}+1)\int_0^t E_s^{\mu_1}(a,u)(r)^2dr.\end{aligned}$$

Similar to (6.60), we can use

$$\begin{aligned}\hbar^2\|a\|_{H^s}^2&\lesssim(\hbar\|(a^{\hbar,0},u^{\hbar,0})\|_{X^s}\|(\tilde{a},\tilde{u})(0)\|_{X^s}\|a^{\hbar,0}\|_{H^s}+\hbar^2\|a^{\hbar,0}\|_{H^s}^2)\\ &\quad+\hbar\|a\|_{H^s}\|(\tilde{a},\tilde{u})\|_Y\|(a,u)\|_{X^s}+\left(\frac{\|(\tilde{a},\tilde{u})\|_Y^{2s+2}+1}{(\mu_1\wedge\mu_2)\mu_1}\right)\int_0^t E_s^{\mu_1}(a,u)(r)^2dr.\end{aligned}$$

Due to the assumptions of Theorem 5 we know that

$$\begin{aligned}\hbar^2\|a\|_{H^s}^2&\lesssim Q^2(\|(\tilde{a},\tilde{u})(0)\|_{X^s}+1)+\hbar\|a\|_{H^s}\|(\tilde{a},\tilde{u})\|_Y\|(a,u)\|_{X^s}\\ &\quad+\left(\frac{\|(\tilde{a},\tilde{u})\|_Y^{2s+2}+1}{(\mu_1\wedge\mu_2)\mu_1}\right)\int_0^t E_s^{\mu_1,\mu_2}(a,u)(r)^2dr.\end{aligned}\tag{6.68}$$

Step 3: Estimate for $\|\partial_t u\|_{H^{s-1}}$. For $\partial_t u$, a similar estimate to (6.58) is given by

$$\begin{aligned}\|\partial_t u\|_{H^{s-1}}&\lesssim(\|\tilde{u}\|_{L^\infty}+\|\tilde{A}\|_{L^\infty})(\|u\|_{H^s}+\|\tilde{A}\|_{\dot{H}^s})+\|u\|_{L^\infty}(\|\tilde{u}\|_{H^s}+\|\tilde{A}\|_{\dot{H}^s})+\|\nabla\tilde{V}\|_{H^{s-1}}\\ &\lesssim(\|\tilde{u}\|_{H^s}+\|\tilde{A}\|_{\dot{H}^s}+\|\tilde{A}\|_{L^\infty})(\|u\|_{H^s}+\|\tilde{A}\|_{\dot{H}^s})+\|(\tilde{a},\tilde{u})\|_Y^2\\ &\lesssim\left(\|(\tilde{a},\tilde{u})\|_Y^{2s+1}+1\right)\left(\|(\tilde{a},\tilde{u})\|_Y^{2s+1}+1+\|(a,u)\|_{X^s}\right).\end{aligned}\tag{6.69}$$

Step 4: Estimate for $\|(a,u)\|_Y$. Since $\|(a,u)\|_Y = \sup_t E_s^{\mu_1,\mu_2}(a,u)(t)$ we have to combine (6.67), (6.68) and (6.69):

$$\begin{aligned}
E_s^{\mu_1,\mu_2}(a,u)(t)^2 \lesssim & \|(a,u)\|_{X^s}^2 + \mu_1^2\hbar^2\|a\|_{H^s}^2 + \mu_2^2\|\partial_t u\|_{H^{s-1}}^2 \\
\lesssim & Q^2(\mu_1^2\|(\tilde a,\tilde u)(0)\|_{X^s}+1) + t\left(\frac{\|(\tilde a,\tilde u)\|_Y^{8s+4}+1}{\mu_1^4}\right) \\
& + \left(\frac{\|(\tilde a,\tilde u)\|_Y^{2s+1}+1}{\mu_1} + \frac{\mu_1\left(\|(\tilde a,\tilde u)\|_Y^{2s+2}+1\right)}{(\mu_1\wedge\mu_2)}\right)\int_0^t E_s^{\mu_1,\mu_2}(a,u)(r)^2 dr \\
& + \mu_1^2\hbar\|a\|_{H^s}\|(\tilde a,\tilde u)\|_Y\|(a,u)\|_{X^s} \\
& + \mu_2^2\left(\|(\tilde a,\tilde u)\|_Y^{4s+2}+1\right)\left(\|(\tilde a,\tilde u)\|_Y^{4s+2}+1+\|(a,u)\|_{X^s}^2\right).
\end{aligned}$$

Using

$$\frac{\|(\tilde a,\tilde u)\|_Y^{2s+1}+1}{\mu_1} + \frac{\mu_1\left(\|(\tilde a,\tilde u)\|_Y^{2s+2}+1\right)}{(\mu_1\wedge\mu_2)} \lesssim \frac{\|(\tilde a,\tilde u)\|_Y^{2s+2}+1}{\mu_1\wedge\mu_2},$$

yields

$$\begin{aligned}
E_s^{\mu_1,\mu_2}(a,u)(t)^2 \lesssim & Q^2(\mu_1^2\|(\tilde a,\tilde u)(0)\|_{X^s}+1) + t\left(\frac{\|(\tilde a,\tilde u)\|_Y^{8s+4}+1}{\mu_1^4}\right) \\
& + \frac{\|(\tilde a,\tilde u)\|_Y^{2s+2}+1}{\mu_1\wedge\mu_2}\int_0^t E_s^{\mu_1,\mu_2}(a,u)(r)^2 dr \\
& + \mu_1^2\hbar\|a\|_{H^s}\|(\tilde a,\tilde u)\|_Y\|(a,u)\|_{X^s} \\
& + \mu_2^2\left(\|(\tilde a,\tilde u)\|_Y^{4s+2}+1\right)\left(\|(\tilde a,\tilde u)\|_Y^{4s+2}+1+\|(a,u)\|_{X^s}^2\right).
\end{aligned}$$

Now suppose that $\|(\tilde a,\tilde u)\|_Y \le L$. Then the estimate above leads to

$$\begin{aligned}
E_s^{\mu_1,\mu_2}(a,u)(t)^2 \lesssim_s & \; Q^2(\mu_1^2 L+1) + \frac{t(L^{8s+4}+1)}{\mu_1^4} \\
& + \left(\frac{L^{2s+2}+1}{\mu_1\wedge\mu_2}\right)\int_0^t E_s^{\mu_1,\mu_2}(a,u)(r)^2 dr \\
& + \mu_2^2(L^{4s+2}+1)^2 + (\mu_1 L + \mu_2^2(L^{4s+2}+1))E_s^{\mu_1,\mu_2}(a,u)(r)^2,
\end{aligned}$$

for some constant C_s. If we choose L large enough and μ_1,μ_2 small enough such that

$$\begin{aligned}
C_s(\mu_1 L + \mu_2^2(L^{4s+2}+1)) &\le 1/2, \\
C_s(Q^2(\mu_1^2 L+1) + \mu_2^2(L^{4s+2}+1)^2) &\le L^2/4,
\end{aligned}$$

we can get

$$E_s^{\mu_1,\mu_2}(a,u)(t)^2 \le \frac{L^2}{2} + 2C_s\left(\left(\frac{L^{2s+2}+1}{\mu_1\wedge\mu_2}\right)\int_0^t E_s^{\mu_1,\mu_2}(a,u)(r)^2 dr + \frac{t(L^{8s+4}+1)}{\mu_1^4}\right).$$

By Gronwall's inequality, we get

$$E_s^{\mu_1,\mu_2}(a,u)(t)^2 \leq \left(\frac{L^2}{2} + \frac{2C(s)(L^{8s+4}+1)}{\mu^4}t\right)\exp\left(2C_s\left(\frac{L^{2s+2}+1}{\mu_1 \wedge \mu_2}\right)t\right).$$

Then there is a T small enough such that

$$\|(a,u)\|_Y^2 = \sup_{t\in[0,T]} E_s^{\mu_1,\mu_2}(a,u)(t)^2 \leq L^2.$$

It follows that $\|(a,u)\|_Y \leq L$. □

Proof of Lemma 6.8. We want to estimate the difference between the image of two solutions $(\tilde{a}^1, \tilde{u}^1)$ and $(\tilde{a}^2, \tilde{u}^2)$ to the linearized system. Let

$$(a^j, u^j) = \Upsilon^{\hbar}_{a^{\hbar,0},u^{\hbar,0}}(\tilde{a}^j, \tilde{u}^j), \quad j = 1, 2.$$

By definition, (a^j, u^j) satisfies

$$\begin{aligned}
\partial_t a_k^j + (\tilde{A}^j + \tilde{u}^j)\cdot\nabla a_k^j + \frac{1}{2}a_k^j \mathrm{div}(\tilde{u}^j + \tilde{A}^j) &= \frac{i\hbar}{2}\Delta a_k^j + \frac{i}{2}\widetilde{\mathbf{B}}_k^{j,i} a_i^j,\\
\partial_t u^j + (\tilde{A}^j + \tilde{u}^j)\cdot\nabla u^j + u^j\cdot\nabla\tilde{A}^j + \nabla(\frac{|\tilde{A}^j|^2}{2} + \tilde{V}^j) &= 0,\\
a_k^j(x,0) &= a_k^{\hbar,0}(x), \quad k = 1, 2\\
u(x,0) &= u^{\hbar,0}(x).
\end{aligned} \tag{6.70}$$

where $\tilde{A}^j, \widetilde{\mathbf{B}}^j$ and $\tilde{V}^j$ are defined as above with every variable added with a superscript j. Define

$$\delta a := a^1 - a^2.$$

The quantities $\delta u, \delta A, \delta V, \delta\mathbf{B}, \delta\rho$ and $\delta\tilde{a}, \delta\tilde{u}, \delta\tilde{A}, \delta\tilde{V}, \delta\widetilde{\mathbf{B}}, \delta\tilde{\rho}$ are similarly defined. Now substract (6.70) with $j = 2$ from (6.65) with $j = 1$:

$$\begin{aligned}
&\partial_t\delta a_j + (\delta\tilde{A} + \delta\tilde{u})\cdot\nabla a_j^1 + (\tilde{A}^2 + \tilde{u}^2)\cdot\nabla\delta a_j\\
&\quad + \frac{1}{2}a_j^1\mathrm{div}(\delta\tilde{u} + \delta\tilde{A}) + \frac{1}{2}\delta a_j \mathrm{div}(u^2 + A^2) = \frac{i}{2}\delta\widetilde{\mathbf{B}}_j^i a_i^1 + \frac{i}{2}\widetilde{\mathbf{B}}_j^{2,i}\delta a_i + \frac{i\hbar}{2}\Delta\delta a_j,\\
&\partial_t\delta u + (\delta\tilde{A} + \delta\tilde{u})\cdot\nabla u^1 + (\tilde{A}^2 + \tilde{u}^2)\cdot\nabla\delta u\\
&+\delta u\cdot\nabla\tilde{A}^1 + u^2\cdot\nabla\delta\tilde{A} + \nabla(\frac{\delta\tilde{A}\cdot(\tilde{A}^1 + \tilde{A}^2)}{2} + \delta\tilde{V}) = 0,\\
&\delta a_j(x,0) = 0,\\
&\delta u(x,0) = 0.
\end{aligned} \tag{6.71}$$

Here, $\delta\tilde{V}$ and $\delta\tilde{A}$ are given by

$$-\Delta\delta\tilde{V} = \delta\tilde{\rho}, \tag{6.72}$$

$$-\Delta\delta\tilde{A} = \hbar\delta\tilde{w} - \hbar\delta\tilde{v} - (\delta\tilde{\rho}\tilde{u}^1 - \tilde{\rho}^2\delta\tilde{u}) + \delta\tilde{\rho}\tilde{A}^2 - \tilde{\rho}^1\delta\tilde{A}. \tag{6.73}$$

Again we have to split the proof into multiple steps since we have to estimate the difference in Y, so we have to estimate $\|(\delta a, \delta u)\|_{X^s}$, $\mu_1 \hbar \|\delta a\|_{H^s}$ and $\mu_2 \|\partial_t \delta u\|_{H^{s-1}}$.

Step 1. Apply ∂^α with $0 \le n(\alpha) \le s-1$ to the first equation in (6.71) which yields

$$\partial_t \partial^\alpha \delta a_j + (\tilde{A}^2 + \tilde{u}^2) \cdot \nabla \partial^\alpha \delta a_j + \mathbf{R}_{d,j} = \frac{i\hbar}{2} \Delta \partial^\alpha \delta a_j,$$

where $\mathbf{R}_{d,j}$ is defined as

$$\begin{aligned}
\mathbf{R}_{d,j} &:= \partial^\alpha((\delta\tilde{A} + \delta\tilde{u}) \cdot \nabla a_j^1) + \partial^\alpha((\tilde{A}^2 + \tilde{u}^2) \cdot \nabla \delta a_j) - (\tilde{A}^2 + \tilde{u}^2) \cdot \nabla \partial^\alpha \delta a_j \\
&+ \frac{1}{2}\partial^\alpha(a_j^1 \mathrm{div}(\delta\tilde{u} + \delta\tilde{A})) + \frac{1}{2}\partial^\alpha(\delta a_j \mathrm{div}(u^2 + A^2)) - \frac{i}{2}\partial^\alpha\left(\delta\widetilde{\mathbf{B}}_j^i a_i^1\right) + \frac{i}{2}\partial^\alpha\left(\widetilde{\mathbf{B}}_j^{2,i} \delta a_i\right) \\
&:= \sum_{j=1}^{7} F_j.
\end{aligned}$$

Then Lemma 6.1 implies

$$\begin{aligned}
\frac{d}{dt}\|\partial^\alpha \delta a_j\|_{L^2} &\lesssim (M(\tilde{a}_i^2, \tilde{u}^2)(t) + \|\tilde{A}^2\|_{W^{1,\infty}})\|\partial^\alpha \delta a_j\|_{L^2} + \|\mathbf{R}_{d,j}\|_{L^2} \\
&\lesssim_{\mu_1,\mu_2,L} \|\partial^\alpha \delta a_j\|_{L^2} + \|\mathbf{R}_{d,j}\|_{L^2}.
\end{aligned}$$

In the second step, we used $(\tilde{a}^2, \tilde{u}^2) \in \mathcal{B}_{s,\mu_1,\mu_2,T,L}$. With the help of the Kato-Ponce inequality and (6.29) we obtain the following estimates. For F_1 in $\mathbf{R}_{d,j}$ we have the following estimates. If $n(\alpha) \ge 1$,

$$\begin{aligned}
\|F_1\|_{L^2} &\lesssim \|\delta\tilde{A} + \delta\tilde{u}\|_{\dot{H}^{n(\alpha)}}\|\nabla a_j^1\|_{L^\infty} + \|a_j^1\|_{H^{s-2}}\|\delta\tilde{A} + \delta\tilde{u}\|_{L^\infty} \\
&\lesssim_{\mu_1,\mu_2,L} \|\delta\tilde{A} + \delta\tilde{u}\|_{\dot{H}^{n(\alpha)}} + \|\delta\tilde{A} + \delta\tilde{u}\|_{L^\infty} \\
&\lesssim_{\mu_1,\mu_2,L} \|\nabla\delta\tilde{A}\|_{H^{s-1}} + \|\delta\tilde{u}\|_{H^s}.
\end{aligned}$$

If $n(\alpha) = 0$ then

$$\begin{aligned}
\|F_1\|_{L^2} &\lesssim \|a_j^1\|_{H^1}\|\delta\tilde{A} + \delta\tilde{u}\|_{L^\infty} \\
&\lesssim_{\mu_1,\mu_2,L} \|\nabla\delta\tilde{A}\|_{H^{s-1}} + \|\delta\tilde{u}\|_{H^s}.
\end{aligned}$$

Now consider $F_2 - F_3$. If $n(\alpha) \ge 1$,

$$\begin{aligned}
\|F_2 - F_3\|_{L^2} &\lesssim \|\tilde{A}^2 + \tilde{u}^2\|_{\dot{H}^{n(\alpha)}}\|\nabla \delta a_j\|_{L^\infty} + \|\delta a_j\|_{H^{s-1}}\|\tilde{A}^2 + \tilde{u}^2\|_{W^{1,\infty}} \\
&\lesssim_{\mu_1,\mu_2,L} \|\delta a_j\|_{H^{s-1}}.
\end{aligned}$$

And if $n(\alpha) = 0$ then $\|F_2 - F_3\|_{L^2} = 0$. For F_4 we have

$$\begin{aligned}
\|F_4\|_{L^2} &\lesssim \|\delta\tilde{u} + \delta\tilde{A}\|_{\dot{H}^{n(\alpha)}}\|\nabla a_j^1\|_{L^\infty} + \|a_j^1\|_{H^{s-1}}\|\delta\tilde{u} + \delta\tilde{A}\|_{W^{1,\infty}} \\
&\lesssim_{\mu_1,\mu_2,L} \|\nabla\delta\tilde{A}\|_{H^{s-1}} + \|\delta\tilde{u}\|_{H^s}.
\end{aligned}$$

For F_5 we have

$$\|F_5\|_{L^2} \lesssim \|u^2 + A^2\|_{\dot{H}^{n(\alpha)}} \|\nabla \delta a_j\|_{L^\infty} + \|\delta a_j\|_{H^{s-1}} \|u^2 + A^2\|_{W^{1,\infty}} \\ \lesssim_{\mu_1,\mu_2,L} \|\delta a_j\|_{H^{s-1}}.$$

For F_6 and F_7 we have

$$\|F_6\|_{L^2} \lesssim \|\delta \widetilde{\mathbf{B}}\|_{H^{s-1}} \|a^1\|_{L^\infty} + \|a^1\|_{H^{s-1}} \|\delta \widetilde{\mathbf{B}}\|_{W^{1,\infty}} \\ \lesssim \|\nabla \delta \tilde{A}\|_{H^{s-1}} \|a^1\|_{L^\infty} + \|a^1\|_{H^{s-1}} \|\nabla \delta \tilde{A}\|_{W^{1,\infty}} \\ \lesssim_{\mu_1,\mu_2,L} \|\nabla \delta \tilde{A}\|_{H^{s-1}},$$

and

$$\|F_7\|_{L^2} \lesssim \|\widetilde{\mathbf{B}}^2\|_{H^{s-3}} \|\delta a\|_{L^\infty} + \|\delta a\|_{H^{s-1}} \|\widetilde{\mathbf{B}}^2\|_{W^{1,\infty}} \\ \lesssim_{\mu_1,\mu_2,L} \|\delta a\|_{H^{s-1}},$$

For $\|\partial^\alpha((\delta\tilde{A} + \delta\tilde{u}) \cdot \nabla a_j^1)\|_{L^2}$ and $\|\partial^\alpha((\tilde{A}^2 + \tilde{u}^2) \cdot \nabla \delta a_j) - (\tilde{A}^2 + \tilde{u}^2) \cdot \nabla \partial^\alpha \delta a_j\|_{L^2}$, we used two different estimates in two different cases, i.e. $n(\alpha) \geq 1$ and $n(\alpha) = 0$, because we need to avoid the term $\|A\|_{L^2}$ (See Remark 12). Collecting the estimates above, similar to (6.31), we can easily prove that

$$\frac{d}{dt}\|\delta a\|_{H^{s-1}} \lesssim_{\mu_1,\mu_2,L} \|\delta a\|_{H^{s-1}} + \|\delta\tilde{u}\|_{H^s} + \|\nabla \delta \tilde{A}\|_{H^{s-1}}.$$

For δu, similar to (6.32), we have

$$\frac{d}{dt}\|\delta u\|_{H^s} \lesssim_{\mu_1,\mu_2,L} \|\delta u\|_{H^s} + \|\delta\tilde{u}\|_{H^s} + \|\nabla \delta \tilde{A}\|_{H^{s-1}} + \|\nabla \delta \tilde{V}\|_{H^s}.$$

For $\delta\tilde{V}$, similar to (6.33), we can prove that

$$\|\nabla \delta V\|_{H^s} \lesssim_{\mu_1,\mu_2,L} \|\delta a\|_{H^{s-1}}.$$

For δA, similar to (6.39) and (6.43), from (6.73) we can derive

$$\|\nabla \delta A\|_{L^2} \lesssim_{\mu_1,\mu_2,L} \|\delta u\|_{L^2} + \|\delta a\|_{L^2} + \hbar \|\nabla \delta a\|_{L^2},$$

and

$$\|\nabla \delta A\|_{H^{s-1}} \lesssim_{\mu_1,\mu_2,L} \|\delta u\|_{H^s} + \|\delta a\|_{H^{s-1}} + \hbar \|\delta a\|_{H^s} + \|\nabla \delta A\|_{L^2} \\ \lesssim_{\mu_1,\mu_2,L} \|\delta u\|_{H^s} + \|\delta a\|_{H^{s-1}} + \hbar \|\delta a\|_{H^s}$$

It follows that

$$\frac{d}{dt}\|(\delta a, \delta u)\|_{X^s} \lesssim_{\mu_1,\mu_2,L} \|(\delta a, \delta u)\|_{X^s} + \|(\delta\tilde{a}, \delta\tilde{u})\|_Y.$$

The integral form of this estimate is

$$\|(\delta a, \delta u)\|_{X^s} \lesssim_{\mu_1,\mu_2,L} \int_0^t \|(\delta a, \delta u)(r)\|_{X^s} dr + t\|(\delta\tilde{a}, \delta\tilde{u})\|_Y. \tag{6.74}$$

This completes the estimate in X^s.

Step 2. Similarly to (6.56), we can prove that

$$\begin{aligned}\hbar^2 \frac{d}{dt}\|\partial_x^\alpha \delta a\|_{L^2}^2 + \hbar^2 \sum_{i=1}^2 \int_\Omega \operatorname{Re}(\delta a_i \partial_x^\alpha \operatorname{div}(\tilde{u}^2)\partial_x^\alpha \delta \overline{a}_i)dx \\ \lesssim_{\mu_1,\mu_2,L} \|(\delta a, \delta u)\|_{X^s}^2 + \|(\delta\tilde{a}, \delta\tilde{u})\|_{Y_{\mu_1,\mu_2}^{T,s}}^2.\end{aligned}$$

Similarly to (6.57), we can prove that

$$\begin{aligned}&-2\hbar \sum_{i=1}^2 \int_\Omega \partial_x^\alpha \tilde{S}\partial_t \operatorname{Im}(\delta\overline{a}_i \partial_x^\alpha \delta a_i)dx - \hbar^2 \sum_{i=1}^2 \int_\Omega \operatorname{Re}(\delta a_i \partial_x^\alpha \operatorname{div}(\tilde{u}^2)\partial_x^\alpha \delta\overline{a}_i)dx \\ &\lesssim_{\mu_1,\mu_2,L} \|(\delta a, \delta u)\|_{X^s}^2 + \|(\delta\tilde{a}, \delta\tilde{u})\|_Y^2.\end{aligned}$$

Then similarly to (6.45), adding the two estimates above yields

$$\hbar^2 \frac{d}{dt}\|\partial_x^\alpha \delta a\|_{L^2}^2 - 2\hbar \sum_{i=1}^2 \int_\Omega \partial_x^\alpha \tilde{S}\partial_t \operatorname{Im}(\delta\overline{a}_i \partial_x^\alpha \delta a_i)dx \lesssim_{\mu_1,\mu_2,L} \|(\delta a, \delta u)\|_{X^s}^2 + \|(\delta\tilde{a}, \delta\tilde{u})\|_Y^2.$$

It follows that

$$\hbar^2 \frac{d}{dt}\|\partial_x^\alpha \delta a\|_{L^2}^2 - 2\hbar \frac{d}{dt} \sum_{i=1}^2 \int_\Omega \partial_x^\alpha \tilde{S} \operatorname{Im}(\delta\overline{a}_i \partial_x^\alpha \delta a_i)dx \lesssim_{\mu_1,\mu_2,L} \|(\delta a, \delta u)\|_{X^s}^2 + \|(\delta\tilde{a}, \delta\tilde{u})\|_Y^2.$$

Integrating from 0 to t gives

$$\begin{aligned}\hbar^2 \|\partial_x^\alpha \delta a\|_{L^2}^2 - 2\hbar \sum_{i=1}^2 \int_\Omega \partial_x^\alpha \tilde{S} \operatorname{Im}(\delta\overline{a}_i \partial_x^\alpha \delta a_i)dx \\ \lesssim_{\mu_1,\mu_2,L} \int_0^t \|(\delta a, \delta u)(r)\|_{X^s}^2 dr + t\|(\delta\tilde{a}, \delta\tilde{u})\|_Y^2.\end{aligned} \tag{6.75}$$

Notice that the second term of the LHS is bounded by

$$\begin{aligned}\left|2\hbar \sum_{i=1}^2 \int_\Omega \partial_x^\alpha \tilde{S} \operatorname{Im}(\delta\overline{a}_i \partial_x^\alpha \delta a_i)dx\right| &\le C_{L,s}\hbar \|(\delta a, \delta u)(r,\cdot)\|_{X^s} \|\partial_x^\alpha \delta a\|_{L^2} \\ &\le \frac{C_{L,s}}{\mu_1}(\|(\delta a, \delta u)\|_{X^s}^2 + \mu_1^2 \hbar^2 \|\partial_x^\alpha \delta a\|_{L^2}^2),\end{aligned}$$

for some constant $C_{L,s}$. Then if we choose μ_1 small enough such that

$$\mu_1 C_{L,s} |\{\alpha : n(\alpha) = s\}| \le \frac{1}{2},$$

we have the following estimate,

$$\sum_{n(\alpha)=s}\mu_1^2\left|2\hbar\sum_{i=1}^{2}\int_\Omega \partial_x^\alpha \tilde{S}\,\mathrm{Im}(\delta\overline{a}_i\partial_x^\alpha\delta a_i)dx\right|\leq\frac{1}{2}(\|(\delta a,\delta u)\|_{X^s}^2+\mu_1^2\hbar^2\|\partial_x^\alpha\delta a\|_{L^2}^2).$$

Summing (6.75) with respect to α yields

$$\begin{aligned}&\hbar^2\|\delta a\|_{\dot{H}^s}^2-\frac{1}{2\mu_1^2}(\|(\delta a,\delta u)(r)\|_{X^s}^2+\mu_1^2\hbar^2\|\partial_x^\alpha\delta a\|_{L^2}^2)\\ \lesssim&\sum_{n(\alpha)=s}\hbar^2\|\partial_x^\alpha\delta a\|_{L^2}^2-2\hbar\sum_{n(\alpha)=s}\sum_{i=1}^{2}\int_\Omega\partial_x^\alpha\tilde{S}\,\mathrm{Im}(\delta\overline{a}_i\partial_x^\alpha\delta a_i)dx\\ \lesssim_{\mu_1,\mu_2,L}&\int_0^t\|(\delta a,\delta u)(r)\|_{X^s}^2dr+t\|(\delta\tilde{a},\delta\tilde{u})\|_Y^2.\end{aligned}$$

Combine this estimate and (6.74) and it follows that

$$\|(\delta a,\delta u)\|_{X^s}^2+\mu_1^2\hbar^2\|\partial_x^\alpha\delta a\|_{L^2}^2\lesssim_{\mu_1,\mu_2,L}\int_0^t\|(\delta a,\delta u)(r)\|_{X^s}^2dr+t\|(\delta\tilde{a},\delta\tilde{u})\|_Y^2.\tag{6.76}$$

Step 3. For $\partial_t\delta u$, we have the following estimate

$$\|\partial_t\delta u\|_{H^{s-1}}\leq C_{\mu_1,L}\left(\|(\delta a,\delta u)\|_{X^s}+\|(\delta\tilde{a},\delta\tilde{u})\|_Y\right)$$

We can choose μ_2 small enough, which depending on μ_1 and L, such that

$$\mu_2C_{\mu_1,L}\leq\frac{1}{4}.$$

Then it is obvious that

$$\mu_2^2\|\partial_t\delta u\|_{H^{s-1}}^2\leq\frac{1}{4}\left(\|(\delta a,\delta u)\|_{X^s}^2+\|(\delta\tilde{a},\delta\tilde{u})\|_Y^2\right).$$

Step 4. Combine this estimate with (6.76) and get

$$\begin{aligned}&\|(\delta a,\delta u)\|_{X^s}^2+\mu_1^2\hbar^2\|\partial_x^\alpha\delta a\|_{L^2}^2+\mu_2^2\|\partial_t\delta u\|_{H^{s-1}}^2\\ \leq&\frac{1}{2}\|(\delta\tilde{a},\delta\tilde{u})\|_Y^2+C_{\mu_1,\mu_2,L}\left(\int_0^t\|(\delta a,\delta u)(r)\|_{X^s}^2dr+t\|(\delta\tilde{a},\delta\tilde{u})\|_Y^2\right).\end{aligned}$$

By Gronwall's inequality, we know that

$$\|(\delta a,\delta u)\|_Y^2\leq\left(\frac{1}{2}+C_{\mu_1,\mu_2,L}T\right)\exp(C_{\mu_1,\mu_2,L}T)\|(\delta\tilde{a},\delta\tilde{u})\|_Y^2.$$

By choosing T small enough we can get (6.66). □

6.5 Behavior at Blowup Time

In this section, we are going to prove Theorem 5.b. We only prove the second part of 5.b, i.e. the blow up of the Pauli-Poisswell-WKB equation (6.4). It is equivalent to Proposition 4. Again, by a similar but simpler argument, we easily obtain the first part in Theorem 5.b, i.e. the blow up of the Euler-Poisswell equation (1.32).

Proposition 4. *Under the assumptions of Proposition 3 there exists a unique local solution* $(a^\hbar, u^\hbar)$ *to the Pauli-Poisswell-WKB equation* (6.4)*. Suppose the solution is defined on a maximal time interval* $[0, T^*)$*. Then we have*

$$\lim_{t \to T^*-} M(a^\hbar, u^\hbar)(t) = +\infty.$$

where M is defined by (6.17)*. Furthermore, when $\hbar$ is small enough, there exists a $K = K_{\hbar,Q,T^*}$, such that*

$$\limsup_{t \to T^*-} \|u^\hbar(t)\|_{W^{1,\infty}} + \|a^\hbar(t)\|_{L^\infty} > K.$$

Here, for fixed Q and T^*, $K_{\hbar,Q,T^*}$ goes to infinity as $\hbar$ goes to zero.

Proof. Suppose that T^* is the maximal time of existence of the solution. Then $E^1_s(t)(a^\hbar, u^\hbar)$ blows up as $t \uparrow T^*$. If $N(a^\hbar, u^\hbar)(t)$ would remain bounded when $t \uparrow T^*$, then by Proposition 2 and under the assumptions of Theorem 5, $E^1_s(a^\hbar, u^\hbar)$ would remain bounded when $t \uparrow T^*$ which is a contradiction. Thus,

$$\lim_{t \uparrow T^*} N(a^\hbar, u^\hbar)(t) = +\infty.$$

On the other hand, it is obvious that $N(a^\hbar, u^\hbar)(t)$ remains bounded when $t < T^*$. It follows that

$$\lim_{t \uparrow T^*} M(a^\hbar, u^\hbar)(t) = +\infty.$$

For the second part we are going to use a bootstrap argument. If for all times

$$\|u^\hbar(t)\|_{W^{1,\infty}} + \|a^\hbar(\cdot, t)\|_{L^\infty} \le K, \tag{6.77}$$

for some constant K to be assigned we also assume that

$$\|a(t)\|_{H^1} + \|a(t)\|_{W^{1,\infty}} + \|a(t)\|_{W^{2,3}} \le K/\hbar, \tag{6.78}$$

for all times. Here we also require that

$$\|a^{\hbar,0}\|_{H^1} + \|a^{\hbar,0}\|_{W^{1,\infty}} + \|a^{\hbar,0}\|_{W^{2,3}} \le K/\hbar. \tag{6.79}$$

Then it follows that for all $0 < t < T^*$,

$$N(a^\hbar, u^\hbar)(t) \le 2K + 1.$$

Since $s > 7/2$, by the Sobolev embedding, we can immediately get

$$\|a(t)\|_{H^1} + \|a(t)\|_{W^{1,\infty}} + \|a(t)\|_{W^{2,3}} \lesssim \|a\|_{H^s},$$

where the constant only depends on s. Due to the a priori estimate we know that

$$\begin{aligned}\|a(t)\|_{H^1} + \|a(t)\|_{W^{1,\infty}} + \|a(t)\|_{W^{2,3}} &\lesssim E_s^1(a^\hbar, u^\hbar)(t) \\ &\lesssim N(a^\hbar, u^\hbar)(t)^{2s+3} Q e^{CN(a^\hbar,u^\hbar)(t)^{2s+3}t} \\ &\leq C(2K+1)^{2s+3} Q e^{C(2K+1)^{2s+3}T^*}. \end{aligned} \tag{6.80}$$

where $C = C_s$. If we have

$$C(2K+1)^{2s+3} Q e^{C(2K+1)^{2s+3}T^*} < \frac{K}{\hbar}, \tag{6.81}$$

then (6.80) is a stronger estimate than (6.78). By the bootstrap argument, we can actually obtain that (6.80) holds true all the time. Then (6.77) implies that $N(a^\hbar, u^\hbar)(t)$ is bounded for all times, which contradicts the fact that T^* is maximal. Thus we only need to show that when $\hbar$ is small enough, there exists $K = K(\hbar, Q, T^*)$ such that (6.81) holds true. And we also require that $K > CQ$. It is easy to see that such a K exists. In fact, we can take

$$K = \frac{1}{2}\left(\left|\log \frac{\sqrt{\hbar}}{CT^*}\right|^{1/(2s+3)} - 1\right).$$

Then the LHS of (6.81) is

$$\frac{CQ\left|\log \frac{\sqrt{\hbar}}{CT^*}\right|}{\sqrt{\hbar}},$$

and the RHS of (6.81) is

$$\frac{1}{2\hbar}\left(\left|\log \frac{\sqrt{\hbar}}{CT^*}\right|^{1/(2s+3)} - 1\right).$$

It is obvious that when $\hbar$ is small enough, (6.81) and (6.79) hold true and K goes to infinity as $\hbar \to 0+$. This finishes the proof. □

6.6 Semiclassical limit

In this section, we are going to prove our main result, Theorem 6. Note that in equation (6.4) the semiclassical parameter $\hbar$ only appears in the terms $(i\hbar/2)\Delta a_j$ and $\hbar w - \hbar v$. In Proposition 2 established the regularity of a and u in $X^s = H^{s-1} \times H^s$ by the a priori estimate. Now we can directly pass to the semiclassical limit $\hbar \to 0$ by taking the difference between $(a^\hbar, u^\hbar)$ and (a, u) in X^s.

Proof of Theorem 6. We split the proof into three parts. In the first part we estimate the difference between the solution $(a^\hbar, u^\hbar)$ of the Pauli-Poisswell-WKB equation and the solution (a, u) of the Euler-Poisswell equation. Here we use the machinery developed so far (i.e. charge and energy conservation and the Poisson estimates). In the second part we show the semiclassical limit of the density $\rho^\hbar$, the current density $J^\hbar$ and the monokinetic Wigner transform $f^\hbar$. In the third part we prove the statement about the time of existence in the theorem.

Let $(a^\hbar, u^\hbar, A^\hbar, V^\hbar)$ denote the solution of (6.4) and (a, u, A, V) the solution of (1.32). Let

$$
\begin{aligned}
\delta a &:= a^\hbar - a, & \delta u &:= u^\hbar - u, & \delta A &:= A^\hbar - A,\\
\delta V &:= V^\hbar - V, & \delta \mathbf{B} &:= \mathbf{B}^\hbar - \mathbf{B}, & \delta \rho &:= \rho^\hbar - \rho.
\end{aligned}
$$

Subtracting (1.32) from (6.4) yields

$$
\begin{aligned}
&\partial_t \delta a + (\delta A + \delta u)\cdot \nabla a_j^\hbar + (A+u)\cdot\nabla \delta a_j\\
&\qquad + \frac{1}{2} a_j^\hbar \operatorname{div}(\delta u + \delta A) + \frac{1}{2}\delta a_j \operatorname{div}(u+A) = \frac{i}{2}\delta \mathbf{B}_j^i a_i^\hbar + \frac{i}{2}\mathbf{B}_j^i \delta a_i + \frac{i\hbar}{2}\Delta a_j^\hbar,\\
&\partial_t \delta u + (\delta A + \delta u)\cdot\nabla u^\hbar + (A+u)\cdot\nabla\delta u + \delta u\cdot\nabla A^\hbar\\
&\qquad + u^\hbar\cdot\nabla\delta A + \nabla\Big(\frac{1}{2}\delta A\cdot(A + A^\hbar) + \delta V\Big) = 0,\\
&\qquad \delta a_j(x,0) = 0\\
&\qquad \delta u(x,0) = 0.
\end{aligned}
\tag{6.82}
$$

Similarly, δV and δA are given by

$$
\begin{aligned}
-\Delta \delta V &= \delta\rho,\\
-\Delta \delta A &= \hbar w^\hbar - \hbar v^\hbar - (\delta\rho u^\hbar - \rho\delta u) + \delta\rho A - \rho^\hbar \delta A.
\end{aligned}
$$

Apply ∂^α with $0 \le n(\alpha) \le s-3$ to the first equation in (6.82), we obtain

$$
\partial_t \partial^\alpha \delta a_j + (A+u)\cdot\nabla\partial^\alpha\delta a_j + \mathbf{R}_d = 0,
$$

with

$$
\begin{aligned}
\mathbf{R}_d :=\ & \partial^\alpha((\delta A + \delta u)\cdot\nabla a_j^\hbar) + \partial^\alpha((A+u)\cdot\nabla\delta a_j) - (A+u)\cdot\nabla\partial^\alpha\delta a_j\\
& + \frac{1}{2}\partial^\alpha(a_j^\hbar \operatorname{div}(\delta u + \delta A)) + \frac{1}{2}\partial^\alpha(\delta a_j \operatorname{div}(u+A))\\
& - \frac{i}{2}\partial^\alpha(\delta\mathbf{B}_j^i a_i^\hbar) + \frac{i}{2}\partial^\alpha(\mathbf{B}_j^i\delta a_i) - \frac{i\hbar}{2}\Delta\partial^\alpha a_j^\hbar.\\
:=\ & \sum_{j=1}^{8} F_j
\end{aligned}
$$

Thanks to Lemma 6.1, we obtain

$$\frac{d}{dt}\|\partial^\alpha \delta a_j\|_{L^2}^2 = \int_\Omega \mathrm{div}(A+u)|\partial^\alpha \delta a_j|^2 dx - \int_\Omega \mathrm{Re}(\mathbf{R}_d \partial^\alpha \overline{\delta a_j}).$$

Step 1: Estimate for $\mathbf{R}_d$. With the help of the Kato-Ponce inequality and (6.29), we get the following estimates. If $n(\alpha)\geq 1$,

$$\begin{aligned}\|F_1\|_{L^2} = \|\partial^\alpha((\delta A+\delta u)\cdot \nabla a_j^\hbar)\|_{L^2} &\lesssim \|\delta A+\delta u\|_{\dot H^{n(\alpha)}}\|\nabla a_j^\hbar\|_{L^\infty} + \|a_j^\hbar\|_{H^{s-2}}\|\delta A+\delta u\|_{L^\infty}\\ &\lesssim_{Q,T} \|\delta A+\delta u\|_{\dot H^{n(\alpha)}} + \|\delta A+\delta u\|_{L^\infty}\\ &\lesssim_{Q,T} \|\nabla \delta A\|_{H^{s-3}} + \|\delta u\|_{H^s},\end{aligned}$$

and

$$\begin{aligned}\|F_2-F_3\|_{L^2} &= \|\partial^\alpha((A+u)\cdot\nabla \delta a_j) - (A+u)\cdot \nabla\partial^\alpha \delta a_j\|_{L^2}\\ &\lesssim \|A+u\|_{\dot H^{n(\alpha)}}\|\nabla \delta a_j\|_{L^\infty} + \|\delta a_j\|_{H^{s-3}}\|A+u\|_{W^{1,\infty}}\\ &\lesssim_{Q,T} \|\delta a_j\|_{H^{s-3}}.\end{aligned}$$

If $n(\alpha)=0$,

$$\begin{aligned}\|F_1\|_{L^2} = \|\partial^\alpha((\delta A+\delta u)\cdot\nabla a_j^\hbar)\|_{L^2} &\lesssim \|a_j^\hbar\|_{H^1}\|\delta A+\delta u\|_{L^\infty}\\ &\lesssim_{Q,T} \|\nabla\delta A\|_{H^{s-3}} + \|\delta u\|_{H^s},\end{aligned}$$

and

$$\|F_2-F_3\|_{L^2} = \|\partial^\alpha((A+u)\cdot\nabla\delta a_j) - (A+u)\cdot\nabla\partial^\alpha\delta a_j\|_{L^2} = 0,$$

Moreover,

$$\begin{aligned}\|F_4\|_{L^2} \lesssim \|\partial^\alpha(a_j^\hbar \mathrm{div}(\delta u+\delta A))\|_{L^2} &\lesssim \|\delta A+\delta u\|_{\dot H^{n(\alpha)}}\|\nabla a_j^\hbar\|_{L^\infty} + \|a_j^\hbar\|_{H^{s-1}}\|\delta A+\delta u\|_{W^{1,\infty}}\\ &\lesssim_{Q,T} \|\nabla\delta A\|_{H^{s-3}} + \|\delta u\|_{H^s}\end{aligned}$$

For F_5 we have

$$\begin{aligned}\|F_5\|_{L^2} \lesssim \|\partial^\alpha(\delta a_j \mathrm{div}(u+A))\|_{L^2} &\lesssim \|A+u\|_{\dot H^{n(\alpha)}}\|\nabla\delta a_j\|_{L^\infty} + \|\delta a_j\|_{H^{s-3}}\|A+u\|_{W^{1,\infty}}\\ &\lesssim_{Q,T} \|\delta a_j\|_{H^{s-3}}.\end{aligned}$$

For F_6 we have

$$\begin{aligned}\|F_6\|_{L^2} \lesssim \|\partial^\alpha(\delta \mathbf{B}_j^i a_i^\hbar)\|_{L^2} &\lesssim \|\delta\mathbf{B}\|_{H^{s-3}}\|a^\hbar\|_{L^\infty} + \|a^\hbar\|_{H^{s-3}}\|\delta\mathbf{B}\|_{W^{1,\infty}}\\ &\lesssim \|\nabla\delta A\|_{H^{s-3}}\|a^\hbar\|_{L^\infty} + \|a^\hbar\|_{H^{s-3}}\|\nabla\delta A\|_{W^{1,\infty}}\\ &\lesssim_{Q,T} \|\nabla\delta A\|_{H^{s-3}}.\end{aligned}$$

For F_7 we have

$$\begin{aligned}\|F_7\|_{L^2} \lesssim \|\partial^\alpha(\mathbf{B}_j^i \delta a_i)\|_{L^2} &\lesssim \|\mathbf{B}\|_{H^{s-3}}\|\delta a\|_{L^\infty} + \|\delta a\|_{H^{s-3}}\|\mathbf{B}\|_{W^{1,\infty}}\\ &\lesssim_{Q,T} \|\delta a\|_{H^{s-3}},\end{aligned}$$

And for F_8, by Proposition 3, we have

$$\|F_8\|_{L^2} \lesssim \hbar \|\Delta \partial^\alpha a_j^\hbar\|_{L^2} \lesssim \hbar \|a^\hbar\|_{H^{s-1}} \lesssim_{Q,T} \hbar.$$

For $\|\partial^\alpha((\delta A + \delta u) \cdot \nabla a_j^\hbar)\|_{L^2}$ and $\|\partial^\alpha((A+u) \cdot \nabla \delta a_j) - (A+u) \cdot \nabla \partial^\alpha \delta a_j\|_{L^2}$, we used two different estimates in two different cases, i.e. $n(\alpha) \geq 1$ and $n(\alpha) = 0$, because we need to avoid the term $\|A\|_{L^2}$ (See Remark 12). Similar to (6.30), we can combine the estimates above together and get

$$\|\mathbf{R}_d\|_{L^2} \lesssim_{Q,T} \|\nabla \delta A\|_{H^{s-3}} + \|\delta u\|_{H^{s-2}} + \|\delta a\|_{H^{s-3}} + \hbar.$$

Step 2: Estimates for $\|\delta u\|_{H^{s-2}}$ and $\|\delta a\|_{H^{s-3}}$. It is obvious that

$$\left| \int_\Omega \operatorname{div}(A+u) |\partial^\alpha \delta a_j|^2 dx \right| \lesssim_{Q,T} \|\delta a\|_{H^{s-3}}^2.$$

Similar to (6.31), it follows that

$$\frac{d}{dt} \|\delta a\|_{H^{s-3}} \lesssim_{Q,T} \|\delta a\|_{H^{s-3}} + \|\delta u\|_{H^{s-2}} + \|\nabla \delta A\|_{H^{s-3}} + \hbar. \tag{6.83}$$

Similarly, for δu, we can obtain

$$\frac{d}{dt} \|\delta u\|_{H^{s-2}} \lesssim_{Q,T} \|\delta u\|_{H^{s-2}} + \|\nabla \delta A\|_{H^{s-3}} + \|\nabla \delta V\|_{H^{s-1}}. \tag{6.84}$$

Step 3: Estimates for δV and δA For δV, similar to (6.33), we can prove that

$$\|\nabla \delta V\|_{H^{s-1}} \lesssim_{Q,T} \|\delta a\|_{H^{s-3}}. \tag{6.85}$$

For δA, similar to (6.39) and (6.43), we can prove that

$$\begin{aligned} \|\nabla \delta A\|_{L^2} &\lesssim_{Q,T} \|\delta u\|_{L^2} + \|\delta a\|_{L^2} + \hbar, \\ \|\nabla \delta A\|_{H^{s-3}} &\lesssim_{Q,T} \hbar + \|\delta u\|_{H^{s-3}} + \|\delta a\|_{H^{s-3}} + \|\nabla \delta A\|_{L^2}. \end{aligned}$$

It follows that

$$\|\nabla \delta A\|_{H^{s-3}} \lesssim_{Q,T} \hbar + \|\delta u\|_{H^{s-3}} + \|\delta a\|_{H^{s-3}}. \tag{6.86}$$

Combining (6.83), (6.84), (6.85) and (6.86), we finally get

$$\frac{d}{dt} (\|\delta a\|_{H^{s-3}} + \|\delta u\|_{H^{s-2}}) \lesssim_{Q,T} \|\delta a\|_{H^{s-3}} + \|\delta u\|_{H^{s-2}} + \hbar.$$

Applying the Gronwall inequality, we immediately get

$$\|\delta a\|_{H^{s-3}} + \|\delta u\|_{H^{s-2}} \lesssim_{Q,T} \|a^{\hbar,0} - a^0\|_{H^{s-3}} + \|u^{\hbar,0} - u^0\|_{H^{s-2}} + \hbar.$$

Due to the the assumptions of Theorem 5 (6.11) immediately follows. This finishes the proof of the first part of Theorem 6.

Now we turn to the second part of Theorem 6, which is the semiclassical limit of the density $\rho^\hbar$, the current $J^\hbar$ and the Wigner transform $f^\hbar$. Recall that $\rho^\hbar$ and $J^\hbar$ are given by

$$\rho^\hbar = |a^\hbar|^2,$$
$$J^\hbar = \hbar w^\hbar - \hbar v^\hbar - \rho^\hbar(u^\hbar + A^\hbar).$$

Since $a^\hbar$ and $u^\hbar$ are uniformly bounded in H^{s-1} and H^s respectively, we can pass to the limit $\hbar \to 0$, which yields

$$\begin{aligned}
\rho^\hbar & \underset{\hbar\to 0}{\longrightarrow} & \rho = |a|^2, \\
\rho^\hbar(u^\hbar + A^\hbar) & \underset{\hbar\to 0}{\longrightarrow} & J = -(\rho u + \rho A), \\
\hbar w^\hbar - \hbar v^\hbar & \underset{\hbar\to 0}{\longrightarrow} & 0.
\end{aligned}$$

in H^{s-3}. The semiclassical limit $J^\hbar \to J$ as $\hbar \to 0$ immediately follows. Finally, we come to the semiclassical of $f^\hbar$. First, by [36], for any fixed $t < T$, there exists a subsequence $\{f^\hbar\}$ (we still denote it by $f^\hbar$ for convenience) and a non-negative Radon measure f, such that

$$f^\hbar \underset{\hbar\to 0}{\rightharpoonup} f \quad \text{in } \mathcal{A}'.$$

It is easy to see that

$$\|\hbar\nabla\Psi^\hbar\|_{L^2} \le C,$$

for some constant C independent of $\hbar$. Due to [59], it follows that

$$\int_{\mathbb{R}^3} f(x,\xi,t)d\xi = \lim_{\hbar\to 0} \rho^\hbar(x,t) = \rho(x,t).$$

It is easy to see that

$$\|(u(x,t) - i\hbar\nabla)\Psi^\hbar\|_{L^2} \underset{\hbar\to 0}{\longrightarrow} 0.$$

Then using the argument in [100], we know that

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} |\xi - u(x,t)|^2 f(x,\xi,t)d\xi dx = 0,$$

and that

$$f(x,\xi,t) = \rho(x,t)\delta(\xi - u(x,t)).$$

It is easy to see that f satisfies (5.17) and (6.12).

The last part of Theorem 6 is proved by a bootstrap argument. For any $T < T^0$, there exists $K = K(T) > 0$, such that for all $0 < t < T$,

$$\|(a,u)(t)\|_{X^s} + N(a,u)(t) \le K.$$

Suppose that

$$E_s^1(a^\hbar, u^\hbar)(t) \le K', \tag{6.87}$$

for all $0 < t < T$ with K' to be determined. By virtue of the first part of Theorem 6, it follows that for all $0 < t < T$,

$$\|(a, u)(t) - (a^\hbar, u^\hbar)(t)\|_{X^{s-2}} \lesssim_{K,K',T} \hbar.$$

Using Sobolev's inequality, we obtain

$$\|u - u^\hbar\|_{W^{1,\infty}} + \|a - a^\hbar\|_{H^1} + \|a - a^\hbar\|_{W^{1,\infty}} + \|a - a^\hbar\|_{W^{2,3}} \lesssim_{K,Q,T} \hbar.$$

When $\hbar$ is small enough, this estimate yields for all $0 < t < T$,

$$\|N(a^\hbar, u^\hbar)(T)\|_{X^s} \leq 2K,$$

By Proposition 2 and the assumptions of Theorem 5 we know that for all $0 < t < T$,

$$E^1_s(a^\hbar, u^\hbar)(t) \leq C(2K)^{2s+3} Q e^{C(2K)^{2s+3}T}.$$

If we take K' larger than $C(2K)^{2s+3} Q e^{C(2K)^{2s+3}T}$, then the above estimate is sharper than (6.87). By the bootstrap argument, we know that (6.87) holds true and $(a^\hbar, u^\hbar)$ does not blow up at T. It follows that for all $T < T^0$,

$$\liminf_{t \to T^\hbar -} T^\hbar \geq T,$$

which immediately leads to (6.13) and finishes the proof. $\square$

A Glossary of notations

$I = [-T, T]$ denotes a time interval.

For a vector space X we use $X^{n\times n}$ to denote its matrix-valued analogue, i.e. $F \in X^{n\times n}$ means that F is a $n \times n$ matrix with components in X.

$\mathcal{M}(\mathbb{R}^d)$ denotes the positive and bounded measures on $\mathbb{R}^d$. A subscript $w*$ indicates that the space is equipped with the weak star topology. We denote the L^p norm by $\|\cdot\|_p$. If M is a $n\times n$ matrix with values in $(L^p)^{n\times n}$, then $\|M\|_p$ denotes $\sum_{i,j} \|M_{ij}\|_p$ and similar for the Sobolev spaces $W^{s,p}$, $s > 0$, $1 \leq p \leq \infty$. We denote the local L^p spaces by L^p_{loc}, similarly for the local Sobolev space H^s_{loc}. $L^{p,\infty}$ denotes the weak L^p space. $W^{k,p}(\mathbb{R}^d)$ are the usual Sobolev spaces and $\mathcal{D}'$ is the space of distributions. $\mathcal{S}$ is the Schwartz space of rapidly decreasing functions and $\mathcal{S}'$ the space of tempered distributions and BC_w is the space of bounded weakly continuous functions. The expression V_- denotes the negative part of V. Tr denotes the trace of a $n \times n$ matrix and tr denotes the trace of a trace class operator on a Hilbert space $\mathcal{H}$.

We denote the L^2 inner product by $\langle f, g\rangle := \int f\overline{g}\mathrm{d}x$. The Fourier transform of a function f is denoted by $\mathcal{F}[f]$ or $\widehat{f}$. $\langle x\rangle$ denotes the Japanese bracket $(1 + |x|^2)^{1/2}$.

Let A be a vector with components A_i where $i = 1, 2, 3$. Then $\nabla \otimes A$ denotes the matrix with components $\partial_i A_j$. We use div or $\nabla\cdot$ for the divergence and $\nabla\times$ for the curl. For a $m \times n$ matrix A and a n-dimensional vector x we use summation convention, i.e.

$$A_i^j x_j := \sum_{j=1}^{n} A_{ij} x_j.$$

For two vector-valued functions $f, g : \mathbb{R}^d \to \mathbb{R}^d$, if f is differentiable, we use $g \cdot \nabla f$ to denote the vector $(\partial^j f_i g_j)_{1\leq i\leq d}$. For a vector of indices $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$, we denote its length by $n(\alpha)$. For $f \in H^s(\Omega)$ and a vector α we denote its derivative by

$$\partial^\alpha f(x) := \partial^{\alpha_1} \dots \partial^{\alpha_n} f(x).$$

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