universität wien

MASTERARBEIT / MASTER'S THESIS

Titel der Masterarbeit / Title of the Master's Thesis

„Quantum vs. Classical Gravity in Bose-Einstein-Condensate Thermodynamics"

verfasst von / submitted by

Thomas Strasser BSc

angestrebter akademischer Grad / in partial fulfilment of the requirements for the degree of

Master of Science (MSc)

Wien, 2023 / Vienna, 2023

Studienkennzahl lt. Studienblatt / degree programme code as it appears on the student record sheet:	A 066 876
Studienrichtung lt. Studienblatt / degree programme as it appears on the student record sheet:	Masterstudium Physics
Betreut von / Supervisor:	Univ. Prof. Dr. Caslav Brukner
Mitbetreut von / Co-Supervisor:	

Abstract

The nature of gravity has been the subject of intense research since even before Isaac Newton. But to this day, neither the insights provided by General Relativity (GR), nor by Quantum Mechanics (QM) seem to be able to answer the question whether it is a quantum operator or a fundamentally classical interaction. Despite best efforts on the theoretical level, what we lack the most is guidance and measurements by experiment. The main problem lies in the weakness of the gravitational force, making desktop experiments almost impossible to realize. Here we derive a Hartree-Fock-Bogoliubov (HFB) theory for a self-gravitating Bose Einstein condensate (BEC) at finite temperatures and provide estimates for desktop experiments to measure the specific heat capacity in order to find signatures of quantum- or classical gravity. We extend HFB-theory for a quantized as well as classical ranged, gravity-like interaction, using a self-consistent Gaussian factorization of the condensed and thermalized states. We combine this ranged interaction with the known results for a repulsive scattering interaction in the Born-approximation, to find a full description of a self-gravitating dilute Bose gas at non-zero temperatures and its thermodynamic observables. Finally, we estimate the evolution of the specific heat capacity c_V and the size and feasibility for such measurements. Our findings show, that the thermodynamics of a BEC exhibit a characteristically different signature depending on whether gravity is a quantum- or a classical operator, and that such experiments are within reach of being conducted at lab-scale in the near future.

Die Natur der Gravitation war bereits vor Isaac Newton Gegenstand intensiver Forschung. Aber bis heute können weder unsere Einsichten aus der Allgemeinen Relativitätstheorie (GR), noch jene aus der Quantenmechanik (QM) die Frage beantworten, ob Gravitation ein Quantenoperator oder eine fundamental klassische Wechselwirkung ist. Trotz bester Bemühungen auf theoretischer Seite ist das, was uns am meisten fehlt, Anleitung durch Experimente und Beobachtungen. Das Hauptproblem liegt in der Schwäche der gravitativen Wechselwirkung, die es beinahe unmöglich macht, Experimente im Labor durchzuführen. Hier leiten wir eine Hartreee-Fock-Bogolyubov-Theorie (HFB) für ein selbst-gravitierendes Bose-Einstein Kondensat bei endlichen Temperaturen her, und geben Schätzungen für Labor-basierte Experimente ab, mithilfe derer die spezifische Wärmekapazität gemessen werden kann, um festzustellen ob es Anzeichen einer Quanten- oder klassischen Gravitation gibt. Wir erweitern HFB-Theorie für eine quantisierte, sowie klassische, langreichweitige, Gravitations-artige Wechselwirkung, indem wir eine selbst-konsistente Gauss'sche Faktorisierung für die kondensierten und thermalisierten Teilchen vornehmen. Wir kombinieren diese langreichweitige Wechselwirkung mit den bekannten Resultaten für eine repulsive Streuwechselwirkung in der Born-Näherung und finden eine volle Beschreibung eines "dünnen" (im Sinne der physikalischen Definition von "dilute") Bose-Gases bei Temperaturen ungleich Null, und deren thermodynamischen Observablen. Zuletzt schätzen wir die Entwicklung der spezifischen Wärmekapazität c_V und die Größe, sowie Machbarkeit, solcher Experimente ab. Unsere Erkenntnisse zeigen, dass die Thermodynamik eines BEC eine charakteristisch unterschiedliche Signatur aufweist, abhängig davon, ob Gravitation ein Quanten- oder klassischer Operator ist, und dass solche Experimente in naher Zukunft in Reichweite sind, was ihre Umsetzbarkeit im Labor betrifft.

I would like to thank my family for their unbounded love and support.
I would not be me if you weren't you.

Contents

1 Introduction

More than a century ago, physics was revolutionized by the discoveries of Quantum Mechanics (QM) [1] and Special- and General Relativity (GR) [2]. Each theory describes its relevant regimes and forces very well - GR for large masses and gravity, and QM for low masses and the other three fundamental forces. Each theory is well tested within its respective limits, delivering predictions which have been confirmed by experiment and observation, time and again. Yet, up to this day, it is unclear how to reconcile the two theories, in order to find a more unified description of nature [3]. Of the many attempts [4], *quantizing gravity* is a popular approach, where QM is seen as a fundamental theory of nature, and it is explored how one could change GR to fit it into the theoretical framework of QM. Questioning, if this is the right approach [5], other researchers have suggested to *gravitize quantum physics* [6], where the principles of GR are upheld to investigate how this leads to possible changes in the theoretical complex of QM. This encompasses so called semi-classical theories of gravity (CG), where the principles of QM are taken to hold, i.e. matter obeys quantum theories, but gravity remains fundamentally classical.

Either way, our hopes of finding an answer simply based on theoretical studies has as of yet been met with disappointment. What we lack most are concrete hints and measurements based on experiment and observation. Hence, it has become an increasingly important part of research to suggest such experiments. Moreover, the scientific community in physics, as well as astrophysics, has been labouring to find experimental and observational setups which show a degree of feasibility that would allow us to conduct such tests within the near future, and without the need to build particle accelerators the size of the Milky Way [7], [8].

In the context of suggesting experimental setups to test for either of the aforementioned options, Bose-Einstein Condensates (BECs) have recently come into focus [9]. Especially, since the first breakthrough realizations of such BECs in experiments in 1995 [10], [11], a large amount of research and improvement in terms of technically manageable experiments with this 4th state of matter has been accomplished. In this thesis, we will also consider a BEC to work out how such signatures of a possibly *quantized gravity* (QG), or fundamentally *classical gravity* (CG), could be spotted. In particular, we will concentrate our efforts on working out the thermodynamic properties and observables at finite temperatures below the critical temperature T_c.

1.1 Theoretical groundwork for quantized- vs. classical gravity

To make some of the statements of the introductory paragraph more explicit, consider the quantization of the weak-field limit of gravity:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,,\ \ |h_{\mu\nu}| << 1 \tag{1}$$

and

$$\mathcal{H}_{int} = -\tfrac{1}{2} T^{\mu\nu} h_{\mu\nu} \tag{2}$$

Without diving too deeply into detail, let us work on the reasonable assumption that matter obeys quantum field theory (QFT), such that $\phi \to \hat{\phi}$ for the fields, and hence generally speaking

$$\hat{T}^{\mu\nu} = \partial_\mu \hat{\phi}\, \partial_\nu \hat{\phi} - \tfrac{1}{2}\eta_{\mu\nu}\, \partial_\rho \hat{\phi}\, \partial_\rho \hat{\phi} + \tfrac{1}{2}\eta_{\mu\nu} m^2 \hat{\phi}^2\,. \tag{3}$$

Accordingly, for classical gravity, we have $-\frac{1}{2}\hat{T}^{\mu\nu} h_{\mu\nu}$, while for a quantized gravity $-\frac{1}{2}\hat{T}^{\mu\nu}\hat{h}_{\mu\nu}$ is conjectured.

Furthermore, we denote the fully classical Interaction-Hamiltonian by

$$\mathcal{H}_{int} = \tfrac{1}{2} \int d^3\mathbf{r}\, \rho(\mathbf{r}) \Phi(\mathbf{r})\,, \tag{4}$$

where $\Phi(\mathbf{r})$ is the classical Newton potential and $\rho(\mathbf{r})$ denotes the classical mass density. In order to arrive at a meaningful formulation one has to consider that for classical gravity one needs to only quantize $\rho(\mathbf{r})$, where $\rho(\mathbf{r}) = m\,\hat{\Psi}^\dagger(\mathbf{r})\hat{\Psi}(\mathbf{r})$ for a BEC, and for a quantized version of gravity, it is necessary to quantize both $\rho(\mathbf{r}$ and $\Phi(\mathbf{r})$. Accordingly, the Interaction-Hamiltonians then read:

$$\begin{aligned} \mathcal{H}_{QG} = &\quad \frac{1}{2} m \int d^3\mathbf{r} : \hat{\Psi}^\dagger(\mathbf{r}) \hat{\Psi}(\mathbf{r}) \hat{\Phi}(\mathbf{r}') : \\ &= -\frac{1}{2} G m^2 \int d^3\mathbf{r}\, d^3\mathbf{r}' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \hat{\Psi}^\dagger(\mathbf{r}) \hat{\Psi}^\dagger(\mathbf{r}') \hat{\Psi}(\mathbf{r}') \hat{\Psi}(\mathbf{r})\,, \end{aligned} \tag{5}$$

where $: \cdots :$ denotes the normal ordered product. For the classical gravity, the Interaction-Hamiltonian is then given by

$$\mathcal{H}_{CG} = m \int d^3\mathbf{r}\, \hat{\Psi}^\dagger(\mathbf{r})\hat{\Psi}(\mathbf{r})\, \Phi[\hat{\Psi}](\mathbf{r}')\,, \tag{6}$$

where $\Phi[\hat{\Psi}](\mathbf{r})$ denotes Φ as a functional of $\hat{\Psi}$ and

$$\hat{\Phi}(\mathbf{r}') = -m \int d^3\mathbf{r}'\, \frac{\hat{\Psi}^\dagger(\mathbf{r}')\hat{\Psi}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \tag{7}$$

in equation (5) is understood to be the quantized version of the solution of Poisson's equation. Then, using the well-known substitution for a BEC, $\hat{\Psi}(\mathbf{r}) = \psi(\mathbf{r})\,\hat{a}$, where $\hat{a}$ denotes the annihilation operator, one arrives at:

$$\begin{aligned} \mathcal{H}_{QG} &= \frac{1}{2}\,\lambda_{QG}\,\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a} \\ \lambda_{QG} &:= -Gm^2 \int d^3\mathbf{r}\, d^3\mathbf{r}'\, \frac{|\psi(\mathbf{r})|^2\,|\psi(\mathbf{r}')|^2}{|\mathbf{r}-\mathbf{r}'|}\,, \end{aligned} \tag{8}$$

for quantized self-gravitational interactions, and

$$\begin{aligned} \mathcal{H}_{CG} &= \lambda_{CG}[\Psi]\,\hat{a}^\dagger\hat{a} \\ \lambda_{CG}[\Psi] &:= Gm \int d^3\mathbf{r}'\, |\psi(\mathbf{r})|^2\, \Phi[\hat{\Psi}](\mathbf{r}')\,, \end{aligned} \tag{9}$$

for classical self-gravitational interactions in the BEC. Please note, that while this description is by no means extensive, it is assumed to hold in the low energy and low momenta regime of a Bose-Einstein condensed phase.

1.2 Outline of this thesis

In the *2nd section* of the thesis we shall start by reviewing the properties of the gravitational potential for a spherically symmetric system, in- and outside of the shell. Then, we will recap the basics of the thermodynamic observables to be discussed and show an exemplary textbook computation of such observables for a quantum harmonic oscillator. We will also give a quick summary of Bose-Einstein condensation to recall the basic approach.

In the *3rd section* we will discuss *Bogoliubov theory* for a zero ranged, i.e. scattering interaction, to properly derive and understand how it works. Based on these results, we will then derive the full *Hartree-Fock-Bogoliubov* (HFB) picture of a ranged interaction in *Section 4*, which is the main theoretical part of this thesis. We will specialize to a gravitational interaction, discuss its *Fourier transform*, and finally combine this with the zero-ranged scattering interaction based on literature results.

Starting with the *specific heat capacity c_V*, we shall discuss feasibility for possible experimental setups in *section 5*, with a particular focus on Orders of Magnitude for the interaction strengths and contributions coming from the normal averages as well as the anomalous averages of HFB theory. We will derive estimates showing a characteristic behaviour depending on which picture of gravity one considers, and which will be of importance when measuring the self-gravitational interaction.

As will become apparent in this theoretical framework, experiments to measure the thermodynamic properties of a self-gravitating BEC, aimed at spotting signatures of whether the nature of gravity is classical or quantum, are reasonably within reach.

2 Preliminary Steps

2.1 Gravitational potential and self-gravitating systems

First, we take a look at the classical definition of the gravitational potential Φ in the Newtonian limit. Recall the gravitational constant $G = 6.674 \cdot 10^{-11}[Nm^2kg^{-2}]$ and the definition of the gravitational potential assuming a sphere of radius R:

$$\Phi(\mathbf{r}) = -G \int dV' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, . \tag{10}$$

We assume the sphere to have a constant mass density ρ, which hence does no longer depend on the coordinate $\mathbf{r}'$. Also, in spherical coordinates, the denominator is

$$\begin{aligned} |\mathbf{r} - \mathbf{r}'| &= (\mathbf{r}' sin\theta' cos\varphi')^2 + (\mathbf{r}' sin\theta' sin\varphi')^2 + (z - \mathbf{r}' cos\theta')^2 \\ &= \mathbf{r}'^2 + z^2 - 2z\mathbf{r}' cos\theta' \end{aligned} \tag{11}$$

and hence

$$\Phi(z) = -G\rho \int_0^{2\pi} d\varphi \int_0^{\mathbf{r}'} d\mathbf{r}' \int_0^{\pi} d\theta' \frac{\mathbf{r}'^2 sin\theta'}{\sqrt{\mathbf{r}'^2 + z^2 - 2z\mathbf{r}' cos\theta'}} \, . \tag{12}$$

Define the distance $s^2 = |\mathbf{r} - \mathbf{r}'| = \mathbf{r}'^2 + z^2 - 2z\mathbf{r}' cos\theta'$ and hold $\mathbf{r}'$ fixed for now (valid as long as one does the θ-integral first in this case), then

$$\begin{aligned} 2sds &= 2z\mathbf{r}' sin\theta' d\theta' \\ \Rightarrow \frac{s}{z} ds &= \mathbf{r}' sin\theta' d\theta' \, . \end{aligned} \tag{13}$$

Rewriting and executing the φ-integral

$$\begin{aligned} \Phi(z) &= -2\pi G\rho \int_0^R d\mathbf{r}' \mathbf{r}' \int_{s_{min}}^{s_{max}} ds \frac{s}{z} \frac{1}{s} \\ &= -\frac{2\pi G\rho}{z} \int_0^R d\mathbf{r}' \mathbf{r}' (s_{max} - s_{min}) \, . \end{aligned} \tag{14}$$

The limits of integration were $0 < \theta' < \pi$ and hence $s_{min}^2 = \mathbf{r}'^2 + z^2 - 2z\mathbf{r}' = (\mathbf{r}' - z)^2$ and $s_{max}^2 = (\mathbf{r}' + z)^2$. Notice that the values of s_{min} define whether we are considering the volume outside of the sphere ($s_{min} > 0$) or inside of it ($s_{min} < 0$). First, let's consider the case of $s_{min} > 0$:

$$\begin{aligned} \Phi(z) &= -\frac{2\pi G\rho}{z} \int_0^R d\mathbf{r}' \mathbf{r}' \Big((\mathbf{r}' + z) - (z - \mathbf{r}') \Big) \\ &= -\frac{2\pi G\rho}{z} \int_0^R d\mathbf{r}' 2\mathbf{r}'^2 \\ &= -\frac{4\pi G\rho}{3z} R^3 \, . \end{aligned} \tag{15}$$

Notice that $4\pi R^3/3$ is nothing but the volume V of the sphere, and since $\rho = M/V$ it follows that $V\rho = M$, and hence

$$\Phi(z) = -\frac{GM}{z} \, . \tag{16}$$

Since we have spherical symmetry, there is nothing special about the z direction, and hence we can rewrite our result as

$$\Phi(\mathbf{r}) = -\frac{GM}{|\mathbf{r}|} \, . \tag{17}$$

Now, considering the case $s_{min} < 0$, i.e. inside the sphere, or more precisely, the gravitational potential of a very thin shell within the sphere, we find

$$\begin{aligned} \Phi(\mathbf{r}) &= \int_0^R d\mathbf{r}' \int_{s_{min}}^{s_{max}} ds \Big(-\frac{2\pi G\rho(\mathbf{r}')\mathbf{r}'}{\mathbf{r}} \Big) \\ &= \int_{s_{min}}^{s_{max}} ds \Big(-\frac{2\pi G\rho R}{\mathbf{r}} \Big) \\ &= -\frac{2\pi G\rho R}{\mathbf{r}} \Big((R + \mathbf{r}) - (R - \mathbf{r}) \Big) \\ &= -4\pi G\rho R \, , \end{aligned} \tag{18}$$

where we have used that in a thin shell the mass density $\rho(\mathbf{r}')$ is zero everywhere, except for $\mathbf{r}'$ close to R, allowing us to set $\mathbf{r}' = R$. Hence the gravitational potential within a spherical shell is constant:

$$\Phi(\mathbf{r}) = -\frac{GM}{R} \,. \tag{19}$$

Considering now the gravitational potential energy of a two-body system

$$U_{grav}(\mathbf{r}) = -\frac{Gm_1 m_2}{|\mathbf{r}_1 - \mathbf{r}_2|} \tag{20}$$

in the light of our results, for a spherical particle-cloud of N particles of (identical) mass m, and hence a mass density of $\rho = mN$, it is easy to see that the potential becomes

$$U_{grav}(\mathbf{r}) = -\frac{Gm^2 N}{R} \,, \tag{21}$$

where $|\mathbf{r}| \leq R$. Let us now make the self-gravitational potential even more explicit. As will be discussed in more detail later, at short distances, the particle interaction is no longer dominated by self-gravitational effects, but by particle-particle interactions. In line with our later considerations, we will assume a repulsive particle self-interaction, i.e. repulsive scattering processes. In the most general case, such an interaction potential can be written as [12]

$$U_A(|\mathbf{r}_i - \mathbf{r}_j|) = \frac{1}{|\mathbf{r}_i - \mathbf{r}_j|_A} = \begin{cases} -\dfrac{1}{|\mathbf{r}_i - \mathbf{r}_j|} & \text{if } |\mathbf{r}_i - \mathbf{r}_j| \geq A \,, \\ +\dfrac{1}{A} & \text{if } |\mathbf{r}_i - \mathbf{r}_j| \leq A \,, \end{cases} \tag{22}$$

where $A \ll R$ is the short-distance cutoff. This will be made more explicit later. The gravitational potential can hence be written as

$$U_{grav} = -Gm^2 \sum_{1\leq i<j\leq N} \frac{1}{|\mathbf{r}_i - \mathbf{r}_j|_A} \,. \tag{23}$$

For future considerations, it makes sense to introduce the dimensionless variable

$$\kappa := \frac{Gm^2 N}{R k_B T} \tag{24}$$

and use it together with

$$\nu(\mathbf{r}_1, \ldots, \mathbf{r}_N) = \frac{1}{N} \sum_{1\leq i<j\leq N} -\frac{1}{|\mathbf{r}_i - \mathbf{r}_j|_A} \,, \tag{25}$$

such that

$$\frac{U_{grav}}{k_B T} = \kappa\nu =: \chi \,. \tag{26}$$

Later, we will use χ for analyzing orders of magnitude of the self-gravitational interaction.

2.2 Observables in the Thermodynamic Limit

It is the goal of this work to derive observables within the thermodynamic limit which can then be measured in a desktop experiment to check for a signature of the self-gravitational interaction of quantized/classical gravity. To this end, we will try to use the following general, well-known, formulae:

- the thermal energy density is defined as $u = \frac{E}{V}$.
- the specific heat is then found using $c_V = \frac{\partial u}{\partial T}$, where T denotes the temperature.

In order to compute the above, one needs to compute the average energy

$$\langle E \rangle = Tr\{\rho\mathcal{H}\} \,, \tag{27}$$

where in thermal equilibrium $\rho = \frac{1}{Z} exp(-\beta\mathcal{H})$. Note that

$$\langle E \rangle = -\frac{\partial ln(Z)}{\partial \beta} \tag{28}$$

with $\beta = 1/k_B T$, and

$$Z = Tr\{exp(-\beta\mathcal{H})\}\,, \tag{29}$$

is the partition function. Then

$$u = -\frac{1}{V}\partial_\beta(lnZ) \tag{30}$$

and

$$c_V = \partial_T\Big(-\frac{1}{V}\partial_\beta(lnZ)\Big)\,. \tag{31}$$

As is known for a classical gas (not condensed into a BEC), the thermodynamic limit for self-gravitating systems does not exist in its usual form ($V \to \infty$, $N \to \infty$, N/V = fixed). The system collapses into a very dense phase which is determined by the short distance (non-gravitational) forces between the particles. One finds though [12], that the thermodynamic functions exist in the *dilute* limit ($V \to \infty$, $N \to \infty$, $N/V^{1/3}$ = fixed), where V stands for the volume of the box containing the gas. In such a limit, the energy E, the free energy and the entropy turn out to be extensive. That is, we find that they take the form of N times a function of

$$\chi := \frac{Gm^2N}{Rk_BT}\,, \tag{32}$$

where χ is an intensive variable, as defined in equation (24). Namely it stays finite when N and $V \propto R^3$ tend to the infinite and hence χ is appropriate for the canonical ensemble. Physical magnitudes as the specific heat, speed of sound, chemical potential and compressibility only depend on χ. The energy, the free energy, the Gibbs free energy and the entropy are of the form of N times a function of χ, which have a finite $N \to \infty$ limit for fixed χ. In other words, the dependence on χ in all these magnitudes is expressed through a single universal function $f(\chi)$.

2.3 Computation of $\langle E\rangle$ & c_V for a quantum harmonic oscillator

The classical harmonic oscillator model (in a chain) is solved by proposing a solution in terms of waves (normal modes of the oscillation):

$$x_k(t) \propto e^{-ikR_n}e^{-i\omega_k t} \tag{33}$$

This solution describes waves propagating in the chain with phase velocity $c = \omega/k$ and group velocity $v_g = \partial\omega/\partial k$ (the speed of sound in the given material). At small k the two velocities are equal, but for large k (for example small spacing between particles) we have $v_g \to 0$. The dispersion relation for such a model reads $\omega(k) = 2\sqrt{\frac{K}{M}}|sin\big(\frac{kd}{2}\big)| =: \omega_0$. The corresponding quantum-mechanical model is found by replacing the position and momentum coordinates in the Hamiltonian by the corresponding operators. Corresponding to the classical solution, we also look for solutions (i.e. eigenvectors that diagonalize the Hamiltonian) in terms of waves. In this basis, x_n and p_n will be expressed as linear combinations of waves with different wavevectors, e.g.

$$x_n = \frac{1}{\sqrt{N}}\sum_k X_k e^{ikR_n}\,. \tag{34}$$

Rewriting the operators X_k and P_k in terms of creation and annihilation operators and substituting them into the Hamiltonian for the harmonic oscillator, one arrives at:

$$\mathcal{H} = \frac{1}{2}\sum_k \omega_k\Big(a_k^\dagger a_k + a_k a_k^\dagger\Big) = \sum_k \omega_k\Big(a_k^\dagger a_k + \frac{1}{2}\Big)\,, \tag{35}$$

where $a^\dagger$, a are the boson creation- and annihilation operators satisfying the canonical commutation relations (CCR)

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij} = 1 \tag{36}$$

for $i = j$, and

$$[\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0 = [\hat{a}_i, \hat{a}_j]\,. \tag{37}$$

The partition function is then computed according to the formalism presented in equations (27) to (31) as follows:

$$\begin{aligned}
Z &= Tr\Big\{e^{-\beta\left(\sum_k \omega_k (a_k^\dagger a_k + \frac{1}{2})\right)}\Big\} \\
&= Tr\Big\{\prod_k e^{-\beta\left(\omega_k (a_k^\dagger a_k + \frac{1}{2})\right)}\Big\} \\
&= \prod_k e^{-\beta\omega_k/2} \sum_n e^{-\beta(\omega_k)^n} \\
&= \prod_k \frac{e^{-\beta\omega_k/2}}{1 - e^{-\beta\omega_k}} \, ,
\end{aligned} \tag{38}$$

where the trace of the operator was rewritten as the sum of its Eigenvalues, $Tr[O] = \sum_{i=1}^{N} \lambda_i$, and the geometric series $\sum_{n=1}^{\infty} q^n = \frac{q}{q-1}$ was used. Subsequent computations follow as described in equations (30) and (31). So for the k-th mode, we can compute

$$\begin{aligned}
\langle E \rangle = \frac{\partial ln(Z)}{\partial \beta} &= \frac{\partial}{\partial \beta} \sum_k \left(\frac{e^{-\beta\omega_k/2}}{1 - e^{-\beta\omega_k}} \right) \\
&= \sum_k \frac{\omega_k \, e^{-\beta\omega_k/2}}{1 - e^{-\beta\omega_k}} \, ,
\end{aligned} \tag{39}$$

and subsequently

$$\begin{aligned}
c_V &= \frac{\partial}{\partial T} \sum_k \left(\frac{\omega_k \, e^{-\beta\omega_k/2}}{1 - e^{-\beta\omega_k}} \right) \\
&= \sum_k \frac{\omega_k^2 \, e^{-\beta\omega_k/2}}{\left(1 - e^{-\beta\omega_k}\right)^2} \, .
\end{aligned} \tag{40}$$

2.4 Bose-Einstein Condensation

In this subsection, we will denote some of the basics of *Bose-Einstein condensation*, as it is found in many textbooks and lecture courses. Our derivation will mainly follow the book by *Pethick & Smith: Bose Einstein Condensation in dilute gases* [13], and we will re-refer to it also in subsequent sections when dealing with properties which are now of "textbook nature".

The *Bose distribution function* for non-interacting Bosons in the i-th single-particle state in thermal equilibrium is given by

$$f_B(i) = \frac{1}{e^{\beta(\epsilon_i - \mu)} - 1} \, , \tag{41}$$

where ϵ_k denotes the energy of the single-particle state, μ is the *chemical potential* and $\beta = 1/k_B T$ is the inverse temperature with Boltzmann constant k_B. The use of the chemical potential μ can be understood as the BEC being in contact with a reservoir or "heat-bath" of such a potential. Furthermore, it is convenient to work in the *grand canonical* formalism, using the grand canonical partition function

$$Z = \sum_{N=0}^{\infty} e^{\beta\mu N} \, Z_N \, , \tag{42}$$

where

$$Z_N = \sum_{k=0}^{\infty} e^{-\beta\epsilon_k} \, . \tag{43}$$

Here, the $\sum_k$ encompasses a complete set of eigenstates of the Hamiltonian of the system with energies ϵ_k. Without any interactions being considered, the Hamiltonian can be written as a sum of single-particle Hamiltonians

$$\mathcal{H} = \sum_i H_i \, . \tag{44}$$

From there, we can write the grand canonical partition function as

$$
\begin{aligned}
Z &= \sum_{n_0} \left(e^{\beta(\mu-\epsilon_0)}\right)^{n_0} \times \sum_{n_1} \left(e^{\beta(\mu-\epsilon_1)}\right)^{n_1} \times \dots \\
&= \prod_i \sum_{n_i} \left(e^{\beta(\mu-\epsilon_i)}\right)^{n_i} .
\end{aligned} \tag{45}
$$

Here, $\{n_i\}$ denotes the set of occupation numbers, which for Bosons means, that $\sum_n$ extends over $n = 0, 1, 2, \dots$. If one intends to calculate the total number of particles, using $N = \beta\, \partial Z/\partial\mu$, one retrieves the Bose distribution function from equation (41)

$$
N = \sum_i n_i = \sum_i \frac{1}{e^{\beta(\epsilon_i-\mu)} - 1} , \tag{46}
$$

where n_i is the average occupation number of the i-th single particle state. Consequently, the lowest energy state has the occupation number

$$
n_0 = \frac{1}{e^{\beta(\epsilon_0-\mu)} - 1} =: N_0 . \tag{47}
$$

As $\mu \longrightarrow \epsilon_0$, this becomes larger and larger and we get the onset of Bose-Einstein condensation. The number of particles which are not condensed, the so called thermalized particles, can then be found with

$$
N_T(T,\mu) = \sum_{i\neq 0} \langle n_i \rangle(T,\mu) = \sum_{i\neq 0} \frac{1}{e^{\beta(\epsilon_i-\mu)} - 1} . \tag{48}
$$

Finally, in the condensed phase, it is known that the occupation of the lowest energy state becomes macroscopic, for temperatures $T < T_c$, which is a characteristic property of Bose-Einstein-condensation. This allows us to write

$$
N_0 \gg N - N_0 . \tag{49}
$$

Moreover, the transition temperature T_c is defined as the highest temperature, where macroscopic occupation of the groundstate occurs.

2.4.1 The uniform ideal Bose gas

Let us now consider the ideal Bose gas in a box of length L, such that the volume is $V = L^3$. Ignoring both, a possible trap and interactions for now, the single particle Hamiltonians become

$$
H = \frac{\mathbf{k}^2}{2m} , \tag{50}
$$

and assuming cyclic boundary conditions to achieve uniformity, we can write the solutions of these Hamiltonians in terms of plane waves

$$
\psi_{\mathbf{k}} = \frac{1}{\sqrt{V}} e^{i\mathbf{kx}} . \tag{51}
$$

In this case, the energy of a state is $\epsilon_k = |\mathbf{k}^2|/2m$. The box also defines the momentum in terms of $n = n_x, n_y, n_z$, $n \in \mathbb{Z}$, and hence the momentum as $\mathbf{k} = 2\pi\hbar n/L$. With $i \longrightarrow \mathbf{k}$, we can now write Z, N_0, N_T as

$$
\begin{aligned}
Z &= \prod_k \sum_{n_k} \left(e^{\beta(\mu-\epsilon_k)}\right)^{n_k} \\
N_0 &= \frac{1}{e^{\beta(\epsilon_0-\mu)} - 1} = \frac{1}{e^{-\beta\mu} - 1} \\
N_T &= \sum_{k\neq 0} \frac{1}{e^{\beta(\epsilon_k-\mu)} - 1} .
\end{aligned} \tag{52}
$$

Note that here we have used the fact that when the number of particles, N, is sufficiently large, we may neglect the zero-point energy and thus equate the lowest energy ϵ_0 to zero. This also implies that the chemical potential $\mu \leq 0$. Assuming that the thermal energy $k_B T$ is much larger than the energy-spacing between the single-particle boson levels, i.e. $k_B T \gg \hbar/2mV^{2/3}$, one can approximate these as continuous, such that $\sum_k \to \int d\mathbf{k}$. Introducing the *fugacity* as $z := e^{\beta\mu}$, we can write

$$N_T = \frac{V}{(2\pi)^4\hbar^3} \int d\mathbf{k}\, \frac{1}{e^{\beta(\epsilon_k-\mu)} - 1} = \frac{V}{\lambda_T^3}\, g_{3/2}\left(e^{\beta\mu}\right), \tag{53}$$

where

$$\begin{aligned} \lambda_T &= \sqrt{\frac{2\pi\hbar^2}{mk_BT}} \\ g_{3/2}(z) &= \frac{2}{\sqrt{\pi}} \int_0^\infty dx\sqrt{x}\frac{1}{z^{-1}e^x - 1}\,. \end{aligned} \tag{54}$$

For $T > T_c$, due to the definition of T_c, we can set $N_T = N$, which gives $g_{3/2}(z) = \lambda_T^3\, N/V$. If $T < T_c$, we set $\mu = 0$ and get

$$N_T = N\left(\frac{T}{T_c}\right)^{3/2}, \tag{55}$$

where

$$T_c := \frac{2\pi\hbar^2}{k_B m}\left(\frac{n}{g_{3/2}(1)}\right), \tag{56}$$

with $g_{3/2}(1) \simeq 2.612$. The number of particles in the condensate is then $N_0 = N - N_T$, or

$$N_0 = N\left(1 - \left(\frac{T}{T_c}\right)^{3/2}\right). \tag{57}$$

The energy of this uniform ideal system can be computed using $E = \sum_k \epsilon_k \langle n_k \rangle$, and hence

$$E = \frac{V}{(2\pi)^4\hbar^3} \int d\mathbf{k}\, \frac{\epsilon_k}{e^{\beta(\epsilon_k-\mu)} - 1} = \frac{3}{2}\frac{V}{\lambda_T^3}\frac{1}{\beta}\, g_{5/2}(X)\,, \tag{58}$$

where

$$X = \begin{cases} z, & \text{if } T > T_c \\ 1, & \text{if } T < T_c \end{cases} \tag{59}$$

and

$$g_{5/2}(z) = \frac{4}{3\sqrt{\pi}} \int_0^\infty dx\, x^{3/2}\frac{1}{z^{-1}e^x - 1}\,. \tag{60}$$

Note that $g_{5/2}(1) = \zeta(5/2) = 1.34149$. Using again that $c_V = \partial E/\partial T$, we get for $T < T_c$

$$c_V = \frac{15\, m^{3/2}\, g_{5/2}(1)}{8\sqrt{2}\,\pi^{3/2}\,\hbar^3}\, V k_B^{5/2}\, T^{3/2}\,. \tag{61}$$

2.4.2 Ideal Bose-gas in a (harmonic oscillator) trap potential

Assuming an isotropic trap with frequencies ω_x, ω_y, ω_z in three dimensions, the single-particle Hamiltonian can be written as

$$H = \frac{\mathbf{k}^2}{2m} + V_{ext}(x)\,, \tag{62}$$

where

$$V_{ext} = \frac{1}{2}m\left(\omega_x^2x^2 + \omega_y^2y^2 + \omega_z^2z^2\right). \tag{63}$$

The eigenvalues of this Hamiltonian are

$$\epsilon_{n_x,n_y,n_z} = \left(n_x + \frac{1}{2}\right)\hbar\omega_x + \left(n_y + \frac{1}{2}\right)\hbar\omega_y + \left(n_z + \frac{1}{2}\right)\hbar\omega_z\,, \tag{64}$$

with $n_x, n_y, n_z \in \mathbb{Z}$. Assuming again that the kinetic energy is much greater than than energy spacing of the trap, we can replace sums by integrals, and so the number of thermal particles is:

$$N_T = \int dn_x dn_y dn_z \, \frac{1}{e^{\beta(\epsilon_{n_x,n_y,n_z}-\mu)} - 1} \tag{65}$$

and hence for $T < T_c$ (the BEC), taking $\mu = \epsilon_0$, using the same steps as above, we get

$$N_0 = N\left(1 - \left(\frac{T}{T_c}\right)^3\right). \tag{66}$$

Not only is this value different to the uniform free gas, but also now

$$T_c = \frac{\hbar\omega_{HO}}{k_B}\left(\frac{N}{\zeta(3)}\right)^{1/3}, \tag{67}$$

where $\omega_{HO} := \left(\omega_x \omega_y \omega_z\right)^{1/3}$. The difference is due to the different behaviour of the density of single-particle states. In terms of the density of states, the number of thermalized particles is given by

$$N_T = \int_0^\infty d\epsilon \, \frac{g(\epsilon)}{e^{\beta\epsilon} - 1}. \tag{68}$$

Now, for a free, three-dimensional uniform gas, where $V_{ext} = 0$, we have

$$g(\epsilon) = \frac{V\, m^{3/2}}{\sqrt{2}\pi^2\, \hbar^3}\sqrt{\epsilon}, \tag{69}$$

while for a gas in a harmonic oscillator trap, we get a density of states equal to

$$g(\epsilon) = \frac{1}{2(\hbar\omega_{HO})^3}\epsilon^2, \tag{70}$$

where we have used the semi-classical approximation for very large N:

$$g(\epsilon) = \int \frac{d\mathbf{x}d\mathbf{p}}{(2\pi\hbar)^3}\delta\left(\epsilon_c - \epsilon_c(\mathbf{x}, \mathbf{p})\right), \tag{71}$$

with $\epsilon_c(\mathbf{x}, \mathbf{p})) = \mathbf{p}^2/2m + V_{ext}(\mathbf{x})$ being the classical energy corresponding to the single-particle Hamiltonian.

The energy of the system is calculated in precisely the same way as before:

$$E = \int_0^\infty d\epsilon \frac{\epsilon\, g(\epsilon)}{z^{-1}\, e^{\beta\epsilon} - 1} = 3\left(\frac{1}{\beta\hbar\omega_{HO}}\right)^3 \frac{1}{\beta}\, g_4(X). \tag{72}$$

Here

$$X = \begin{cases} z, & \text{if } T > T_c \\ 1, & \text{if } T < T_c \end{cases} \tag{73}$$

and consequently

$$c_V = \frac{12\, k_B^4 T^3}{(\hbar\omega_{HO})^3}\, g_4(1), \tag{74}$$

for $T < T_c$. Note that $g_4(1) = \zeta(4) = 1.08232$. This looks quite different to the uniform box case, but a way of relating them is to note that the size of the condensate is fixed by the harmonic oscillator length $a_{HO} = \sqrt{\hbar/m\omega_{HO}}$, which plays the role of L in the uniform box case. For interacting particles it is not so straightforward since the particles are no longer independent. Instead, researchers have tended to use the Bogoliubov approximation, or extensions thereof, such as the Hartree-Fock- and Popov-approximations, where one can describe the system in terms of an ideal gas of quasi-particles, with varying degrees of accuracy. In the following sections, we shall discuss the textbook version of Bogoliubov theory for "hard-ball" scattering, and then, in the main part of this thesis, derive the full *Hartree-Fock-Bogoliubov*-theory (HFB) for a ranged (gravitational) interaction. We shall work out the difference between gravity being a quantum-operator, like scattering, or a fundamentally classical interaction, as in the *Schrödinger-Newton*-formalism.

3 Bogoliubov Theory

In this section, we shall go through what became known as *Bogoliubov Theory*. N.N. Bogoliubov (or Bogolyubov in some sources) was the first to show [14], in 1947, that when we are concerned with the ground state for the weakly interacting dilute boson gas an interaction energy operator of the form

$$H_{int} = \frac{U}{2V} \sum_{p_1+p_2=p_3+p_4} a^{\dagger}_{p_1} a^{\dagger}_{p_2} a_{p_3} a_{p_4} \tag{75}$$

can be significantly simplified. He introduced a linear symplectic transformation on phase space which makes it possible to carry out diagonalization and obtain the energy spectrum. This idea has now become a standard textbook treatment, and accordingly, we also follow what is found in sources like [13], [15] (Books 1-4) or [16], to name just a few.

The simplification is based on the idea that in the ground state, the "condensed phase", the particles are in the lowest energy level, and this energy is taken to be zero. Considering a *dilute gas*, where $na_s^3 \ll 1$, a_s denoting the *s-wave scattering length*, the interactions are so "weak", and predominantly two-body only, that its ground state will only differ slightly from the state of an ideal gas. This implies, that the occupation number of the ground-state will still exceed the number of particles in excited states considerably, such that $N_0 \gg N - N_0$. This idea, once properly understood, will also be the starting point of our later considerations, which we shall then extend to the *Hartree-Fock-Bogoliubov* method.

3.1 Two Body Scattering

Let's denote the scattering length in the Born approximation, without proof, by

$$a_s = \frac{\mu}{2\pi\hbar^2} \int d\mathbf{r}\, U(\mathbf{r})\,, \tag{76}$$

which shows that the true low-energy scattering behaviour may be obtained by using an effective two-body interaction U_{eff}, having the property

$$\int U_{eff} = \frac{2\pi\hbar^2 a_s}{\mu} \equiv U_0 \tag{77}$$

where $\mu = \frac{m_1 m_2}{m_1+m_2}$ denotes the reduced mass (center of mass frame), which for particles of the same mass of course is $\mu = \frac{m}{2}$ and hence

$$U_0 = \frac{4\pi\hbar^2 a_s}{m}\,. \tag{78}$$

It is a useful fact for calculating low-energy properties of the system, that the true interatomic potential, which in general has a complicated dependence on interparticle separation, may be replaced by an effective interaction proportional to the scattering length. In position space, the effective interaction between two particles situated at points $\mathbf{r}$ and $\mathbf{r}'$ is hence often written as $U_{eff}(\mathbf{r}, \mathbf{r}') = U_0\, \delta(\mathbf{r} - \mathbf{r}')$, see also [13] (Sec. 5.2.1, pg. 133 ff.).

3.2 Bogoliubov's simplification of the Interaction

To begin with, let's consider the following Hamiltonian for low-energy, low-momenta scattering processes as described above,

$$\mathcal{H}_{SC} = \frac{U_0}{2V} \sum_{p,-p} a^{\dagger}_p a^{\dagger}_{-p} a_p a_{-p}\,, \tag{79}$$

where the bosonic creation- and annihilation operators are again defined as in equations (36) and (37). In the center of mass frame of the scattering processes, we can use the subscripts p and $-p$. Furthermore, it is now time to make use of the so-called *Bogoliubov approximation*, which means that we consider the fact the the number of particles in the ground state N_0 is much larger than the number of particles in states of elementary excitations, $N_0 \gg N - N_0$, where N denotes the total particle number, which is conserved. Since the matrix elements of the Bose operators a_i are equal to $\sqrt{N_i}$ we can clearly neglect the interactions of particles above the zero energy level with each other, and simply take into account only the interactions of condensate particles with one another and with the excited particles. This means that we need only retain the following terms in the sum of equation (79)

$$\mathcal{H}_{SC} = \frac{U_0}{2V}\left[a_0^\dagger a_0^\dagger a_0 a_0 + \sum_{p\neq 0}\left(2a_p^\dagger a_0^\dagger a_p a_0 + 2a_0^\dagger a_{-p}^\dagger a_0 a_{-p} + a_p^\dagger a_{-p}^\dagger a_0 a_0 + a_0^\dagger a_0^\dagger a_p a_{-p}\right)\right]. \tag{80}$$

In light of the fact the N_0 is a very large number, it is justified to regard the operators $a_0^\dagger$ an a_0 as simple numbers and replace them by $\sqrt{N_0}$. One finds

$$\mathcal{H}_{SC} = \frac{U_0}{2V}\left[N_0^2 + 2N_0\sum_{p\neq 0}\left(a_{-p}^\dagger a_{-p} + a_p^\dagger a_p\right) + N_0\sum_{p\neq 0}\left(a_p^\dagger a_{-p}^\dagger + a_p a_{-p}\right)\right]. \tag{81}$$

Obviously, one also has to include the kinetic term, denoted here by $\epsilon_p^0 := p^2/2m$, and hence the full scattering Hamiltonian reads

$$\begin{aligned}\mathcal{H}_{SC} &= \frac{N_0^2 U_0}{2V} + \sum_{p\neq 0}\left(\epsilon_p^0 a_p^\dagger a_p + \frac{N_0 U_0}{V}\left(a_{-p}^\dagger a_{-p} + a_p^\dagger a_p\right)\right) + \frac{N_0 U_0}{2V}\sum_{p\neq 0}\left(a_p^\dagger a_{-p}^\dagger + a_p a_{-p}\right)\\ &= \frac{N_0^2 U_0}{2V} + \sum_{p\neq 0}\left(\epsilon_p^0 + 2\frac{N_0 U_0}{V}\right)a_p^\dagger a_p + \frac{N_0 U_0}{2V}\sum_{p\neq 0}\left(a_p^\dagger a_{-p}^\dagger + a_p a_{-p}\right).\end{aligned} \tag{82}$$

To understand the step from the first to the second line, consider an effective interaction $U(\mathbf{r})$ with non-zero range and its Fourier transform, such that

$$U(\mathbf{p}) = \int d\mathbf{r}\; U(\mathbf{r})\, exp\big(-i\mathbf{pr}/\hbar\big), \tag{83}$$

which implies that for a contact interaction the Fourier transform $U(\mathbf{p})$ is independent of $\mathbf{p}$, and hence $\left(a_{-p}^\dagger a_{-p} + a_p^\dagger a_p\right) = 2a_p^\dagger a_p$. Furthermore, considering that $N = N_0 + \frac{1}{2}\sum_{p\neq 0}(a_{-p}^\dagger a_{-p} + a_p^\dagger a_p)$, we can replace the N_0 in the first term by N, to give $\frac{N^2 U_0}{2V} + \ldots$, which is permissible since the fluctuation in the particle number is small. Moreover, let's denote the condensate density by $n_0 = N_0/V$ and the total density by $n = N/V$. Since we consider states differing little from the state with all particles in the condensed phase it makes no difference whether the condensate density or the total density appears in the terms in the sum. Collecting our arguments, we can write down the fully symmetrical version of the scattering Hamiltonian as

$$\mathcal{H}_{SC} = \frac{n^2 U_0}{2} + \sum_{p\neq 0}\left(\frac{1}{2}\epsilon_p^0 + n_0 U_0\right)\left(a_{-p}^\dagger a_{-p} + a_p^\dagger a_p\right) + \frac{n_0 U_0}{2}\sum_{p\neq 0}\left(a_p^\dagger a_{-p}^\dagger + a_p a_{-p}\right) \tag{84}$$

Following *Pethick & Smith* [13], we denote $a^\dagger$ and a as creating/annihilating bosons with momentum $\mathbf{p}$ and $b^\dagger$, b creating/annihilating bosons with momentum $-\mathbf{p}$, we can rewrite it as

$$h_{sc} = \epsilon_0\big(a^\dagger a + b^\dagger b\big) + \epsilon_1\big(a^\dagger b^\dagger + b\,a\big), \tag{85}$$

where ϵ_0 and ϵ_1 are considered numbers $\in \mathbb{C}$. The operators a and b are required to obey the canonical commutation relations

$$[a, a^\dagger] = 1 = [b, b^\dagger] \;\text{ and 0 otherwise.} \tag{86}$$

3.3 Bogoliubov Transformation

It was Bogoliubov's idea to apply a linear transformation by introducing a set of new operators α and β, s.t.

$$\begin{aligned}\alpha = ua + vb^\dagger\;,\;\; \beta = ub + va^\dagger\\ \alpha^\dagger = u^* a^\dagger + v^* b\;,\;\; \beta^\dagger = u^* b^\dagger + v^* a\,,\end{aligned} \tag{87}$$

for $u, v \in \mathbb{C}$ and * denoting the complex conjugate. This new set of operators is also required to obey the canonical commutation relations, such that

$$[\alpha, \alpha^\dagger] = 1 = [\beta, \beta^\dagger] \text{ and 0 otherwise.} \tag{88}$$

To determine u and v, we insert (87) in (88) and use the CCR from (86)

$$\begin{aligned}
[\alpha, \alpha^\dagger] &= [ua + vb^\dagger, u^* a^\dagger + v^* b] \\
&= \Big(\big(ua + vb^\dagger\big)\big(u^* a^\dagger + v^* b\big) - \big(u^* a^\dagger + v^* b\big)\big(ua + vb^\dagger\big) \Big) \\
&= u^2 aa^\dagger + uv^* ab + vu^* b^\dagger a^\dagger + v^2 b^\dagger b - u^2 a^\dagger a - u^* v a^\dagger b^\dagger - v^* uba - v^2 bb^\dagger \\
&= \big(u^2 aa^\dagger - u^2 a^\dagger a\big) + \big(uv^* ab - v^* uba\big) + \big(vu^* b^\dagger a^\dagger - u^* v a^\dagger b^\dagger\big) - \big(v^2 bb^\dagger - v^2 b^\dagger b\big) \\
&= u^2 [a, a^\dagger] + uv^* [a, b] + vu^* [b^\dagger, a^\dagger] - v^2 [b, b^\dagger] \\
&= u^2 - v^2 = 1 \text{ by requirement.}
\end{aligned} \tag{89}$$

Since this form is suggestive of the hyperbolic identity $cosh^2 x - sinh^2 x = 1$ the constants u and v can be re-parametrized as

$$\begin{aligned}
u &= e^{i\theta_1} cosh\, \mathbf{r} \\
v &= e^{i\theta_2} sinh\, \mathbf{r}\,,
\end{aligned} \tag{90}$$

which is interpreted as a *linear symplectic transformation* of the phase space. The *symplectic* property ensures that the phase space volume (or area) is preserved, and is an even stronger condition than *Liouville's theorem*. In other words, the *Bogoliubov Transformation* allows us to diagonalize the Hamiltonian while keeping the phase space area preserved. This will be discussed again in Section (4). The inverse transformation is then given by

$$a = u\alpha + v\beta^\dagger \ , \ b = u\beta + v\alpha^\dagger \ . \tag{91}$$

Inserting this into equation (85), we get

$$\tilde{h}_{sc} = 2v^2 \epsilon_0 - 2uv\epsilon_1 + \big(\epsilon_0(u^2 + v^2) - 2uv\epsilon_1\big)\big(\alpha^\dagger \alpha + \beta^\dagger \beta\big) + \big(\epsilon_1(u^2 + v^2) - 2uv\epsilon_0\big)\big(\alpha\beta + \beta^\dagger \alpha^\dagger\big)\,, \tag{92}$$

where the last term can be made to vanish by choosing u and v so that its coefficient is zero, i.e.

$$\epsilon_1(u^2 + v^2) - 2uv\epsilon_0 = 0\,. \tag{93}$$

Using the parametrization from (90) one then finds

$$tanh\, 2\mathbf{r} = \frac{\epsilon_1}{\epsilon_0} \tag{94}$$

and by defining

$$\epsilon = \sqrt{\epsilon_0^2 - \epsilon_1^2}\,, \tag{95}$$

solving in terms of the ratio ϵ_1/ϵ_0 and inserting into equation (92) one finally gets

$$\tilde{h}_{sc} = \epsilon\big(\alpha^\dagger \alpha + \beta^\dagger \beta\big) + \epsilon - \epsilon_0\,. \tag{96}$$

The ground-state energy is $\epsilon - \epsilon_0$, which is negative, and the excited states correspond to the addition of two independent kinds of bosons with energy ϵ, created by the operators $\alpha^\dagger$ and $\beta^\dagger$. For $\epsilon \in \mathbb{R}$ we must have $|\epsilon_0| > |\epsilon_1|$. If $|\epsilon_0| < |\epsilon_1|$, the excitation energy is imaginary, corresponding to an instability of the system.

We can bring the Hamiltonian in (84) into diagonal form, using the transformation

$$a_p = u_p \alpha_p - v_p \alpha_{-p}^\dagger\,, \quad a_{-p} = u_p \alpha_{-p} - v_p \alpha_p^\dagger\,, \tag{97}$$

where a_p corresponds to a and a_{-p} to b, α_p to α and α_{-p} to β. Inserting it, we get

$$\mathcal{H}_{SC} = \frac{n^2 U_0}{2} + \sum_{p\,(p\neq 0)} \epsilon_p \alpha_p^\dagger \alpha_p - \frac{1}{2} \sum_{p\,(p\neq 0)} \big(\epsilon_p^0 + n_0 U_0 - \epsilon_p\big)\,, \tag{98}$$

where

$$\epsilon_p = \sqrt{\left(\epsilon_p^0 + n_0 U_0\right)^2 - \left(n_0 U_0\right)^2} = \sqrt{\left(\epsilon_p^0\right)^2 + 2\epsilon_p^0 n_0 U_0}\,, \tag{99}$$

which is known as the *Bogoliubov dispersion relation*. Note, that for small p, the energy can be written as $\epsilon_p = sp$, where

$$s^2 = \frac{n_0 U_0}{m}\,. \tag{100}$$

3.4 Ground state energy

To make a consistent calculation of the ground-state energy one must use an effective interaction $U(p_c)$ in which all intermediate states with momenta in excess of some cut-off value p_c are taken into account, and then evaluate the energy omitting all intermediate states in the sum which have momenta in excess of this cut-off. The ground state energy is therefore

$$E_0 = \frac{n^2 U(p_c)}{2} - \frac{1}{2} \sum_{p\ (p<p_c)} \left(\epsilon_p^0 + n_0 U_0 - \epsilon_p\right), \tag{101}$$

where the effective interaction for small p_c and zero energy is given by

$$U(p_c) = U_0 + \frac{U_0^2}{V} \sum_{p\ (p<p_c)} \frac{1}{2\epsilon_p^0}\,, \tag{102}$$

which when inserted into the above expression gives

$$E_0 = \frac{n^2 U_0}{2} - \frac{1}{2} \sum_{p\ (p<p_c)} \left(\epsilon_p^0 + n_0 U_0 - \epsilon_p - \frac{\left(n_0 U_0\right)^2}{2\epsilon_p^0}\right). \tag{103}$$

If one chooses the cut-off momentum to be large compared with ms but small compared with $\hbar/a$ the result does not depend on p_c, and, using the fact that $n_0 \sim n$, one finds

$$\begin{aligned} \frac{E_0}{V} &= \frac{n^2 U_0}{2} + \frac{8}{15\pi^2}\left(\frac{ms}{\hbar}\right)^3 ms^2 \\ &= \frac{n^2 U_0}{2}\left(1 + \frac{128}{15\pi^{1/2}}\left(na^3\right)^{1/2}\right) \end{aligned} \tag{104}$$

The first form of the correction term indicates that the order of magnitude of the energy change is the number of states having wave numbers less than the inverse coherence length, times the typical energy of an excitation with this wave number, as one would expect. This result was first obtained by Lee and Yang [17].

Remark: Utilizing the result from equation (78), one can estimate the number of particles in states of elementary excitations by

$$\frac{n_{ex}}{n} = \frac{8}{3\sqrt{\pi}}\left(na^3\right)^{1/2}. \tag{105}$$

Remark: It is possible to rewrite (98), based on the (previously made) assumption that our BEC is homogeneous (i.e. stationary) and hence translationally invariant, and that initial and final states have the same total momentum, allowing for a plane wave Ansatz, to read

$$\mathcal{H}_{SC} = \frac{U_0}{2} n_0^2 + \sum_{p\neq 0}\left(\epsilon_p^0 + 2n_0 U_0\right) n_{ex} + \frac{U_0}{V} n_{ex}^2 \tag{106}$$

which is already in similar form to what we are about to derive. The connection to how and which terms are taken into consideration will be made clearer in the following section.

As discussed even more extensively in the textbooks referenced at the beginning of this section, applying the methods shown above leads to an important conclusion: it shows that the addition of an elementary excitation of momentum **p** is the superposition of the addition of a particle of momentum **p** together with the removal of a particle from the condensate, and the removal of a particle with momentum $-\mathbf{p}$ accompanied by the addition of a particle to the condensate. The physical character of long-wavelength excitations may be seen by using the fact that $u \sim v$ in this limit, which confirms the phononic nature of long-wavelength excitations.

The intuitive picture is, that it is not the individual particles which are propagating, but the excitation itself. This corresponds precisely to the physical characteristics of a wave propagating through some medium, for example a sound wave. Within this analogy, the excitation spectrum is also called *Bogoliubov dispersion relation*, and c_s can be derived as the "speed of sound" at which the excitations are propagating through the BEC. In Bogoliubov theory

$$c_s = \sqrt{\frac{n_0 \, U_0}{m}} \, . \tag{107}$$

In the following section (4), we will see how taking into account finite temperatures and different interactions will change the behaviour of the phononic excitations, influencing the Eigenenergies and consequently the thermodynamic observables.

4 Self-gravitating BECs at finite temperatures

We shall now turn our attention to a Bose-gas at *finite temperatures*, so $0 \neq T < T_c$. While again considering textbook results for *contact interactions*, we shall focus on ranged interactions, and more specifically, *self-gravitational* ones.

The key issue in describing a finite temperature Bose gas is the fact, that when a Bose–Einstein condensate is created in a harmonic or similar kind of trap, it has two clearly identifiable parts: the condensate itself and a cloud of non-condensed thermal atoms which surrounds the condensate, but also penetrates it to a certain extent. While the distinction seems obvious, there is no precise way of specifying a boundary between the two. In order to develop a theoretical description of the BEC at finite temperatures, this distinction is maybe not mandatory. Yet, it has been included in one way or another in the various theoretical models, and not all of them are straightforwardly comparable.

The Hartree–Fock–Bogoliubov (HFB) method, which will primarily be used in this chapter, is an approximate treatment of Bose–Einstein condensation, which builds on the Bose broken symmetry approach to Bogoliubov's method that we described in the previous subsection. This allows one to separate the condensate from the thermal cloud in a way that is intuitively natural.

The approximation method used is based on factorization assumptions for mean values, such as $\langle \Psi^\dagger \Psi \Psi \rangle = \langle \Psi^\dagger \Psi \rangle \langle \Psi \rangle$, motivated by *Wick's theorem*, which states that at equilibrium an average over multiple operators can be approximated by sums of averages of pairwise contracted operators [18]. The result though, is not a systematic perturbation theory, and some contradictions arise, a fact clarified by Griffin [19].

As Anderson [20] summarizes in agreement with many textbooks, there are other methods of computing the behaviour of a BEC at finite temperatures. Variational approaches, Feynman Perturbation theory and Wilson Renormalization Group theory, as well as extensions to the HFB approximations using Popov's ideas and the improvements presented by *Zaremba, Nikuni* and *Griffin* [21] on the basis of the Hartree–Fock–Bogoliubov theory in the Popov approximation, all provide means to study characteristic processes of a BEC approaching T_c.

4.1 Separation into condensate and non-condensate terms

Following above mentioned references, as well as Giorgini, Pitaevskii & Stringari [22], we start by denoting the grand-canonical Hamiltonian written in terms of the creation and annihilation particle field operators $\Psi^\dagger$ and Ψ, for our two cases. Note that, since we are now considering the finite-temperature regime, we can no longer set the chemical potential μ to zero, but have to include it:

$$
\begin{aligned}
K_{CG} = H_{CG} - \mu N &= \int d^3\mathbf{r}\, \Psi^\dagger(\mathbf{r}) \left[-\frac{\hbar^2\nabla^2}{2m} + V_{ext}(\mathbf{r}) - \mu + m\Phi[\Psi](\mathbf{r}) \right] \Psi(\mathbf{r}) + \frac{U_0}{2} \int d^3\mathbf{r}\, \Psi^\dagger(\mathbf{r})\Psi^\dagger(\mathbf{r})\Psi(\mathbf{r})\Psi(\mathbf{r}) \\
K_{QG} = H_{QG} - \mu N &= \int d^3\mathbf{r}\, \Psi^\dagger(\mathbf{r}) \left[-\frac{\hbar^2\nabla^2}{2m} + V_{ext}(\mathbf{r}) - \mu \right] \Psi(\mathbf{r}) + \frac{1}{2} m\ :\Psi^\dagger(\mathbf{r})\Psi(\mathbf{r})\hat{\Phi}(\mathbf{r}): + \frac{U_0}{2} \int d^3\mathbf{r}\, \Psi^\dagger(\mathbf{r})\Psi^\dagger(\mathbf{r})\Psi(\mathbf{r})\Psi(\mathbf{r})
\end{aligned}
\tag{108}
$$

where the $: \cdots :$ stands for a normal ordering prescription, (CG) stands for the *fundamentally classical*, (QG) for the *quantized* gravity picture, and

$$
\hat{\Phi}(\mathbf{r}) = -Gm \int d^3\mathbf{r}'\, \frac{\Psi^\dagger(\mathbf{r}')\Psi(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \tag{109}
$$

is understood to be the quantized version of Poisson's equation [23]. In contrast, depending on the chosen CG theory, Φ is a certain quantum average of this expression. To meaningfully denote classical gravity within the probabilistic framework of quantum mechanics, we will employ results from Schroedinger-Newton theory [9]. It is the Newtonian limit of the semi-classical CG theory, $\Phi = \langle \Phi \rangle$. Setting $\mathcal{L} = -\frac{\hbar^2\nabla^2}{2m} + V_{ext}(\mathbf{r})$, one can write

$$K_{CG} = \int d^3\mathbf{r}\, \Psi^\dagger(\mathbf{r}) \left[\mathcal{L} - \mu - Gm^2 \int d^3\mathbf{r}' \frac{\Psi^\dagger(\mathbf{r}) \left\langle \Psi^\dagger(\mathbf{r}')\Psi(\mathbf{r}') \right\rangle \Psi(\mathbf{r})}{|\mathbf{r} - \mathbf{r}'|} \right] \Psi(\mathbf{r}) + \frac{U_0}{2} \int d^3\mathbf{r}\, \Psi^\dagger(\mathbf{r})\Psi^\dagger(\mathbf{r})\Psi(\mathbf{r})\Psi(\mathbf{r}) \,,$$
$$K_{QG} = \int d^3\mathbf{r}\, \Psi^\dagger(\mathbf{r}) \left[\mathcal{L} - \mu \right] \Psi(\mathbf{r}) - \frac{1}{2} Gm^2 \int d^3\mathbf{r}' \frac{\Psi^\dagger(\mathbf{r})\Psi^\dagger(\mathbf{r}')\Psi(\mathbf{r}')\Psi(\mathbf{r})}{|\mathbf{r} - \mathbf{r}'|} + \frac{U_0}{2} \int d^3\mathbf{r}\, \Psi^\dagger(\mathbf{r})\Psi^\dagger(\mathbf{r})\Psi(\mathbf{r})\Psi(\mathbf{r}) \,, \tag{110}$$

where we have used the same approximation for the scattering term (last on the RHS), namely that we are assuming a contact interaction with s-wave scattering length a_s at $\mathbf{r} = \mathbf{r}'$, as denoted in the previous section.

4.1.1 Separation of the fields

Consider now again the ground state of the BEC, characterized by a wavefunction ϕ_0, composed of N_0 atoms such that the quantum state can be written as $|\phi_0, N_0\rangle$. The field operator $\Psi^{(\dagger)}(\mathbf{r})$ can then be rewritten in the form of creation and annihilation ladder operators $a_i^{(\dagger)}$:

$$\Psi(\mathbf{r}) = \phi_0(\mathbf{r})\; a_0 + \sum_{k\neq 0}^{\infty} \phi_k(\mathbf{r})\; a_k = \tilde{\phi}(\mathbf{r}) + \tilde{\psi}(\mathbf{r}) \,, \tag{111}$$

of the fields, where the $\tilde{\psi}(\mathbf{r})$ term represents thermalized particles. This is suggestive of *weak-field perturbation theory* or a "linearization" of the problem, and indeed that is precisely what it is. Moreover, because $N_0 \gg N - N_0$, the notion of a "background field" plus a small perturbative component becomes a very intuitive approach.

Note that the ϕ_k are orthogonal to ϕ_0 and $a_0\,|\phi_0, N_0\rangle = \sqrt{N_0}\,|\phi_0, N_0 - 1\rangle$, so by construction

$$\int d^3\mathbf{r}\; \tilde{\phi}^*(\mathbf{r})\, \tilde{\psi}(\mathbf{r}) = 0 \,. \tag{112}$$

If one now applies the decomposition from (111) to the Hamiltonians, one can rewrite them as the sum of

$$K = K_0 + K_1 + K_1^\dagger + K_2 \,, \tag{113}$$

which are structured as follows:

$$\begin{aligned} K_0 &= \int d^3\mathbf{r} \quad \tilde{\phi}^* \Big[\dots \Big] \tilde{\phi} \\ K_1 &= \int d^3\mathbf{r} \quad \tilde{\psi}^\dagger \Big[\dots \Big] \tilde{\phi} \\ K_1^\dagger &= \int d^3\mathbf{r} \quad \tilde{\phi}^* \Big[\dots \Big] \tilde{\psi} \\ K_2 &= \int d^3\mathbf{r} \quad \tilde{\psi}^\dagger \Big[\dots \Big] \tilde{\psi} \end{aligned} \tag{114}$$

and so the terms linear in $\tilde{\psi}$ are

$$K_1 + K_1^\dagger = \int d^3\mathbf{r}\; \tilde{\psi}^\dagger \Big[\dots \Big] \tilde{\phi} + \int d^3\mathbf{r}\; \tilde{\phi}^* \Big[\dots \Big] \tilde{\psi} \,. \tag{115}$$

Now, if $\tilde{\phi}$ is chosen to be the ground-state solution of

$$\left[-\frac{\hbar^2 \nabla^2}{2m} + V_{ext}(\mathbf{r}) + \dots \right] \tilde{\phi} = \epsilon_0(N_0)\tilde{\phi} \tag{116}$$

then by virtue of equation (112), the second term on the RHS in (115) vanishes identically. To the extent that $N_0 \gg 1$, the same will be true of the first term to a good approximation, and as a result, there are no single-particle excitations from the N_0-subspace.

4.1.2 Decomposition of the Interactions

Let's now look at both the local, as well as the non-local, interactions. Decomposing the interactions which are quartic in the fields, i.e. the contact- and the quantized self-gravitational-interaction, one gets, first for the contact interaction:

$$\begin{aligned}\Psi^\dagger(\mathbf{r})\Psi^\dagger(\mathbf{r})\Psi(\mathbf{r})\Psi(\mathbf{r}) = |\tilde{\phi}|^4 + 2|\tilde{\phi}|^2\tilde{\phi}^*\tilde{\psi} + 2|\tilde{\phi}|^2\tilde{\phi}\tilde{\psi}^\dagger \\ + 4|\tilde{\phi}|^2\tilde{\psi}^\dagger\tilde{\psi} + \tilde{\phi}^{*2}\tilde{\psi}\tilde{\psi} + \tilde{\phi}^2\tilde{\psi}^\dagger\tilde{\psi}^\dagger \\ + 2\tilde{\phi}^*\tilde{\psi}^\dagger\tilde{\psi}\psi + 2\tilde{\phi}\tilde{\psi}^\dagger\tilde{\psi}^\dagger\tilde{\psi} + \tilde{\psi}^\dagger\tilde{\psi}^\dagger\tilde{\psi}\tilde{\psi}\,,\end{aligned} \tag{117}$$

where the fact that $\mathbf{r} = \mathbf{r}'$ has been used to summarize a few terms. For the quantized self-gravitational-interaction, where non-locality prevents us from such summarizations, we find:

$$\begin{aligned}\Psi^\dagger(\mathbf{r})\Psi^\dagger(\mathbf{r}')\Psi(\mathbf{r})\Psi(\mathbf{r}') = \tilde{\phi}^*(\mathbf{r})\tilde{\phi}^*(\mathbf{r}')\tilde{\phi}(\mathbf{r}')\tilde{\phi}(\mathbf{r}) + \tilde{\phi}^*(\mathbf{r})\tilde{\phi}^*(\mathbf{r}')\tilde{\phi}(\mathbf{r}')\tilde{\psi}(\mathbf{r}) \\ + \tilde{\phi}^*(\mathbf{r})\tilde{\phi}^*(\mathbf{r}')\tilde{\psi}(\mathbf{r}')\tilde{\phi}(\mathbf{r}) + \tilde{\phi}^*(\mathbf{r})\tilde{\phi}^*(\mathbf{r}')\tilde{\psi}(\mathbf{r}')\tilde{\psi}(\mathbf{r}) \\ + \tilde{\phi}^*(\mathbf{r})\tilde{\psi}^\dagger(\mathbf{r}')\tilde{\phi}(\mathbf{r}')\tilde{\phi}(\mathbf{r}) + \tilde{\phi}^*(\mathbf{r})\tilde{\psi}^\dagger(\mathbf{r}')\tilde{\phi}(\mathbf{r}')\tilde{\psi}(\mathbf{r}) \\ + \tilde{\phi}^*(\mathbf{r})\tilde{\psi}^\dagger(\mathbf{r}')\tilde{\psi}(\mathbf{r}')\tilde{\phi}(\mathbf{r}) + \tilde{\phi}^*(\mathbf{r})\tilde{\psi}^\dagger(\mathbf{r}')\tilde{\psi}(\mathbf{r}')\tilde{\psi}(\mathbf{r}) \\ + \tilde{\psi}^\dagger(\mathbf{r})\tilde{\phi}^*(\mathbf{r}')\tilde{\phi}(\mathbf{r}')\tilde{\phi}(\mathbf{r}) + \tilde{\psi}^\dagger(\mathbf{r})\tilde{\phi}^*(\mathbf{r}')\tilde{\phi}(\mathbf{r}')\tilde{\psi}(\mathbf{r}) \\ + \tilde{\psi}^\dagger(\mathbf{r})\tilde{\phi}^*(\mathbf{r}')\tilde{\psi}(\mathbf{r}')\tilde{\phi}(\mathbf{r}) + \tilde{\psi}^\dagger(\mathbf{r})\tilde{\phi}^*(\mathbf{r}')\tilde{\psi}(\mathbf{r}')\tilde{\psi}(\mathbf{r}) \\ + \tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}^\dagger(\mathbf{r}')\tilde{\phi}(\mathbf{r}')\tilde{\phi}(\mathbf{r}) + \tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}^\dagger(\mathbf{r}')\tilde{\phi}(\mathbf{r}')\tilde{\psi}(\mathbf{r}) \\ + \tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}^\dagger(\mathbf{r}')\tilde{\psi}(\mathbf{r}')\tilde{\phi}(\mathbf{r}) + \tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}^\dagger(\mathbf{r}')\tilde{\psi}(\mathbf{r}')\tilde{\psi}(\mathbf{r})\,.\end{aligned} \tag{118}$$

Based on the arguments presented in (112) to (116), we are left with terms containing either $\tilde{\phi}^*(\mathbf{r})\tilde{\phi}(\mathbf{r})$ or $\tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}(\mathbf{r})$. Yet, the Hamiltonian is not quadratic, and we must find an approximation to achieve this - the so called *self-consistent quadratic approximation*-method. As Griffin notes in [19], the only approximation that has to be made is in how we treat the fields in terms of expectation values. Let's first take a proper look at the case of a zero-range interaction.

4.2 Self-consistent quadratic approximations (Wick's Theorem)

The decomposed Hamiltonians are still not quadratic, but contain terms of quartic order. To come to grips with them, it has become commonplace in the literature (see references below) to approximate the terms from (117) and (118) using a so called *self-consistent quadratic approximation* method. This approach is motivated by *Wick's Theorem* and based on the condition that all states are Gaussian initially.

4.2.1 Self-consistent quadratic approximation for zero-range interactions

The HFB method is hinged on a description of the field operator as the sum of a c-number condensate wave-function $\tilde{\phi}(\mathbf{r})$, which becomes the order-parameter of the condensed phase, and an operator $\tilde{\psi}(\mathbf{r})$. Following [24], we first define the following quantities:

$$\begin{aligned}\langle\Psi^\dagger(\mathbf{r})\Psi(\mathbf{r})\rangle &= \mathbf{n}(\mathbf{r}) = n_0(\mathbf{r}) + n_T(\mathbf{r}) \\ \langle\Psi(\mathbf{r})\Psi(\mathbf{r})\rangle &= \mathbf{m}(\mathbf{r})\,,\end{aligned} \tag{119}$$

where $\mathbf{n}(\mathbf{r})$ denotes the total particle number-density, and $\mathbf{m}(\mathbf{r})$ is often called *anomalous contribution* - in matrix jargon these are *off-diagonal* terms of the order $N-2 \simeq N$ for very large N. From this we can write down:

$$\begin{aligned}\langle\tilde{\phi}^*(\mathbf{r})\tilde{\phi}(\mathbf{r})\rangle &= |\tilde{\phi}(\mathbf{r})|^2 = n_0(\mathbf{r}) \\ \langle\tilde{\phi}^*(\mathbf{r})\tilde{\phi}^*(\mathbf{r})\rangle &= (\tilde{\phi}^*)^2(\mathbf{r}) \\ \langle\tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}(\mathbf{r})\rangle &= \mathbf{n}(\mathbf{r}) - |\tilde{\phi}(\mathbf{r})|^2 = n_T(\mathbf{r}) \\ \langle\tilde{\psi}(\mathbf{r})\tilde{\psi}(\mathbf{r})\rangle &= \mathbf{m}(\mathbf{r}) - \tilde{\phi}^2(\mathbf{r}) = m_T(\mathbf{r})\,.\end{aligned} \tag{120}$$

Now, we shall look again at the decomposition of the interaction in equation (117) and simplify terms cubic and quartic in $\tilde{\psi}$. For the contact interaction, where $\mathbf{r} = \mathbf{r}'$, we then have:

$$\begin{aligned}\tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}(\mathbf{r})\tilde{\psi}(\mathbf{r}) &= 2\,\langle\tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}(\mathbf{r})\rangle\,\tilde{\psi}(\mathbf{r}) + \langle\tilde{\psi}(\mathbf{r})\tilde{\psi}(\mathbf{r})\rangle\,\tilde{\psi}^\dagger(\mathbf{r})\\ &= 2\,n_T(\mathbf{r})\,\tilde{\psi}(\mathbf{r}) + m_T(\mathbf{r})\,\tilde{\psi}^\dagger(\mathbf{r})\\ \tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}(\mathbf{r}) &= 2\,\langle\tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}(\mathbf{r})\rangle\,\tilde{\psi}(\mathbf{r}) + \langle\tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}^\dagger(\mathbf{r})\rangle\,\tilde{\psi}(\mathbf{r})\\ &= 2\,n_T(\mathbf{r})\,\tilde{\psi}(\mathbf{r}) + m_T^*(\mathbf{r})\,\tilde{\psi}(\mathbf{r})\end{aligned} \tag{121}$$

and for the quartic term:

$$\begin{aligned}\tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}(\mathbf{r})\tilde{\psi}(\mathbf{r}) = {} & 4\,\langle\tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}(\mathbf{r})\rangle\,\tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}(\mathbf{r})\\ &+ \langle\tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}^\dagger(\mathbf{r})\rangle\,\tilde{\psi}(\mathbf{r})\tilde{\psi}(\mathbf{r})\\ &+ \langle\tilde{\psi}(\mathbf{r})\tilde{\psi}(\mathbf{r})\rangle\,\tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}^\dagger(\mathbf{r})\\ &- 2\,\langle\tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}(\mathbf{r})\rangle\,\langle\tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}(\mathbf{r})\rangle\\ &- \langle\tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}^\dagger(\mathbf{r})\rangle\,\langle\tilde{\psi}(\mathbf{r})\tilde{\psi}(\mathbf{r})\rangle\\ = {} & 4\,n_T(\mathbf{r})\,\tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}(\mathbf{r})\\ &+ m_T^*(\mathbf{r})\,\tilde{\psi}(\mathbf{r})\tilde{\psi}(\mathbf{r})\\ &+ m_T(\mathbf{r})\,\tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}^\dagger(\mathbf{r})\\ &- 2n_T^2 - m_T^2\,.\end{aligned} \tag{122}$$

In equation (122), the last two terms in either decomposition pop up because we can approximately write:

$$\langle\tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}(\mathbf{r})\tilde{\psi}(\mathbf{r})\rangle \simeq 2\,\langle\tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}(\mathbf{r})\rangle\,\langle\tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}(\mathbf{r})\rangle + \langle\tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}^\dagger(\mathbf{r})\rangle\,\langle\tilde{\psi}(\mathbf{r})\tilde{\psi}(\mathbf{r})\rangle\,. \tag{123}$$

As mentioned before, this entire approximation is motivated by Wick's theorem, which states that at equilibrium an average over multiple operators can be approximated by sums of averages of pairwise contracted operators as above, given that states are initially Gaussian. The approximation maintains correlations of non-condensate operators only to quadratic order. Along the same reasoning they do not appear in equation (121), since by construction $\langle\tilde{\psi}(\mathbf{r})\rangle = 0$, rendering the decomposition $\langle\tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}(\mathbf{r})\tilde{\psi}(\mathbf{r})\rangle = 0$.

While one could make things simpler, applying the *Popov-approximation*, dropping all terms containing anomalous contributions of the form $\langle\tilde{\psi}(\mathbf{r})\tilde{\psi}(\mathbf{r})\rangle$, or as denoted in equation (120) above, $m_T(\mathbf{r})$, and their respective complex conjugates, we shall not do so at this point. To the contrary, we will show that in the thermodynamic limit, the *anomalous contributions* become an important factor.

The entire topic around the justification of the method is discussed in great detail in [19] for zero-ranged interactions, and yields - when dropping the m_T-terms (Popov) - a gapless energy-spectrum from $T \ll T_c$ up until the region close to T_c. Proukakis [18] mentions, though, that in the case of Bose–Einstein condensation of neutral bosonic atoms, where the condensate mean field $\tilde{\phi}(\mathbf{r})$ is the dominant parameter, the anomalous average plays a more minor role for effectively repulsive interactions between condensate atoms; such a contribution, which does actually become crucial in the presence of attractive effective interactions and molecular BECs, should however not be dismissed a priori even for repulsive interactions. We agree with the assessment and will discuss the importance of the off-diagonal terms in section (5).

In the next step one has to now insert the decompositions above into equations (114). We find, for the contact interaction, in agreement with [22], that the groundstate contribution K_0 reads:

$$K_0 = \int d^3\mathbf{r}\,\tilde{\phi}^*(\mathbf{r})\left(\mathcal{L}-\mu+\frac{U_0}{2}n_0(\mathbf{r})\right)\tilde{\phi}(\mathbf{r}) - U_0\int d^3\mathbf{r}\,\langle\tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}(\mathbf{r})\rangle\,\langle\tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}(\mathbf{r})\rangle - \frac{U_0}{2}\int d^3\mathbf{r}\,\langle\tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}^\dagger(\mathbf{r})\rangle\,\langle\tilde{\psi}(\mathbf{r})\tilde{\psi}(\mathbf{r})\rangle\,. \tag{124}$$

As already shown (see equation (116) and (112)), the contributions coming from K_1 and $K_1^\dagger$ vanish identically, and all that remains to be included is

$$K_2 = \int d^3\mathbf{r}\,\tilde{\psi}^\dagger(\mathbf{r})\Big(\mathcal{L}-\mu+2U_0\big(n_0(\mathbf{r})+n_T(\mathbf{r})\big)\Big)\tilde{\psi}(\mathbf{r}) + \frac{U_0}{2}\int d^3\mathbf{r}\,\Big(m_T(\mathbf{r})\,\tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}^\dagger(\mathbf{r}) + m_T^*(\mathbf{r})\,\tilde{\psi}(\mathbf{r})\tilde{\psi}(\mathbf{r})\Big) \tag{125}$$

4.2.2 Self-consistent quadratic approximation for quantized ranged interactions

Let us now take a look at a ranged interaction, $\mathbf{r} \neq \mathbf{r}'$. We need to look back at equation (118) for the decomposition of the fully quantized ranged interaction terms. These can be simplified in the same way as for the zero-range interaction, except that now $\mathbf{r} \neq \mathbf{r}'$, leading to slightly different results, as can be seen, for example, for terms of the form

$$\begin{aligned} \tilde{\phi}(\mathbf{r}) \left(\tilde{\psi}^\dagger(\mathbf{r})\, \tilde{\psi}^\dagger(\mathbf{r}')\, \tilde{\psi}(\mathbf{r}')\right) &= \tilde{\phi}(\mathbf{r}) \left(\left\langle \tilde{\psi}^\dagger(\mathbf{r}')\, \tilde{\psi}(\mathbf{r}')\right\rangle \tilde{\psi}^\dagger(\mathbf{r}) + \left\langle \tilde{\psi}^\dagger(\mathbf{r})\, \tilde{\psi}(\mathbf{r}')\right\rangle \tilde{\psi}^\dagger(\mathbf{r}') + \left\langle \tilde{\psi}^\dagger(\mathbf{r})\, \tilde{\psi}^\dagger(\mathbf{r}')\right\rangle \tilde{\psi}(\mathbf{r}')\right) \\ &= \tilde{\phi}(\mathbf{r}) \left(n_T(\mathbf{r}')\, \tilde{\psi}^\dagger(\mathbf{r}) + G_T(\mathbf{r},\mathbf{r}')\, \tilde{\psi}^\dagger(\mathbf{r}') + M_T^*(\mathbf{r},\mathbf{r}')\tilde{\psi}(\mathbf{r}')\right), \end{aligned} \tag{126}$$

where we have defined $G_T(\mathbf{r},\mathbf{r}') := \left\langle \tilde{\psi}^\dagger(\mathbf{r})\, \tilde{\psi}(\mathbf{r}')\right\rangle$ and $M_T(\mathbf{r},\mathbf{r}') := \left\langle \tilde{\psi}^\dagger(\mathbf{r})\, \tilde{\psi}^\dagger(\mathbf{r}')\right\rangle$. Note that there exist four such terms but conveniently they all drop out due to the vanishing contributions from K_1 and $K_1^\dagger$. For the quartic term, the quadratic approximation yields

$$\begin{aligned} \tilde{\psi}^\dagger(\mathbf{r})\, \tilde{\psi}^\dagger(\mathbf{r}')\, \tilde{\psi}(\mathbf{r})\, \tilde{\psi}(\mathbf{r}') &= \left\langle \tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}(\mathbf{r})\right\rangle \tilde{\psi}^\dagger(\mathbf{r}')\, \tilde{\psi}(\mathbf{r}') + \left\langle \tilde{\psi}^\dagger(\mathbf{r}')\tilde{\psi}(\mathbf{r}')\right\rangle \tilde{\psi}^\dagger(\mathbf{r})\, \tilde{\psi}(\mathbf{r}) \\ &+ \left\langle \tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}(\mathbf{r}')\right\rangle \tilde{\psi}^\dagger(\mathbf{r}')\, \tilde{\psi}(\mathbf{r}) + \left\langle \tilde{\psi}^\dagger(\mathbf{r}')\tilde{\psi}(\mathbf{r})\right\rangle \tilde{\psi}^\dagger(\mathbf{r})\, \tilde{\psi}(\mathbf{r}') \\ &+ \left\langle \tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}^\dagger(\mathbf{r}')\right\rangle \tilde{\psi}(\mathbf{r}')\, \tilde{\psi}(\mathbf{r}) + \left\langle \tilde{\psi}(\mathbf{r}')\tilde{\psi}(\mathbf{r})\right\rangle \tilde{\psi}^\dagger(\mathbf{r})\, \tilde{\psi}^\dagger(\mathbf{r}') \\ &- \left\langle \tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}(\mathbf{r}')\right\rangle \left\langle \tilde{\psi}^\dagger(\mathbf{r}')\tilde{\psi}(\mathbf{r})\right\rangle - \left\langle \tilde{\psi}^\dagger(\mathbf{r}')\tilde{\psi}(\mathbf{r}')\right\rangle \left\langle \tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}(\mathbf{r})\right\rangle \\ &- \left\langle \tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}^\dagger(\mathbf{r}')\right\rangle \left\langle \tilde{\psi}(\mathbf{r}')\tilde{\psi}(\mathbf{r})\right\rangle \end{aligned} \tag{127}$$

$$\begin{aligned} &= n_T(\mathbf{r})\, \tilde{\psi}^\dagger(\mathbf{r}')\, \tilde{\psi}(\mathbf{r}') + n_T(\mathbf{r}')\tilde{\psi}^\dagger(\mathbf{r})\, \tilde{\psi}(\mathbf{r}) \\ &+ G_T(\mathbf{r},\mathbf{r}')\, \tilde{\psi}^\dagger(\mathbf{r}')\, \tilde{\psi}(\mathbf{r}) + G_T(\mathbf{r}',\mathbf{r})\, \tilde{\psi}^\dagger(\mathbf{r})\, \tilde{\psi}(\mathbf{r}') \\ &+ M_T^*(\mathbf{r},\mathbf{r}')\, \tilde{\psi}(\mathbf{r}')\, \tilde{\psi}(\mathbf{r}) + M_T(\mathbf{r}',\mathbf{r})\, \tilde{\psi}^\dagger(\mathbf{r})\, \tilde{\psi}^\dagger(\mathbf{r}') \\ &- \text{pairwise averages}\,. \end{aligned}$$

Then, after sorting terms, the factorized grand canonical Hamiltonian consists of

$$\begin{aligned} K_0 = &\int d^3\mathbf{r}\, \tilde{\phi}^*(\mathbf{r}) \left(\mathcal{L}-\mu\right) \tilde{\phi}(\mathbf{r}) - \frac{Gm^2}{2} \int d^3\mathbf{r} \int d^3\mathbf{r}'\, \frac{1}{|\mathbf{r}-\mathbf{r}'|} \Big[\tilde{\phi}^*(\mathbf{r})\tilde{\phi}^*(\mathbf{r}')\tilde{\phi}(\mathbf{r}')\tilde{\phi}(\mathbf{r}) \\ &- \left\langle \tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}(\mathbf{r}')\right\rangle \left\langle \tilde{\psi}^\dagger(\mathbf{r}')\tilde{\psi}(\mathbf{r})\right\rangle - \left\langle \tilde{\psi}^\dagger(\mathbf{r}')\tilde{\psi}(\mathbf{r}')\right\rangle \left\langle \tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}(\mathbf{r})\right\rangle - \left\langle \tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}^\dagger(\mathbf{r}')\right\rangle \left\langle \tilde{\psi}(\mathbf{r}')\tilde{\psi}(\mathbf{r})\right\rangle\Big] \end{aligned} \tag{128}$$

$$\begin{aligned} = &\int d^3\mathbf{r}\, \tilde{\phi}^*(\mathbf{r}) \left(\mathcal{L}-\mu\right) \tilde{\phi}(\mathbf{r}) - \frac{Gm^2}{2} \int d^3\mathbf{r} \int d^3\mathbf{r}'\, \frac{1}{|\mathbf{r}-\mathbf{r}'|} \Big[\tilde{\phi}^*(\mathbf{r})\tilde{\phi}(\mathbf{r})n_0(\mathbf{r}') \\ &- G_T(\mathbf{r},\mathbf{r}')\, G_T(\mathbf{r}',\mathbf{r})\ - G_T(\mathbf{r}',\mathbf{r}')\, G_T(\mathbf{r},\mathbf{r}) - M_T^*(\mathbf{r},\mathbf{r}')\, M_T(\mathbf{r}',\mathbf{r})\Big] \end{aligned}$$

and

$$\begin{aligned} K_2 = &\int d^3\mathbf{r}\, \tilde{\psi}^\dagger(\mathbf{r}) \left(\mathcal{L}-\mu\right) \tilde{\psi}(\mathbf{r}) - Gm^2 \int d^3\mathbf{r} \int d^3\mathbf{r}'\, \frac{1}{|\mathbf{r}-\mathbf{r}'|} \Big[\tilde{\psi}^\dagger(\mathbf{r})\, \tilde{\psi}(\mathbf{r})\Big(\tilde{\phi}^*(\mathbf{r}')\tilde{\phi}(\mathbf{r}') + \left\langle \tilde{\psi}^\dagger(\mathbf{r}')\tilde{\psi}(\mathbf{r}')\right\rangle\Big) \\ &+ \tilde{\psi}^\dagger(\mathbf{r}')\, \tilde{\psi}(\mathbf{r})\Big(\tilde{\phi}^*(\mathbf{r})\tilde{\phi}(\mathbf{r}') + \left\langle \tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}(\mathbf{r}')\right\rangle\Big) + \frac{1}{2}\Big(\tilde{\psi}^\dagger(\mathbf{r})\left\langle \tilde{\psi}(\mathbf{r}')\tilde{\psi}(\mathbf{r})\right\rangle\tilde{\psi}^\dagger(\mathbf{r}') + \tilde{\psi}(\mathbf{r}')\left\langle \tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}^\dagger(\mathbf{r}')\right\rangle\tilde{\psi}(\mathbf{r})\Big)\Big] \end{aligned}$$

$$\begin{aligned} = &\int d^3\mathbf{r}\, \tilde{\psi}^\dagger(\mathbf{r}) \left(\mathcal{L}-\mu\right) \tilde{\psi}(\mathbf{r}) - Gm^2 \int d^3\mathbf{r} \int d^3\mathbf{r}'\, \frac{1}{|\mathbf{r}-\mathbf{r}'|} \Big[\tilde{\psi}^\dagger(\mathbf{r})\, \tilde{\psi}(\mathbf{r})\Big(n_0(\mathbf{r}') + n_T(\mathbf{r}')\Big) \\ &+ \tilde{\psi}^\dagger(\mathbf{r}')\, \tilde{\psi}(\mathbf{r})\Big(G_0(\mathbf{r},\mathbf{r}') + G_T(\mathbf{r},\mathbf{r}')\Big) + \frac{1}{2}\Big(M_T(\mathbf{r}',\mathbf{r})\, \tilde{\psi}^\dagger(\mathbf{r})\, \tilde{\psi}^\dagger(\mathbf{r}') + M_T^*(\mathbf{r},\mathbf{r}')\, \tilde{\psi}(\mathbf{r}')\, \tilde{\psi}(\mathbf{r})\Big)\Big]\,. \end{aligned} \tag{129}$$

Note that in the first two terms of the interaction part on the RHS of equation (129), symmetry leads to a doubling of contributions when $\mathbf{r}$ is exchanged for $\mathbf{r}'$. It is easy to check whether this is correct by simply changing the ranged interaction potential back to the zero-range potential and setting $\mathbf{r} = \mathbf{r}'$, upon which we recover the known HFB-Hamiltonians denoted in equations (124) and (125).

4.3 Denoting Classical Gravity in the Schrödinger-Newton picture for HFB-Theory

For the following considerations, we will constrain ourselves to the so called *Schrödinger-Newton* formulation of gravity. Intuitively, it seems natural to incorporate a classical operator into the probabilistic framework of quantum theory by simply denoting it as its expectation value. A more rigorous reasoning for why this works can be found in [9]. Now, if the Schrödinger–Newton equation is considered as a fundamental equation, there is a corresponding N-body equation that was already given by Diósi [25], and can be derived from semiclassical gravity in the same way as the one-particle equation:

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\Psi(t,\mathbf{x}_1,\dots,\mathbf{x}_N) = \Bigg(& -\sum_{j=1}^{N}\frac{\hbar^2}{2m_j}\nabla_j^2 + V_{ext}(x) \\ & - G\sum_{j,k=1}^{N} m_j m_k \int \mathrm{d}^3\mathbf{y}_1\cdots\mathrm{d}^3\mathbf{y}_N\,\frac{|\Psi(t,\mathbf{y}_1,\dots,\mathbf{y}_N)|^2}{|\mathbf{x}_j-\mathbf{y}_k|}\Bigg)\Psi(t,\mathbf{x}_1,\dots,\mathbf{x}_N)\,, \end{aligned} \tag{130}$$

where the required boundary condition is, that the gravitational potential goes to zero as $\mathbf{r}$ goes to infinity. We will simplify our model by taking the gravitational potential $\Phi = \langle\Phi\rangle$. Following [9], let $\Omega_N(t,\mathbf{r}_1,\dots,\mathbf{r}_N)$ be the N-particle wave-function

$$|\Omega_N\rangle = \frac{1}{\sqrt{N!}}\left(\int\prod_{i=1}^{N} d^3r_i\right)\Omega_N(t,\mathbf{r}_1,\dots,\mathbf{r}_N)\,\Psi^\dagger(\mathbf{r_1})\cdots\Psi^\dagger(\mathbf{r_N})\,|0\rangle\,. \tag{131}$$

The expectation value is then

$$\langle\Omega(\mathbf{r})|\Psi^\dagger(\mathbf{r})\Psi(\mathbf{r})|\Omega(\mathbf{r})\rangle = \sum_{j=1}^{N}\left(\int\prod_{j\neq i,i=1}^{N} d^3r_i\right)|\Omega_N(t,\mathbf{r}_1,\dots,\mathbf{r}_{j-1},\mathbf{r},\mathbf{r}_{j+1},\dots,\mathbf{r}_N)|^2 \tag{132}$$

which leads to [23]

$$H[\Omega](t) = \int d^3\mathbf{r}\,\Psi^\dagger(\mathbf{r})\Big(\mathcal{L}-\mu\Big)\Psi(\mathbf{r}) - Gm^2\int d^3\mathbf{r}'\,\frac{1}{|\mathbf{r}-\mathbf{r}'|}\Psi^\dagger(\mathbf{r})\Psi(\mathbf{r})\langle\Omega(t)|\Psi^\dagger(\mathbf{r}')\Psi(\mathbf{r}')\,|\Omega(t)\rangle \tag{133}$$

At equilibrium, the static system is then described by

$$K_{CG} = \int d^3\mathbf{r}\,\Psi^\dagger(\mathbf{r})\Big[\mathcal{L}-\mu\Big]\Psi(\mathbf{r}) - Gm^2\int d^3\mathbf{r}'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\Psi^\dagger(\mathbf{r})\,\Phi[\Psi](\mathbf{r}')\,\Psi(\mathbf{r})\,. \tag{134}$$

Here, the notation $\Phi[\Psi](\mathbf{r}')$ alludes to the "non-linear nature". In other words Ψ is necessary to determine Φ. This is often referred to as a wavefunction "self-interaction" since, in the first quantization picture, the wavefunction of a single particle will now interact with itself, something that can never occur in a quantum theory of gravity. By treating $\Phi = \langle\Phi\rangle$ we immediately see that the Hamiltonian remains quadratic in the self-gravitational interaction. This can be seen also by decomposing Ψ as before, such that

$$\Psi^\dagger\Psi \to (\tilde{\phi}+\tilde{\psi})^\dagger(\tilde{\phi}+\tilde{\psi}) = \tilde{\phi}^*\tilde{\phi} + \tilde{\phi}^*\tilde{\psi} + \tilde{\psi}^\dagger\tilde{\phi} + \tilde{\psi}^\dagger\tilde{\psi}\,. \tag{135}$$

Using our results from previous sections, we can write down the contributing Hamitonians as

$$K_0 = \int d^3\mathbf{r}\,\tilde{\phi}^*(\mathbf{r})\Big[\mathcal{L}-\mu\Big]\tilde{\phi}(\mathbf{r}) - Gm^2\int d^3\mathbf{r}'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\tilde{\phi}^*(\mathbf{r})\left(\Phi[\phi](\mathbf{r}')+\Phi[\psi](\mathbf{r}')\right)\tilde{\phi}(\mathbf{r}) \tag{136}$$

and

$$K_2 = \int d^3\mathbf{r}\,\tilde{\psi}^\dagger(\mathbf{r})\Big[\mathcal{L}-\mu\Big]\tilde{\psi}(\mathbf{r}) - Gm^2\int d^3\mathbf{r}'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\tilde{\psi}^\dagger(\mathbf{r})\left(\Phi[\phi](\mathbf{r}')+\Phi[\psi](\mathbf{r}')\right)\tilde{\psi}(\mathbf{r})\,. \tag{137}$$

Clearly, at least within this treatment, no other terms arise in the interaction.

4.4 Phase Space $\mathbb{R}^{2n}$

Bose-Einstein condensation, as a second order phase transition, is a phenomenon most often best observed in phase space, rather than in configuration space. It hence makes sense to look at the Fourier transform of equations (128) and (129), as well as (136) and (137). Recall our already made assumptions, regarding translational symmetry - a homogeneous system - the Fourier transform of the order parameter and the thermalized wavefunction can be written as

$$\hat{\Psi}(\mathbf{r}) = \frac{1}{V^{1/2}} \sum_k e^{ik\mathbf{r}/\hbar} \hat{a}_k = \frac{V^{1/2}}{(2\pi\hbar)^3} \int d\mathbf{k}\, e^{ik\mathbf{r}/\hbar} \hat{a}_k \ , \tag{138}$$

and its respective hermitian conjugate $\hat{\Psi}^\dagger(\mathbf{r}) \rightarrow \hat{a}_k^\dagger$. Using the decomposition from (111), which is in agreement with the Bogoliubov mean-field approximation of the ground state for $\Phi_0 \rightarrow a_0 \simeq \sqrt{N_0}$, and the respective thermalized states, we can write an arbitrary non-local interaction as

$$H_I = \frac{1}{2} \sum_{k_1,k_2,k_3,k_4} \langle k_1, k_2 | V_I(\mathbf{q}) | k_3, k_4 \rangle \, a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4} \ , \tag{139}$$

with matrix elements

$$\langle k_1, k_2 | V_I(\mathbf{q}) | k_3, k_4 \rangle = \frac{1}{V^2} \int d\mathbf{r} \int d\mathbf{r}' \, e^{-i(k_1-k_4)\mathbf{r}} \, e^{-i(k_2-k_3)\mathbf{r}'} \, V_I(\mathbf{r}, \mathbf{r}') \ . \tag{140}$$

For a homogeneous system the interaction depends only on the difference between $\mathbf{r} - \mathbf{r}'$, $V_I(\mathbf{r}, \mathbf{r}') = V_I(|\mathbf{r} - \mathbf{r}'|)$. The Fourier transform of the interaction potential is computed by twice integrating by parts, requiring as boundary conditions that V_I goes to 0 as $|\mathbf{r} - \mathbf{r}'| \rightarrow \infty$. For the sake of generality, let's consider a ranged potential of the form $V_I(\mathbf{r}, \mathbf{r}') = \frac{C}{|x|}$, where $|x| = |\mathbf{r} - \mathbf{r}'|$ and C is an arbitrary constant for the moment, $C < 0$, (but really, later, $C = -Gm^2$). Then

$$\begin{aligned} \mathcal{F}(V_I) &= C \int d\mathbf{x} \frac{1}{|\mathbf{x}|} e^{-iqx/\hbar} \\ &= C \left[-\frac{\hbar}{iq} \frac{1}{|\mathbf{x}|} e^{-iqx/\hbar} \Bigg|_{-\infty}^{+\infty} + \frac{\hbar}{iq} \int d\mathbf{x} \left(\frac{d}{d\mathbf{x}} \frac{1}{|\mathbf{x}|} \right) e^{-iqx/\hbar} \right] . \end{aligned} \tag{141}$$

Due to the chosen boundary conditions, the first term is equal to 0 in the limit of $|x| \rightarrow \infty$. Repeating the procedure gives

$$= C \left[\frac{\hbar^2}{q^2} \left(\frac{d}{d\mathbf{x}} \frac{1}{|\mathbf{x}|} \right) e^{-iqx/\hbar} \Bigg|_{-\infty}^{+\infty} + \frac{\hbar^2}{q^2} \int d\mathbf{x} \left(\frac{d^2}{d\mathbf{x}^2} \frac{1}{|\mathbf{x}|} \right) e^{-iqx/\hbar} \right] , \tag{142}$$

where the first term on the RHS again goes to zero for above mentioned reasons. Now using the well known result $\frac{d^2}{dx^2} \frac{1}{|x|} = \triangle \frac{1}{|x|} = 4\pi\, \delta^d(x - x')$, we have

$$\mathcal{F}\big(V_I(\mathbf{r}, \mathbf{r}')\big) = V_I(\mathbf{q}) = 4\pi C \frac{\hbar^2}{\mathbf{q}^2} \ . \tag{143}$$

We can now simplify the matrix elements

$$\langle k_1, k_2 | V_I(\mathbf{q}) | k_3, k_4 \rangle = \frac{1}{V^2} V_I(\mathbf{q}) \, \delta_{\mathbf{q}, k_1 - k_4} \, \delta_{\mathbf{q}, -k_2 + k_3} \tag{144}$$

and by setting $k_1 = k$, $k_2 = k'$, $k_3 = k' + q$, and $k_4 = k - q$, the two-body interaction becomes

$$V_I = \frac{1}{2V} V_I(\mathbf{q}) \, a_k^\dagger a_{k'}^\dagger a_{k'+\mathbf{q}} a_{k-\mathbf{q}} \ . \tag{145}$$

Considering conservation of momentum, this can be seen like a scattering process with two incoming particles of momenta k and k', and two outgoing particles of momenta $k' + \mathbf{q}$, $k - \mathbf{q}$, where k denotes the wave vector in a plane-wave basis. We can further simplify this by taking another good look at the potential of the long-range interaction in phase-space. Since for a translationally invariant system, we have

$V_I(\mathbf{r},\mathbf{r}') = V_I(|\mathbf{r}-\mathbf{r}'|)$, we can see what this constraint implies in phase space by rewriting equation (140) as

$$V_I(\mathbf{r},\mathbf{r}') = \int \frac{d\mathbf{k}}{(2\pi)^3} \int \frac{d\mathbf{k}'}{(2\pi)^3} V_I(\mathbf{k},\mathbf{k}')\, e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}')} e^{i(\mathbf{k}'+\mathbf{k})\mathbf{r}'} . \tag{146}$$

Since this has to be a function of $\mathbf{r}-\mathbf{r}'$, it is obvious from the factor $e^{i(\mathbf{k}'+\mathbf{k})\mathbf{r}'}$ that any reference to the absolute value of $\mathbf{r}'$ can only vanish, if $\mathbf{k}' = -\mathbf{k}$. Hence $V_I(\mathbf{k},\mathbf{k}') \propto \delta(\mathbf{k}+\mathbf{k}')$ or in short $V_I(\mathbf{r},\mathbf{r}') \to V_I(\mathbf{k},-\mathbf{k})$. For correlation functions, for instance, this means

$$G(\mathbf{r},\mathbf{r}') = \langle \hat{\Psi}^\dagger(\mathbf{r})\hat{\Psi}(\mathbf{r}')\rangle \longrightarrow G(\mathbf{k}) = \langle \hat{\Psi}^\dagger(\mathbf{k})\hat{\Psi}(-\mathbf{k})\rangle . \tag{147}$$

Another way to see this is by noting that in a translationally invariant (a.k.a. stationary) system, we must have

$$G(\mathbf{r},\mathbf{r}') = G(\mathbf{r}+c,\mathbf{r}'+c) , \tag{148}$$

where c is some constant vector. Therefore, it must also be true that

$$G(\mathbf{r},\mathbf{r}') = G(\mathbf{r}-\mathbf{r}',0) =: G(\mathbf{r}-\mathbf{r}') , \tag{149}$$

since the correlation only depends on $|\mathbf{r}-\mathbf{r}'|$. Performing the Fourier-transformation

$$\begin{aligned}
\tilde{G}(\mathbf{p},\mathbf{q}) &= \frac{1}{(2\pi)^d} \int\int d^d\mathbf{r}\, d^d\mathbf{r}'\, e^{i\mathbf{p}\mathbf{r}+i\mathbf{q}\mathbf{r}'} G(\mathbf{r}-\mathbf{r}') \\
&= \frac{1}{(2\pi)^d} \int\int d^d u\, d^d\mathbf{r}'\, e^{i\mathbf{p}u}\, e^{i(\mathbf{p}+\mathbf{q})\mathbf{r}'} G(u) \\
&= \tilde{G}(\mathbf{p})\, \delta(\mathbf{p}+\mathbf{q}) ,
\end{aligned} \tag{150}$$

which forces $\mathbf{q} = -\mathbf{p}$, and renders $\tilde{G}(\mathbf{p})$ diagonal in phase-space [26].

4.4.1 Quantized Gravity in Phase Space $\mathbb{R}^{2n}$

Let's look back at equations (128) and (129) and Fourier-transform the expressions term by term. Recall that we have assumed *homogeneity*, and *isotropy* initially. In this case, we have:

$$\psi_{\boldsymbol{k}}(\boldsymbol{r}) = e^{i\boldsymbol{k}.\boldsymbol{r}} , \tag{151}$$

(ignoring dimensional factors of the volume V), and $n_0(\boldsymbol{r})$ and $G_0(\boldsymbol{r},\boldsymbol{r}')$ are independent of $\boldsymbol{r}$ and $\boldsymbol{r}'$, and we can also take $G_T(\boldsymbol{r},\boldsymbol{r}') = G_T(\boldsymbol{r}-\boldsymbol{r}')$. We Fourier transform the following:

$$\begin{aligned}
V_I(\boldsymbol{r}-\boldsymbol{r}') &= \int \frac{d^3\boldsymbol{q}}{(2\pi)^3} \tilde{V}_I(\boldsymbol{q}) e^{i\boldsymbol{q}.(\boldsymbol{r}-\boldsymbol{r}')} \\
n_T(\boldsymbol{r}) &= \int \frac{d^3\boldsymbol{p}'}{(2\pi)^3} \tilde{n}_T(\boldsymbol{p}') e^{i\boldsymbol{p}'.\boldsymbol{r}} \\
G_T(\boldsymbol{r}-\boldsymbol{r}') &= \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \tilde{G}_T(\boldsymbol{p}) e^{i\boldsymbol{p}.(\boldsymbol{r}-\boldsymbol{r}')} .
\end{aligned} \tag{152}$$

In this case, the interaction Hamiltonian is:

$$\begin{aligned}
\hat{K}_{int} &= -Gm^2 \int d^3\boldsymbol{r}\, d^3\boldsymbol{r'} \int \frac{d^3\boldsymbol{q}}{(2\pi)^3} \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{d^3\boldsymbol{k'}}{(2\pi)^3} \tilde{V}_I(\boldsymbol{q}) e^{i\boldsymbol{q}(\boldsymbol{r}-\boldsymbol{r'})} \Big[\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k'}} e^{i\boldsymbol{r}(\boldsymbol{k'}-\boldsymbol{k})}\Big(n_0 + \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\tilde{n}_T(\boldsymbol{p})e^{i\boldsymbol{p}\boldsymbol{r'}}\Big) \\
&\quad + \hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k'}} e^{-i\boldsymbol{r'}\boldsymbol{k}} e^{i\boldsymbol{r}\boldsymbol{k'}}\Big(\tilde{G}_0 + \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} e^{i\boldsymbol{p}(\boldsymbol{r}-\boldsymbol{r'})}\tilde{G}_T(\boldsymbol{p})\Big) + \cdots\Big] \\
&= -Gm^2 \int \frac{d^3\boldsymbol{q}}{(2\pi)^3} \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{d^3\boldsymbol{k'}}{(2\pi)^3}\tilde{V}_I(\boldsymbol{q})\Big[\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k'}}\Big(n_0(2\pi)^6\delta^{(3)}(\boldsymbol{q}+\boldsymbol{k'}-\boldsymbol{k})\delta^{(3)}(\boldsymbol{q}) + (2\pi)^6\delta^{(3)}(\boldsymbol{q}+\boldsymbol{k'}-\boldsymbol{k}) \\
&\quad \delta^{(3)}(-\boldsymbol{q}+\boldsymbol{p})\tilde{n}_T(\boldsymbol{p})\Big) \\
&\quad + \hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k'}}\Big(\tilde{G}_0(2\pi)^6\delta^{(3)}(\boldsymbol{q}+\boldsymbol{k'})\delta^{(3)}(-\boldsymbol{q}-\boldsymbol{k}) + (2\pi)^6\delta^{(3)}(\boldsymbol{q}+\boldsymbol{k'}+\boldsymbol{p})\delta^{(3)}(-\boldsymbol{q}-\boldsymbol{k}-\boldsymbol{p})\tilde{G}_T(\boldsymbol{p})\Big) + \cdots\Big], \\
&= -Gm^2 \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\Big[\tilde{V}_I(0)n_0\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}} + \int \frac{d^3\boldsymbol{q}}{(2\pi)^3}\tilde{V}_I(\boldsymbol{q})\tilde{n}_T(\boldsymbol{q})\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}-\boldsymbol{q}} \\
&\quad + \tilde{V}_I(-\boldsymbol{k})\tilde{G}_0\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}} + \int \frac{d^3\boldsymbol{q}}{(2\pi)^3}\tilde{V}_I(\boldsymbol{q})\tilde{G}_T(-\boldsymbol{q}-\boldsymbol{k})\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}} + \cdots\Big], \\
&= -Gm^2 \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\Big[\tilde{V}_I(0)n_0\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}} + \int \frac{d^3\boldsymbol{q}}{(2\pi)^3}\tilde{V}_I(q)\tilde{n}_T(\boldsymbol{q})\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}-\boldsymbol{q}} \\
&\quad + \tilde{V}_I(k)\tilde{G}_0\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}} + \int \frac{d^3\boldsymbol{q}}{(2\pi)^3}\tilde{V}_I(q)\tilde{G}_T(\boldsymbol{q}+\boldsymbol{k})\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}} + \cdots\Big].
\end{aligned} \tag{153}$$

and for the *off-diagonal* terms

$$\hat{K}_{Int,m} = -Gm^2 \int d^3\mathbf{r}\, d^3\mathbf{r'}\, V_I(|\mathbf{r}-\mathbf{r'}|)\Big[\ldots + \frac{1}{2}\Big(m_T(\mathbf{r'},\mathbf{r})\,\tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}^\dagger(\mathbf{r'}) + m_T^*(\mathbf{r},\mathbf{r'})\,\tilde{\psi}(\mathbf{r'})\tilde{\psi}(\mathbf{r})\Big)\Big]. \tag{154}$$

Introducing the additional notation

$$m_T(\mathbf{r},\mathbf{r'}) = m_T(\mathbf{r}-\mathbf{r'}) = \int \frac{d^3\mathbf{l}}{(2\pi)^3} m_T(\mathbf{l})\, e^{i\mathbf{l}(\mathbf{r}-\mathbf{r'})} \tag{155}$$

and performing the **r**- & **k**′-integrals after rearranging the terms in the exponentials:

$$\begin{aligned}
&= -Gm^2 \int d^3\mathbf{r}\, d^3\mathbf{r'} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{d^3\mathbf{k'}}{(2\pi)^3}\, V_I(\mathbf{q})\, e^{i\mathbf{q}(\mathbf{r}-\mathbf{r'})}\Big[\ldots + \frac{1}{2}\Big(m_T(\mathbf{l})\, e^{i\mathbf{l}(\mathbf{r'}-\mathbf{r})}\, a^\dagger_{\mathbf{k}} e^{-i\mathbf{k}\mathbf{r}}\, a^\dagger_{\mathbf{k'}} e^{-i\mathbf{k'}\mathbf{r'}} \\
&\qquad + m_T^*(\mathbf{l'})\, e^{-i\mathbf{l'}(\mathbf{r}-\mathbf{r'})}\, a_{\mathbf{k'}} e^{i\mathbf{k'}\mathbf{r'}}\, a_{\mathbf{k}} e^{i\mathbf{k}\mathbf{r}}\Big)\Big] \\
&= -Gm^2 \int \frac{d^3\mathbf{q}}{(2\pi)^3} \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{d^3\mathbf{k'}}{(2\pi)^3}\, V_I(\mathbf{q})\Big[\ldots + \frac{1}{2}\Big(m_T(\mathbf{l})\, a^\dagger_{\mathbf{k}} a^\dagger_{\mathbf{k'}}\, (2\pi)^6\, \delta(-\mathbf{q}-\mathbf{k}+\mathbf{l})\, \delta(\mathbf{q}-\mathbf{k'}-\mathbf{l}) \\
&\qquad + m_T^*(\mathbf{l'})\, a_{\mathbf{k'}} a_{\mathbf{k}}\, (2\pi)^6\, \delta(\mathbf{q}+\mathbf{k}-\mathbf{l})\, \delta(-\mathbf{q}+\mathbf{k'}+\mathbf{l})\Big)\Big] \\
&= -Gm^2 \int \frac{d^3\mathbf{q}}{(2\pi)^3} \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, V_I(\mathbf{q})\Big[\ldots + \frac{1}{2}\Big(a^\dagger_{\mathbf{k}} a^\dagger_{-\mathbf{k}}\, m_T(\mathbf{k}) + a_{-\mathbf{k}} a_{\mathbf{k}}\, m_T^*(-\mathbf{k})\Big)\Big].
\end{aligned} \tag{156}$$

Altogether, the Fourier-transformed Hamiltonian takes the form:

$$\begin{aligned}
H_0 &= \Big(V_{ext} - \mu\Big)\, n_0 + \frac{1}{2} V_G\Big|_{k=0}\, n_0^2 \\
H_2 &= \Big(-\tfrac{\hbar^2\mathbf{k}^2}{2m} + V_{ext} - \mu\Big)\, a^\dagger_{\mathbf{k}} a_{\mathbf{k}} \\
&\quad + V_G(0)\, n_0\, a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + V_G(\mathbf{k})\,\Big[n_T(\mathbf{k}) a^\dagger_{\mathbf{k}} a_{\mathbf{k}} \\
&\quad + \Big(G_0 + G_T^{(1)}(\mathbf{q}+\mathbf{k})\Big)\, a^\dagger_{\mathbf{k}} a_{-\mathbf{k}} \\
&\quad + \frac{1}{2}\Big(m_T(\mathbf{k})\, a^\dagger_{\mathbf{k}} a^\dagger_{-\mathbf{k}} + m_T^*(-\mathbf{k})\, a_{-\mathbf{k}} a_{\mathbf{k}}\Big)\Big].
\end{aligned} \tag{157}$$

The main difference to the scattering HFB-Hamiltonian is how $\frac{1}{2}$ of the contributions $(n_0 + n_T)$ has now become a correlation function $G^{(1)}(\mathbf{r}, \mathbf{r}') = \phi^*(\mathbf{r})\phi(\mathbf{r}') + G_T^{(1)}(\mathbf{r}', \mathbf{r})$. One can diagonalize H_2 by means of a *Bogoliubov Transformation*, to find the Eigenenergies of the system.

Please also note, that the Fourier transform of the interaction potential shows a singularity at $\mathbf{k} = 0$. To deal with this singularity in a physically meaningful way, recall how the potential was most generally denoted in equation (22). Reminding ourselves that the scattering length a_s is effectively the "size of the scatterer", it becomes clear, that the cutoff defined in (22) is equivalent to this physical entity, because the zero-range interaction becomes dominant at this length scale. Hence, $|\mathbf{r} - \mathbf{r}'|$ is bounded from below by a_s which implies

$$V_G(\mathbf{k})\Big|_{\mathbf{k}=0} = -Gm^2\frac{1}{a_s} =: V(0) \quad , \text{for } |\mathbf{r} - \mathbf{r}'| \leq a_s\,. \tag{158}$$

As a final sanity check, we now compare this to the approximations in [13] and [21]. In the former case, we neglect the thermal cloud and take $n_T = 0 = G_T$, as well as taking $G_0 = n_0$, giving:

$$\hat{K}_{int} \approx -Gm^2 \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\left[n_0\left(\tilde{V}_I(0) + \tilde{V}_I(k)\right)\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}} + \cdots\right]. \tag{159}$$

This matches the first term of (8.10) in [13]. The first part is the Hartree interaction (where the particle effectively interacts with the whole condensate) and the second is the Fock interaction (where the particle interacts with a single particle in the condensate).

On the other hand, for [21] we must keep the thermal cloud but set $G_0 = n_0$, $G_T = n_T$ and take the contact approximation where we just set $V_I(q) = V_I(0)$ - we then have:

$$\hat{K}_{int} \approx -2Gm^2 \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\left[\tilde{V}_I(0)(n_0 + n_T)\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}} + \cdots\right], \tag{160}$$

which matches the second term in (3.29) in [21] (for a uniform system). Since we have taken a contact potential, the Hartree and Fock interactions are equal [13]. These interactions can also be derived by considering a uniform system in just the Bogoliubov approximation.

4.4.2 Bogoliubov Transformation

As already discussed in the previous section, the *Bogoliubov Transformation* is a linear, symplectic transformation on $\mathbb{R}^{2n}$. Let us adapt notation as before for the sake of clarity, and relabel:

$$\begin{aligned} a_{\mathbf{k}} &\rightarrow a \quad , \quad a^\dagger_{\mathbf{k}} \rightarrow a^\dagger \\ a_{-\mathbf{k}} &\rightarrow b \quad , \quad a^\dagger_{-\mathbf{k}} \rightarrow b^\dagger \end{aligned} \tag{161}$$

Then H_2 reads:

$$\begin{aligned} H_2 = &\left(-\tfrac{\hbar^2\mathbf{k}^2}{2m} + V_{ext} - \mu\right)a^\dagger a \\ &+ V_G(0)\,n_0\,a^\dagger a + V_G(\mathbf{k})\,\Big[n_T(\mathbf{k})a^\dagger a \\ &+ \left(G_0 + G_T^{(1)}(\mathbf{q}+\mathbf{k})\right)a^\dagger b \\ &+ \frac{1}{2}\Big(m_T(\mathbf{k})\,a^\dagger b^\dagger + m_T^*(-\mathbf{k})\,ba\Big)\Big]\,. \end{aligned} \tag{162}$$

If we set $\mathbf{r} = \mathbf{r}'$ and hence the b in the 3^{rd} line of equation (162) back to a, we recover the usual form for a *zero-range interaction*, as for instance denoted in [13]. Defining the *quasiparticle creation- & annihilation operators* as

$$\begin{aligned} \beta_{\mathbf{k}} &\rightarrow \alpha \\ \beta_{-\mathbf{k}} &\rightarrow \beta\,, \end{aligned} \tag{163}$$

which are required to satisfy the canonical commutation relations $[\alpha, \alpha^\dagger] = [\beta, \beta^\dagger] = \delta_{ij}$ and 0 else, the *Bogoliubov Transformation* can be denoted as

$$\begin{aligned} a &= u\alpha - v\beta^\dagger \quad , \quad b = u\beta - v\alpha^\dagger \\ a^\dagger &= u\alpha^\dagger - v\beta \quad , \quad b^\dagger = u\beta^\dagger - v\alpha \end{aligned} \tag{164}$$

and requiring that

$$u_{\mathbf{k}}^2 - v_{\mathbf{k}}^2 = 1\,, \tag{165}$$

where without loss of generality $u, v \in \mathbb{R}$, since their phases are arbitrary. Then contributions coming from $a^\dagger a$ transform like

$$\begin{aligned} \langle n_k | a^\dagger a | n_k \rangle &= \langle n_k | \big(u\alpha^\dagger - v\beta\big) \big(u\alpha - v\beta^\dagger\big) | n_k \rangle \\ &= \langle n_k |\, u^2\, \alpha^\dagger \alpha - uv\, \alpha^\dagger \beta^\dagger - vu\, \beta\alpha + v^2\, \beta\beta^\dagger | n_k \rangle \\ &= \langle n_k |\, u^2\, \alpha^\dagger \alpha - uv\, \alpha^\dagger \beta^\dagger - vu\, \beta\alpha + v^2\, \big(\beta\beta^\dagger - \beta^\dagger\beta\big) + v^2\, \beta^\dagger\beta | n_k \rangle \\ &= \langle n_k |\, u^2\, \alpha^\dagger \alpha - uv\, \alpha^\dagger \beta^\dagger - vu\, \beta\alpha + v^2 \mathbb{I} + v^2\, \beta^\dagger\beta | n_k \rangle \\ &= \sum_k v_k^2 + n_k (u^2 + v^2) \end{aligned} \tag{166}$$

and

$$\begin{aligned} \langle n_k | a^\dagger b | n_k \rangle &= \langle n_k | \big(u\alpha^\dagger - v\beta\big) \big(u\beta - v\alpha^\dagger\big) | n_k \rangle \\ &= \langle n_k |\, u^2\, \alpha^\dagger \beta - uv\, \alpha^\dagger \alpha^\dagger - vu\, \beta\beta + v^2 \mathbb{I} + v^2\, \alpha^\dagger\beta | n_k \rangle \\ &= \sum_k v_k^2 + n_k (u^2 + v^2) \end{aligned} \tag{167}$$

and the anomalous contributions transform as

$$\begin{aligned} \langle n_k | a^\dagger b^\dagger | n_k \rangle &= \langle n_k | \big(u\alpha^\dagger - v\beta\big) \big(u\beta^\dagger - v\alpha\big) | n_k \rangle \\ &= \langle n_k |\, u^2\, \alpha^\dagger \beta^\dagger - uv\, \alpha^\dagger \alpha - vu\, \beta\beta^\dagger + v^2\, \beta\alpha | n_k \rangle \\ &= -\sum_k uv\big(2n_k + 1\big) \end{aligned} \tag{168}$$

and of course the same result is true for $\langle n_k | ba | n_k \rangle = -\sum_k uv\big(2n_k + 1\big)$. When looking at the transformed contributions, one must keep in mind that the α and β annihilation- and creation-operators do not represent single particle states, but collective excitations in the respective $\mathbf{k}$-modes. This again represents the sound-like propagation of the "excited waves" through the BEC.

Let us consider again the contribution coming from the correlation function in equation (167). Following the arguments of Penrose & Onsager [27], we look back at the definition in equations (111) and (113). From it, it immediately follows that

$$G^{(1)}(\mathbf{r}, \mathbf{r}') = N\, \phi^*(\mathbf{r})\, \phi(\mathbf{r}') \tag{169}$$

and therefore

$$\begin{aligned} &\Rightarrow \int d^3\mathbf{r}' G^{(1)}(\mathbf{r}, \mathbf{r}')\, \phi(\mathbf{r}') = N\, \phi(\mathbf{r}) \\ &\Rightarrow \int d^3\mathbf{r}' G^{(1)}(\mathbf{r}, \mathbf{r}')\, \psi_k(\mathbf{r}') = 0\,. \end{aligned} \tag{170}$$

The other $\psi_k(\mathbf{r})$ have eigenvalues n_k, which are of $O(1)$, and hence the correlation function can be decomposed as

$$G^{(1)}(\mathbf{r}, \mathbf{r}') = N_0\, \phi^*(\mathbf{r})\, \phi(\mathbf{r}') + \sum_k n_k\, \psi_k^\dagger(\mathbf{r})\, \psi_k(\mathbf{r}')\,. \tag{171}$$

Furthermore, since equation (170) corresponds to a Hermitian operator eigenvalue equation, the eigenvalues are real, and we can take the eigenfunctions to form a complete orthonormal set. This, together with the arguments presented in equations (148) and (150), shows that H_2 for a ranged interaction contributes to the energy in precisely the same way as a zero-ranged one, up to a coupling factor, after applying the Bogoliubov transformations. This convenient result makes it easy to rewrite H_{HFB} including both zero- and long-ranged interactions in the fully quantized version.

Therefore, we denote:

$$V_I = U_0 + V_G \tag{172}$$

where

$$\begin{aligned} U_0 &= 4\pi\hbar^2 \frac{a_s}{m}\,, \\ V_G &= -4\pi\hbar^2 \frac{Gm^2}{k^2} \end{aligned} \tag{173}$$

with a_s denoting the *s-wave scattering length*, as usual, G the gravitational constant and m the particle mass.

$$\Rightarrow V_I = 4\pi\hbar^2 \left(\frac{a_s}{m} - \frac{Gm^2}{k^2} \right). \tag{174}$$

Furthermore, we denote the external (harmonic oscillator potential) by V_{ext}, as before. Let us recall the Hamiltonian of the zero-range scattering interaction, which reads

$$\begin{aligned} H_s =& \left(V_{ext} - \mu\right) n_0 + \frac{1}{2} U_0 \, n_0^2 + \sum_{k\neq 0} \frac{1}{2} \left(-\frac{\hbar^2 k^2}{2m} + V_{ext} - \mu \right) \left(a_k^\dagger a_k + a_{-k}^\dagger a_{-k} \right) \\ &+ \sum_{k \neq 0} U_0 \left(n_0 + n_T \right) \left(a_k^\dagger a_k + a_{-k}^\dagger a_{-k} \right) + \frac{1}{2} \sum_{k\neq 0} U_0 \left(m_k a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger + m_{-k}^* a_{-\mathbf{k}} a_{\mathbf{k}} \right) \\ =& \left(V_{ext} - \mu\right) n_0 + \frac{1}{2} U_0 \, n_0^2 + \sum_{k\neq 0} \left(-\frac{\hbar^2 k^2}{2m} + V_{ext} - \mu \right) a_k^\dagger a_k \\ &+ \sum_{k\neq 0} 2\, U_0 \left(n_0 + n_T \right) a_k^\dagger a_k + \frac{1}{2} \sum_{k\neq 0} U_0 \left(m_k a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger + m_{-k}^* a_{-\mathbf{k}} a_{\mathbf{k}} \right) \end{aligned} \tag{175}$$

Then, in the light of above results, we can denote the full Hamiltonian, including scattering and quantized gravitational interaction, as

$$\begin{aligned} \mathcal{H}_{QG} =& \left(V_{ext} - \mu\right) n_0 + \frac{1}{2} \left(U_0 + V_G\big|_{k=0} \right) n_0^2 + \sum_{k\neq 0} \left(-\frac{\hbar^2 k^2}{2m} + V_{ext} - \mu \right) a_k^\dagger a_k \\ &+ \sum_{k\neq 0} \Big[V(0)\, n_0 + V(\mathbf{k}) \left(n_0 + 2n_T \right) + 2U_0 \left(n_0 + n_T \right) \Big] a_k^\dagger a_k \\ &+ \sum_{k\neq 0} \frac{1}{2} \Big(U_0 + V_G(\mathbf{k}) \Big) \left(m_k a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger + m_{-k}^* a_{-\mathbf{k}} a_{\mathbf{k}} \right). \end{aligned} \tag{176}$$

Finally, the *Bogoliubov Transformation* for *Bosons* amounts to solving the Eigenvalue problem

$$\epsilon_{k,QG}^2 = \left(-\frac{\hbar^2 k^2}{2m} + V_{ext} - \mu + V(0)\, n_0 + V(\mathbf{k}) \left(n_0 + 2n_T \right) + 2U_0 \left(n_0 + n_T \right) \right)^2 - \Big(m_k \left(U_0 + V_G(\mathbf{k}) \right) \Big)^2. \tag{177}$$

A quick look back: in equation (90) we chose u and v as $u_k = cosh(\theta_k)$ and $v_k = sinh(\theta_k)$. Considering the additional condition that the Bogoliubov Transformation serves to make the coefficients $\beta^\dagger \beta^\dagger$ and $\beta\beta$ vanish, θ_k must be chosen appropriately. From this, one can uniquely determine the coefficients as

$$\begin{aligned} u_{k,QG}^2 &= \frac{1}{2} \left(\frac{-\frac{\hbar^2 k^2}{2m} + V_{ext} - \mu + V(0)\, n_0 + V(\mathbf{k}) \left(n_0 + 2n_T \right) + 2U_0 \left(n_0 + n_T \right)}{\epsilon_k} + 1 \right) \\ v_{k,QG}^2 &= \frac{1}{2} \left(\frac{-\frac{\hbar^2 k^2}{2m} + V_{ext} - \mu + V(0)\, n_0 + V(\mathbf{k}) \left(n_0 + 2n_T \right) + 2U_0 \left(n_0 + n_T \right)}{\epsilon_k} - 1 \right). \end{aligned} \tag{178}$$

We arrive at the diagonalized Hamiltonian (also taking the *ensemble average*):

$$\begin{aligned}\langle\mathcal{H}_{QG}\rangle &= \big(V_{ext}-\mu\big)N_0 + \frac{1}{2}\Big(U_0 + V_G\big|_{k=0}\Big)\, n_0^2 \\ &+ \sum_{k\neq 0}\left[\epsilon_k - \bigg(-\tfrac{\hbar^2k^2}{2m} + V_{ext} - \mu + V(0)\, n_0 + V(\mathbf{k})\big(n_0 + 2n_T\big) + 2U_0\big(n_0+n_T\big)\bigg)\right] + \sum_{k\neq 0}\epsilon_k\alpha_k^\dagger\alpha_k\end{aligned} \tag{179}$$

4.4.3 Fundamentally Classical Gravity in Phase Space $\mathbb{R}^{2n}$

Starting at equations (135) and (136), which we shall consider again using our results from the previous sections, we compute (explicitly for the ranged interaction part):

$$\begin{aligned}\hat{K}_{int} &= -Gm^2\int d^3\mathbf{r}\, d^3\mathbf{r}'\int\frac{d^3\mathbf{q}}{(2\pi)^3}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{d^3\mathbf{k}'}{(2\pi)^3}\, V_I(\mathbf{q})\, e^{i\mathbf{q}(\mathbf{r}-\mathbf{r}')}\left[a_\mathbf{k}^\dagger e^{-i\mathbf{kr}}\, a_{\mathbf{k}'}e^{i\mathbf{k}'\mathbf{r}}\Big(n_0(\mathbf{r}') + \int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\tilde{n}(\mathbf{p})\, e^{i\mathbf{pr}'}\Big)\right] \\ &= -Gm^2\int\frac{d^3\mathbf{q}}{(2\pi)^3}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{d^3\mathbf{k}'}{(2\pi)^3}\, V_I(\mathbf{q})\left[a_\mathbf{k}^\dagger a_{\mathbf{k}'}(2\pi)^6\delta(\mathbf{q}-\mathbf{k}+\mathbf{k}')\,\Big(n_0\,\delta(\mathbf{q}) + \tilde{n}(\mathbf{p})\,\delta(\mathbf{p}-\mathbf{q})\Big)\right] \\ &= -Gm^2\int\frac{d^3\mathbf{k}}{(2\pi)^3}\left[V(0)\, n_0\, a_\mathbf{k}^\dagger a_\mathbf{k} + \int\frac{d^3\mathbf{q}}{(2\pi)^3}V_I(\mathbf{q})\tilde{n}_T(\mathbf{q})\, a_\mathbf{k}^\dagger a_{\mathbf{k}-\mathbf{q}}\right]\end{aligned} \tag{180}$$

The Fourier-transformed self-gravitational interaction can hence be denoted as

$$\begin{aligned}H_{int,CG} &= V_G(0)\Big(a_0^\dagger a_0\langle a_0^\dagger a_0\rangle + a_0^\dagger a_0\langle a_{-\mathbf{k}}^\dagger a_{-\mathbf{k}}\rangle\Big) + \sum_{k\neq 0}V_G(0)\, a_\mathbf{k}^\dagger a_\mathbf{k}\langle a_0^\dagger a_0\rangle + V_G(\mathbf{k})a_\mathbf{k}^\dagger a_\mathbf{k}\langle a_{-\mathbf{k}}^\dagger a_{-\mathbf{k}}\rangle \\ &= V_G(0)\Big(N_0^2 + N_0\, n_T(-\mathbf{k})\Big) + \sum_{k\neq 0}\Big(V_G(0)n_0 + V_G(\mathbf{k})n_T(\mathbf{k})\Big)a_\mathbf{k}^\dagger a_\mathbf{k}\,,\end{aligned} \tag{181}$$

where we have again used the definitions from equation (120). If we now add the scattering interaction terms, which are precisely the ones known from HFB-theory, we get:

$$\begin{aligned}\mathcal{H}_{CG} = \Big(V_{ext}-\mu\Big)\, n_0 + \frac{1}{2}U_0\, n_0^2 + V_G(0)\, n_0\,\big(n_0+n_T\big) &+ \sum_{k\neq 0}\bigg(-\frac{\hbar^2k^2}{2m} + V_{ext} - \mu\bigg)\, a_k^\dagger a_k \\ &+ \sum_{k\neq 0}\Big[V(0)\, n_0 + V(\mathbf{k})n_T + 2U_0\big(n_0+n_T\big)\Big]a_k^\dagger a_k \\ &+ \sum_{k\neq 0}\frac{1}{2}U_0\,\Big(m_k a_\mathbf{k}^\dagger a_{-\mathbf{k}}^\dagger + m_{-k}^* a_{-\mathbf{k}}a_\mathbf{k}\Big)\,,\end{aligned} \tag{182}$$

where we again denoted the zero-range scattering interaction potential as U_0 and the fundamentally classical gravitational potential as V_G, as above. Solving the Eigenvalue problem we find

$$\epsilon_{k,CG}^2 = \bigg(-\frac{\hbar^2k^2}{2m} + V_{ext} - \mu + V(0)\, n_0 + V(\mathbf{k})n_T + 2U_0\big(n_0+n_T\big)\bigg)^2 - \Big(m_k U_0\Big)^2\,, \tag{183}$$

and the ensemble average reads

$$\begin{aligned}\langle\mathcal{H}_{CG}\rangle &= \Big(V_{ext}-\mu\Big)\, n_0 + \frac{1}{2}U_0\, n_0^2 + V_G\Big|_{k=0} n_0\,\big(n_0+n_T\big) \\ &+ \sum_{k\neq 0}\left[\epsilon_k - \bigg(-\tfrac{\hbar^2k^2}{2m} + V_{ext} - \mu + V(0)\, n_0 + V(\mathbf{k})n_T + 2U_0\big(n_0+n_T\big)\bigg)\right] + \sum_{k\neq 0}\epsilon_k\alpha_k^\dagger\alpha_k\,.\end{aligned} \tag{184}$$

5 Thermodynamic Observables & Experimental Feasibility

Noticing that the ensemble average $\langle \mathcal{H} \rangle = \langle E \rangle$, and the relation $\langle \alpha_k^\dagger \alpha_k \rangle = \int dk\, 1/(exp(\beta\epsilon_k) - 1) =: f_k$, we can now easily compute the *specific heat capacity* c_V:

$$c_V = \frac{\partial}{\partial T} \langle E \rangle = \sum_{k \neq 0} \frac{\epsilon_k^2\, e^{-\beta\epsilon_k}}{k_B T^2\,(1 - e^{-\beta\epsilon_k})^2}\,, \tag{185}$$

as expected. From this, it is easy to see, that the only difference in the contributions coming from the gravitational interaction lies encoded in the ϵ_k, and hence a direct comparison between the two energies can help clear the picture. Let us write them down again, for clarity:

$$\begin{aligned} \epsilon_{k,QG}^2 &= \left(-\frac{\hbar^2 k^2}{2m} + V_{ext} - \mu + V(0)\, n_0 + V(\mathbf{k})\big(n_0 + 2n_T\big) + 2U_0\big(n_0 + n_T\big) \right)^2 - \Big(m_k \Big(U_0 + V_G(\mathbf{k}) \Big) \Big)^2 \\ \epsilon_{k,CG}^2 &= \left(-\frac{\hbar^2 k^2}{2m} + V_{ext} - \mu + V(0)\, n_0 + V(\mathbf{k}) n_T + 2U_0\big(n_0 + n_T\big) \right)^2 - \Big(m_k U_0 \Big)^2 . \end{aligned} \tag{186}$$

5.1 Magnitude of the normal and anomalous contributions in ϵ_k

Let us now turn our attention to contributions coming from the occupation numbers n_k, and the anomalous contributions from m_k. As we will see shortly, this is a key estimate with respect to the behaviour of the BEC as we approach T_c and hence the difference in *specific heat capacity* c_V for each of the two gravity models. For simplicity, we introduce the notation

$$\epsilon_k = \sqrt{\big(\mathcal{L} - \mu + I_n(\mathbf{k})\big)^2 - \big(I_m(\mathbf{k})\big)^2}\,, \tag{187}$$

where $\mathcal{L} = \frac{\hbar^2 k^2}{2m} + V_{ext}$ again, and the $I_{n(m)}(\mathbf{k})$ generalize the interactions. Assume equilibrium of the repulsive and the attractive interaction potentials, such that $|U_0| \simeq |V_G|$. Define:

$$\begin{aligned} \tilde{I}_n &:= U_0(n_0 + n_T) = V_G(n_0 + n_T) \\ \tilde{I}_m &:= U_0 m_T = V_G m_T \end{aligned} \tag{188}$$

while also tacitly assuming $|V(0)| \simeq |V(\mathbf{k})|$. Accordingly, we can denote (up to prefactors)

$$\begin{aligned} u_k^2 &= \tfrac{1}{2}\Big[\big(\mathcal{L} - \mu + I_n(\mathbf{k})\big)/\epsilon_k + 1 \Big] \\ v_k^2 &= \tfrac{1}{2}\Big[\big(\mathcal{L} - \mu + I_n(\mathbf{k})\big)/\epsilon_k - 1 \Big] \end{aligned} \tag{189}$$

in correspondence to equation (178). Note that $n_k, m_k \in \mathbb{R}$ but while $n_k > 0$, we have that $m_k < 0$. If one examines the asymptotic behaviour of the two contributions, as for instance in Davoudi et al. [28], one finds

$$\lim_{k \to 0} \big(n_k = m_k \big) = \big(2 f_{\mathbf{k}} + 1 \big) \lim_{k \to 0} \left[\frac{I_n(\mathbf{k})}{2\epsilon_k} \right], \tag{190}$$

where $f_{\mathbf{k}}$ corresponds to the expression given at the beginning of this section. Yukalov and Kleinert [29] make this even more explicit and demonstrate the behaviour of the two contributions in the long- and short-wavelength limit. Here, we will just quote their results, as

$$\begin{aligned} \big| m_k \big| &\simeq n_k \quad \text{for } k \to 0 \\ \big| m_k \big| &\gg n_k \quad \text{for } k \to \infty\,. \end{aligned} \tag{191}$$

Not only does this imply that the *anomalous contributions* m_k are always important, but it is also a key result for our analysis of the two ϵ_k's. If we look at equation (186) again, inserting the appropriate prefactors according to our definitions above, we can make the following estimation:

$$\begin{aligned} \epsilon_{k,QG}^2 &= \Big(\mathcal{L} - \mu + 4\tilde{I}_n(\mathbf{k}) \Big)^2 - \Big(2\tilde{I}_m(\mathbf{k}) \Big)^2 \\ \epsilon_{k,CG}^2 &= \Big(\mathcal{L} - \mu + 3\tilde{I}_n(\mathbf{k}) \Big)^2 - \Big(\tilde{I}_m(\mathbf{k}) \Big)^2, \end{aligned} \tag{192}$$

where we have taken into account that for $n \simeq 10^{21}$ particles $|U_0| \simeq |V_G|$, as shown above. Then, in the long-wavelength limit, where $\epsilon_k \ll T$, we can set $\tilde{I}_n(\mathbf{k}) \simeq \tilde{I}_m(\mathbf{k})$. If we neglect the trap potential, $V_{ext} = 0$, and assume $\mu \longrightarrow 0$ as $k \longrightarrow 0$, we can make a rough estimate of the ratio of the two energies as

$$\lim_{k\to 0} \left[\frac{\epsilon_{k,QG}}{\epsilon_{k,CG}}\right] \simeq \frac{3}{2} \,. \tag{193}$$

For the short-wavelength limit, where $\epsilon_k \gg T$ and hence $|m_k| \gg n_k$, we will neglect the terms only containing $\tilde{I}_n(\mathbf{k})$ to get

$$\begin{aligned}\epsilon^2_{k,QG} &= \left(\mathcal{L} - \mu\right)^2 - \left(2\tilde{I}_m(\mathbf{k})\right)^2 \\ \epsilon^2_{k,CG} &= \left(\mathcal{L} - \mu\right)^2 - \left(\tilde{I}_m(\mathbf{k})\right)^2 ,\end{aligned} \tag{194}$$

leading to

$$\lim_{k\to \infty} \left[\frac{\epsilon_{k,QG}}{\epsilon_{k,CG}}\right] < 1 \,, \quad \text{for } \tilde{I}_m < \left(\mathcal{L} - \mu\right) . \tag{195}$$

This behaviour will give a characteristic signature of the difference between the two pictures of *classical*- versus *quantized gravity* in the thermodynamic & large-volume limit.

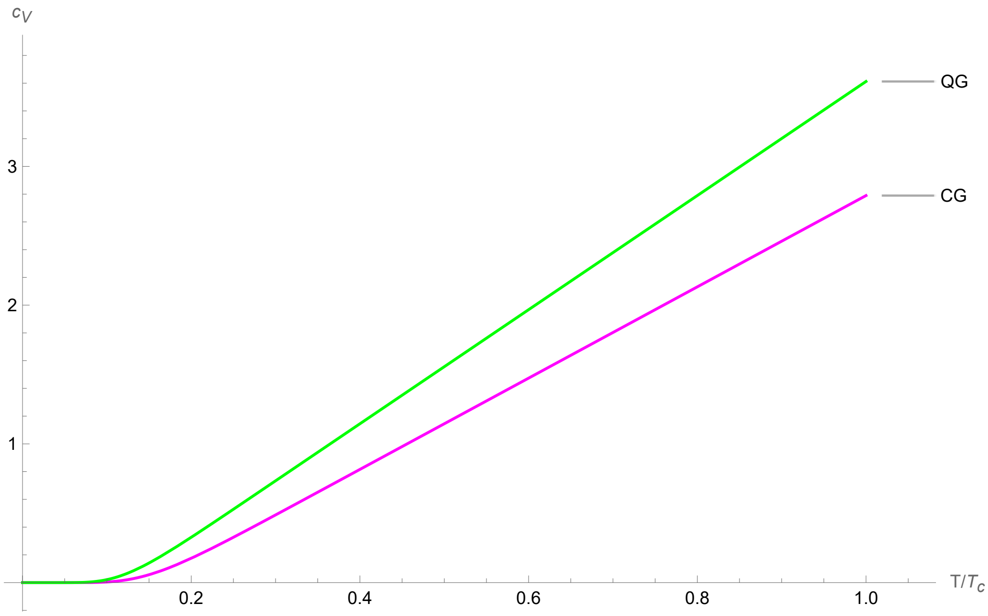

Figure 1: The evolution of the specific heat capacity c_V , with $\frac{\epsilon^2_{k,QG}}{\epsilon^2_{k,CG}} = 0.66$

5.2 Consideration of the trap & BEC-size

To complete our discussion of the thermodynamic observables and experimental feasibility, let's briefly consider the influence of a harmonic oscillator trap as has been proposed throughout this work. Let's fix the trap frequency $\omega_0 = 100Hz$. Starting from there, with T fixed at 10^{-9} Kelvin, we can compute that

$$\frac{\hbar\omega}{k_B T} \propto \frac{O(10^{-34})\, O(10^2)}{O(10^{-23})\, O(10^{-9})} \propto O(1) \,. \tag{196}$$

Note that $\hbar\omega/k_B T$ is dimensionless. Now we check for which particle number the self-gravitational interaction will become of the same order of magnitude. So we write, again in dimensionless form as mentioned already in section [2] in equation (26):

$$1 = \frac{Gm^2N^2}{Rk_BT} \;\Leftrightarrow\; N = \left(\frac{Rk_BT}{Gm^2}\right)^{\frac{1}{2}} \propto 10^{15} \,, \tag{197}$$

where we have assumed a volume of $1m^3$. This also gives us a first rough estimate regarding the size of the BEC which would be necessary to tune the repulsive and the attractive interactions to be of the same order of magnitude. We estimate $N > 10^{15}$ for the self-gravitational interaction to become visible clearly in the BEC and advise striving for even larger N condensates.

6 Summary & Outlook

In the present work, we have considered a self-gravitating Bose-Einstein condensate at finite temperatures, aiming to provide a theoretical foundation for future experiments measuring its thermodynamic properties to identify whether the fundamental nature of gravity shows a characteristic signature. To this end, we have derived a full Hartree-Fock-Bogoliubov theory for a gravity-like ranged interaction, considering both a quantized gravity picture, as well as a classical one. We discussed the specific heat capacity at constant volume, c_V, and provided estimates for the size where the effects of the self-gravitational interaction would become clearly visible.

Our calculations show, that such experiments are indeed possible, and the signature of quantum- vs. classical-gravity should be discernible. With improvements being made regarding the size of BEC-experiments, we are hopeful our projections will be within reach in the very near future. We are positive, that precision measurements of thermodynamic properties of a self-gravitating BEC can provide a new angle to resolve the long-standing issue of whether gravity shows signs of ”quantum-ness” or is a fundamentally classical type of interaction.

The foundations laid out in this work provide solid ground to spawn future considerations. We would like to note, that the Hartree-Fock-Bogoliubov theory derived herein still suffers from the problem of showing a gap in the energy spectrum, in direct contradiction to the Hugenholtz-Pines-theorem [30], which is known to be rigorously true. This can be amended though, improving the current theory along the lines suggested for example by Yukalov & Kleinert [29]. Once a gapless-spectrum-theory is derived, explicit calculations of the shift in critical temperature T_c, with respect to the classical- and quantized-gravity picture should help to further clarify such precisely measurable thermodynamic observables.

On the experimental side, while such measurements can surely be carried out in a ground-based lab, it makes sense to consider conducting those tests as far away from Earth’s gravity well as possible, to ensure minimum interference with the gravitational field of our planet. Indeed, Bose-Einstein condensation has been observed successfully aboard the ISS [31]. This, together with ongoing technological improvements regarding the precision of measurements, leaves us hopeful of being able to carry out the suggested experiments in the near future.

References

[1] Paul Adrien Maurice Dirac. *The Principles of Quantum Mechanics*. International series of monographs on physics. Clarendon Press, 1981. ISBN: 9780198520115.

[2] Albert Einstein. “Zur Elektrodynamik bewegter Körper. (German) [On the electrodynamics of moving bodies]”. In: *Annalen der Physik* 322.10 (1905), pp. 891–921. DOI: `http://dx.doi.org/10.1002/andp.19053221004`.

[3] Carlo Rovelli. “Notes for a brief history of quantum gravity”. In: *arXiv e-prints*, gr-qc/0006061 (June 2000), gr–qc/0006061. arXiv: `gr-qc/0006061 [gr-qc]`.

[4] *Approaches to Quantum Gravity: Toward a New Understanding of Space, Time and Matter*. Cambridge University Press, 2009. DOI: `10.1017/CBO9780511575549`.

[5] S. Carlip. “Is quantum gravity necessary?” In: *Classical and Quantum Gravity* 25.15, 154010 (Aug. 2008), p. 154010. DOI: `10.1088/0264-9381/25/15/154010`. arXiv: `0803.3456 [gr-qc]`.

[6] R.J. Gleiser, C.N. Kozameh, and O.M. Moreschi. *General Relativity and Gravitation 1992, Proceedings of the Thirteenth INT Conference on General Relativity and Gravitation, held at Cordoba, Argentina, 28 June - July 4 1992*. Taylor & Francis, 1993. ISBN: 9780750302616. URL: `https://books.google.at/books?id=FFeyTEU10BAC`.

[7] Richard Howl, Roger Penrose, and Ivette Fuentes. “Exploring the unification of quantum theory and general relativity with a Bose-Einstein condensate”. In: *New Journal of Physics* 21.4, 043047 (Apr. 2019), p. 043047. DOI: `10.1088/1367-2630/ab104a`. arXiv: `1812.04630 [quant-ph]`.

[8] Richard Howl et al. “Non-Gaussianity as a Signature of a Quantum Theory of Gravity”. In: *PRX Quantum* 2 (1 Feb. 2021), p. 010325. DOI: `10.1103/PRXQuantum.2.010325`. URL: `https://link.aps.org/doi/10.1103/PRXQuantum.2.010325`.

[9] Mohammad Bahrami et al. “The Schrödinger-Newton equation and its foundations”. In: *New Journal of Physics* 16.11, 115007 (Nov. 2014), p. 115007. DOI: `10.1088/1367-2630/16/11/115007`. arXiv: `1407.4370 [quant-ph]`.

[10] M. H. Anderson et al. “Observation of Bose-Einstein Condensation in a Dilute Atomic Vapor”. In: *Science* 269.5221 (1995), pp. 198–201. DOI: `10.1126/science.269.5221.198`. eprint: `https://www.science.org/doi/pdf/10.1126/science.269.5221.198`. URL: `https://www.science.org/doi/abs/10.1126/science.269.5221.198`.

[11] K. B. Davis et al. “Bose-Einstein Condensation in a Gas of Sodium Atoms”. In: *Phys. Rev. Lett.* 75 (22 Nov. 1995), pp. 3969–3973. DOI: `10.1103/PhysRevLett.75.3969`. URL: `https://link.aps.org/doi/10.1103/PhysRevLett.75.3969`.

[12] H. J. de Vega and N. Sanchez. “Statistical Mechanics of the self-gravitating gas: thermodynamic limit, phase diagrams and fractal structures”. In: *arXiv e-prints*, astro-ph/0505561 (May 2005), astro–ph/0505561. arXiv: `astro-ph/0505561 [astro-ph]`.

[13] C. J. Pethick and H. Smith. *Bose–Einstein Condensation in Dilute Gases*. 2nd ed. Cambridge University Press, 2008. DOI: `10.1017/CBO9780511802850`.

[14] NN Bogolyubov. “SSSR, Ser. Fiz. 11, 77 (1947) NN Bogolyubov”. In: *J. Phys. USSR* 11 (1947), p. 23.

[15] Crispin W. Gardiner and Peter Zoller. “The Quantum World of Ultra-Cold Atoms and Light Book 1: Foundations of Quantum Optics”. In: 2014.

[16] Michael El-Batanouny. *Advanced Quantum Condensed Matter Physics: One-Body, Many-Body, and Topological Perspectives*. Cambridge University Press, 2020. DOI: `10.1017/9781108691291`.

[17] T. D. Lee and C. N. Yang. “Many-Body Problem in Quantum Mechanics and Quantum Statistical Mechanics”. In: *Phys. Rev.* 105 (3 Feb. 1957), pp. 1119–1120. DOI: `10.1103/PhysRev.105.1119`. URL: `https://link.aps.org/doi/10.1103/PhysRev.105.1119`.

[18] Nick P. Proukakis and Brian Jackson. “PHD TUTORIAL: Finite-temperature models of Bose Einstein condensation”. In: *Journal of Physics B Atomic Molecular Physics* 41.20, 203002 (Oct. 2008), p. 203002. DOI: `10.1088/0953-4075/41/20/203002`. arXiv: `0810.0210 [cond-mat.other]`.

[19] A. Griffin. “Conserving and gapless approximations for an inhomogeneous Bose gas at finite temperatures”. In: *Phys. Rev. B* 53 (14 Apr. 1996), pp. 9341–9347. DOI: `10.1103/PhysRevB.53.9341`. URL: `https://link.aps.org/doi/10.1103/PhysRevB.53.9341`.

[20] Jens O. Andersen. “Theory of the weakly interacting Bose gas”. In: *Rev. Mod. Phys.* 76 (2 July 2004), pp. 599–639. DOI: 10.1103/RevModPhys.76.599. URL: https://link.aps.org/doi/10.1103/RevModPhys.76.599.

[21] Allan Griffin, Tetsuro Nikuni, and Eugene Zaremba. *Bose-Condensed Gases at Finite Temperatures*. Cambridge University Press, 2009. DOI: 10.1017/CBO9780511575150.

[22] S. Giorgini, L. P. Pitaevskii, and S. Stringari. “Thermodynamics of a trapped Bose-condensed gas”. In: *Journal of Low Temperature Physics* 109.1-2 (Oct. 1997), pp. 309–355. DOI: 10.1007/BF02396737. arXiv: cond-mat/9704014 [cond-mat].

[23] Richard Howl et al. “Non-Gaussianity as a signature of a quantum theory of gravity”. In: *PRX Quantum* 2 (2021), p. 010325. DOI: 10.1103/PRXQuantum.2.010325. arXiv: 2004.01189 [quant-ph].

[24] Crispin W. Gardiner and Peter Zoller. “The Quantum World of Ultra-Cold Atoms and Light Book III: Ultra-cold Atoms”. In: 2017.

[25] L. Diósi. “Gravitation and quantum-mechanical localization of macro-objects”. In: *Physics Letters A* 105.4-5 (Oct. 1984), pp. 199–202. DOI: 10.1016/0375-9601(84)90397-9. arXiv: 1412.0201 [quant-ph].

[26] Robert J. Adler and Jonathan E. Taylor. “Random Fields and Geometry”. In: 2007.

[27] Oliver Penrose and Lars Onsager. “Bose-Einstein Condensation and Liquid Helium”. In: *Phys. Rev.* 104 (3 Nov. 1956), pp. 576–584. DOI: 10.1103/PhysRev.104.576. URL: https://link.aps.org/doi/10.1103/PhysRev.104.576.

[28] B. Davoudi, A. Minguzzi, and M. P. Tosi. “Hartree-Fock-Bogoliubov theory of a charged Bose gas at finite temperature”. In: *prb* 65.14, 144507 (Apr. 2002), p. 144507. DOI: 10.1103/PhysRevB.65.144507. arXiv: cond-mat/0110175 [cond-mat.stat-mech].

[29] V. I. Yukalov and H. Kleinert. “Gapless Hartree-Fock-Bogoliubov approximation for Bose gases”. In: *pra* 73.6, 063612 (June 2006), p. 063612. DOI: 10.1103/PhysRevA.73.063612. arXiv: cond-mat/0606484 [cond-mat.stat-mech].

[30] N. M. Hugenholtz and D. Pines. “Ground-State Energy and Excitation Spectrum of a System of Interacting Bosons”. In: *Phys. Rev.* 116 (3 Nov. 1959), pp. 489–506. DOI: 10.1103/PhysRev.116.489. URL: https://link.aps.org/doi/10.1103/PhysRev.116.489.

[31] David C. Aveline et al. “Observation of Bose-Einstein condensates in an Earth-orbiting research lab”. In: *nat* 582.7811 (June 2020), pp. 193–197. DOI: 10.1038/s41586-020-2346-1.