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Abstract

Biological membranes naturally occur in a fascinating variety of shapes. The most prominent example is the biconcave equilibrium configuration of red blood cells. In a continuum mechanical approach, biomembranes can be described as surfaces that minimize the Canham-Helfrich elastic bending energy. This quadratic curvature functional generalizes the Willmore energy by taking different bending rigidities and the presence of a spontaneous curvature parameter into account. Minimization has to be performed under constraints on membrane area and enclosed volume.

This thesis focuses on two different aspects of variational models involving the Canham-Helfrich energy. First, we study heterogeneous biomembranes, where we are interested in the influence of the varying material parameters on the existence of minimizers and their properties. We present results for a one-dimensional Canham-Helfrich model, describing a planar elastic curve whose stiffness parameter is modulated by an additional phase density. Moreover, we consider multiphase biomembranes with sharp phase-interfaces, where we prove existence of minimizers within the geometric measure theoretical framework of curvature varifolds. The second focus of this thesis is to analyze the Canham-Helfrich gradient flow, which governs the time-evolution of biomembrane configurations towards equilibrium. Based on our varifold setting, we establish existence of global solutions via the Generalized Minimizing Movements method and also discuss the evolution of more regular surfaces. In both aspects addressed in this thesis, the material parameter bounds we require for existence of solutions are consistent with the available experimental data on biological membranes.

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Besides my thesis supervisors, I am grateful to all my additional co-authors in publications: I thank Gaspard Jankowiak for sharing his knowledge in mathematical biology and numerics, Luca Lussardi for introducing me to geometric measure theory, and Martin Kružík for his expertise in gradient flows and continuum mechanics. I greatly enjoyed our joint research work. My thanks also go to all my past and present colleagues from Ulisse Stefanelli's research group *Applied Mathematics and Modeling*, especially to my office-mates. I benefited a lot from their community, creating a friendly learning environment. In particular, I am indebted to Matteo Tommasini, not only for his support concerning any administrative or computer issues, but also for helping me better understand Grassmannians from the viewpoint of algebraic geometry. I also thank my mentor Michael Kunzinger for his support within the framework of the Vienna School of Mathematics (VSM). Of course, my thesis research would not have been possible without financial resources and I am grateful to all project leaders for their support. Specifically, I acknowledge the funding received from the Austrian Science Fund (FWF) through the SFB *Taming complexity in partial differential systems* lead by Ulisse Stefanelli and the Doctoral program DK *Dissipation and dispersion in nonlinear partial differential equations* lead by Ansgar Jüngel. During my PhD, I also got funded by the DFG-FWF international joint project *Variational Modeling of Molecular Geometries* lead by Manuel Friedrich and Ulisse Stefanelli. My travels to Prague to collaborate with Martin Kružík were reimbursed through two WTZ programs of the Agency for Education and Internationalization (OeAD) that were coordinated by Elisa Davoli and Ulisse Stefanelli.

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Chapter 1

Introduction

1.1 Motivation

Biological membranes (biomembranes) are single or double layers of lipids. In addition, a variety of different molecules can also be included in the membranes. The most important examples of biomembranes are cell membranes, which consist of a double layer of phospholipids with embedded cholesterol and proteins. This thesis is motivated by the fascinating variety of biomembrane shapes, a paramount example being the biconcave discocytes of human red blood cells.

The equilibrium configurations of homogeneous biological membranes can be modeled as surfaces M in $\mathbb{R}^3$ that minimize the **Canham-Helfrich energy**

$$E_{\text{CH}}(M) := \int_M \left(\frac{\beta}{2} (H - H_0)^2 + \gamma K \right) d\mathcal{H}^2 \quad (1.1)$$

under constraints on area and enclosed volume. Here, H and K denote the mean curvature and the Gauss curvature of M respectively. The material parameters are the bending rigidities $\beta > 0$ and $\gamma < 0$, and the spontaneous curvature parameter $H_0 \in \mathbb{R}$.

An even richer morphology can be observed in case of **heterogeneous membranes**, that is, biomembranes with spatially varying composition due to the presence of different lipids or other membrane molecules. For these membranes, a coupling effect between membrane geometry and heterogeneity arises. Within the Canham-Helfrich variational model, to this date, the influence of heterogeneity has been mainly studied analytically in the rotationally symmetric regime or for configurations close to a sphere.

We are also interested in the evolution of biomembranes towards equilibrium. In a viscous environment, this may be modeled by the **gradient flow dynamics** associated to the Canham-Helfrich energy. The available analytic results for this evolution problem usually break down at singularities including topology changes and therefore are not able to cover important biomembrane morphology changes like cell division.

1.2 Research questions

In this thesis, we analyze the extension of the Canham-Helfrich model to heterogeneous biomembranes, as well as the Canham-Helfrich gradient flow. Both problems are addressed in a low-regularity regime, by employing techniques from geometric measure theory.

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The classical Canham-Helfrich energy (1.1) with constant material parameters cannot capture the rich geometry of heterogeneous biomembranes, as for example the two-phase configuration displayed in Figure 2.7. Spatial heterogeneities influence the shape by locally changing the parameters β , γ , and H_0 . Thereby, the composition of the membrane will affect its geometry. Moreover, since lipid membranes behave as two-dimensional fluids, the constituents are not fixed within the membrane but can migrate due to surface diffusion and electrochemical dynamics. The result is an interplay between molecule concentrations, influencing the local membrane shape, and geometry, which influences the concentrations of membrane molecules. Due to this coupling effect between geometry and composition in fluid membranes, the energy has to be optimized simultaneously with respect to shape and composition.

In a continuum approach, the composition of heterogeneous membranes can be described by a locally integrable density

$$\rho: M \rightarrow \mathbb{R}.$$

For example, ρ can represent the fraction of a certain type of lipid or protein relative to another type. The presence of such multiple phases of the membrane material gives rise to an additional energy contribution related to the phases, so that the Canham-Helfrich energy for multiphase membranes takes the form

$$E(\rho, M) := \int_M \left(\frac{\beta(\rho)}{2} (H - H_0(\rho))^2 + \gamma(\rho) K \right) d\mathcal{H}^2 + E_{\text{phase}}(\rho, M). \quad (1.2)$$

This energy has to be minimized with respect to the shape M and the density ρ , under constraints on area, enclosed volume, and phase mass.

Our research questions are as follows:

- **Existence:** For which values of the model parameters do minimizers exist?
- **Solution properties:** Are the minimizers nontrivial, i.e., different from a sphere with constant density? What is the regularity of minimizers?

The gradient flow dynamics describes the steepest energy-descent evolution towards equilibrium. In case of the Canham-Helfrich energy, the gradient flow equation is *formally* a partial differential equation (PDE) of the form

$$v = -\nabla E_{\text{CH}}(M), \quad (1.3)$$

where v denotes the velocity field of the time-dependent surface M and the derivative operator ∇ has to be suitably defined.

In this regard, our main question concerns:

- **Evolution:** Define a solution concept for the evolution towards equilibrium that allows for low regularity and nontrivial topologies.

These above research questions are answered by the main results of the thesis, which are collected in three papers. Specifically, the publications [20] and [22] address the minimization of the multiphase Canham-Helfrich energy (1.2), whereas [21] introduces a low regularity solution concept for the gradient flow evolution (1.3) and discusses its properties.

1.3 Outline and main results

In this section, we describe the outline of the thesis and summarize the main results. In the remaining Section 1.4 of **Chapter 1** we list some relevant notation.

In the preliminary **Chapter 2**, we collect background information on biological membranes and introduce the Canham-Helfrich model in more detail (Section 2.1). Then, we present the relevant tools from calculus of variations and geometric measure theory in application to geometric energy functionals involving curvature (Section 2.2).

In **Chapter 3**, we study a variational model for elastic curves with a density-dependent stiffness parameter, which is inspired by the Canham-Helfrich model for heterogeneous biomembranes. We prove existence of minimizers and analyze the bifurcation structure of solutions. This chapter consists of my publication [20] in collaboration with GASPARD JANKOWIAK, CHRISTIAN SCHMEISER, and ULISSE STEFANELLI, which appeared in the *European Journal of Applied Mathematics*. The main results are summarized in Subsection 1.3.1.

In **Chapter 4**, we establish existence of minimizers for the Canham-Helfrich energy of multiphase biomembranes with sharp phase-interfaces. We work in the framework of oriented curvature varifolds, which allows for minimizers without symmetry restrictions and of general topology. This chapter consists of my paper [22] in collaboration with LUCA LUSSARDI and ULISSE STEFANELLI, published in the journal *Calculus of Variations and Partial Differential Equations*. A synopsis is given in Subsection 1.3.2.

In **Chapter 5**, we address the gradient flow evolution problem for the Canham-Helfrich energy in the setting of varifolds. This chapter consists of my submitted work [21] in collaboration with MARTIN KRUŽÍK and ULISSE STEFANELLI, which is available on *ArXiv*. We describe the main findings in Subsection 1.3.3.

Finally, we list the **References** and provide the German abstract (**Zusammenfassung**).

1.3.1 Chapter 3: Elastic curves

In [20] (Chapter 3), we discuss a Canham-Helfrich type model for elastic curves with density, which are a one-dimensional analog of heterogeneous biomembranes.

We consider closed planar curves γ , with the parametrization

$$\gamma: [0, L] \rightarrow \mathbb{R}^2$$

for $L > 0$. The curves have fixed length, which reflects the area constraint for vesicles. In order to describe heterogeneity, we introduce a scalar density ρ defined on γ with given total mass $\int_\gamma \rho = M > 0$. Taking advantage of the arclength parametrization allows us to model the density as a function on the parameter interval as well, i.e.,

$$\rho: [0, L] \rightarrow \mathbb{R}.$$

In this setting, the multiphase Canham-Helfrich energy (1.2) reduces to

$$E(\rho, \gamma) = \frac{1}{2} \int_0^L \beta(\rho(s)) (\kappa(s) - \kappa_0(\rho(s)))^2 ds + E_{\text{phase}}(\rho, \gamma).$$

Here, κ is the curvature of γ , which in arclength parametrization is given by $\kappa = \dot{\gamma}^\perp \cdot \ddot{\gamma} = \dot{\theta}$ with the inclination angle θ . The functions $\beta: \mathbb{R} \rightarrow (0, \infty)$ and $\kappa_0: \mathbb{R} \rightarrow \mathbb{R}$ describe the density-dependent bending rigidity (stiffness) and spontaneous curvature parameter respectively. We

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focus on the density-modulating effect of β , which we assume to be smooth and bounded away from zero, see (3.3). Adopting a phase energy E_{phase} of the form $\frac{\mu}{2} \int_0^L \dot{\rho}^2(s) \, ds$ with a coupling constant $\mu \in \mathbb{R}$ leads to the energy (3.1), namely,

$$E_\mu(\rho, \gamma) = \frac{1}{2} \int_0^L (\beta(\rho(s)) \kappa^2(s) + \mu \dot{\rho}^2(s)) \, ds.$$

We investigate the minimization problem for E_μ in a Sobolev space setting, i.e.,

$$\min_{(\rho, \gamma) \in \mathcal{R} \times \mathcal{A}} E_\mu(\rho, \gamma),$$

cf. (3.8), with admissibility classes $\mathcal{R} \subset H^1(0, L)$ if $\mu \neq 0$ (or $\mathcal{R} \subset L^1(0, L)$ if $\mu = 0$) and $\mathcal{A} \subset H^2((0, L); \mathbb{R}^2)$. The precise definitions of $\mathcal{R}$ and $\mathcal{A}$ are given in (3.6) and (3.4) respectively, where in view of the closedness of the curves, $(0, L)$ is replaced by the one-dimensional torus, see Section 3.2. Our results include the

- Existence of nontrivial minimizers for $\mu > 0$,
- Bifurcation structure of solutions as $\mu \rightarrow 0$,

which we explain in more detail in the following paragraphs. Before that, we briefly comment on the references (see also Section 3.1). Classical elastic curves without density are a well-established field in differential geometry, see for example [64, 99]. Yet, results for heterogeneous elastic curves are rather scarce. The only previous work seems to be [51], where the sharp-interface limit of heterogeneous planar elastic curves has been studied, focusing on the effect of spontaneous curvature. Anisotropic energies for planar curves have been recently considered in [19, 82].

After online-publication of our article [20], the gradient flow dynamics for elastic curves with density, additionally including a variable spontaneous curvature, has been analyzed in [31].

Nontrivial minimizers

In Section 3.3, we argue that our model is able to capture configurations that are one-dimensional analogs to the nontrivial shapes observed for heterogeneous biomembranes, for example in [12] (see Figure 2.7). For this purpose, we need to ensure that our variational model allows for nontrivial minimizers. This turns out to be indeed true, provided that $\mu > 0$ is not too large.

More specifically, let us start by remarking that in case of $\mu \leq 0$, the minimization problem is either trivial or unsolvable. In fact, if $\mu < 0$, then E_μ is not bounded from below and thus can not be minimized. In case of $\mu = 0$, we prove that if the integrand

$$(\rho, \kappa) \mapsto \Phi(\rho, \kappa) = \frac{1}{2} \beta(\rho) \kappa^2$$

of E_0 is strictly convex, then the only solution of $\min_{(\rho, \gamma) \in \mathcal{R} \times \mathcal{A}} E_0(\rho, \gamma)$ is the circle γ_0 with curvature

$$\kappa_0 = 2\pi/L$$

and with constant density

$$\rho_0 = M/L,$$

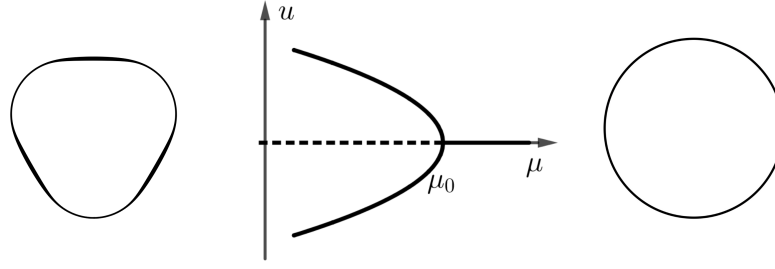


Figure 1.1: Illustration of the bifurcation structure of solutions $u = (\rho, \gamma)$ of the form of a supercritical pitchfork bifurcation. As the bifurcation parameter $\mu > 0$ approaches zero and the critical value μ_0 is passed, other solutions (like the curve γ with variable density ρ , whose value is represented by the line thickness; left) emerge in addition to the trivial solution (the circle γ_0 with constant density ρ_0 ; right). The trivial solution then gets unstable (dashed line), whereas the new solution branches are stable (solid line).

which is uniquely determined up to rigid motions. Therefore, nontrivial minimizers of E_0 do not occur in this regime. However, if the integrand is not convex, then E_0 may not have minimizers in $\mathcal{R} \times \mathcal{A}$ at all (Proposition 3.3.1). Thus, the minimization problem for E_μ is of limited interest if $\mu \leq 0$. Henceforth, assuming $\mu > 0$, we establish existence of minimizers of E_μ in $\mathcal{R} \times \mathcal{A}$ via the Direct Method (Proposition 3.3.2). Since the additional phase energy $\frac{\mu}{2} \int_0^L \dot{\rho}^2(s) ds$ is nonnegative and zero at ρ_0 , which is the only admissible constant density-function, we also know that the minimizer of E_μ will be trivial whenever Φ is convex. We finally prove that if μ exceeds a certain threshold $\mu_0 > 0$, then the only minimizer is again the trivial one (Proposition 3.3.3).

Bifurcations

We study the nontrivial minimizers by investigating the bifurcation structure of solutions to the Euler-Lagrange equations associated to E_μ with $\mu > 0$. As μ approaches zero, new solutions $u = (\rho, \gamma)$ bifurcate from the trivial one $u_0 = (\rho_0, \gamma_0)$ in the form of a pitchfork bifurcation as is illustrated in Figure 1.1. We provide analytical results (Section 3.4) which are confirmed by numerical simulations (Section 3.5).

Stationarity of the Lagrangian associated to E_μ leads to the Euler-Lagrange equations (3.18), (3.19), namely,

$$\begin{aligned} \mu \ddot{\rho}(s) - \frac{1}{2} \beta'(\rho(s)) \dot{\theta}^2(s) &= \lambda_M, \\ \frac{d}{ds} \left(\beta(\rho(s)) \dot{\theta}(s) \right) &= -\lambda_x \sin \theta(s) + \lambda_y \cos \theta(s), \end{aligned}$$

which are a system of second-order nonlinear ordinary differential equations for the density ρ and the inclination angle θ , representing the curve γ . The Lagrange multipliers λ_M and λ_x, λ_y correspond to the mass and closedness constraints (3.21). The system is completed by the boundary conditions (3.20) which ensure periodicity of solutions and eliminate its invariances with respect to rotations in $\mathbb{R}^2$ as well as shifts in the argument s , cf. (3.22).

Our goal is to determine for which specific form of the stiffness coefficient β nontrivial solutions emerge as the bifurcation parameter μ approaches zero. To address this question, we linearize the system of Euler-Lagrange equations around its trivial solution. We then find that three different cases of nontrivial solutions can occur (Proposition 3.4.1). In particular, we identify

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the critical values

$$\mu = \mu_j(m, h) \quad (j \in \mathbb{N})$$

corresponding to each case, which only depend on the first- and second-order Taylor expansion coefficients of the stiffness,

$$\beta(\rho) \approx \rho_0 + m(\rho - \rho_0) + \frac{h}{2}(\rho - \rho_0)^2.$$

The conditions for the existence of nontrivial solutions are in accordance with the requirement of nonconvexity of Φ that we identified in Section 3.3. Moreover, we determine the local structure of the bifurcations from the solvability of the second- and third-order perturbations of the system of Euler-Lagrange equations (Subsection 3.4.3), which are confirmed by energy considerations (Subsection 3.4.4). We find that the bifurcations are locally of pitchfork type and identify parameter regimes, where the nontrivial solutions are stable and decrease energy (supercritical pitchfork, cf. Figure 1.1) or are unstable and increase energy (subcritical pitchfork).

The precise values of the bifurcation parameters μ_j as well as the stability regions are validated by numerically solving the nonlinear Euler-Lagrange system (Section 3.5). However, the simulations also reveal a much richer global bifurcation structure, which is presently not captured by our local analytical results. Finally, we provide illustrations of some configurations of curves with density-modulated stiffness in the different parameter regimes (Figure 3.6).

1.3.2 Chapter 4: Varifold minimizers

In [22] (Chapter 4), we establish the existence of minimizers in the weak setting of varifolds of the Canham-Helfrich model for multiphase biomembranes with sharp phase-interfaces.

A multiphase vesicle with $N \in \mathbb{N}$, $N \geq 2$ different phases that are separated by sharp interfaces can be described as a closed (i.e., compact without boundary) orientable surface $M \subset \mathbb{R}^3$, which is the union

$$M = \bigcup_{i=1}^N M_i$$

of orientable surfaces $M_1, \dots, M_N$. These represent the individual phases of the membrane configuration. In case the phases touch, we require that they only intersect at their boundaries, so that $M_i \cap M_j = \partial M_i \cap \partial M_j$ for $1 \leq i \neq j \leq N$. In particular, we allow for phases without boundary, $\partial M_i = \emptyset$, which are disconnected from the other phases. The common phase boundary is given by

$$\Gamma = \bigcup_{i=1}^N \partial M_i,$$

which is a curve on M . Moreover, since M is closed, the surfaces M_i must be such that their union has no boundary,

$$\partial M = \emptyset.$$

The sharp-interface, multiphase Canham-Helfrich functional

$$E_{\text{CH},N}(M_1, \dots, M_N) = \sum_{i=1}^N E_{\text{CH}}^i(M_i) + \sigma \mathcal{H}^1(\Gamma)$$

consists of the sum of all individual Canham-Helfrich contributions (1.1), namely,

$$E_{\text{CH}}^i(M_i) = \int_{M_i} \left(\frac{\beta^i}{2} (H - H_0^i)^2 + \gamma^i K \right) d\mathcal{H}^2,$$

and the line tension energy $\sigma \mathcal{H}^1(\Gamma) = \sigma \int_{\Gamma} d\mathcal{H}^1$ on phase boundaries Γ (cf. (4.3) in the case $N = 2$). Here, β^i, γ^i , and H_0^i are the material parameters of the phase M_i and $\sigma > 0$ denotes the line tension coefficient. The energy $E_{\text{CH},N}$ can be considered as a special case of the multiphase energy (1.2) for a piecewise constant phase density $\rho: M \rightarrow \mathbb{R}$ and a phase energy E_{phase} given by the line tension energy along the interfaces.

We address the minimization of $E_{\text{CH},N}$ restricted to multiphase configurations $(M_1, \dots, M_N)$ satisfying enclosed-volume and phase-area constraints. For $N = 1$, this reduces to the Canham-Helfrich problem for homogeneous membranes. The latter has been solved with the Direct Method for axisymmetric surfaces with genus zero or one in [27], as well as for $C^{1,1}$ regular surfaces with uniform curvature bounds in [34]. Existence of homogeneous Canham-Helfrich minimizers for genus-zero surfaces without symmetry restriction has been recently established in [41, 78]. However, for the two- or multiphase Canham-Helfrich problem, i.e., for $N \geq 2$, analytic results previously have only been obtained in the axisymmetric regime with fixed genus [26, 52, 53]. A more detailed review of the literature is given in Section 4.1.

After this work was published in [22], the sharp-interface limit of multiphase membranes of almost spherical shape has been studied in [42].

In contrast to these existence results we require neither symmetry nor connectedness, and we do not fix the genus of the membrane. In particular, our model includes configurations arising from budding transitions (see Figure 2.5) as well as other equilibria of nonspherical topology, which are observed in experiments. To allow for such generality of minimizers, we consider the variational problem in the setting of varifolds. Varifolds are specific Radon measures that extend the class of differentiable submanifolds of $\mathbb{R}^n$ to a space with favorable compactness properties with respect to the weak-* convergence of measures. They provide a suitable framework to tackle geometric variational problems with the Direct Method.

Our main results are:

- Varifold formulation of the Canham-Helfrich model,
- Existence of multiphase minimizers,
- Solution regularity and diameter bounds.

Varifold formulation

Our first task is to generalize the problem to the varifold setting. The Canham-Helfrich energy (1.1) of a surface M involves its curvature and, through the spontaneous curvature parameter H_0 , also depends on its orientation. Moreover, the line tension energy is proportional to the length of the boundaries of the individual phases M_i as well. For these reasons we need to consider a specific class of varifolds, which encompasses notions of curvature, orientation, and boundary at the same time. Working in a fixed open set $\Omega \subset \mathbb{R}^3$, we introduce in (4.9) the class of oriented curvature varifolds with boundary,

$$AV_2^o(\Omega) = \{V \in IV_2^o(\Omega) : q_{\#}V \in AV_2(\Omega)\}.$$

This definition combines Hutchinson's oriented varifolds $IV_2^o(\Omega) \subset \mathcal{M}(\Omega \times G_{2,3}^o)$ with Mantegazza's curvature varifolds with boundary, $AV_2(\Omega) \subset \mathcal{M}(\Omega \times G_{2,3})$ [54, 73]. The varifold $q_{\#}V$ denotes the push-forward of V by the covering map q between oriented and unoriented Grassmannians, $G_{2,3}^o$ and $G_{2,3}$. The map is given by

$$q: \Omega \times G_{2,3}^o \rightarrow \Omega \times G_{2,3}, \quad q(x, \xi) = (x, P),$$

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where $P = (P_{ij})_{i,j=1}^3 \in G_{2,3}$ with components $P_{ij} = \delta_{ij} - \nu_i \nu_j$ is the tangential projection to the plane through the origin in $\mathbb{R}^3$ that has the unit normal $\nu = *\xi \in \mathbb{S}^2$ corresponding to the orientation two-vector $\xi \in G_{2,3}^o$. In the following, we will also write $q(\xi) = P$ for notational simplicity. In the spirit of Hutchinson's measure-function pairs, we introduce the (weak-*) convergence of oriented curvature varifolds,

$$V_n \rightharpoonup^* V \quad (n \rightarrow \infty) \quad \text{in} \quad AV_2^o(\Omega),$$

as the (weak-*) convergences $V_n \rightharpoonup^* V$ in $\mathcal{M}(\Omega \times G_{2,3}^o)$ and $A^{q_\# V_n}(q_\# V_n) \rightharpoonup^* A^{q_\# V}(q_\# V)$ in $\mathcal{M}^3(\Omega \times G_{2,3})$. Here,

$$A^{q_\# V} = (A_{ijk}^{q_\# V})_{i,j,k=1}^3 \in L_{\text{loc}, q_\# V}^1(\Omega \times G_{2,3}; \mathbb{R}^{3 \times 3 \times 3})$$

denote Mantegazza's curvature functions of $q_\# V$.

Representing the membrane M by the oriented curvature varifold $V \in AV_2^o(\Omega)$ and expressing the classical Canham-Helfrich energy (1.1) in terms of the curvature functions leads us to the generalized Canham-Helfrich functional (4.11), namely,

$$F_{\text{CH}}: AV_2^o(\Omega) \rightarrow \mathbb{R}, \quad F_{\text{CH}}(V) = \int_{\Omega \times G_{2,3}^o} f_{\text{CH}}(\xi, A^{q_\# V}(x, q(\xi))) \, dV(x, \xi),$$

with $f_{\text{CH}}: G_{2,3}^o \times \mathbb{R}^{3 \times 3 \times 3} \rightarrow \mathbb{R}$ given by

$$f_{\text{CH}}(\xi, A) = \sum_{i=1}^3 \left(\frac{\beta}{2} \left(\sum_{j=1}^3 A_{jij} - (*\xi)_i H_0 \right)^2 + \frac{\gamma}{2} \left(\sum_{j=1}^3 A_{jij} \right)^2 - \frac{\gamma}{4} \sum_{j,k=1}^3 A_{ijk}^2 \right).$$

The configuration of a multiphase membrane with N sharply separated phases is described by the list of oriented curvature varifolds $(V^1, \dots, V^N) \in AV_2^o(\Omega)^N$. The corresponding multiphase energy reads

$$F_{\text{CH},N}(V^1, \dots, V^N) = \sum_{i=1}^N \left(F_{\text{CH}}^i(V^i) + \sigma^i \|\partial(q_\# V^i)\|_{\text{TV}}(\Omega \times G_{2,3}) \right),$$

where, analogously to the classical setting, F_{CH}^i denotes the energy with material parameters β^i, γ^i, H_0^i and $\sigma^i > 0$ are the line tension coefficients of the phase boundaries ($i = 1, \dots, N$). This extends the classical two-phase model which can be recovered via a single line tension coefficient.

Minimization has to be performed under constraints on phase mass (area), enclosed volume, and membrane geometry, which have to be translated to the varifold setting. The requirement of fixed membrane area is replaced by fixing the total mass

$$\mu_V(\Omega) = V(\Omega \times G_{2,3}^o)$$

of the varifold. The volume constraint is formulated by fixing the value of

$$\text{encvol}(V) = \frac{1}{3} \int_{\Omega \times G_{2,3}^o} x \cdot (*\xi) \, dV(x, \xi).$$

In the multiphase case, one further has to guarantee that the membrane represented by

$$V = \sum_{i=1}^N V^i$$

is closed, i.e., V has no boundary, and that the individual phases V_i and V_j ($i \neq j$) overlap at most at their boundaries. The varifold formulation of the multiphase Canham-Helfrich model thus consists of the minimization of $F_{\text{CH},N}(V^1, \dots, V^N)$ for $V^i \in AV_2^o(\Omega)$ ($i = 1, \dots, N$) under the constraints

- (i) Mass: $\mu_{V^i}(\Omega) = m_i \quad (i = 1, \dots, N),$
- (ii) Volume: $\text{encvol}(\sum_{i=1}^N V^i) = e,$
- (iii) Closedness: $\partial(\sum_{i=1}^N q_i V^i) = 0,$
- (iv) Overlap: $\sum_{i,j=1, i \neq j}^N (\mu_{V^i} \otimes \mu_{V^j})(D_{\Omega,\varepsilon}) \leq \varepsilon^3/\varepsilon_0$ for all $\varepsilon \in (0, \varepsilon_0).$

Here, $m_i, e, \varepsilon_0 > 0$ are fixed model parameters. To describe boundary-overlap for varifolds in a way compatible with varifold convergence (see Lemma 4.4.4), we consider the scaling of the product measure $\mu_{V^i} \otimes \mu_{V^j}$ on

$$D_{\Omega,\varepsilon} = \{(x, y) \in \Omega \times \Omega : |x - y| < \varepsilon\}$$

with ε . As we detail in Lemma 4.4.3 (and illustrate in Figures 4.1, 4.2, and 4.3), Condition (iv) captures the fact that, in case of phases corresponding to (countably two-rectifiable) surfaces, the intersection of M_i and M_j for $i \neq j$ must have zero area, i.e., zero two-dimensional Hausdorff measure $\mathcal{H}^2(M_i \cap M_j) = 0$, but may still be lower dimensional, for instance one- or zero-dimensional, i.e., $\mathcal{H}^p(M_i \cap M_j) \in (0, \infty)$ for $p = 1$ or $p = 0$. In particular, (iv) guarantees that V^i and V^j are mutually singular measures (Lemma 4.4.5), which implies that the boundary of their sum is simply the sum of their boundaries (Lemma 4.4.6).

Multiphase minimizers

In our main result, Theorem 4.4.9, we establish existence of multiphase varifold minimizers $V = \sum_{i=1}^N V^i$ of $F_{\text{CH},N}$. We argue by the Direct Method, which consists in establishing lower semicontinuity of the energy and compactness of minimizing sequences. The proof gradually progresses from results for single-phase and two-phase membranes to the multiphase case.

First, we show that if the bending rigidities β and γ satisfy the bounds (4.13),

$$-\frac{6}{5}\beta < \gamma < 0,$$

then the integrand f_{CH} is strictly convex in the curvature variables (Proposition 4.3.1). This allows us to apply Hutchinson's results on functionals of measure-function pairs to establish lower semicontinuity of F_{CH} under convergence in $AV_2^o(\Omega)$ (Theorem 4.3.2), i.e.,

$$F_{\text{CH}}(V) \leq \liminf_{n \rightarrow \infty} F_{\text{CH}}(V_n) \quad \text{whenever} \quad V_n \rightharpoonup^* V \quad \text{in} \quad AV_2^o(\Omega).$$

By the convergence of boundary measures $\partial(q_i V_n^i) \rightharpoonup^* \partial(q_i V^i)$ ($n \rightarrow \infty$) and the lower semicontinuity of the total variation norm $\partial(q_i V^i) \mapsto \|\partial(q_i V^i)\|_{\text{TV}}(\Omega \times G_{2,3})$, we thus have lower semicontinuity of the multiphase energy $F_{\text{CH},N}$ in the arguments $(V^1, \dots, V^N)$ as well.

Next, we introduce the class of admissible varifolds (4.15),

$$\mathcal{A} = \{V \in AV_2^o(\Omega) : \text{supp } \mu_V \subset Q, \mathbb{M}_{\partial T_V}(\Omega) \leq m^o\}.$$

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Here, $Q \subset\subset \Omega$ is a fixed compact set with nonempty interior and the fixed constant $m^o > 0$ controls the mass of the boundary of the current T_V associated to the oriented varifold V . We show that $\mathcal{A}$ is closed in $AV_2^o(\Omega)$ (Lemma 4.4.1).

The proof of compactness of minimizing sequences $(V_n^1, \dots, V_n^N)_{n \in \mathbb{N}}$ for $V_n^i \in \mathcal{A}$ with respect to convergence in $AV_2^o(\Omega)$ as $n \rightarrow \infty$ relies on the first variation bounds (4.6) given in Lemma 4.2.3,

$$\|\delta(q_\# V_n^i)\|_{\text{TV}}(\Omega) \leq \sqrt{3 \mu_{V^i}(\Omega)} \|A^{q_\# V_n^i}\|_{L^2_{q_\# V_n^i}(\Omega \times G_{2,3})} + \|\partial(q_\# V_n^i)\|_{\text{TV}}(\Omega \times G_{2,3}),$$

and the curvature bounds (4.14) established in Proposition 4.3.1,

$$\|A^{q_\# V_n^i}\|_{L^2_{q_\# V_n^i}(\Omega \times G_{2,3})}^2 \leq c_1 (F_{\text{CH}}^i(V_n^i) + c_2 \mu_{V_n^i}(\Omega))$$

where $c_1 > 0$ and $c_2 \geq 0$, the latter being a further consequence of the convexity of f_{CH} . Together with the bounded current mass in the definition of $\mathcal{A}$, these estimates allow us to combine Hutchinson's compactness theorem for oriented varifolds and Mantegazza's compactness theorem for curvature varifolds with boundary (cf. Theorems 4.2.6 and 4.2.4 respectively).

Finally, verifying closedness of the constraints (i)–(iv) with respect to convergence in $AV_2^o(\Omega)$ completes the proof of existence of multiphase varifold minimizers (Theorem 4.4.9).

Solution properties

Our multiphase varifold minimizers $V = \sum_{i=1}^N V^i$ are Radon measures, so it is natural to ask whether they are regular enough to represent surfaces with classical curvature functions. In Lemma 4.4.10 we give a first answer in this direction: The fact that we work with (oriented) curvature varifolds V^i implies that they are C^2 -rectifiable, i.e., their spatial measure μ_{V^i} is supported on a countable union of C^2 surfaces, and the same holds for the union V .

Moreover, we argue in Subsection 4.4.2 that the closedness condition (iii) and the overlap condition (iv) in our model imply that the interface conormals must be antiparallel. This fact rules out the presence of kinks at phase-interfaces.

Finally, in Lemma 4.4.12 and Lemma 4.4.13, we establish the following lower and upper diameter bounds for the spatial supports of the phases V^i ($1 \leq i \leq N$):

$$0 < \frac{c \mu_{V^i}(\Omega)}{\sqrt{\mu_{V^i}(\Omega) 4\mathcal{W}(V^i)} + \|\partial(q_\# V^i)\|_{\text{TV}}(\Omega \times G_{2,3})} \leq \text{diam}(\text{supp } \mu_{V^i}) \leq C \sqrt{\mu_V(\Omega) \mathcal{W}(V)} < \infty$$

with constants $c, C > 0$. Here,

$$\mathcal{W}(V) = \frac{1}{4} \int_{\Omega} |\bar{H}^{q_\# V}|^2 d\mu_V$$

is the Willmore energy of V , which by $\bar{H}^{q_\# V} = (\sum_{j=1}^3 A_{jij}^{q_\# V})_{i=1}^3$, $|\bar{H}^{q_\# V}|^2 \leq 3|A^{q_\# V}|^2$, and the curvature bounds (4.14) is finite for all minimizers V^i as well as for their sum $V = \sum_{i=1}^N V^i$. Note that in Chapter 4 and in [22] we employ a different normalization, $\mathcal{W}(V) = \int_{\Omega} |\bar{H}^{q_\# V}|^2 d\mu_V$, and write $\widetilde{\mathcal{W}}(V) = \frac{1}{4} \int_{\Omega} |\bar{H}^{q_\# V}|^2 d\mu_V$.

The lower diameter bound guarantees that no phase with $\mu_{V^i} = m_i > 0$ degenerates by shrinking to a point. The upper bound excludes that the minimizers expand to infinity, which further allows us to enlarge our admissibility class $\mathcal{A}$ by removing the support restriction $\text{supp } \mu_V \subset Q$.

1.3.3 Chapter 5: Canham-Helfrich flow

In [21] (Chapter 5), we tackle the gradient flow evolution for the Canham-Helfrich energy in the setting of varifolds.

The Canham-Helfrich flow describes the overdamped evolution of a biomembrane towards its equilibrium configuration if external forces are neglected. We address this evolution problem variationally via the Generalized Minimizing Movements (GMMs) method [3, 5, 35], working in a Wasserstein space of varifolds.

Our principal findings are as follows:

- Definition of the Canham-Helfrich flow of varifolds,
- Existence under additional constraints,
- Evolution of multiply covered regular surfaces and conservation of multiplicity.

Previous results on the Canham-Helfrich flow have been obtained in the case of smooth immersions of closed surfaces, where the classical L^2 gradient flow takes the form of a PDE. By extending techniques developed for the Willmore flow [59–62, 88, 89, 96], analytical results for the Canham-Helfrich flow have been presented in [16, 57, 69, 80]. They include short-time existence, uniqueness, and a lifespan estimate in terms of the initial curvature. Moreover, long-time existence is guaranteed if the initial surface is close to a sphere, but otherwise, singularities may occur in finite time as well. A more exhaustive description of the literature on Canham-Helfrich flow is provided in Section 5.1.

In contrast to these results, our varifold solution does not require small initial energy and can also handle the possible occurrence of singularities. In particular, topology changes like the budding transformation (cf. Figure 2.5) can be treated.

Flow of varifolds

First we introduce a solution concept for the Canham-Helfrich flow in terms of GMMs in a Wasserstein space of varifolds (Definition 5.4.1). Here, based on the varifold formulation of the energy $F_{\text{CH}}(V)$ established in (4.11) of our previous work [22] (Chapter 4), we consider the evolution problem for oriented two-varifolds

$$V \in \mathcal{V} \subset \mathcal{M}(\mathbb{R}^3 \times \mathbb{S}^2)$$

given by (5.14). Elements of $\mathcal{V}$ are probability measures normalized to the fixed mass $m_0 > 0$ and have compact support. In view of modeling the evolution of homogeneous closed membranes, we consider the extended Canham-Helfrich energy

$$G_{\text{CH}}: \mathcal{V} \rightarrow (-\infty, \infty], \quad G_{\text{CH}}(V) = \begin{cases} F_{\text{CH}}(V), & V \in \mathcal{A} \\ \infty, & \text{else} \end{cases}$$

with the class of admissible varifolds given by (5.15), namely

$$\mathcal{A} = \{V \in \mathcal{V} : V \in AV_2^o(\mathbb{R}^3), \partial(q_\# V) = 0, \partial T_V = 0\}.$$

This is a special case of the class (4.15) when restricting to varifolds without boundaries.

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When equipped with the p -Wasserstein distance W_p for any fixed $p \in [1, \infty)$, the space of varifolds $\mathcal{V}$ turns into a complete separable metric space. This allows us to introduce a global-in-time solution of the varifold Canham-Helfrich flow by the method of GMMs: Starting from an initial state $V^0 \in \mathcal{V}$ with $G_{\text{CH}}(V^0) < \infty$ and a discretization of the time interval with step size $\tau > 0$, one recursively constructs discrete solutions $(V_\tau^n)_{n \in \mathbb{N}_0}$ by setting $V_\tau^0 = V^0$ and choosing V_τ^n ($n \geq 1$) as minimizers of the energy augmented by a metric dissipation term,

$$V \mapsto G_{\text{CH},\tau}(V; V_\tau^{n-1}) = G_{\text{CH}}(V) + \frac{1}{2\tau} W_p^2(V, V_\tau^{n-1}).$$

Then, one interpolates the sequence $(V_\tau^n)_{n \in \mathbb{N}_0}$, obtains a piecewise constant solution $\bar{V}_\tau: [0, \infty) \rightarrow \mathcal{V}$, and eventually passes to the limit as $\tau \rightarrow 0$ to arrive at the continuous flow

$$V: [0, \infty) \rightarrow \mathcal{V}.$$

In our main result (Theorem 5.4.2) we establish the existence of a GMMs solution associated to G_{CH} starting from any admissible initial configuration $V^0 \in \mathcal{A}$ with finite energy. In particular, the solution satisfies $V \in AC_{\text{loc}}^2([0, \infty); \mathcal{V})$ with

$$V(t) \in \mathcal{A} \quad \text{and} \quad G_{\text{CH}}(V(t)) + \frac{1}{2} \int_0^t |V'|^2(r) \, dr \leq G_{\text{CH}}(V^0)$$

for all $0 \leq t < \infty$, where $|V'| \in L_{\text{loc}}^2(0, \infty)$ denotes the metric derivative of V . This result holds under the parameter bounds $-\frac{6}{5}\beta < \gamma < 0$ (5.6), which are the same as (4.13) in our previous work [22] (Chapter 4). In the proof, we take advantage of the equivalence between the convergence in the Wasserstein topology and the weak-* convergence of varifolds.

Furthermore, we show in Proposition 5.4.4 that the spatial support of our GMMs solution $t \mapsto V(t)$ satisfies the diameter bounds

$$c_0 \leq \text{diam}(\text{supp } \mu_{V(t)}) \leq c^0$$

for all $0 \leq t < \infty$, where the constants $c_0, c^0 > 0$ only depend on V^0 and on the parameters m_0, β, γ , and H_0 . As the upper bound c^0 is independent of the choice of the compact support required in $\mathcal{V}$, the latter eventually turns out to be an unnecessary assumption.

Additional constraints

In Corollary 5.4.3, we remark that the varifold Canham-Helfrich flow still exists under the presence of additional constraints, provided that they are closed in $\mathcal{A}$ with respect to the weak-* topology. This in particular allows us to define the flow of varifolds with fixed enclosed volume (5.18),

$$\text{encvol}(V) = \frac{1}{3} \int_{\mathbb{R}^3 \times \mathbb{S}^2} x \cdot \nu \, dV(x, \nu) = v_0 > 0,$$

which is part of the original Canham-Helfrich variational model for biomembranes. Moreover, we show that Corollary 5.4.3 implies the existence of the Canham-Helfrich flow of axisymmetric varifolds, upon restricting the admissibility class to varifolds V such that

$$\Psi_\# V = V,$$

with $\Psi: \mathbb{R}^3 \times \mathbb{S}^2 \rightarrow \mathbb{R}^3 \times \mathbb{S}^2$ being the reflection map about a fixed plane through the origin and $\Psi_\#$ denoting its induced push-forward. Furthermore, the flow can be constrained to rotationally symmetric varifolds as well.

Multiply covered surfaces

In order to address the evolution of more regular surfaces, we consider the subclass (5.19) of admissible varifolds with constant multiplicity $k \in \mathbb{N}$ that are supported on uniformly $C^{1,1}$ regular connected closed embedded surfaces of genus $g \in \mathbb{N}_0$:

$$\mathcal{A}_k^g = \left\{ V = V[M, \nu, k, 0] \in \mathcal{A} : M = \psi(\Sigma^g) \text{ for an embedding } \psi \in C^{1,1}(\Sigma^g; \mathbb{R}^3) \right. \\ \left. \text{with } \|\psi\|_{C^{1,1}(\Sigma^g)} \leq L, \nu \text{ is the outer unit normal vector to } M \right\}.$$

We also introduce the set of regular genus- g surfaces with arbitrary integer multiplicity in (5.20):

$$\mathcal{A}^g = \{ V \in \mathcal{A} : \exists k \in \mathbb{N} \text{ such that } V \in \mathcal{A}_k^g \}.$$

The requirement of constant multiplicity comes naturally by the zero boundary current of admissible varifolds and Federer's Constancy Theorem (Theorem 5.2.1). In Lemma 5.5.5, we verify closedness of both classes in $\mathcal{A}$, so that the evolution exists by Corollary 5.4.3.

Restricting to surfaces with zero genus ($g = 0$), we establish stationarity of spheres with respect to the flow. More precisely, we prove in Proposition 5.5.1 that the k -fold covered sphere with mass m_0 uniquely minimizes the Canham-Helfrich energy in $\mathcal{A}_k^0$, provided that the spontaneous curvature H_0 lies within certain bounds.

In Proposition 5.5.4, we present an estimate on the multiplicity k of $V \in \mathcal{A}_k^g$ in terms of its Canham-Helfrich energy,

$$k \leq \bar{k}(F_{\text{CH}}(V)),$$

which holds whenever $0 < -2\beta < \gamma(1 - g)$ and $\gamma < 0$. The function $\bar{k} : \mathbb{R} \rightarrow \mathbb{N}$ is nondecreasing and only depends on β, γ , and H_0 . This estimate is analogous to the Li-Yau inequality for the Willmore energy of a varifold, $k \leq \frac{1}{4\pi} \mathcal{W}(V)$.

Finally, in Theorem 5.5.6, we show that the varifold Canham-Helfrich flow preserves multiplicity. Starting from $V^0 \in \mathcal{A}_k^g$ with finite energy, the flow restricted to regular surfaces with arbitrary multiplicity $\mathcal{A}^g$ still satisfies

$$V(t) \in \mathcal{A}_k^g$$

for all $0 \leq t < \infty$. The proof is based on our Li-Yau-type estimate and a contradiction argument, which requires a technical lemma on the Wasserstein distance of multiply covered surfaces (Lemma 5.5.7).

1.4 List of notation

We briefly list the most important notation employed in this Introduction (Chapter 1) and in the Preliminaries (Chapter 2). Further notation will be explained in the subsequent chapters.

	Geometry of curves (Subsections 1.3.1, 2.2.1; Chapter 3)
$\gamma; \Gamma$	curve in $\mathbb{R}^2$; in $\mathbb{R}^3$
γ_0	circle with radius $L/2\pi$
κ	scalar curvature
L	total length
s	arclength
θ	inclination angle

Geometry of surfaces

M, Σ	surface in $\mathbb{R}^3$
∂M	boundary
g	genus
H	mean curvature
$\bar{H}$	mean curvature vector
K	Gauss curvature
κ_1, κ_2	principal curvatures
ν	normal vector
$\bar{\nu}, \nu^{\partial M}$	conormal vector of ∂M
II	second fundamental form
$T_x M$	tangent space to M at $x \in M$

Varifolds and currents

V	varifold
A^V	curvature functions
∂V	boundary measure
δV	first variation
$\mu_V, \pi_{\sharp} V$	mass measure
$q_{\sharp} V$	unoriented counterpart associated to V
T_V	current associated to V
$\mathbb{M}_{\partial T_V}$	mass of the boundary current of T_V
$\theta, \theta^{\pm}$	multiplicities
P	projection matrix, element of $G_{m,n}$
ξ	orientation, element of $G_{m,n}^o$

Sets

$\mathcal{A}$	admissible planar curves (Subsection 1.3.1, Chapter 3)
$\mathcal{A}, \mathcal{A}^g, \mathcal{A}_k^g$	admissible varifolds
AV_m	curvature m -varifolds with boundary
AV_m^o	oriented curvature m -varifolds with boundary
$C; C_c$	continuous functions; compactly supported
$C^k; C_c^k$	k -times differentiable functions ($k \in \mathbb{N}$); compactly supported
$G_{m,n}$	Grassmannian of all m -dimensional linear subspaces of $\mathbb{R}^n$ ($m, n \in \mathbb{N}$)
$G_{m,n}^o$	oriented Grassmannian
H^k	L^2 -based Sobolev space $W^{k,2}$ ($k \in \mathbb{N}$)
IV_m	integral m -varifolds
IV_m^o	oriented integral m -varifolds
$L^p; L_{\mu}^p$	Lebesgue space ($p \in [1, \infty]$); with respect to the measure μ
$\mathcal{M}; \mathcal{P}$	(nonnegative) Radon measures; with fixed mass $m_0 > 0$
$\mathcal{M}^n, \mathcal{M}(\cdot; \mathbb{R}^n)$	vector-valued Radon measures ($n \in \mathbb{N}$)
Q	compact domain in $\mathbb{R}^3$
$\mathcal{R}$	admissible densities of heterogeneous curves (Subsection 1.3.1, Chapter 3)
$\mathbb{S}^2$	two-dimensional unit sphere in $\mathbb{R}^3$
$W^{k,p}$	L^p -based Sobolev space ($p \in [1, \infty]$, $k \in \mathbb{N}$)
Ω	open domain in $\mathbb{R}^n$

Functions, measures, norms	
Area	area functional of a surface
$\chi_B, 1_B$	characteristic function of a set $B \subset \mathbb{R}^n$
diam	diameter of a set in $\mathbb{R}^n$
dist	distance between sets in $\mathbb{R}^n$
Dom	domain of a function
E, E_{EB}, E_μ	elastica energies of curves (Subsections 1.3.1, 2.2.1, Chapter 3)
E_{phase}	phase energy
$E, E_{\text{CH}}, E_{\text{CH},N}$	Canham-Helfrich energies of surfaces
$F_{\text{CH}}, F_{\text{CH},N}; G_{\text{CH}}$	Canham-Helfrich energies of varifolds; probability measures
$\mathcal{H}^k$	k -dimensional Hausdorff measure ($k \in \mathbb{N}_0$)
ρ	phase density
supp	support of a function or measure
v	velocity
Vol; encvol	volume enclosed by a surface; by a varifold
$\mathcal{W}$	Willmore energy
W_p	p -Wasserstein distance ($p \in [1, \infty]$)
$ \cdot $	absolute value in $\mathbb{R}^n$ or Frobenius norm for matrices in $\mathbb{R}^{n \times n}$
$\ \cdot\ _X$	norm in the Banach space X
$\ \cdot\ _{\text{TV}}, \ \cdot\ $	total variation norm
$\lfloor$	restriction of a measure
Scalar parameters	
a_0	area
α	surface tension
β, γ	bending rigidities/stiffnesses (possibly functions of ρ)
$C, c, c_0, c^0; c_1, c_2$	constants; in curvature bounds (Lemma 2.2.1, Proposition 4.3.1)
h	second-order Taylor coefficient of β (Subsection 1.3.1, Chapter 3)
H_0	spontaneous curvature (possibly a function of ρ)
κ_0	constant curvature $2\pi/L$ of the circle γ_0 (Subsection 1.3.1, Chapter 3)
L	length of the curve γ (Subsections 1.3.1, 2.2.1; Chapter 3)
L	bound for the uniform $C^{1,1}$ regularity (Subsection 1.3.3, Chapter 5)
m	first-order Taylor coefficient of β (Subsection 1.3.1, Chapter 3)
m_0, m_i, m	mass
M	mass of a curve γ with density ρ (Subsection 1.3.1, Chapter 3)
μ, μ_j	bifurcation parameters (Subsection 1.3.1, Chapter 3)
N	number of phases
p	osmotic pressure difference
ρ_0	constant mass density M/L of the circle γ_0 (Subsection 1.3.1, Chapter 3)
σ, σ^i	line tension coefficient
t	time
τ	time step
v_0, e	enclosed volume

Chapter 2

Preliminaries

In this preliminary chapter, we present basic properties of biological membranes and their mathematical modeling (Section 2.1) and provide some background material on calculus of variations and geometric measure theory (Section 2.2).

2.1 Modeling biological membranes

The most prominent examples of biological membranes are cell membranes. We briefly discuss their role for biological processes and comment on their physical properties (Subsection 2.1.1), before we introduce the Canham-Helfrich variational model for their shapes (Subsection 2.1.2). Then we explain how this model has to be adapted in order to capture configurations of heterogeneous biomembranes (Subsection 2.1.3) and finally present the evolution problem for membranes (Subsection 2.1.4). Here, we mainly follow the textbook [17] on cell membranes and the review paper [93] on vesicle configurations.

2.1.1 Cell membranes

The basic entities of living organisms are **cells**. They feature a great variety of shapes and structures, as illustrated in Figure 2.1, which shows a schematic model of an animal cell. A cell consists of a plasma membrane filled with the fluid cytoplasm and contains a nucleus and organelles. The nucleus holds the DNA and is bounded by a double membrane with pores. It is surrounded by the endoplasmic reticulum, a highly branched surface where proteins are synthesized. Further organelles are the Golgi apparatus, which sorts the proteins, and several mitochondria, which produce the cell's energy. The plasma membrane, which shields the cell from its environment, is sometimes reinforced by an interior cytoskeleton network. In contrast to animal cells, plant cells are additionally protected by an outer cell wall made of cellulose.

The outer plasma membrane and the boundaries of the interior units of the cell consist of **biological membranes**, for short, **biomembranes**. This material is able to fold into complicated structures in the organelles and to undergo dynamical morphology changes like in exocytosis processes or cell division, without losing its stability. This shows that biomembranes are robust and flexible at the same time. In fact, from a viewpoint of continuum mechanics, biological membranes behave like two-dimensional incompressible viscous fluids with an elastic response to bending forces. This combination of fluid- and solid-mechanics features their fascinating material properties.

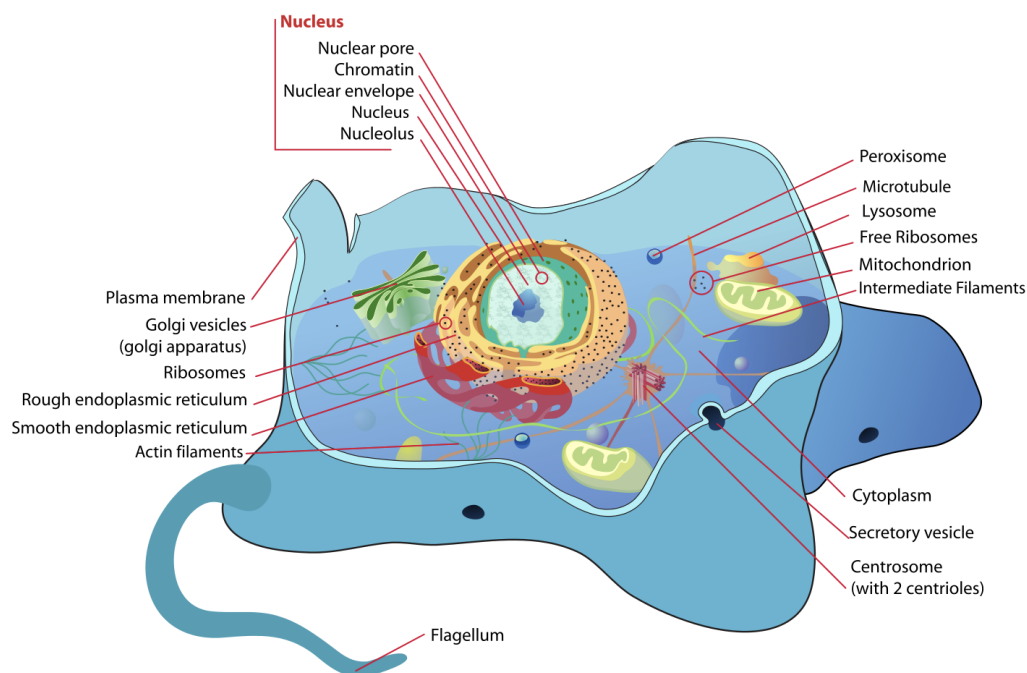


Figure 2.1: Illustration of an animal cell. [Source: LadyofHats, Public domain via Wikimedia Commons]

The mechanical characteristics of biological membranes stem from their special molecular structure. When zooming in, one discovers that biological membranes form a “fluid mosaic”, a term coined by SINGER & NICOLSON [97]. As illustrated in Figure 2.2, the basic building blocks are phospholipid molecules. However, cell membranes generally also contain other molecules like cholesterol, embedded proteins, and have attached sugars. Phospholipids have a hydrophilic head group and two hydrophobic tails of hydrocarbon chains. Therefore, in order to shield the tails from water, these molecules tend to assemble to bilayer sheets when put in an aqueous solution. As any exposed edges are energetically unfavorable, these lipid membranes form closed containers. Such lipid bilayer (or monolayer) **vesicles** can be synthesized in the laboratory and observed with optical microscopes. Therefore, they act as a model system, whose study allows us to understand the influence of biomembrane mechanics on the cell’s morphology.

The thickness of lipid bilayers is about 4–5 nm, whereas the size of vesicles and cells typically is in the μm -range. Due to this separation of length scales by three orders of magnitude, many modeling questions can be addressed by approximating biomembranes as **surfaces**. The special molecular behavior of biomembranes gives rise to two additional geometric constraints. Firstly, the individual molecules in a lipid bilayer tend to be at a fixed distance to each other due to electrochemical repulsion of van-der-Waals type. Moreover, the number of molecules in each sheet of the bilayer is almost constant in experimentally relevant time-scales of minutes to hours. For these reasons biomembranes can be assumed to have fixed **area**. This property of area conservation sets biological membranes apart from soap films or other interfaces between two materials, which tend to minimize their area instead. Secondly, due to the osmotic pressure equilibrium between inside and outside fluids, the **volume** enclosed by the membrane can be considered as fixed as well. The nature of the electrochemical forces also allows membrane molecules to shift past each other with very low shear resistance, which explains the fluidity of lipid mono- and bilayers. Furthermore, the membrane’s resistance to stretching is much larger than its resistance to bending. Therefore, biomembrane material mechanically behaves as an **incompressible two-dimensional viscous fluid with bending elasticity**.

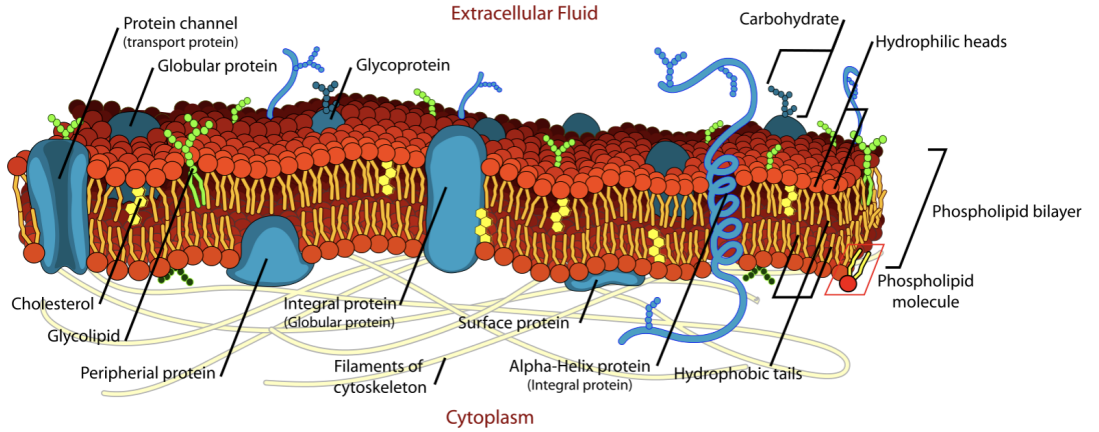


Figure 2.2: Illustration of a cell membrane as a fluid mosaic. [Source: LadyofHats, Public domain via Wikimedia Commons]

In equilibrium, vesicles and simple cells often adopt smooth and symmetrical shapes. A prominent example is that of a human **red blood cell**, which is normally a biconcave discocyte (Figure 2.3). Note however that in different environments the equilibrium shape is different as well: When immersed in fluids with high salt concentration (hypertonic environment), the cells shrink to get a spiky appearance, whereas in low concentration (hypotonic environment) they inflate and become almost spherical, see Figure 2.4.

Another famous shape transformation is the **budding transition** in lipid vesicles, which is similar to exocytosis or cell division. Vesicle budding occurs due to thermal expansion, because the area grows faster than the enclosed volume. As the temperature of the environment increases, spherical vesicles become initially prolate axisymmetric ellipsoids and then pear-like, before the necks eventually close and spherical buds form, see Figure 2.5. The counterpart of vesicle budding or exocytosis is the **fusion** of vesicles or endocytosis. This process occurs for example when viruses, which may feature a lipid membrane as well, infect cells.

2.1.2 The Canham-Helfrich model

Inspired by the shapes of red blood cells, in the early 1970s, CANHAM [25] and HELFRICH [50] proposed that equilibrium biomembrane shapes are surfaces M of minimal elastic bending energy. This is now called the Canham-Helfrich energy and is given by (1.1), namely,

$$E_{\text{CH}}(M) = \int_M \left(\frac{\beta}{2} (H - H_0)^2 + \gamma K \right) d\mathcal{H}^2.$$

In this surface integral, $d\mathcal{H}^2$ denotes the area element expressed in terms of the two-dimensional Hausdorff measure, H and K are the mean and Gauss curvatures of M (see Subsection 2.2.1), $\beta > 0$ and $\gamma < 0$ are the bending rigidities (or stiffnesses) of the membrane material, and $H_0 \in \mathbb{R}$ is the spontaneous curvature parameter. The latter was introduced by Helfrich to represent a possible asymmetry between the bilayer leaflets, which may be caused by a difference in their composition or by different conditions in their surroundings.

The classical **Canham-Helfrich problem** consists in the minimization of the Canham-Helfrich energy constrained to smooth closed orientable surfaces $M \subset \mathbb{R}^3$ with fixed area and fixed enclosed volume,

$$\text{Area}(M) = a_0 \quad \text{and} \quad \text{Vol}(M) = v_0, \quad (2.1)$$

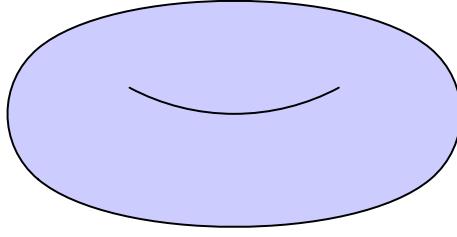


Figure 2.3: Illustration of human red blood cell with typical biconcave shape.

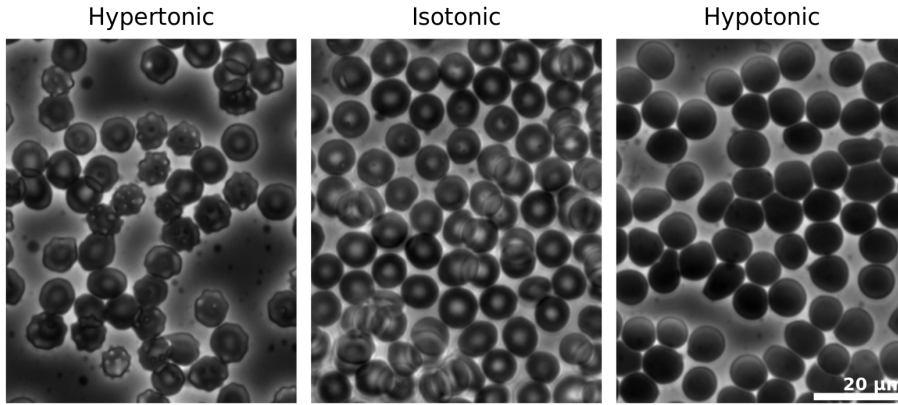


Figure 2.4: Microscopy images of different shapes of human red blood cells: Wrinkled echinocytes in a hypertonic environment (left), normal biconcave discocytes in an isotonic environment (middle), and inflated spherocytes in a hypotonic environment (right). [Source: Zephyris, CC BY-SA 3.0 via Wikimedia Commons]

for given numbers $a_0, v_0 > 0$. We fix the orientation of M by choosing

$$\nu: M \rightarrow \mathbb{S}^2$$

to denote its outer unit normal vector field. The area of M simply is its complete integral (or two-dimensional Hausdorff measure),

$$\text{Area}(M) := \int_M d\mathcal{H}^2 = \mathcal{H}^2(M). \quad (2.2)$$

If $M = \partial B$ for a domain $B \subset \mathbb{R}^3$, the volume enclosed by M is given by $\text{Vol}(M) = \int_B dx$. With $\text{div}(x) = 3$, the divergence theorem implies $\int_B dx = \frac{1}{3} \int_B \text{div}(x) dx = \frac{1}{3} \int_{\partial B} x \cdot \nu(x) d\mathcal{H}^2(x)$. Consequently, the enclosed volume can be expressed as the surface integral

$$\text{Vol}(M) := \frac{1}{3} \int_M x \cdot \nu(x) d\mathcal{H}^2(x). \quad (2.3)$$

Remark 2.1.1 (Other variational models for lipid bilayer membranes, cf. [93]). The Canham-Helfrich model, which is also called the *spontaneous curvature model*, has been extended in various ways to take the bilayer architecture into account. Roughly speaking, bending a layer of finite thickness causes compression of the material on one side and dilatation on the other side, which results in an area difference. In the so-called *area-difference elasticity model*, the area-difference between the neutral surfaces of the two monolayers must be close to an optimal

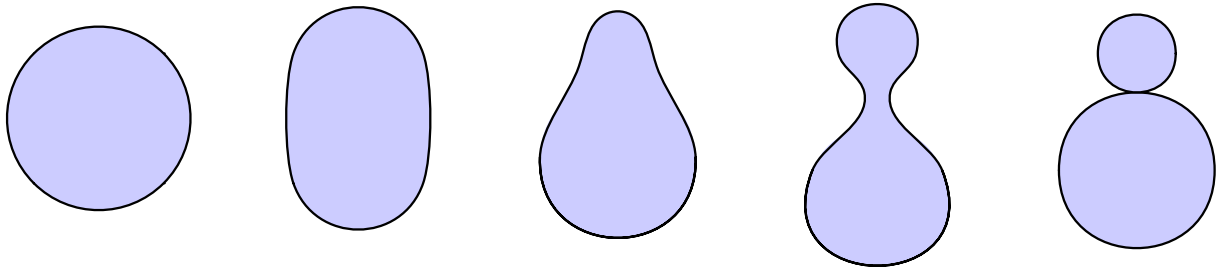


Figure 2.5: Illustration of the budding transformation of vesicles, which has been observed experimentally, cf. [93, p. 17]. As temperature increases, starting from a spherical shape (left), the configuration changes to a rotation ellipsoid and pear-like shape, whose neck gets narrower and closes to eventually form two separated spherical vesicles (right). Conversely, if read from right to left, the sequence of images illustrate the fusion of vesicles.

value, whereas in the *bilayer-couple model*, this difference is constrained to be fixed. These models are nonlocal, as they involve the integrated mean curvature, $\int_M H \, d\mathcal{H}^2$.

The classical Canham-Helfrich problem is a **geometric variational problem** (see also Subsection 2.2.1), as the energy only depends on the shape of the surface and not on its specific parametrization, nor on its position or orientation in space. In other words, the energy and all constraints are invariant under reparametrizations, translations, and rotations.

In order to define admissible surfaces, the area and volume constraint data (2.1) must respect the isoperimetric inequality, $6\sqrt{\pi} \operatorname{Vol}(M) \leq \sqrt{\operatorname{Area}(M)^3}$. Equality only holds if M is a sphere, which has the largest volume for given area. The **isoperimetric ratio (reduced volume)** is the quotient of the volume enclosed by M to the volume of a sphere with the same surface area as M ,

$$\iota(M) := \frac{6\sqrt{\pi} \operatorname{Vol}(M)}{\sqrt{\operatorname{Area}(M)^3}} = \frac{\operatorname{Vol}(M)}{\frac{4}{3}\pi R_0^3}, \quad (2.4)$$

where $R_0 = \sqrt{\frac{\operatorname{Area}(M)}{4\pi}}$. Isoperimetry implies that $\iota(M) \in (0, 1]$.

By the GAUSS-BONNET Theorem applied to smooth closed embedded surfaces M , the integral of the Gauss curvature is a topological constant,

$$\int_M K \, d\mathcal{H}^2 = 4\pi(1 - g).$$

Here the genus $g \in \mathbb{N}_0$ is the number of handles of M . Therefore, if one restricts to surfaces with fixed genus, the last term in the Canham-Helfrich energy E_{CH} (1.1) can be omitted, which actually corresponds to setting $\gamma = 0$. In this case, one only needs to minimize

$$E_{\text{CH}}^0(M) := \int_M (H - H_0)^2 \, d\mathcal{H}^2.$$

The **Willmore energy** is a famous special case of the Canham-Helfrich energy for $H_0 = 0$,

$$\mathcal{W}(M) := \frac{1}{4} \int_M H^2 \, d\mathcal{H}^2. \quad (2.5)$$

Its global minimizer among closed surfaces (without constraints) is the sphere with energy 4π .

Minimization problems involving the Willmore energy are well studied. Yet, many existence results take advantage of its conformal invariance, which is invariance under angle-preserving

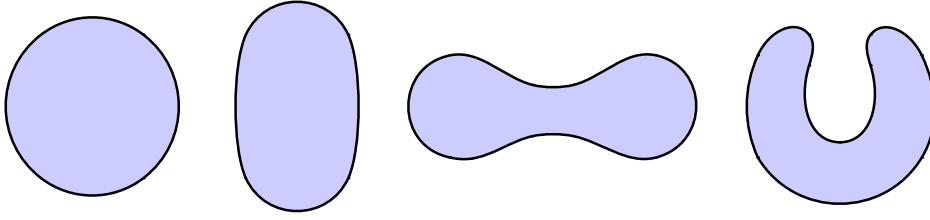


Figure 2.6: Illustration of axisymmetric shapes M of minimal Willmore energy for different isoperimetric ratios $\iota(M) = \iota_0$ similar to those numerically obtained in [93, p. 46]: Starting from $\iota_0 = 1$, where the minimizer is a sphere (left), the optimal axisymmetric shapes are prolate rotation ellipsoids for $\iota_0 > 0.65$, then biconcave discocytes in the range of about $0.65 > \iota_0 > 0.59$ (in fact, the reduced volume of red blood cells is about $\iota_0 = 0.64$), and eventually stomatocytes (right) for $0 < \iota_0 < 0.59$.

transformations Ψ of $\mathbb{R}^3$, that is, a surface M and its transformation $M' := \Psi(M)$ have the same energy. By LIOUVILLE's Theorem, conformal maps Ψ are compositions of translations, rotations, reflections, scalings, and inversions. However, the spontaneous curvature in the Canham-Helfrich model introduces a natural length-scale, because the energy is minimal for spheres of radius $R = -2/H_0$ (if $H_0 < 0$). Hence, the Canham-Helfrich energy (5.1) is not scale invariant, unless $H_0 = 0$, that is, in the Willmore case. Indeed, the energy

$$E_{\text{CH}}|_{H_0=0}(M) = \int_M \left(\frac{\beta}{2} H^2 + \gamma K \right) d\mathcal{H}^2$$

of a surface M and of its scaling $M' := aM$ ($a > 0$) is the same, because the curvatures of M and M' are related by $H' = H/a$ and $K' = K/a^2$ whereas the area element of M' involves the factor a^2 . Therefore, instead of constraining both, $\text{Area}(M)$ and $\text{Vol}(M)$, it is sufficient to consider surfaces with fixed isoperimetric ratio (2.4) when minimizing $E_{\text{CH}}|_{H_0=0}$.

A necessary condition for minimality (more precisely, stationarity) of the Canham-Helfrich energy is the Euler-Lagrange equation, which in the context of geometric energies is called **shape equation**. Taking the area and the volume constraint into account by the method of Lagrange multipliers, one considers stationarity of the penalized functional

$$E_{\text{CH}}^{\alpha,p}(M) := E_{\text{CH}}(M) + \alpha (\text{Area}(M) - a_0) + p (\text{Vol}(M) - v_0). \quad (2.6)$$

The Lagrange multipliers for the area and volume constraint can be interpreted as the surface tension α and pressure difference p respectively. Stationarity of $E_{\text{CH}}^{\alpha,p}$ results in the shape equation

$$\beta \left(\Delta H + \frac{1}{2}(H - H_0)(H^2 - 4K + HH_0) \right) + \alpha H + p = 0,$$

where Δ denotes the LAPLACE-BELTRAMI operator on M . This is a fourth-order quasilinear elliptic PDE for the surface parametrization. Upon neglecting bending forces by setting $\beta = 0$, the shape equation reduces to Laplace's equation $\alpha H + p = 0$, which describes the equilibrium of soap films.

Various numerical studies are available presenting solutions similar to observed biomembrane shapes. In particular, for certain parameters, one obtains biconcave discocytes like those of red blood cells. By restricting to rotationally symmetric membranes, the shape equation reduces to a system of ordinary differential equations. The bifurcation of initially spherical solutions when the pressure increases has been studied already by Helfrich in [50]. Actually, shapes similar to red blood cell discocytes already emerge in the simple case of minimizing the Willmore energy

(2.5) under the isoperimetry constraint $\iota(M) = \iota_0$ (2.4) and restricting to axisymmetric surfaces, see Figure 2.6. If rotational symmetry is not imposed, one also finds other stationary shapes such as vesicles of starfish type geometry, see [93].

2.1.3 Heterogeneous membranes

In reality, cell membranes are not homogeneous but are a fluid mosaic containing additional lipids and various embedded proteins (cf. Figure 2.2). The goal is to understand the role of these constituents regarding stable membrane configurations, which can indeed be very pronounced. In experiments with giant unilamellar vesicles, e.g., [12], one observes heterogeneous membranes similar to the one illustrated in Figure 2.7. Here, the different shades of blue represent different phases and one sees that curvature is clearly coupled to composition and vice versa. Such a coupling effect in heterogeneous biomembranes was first proposed by LEIBLER [66].

In order to incorporate the effect of a variable **phase density**

$$\rho: M \rightarrow \mathbb{R}$$

into the Canham-Helfrich model, one has to let the membrane material parameters depend on ρ as well. The multiphase energy thus is given by (1.2), namely,

$$E(\rho, M) = \int_M \left(\frac{\beta(\rho)}{2} (H - H_0(\rho))^2 + \gamma(\rho) K \right) d\mathcal{H}^2 + E_{\text{phase}}(\rho, M).$$

This energy also includes an additional phase energy E_{phase} taking into account phase interaction effects. The fluidity of biomembranes allows the membrane molecules to move to a spot which best suits their preferred curvature. Therefore, one has to minimize the functional E with respect to both shape and density. Moreover, in addition to the area and volume constraints $\text{Area}(M) = a_0$ and $\text{Vol}(M) = v_0$ in (2.1), due to conservation of mass, one needs to require that the phase mass is a fixed constant $m_0 \in \mathbb{R}$ as well, namely,

$$\int_M \rho d\mathcal{H}^2 = m_0.$$

In case of **sharp phase-interfaces** one does not need a continuous phase density ρ , but can formulate the energy as a functional dependent on surfaces representing the individual phases. In the classical setting, this multiphase model was introduced by JÜLICHER & LIPOWSKI [55] to describe domain-induced budding. In case of Figure 2.7, budding means that the dark blue parts would pinch off to form disconnected vesicles. The multiphase energy then consists of the sum of the individual Canham-Helfrich energies plus a phase energy that penalizes the length of the interface by its line tension.

In case of two distinct phases modeled by the compact orientable surfaces M_1 and M_2 , the sharp-interface two-phase Canham-Helfrich energy reads

$$E_{\text{CH},2}(M_1, M_2) = \sum_{i=1}^2 \underbrace{\int_{M_i} \left(\frac{\beta^i}{2} (H - H_0^i)^2 + \gamma^i K \right) d\mathcal{H}^2}_{\text{Canham-Helfrich energy } E_{\text{CH}}^i(M_i)} + \underbrace{\sigma \int_{M_1 \cap M_2} d\mathcal{H}^1}_{\text{Line tension energy, } \sigma > 0}.$$

In particular, the constraint on phase mass reduces to conservation of the individual phase areas

$$\text{Area}(M_1) = m_1 \quad \text{and} \quad \text{Area}(M_2) = m_2,$$

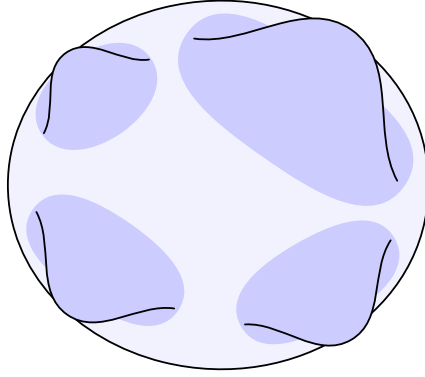


Figure 2.7: Illustration of a giant unilamellar vesicle showing two distinct phases separated by sharp phase-interfaces, as observed in [12].

where $m_1, m_2 \in \mathbb{R}$ are fixed. The generalization to multiphase energies $E_{CH,N}(M_1, \dots, M_N)$ of membranes

$$M = \bigcup_{i=1}^N M_i$$

featuring $N \in \mathbb{N}$ different phases is straightforward. The **multiphase variational problem** is the following: Minimize the energy $E_{CH,N}$ among compact orientable surfaces M_i ($i = 1, \dots, N$) possibly with boundary, such that the phase masses (areas) are fixed, the total enclosed volume is fixed, their union is a closed membrane, and they overlap precisely at their boundaries, i.e.,

- (i) Mass: $\text{Area}(M_i) = m_i$ ($1 \leq i \leq N$),
- (ii) Volume: $\text{Vol}(\bigcup_{i=1}^N M_i) = v_0$,
- (iii) Closedness: $\partial(\bigcup_{i=1}^N M_i) = \emptyset$,
- (iv) Overlap: $M_i \cap M_j = \partial M_i \cap \partial M_j$ ($1 \leq i \neq j \leq N$),

for given data $m_i > 0$ and $v_0 > 0$.

In [20], reproduced in Chapter 3, we address a one-dimensional analog of the Canham-Helfrich problem with continuous phase density, namely elastic curves with density-modulated stiffness. In [22], reproduced in Chapter 4, we consider multiphase membranes with sharp phase-interfaces and establish existence of minimizers in a varifold setting, without any restrictions on their symmetry.

2.1.4 Membrane evolution

Biological membranes are viscous fluids confined to an inextensible surface. Accordingly, their dissipative fluid dynamics is governed by the **incompressible Navier-Stokes equations**

$$\begin{cases} \rho_0 D_t v - \nabla \cdot S &= f, \\ \nabla \cdot v &= 0, \end{cases}$$

with the twist that these equations are posed on an a priori unknown, time evolving surface

$$t \mapsto M(t) \quad \text{for } t \in [0, \infty).$$

Here, $v(\cdot, t): M(t) \rightarrow \mathbb{R}^3$ denotes the spatial (Eulerian) fluid velocity, ρ_0 is the surface mass density, D_t is the material time derivative operator, and $\nabla \cdot$ stands for the surface divergence operator. The surface stress tensor S depends on v and on the membrane viscosity via the BOUSSINESQ-SCRIVEN constitutive law [92]. The forcing f accounts for the pressure and the viscous drag exerted by the surrounding bulk fluid, as well as for the geometric bending force given by the variational derivative of the Canham-Helfrich energy E_{CH} (1.1). Starting from an initial surface

$$M(0) = M_0$$

with velocity $v(\cdot, 0) = v_0$, the evolution of the membrane shape $M(t)$ is determined by the normal component $v \cdot \nu$ of the velocity.

Remark 2.1.2 (Evolution of heterogeneous membranes). Heterogeneous membranes evolve not only by changing their shape, but also by continuously rearranging the local concentration of their constituents. The associated scalar phase density

$$\rho(\cdot, t): M(t) \rightarrow \mathbb{R}$$

is the solution of the *surface diffusion equation* $D_t \rho = \nabla \cdot F$ posed on the evolving surface. The flux F features the variational derivative of the Canham-Helfrich energy (1.2) with respect to the density ρ . In turn, the density influences the shape by altering the force f in the Navier-Stokes equations. This coupled nonlinear PDE system can be variationally derived from ONSAGER's principle of *maximum energy dissipation*, see [6].

When one restricts the dynamics to the quasistatic (overdamped) relaxation to equilibrium, acceleration terms are neglected. In this regime, and disregarding external forces, biomembrane evolution reduces to a **gradient flow** (see Subsection 2.2.3), which formally is given by the PDE (1.3), namely,

$$v = -\nabla E_{\text{CH}}(M).$$

In [21], reproduced in Chapter 5, we discuss the gradient flow for the Canham-Helfrich energy in a varifold setting.

2.2 Calculus of variations and geometric measure theory

In this thesis, the configurations of biomembranes are described in a variational approach, that is, by minimizing their energy. In this section, we introduce this **variational method** and its application to geometric variational problems. After collecting some basic notions for curves and surfaces, we explain the Direct Method and provide a proof of the classical curvature bounds for the Canham-Helfrich energy (Subsection 2.2.1). We then give a brief introduction to concepts of geometric measure theory with emphasis on varifolds (Subsection 2.2.2). This material is mainly based on the lecture notes [94] and the research papers [54, 73]. Finally, we introduce the concept of gradient flows, focusing on their application to geometric energies and on their formulation in a metric space setting (Subsection 2.2.3), where we follow the textbook [5].

2.2.1 Geometric variational problems and the Direct Method

Geometric variational problems are variational problems for **geometric energy functionals**. These are energies of geometric objects, that solely depend on their shape, but not on their position, nor orientation in space, nor on the choice of their parametrization. We present basic notions for energies of planar curves and of surfaces in three-dimensional space, that in particular involve curvature.

Curves and surfaces

A **curve** γ in the plane is an equivalence class of parametrizations $\gamma: [a, b] \rightarrow \mathbb{R}^2$. The derivative $\dot{\gamma}$ defines its tangent vector and the length of γ is given by

$$L(\gamma) = \mathcal{H}^1(\gamma) = \int_{\gamma} d\mathcal{H}^1 = \int_a^b |\dot{\gamma}(t)| dt.$$

The **scalar curvature** $\kappa: [a, b] \rightarrow \mathbb{R}$ measures the change of direction of $\dot{\gamma}$ and is given by

$$\kappa = \det(\dot{\gamma}, \ddot{\gamma})/|\dot{\gamma}|^3.$$

The simplest quadratic curvature energy for curves is the classical EULER-BERNOULLI **elastica functional**

$$E_{\text{EB}}(\gamma) = \frac{1}{2} \int_{\gamma} \kappa^2 d\mathcal{H}^1 = \frac{1}{2} \int_0^L |\ddot{\gamma}(s)|^2 ds,$$

which describes the elastic bending energy of a thin inextensible rod. The second formula only holds for curves in arclength parametrization, which have length $L > 0$ and $\kappa^2 = |\ddot{\gamma}|^2$. The only smooth closed stationary curves of fixed length are the circle and Bernoulli's lemniscate (Figure-Eight curve), the circle being the global minimizer of E_{EB} , cf. [64, 99]. For further notation and preliminaries on planar curves we refer to Subsection 3.2.1.

A **surface** M is a two-dimensional submanifold of $\mathbb{R}^3$. The analog of the length of a curve is the surface area (2.2), namely,

$$\text{Area}(M) = \mathcal{H}^2(M) = \int_M d\mathcal{H}^2.$$

The associated minimization problem with Dirichlet data is known as the PLATEAU problem, whose solutions spanning a given boundary curve are models of soap films.

More complicated geometric variational problems involve energies that depend on the surface unit normal $\nu: M \rightarrow \mathbb{S}^2$ or on its derivatives, which define the curvatures. A surface possesses two principal curvatures, κ_1 and κ_2 , which are the eigenvalues of the **shape operator** $-\nabla^P \nu$. Here, $\nabla^P := (\sum_{j=1}^3 P_{ij} \partial_j)_{i=1}^3$ denotes the tangential derivative, where $P_{ij} = \delta_{ij} - \nu_i \nu_j$ are the components of the tangential projection matrix. The **second fundamental form** of M reads

$$II = -\nabla^P \nu \otimes \nu.$$

The scalar **mean curvature** and the **Gauss curvature** of M can be expressed in terms of the principal curvatures as

$$H = \kappa_1 + \kappa_2 \quad \text{and} \quad K = \kappa_1 \kappa_2.$$

The **mean curvature vector** is given by

$$\bar{H} = H\nu.$$

The surface analog of the elastica energy is the WILLMORE energy (2.5), namely,

$$\mathcal{W}(M) = \frac{1}{4} \int_M H^2 d\mathcal{H}^2.$$

This energy has first been considered in the 19th century as a model for thin elastic plates, then its conformal invariance was studied in the 1920s, before it was eventually rediscovered in the early 1960s, cf. [101]. The unique global minimizer of $\mathcal{W}$ among smooth closed surfaces is the sphere $\mathbb{S}^2$, for which $\mathcal{W}(\mathbb{S}^2) = 4\pi$.

The Direct Method

The **Direct Method of the Calculus of Variations** is a standard method to prove existence of minimizers of an energy functional. It relies on identifying a topology, where one has compactness of minimizing sequences and lower semicontinuity of the energy. More precisely, let the energy

$$E: S \rightarrow (-\infty, \infty]$$

be defined on a topological space S and assume that $\inf_S E > -\infty$. Then there exists a **minimizing sequence** $(u_n)_{n \in \mathbb{N}} \subset S$, namely,

$$E(u_n) \rightarrow \inf_S E \quad (n \rightarrow \infty).$$

If one can show

- **(Sequential) compactness:**

$(u_n)_{n \in \mathbb{N}}$ admits a convergent subsequence $u_{n_k} \xrightarrow{S} u \in S$ ($k \rightarrow \infty$), and

- **(Sequential) lower semicontinuity:**

$E(v) \leq \liminf_{n \rightarrow \infty} E(v_n)$ holds for all $(v_n)_{n \in \mathbb{N}} \subset S$ with $v_n \xrightarrow{S} v \in S$ ($n \rightarrow \infty$),

then u is a minimizer of E . Indeed,

$$\inf_S E = \lim_{n \rightarrow \infty} E(u_n) = \lim_{k \rightarrow \infty} E(u_{n_k}) = \liminf_{k \rightarrow \infty} E(u_{n_k}) \geq E(u) \geq \inf_S E$$

implies the equality $E(u) = \inf_S E$ showing that u is a minimizer, namely,

$$E(u) = \min_S E.$$

A suitable topology balancing compactness and lower semicontinuity is often inferred from a-priori bounds on the energy.

Classical curvature bounds for the Canham-Helfrich energy

The Canham-Helfrich energy (1.1) of a smooth surface features the following a-priori bounds on curvature.

Lemma 2.2.1 (Classical curvature bounds). *Let $M \subset \mathbb{R}^3$ be a compact orientable C^2 surface, $H_0 \in \mathbb{R}$, and $-2\beta < \gamma < 0$. Then*

$$\|H\|_{L^2(M)}^2 = \int_M (\kappa_1^2 + \kappa_2^2) \, d\mathcal{H}^2 \leq c_1 (E_{\text{CH}}(M) + c_2 \text{Area}(M))$$

for $c_1 > 0$, $c_2 \geq 0$ that only depend on β, γ , and H_0 . Moreover, $c_2 = 0$ if and only if $H_0 = 0$.

Proof. It suffices to show the inequality $\frac{\beta}{2}(H - H_0)^2 + \gamma K \geq \frac{1}{c_1}|H|^2 - c_2$, which upon integration yields the claim, namely,

$$\begin{aligned} E_{\text{CH}}(M) &= \int_M \left(\frac{\beta}{2}(H - H_0)^2 + \gamma K \right) \, d\mathcal{H}^2 \\ &\geq \frac{1}{c_1} \int_M |H|^2 \, d\mathcal{H}^2 - c_2 \mathcal{H}^2(M) = \frac{1}{c_1} \|H\|_{L^2(M)}^2 - c_2 \text{Area}(M). \end{aligned}$$

2 Preliminaries

Let $\varepsilon > 0$ which will be further specified below. By Young's inequality $-2HH_0 \geq -\varepsilon H^2 - H_0^2/\varepsilon$,

$$\begin{aligned} \frac{\beta}{2}(H - H_0)^2 + \gamma K &= \frac{\beta}{2}(H^2 - 2HH_0 + H_0^2) + \gamma K \\ &\geq \beta(1 - \varepsilon)\frac{H^2}{2} + \beta(1 - 1/\varepsilon)\frac{H_0^2}{2} + \gamma K \\ &= \beta(1 - \varepsilon)\frac{|II|^2}{2} + (\beta(1 - \varepsilon) + \gamma)K + \beta(1 - 1/\varepsilon)\frac{H_0^2}{2}, \end{aligned}$$

where the last step follows from $H^2/2 = (\kappa_1^2 + \kappa_2^2)/2 + \kappa_1\kappa_2 = |II|^2/2 + K$. The estimate

$$(\beta(1 - \varepsilon) + \gamma)K \geq -|\beta(1 - \varepsilon) + \gamma|(\kappa_1^2 + \kappa_2^2)/2 = -|\beta(1 - \varepsilon) + \gamma||II|^2/2$$

then leads to the desired inequality

$$\frac{\beta}{2}(H - H_0)^2 + \gamma K \geq \underbrace{\frac{1}{2}(\beta(1 - \varepsilon) - |\beta(1 - \varepsilon) + \gamma|)}_{=: 1/c_1}|II|^2 + \underbrace{\frac{\beta}{2}(1 - 1/\varepsilon)H_0^2}_{=: -c_2},$$

because by the assumption $-2\beta < \gamma < 0$ there exists $\varepsilon > 0$ such that $c_1 > 0$ and $c_2 \geq 0$. Indeed, since $-2\beta < \gamma < 0 \iff 0 < 2\beta + \gamma < 2\beta \iff 0 < 1 + \frac{\gamma}{2\beta} < 1$ we can choose

$$0 < \varepsilon < 1 + \frac{\gamma}{2\beta} < 1,$$

which in particular implies $\beta(1 - \varepsilon) > -\gamma/2 > 0$. Consequently, $\beta(1 - \varepsilon) + \gamma > -\gamma/2 + \gamma = \gamma/2$, which with $\gamma < 0$ shows that the argument of $|\cdot|$ in the expression defining $1/c_1$ may take both signs. Then $c_1 > 0$ follows from

$$\frac{1}{c_1} = \begin{cases} \frac{1}{2}(\beta(1 - \varepsilon) - (\beta(1 - \varepsilon) + \gamma)) = -\frac{\gamma}{2} > 0, & \text{for } \beta(1 - \varepsilon) + \gamma \geq 0, \\ \frac{1}{2}(\beta(1 - \varepsilon) - (-\beta(1 - \varepsilon) - \gamma)) = \beta(1 - \varepsilon) + \frac{\gamma}{2} > -\frac{\gamma}{2} + \frac{\gamma}{2} = 0, & \text{for } \beta(1 - \varepsilon) + \gamma < 0. \end{cases}$$

Nonnegativity of c_2 can be seen from

$$c_2 = -\frac{\beta}{2}(1 - 1/\varepsilon)H_0^2 = \frac{\beta(1 - \varepsilon)}{2\varepsilon}H_0^2 > -\frac{\gamma}{4\varepsilon}H_0^2 \geq 0$$

and the definition of c_2 implies that $c_2 = 0$ if and only if $H_0 = 0$, which completes the proof. $\square$

An alternative proof is given in [27, Lemma 1].

The L^2 curvature bounds of Lemma 2.2.1 are not enough to guarantee compactness of smooth immersions

$$\varphi \in C^1(\Sigma; \mathbb{R}^3) \quad \text{with} \quad \varphi(\Sigma) = M,$$

because the Sobolev embedding $W^{2,p} \hookrightarrow C^1$ on the two-dimensional domain Σ requires L^p curvature bounds with $p > 2$. For this reason, the Direct Method does not directly apply in a “parametric approach”, where surfaces are represented by immersions in Sobolev spaces. Alternatively one can consider the energy as a functional on a class of surfaces, the “ambient approach”. A particularly suitable class are varifolds, which will be introduced in the forthcoming Subsection 2.2.2.

Remark 2.2.2 (Counterexample, cf. [49]). With the topology of varifolds, there exists a counterexample to lower semicontinuity of the Canham-Helfrich functional for $\beta = 2, \gamma = 0$, and $H_0 = 1$, i.e., for the energy $E_{\text{CH}}^0(M) = \int_M (H - 1)^2 d\mathcal{H}^2$ by GROSSE-BRAUCKMANN [49]: There exists a sequence of unbounded surfaces $(M_k)_{k \in \mathbb{N}}$ immersed in $\mathbb{R}^3$ that have constant mean curvature $H = 1$ so that $E_{\text{CH}}^0(M_k) = 0$, but which converges in the sense of varifolds to a plane M with multiplicity two. Consequently, $\lim_{k \rightarrow \infty} E_{\text{CH}}^0(M_k) = 0 < E_{\text{CH}}(M) = \infty$, which contradicts lower semicontinuity.

2.2.2 An introduction to varifolds

Varifolds are a concept of geometric measure theory that generalize differentiable submanifolds of Euclidean spaces or of other manifolds. They were introduced in the 1960s by YOUNG and ALMGREN, with the very aim of tackling geometric variational problems by the Direct Method. This also explains their name, which is a concatenation of “variational manifolds”.

Let $m, n \in \mathbb{N}$ with $1 \leq m \leq n$ and $\Omega \subset \mathbb{R}^n$ be open. By $G_{m,n}$ we denote the set of all m -dimensional linear subspaces of $\mathbb{R}^n$, the **Grassmannian**. Elements $P \in G_{m,n}$ can be represented by the orthogonal projection matrices $P \in \mathbb{R}^{n \times n}$ on the linear subspace in $G_{m,n}$.

Definition 2.2.3 (Varifolds). An m -varifold in Ω is a Radon measure $V \in \mathcal{M}(\Omega \times G_{m,n})$.

The **mass** of a varifold is the induced Radon measure $\mu_V := \pi_\# V \in \mathcal{M}(\Omega)$, that is, the push-forward by $\pi: \Omega \times G_{m,n} \rightarrow \Omega$, $\pi(x, P) := x$, so that $\mu_V(B) = V(B \times G_{m,n})$ for all Borel sets $B \subseteq \Omega$. In particular, the total mass of V is

$$\mu_V(\Omega) = \int_{\Omega} d\mu_V(x) = \int_{\pi(\Omega \times G_{m,n})} d(\pi_\# V)(x) = \int_{\Omega \times G_{m,n}} dV(x, P) = V(\Omega \times G_{m,n}).$$

By RIESZ' Theorem, Radon measures are in duality to continuous functions with compact support. Thus varifolds are characterized by their action on test functions $\varphi \in C_c(\Omega \times G_{m,n})$, namely,

$$V(\varphi) = \int_{\Omega \times G_{m,n}} \varphi(x, P) dV(x, P).$$

Convergence of varifolds is the **weak-* convergence** of Radon measures given by

$$V_k \rightharpoonup^* V \ (k \rightarrow \infty) : \Longleftrightarrow V_k(\varphi) \rightarrow V(\varphi) \ (k \rightarrow \infty) \quad \text{for all } \varphi \in C_c(\Omega \times G_{m,n}).$$

Being just Radon measures, varifolds can be very irregular. A subclass of varifolds has more similarity to classical submanifolds: These are the integral varifolds, which essentially consist of countably rectifiable sets with integer multiplicity.

A set $M \subset \mathbb{R}^n$ is called **countably m -rectifiable**, if it has finite m -dimensional Hausdorff measure and, up to a set of Hausdorff measure zero, coincides with a countable union of C^1 submanifolds:

$$\mathcal{H}^m(M) < \infty \quad \text{and} \quad \mathcal{H}^m(M \setminus \bigcup_{j=1}^{\infty} M_j) = 0$$

for embedded m -dimensional C^1 submanifolds $M_j \subset \mathbb{R}^n$. A countably m -rectifiable set M can also be characterized by the existence of **approximate tangent spaces** $T_x M$ at $\mathcal{H}^m$ -a.e. points $x \in M$, which are defined by $\lim_{r \rightarrow 0} \int_{\eta_{x,r}(M)} \phi(y) d\mathcal{H}^m(y) = \int_{T_x M} \phi(y) d\mathcal{H}^m(y)$ for all $\phi \in C_c(\mathbb{R}^n)$ and $\mathcal{H}^m$ -a.e. $x \in M$, where $\eta_{x,r}(\cdot) := \frac{\cdot - x}{r}$ is the blow-up of M as $r \rightarrow 0$.

Definition 2.2.4 (Integral varifolds, IV_m). An integral m -varifold is an equivalence class

$$V = V[M, \theta] \in IV_m(\Omega)$$

with representatives defined by $dV(x, P) = \theta(x) d(\mathcal{H}^m \llcorner M)(x) \otimes \delta_{T_x M}(P)$, i.e.,

$$V(\varphi) = \int_{\Omega \times G_{m,n}} \varphi(x, P) dV(x, P) = \int_M \varphi(x, T_x M) \theta(x) d\mathcal{H}^m(x) \quad \text{for all } \varphi \in C_c(\Omega \times G_{m,n}),$$

where $M \subset \Omega$ is countably m -rectifiable and the **multiplicity** $\theta \in L^1_{\text{loc}, \mathcal{H}^m \llcorner M}(M)$ has values $\theta(x) \in \mathbb{N}$ for $\mathcal{H}^m$ -a.e. $x \in M$. Equivalence holds if M and M' have $\mathcal{H}^m$ -negligible symmetric difference, i.e., $\mathcal{H}^m(M \Delta M') = 0$, and $\theta(x) = \theta'(x)$ for $\mathcal{H}^m$ -a.e. in $x \in M \cap M'$.

Any smoothly embedded submanifold M of $\mathbb{R}^n$ corresponds to an integral varifold with multiplicity $\theta \equiv 1$, i.e., $V = V[M, 1]$. The role of the multiplicity can be seen from the following example: If two surfaces approach each other and eventually partly overlap, their varifold limit will be given by the limit surface, but with $\theta(x) = 2$ for x in the overlap region.

In contrast to general varifolds $V \in \mathcal{M}(\Omega \times G_{m,n})$, a bounded mass $\mu_V(\Omega) < \infty$ is not enough for the compactness of integral varifolds $V \in IV_m(\Omega)$. Qualitatively, this is clear, as one can curl up a surface of fixed area to a point corresponding to the zero varifold. The missing ingredient was identified by ALLARD [1] and is a bound on the first variation, which is related to the mean curvature.

The **first variation** of $V \in \mathcal{M}(\Omega \times G_{m,n})$ is the linear functional

$$\delta V: C_c^1(\Omega; \mathbb{R}^n) \rightarrow \mathbb{R}, \quad \delta V(u) := \int_{\Omega \times G_{m,n}} \operatorname{div}^P u(x) \, dV(x, P)$$

where $\operatorname{div}^P u := \sum_{i,j=1}^n P_{ij} \partial_j u_i$ is the tangential divergence. Its total variation norm is given by

$$\|\delta V\|_{\text{TV}}(B) := \sup \{ \delta V(u) : u \in C_c^1(\Omega; \mathbb{R}^n), \|u\|_{L^\infty(\Omega)} \leq 1, \operatorname{supp}(u) \subset\subset B \}$$

for all Borel sets $B \subseteq \Omega$.

Theorem 2.2.5 (Allard's compactness theorem, [1]). *If $(V_h)_{h \in \mathbb{N}} \subset IV_m(\Omega)$ and*

$$\mu_{V_h}(\Omega) + \|\delta V_h\|_{\text{TV}}(\Omega) < \infty$$

uniformly in h , then there exists $V \in IV_m(\Omega)$ and a subsequence $(V_{h_k})_{k \in \mathbb{N}}$ with $V_{h_k} \rightharpoonup^* V$ as $k \rightarrow \infty$.

If $\|\delta V\|_{\text{TV}}(\Omega) < \infty$ holds, δV gives rise to a Radon measure on Ω by Riesz' Theorem. The RADON-NIKODYM Theorem then allows us to split δV in an absolutely continuous part and a singular part with respect to the mass measure μ_V :

$$\delta V = (\delta V)_{\text{ac}} + (\delta V)_{\text{sing}} = -\bar{H}^V \mu_V - \sigma^V.$$

The density $\bar{H}^V := -\frac{d(\delta V)}{d\mu_V} \in L_{\text{loc}, \mu_V}^1(\Omega; \mathbb{R}^n)$ defines Allard's **generalized mean curvature vector** and the singular part can be interpreted as a **boundary measure** $\sigma^V \in \mathcal{M}^n(\Omega)$. This terminology is justified by applying δV to a vector-valued test function $u \in C_c^1(\Omega; \mathbb{R}^n)$, namely,

$$\delta V(u) = - \int_{\Omega} \bar{H}^V \cdot u \, d\mu_V - \int_{\Omega} u \cdot d\sigma^V,$$

which is similar to the classical divergence theorem for submanifolds with boundary.

In order to formulate geometric variational problems involving more components of the second fundamental form than the mean curvature vector, as in particular is the case for the Canham-Helfrich energy, Allard's mean curvature vector is not enough. To treat these variational problems with varifolds, one needs the concept of curvature varifolds. This subclass of varifolds was first introduced by HUTCHINSON [54] and MANTEGAZZA [73] extended the definition to include surfaces with boundary.

Definition 2.2.6 (Curvature varifolds with boundary, AV_m). A curvature m -varifold with boundary, $V \in AV_m(\Omega)$, is an integral m -varifold $V \in IV_m(\Omega)$ such that there exist

$$A^V \in L_{\text{loc}, V}^1(\Omega \times G_{m,n}; \mathbb{R}^{n \times n \times n}) \quad \text{and} \quad \partial V \in \mathcal{M}^n(\Omega \times G_{m,n})$$

with

$$\int_{\Omega \times G_{m,n}} \sum_{j=1}^n \left(P_{ij} \partial_j \varphi + \sum_{k=1}^n (\partial_{P_{jk}} \varphi) A_{ijk}^V + A_{jij}^V \varphi \right) dV = - \int_{\Omega \times G_{m,n}} \varphi d(\partial V)_i$$

for all $\varphi \in C_c^1(\Omega \times G_{m,n})$ and $1 \leq i \leq n$.

For $V = V[M, 1]$ with a smooth hypersurface M in $\mathbb{R}^n$ with boundary ∂M , the **generalized curvature functions** A^V and the **boundary measure** ∂V reduce to their classical counterparts. Indeed, in this case, the components of $A = A^V$ are the tangential derivatives of the projection matrices, namely,

$$A_{ijk} = \partial_i^P P_{jk} = \sum_{l=1}^n (P_{il} \partial_l) P_{jk} = \Pi_{ik}^j + \Pi_{ij}^k,$$

the components $\Pi_{ij}^k = -(\partial_j^P \nu_i) \nu_k$ of the second fundamental form can be recovered from

$$\Pi_{ij}^k = \sum_{l=1}^n A_{ikl} P_{lj},$$

and the components of the mean curvature vector $\bar{H} = H\nu$ read

$$\bar{H}_i = \sum_{l=1}^n \Pi_{ll}^i = \sum_{l=1}^n A_{lil}$$

for $1 \leq i, j, k \leq n$. Moreover, the boundary measure $\partial V \in \mathcal{M}^n(\Omega \times G_{m,n})$ induces the measure

$$\pi_{\#}(\partial V) = \nu^{\partial M} \mathcal{H}^{n-2} \llcorner (\partial M) = \nu^{\partial M} \delta_{\partial M} \in \mathcal{M}^n(\Omega),$$

where $\nu^{\partial M}$ stands for the inner conormal vector of M , i.e., the inwards-pointing normal to ∂M .

For further results on geometric measure theory, in particular definitions and notations for **oriented varifolds** and **currents**, as well as their compactness properties with respect to weak-* convergence, we refer to the Subsections 4.2.3, 4.2.4, and 5.2.2.

2.2.3 Gradient flows

Given an energy and an initial state, a **gradient flow** describes the ensuing evolution towards equilibrium along a trajectory of fastest energy decrease. The gradient flow associated to the Canham-Helfrich energy is referred to as the **Canham-Helfrich flow**. This models the evolution of a biological membrane that viscously relaxes towards its equilibrium configuration, without the presence of any additional force. In case of a homogeneous membrane, this is a purely geometric flow.

In its classical formulation, a geometric gradient flow of submanifolds is defined in terms of their parametrizations. Restricting to surfaces immersed in $\mathbb{R}^3$, let

$$\varphi: \Sigma \times [0, \infty) \rightarrow \mathbb{R}^3$$

with $\Sigma \subset \mathbb{R}^2$ open (locally) parametrize a smooth surface evolving in time,

$$M(t) = \varphi(\Sigma, t) = \varphi_t(\Sigma)$$

for $0 \leq t < \infty$. A geometric energy of surfaces, E , can be seen as a functional on a class of admissible parametrizations $\varphi_t := \varphi(\cdot, t) \in \mathcal{A} \subset C^k(\Sigma; \mathbb{R}^3)$ for some suitable $k \in \mathbb{N}$.

Definition 2.2.7 (L^2 gradient flow). The L^2 gradient flow associated to E and starting from the initial surface $M_0 = \varphi_0(\Sigma)$ with $\varphi_0 \in \mathcal{A}$ is the solution $t \mapsto \varphi_t \in \mathcal{A}$ to the initial value problem

$$\begin{cases} \dot{\varphi} = -\nabla E(\varphi) & \text{on } \Sigma \times [0, \infty), \\ \varphi(\cdot, 0) = \varphi_0 & \text{on } \Sigma. \end{cases}$$

Here, $\dot{\varphi} = \partial_t \varphi$ is the time-derivative and ∇E is the L^2 gradient of E defined by expressing the first variation (Gâteaux derivative) δE in terms of the L^2 inner product, namely,

$$\delta E(\varphi)(\eta) := \left. \frac{d}{d\varepsilon} E(\varphi + \varepsilon \eta) \right|_{\varepsilon=0} = \langle \nabla E(\varphi), \eta \rangle_{L^2(\Sigma)} \quad \text{for all } \eta \in C_c^\infty(\Sigma; \mathbb{R}^3),$$

where

$$\langle u, v \rangle_{L^2(\Sigma)} := \int_{\Sigma} u(x) \cdot v(x) \, d\mathcal{H}^2(x).$$

Reparametrization invariance of geometric energies implies that, for all $0 \leq t < \infty$, the spatial velocity

$$v(\cdot, t) = \dot{\varphi}(\varphi_t^{-1}(\cdot), t)$$

is orthogonal to $M(t)$, that is, $v(\cdot, t)$ is aligned with the unit normal $\nu(\cdot, t): M(t) \rightarrow \mathbb{R}^3$.

In case of the Canham-Helfrich flow, there are two possibilities to incorporate the area and volume constraints (2.1). Either, one considers them as globally fixed quantities and defines a projected gradient flow [80, 88, 89], or one treats them as local constraints by working with the penalized energy $E_{\text{CH}}^{\alpha, p}$ introduced in (2.6). In the second approach, the **Canham-Helfrich flow equation** (1.3) takes the form of the following fourth-order quasilinear parabolic PDE:

$$\begin{aligned} v &= -\nabla E_{\text{CH}}^{\alpha, p} \\ &= -\beta \left(\Delta H + \frac{1}{2}(H - H_0)(H^2 - 4K + HH_0) \right) \nu - \alpha H \nu - p \nu. \end{aligned}$$

The disadvantage of this classical PDE formulation is that it can not be extended beyond singularities, for example when topology changes occur as in the budding transformation. An idea to overcome this problem is to generalize the notion of a gradient flow to spaces of generalized surfaces, in particular, varifolds. As we elaborate in Chapter 5, these spaces turn out to be metrizable thanks to the mass constraint and we may therefore apply concepts for gradient flows in metric spaces. In particular, we employ the **Generalized Minimizing Movements (GMMs)** method introduced by DE GIORGI [35]. This time-discrete approximation, which is also known as the incremental minimization scheme, can be seen as a variational extension of the implicit Euler-scheme to the metric space setting. Actually, the GMMs approach has a solid tradition in case of the mean curvature flow, starting with the work of ALMGREN, TAYLOR, & WANG [2].

Following [5, Ch. 2], we consider a complete metric space (S, d) and a functional

$$E: S \rightarrow (-\infty, \infty],$$

which is bounded from below and proper, i.e., $E \not\equiv \infty$. The associated gradient flow evolution in a time interval $[0, T]$ and starting from

$$u^0 \in \text{Dom}(E) := \{u \in S : E(u) < \infty\}$$

is a curve $u: [0, T] \rightarrow S$ with $u(0) = u^0$, along which E decreases most rapidly. We are interested in global solutions where the time-interval is $[0, \infty)$.

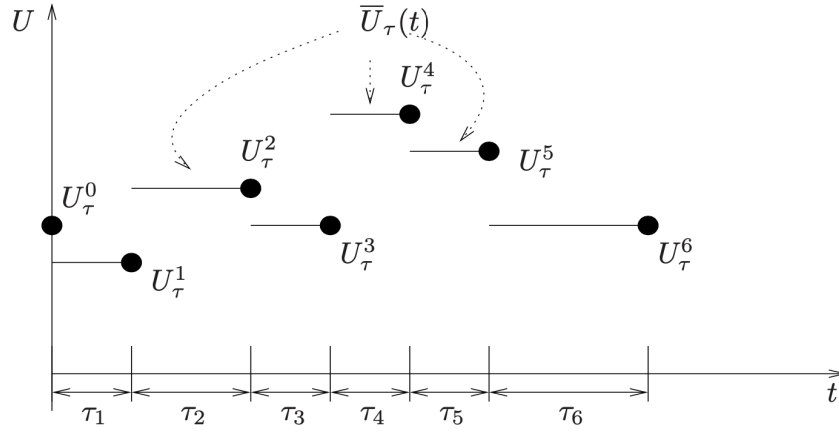


Figure 2.8: The discrete solution $(U_\tau^n)_{n \in \mathbb{N}_0}$ in the minimizing movement scheme (in the case of a variable mesh size) and the corresponding piecewise constant interpolant $t \mapsto \bar{U}_\tau(t)$. [Source: [5]]

Definition 2.2.8 (GMMs, [5, Def. 2.0.6]). A curve

$$u: [0, \infty) \rightarrow S$$

is called a GMM associated to E starting from u^0 , if there exists a sequence $(\mathcal{P}_{\tau_m})_{m \in \mathbb{N}}$ of partitions with fineness $\tau_m \rightarrow 0$ ($m \rightarrow \infty$), such that for all $t \geq 0$,

$$\bar{U}_{\tau_m}(t) \xrightarrow{(S,d)} u(t).$$

Here, as illustrated in Figure 2.8,

$$\mathcal{P}_\tau := \{0, \tau, 2\tau, \dots\}$$

is a partition of $[0, \infty)$ with steps of size $\tau > 0$ (or with steps of various lengths but common lower bound $\tau > 0$) and

$$\bar{U}_\tau: [0, \infty) \rightarrow S$$

is the piecewise constant interpolant of the sequence $(U_\tau^n)_{n \in \mathbb{N}_0}$ of **discrete minimizers**, which are iteratively constructed by

$$U_\tau^0 := u^0 \quad \text{and} \quad U_\tau^n \in \operatorname{argmin}_S E_\tau(\cdot; U_\tau^{n-1}),$$

with $E_\tau(\cdot; U^{n-1}): S \rightarrow (-\infty, \infty]$ being the augmented functional

$$u \mapsto E_\tau(u; U^{n-1}) := E(u) + \frac{1}{2\tau} d^2(u, U^{n-1}).$$

Remark 2.2.9 (GMMs in Euclidean spaces). If $S = \mathbb{R}^N$ for some $N \in \mathbb{N}$, then a GMM u is the solution to the classical gradient flow system of ordinary differential equations given by $u(0) = u^0$ and $\dot{u}(t) = -\nabla E(u(t))$ for $t \geq 0$. The definition of discrete solutions $(U_\tau^n)_{n \in \mathbb{N}_0}$ corresponds to the implicit Euler scheme.

Any GMMs evolution u decreases the energy, i.e.,

$$E(u(t)) \leq E(u^0) \quad \text{for all } t \geq 0.$$

Hence, if the flow starts from a minimizer, i.e., $E(u^0) = \min_S E$, then it is stationary, i.e.,

$$u(t) = u^0 \quad \text{for all } t \geq 0.$$

2 Preliminaries

Theorem 2.2.10 (Existence of GMMs, [5, Prop. 2.2.3]). *If the sublevel sets of E are compact, then, starting from $u^0 \in \text{Dom}(E)$, there exists a GMMs solution $u \in AC_{\text{loc}}^2([0, \infty); S)$.*

Here, sublevel sets of E are given by

$$\{E \leq c\} = \{v \in S : E(v) \leq c\}$$

for any $c \in \mathbb{R}$. Notice that as S is metric, compactness of sublevels $\{E \leq c\}$ implies their closedness, which in turn is equivalent to the lower semicontinuity of E . Moreover, $AC_{\text{loc}}^2([0, \infty); S)$ denotes the class of **locally L^2 -absolutely continuous curves**, that is, maps $u : [0, \infty) \rightarrow S$ such that, for all finite intervals $(a, b) \subset [0, \infty)$, there exists $w \in L^2(a, b)$ with

$$d(u(t_1), u(t_2)) \leq \int_{t_1}^{t_2} w(t) dt$$

for all $a < t_1 \leq t_2 < b$. Any $u \in AC_{\text{loc}}^2([0, \infty); S)$ possesses a **metric derivative** $|u'| \in L_{\text{loc}}^2(0, \infty)$ given by

$$|u'| (t) := \lim_{s \rightarrow t} \frac{d(u(s), u(t))}{|s - t|} \quad \text{for a.e. } t \in (0, \infty).$$

Furthermore, we have the following energy inequality, which quantifies the dissipation of energy during the gradient flow evolution:

Lemma 2.2.11 (Energy inequality). *Under the assumptions of Theorem 2.2.10, the GMMs solution u satisfies*

$$E(u(t)) + \frac{1}{2} \int_0^t |u'|^2(r) dr \leq E(u^0) \quad (2.7)$$

for all $0 \leq t < \infty$.

Proof. Let $(U^n)_{n \in \mathbb{N}_0}$ be a sequence of discrete minimizers for a fixed time-step $\tau = \tau_m$. From minimality of U^n we have

$$E_\tau(U^n; U^{n-1}) = E(U^n) + \frac{1}{2\tau} d^2(U^n, U^{n-1}) \leq E_\tau(U^{n-1}; U^{n-1}) = E(U^{n-1}).$$

Iterating this inequality from $i \in \mathbb{N}_0$ to $j \in \mathbb{N}$ with $i < j$ results in the discrete energy estimate $E(U^j) + \frac{1}{2\tau} \sum_{n=i+1}^j d^2(U^n, U^{n-1}) \leq E(U^i)$, or

$$E(U^i) - E(U^j) \geq \frac{1}{2\tau} \sum_{n=i+1}^j d^2(U^n, U^{n-1}).$$

Let $\bar{w}_\tau : [0, \infty) \rightarrow [0, \infty)$ be the piecewise constant function with values $\bar{w}_\tau(t_n) := \frac{d(U^n, U^{n-1})}{\tau}$ in the time intervals $(t_{n-1}, t_n]$ for $t_n = n\tau$, $n \in \mathbb{N}$. With $U^n = \bar{U}_\tau(t_n)$ (cf. Figure 2.8), the estimate takes the form

$$E(\bar{U}_\tau(t_i)) - E(\bar{U}_\tau(t_j)) \geq \frac{1}{2} \sum_{n=i+1}^j \bar{w}_\tau^2(t_n) \tau = \frac{1}{2} \int_{t_i}^{t_j} \bar{w}_\tau^2(r) dr.$$

With $t_i =: s < t := t_j$, this gives

$$\frac{1}{2} \int_s^t \bar{w}_\tau^2(r) dr \leq E(\bar{U}_\tau(s)) - E(\bar{U}_\tau(t)) \leq E(u^0) - \inf_S E,$$

showing that there exists a (nonrelabelled) subsequence with $\bar{w}_\tau \rightharpoonup^* w$ in $L^2_{\text{loc}}(s, t)$ ($\tau \rightarrow 0$). In particular, by the triangle inequality,

$$d(\bar{U}_\tau(s), \bar{U}_\tau(t)) = d(U^i, U^j) \leq \sum_{n=i+1}^j d(U^n, U^{n-1}) = \sum_{n=i+1}^j \tau w_\tau(t_n) = \int_s^t \bar{w}_\tau(r) \, dr.$$

Then the convergences $\bar{U}_\tau(t) \xrightarrow{(S,d)} u(t)$ and $\bar{U}_\tau(s) \xrightarrow{(S,d)} u(s)$ for $\tau \rightarrow 0$ imply that the limit of the previous inequality is $d(u(s), u(t)) \leq \int_s^t w(r) \, dr$, showing that $w \in L^2_{\text{loc}}(s, t)$ is an upper bound for the metric derivative, namely

$$|u'(r)| \leq w(r)$$

for a.e. $r \in (s, t)$. For $s = 0$, the discrete energy estimate reads

$$E(u^0) - E(\bar{U}_\tau(t)) \geq \frac{1}{2} \int_0^t \bar{w}_\tau^2(r) \, dr.$$

Upon rearranging terms to $E(\bar{U}_\tau(t)) + \frac{1}{2} \int_0^t \bar{w}_\tau^2(r) \, dr \leq E(u^0)$ and employing lower semicontinuity,

$$E(u(t)) \leq \liminf_{\tau \rightarrow 0} E(\bar{U}_\tau(t)),$$

we obtain the energy inequality (2.7) as $\tau \rightarrow 0$. $\square$

Remark 2.2.12 (Energy identities under higher regularity, [5]). If in addition to the assumptions of Theorem 2.2.10, E is continuous, then one can show that

$$E(u(t)) + \frac{1}{2} \int_s^t |u'|^2(r) \, dr \leq E(u(s))$$

holds for all $0 \leq s < t < \infty$. Moreover, if the local slope $|\partial E|$ is lower semicontinuous and a strong upper gradient for E , then u is a *curve of maximal slope* and satisfies the *energy equality*

$$E(u(t)) + \frac{1}{2} \int_s^t (|u'|^2(r) + |\partial E|^2(u(r))) \, dr = E(u(s)),$$

for all $0 \leq s < t < \infty$, see [5, Definitions 1.2.1, 1.2.4, 1.3.2; Theorem 2.3.3].

Chapter 3

Bifurcation of elastic curves with modulated stiffness

This chapter consists of my publication [20] with GASPARD JANKOWIAK, CHRISTIAN SCHMEISER, and ULISSE STEFANELLI.

Abstract

We investigate the equilibrium configurations of closed planar elastic curves of fixed length, whose stiffness, also known as the bending rigidity, depends on an additional density variable. The underlying variational model relies on the minimization of a bending energy with respect to shape and density and can be considered as a one-dimensional analog of the Canham-Helfrich model for heterogeneous biological membranes. We present a generalized Euler-Bernoulli elastica functional featuring a density-dependent stiffness coefficient. In order to treat the inherent nonconvexity of the problem we introduce an additional length scale in the model by means of a density gradient term. We derive the system of Euler-Lagrange equations and study the bifurcation structure of solutions with respect to the model parameters. Both analytical and numerical results are presented.

3.1 Introduction

We investigate the equilibrium configurations of elastic curves featuring an additional scalar density variable which influences the bending rigidity. Our interest is motivated by the variational modelization of the *shapes of biological membranes*, originally proposed by CANHAM [25] and HELFRICH [50] to explain the characteristic biconcave shape of a human red blood cell. According to this model, the equilibrium membrane shape Σ minimizes the bending energy

$$E_{\text{CH}}(\Sigma) = \int_{\Sigma} \left(\frac{\beta}{2} (H - H_0)^2 + \gamma K \right) dS$$

under suitable constraints on membrane area and enclosed volume. Here, Σ is a smooth closed surface embedded in $\mathbb{R}^3$, H and K are the mean and the Gauss curvature of Σ , respectively, and the material parameters comprise the stiffnesses (bending rigidities) $\beta > 0$, $\gamma < 0$ as well as the spontaneous curvature $H_0 \in \mathbb{R}$. The material parameters of heterogeneous biomembranes are assumed to depend on the variable membrane composition, which is described by

a scalar function $\rho: \Sigma \rightarrow \mathbb{R}$ which we interpret as a density of fixed total mass. On the other hand, the geometry of the membrane influences the distribution of the density ρ , which originates a *coupling effect between curvature and composition*. Indeed, the energy for heterogeneous biomembranes has to be minimized with respect to both membrane geometry Σ and composition ρ simultaneously. Configurations featuring this coupling have been experimentally observed, e.g. example by BAUMGART, HESS, & WEBB [12] in case of giant unilamellar vesicles. Furthermore, the coupling effect also plays an essential role the dynamic morphology changes of cells, where special curved membrane proteins are involved, cf. MCMAHON & GALLOP [76].

Results on the mathematical analysis of the variational problem for heterogeneous biomembranes have been obtained by CHOKSI, MORANDOTTI, & VENERONI [26] as well as HELMERS [53], who proved the existence of multiphase minimizers in the axisymmetric regime. By dropping the symmetry restriction, existence of multiphase minimizers has been recently obtained by [22] in the weak setting of *varifolds*. For a collection of recent results on both single- and multiphase Canham-Helfrich models, the reader is referred to [11, 22, 36, 41, 42, 70, 78, 84, 102].

To the best of our knowledge, proving existence of minimizers for membranes featuring continuous phase densities and general material parameter models is an open problem. We move a first step in this direction in the present paper, by focusing on the lower-dimensional setting of curves instead. A classical elastic curve in the plane, $\gamma: [0, L] \rightarrow \mathbb{R}^2$, minimizes the EULER-BERNOULLI elastic bending energy (also known as the WILLMORE energy)

$$E_{\text{EB}}(\gamma) = \frac{1}{2} \int_{\gamma} \kappa^2 \, ds,$$

where κ is the scalar curvature of γ . The stationary points are called *elasticae* and can be analytically described in terms of elliptic functions. As was already clear to Euler, the only closed elasticae of fixed length in the plane are the circle and Bernoulli's Figure 8 curve, the single covered circle being the unique global minimizer of E_{EB} , see for example TRUESDELL [99] and LANGER & SINGER [64].

We now modify the setting by taking the additional scalar density ρ into the picture. The density ρ modulates the elastic behavior of the curve. For this purpose, we consider the following elastic bending energy with *density-modulated stiffness*,

$$E_0(\rho, \gamma) = \frac{1}{2} \int_{\gamma} \beta(\rho) \kappa^2 \, ds.$$

Our interest lies on the effects of the variable stiffness β and we dispense with the spontaneous curvature H_0 , for simplicity. In order to take into account the coupling between shape and composition, we have to minimize E_0 with respect to both γ and ρ . Admissible curves γ are asked to be planar, regular, C^1 -closed, and have fixed length L , whereas admissible densities ρ are required to have fixed mass $\int_{\gamma} \rho \, ds = M$.

The application of the Direct Method for the minimization of E_0 calls for checking lower semi-continuity with respect to weak topologies, which in turn asks for the convexity of the integrand of E_0 . Yet, if such convexity is imposed, only the trivial minimizer exists, namely the constant density $\rho_0 = M/L$ on a circle with curvature $\kappa_0 = 2\pi/L$. This however is insufficient for describing the rich geometric morphologies that can be observed in biological membranes.

In the following, we will therefore not assume convexity of the integrand of E_0 . This lack of convexity may however lead to nonexistence of minimizers, see Section 3.3 below. We are hence forced to consider a regularized energy E_{μ} , featuring an additional length scale in terms of a

gradient term in ρ , namely,

$$E_\mu(\rho, \gamma) = \frac{1}{2} \int_\gamma (\beta(\rho) \kappa^2 + \mu \dot{\rho}^2) \, ds, \quad (3.1)$$

where $\dot{\rho} = \frac{d\rho}{ds}$. The parameter μ may be physically interpreted as the diffusivity of the density, cf. (3.7). For μ large, the only minimizer is the trivial one, see Proposition 3.3.3. By lowering μ one observes the onset of bifurcations from the trivial state. The main focus of this study is the rigorous bifurcation analysis in terms of μ . We analytically classify the bifurcation behavior of solutions of the Euler-Lagrange equations of E_μ . Moreover, we provide an exhaustive suite of numerical experiments, illustrating the distinguished patterning of minimizers of E_μ , depending on μ .

A variational model for planar elastic curves with density has also been studied by HELMERS [51]. He focused on the effect of spontaneous curvature and established a Γ -convergence result to the sharp interface limit. Let us mention also the recent work by PALMER & PÁMPANO [82], who presented analysis and numerics for the shapes of elastic rods with anisotropic bending energies.

We conclude this introduction by presenting the outline of this study. In Section 3.2, we briefly describe the mathematical setting and explain our notation. Section 3.3 is devoted to the justification of our model by existence and nonexistence results for minimizers. In Section 3.4, we analytically discuss the local bifurcation structure of solutions to the associated Euler-Lagrange equations. Numerical results for the bifurcation branches as well as for the configurations of the curves are presented in Section 3.5. Finally, Section 3.6 summarizes our findings.

3.2 Mathematical setting

We devote this section to make the mathematical setting precise and fix notation.

3.2.1 Notation and preliminaries on curves

We collect some basic information on curves [38]. In the following, we will consider *closed planar curves* $\gamma \in H^2(\mathbb{T}_L)^2$, where $\mathbb{T}_L := \mathbb{R}/L\mathbb{Z}$ is the one-dimensional torus with period $L > 0$. The fact that $H^2(\mathbb{T}_L) \subset C^1(\mathbb{T}_L)$ ensures that $\gamma: [0, L] \rightarrow \mathbb{R}^2$ represents a C^1 -closed curve and $\gamma(0) = \gamma(L)$ and $\dot{\gamma}(0) = \dot{\gamma}(L)$. We systematically assume γ to be parametrized by arclength s , namely, $|\dot{\gamma}| = 1$. This induces that $\ddot{\gamma} \in L^2(\mathbb{T}_L)^2$ is orthogonal to $\dot{\gamma}$. The normal vector n to the curve is defined pointwise by counterclockwise rotating $\dot{\gamma}$ by $\pi/2$. That is, by denoting $\gamma(s) = (x(s), y(s))$, $n(s) = \dot{\gamma}(s)^\perp := (-\dot{y}(s), \dot{x}(s))$. The rate of change of $\dot{\gamma}$ in direction n is measured by the scalar *curvature* $\kappa = n \cdot \ddot{\gamma} = \det(\dot{\gamma}, \ddot{\gamma}) \in L^2(\mathbb{T}_L)$ of the curve, so that $\ddot{\gamma} = (n \cdot \ddot{\gamma}) n = \kappa n$.

The *inclination angle* $\theta \in L^2(\mathbb{T}_L)$ is the angle between the x -axis and the tangent $\dot{\gamma}$, that is $\dot{\gamma} = (\dot{x}, \dot{y}) = (\cos \theta, \sin \theta)$. Note that in even for smooth γ , θ is discontinuous on $\mathbb{T}_L$. However, the map $s \mapsto \theta(s) - \frac{2\pi}{L} I s$ is an element of $H^1(\mathbb{T}_L)$, where the *rotation index* of the curve $I \in \mathbb{Z}$ counts the number of complete turns of $\dot{\gamma}$ according to the standard orientation, see below. The curvature function $\kappa \in L^2(\mathbb{T}_L)$ uniquely determines the curve $\gamma \in H^2(\mathbb{T}_L)^2$ up to translations and rotations in $\mathbb{R}^2$ [38, Section 1-5, pp. 19, 24, and Section 1-7, p. 36]. In particular, if $|\dot{\gamma}| = 1$, then the following holds:

$$\kappa = \dot{\theta}, \quad \theta(s') = \theta(0) + \int_0^{s'} \kappa(s'') \, ds'', \quad \gamma(s) = \begin{pmatrix} x(s) \\ y(s) \end{pmatrix} = \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} + \int_0^s \begin{pmatrix} \cos \theta(s') \\ \sin \theta(s') \end{pmatrix} \, ds'. \quad (3.2)$$

3 Bifurcation of elastic curves with modulated stiffness

Identifying all curves whose images only differ by isometries in $\mathbb{R}^2$, one may adapt the coordinate system to $x(0) = y(0) = \theta(0) = 0$, corresponding indeed to the choice $\gamma(0) = (0, 0)$ and $\dot{\gamma}(0) = (1, 0)$. A curve $\gamma \in H^2(\mathbb{T}_L)^2$ parametrized by arclength satisfies the following identities:

$$\begin{aligned} 0 &= \gamma(L) - \gamma(0) = \int_0^L \dot{\gamma}(s) \, ds = \int_0^L \begin{pmatrix} \cos \theta(s) \\ \sin \theta(s) \end{pmatrix} \, ds = \int_0^L \begin{pmatrix} \cos(\theta(0) + \int_0^s \kappa(t) \, dt) \\ \sin(\theta(0) + \int_0^s \kappa(t) \, dt) \end{pmatrix} \, ds, \\ 0 &= \dot{\gamma}(L) - \dot{\gamma}(0) = (\cos \theta(L) - \cos \theta(0), \sin \theta(L) - \sin \theta(0)). \end{aligned}$$

The latter is equivalent to $\theta(L) - \theta(0) = \int_0^L \kappa(s) \, ds = 2\pi I$. A curve $\gamma: [0, L] \rightarrow \mathbb{R}^2$ is called *simple* if it is an injective map and *regular*, if it is C^1 and $\dot{\gamma}(t) \neq 0$ for all $t \in [0, L]$. By the Theorem of Turning Tangents [38, Section 5-6, Theorem 2, p. 396], a simple C^1 -closed regular planar positively oriented C^1 curve has rotation index $I = 1$. This allows us to represent a simple C^1 -closed curve $\gamma \in H^2(\mathbb{T}_L)^2$ parametrized by arclength by its inclination angle θ , granted that $\theta - \frac{2\pi}{L} I s \in H^1(\mathbb{T}_L)$ and

$$\theta(0) = 0, \quad \theta(L) = 2\pi \quad \text{and} \quad \int_0^L \begin{pmatrix} \cos \theta(s) \\ \sin \theta(s) \end{pmatrix} \, ds = 0,$$

or by its curvature $\kappa \in L^2(\mathbb{T}_L)$, additionally satisfying

$$\int_0^L \kappa(s) \, ds = 2\pi \quad \text{and} \quad \int_0^L \begin{pmatrix} \cos(\int_0^s \kappa(t) \, dt) \\ \sin(\int_0^s \kappa(t) \, dt) \end{pmatrix} \, ds = 0.$$

Eventually, note that by requiring a planar curve to be closed restricts the possible curvature functions. According to the Four Vertex Theorem [38, Section 1-7, Theorem 2, p. 37], a smooth simple closed regular planar curve has either constant curvature (i.e. is a circle) or the curvature function possesses at least four vertices, i.e. two local minima and two local maxima. The converse statement is given in [30]: every continuous function which either is a nonzero constant or has at least four vertices is the curvature of a simple closed regular planar curve.

3.2.2 Elastic energies with modulated stiffness

We consider planar curves $\gamma \in H^2(\mathbb{T}_L)^2$ parametrized by arclength. With no loss of generality, we will assume from now on the length L of the curve to be 2π . The scalar density field $\rho: [0, 2\pi] \rightarrow \mathbb{R}$ is considered to be a function of the arclength of the curve. Moreover, we are given a density-modulated stiffness

$$\beta \in C^2(\mathbb{R}) \quad \text{with} \quad \inf \beta =: \beta_m > 0. \quad (3.3)$$

In the following, we will assume (3.3) to hold throughout, without explicit mention. Note that however some results in this section are valid under weaker conditions on β as well.

Admissible curves are defined as elements of the set

$$\mathcal{A} := \left\{ \gamma \in H^2(\mathbb{T}_{2\pi})^2 : \begin{array}{l} |\dot{\gamma}| = 1, \gamma(0) = \gamma(2\pi) = (0, 0), \dot{\gamma}(0) = \dot{\gamma}(2\pi) = (1, 0), \\ \int_0^{2\pi} \det(\dot{\gamma}(s), \ddot{\gamma}(s)) \, ds = 2\pi \end{array} \right\}. \quad (3.4)$$

In particular, admissible curves are planar, arclength parametrized, and C^1 -closed. Note that we are not enforcing injectivity of γ (i.e. γ being simple) and we just require the weaker condition $I = 1$. This simplifies our tractation, having no effect on the bifurcation result (Section 3.4).

By the representation theorem for plane curves, any admissible curve $\gamma \in \mathcal{A}$ can be recovered from its inclination angle θ or its curvature κ . Correspondingly, we can equivalently indicate admissible curves as

$$\mathcal{A} = \left\{ \theta \in L^2(\mathbb{T}_{2\pi}) : \theta - s \in H^1(\mathbb{T}_{2\pi}), \int_0^{2\pi} \begin{pmatrix} \cos \theta(s) \\ \sin \theta(s) \end{pmatrix} ds = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \theta(0) = 0 \right\} \quad (3.5)$$

or

$$\mathcal{A} = \left\{ \kappa \in L^2(\mathbb{T}_{2\pi}) : \int_0^{2\pi} \begin{pmatrix} \cos \left(\int_0^s \kappa(t) dt \right) \\ \sin \left(\int_0^s \kappa(t) dt \right) \end{pmatrix} ds = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \int_0^{2\pi} \kappa(s) ds = 2\pi \right\}.$$

The abuse of notation in defining the set $\mathcal{A}$ is motivated by the above-mentioned equivalence of the representations via γ , θ , and κ , up to fixing $\gamma(0) = (0, 0)$ and $\dot{\gamma}(0) = (1, 0)$ or $\theta(0) = 0$.

Admissible densities ρ are asked to have fixed total mass. By possibly redefining β , one may assume such mass to be 2π , which simplifies notation. Given the parameter $\mu \in [0, \infty)$, we define

$$\mathcal{R} := \left\{ \rho \in L^1(\mathbb{T}_{2\pi}) : \mu\rho \in H^1(\mathbb{T}_{2\pi}), \int_0^{2\pi} \rho(s) ds = 2\pi \right\}. \quad (3.6)$$

For the sake of simplicity, we do not restrict the values of ρ to be nonnegative, which would however be sensible, for ρ is interpreted as a density. Note that this simplification has no effect on the bifurcation results, which are actually addressing a neighborhood of the trivial state only, where ρ is constant and positive. The *elastic energy with modulated stiffness* is defined as

$$E_\mu(\rho, \gamma) := \int_0^{2\pi} \left(\frac{1}{2} \beta(\rho) \dot{\gamma}^2 + \frac{\mu}{2} \dot{\rho}^2 \right) ds. \quad (3.7)$$

Note that the energy E_μ can be equivalently rewritten as $E_\mu(\rho, \gamma) = E_\mu(\rho, \theta) = E_\mu(\rho, \kappa)$, again by abusing notation.

We identify elastic curves with modulated stiffness as minimizers of E_μ . In particular, we consider the following minimization problem

$$\min_{(\rho, \gamma) \in \mathcal{R} \times \mathcal{A}} E_\mu(\rho, \gamma). \quad (3.8)$$

In contrast to the classical Euler-Bernoulli model for elasticae [64, 99], which is a purely geometric variational problem, here the density plays an active role in the selection of the optimal geometry.

3.3 Existence and nonexistence

As mentioned in the Introduction 3.1, the minimization of E_0 turns out to be of limited interest. Indeed, if the integrand

$$\Phi(\rho, \kappa) = \frac{1}{2} \beta(\rho) \kappa^2$$

is strictly convex, problem (3.8) for $\mu = 0$ admits only the trivial solution

$$(\rho_0, \kappa_0) := (1, 1). \quad (3.9)$$

This can be directly checked via Jensen's inequality by computing, for any $(\rho, \kappa) \in \mathcal{R} \times \mathcal{A}$,

$$\begin{aligned} E_0(\rho, \kappa) &= \int_0^{2\pi} \Phi(\rho, \kappa) ds \\ &\geq 2\pi \Phi \left(\frac{1}{2\pi} \int_0^{2\pi} \rho ds, \frac{1}{2\pi} \int_0^{2\pi} \kappa ds \right) = 2\pi \Phi(1, 1) = E_0(\rho_0, \kappa_0) \end{aligned}$$

where the inequality is strict whenever ρ or κ are not constant, namely, whenever $(\rho, \kappa) \neq (\rho_0, \kappa_0)$. Let us mention that the integrand Φ is strictly convex if and only if

$$\beta'' > 0 \quad \text{and} \quad \beta''\beta > 2(\beta')^2. \quad (3.10)$$

In order to allow the complex geometrical patterning of biological shapes to possibly be described by the minimization problem (3.8), one is hence forced to dispense of (3.10), for in that case the only minimizer of E_0 (and, a fortiori E_μ) would be the trivial one (ρ_0, κ_0) . In the setting of our bifurcation results, our choices for β will then fulfill

$$\beta''(\rho) \leq 0 \quad \text{or} \quad \beta''(\rho)\beta(\rho) \leq 2(\beta'(\rho))^2 \quad \text{for some } \rho \geq 0, \quad (3.11)$$

at least in a neighborhood of the trivial state ρ_0 .

On the other hand, lacking convexity of the integrand Φ , the energy E_0 fails to be weakly lower semicontinuous on $\mathcal{R} \times \mathcal{A}$, e.g. [47, Thm. 5.14], and existence of minimizers may genuinely fail. We collect a remark in this direction in the following.

Proposition 3.3.1 (No minimizers for E_0). *Assume that*

$$\beta(0) < \beta(\rho) \quad \forall \rho > 0. \quad (3.12)$$

Then, the minimization problem (3.8) with $\mu = 0$ admits no solution.

Before moving to the proof, let us point out that condition (3.12) implies in particular that Φ is not convex. Indeed, if Φ were convex, one could take any $\rho > 0$ and $\lambda \in (0, 1)$ and compute

$$\begin{aligned} \frac{1}{2}\beta(\rho)\kappa_0^2 &= \lim_{\lambda \rightarrow 1} \Phi(\lambda(0, \kappa_0/\lambda) + (1-\lambda)(\rho/(1-\lambda), 0)) \\ &\leq \lim_{\lambda \rightarrow 1} \left(\lambda\Phi(0, \kappa_0/\lambda) + (1-\lambda)\Phi(\rho/(1-\lambda), 0) \right) = \lim_{\lambda \rightarrow 1} \lambda\Phi(0, \kappa_0/\lambda) = \frac{1}{2}\beta(0)\kappa_0^2, \end{aligned}$$

contradicting (3.12). Note that the role of the value κ_0 in the latter computation is immaterial as one can argue with any $\kappa \neq 0$.

Proof of Proposition 3.3.1. Let us show that E_0 cannot be minimized on $\mathcal{R} \times \mathcal{A}$. We firstly remark that

$$E_0(\rho, \kappa) \stackrel{(3.12)}{\geq} E_0(0, \kappa) \stackrel{\text{Jensen}}{\geq} E_0(0, \kappa_0) \quad \forall (\rho, \kappa) \in \mathcal{R} \times \mathcal{A}. \quad (3.13)$$

In fact, the first inequality is strict as soon as $\rho\kappa \not\equiv 0$ almost everywhere while the second one is strict as soon as κ is not constantly equal to κ_0 (recall that $\beta(0) > 0$). For all $\lambda \in (0, 1)$, we now define

$$\begin{aligned} \rho_\lambda(s) &= \begin{cases} 0 & \text{for } s \in [0, \lambda\pi] \\ \rho_0/(1-\lambda) & \text{for } s \in (\lambda\pi, \pi] \\ 0 & \text{for } s \in (\pi, (1+\lambda)\pi] \\ \rho_0/(1-\lambda) & \text{for } s \in ((1+\lambda)\pi, 2\pi], \end{cases} \\ \kappa_\lambda(s) &= \begin{cases} \kappa_0/\lambda & \text{for } s \in [0, \lambda\pi] \\ 0 & \text{for } s \in (\lambda\pi, \pi] \\ \kappa_0/\lambda & \text{for } s \in (\pi, (1+\lambda)\pi] \\ 0 & \text{for } s \in ((1+\lambda)\pi, 2\pi]. \end{cases} \end{aligned}$$

We now check that $(\rho_\lambda, \kappa_\lambda) \in \mathcal{R} \times \mathcal{A}$. Indeed,

$$\int_0^{2\pi} \rho_\lambda \, ds = 2\pi(1-\lambda) \frac{\rho_0}{1-\lambda} = 2\pi\rho_0 = 2\pi, \quad \int_0^{2\pi} \kappa_\lambda \, ds = 2\lambda\pi \frac{\kappa_0}{\lambda} = 2\pi\kappa_0 = 2\pi.$$

Moreover, by letting $K_\lambda(s) = \int_0^s \kappa_\lambda(r) \, dr$, namely,

$$K_\lambda(s) = \begin{cases} \kappa_0 s / \lambda & \text{for } s \in [0, \lambda\pi] \\ \kappa_0 \pi & \text{for } s \in (\lambda\pi, \pi] \\ \kappa_0(s - (1-\lambda)\pi) / \lambda & \text{for } s \in (\pi, (1+\lambda)\pi] \\ \kappa_0 2\pi & \text{for } s \in ((1+\lambda)\pi, 2\pi], \end{cases}$$

we can compute

$$\begin{aligned} \int_0^{2\pi} \cos \left(\int_0^s \kappa_\lambda(r) \, dr \right) \, ds &= \int_0^{2\pi} \cos(K_\lambda(s)) \, ds \\ &= \int_0^{\lambda\pi} \cos(\kappa_0 s / \lambda) \, ds + \int_{\lambda\pi}^{\pi} \cos(\kappa_0 \pi) \, ds \\ &\quad + \int_{\pi}^{(1+\lambda)\pi} \cos(\kappa_0(s - (1-\lambda)\pi) / \lambda) \, ds + \int_{(1+\lambda)\pi}^{2\pi} \cos(2\kappa_0 \pi) \, ds \\ &= \frac{\lambda}{\kappa_0} \sin(\kappa_0 \pi) - \frac{\lambda}{\kappa_0} \sin 0 + \cos(\kappa_0 \pi)(1-\lambda)\pi \\ &\quad + \frac{\lambda}{\kappa_0} \sin(2\kappa_0 \pi) - \frac{\lambda}{\kappa_0} \sin(\kappa_0 \pi) + \cos(2\kappa_0 \pi)(1-\lambda)\pi \\ &= \lambda \sin \pi - \lambda \sin 0 + (\cos \pi)(1-\lambda)\pi + \lambda \sin 2\pi - \lambda \sin \pi + (\cos 2\pi)(1-\lambda)\pi = 0, \end{aligned}$$

and analogously

$$\begin{aligned} \int_0^{2\pi} \sin \left(\int_0^s \kappa_\lambda(r) \, dr \right) \, ds &= \int_0^{2\pi} \sin(K_\lambda(s)) \, ds \\ &= -\lambda \cos \pi + \lambda \cos 0 + (\sin \pi)(1-\lambda)\pi - \lambda \cos 2\pi + \lambda \cos \pi + (\sin 2\pi)(1-\lambda)\pi = 0. \end{aligned}$$

The latter ensures in particular that $(\rho_\lambda, \kappa_\lambda) \in \mathcal{R} \times \mathcal{A}$.

Let us now compute

$$E_0(\rho_\lambda, \kappa_\lambda) = \lambda E_0(0, \kappa_0 / \lambda) + (1-\lambda) E_0(\rho / (1-\lambda), 0) = \lambda E_0(0, \kappa_0 / \lambda)$$

and note that $E_0(\rho_\lambda, \kappa_\lambda) \rightarrow E_0(0, \kappa_0)$ as $\lambda \rightarrow 1$. Owing to (3.13), this entails that $E_0(\rho_\lambda, \kappa_\lambda)$ is an infimizing sequence on $\mathcal{R} \times \mathcal{A}$. On the other hand, the value $E_0(0, \kappa_0)$ cannot be reached in $\mathcal{R} \times \mathcal{A}$. Indeed, assume by contradiction to have $(\rho, \kappa) \in \mathcal{R} \times \mathcal{A}$ with $E_0(\rho, \kappa) = E_0(0, \kappa_0)$. Recalling (3.13), we have that $\rho\kappa = 0$ almost everywhere and $\kappa = \kappa_0$. This entails that $\rho = 0$ almost everywhere so that necessarily $(\rho, \kappa) = (0, \kappa_0)$, which however does not belong to $\mathcal{R} \times \mathcal{A}$. $\square$

Despite the lack of lower semicontinuity and the possible nonexistence of minimizers of variational problems, in some cases information may still be retrieved by analyzing the structure of infimizing sequences, see [7]. This perspective seems however to be of little relevance here. Assume (ρ, κ) to be a minimizer of E_0 in $\mathcal{R} \times \mathcal{A}$ and let $(\rho_\#, \kappa_\#)$ denote its periodic extension to $\mathbb{R}$. Let the *fine-scaled* trajectories

$$\rho_n(s) = \rho_\#(ns), \quad \kappa_n(s) = \kappa_\#(ns) \quad \forall s \in [0, 2\pi]$$

be defined. One may check that $(\rho_n, \kappa_n) \in \mathcal{R} \times \mathcal{A}$ as well and that $E_0(\rho_n, \kappa_n) = E_0(\rho, \kappa)$, so that all (ρ_n, κ_n) are minimizers (infimizing, in particular). On the other hand, (ρ_n, κ_n) weakly converges to its mean (ρ_0, κ_0) . This shows that, the limiting behavior of infimizing sequences may deliver scant information, for we recover the trivial state.

These facts motivate our interest for focusing on the case $\mu > 0$ in the minimization problem (3.8). In contrast to the case $\mu = 0$ of Proposition 3.3.1, energy E_μ can be minimized in $\mathcal{R} \times \mathcal{A}$ for all $\mu > 0$.

Proposition 3.3.2 (Existence for positive μ). *Let $\mu > 0$. Then, the minimization problem (3.8) admits a solution.*

Proof. This is an immediate application of the Direct Method. Let $(\rho_n, \kappa_n) \in \mathcal{R} \times \mathcal{A}$ be an infimizing sequence for E_μ (such a sequence exists, for $E_\mu(\rho_0, \kappa_0) > -\infty$). We can assume with no loss of generality that $\sup E_\mu(\rho_n, \kappa_n) < \infty$. In particular, as $\beta \geq \beta_m > 0$ we have that ρ_n and κ_n are uniformly bounded in $H^1(\mathbb{T}_{2\pi})$ and in $L^2(\mathbb{T}_{2\pi})$, respectively. This implies, at least for a not relabeled subsequence, that $\rho_n \rightharpoonup \rho$ in $H^1(\mathbb{T}_{2\pi})$ hence strongly in $C(\mathbb{T}_{2\pi})$ and $\kappa_n \rightharpoonup \kappa$ in $L^2(\mathbb{T}_{2\pi})$. We can hence pass to the limit in the relations

$$\int_0^{2\pi} \rho_n \, ds = 2\pi, \quad \int_0^{2\pi} \begin{pmatrix} \cos\left(\int_0^s \kappa_n(t) \, dt\right) \\ \sin\left(\int_0^s \kappa_n(t) \, dt\right) \end{pmatrix} ds = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \int_0^{2\pi} \kappa_n \, ds = 2\pi$$

and obtain that $(\rho, \kappa) \in \mathcal{R} \times \mathcal{A}$ as well. Moreover, $\beta(\rho_n) \rightarrow \beta(\rho)$ strongly in $C(\mathbb{T}_{2\pi})$ as β is locally Lipschitz continuous. This implies that $(\beta(\rho_n))^{1/2} \kappa_n \rightharpoonup (\beta(\rho))^{1/2} \kappa$ in $L^2(\mathbb{T}_{2\pi})$ and lower semicontinuity ensures that $E_\mu(\rho, \kappa) \leq \liminf_{n \rightarrow \infty} E_\mu(\rho_n, \kappa_n) = \inf E_\mu$, so that (ρ, κ) is a solution of problem (3.8). $\square$

The parameter μ is a datum of the problem and it is in particular related to the characteristic length scale at which ρ changes along the curve. If μ is chosen to be large compared with the length of the curve, the minimizer is again forced to be trivial. Let us make these heuristics precise in the following.

Proposition 3.3.3 (Trivial minimizer for μ large). *For μ large enough, the trivial state (ρ_0, θ_0) is the unique solution of the minimization problem (3.8).*

Proof. We structure the proof into two steps. In Step 1 we show that, for μ large, the trivial state

$$u_0 = (\rho_0, \theta_0)$$

with $\rho_0 = 1$ and $\theta_0(s) = s$ is a strict minimizer in a neighborhood which is independent of μ . In Step 2, we prove that all minimizers converge to u_0 in the H^1 norm as $\mu \rightarrow \infty$. The combination of these two steps entails then that all minimizers $u = (\rho, \theta)$ necessarily coincide with u_0 for μ sufficiently large, for they are arbitrarily close to u_0 (Step 2) which is locally the unique minimizer (Step 1).

Step 1: The trivial state is a strict local minimizer. Let us check that, for μ large enough, the second variation $\delta^2 E_\mu(u_0)$ of $E_\mu(u) = \frac{1}{2} \int_0^{2\pi} (\beta(\rho) \dot{\theta}^2 + \mu \dot{\rho}^2) \, ds$ is positive. Indeed, for the arbitrary directions $u_1 = (\rho_1, \theta_1)$ and $\tilde{u}_1 = (\tilde{\rho}_1, \tilde{\theta}_1)$, we can compute

$$\begin{aligned} & \delta^2 E_\mu(u_0)(u_1, \tilde{u}_1) \\ &= \int_0^{2\pi} \left(\frac{1}{2} \beta''(\rho_0) \dot{\theta}_0^2 \rho_1 \tilde{\rho}_1 + \beta'(\rho_0) \dot{\theta}_0 \left(\dot{\theta}_1 \tilde{\rho}_1 + \rho_1 \dot{\tilde{\theta}}_1 \right) + \beta(\rho_0) \dot{\theta}_1 \dot{\tilde{\theta}}_1 + \mu^2 \dot{\rho}_1 \dot{\tilde{\rho}}_1 \right) ds, \end{aligned} \quad (3.14)$$

which is uniformly continuous around u_0 . In particular, with $\dot{\theta}_0 = 1$ and rearranging terms,

$$\delta^2 E_\mu(u_0)(u_1, u_1) = \int_0^{2\pi} \left(\mu \dot{\rho}_1^2 + \beta(\rho_0) \dot{\theta}_1^2 + 2\beta'(\rho_0) \rho_1 \dot{\theta}_1 + \frac{\beta''(\rho_0)}{2} \rho_1^2 \right) ds.$$

By integrating by parts and using the Cauchy-Schwarz inequality in the third term, we get

$$\begin{aligned} \delta^2 E_\mu(u_0)(u_1, u_1) &\geq \mu \int_0^{2\pi} \dot{\rho}_1^2 ds + \beta(\rho_0) \int_0^{2\pi} \dot{\theta}_1^2 ds \\ &\quad - \left(\frac{4\beta'(\rho_0)^2}{C\beta(\rho_0)} \right)^{\frac{1}{2}} \|\dot{\rho}_1\|_{L^2(0,2\pi)} (C\beta(\rho_0))^{\frac{1}{2}} \|\theta_1\|_{L^2(0,2\pi)} - \frac{|\beta''(\rho_0)|}{2} \int_0^{2\pi} \rho_1^2 ds, \end{aligned}$$

where C is the Poincaré constant on $(0, 2\pi)$. Using again Poincaré's inequality to bound the second and last term in the right-hand side above and Young's inequality for the third term, we are left with

$$\delta^2 E_\mu(u_0)(u_1, u_1) \geq \left(\mu - \frac{2\beta'(\rho_0)^2}{C\beta(\rho_0)} - \frac{|\beta''(\rho_0)|}{2C} \right) \int_0^{2\pi} \dot{\rho}_1^2 ds + \frac{\beta(\rho_0)}{2} \int_0^{2\pi} \dot{\theta}_1^2 ds,$$

which is positive for

$$\mu > \frac{2\beta'(\rho_0)^2}{C\beta(\rho_0)} + \frac{|\beta''(\rho_0)|}{2C}.$$

As $\delta^2 E_\mu(u_0)$ is positive, u_0 minimizes E_μ on some neighborhood $U_\mu \subset \mathcal{R} \times \mathcal{A}$ for $\mu \geq \mu_0$ and for some $\mu_0 > 0$. Since E_μ is increasing in μ and $E_\mu(u_0)$ does not depend on μ , U_μ may be taken to be increasing in μ as well. Thus, u_0 minimizes E_μ on U_{μ_0} for all $\mu \geq \mu_0$.

Step 2: Global minimizers converge to the trivial state. We next prove that, for any $\delta > 0$ there exists $\mu_c > 0$ such that for any $\mu > \mu_c$, any global minimizer (ρ, θ) of E_μ is such that

$$\|\rho - \rho_0\|_{L^2(0,2\pi)} + \|\theta - \theta_0\|_{L^2(0,2\pi)} \lesssim \|\dot{\rho}\|_{L^2(0,2\pi)} + \|\dot{\theta} - \dot{\theta}_0\|_{L^2(0,2\pi)} < \delta \quad (3.15)$$

where we use the sign $\lesssim$ to indicate the implicit occurrence of a constant just depending on data. In fact, we have that

$$\begin{aligned} E_\mu(\rho_0, \theta_0) &\geq E_\mu(\rho, \theta) = \int_0^{2\pi} \left(\frac{1}{2} \beta(\rho) \dot{\theta}^2 + \frac{\mu}{2} \dot{\rho}^2 \right) ds \\ &\geq \int_0^{2\pi} \left(\frac{1}{2} \beta_m \dot{\theta}^2 + \frac{\mu}{2} \dot{\rho}^2 \right) ds \geq \int_0^{2\pi} \left(\frac{1}{2} \beta_m \dot{\theta}_0^2 + \frac{\mu}{2} \dot{\rho}^2 \right) ds, \end{aligned} \quad (3.16)$$

since θ_0 minimizes the Dirichlet energy $\int_0^{2\pi} \dot{\theta}^2 ds$ under the conditions $\theta(0) = 0$, $\theta(2\pi) = 2\pi$. Since $E_\mu(\rho_0, \theta_0) = \frac{1}{2} \int_0^{2\pi} \beta(\rho_0) \dot{\theta}_0^2 ds = \pi \beta(\rho_0) < \infty$, both terms in the above right-hand side are bounded. We hence deduce that $\int_0^{2\pi} \dot{\theta}^2 ds$ is bounded uniformly in μ and $\int_0^{2\pi} \dot{\rho}^2 ds = O(\mu^{-1}) = o(\mu^{-1/2})$ as $\mu \rightarrow \infty$, so that there exists $\mu_1 > 0$ such that for $\mu > \mu_1$ we have $\int_0^{2\pi} \dot{\rho}^2 ds < \delta/2$. Now, $\rho \in \mathcal{R}$ implies $\int_0^{2\pi} (\rho - \rho_0) ds = 0$ and by the Poincaré inequality as well as the continuous embedding in $L^\infty(0, 2\pi)$,

$$\|\rho - \rho_0\|_{L^\infty(0,2\pi)} \lesssim \|\dot{\rho}\|_{L^2(0,2\pi)} = o(\mu^{-1/4}),$$

and, by the local Lipschitz continuity of β ,

$$\|\beta(\rho) - \beta(\rho_0)\|_{L^\infty(0,2\pi)} = o(\mu^{-1/4}).$$

This allows us to refine estimate (3.16) as follows:

$$\begin{aligned} E_\mu(\rho_0, \theta_0) &\geq E_\mu(\rho, \theta) \geq \int_0^{2\pi} \left(\frac{1}{2} \beta(\rho_0) \dot{\theta}_0^2 + \frac{\mu}{2} \dot{\rho}^2 \right) ds + o(\mu^{-1/4}) \\ &= E_\mu(\rho_0, \theta_0) + \frac{\mu}{2} \int_0^{2\pi} \dot{\rho}^2 ds + o(\mu^{-1/4}), \end{aligned} \quad (3.17)$$

from which we get $\lim_{\mu \rightarrow \infty} \mu \int_0^{2\pi} \dot{\rho}^2 ds = 0$, and then

$$\lim_{\mu \rightarrow \infty} E_\mu(\rho, \theta) = \lim_{\mu \rightarrow \infty} \int_0^{2\pi} \frac{1}{2} \beta(\rho) \dot{\theta}^2 ds = E_\mu(\rho_0, \theta_0).$$

Finally, we control

$$\left| \int_0^{2\pi} \frac{1}{2} \beta(\rho) \dot{\theta}^2 ds - E_\mu(\rho_0, \theta_0) \right| = \left| \int_0^{2\pi} \frac{\beta(\rho_0)}{2} (\dot{\theta}^2 - \dot{\theta}_0^2) ds + o(\mu^{-1/4}) \right|,$$

so to prove that $\lim_{\mu \rightarrow \infty} \int_0^{2\pi} \dot{\theta}^2 ds = \lim_{\mu \rightarrow \infty} \int_0^{2\pi} \dot{\theta}_0^2 ds = 2\pi$. This is enough to conclude that

$$\lim_{\mu \rightarrow \infty} \|\dot{\theta} - \dot{\theta}_0\|_{L^2(0, 2\pi)} = 0.$$

We can then choose μ_2 such that, for $\mu > \mu_2$, $\|\dot{\theta} - \dot{\theta}_0\|_{L^2(0, 2\pi)} < \delta/2$ and set $\mu_c = \max\{\mu_1, \mu_2\}$ for which the second inequality in (3.15) holds. The first inequality follows from Poincaré's inequality. $\square$

We present now a symmetry result which will turn out useful later on, when interpreting the numerical findings.

Proposition 3.3.4 (Symmetry of E_μ). *If (ρ, θ) is a local minimizer of E_μ for β , then*

$$(2\rho_0 - \rho, \theta)$$

is a local minimizer of E_μ for $\tilde{\beta}$, defined as $\tilde{\beta}(\rho) = \beta(2\rho_0 - \rho)$.

Proof. The integrand is unchanged by this transformation, so that the first and second variations of E_μ at (ρ, θ) and $(2\rho_0 - \rho, \theta)$ when considering respectively β and $\tilde{\beta}$ are identical. $\square$

3.4 Bifurcation analysis

By Proposition 3.3.3, the circle with constant density is the global minimizer of the energy E_μ (3.1) for large enough values of the diffusivity μ . In this section, candidates for nontrivial minimizers are constructed by bifurcation from this trivial critical point with decreasing $\mu > 0$ as bifurcation parameter. The analysis will be based on the Euler-Lagrange equations of a suitable Lagrangian, incorporating the constraints of closedness of the curve and of given total mass. Additional auxiliary conditions will eliminate symmetries resulting from arbitrary positioning of the curve in the plane.

3.4.1 Euler-Lagrange equations

We introduce the Lagrange multipliers $(\lambda_x, \lambda_y) \in \mathbb{R}^2$ for the closedness constraint and $\lambda_M \in \mathbb{R}$ for the mass constraint (cf. the definitions (3.5) and (3.6) of admissible θ and ρ respectively), and define the Lagrangian

$$\mathcal{L}(\bar{u}) := \frac{1}{2} \int_0^{2\pi} \left(\beta(\rho) \dot{\theta}^2 + \mu \dot{\rho}^2 \right) ds + \int_0^{2\pi} (\lambda_x \cos \theta + \lambda_y \sin \theta + \lambda_M (\rho - 1)) ds ,$$

with

$$\bar{u} := (\rho, \theta, \lambda_x, \lambda_y, \lambda_M).$$

Here, the maps $s \mapsto \rho(s)$ and $s \mapsto \theta(s) - s$ are elements of $H^1(\mathbb{T}_{2\pi})$, as well as the multipliers $(\lambda_x, \lambda_y, \lambda_M)$, which can be considered as constant maps. Critical points of the energy subject to the constraints solve the *Euler-Lagrange equations*

$$\mu \ddot{\rho} - \frac{1}{2} \beta'(\rho) \dot{\theta}^2 = \lambda_M , \quad (3.18)$$

$$\frac{d}{ds} \left(\beta(\rho) \dot{\theta} \right) = -\lambda_x \sin \theta + \lambda_y \cos \theta , \quad (3.19)$$

along with the *boundary conditions*

$$\rho(2\pi) - \rho(0) = 0, \quad \dot{\rho}(2\pi) - \dot{\rho}(0) = 0, \quad \theta(0) = 0, \quad \theta(2\pi) = 2\pi , \quad (3.20)$$

and mass and closedness *constraints*

$$\int_0^{2\pi} \rho ds = 2\pi, \quad \int_0^{2\pi} \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} ds = \begin{pmatrix} 0 \\ 0 \end{pmatrix} , \quad (3.21)$$

respectively. For every solution, new solutions can be produced by arbitrary shifts in s . In order to eliminate this degree of freedom we add the condition

$$\rho(0) = 1 , \quad (3.22)$$

where we note that 1 is the average value of ρ by the mass constraint, which is assumed by every continuous solution. One symmetry remains: The problem is still invariant under the flip symmetry $s \leftrightarrow -s$ (with $\theta \leftrightarrow -\theta$, $\lambda_y \leftrightarrow -\lambda_y$).

The *trivial solution* of (3.18)–(3.22),

$$\bar{u}_0 := (\rho_0, \theta_0, \lambda_{x0}, \lambda_{y0}, \lambda_{M0}), \quad (3.23)$$

which comprises the minimizer $u_0 = (\rho_0, \theta_0)$ of E_μ for large enough μ (see Proposition 3.3.3), is the unit circle with constant density:

$$\theta_0(s) = s, \quad \rho_0(s) = 1, \quad \lambda_{x0} = \lambda_{y0} = 0, \quad \lambda_{M0} = -\frac{1}{2} \beta'(1). \quad (3.24)$$

For the stiffness coefficient β we shall assume the following local behavior close to the trivial solution:

$$\beta(\rho) = 1 + m(\rho - 1) + h \frac{(\rho - 1)^2}{2} + O((\rho - 1)^5) \quad \text{as } \rho \rightarrow 1. \quad (3.25)$$

The bifurcation behaviour will be characterized in terms of the Taylor coefficients $m, h \in \mathbb{R}$. The complexity of the bifurcation computations below have motivated the simplifying assumption that the third- and fourth-order coefficients vanish. Including these higher-order terms would only alter the sub-/supercritical nature of the bifurcation, but not the critical values for the parameter μ .

3.4.2 Linearization around the trivial state

In terms of a small correction

$$\bar{u}_1 := \bar{u} - \bar{u}_0, \quad \bar{u}_1 = (\rho_1, \theta_1, \lambda_{x1}, \lambda_{y1}, \lambda_{M1}), \quad (3.26)$$

the linearization of problem (3.18)–(3.22) reads (using (3.25))

$$\mu \ddot{\rho}_1(s) - m \dot{\theta}_1(s) - \frac{h}{2} \rho_1(s) - \lambda_{M1} = f(s), \quad (3.27)$$

$$\frac{d}{ds} \left(\dot{\theta}_1(s) + m \rho_1(s) \right) + \lambda_{x1} \sin s - \lambda_{y1} \cos s = g(s), \quad (3.28)$$

subject to the boundary conditions

$$\rho_1(2\pi) - \rho_1(0) = 0, \quad \dot{\rho}_1(2\pi) - \dot{\rho}_1(0) = 0, \quad \theta_1(0) = 0, \quad \theta_1(2\pi) = 0, \quad (3.29)$$

the constraints

$$\int_0^{2\pi} \rho_1(s) \, ds = 0, \quad \int_0^{2\pi} \begin{pmatrix} -\sin s \\ \cos s \end{pmatrix} \theta_1(s) \, ds = \begin{pmatrix} \alpha_x \\ \alpha_y \end{pmatrix}, \quad (3.30)$$

and the auxiliary condition

$$\rho_1(0) = 0. \quad (3.31)$$

The inhomogeneities $f(s), g(s), \alpha_x, \alpha_y$ can be interpreted as nonlinear corrections.

Proposition 3.4.1 (Solution of the homogeneous linearized system). *A nonzero solution of (3.27)–(3.31) with $f = g = \alpha_x = \alpha_y = 0$, $\mu > 0$, and $m, h \in \mathbb{R}$, only exists in the following cases:*

Case 1: *There exists $j \in \mathbb{N}$, $j \geq 2$, such that $\mu = \mu_j(m, h) := \frac{1}{j^2} (m^2 - \frac{h}{2}) \neq -\frac{h}{2}$. The space of solutions is one-dimensional and given by*

$$\begin{aligned} \rho_1(s) &= a_1 \sin(js), \quad \theta_1(s) = \frac{a_1 m}{j} (\cos(js) - 1), \\ \lambda_{x1} &= \lambda_{y1} = \lambda_{M1} = 0, \quad a_1 \in \mathbb{R}. \end{aligned} \quad (3.32)$$

Case 2: $\mu = \mu_1(h) := -\frac{h}{2} \neq \frac{1}{j^2} (m^2 - \frac{h}{2})$ for all $j \in \mathbb{N}$, $j \geq 2$. *The space of solutions is one-dimensional and given by*

$$\rho_1(s) = b_1 \sin s, \quad \theta_1(s) = 0, \quad \lambda_{x1} = \lambda_{M1} = 0, \quad \lambda_{y1} = b_1 m, \quad b_1 \in \mathbb{R}. \quad (3.33)$$

Case 3: *There exists $j \in \mathbb{N}$, $j \geq 2$, such that $\mu = \mu_j(m, h) = \mu_1(h)$. The space of solutions is two-dimensional and given by*

$$\begin{aligned} \rho_1(s) &= a_1 \sin(js) + b_1 \sin s, \quad \theta_1(s) = \frac{a_1 m}{j} (\cos(js) - 1), \\ \lambda_{x1} &= \lambda_{M1} = 0, \quad \lambda_{y1} = b_1 m, \quad a_1, b_1 \in \mathbb{R}. \end{aligned}$$

Remark 3.4.2. 1. As expected, bifurcations only occur under the condition (see (3.11))

$$2m^2 - h = 2\beta'(1)^2 - \beta(1)\beta''(1) > 0.$$

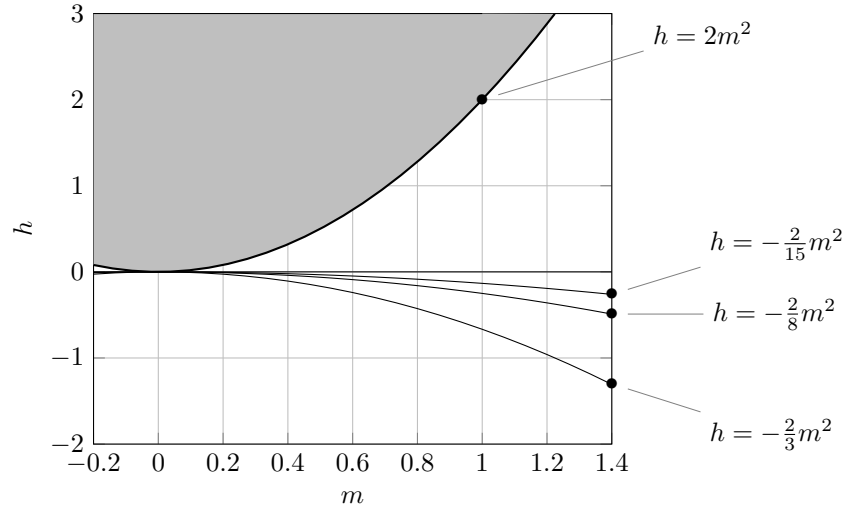


Figure 3.1: Regions of different bifurcation behaviour in the (m, h) -plane according to Proposition 3.4.1: No bifurcations above the parabola $h = 2m^2$. Only Case 1 bifurcations between the parabola and the m -axis. Case 1 and Case 2 bifurcations below the m axis, with codimension-two bifurcations (Case 3) on the parabolas $h = -2m^2/(j^2 - 1)$, $j \geq 2$. The first bifurcation is a Case 2 bifurcation below the parabola $h = -2m^2/3$, and a Case 1 bifurcation with $j = 2$ otherwise.

2. For Case 1 solutions, the curvature correction $\kappa_1 = \dot{\theta}_1$ satisfies $\kappa_1 = -m\rho_1$. This means that the sign of $m = \beta'(1)$ decides if curvature maxima coincide with density maxima ($m < 0$) or with density minima ($m > 0$), which is not a surprising result. The condition $j \geq 2$ is a manifestation of the Four Vertex Theorem (see Section 3.2).
3. Case 2 solutions exhibit only one maximum and one minimum of the density without any effect on the circular shape of the curve. The numerical computations reported in Section 3.5 show, however, that these solutions initiate bifurcating branches strongly deviating from the circular shape far enough from the bifurcation point.
4. Whereas Cases 1 and 2 correspond to codimension-one bifurcations, Case 3 represents a bifurcation of codimension two, whose nonlinear structure will not be analyzed in the following.
5. In a bifurcation scenario, where the values of m and h are fixed and the value of μ is decreased, there are several situations. For $h \geq 2m^2$ no bifurcations occur by convexity (see above). For $0 \leq h < 2m^2$ an infinite series of Case 1 bifurcations occurs at the bifurcation values μ_j , $j \geq 2$, with the first one at $\mu = \mu_2$. For $h < 0$, apart from the bifurcations at $\mu = \mu_2, \mu_3, \dots$, there is also a Case 2 bifurcation at $\mu = \mu_1$. There are two subcases concerning the question, which bifurcation occurs first, determined by the criterion

$$\mu_1 > \mu_2 \iff h < -\frac{2}{3}m^2.$$

Codimension-two bifurcations (Case 3) occur whenever $\mu_1 = \mu_j$, i.e. $h = -2m^2/(j^2 - 1)$ for some $j \geq 2$. These observations are illustrated in Figure 3.1 in the (m, h) -plane and in Figure 3.2 in the (h, μ) -plane.

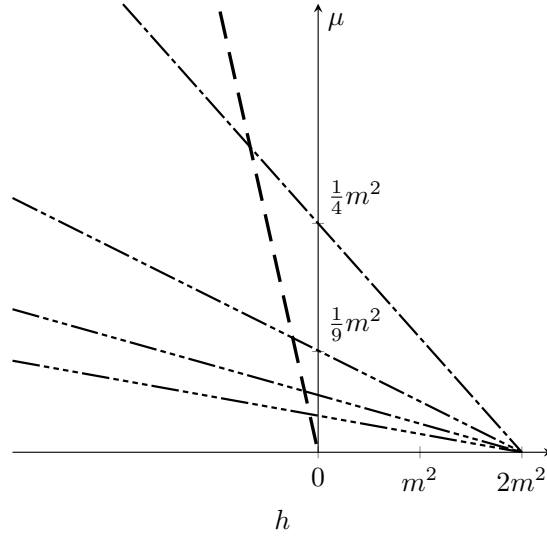


Figure 3.2: Critical values $\mu = \mu_j(m, h)$ as a function of h for Case 1 (thin: $\mu_j = \frac{1}{j^2}(m^2 - \frac{h}{2})$ and $j \geq 2$) and Case 2 (bold: $\mu = \mu_1 = -\frac{h}{2}$) according to Proposition 3.4.1. The intersection points correspond to (the degenerate) Case 3 which is not studied in this work. The type of dashes indicate the value of $j \in \mathbb{N}$: — — for $j = 1$, — - — for $j = 2$, etc.

Proof. By the smoothness of solutions of ordinary differential equations we can employ Fourier representation and write

$$\rho_1(s) = \sum_{k \in \mathbb{Z}} \hat{\rho}_{1,k} e^{iks}, \quad \theta_1(s) = \sum_{k \in \mathbb{Z}} \hat{\theta}_{1,k} e^{iks}.$$

We keep the inhomogeneities for the moment, since this will be useful for the proof of the following result, and we use their Fourier series

$$f(s) = \sum_{k \in \mathbb{Z}} \hat{f}_k e^{iks}, \quad g(s) = \sum_{k \in \mathbb{Z}} \hat{g}_k e^{iks}.$$

The constraints (3.30) imply

$$\hat{\rho}_{1,0} = 0, \quad \hat{\theta}_{1,\pm 1} = \frac{1}{2\pi}(\alpha_y \pm i\alpha_x). \quad (3.34)$$

Comparing Fourier coefficients for $k = 0$ in (3.27), (3.28) implies

$$\lambda_{M1} = -\hat{f}_0 \quad \text{and} \quad \hat{g}_0 = 0, \quad (3.35)$$

where the latter has to be seen as a solvability condition for the inhomogeneous problem. Coefficients for $k = \pm 1$ in (3.27) and (3.28) give

$$-(\mu - \mu_1) \hat{\rho}_{1,\pm 1} = \hat{f}_{\pm 1} - \frac{m}{2\pi}(\alpha_x \mp i\alpha_y), \quad (3.36)$$

$$\pm mi\hat{\rho}_{1,\pm 1} \mp \frac{i}{2}\lambda_{x1} - \frac{1}{2}\lambda_{y1} = \hat{g}_{\pm 1} + \frac{1}{2\pi}(\alpha_y \pm i\alpha_x). \quad (3.37)$$

For coefficients with $|k| \geq 2$, we obtain

$$-\left(\mu k^2 + \frac{h}{2}\right) \hat{\rho}_{1,k} - ikm\hat{\theta}_{1,k} = \hat{f}_k, \quad -k^2\hat{\theta}_{1,k} + ikm\hat{\rho}_{1,k} = \hat{g}_k,$$

implying

$$-k^2(\mu - \mu_{|k|})\widehat{\rho}_{1,k} = \widehat{f}_k - \frac{im}{k}\widehat{g}_k, \quad \widehat{\theta}_{1,k} - \frac{im}{k}\widehat{\rho}_{1,k} = -\frac{1}{k^2}\widehat{g}_k. \quad (3.38)$$

The results follow immediately from the homogeneous ($f = g = \alpha_x = \alpha_y = 0$) versions of (3.34)–(3.38), using the auxiliary conditions (3.31). $\square$

Lemma 3.4.3 (Solvability conditions). *Case 1. Problem (3.27)–(3.31) with $0 < \mu = \mu_j \neq \mu_1$, $j \geq 2$, has a solution if and only if*

$$\begin{aligned} j \int_0^{2\pi} f(s) \cos(js) \, ds &= m \int_0^{2\pi} g(s) \sin(js) \, ds, \\ j \int_0^{2\pi} f(s) \sin(js) \, ds &= -m \int_0^{2\pi} g(s) \cos(js) \, ds, \quad \text{and} \quad \int_0^{2\pi} g(s) \, ds = 0. \end{aligned} \quad (3.39)$$

Case 2. Problem (3.27)–(3.31) with $0 < \mu = \mu_1 \neq \mu_j$, $\forall j \geq 2$, has a solution if and only if

$$\int_0^{2\pi} f(s) \cos s \, ds = m\alpha_x, \quad \int_0^{2\pi} f(s) \sin s \, ds = m\alpha_y, \quad \text{and} \quad \int_0^{2\pi} g(s) \, ds = 0. \quad (3.40)$$

Proof. For Case 1, the solvability conditions follow from (3.35), (3.38), and for Case 2 from (3.35), (3.36). $\square$

3.4.3 Asymptotic expansion around bifurcation points

For the codimension-one bifurcations identified above (Cases 1 and 2 in Proposition 3.4.1), the existence of bifurcating solution branches is guaranteed by general results on bifurcations from simple eigenvalues [29]. The local shape of these branches will be analyzed by perturbation expansions. By the presence of a flip symmetry in problem (3.18)–(3.22), pitchfork bifurcations can be expected, at least generically. For the bifurcation at $\mu = \mu_j$ ($j \in \mathbb{N}$), we therefore introduce

$$\mu = \mu_j - \sigma A^2, \quad 0 < A \ll 1, \quad \sigma \in \{1, -1\}.$$

The small parameter A measures the distance from the bifurcation point, whereas the sign σ , to be determined by the analysis, tells us whether the bifurcation is *supercritical* for $\sigma > 0$ or *subcritical* for $\sigma < 0$. This convention is in line with the scenario of decreasing μ (see Remark 3.4.2, 5.). The solution $\bar{u} = (\rho, \theta, \lambda_x, \lambda_y, \lambda_M)$ of (3.18)–(3.22) will be approximated by an asymptotic expansion

$$\bar{u} = \bar{u}_0 + A\bar{u}_1 + A^2\bar{u}_2 + A^3\bar{u}_3 + O(A^4), \quad (3.41)$$

$A \rightarrow 0$, where the reason for going up to the third order will become apparent below.

Remark 3.4.4 (Bifurcation diagram for classical elasticae). The expectation of pitchfork bifurcations and, thus, the ansatz (3.41) can also be motivated by the bifurcation diagram for classical elasticae (e.g. [74, Ch. 7]). The diagram shows an infinite series of bifurcations similar to the series of Case 1 bifurcations for (3.18)–(3.22). In the classical elastica problem all these bifurcations are supercritical pitchforks.

The notation $\bar{u}_k := (\rho_k, \theta_k, \lambda_{xk}, \lambda_{yk}, \lambda_{Mk})$ for $k \in \mathbb{N}$ in (3.41) is consistent with the Definitions (3.23) of $\bar{u}_0$ and (3.26) of $\bar{u}_1$ given above. We recall that $\bar{u}_0$ is the trivial rotationally symmetric solution of the Euler-Lagrange equations (3.18)–(3.22). The first correction $\bar{u}_1$ satisfies the homogeneous version ($f = g = \alpha_x = \alpha_y = 0$) of the linearized problem (3.27)–(3.31) with $\mu = \mu_j$,

whose solution is unique up to a scalar constant (a_1 in Case 1 and b_1 in Case 2). The problems for $\bar{u}_2$ and $\bar{u}_3$ are determined by substituting the ansatz (3.41) into (3.18)–(3.22), expanding the nonlinearities and comparing coefficients of A^2 and A^3 . Both $\bar{u}_2$ and $\bar{u}_3$ solve inhomogeneous versions of the linearized problem (3.27)–(3.31) with $\mu = \mu_j$ and the inhomogeneities

$$f_2 = \frac{m}{2}\dot{\theta}_1^2 + h\rho_1\dot{\theta}_1, \quad (3.42)$$

$$g_2 = -\frac{d}{ds}\left(m\rho_1\dot{\theta}_1 + \frac{h}{2}\rho_1^2\right) - \theta_1(\lambda_{x1}\cos s + \lambda_{y1}\sin s), \quad (3.43)$$

$$\begin{pmatrix} \alpha_{x2} \\ \alpha_{y2} \end{pmatrix} = \frac{1}{2} \int_0^{2\pi} \theta_1^2 \begin{pmatrix} \cos s \\ \sin s \end{pmatrix} ds, \quad (3.44)$$

for $\bar{u}_2$, and

$$f_3 = \sigma\ddot{\rho}_1 + m\dot{\theta}_1\dot{\theta}_2 + \frac{h}{2}\left(2\rho_2\dot{\theta}_1 + 2\rho_1\dot{\theta}_2 + \rho_1\dot{\theta}_1^2\right), \quad (3.45)$$

$$\begin{aligned} g_3 = & -\frac{d}{ds}\left(m\rho_1\dot{\theta}_2 + m\rho_2\dot{\theta}_1 + \frac{h}{2}\rho_1^2\dot{\theta}_1 + h\rho_1\rho_2\right) \\ & - \theta_1(\lambda_{x2}\cos s + \lambda_{y2}\sin s) - \theta_2(\lambda_{x1}\cos s + \lambda_{y1}\sin s) \\ & + \frac{1}{2}\theta_1^2(\lambda_{x1}\sin s - \lambda_{y1}\cos s), \end{aligned} \quad (3.46)$$

$$\begin{pmatrix} \alpha_{x3} \\ \alpha_{y3} \end{pmatrix} = \int_0^{2\pi} \begin{pmatrix} \theta_1\theta_2\cos s - \frac{1}{6}\theta_1^3\sin s \\ \theta_1\theta_2\sin s + \frac{1}{6}\theta_1^3\cos s \end{pmatrix} ds, \quad (3.47)$$

for $\bar{u}_3$. Note that the inhomogeneities depend on lower-order terms. So the terms in the asymptotic expansion (3.41) can be computed recursively. However, this comes with two problems, which are connected: the solution of the linearized problem is not unique (see Proposition 3.4.1) and it does not have a solution for arbitrary inhomogeneities (see Lemma 3.4.3). The strategy is to recover the lacking information for uniqueness from the solvability conditions for higher-order problems. It will turn out (as a consequence of the above mentioned flip symmetry) that the inhomogeneities (3.42)–(3.44) of the second-order problem satisfy the solvability conditions, no matter what the value of the missing first-order constant (a_1 in Case 1 and b_1 in Case 2) is. This is the reason why the third-order problem has to be considered, whose solvability condition will provide an equation for the missing first-order constant. In the following, the essential results of these straightforward but lengthy computations will be given. They have been carried out manually and checked with the help of MATHEMATICA.

Case 1 Bifurcations

The goal is to determine the value of the constant a_1 in the first-order correction $\bar{u}_1$ of the expansion (3.41), given in (3.32). The first step is the computation of the second-order terms.

Lemma 3.4.5 (Case 1: Second-order solution). *Let $j \geq 2$ and $\bar{u}_1$ be given by (3.32). Then every solution of (3.27)–(3.31), with $0 < \mu = \mu_j \neq \mu_1$ and the inhomogeneities given by (3.42)–(3.44), can be written as*

$$\begin{aligned} \rho_2(s) &= a_2 \sin(js) + \frac{a_1^2(m(m^2 - h))}{2(2m^2 - h)}(\cos(2js) - \cos(js)), \\ \theta_2(s) &= a_2 \frac{m}{j}(\cos(js) - 1) - \frac{a_1^2(6m^4 - 6m^2h + h^2)}{8j(2m^2 - h)}\sin(2js), \quad a_2 \in \mathbb{R}, \\ \lambda_{x2} = \lambda_{y2} &= 0, \quad \lambda_{M2} = -\frac{a_1^2m(m^2 - 2h)}{4}. \end{aligned} \quad (3.48)$$

Lemma 3.4.6 (Case 1: The missing constant). *Let $j \geq 2$, $0 < \mu_j \neq \mu_1$, and $\bar{u}_1, \bar{u}_2$ be given by (3.32), (3.48). Then the inhomogeneities (3.45)–(3.47) satisfy the solvability conditions (3.39), if and only if*

$$a_1 \left(j^2 \sigma - a_1^2 \frac{Z(m, h)}{8(2m^2 - h)} \right) = 0 \quad \text{with} \quad Z(m, h) := -14m^6 + 36m^4h - 18m^2h^2 + h^3. \quad (3.49)$$

This shows that Case 1 bifurcations are pitchforks if and only if $Z(m, h) \neq 0$. The amplitude of the first-order term along the bifurcating branch is determined by the nontrivial solutions of (3.49):

$$a_1^2 = \frac{8j^2 \sigma (2m^2 - h)}{Z(m, h)}, \quad (3.50)$$

which shows for the criticality $\sigma = \text{sign } Z(m, h)$ that the bifurcation is *supercritical* for $Z(m, h) > 0$ and *subcritical* for $Z(m, h) < 0$. Writing $Z(m, h) = m^6(z^3 - 18z^2 + 36z - 14)$ with $z := h/m^2 < 2$ shows that $Z(m, h) > 0$ and $2m^2 > h$ are equivalent to

$$z_1 < z < z_2 \quad \text{with} \quad z_1 \approx 0.52 \quad \text{and} \quad z_2 \approx 1.71. \quad (3.51)$$

Consequently, a supercritical pitchfork bifurcation occurs in the parabolic region

$$\{(m, h) \in \mathbb{R}^2 : z_1 m^2 < h < z_2 m^2\}.$$

Conversely, if (m, h) is such that $Z(m, h) < 0$, which holds for $h < z_1 m^2$ or $2m^2 > h > z_2 m^2$, then the bifurcation is subcritical. The situation is illustrated in Figure 3.3. Note that the criticality is independent of j . For a fixed pair (m, h) the whole series of Case 1 bifurcations has the same criticality.

Case 2 Bifurcations

Lemma 3.4.7 (Case 2: Second-order solution). *Let $\bar{u}_1$ be given by (3.33). Then every solution of (3.27)–(3.31), with $0 < \mu = \mu_1 \neq \mu_j$, $\forall j \geq 2$, and the inhomogeneities given by (3.42)–(3.44), can be written as*

$$\begin{aligned} \rho_2(s) &= b_2 \sin s + \frac{b_1^2 m h}{2(2m^2 + 3h)} (\cos(2s) - \cos s), \quad b_2 \in \mathbb{R}, \\ \theta_2(s) &= \frac{b_1^2 3h^2}{8(2m^2 + 3h)} \sin(2s), \\ \lambda_{x2} &= -\frac{b_1^2 m^2 h}{2(2m^2 + 3h)}, \quad \lambda_{y2} = b_2 m, \quad \lambda_{M2} = 0. \end{aligned} \quad (3.52)$$

Lemma 3.4.8 (Case 2: The missing constant). *Let $0 < \mu_1 \neq \mu_j$, $\forall j \geq 2$, and $\bar{u}_1, \bar{u}_2$ be given by (3.33), (3.52). Then the inhomogeneities (3.45)–(3.47) satisfy the solvability conditions (3.40), if and only if*

$$b_1 \left(\sigma + b_1^2 \frac{3h^3}{8(2m^2 + 3h)} \right) = 0. \quad (3.53)$$

This shows that Case 2 bifurcations are pitchforks, since

$$b_1^2 = -\frac{8\sigma(2m^2 + 3h)}{3h^3}$$

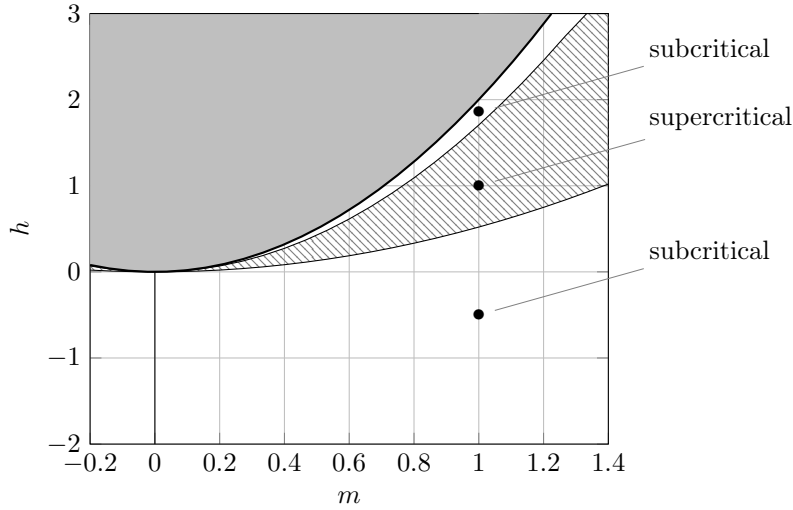


Figure 3.3: Contour plot of $Z(m, h)$ given in (3.49). The solution in Case 1 has the structure of a supercritical pitchfork bifurcation whenever $Z(m, h) > 0$ and $h < 2m^2$. These conditions define the crosshatched region $z_1 m^2 < h < z_2 m^2$ below the parabola $h = 2m^2$ (black line); $z_1 \approx 0.52$ and $z_2 \approx 1.71$, see (3.51). Conversely, if $Z(m, h) < 0$ which is true when $h < z_1 m^2$ or in the narrow white region given by $z_2 m^2 < h < 2m^2$, then the bifurcation is subcritical.

is finite by $\mu_1 = -h/2 > 0$ and nonvanishing by $2m^2 + 3h = 8(\mu_2 - \mu_1) \neq 0$. The bifurcation is subcritical for $2m^2 + 3h < 0$, i.e., when the Case 2 bifurcation is the first one for decreasing μ (see Remark 3.4.2, 5.). It is supercritical for $2m^2 + 3h > 0$. It is also noteworthy that the circular shape of the trivial solution curve is now perturbed at the order of A^2 with the perturbation given in (3.52). The leading order density perturbation ρ_1 has both its extrema coinciding either with the maxima of the curvature perturbation $\dot{\theta}_2$ (in the subcritical case) or with its minima (in the supercritical case). This can be verified numerically, see Cases (iv) and (v) in Section 3.5.3 and Figure 3.6 for $j = 1$.

3.4.4 Energy and stability

As an indicator for the stability of bifurcating solutions, we investigate the changes of the energy (3.1) along bifurcating branches. For this purpose, we substitute $\mu = \mu_j - \sigma A^2$ ($j \in \mathbb{N}$) and the asymptotic expansion (3.41) of the bifurcating solution into the energy and re-expand:

$$E_\mu(\rho, \theta) = \frac{1}{2} \int_0^{2\pi} (\beta(\rho) \dot{\theta}^2 + \mu \dot{\rho}^2) ds = E_0 + A E_1 + A^2 E_2 + A^3 E_3 + A^4 E_4 + O(A^5),$$

$A \rightarrow 0$. For the coefficients we obtain (all computations of this section were again verified with MATHEMATICA)

$$\begin{aligned} E_0 &= \frac{1}{2} \int_0^{2\pi} ds = \pi = E_{\mu_j}(\rho_0, \theta_0) \quad (j \in \mathbb{N}), \\ E_1 &= \frac{1}{2} \int_0^{2\pi} (m \rho_1 + 2 \dot{\theta}_1) ds = 0, \\ E_2 &= \frac{1}{2} \int_0^{2\pi} \left(2m \rho_1 \dot{\theta}_1 + \frac{h}{2} \rho_1^2 + \dot{\theta}_1^2 + \mu_j \dot{\rho}_1^2 \right) ds, \end{aligned}$$

$$\begin{aligned}
 E_3 &= \frac{1}{2} \int_0^{2\pi} \left(2m\rho_1\dot{\theta}_2 + 2m\rho_2\dot{\theta}_1 + m\rho_1\dot{\theta}_1^2 + h\rho_1\rho_2 + h\rho_1^2\dot{\theta}_1 + 2\dot{\theta}_1\dot{\theta}_2 + 2\mu_j\dot{\rho}_1\dot{\rho}_2 \right) ds, \\
 E_4 &= \frac{1}{2} \int_0^{2\pi} \left(2m\rho_1\dot{\theta}_3 + 2m\rho_3\dot{\theta}_1 + m\rho_2\dot{\theta}_1^2 + 2m\rho_2\dot{\theta}_2 + 2m\rho_1\dot{\theta}_1\dot{\theta}_2 + \frac{h}{2}\rho_2^2 + h\rho_1\rho_3 \right. \\
 &\quad \left. + 2h\rho_1\rho_2\dot{\theta}_1 + \frac{h}{2}\rho_1^2\dot{\theta}_1^2 + h\rho_1^2\dot{\theta}_2 + 2\dot{\theta}_1\dot{\theta}_3 + \dot{\theta}_2^2 + \mu_j\dot{\rho}_2^2 + 2\mu_j\dot{\rho}_1\dot{\rho}_3 - \sigma\dot{\rho}_1^2 \right) ds.
 \end{aligned}$$

Lemma 3.4.9 (Energy expansion). *Let $j \in \mathbb{N}$ and let $\bar{u}_1, \bar{u}_2$ be given by (3.32), (3.48) in Case 1 for $j \geq 2$, or by (3.33), (3.52) in Case 2 for $j = 1$. Let the constants a_1 in Case 1 or b_1 in Case 2 be chosen such that the solvability conditions of Lemma 3.4.6 in Case 1 or of Lemma 3.4.8 in Case 2 hold. Let $\bar{u}_3$ be a solution of the inhomogeneous linearized problem (3.27)–(3.31) with $\mu = \mu_j$ and the corresponding third-order inhomogeneities $f = f_3, g = g_3, \alpha_x = \alpha_{x3}$, and $\alpha_y = \alpha_{y3}$ given in (3.45)–(3.47). Then the coefficients in the energy expansion above satisfy*

$$E_1 = E_2 = E_3 = 0$$

in both cases, as well as

$$E_4 = -\frac{2\pi j^4(2m^2 - h)}{Z(m, h)} \quad (3.54)$$

in Case 1 with the notation of Lemma 3.4.6. In Case 2 we have

$$E_4 = \frac{2\pi(3h + 2m^2)}{3h^3}.$$

As expected, the sign of E_4 goes with criticality of the bifurcating branch. Stability is gained ($E_4 < 0$) along supercritical branches and lost ($E_4 > 0$) along subcritical branches. In particular, this can be expected to determine the stability of the branch corresponding to the first bifurcation for decreasing μ .

3.5 Numerical continuation of bifurcation branches

3.5.1 Discretization

The Euler-Lagrange equations (3.18) and (3.19) are discretized by finite differences as follows. For $N \in \mathbb{N}$ we discretize the interval $[0, L]$ by introducing $\Delta s = L(N - 1)^{-1}$ and $s_i = i\Delta s$, $0 \leq i \leq N - 1$, which naturally leads to the (abuse of) notation

$$\rho = (\rho_i)_{i=1}^{N-1}, \quad \theta = (\theta_i)_{i=1}^{N-1} \quad \text{with} \quad \rho_i = \rho(s_i), \quad \theta_i = \theta(s_i).$$

This can be thought of as considering a polygonal approximation of the curve γ , where θ_i is the angle of the i th side and where ρ_i is a piecewise constant approximation of ρ on that side (i.e., ρ_i is not associated to a vertex).

Using the notation $u = (\rho, \theta)$ and $\Lambda = (\lambda_x, \lambda_y, \lambda_M)$, we propose the following natural finite differences approximation for (3.18) and (3.19), respectively:

$$EL_\rho(u, \Lambda) := \mu \left(\frac{\rho_{i-1} - 2\rho_i + \rho_{i+1}}{\Delta s^2} \right) - \frac{1}{2}\beta'(\rho_i) \left(\frac{\rho_{i+1} - \rho_{i-1}}{2\Delta s} \right)^2 - \lambda_M = 0, \quad (3.55)$$

$$\begin{aligned}
 EL_\theta(u, \Lambda) &:= \frac{1}{\Delta s} \left(\beta \left(\frac{\rho_{i+1} + \rho_i}{2} \right) \left(\frac{\theta_{i+1} - \theta_i}{\Delta s} \right) - \beta \left(\frac{\rho_i + \rho_{i-1}}{2} \right) \left(\frac{\theta_i - \theta_{i-1}}{\Delta s} \right) \right) \\
 &\quad + \lambda_x \sin \theta_i - \lambda_y \cos \theta_i = 0, \quad (3.56)
 \end{aligned}$$

for $0 \leq i \leq N-1$.

To remove the degree of freedom associated to solid rotations, we set $\theta(0) = 0$ at the continuous level, see (3.20). This is reflected by the choice $\theta_0 = 0$ at the discrete level. We also need to provide values for indices $i = -1, N$. Again by periodicity we set $\rho_{-1} = \rho_{N-1}$, $\rho_N = \rho_0$, $\theta_{-1} = \theta_{N-1} - 2\pi$, and $\theta_N = \theta_0 + 2\pi$. Thus, we only consider (3.55) for $0 \leq i < N-1$ and (3.56) for $0 < i < N-1$.

The mass and closedness constraints (3.21) can be naturally approximated as

$$C_M(u, \Lambda) := \Delta s \sum_{i=0}^{N-1} \rho_i - M = 0 \quad \text{and} \quad \begin{pmatrix} C_x \\ C_y \end{pmatrix}(u, \Lambda) := \Delta s \sum_{i=0}^{N-1} \begin{pmatrix} \cos \theta_i \\ \sin \theta_i \end{pmatrix} = 0.$$

We are left with a system of $2N+1$ nonlinear equations for $\bar{u} = (u, \Lambda)$,

$$r(\bar{u}) := (EL_\rho, EL_\theta, C_x, C_y, C_M)(\bar{u}) = 0,$$

which we propose to solve numerically using a damped Newton method: If we assume $\bar{u}^k = (u^k, \Lambda^k)$ ($k \in \mathbb{N}_0$) to be known, we look for $\bar{u}^{k+1}$ as a solution to

$$J(\bar{u}^k)(\bar{u}^{k+1} - \bar{u}^k) = -\eta r(\bar{u}^k), \quad (3.57)$$

where J is the Jacobian of r and $\eta \leq 1$ is the damping parameter with $\eta = 1$ corresponding to the standard Newton's method.

Continuation of branches

To follow numerically the bifurcation branches, one can pick some μ close to the critical value μ_j (see Proposition 3.4.1) and take as initial value a perturbation of the trivial solution (corresponding to the circle with homogeneous ρ). The position of μ relative to the critical value and the amplitude of the bifurcation are given precisely by the results of Section 3.4. Solving (3.57) yields a numerical approximation of a critical point, which can be used as initial condition for neighbouring values of μ . By iterating this process, one can move along the branch, provided that

- (1) the branch is locally smooth (for example, this is not the case when ρ hits zeros of β , where one could expect the branch to terminate),
- (2) the features of the solution can be resolved by the discretization level with the chosen value of N .

3.5.2 Choice of parameters

In the following, we will consider a number of different situations, depending on the choice of parameters (m, h) modulating the stiffness coefficient β , which will be of the form (3.25) with $\beta_0 = 1$, namely

$$\beta(\rho) = 1 + m(\rho - \rho_0) + \frac{h}{2}(\rho - \rho_0)^2.$$

As before we will take $M = L = 2\pi$, so that $\rho_0 = 1$. We consider six sets of parameters, covering the different cases given in Proposition 3.4.1:

- (i) $(m, h) = (1, 1.85)$ corresponding to Case 1 with $\sigma = -1$ (subcritical bifurcation),
- (ii) $(m, h) = (1, 1)$ corresponding to Case 1 with $\sigma = 1$ (supercritical bifurcation),
- (iii) $(m, h) = (1, 1/4)$ corresponding to Case 1 with $\sigma = -1$ (subcritical bifurcation),
- (iv) $(m, h) = (1, -1/2)$ corresponding to Case 1 with $\sigma = -1$ (subcritical bifurcation) and supercritical Case 2,
- (v) $(m, h) = (1, -2)$ corresponding to Case 1 with $\sigma = -1$ (subcritical bifurcation) and subcritical Case 2,
- (vi) $(m, h) = (0, -1)$, which is similar to (v) in the special choice $m = 0$. At first order, γ should remain a circle, including for Case 1, as the correction coefficient $\theta_1 \equiv 0$ for $m = 0$, see (3.32).

The parameters in (i) – (vi) are represented in Figure 3.4 and the corresponding results are presented in Figures 3.5 and 3.6.

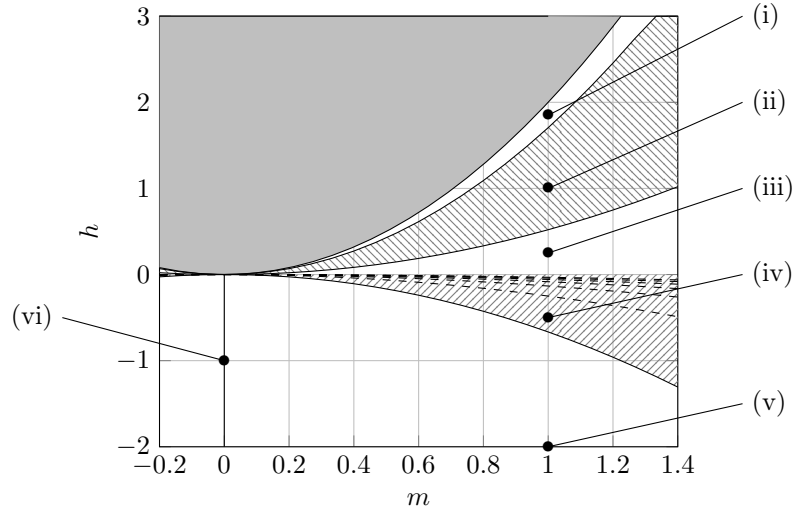


Figure 3.4: The different sets of model parameters (m, h) for the stiffness coefficient β given by (3.25). According to the bifurcation analysis (Section 3.4), the gray region corresponds to parameters which have no critical points except the trivial solution. The crosshatched region corresponds to supercritical bifurcations (Case 1 for $h > 0$, Case 2 for $h < 0$) and the plain white region to subcritical bifurcations. The dashed parabolas indicate where Case 3 occurs, for j up to 8.

3.5.3 Results

The method described above (Section 3.5.1) was implemented in Julia [14]. Figure 3.5 presents the bifurcating branches both in terms of the amplitude of the density ρ and in terms of the energy E_μ . It offers a partial confirmation of the results of Section 3.4 in that:

- For the parameter case (ii), $j \geq 2$ and (iv), $j = 1$, the bifurcation appears supercritical, i.e., the branch bifurcates to the left of the critical μ . Additionally, the energy decreases

close to the trivial state. These branches offer critical points of E_μ which are candidates to be global minimizers.

- For all other parameter cases, the bifurcation is subcritical, i.e., the branch bifurcates to the right of the critical μ , and the energy initially increases as one gets further from the trivial state.

Interestingly, the parameter cases (i) and (iv) feature turning points, where the derivative of E_μ along the branch seems to change sign. In parameter case (i) it becomes negative, leading to critical points of lower energy with respect to the trivial state, and potentially global minimizers. This fact precludes uniqueness of minimizers of E_μ in general.

We were able to track an additional branch in parameter cases (i) to (iii), which seem to bifurcate from the $j = 2$ branch. No analytical results are available at this point, but we can make the following observations. The corresponding shapes, presented in gray in Figure 3.6, look like the ones obtained for $j = 1$ in the parameter cases (iv) to (vi). This justifies the placement in the first column, although j has no meaning for this branch. In the parameter cases (i) and (iii), it bifurcates from the $j = 2$ branch with decreasing energy for the choice of parameter considered. Parameter case (ii) is a bit different, in that the bifurcation leads to critical points of higher energy, although the branch features a turning point, after which E_μ starts decreasing and eventually becomes smaller than for the $j = 2$ branch, for a given value of μ .

Other features of the critical points further along the branch can be seen in Figure 3.6. For the parameter cases (i) to (v) and $j > 1$, one can identify the value of j with the number of flatter sections in each closed curve. These correspond to higher values of ρ , which agree with the fact that for all choices of parameters presented here, $m \geq 0$. This can be roughly thought as higher values of ρ penalizing higher values of the curvature $\dot{\theta}$. For $j = 1$ or in Case (vi), the situation is different, since the first-order correction $\theta_1 \equiv 0$. If one goes further in the expansion, one can expect that the next order correction θ_2 have the form $\cos(2js)$, that is half the period of ρ_1 . This could explain that in these cases, ρ seems to have j maxima when $\dot{\theta}$ has $2j$.

Additionally, *far* from the bifurcation point and after potential turning points, one can distinguish the parameter cases (i) and (ii) from (iii) to (vi). For the former, μ decreases along the branch, E_μ decreases and ρ seems to concentrate on flat sections. For the latter, the situation is the opposite: μ increases along the branch, E_μ increases and ρ stays rather smooth.

Remark 3.5.1. In Proposition 3.3.3 it is stated that for β bounded away from 0, only the trivial state (ρ_0, θ_0) is a minimizer of E_μ . The branches in Figure 3.5 which seem to continue far to large values of μ have an energy clearly larger than $\pi = E_\mu(\rho_0, \theta_0)$. We also recall that for the results presented here, the choice of β is quadratic, and thus not bounded away from 0. There is then no contradiction of our analysis.

A systematic study of the stability in terms of the energy would be interesting, although probably necessarily limited to numerics, as it would help identifying local minimizers. Such an investigation is however out of the scope of this paper.

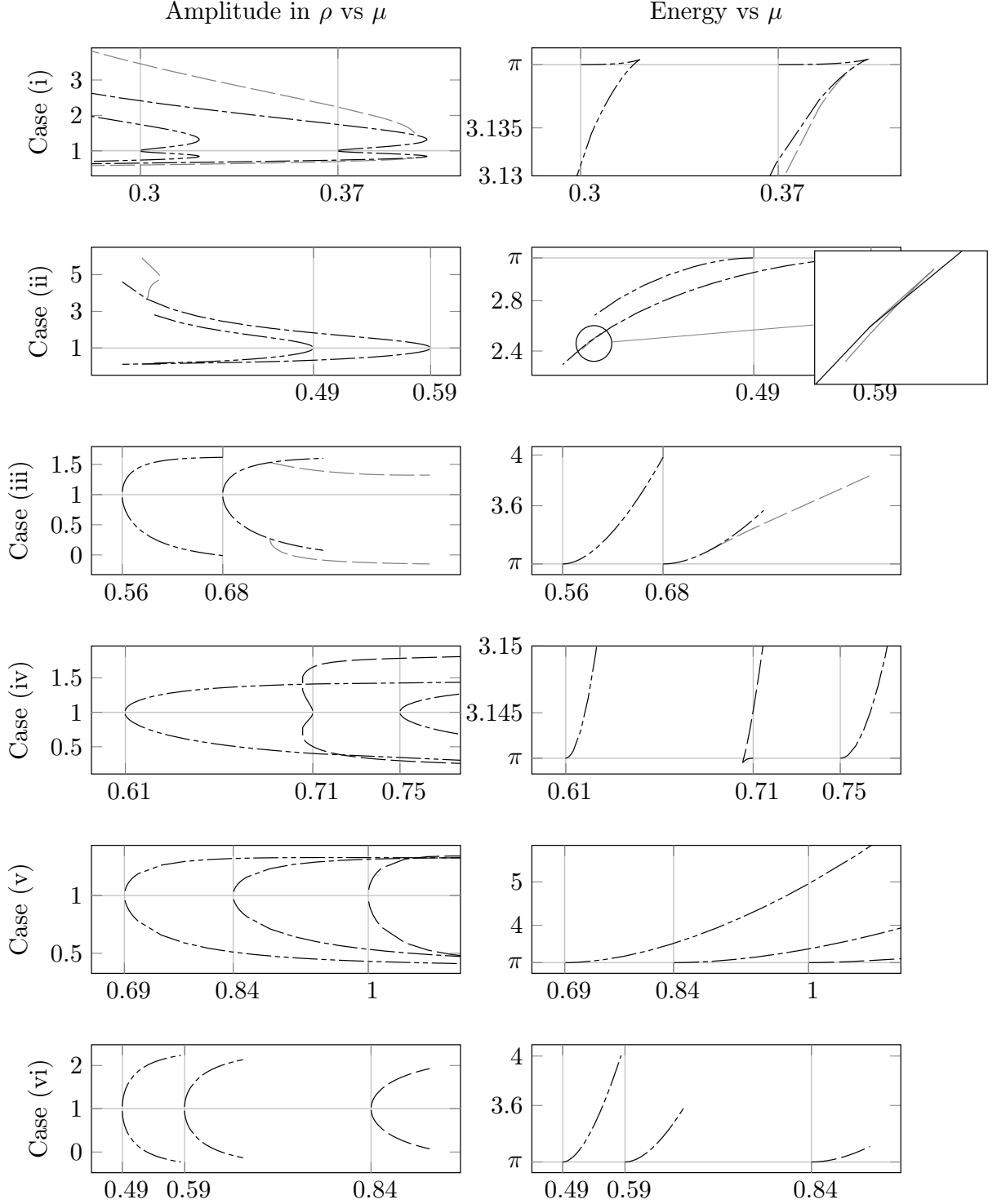


Figure 3.5: Numerical results for the parameter cases (i) to (vi), in columns, for $j = 1, 2, 3$. The first column shows the amplitude in ρ , where the lower and greatest values of ρ are represented. The horizontal gray line corresponds to the trivial solution, for which $\rho = 1$. The second column shows the energy E_μ , with the horizontal gray line again corresponding to the trivial solution, for which $E_\mu = \pi$. The dashes indicate the value of j for each branch: — for $j = 1$ (absent in (i) to (iii)), - - for $j = 2$, - . - for $j = 3$. The gray vertical lines indicate the theoretical critical values for μ given in Proposition 3.4.1. In the parameter cases (i) to (iii), the secondary bifurcation branch is plotted in gray. As detailed in Section 3.4.3, at a supercritical (resp. subcritical) bifurcation point, the branch will appear for values of μ greater (resp. lower) than the critical value.

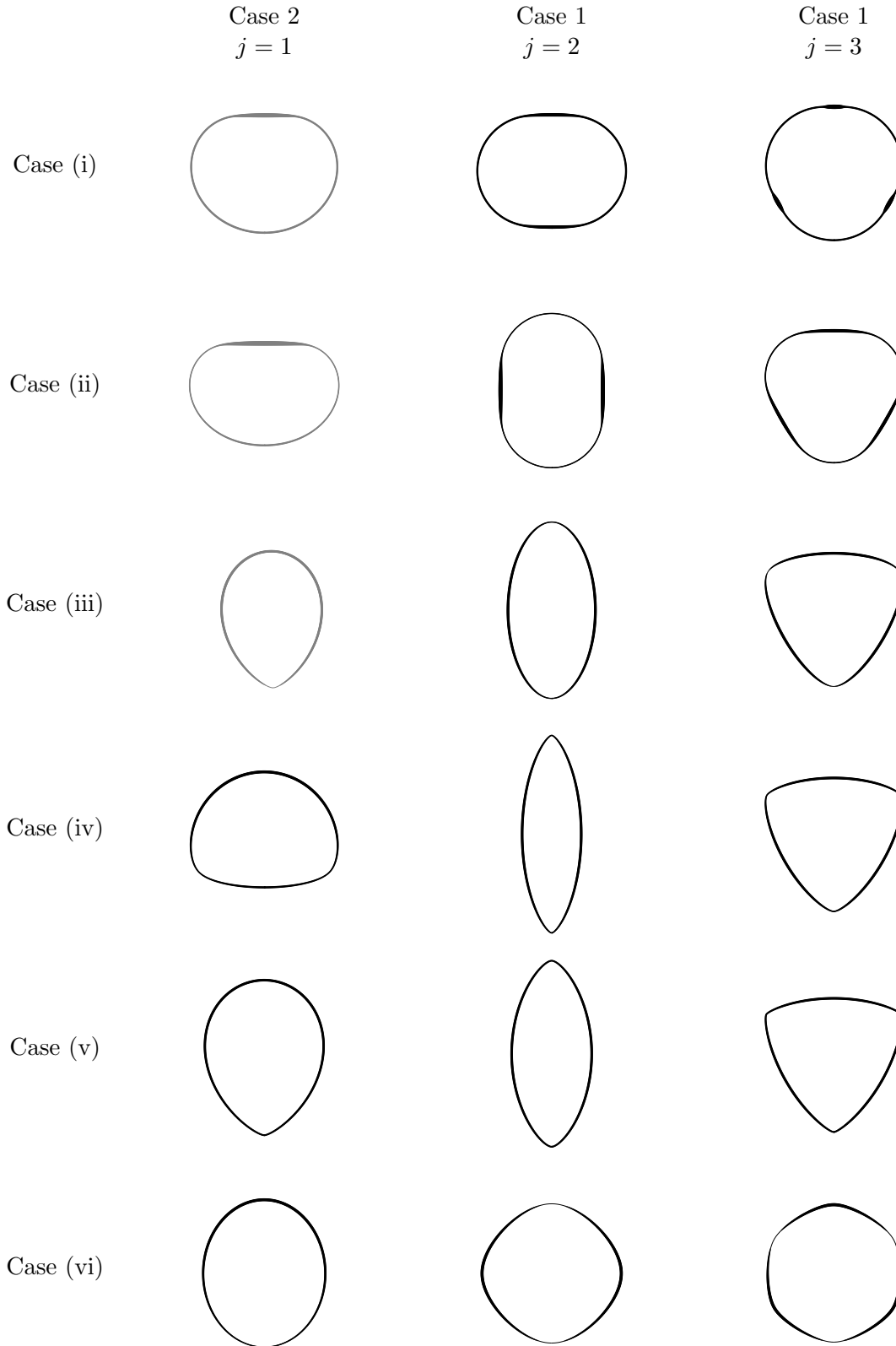


Figure 3.6: The shapes corresponding to each case, with $j = 1, 2, 3$ increasing with each column. These correspond to the last point computed on the branches shown in Figure 3.5. In the parameter cases (i) to (iii), the shapes in the first column are in gray, as they do not correspond to branches bifurcating from the trivial state, and j is not defined in this case. They are placed in the first column due to their resemblance to shapes obtained for $h < 0$, in parameter cases (iv) to (vi). Thicker lines denote larger values of ρ .

3.6 Conclusion

To describe elastic curves in the plane, we introduced a regularized Canham-Helfrich type functional which includes a density-modulated stiffness β . We proved that the associated minimization problem has a solution if the regularization parameter is positive. If not, the problem has no solution in general. Conditions on the first derivatives of β were derived so that the problem has non trivial solutions. In this case, a bifurcation analysis around the trivial solution was performed, the regularization parameter playing the role of the bifurcation parameter. A family of both subcritical and supercritical pitchfork bifurcations was found, depending on the choice of β . This is in contrast to classical elastic curves, which display supercritical bifurcations only. An expansion of the energy confirmed that subcritical (resp. supercritical) solutions correspond to a gain (resp. a loss) of energy compared to the trivial state. This analysis was completed by numerical continuation of the bifurcating branches, which confirmed the theoretical findings. Secondary bifurcations and turning points – found numerically – testify of the intricate mathematical structure of the model. In particular no uniqueness should be expected for the minimization problem, except for large regularization.

Acknowledgments

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Chapter 4

Existence of varifold minimizers for the multiphase Canham-Helfrich functional

This chapter consists of my paper [22] with LUCA LUSSARDI and ULISSE STEFANELLI.

Abstract

We address the minimization of the Canham-Helfrich functional in presence of multiple phases. The problem is inspired by the modelization of heterogeneous biological membranes, which may feature variable bending rigidities and spontaneous curvatures. With respect to previous contributions, no symmetry of the minimizers is here assumed. Correspondingly, the problem is reformulated and solved in the weaker frame of oriented curvature varifolds. We present a lower semicontinuity result and prove existence of single- and multiphase minimizers under area and enclosed-volume constraints. Additionally, we discuss regularity of minimizers and establish lower and upper diameter bounds.

4.1 Introduction

The equilibrium shape of a biological membrane is described by the classical CANHAM [25] & HELFRICH [50] theory as the result of the constrained minimization of its bending energy. Given a smooth closed surface M in $\mathbb{R}^3$, its *Canham-Helfrich energy* reads

$$E_{\text{CH}}(M) = \int_M \left(\frac{\beta}{2} (H - H_0)^2 + \gamma K \right) d\mathcal{H}^2 \quad (4.1)$$

where β, γ , and $H_0 \in \mathbb{R}$ are given material constants, H is the mean curvature of M , and K is the Gauss curvature of M . The minimization is constrained to surfaces of fixed area and fixed enclosed volume. The preservation of area reflects the inextensibility of the membrane material and the volume constraint arises from the osmotic pressure balance between the vesicle and its surroundings [17]. In contrast to the classical *Willmore energy* [62, 87, 91, 95]

$$\mathcal{W}(M) = \int_M H^2 d\mathcal{H}^2, \quad (4.2)$$

4 Existence of varifold minimizers for the multiphase Canham-Helfrich functional

the Canham-Helfrich energy (4.1) for $H_0 \neq 0$ implicitly depends on the orientation of M as well, as its choice determines the sign of H .

Biomembranes consist of single or double layers of lipids and frequently also contain other molecules. For example, cell membranes are lipid bilayers with embedded cholesterol and proteins. The molecule concentration locally alters the mechanical properties of the membrane by changing the bending rigidities β, γ and the spontaneous curvature H_0 [50]. Thereby, the composition of the membrane affects its shape. Moreover, since lipid membranes behave as two-dimensional fluids [97], the constituents are not fixed within the membrane but rather can migrate due to surface diffusion and electrochemical dynamics. The result is an interplay between molecule concentrations, influencing the local membrane geometry, and geometry, influencing concentrations [17, 76].

In specific situations, biomembranes with well-separated domains of characteristic composition are observed [12]. In the case of two of such *phases*, one can model the membrane as the union of compact surfaces M_1 and M_2 in $\mathbb{R}^3$ that overlap precisely at their boundary. The associated energy is the sum of the bending energies (4.1) of the individual phases, augmented by an additional *line tension* contribution penalizing phase-interfaces [55]. The membrane morphology is hence described by the minimization of the *sharp-interface, two-phase Canham-Helfrich energy*

$$E_{\text{CH},2}(M_1, M_2) = \sum_{i=1,2} \int_{M_i} \left(\frac{\beta^i}{2} (H - H_0^i)^2 + \gamma^i K \right) d\mathcal{H}^2 + \sigma \int_{\Gamma_{1,2}} d\mathcal{H}^1. \quad (4.3)$$

Here, β^i , γ^i , and H_0^i are the material parameters of the individual phases M_i ($i = 1, 2$), $\sigma > 0$ is the line-tension coefficient, and the phase boundary $\Gamma_{1,2} = M_1 \cap M_2$ is assumed to be a union of closed curves. By assuming $\partial M = \emptyset$ one has that the two-phase membrane $M = M_1 \cup M_2$ is closed.

In addition to the constraint on the volume enclosed by M , one fixes the areas of the individual phases M_1 and M_2 . This two-phase model can be extended to more phases by letting $M = \bigcup_{i=1}^N M_i$, adding all individual Canham-Helfrich contributions, and penalizing all interfaces.

The focus of this paper is on investigating existence of equilibrium solutions associated to the multiphase Canham-Helfrich functional (4.3) under enclosed-volume and phase-area constraints. This variational problem has already been addressed by CHOKSI, MORANDOTTI, & VENERONI [26, 27], who nevertheless focused on the case of *axisymmetric* membranes and phases. This assumption allowed them to reformulate the problem in terms of generating curves in the plane, a fact that is crucially exploited in the analysis. Still, by imposing rotational symmetry, many biologically relevant shapes may be ruled out [12].

Our goal is hence to remove this restriction and to study multiphase minimizers without a priori assumptions on their symmetry. The price to pay for such generality is that the strong formulation (4.3) has to be relaxed. In particular, we resort to rephrasing the problem in the general framework of *curvature varifolds* [54, 73], which are a geometric measure theory generalization of surfaces [1, 94]. This asks for defining a corresponding *generalized Canham-Helfrich functional*, see (4.11) below. Such generalization is not entirely new. In fact, the minimization of geometric functionals involving curvature was HUTCHINSON's [54] motivation to introduce curvature varifolds in the first place. Notice that the Willmore functional (4.2) is readily defined for integral varifolds with locally bounded first variation in the sense of ALLARD [1], because its integrand can be expressed solely in terms of the mean curvature vector. However, the density of the Canham-Helfrich functional (4.1) with $\gamma \neq 0$ contains more components of the second fundamental form and, with $H_0 \neq 0$, also depends on the orientation. For these reasons and in view of the presence of interfaces in the multiphase energy (4.3), we combine here

Hutchinson's [54] oriented varifolds and MANTEGAZZA's [73] curvature varifolds with boundary, formulating the variational problem in the setting of *oriented curvature varifolds with boundary*, see (4.9).

Our main result is the existence of minimizers for the varifold reformulation (4.11) of the Canham-Helfrich functional, both in the single- and in the multiphase regime. We proceed along the blueprint of the classical Direct Method. Lower semicontinuity with respect to curvature varifold convergence follows from convexity, upon checking that the energy bounds the L^2 -norm of the second fundamental form. Such convexity holds under the qualification on the material parameters $-6\beta < 5\gamma < 0$, see (4.13), which is more restrictive than the one from [26, 27], but is still in the range of experimental data, see [13, Formula (1.4)] and the references therein. The existence of a single-phase minimizer with fixed area and fixed enclosed volume is proved in Theorem 4.4.2. The multiphase case is tackled by additionally imposing phases to have individually fixed areas and not to overlap. Properly reformulated in the varifold setting, these conditions still allow for proving an existence result, see Theorem 4.4.9. Eventually, we show that both single- and multiphase minimizers are C^2 -rectifiable (Lemma 4.4.10) and have bounded Willmore energy (Lemma 4.4.11). This last fact allows us to prove that the diameter of minimizers is bounded from above as well as away from zero (Lemmas 4.4.12 and 4.4.13). In the case of multiple phases, the lower bounds on the diameter apply to each individual phase.

We would like to put our contribution in perspective by briefly reviewing the available literature on the minimization of Canham-Helfrich functionals, that is, (4.1) or (4.3) for $H_0 \neq 0$. As already mentioned, the axisymmetric case has been discussed in [27] for single phases and in [26] in the case of multiple phases with sharp interfaces. Also under rotational-symmetry assumptions, HELMERS [52, 53] adopts a surface phase-field approach and presents Γ -convergence results to the sharp-interface limit for two-phase membranes with possibly discontinuous tangent vectors. To the best of our knowledge, [26, 52, 53] currently are the only analysis results for multiphase Canham-Helfrich variational problems. Single-phase minimizers without symmetry assumptions have been recently addressed by EICHMANN [40, 41] and MONDINO & SCHARER [78]. In [40] specific Dirichlet boundary conditions are considered, by which the problem can be formulated in terms of branched immersions. The analysis in [41] focuses on closed immersions with prescribed genus g whereas [78] deals with the case of weak and possibly branched immersions of a 2-sphere in $\mathbb{R}^3$ (which has $g = 0$). The minimization under additional confinement and connectedness constraints is discussed by MÜLLER & RÖGER [79] and DONDL & WOJTOWYTSCH [39, 102], though with emphasis on the Willmore energy (4.2). In this context we also mention the recent contribution by NOVAGA & POZZETTA [81], where Mantegazza's curvature varifolds with boundary are employed to study connected Willmore minimizers for Dirichlet data. Based on currents instead of varifolds, DELLADIO [37] introduced special generalized Gauss graphs as an alternative theoretical framework for the minimization of curvature-dependent functionals. Concerning approximations for the single-phase Canham-Helfrich variational problem (4.1), a bulk diffuse-interface formulation is addressed in BELLETTINI & MUGNAI [13]. Moreover, the Canham-Helfrich energy is obtained as limit of micro- and mesoscopic energies by PELETIER & RÖGER [84], see also [70–72]. The multiphase Canham-Helfrich variational problem has also been studied numerically; for example, BARRETT, GARCKE, & NÜRNBERG [10, 11] present a finite element approach to the gradient flow formulation of the evolution problem associated to the two-phase energy (4.3).

Before closing this introduction, let us mention that we do not assume here that the genus of the membrane is prescribed. Indeed, allowing for equilibrium shapes with different topology is beneficial from a modeling perspective. Due to the phase-interface penalization in the energy (4.3), a union of several disconnected single-phase membranes might have the lowest energy in

certain parameter regimes. This situation is natural as it corresponds to vesicle budding or cell division. On the contrary, by prescribing g for closed single-phase membranes M , one would drop the Gauss curvature term in the Canham-Helfrich functional (4.1), because its integral would be a topological invariant by the Gauss-Bonnet Theorem: $\int_M K \, d\mathcal{H}^2 = 4\pi(1 - g)$. Yet, when dealing with heterogeneous membranes, the bending modulus γ will generally not be constant and its contribution plays an essential role in our model.

The paper is organized as follows. Some preliminary material including notation is gathered in Section 4.2. The varifold formulation of the Canham-Helfrich problem is detailed in Section 4.3, where we also discuss lower semicontinuity (Theorem 4.3.2). The existence of single-phase (Theorem 4.4.2) and multiphase minimizers (Theorem 4.4.9) is eventually proved in Section 4.4, which we conclude with some comments on regularity and diameter bounds for minimizers.

4.2 Notation and preliminaries

4.2.1 Notation for measures and norms

Let X be a locally compact and separable metric space, which in the applications of this paper can be identified with a nonempty Euclidean subset $X \subset \mathbb{R}^\alpha$ for some $\alpha \in \mathbb{N}$. The class of nonnegative (finite) Radon measures on X is denoted by $\mathcal{M}(X)$ and we write $\mathcal{M}^\beta(X)$ with $\beta \in \mathbb{N}$ for its $\mathbb{R}^\beta$ -valued counterpart. We write $\|\mu\| \in \mathcal{M}(X)$ for the total variation measure of $\mu \in \mathcal{M}^\beta(X)$. Depending on the context, $|\cdot|$ stands for the absolute value, the Euclidean norm, the comass norm (see Section 4.2.3), or the Frobenius norm. For example, if $A \in \mathbb{R}^{n \times n \times n}$ with $n \in \mathbb{N}$, then $|A|^2 = \sum_{i,j,k=1}^n A_{ijk}^2$. By $B_r(x) := \{y \in \mathbb{R}^n : |x - y| < r\}$ we denote the open ball of radius $r > 0$ and centered at $x \in \mathbb{R}^n$. If $M \subset \mathbb{R}^n$ then its diameter is given by $\text{diam}(M) := \sup_{x,y \in M} |x - y|$ and $\text{dist}(x, M) := \inf_{y \in M} |x - y|$ defines the distance of $x \in \mathbb{R}^n$ to M . The characteristic function of a set $U \subset X$ denoted $\chi_U: X \rightarrow \{0, 1\}$ is defined via $\chi_U(u) = 1$ if $u \in U$ and $\chi_U(u) = 0$ otherwise. If $\mu \in \mathcal{M}(X)$ then $L_\mu^p(X; \mathbb{R}^\beta)$ with $p \in [1, \infty)$ and $\beta \in \mathbb{N}$ is the space of equivalence classes of μ -measurable functions $f: X \rightarrow \mathbb{R}^\beta$ with finite norm $\|f\|_{L_\mu^p(X)} := \left(\int_X |f(x)|^p \, d\mu(x) \right)^{\frac{1}{p}}$ and we write $f \in L_{\text{loc}, \mu}^p(X; \mathbb{R}^\beta)$ whenever $\|f\|_{L_\mu^p(Q)} < \infty$ for all compact sets $Q \subset\subset X$.

If $\mu_h, \mu \in \mathcal{M}(X)$ and we let $h \rightarrow \infty$, then we say $\mu_h \rightharpoonup^* \mu$ in $\mathcal{M}(X)$ whenever $\mu_h(\varphi) = \int_X \varphi \, d\mu_h \rightarrow \int_X \varphi \, d\mu = \mu(\varphi)$ for all continuous functions with compact support, $\varphi \in C_c(X)$. The convergence of vector-valued Radon measures $\mu_h \rightharpoonup^* \mu$ in $\mathcal{M}^\beta(X)$ is defined analogously upon testing on $\varphi \in C_c(X; \mathbb{R}^\beta)$ with duality $\mu(\varphi) = \int_X \langle \varphi, d\mu \rangle$. Here $\langle \cdot, \cdot \rangle$ stands for the Euclidean inner product of vectors, but we use the same bracket to indicate the duality in the context of multivectors (again Section 4.2.3). For $\mu \in \mathcal{M}(X)$ and a μ -measurable map $f: X \rightarrow Y$ with another locally compact and separable metric space Y , the push-forward measure of μ by f is defined by $(f_\# \mu)(E) := \mu(f^{-1}(E))$ for all Borel sets $E \subset Y$. If f is continuous and proper, meaning $f^{-1}(Q)$ is compact for all compact $Q \subset\subset Y$, then $f_\# \mu \in \mathcal{M}(Y)$.

Further notation will be explained in the remainder of this section and in the main text.

4.2.2 Convergence as measure-function pairs

In this paragraph we review the notion of measure-function pair convergence and the corresponding lower semicontinuity of integral functionals. For details we refer to Hutchinson [54].

A *measure-function pair* (μ, B) over $X \subseteq \mathbb{R}^\alpha$ with values in $\mathbb{R}^\beta$ ($\alpha, \beta \in \mathbb{N}$) consists of a measure

$\mu \in \mathcal{M}(X)$ and a function $B \in L^1_{\text{loc},\mu}(X; \mathbb{R}^\beta)$. Convergence

$$(\mu_h, B_h) \rightharpoonup^* (\mu, B)$$

as measure-function pairs is defined by the two conditions

$$\mu_h \rightharpoonup^* \mu \quad \text{in } \mathcal{M}(X) \quad \text{and} \quad B_h \mu_h \rightharpoonup^* B \mu \quad \text{in } \mathcal{M}^\beta(X).$$

Theorem 4.2.1 (Integral functionals of measure-function pairs). *Let $X \subseteq \mathbb{R}^\alpha$ be nonempty and $f: X \times \mathbb{R}^\beta \rightarrow \mathbb{R}$ satisfy*

- (a) *f is continuous,*
- (b) *$f(y, \cdot)$ is convex for any $y \in X$, and*
- (c) *$f(y, B) \geq c|B|^2 - d$ holds for any $(y, B) \in X \times \mathbb{R}^\beta$ and for some $c, d > 0$.*

Let (μ_h, B_h) be a sequence of measure-function pairs over $X \subseteq \mathbb{R}^\alpha$ with values in $\mathbb{R}^\beta$ and let J be the functional

$$(\mu_h, B_h) \mapsto J(\mu_h, B_h) := \int_X f(y, B_h(y)) \, d\mu_h(y).$$

Then, with $h \rightarrow \infty$, the following assertions hold:

- (i) (Compactness) *If there exists $\mu \in \mathcal{M}(X)$ such that $\mu_h \rightharpoonup^* \mu$ in $\mathcal{M}(X)$ and if $J(\mu_h, B_h)$ is bounded, then there exists $B \in L^1_{\text{loc},\mu}(X; \mathbb{R}^\beta)$, such that up to a subsequence (not relabeled), $(\mu_h, B_h) \rightharpoonup^* (\mu, B)$ as measure-function pairs;*
- (ii) (Lower semicontinuity) *If $(\mu_h, B_h) \rightharpoonup^* (\mu, B)$ as measure-function pairs, then*

$$J(\mu, B) \leq \liminf_{h \rightarrow \infty} J(\mu_h, B_h).$$

4.2.3 Currents

We briefly recall some points of the theory of currents. For details see [94].

Let $\mathcal{D}^m(\mathbb{R}^n)$ be the set of all smooth m -forms with compact support in $\mathbb{R}^n$, where $m \in \mathbb{N}_0$ and $n \in \mathbb{N}$ with $m \leq n$. The (topological) dual space of $\mathcal{D}^m(\mathbb{R}^n)$ is the space of all m -currents on $\mathbb{R}^n$, denoted by $\mathcal{D}_m(\mathbb{R}^n)$. In particular, 0-currents coincide with the distributions on $\mathbb{R}^n$.

In accordance with the dual nature of $\mathcal{D}_m(\mathbb{R}^n)$, if $T_h, T \in \mathcal{D}_m(\mathbb{R}^n)$ we say that $T_h \rightharpoonup^* T$ if $T_h(\omega) \rightarrow T(\omega)$ for all $\omega \in \mathcal{D}^m(\mathbb{R}^n)$. If $m \geq 1$ then the boundary $\partial T \in \mathcal{D}_{m-1}(\mathbb{R}^n)$ of $T \in \mathcal{D}_m(\mathbb{R}^n)$ is defined via Stokes' theorem as $\partial T(\omega) := T(d\omega)$, where $d\omega$ denotes the exterior differential of the $(m-1)$ -form ω . The mass of $T \in \mathcal{D}_m(\mathbb{R}^n)$ is its dual norm

$$\mathbb{M}_T := \sup_{\omega \in \mathcal{D}^m(\mathbb{R}^n), \|\omega\|_{L^\infty(\mathbb{R}^n)} \leq 1} T(\omega),$$

where $\|\omega\|_{L^\infty(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n} |\omega(x)|$ with $|\omega(x)| = \sum_{1 \leq i_1 < \dots < i_m \leq n} \langle \omega(x), e_{i_1} \wedge \dots \wedge e_{i_m} \rangle$ for basis vectors e_i of $\mathbb{R}^n$, the comass norm of the m -covector $\omega(x)$. The mass $\mathbb{M}_T(\Omega)$ of $T \in \mathcal{D}_m(\Omega)$ with $\Omega \subset \mathbb{R}^n$ open is defined analogously upon restricting to test forms with $\text{supp}(\omega) \subset \subset \Omega$.

In order to define rectifiable currents we recall the notion of a rectifiable set. Let M be a Borel subset of $\mathbb{R}^n$ and denote the m -dimensional Hausdorff measure in $\mathbb{R}^n$ by $\mathcal{H}^m$. We say that M is

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(countably) m -rectifiable, if $M = M_0 \cup \bigcup_{k=1}^{\infty} M_k$ where $\mathcal{H}^m(M_0) = 0$ and M_k is contained in the image of a Lipschitz function $f_k: \mathbb{R}^m \rightarrow \mathbb{R}^n$ for all $k \in \mathbb{N}$. It turns out that for $\mathcal{H}^m$ -a.e. $x \in M$ there exists an *approximate tangent space*, which we denote by $T_x M$ as in the smooth setting.

We say that $T \in \mathcal{D}_m(\mathbb{R}^n)$ is an m -rectifiable current with integer multiplicity if there exists

- (a) an m -rectifiable set M in $\mathbb{R}^n$,
- (b) an *orientation* of M , that is, a Borel map ξ defined on M that to $\mathcal{H}^m$ -a.e. $x \in M$ assigns a unit simple m -vector $\xi(x)$ which spans $T_x M$, and
- (c) an integer-valued *multiplicity*, that is, a locally $(\mathcal{H}^m \llcorner M)$ -summable function $j: M \rightarrow \mathbb{Z}$, where $\mathcal{H}^m \llcorner M$ denotes the $\mathcal{H}^m$ measure restricted to M ,

such that

$$T(\omega) = \int_M \langle \omega(x), \xi(x) \rangle j(x) \, d\mathcal{H}^m(x) \quad \text{for all } \omega \in \mathcal{D}^m(\mathbb{R}^n).$$

In this case we write $T = [M, \xi, j]$. Finally we say that $T \in \mathcal{D}_m(\mathbb{R}^n)$ is an *integral m -current* if both T and ∂T are rectifiable currents with integer multiplicity.

The theory of currents features the following compactness and lower semicontinuity results, see again [94]:

Theorem 4.2.2 (Federer-Fleming). *Let (T_h) be a sequence of integral m -currents on $\mathbb{R}^n$ such that $\mathbb{M}_{T_h} + \mathbb{M}_{\partial T_h}$ is bounded and let $h \rightarrow \infty$. Then, up to a subsequence (not relabeled), $T_h \rightharpoonup^* T$ in $\mathcal{D}_m(\mathbb{R}^n)$ and $\partial T_h \rightharpoonup^* \partial T$ in $\mathcal{D}_{m-1}(\mathbb{R}^n)$, where the limit T is an integral m -current. Moreover we have $\mathbb{M}_T \leq \liminf_{h \rightarrow \infty} \mathbb{M}_{T_h}$ and $\mathbb{M}_{\partial T} \leq \liminf_{h \rightarrow \infty} \mathbb{M}_{\partial T_h}$.*

The main insight of the Federer-Fleming Theorem 4.2.2 is not sequential compactness, namely existence of convergent subsequences for integral currents with bounded mass, but the fact that the limit currents are again integral (closedness). This remark applies to the forthcoming Theorems 4.2.4 and 4.2.6 as well.

4.2.4 Varifolds

This paragraph is dedicated to a review of the theory of curvature varifolds, which are the weak version of surfaces that we need. For details we refer to [1, 54, 94].

Let $\Omega \subset \mathbb{R}^n$ be open and nonempty. Let $m, n \in \mathbb{N}$ with $1 \leq m \leq n$. An m -varifold in Ω is a Radon measure

$$V \in \mathcal{M}(\Omega \times G_{m,n}).$$

Here $G_{m,n}$ denotes the *Grassmannian*, which consists of all m -dimensional linear subspaces of $\mathbb{R}^n$. An element $P \in G_{m,n}$ is identified with its associated orthogonal projection matrix $P \in \mathbb{R}^{n \times n}$. This allows us to view $\Omega \times G_{m,n}$ as a subset of some Euclidean space.

The push-forward measure of V under the canonical projection map $\pi: \Omega \times G_{m,n} \rightarrow \Omega$, $\pi(x, P) = x$ defines the *mass measure* (or *weight*) of the varifold,

$$\mu_V := \pi_{\#} V \in \mathcal{M}(\Omega).$$

The *mass* of V then reads

$$\mu_V(\Omega) = \int_{\Omega} d\mu_V = \int_{\Omega \times G_{m,n}} dV = V(\Omega \times G_{m,n}).$$

As Radon measures, varifolds are determined through their action on continuous functions with compact support in $\Omega \times G_{m,n}$. Based on this duality one can introduce the following useful subclass: An *integral m -varifold* in Ω is given by

$$V(\varphi) = \int_M \varphi(x, T_x M) \theta(x) d\mathcal{H}^m(x) \quad \text{for all } \varphi \in C_c(\Omega \times G_{m,n}),$$

where $M \subset \Omega$ is (countably) m -rectifiable and $\theta: M \rightarrow \mathbb{N}$, called *multiplicity*, is locally summable w.r.t. $\mathcal{H}^m \llcorner M$. In this case we write $V = v(M, \theta)$ and we denote the space of all integral m -varifolds in Ω by $IV_m(\Omega)$. Its elements can be seen as the geometric measure theory analog to the m -dimensional differentiable submanifolds in Ω .

By a clever application of the divergence theorem for manifolds, Hutchinson [54], and later Mantegazza [73], generalized the concept of the second fundamental form to the varifold setting. For our convenience we will follow Mantegazza's approach since it also gives a good definition of the boundary, see [73] for details. We say that $V \in IV_m(\Omega)$ is a *curvature varifold with boundary* and write $V \in AV_m(\Omega)$, if there exists

$$A^V \in L^1_{\text{loc}, V}(\Omega \times G_{m,n}; \mathbb{R}^{n \times n \times n}) \quad \text{and} \quad \partial V \in \mathcal{M}^n(\Omega \times G_{m,n})$$

such that

$$\int_{\Omega \times G_{m,n}} (P_{ij} \partial_j \varphi + (\partial_{P_{jk}} \varphi) A^V_{ijk} + A^V_{jij} \varphi)(x, P) dV(x, P) = - \int_{\Omega \times G_{m,n}} \varphi(x, P) d(\partial V)_i(x, P)$$

for all $\varphi \in C^1_c(\Omega \times G_{m,n})$ and $1 \leq i \leq n$. Here and in the following we employ summation convention, that is, we take the sum over all indices that occur twice. The curvature functions A^V give rise to the *generalized second fundamental form* $\Pi^V \in L^1_{\text{loc}, V}(\Omega \times G_{m,n}; \mathbb{R}^{n \times n \times n})$ via

$$(\Pi^V(\cdot, P))_{ij}^k := A^V_{ikl}(\cdot, P) P_{lj}$$

and ∂V is the *boundary measure* of the varifold. The associated *generalized mean curvature vector* $\bar{H}^V \in L^1_{\text{loc}, V}(\Omega \times G_{m,n}; \mathbb{R}^n)$ has the components

$$\bar{H}^V_i := A^V_{jij}.$$

Allard's *first variation* [1, 94] of a varifold V is the linear functional $\delta V: C^1_c(\Omega; \mathbb{R}^n) \rightarrow \mathbb{R}$,

$$\delta V(X) = \int_{\Omega \times G_{m,n}} \text{div}^P X(x) dV(x, P)$$

with the tangential divergence

$$\text{div}^P X := P_{ij} \partial_j X_i.$$

Insertion of $\varphi(x, P) = X_i(x)$ with $X \in C^1_c(\Omega; \mathbb{R}^n)$ in Mantegazza's definition and summation over $1 \leq i \leq n$ reveals that $V \in AV_m(\Omega)$ satisfies the *first variation formula* [73]

$$\delta V = \langle -\pi_{\#}(\bar{H}^V V + \partial V), \cdot \rangle, \quad (4.4)$$

that is,

$$\delta V(X) = - \int_{\Omega \times G_{m,n}} \bar{H}^V_i(x, P) X_i(x) dV(x, P) - \int_{\Omega \times G_{m,n}} X_i(x) d(\partial V)_i(x, P).$$

Formula (4.4) shows that a curvature varifold $V \in AV_m(\Omega)$ has locally bounded first variation. In particular, $\|\delta V\| \in \mathcal{M}(\Omega)$ with the total variation measure given by

$$\|\delta V\|(\Omega') := \sup_{X \in C^1_c(\Omega; \mathbb{R}^n), \text{supp}(X) \subset \subset \Omega', \|X\|_{L^\infty(\Omega)} \leq 1} \delta V(X)$$

for open subsets $\Omega' \subset \Omega$. Next we prove some preliminary estimates that we will need.

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Lemma 4.2.3 (First variation bounds). *Let $V \in AV_m(\Omega)$. Then we have*

$$|\bar{H}^V|^2 \leq n|A^V|^2. \quad (4.5)$$

Moreover, if in addition $A^V \in L_V^2(\Omega \times G_{m,n}; \mathbb{R}^{n \times n \times n})$, then

$$\|\delta V\|(\Omega) \leq \sqrt{n \mu_V(\Omega)} \|A^V\|_{L_V^2(\Omega \times G_{m,n})} + \|\partial V\|(\Omega \times G_{m,n}). \quad (4.6)$$

Proof. By Formula (4.4)

$$\begin{aligned} \|\delta V\|(\Omega) &= \| -\pi_{\sharp} (\bar{H}^V V + \partial V) \|(\Omega) \leq \|\bar{H}^V V + \partial V\|(\Omega \times G_{m,n}) \\ &\leq \|\bar{H}^V V\|(\Omega \times G_{m,n}) + \|\partial V\|(\Omega \times G_{m,n}) \end{aligned}$$

where we notice that

$$\|\bar{H}^V V\|(\Omega \times G_{m,n}) = \int_{\Omega \times G_{m,n}} |\bar{H}^V| dV = \|\bar{H}^V\|_{L_V^1(\Omega \times G_{m,n})}.$$

The estimate (4.5) is easily obtained:

$$|\bar{H}^V|^2 = \sum_{i=1}^n \left(\bar{H}_i^V \right)^2 = \sum_{i=1}^n \left(\sum_{j=1}^n A_{jij}^V \right)^2 \leq \sum_{i=1}^n \left(n \sum_{j=1}^n (A_{jij}^V)^2 \right) \leq n \sum_{i,j,k=1}^n (A_{ijk}^V)^2 = n|A^V|^2.$$

Consequently, we have

$$\|\bar{H}^V\|_{L_V^1(\Omega \times G_{m,n})} = \int_{\Omega \times G_{m,n}} |\bar{H}^V| dV \leq \sqrt{n} \int_{\Omega \times G_{m,n}} |A^V| dV = \sqrt{n} \|A^V\|_{L_V^1(\Omega \times G_{m,n})},$$

showing that $A^V \in L_V^1(\Omega \times G_{m,n}; \mathbb{R}^{n \times n \times n})$ implies $\bar{H}^V \in L_V^1(\Omega \times G_{m,n}; \mathbb{R}^n)$. If A^V is square-integrable then Hölder's inequality gives

$$\begin{aligned} \|A^V\|_{L_V^1(\Omega \times G_{m,n})} &= \int_{\Omega \times G_{m,n}} |A^V| dV = \int_{\Omega} |A^V| d\mu_V \\ &\leq \left(\int_{\Omega} d\mu_V \right)^{1/2} \left(\int_{\Omega} |A^V|^2 d\mu_V \right)^{1/2} = (\mu_V(\Omega))^{1/2} \|A^V\|_{L_V^2(\Omega \times G_{m,n})}, \end{aligned}$$

and thus

$$\|\bar{H}^V\|_{L_V^1(\Omega \times G_{m,n})} \leq \sqrt{n \mu_V(\Omega)} \|A^V\|_{L_V^2(\Omega \times G_{m,n})}.$$

With $\|\delta V\|(\Omega) \leq \|\bar{H}^V\|_{L_V^1(\Omega \times G_{m,n})} + \|\partial V\|(\Omega \times G_{m,n})$ from above this yields (4.6). $\square$

One of the main results by Mantegazza [73] is the following compactness (closedness) theorem.

Theorem 4.2.4 (Compactness for curvature varifolds). *Let $p > 1$ and let (V_h) be a sequence in $AV_m(\Omega)$. Assume that*

$$\mu_{V_h}(\Omega) + \|A^{V_h}\|_{L_{V_h}^p(\Omega \times G_{m,n})}^p + \|\partial V_h\|(\Omega \times G_{m,n}) \leq c$$

for some $c > 0$. Then there exists $V \in AV_m(\Omega)$, such that up to a subsequence (not relabeled), $V_h \rightharpoonup^* V$ in $AV_m(\Omega)$. Moreover, $\partial V_h \rightharpoonup^* \partial V$ in $\mathcal{M}^n(\Omega \times G_{m,n})$.

Here, *curvature varifold convergence*

$$V_h \rightharpoonup^* V \quad \text{in} \quad \mathcal{M}(\Omega)$$

is defined as the measure-function pairs convergence $(V_h, A^{V_h}) \rightharpoonup^* (V, A^V)$ over $\Omega \times G_{m,n}$ with values in $\mathbb{R}^{n \times n \times n}$, cf. Section 4.2.2.

Varifolds including orientation were introduced by Hutchinson [54]. An *oriented m -varifold* in Ω is a Radon measure

$$V \in \mathcal{M}(\Omega \times G_{m,n}^o),$$

with the *oriented Grassmannian* $G_{m,n}^o$ given by the set of all oriented m -dimensional linear subspaces of $\mathbb{R}^n$. The elements $\xi \in G_{m,n}^o$ can be represented by m -vectors in $\mathbb{R}^n$. The two-fold covering map

$$q: G_{m,n}^o \rightarrow G_{m,n}, \quad q(\pm\xi) = P$$

allows us to define the *unoriented counterpart* of the varifold V as the push-forward measure of V by $\text{id}_{\mathbb{R}^n} \otimes q: \Omega \times G_{m,n}^o \rightarrow \Omega \times G_{m,n}$, which is a continuous and proper map. We shorten the notation and write

$$q_{\#}V := (\text{id}_{\mathbb{R}^n} \otimes q)_{\#}V \in \mathcal{M}(\Omega \times G_{m,n}).$$

The masses of V and $q_{\#}V$ coincide:

$$\mu_{q_{\#}V}(\Omega) = \int_{\Omega \times G_{m,n}^o} d(q_{\#}V) = \int_{\Omega \times G_{m,n}} dV = \mu_V(\Omega).$$

Lemma 4.2.5 (Convergence of oriented and unoriented varifolds). *Let $h \rightarrow \infty$. If $V_h \rightharpoonup^* V$ in $\mathcal{M}(\Omega \times G_{m,n}^o)$ then $q_{\#}V_h \rightharpoonup^* q_{\#}V$ in $\mathcal{M}(\Omega \times G_{m,n})$.*

Proof. Let $\varphi \in C_c(\Omega \times G_{m,n})$ and consider $\psi \in C_c(\Omega \times G_{m,n}^o)$ given by

$$\psi(x, \xi) := \varphi(x, q(\xi)) = \varphi(x, q(-\xi)) = \psi(x, -\xi).$$

Then, with $P = q(\pm\xi)$,

$$\begin{aligned} (q_{\#}V_h)(\varphi) &= \int_{\Omega \times G_{m,n}} \varphi(x, P) d(q_{\#}V_h)(x, P) = \frac{1}{2} \int_{\Omega \times G_{m,n}^o} \varphi(x, q(\xi)) dV_h(x, \xi) = \frac{1}{2} V_h(\psi) \\ &\rightarrow \frac{1}{2} V(\psi) = \frac{1}{2} \int_{\Omega \times G_{m,n}^o} \varphi(x, q(\xi)) dV(x, \xi) = \int_{\Omega \times G_{m,n}} \varphi(x, P) d(q_{\#}V)(x, P) = (q_{\#}V)(\varphi), \end{aligned}$$

where the factor $\frac{1}{2}$ comes from the symmetry of ψ . □

The classical divergence theorem for closed hypersurfaces of $\mathbb{R}^n$ motivates us to assign an *enclosed volume* to an oriented $(n-1)$ -varifold $V \in \mathcal{M}(\Omega \times G_{n-1,n}^o)$ by

$$\text{encvol}(V) := \frac{1}{n} \int_{\Omega \times G_{n-1,n}^o} x \cdot (*\xi) dV(x, \xi).$$

Here, $*$ stands for the Hodge operator from exterior calculus, which is a bijection between m - and $(n-m)$ -(co)-vectors in $\mathbb{R}^n$, $0 \leq m \leq n$. In particular, in the relevant case $n = 3$ and $m = 2$, we have $*(a \wedge b) = a \times b$ for $a, b \in \mathbb{R}^3$. Linearity of encvol directly implies that

$$V_h \rightharpoonup^* V \text{ in } \mathcal{M}(\Omega \times G_{n-1,n}^o), \text{ supp}(V_h) \text{ compact} \implies \text{encvol}(V_h) \rightarrow \text{encvol}(V). \quad (4.7)$$

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Similarly to an integral varifold, an *oriented integral m -varifold* in Ω is defined by

$$V(\psi) := \int_M (\psi(x, \xi(x))\theta^+(x) + \psi(x, -\xi(x))\theta^-(x)) \, d\mathcal{H}^m(x) \quad \text{for all } \psi \in C_c(\Omega \times G_{m,n}^o),$$

where $M \subset \Omega$ is (countably) m -rectifiable, $\theta^\pm: M \rightarrow \mathbb{N}_0$, called *multiplicities*, are locally summable w.r.t. $\mathcal{H}^m \llcorner M$, and ξ is the orientation as for a rectifiable current (cf. Section 4.2.3). We will use the notation $V = v^o(M, \xi, \theta^+, \theta^-)$ to indicate such a varifold and we denote by $IV_m^o(\Omega)$ the space of all oriented integral m -varifolds in Ω . The unoriented counterpart of V is given by $q_\# V = v(M, \theta^+ + \theta^-) \in IV_m(\Omega)$. Moreover, one can associate to V the integer multiplicity m -rectifiable current $T_V := [M, \xi, \theta^+ - \theta^-]$. As $\theta^\pm$ take values in the nonnegative integers, the current mass is bounded by the mass of V :

$$\mathbb{M}_{T_V}(\Omega) = \int_M |\theta^+ - \theta^-| \, d\mathcal{H}^m \leq \int_M (\theta^+ + \theta^-) \, d\mathcal{H}^m = \mu_{q_\# V}(\Omega) = \mu_V(\Omega). \quad (4.8)$$

In [54], Hutchinson presented the following compactness (closedness) result.

Theorem 4.2.6 (Compactness for oriented varifolds). *Let (V_h) be a sequence in $IV_m^o(\Omega)$ with*

$$\mu_{V_h}(\Omega) + \|\delta(q_\# V_h)\|(\Omega) + \mathbb{M}_{\partial T_{V_h}}(\Omega) \leq c$$

for some $c > 0$. Then there exists $V \in IV_m^o(\Omega)$, such that up to a subsequence (not relabeled), $V_h \rightharpoonup^* V$ in $\mathcal{M}(\Omega \times G_{m,n}^o)$.

We eventually introduce the class of *oriented curvature m -varifolds*,

$$AV_m^o(\Omega) := \{V \in IV_m^o(\Omega) : q_\# V \in AV_m(\Omega)\}. \quad (4.9)$$

Given $V_h, V \in AV_m^o(\Omega)$ and $h \rightarrow \infty$, we say that

$$V_h \rightharpoonup^* V \text{ in } AV_m^o(\Omega)$$

if $V_h \rightharpoonup^* V$ in $\mathcal{M}(\Omega \times G_{m,n}^o)$ and $q_\# V_h \rightharpoonup^* q_\# V$ in $AV_m(\Omega)$.

If $M \subset \Omega$ is a smooth hypersurface of $\mathbb{R}^n$ with unit normal $\nu: M \rightarrow \mathbb{S}^{n-1}$ we associate to it the canonical oriented curvature $(n-1)$ -varifold

$$V_M := v^o(M, \xi, 1, 0) \in AV_{n-1}^o(\Omega), \quad \nu = *\xi. \quad (4.10)$$

Using smooth differential geometry (see for instance Hutchinson [54]) it can be verified that

$$(A^{q_\# V_M})_{ijk} = A_{ijk} := P_{il} \partial_l P_{jk} \quad \text{and} \quad \bar{H}^{q_\# V_M} = \bar{H} := H\nu,$$

where $P = \mathbb{I} - \nu \otimes \nu$ is the tangential projection matrix and H is the scalar mean curvature of M . In particular, in case of smooth surfaces M in $\mathbb{R}^3$ with κ_1, κ_2 denoting their principal curvatures, we have $H = \kappa_1 + \kappa_2$ and $K = \kappa_1 \kappa_2$ defines the Gauss curvature of M . Since $\frac{1}{2}(\kappa_1 + \kappa_2)^2 - \frac{1}{2}(\kappa_1^2 + \kappa_2^2) = \kappa_1 \kappa_2$, we have

$$K = \frac{1}{2}|\bar{H}|^2 - \frac{1}{2}|\bar{H}|^2 = \frac{1}{2}|\bar{H}|^2 - \frac{1}{4}|A|^2.$$

For the second equality we employed $|A|^2 = 2|\bar{H}|^2$, which is a consequence of $A_{ijk} = \bar{H}_{ik}^j + \bar{H}_{ij}^k$ and the mapping properties of the second fundamental form $\bar{H} = -(\nabla^P \nu) \otimes \nu$, i.e. in components, $\bar{H}_{ij}^k = -(\partial_j^P \nu_i) \nu_k$ with the tangential partial derivative $\partial_j^P = P_{jk} \partial_k$.

4.3 The Canham-Helfrich functional for oriented curvature varifolds

We introduce a formulation of the Canham-Helfrich energy in the framework of oriented curvature varifolds, see (4.9). Let M be a compact smooth surface embedded in an open subset $\Omega \subset \mathbb{R}^3$ and assume for the time being that $V \in AV_2^o(\Omega)$ is the associated varifold (4.10). Then the quantities

$$\bar{H}_i = H \nu_i = \sum_{j=1}^3 A_{jij} \quad (i = 1, 2, 3), \quad K = \frac{1}{2}|\bar{H}|^2 - \frac{1}{4}|A|^2, \quad \text{and} \quad \nu = *\xi$$

are defined for V and one can rewrite the integrand of the functional (4.1) as

$$\begin{aligned} \frac{\beta}{2}(H - H_0)^2 + \gamma K &= \frac{\beta}{2}|\bar{H} - (*\xi)H_0|^2 + \frac{\gamma}{2}|\bar{H}|^2 - \frac{\gamma}{4}|A|^2 \\ &= \sum_{i=1}^3 \left(\frac{\beta}{2}(\bar{H}_i - (*\xi)_i H_0)^2 + \frac{\gamma}{2}\bar{H}_i^2 - \frac{\gamma}{4} \sum_{j,k=1}^3 A_{ijk}^2 \right). \end{aligned}$$

The latter suggests to define the *generalized Canham-Helfrich functional* as

$$F_{\text{CH}}: AV_2^o(\Omega) \rightarrow \mathbb{R}, \quad F_{\text{CH}}(V) := \int_{\Omega \times G_{2,3}^o} f_{\text{CH}}(\xi, A^{q\sharp V}(x, q(\xi))) \, dV(x, \xi), \quad (4.11)$$

where the density $f_{\text{CH}}: G_{2,3}^o \times \mathbb{R}^{3 \times 3 \times 3} \rightarrow \mathbb{R}$ is given by

$$f_{\text{CH}}(\xi, A) := \sum_{i=1}^3 \left(\frac{\beta}{2} \left(\left(\sum_{j=1}^3 A_{jij} \right) - (*\xi)_i H_0 \right)^2 + \frac{\gamma}{2} \left(\sum_{j=1}^3 A_{jij} \right)^2 - \frac{\gamma}{4} \sum_{j,k=1}^3 A_{ijk}^2 \right).$$

Recall from Section 4.2.4 that the elements of $AV_2^o(\Omega)$ are oriented integral 2-varifolds

$$V = v^o(M, \xi, \theta^+, \theta^-) \in IV_2^o(\Omega)$$

in the sense of Hutchinson [54], whose unoriented counterparts

$$q_{\sharp}V = v(M, \theta^+ + \theta^-) \in AV_2(\Omega)$$

are curvature 2-varifolds in the sense of Mantegazza [73]. Let

$$P: M \rightarrow G_{2,3} \quad \text{and} \quad \pm \xi: M \rightarrow G_{2,3}^o$$

denote the mappings that assign the approximate tangent space $P(x) := T_x M$ and its oriented counterparts $\pm \xi(x)$ respectively to $(\mathcal{H}^2 \llcorner M)$ -a.e. $x \in M$. Then P and ξ are related by the covering map $q: G_{2,3}^o \rightarrow G_{2,3}$, namely $P(x) = q(\xi(x)) = q(-\xi(x))$. It follows that the generalized Canham-Helfrich functional (4.11) takes the form

$$\begin{aligned} F_{\text{CH}}(V) &= \int_M \left(f_{\text{CH}}(\xi(x), A^{q\sharp V}(x, P(x))) \theta^+(x) + f_{\text{CH}}(-\xi(x), A^{q\sharp V}(x, P(x))) \theta^-(x) \right) d\mathcal{H}^2(x) \\ &= \sum_{i=1}^3 \int_M \left(\frac{\beta}{2} \left(\bar{H}_i^{q\sharp V}(\cdot, P) - (*\xi)_i H_0 \right)^2 \theta^+ + \frac{\beta}{2} \left(\bar{H}_i^{q\sharp V}(\cdot, P) + (*\xi)_i H_0 \right)^2 \theta^- \right. \\ &\quad \left. + \left(\frac{\gamma}{2} (\bar{H}_i^{q\sharp V}(\cdot, P))^2 - \frac{\gamma}{4} \sum_{j,k=1}^3 (A_{ijk}^{q\sharp V}(\cdot, P))^2 \right) (\theta^+ + \theta^-) \right) d\mathcal{H}^2, \quad (4.12) \end{aligned}$$

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where for $\mathcal{H}^2$ -a.e. $x \in M$ and $i = 1, 2, 3$,

$$\bar{H}_i^{q_i V}(x, P(x)) = \sum_{j=1}^3 A_{jij}^{q_i V}(x, P(x)).$$

We want to investigate now coercivity and lower semicontinuity of F_{CH} with respect to the convergence in $AV_2^o(\Omega)$. Let us point out that the lower semicontinuity of functionals of Canham-Helfrich type is a delicate issue, for counterexamples in specific situations are known [49, Rem. (ii), p. 550].

Firstly, we first establish conditions on the material parameters β and γ that guarantee strict convexity of the energy density in the curvature variables.

Proposition 4.3.1 (Strict convexity of f_{CH}). *If β and γ satisfy the relation*

$$-\frac{6}{5}\beta < \gamma < 0, \quad (4.13)$$

then $f_{\text{CH}}(\xi, \cdot): \mathbb{R}^{3 \times 3 \times 3} \rightarrow \mathbb{R}$ is strictly convex for all $\xi \in G_{2,3}^o$. Moreover, there exist $c_1 > 0$ and $c_2 \geq 0$, with $c_2 = 0$ only if $H_0 = 0$, such that

$$\|A^{q_i V}\|_{L_{q_i V}^2(\Omega \times G_{2,3})}^2 \leq c_1 (F_{\text{CH}}(V) + c_2 \mu_V(\Omega)) \quad \text{for all } V \in AV_2^o(\Omega). \quad (4.14)$$

Proof. We split f_{CH} in the sum of quadratic and linear terms in A . By using the fact that $\sum_{i,j,k=1}^3 A_{ijk}^2 = \sum_{i,j,k=1}^3 A_{jik}^2$ we write

$$\begin{aligned} f_{\text{CH}}(\xi, A) &= \sum_{i=1}^3 \left(\frac{\beta}{2} \left(\left(\sum_{j=1}^3 A_{jij} \right) - (*\xi)_i H_0 \right)^2 + \frac{\gamma}{2} \left(\sum_{j=1}^3 A_{jij} \right)^2 - \frac{\gamma}{4} \sum_{j,k=1}^3 A_{ijk}^2 \right) \\ &= \sum_{i=1}^3 \underbrace{\left(\frac{\beta+\gamma}{2} \left(\sum_{j=1}^3 A_{jij} \right)^2 - \frac{\gamma}{4} \sum_{j,k=1}^3 A_{jik}^2 \right)}_{=: f(a^i)} + \sum_{i=1}^3 \underbrace{\left(-\beta \left(\sum_{j=1}^3 A_{jij} \right) (*\xi)_i H_0 + \frac{\beta}{2} (*\xi)_i^2 H_0^2 \right)}_{=: l_i(\xi, a^i)} \\ &= \sum_{i=1}^3 (f(a^i) + l_i(\xi, a^i)). \end{aligned}$$

For the last equality we introduced

$$a^i := (a_{11}^i, a_{22}^i, a_{33}^i, a_{12}^i, a_{13}^i, a_{21}^i, a_{23}^i, a_{31}^i, a_{32}^i) \in \mathbb{R}^9$$

with $a_{jk}^i := A_{jik}$. Let us now check that $f: \mathbb{R}^9 \rightarrow \mathbb{R}$ is strictly convex. Indeed, one has that

$$\begin{aligned} f(a) &= \frac{\beta+\gamma}{2} \left(\sum_{j=1}^3 a_{jj} \right)^2 - \frac{\gamma}{4} \sum_{j,k=1}^3 a_{jk}^2 \\ &= \frac{\beta+\gamma}{2} (a_{11}^2 + a_{22}^2 + a_{33}^2 + 2a_{11}a_{22} + 2a_{11}a_{33} + 2a_{22}a_{33}) \\ &\quad - \frac{\gamma}{4} (a_{11}^2 + a_{22}^2 + a_{33}^2 + a_{12}^2 + a_{13}^2 + a_{23}^2 + a_{21}^2 + a_{31}^2 + a_{32}^2) \\ &= \frac{2\beta+\gamma}{4} (a_{11}^2 + a_{22}^2 + a_{33}^2) + (\beta+\gamma)(a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33}) \\ &\quad - \frac{\gamma}{4} (a_{12}^2 + a_{13}^2 + a_{23}^2 + a_{21}^2 + a_{31}^2 + a_{32}^2). \end{aligned}$$

By computing partial derivatives one obtains

$$\begin{aligned} \frac{\partial f}{\partial a_{11}} &= \frac{2\beta + \gamma}{2} a_{11} + (\beta + \gamma)(a_{22} + a_{33}), & \frac{\partial f}{\partial a_{12}} &= -\frac{\gamma}{2} a_{12}, \\ \frac{\partial^2 f}{\partial a_{11}^2} &= \frac{2\beta + \gamma}{2}, & \frac{\partial^2 f}{\partial a_{11} \partial a_{22}} &= \beta + \gamma, & \frac{\partial^2 f}{\partial a_{12}^2} &= -\frac{\gamma}{2} \end{aligned}$$

and analogously for the other components. The Hessian $D^2 f(a) \in \mathbb{R}^{9 \times 9}$ then reads

$$D^2 f(a) = \begin{pmatrix} \frac{2\beta + \gamma}{2} & \beta + \gamma & \beta + \gamma & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta + \gamma & \frac{2\beta + \gamma}{2} & \beta + \gamma & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta + \gamma & \beta + \gamma & \frac{2\beta + \gamma}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\gamma}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{\gamma}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\gamma}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\gamma}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\gamma}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\gamma}{2} \end{pmatrix}.$$

Its eigenvalues are

$$\lambda_1 = \frac{1}{2}(6\beta + 5\gamma) \quad \text{and} \quad \lambda_2 = \lambda_3 = \dots = \lambda_9 = -\frac{\gamma}{2}.$$

Thus the Hessian is positive definite ($\lambda_k > 0$ for $1 \leq k \leq 9$) if and only if $6\beta + 5\gamma > 0$ and $\gamma < 0$, which is equivalent to $-6\beta < 5\gamma < 0$, namely (4.13). The integrand $f_{\text{CH}}(\xi, \cdot)$ is hence a strictly convex second-order polynomial for all $\xi \in G_{2,3}^o$. Moreover, ξ only occurs explicitly in the term $\sum_{i=1}^3 l_i(\xi, a^i)$, which is smooth, being a second-order polynomial in the components of ξ and vanishes if $H_0 = 0$. Consequently, there exist a constant $c_1 > 0$ and continuous maps $c_2^\pm: G_{2,3}^o \rightarrow [0, \infty)$, such that

$$f_{\text{CH}}(\xi, A) \geq \frac{1}{c_1} |A|^2 - c_2^+(\xi), \quad f_{\text{CH}}(-\xi, A) \geq \frac{1}{c_1} |A|^2 - c_2^-(\xi).$$

Since $\theta^\pm(x) \geq 0$ for $\mathcal{H}^2$ -a.e. $x \in M$, from (4.12) we then get

$$\begin{aligned} f_{\text{CH}}(\xi, A)\theta^+ + f_{\text{CH}}(-\xi, A)\theta^- &\geq \left(\frac{1}{c_1} |A|^2 - c_2^+(\xi) \right) \theta^+ + \left(\frac{1}{c_1} |A|^2 - c_2^-(\xi) \right) \theta^- \\ &= \frac{1}{c_1} |A|^2 (\theta^+ + \theta^-) - (c_2^+(\xi)\theta^+ + c_2^-(\xi)\theta^-) \\ &\geq \frac{1}{c_1} |A|^2 (\theta^+ + \theta^-) - c_2 (\theta^+ + \theta^-) = \left(\frac{1}{c_1} |A|^2 - c_2 \right) (\theta^+ + \theta^-) \end{aligned}$$

where $c_2 := \max_{\xi \in G_{2,3}^o} \max\{c_2^+(\xi), c_2^-(\xi)\} \geq 0$. Therefore, we conclude that

$$\begin{aligned} F_{\text{CH}}(V) &\geq \int_M \left(\frac{1}{c_1} |A^{q^\# V}(x, q(\xi(x)))|^2 - c_2 \right) (\theta^+ + \theta^-)(x) \, d\mathcal{H}^2(x) \\ &= \int_{\Omega \times G_{2,3}} \left(\frac{1}{c_1} |A^{q^\# V}(x, P)|^2 - c_2 \right) d(q^\# V)(x, P) \\ &= \frac{1}{c_1} \|A^{q^\# V}\|_{L_{q^\# V}^2(\Omega \times G_{2,3})}^2 - c_2 \mu_V(\Omega) \end{aligned}$$

and the estimate (4.14) is proved. $\square$

4 Existence of varifold minimizers for the multiphase Canham-Helfrich functional

Secondly, we state the main result of this section:

Theorem 4.3.2 (Lower semicontinuity). *Assume (4.13). Then F_{CH} is lower semicontinuous with respect to the convergence in $AV_2^o(\Omega)$.*

Proof. Let $V_h \rightharpoonup^* V$ in $AV_2^o(\Omega)$ and assume with no loss of generality that

$$\sup_h F_{\text{CH}}(V_h) < \infty.$$

We have that $V_h \rightharpoonup^* V$ converges as oriented varifolds in $\mathcal{M}(\Omega \times G_{2,3}^o)$ and $(q_\# V_h, A^{q_\# V_h}) \rightharpoonup^* (q_\# V, A^{q_\# V})$ converges as measure-function pairs over $\Omega \times G_{2,3}$. We observe that

$$F_{\text{CH}}(V) = \int_{\Omega \times G_{2,3}^o} f_{\text{CH}}(\xi, A^{q_\# V}(x, q(\xi))) dV(x, \xi) = \int_{\Omega \times G_{2,3}^o} f_{\text{CH}}(\xi, B(x, \xi)) dV(x, \xi)$$

where the function $B: \Omega \times G_{2,3}^o \rightarrow \mathbb{R}^{3 \times 3 \times 3}$ is given by

$$B(x, \xi) := A^{q_\# V}(x, q(\xi)).$$

We now pass to the limit with respect to the measure-function pair (V, B) over $\Omega \times G_{2,3}^o$ with values in $\mathbb{R}^{3 \times 3 \times 3}$. The symmetries

$$B(x, \xi) = A^{q_\# V}(x, q(\xi)) = A^{q_\# V}(x, q(-\xi)) = B(x, -\xi)$$

and

$$B_h(x, \xi) := A^{q_\# V_h}(x, q(\xi)) = A^{q_\# V_h}(x, q(-\xi)) = B_h(x, -\xi)$$

allow us to infer that

$$(V_h, B_h) \rightharpoonup^* (V, B)$$

in the sense of measure-function pair convergence over $\Omega \times G_{2,3}^o$. The integrand f_{CH} is continuous on $G_{2,3}^o \times \mathbb{R}^{3 \times 3 \times 3}$ and a strictly convex quadratic function in the second variable by Proposition 4.3.1. In particular, there exist $c_1, c_2 > 0$ such that for all $\xi \in G_{2,3}^o$

$$f_{\text{CH}}(\xi, B) \geq \frac{1}{c_1} |B|^2 - c_2.$$

Therefore, Theorem 4.2.1 (ii) applies and we obtain

$$\begin{aligned} F_{\text{CH}}(V) &= \int_{\Omega \times G_{2,3}^o} f_{\text{CH}}(\xi, B(x, \xi)) dV(x, \xi) \\ &\leq \liminf_{h \rightarrow \infty} \int_{\Omega \times G_{2,3}^o} f_{\text{CH}}(\xi, B_h(x, \xi)) dV_h(x, \xi) \\ &= \liminf_{h \rightarrow \infty} F_{\text{CH}}(V_h), \end{aligned}$$

which concludes the proof. $\square$

4.4 Existence of minimizers

In order to establish the existence of minimizers of F_{CH} among varifolds with fixed total mass, we assume that the spatial support of the admissible varifolds is contained in a fixed compact set. Additionally, to guarantee that weak limits are still orientable, we require that the mass of the boundary current of the admissible varifolds is bounded by a fixed constant.

Lemma 4.4.1 (Closedness of admissible varifolds). *Let $Q \subset\subset \Omega$ have nonempty interior and let $m^o > 0$. Then the class*

$$\mathcal{A} := \{V \in AV_2^o(\Omega) : \text{supp } \mu_V \subset Q, \mathbb{M}_{\partial T_V}(\Omega) \leq m^o\} \quad (4.15)$$

is closed in $AV_2^o(\Omega)$.

Proof. Consider $V_h \in \mathcal{A}$ with $V_h \rightharpoonup^* V$ in $AV_2^o(\Omega)$ (4.9) as $h \rightarrow \infty$. With the convergence of the mass measures, $\mu_{V_h} \rightharpoonup^* \mu_V$ in $\mathcal{M}(\Omega)$, standard results from measure theory then give

$$V_h \rightharpoonup^* V \text{ in } AV_2^o(\Omega), \quad \text{supp } \mu_{V_h} \subset Q \implies \text{supp } \mu_V \subset Q.$$

Moreover, the 2-currents T_{V_h} and T_V corresponding to V_h and V respectively are 2-rectifiable with integer multiplicity. We have

$$V_h \rightharpoonup^* V \text{ in } AV_2^o(\Omega) \implies T_{V_h} \rightharpoonup^* T_V \text{ in } \mathcal{D}_2(\Omega) \implies \partial T_{V_h} \rightharpoonup^* \partial T_V \text{ in } \mathcal{D}_1(\Omega).$$

The masses $\mathbb{M}_{T_{V_h}}(\Omega)$ are bounded by virtue of estimate (4.8). With $\mathbb{M}_{\partial T_{V_h}}(\Omega) \leq m^o$, the lower semicontinuity of the mass of currents (Theorem 4.2.2) then yields

$$\mathbb{M}_{\partial T_V}(\Omega) \leq \liminf_{h \rightarrow \infty} \mathbb{M}_{\partial T_{V_h}}(\Omega) \leq m^o,$$

completing the proof. □

4.4.1 Single-phase membranes

In this paragraph we present an existence result of varifold-minimizers for the single-phase Canham-Helfrich energy (4.11), including the constraints of fixed surface area and enclosed volume,

$$m > 0 \quad \text{and} \quad e > 0. \quad (4.16)$$

In order not to rule out embedded smooth solutions, we further assume that m and e satisfy the isoperimetric inequality in $\mathbb{R}^3$,

$$(6\sqrt{\pi}e)^{1/3} \leq m^{1/2}.$$

Equality among smooth closed embedded surfaces is achieved by a sphere.

Theorem 4.4.2 (Existence). *Assume (4.13) and (4.16). Then there exists a solution of*

$$\min\{F_{\text{CH}}(V) : V \in \mathcal{A}, \mu_V(\Omega) = m, \partial(q_{\#}V) = 0, \text{encvol}(V) = e\}.$$

Proof. As varifolds $V \in \mathcal{A}$ with $\mu_V(\Omega) = m$, $\partial(q_{\#}V) = 0$, and $\text{encvol}(V) = e$ exist (think of ellipsoids), one can find a minimizing sequence (V_h) for F_{CH} , i.e.,

$$\lim_{h \rightarrow \infty} F_{\text{CH}}(V_h) = \inf\{F_{\text{CH}}(V) : V \in \mathcal{A}, \mu_V(\Omega) = m, \partial(q_{\#}V) = 0, \text{encvol}(V) = e\},$$

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where $V_h \in \mathcal{A}$ and satisfies $\mu_{V_h}(\Omega) = m$, $\partial(q_\# V_h) = 0$, $\text{encvol}(V_h) = e$. By Proposition 4.3.1,

$$\|A^{q_\# V_h}\|_{L^2_{q_\# V_h}(\Omega \times G_{2,3})}^2 \leq c' (F_{\text{CH}}(V_h) + \mu_{V_h}(\Omega)),$$

for some $c' > 0$, which with (4.6) yields

$$\begin{aligned} \|\delta(q_\# V_h)\|(\Omega) &\leq \sqrt{3\mu_{V_h}(\Omega)} \|A^{q_\# V_h}\|_{L^2_{q_\# V_h}(\Omega \times G_{2,3})} + \underbrace{\|\partial(q_\# V_h)\|(\Omega \times G_{2,3})}_{=0} \\ &\leq c\sqrt{\mu_{V_h}(\Omega)} \sqrt{F_{\text{CH}}(V_h) + \mu_{V_h}(\Omega)} \end{aligned} \quad (4.17)$$

for some $c > 0$. From $\mu_{V_h}(\Omega) = m$ and the minimality of V_h we thus obtain

$$\|\delta(q_\# V_h)\|(\Omega) \leq c\sqrt{m} \sqrt{F_{\text{CH}}(V_h) + m} < \infty \quad \text{uniformly in } h.$$

By the compactness Theorem 4.2.6,

$$\mu_{V_h}(\Omega) + \|\delta(q_\# V_h)\|(\Omega) + \mathbb{M}_{\partial T_{V_h}}(\Omega) < \infty \implies \exists V_{h'} \rightharpoonup^* V \text{ in } \mathcal{M}(\Omega \times G_{2,3}^o), \quad V \in IV_2^o(\Omega).$$

Consequently, by Lemma 4.2.5, the subsequence $(V_{h'})$ also satisfies

$$q_\# V_{h'} \rightharpoonup^* q_\# V \text{ in } \mathcal{M}(\Omega \times G_{2,3}), \quad q_\# V \in IV_2(\Omega).$$

By the compactness Theorem 4.2.4,

$$\mu_{q_\# V_{h'}}(\Omega) + \|A^{q_\# V_{h'}}\|_{L^2_{q_\# V_{h'}}(\Omega \times G_{2,3})}^2 + \underbrace{\|\partial(q_\# V_{h'})\|(\Omega \times G_{2,3})}_{=0} < \infty \implies q_\# V_{h'} \rightharpoonup^* \tilde{V} \text{ in } AV_2(\Omega).$$

Since convergence in $AV_2(\Omega)$ and $IV_2^o(\Omega)$ imply convergence as Radon measures $\mathcal{M}(\Omega \times G_{2,3})$, uniqueness of limits yields that $\tilde{V} = q_\# V$. Consequently, the minimizing sequence V_h has a subsequence which converges in $\mathcal{A}$, i.e.

$$V_{h'} \rightharpoonup^* V \text{ in } AV_2^o(\Omega).$$

The limit V also satisfies the constraints on admissibility, mass, zero boundary, and enclosed volume. Indeed, $V \in \mathcal{A}$ follows from closedness of $\mathcal{A}$ established in Lemma 4.4.1. The compact support of admissible varifolds allows us to infer $\mu_V(\Omega) = m$ via results from measure theory. The condition on the boundary measure $\partial(q_\# V) = 0$ follows from Mantegazza's Theorem 4.2.4, according to which $\partial(q_\# V_{h'}) \rightharpoonup^* \partial(q_\# V)$ in $\mathcal{M}^3(\Omega \times G_{2,3})$. Finally, $\text{encvol}(V) = e$ holds by virtue of (4.7). Moreover, Theorem 4.3.2 shows that F_{CH} is lower semicontinuous on $AV_2^o(\Omega)$. Therefore, by the direct method, the limit $V \in \mathcal{A}$ is a minimizer. $\square$

4.4.2 Sharp-interface multiphase membranes

We introduce the two-phase energy

$$F_{\text{CH},2}(V^1, V^2) := F_{\text{CH}}^1(V^1) + F_{\text{CH}}^2(V^2) + \sigma^1 \|\partial(q_\# V^1)\|(\Omega \times G_{2,3}) + \sigma^2 \|\partial(q_\# V^2)\|(\Omega \times G_{2,3})$$

for $V^1, V^2 \in \mathcal{A}$ and some $\sigma^1, \sigma^2 > 0$. Here, F_{CH}^1 and F_{CH}^2 are defined as in (4.12) with parameters β^1, γ^1 and β^2, γ^2 , respectively. The two varifolds V^1 and V^2 correspond to the individual phases.

In order to model sharp interfaces, the spatial supports of V^1 and V^2 are required not to overlap on $\mathcal{H}^2$ -nonnegligible sets. According to definition (4.15), admissible varifolds $V_i \in \mathcal{A}$ ($i = 1, 2$) are integral, whence their weight measures are of the form

$$\mu_i := \mu_{V^i} = \theta_i \mathcal{H}^2 \llcorner M_i \in \mathcal{M}(\Omega)$$

with $M_i \subset \Omega$ countably 2-rectifiable, $\mathcal{H}^2$ -measurable and with positive, integer-valued multiplicity $\theta_i \in L^1_{\text{loc}; \mathcal{H}^2 \llcorner M_i}(M_i; \mathbb{N})$. Overlap is quantified in terms of the scaling of the product-mass $\mu_1 \otimes \mu_2 \in \mathcal{M}(\Omega \times \Omega)$ evaluated on the open ε -neighborhood

$$D_{\Omega, \varepsilon} := \{(x, y) \in \Omega \times \Omega : |x - y| < \varepsilon\}$$

of the diagonal in $\Omega \times \Omega$: We say that the varifolds $V_1, V_2 \in \mathcal{A}$ satisfy the *no-overlap condition* with $\varepsilon_0 > 0$, if

$$(\mu_1 \otimes \mu_2)(D_{\Omega, \varepsilon}) \leq \varepsilon^3 / \varepsilon_0 \quad \text{for all } \varepsilon \in (0, \varepsilon_0). \quad (4.18)$$

The next lemma shows that this condition rules out $\mathcal{H}^2$ -overlap of M_1 and M_2 , still allowing $\mathcal{H}^1$ - or $\mathcal{H}^0$ -overlap.

Lemma 4.4.3 (No-Overlap). *Let $\mu_i = \theta_i \mathcal{H}^2 \llcorner M_i \in \mathcal{M}(\Omega)$ be as above and have fixed mass $m_i = \mu_i(\Omega) > 0$ for $i = 1, 2$. Then, by letting $S_\varepsilon := \{x \in M_1 : \text{dist}(x, M_2) < \varepsilon\}$ we have that*

$$(\mu_1 \otimes \mu_2)(D_{\Omega, \varepsilon}) = \int_{S_\varepsilon} \mu_2(B_\varepsilon(x)) \, d\mu_1(x) \quad (4.19)$$

and the following statements hold:

- (i) *If there exists $c > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, $\mu_2(B_\varepsilon(x)) \geq c\varepsilon^2$ for all $x \in M_2$ and $\mathcal{H}^2(M_1 \cap M_2) > 0$, then condition (4.18) is not satisfied.*
- (ii) *If there exists $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, $\mu_2(B_\varepsilon(x)) \leq C\varepsilon^2$ for all $x \in S_\varepsilon$, $\mu_1(S_\varepsilon) \leq \varepsilon/(C\varepsilon_0)$, and $0 < \mathcal{H}^m(M_1 \cap M_2) < \infty$ with $m \in [0, 1]$, then (4.18) is fulfilled.*

Analogous statements hold if μ_1 and μ_2 are interchanged.

See Figures 4.1, 4.2, and 4.3 for some illustration of the statement above and its proof.

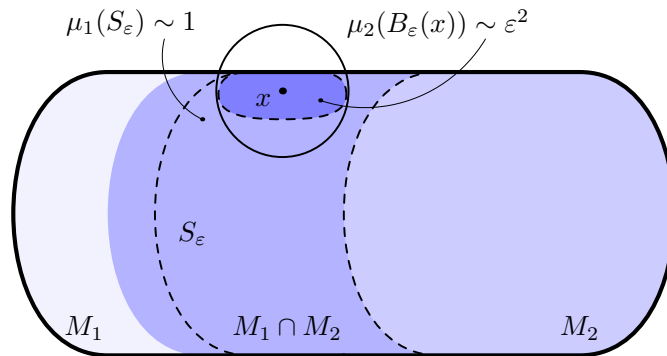
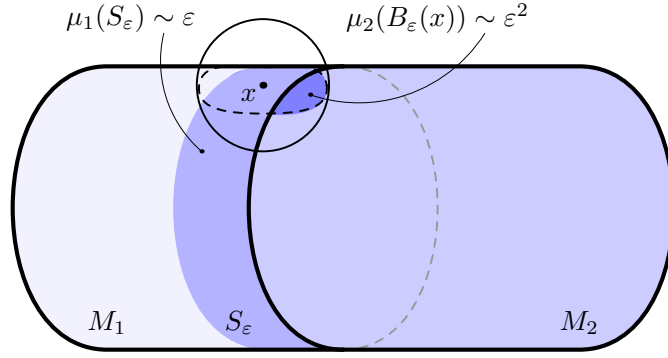
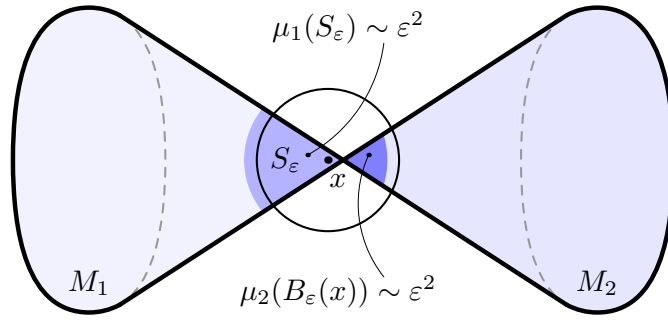


Figure 4.1: $\mathcal{H}^2$ -overlap, Lemma 4.4.3(i).


 Figure 4.2: $\mathcal{H}^1$ -overlap, Lemma 4.4.3(ii) with $m = 1$.

 Figure 4.3: $\mathcal{H}^0$ -overlap, Lemma 4.4.3(ii) with $m = 0$.

Proof. Let $\varepsilon > 0$. The fixed total masses imply that $\mu_i/m_i \in \mathcal{M}(\Omega)$ and $(\mu_1 \otimes \mu_2)/(m_1 m_2) \in \mathcal{M}(\Omega \times \Omega)$ are probability measures. By disintegration for probability measures (see [4, Thm. 2.28, p. 57] or [5, Thm. 5.3.1, p. 121]) and $|x - y| < \varepsilon$ being equivalent to $y \in B_\varepsilon(x)$, we obtain

$$(\mu_1 \otimes \mu_2)(D_{\Omega, \varepsilon}) = \int_{D_{\Omega, \varepsilon}} d(\mu_1 \otimes \mu_2) = \int_{\Omega} \mu_2(B_\varepsilon(x)) d\mu_1(x) = \int_{M_1} \mu_2(B_\varepsilon(x)) d\mu_1(x).$$

It holds that

$$\mu_2(B_\varepsilon(x)) = 0 \quad \text{for all } x \in \Omega, \text{ dist}(x, M_2) \geq \varepsilon.$$

With $S_\varepsilon = \{x \in M_1 : \text{dist}(x, M_2) < \varepsilon\}$, this implies that $\mu_2(B_\varepsilon(x)) = 0$ for all $x \in M_1 \setminus S_\varepsilon$, which shows (4.19).

To prove statement (i), assume that $\mathcal{H}^2(M_1 \cap M_2) > 0$. With $S_\varepsilon \supseteq M_1 \cap M_2$ and the lower bound $\mu_2(B_\varepsilon(x)) \geq c\varepsilon^2$ for $x \in M_2$ we arrive at the estimate

$$\begin{aligned} (\mu_1 \otimes \mu_2)(D_{\Omega, \varepsilon}) &= \int_{S_\varepsilon} \mu_2(B_\varepsilon(x)) d\mu_1(x) \geq \int_{M_1 \cap M_2} \mu_2(B_\varepsilon(x)) d\mu_1(x) \\ &\geq c\varepsilon^2 \int_{M_1 \cap M_2} \underbrace{\theta_1(x)}_{\geq 1} d\mathcal{H}^2(x) \geq c\varepsilon^2 \mathcal{H}^2(M_1 \cap M_2). \end{aligned}$$

This gives a contradiction to (4.18) for any $\varepsilon_0 > 0$, as the choice

$$0 < \varepsilon < \min\{1, \varepsilon_0, \varepsilon_0 c \mathcal{H}^2(M_1 \cap M_2)\}$$

yields $c \mathcal{H}^2(M_1 \cap M_2) > \varepsilon/\varepsilon_0$ and thus $(\mu_1 \otimes \mu_2)(D_{\Omega, \varepsilon}) \geq c\varepsilon^2 \mathcal{H}^2(M_1 \cap M_2) > \varepsilon^3/\varepsilon_0$.

Under the assumptions in (ii), namely $\mu_2(B_\varepsilon(x)) \leq C\varepsilon^2$ for $x \in S_\varepsilon$ and $\mu_1(S_\varepsilon) \leq \varepsilon/(C\varepsilon_0)$, we obtain

$$(\mu_1 \otimes \mu_2)(D_{\Omega,\varepsilon}) = \int_{S_\varepsilon} \mu_2(B_\varepsilon(x)) \, d\mu_1(x) \leq C\varepsilon^2 \mu_1(S_\varepsilon) \leq C\varepsilon^2 \varepsilon/(C\varepsilon_0) = \varepsilon^3/\varepsilon_0$$

so that (4.18) holds. $\square$

The no-overlap condition (4.18) is closed with respect to oriented curvature varifold convergence, which is crucial for including it as a constraint in the direct method:

Lemma 4.4.4 (Closedness of the no-overlap condition). *Let $V_h^i \rightharpoonup^* V^i$ in $AV_2^o(\Omega)$ for $i = 1, 2$. If all V_h^1 and V_h^2 satisfy (4.18) with $\varepsilon_0 > 0$, then this is also true for the limits V^1 and V^2 .*

Proof. The convergence $V_h^i \rightharpoonup^* V^i$ in $AV_2^o(\Omega)$ implies $\mu_{V_h^i} \rightharpoonup^* \mu_{V^i}$ in $\mathcal{M}(\Omega)$. The product of Radon measures in $\mathcal{M}(\Omega)$ commutes with weak-* limits [18, Thm. 8.4.10, p. 198], namely,

$$\mu_{V_h^1} \rightharpoonup^* \mu_{V^1}, \quad \mu_{V_h^2} \rightharpoonup^* \mu_{V^2} \quad \text{in } \mathcal{M}(\Omega) \implies \mu_{V_h^1} \otimes \mu_{V_h^2} \rightharpoonup^* \mu_{V^1} \otimes \mu_{V^2} \quad \text{in } \mathcal{M}(\Omega \times \Omega).$$

Let $\varepsilon \in (0, \varepsilon_0)$. By lower semicontinuity of measures on the open set $D_{\Omega,\varepsilon}$ and by (4.18) for V_h^1 and V_h^2 , we obtain

$$(\mu_{V^1} \otimes \mu_{V^2})(D_{\Omega,\varepsilon}) \leq \liminf_{h \rightarrow \infty} (\mu_{V_h^1} \otimes \mu_{V_h^2})(D_{\Omega,\varepsilon}) \leq \varepsilon^3/\varepsilon_0,$$

which proves the claim. $\square$

Note that the no-overlap condition (4.18) implies mutual singularity of varifolds. We have the following.

Lemma 4.4.5 (Mutual singularity). *If $V^1, V^2 \in AV_2(\Omega)$ satisfy (4.18) and, for $i = 1$ or 2 , there exists $c > 0$ with $\mu_i(B_\varepsilon(x)) \geq c\varepsilon^2$ for all $x \in M_i$, $\varepsilon \in (0, \varepsilon_0)$, then $V^1 \perp V^2$ in $\mathcal{M}(\Omega \times G_{2,3})$.*

Proof. By Lemma 4.4.3 (i), $\mu_i = \mu_{V^i} = \theta_i \mathcal{H}^2 \llcorner M_i$ ($i = 1, 2$) satisfy $\mathcal{H}^2(M_1 \cap M_2) = 0$. Therefore, as $\theta_1 \in L_{\text{loc}; \mathcal{H}^2 \llcorner M_1}^1(M_1; \mathbb{N})$ and $M_2 \cap (\Omega \setminus M_2) = \emptyset$, we have

$$\mu_1(M_2) = \int_{M_1 \cap M_2} \theta_1 \, d\mathcal{H}^2 = 0 \quad \text{and} \quad \mu_2(\Omega \setminus M_2) = \int_{M_2 \cap (\Omega \setminus M_2)} \theta_2 \, d\mathcal{H}^2 = 0.$$

Thus μ_1 and μ_2 are separated by the Borel set M_2 (and also M_1 by an analogous argument). This shows that $\mu_1 \perp \mu_2$ in $\mathcal{M}(\Omega)$, which yields $V^1 \perp V^2$ in $\mathcal{M}(\Omega \times G_{2,3})$. $\square$

For modeling closed two-phase membranes, the individual phases are required to have matching boundaries, or equivalently, the boundary of the sum (union) of V^1 and V^2 must be empty. The sum of mutually singular varifolds features curvature functions that can be expressed in terms of characteristic functions of individual supports; the boundary measure of the sum is simply the sum of the individual boundary measures. Indeed, we have the following.

Lemma 4.4.6 (Sum of mutually singular curvature varifolds). *If $V^1, V^2 \in AV_2(\Omega)$ satisfy $V^1 \perp V^2$, then $V^1 + V^2 \in AV_2(\Omega)$ with the curvature functions*

$$A^{V^1+V^2} := A^{V^1} \chi_{\text{supp}(V^1)} + A^{V^2} \chi_{\text{supp}(V^2)} \tag{4.20}$$

and the boundary measure

$$\partial(V^1 + V^2) := \partial V^1 + \partial V^2. \tag{4.21}$$

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Proof. We show that the sum $V^1 + V^2$ satisfies Mantegazza's integration by parts formula: Let $\varphi \in C_c^1(\Omega \times G_{2,3})$. Then $A^{V^1+V^2} = A^{V^1} \chi_{\text{supp}(V^1)} + A^{V^2} \chi_{\text{supp}(V^2)}$ yields

$$\begin{aligned} & \int_{\Omega \times G_{2,3}} \left(P_{ij} \partial_j \varphi + (\partial_{P_{jk}} \varphi) A_{ijk}^{V^1+V^2} + \varphi A_{jij}^{V^1+V^2} \right) d(V^1 + V^2) \\ &= \int_{\Omega \times G_{2,3}} \left(P_{ij} \partial_j \varphi + (\partial_{P_{jk}} \varphi) A_{ijk}^{V^1} + \varphi A_{jij}^{V^1} \right) dV^1 \\ & \quad + \int_{\Omega \times G_{2,3}} \left(P_{ij} \partial_j \varphi + (\partial_{P_{jk}} \varphi) A_{ijk}^{V^2} + \varphi A_{jij}^{V^2} \right) dV^2 \\ &= - \int_{\Omega \times G_{2,3}} \varphi d(\partial V^1)_i - \int_{\Omega \times G_{2,3}} \varphi d(\partial V^2)_i \\ &= - \int_{\Omega \times G_{2,3}} \varphi d(\partial V^1 + \partial V^2)_i = - \int_{\Omega \times G_{2,3}} \varphi d(\partial(V^1 + V^2))_i, \end{aligned}$$

where the last equality follows from $\partial V^1 + \partial V^2 = \partial(V^1 + V^2)$. $\square$

Remark 4.4.7 (Sum of general curvature varifolds). For $V^1, V^2 \in AV_m(\Omega)$ with $\Omega \subset \mathbb{R}^n$ open we have (4.21) and $A^{V^1+V^2} = f^1 A^{V^1} + f^2 A^{V^2}$ where $f^i = dV^i / d(V^1 + V^2) \in L_{\text{loc}, V^1+V^2}^1(\Omega \times G_{m,n})$ for $i = 1, 2$. The density (Radon-Nikodym derivative) f^i always exists because of the absolute continuity $V^i \ll V^1 + V^2$, as for all $U \subseteq \Omega \times G_{m,n}$ open, $(V^1 + V^2)(U) = 0$ implies $V^i(U) = 0$, for $i = 1, 2$. In case of mutual singularity, we recover the specific case of (4.20), for the densities reduce to the characteristic functions $f^i = \chi_{\text{supp}(V^i)}$.

We note that requiring a zero boundary measure of the sum $V^1 + V^2$ rules out kinks at phase-boundaries. Indeed, consider two compact C^2 surfaces M_1 and M_2 in Ω with associated curvature varifolds

$$V^i := v(M_i, 1) \in AV_2(\Omega)$$

for $i = 1, 2$. Their Mantegazza and Allard boundary measures are given by

$$\partial V^i = \delta_{\partial M_i} \otimes \bar{\nu}_i \delta_{q(*\bar{\nu}_i)} \quad \text{and} \quad \pi_{\sharp}(\partial V^i) = \bar{\nu}_i \delta_{\partial M_i}$$

respectively. Here $\bar{\nu}_i := \nu^{\partial M_i}$ denotes the inwards-pointing unit normal of ∂M_i and $q(*\bar{\nu}_i) \in G_{2,3}$ is the corresponding unoriented tangent plane. Assume that the sets only overlap at their common boundary

$$\partial M_1 = \partial M_2 = M_1 \cap M_2 \neq \emptyset$$

and satisfy the no-overlap condition (4.18) (thanks to Lemma 4.4.3 this is the case if M_1 and M_2 have minimal distance ε_0 elsewhere; cf. Figure 4.2). Then their sum reads

$$V^1 + V^2 = v(M_1 \cup M_2, 1)$$

and with (4.21) satisfies $\partial(V^1 + V^2) = \partial V^1 + \partial V^2 = \delta_{M_1 \cap M_2} \otimes (\bar{\nu}_1 \delta_{q(*\bar{\nu}_1)} + \bar{\nu}_2 \delta_{q(*\bar{\nu}_2)})$ and $\pi_{\sharp}(\partial(V^1 + V^2)) = (\bar{\nu}_1 + \bar{\nu}_2) \delta_{M_1 \cap M_2}$. Consequently, if

$$\partial(V^1 + V^2) = 0,$$

then $\bar{\nu}_1 = -\bar{\nu}_2$. Thus the unit normals must be antiparallel, showing that $M_1 \cup M_2$ is C^1 and no kinks occur.

We are in the position to present the main results of this section.

Theorem 4.4.8 (Existence, two-phase case). *Fix $m_1, m_2, \varepsilon_0, e > 0$ and let the bounds (4.13) hold true both for β^1, γ^1 and β^2, γ^2 . Then there exists a pair $V^1, V^2 \in \mathcal{A}$ solving*

$$\min \left\{ F_{\text{CH},2}(V^1, V^2) : V^1, V^2 \in \mathcal{A}, \quad \mu_{V^1}(\Omega) = m_1, \quad \mu_{V^2}(\Omega) = m_2, \quad \partial(q_{\#}V^1 + q_{\#}V^2) = 0 \right. \\ \left. (\mu_{V^1} \otimes \mu_{V^2})(D_{\Omega,\varepsilon}) \leq \varepsilon^3/\varepsilon_0 \text{ for all } \varepsilon \in (0, \varepsilon_0), \quad \text{encvol}(V^1 + V^2) = e \right\}.$$

Proof. We essentially follow the argument for the single-phase case (Theorem 4.4.2), by letting $V = V^i$ for $i = 1, 2$. In order to reproduce estimate (4.17), we use now

$$\|\partial(q_{\#}V^i)\|(\Omega \times G_{2,3}) \leq c F_{\text{CH},2}(V^1, V^2).$$

Thus the minimizing $V_h^i \in \mathcal{A}$ ($i = 1, 2$) enjoy the first variation bounds

$$\|\delta(q_{\#}V_h^i)\|(\Omega) \leq \sqrt{3 \mu_{V_h^i}(\Omega)} \|A^{q_{\#}V_h^i}\|_{L^2_{q_{\#}V_h^i}(\Omega \times G_{2,3})} + \|\partial(q_{\#}V_h^i)\|(\Omega \times G_{2,3}) \\ \leq c' \left(\sqrt{\mu_{V_h^i}(\Omega)} \sqrt{F_{\text{CH},2}(V^1, V^2) + \mu_{V_h^i}(\Omega) + F_{\text{CH},2}(V^1, V^2)} \right) < \infty.$$

Lemma 4.4.3 gives closedness for the no-overlap condition. We can calculate the boundary measure of the sum of curvature varifolds via formula (4.21) and by Theorem 4.2.4 we obtain

$$\partial(q_{\#}V_h^1 + q_{\#}V_h^2) = \partial(q_{\#}V_h^1) + \partial(q_{\#}V_h^2) \rightharpoonup^* \partial(q_{\#}V^1) + \partial(q_{\#}V^2) = \partial(q_{\#}V^1 + q_{\#}V^2).$$

This shows conservation of the zero boundary measure, because if $\partial(q_{\#}V_h^1 + q_{\#}V_h^2) = 0$, then also $\partial(q_{\#}V^1 + q_{\#}V^2) = 0$. We note that the enclosed volume is defined for the sum $V^1 + V^2$ of oriented varifolds $\mathcal{M}(\Omega \times G_{2,3}^o)$. Therefore, the volume condition is stable under varifold limits also in the multiphase case. $\square$

The existence result directly extends to sharp-interface N -phase membranes with energy

$$F_{\text{CH},N}(V^1, \dots, V^N) := \sum_{i=1}^N (F_{\text{CH}}^i(V^i) + \sigma^i \|\partial(q_{\#}V^i)\|)$$

for $V^i \in \mathcal{A}$ and $\sigma^i > 0$ ($i = 1, \dots, N$).

Theorem 4.4.9 (Existence, multiphase case). *Let $m_i > 0$ for $i = 1, \dots, N$ and $\varepsilon_0, e > 0$ be fixed. If $-6\beta^i < 5\gamma^i < 0$, for $i = 1, \dots, N$, then there exists a solution $(V^i)_{i=1}^N$ of*

$$\min \left\{ F_{\text{CH},N}(V^1, \dots, V^N) : V^i \in \mathcal{A}, \quad \mu_{V^i}(\Omega) = m_i \quad (i = 1, \dots, N), \quad \partial(\sum_{i=1}^N q_{\#}V^i) = 0, \right. \\ \left. \sum_{i,j=1, i \neq j}^N (\mu_{V^i} \otimes \mu_{V^j})(D_{\Omega,\varepsilon}) \leq \varepsilon^3/\varepsilon_0 \text{ for all } \varepsilon \in (0, \varepsilon_0), \quad \text{encvol}(\sum_{i=1}^N V^i) = e \right\}.$$

As for the single-phase case, the area and enclosed-volume bounds m_i and e are required to respect the isoperimetric inequality

$$(6\sqrt{\pi}e)^{1/3} \leq \left(\sum_{i=1}^N m_i \right)^{1/2}$$

to include smooth multiphase minimizers.

4.4.3 Regularity and diameter bounds

We collect here some remarks on all minimizers V for the single-phase problem (Theorem 4.4.2) and on all individual phases V^i of minimizers (or their sum) for the sharp-interface multiphase problem (Theorem 4.4.9).

Lemma 4.4.10 (C^2 rectifiability). *Let $V \in AV_2^o(\Omega)$. Then, there exists a countable collection $(\Sigma_i)_{i=1}^\infty$ of C^2 surfaces in $\mathbb{R}^3$ such that*

$$\mu_V(\Omega \setminus \bigcup_{i=1}^\infty \Sigma_i) = 0.$$

Moreover, for each Σ_i , the second fundamental forms of V and Σ_i μ_V -a.e. agree on $\Omega \cap \Sigma_i$.

Proof. It suffices to observe $q_\# V \in AV_2(\Omega)$ and apply [77, Theorem 1]. $\square$

In order to proceed further, we remark that all minimizers have finite Willmore energy (4.2), which in the context of varifolds (and up to normalization) is given by

$$\mathcal{W}(W) := \int_{\Omega} |\bar{H}^W|^2 d\mu_W = \int_{\Omega \times G_{2,3}} |\bar{H}^W(x, P)|^2 dW(x, P).$$

Lemma 4.4.11 (Bound on the Willmore energy). *Assume (4.13). Then, for all $V \in AV_2^o(\Omega)$ we have*

$$\mathcal{W}(V) := \mathcal{W}(q_\# V) \leq 3c_1 (F_{\text{CH}}(V) + c_2 \mu_V(\Omega)),$$

where c_1 and c_2 are the constants from Proposition 4.3.1. In particular, if $\mu_V(\Omega) < \infty$ and $F_{\text{CH}}(V) < \infty$, one deduces $\mathcal{W}(V) < \infty$.

Proof. Letting $W := q_\# V \in AV_2(\Omega)$, the algebraic estimate $|\bar{H}^W|^2 \leq 3|A^W|^2$, see (4.5), gives

$$\mathcal{W}(W) \leq 3 \int_{\Omega \times G_{2,3}} |A^W|^2 dW = 3 \|A^W\|_{L_W^2(\Omega \times G_{2,3})}^2.$$

Relation (4.14) ensures that $\|A^{q_\# V}\|_{L_{q_\# V}^2(\Omega \times G_{2,3})}^2 \leq c_1 (F_{\text{CH}}(V) + c_2 \mu_V(\Omega))$, so that the assertion follows. $\square$

Estimates on the Willmore energy are related to embeddedness. Indeed, in case of a smooth immersion $\psi: \Sigma \rightarrow \mathbb{R}^3$ of a compact surface, if there exists $p \in \mathbb{R}^3$ such that $\psi^{-1}(p) = \{x_1, \dots, x_k\}$ (distinct points), then one has the Li-Yau inequality [68, Theorem 6]

$$\widetilde{\mathcal{W}}(\Sigma) \geq 4\pi k, \tag{4.22}$$

where $\widetilde{\mathcal{W}}(\Sigma) = \frac{1}{4}\mathcal{W}(\Sigma)$ is the Willmore energy (4.2) normalized to $\widetilde{\mathcal{W}}(\mathbb{S}^2) = 4\pi$. The consequences are as follows (see also [62]):

- (i) $\widetilde{\mathcal{W}}(\Sigma) \geq 4\pi$ for embedded surfaces Σ ; equality holds iff $\Sigma = \mathbb{S}^2$.
- (ii) $\widetilde{\mathcal{W}}(\Sigma) \geq 8\pi$ for surfaces Σ with self-intersections.
- (iii) If $\widetilde{\mathcal{W}}(\Sigma) < 8\pi$ then Σ must be embedded.

The next two lemmas exclude that the Cahn-Helfrich varifold-minimizers collapse to a single point or expand to infinity by estimating their diameter. The statements are a varifold analog of the diameter bounds given in [95] for smooth connected compact hypersurfaces Σ in $\mathbb{R}^n$: If $\partial\Sigma = \emptyset$, then [95, Lemma 1.1] shows

$$\sqrt{\frac{|\Sigma|}{\mathcal{W}(\Sigma)}} \leq \text{diam}(\Sigma) \leq C\sqrt{|\Sigma| \mathcal{W}(\Sigma)}.$$

For surfaces in $\mathbb{R}^3$ it has been proven in [98, Lemma 1] that $C = 2/\pi$. Moreover, if Σ is a smooth connected compact hypersurface in $\mathbb{R}^n$ whose boundary $\partial\Sigma \neq \emptyset$ has finitely many connected components Γ_i , then one has the upper bound [95, Lemma 1.2]

$$\text{diam}(\Sigma) \leq C \left(\int_{\Sigma} (\kappa_1^2 + \kappa_2^2)^{1/2} + \sum_i \text{diam}(\Gamma_i) \right).$$

Lemma 4.4.12 (Lower diameter bound). *Assume (4.13) and let $V \in AV_2^o(\Omega)$ satisfy $\mu_V(\Omega) = m > 0$, $\|\partial(q_{\sharp}V)\|(\Omega \times G_{2,3}) < \infty$, and $F_{\text{CH}}(V) < \infty$. Then, $\text{diam}(\text{supp } \mu_V) > 0$.*

Proof. We argue along the lines of [103, Theorem 3.1.1] but generalize the proof to varifolds with nonzero Mantegazza boundary measure. With no loss of generality, let $0 \in \text{supp } \mu_V$ and consider $X \in C_c^1(\Omega; \mathbb{R}^3)$ with

$$X(x) = x \quad \text{on} \quad \text{supp } \mu_V \quad \text{and} \quad \|X\|_{L^\infty(\Omega)} = \sup_{x \in \Omega} |X(x)| \leq c_\infty \text{diam}(\text{supp } \mu_V)$$

for some $c_\infty > 1$. Then, for all (projections on) planes $P \in G_{2,3}$ and $x \in \text{supp } \mu_V$,

$$\text{div}^P X(x) = P_{ij} \partial_j X_i(x) = 2.$$

Consequently, Mantegazza's first variation formula (4.4) gives (here we write V instead of $q_{\sharp}V = v(M, \theta^+ + \theta^-)$ to shorten the expressions)

$$\begin{aligned} 2\mu_V(\Omega) &= \int_{\Omega} \text{div}^{P(x)} X(x) \, d\mu_V(x) \\ &= - \int_{\Omega} \langle X(x), \bar{H}^V(x, P(x)) \rangle \, d\mu_V(x) - \int_{\Omega \times G_{2,3}} \langle X(x), d(\partial V)(x, P) \rangle \\ &\leq \left(\int_{\Omega} |X|^2 \, d\mu_V \right)^{1/2} \left(\int_{\Omega} |\bar{H}^V(\cdot, P)|^2 \, d\mu_V \right)^{1/2} + \int_{\Omega \times G_{2,3}} |\langle X, d(\partial V) \rangle| \\ &\leq \|X\|_{L^\infty(\Omega)} \left(\int_{\Omega} d\mu_V \right)^{1/2} \left(\int_{\Omega \times G_{2,3}} |\bar{H}^V|^2 \, dV \right)^{1/2} + \|X\|_{L^\infty(\Omega)} \int_{\Omega \times G_{2,3}} d\|\partial V\| \\ &\leq c_\infty \text{diam}(\text{supp } \mu_V) \left(\left(\mu_V(\Omega) \int_{\Omega \times G_{2,3}} |\bar{H}^V|^2 \, dV \right)^{1/2} + \|\partial V\|(\Omega \times G_{2,3}) \right). \end{aligned}$$

Hence

$$2\mu_V(\Omega) \leq c_\infty \text{diam}(\text{supp } \mu_V) \left(\sqrt{\mu_V(\Omega) \mathcal{W}(V)} + \|\partial V\|(\Omega \times G_{2,3}) \right).$$

As $\mu_V(\Omega) = m > 0$, $\|\partial V\|(\Omega \times G_{2,3}) < \infty$, and $\mathcal{W}(V) < \infty$ (see Lemma 4.4.11) one has the lower bound

$$0 < \frac{c \mu_V(\Omega)}{\sqrt{\mu_V(\Omega) \mathcal{W}(V)} + \|\partial V\|(\Omega \times G_{2,3})} \leq \text{diam}(\text{supp } \mu_V)$$

with $c := 2/c_\infty > 0$ and the statement follows. $\square$

4 Existence of varifold minimizers for the multiphase Canham-Helfrich functional

Note that the statement of Lemma 4.4.12 remains true if the condition $\mu_V(\Omega) = m$ is replaced by $\mu_V(\Omega) > 0$.

Lemma 4.4.13 (Upper diameter bound). *Let $\Omega = \mathbb{R}^3$ and, in addition to the assumptions of Lemma 4.4.12, assume $\partial(q_\sharp V) = 0$. Then $\text{diam}(\text{supp } \mu_V) < \infty$.*

Proof. The upper diameter bound follows from [95, Lemma 1.1] or [98, Lemma 1] for smooth surfaces, which can be generalized to the varifold setting [103, Theorem 3.1.1]. $\square$

The latter result is based on a refined version of Li-Yau inequality (4.22), which seems not to hold for varifolds with boundary. Consequently, as it stands, the upper bound only directly applies to minimizers V of the single-phase problem, but not to the individual phases V^i in the multiphase case. However, by formula (4.21) and Theorem 4.4.9 the sum $\sum_{i=1}^N q_\sharp V^i$ (corresponding to the union of sets) is a varifold without boundary, for which the upper diameter bound holds. A fortiori, also the individual phases have bounded diameter.

The upper diameter bound eventually allows us to remove the constraint $\text{supp } \mu_V \subset Q$ for a given compact $Q \subset \subset \Omega$ from the definition of admissible varifolds $V \in \mathcal{A}$ for Canham-Helfrich minimizers, see (4.15).

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Chapter 5

Generalized Minimizing Movements for the varifold Canham-Helfrich flow

This chapter consists of my submitted work [21] with MARTIN KRUŽÍK and ULISSE STEFANELLI.

Abstract

The gradient flow of the Canham-Helfrich functional is tackled via the Generalized Minimizing Movements approach. We prove the existence of solutions in Wasserstein spaces of varifolds, as well as upper and lower diameter bounds. In the more regular setting of multiply covered $C^{1,1}$ surfaces, we provide a Li-Yau-type estimate for the Canham-Helfrich energy and prove the conservation of multiplicity along the evolution.

5.1 Introduction

Minimizers of the *Canham-Helfrich energy*

$$E_{\text{CH}}(M) = \int_M \left(\frac{\beta}{2} (H - H_0)^2 + \gamma K \right) d\mathcal{H}^2 \quad (5.1)$$

can be seen as models for the equilibrium shapes of single-phase biological membranes [25, 50]. The membrane is represented by the closed, orientable C^2 surface M in $\mathbb{R}^3$ with H and K respectively denoting its mean and Gauss curvature. The material properties of the membrane material are encoded in the bending rigidities $\beta > 0$, $\gamma < 0$ and in the spontaneous curvature $H_0 \in \mathbb{R}$ and minimization is usually performed under area and enclosed-volume constraint.

We are interested in studying the *gradient flow dynamics* associated to the Canham-Helfrich energy E_{CH} . This can be expected to schematically describe the dissipative evolution of single-phase biomembranes in a viscous environment, possibly up to convergence to equilibrium over time. Our aim is to tackle such evolution in the weak setting of varifolds, building on the recent varifold approach to the minimization of the Canham-Helfrich energy in [22] (Chapter 4). Compared with stronger approaches, working with varifolds allows us to overcome the bottleneck represented by possible singularity formation and to obtain a global existence theory. The price

to pay for this is the weakness of the evolution notion, as we resort in considering variational solutions in DE GIORGI'S *Generalized Minimizing Movements* (GMMs) sense [3, 5, 35].

Before presenting our results in detail, let us sketch a brief review of the related literature. In the stationary case, the minimization of the Canham-Helfrich energy has been investigated in a number of different settings. The case of axisymmetric surfaces is considered by CHOKSI, VENERONI, & MORANDOTTI [26, 27], also in connection with the onset of two different phases on the surface. Minimization in the class of uniform $C^{1,1}$ surfaces for a general class of geometric functionals including E_{CH} is analyzed by DALPHIN [33, 34]. Within the so-called parametric approach, the existence of minimizers in the framework of weak immersions is proved by MONDINO & SCHARRER [78]. In the weak ambient approach of oriented varifolds one has to record the result by EICHMANN [41] and that of the already mentioned [22] (Chapter 4). In the latter, the multiphase case is also considered. Eventually, minimization in the setting of generalized Gauss graphs is studied by KUBIN, LUSSARDI, & MORANDOTTI [58]. A Li-Yau-type inequality for the Canham-Helfrich energy has been recently obtained by RUPP & SCHARRER [90].

In the evolutionary case, local existence results for the classical L^2 gradient flow of the Canham-Helfrich functional were obtained by KOHSAKA & NAGASAWA [57] and NAGASAWA & YI [80], see also LIU [69] for the estimate of the lifespan of a smooth solution. In the setting of vesicles in a viscous environment, i.e., in combination with fluid dynamics surrounding the volume enclosed by M , we mention the local well-posedness results by KÖHNE & LENGELER [56, 67] and WANG, ZHANG, & ZHANG [100]. Moreover, we refer to ELLIOTT & STINNER [43], BARRETT, GARCKE, & NÜRNBERG [10, 11, 48], and ELLIOTT & HATCHER [42] for the finite element approximation of gradient-flow evolution of two-phase biomembranes.

The special case of the *Willmore* flow corresponds to the choice $H_0 = 0$ and has also been specifically considered. We refer to SIMONETT [96] for global existence in the vicinity of spheres and KUWERT & SCHÄTZLE for local existence in case of small initial energy [59] and a lifespan estimate in terms of local initial curvature [60]. A proof of convergence to the sphere for initial Willmore energy smaller than 8π can be found in [61], see also the review [62]. KUWERT & SCHEUER [63] prove stability estimates along the flow. The parametric approach to the Willmore flow has been treated by PALMURELLA & RIVIÈRE [83] who establish global well-posedness for small initial energy of the so called *conformal Willmore flow* and convergence to the sphere. The Willmore flow in the vicinity of the torus has been considered by DALL'ACQUA, MÜLLER, SCHÄTZLE, & SPENER [32] and the flow under additional volume or isoperimetric constraint is studied by RUPP [88, 89]. Discussions on singularity formation can be found in MCCOY & WHEELER [75] and BLATT [15, 16], numerical approximations are found in BARRETT, GARCKE, & NÜRNBERG [8, 9]. Finally, let us refer to COLLI & LAURENÇOT [28], RÄTZ & RÖGER [86], and FEI & LIU [46] for the phase-field approach to the Willmore flow under different settings.

The focus of this work is on the GMMs approach to the Canham-Helfrich functional (5.1) in the varifold setting [22] (Chapter 4), formulated as a functional G_{CH} of probability measures. The very weak varifold setting is instrumental for proving the existence of equilibria without a-priori structural assumptions on the minimizers. In particular, it is well-suited to handle the possible onset of singularities while evolving far from stationary points.

The GMMs evolution notion rests on a limit passage in time-discrete approximations V^n , which are obtained from some given initial state V^0 and time step $\tau > 0$ via the successive in $n \in \mathbb{N}$ minimization of the incremental functional

$$V \mapsto G_{\text{CH}}(V) + \frac{1}{2\tau} W_p^2(V, V^{n-1}),$$

see (5.16). The GMMs $[0, \infty) \ni t \mapsto V(t)$ are then pointwise limits of subsequences of (piecewise

constant in time interpolants) of time-discrete approximations V^n as $\tau \rightarrow 0$, see Definition 5.4.1. Note that the incremental functional features the interplay of energy minimization and distance-control from the previous discrete state V^{n-1} . In particular, we consider here the Wasserstein distance W_p with $p \in [1, \infty)$ between the two varifolds V and V^{n-1} , seen as Radon measures with equal mass. This point distinguishes our analysis from previous contributions, where L^2 gradient flows on different parametrization settings are considered instead.

The plan of the paper is as follows. After providing preliminaries in Section 5.2 and recalling some material on the stationary case in Section 5.3, we prove global existence of GMMs for G_{CH} in Section 5.4. Contrary to previous results on Canham-Helfrich flow, no smallness condition on the energy of the initial state is required (Theorem 5.4.2). GMMs are absolutely continuous curves with respect to the metric W_p and the Canham-Helfrich energy does not increase along the evolution. This entails that GMMs satisfy lower and upper diameter bounds (Proposition 5.4.4). Moreover, we prove that the flow can be constrained to fulfill additional properties, and still admit the existence of a GMMs solution (Corollary 5.4.3). This allows us to consider an enclosed-volume constraint or to reduce to reflection- or axisymmetric varifolds. In the same spirit, evolution can be constrained to multiply covered surfaces as well, as long as uniform $C^{1,1}$ regularity is enforced as in [34], see also [65]. We discuss this more regular case in Section 5.5, where we prove that GMMs exist for any given genus and that the multiplicity of the initial surface is conserved along the flow (Theorem 5.5.6). The minimality (hence stationarity) of multiply covered spheres is analyzed in Subsection 5.5.2. Moreover, we present an a-priori bound on the multiplicity in terms of the Canham-Helfrich energy (Proposition 5.5.4). Such Li-Yau-type bound can be compared with the recent one from [90] where however no Gauss term is considered, i.e., $\gamma = 0$. Eventually, we discuss in Subsection 5.5.6 the possibility of weakening the metric W_p and consider instead the Wasserstein distance between the surfaces, here seen as restricted two-dimensional Hausdorff $\mathcal{H}^2$ measures in space. This choice still allows for existence of GMMs of multiply covered surfaces, as long as the multiplicity is conserved. We conclude by briefly addressing more general curvature functionals than the Canham-Helfrich energy.

5.2 Notation and preliminaries

5.2.1 Measures and perimeter

Let X be a locally compact and separable metric space and $N \in \mathbb{N}$. The dual space

$$\mathcal{M}^N(X) := C_c(X; \mathbb{R}^N)^*$$

of compactly supported vector functions on X is the space of vector-valued *Radon measures* on X having N components. It is normed by the total variation $\|\cdot\|_{\text{TV}}(X)$ and equipped with the *weak-* topology*: $\mu_n \rightharpoonup^* \mu$ ($n \rightarrow \infty$) if

$$\int_X \varphi(x) d\mu_n(x) \rightarrow \int_X \varphi(x) d\mu(x) \quad \text{for all} \quad \varphi \in C_c(X; \mathbb{R}^N).$$

The *support* of $\mu \in \mathcal{M}^N(X)$, denoted by $\text{supp}(\mu)$, is the closed set of points $x \in X$ such that $\|\mu\|_{\text{TV}}(U) > 0$ for all neighborhoods U of x . The *restriction* of μ to $B \subset X$ is the measure $\mu \llcorner B$ given by

$$(\mu \llcorner B)(A) := \mu(A \cap B)$$

for all Borel sets $A \subset X$. For $\mu \in \mathcal{M}^N(X)$ and a μ -measurable map $f: X \rightarrow Y$, where Y is another locally compact and separable metric space, the *push-forward measure* of μ by f is

defined by

$$(f_{\#}\mu)(A) := \mu(f^{-1}(A))$$

for all Borel sets $A \subset Y$. If f is continuous and such that $f^{-1}(B)$ is compact for all compact $B \subset Y$, then $f_{\#}\mu \in \mathcal{M}^N(Y)$. In this case, $\text{supp}(f_{\#}\mu) = f(\text{supp}(\mu))$ and $\int_Y g(y) d(f_{\#}\mu)(y) = \int_X (g \circ f)(x) d\mu(x)$ hold for all measurable $g: Y \rightarrow \mathbb{R}^N$. We omit N from the notation if $N = 1$. The scalar Radon measure $\mu \in \mathcal{M}(X)$ is called *positive* if $\mu(A) \geq 0$ for every Borel set $A \subset X$. In this case, $\mu = \|\mu\|_{\text{TV}}$. For $1 \leq p < \infty$, $L^p_\mu(X; \mathbb{R}^N)$ denotes the Lebesgue space of all μ -measurable functions $v: X \rightarrow \mathbb{R}^N$ such that

$$\|v\|_{L^p_\mu(X)} := \left(\int_X |v(x)|^p d\mu(x) \right)^{1/p} < \infty.$$

If $X \subset \mathbb{R}^d$ for $d \in \mathbb{N}$ and μ is the d -dimensional Lebesgue measure, we simply write dx instead of $d\mu(x)$.

Let $E \subset \mathbb{R}^d$ be a Lebesgue measurable set. We say that E is of *finite perimeter* $\text{Per}(E)$, if there exists a Radon measure $\mu^E \in \mathcal{M}^d(\mathbb{R}^d)$ such that $\int_E (\text{div } \varphi) dx = \int_{\mathbb{R}^d} \varphi d\mu^E$ for every $\varphi \in C_c^1(\mathbb{R}^d; \mathbb{R}^d)$ with

$$\text{Per}(E) := \|\mu^E\|_{\text{TV}}(\mathbb{R}^d) < \infty.$$

The *relative perimeter* of E in a Borel set $A \subset \mathbb{R}^d$ is $\text{Per}(E; A) := \|\mu^E\|_{\text{TV}}(A)$. By definition, $\mu^E = -D1_E$, where D stands for the weak derivative and $1_E: \mathbb{R}^d \rightarrow \{0, 1\}$ is the characteristic function of E . With $B_r(x)$ we indicate the open ball

$$B_r(x) := \{y \in \mathbb{R}^d : |x - y| < r\}$$

centered at $x \in \mathbb{R}^d$ and of radius $r > 0$. The set of all points $x \in \text{supp}(\mu^E)$ such that $\nu(x) := \lim_{r \rightarrow 0} \mu^E(B_r(x)) / |\mu^E(B_r(x))|$ exists and satisfies $\nu(x) \in \mathbb{S}^{d-1}$ is called the *reduced boundary* of E , denoted by $\partial^* E$. The field ν defined in this way is called the (measure-theoretic outer unit) *normal* to E .

5.2.2 Varifolds and currents

Let $m \in \mathbb{N}_0$ with $0 \leq m \leq d$ (note however that we will only need the case $d = 3$ and $m = 2$ later on). The *Grassmannian* $G_{m,d}$ is the set of all m -dimensional linear subspaces of $\mathbb{R}^d$. Elements of $G_{m,d}$ are identified with the corresponding orthogonal projections $P \in \mathbb{R}^{d \times d}$ on the m -dimensional subspaces. The *oriented Grassmannian*, $G_{m,d}^o$, is the set of all oriented m -dimensional linear subspaces in $\mathbb{R}^d$. Its elements are represented by m -vectors ξ in $\mathbb{R}^d$.

An *m-varifold* (later *varifold*, for short) in $\mathbb{R}^d$ is a positive Radon measure

$$V \in \mathcal{M}(\mathbb{R}^d \times G_{m,d}).$$

Similarly, an *oriented m-varifold* in $\mathbb{R}^d$ is defined on the oriented Grassmannian, i.e., $V \in \mathcal{M}(\mathbb{R}^d \times G_{m,d}^o)$. The *mass* of the varifold V (either oriented or not) is the positive Radon measure $\mu_V \in \mathcal{M}(\mathbb{R}^d)$ given by

$$\mu_V(\Omega) := V(\Omega \times G_{m,d})$$

for all Borel sets $\Omega \subset \mathbb{R}^d$. We introduce the *two-fold covering map*

$$q: \mathbb{R}^d \times G_{m,d}^o \rightarrow \mathbb{R}^d \times G_{m,d}, \quad q(x, \pm\xi) = (x, P),$$

where P is the projection map onto the linear subspace spanned by $\pm\xi$. Then, to any oriented varifold $V \in \mathcal{M}(\mathbb{R}^d \times G_{m,d}^o)$, we can associate a varifold $q_\#V \in \mathcal{M}(\mathbb{R}^d \times G_{m,d})$ given by the push-forward of V under q , i.e., for all $\varphi \in C_c(\mathbb{R}^d \times G_{m,d})$,

$$(q_\#V)(\varphi) = \int_{\mathbb{R}^d \times G_{m,d}} \varphi(x, P) d(q_\#V)(x, P) = \int_{\mathbb{R}^d \times G_{m,d}^o} \varphi(q(x, \xi)) dV(x, \xi).$$

In addition, V gives rise to the m -current $T_V \in \mathcal{D}_m(\mathbb{R}^d)$ defined by

$$T_V(\omega) := \int_{\mathbb{R}^d \times G_{m,d}^o} \langle \omega(x), \xi \rangle dV(x, \xi)$$

for all $\omega \in \mathcal{D}^m(\mathbb{R}^d)$, the smooth, compactly supported m -forms in $\mathbb{R}^d$. Recall that a *general m -current* in $\mathbb{R}^d$, denoted by $T \in \mathcal{D}_m(\mathbb{R}^d)$, is an element of the dual space to $\mathcal{D}^m(\mathbb{R}^d)$. Its *mass* in $\Omega \subset \mathbb{R}^d$ is defined by the dual norm

$$\mathbb{M}_T(\Omega) := \sup\{T(\omega) : \omega \in \mathcal{D}^m(\mathbb{R}^d), \text{supp}(\omega) \subset \Omega, \|\omega\|_\infty \leq 1\},$$

where $\|\omega\|_\infty = \sup_{x \in \mathbb{R}^d} |\omega(x)|$. If $\Omega = \mathbb{R}^d$ we simply write $\mathbb{M}_T$. The *boundary* ∂T of $T \in \mathcal{D}_m(\mathbb{R}^d)$ is the $(m-1)$ -current given by

$$\partial T(\eta) := T(d\eta) \quad \text{for all } \eta \in \mathcal{D}^{m-1}(\mathbb{R}^d).$$

A set $M \subset \mathbb{R}^d$ ($0 \leq m \leq d$) is called *m -rectifiable* (see, e.g., [94]) if its m -dimensional Hausdorff measure $\mathcal{H}^m(M)$ is finite and if, up to a $\mathcal{H}^m$ -zero set M_0 , it is contained in a countable union of disjoint images M_i of Lipschitz maps from $\mathbb{R}^m$ to $\mathbb{R}^d$, i.e.,

$$M \subset \left(\bigcup_{i=1}^\infty M_i\right) \cup M_0 \quad \text{with} \quad \mathcal{H}^m(M_0) = 0.$$

If M is m -rectifiable, then its cone of *approximate tangent vectors* [85, p. 7] is an m -dimensional plane for $\mathcal{H}^m$ -almost all $x \in M$, which we will denote by $T_x M$ as in the smooth setting. An *orientation* ξ^M of an m -rectifiable set $M \subset \mathbb{R}^d$ is a measurable choice of orientation for each tangent space $T_x M$. If $T_x M$ is spanned by $(\tau_i(x))_{i=1}^m$ with $\tau_i(x) \in \mathbb{R}^d$, then, for a.e. $x \in M$, one has the two choices

$$\pm \xi^M(x) := \pm \tau_1(x) \wedge \cdots \wedge \tau_m(x) = (T_x M)^\pm \in G_{m,d}^o.$$

If $m = d - 1$, we can identify the orientation $\xi^M(x)$ with the unit normal vector $\nu^M(x) \in \mathbb{S}^{d-1}$. In particular, in the case $m = 2$ and $d = 3$, which is the relevant one in our case, we will apply the covering map in the form $q: \mathbb{R}^3 \times \mathbb{S}^2 \rightarrow \mathbb{R}^3 \times G_{2,3}$, $q(x, \pm\nu) = (x, P)$, where $P \in \mathbb{R}^{3 \times 3}$ is the projection map onto the plane orthogonal to $\pm\nu$.

Let $M \subset \mathbb{R}^d$ be m -rectifiable with orientation ξ^M . A *rectifiable m -current* $T \in \mathcal{D}_m(\mathbb{R}^d)$ with *integer multiplicity* $\theta \in L^1_{\mathcal{H}^m \llcorner M}(\mathbb{R}^d; \mathbb{Z})$ is defined as

$$T(\omega) = \int_M \langle \omega(x), \xi^M(x) \rangle \theta(x) d\mathcal{H}^m(x)$$

for all $\omega \in \mathcal{D}^m(\mathbb{R}^d)$. This current will be denoted by $T = T[M, \xi^M, \theta]$ to emphasize that it is determined by the triple M, ξ^M, θ . Its mass corresponds to

$$\mathbb{M}_T = \int_M |\theta(x)| d\mathcal{H}^m(x).$$

A rectifiable current T with $\mathbb{M}_T < \infty$ is called an *integral current*, if in addition $\mathbb{M}_{\partial T} < \infty$. If an integral current is supported on a smooth submanifold and its boundary is supported on the boundary of this submanifold, then the current must have constant multiplicity. This fact is known as *Federer's Constancy Theorem*, which actually also holds in the more general case of Lipschitz submanifolds [85, Cor. 1.54].

Theorem 5.2.1 (Constancy, [45, Sec. 4.1.31]). *Let $M \subset \mathbb{R}^d$ be an m -dimensional, connected C^1 submanifold of $\mathbb{R}^d$ with boundary and assume that M is oriented by $\xi^M \in C(M; G_{m,d}^o)$. If an integral current $T \in \mathcal{D}_m(\mathbb{R}^d)$ satisfies $\text{supp}(T) \subset M$ and $\text{supp}(\partial T) \subset \partial M$, then T has a constant multiplicity $\theta = c \in \mathbb{Z}$, i.e.,*

$$T(\omega) = c \int_M \langle \omega, \xi^M \rangle d\mathcal{H}^m \quad \text{for all } \omega \in \mathcal{D}^m(\mathbb{R}^d).$$

Let $M \subset \mathbb{R}^d$ be m -rectifiable and $\theta \in L^1_{\mathcal{H}^m \llcorner M}(\mathbb{R}^d; \mathbb{N})$. An *integral varifold* $V = V[M, \theta]$ is a varifold defined for all $\varphi \in C_c(\mathbb{R}^d \times G_{m,d})$ by the formula

$$V(\varphi) = \int_M \varphi(x, T_x M) \theta(x) d\mathcal{H}^m(x).$$

Equivalently, we will also write $dV(x, P) = \theta(x) d(\mathcal{H}^m \llcorner M)(x) \otimes \delta_{T_x M}(P)$. The set of all integral m -varifolds on $\mathbb{R}^d$ is denoted by $IV_m(\mathbb{R}^d)$. The mass of $V = V[M, \theta]$ corresponds to

$$\mu_V(\mathbb{R}^d) = \int_M \theta(x) d\mathcal{H}^m(x).$$

If ξ^M is an orientation of M and $\theta^\pm \in L^1_{\mathcal{H}^m \llcorner M}(\mathbb{R}^d; \mathbb{N}_0)$, then an oriented varifold V defined for all $\psi \in C_c(\mathbb{R}^d \times G_{m,d}^o)$ by the formula

$$V(\psi) = \int_M (\theta^+(x) \psi(x, \xi^M(x)) + \theta^-(x) \psi(x, -\xi^M(x))) d\mathcal{H}^m(x)$$

is called an *oriented integral varifold* and is denoted by $V = V[M, \xi^M, \theta^+, \theta^-] \in IV_m^o(\mathbb{R}^d)$. We also write $dV(x, \xi) = d(\mathcal{H}^m \llcorner M)(x) \otimes (\theta^+(x) \delta_{\xi^M(x)}(\xi) + \theta^-(x) \delta_{-\xi^M(x)}(\xi))$. The mass of V corresponds to

$$\mu_V(\mathbb{R}^d) = \int_M (\theta^+ + \theta^-)(x) d\mathcal{H}^m(x).$$

If

$$V = V[M, \xi^M, \theta^+, \theta^-] \in IV_m^o(\mathbb{R}^d),$$

then

$$q_\# V = q_\# V[M, \theta^+ + \theta^-] \in IV_m(\mathbb{R}^d) \quad \text{and} \quad T_V = T_V[M, \xi^M, \theta^+ - \theta^-] \in \mathcal{D}_m(\mathbb{R}^d).$$

We say that $V \in IV_m(\mathbb{R}^d)$ is a *curvature varifold with boundary* and write $V \in AV_m(\mathbb{R}^d)$, if there exists

$$A^V \in L^1_{\text{loc}, V}(\mathbb{R}^d \times G_{m,d}; \mathbb{R}^{d \times d \times d}) \quad \text{and} \quad \partial V \in \mathcal{M}^d(\mathbb{R}^d \times G_{m,d})$$

such that

$$\begin{aligned} & \int_{\mathbb{R}^d \times G_{m,d}} \sum_{j=1}^d \left(P_{ij} \partial_j \varphi(x, P) + \sum_{k=1}^d (\partial_{P_{jk}} \varphi(x, P)) A_{ijk}^V(x, P) + A_{jij}^V(x, P) \varphi(x, P) \right) dV(x, P) \\ &= - \int_{\mathbb{R}^d \times G_{m,d}} \varphi(x, P) d(\partial V)_i(x, P) \end{aligned} \quad (5.2)$$

for all $\varphi \in C_c^1(\mathbb{R}^d \times G_{m,d})$ and $1 \leq i \leq d$. The *curvature functions* A_{ijk}^V ($1 \leq i, j, k \leq d$) give rise to the *generalized second fundamental form* $\Pi^V \in L_{\text{loc},V}^1(\mathbb{R}^d \times G_{m,d}; \mathbb{R}^{d \times d \times d})$ via

$$(\Pi^V(\cdot, P))_{ij}^k := \sum_{l=1}^d A_{ikl}^V(\cdot, P) P_{lj}$$

and the vector-valued Radon measure ∂V is called the *boundary measure*. The set of *oriented curvature varifolds*, $AV_m^o(\mathbb{R}^d)$, is given by all $V \in IV_m^o(\mathbb{R}^d)$ such that $q_\# V \in AV_m(\mathbb{R}^d)$.

5.2.3 Compactness

Let us recall the *compactness theorems* for oriented integral varifolds by Hutchinson [54] and for curvature varifolds with boundary by Mantegazza [73]. Let $\Omega \subset \mathbb{R}^d$ be open. The classes of varifolds defined on Ω are denoted by $IV_m(\Omega)$, $IV_m^o(\Omega)$, and $AV_m(\Omega)$.

Theorem 5.2.2 (Compactness, [54, Thm. 3.1]). *The set*

$$\{V \in IV_m^o(\Omega) : \mu_V(\Omega) + \|\delta(q_\# V)\|_{\text{TV}}(\Omega) + \mathbb{M}_{\partial T_V}(\Omega) < \infty\}$$

is compact in the weak- topology of $\mathcal{M}(\Omega \times G_{m,d}^o)$.*

Here, $\delta(q_\# V) : C_c^1(\Omega; \mathbb{R}^d) \rightarrow \mathbb{R}$ stands for the first variation [1] of the varifold $q_\# V$, which can be estimated in terms of mass, curvature, and boundary [22, Lemma 2.3] (Chapter 4, Lemma 4.2.3),

$$\|\delta(q_\# V)\|_{\text{TV}}(\Omega) \leq \sqrt{d \mu_V(\Omega)} \|A^{q_\# V}\|_{L_{q_\# V}^2(\Omega \times G_{m,d})} + \|\partial(q_\# V)\|_{\text{TV}}(\Omega \times G_{m,d}).$$

The *curvature varifold convergence*

$$V_n \rightharpoonup^* V \quad \text{in} \quad AV_m(\Omega) \quad (n \rightarrow \infty)$$

is defined as the *measure-function pairs convergence* $(V_n, A^{V_n}) \rightharpoonup^* (V, A^V)$: $V_n \rightharpoonup^* V$ in $\mathcal{M}(\Omega \times G_{m,d})$ and $A^{V_n} V_n \rightharpoonup^* A^V V$ in $\mathcal{M}^{d \times d \times d}(\Omega \times G_{m,d})$, cf. [54]. In particular, if $V_n \rightharpoonup^* V$ in $AV_m(\Omega)$ then, by definition (5.2), $\partial V_n \rightharpoonup^* \partial V$ in $\mathcal{M}^d(\Omega \times G_{m,d})$.

Theorem 5.2.3 (Compactness, [73, Thm. 6.1]). *The set*

$$\left\{ V \in AV_m(\Omega) : \mu_V(\Omega) + \|A^V\|_{L_V^2(\Omega \times G_{m,d})}^2 + \|\partial V\|_{\text{TV}}(\Omega \times G_{m,d}) < \infty \right\}$$

is compact with respect to curvature varifold convergence.

5.2.4 Wasserstein space

Let (X, d_X) be a complete separable metric space and

$$\mathcal{P}(X) := \{\mu \in \mathcal{M}(X) : \mu(X) = m_0\}$$

indicate the set of positive Radon measures with fixed mass $m_0 > 0$. For $p \in [1, \infty)$, let

$$\mathcal{P}_p(X) := \left\{ \mu \in \mathcal{P}(X) : \int_X d_X^p(x, x_0) \, d\mu(x) < \infty \right\}$$

for some $x_0 \in X$ denote its subspace of measures with *finite p -th moment* and recall that $\mathcal{P}_p(X) = \mathcal{P}(X)$ for all p if X is compact. The p -Wasserstein distance between $\mu, \tilde{\mu} \in \mathcal{P}_p(X)$ is defined by

$$W_p(\mu, \tilde{\mu}) := \left(\min_{\lambda \in \Pi(\mu, \tilde{\mu})} \int_{X \times X} d_X^p(x, \tilde{x}) \, d\lambda(x, \tilde{x}) \right)^{1/p},$$

where $\Pi(\mu, \tilde{\mu}) := \{\lambda \in \mathcal{M}(X \times X) : (\pi_1)_\# \lambda = \mu, (\pi_2)_\# \lambda = \tilde{\mu}\}$ is the set of all couplings with marginals μ and $\tilde{\mu}$. The metric space $(\mathcal{P}_p(X), W_p)$ is called the p -Wasserstein space on (X, d_X) . If $\mu, \tilde{\mu} \in \mathcal{P}_1(X)$ and have bounded support, then by [18, Thm. 8.10.45] and [5, (7.1.2)],

$$W_1(\mu, \tilde{\mu}) = \sup \left\{ \int_X f \, d(\mu - \tilde{\mu}) : f \in \text{Lip}_1(X) \right\},$$

which is known as the *Kantorovich-Rubinshtein distance* (or *modified bounded Lipschitz distance*). Here, Lip_1 stands for Lipschitz continuous functions with unit Lipschitz constant. This representation renders the W_1 distance particularly useful for applications, see [23, 24].

The metric space $(\mathcal{P}_p(X), W_p)$ is complete and separable [5, Prop. 7.1.5]. Moreover, W_p generates the weak- $*$ topology on $\mathcal{P}_p(X)$: If $(\mu_n) \subset \mathcal{P}_p(X)$, $\mu \in \mathcal{P}_p(X)$, and $n \rightarrow \infty$, then

$$W_p(\mu_n, \mu) \rightarrow 0 \iff \mu_n \rightharpoonup^* \mu \text{ and } \int_X f \, d\mu_n \rightarrow \int_X f \, d\mu \text{ for } f \in C(X) \text{ with } p\text{-growth,}$$

i.e., $|f| \leq c(1 + d_X^p(\cdot, x_0))$ for some $x_0 \in X$ and $c > 0$.

5.3 The Canham-Helfrich functional

We devote this section to recalling some results for the stationary case. In particular, by restricting to $d = 3$ and $m = 2$ we follow [22] (Chapter 4) and introduce the generalization of the classical Canham-Helfrich energy (5.1) to oriented curvature two-varifolds

$$F_{\text{CH}} : AV_2^o(\mathbb{R}^3) \rightarrow \mathbb{R}, \quad F_{\text{CH}}(V) := \int_{\mathbb{R}^3 \times \mathbb{S}^2} f_{\text{CH}}(\nu, A^{q_\# V}(q(x, \nu))) \, dV(x, \nu), \quad (5.3)$$

where the integrand $f_{\text{CH}} : \mathbb{S}^2 \times \mathbb{R}^{3 \times 3 \times 3} \rightarrow \mathbb{R}$ is given by

$$f_{\text{CH}}(\nu, A) := \sum_{i=1}^3 \left(\frac{\beta}{2} \left(\sum_{j=1}^3 A_{jij} - \nu_i H_0 \right)^2 + \frac{\gamma}{2} \left(\sum_{j=1}^3 A_{jij} \right)^2 - \frac{\gamma}{4} \sum_{j,k=1}^3 A_{ijk}^2 \right).$$

The form of the integrand in (5.3) can be deduced from the identity $(H - H_0)^2 = |\bar{H} - \nu H_0|^2 = \sum_{i=1}^3 (\bar{H}_i - \nu_i H_0)^2$ by replacing the mean curvature vector $\bar{H} = H\nu$ by its *varifold analog*

$$\bar{H}^{q_\# V} := \left(\sum_{j=1}^3 A_{jij}^{q_\# V} \right)_{i=1}^3.$$

Note that the squared norm of the second fundamental form of a curvature varifold is given by

$$|\Pi^{q_\# V}|^2 = \frac{1}{2} |A^{q_\# V}|^2 = \frac{1}{2} \sum_{i,j,k=1}^3 (A_{ijk}^{q_\# V})^2,$$

which allows us to define the *generalized Gauss curvature*

$$K^{q_\# V} := \frac{1}{2} (|\bar{H}^{q_\# V}|^2 - |\Pi^{q_\# V}|^2) = \frac{1}{2} |\bar{H}^{q_\# V}|^2 - \frac{1}{4} |A^{q_\# V}|^2. \quad (5.4)$$

Since $q(x, \nu) = (x, P) = q(x, -\nu)$, it follows that $(x, \nu) \mapsto A^{q\sharp V}(q(x, \nu))$ does not depend on the sign of ν and reversing orientation has the sole effect of changing the sign in front of H_0 .

We identify the *area* of an oriented varifold $V \in \mathcal{M}(\mathbb{R}^3 \times \mathbb{S}^2)$ with its mass, namely, $\mu_V(\mathbb{R}^3) = V(\mathbb{R}^3 \times \mathbb{S}^2)$. For $m_0 > 0$ fixed, at the stationary level we are interested in varifolds $V \in AV_2^o(\mathbb{R}^3)$ minimizing F_{CH} under the mass constraint

$$\mu_V(\mathbb{R}^3) = m_0. \quad (5.5)$$

5.3.1 Varifold minimizers

Existence of varifold minimizers follows by the Direct Method, exploiting compactness results for oriented and curvature varifolds (see Theorems 5.2.2 and 5.2.3) together with the lower semicontinuity [22, Thm. 3.2] (Chapter 4, Thm. 4.3.2),

$$F_{\text{CH}}(V) \leq \liminf_{n \rightarrow \infty} F_{\text{CH}}(V_n) \quad \text{for } V_n \rightharpoonup^* V \text{ } (n \rightarrow \infty) \text{ in } AV_2^o(\mathbb{R}^3),$$

where $V_n \rightharpoonup^* V$ is the curvature varifold convergence, namely, $V_n \rightharpoonup^* V$ in $\mathcal{M}(\mathbb{R}^3 \times \mathbb{S}^2)$ and $(q\sharp V_n, A^{q\sharp V_n}) \rightharpoonup^* (q\sharp V, A^{q\sharp V})$ as measure-function pairs. A key ingredient in the mentioned lower-semicontinuity proof is the convexity of the integrand $f_{\text{CH}}(\nu, \cdot)$, which holds under the following condition on the parameters:

$$-\frac{6}{5}\beta < \gamma < 0. \quad (5.6)$$

In fact, such convexity gives rise to the *curvature bound* [22, Prop. 3.1] (Chapter 4, Prop. 4.3.1)

$$\|A^{q\sharp V}\|_{L_{q\sharp V}^2(\mathbb{R}^3 \times G_{2,3})}^2 \leq c_1 (F_{\text{CH}}(V) + c_2 \mu_V(\mathbb{R}^3)) \quad (5.7)$$

for all $V \in AV_2^o(\mathbb{R}^3)$, with constants $c_1 > 0$, $c_2 \geq 0$ depending on data (in particular, $c_2 = 0$ if $H_0 = 0$). An early consequence of (5.7) is that the Canham-Helfrich energy F_{CH} is bounded from below in terms of the mass.

5.3.2 Lower bound in terms of the Willmore energy

A special instance of the classical Canham-Helfrich energy is the *Willmore energy*

$$\mathcal{W}(M) := \frac{1}{4} \int_M H^2 d\mathcal{H}^2,$$

corresponding indeed to the choice $H_0 = 0$, $\beta = 1/2$, and $\gamma = 0$ in (5.1). If $M \subset \mathbb{R}^3$ is an immersed closed surface one has $\mathcal{W}(M) \geq 4\pi$ with equality only for the sphere $M = \mathbb{S}_R^2$ of any radius $R > 0$ [101].

Without changing notation, one can define the Willmore energy of an (oriented) curvature varifold $V \in AV_2^o(\mathbb{R}^3 \times \mathbb{S}^2)$ as

$$\mathcal{W}(V) := \frac{1}{4} \int_{\mathbb{R}^3 \times G_{2,3}} |\bar{H}^{q\sharp V}(x, P)|^2 d(q\sharp V)(x, P) = \frac{1}{4} \int_{\mathbb{R}^3 \times \mathbb{S}^2} |\bar{H}^{q\sharp V}(q(x, \nu))|^2 dV(x, \nu),$$

which for $V = V[M, \nu^M, \theta^+, \theta^-]$ reduces to

$$\mathcal{W}(V) = \frac{1}{4} \int_{\mathbb{R}^3} |\bar{H}^{q\sharp V}|^2 d\mu_V = \frac{1}{4} \int_M |\bar{H}^{q\sharp V}|^2 (\theta^+ + \theta^-) d\mathcal{H}^2. \quad (5.8)$$

The latter controls the multiplicity of the varifold $q_{\sharp}V = q_{\sharp}V[M, \theta^+ + \theta^-]$ via the *Li-Yau inequality* [68] (see [62] for its generalization to integral varifolds),

$$\mathcal{W}(V) \geq 4\pi(\theta^+ + \theta^-). \quad (5.9)$$

The algebraic estimate $|\bar{H}^{q_{\sharp}V}|^2 = \sum_{i=1}^3 \left(\sum_{j=1}^3 A_{jij}^{q_{\sharp}V} \right)^2 \leq 3 \sum_{i,j,k=1}^3 \left(A_{ijk}^{q_{\sharp}V} \right)^2 = 3|A^{q_{\sharp}V}|^2$, cf. [22, Lemma 2.3] (Chapter 4, Lemma 4.2.3), entails $4\mathcal{W}(V) \leq 3\|A^{q_{\sharp}V}\|_{L^2_{q_{\sharp}V}(\mathbb{R}^3 \times G_{2,3})}^2$. From this, under assumption (5.6), the curvature bounds (5.7) imply the control

$$4\mathcal{W}(V) \leq 3c_1 (F_{\text{CH}}(V) + c_2 \mu_V(\mathbb{R}^3)), \quad (5.10)$$

which, in combination with (5.9), in turn gives a multiplicity bound in terms of the Canham-Helfrich energy, i.e., a Li-Yau-type inequality. Let us recall again that Li-Yau inequalities for the Canham-Helfrich functional have been recently obtained in [90], where, however, the Gauss term is not considered.

In the following, we derive an alternative estimate in terms of the Willmore energy with explicit constants. Recall that $V = V[M, \nu^M, \theta^+, \theta^-]$ is given by

$$\mathrm{d}V(x, \nu) = \mathrm{d}(\mathcal{H}^2 \llcorner M)(x) \otimes (\theta^+(x) \delta_{\nu^M(x)}(\nu) + \theta^-(x) \delta_{-\nu^M(x)}(\nu)), \quad (5.11)$$

with an $\mathcal{H}^2$ -measurable, countably rectifiable set $M \subset \mathbb{R}^3$ with orientation $\pm \nu^M(x)$ (corresponding to $(T_x M)^{\pm}$) and locally integrable multiplicities $\theta^{\pm}(x) \in \mathbb{N}$ for $(\mathcal{H}^2 \llcorner M)$ -a.e. $x \in \mathbb{R}^3$. Moreover, if V is an oriented curvature varifold, i.e., $V \in AV_2^o(\mathbb{R}^3 \times \mathbb{S}^2)$, then the energy (5.3) can be written as

$$\begin{aligned} F_{\text{CH}}(V) &= \int_{\mathbb{R}^3 \times \mathbb{S}^2} \left(\frac{\beta}{2} |\bar{H}^{q_{\sharp}V}(q(x, \nu)) - \nu H_0|^2 + \gamma K^{q_{\sharp}V}(q(x, \nu)) \right) \mathrm{d}V(x, \nu) \\ &= \int_M \left(\left(\frac{\beta}{2} |\bar{H}^{q_{\sharp}V}(q(x, \nu^M(x))) - \nu^M(x) H_0|^2 + \gamma K^{q_{\sharp}V}(q(x, \nu^M(x))) \right) \theta^+(x) \right. \\ &\quad \left. + \left(\frac{\beta}{2} |\bar{H}^{q_{\sharp}V}(q(x, -\nu^M(x))) + \nu^M(x) H_0|^2 + \gamma K^{q_{\sharp}V}(q(x, -\nu^M(x))) \right) \theta^-(x) \right) \mathrm{d}\mathcal{H}^2(x). \end{aligned}$$

Taking into account that $q(x, -\nu^M) = q(x, \nu^M)$,

$$|\bar{H}^{q_{\sharp}V} \mp \nu^M H_0|^2 = |\bar{H}^{q_{\sharp}V}|^2 \mp 2\bar{H}^{q_{\sharp}V} \cdot \nu^M H_0 + |\nu^M|^2 H_0^2,$$

and $|\nu^M|^2 = 1$, this reduces to

$$\begin{aligned} F_{\text{CH}}(V) &= \int_M \left(\frac{\beta}{2} (|\bar{H}^{q_{\sharp}V}|^2 + H_0^2) + \gamma K^{q_{\sharp}V} \right) (\theta^+ + \theta^-) \mathrm{d}\mathcal{H}^2 \\ &\quad - \beta H_0 \int_M \bar{H}^{q_{\sharp}V} \cdot \nu^M (\theta^+ - \theta^-) \mathrm{d}\mathcal{H}^2, \end{aligned} \quad (5.12)$$

where the argument $q(\cdot, \nu^M(\cdot))$ of $\bar{H}^{q_{\sharp}V}$ and $K^{q_{\sharp}V}$ has been omitted for conciseness. This representation of the Canham-Helfrich energy allows deducing another lower bound of F_{CH} in terms of the Willmore energy.

Lemma 5.3.1 (Lower bound for the energy). *Let $\beta > 0$ and $\gamma, H_0 \in \mathbb{R}$. Then, for varifolds $V = V[M, \nu^M, \theta^+, \theta^-] \in AV_2^o(\mathbb{R}^3)$ with mass $\mu_V(\mathbb{R}^3) = m_0 > 0$,*

$$F_{\text{CH}}(V) \geq 2\beta \left(\sqrt{\frac{\mathcal{W}(V)}{m_0}} - \frac{|H_0|}{2} \right)^2 m_0 + \gamma \int_M K^{q_{\sharp}V} (\theta^+ + \theta^-) \mathrm{d}\mathcal{H}^2. \quad (5.13)$$

Proof. We use the short-hand notation $H = \bar{H}^{q_\# V} \cdot \nu^M$ and omit everywhere the superscript $(\cdot)^{q_\# V}$ for brevity. Moving from (5.12) we get

$$\begin{aligned} F_{\text{CH}}(V) &= \int_M \left(\frac{\beta}{2} (H^2 + H_0^2) + \gamma K \right) (\theta^+ + \theta^-) d\mathcal{H}^2 - \beta H_0 \int_M H(\theta^+ - \theta^-) d\mathcal{H}^2 \\ &= 2\beta \left(\frac{1}{4} \int_M H^2(\theta^+ + \theta^-) d\mathcal{H}^2 - \frac{H_0}{2} \int_M H(\theta^+ - \theta^-) d\mathcal{H}^2 + \frac{H_0^2}{4} m_0 \right) \\ &\quad + \gamma \int_M K(\theta^+ + \theta^-) d\mathcal{H}^2, \end{aligned}$$

where we used the mass constraint $\mu_V(\mathbb{R}^3) = \int_M (\theta^+ + \theta^-) d\mathcal{H}^2 = m_0$ (5.5) for the second equality. As $\mathcal{W}(V) = \frac{1}{4} \int_M H^2(\theta^+ + \theta^-) d\mathcal{H}^2$ from (5.8) and $\theta^+, \theta^- \geq 0$, the second term in the right-hand side can be estimated as

$$\begin{aligned} \frac{H_0}{2} \int_M H(\theta^+ - \theta^-) d\mathcal{H}^2 &\leq |H_0| \int_M \frac{|H|}{2} |\theta^+ - \theta^-| d\mathcal{H}^2 \leq |H_0| \int_M \frac{|H|}{2} (\theta^+ + \theta^-) d\mathcal{H}^2 \\ &\leq |H_0| \sqrt{\frac{1}{4} \int_M H^2(\theta^+ + \theta^-) d\mathcal{H}^2} \sqrt{\int_M (\theta^+ + \theta^-) d\mathcal{H}^2} = |H_0| \sqrt{\mathcal{W}(V)} \sqrt{m_0}. \end{aligned}$$

This gives the lower bound

$$F_{\text{CH}}(V) \geq 2\beta \left(\mathcal{W}(V) - |H_0| \sqrt{\mathcal{W}(V)} \sqrt{m_0} + \frac{H_0^2}{4} m_0 \right) + \gamma \int_M K(\theta^+ + \theta^-) d\mathcal{H}^2,$$

where the first term can be rewritten as

$$\mathcal{W}(V) - |H_0| \sqrt{\mathcal{W}(V)} \sqrt{m_0} + \frac{H_0^2}{4} m_0 = \left(\sqrt{\mathcal{W}(V)} - \frac{|H_0|}{2} \sqrt{m_0} \right)^2 = \left(\sqrt{\frac{\mathcal{W}(V)}{m_0}} - \frac{|H_0|}{2} \right)^2 m_0,$$

which completes the proof. $\square$

5.4 Canham-Helfrich flow of varifolds

We now turn to the evolutionary case and consider the gradient flow of the Canham-Helfrich energy in the setting of curvature varifolds as in [22] (Chapter 4). In Subsection 5.4.1, we show global existence of a solution using the GMMs approach. We discuss restricted flows in Subsection 5.4.2 and, finally, provide diameter bounds in Subsection 5.4.3.

To ease notations, the evolution will first be constrained to take place in a given compact set $Q \subset \subset \mathbb{R}^3$. By fixing the mass $m_0 > 0$ we let

$$\mathcal{V} := \{V \in \mathcal{M}(\mathbb{R}^3 \times \mathbb{S}^2) : \mu_V(\mathbb{R}^3) = V(\mathbb{R}^3 \times \mathbb{S}^2) = m_0, \text{supp}(V) \subset Q \times \mathbb{S}^2\}, \quad (5.14)$$

where we recall that $\mathcal{M}(\mathbb{R}^3 \times \mathbb{S}^2)$ indicates the set of Radon measures on $\mathbb{R}^3 \times \mathbb{S}^2$. We prescribe the class $\mathcal{A}$ of *admissible varifolds* by

$$\mathcal{A} := \{V \in \mathcal{V} : V \in AV_2^o(\mathbb{R}^3), \partial(q_\# V) = 0, \partial T_V = 0\}. \quad (5.15)$$

The class $\mathcal{A}$ is closed in $AV_2^o(\mathbb{R}^3)$ with respect to curvature varifold convergence. Indeed, let $V_n \in \mathcal{A}$ and $V \in AV_2^o(\mathbb{R}^3)$ be such that $V_n \rightharpoonup^* V$ in $AV_2^o(\mathbb{R}^3)$ as $n \rightarrow \infty$. One readily gets that $\mu_V(\mathbb{R}^3) = m_0$, $\text{supp}(\mu_V) \subset Q$. Moreover, curvature varifold convergence implies that

$\partial(q_{\#}V_n) \rightharpoonup^* \partial(q_{\#}V)$ in $\mathcal{M}^3(\mathbb{R}^3 \times G_{2,3})$, which with $\partial(q_{\#}V_n) = 0$ gives the condition $\partial(q_{\#}V) = 0$. As shown in [22, Lemma 4.1] (Chapter 4, Lemma 4.4.1), the limit V also satisfies $\partial T_V = 0$.

Let us note that the requirement $\partial T_V = 0$ in $\mathcal{A}$ could be relaxed to $\mathbb{M}_{\partial T_V}(\mathbb{R}^3) \leq m^c$ for some fixed $m^c \geq 0$, not affecting the analysis. We restrict ourselves to the choice $m^c = 0$, however, for it is well-adapted to the modeling of single-phase membranes without boundary. Moreover, it simplifies the notation and it will be useful in the case of multiply covered smooth surfaces (Section 5.5), where we can take advantage of Federer's Constancy Theorem 5.2.1.

In order to enforce the evolution in $\mathcal{A}$, we additionally constrain the energy by defining

$$G_{\text{CH}}: \mathcal{V} \rightarrow (-\infty, \infty], \quad G_{\text{CH}}(V) := \begin{cases} F_{\text{CH}}(V), & V \in \mathcal{A} \\ \infty, & \text{else.} \end{cases}$$

We now endow $\mathcal{V}$ with the Wasserstein metric W_p . To this aim, we fix $p \in [1, \infty)$ and follow the construction explained in Subsection 5.2.4 with $(X, d_X) = (\mathbb{R}^3 \times \mathbb{S}^2, d)$ where d is the metric defined as

$$d((x, \nu), (\tilde{x}, \tilde{\nu})) := |x - \tilde{x}| + |\nu - \tilde{\nu}|$$

(other equivalent metrics on $\mathbb{R}^3 \times \mathbb{S}^2$ could be considered as well). Note that $W_p(V, \tilde{V}) < \infty$ for all $V, \tilde{V} \in \mathcal{V}$, for the supports of μ_V and $\mu_{\tilde{V}}$ are contained in the bounded set Q . We have that

$$W_p^p(V, \tilde{V}) = \begin{cases} \min_{\lambda \in \Pi(V, \tilde{V})} \int_{(Q \times \mathbb{S}^2)^2} (|x - \tilde{x}| + |\nu - \tilde{\nu}|)^p d\lambda((x, \nu), (\tilde{x}, \tilde{\nu})) & \text{for } p > 1, \\ \sup \left\{ \int_{Q \times \mathbb{S}^2} f d(V - \tilde{V}) : f \in \text{Lip}_1(Q \times \mathbb{S}^2) \right\} & \text{for } p = 1. \end{cases}$$

The actual choice of p is irrelevant for the analysis and can be possibly adjusted to different modeling or computational needs.

Note that the Wasserstein metric $W_p(V, \tilde{V})$ controls the distance of the varifolds V and $\tilde{V}$ both in space position and in orientation. Such strong control is necessary, for two varifolds sharing the same support in $\mathbb{R}^3$ could still differ in their orientation. In the more regular case of multiply covered smooth surfaces, we will be able to consider a weaker metric as well, featuring just the Wasserstein distance of the supports of μ_V and $\mu_{\tilde{V}}$ in $\mathbb{R}^3$, see Section 5.5.

5.4.1 Generalized Minimizing Movements (GMMs)

We are now ready to introduce our concept of evolution. Given an *initial state* $V^0 \in \mathcal{V}$ with $G_{\text{CH}}(V^0) < \infty$ and a time step $\tau > 0$, we recursively define *discrete minimizers* $(V_\tau^n)_{n \in \mathbb{N}_0}$ by $V_\tau^0 := V^0$ and $V_\tau^n \in \arg\min_{\mathcal{V}} G_{\text{CH}, \tau}(\cdot; V_\tau^{n-1})$ for $n \in \mathbb{N}$, where

$$G_{\text{CH}, \tau}(\cdot; V_\tau^{n-1}): \mathcal{V} \rightarrow (-\infty, \infty]$$

is the *incremental functional*

$$G_{\text{CH}, \tau}(V; V_\tau^{n-1}) := G_{\text{CH}}(V) + \frac{1}{2\tau} W_p^2(V, V_\tau^{n-1}). \quad (5.16)$$

Define $\bar{V}_\tau: [0, \infty) \rightarrow \mathcal{V}$ to be the piecewise constant interpolant given by $\bar{V}_\tau(0) := V^0$ and $\bar{V}_\tau(t) := V_\tau^n$ for $t \in ((n-1)\tau, n\tau]$. The choice of the power 2 in the Wasserstein term above corresponds to the case of *gradient flows*. Replacing $W_p^2(V, V_\tau^{n-1})$ by $W_p^p(V, V_\tau^{n-1})$ would require just minor notational adaptations and would correspond to the case of *doubly nonlinear flows* instead [5].

We define our evolution notion as follows, see [5, Def. 2.0.6].

Definition 5.4.1 (Generalized Minimizing Movements). A curve $V: [0, \infty) \rightarrow \mathcal{V}$ is called a *Generalized Minimizing Movement (GMM)* for the varifold Canham-Helfrich flow if there exists a sequence $\tau_m \rightarrow 0$ as $m \rightarrow \infty$, such that

$$\bar{V}_{\tau_m}(t) \rightarrow V(t)$$

with respect to W_p for all $t \in [0, \infty)$.

A curve $V: [0, \infty) \rightarrow \mathcal{V}$ is said to be *locally L^2 -absolutely continuous* if there exists $w \in L^2_{\text{loc}}(0, \infty)$ such that

$$W_p(V(t_1), V(t_2)) \leq \int_{t_1}^{t_2} w(t) dt \quad (5.17)$$

for all $0 \leq t_1 < t_2 < \infty$. In this case, we write $V \in AC^2_{\text{loc}}([0, \infty); \mathcal{V})$. Given a locally L^2 -absolutely continuous curve V , the *metric derivative*

$$|V'| (t) := \lim_{s \rightarrow t} \frac{W_p(V(s), V(t))}{|s - t|}$$

can be proved to exist for almost every $t \geq 0$ [5, Ch. 1]. The function $[0, \infty) \ni t \mapsto |V'| (t)$ belongs to $L^2_{\text{loc}}(0, \infty)$ and is minimal among the functions w fulfilling (5.17).

We start by recording a first existence result.

Theorem 5.4.2 (Existence of GMMs). Assume (5.6) and let $V^0 \in \mathcal{A}$ with $G_{\text{CH}}(V^0) < \infty$. Then, there exists a GMM $V \in AC^2_{\text{loc}}([0, \infty); \mathcal{V})$ associated to G_{CH} starting from $V(0) = V^0$. Moreover, for all $t \geq 0$,

$$G_{\text{CH}}(V(t)) \leq G_{\text{CH}}(V^0) \quad \text{and} \quad V(t) \in \mathcal{A}.$$

Proof. The functional $G_{\text{CH}}: \mathcal{V} \rightarrow (-\infty, \infty]$ is defined on the complete metric space $(\mathcal{V}, W_p)$ with domain contained in $\mathcal{A}$. By assuming (5.6), namely $-6\beta/5 < \gamma < 0$, the Canham-Helfrich integrand $f_{\text{CH}}(\nu, \cdot)$ is strictly convex for all $\nu \in \mathbb{S}^2$ and the curvature bound (5.7)

$$\|A^{q\#}V\|_{L^2_{q\#}V(\mathbb{R}^3 \times G_{2,3})}^2 \leq c_1 (G_{\text{CH}}(V) + c_2 \mu_V(\mathbb{R}^3))$$

holds. Owing to the fact that the W_p -metric topology and the weak-* topology of varifolds are equivalent, we deduce from [22, Thm. 3.2] (Chapter 4, Thm. 4.3.2) that G_{CH} is lower semicontinuous in $(\mathcal{V}, W_p)$. On the other hand, Theorems 5.2.2 and 5.2.3 ensure that the sublevel sets of G_{CH} are compact. We are hence in the setting of [5, Prop. 2.2.3] and the existence of a GMM $V \in AC^2_{\text{loc}}([0, \infty); \mathcal{V})$ ensues for all $V^0 \in \mathcal{A}$ with $G_{\text{CH}}(V^0) < \infty$. Moreover, the solution V satisfies the energy inequality

$$G_{\text{CH}}(V(t)) + \frac{1}{2} \int_0^t |V'|^2(r) dr \leq G_{\text{CH}}(V^0)$$

for all $t \geq 0$ (cf. Lemma 2.2.11). In particular, we have $G_{\text{CH}}(V(t)) \leq G_{\text{CH}}(V^0) < \infty$, which in turn entails that $V(t) \in \mathcal{A}$. $\square$

By their definition, GMMs starting from a minimizer V^0 of the Canham-Helfrich functional are stationary, i.e.,

$$V(t) = V^0$$

for all $t \geq 0$ whenever $V^0 \in \operatorname{argmin}_{\mathcal{V}} G_{\text{CH}}$.

5.4.2 Restricted flows

In the following, we will be interested in considering GMMs in more restrictive settings. This can be easily accommodated within the above theory by simply constraining the functional G_{CH} to a closed subset of the admissible class $\mathcal{A}$. More precisely, we have the following corollary to Theorem 5.4.2.

Corollary 5.4.3 (Existence of restricted GMMs). *Assume (5.6) and let $\tilde{\mathcal{A}} \subset \mathcal{A}$ be closed with respect to the weak-* topology. Then, starting from $V^0 \in \tilde{\mathcal{A}}$ with $G_{\text{CH}}(V^0) < \infty$, there exists a GMMs solution associated to*

$$\tilde{G}_{\text{CH}}(V) := \begin{cases} G_{\text{CH}}(V), & V \in \tilde{\mathcal{A}}, \\ \infty, & \text{else} \end{cases}$$

with $V(t) \in \tilde{\mathcal{A}}$ for all $t \geq 0$.

The main application of Corollary 5.4.3 is that of ensuring the existence of GMMs in the regular setting of multiply covered smooth surfaces in Section 5.5. In the remainder of this subsection, we apply the corollary to demonstrate the possibility of including a volume constraint or enforcing symmetry.

Volume constraint

The modeling of biomembranes usually requires to constrain the minimization of the Canham-Helfrich energy to surfaces with fixed enclosed volume. This can be accomplished in the varifold setting as well.

We introduce the *enclosed volume* of an oriented varifold $V \in IV_2^o(\mathbb{R}^3)$ by

$$\text{encvol}(V) := \frac{1}{3} \int_{\mathbb{R}^3 \times \mathbb{S}^2} x \cdot \nu \, dV(x, \nu). \quad (5.18)$$

This definition is inspired by the smooth setting: if $V = V[M, \nu^M, 1, 0]$ such that $M = \partial E$ for a C^1 domain $E \subset \mathbb{R}^3$ with outer unit normal ν^M , as $\text{div}(x) = 3$ the divergence theorem yields

$$\text{encvol}(V) = \frac{1}{3} \int_M x \cdot \nu^M(x) \, d\mathcal{H}^2(x) = \frac{1}{3} \int_E \text{div}(x) \, dx = \int_E dx = |E| =: \text{vol}(M).$$

To include the volume constraint we restrict the varifold evolution to the set

$$\tilde{\mathcal{A}} := \{V \in \mathcal{A} : \text{encvol}(V) = v_0\}$$

for some fixed volume $v_0 > 0$. In order not to rule out classical smooth solutions by violating the isoperimetric constraint, we require $36\pi v_0^2 \leq m_0^3$. As the functional $V \mapsto \text{encvol}(V)$ from (5.18) is linear, the set $\tilde{\mathcal{A}}$ above is closed with respect to the weak-* convergence. Corollary 5.4.3 thus gives existence of restricted GMMs respecting a volume constraint.

Symmetry

Starting from a symmetric initial configuration V^0 , it is unclear if GMMs conserve symmetry. The lack of uniqueness of GMMs and the fact that minimizers of the Canham-Helfrich energy are not necessarily spheres suggest that symmetry conservation may not hold.

On the other hand, one can use Corollary 5.4.3 to enforce additionally symmetry by means of a constraint. Let us firstly discuss the case of *reflection symmetry*. Without loss of generality, we assume that we reflect about a plane $P \subset \mathbb{R}^3$ through the origin and that the compact $Q \subset \subset \mathbb{R}^3$ in (5.14) is symmetric with respect to P . The associated reflection mapping $\Psi: \mathbb{R}^3 \times \mathbb{S}^2 \rightarrow \mathbb{R}^3 \times \mathbb{S}^2$ is defined by

$$\Psi(x, \nu) = (Sx, S\nu) \quad \text{with} \quad S \in O(3), \quad \det S = -1.$$

The reflection of $V \in \mathcal{M}(\mathbb{R}^3 \times \mathbb{S}^2)$ is the push-forward measure $\Psi_\# V \in \mathcal{M}(\mathbb{R}^3 \times \mathbb{S}^2)$. As reflection is linear, the class of reflection-symmetric admissible varifolds,

$$\tilde{\mathcal{A}} := \{V \in \mathcal{A} : \Psi_\# V = V\},$$

is weakly-* closed in $\mathcal{A}$. One can hence apply Corollary 5.4.3 and obtain that, starting from a reflection-symmetric configuration $V_0 \in \tilde{\mathcal{A}}$, there exists a reflection-symmetric GMMs solution $V(t) \in \tilde{\mathcal{A}}$ for all $t \geq 0$.

The case of *rotational symmetry* around a fixed axis $(x_a, \nu_a) \in \mathbb{R}^3 \times \mathbb{S}^2$ can be treated analogously. Without loss of generality, we choose x_a to be the x_3 -axis in $\mathbb{R}^3$ and ν_a to be the north pole in $\mathbb{S}^2$, i.e., $\nu_a = (0, 0, 1)$. Moreover, we assume that Q is rotationally symmetric around x_a . Let $\Psi^\alpha: \mathbb{R}^3 \times \mathbb{S}^2 \rightarrow \mathbb{R}^3 \times \mathbb{S}^2$ given by

$$\Psi^\alpha(x, \nu) = (R^\alpha x, R^\alpha \nu) \quad \text{with} \quad R^\alpha \in SO(3)$$

denote the anticlockwise rotations about an angle $\alpha \in [0, 2\pi)$ around the axes x_a, ν_a respectively. A measure $V \in \mathcal{M}(\mathbb{R}^3 \times \mathbb{S}^2)$ is rotationally symmetric if for all $\alpha \in [0, 2\pi)$ there exists a rotation Ψ^α such that $\Psi^\alpha_\# V = V$. Any rotation $R^\alpha \in SO(3)$ can be represented as composition of two reflections, namely $R^\alpha = S_1 \circ S_2$ for $S_i \in O(3)$ with $\det S_i = -1$ ($i = 1, 2$), so that $\Psi^\alpha = \Psi_1 \circ \Psi_2$ for $\Psi_i(x, \nu) := (S_i x, S_i \nu)$, and hence the class of rotationally symmetric varifolds,

$$\tilde{\mathcal{A}} := \{V \in \mathcal{A} : \Psi^\alpha_\# V = V \text{ for all } \alpha \in [0, 2\pi)\},$$

can be readily checked to be weakly-* closed as well. Therefore, Corollary 5.4.3 applies, showing existence of rotationally symmetric GMMs, starting from a rotationally symmetric $V_0 \in \tilde{\mathcal{A}}$.

5.4.3 Diameter bounds

Although our notion of Canham-Helfrich flow is rather weak, we can show that solutions do not degenerate, for the GMMs of Definition 5.4.1 do not shrink to points nor expand indefinitely.

Proposition 5.4.4 (Diameter bounds). *Assume (5.6) and let $t \mapsto V(t)$ be a GMMs solution. Then, there exist constants $c_0, c^0 > 0$ depending only on the parameters V^0, m_0, β, γ , and H_0 , such that*

$$c_0 \leq \text{diam}(\text{supp } \mu_{V(t)}) \leq c^0 \quad \forall t \geq 0.$$

Proof. We start with the lower bound. By [22, Lemma 4.12] (Chapter 4, Lemma 4.4.12), an oriented curvature varifold $V \in AV_2^o(\mathbb{R}^3)$ satisfies

$$\text{diam}(\text{supp}(\mu_V)) \geq \frac{c \mu_V(\mathbb{R}^3)}{\sqrt{\mu_V(\mathbb{R}^3) 4\mathcal{W}(V) + \|\partial(q_\# V)\|_{\text{TV}}(\mathbb{R}^3 \times G_{2,3})}}$$

with $c > 0$. As $V \in \mathcal{A}$, we have $\mu_V(\mathbb{R}^3) = m_0$ and $\partial(q_\# V) = 0$, so that the right-hand side is bounded below by $\sqrt{\frac{m_0}{4\mathcal{W}(V(t))}}$. Moreover, $4\mathcal{W}(V(t)) \leq 3c_1(F_{\text{CH}}(V(t)) + c_2 m_0)$ by (5.10).

As a GMM satisfies $V(t) \in \mathcal{A}$ and $F_{\text{CH}}(V(t)) = G_{\text{CH}}(V(t)) \leq G_{\text{CH}}(V^0)$ for all $t \geq 0$ by the energy-decreasing property, this implies $4\mathcal{W}(V(t)) \leq 3c_1 (G_{\text{CH}}(V^0) + c_2 m_0)$. Consequently we obtain the lower bound

$$\text{diam}(\text{supp } \mu_{V(t)}) \geq \sqrt{\frac{m_0}{3c_1 (G_{\text{CH}}(V^0) + c_2 m_0)}} := c_0 > 0.$$

The starting point for the upper bound is the generalization to curvature varifolds without boundary of the estimate $\text{diam}(\text{supp } \mu_V) \leq \frac{2}{\pi} \sqrt{\mu_V(\mathbb{R}^3) \mathcal{W}(V)}$ [95, 98]. Then, similar arguments as for the lower bound show that

$$\text{diam}(\text{supp } \mu_{V(t)}) \leq \frac{1}{\pi} \sqrt{3m_0 c_1 (G_{\text{CH}}(V^0) + c_2 m_0)} =: c^0 > 0$$

for all $t \geq 0$, proving the claim. $\square$

Note in particular that the upper bound c^0 is independent of the choice of the bounding compact Q . This proves that restricting to Q is in fact not necessary. This allows us to omit Q in the sequel, i.e., we replace $\mathcal{V}$ introduced in (5.14) by $\mathcal{P}(\mathbb{R}^3 \times \mathbb{S}^2)$, see Subsection 5.2.4.

5.5 Canham-Helfrich flow of multiply covered regular surfaces

Let us now turn our attention to the case of multiply covered smooth surfaces, which amounts to constrain the evolution to the class of varifolds with smooth supports in $\mathbb{R}^3$. Such more regular setting allows for finer results. In fact, one can prove that the multiplicity is preserved along the evolution. On this basis, one has the possibility of considering a weaker setting where the Wasserstein distance between varifolds, which is rather strong, is replaced by the more natural Wasserstein distance between supports in space.

We start by introducing some notation in Subsection 5.5.1. Then, we discuss the reference case of multiply covered spheres in Subsection 5.5.2. In particular, we assess the minimality of multiply covered spheres in dependence of the given mass constraint m_0 and spontaneous curvature H_0 . We then present a Li-Yau-like bound on the multiplicity in terms of the Canham-Helfrich energy for general multiply covered surfaces in Subsection 5.5.3. In order to possibly consider a restricted flow in the spirit of Corollary 5.4.3 for multiply covered surfaces, we verify closedness of the corresponding classes in Subsection 5.5.4. An argument for multiplicity conservation during the flow is presented in Subsection 5.5.5. Eventually, we discuss the case of the weaker evolution setting given by the Wasserstein distance between the supports in Subsection 5.5.6 and we comment on some generalizations in Subsection 5.5.7.

5.5.1 Multiply covered surfaces

In the remainder of the paper, we restrict to varifolds $V = V[M, \nu, \theta^+, \theta^-]$ that are supported on closed (i.e., compact without boundary), connected $C^{1,1}$ surfaces $M \subset \mathbb{R}^3$, that is, embedded C^1 surfaces M with Lipschitz continuous unit normal vector $\nu := \nu^M: M \rightarrow \mathbb{S}^2$. In this setting, by Rademacher's theorem, $\bar{H}^{q\sharp V}$, $|A^{q\sharp V}|^2$, and $K^{q\sharp V}$ are defined almost everywhere on M .

By the admissibility condition of zero boundary current in the Definition (5.15) of $\mathcal{A}$, i.e., $\partial T_V = 0$, Federer's Constancy Theorem for smoothly supported currents (Theorem 5.2.1) entails that the density $\theta^+ - \theta^-$ of T_V is constant. This gives $\bar{\theta}_0 \in \mathbb{Z}$ such that $(\mathcal{H}^2 \llcorner M)$ -a.e.,

$$\theta^+ - \theta^- = \bar{\theta}_0.$$

In this setting, the Canham-Helfrich functional (5.12) reduces to

$$F_{\text{CH}}(V) = \int_M \left(\frac{\beta}{2} (H^2 + H_0^2) + \gamma K \right) (\theta^+ + \theta^-) d\mathcal{H}^2 - \beta \bar{\theta}_0 H_0 \int_M H d\mathcal{H}^2.$$

It follows that $F_{\text{CH}}(V)$ will be small for large $\bar{\theta}_0$, provided that $H_0 \int_M H d\mathcal{H}^2 > 0$. This favorable case occurs for example if H does not change sign (i.e., M is mean convex) and $H_0 H > 0$. Otherwise, if $H_0 \int_M H d\mathcal{H}^2 < 0$, a negative multiplicity $\bar{\theta}_0$ is favored, corresponding to an orientation flip.

This suggests to restrict with no loss of generality to multiplicities of the form

$$\theta^+ = \bar{\theta}_0 = k \in \mathbb{N} \quad \text{and} \quad \theta^- = 0.$$

In order to ensure the closure of this more regular class of configurations, we follow [34] and further restrict to surfaces of *uniform* $C^{1,1}$ regularity with given genus $g \in \mathbb{N}_0$. More precisely, fix $L > 0$ and a smooth connected closed surface Σ^g embedded in $\mathbb{R}^3$ and with genus $g \in \mathbb{N}_0$. Then, for $k \in \mathbb{N}$, we consider the subclass of admissible varifolds given by

$$\begin{aligned} \mathcal{A}_k^g := & \left\{ V = V[M, \nu, k, 0] \in \mathcal{A} : M = \psi(\Sigma^g) \text{ for an embedding } \psi \in C^{1,1}(\Sigma^g; \mathbb{R}^3) \right. \\ & \left. \text{with } \|\psi\|_{C^{1,1}(\Sigma^g)} \leq L, \nu \text{ is the outer unit normal vector to } M \right\}. \end{aligned} \quad (5.19)$$

We will also employ the notation

$$\mathcal{A}^g := \{V \in \mathcal{A} : \exists k \in \mathbb{N} \text{ such that } V \in \mathcal{A}_k^g\} \quad (5.20)$$

to indicate multiply covered surfaces of genus g . Recall that a differentiable map $\psi : \Sigma^g \rightarrow \mathbb{R}^3$ is an embedding if it is injective and $\nabla \psi$ has rank two. Moreover,

$$\|\psi\|_{C^{1,1}(\Sigma^g)} := \sup_{x \in \Sigma^g} \left(|\psi(x)| + |\nabla \psi(x)| + \sup_{x' \neq x \in \Sigma^g} \frac{|\nabla \psi(x) - \nabla \psi(x')|}{|x - x'|} \right).$$

The constraint $\|\psi\|_{C^{1,1}(\Sigma^g)} \leq L$ entails that elements $V = V[M, \nu, k, 0] \in \mathcal{A}_k^g$ have a uniform curvature bound of their spatial supports $M \subset \mathbb{R}^3$. Equivalently, the sets M satisfy a uniform ball condition or are sets of positive reach [34, 44].

For $V \in \mathcal{A}_k^g$ we have that

$$G_{\text{CH}}(V) = F_{\text{CH}}(V) = k E_{\text{CH}}(M),$$

with

$$E_{\text{CH}}(M) = \int_M \left(\frac{\beta}{2} (H - H_0)^2 + \gamma K \right) d\mathcal{H}^2 = \frac{\beta}{2} \int_M (H - H_0)^2 d\mathcal{H}^2 + 4\pi\gamma(1 - g)$$

by the Gauss-Bonnet Theorem. Note that the existence of smooth closed surfaces M of fixed genus $g \in \mathbb{N}$ minimizing E_{CH} can be proved in some parameter range [27, 34, 78, 90].

5.5.2 Stationarity of spheres

Before moving on, let us collect some remarks on the case where M is a sphere. We thus restrict to the class $\mathcal{A}_k^0$ in the following. As GMMs starting from a Canham-Helfrich energy minimizer

are stationary, we investigate under which conditions we have minimality of multiply covered spheres.

By $\mathbb{S}_R^2$ we denote a sphere with radius $R > 0$, centered at some $x_0 \in \mathbb{R}^3$, and equipped with its outer unit normal $\nu: \mathbb{S}_R^2 \rightarrow \mathbb{S}^2$. In particular, $\mathbb{S}_R^2$ has the constant mean curvature $H = -2/R$ and the constant Gauss curvature $K = 1/R^2$. A *multiply covered sphere* is an oriented varifold of the form

$$V_{\mathbb{S}_R^2} := V[\mathbb{S}_R^2, \nu, \theta_0, 0]$$

with multiplicity $\theta_0: \mathbb{S}_R^2 \rightarrow \mathbb{N}$. Its mass is given by $\mu_{V_{\mathbb{S}_R^2}}(\mathbb{R}^3) = \int_{\mathbb{S}_R^2} \theta_0 d\mathcal{H}^2 = m_0 > 0$ and its energy (5.3) takes the form

$$\begin{aligned} F_{\text{CH}}(V_{\mathbb{S}_R^2}) &= \frac{\beta}{2} \left(-\frac{2}{R} - H_0 \right)^2 m_0 + \frac{\gamma}{R^2} m_0 = 2\beta m_0 \left(\frac{1}{R} + \frac{H_0}{2} \right)^2 + \frac{\gamma}{R^2} m_0 \\ &= \left(\frac{2\beta + \gamma}{R^2} + \frac{2\beta H_0}{R} + \frac{\beta H_0^2}{2} \right) m_0 = 2\beta m_0 \left(\frac{1 + \frac{\gamma}{2\beta}}{R^2} + \frac{H_0}{R} + \frac{H_0^2}{4} \right). \end{aligned} \quad (5.21)$$

In case of a constant multiplicity $\theta_0(x) = k \in \mathbb{N}$ for a.e. $x \in \mathbb{S}_R^2$, the mass of $V_{\mathbb{S}_R^2}$ reads

$$\mu_{V_{\mathbb{S}_R^2}}(\mathbb{R}^3) = 4\pi k R^2 = m_0,$$

which relates the radius and the multiplicity. Consequently, for fixed mass, the only multiply covered sphere (up to isometries) with constant multiplicity $k \in \mathbb{N}$ is the *k-covered sphere*

$$S_k := V_{\mathbb{S}_{R_k}^2} = V[\mathbb{S}_{R_k}^2, \nu, k, 0] \quad \text{with} \quad R_k := \sqrt{\frac{m_0}{4\pi k}} = \frac{R_1}{\sqrt{k}}, \quad R_1 := \sqrt{\frac{m_0}{4\pi}}.$$

Since $\mathbb{S}_{R_k}^2$ is just a scaling of $\mathbb{S}^2$, we have $S_k \in \mathcal{A}_k^0$, at least for k small with respect to the uniform curvature bound L in (5.19). The energy of S_k is a function of the multiplicity k alone,

$$F_{\text{CH}}(S_k) = 2\beta m_0 \left(\frac{1 + \frac{\gamma}{2\beta}}{R_1^2} k + \frac{H_0}{R_1} \sqrt{k} + \frac{H_0^2}{4} \right). \quad (5.22)$$

For $\beta = 1/2$, $H_0 = 0$, and $\gamma = 0$, this reduces to the Willmore energy of a k -covered sphere, $\mathcal{W}(S_k) = \frac{1}{4} \left(\frac{2}{R_k} \right)^2 m_0 = 4\pi k$, which corresponds to equality in the Li-Yau estimate (5.9).

The following lemma shows that the k -covered sphere is the only Canham-Helfrich minimizer among multiply covered surfaces of genus zero, provided that $|H_0|$ is small enough.

Proposition 5.5.1 (Spheres are unique minimizers for fixed multiplicity). *Let $\beta > 0$, $\gamma \in \mathbb{R}$, and $k \in \mathbb{N}$ such that $S_k \in \mathcal{A}_k^0$. If $H_0 \leq 0$ and satisfies*

$$|H_0| \leq 2/R_k = \sqrt{16\pi k/m_0},$$

then S_k uniquely minimizes F_{CH} on $\mathcal{A}_k^0$. The statement is also true if $H_0 > 0$, if S_k is defined with the inner normal. In case $H_0 = 0$, S_k is a minimizer independently of its orientation.

Proof. Let $V \in \mathcal{A}_k^0$ be arbitrary. Starting from the lower bound (5.13) given in Lemma 5.3.1, we aim at deducing a new lower bound that is independent of V . Since $\theta^+ + \theta^- = k \in \mathbb{N}$ and M is a closed $C^{1,1}$ surface of genus $g = 0$, the Gauss-Bonnet Theorem applies:

$$\gamma \int_M K^{q\sharp V} (\theta^+ + \theta^-) d\mathcal{H}^2 = \gamma k \int_M K^{q\sharp V} d\mathcal{H}^2 = 4\pi \gamma k (1 - g) = 4\pi \gamma k. \quad (5.23)$$

Combining the assumption $\frac{|H_0|}{2} \leq \frac{1}{R_k} = \sqrt{\frac{4\pi k}{m_0}}$ with $\sqrt{\frac{\mathcal{W}(V)}{m_0}} \geq \sqrt{\frac{4\pi k}{m_0}}$, from the Li-Yau estimate (5.9), shows that

$$\sqrt{\frac{\mathcal{W}(V)}{m_0}} - \frac{|H_0|}{2} \geq \sqrt{\frac{4\pi k}{m_0}} - \frac{|H_0|}{2} \geq 0.$$

With (5.13) and (5.23), this gives

$$F_{\text{CH}}(V) \geq 2\beta m_0 \left(\sqrt{\frac{\mathcal{W}(V)}{m_0}} - \frac{|H_0|}{2} \right)^2 + \gamma \int_M K^{\sharp V} (\theta^+ + \theta^-) d\mathcal{H}^2 \geq 2\beta m_0 \left(\sqrt{\frac{4\pi k}{m_0}} - \frac{|H_0|}{2} \right)^2 + 4\pi \gamma k.$$

This lower bound is uniquely attained at S_k . In fact, by (5.21) and $H_0 \leq 0$ we get

$$F_{\text{CH}}(S_k) = 2\beta m_0 \left(\frac{1}{R_k} + \frac{H_0}{2} \right)^2 + \frac{\gamma}{R_k^2} m_0 = 2\beta m_0 \left(\sqrt{\frac{4\pi k}{m_0}} - \frac{|H_0|}{2} \right)^2 + 4\pi \gamma k.$$

This proves that S_k is the unique minimizer in $\mathcal{A}_k^0$, because equality in the Li-Yau estimate (5.9) holds only for spheres. Choosing ν as the inner normal just reverses the sign in front of H_0 and the assertion follows as above. Finally, we note that for $H_0 = 0$, the assumption $|H_0|/2 \leq 1/R_k$ is trivially satisfied. $\square$

Proposition 5.5.1 is an extension of an argument by DALPHIN [33, Chapter 5] to the multilayered setting. There, also counterexamples to sphere minimality are explicitly constructed in classes of axisymmetric surfaces (cigars or stomatocytes), if the bound on H_0 is violated. Of course, the same counterexamples apply to our multiply covered case.

Let us now drop the prescription of the multiplicity while still asking $0 \leq -H_0 \leq \sqrt{16\pi/m_0}$, so that any minimizer of F_{CH} in $\mathcal{A}^0$ (5.20) is necessarily a sphere. Our aim is then to determine which sphere is actually the absolute minimizer across multiplicities. We emphasize that due to the possibly nontrivial multiplicity, the Gauss-term (5.23) is variable as well and, unless $\gamma = 0$, can not be neglected in the optimization process. We have the following.

Proposition 5.5.2 (Optimal spheres). *Let $-2\beta < \gamma \leq 0$, $0 \leq -H_0 \leq \sqrt{16\pi/m_0}$, and define*

$$k_* := \frac{m_0 H_0^2}{16\pi} \frac{1}{(1 + \frac{\gamma}{2\beta})^2}, \quad (5.24)$$

$$Y_* := (\sqrt{\lfloor k_* \rfloor} - \sqrt{\lceil k_* \rceil}) H_0 - \sqrt{\frac{4\pi}{m_0}} \left(1 + \frac{\gamma}{2\beta} \right). \quad (5.25)$$

If $k_* \in \mathbb{N}$, then the unique minimizer of F_{CH} on $\mathcal{A}^0$ is given by S_{k_*} . If $k_* \in [0, 1]$ then the unique minimizer is S_1 . If $k_* \in (1, \infty) \setminus \mathbb{N}$, then the unique minimizer is $S_{\lfloor k_* \rfloor}$ for $Y_* < 0$ and $S_{\lceil k_* \rceil}$ for $Y_* > 0$. In case $k_* \in (1, \infty) \setminus \mathbb{N}$ and $Y_* = 0$, the two spheres $S_{\lfloor k_* \rfloor}$ and $S_{\lceil k_* \rceil}$ are the sole minimizers.

Before giving the proof of the proposition, let us note that the assumption $-2\beta < \gamma \leq 0$ implies $0 < 1 + \frac{\gamma}{2\beta} \leq 1$. Moreover, under the smallness condition on $|H_0|$ of the statement, one has that $k_* \leq (1 + \frac{\gamma}{2\beta})^{-2}$. In particular, the multiplicity of the optimal sphere is bounded above in terms of the parameters β and γ only. Yet, the determination of the actual minimizing sphere S_k via the sign of Y_* does depend on the parameters m_0 and H_0 as well.

Proof of Proposition 5.5.2. Looking back to (5.22), we have that $k \mapsto F_{\text{CH}}(S_k)$ is the restriction to $\mathbb{N}$ of the smooth, strictly convex, and unbounded function $f: (0, \infty) \rightarrow \mathbb{R}$ given by

$$f(r) := 2\beta m_0 \left(\frac{1 + \frac{\gamma}{2\beta}}{R_1^2} r + \frac{H_0}{R_1} \sqrt{r} + \frac{H_0^2}{4} \right).$$

As f is uniquely minimized at $r = k_*$, one has that $\mathbb{N} \ni k \mapsto F_{\text{CH}}(S_k)$ is minimized either in $k = k_*$ if $k_* \in \mathbb{N}$, in $k = 1$ if $0 \leq k_* \leq 1$, and otherwise in $k = \lfloor k_* \rfloor$, in $k = \lceil k_* \rceil$, or in both of the latter. These assertions follow directly by comparing $F_{\text{CH}}(S_{\lfloor k_* \rfloor})$ and $F_{\text{CH}}(S_{\lceil k_* \rceil})$ via the sign of $Y_* = \frac{R_1}{2\beta m_0} (F_{\text{CH}}(S_{\lfloor k_* \rfloor}) - F_{\text{CH}}(S_{\lceil k_* \rceil})) = \frac{R_1}{2\beta m_0} (f(\lfloor k_* \rfloor) - f(\lceil k_* \rceil))$, which by $R_1 = \sqrt{\frac{m_0}{4\pi}}$ and $\lfloor k_* \rfloor - \lceil k_* \rceil = -1$ takes the form (5.25). $\square$

Let us close this subsection by discussing the specific case of the single-covered sphere. We have the following.

Corollary 5.5.3 (Optimality of the single-covered sphere). *Let $-2\beta < \gamma \leq 0$ and*

$$0 \leq -H_0 < \sqrt{\frac{4\pi}{m_0}} \left(1 + \frac{\gamma}{2\beta}\right) (1 + \sqrt{2}).$$

Then, S_1 is the unique minimizer of F_{CH} on $\mathcal{A}^0$.

Indeed, under the condition on H_0 one readily checks that $-H_0 < \sqrt{16\pi/m_0}$ as well as that k_* from (5.24) and Y_* from (5.25) fulfill $k_* < 2$ and $Y_* < 0$, respectively. Hence, the assertion of Corollary 5.5.3 follows from Proposition 5.5.2.

5.5.3 Li-Yau estimate

We now derive a multiplicity bound in terms of the Canham-Helfrich energy, reminiscent of the Li-Yau inequality $4\pi k \leq \mathcal{W}(V)$ (5.9) for the Willmore energy of an integral varifold $V = V[M, k]$. The estimate below can be compared with the recent [90], where some alternative Li-Yau inequality is derived for the case $\gamma = 0$.

We restrict to varifolds V in the class $\mathcal{A}_k^g$ (5.19), for we make a crucial use of the Gauss-Bonnet Theorem. We have the following.

Proposition 5.5.4 (Multiplicity bound). *Let $V \in \mathcal{A}_k^g$, $H_0 \in \mathbb{R}$, $\beta > 0$, $\gamma < 0$, and assume that $-2\beta < \gamma(1 - g)$. Then, there exists a nondecreasing function $\bar{k}: \mathbb{R} \rightarrow \mathbb{N}$ depending on the parameters β , γ , and H_0 , but independent of the genus g , such that*

$$k \leq \bar{k}(F_{\text{CH}}(V)).$$

The proposition ensures that the multiplicity of any $V \in \mathcal{A}^g$ can be bounded in terms of its Canham-Helfrich energy, as long as $0 < -2\beta < \gamma(1 - g)$. Note that the latter holds for any genus $g \in \mathbb{N}_0$ if $-2\beta < \gamma < 0$ or (5.6) holds. In fact, with (5.6), i.e., $-\frac{6}{5}\beta < \gamma < 0$, we also have $-2\beta < \gamma < 0$, because $-2\beta = -\frac{5}{3}\frac{6}{5}\beta < \frac{5}{3}\gamma < \gamma < 0$. This immediately gives $-2\beta < \gamma(1 - g)$ in the cases $g = 0$ and $g = 1$, namely $-2\beta < \gamma$ and $-2\beta < 0$ respectively. If $g > 1$ then $1 - g < 0$, so that $\gamma < \gamma(1 - g)$ for $\gamma < 0$. Therefore $-2\beta < \gamma < 0$ implies $-2\beta < \gamma(1 - g)$ in this case as well.

Proof of Proposition 5.5.4. By Lemma 5.3.1, any oriented curvature varifold $V = V[M, \nu, \theta^+, \theta^-] \in \text{AV}_2^o(\mathbb{R}^3)$ with mass $\mu_V(\mathbb{R}^3) = m_0$ satisfies

$$F_{\text{CH}}(V) \geq 2\beta \left(\mathcal{W}(V) - |H_0| \sqrt{\mathcal{W}(V)} \sqrt{m_0} + \frac{H_0^2}{4} m_0 \right) + \gamma \int_M K(\theta^+ + \theta^-) d\mathcal{H}^2,$$

where the Gauss curvature $K = K^{q\sharp V}$ is given by (5.4). For $V \in \mathcal{A}_k^g$, we have $\theta^+ = k$, $\theta^- = 0$, and M is a $C^{1,1}$ surface, for which Gauss-Bonnet applies:

$$\int_M K(\theta^+ + \theta^-) d\mathcal{H}^2 = k \int_M K d\mathcal{H}^2 = 4\pi(1 - g)k.$$

By Young's inequality,

$$|H_0| \sqrt{\mathcal{W}(V)} \sqrt{m_0} \leq \varepsilon \mathcal{W}(V) + \frac{H_0^2 m_0}{4\varepsilon}$$

with $\varepsilon > 0$. With the classical Li-Yau inequality $\mathcal{W}(V) \geq 4\pi k$, the estimate then gives

$$\begin{aligned} F_{\text{CH}}(V) &\geq 2\beta \left((1-\varepsilon)\mathcal{W}(V) + \left(1 - \frac{1}{\varepsilon}\right) \frac{H_0^2 m_0}{4} \right) + 4\pi\gamma(1-g)k \\ &\geq 4\pi(2\beta(1-\varepsilon) + \gamma(1-g))k + \beta \left(1 - \frac{1}{\varepsilon}\right) \frac{H_0^2 m_0}{2}. \end{aligned}$$

Alternatively, one can deduce this lower bound from $F_{\text{CH}}(V) = kE_{\text{CH}}(M)$ and $k\mathcal{H}^2(M) = m_0$, starting from the analog of Lemma 5.3.1 in the classical case.

Thanks to the assumptions $\beta > 0$, $\gamma < 0$, and $2\beta + \gamma(1-g) > 0$, we can choose

$$0 < \varepsilon < \frac{1}{2\beta}(2\beta + \gamma(1-g)) = 1 + \frac{\gamma(1-g)}{2\beta} \begin{cases} = 1 + \frac{\gamma}{2\beta}, & g = 0, \\ = 1, & g = 1, \\ > 1, & g \geq 2, \end{cases}$$

such that $2\beta\varepsilon < 2\beta + \gamma(1-g)$. Then $2\beta(1-\varepsilon) + \gamma(1-g) = 2\beta + \gamma(1-g) - 2\beta\varepsilon > 0$ and we obtain the multiplicity bound

$$k \leq \frac{F_{\text{CH}}(V) - \beta \left(1 - \frac{1}{\varepsilon}\right) \frac{H_0^2 m_0}{2}}{4\pi(2\beta(1-\varepsilon) + \gamma(1-g))}.$$

With the choice $\varepsilon = \frac{1}{4\beta}(2\beta + \gamma(1-g))$ for $g = 0$ or 1 and $\varepsilon = 1$ for $g \geq 2$ we get

$$k \leq \begin{cases} \frac{1}{4\pi} \left(\frac{2}{2\beta+\gamma} F_{\text{CH}}(V) + \beta \frac{2\beta-\gamma}{(2\beta+\gamma)^2} H_0^2 m_0 \right) =: \bar{k}_0, & g = 0, \\ \frac{1}{4\pi} \left(\frac{1}{\beta} F_{\text{CH}}(V) + \frac{H_0^2 m_0}{2} \right) =: \bar{k}_1, & g = 1, \\ \frac{F_{\text{CH}}(V)}{4\pi\gamma(1-g)} \leq -\frac{F_{\text{CH}}(V)}{4\pi\gamma} =: \bar{k}_2, & g \geq 2. \end{cases}$$

The inequality in case $g \geq 2$ follows from $\gamma < 0$, by which $\gamma(1-g) \geq -\gamma$, and the fact that $F_{\text{CH}}(V) \geq 0$ in this case. Thus, the next largest integer of the maximum of these values gives the desired upper multiplicity bound that is increasing in $F_{\text{CH}}(V)$,

$$\bar{k}(F_{\text{CH}}(V)) := \lceil \max\{\bar{k}_0, \bar{k}_1, \bar{k}_2\} \rceil.$$

We note that $\bar{k}(F_{\text{CH}}(V))$, defining $\bar{k}: \mathbb{R} \rightarrow \mathbb{N}$, depends on β , γ , and H_0 but not on g . $\square$

5.5.4 Closedness of regular classes

We now check that Corollary 5.4.3 can be applied to the subclasses of admissible varifolds corresponding to multiply covered smooth surfaces. In particular, this amounts to proving that the classes $\mathcal{A}^g$ (5.20) and $\mathcal{A}_k^g$ (5.19) are closed in $\mathcal{A}$ (5.15). We have the following.

Lemma 5.5.5 (Closedness). *Let $(V_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ and $V \in \mathcal{A}$ with $V_n \rightharpoonup^* V$ as $n \rightarrow \infty$.*

- (i) *If $(V_n)_{n \in \mathbb{N}} \subset \mathcal{A}^g$ then $V \in \mathcal{A}^g$.*
- (ii) *Let $k \in \mathbb{N}$. If $(V_n)_{n \in \mathbb{N}} \subset \mathcal{A}_k^g$ then $V \in \mathcal{A}_k^g$.*

Proof. We first prove (ii). Let $V_n \in \mathcal{A}_k^g$ and $V \in \mathcal{A}$ be such that $V_n \rightharpoonup^* V$ as varifolds. By definition, $V_n = V_n[M_n, \nu^{M_n}, k, 0]$ and by compactness of the uniform $C^{1,1}$ surfaces M_n [34, 44], there exists a uniform $C^{1,1}$ surface M such that (a not relabeled subsequence satisfies) $M_n \rightarrow M$ ($n \rightarrow \infty$) in the Hausdorff distance. This gives rise to the varifold

$$\tilde{V} := \tilde{V}[M, \nu^M, k, 0] \in \mathcal{A}_k^g.$$

By uniqueness of varifold limits $\tilde{V}$ coincides with V (which also implies that the whole sequence M_n converges). This shows that $V \in \mathcal{A}_k^g$, completing the proof of (ii).

In order to show (i), we observe that by (5.20), elements $V_n \in \mathcal{A}^g$ are of the form $V_n = V_n[M_n, \nu^{M_n}, j_n, 0]$ for some $j_n \in \mathbb{N}$. The mass constraint implies that

$$j_n = \frac{m_0}{\mathcal{H}^2(M_n)}.$$

As before, $M_n \rightarrow M$ ($n \rightarrow \infty$) in the Hausdorff sense. The uniform $C^{1,1}$ regularity implies that oscillations are controlled and $\mathcal{H}^2(M_n) \rightarrow \mathcal{H}^2(M)$. Therefore, we have

$$j_n = \frac{m_0}{\mathcal{H}^2(M_n)} \rightarrow \frac{m_0}{\mathcal{H}^2(M)} =: j.$$

This gives rise to the varifold $\tilde{V} = \tilde{V}[M, \nu^M, j, 0]$, which must coincide with V , as in the proof of (ii), showing that $V \in \mathcal{A}^g$. $\square$

5.5.5 Conservation of multiplicity

Corollary 5.4.3 and Proposition 5.5.5 ensure that, starting from an initial state $V^0 \in \mathcal{A}_k^g$, there exists a restricted GMMs solution for the varifold Canham-Helfrich flow in $\mathcal{A}_k^g$. In this section, we prove a stronger fact, namely, that multiplicity is conserved by GMMs in $\mathcal{A}^g$, without the need to be constrained.

Theorem 5.5.6 (Conservation of multiplicity). *Assume (5.6) and let $V^0 \in \mathcal{A}_k^g$ satisfy $G_{\text{CH}}(V^0) < \infty$. Let V be a GMMs solution restricted to $\mathcal{A}^g$. Then*

$$V(t) \in \mathcal{A}_k^g$$

for all $t \geq 0$, i.e., the flow conserves multiplicity.

Proof. Let us start by remarking that, thanks to Proposition 5.5.4, any $V \in \mathcal{A}^g$ actually satisfies $V \in \mathcal{A}_k^g$ for some $k \leq \bar{k}(G_{\text{CH}}(V))$. Let V be a restricted GMMs solution in $\mathcal{A}^g$ with $V(0) = V^0$. We know by the closedness of $\mathcal{A}^g$ from Lemma 5.5.5 (i) and by Corollary 5.4.3 that such GMMs exist. Let $(V_m^n)_{m,n \in \mathbb{N}} \subset \mathcal{A}^g$ be a corresponding sequence of discrete minimizers with time step $\tau_m > 0$ for $m \in \mathbb{N}$ with $\tau_m \rightarrow 0$. Then $G_{\text{CH}}(V^0) \geq G_{\text{CH}}(V_m^n)$ and, since the function $\bar{k}$ from Proposition 5.5.4 is nondecreasing, it follows that

$$\bar{k}(G_{\text{CH}}(V_m^n)) \leq \bar{k}(G_{\text{CH}}(V^0)) =: \bar{k}_0.$$

Next, we show that $V_m^n \in \mathcal{A}_k^g$ by induction on $n \in \mathbb{N}$. Assume that $V_m^{n-1} \in \mathcal{A}_k^g$ and let by contradiction $V_m^n \in \mathcal{A}_j^g$ with $j \neq k$. Lemma 5.5.7 below ensures that

$$W_p(V_m^n, V_m^{n-1}) \geq \alpha(j, k) > 0. \quad (5.26)$$

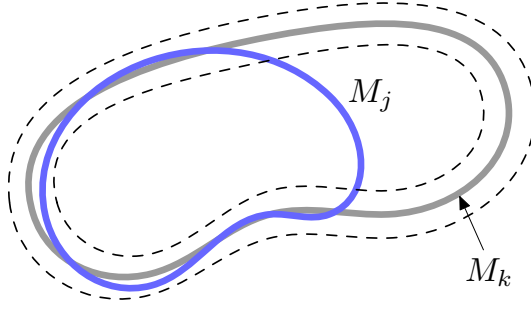


Figure 5.1: Illustration of the argument of Lemma 5.5.7 for $j > k$. The surface M_j (blue) must leave the tubular neighbourhood (dashed line) around the surface M_k (grey).

By defining $\bar{\alpha} := \min_{1 \leq j \neq k \leq \bar{k}_0} \alpha(j, k) > 0$ and $f_j := \inf_{\mathcal{A}_j^g} G_{\text{CH}}$ we hence have

$$G_{\text{CH}}(V^0) \geq G_{\text{CH}}(V_m^{n-1}) \geq G_{\text{CH}}(V_m^n) + \frac{1}{2\tau_m} W_p^2(V_m^n, V_m^{n-1}) \geq f_j + \frac{\bar{\alpha}^2}{2\tau_m}.$$

This leads to a contradiction as soon as the time step τ_m is small enough, namely, if

$$\tau_m < \frac{\bar{\alpha}^2}{2(G_{\text{CH}}(V^0) - f_j)}.$$

Therefore, all discrete minimizers must satisfy $V_m^n \in \mathcal{A}_k^g$ for τ_m small enough. Of course, the same holds true for the piecewise constant interpolants $\bar{V}_{\tau_m}(t)$ for all $t \geq 0$. Finally, the closedness of $\mathcal{A}_k^g$ from Proposition 5.5.5 (ii) shows that multiplicity k is preserved when taking the limit $\tau_m \rightarrow 0$ as well, i.e., $\bar{V}_{\tau_m}(t) \rightarrow V(t) \in \mathcal{A}_k^g$ for all $t \geq 0$, which completes the proof. $\square$

We are hence left with checking the following technical Lemma, which is used in the argument of Theorem 5.5.6 for formula (5.26). In particular, we show a uniform control from below of the Wasserstein distance between two multiply covered surfaces with different multiplicities.

Lemma 5.5.7 (Distance threshold). *For all $j, k \in \mathbb{N}$ with $j \neq k$ there exists $\alpha = \alpha(j, k) > 0$ such that*

$$W_p(V_j, V_k) \geq \alpha$$

for all $V_j \in \mathcal{A}_j^g$ and $V_k \in \mathcal{A}_k^g$.

Proof. As the statement is symmetric in j and k , with no loss of generality we may assume that $j > k$. We proceed by contradiction and assume that for all $\delta > 0$ there exist $V_j \in \mathcal{A}_j^g$ and $V_k \in \mathcal{A}_k^g$ with

$$W_p(V_j, V_k) < \delta^2. \quad (5.27)$$

We show that condition (5.27) leads to a contradiction if δ is chosen to be small enough. More precisely, we prove that one can find $0 < \delta_* < 1$ depending only on k and j , as well as on the data m_0 , L , and p , such that for any $0 < \delta < \delta_*$ no varifolds $V_j \in \mathcal{A}_j^g$ and $V_k \in \mathcal{A}_k^g$ exist such that (5.27) holds.

Along the proof, we use the notation $V_j = V[M_j, \tilde{\nu}, j, 0]$ and $V_k = V[M_k, \nu, k, 0]$ for varifolds in $\mathcal{A}_j^g$ and $\mathcal{A}_k^g$, respectively. Moreover, we indicate by $C_i > 0$, $i = 0, \dots, 7$, some constants depending solely on the data, in particular possibly on L . To start with, let us note that there

exists $C_0 > 0$ such that any $V \in \mathcal{A}^g$ is such that the corresponding spatial support M in $\mathbb{R}^3$ fulfills the lower bound

$$\text{vol}(M) \geq C_0,$$

where $\text{vol}(M)$ indicates the volume of the region which is enclosed by M , cf. Subsection 5.4.2.

We subdivide the proof into three steps. At first, we show that any surface which is completely contained in a sufficiently small neighborhood of M_k must have at least the area of M_k , up to a small error. Since

$$\mathcal{H}^2(M_j) < \mathcal{H}^2(M_k)$$

by $j > k$ and by the mass constraint, this implies that M_j cannot be completely contained in such a small neighborhood of M_k , see Figure 5.1. In the second step, we give a lower bound on the area of M_j which is not contained in the small neighborhood of M_k . Eventually, in Step 3 we show that such a lower bound contradicts condition (5.27).

Step 1: A lower bound on the area. We start by checking that one can find $\delta_0 > 0$ and $C_1 > 0$ such that

$$\begin{aligned} \forall 0 < \delta < \delta_0, \forall V_k \in \mathcal{A}_k^g, \forall E \subset \mathbb{R}^3 \text{ of finite perimeter with } |E| \geq C_0 : \\ \partial^* E \subset M_k^\delta := M_k + B_\delta(0) \Rightarrow \mathcal{H}^2(\partial^* E) \geq \mathcal{H}^2(M_k) - C_1 \delta, \end{aligned} \quad (5.28)$$

where we recall the notation for the open ball $B_\delta(0) = \{x \in \mathbb{R}^3 : |x| < \delta\}$.

Let us start by choosing $\delta_1 > 0$ so small that the projection π_{M_k} onto M_k is well-defined on $M_k^{\delta_1}$, for any $V_k \in \mathcal{A}_k^g$. As M_k is a set of positive reach, such a δ_1 exists and depends on L only. Then, one can find $C_2 > 0$ such that

$$\text{vol}(M_k^\delta) \leq C_2 \delta \mathcal{H}^2(M_k) \quad \forall 0 < \delta < \delta_1. \quad (5.29)$$

Note that C_2 depends on L and that the latter inequality cannot be guaranteed if the uniform $C^{1,1}$ regularity of M_k is dropped. Next we choose $0 < \delta_0 < \min\{C_0 k / (C_2 m_0), \delta_1\}$, which implies that

$$\partial^* E \subset M_k^{\delta_0} \Rightarrow \pi_{M_k}(\partial^* E) \equiv M_k.$$

Indeed, if $\pi_{M_k}(\partial^* E) \neq M_k$, the fact that $\partial^* E \subset M_k^{\delta_0}$ would imply that the whole E is contained in $M_k^{\delta_0}$. In combination with inequality (5.29) this would require that

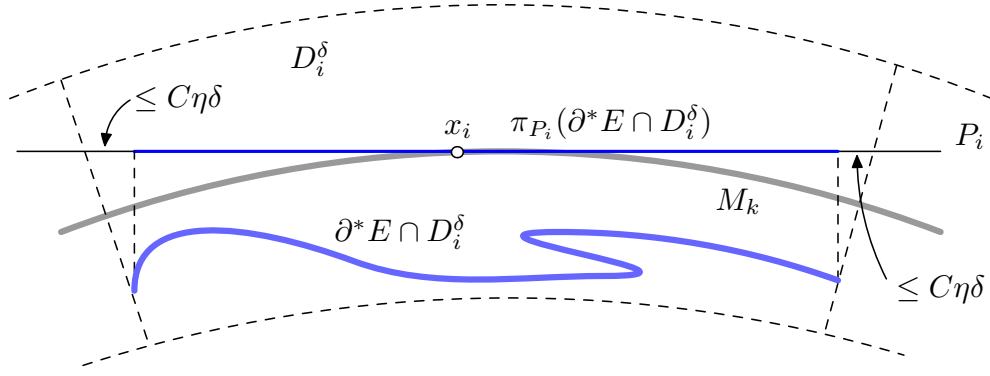
$$C_0 \leq |E| \leq \text{vol}(M_k^{\delta_0}) \leq C_2 \delta_0 \mathcal{H}^2(M_k) = C_2 \delta_0 \frac{m_0}{k} < C_0,$$

which is a contradiction.

Let $\xi: U \rightarrow \Sigma^g$ be a fixed smooth a.e. injective parametrization of $\Sigma^g \subset \mathbb{R}^3$ with $U = [0, 1]^2$. We can cover U up to a negligible set by the union of disjoint open sets $\{U_i\}_{i=1}^{N_\eta}$ on which ξ is injective, each having diameter smaller than some given $\eta > 0$, to be fixed below. Note that if η is small enough, such a covering can be chosen in such a way that $N_\eta \leq 2/\eta^2$ (take disjoint open squares of side η , where we have $N_\eta \leq \lceil 1/\eta \rceil^2$, hence $N_\eta \leq 2/\eta^2$ for $0 < \eta < (\sqrt{5} - 1)/2$).

Given $V_k \in \mathcal{A}_k^g$ with $M_k = \psi(\Sigma^g)$ as in (5.19), define $\phi := \psi \circ \xi: U \rightarrow M_k$ and recall that the normal at $x \in M_k$ is indicated by $\nu(x)$. We let $u_i \in U_i$ be fixed and define $P_i \subset \mathbb{R}^3$ to be the tangent plane to M_k at the point $x_i = \phi(u_i)$. Moreover, for all $0 < \delta < \delta_0$ we define the δ -neighborhood of $\phi(U_i)$ in the normal direction, namely,

$$D_i^\delta := \{\tilde{x} = x + h \nu(x) \in \mathbb{R}^3 : x \in \phi(U_i), |h| < \delta\},$$


 Figure 5.2: The notation of *Step 1* in the proof of Lemma 5.5.7.

see Figure 5.2. We observe that

$$\mathcal{H}^2(\partial^* E \cap D_i^\delta) \geq \mathcal{H}^2(\pi_{P_i}(\partial^* E \cap D_i^\delta)), \quad (5.30)$$

for orthogonal projections to a plane decrease the perimeter. Next, we bound $\mathcal{H}^2(\pi_{P_i}(\partial^* E \cap D_i^\delta))$ by the area of $P_i \cap D_i^\delta$, up to a small error. More precisely, we exploit the Lipschitz continuity of $x \mapsto \nu(x)$ in order to get that

$$\sup_{y \in P_i \cap D_i^\delta} \inf_{x \in \pi_{P_i}(\partial^* E \cap D_i^\delta)} |x - y| \leq C_3 \eta \delta, \quad (5.31)$$

where $C_3 > 0$ depends on L only. By possibly choosing another constant $C_4 > 0$, again depending on L , we can check that the length of the boundary $\partial(P_i \cap D_i^\delta) \subset P_i$ is bounded by $C_4 \eta$. Inequality (5.31) hence implies that

$$\mathcal{H}^2(\pi_{P_i}(\partial^* E \cap D_i^\delta)) \geq \mathcal{H}^2(P_i \cap D_i^\delta) - C_5 \eta^2 \delta, \quad (5.32)$$

where $C_5 > 0$ is just depending on L . Eventually, one can estimate the area of $P_i \cap D_i^\delta$ via that of $\phi(U_i)$. In fact, the surface $P_i \cap D_i^\delta$ can be parametrized over U_i via

$$\tilde{\phi}(u) = \phi(u) + h(u)\nu(\phi(u))$$

where the C^1 function $h: U_i \rightarrow (-\delta, \delta)$ is uniquely determined by asking $P_i \cap D_i^\delta = \tilde{\phi}(U_i)$. Owing to the smoothness of M_k we can show that the infinitesimal area elements on $\phi(U_i)$ and $P_i \cap D_i^\delta$ are comparable. For this purpose, we introduce the short-hand notation

$$a := \partial_1 \phi \times \partial_2 \phi: U_i \rightarrow \mathbb{R}^3 \quad \text{and} \quad \tilde{a} := \partial_1 \tilde{\phi} \times \partial_2 \tilde{\phi}: U_i \rightarrow \mathbb{R}^3.$$

As by definition $\tilde{a} = (\partial_1 \phi + (\partial_1 h)\nu + h(\nabla \nu)\partial_1 \phi) \times (\partial_2 \phi + (\partial_2 h)\nu + h(\nabla \nu)\partial_2 \phi)$, we get

$$a \cdot \tilde{a} = |a|^2 + b \cdot a$$

for $b := \partial_1 \phi \times h(\nabla \nu)\partial_2 \phi + h(\nabla \nu)\partial_1 \phi \times \partial_2 \phi + h^2(\nabla \nu)\partial_1 \phi \times (\nabla \nu)\partial_2 \phi$. We can hence find $C_6 > 0$ such that $|\tilde{a}| \leq C_6$ and $|b| \leq C_6 \delta$. As $x \mapsto \frac{a(x)}{|a(x)|} = \nu(x)$ is Lipschitz continuous we get $|\frac{a}{|a|} - \frac{\tilde{a}}{|\tilde{a}|}| \leq C_7 \eta$. The latter in particular entails that

$$C_7 \eta \geq \left| \left(\frac{a}{|a|} - \frac{\tilde{a}}{|\tilde{a}|} \right) \cdot \frac{\tilde{a}}{|\tilde{a}|} \right| = \left| \frac{a \cdot \tilde{a}}{|a| |\tilde{a}|} - 1 \right| = \left| \frac{|a|^2 + b \cdot a}{|a| |\tilde{a}|} - 1 \right|.$$

By multiplying by $|\tilde{a}|$ we obtain $\left| |a| - |\tilde{a}| + \frac{b \cdot a}{|a|} \right| \leq C_7 \eta |\tilde{a}|$. Eventually, by using the bounds $|\tilde{a}| \leq C_6$ and $|b| \leq C_6 \delta$, we conclude that

$$||\partial_1 \phi \times \partial_2 \phi| - |\partial_1 \tilde{\phi} \times \partial_2 \tilde{\phi}|| \leq C_6 \delta + C_6 C_7 \eta.$$

This implies

$$\begin{aligned} \mathcal{H}^2(P_i \cap D_i^\delta) &= \int_{U_i} |\partial_1 \tilde{\phi} \times \partial_2 \tilde{\phi}| \geq \int_{U_i} (|\partial_1 \phi \times \partial_2 \phi| - (C_6 \delta + C_6 C_7 \eta)) \\ &= \mathcal{H}^2(\phi(U_i)) - (C_6 \delta + C_6 C_7 \eta) \mathcal{H}^2(U_i). \end{aligned} \quad (5.33)$$

By combining inequalities (5.30), (5.32)-(5.33), and summing for $i = 1, \dots, N_\eta$, we get

$$\begin{aligned} \mathcal{H}^2(\partial^* E) &\geq \mathcal{H}^2(M_k) - C_5 \eta^2 \delta N_\eta - (C_6 \delta + C_6 C_7 \eta) \mathcal{H}^2(U) \\ &\geq \mathcal{H}^2(M_k) - 2C_5 \delta - (C_6 \delta + C_6 C_7 \eta). \end{aligned}$$

It hence suffices to let $\eta = \delta$ and define $C_1 := 2C_5 + C_6 + C_6 C_7$ to obtain (5.28).

Step 2: Uniform bound of the excess area. Given $V_j \in \mathcal{A}_j^g$, we apply (5.28) to $E \subset \mathbb{R}^3$ with $\partial E = M_j$. Consequently, if $M_j \subset M_k^\delta$, then

$$\frac{m_0}{j} = \mathcal{H}^2(M_j) \geq \mathcal{H}^2(M_k) - C_1 \delta > \frac{m_0}{k} - C_1 \delta_0$$

is true for any $0 < \delta < \delta_0$. Recalling $j > k$, this leads to a contradiction as soon as

$$C_1 \delta_0 < \frac{m_0}{k} - \frac{m_0}{j}. \quad (5.34)$$

Under the latter assumption, we have hence proved that $M_j \not\subset M_k^\delta$ for any $\delta < \delta_0$ and any $V_j \in \mathcal{A}_j^g$ and $V_k \in \mathcal{A}_k^g$.

We now aim at refining this argument by showing that, for δ small enough, the area of the portion of M_j which is not contained in M_k^δ can be uniformly bounded from below. More precisely, given any bounded open set $A \subset \mathbb{R}^3$, we show that there exists $0 < \delta_* < \delta_0$ such that

$$\forall V_k \in \mathcal{A}_k^g, \forall E \subset \mathbb{R}^3 \text{ with } \text{Per}(E; A) \leq \frac{m_0}{j}, |E| \geq C_0 : \quad \text{Per}(E; (\overline{M_k^\delta})^c) \geq \delta_*. \quad (5.35)$$

Statement (5.35) can be checked by contradiction: assume that for all $0 < \delta < \delta_0$ there exists $E_\delta \subset \mathbb{R}^3$ with $\text{Per}(E_\delta; A) \leq m_0/j$, $|E_\delta| \geq C_0$, and $\text{Per}(E_\delta; (\overline{M_k^\delta})^c) < \delta$. By compactness in the class of finite-perimeter sets we find a not relabeled subsequence $(E_\delta)_\delta$ and a finite perimeter set $E \subset \mathbb{R}^3$ such that, as $\delta \rightarrow 0$, the characteristic functions 1_{E_δ} converge to 1_E in $L^1(\mathbb{R}^3)$ and the Radon measures $\mu_\delta := -D1_{E_\delta}$ fulfill $\mu_\delta \rightharpoonup^* \mu = -D1_E$. This implies that $\text{vol}(E) \geq C_0$ and $\text{Per}(E; A) \leq m_0/j$, as well. Moreover, for all $0 < \bar{\delta} < \delta_0$ we have that

$$\text{Per}(E; (\overline{M_k^{\bar{\delta}}})^c) \leq \liminf_{\delta \rightarrow 0} \text{Per}(E_\delta; (\overline{M_k^\delta})^c) \leq \liminf_{\delta \rightarrow 0} \text{Per}(E_\delta; (\overline{M_k^\delta})^c) = 0.$$

This shows that $\partial^* E \subset M_k^{\bar{\delta}}$. By applying estimate (5.28) we deduce that $\text{Per}(E; M_k^{\delta_1}) \geq \mathcal{H}^2(M_k) - C_1 \bar{\delta}$ which leads to a contradiction as

$$\frac{m_0}{j} \geq \text{Per}(E; M_k^{\delta_1}) \geq \mathcal{H}^2(M_k) - C_1 \bar{\delta} = \frac{m_0}{k} - C_1 \bar{\delta} \stackrel{(5.34)}{>} \frac{m_0}{j}.$$

This proves that there exists $0 < \delta_* < \delta_1$ such that (5.35) holds. In particular, for all $V_j \in \mathcal{A}_j^g$ and $V_k \in \mathcal{A}_k^g$ we have that

$$\mathcal{H}^2(M_j \cap (\overline{M_k^{\delta_*}})^c) \geq \delta_*. \quad (5.36)$$

Step 3: Conclusion of the proof. Let $V_j \in \mathcal{A}_j^g$ and $V_k \in \mathcal{A}_k^g$ fulfill condition (5.27) for some $\delta < \delta_*$. Denote by $\lambda \in \Pi(V_k, V_j) \subset \mathcal{M}((\mathbb{R}^3 \times \mathbb{S}^2)^2)$ an optimal transport plan between V_k and V_j , i.e.,

$$W_p^p(V_j, V_k) = \int_{(\mathbb{R}^3 \times \mathbb{S}^2)^2} d^p((x, \nu), (\tilde{x}, \tilde{\nu})) \, d\lambda((x, \nu), (\tilde{x}, \tilde{\nu})),$$

where we recall that $d((x, \nu), (\tilde{x}, \tilde{\nu})) = |x - \tilde{x}| + |\nu - \tilde{\nu}|$. Consider the δ -neighborhood N^δ of the diagonal in $(\mathbb{R}^3 \times \mathbb{S}^2)^2$ given by

$$N^\delta := \{(y, \tilde{y}) \in (\mathbb{R}^3 \times \mathbb{S}^2)^2 : d(y, \tilde{y}) < \delta\}.$$

As $1 \leq \delta^{-p} d^p(y, \tilde{y})$ for all $(y, \tilde{y}) \in (N^\delta)^c$ we can estimate the measure of the complement $(N^\delta)^c$ by a classical Chebyshev-type argument:

$$\lambda((N^\delta)^c) = \int_{(N^\delta)^c} d\lambda \leq \delta^{-p} \int_{(\mathbb{R}^3 \times \mathbb{S}^2)^2} d^p(y, \tilde{y}) \, d\lambda(\tilde{y}, y) = \delta^{-p} W_p^p(V_j, V_k) \stackrel{(5.27)}{<} \delta^p. \quad (5.37)$$

Next we observe that $\tilde{x} \in M_j \cap (\overline{M_k^\delta})^c$ implies $|\tilde{x} - x| \geq \delta$ and thus $((x, \nu(x)), (\tilde{x}, \tilde{\nu}(\tilde{x}))) \in (N^\delta)^c$ for all $x \in M_k$. With

$$Z := \{((x, \nu(x)), (\tilde{x}, \tilde{\nu}(\tilde{x}))) \in (\mathbb{R}^3 \times \mathbb{S}^2)^2 : x \in M_k, \tilde{x} \in M_j \cap (\overline{M_k^\delta})^c\} \subset (N^\delta)^c$$

we hence have that

$$\mathcal{H}^2(M_j \cap (\overline{M_k^\delta})^c) \leq \lambda(Z) \leq \lambda((N^\delta)^c) \stackrel{(5.37)}{<} \delta^p. \quad (5.38)$$

Eventually, we combine inequalities (5.36) and (5.38) to get

$$\delta \geq \delta^p \stackrel{(5.38)}{>} \mathcal{H}^2(M_j \cap (\overline{M_k^\delta})^c) \geq \mathcal{H}^2(M_j \cap (\overline{M_k^{\delta_*}})^c) \stackrel{(5.36)}{\geq} \delta_* > \delta$$

which is a contradiction. $\square$

5.5.6 Existence of GMMs under a weaker metric

In the regular case, one can prove the existence of GMMs for the Canham-Helfrich functional under the weaker metric given by the Wasserstein distance of the spatial supports. Indeed, let $V^0, V^1 \in \mathcal{P}(\mathbb{R}^3 \times \mathbb{S}^2)$ and set

$$\mu^0 := \mu_{V^0}, \quad \mu^1 := \mu_{V^1} \in \mathcal{P}(\mathbb{R}^3)$$

Then, one can prove that the varifold Wasserstein distance, $W_p(V^1, V^0)$, controls the Wasserstein distance between spatial Radon measures, $\overline{W}_p(\mu^1, \mu^0)$, in the following sense

$$W_p(V^1, V^0) \geq \overline{W}_p(\mu^1, \mu^0).$$

Indeed, with $\lambda \in \Pi(V^1, V^0)$ realizing the minimum in the definition of $W_p(V^1, V^0)$, we get

$$\begin{aligned}
 W_p^p(V^1, V^0) &= \int_{(\mathbb{R}^3 \times \mathbb{S}^2)^2} (|x - \tilde{x}| + |\nu - \tilde{\nu}|)^p d\lambda((x, \nu), (\tilde{x}, \tilde{\nu})) \\
 &= \int_{(\mathbb{R}^3 \times \mathbb{S}^2)^2} |x - \tilde{x}|^p d\lambda((x, \nu), (\tilde{x}, \tilde{\nu})) + \underbrace{\sum_{k=1}^p \binom{p}{k} \int_{(\mathbb{R}^3 \times \mathbb{S}^2)^2} |x - \tilde{x}|^{p-k} |\nu - \tilde{\nu}|^k d\lambda((x, \nu), (\tilde{x}, \tilde{\nu}))}_{\geq 0} \\
 &\geq \int_{\mathbb{R}^3 \times \mathbb{R}^3} |x - \tilde{x}|^p d(\pi_{\#}^{\mathbb{R}^3} \lambda)(x, \tilde{x}) \underbrace{\int_{\mathbb{S}^2 \times \mathbb{S}^2} d(\pi_{\#}^{\mathbb{S}^2} \lambda)(\nu, \tilde{\nu})}_{=1} \geq \overline{W}_p^p(\mu^1, \mu^0),
 \end{aligned}$$

as the $\mathbb{R}^3 \times \mathbb{S}^2$ -marginals V^0 and V^1 of λ are oriented varifolds of the form (5.11) and the $\mathbb{R}^3 \times \mathbb{R}^3$ -marginal $\pi_{\#}^{\mathbb{R}^3} \lambda \in \mathcal{M}(\mathbb{R}^3 \times \mathbb{R}^3)$ is a possible choice in $\Pi(\mu^1, \mu^0)$.

On the other hand, as soon as multiplicity is fixed to some $k \in \mathbb{N}$ and we restrict to regular varifolds $V^1, V^0 \in \mathcal{A}_k^g$ given by (5.19) which have fixed orientation, we have that

$$\overline{W}_p(\mu^1, \mu^0) = 0 \implies W_p(V^1, V^0) = 0. \quad (5.39)$$

Indeed, since $\mu^0 = k \mathcal{H}^2 \llcorner M_0$ and $\mu^1 = k \mathcal{H}^2 \llcorner M_1$ with regular surfaces M_0 and M_1 respectively, $\overline{W}_p(\mu^1, \mu^0) = 0$ implies that $M_0 = M_1 =: M$ and $W_p(V^1, V^0) = 0$ then follows from the equality $V^0 = V[M, \nu^M, k, 0] = V^1$.

The nondegeneracy condition (5.39) is instrumental in proving the existence of GMMs, because it qualifies $\overline{W}_p$ as a distance on $\mathcal{A}_k^g$. Correspondingly, the existence result from [5, Prop. 2.2.3] can still be applied to prove an existence result analogous to Theorem 5.4.2. However, note that condition (5.39) hinges on the conservation of multiplicity and that the proof of Theorem 5.5.6, in particular the argument of Lemma 5.5.7, requires the control of the stronger distance $W_p(V^1, V^0)$. This amounts to say that, effectively, we can resort to the weaker metric $\overline{W}_p(\mu^1, \mu^0)$ just in the frame of GMMs restricted a priori to $\mathcal{A}^g$, namely, multiply covered uniformly regular $C^{1,1}$ surfaces.

5.5.7 More general curvature functionals

Most results of this section can also be established for a larger class of geometric functionals. In particular, we could prove the existence of GMMs for more general curvature energies of the form

$$D(M) := \int_M f(x, \nu(x), H(x), K(x)) d\mathcal{H}^2(x).$$

Here, M is a uniformly regular $C^{1,1}$ surface in $\mathbb{R}^3$, as in (5.19), and the integrand f is continuous and convex in its last two arguments. This class of curvature functionals has already been considered in [34] concerning equilibrium problems. In particular, it has been verified there that D admits minimizers in $\mathcal{A}_1^g$ for any fixed genus $g \in \mathbb{N}$, also under a volume constraint.

The analysis in [34] hinges upon the compactness of $\mathcal{A}_1^g$ and on the lower semicontinuity of D . These same tools ensure the existence of a GMM for D with respect to the Wasserstein metric W_p . The adjustments required are mainly notational. The only point deserving some attention is the a-priori multiplicity bound in terms of the functional, for a Li-Yau inequality may fail in degenerate cases. Still, a bound on the multiplicity can be directly deduced from the uniform $C^{1,1}$ regularity, i.e., depending on the constant L in the definition (5.19) of $\mathcal{A}_k^g$. By assuming

such a-priori bound on the multiplicity, the argument of Theorem 5.5.6 still holds. In particular, multiplicity is conserved along the evolution. Moreover, in the weaker metric setting given by the Wasserstein distance $\overline{W}_p$ on the spatial supports, one can still prove existence of a restricted GMMs solution to $\mathcal{A}_k^g$ by constraining the multiplicity.

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Zusammenfassung

Biologische Membranen besitzen eine erstaunliche natürliche Formenvielfalt. Das bekannteste Beispiel ist die bikonkave Gleichgewichtskonfiguration von roten Blutkörperchen. Im Rahmen der Kontinuumsmechanik können Biomembranen als Flächen minimaler elastischer Biegun­gsenergie beschrieben werden. Bei dieser sogenannten Canham-Helfrich-Energie handelt es sich um ein quadratisches Krümmungsfunktional, welches die Willmore-Energie verallgemeinert, indem unterschiedliche Biegungssteifigkeiten und die spontane Krümmung berücksichtigt werden. Die Minimierung erfolgt unter Bedingungen an den Flächeninhalt und an das eingeschlossene Volumen der Membran.

Diese Doktorarbeit konzentriert sich auf zwei verschiedene Aspekte von auf der Canham-Helfrich-Energie basierenden variationellen Modellen für Biomembranen. Zunächst behandeln wir heterogene Membranen, wobei unser Interesse dem Einfluss der veränderlichen Materialparameter auf die Existenz von Minimierern und deren Eigenschaften gilt. Hier präsentieren wir Resultate für ein eindimensionales Canham-Helfrich-Modell für ebene elastische Kurven, deren Steifigkeit von einer Phasendichte abhängt. Dann betrachten wir Mehrphasenmembranen mit scharfen Phasengrenzen und zeigen die Existenz von Energie-minimierenden Krümmungsvarifaltigkeiten, einem Konzept aus der geometrischen Maßtheorie. Das zweite Thema der Dissertation ist die Analyse des Canham-Helfrich-Gradientenflusses, welcher die zeitliche Evolution von Biomembrankonfigurationen hin zu ihrem Gleichgewichtszustand beschreibt. Basierend auf unserem Varifaltigkeitsmodell können wir die Existenz globaler Lösungen mit der Methode der Generalized Minimizing Movements zeigen und weiters die Evolution von Flächen mit höherer Regularität diskutieren. Für beide in dieser Arbeit behandelten Aspekte gilt, dass die für die Existenz erforderlichen Parameterbereiche im Einklang mit experimentell gewonnenen Daten für biologische Membranen stehen.