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Hyperseries and generalized real analytic functions

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Diksha Tiwari

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Contents

Chapter 1

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Chapter 2

Abstract

The main goal of this PhD thesis has been to start developing the theory of *generalized real analytic functions* (GRAF). The framework is that of generalized smooth functions, a branch of Colombeau theory where generalized functions share many nonlinear properties with ordinary smooth functions, like the closure with respect to composition, a good integration theory, and several classical theorems of calculus. Unlike classical Colombeau theory of real analytic functions, the basic idea to define GRAF is not to use a characterization by growth of derivatives, but the notion of *hyperseries*, i.e. of series of Colombeau generalized numbers extended over the set of *hypernatural numbers* ${}^{\rho}\widetilde{\mathbb{N}}$ (generalized numbers having an integer representative). In fact, since the ring ${}^{\rho}\widetilde{\mathbb{R}}$ of Robinson-Colombeau is non-Archimedean and Cauchy complete, a classical series $\sum_{n=0}^{+\infty} a_n$ of generalized numbers $a_n \in {}^{\rho}\widetilde{\mathbb{R}}$ is convergent if and only if $a_n \to 0$ in the sharp topology. This property does not permit us to generalize several classical results in the study of analytic generalized functions. The notion of hyperseries naturally needs the preliminary concept of hyperlimit, i.e. of limit $n \to +\infty$ for $n \in {}^{\rho}\widetilde{\mathbb{N}}$. Both for studying monotone hyperlimits and the radius of convergence of hyper-power series, we had to consider that $\widetilde{\mathbb{R}}$ is necessarily not order-complete, and we therefore studied the notions of close supremum and infimum in order to generalize the classical theorems for hyperlimits and hyper superior and inferior limits. The corresponding notion of hyperseries allows us to extend numerous classical results which do not hold using classical series in a non-Archimedean framework: all the classical convergence tests and examples, Cauchy product and Mertens' theorem and even the study of divergent infinite hyper sums. These foundations allow us to develop the notion of GRAF by studying hyper-power series and proving classical results such as algebraic operations, composition and inversion of hyper-power series. We also considered term by term derivation and integration and the characterization by local uniform upper bounds of derivatives. On the contrary with respect to the classical use of series, we can recover several classical examples in a non-infinitesimal set of convergence and all Colombeau real analytic functions. The notion of generalized real analytic function reveals to be less rigid with respect to the classical

one, e.g. including classical non-analytic smooth functions with flat points and several distributions such as the Dirac delta. For this reason, the classical identity theorem does not hold in general and we proved only a sufficient condition. On the other hand, this better flexibility is important, because we eventually aim to generalize, in a future work, the Cauchy-Kowalevski theorem to a wider class of generalized functions.

Chapter 3

Kurzfassung

Das Ziel dieser PhD Dissertation ist die Entwicklung einer Theorie der verallgemeinerten (engl. generalized) reell analytischen Funktionen (GRAF). Den Rahmen dafür bilden verallgemeinerte glatte Funktionen, ein Teilgebiet der Colombeau Theorie, welche zahlreiche nicht-lineare Eigenschaften mit gewöhnlichen glatten Funktionen teilen, wie zum Beispiel Abgeschlossenheit bezüglich Komposition, eine vielfältige Integrationstheorie, sowie mehrere klassische Theoreme der Analysis. Im Gegensatz zu der klassischen Colombeau Theorie der reell analytischen Funktionen, ist die Grundidee in der Definition der GRAF nicht eine Charakterisierung durch das Wachstum der Ableitungen, sondern das Prinzip einer Hyperreihe, d.h. einer Reihe bestehend aus verallgemeinerten Colombeau Zahlen, erweitert über der Menge der hypernatürlichen Zahlen ${}^{\rho}\widetilde{\mathbb{N}}$ (verallgemeinerte Zahlen mit einer ganzzahligen Repräsentation). Da der Robinson–Colombeau Ring ${}^{\rho}\widetilde{\mathbb{R}}$ nicht-Archimedisch und Cauchyvollständig ist, konvergieren klassische Reihen der Form $\sum_{n=0}^{+\infty} a_n$ mit $a_n \in {}^{\rho}\widetilde{\mathbb{R}}$ dann und nur dann wenn $a_n \to 0$ in der sogenannten *sharp topology*. Diese Eigenschaft hindert uns mehrere klassische Resultate auf die verallgemeinerten analytischen Funktionen zu erweitern. Das Prinzip einer Hyperreihe benötigt das Konzept eines Hyperlimits, d.h. eines Limits $n \to +\infty$, für $n \in {}^{\rho}\widetilde{\mathbb{N}}$. Für die Untersuchung monotoner Hyperlimiten und Konvergenzradien von Hyperpotenzreihen mussten wir berücksichtigen dass $\widetilde{\mathbb{R}}$ nicht ordnungsvollständig ist. Daher befassten wir uns mit den Konzepten eines abgeschlossenen Supremums und Infimum um klassische Theoreme auf Hyperlimiten, sowie Hyperlimes superior und inferior zu verallgemeinern. Das dazugehörige Konzept einer Hyperreihe erlaubt es uns zahlreiche klassische Resultate zu erweitern, die für klassische Folgen in einem nicht-Archimedischen Rahmen nicht gelten: alle klassischen Konvergenzkriterien, die Cauchy-Produktformel, den Satz von Mertens und sogar die Untersuchung divergierender unendlicher Hyperreihen. Diese Grundbegriffe erlauben es uns das Konzept von GRAF durch Hyperpotenzreihen zu untersuchen, sowie klassische Resultate zu beweisen, wie algebraische Operationen, Komposition und Inversion von Hyperpotenzreihen. Wir behandeln ebenfalls gliedweise Differentiation, Integration und die Charakterisierung durch lokal gleichmäßige obere Schranken der Ableitungen. Im Gegensatz zum klassischen Gebrauch von Reihen, können wir mehrere klassische Beispiele in einer nicht-infinitesimale Konvergenzmenge, sowie alle Colombeau reell analytische Funktionen behandeln. Das Konzept einer verallgemeinerten reell analytischen Funktion ist weniger starr im Vergleich zu ihrem klassischen Analogon, z. Bsp. beinhaltet dieses klassisch nicht-analytische Funktionen mit flachen Punkten, sowie mehrere Distributionen wie die Dirac Delta-Distribution. Aus diesem Grund gilt im Allgemeinen das klassische Identitätstheorem nicht, und wir beweisen nur eine hinreichende Bedingung. Andererseits ist diese größere Flexibilität wichtig, weil wir letztendlich die Absicht haben, in einer künftigen Arbeit, das Cauchy–Kowalevski Theorem für eine breitere Klasse an verallgemeinerten Funktionen zu beweisen.

Chapter 4

Introduction

This is a collective dissertation, and it resulted in the publication of [68, 92, 93], corresponding resp. to Chap. 5, 6 and 7. In order to start working with generalized numbers and generalized smooth functions, the candidate D. Tiwari and A. Mukhammadiev worked together on the first part of [68], even if their work immediately already diverged in the remaining part of [68], where A. Mukhammadiev studied the properties of supremum, infimum and hyperlimits, whereas D. Tiwari focused on the definitions of hyper superior and inferior limits and their properties. The main problem of A. Mukhammadiev's Ph.D. thesis [67] was then devoted to hyperfinite Fourier transform, whereas D. Tiwari's dissertation is centered on the theory of generalized real analytic functions. For both works, the basic notions of hyperlimits, supremum, infimum and hyper superior and inferior limits were hence essential.

Generalized smooth functions theory (GSF; see e.g. [41]) is a branch of Colombeau theory where generalized functions share a number of fundamental properties with smooth functions, in particular with respect to composition and nonlinear operations. The general framework is that of non-Archimedean mathematics, i.e. the study of rings, usually extending the real field, that contain infinitesimal and/or infinite numbers and their use e.g. in mathematical analysis, differential geometry and mathematical physics. Non-Archimedean mathematics includes topics and researchers, among others, such as nonstandard analysis, synthetic differential geometry, the Levi-Civita field. (cf. [63, 56, 89]), etc. The main aim of this dissertation is showing the flexibility of a non-Archimedean framework, such as the Robinson-Colombeau ring of generalized numbers (see chapter 5), by strongly extending classical results of analysis of real analytic generalized functions. In general, a key concept of non-Archimedean analysis is that extending the real field $\mathbb{R}$ into a ring containing infinitesimals and infinite numbers could eventually lead to the solution of non trivial problems. This is the case, e.g., of Colombeau theory, where nonlinear generalized functions can be viewed as set-theoretical maps on domains consisting of generalized points of the non-Archimedean ring $\widetilde{\mathbb{R}}$. This orientation has become increasingly important in recent years and hence it has led to the study of preliminary notions of $\widetilde{\mathbb{R}}$ (cf., e.g., [74, 5, 3, 75, 4, 6, 95, 64, 41]).

In particular, the sharp topology on $\widetilde{\mathbb{R}}$ (cf., e.g., [34, 81, 82] and chapter 5) is the appropriate notion to deal with continuity of this class of generalized functions having infinite derivatives and for a suitable concept of well-posedness. This topology necessarily has to deal with balls $B_r(c) = \left\{x \in \widetilde{\mathbb{R}} \mid |x-c| < r\right\}$ having infinitesimal and invertible radii $r \in \widetilde{\mathbb{R}}_{>0}$, and thus, e.g., $\frac{1}{n^2} \not\to 0$ if $n \to +\infty$, $n \in \mathbb{N}$, because we never have $\mathbb{R}_{>0} \ni \frac{1}{n^2} < r$ if r is infinitesimal. Thereby, the general term of the series $\sum_{n=1}^{+\infty} \frac{1}{n^2}$ does not converge to zero in this topology. A related unusual property related to the sharp topology can be derived from the following inequalities (where $m \in \mathbb{N}$, $n \in \mathbb{N}_{\le m}$, $r \in \widetilde{\mathbb{R}}_{>0}$ is an infinitesimal number, and $|x_{k+1} - x_k| \le r^2$)

$$|x_m - x_n| \le |x_m - x_{m-1}| + \ldots + |x_{n+1} - x_n| \le (m-n)r^2 < r,$$

which imply that $(x_n)_{n\in\mathbb{N}} \in \widetilde{\mathbb{R}}^{\mathbb{N}}$ is a Cauchy sequence if and only if $|x_{n+1} - x_n| \to 0$. As a consequence, a series of CGN

$$\sum_{n\in\mathbb{N}} a_n \text{ converges} \iff a_n \to 0 \text{ (in the sharp topology)}. \tag{4.0.1}$$

(see [68, 92]; actually, this is a well-known property of every ultrametric space, cf., e.g., [55, 81]). Naturally, this has several counter-intuitive consequences (arising from differences with the classical theory) when we have to deal with the study of series, analytic generalized functions, or sigma-additivity and classical limit theorems in integration of generalized functions (cf., e.g., [79, 94, 42]). E.g. if $\sum_{n\in\mathbb{N}} \frac{x^n}{n!}$ converges then necessarily x must be an infinitesimal number. This implies that analytic Colombeau generalized functions (CGF) cannot be expressed using their Taylor series with the usual radius of convergence. On the contrary, we proved that $\sum_{n\in\widetilde{\mathbb{N}}} \frac{x^n}{n!} = e^x$ for all $x \in {}^\rho\widetilde{\mathbb{R}}$ where $e^x \in {}^\rho\widetilde{\mathbb{R}}$. Unlike classical Colombeau theory of real analytic functions, the basic idea of this dissertation is to define *generalized real analytic functions* (GRAF) not using a characterization by growth of derivatives, but the notion of *hyperseries*, i.e. of series of Colombeau generalized numbers extended over the set of *hypernatural numbers* ${}^\rho\widetilde{\mathbb{N}}$ (generalized numbers having an integer representative) so that, e.g., $\frac{1}{n^2} < r$ for all infinite $n \in {}^\rho\widetilde{\mathbb{N}}$ sufficiently large. We thus expect to obtain a more flexible and general notion of (power) series.

One of the aims of this dissertation is hence to solve this kind of counter-intuitive properties so as to arrive at useful notions for the theory of generalized functions. In order to settle this problem, it is important to generalize the role of the net (ε), as used in Colombeau theory, into a more general $\rho = (\rho_\varepsilon) \to 0$ (which is called a *gauge*), and hence to generalize $\widetilde{\mathbb{R}}$ into the ring ${}^\rho\widetilde{\mathbb{R}}$ (see Def. 1.Chap. 5). We then introduce the set of *hypernatural numbers* as

$${}^\rho\widetilde{\mathbb{N}} := \left\{[n_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}} \mid n_\varepsilon \in \mathbb{N} \quad \forall\varepsilon\right\}.$$

The notion of sequence is therefore substituted with that of *hypersequence*, as a map $(x_n)_n : {}^{\sigma}\widetilde{\mathbb{N}} \longrightarrow {}^{\rho}\widetilde{\mathbb{R}}$, where σ is, generally speaking, another gauge. As we will see, (cf. example 27.Chap. 5) only in this way we are able to prove e.g. that $\frac{1}{\log n} \to 0$ in ${}^{\rho}\widetilde{\mathbb{R}}$ as $n \in {}^{\sigma}\widetilde{\mathbb{N}}$ but only for a suitable gauge σ (depending on ρ), whereas this limit does not exist if $\sigma = \rho$.

Finally, the notions of supremum and infimum are naturally linked to the notion of limit of a monotonic (hyper)sequence. Being an ordered set, ${}^{\rho}\widetilde{\mathbb{R}}$ already has a definition of, let us say, supremum as least upper bound. However, as already preliminary studied and proved by [31], this definition does not fit well with topological properties of ${}^{\rho}\widetilde{\mathbb{R}}$ because generalized numbers $[x_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}}$ can actually jump as $\varepsilon \to 0^+$ (see Chap. 5). It is well known that in $\mathbb{R}$ we have $m = \sup(S)$ if and only if m is an upper bound of S and

$$\forall r \in \mathbb{R}_{>0}\, \exists s \in S : \; m - r \leq s. \tag{4.0.2}$$

This could be generalized into the notion of *close supremum* in ${}^{\rho}\widetilde{\mathbb{R}}$, as in [31], that results into better topological properties, see Sec. 4.Chap. 5. The ideas presented in this dissertation, which is self-contained, can surely be useful to explore similar ideas in other non-Archimedean settings, such as [14, 12, 85, 77, 55].

The study of supremum, infimum and hyperlimits of Colombeau generalized numbers (CGN) was carried out in [68] with the aim to subsequently apply it to introduce a corresponding notion of hyperseries. The point of view of chapter 6 is that in a non-Archimedean ring such as ${}^{\rho}\widetilde{\mathbb{R}}$, the notion of *hyperseries* $\sum_{n \in {}^{\rho}\widetilde{\mathbb{N}}} a_n$, i.e. where we sum over the set of hyperfinite natural numbers ${}^{\rho}\widetilde{\mathbb{N}}$, yields results which are more closely related to the classical ones, e.g. in studying analytic functions, sigma additivity and limit theorems for integral calculus, or in possible generalization of the Cauchy-Kowalevski theorem to generalized smooth functions.

Considering the theory of analytic CGF as developed in [79] for the real case and in [94] for the complex one, it is worth to mention that several properties have been proved in both cases: closure with respect to composition, integration over homotopic paths, Cauchy integral theorem, existence of analytic representatives u_ε, a real analytic CGF is identically zero if it is zero on a set of positive Lebesgue measure, etc. (cf. [94, 79, 72] and references therein). On the other hand, even if in [94] it is also proved that each complex analytic CGF can be written as a Taylor series, necessarily this result holds only in an infinitesimal neighborhood of each point. The impossibility to extend this property to a finite neighborhood is due to property (4.0.1) and is hence closely related to the approach we follow in the chapter 6.

We refer to [68] and chapter 5 for notions such as the ring of Robinson-Colombeau, subpoints, hypernatural numbers, supremum, infimum and hyperlimits. See also [64] for a more general approach to asymptotic gauges.

In [92] and chapter 6, the notion of hyperseries allows us to extend numerous classical results which do not hold using classical series in a non-Archimedean framework: all the classical convergence tests and examples, Cauchy product and Mertens' theorem and even the study of divergent infinite hyper sums.

These foundations allow us to develop the notion of hyper-power series in [93] and in chapter 7. As we will see, this yields results which are more closely related to classical ones, such as, e.g. the equality ${}^{\rho}\sum_{n\in{}^{\rho}\widetilde{\mathbb{N}}}\frac{x^n}{n!}=e^x$ that holds for all $x\in{}^{\rho}\widetilde{\mathbb{R}}$ where the exponential is moderate, i.e. if $|x|\le\log\left(\mathrm{d}\rho^{-R}\right)$ for some $R\in\mathbb{N}$ (here $\mathrm{d}\rho:=[\rho_\varepsilon]\in{}^{\rho}\widetilde{\mathbb{R}}$). On the other hand, we will see that classical smooth but non-analytic functions, e.g. smooth functions with flat points, and Schwartz distributions like the Dirac delta, are now included in the related notion of GRAF. This implies that necessarily we cannot have a trivial generalization of the identity theorem (see e.g. [57, Cor. 1.2.6, 1.2.7]) but, on the contrary, only a suitable sufficient condition (see Thm. 40.Chap. 7 below). The notion of generalized real analytic function hence reveals to be less rigid than the classical concept, by including a large family of non-trivial generalized functions (e.g. Dirac delta δ, Heaviside function H, but also powers δ^k, $k\in\mathbb{N}$, and compositions $\delta\circ\delta$, $\delta^k\circ H^h$, $H^h\circ\delta^k$, etc., for $h,k\in\mathbb{N}$.

Conversely, GRAF preserve a lot of classical results: they can be thought of as infinitely long polynomials $f(x)={}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}}a_n(x-c)^n$, with uniquely determined coefficient $a_n=\frac{f^{(n)}(c)}{n!}$, they can be added, multiplied, composed, differentiated, integrated term by term, are closed with respect to inverse function, etc. This lays the foundation for a potential interesting generalization of the Cauchy-Kowalevski theorem which is able to include many non-analytic (but generalized real analytic) functions. However, we did not develop this theorem in this thesis, but we plan to prove it in a future work.

Some conclusions we can summarize at the closure of this thesis are the following:

1. In chapter 5 we have shown how to deal with several deficiencies of the ring of Robinson-Colombeau generalized numbers ${}^{\rho}\widetilde{\mathbb{R}}$: trichotomy law for the order relations $\le$ and $<$, existence of supremum and infimum and limits of sequences with a topology generated by infinitesimal radii. In each case, we obtain a faithful generalization of the classical case of real numbers. We think that some of the ideas we presented in this article can inspire similar works in other non-Archimedean settings such as (constructive) nonstandard analysis, p-adic analysis, the Levi-Civita field, surreal numbers, etc.

2. One of the main goals of chapter 6 is to show that when dealing with non-Archimedean Cauchy complete rings, it can be worth to consider summations over infinite natural numbers instead of classical series indexed by $n\in\mathbb{N}$. As we proved for the Robinson-Colombeau ring ${}^{\rho}\widetilde{\mathbb{R}}$, this allows us to extend numerous classical results which do not hold using classical series in a non-Archimedean framework. We tried to motivate in a clear way why we have to consider two gauges σ and ρ and, for each result, what relationships we have to consider among them., a possible summary could be: for meaningful examples of convergent series, for all convergence tests, one can assume $\rho^*\le\sigma\le\rho^*$ (where $\rho^*\le\sigma$ means $\rho_\varepsilon^Q\le\sigma_\varepsilon$ for some

$Q > 0$ and for ε small, similarily for $\sigma \le \rho^*$; see Def. 2.Chap. 6). For ε-wise convergence, to have larger domains (see Example 11.10.Chap. 6), one can assume $\sigma \le \rho^*$. For divergent hypersums, one must assume $\sigma \ge \rho^*$ and work in ${}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$. Intuitively, to get

$$\left| \sum_{n=0}^{N} a_n - s \right| \le \mathrm{d}\rho^q \tag{4.0.3}$$

one is forced to "sum up" a sufficiently large number $N \in {}^{\sigma}\widetilde{\mathbb{N}}$ of summands; how much this number has to be "large" depends on the relationships between the two gauges σ and ρ whereas, if this is not the case, σ can be "too large" with respect to ρ (i.e. σ^{-1} "too small" with respect to ρ^{-1}, e.g. we could have $\sigma_\varepsilon = \log\left(-\log \rho_\varepsilon\right)^{-1}$, see Example 8.Chap. 6) and we cannot assure to obtain (4.0.3) because we do not sum up a sufficient number of summands with respect to $\mathrm{d}\rho^q$. Since non-Archimedean rings are not Dedekind-complete, we have been forced to substitute the least-upper-bound property using the weaker Cauchy completeness. Once again, we think that several of the proofs we presented here can be generalized to other non-Archimedean Cauchy complete settings.

3. Sometimes, e.g. in the study of PDE, the class of real analytic functions is described as a too rigid set of solutions. In spite of their good properties with respect to algebraic operations, composition, differentiation, integration, inversion, etc., this rigidity is essentially well represented by the identity principle that necessarily excludes e.g. solitons with compact support or interesting generalized functions. Thanks to Thm. 39.Chap. 7, which states that the interior of the set of zeros of GRAF is a clopen set of ${}^{\rho}\widetilde{\mathbb{R}}$, we can state that this rigidity is due to the banishing of non-Archimedean numbers from mathematical analysis. The use of hyperseries allows one to recover all these features including also interesting non trivial generalized functions and compactly supported functions. This paves the way for an interesting generalization of the Cauchy-Kowalevski theorem for GRAF that we intend to develop in a subsequent work. Its proof can be approached by trying a generalization of the classical method of majorants, or using the Picard-Lindelöf theorem for PDE with GSF and then using characterization Thm. 37.Chap. 7 to show that the GSF solution is actually a GRAF.

Chapter 5

Supremum, infimum and hyperlimits in the non-Archimedean ring of Colombeau generalized numbers

By A. Mukhammadiev, D. Tiwari, G. Apaaboah, P. Giordano

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Supremum, infimum and hyperlimits in the non-Archimedean ring of Colombeau generalized numbers

A. Mukhammadiev[1] · D. Tiwari[1] · G. Apaaboah[2] · P. Giordano[1]

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Abstract

It is well-known that the notion of limit in the sharp topology of sequences of Colombeau generalized numbers $\widetilde{\mathbb{R}}$ does not generalize classical results. E.g. the sequence $\frac{1}{n} \not\to 0$ and a sequence $(x_n)_{n\in\mathbb{N}}$ converges *if* and only if $x_{n+1} - x_n \to 0$. This has several deep consequences, e.g. in the study of series, analytic generalized functions, or sigma-additivity and classical limit theorems in integration of generalized functions. The lacking of these results is also connected to the fact that $\widetilde{\mathbb{R}}$ is necessarily not a complete ordered set, e.g. the set of all the infinitesimals has neither supremum nor infimum. We present a solution of these problems with the introduction of the notions of hypernatural number, hypersequence, close supremum and infimum. In this way, we can generalize all the classical theorems for the hyperlimit of a hypersequence. The paper explores ideas that can be applied to other non-Archimedean settings.

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P. Giordano
paolo.giordano@univie.ac.at

A. Mukhammadiev
akbarali.mukhammadiev@univie.ac.at

D. Tiwari
diksha.tiwari@univie.ac.at

G. Apaaboah
apaaboag@etu.univ-grenoble-alpes.fr

[1] University of Vienna, Wien, Austria

[2] University Grenoble Alpes, Saint-Martin-d'Héres, France

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1 Introduction

A key concept of non-Archimedean analysis is that extending the real field $\mathbb{R}$ into a ring containing infinitesimals and infinite numbers could eventually lead to the solution of non trivial problems. This is the case, e.g., of Colombeau theory, where nonlinear generalized functions can be viewed as set-theoretical maps on domains consisting of generalized points of the non-Archimedean ring $\widetilde{\mathbb{R}}$. This orientation has become increasingly important in recent years and hence it has led to the study of preliminary notions of $\widetilde{\mathbb{R}}$ (cf., e.g., [1–4,11,15–17,25]; see below for a self-contained introduction to the ring of Colombeau generalized numbers $\widetilde{\mathbb{R}}$).

In particular, the sharp topology on $\widetilde{\mathbb{R}}$ (cf., e.g., [9,21,22] and below) is the appropriate notion to deal with continuity of this class of generalized functions and for a suitable concept of well-posedness. This topology necessarily has to deal with balls having infinitesimal radius $r \in \widetilde{\mathbb{R}}$, and thus $\frac{1}{n} \not\to 0$ if $n \to +\infty$, $n \in \mathbb{N}$, because we never have $\mathbb{R}_{>0} \ni \frac{1}{n} < r$ if r is infinitesimal. Another unusual property related to the sharp topology can be derived from the following inequalities (where $m \in \mathbb{N}$, $n \in \mathbb{N}_{\le m}$, $r \in \widetilde{\mathbb{R}}_{>0}$ is an infinitesimal number, and $|x_{k+1} - x_k| \le r^2$)

$$|x_m - x_n| \le |x_m - x_{m-1}| + \ldots + |x_{n+1} - x_n| \le (m-n)r^2 < r,$$

which imply that $(x_n)_{n\in\mathbb{N}} \in \widetilde{\mathbb{R}}^{\mathbb{N}}$ is a Cauchy sequence if and only if $|x_{n+1} - x_n| \to 0$ (actually, this is a well-known property of every ultrametric space, cf., e.g., [14,21]). Naturally, this has several counter-intuitive consequences (arising from differences with the classical theory) when we have to deal with the study of series, analytic generalized functions, or sigma-additivity and classical limit theorems in integration of generalized functions (cf., e.g., [12,19,24]).

One of the aims of the present article is to solve this kind of counter-intuitive properties so as to arrive at useful notions for the theory of generalized functions. In order to settle this problem, it is important to generalize the role of the net (ε), as used in Colombeau theory, into a more general $\rho = (\rho_\varepsilon) \to 0$ (which is called a *gauge*), and hence to generalize $\widetilde{\mathbb{R}}$ into some ${}^{\rho}\widetilde{\mathbb{R}}$ (see Definition 1). We then introduce the set of *hypernatural numbers* as

$$\widetilde{\mathbb{N}} := \left\{ [n_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}} \mid n_\varepsilon \in \mathbb{N} \quad \forall \varepsilon \right\},$$

so that it is natural to expect that $\frac{1}{n} \to 0$ in the sharp topology if $n \to +\infty$ with $n \in {}^{\rho}\widetilde{\mathbb{N}}$, because now n can also take infinite values. The notion of sequence is therefore substituted with that of *hypersequence*, as a map $(x_n)_n : {}^{\sigma}\widetilde{\mathbb{N}} \longrightarrow {}^{\rho}\widetilde{\mathbb{R}}$, where σ is, generally speaking, another gauge. As we will see, (cf. Example 27) only in this way we are able to prove e.g. that $\frac{1}{\log n} \to 0$ in ${}^\rho\widetilde{\mathbb{R}}$ as $n \in {}^\sigma\widetilde{\mathbb{N}}$ but only for a suitable gauge σ (depending on ρ), whereas this limit does not exist if $\sigma = \rho$.

Finally, the notions of supremum and infimum are naturally linked to the notion of limit of a monotonic (hyper)sequence. Being an ordered set, ${}^\rho\widetilde{\mathbb{R}}$ already has a definition of, let us say, supremum as least upper bound. However, as already preliminary studied and proved by [8], this definition does not fit well with topological properties of ${}^\rho\widetilde{\mathbb{R}}$ because generalized numbers $[x_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}$ can actually jump as $\varepsilon \to 0^+$ (see Sect. 4). It is well known that in $\mathbb{R}$ we have $m = \sup(S)$ if and only if m is an upper bound of S and

$$\forall r \in \mathbb{R}_{>0}\, \exists s \in S : \ m - r \le s. \tag{1.1}$$

This could be generalized into the notion of *close supremum* in ${}^\rho\widetilde{\mathbb{R}}$, generalizing [8], that results into better topological properties, see Sect. 4. The ideas presented in the present article, which is self-contained, can surely be useful to explore similar ideas in other non-Archimedean settings, such as [5,6,14,18,23].

2 The Ring of Robinson Colombeau and the hypernatural numbers

In this section, we introduce our non-Archimedean ring of scalars and its subset of hypernatural numbers. For more details and proofs about the basic notions introduced here, the reader can refer e.g. to [7,12,13].

As we mentioned above, in order to accomplish the theory of hyperlimits, it is important to generalize Colombeau generalized numbers by taking an arbitrary asymptotic scale instead of the usual $\rho_\varepsilon = \varepsilon$:

Definition 1 Let $\rho = (\rho_\varepsilon) \in (0, 1]^I$ be a net such that $(\rho_\varepsilon) \to 0$ as $\varepsilon \to 0^+$ (in the following, such a net will be called a *gauge*), then

(i) $\mathcal{I}(\rho) := \left\{(\rho_\varepsilon^{-a}) \mid a \in \mathbb{R}_{>0}\right\}$ is called the *asymptotic gauge* generated by ρ.

(ii) If $\mathcal{P}(\varepsilon)$ is a property of $\varepsilon \in I$, we use the notation $\forall^0\varepsilon : \mathcal{P}(\varepsilon)$ to denote $\exists\varepsilon_0 \in I\, \forall\varepsilon \in (0, \varepsilon_0] : \mathcal{P}(\varepsilon)$. We can read $\forall^0\varepsilon$ as *for* ε *small*.

(iii) We say that a net $(x_\varepsilon) \in \mathbb{R}^I$ *is* ρ*-moderate*, and we write $(x_\varepsilon) \in \mathbb{R}_\rho$ if

$$\exists (J_\varepsilon) \in \mathcal{I}(\rho) : \ x_\varepsilon = O(J_\varepsilon) \text{ as } \varepsilon \to 0^+,$$

i.e., if

$$\exists N \in \mathbb{N}\, \forall^0\varepsilon : \ |x_\varepsilon| \le \rho_\varepsilon^{-N}.$$

(iv) Let $(x_\varepsilon), (y_\varepsilon) \in \mathbb{R}^I$, then we say that $(x_\varepsilon) \sim_\rho (y_\varepsilon)$ if

$$\forall (J_\varepsilon) \in \mathcal{I}(\rho) : \ x_\varepsilon = y_\varepsilon + O(J_\varepsilon^{-1}) \text{ as } \varepsilon \to 0^+,$$

that is if

$$\forall n \in \mathbb{N}\, \forall^0\varepsilon : \ |x_\varepsilon - y_\varepsilon| \le \rho_\varepsilon^n.$$

This is a congruence relation on the ring $\mathbb{R}_\rho$ of moderate nets with respect to pointwise operations, and we can hence define

$$\widetilde{\mathbb{R}} := \mathbb{R}_\rho / \sim_\rho,$$

which we call *Robinson-Colombeau ring of generalized numbers*. This name is justified by [7,20]: Indeed, in [20] A. Robinson introduced the notion of moderate and negligible nets depending on an arbitrary fixed infinitesimal ρ (in the framework of nonstandard analysis); independently, J.F. Colombeau, cf. e.g. [7] and references therein, studied the same concepts without using nonstandard analysis, but considering only the particular infinitesimal (ε).

(v) In particular, if the gauge $\rho = (\rho_\varepsilon)$ is non-decreasing, then we say that ρ is a *monotonic gauge*. Clearly, considering a monotonic gauge narrows the class of moderate nets: e.g. if $\lim_{\varepsilon \to \frac{1}{k}} x_\varepsilon = +\infty$ for all $k \in \mathbb{N}_{>0}$, then $(x_\varepsilon) \notin \mathbb{R}_\rho$ for any monotonic gauge ρ.

In the following, ρ will always denote a net as in Definition 1, even if we will sometimes omit the dependence on the infinitesimal ρ, when this is clear from the context. We will also use other directed sets instead of I: e.g. $J \subseteq I$ such that 0 is a closure point of J, or $I \times \mathbb{N}$. The reader can easily check that all our constructions can be repeated in these cases.

We also recall that we write $[x_\varepsilon] \le [y_\varepsilon]$ if there exists $(z_\varepsilon) \in \mathbb{R}^I$ such that $(z_\varepsilon) \sim_\rho 0$ (we then say that (z_ε) is *ρ-negligible*) and $x_\varepsilon \le y_\varepsilon + z_\varepsilon$ for ε small. Equivalently, we have that $x \le y$ if and only if there exist representatives $[x_\varepsilon] = x$ and $[y_\varepsilon] = y$ such that $x_\varepsilon \le y_\varepsilon$ for all ε.

Although the order $\le$ is not total, we still have the possibility to define the infimum $[x_\varepsilon] \wedge [y_\varepsilon] := [\min(x_\varepsilon, y_\varepsilon)]$, the supremum $[x_\varepsilon] \vee [y_\varepsilon] := [\max(x_\varepsilon, y_\varepsilon)]$ of a finite number of generalized numbers. Henceforth, we will also use the customary notation ${}^\rho\widetilde{\mathbb{R}}^*$ for the set of invertible generalized numbers, and we write $x < y$ to say that $x \le y$ and $x - y \in {}^\rho\widetilde{\mathbb{R}}^*$. Our notations for intervals are: $[a, b] := \{x \in {}^\rho\widetilde{\mathbb{R}} \mid a \le x \le b\}$, $[a, b]_{\mathbb{R}} := [a, b] \cap \mathbb{R}$. Finally, we set $\mathrm{d}\rho := [\rho_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}$, which is a positive invertible infinitesimal, whose reciprocal is $\mathrm{d}\rho^{-1} = [\rho_\varepsilon^{-1}]$, which is necessarily a strictly positive infinite number.

The following result is useful to deal with positive and invertible generalized numbers. For its proof, see e.g. [2,3,12,13].

Lemma 2 *Let $x \in {}^\rho\widetilde{\mathbb{R}}$. Then the following are equivalent:*

(i) *x is invertible and $x \ge 0$, i.e. $x > 0$.*
(ii) *For each representative $(x_\varepsilon) \in \mathbb{R}_\rho$ of x we have $\forall^0 \varepsilon : \ x_\varepsilon > 0$.*
(iii) *For each representative $(x_\varepsilon) \in \mathbb{R}_\rho$ of x we have $\exists m \in \mathbb{N}\, \forall^0 \varepsilon : \ x_\varepsilon > \rho_\varepsilon^m$.*
(iv) There exists a representative $(x_\varepsilon) \in \mathbb{R}_\rho$ of x such that $\exists m \in \mathbb{N}\, \forall^0 \varepsilon : \ x_\varepsilon > \rho_\varepsilon^m$.

2.1 The language of subpoints

The following simple language allows us to simplify some proofs using steps recalling the classical real field $\mathbb{R}$. We first introduce the notion of *subpoint*:

Definition 3 For subsets $J, K \subseteq I$ we write $K \subseteq_0 J$ if 0 is an accumulation point of K and $K \subseteq J$ (we read it as: K *is co-final in* J). Note that for any $J \subseteq_0 I$, the constructions introduced so far in Definition 1 can be repeated using nets $(x_\varepsilon)_{\varepsilon\in J}$. We indicate the resulting ring with the symbol ${}^\rho\widetilde{\mathbb{R}}^n|_J$. More generally, no peculiar property of $I = (0, 1]$ will ever be used in the following, and hence all the presented results can be easily generalized considering any other directed set. If $K \subseteq_0 J$, $x \in {}^\rho\widetilde{\mathbb{R}}^n|_J$ and $x' \in {}^\rho\widetilde{\mathbb{R}}^n|_K$, then x' is called a *subpoint* of x, denoted as $x' \subseteq x$, if there exist representatives $(x_\varepsilon)_{\varepsilon\in J}$, $(x'_\varepsilon)_{\varepsilon\in K}$ of x, x' such that $x'_\varepsilon = x_\varepsilon$ for all $\varepsilon \in K$. In this case we write $x' = x|_K$, $\text{dom}(x') := K$, and the restriction $(-)|_K : {}^\rho\widetilde{\mathbb{R}}^n \longrightarrow {}^\rho\widetilde{\mathbb{R}}^n|_K$ is a well defined operation. In general, for $X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ we set $X|_J := \{x|_J \in {}^\rho\widetilde{\mathbb{R}}^n|_J \mid x \in X\}$.

In the next definition, we introduce binary relations that hold only *on subpoints*. Clearly, this idea is inherited from nonstandard analysis, where cofinal subsets are always taken in a fixed ultrafilter.

Definition 4 Let $x, y \in {}^\rho\widetilde{\mathbb{R}}$, $L \subseteq_0 I$, then we say

(i) $x <_L y \ :\Longleftrightarrow\ x|_L < y|_L$ (the latter inequality has to be meant in the ordered ring ${}^\rho\widetilde{\mathbb{R}}|_L$). We read $x <_L y$ as "*x is less than y on L*".
(ii) $x <_\text{s} y \ :\Longleftrightarrow\ \exists L \subseteq_0 I : \ x <_L y$. We read $x <_\text{s} y$ as "*x is less than y on subpoints*".

Analogously, we can define other relations holding only on subpoints such as e.g.: $\in_\text{s}$, $\le_\text{s}$, $=_\text{s}$, $\subseteq_\text{s}$, etc.

For example, we have

$$
\begin{aligned}
x \le y &\iff \forall L \subseteq_0 I : \ x \le_L y \\
x < y &\iff \forall L \subseteq_0 I : \ x <_L y,
\end{aligned}
$$

the former following from the definition of $\le$, whereas the latter following from Lemma 2. Moreover, if $\mathcal{P}\{x_\varepsilon\}$ is an arbitrary property of x_ε, then

$$
\neg\left(\forall^0\varepsilon : \ \mathcal{P}\{x_\varepsilon\}\right) \iff \exists L \subseteq_0 I\, \forall\varepsilon \in L : \ \neg\mathcal{P}\{x_\varepsilon\}. \tag{2.1}
$$

Note explicitly that, generally speaking, relations on subpoints, such as $\le_\text{s}$ or $=_\text{s}$, do not inherit the same properties of the corresponding relations for points. So, e.g., both $=_\text{s}$ and $\le_\text{s}$ are not transitive relations.

The next result clarifies how to equivalently write a negation of an inequality or of an equality using the language of subpoints.

Lemma 5 *Let $x, y \in {}^\rho\widetilde{\mathbb{R}}$, then*

(i) $x \not\le y \iff x >_\text{s} y$
(ii) $x \not< y \iff x \ge_\text{s} y$
(iii) $x \ne y \iff x >_\text{s} y$ *or* $x <_\text{s} y$

Proof (i) $\Leftarrow$: The relation $x >_\text{s} y$ means $x|_L > y|_L$ for some $L \subseteq_0 I$. By Lemma 2 for the ring ${}^\rho\widetilde{\mathbb{R}}|_L$, we get $\forall^0\varepsilon \in L : \ x_\varepsilon > y_\varepsilon$, where $x = [x_\varepsilon]$, $y = [y_\varepsilon]$ are any representatives of x, y resp. The conclusion follows by (2.1)

(i) $\Rightarrow$: Take any representatives $x = [x_\varepsilon]$, $y = [y_\varepsilon]$. The property

$$\forall q \in \mathbb{R}_{>0}\, \forall^0 \varepsilon : \; x_\varepsilon \le y_\varepsilon + \rho_\varepsilon^q$$

for $q \to +\infty$ implies $x \le y$. We therefore have

$$\exists q \in \mathbb{R}_{>0}\, \exists L \subseteq_0 I\, \forall \varepsilon \in L : \; x_\varepsilon > y_\varepsilon + \rho_\varepsilon^q,$$

i.e. $x >_L y$.

(ii) $\Rightarrow$: We have two cases: either $x - y$ is not invertible or $x \not\le y$. In the former case, the conclusion follows from [13, Thm. 1.2.39]. In the latter one, it follows from (i).

(ii) $\Leftarrow$: By contradiction, if $x < y$ then $x =_L y$ for some $L \subseteq_0 I$, which contradicts the invertibility of $x - y$.

(iii) $\Rightarrow$: By contradiction, assume that $x \not>_{\mathrm{s}} y$ and $x \not<_{\mathrm{s}} y$. Then (i) would yield $x \le y$ and $y \le x$, and hence $x = y$. The opposite implication directly follows by contradiction. □

Using the language of subpoints, we can write different forms of dichotomy or trichotomy laws for inequality. The first form is the following

Lemma 6 *Let $x, y \in {}^\rho\widetilde{\mathbb{R}}$, then*

(i) $x \le y$ *or* $x >_{\mathrm{s}} y$
(ii) $\neg(x >_{\mathrm{s}} y$ *and* $x \le y)$
(iii) $x = y$ *or* $x <_{\mathrm{s}} y$ *or* $x >_{\mathrm{s}} y$
(iv) $x \le y \;\Rightarrow\; x <_{\mathrm{s}} y$ *or* $x = y$
(v) $x \le_{\mathrm{s}} y \iff x <_{\mathrm{s}} y$ *or* $x =_{\mathrm{s}} y$.

Proof (i) and (ii) follows directly from Lemma 5. To prove (iii), we can consider that $x >_{\mathrm{s}} y$ or $x \not>_{\mathrm{s}} y$. In the second case, Lemma 5 implies $x \le y$. If $y \le x$ then $x = y$; otherwise, once again by Lemma 5, we get $x <_{\mathrm{s}} y$. To prove (iv), assume that $x \le y$ but $x \not<_{\mathrm{s}} y$, then $x \ge y$ by Lemma 5.(i) and hence $x = y$. The implication $\Leftarrow$ of (v) is trivial. On the other hand, if $x \le_{\mathrm{s}} y$ and $x \not<_{\mathrm{s}} y$, then $y \le x$ from Lemma 5.(i), and hence $x =_{\mathrm{s}} y$. □

As usual, we note that these results can also be trivially repeated for the ring ${}^\rho\widetilde{\mathbb{R}}|_L$. So, e.g., we have $x \not\le_L y$ if and only if $\exists J \subseteq_0 L : \; x >_J y$, which is the analog of Lemma 5.(i) for the ring ${}^\rho\widetilde{\mathbb{R}}|_L$.

The second form of trichotomy (which for ${}^\rho\widetilde{\mathbb{R}}$ can be more correctly named as *quadrichotomy*) *is* stated as follows:

Lemma 7 *Let $x = [x_\varepsilon]$, $y = [y_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}$, then*

(i) $x \le y$ *or* $x \ge y$ *or* $\exists L \subseteq_0 I : \; L^c \subseteq_0 I, \; x \ge_L y$ *and* $x \le_{L^c} y$
(ii) *If for all $L \subseteq_0 I$ the following implication holds*

$$x \le_L y, \; \text{or } x \ge_L y \;\Rightarrow\; \forall^0 \varepsilon \in L : \; \mathcal{P}\{x_\varepsilon, y_\varepsilon\}, \tag{2.2}$$

then $\forall^0 \varepsilon : \; \mathcal{P}\{x_\varepsilon, y_\varepsilon\}$.

(iii) *If for all $L \subseteq_0 I$ the following implication holds*

$$x <_L y, \text{ or } x >_L y \text{ or } x =_L y \Rightarrow \forall^0 \varepsilon \in L : \mathcal{P}\{x_\varepsilon, y_\varepsilon\}, \tag{2.3}$$

then $\forall^0 \varepsilon : \mathcal{P}\{x_\varepsilon, y_\varepsilon\}$.

Proof (i) : if $x \not\leq y$, then $x >_s y$ from Lemma 5.(i). Let $[x_\varepsilon] = x$ and $[y_\varepsilon] = y$ be two representatives, and set $L := \{\varepsilon \in I \mid x_\varepsilon \geq y_\varepsilon\}$. The relation $x >_s y$ implies that $L \subseteq_0 I$. Clearly, $x \geq_L y$ (but note that in general we cannot prove $x >_L y$). If $L^c \not\subseteq_0 I$, then $(0, \varepsilon_o] \subseteq L$ for some ε_0, i.e. $x \geq y$. On the contrary, if $L^c \subseteq_0 I$, then $x \leq_{L^c} y$.

(ii): Property (i) states that we have three cases. If $x_\varepsilon \leq y_\varepsilon$ for all $\varepsilon \leq \varepsilon_0$, then it suffices to set $L := (0, \varepsilon_0]$ in (2.2) to get the claim. Similarly, we can proceed if $x \geq y$. Finally, if $x \geq_L y$ and $x \leq_{L^c} y$, then we can apply (2.2) both with L and L^c to obtain

$$\forall^0 \varepsilon \in L : \mathcal{P}\{x_\varepsilon, y_\varepsilon\}$$
$$\forall^0 \varepsilon \in L^c : \mathcal{P}\{x_\varepsilon, y_\varepsilon\},$$

from which the claim directly follows.

(iii): By contradiction, assume

$$\forall \varepsilon \in L : \neg\mathcal{P}\{x_\varepsilon, y_\varepsilon\}, \tag{2.4}$$

for some $L \subseteq_0 I$. We apply (i) to the ring ${}^\rho\widetilde{\mathbb{R}}|_L$ to obtain the following three cases:

$$x \leq_L y \text{ or } x \geq_L y \text{ or } \exists J \subseteq_0 L : J^c \subseteq_0 L, \ x \geq_J y \text{ and } x \leq_{J^c} y. \tag{2.5}$$

If $x \leq_L y$, by Lemma 6.(iv) for the ring ${}^\rho\widetilde{\mathbb{R}}|_L$, this case splits into two sub-cases: $x =_L y$ or $\exists K \subseteq_0 L : x <_K y$. If the former holds, using (2.3) we get $\mathcal{P}\{x_\varepsilon, y_\varepsilon\}$ $\forall^0 \varepsilon \in L$, which contradicts (2.4). If $x <_K y$, then $K \subseteq_0 I$ and we can apply (2.4) with K to get $\mathcal{P}\{x_\varepsilon, y_\varepsilon\}$ $\forall^0 \varepsilon \in K$, which again contradicts (2.4) because $K \subseteq_0 L$. Similarly we can proceed with the other three cases stated in (2.5). □

Property Lemma 7.(ii) represents a typical replacement of the usual dichotomy law in $\mathbb{R}$: for arbitrary $L \subseteq_0 I$, we can assume to have two cases: either $x \leq_L y$ or $x \geq_L y$. If in both cases we are able to prove $\mathcal{P}\{x_\varepsilon, y_\varepsilon\}$ for $\varepsilon \in L$ small, then we always get that this property holds for all ε small. Similarly, we can use the strict trichotomy law stated in (iii).

2.2 Inferior, superior and standard parts

Other simple tools that we can use to study generalized numbers of ${}^\rho\widetilde{\mathbb{R}}$ are the *inferior* and *superior parts* of a number. Only in this section of the article, we assume that ρ is a monotonic gauge.

Definition 8 Let $x = [x_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}$ be a generalized number, then:

(i) If $\exists L \in \mathbb{R}: \ L \le x$, then $x_{\mathrm{i}} := \left[\inf_{e\in(0,\varepsilon]} x_e\right]$ is called the *inferior part of* x.
(ii) If $\exists U \in \mathbb{R}: \ x \le U$, then $x_{\mathrm{s}} := \left[\sup_{e\in(0,\varepsilon]} x_e\right]$ is called the *superior part of* x.

Moreover, we set:

(iii) $x_{\mathrm{i}}^{\circ} := \liminf_{\varepsilon\to 0^+} x_\varepsilon \in \mathbb{R}\cup\{\pm\infty\}$, where $[x_\varepsilon] = x$ is any representative of x, is called the *inferior standard part* of x. Note that if $\exists x_{\mathrm{i}}$, i.e. if x is finitely bounded from below, then $(x_{\mathrm{i}})^{\circ} = x_{\mathrm{i}}^{\circ} \in \mathbb{R}$ and $x_{\mathrm{i}}^{\circ} \ge x_{\mathrm{i}}$.
(iv) $x_{\mathrm{s}}^{\circ} := \limsup_{\varepsilon\to 0^+} x_\varepsilon \in \mathbb{R}\cup\{\pm\infty\}$, where $[x_\varepsilon] = x$ is any representative of x, is called the *superior standard part* of x. Note that if $\exists x_{\mathrm{s}}$, i.e. if x is finitely bounded from above, then $(x_{\mathrm{s}})^{\circ} = x_{\mathrm{s}}^{\circ} \in \mathbb{R}$ and $x_{\mathrm{s}}^{\circ} \le x_{\mathrm{s}}$.

Note that, since $\rho = (\rho_\varepsilon)$ is non-decreasing, if $[x'_\varepsilon] = x$ is another representative, then for all $e \in (0,\varepsilon]$, we have $x'_e \le x_e + \rho_e^n \le x_e + \rho_\varepsilon^n \le \rho_\varepsilon^n + \sup_{e\in(0,\varepsilon]} x_e$ and hence $\sup_{e\in(0,\varepsilon]} x'_e \le \rho_\varepsilon^n + \sup_{e\in(0,\varepsilon]} x_e$. This shows that inferior and superior parts, when they exist, are well-defined. Moreover, if (z_ε) is negligible, then $\limsup_{\varepsilon\to 0^+}(x_\varepsilon + z_\varepsilon) \le \limsup_{\varepsilon\to 0^+} x_\varepsilon + 0$, which shows that x_{s}° is well-defined (similarly for x_{i}° using super-additivitiy of lim inf).

Clearly, $x_{\mathrm{i}} \le x \le x_{\mathrm{s}}$ and $x_{\mathrm{i}}^{\circ} \le x_{\mathrm{s}}^{\circ}$. We have that the generalized number x is near-standard if and only if $x_{\mathrm{i}}^{\circ} = x_{\mathrm{s}}^{\circ} =: x^{\circ} \in \mathbb{R}$; it is infinitesimal if and only if $\exists\, x^{\circ} = 0$; it is a positive infinite number if and only if $x_{\mathrm{i}}^{\circ} = x_{\mathrm{s}}^{\circ} =: x^{\circ} = +\infty$ (the same for negative infinite numbers); it is a finite number if and only if $x_{\mathrm{i}}^{\circ}, x_{\mathrm{s}}^{\circ} \in \mathbb{R}$. Finally, there always exist $x', x'' \subseteq x$ such that $x' \approx x_{\mathrm{i}}^{\circ}$ and $x'' \approx x_{\mathrm{s}}^{\circ}$, where $x \approx y$ means that $x - y$ is infinitesimal (i.e. $|x-y| \le r$ for all $r \in \mathbb{R}_{>0}$ or, equivalently, $\lim_{\varepsilon\to 0^+} x_\varepsilon - y_\varepsilon = 0$ for all $[x_\varepsilon] = x$, $[y_\varepsilon] = y$). Therefore, any generalized number in ${}^{\rho}\widetilde{\mathbb{R}}$ is either finite or some of its subpoints is infinite; in the former case, some of its subpoints is near standard.

2.3 Topologies on ${}^{\rho}\widetilde{\mathbb{R}}^n$

On the ${}^{\rho}\widetilde{\mathbb{R}}$-module ${}^{\rho}\widetilde{\mathbb{R}}^n$ we can consider the natural extension of the Euclidean norm, i.e. $|[x_\varepsilon]| := [|x_\varepsilon|] \in {}^{\rho}\widetilde{\mathbb{R}}$, where $[x_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}}^n$. Even if this generalized norm takes values in ${}^{\rho}\widetilde{\mathbb{R}}$, it shares some essential properties with classical norms:

$$
\begin{aligned}
&|x| = x \vee (-x)\\
&|x| \ge 0\\
&|x| = 0 \Rightarrow x = 0\\
&|y\cdot x| = |y|\cdot|x|\\
&|x+y| \le |x| + |y|\\
&||x| - |y|| \le |x-y|.
\end{aligned}
$$

It is therefore natural to consider on ${}^{\rho}\widetilde{\mathbb{R}}^n$ topologies generated by balls defined by this generalized norm and a set of radii:

Definition 9 We say that $\mathfrak{R}$ is a *set of radii* if

(i) $\mathfrak{R} \subseteq {}^{\rho}\widetilde{\mathbb{R}}^{*}_{\ge 0}$ is a non-empty subset of positive invertible generalized numbers.

(ii) For all $r, s \in \mathfrak{R}$ the infimum $r \wedge s \in \mathfrak{R}$.
(iii) $k \cdot r \in \mathfrak{R}$ for all $r \in \mathfrak{R}$ and all $k \in \mathbb{R}_{>0}$.

Moreover, if $\mathfrak{R}$ is a set of radii and $x, y \in {}^{\rho}\widetilde{\mathbb{R}}$, then:

(i) We write $x <_{\mathfrak{R}} y$ if $\exists r \in \mathfrak{R} : r \le y - x$.
(ii) $B_r^{\mathfrak{R}}(x) := \left\{ y \in {}^{\rho}\widetilde{\mathbb{R}}^n \mid |y - x| <_{\mathfrak{R}} r \right\}$ for any $r \in \mathfrak{R}$.
(iii) $B_\rho^{\mathrm{E}}(x) := \{ y \in \mathbb{R}^n \mid |y - x| < \rho \}$, for any $\rho \in \mathbb{R}_{>0}$, denotes an ordinary Euclidean ball in $\mathbb{R}^n$.

For example, ${}^{\rho}\widetilde{\mathbb{R}}^*_{\ge 0}$ and $\mathbb{R}_{>0}$ are sets of radii.

Lemma 10 *Let $\mathfrak{R}$ be a set of radii and $x, y, z \in {}^{\rho}\widetilde{\mathbb{R}}$, then*

(i) $\neg (x <_{\mathfrak{R}} x)$.
(ii) *$x <_{\mathfrak{R}} y$ and $y <_{\mathfrak{R}} z$ imply $x <_{\mathfrak{R}} z$.*
(iii) $\forall r \in \mathfrak{R} : \ 0 <_{\mathfrak{R}} r$.

The relation $<_{\mathfrak{R}}$ has better topological properties as compared to the usual strict order relation $x \le y$ and $x \ne y$ (a relation that we will therefore never use) because of the following result:

Theorem 11 *The set of balls $\left\{ B_r^{\mathfrak{R}}(x) \mid r \in \mathfrak{R},\ x \in {}^{\rho}\widetilde{\mathbb{R}}^n \right\}$ generated by a set of radii $\mathfrak{R}$ is a base for a topology on ${}^{\rho}\widetilde{\mathbb{R}}^n$.*

Henceforth, we will only consider the sets of radii ${}^{\rho}\widetilde{\mathbb{R}}^*_{\ge 0} = {}^{\rho}\widetilde{\mathbb{R}}_{>0}$ and $\mathbb{R}_{>0}$ and will use the simplified notation $B_r(x) := B_r^{\mathfrak{R}}(x)$ if $\mathfrak{R} = {}^{\rho}\widetilde{\mathbb{R}}_{>0}$. The topology generated in the former case is called *sharp topology*, whereas the latter is called *Fermat topology*. We will call *sharply open set* any open set in the sharp topology, and *large open set* any open set in the Fermat topology; clearly, the latter is coarser than the former. It is well-known (see e.g. [2,3,9,10,12] and references therein) that this is an equivalent way to define the sharp topology usually considered in the ring of Colombeau generalized numbers. We therefore recall that the sharp topology on ${}^{\rho}\widetilde{\mathbb{R}}^n$ is Hausdorff and Cauchy complete, see e.g. [2,10].

2.4 Open, closed and bounded sets generated by nets

A natural way to obtain sharply open, closed and bounded sets in ${}^{\rho}\widetilde{\mathbb{R}}^n$ is by using a net (A_ε) of subsets $A_\varepsilon \subseteq \mathbb{R}^n$. We have two ways of extending the membership relation $x_\varepsilon \in A_\varepsilon$ to generalized points $[x_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}}^n$ (cf. [11,17]):

Definition 12 Let (A_ε) be a net of subsets of $\mathbb{R}^n$, then

(i) $[A_\varepsilon] := \left\{ [x_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}}^n \mid \forall^0 \varepsilon : \ x_\varepsilon \in A_\varepsilon \right\}$ is called the *internal set* generated by the net (A_ε).
(ii) Let (x_ε) be a net of points of $\mathbb{R}^n$, then we say that $x_\varepsilon \in_\varepsilon A_\varepsilon$, and we read it as (x_ε) *strongly belongs to* (A_ε), if

  (i) $\forall^0 \varepsilon : \ x_\varepsilon \in A_\varepsilon$.
  (ii) If $(x'_\varepsilon) \sim_\rho (x_\varepsilon)$, then also $x'_\varepsilon \in A_\varepsilon$ for ε small.

Moreover, we set $\langle A_\varepsilon \rangle := \left\{ [x_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}^n \mid x_\varepsilon \in_\varepsilon A_\varepsilon \right\}$, and we call it the *strongly internal set* generated by the net (A_ε).

(iii) We say that the internal set $K = [A_\varepsilon]$ is *sharply bounded* if there exists $M \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ such that $K \subseteq B_M(0)$.

(iv) Finally, we say that the net (A_ε) is *sharply bounded* if there exists $N \in \mathbb{R}_{>0}$ such that $\forall^0 \varepsilon \, \forall x \in A_\varepsilon : \ |x| \le \rho_\varepsilon^{-N}$.

Therefore, $x \in [A_\varepsilon]$ if there exists a representative $[x_\varepsilon] = x$ such that $x_\varepsilon \in A_\varepsilon$ for ε small, whereas this membership is independent from the chosen representative in case of strongly internal sets. An internal set generated by a constant net $A_\varepsilon = A \subseteq \mathbb{R}^n$ will simply be denoted by $[A]$.

The following theorem (cf. [11,17] for the case $\rho_\varepsilon = \varepsilon$, and [12] for an arbitrary gauge) shows that internal and strongly internal sets have dual topological properties:

Theorem 13 *For $\varepsilon \in I$, let $A_\varepsilon \subseteq \mathbb{R}^n$ and let $x_\varepsilon \in \mathbb{R}^n$. Then we have*

(i) $[x_\varepsilon] \in [A_\varepsilon]$ *if and only if* $\forall q \in \mathbb{R}_{>0} \, \forall^0 \varepsilon : \ d(x_\varepsilon, A_\varepsilon) \le \rho_\varepsilon^q$. *Therefore* $[x_\varepsilon] \in [A_\varepsilon]$ *if and only if* $[d(x_\varepsilon, A_\varepsilon)] = 0 \in {}^\rho\widetilde{\mathbb{R}}$.

(ii) $[x_\varepsilon] \in \langle A_\varepsilon \rangle$ *if and only if* $\exists q \in \mathbb{R}_{>0} \, \forall^0 \varepsilon : \ d(x_\varepsilon, A_\varepsilon^c) > \rho_\varepsilon^q$, *where* $A_\varepsilon^c := \mathbb{R}^n \setminus A_\varepsilon$. *Therefore, if* $(d(x_\varepsilon, A_\varepsilon^c)) \in \mathbb{R}_\rho$, *then* $[x_\varepsilon] \in \langle A_\varepsilon \rangle$ *if and only if* $[d(x_\varepsilon, A_\varepsilon^c)] > 0$.

(iii) $[A_\varepsilon]$ *is sharply closed.*

(iv) $\langle A_\varepsilon \rangle$ *is sharply open.*

(v) $[A_\varepsilon] = [\boldsymbol{cl}\,(A_\varepsilon)]$, *where* $\boldsymbol{cl}\,(S)$ *is the closure of* $S \subseteq \mathbb{R}^n$.

(vi) $\langle A_\varepsilon \rangle = \langle \boldsymbol{int}(A_\varepsilon) \rangle$, *where* $\mathrm{int}\,(S)$ *is the interior of* $S \subseteq \mathbb{R}^n$.

For example, it is not hard to show that the closure in the sharp topology of a ball of center $c = [c_\varepsilon]$ and radius $r = [r_\varepsilon] > 0$ is

$$\overline{B_r(c)} = \left\{ x \in {}^\rho\widetilde{\mathbb{R}}^d \mid |x - c| \le r \right\} = \left[\overline{B^{\mathrm{E}}_{r_\varepsilon}(c_\varepsilon)} \right], \tag{2.6}$$

whereas

$$B_r(c) = \left\{ x \in {}^\rho\widetilde{\mathbb{R}}^d \mid |x - c| < r \right\} = \langle B^{\mathrm{E}}_{r_\varepsilon}(c_\varepsilon) \rangle.$$

Using internal sets and adopting ideas similar to those used in proving Lemma 7, we also have the following form of dichotomy law:

Lemma 14 *For $\varepsilon \in I$, let $A_\varepsilon \subseteq \mathbb{R}^n$ and let $x = [x_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}^n$. Then we have:*

(i) $x \in [A_\varepsilon]$ *or* $x \in [A_\varepsilon^c]$ *or* $\exists L \subseteq_0 I : \ L^c \subseteq_0 I, \ x \in_L [A_\varepsilon], \ x \in_{L^c} [A_\varepsilon^c]$

(ii) *If for all* $L \subseteq_0 I$ *the following implication holds*

$$x \in_L [A_\varepsilon] \text{ or } x \in_L [A_\varepsilon^c] \ \Rightarrow \ \forall^0 \varepsilon \in L : \ \mathcal{P}\{x_\varepsilon\},$$

then $\forall^0 \varepsilon : \ \mathcal{P}\{x_\varepsilon\}$.

Proof (i): If $x \notin [A_\varepsilon^c]$, then $x_\varepsilon \in A_\varepsilon$ for all $\varepsilon \in K$ and for some $K \subseteq_0 I$. Set $L := \{\varepsilon \in I \mid x_\varepsilon \in A_\varepsilon\}$, so that $K \subseteq L \subseteq_0 I$. We have $x \in_L [A_\varepsilon]$. If $L^c \not\subseteq_0 I$, then $(0, \varepsilon_0] \subseteq L$ for some ε_0, i.e. $x \in [A_\varepsilon]$. On the contrary, if $L^c \subseteq_0 I$, then $x \in_{L^c} [A_\varepsilon^c]$.

(ii): We can proceed as in the proof of Lemma 7.(ii) using (i). □

3 Hypernatural numbers

We start by defining the set of hypernatural numbers in ${}^{\rho}\widetilde{\mathbb{R}}$ and the set of ρ-moderate nets of natural numbers:

Definition 15 We set

(i) ${}^{\rho}\widetilde{\mathbb{N}} := \left\{[n_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}} \mid n_\varepsilon \in \mathbb{N} \quad \forall\varepsilon\right\}$
(ii) $\mathbb{N}_\rho := \left\{(n_\varepsilon) \in \mathbb{R}_\rho \mid n_\varepsilon \in \mathbb{N} \quad \forall\varepsilon\right\}$.

Therefore, $n \in {}^{\rho}\widetilde{\mathbb{N}}$ if and only if there exists $(x_\varepsilon) \in \mathbb{R}_\rho$ such that $n = [\mathrm{int}(|x_\varepsilon|)]$. Clearly, $\mathbb{N} \subset {}^{\rho}\widetilde{\mathbb{N}}$. Note that the integer part function $\mathrm{int}(-)$ is not well-defined on ${}^{\rho}\widetilde{\mathbb{R}}$. In fact, if $x = 1 = \left[1 - \rho_\varepsilon^{1/\varepsilon}\right] = \left[1 + \rho_\varepsilon^{1/\varepsilon}\right]$, then $\mathrm{int}\left(1 - \rho_\varepsilon^{1/\varepsilon}\right) = 0$ whereas $\mathrm{int}\left(1 + \rho_\varepsilon^{1/\varepsilon}\right) = 1$, for ε sufficiently small. Similar counter examples can be set for floor and ceiling functions. However, the nearest integer function is well defined on ${}^{\rho}\widetilde{\mathbb{N}}$, as proved in the following

Lemma 16 *Let $(n_\varepsilon) \in \mathbb{N}_\rho$ and $(x_\varepsilon) \in \mathbb{R}_\rho$ be such that $[n_\varepsilon] = [x_\varepsilon]$. Let* $\mathrm{rpi} : \mathbb{R} \longrightarrow \mathbb{N}$ *be the function rounding to the nearest integer with tie breaking towards positive infinity, i.e. rpi$(x) = \lfloor x + \frac{1}{2} \rfloor$. Then* $\mathrm{rpi}(x_\varepsilon) = n_\varepsilon$ *for ε small. The same result holds using* $\mathrm{rni} : \mathbb{R} \longrightarrow \mathbb{N}$*, the function rounding half towards $-\infty$.*

Proof We have $\mathrm{rpi}(x) = \lfloor x + \frac{1}{2} \rfloor$, where $\lfloor - \rfloor$ is the floor function. For ε small, $\rho_\varepsilon < \frac{1}{2}$ and, since $[n_\varepsilon] = [x_\varepsilon]$, always for ε small, we also have $n_\varepsilon - \rho_\varepsilon + \frac{1}{2} < x_\varepsilon + \frac{1}{2} < n_\varepsilon + \rho_\varepsilon + \frac{1}{2}$. But $n_\varepsilon \le n_\varepsilon - \rho_\varepsilon + \frac{1}{2}$ and $n_\varepsilon + \rho_\varepsilon + \frac{1}{2} < n_\varepsilon + 1$. Therefore $\lfloor x_\varepsilon + \frac{1}{2} \rfloor = n_\varepsilon$. An analogous argument can be applied to $\mathrm{rni}(-)$. □

Actually, this lemma does not allow us to define a *nearest integer* function $\mathrm{ni} : {}^{\rho}\widetilde{\mathbb{N}} \longrightarrow \mathbb{N}_\rho$ as $\mathrm{ni}([x_\varepsilon]) := \mathrm{rpi}(x_\varepsilon)$ because if $[x_\varepsilon] = [n_\varepsilon]$, the equality $n_\varepsilon = \mathrm{rpi}(x_\varepsilon)$ holds only for ε small. A simpler approach is to choose a representative $(n_\varepsilon) \in \mathbb{N}_\rho$ for each $x \in {}^{\rho}\widetilde{\mathbb{N}}$ and to define $\mathrm{ni}(x) := (n_\varepsilon)$. Clearly, we must consider the net $(\mathrm{ni}(x)_\varepsilon)$ only for ε small, such as in equalities of the form $x = \left[\mathrm{ni}(x)_\varepsilon\right]$. This is what we do in the following

Definition 17 The *nearest integer function* $\mathrm{ni}(-)$ is defined by:

(i) $\mathrm{ni} : {}^{\rho}\widetilde{\mathbb{N}} :\longrightarrow \mathbb{N}_\rho$
(ii) If $[x_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{N}}$ and $\mathrm{ni}\left([x_\varepsilon]\right) = (n_\varepsilon)$ then $\forall^0\varepsilon :\ n_\varepsilon = \mathrm{rpi}(x_\varepsilon)$.

In other words, if $x \in {}^{\rho}\widetilde{\mathbb{N}}$, then $x = [\mathrm{ni}(x)_\varepsilon]$ and $\mathrm{ni}(x)_\varepsilon \in \mathbb{N}$ for all ε. Another possibility is to formulate Lemma 16 as

$$[x_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{N}} \quad \Longleftrightarrow \quad [x_\varepsilon] = [\mathrm{rpi}(x_\varepsilon)].$$

Therefore, without loss of generality we may always suppose that $x_\varepsilon \in \mathbb{N}$ whenever $[x_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{N}}$.

Remark 18 (i) ${}^{\sigma}\widetilde{\mathbb{N}}$, with the order $\le$ induced by ${}^{\sigma}\widetilde{\mathbb{R}}$, is a directed set; it is closed with respect to sum and product although recursive definitions using ${}^{\sigma}\widetilde{\mathbb{N}}$ are not possible.

(ii) In ${}^{\sigma}\widetilde{\mathbb{N}}$ we can find several chains (totally ordered subsets) such as: $\mathbb{N}$, $\mathbb{N}\cdot[\mathrm{int}(\rho_\varepsilon^{-k})]$ for a fixed $k\in\mathbb{N}$, $\{[\mathrm{int}(\rho_\varepsilon^{-k})]\mid k\in\mathbb{N}\}$.

(iii) Generally speaking, if $m,n\in{}^{\rho}\widetilde{\mathbb{N}}$, $m^n\notin{}^{\rho}\widetilde{\mathbb{N}}$ because the net $\left(m_\varepsilon^{n_\varepsilon}\right)$ can grow faster than any power (ρ_ε^{-K}). However, if we take two gauges σ, ρ satisfying $\sigma\le\rho$, using the net $\left(\sigma_\varepsilon^{-1}\right)$ we can measure infinite nets that grow faster than (ρ_ε^{-K}) because $\sigma_\varepsilon^{-1}\ge\rho_\varepsilon^{-1}$ for ε small. Therefore, we can take $m,n\in{}^{\sigma}\widetilde{\mathbb{N}}$ such that $(\mathrm{ni}(m)_\varepsilon)$, $(\mathrm{ni}(n)_\varepsilon)\in\mathbb{R}_\rho$; we think at m,n as σ-hypernatural numbers growing at most polynomially with respect to ρ. Then, it is not hard to prove that if ρ is an arbitrary gauge, and we consider the auxiliary gauge $\sigma_\varepsilon:=\rho_\varepsilon^{e^{1/\rho_\varepsilon}}$. then $m^n\in{}^{\sigma}\widetilde{\mathbb{N}}$.

(iv) If $m\in{}^{\rho}\widetilde{\mathbb{N}}$, then $1^m:=\left[(1+z_\varepsilon)^{m_\varepsilon}\right]$, where (z_ε) is ρ-negligible, is well defined and $1^m=1$. In fact, $\log(1+z_\varepsilon)^{m_\varepsilon}$ is asymptotically equal to $m_\varepsilon z_\varepsilon\to0$, and this shows that $\left((1+z_\varepsilon)^{m_\varepsilon}\right)$ is moderate. Finally, $|(1+z_\varepsilon)^{m_\varepsilon}-1|\le|z_\varepsilon|\,m_\varepsilon(1+z_\varepsilon)^{m_\varepsilon-1}$ by the mean value theorem.

4 Supremum and Infimum in ${}^{\rho}\widetilde{\mathbb{R}}$

To solve the problems we explained in the introduction of this article, it is important to generalize at least two main existence theorems for limits: the Cauchy criterion and the existence of a limit of a bounded monotone sequence. The latter is clearly related to the existence of supremum and infimum, which cannot be always guaranteed in the non-Archimedean ring ${}^{\rho}\widetilde{\mathbb{R}}$. As we will see more clearly later (see also [8]), to arrive at these existence theorems, the notion of supremum, i.e. the least upper bound, is not the correct one. More appropriately, we can associate a notion of close supremum (and close infimum) to every topology generated by a set of radii (see Definition 9).

Definition 19 Let $\mathfrak{R}$ be a set of radii and let τ be the topology on ${}^{\rho}\widetilde{\mathbb{R}}$ generated by $\mathfrak{R}$. Let $P\subseteq{}^{\rho}\widetilde{\mathbb{R}}$, then we say that τ *separates points of* P if

$$\forall p,q\in P:\ p\neq q\ \Rightarrow\ \exists A,B\in\tau:\ p\in A,\ q\in B,\ A\cap B=\emptyset,$$

i.e. if P with the topology induced by τ is Hausdorff.

Definition 20 Let τ be a topology on ${}^{\rho}\widetilde{\mathbb{R}}$ generated by a set of radii $\mathfrak{R}$ that separates points of $P\subseteq{}^{\rho}\widetilde{\mathbb{R}}$ and let $S\subseteq{}^{\rho}\widetilde{\mathbb{R}}$. Then, we say that σ is (τ,P)-*supremum of* S if

(i) $\sigma\in P$;
(ii) $\forall s\in S:\ s\le\sigma$;
(iii) σ is a point of closure of S in the topology τ, i.e. if $\forall A\in\tau:\ \sigma\in A\ \Rightarrow\ \exists\bar{s}\in S\cap A$.

Similarly, we say that ι is (τ,P)-*infimum of* S if

(i) $\iota\in P$;
(ii) $\forall s\in S:\ \iota\le s$;
(iii) ι is a point of closure of S in the topology τ, i.e. if $\forall A\in\tau:\ \iota\in A\ \Rightarrow\ \exists\bar{s}\in S\cap A$.

In particular, if τ is the sharp topology and $P={}^{\rho}\widetilde{\mathbb{R}}$, then following [8], we simply call the (τ,P)-supremum, the *close supremum* (the adjective close will be omitted if it will be clear from the context) or the *sharp supremum* if we want to underline the dependency on the topology. Analogously, if τ is the Fermat topology and $P = \mathbb{R}$, then we call the (τ, P)-supremum the *Fermat supremum*. Note that (iii) implies that if σ is (τ, P)-supremum of S, then necessarily $S \neq \emptyset$.

Remark 21 (i) Let $S \subseteq {}^{\rho}\widetilde{\mathbb{R}}$, then from Definition 9 and Theorem 11 we can prove that σ is the (τ, P)-supremum of S if and only if

(a) $\forall s \in S : \ s \le \sigma$;
(b) $\forall r \in \mathfrak{R}\, \exists \bar{s} \in S : \ \sigma - r \le \bar{s}$.

In particular, for the sharp supremum, (b) is equivalent to

$$\forall q \in \mathbb{N}\, \exists \bar{s} \in S : \ \sigma - \mathrm{d}\rho^{q} \le \bar{s}. \tag{4.1}$$

In the following of this article, we will also mainly consider the sharp topology and the corresponding notions of sharp supremum and infimum.

(ii) If there exists the sharp supremum σ of $S \subseteq {}^{\rho}\widetilde{\mathbb{R}}$ and $\sigma \notin S$, then from (4.1) it follows that S is necessarily an infinite set. In fact, applying (4.1) with $q_1 := 1$ we get the existence of $\bar{s}_1 \in S$ such that $\sigma - \mathrm{d}\rho^{q_1} < \bar{s}_1$. We have $\bar{s}_1 \neq \sigma$ because $\sigma \notin S$. Hence, Lemma 5.(iii) and Definition 20.(ii) yield that $\bar{s}_1 <_{\mathrm{s}} \sigma$. Therefore, $\sigma - \bar{s}_1 \ge_{\mathrm{s}} \mathrm{d}\rho^{q_2}$ for some $q_2 > q_1$. Applying again (4.1) we get $\sigma - \mathrm{d}\rho^{q_2} < \bar{s}_2$ for some $\bar{s}_2 \in S \setminus \{\bar{s}_1\}$. Recursively, this process proves that S is infinite. On the other hand, if $S = \{s_1, \ldots, s_n\}$ and $s_i = [s_{i\varepsilon}]$, then $\sup([\{s_{1\varepsilon}, \ldots, s_{n\varepsilon}\}]) = s_1 \vee \ldots \vee s_n$. In fact, $s_1 \vee \ldots \vee s_n = \left[\max_{i=1,\ldots,n} s_{n\varepsilon}\right] \in [\{s_{1\varepsilon}, \ldots, s_{n\varepsilon}\}]$.

(iii) If $\exists \sup(S) = \sigma$, then there also exists the $\sup(\mathrm{interl}(S)) = \sigma$, where (see [17]) we recall that

$$\mathrm{interl}(S) := \left\{ \sum_{j=1}^{m} e_{S_j} s_j \mid m \in \mathbb{N}, \ S_j \subseteq_0 I, \ s_j \in S \ \forall j \right\}, \ e_S := [1_S] \in {}^{\rho}\widetilde{\mathbb{R}}$$

(1_S is the characteristic function of $S \subseteq I$). This follows from $S \subseteq \mathrm{interl}(S)$. Vice versa, if $\exists \sup(\mathrm{interl}(S)) = \sigma$ and $\mathrm{interl}(S) \subseteq S$ (e.g. if S is an internal or strongly internal set), then also $\exists \sup(S) = \sigma$.

Theorem 22 *There is at most one sharp supremum of S, which is denoted by* $\sup(S)$.

Proof Assume that σ_1 and σ_2 are supremum of S. That is Definition 20.(ii) and (4.1) hold both for σ_1, σ_2. Then, for all fixed $q \in \mathbb{N}$, there exists $\bar{s}_2 \in S$ such that $\sigma_2 - \mathrm{d}\rho^{q} \le \bar{s}_2$. Hence $\bar{s}_2 \le \sigma_1$ because $\bar{s}_2 \in S$. Analogously, we have that $\sigma_1 - \mathrm{d}\rho^{q} \le \bar{s}_1 \le \sigma_2$ for some $\bar{s}_1 \in S$. Therefore, $\sigma_2 - \mathrm{d}\rho^{q} \le \sigma_1 \le \sigma_2 + \mathrm{d}\rho^{q}$, and this implies $\sigma_1 = \sigma_2$ since $q \in \mathbb{N}$ is arbitrary. □

In [8], the notation $\overline{\sup}(S)$ is used for the close supremum. On the other hand, we will *never* use the notion of supremum as least upper bound. For these reasons, we prefer to use the simpler notation $\sup(S)$. Similarly, we use the notation $\inf(S)$ for the close (or sharp) infimum. From Rem. 21.(a) and (b) it follows that

$$\inf(S) = -\sup(-S) \tag{4.2}$$

in the sense that the former exists if and only if the latter exists and in that case they are equal. For this reason, in the following we only study the supremum.

Example 23 (i) Let $K = [K_\varepsilon] \Subset_f {}^\rho\widetilde{\mathbb{R}}$ be a functionally compact set (cf. [10]), i.e. $K \subseteq B_M(0)$ for some $M \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ and $K_\varepsilon \Subset \mathbb{R}$ for all ε. We can then define $\sigma_\varepsilon := \sup(K_\varepsilon) \in K_\varepsilon$. From $K \subseteq B_M(0)$, we get $\sigma := [\sigma_\varepsilon] \in K$. It is not hard to prove that $\sigma = \sup(K) = \max(K)$. Analogously, we can prove the existence of the sharp minimum of K.

(ii) If $S = (a, b)$, where $a, b \in {}^\rho\widetilde{\mathbb{R}}$ and $a \le b$, then $\sup(S) = b$ and $\inf(S) = a$.

(iii) If $S = \left\{\frac{1}{n} \mid n \in {}^\rho\widetilde{\mathbb{N}}\right\}$, then $\inf(S) = 0$.

(iv) Like in several other non-Archimedean rings, both sharp supremum and infimum of the set D_∞ of all infinitesimals do not exist. In fact, by contradiction, if σ were the sharp supremum of D_∞, then from (4.1) for $q = 1$ we would get the existence of $\bar{h} \in D_\infty$ such that $\sigma \le \bar{h} + \mathrm{d}\rho$. But then $\sigma \in D_\infty$, so also $2\sigma \in D_\infty$. Therefore, we get $2\sigma \le \sigma$ because σ is an upper bound of D_∞, and hence $\sigma = 0 \ge \mathrm{d}\rho$, a contradiction. Similarly, one can prove that there does not exist the infimum of this set.

(v) Let $S = (0, 1)_{\mathbb{R}} = \{x \in \mathbb{R} \mid 0 < x < 1\}$, then clearly $\sigma = 1$ is the Fermat supremum of S whereas there does not exist the sharp supremum of S. Indeed, if $\sigma = \sup(S)$, then $s \le \sigma \le \bar{s} + \mathrm{d}\rho$ for all $s \in S$ and for some $\bar{s} \in S$. Taking any $s \in (\bar{s}, 1)_{\mathbb{R}} \subseteq S$ we get $s \le \sigma \le \bar{s} + \mathrm{d}\rho$, which, for $\varepsilon \to 0$, implies $s \le \bar{s}$ because $s, \bar{s} \in \mathbb{R}$. This contradicts $s \in (\bar{s}, 1)$. In particular, 1 is not the sharp supremum. This example shows the importance of Definition 20, i.e. that the best notion of supremum in a non-Archimedean setting depends on a fixed topology.

(vi) Let $S = (0, 1) \cup \{\hat{s}\}$ where $\hat{s}|_L = 2$, $\hat{s}|_{L^c} = \frac{1}{2}$, $L \subseteq_0 I$, $L^c \subseteq_0 I$, then $\nexists \sup(S)$. In fact, if $\exists\sigma := \sup(S)$, then $\sigma|_L \ge \hat{s}|_L = 2$ and $\sigma|_{L^c} = 1$. Assume that $\exists\bar{s} \in S : \ \sigma - \mathrm{d}\rho \le \bar{s}$, then $2 - \mathrm{d}\rho|_L \le \sigma|_L - \mathrm{d}\rho|_L \le \bar{s}|_L$. Thereby, $\bar{s}|_L > \frac{3}{2}$ and hence $\bar{s} \notin (0, 1)$ and $\bar{s} = \hat{s}$. We hence get $\sigma|_{L^c} - \mathrm{d}\rho|_{L^c} \le \hat{s}|_{L^c}$, i.e. $1 - \mathrm{d}\rho|_{L^c} \le \frac{1}{2}$, which is impossible. We can intuitively say that the subpoint $\hat{s}|_L$ creates a "ε-hole" (i.e. a "hole" only for some ε) on the right of S and hence S is not "an ε-continuum" on this side. Finally note that the point $u|_L := 2$ and $u|_{L^c} := 1$ is the least upper bound of S.

Lemma 24 *Let $A, B \subseteq {}^\rho\widetilde{\mathbb{R}}$, then*

(i) $\forall\lambda \in {}^\rho\widetilde{\mathbb{R}}_{>0} : \ \sup(\lambda A) = \lambda \sup(A)$*, in the sense that one supremum exists if and only if the other one exists, and in that case they coincide;*

(ii) $\forall\lambda \in {}^\rho\widetilde{\mathbb{R}}_{<0} : \ \sup(\lambda A) = \lambda \inf(A)$*, in the sense that one supremum/infimum exists if and only if the other one exists, and in that case they coincide;*

Moreover, if $\exists \sup(A), \sup(B)$*, then:*

(iii) *If* $A \subseteq B$*, then* $\sup(A) \le \sup(B)$*;*

(iv) $\sup(A + B) = \sup(A) + \sup(B)$*;*

(v) *If* $A, B \subseteq {}^\rho\widetilde{\mathbb{R}}_{\ge 0}$*, then* $\sup(A \cdot B) = \sup(A) \cdot \sup(B)$.

Proof (i): If $\exists \sup(\lambda A)$, then we have $a \le \frac{1}{\lambda}\sup(\lambda A)$ for all $a \in A$. For all $q \in \mathbb{N}$, we can find $\bar{a} \in A$ such that $\sup(\lambda A) - \lambda\bar{a} \le \mathrm{d}\rho^q$. Thereby, $\frac{1}{\lambda}\sup(\lambda A) - \bar{a} \le \frac{1}{\lambda}\mathrm{d}\rho^q \to 0$

as $q \to +\infty$ because λ is moderate. This proves that $\exists \sup(A) = \frac{1}{\lambda}\sup(\lambda A)$. Similarly, we can prove the opposite implication.

(ii): From (i) and (4.2) we get: $\sup(\lambda A) = \sup(-\lambda(-A)) = -\lambda \sup(-A) = \lambda \inf(A)$.

(iii): By contradiction, using Lemma 5.(i), if $\sup(A) >_L \sup(B)$ for some $L \subseteq_0 I$, then $\sup(A) - \sup(B) >_L \mathrm{d}\rho^q$ for some $q \in \mathbb{N}$ by Lemma 2 for the ring ${}^\rho\widetilde{\mathbb{R}}|_L$. Property (4.1) yields $\sup(A) - \mathrm{d}\rho^q \le \bar{a}$ for some $\bar{a} \in A$, and $\bar{a} \le \sup(B)$ because $A \subseteq B$. Thereby, $\sup(A) - \sup(B) \le \mathrm{d}\rho^q$, which implies $\mathrm{d}\rho^q <_L \mathrm{d}\rho^q$, a contradiction.

(iv) and (v) follow easily from Definition 20.(ii) and (4.1). □

In the next section, we introduce in the non-Archimedean framework ${}^\rho\widetilde{\mathbb{R}}$ how to approximate $\sup(S)$ of $S \subseteq {}^\rho\widetilde{\mathbb{R}}$ using points of S and upper bounds, and the non-Archimedean analogous of the notion of upper bound.

5 The hyperlimit of a hypersequence

5.1 Definition and examples

Definition 25 A map $x : {}^\sigma\widetilde{\mathbb{N}} \longrightarrow {}^\rho\widetilde{\mathbb{R}}$, whose domain is the set of hypernatural numbers ${}^\sigma\widetilde{\mathbb{N}}$ is called a $(\sigma-)$ *hypersequence* (of elements of ${}^\rho\widetilde{\mathbb{R}}$). The values $x(n) \in {}^\rho\widetilde{\mathbb{R}}$ at $n \in {}^\sigma\widetilde{\mathbb{N}}$ of the function x are called *terms* of the hypersequence and, as usual, denoted using an index as argument: $x_n = x(n)$. The hypersequence itself is denoted by $(x_n)_{n\in{}^\sigma\widetilde{\mathbb{N}}}$, or simply $(x_n)_n$ if the gauge on the domain is clear from the context. Let σ, ρ be two gauges, $x : {}^\sigma\widetilde{\mathbb{N}} \longrightarrow {}^\rho\widetilde{\mathbb{R}}$ be a hypersequence and $l \in {}^\rho\widetilde{\mathbb{R}}$. We say that l is *hyperlimit* of $(x_n)_n$ as $n \to \infty$ and $n \in {}^\sigma\widetilde{\mathbb{N}}$, if

$$\forall q \in N\, \exists M \in {}^\sigma\widetilde{\mathbb{N}}\, \forall n \in {}^\sigma\widetilde{\mathbb{N}}_{\ge M} : \ |x_n - l| < \mathrm{d}\rho^q.$$

In the following, if not differently stated, ρ and σ will always denote two gauges and $(x_n)_n$ a σ-hypersequence of elements of ${}^\rho\widetilde{\mathbb{R}}$. Finally, if $\sigma_\varepsilon \ge \rho_\varepsilon$, at least for all ε small, we simply write $\sigma \ge \rho$.

Remark 26 In the assumption of Definition 25, let $k \in {}^\rho\widetilde{\mathbb{R}}_{>0}$, $N \in \mathbb{N}$, then the following are equivalent:

(i) $l \in {}^\rho\widetilde{\mathbb{R}}$ is the hyperlimit of $(x_n)_n$ as $n \in {}^\sigma\widetilde{\mathbb{N}}$.
(ii) $\forall \eta \in {}^\rho\widetilde{\mathbb{R}}_{>0}\, \exists M \in {}^\sigma\widetilde{\mathbb{N}}\, \forall n \in {}^\sigma\widetilde{\mathbb{N}}_{\ge M} : \ |x_n - l| < \eta$.
(iii) Let $U \subseteq {}^\rho\widetilde{\mathbb{R}}$ be a sharply open set, if $l \in U$ then $\exists M \in {}^\sigma\widetilde{\mathbb{N}}\, \forall n \in {}^\sigma\widetilde{\mathbb{N}}_{\ge M} : \ x_n \in U$.
(iv) $\forall q \in \mathbb{N}\, \exists M \in {}^\sigma\widetilde{\mathbb{N}}\, \forall n \in {}^\sigma\widetilde{\mathbb{N}}_{\ge M} : \ |x_n - l| < k \cdot \mathrm{d}\rho^q$.
(v) $\forall q \in \mathbb{N}\, \exists M \in {}^\sigma\widetilde{\mathbb{N}}\, \forall n \in {}^\sigma\widetilde{\mathbb{N}}_{\ge M} : \ |x_n - l| < \mathrm{d}\rho^{q-N}$.

Directly by the inequality $|l_1 - l_2| \le |l_1 - x_n| + |l_2 - x_n| \le 2\mathrm{d}\rho^{q+1} < \mathrm{d}\rho^q$ (or by using that the sharp topology on ${}^\rho\widetilde{\mathbb{R}}$ is Hausdorff) it follows that there exists at most one hyperlimit, so that we can use the notation

$${}^\rho\lim_{n\in{}^\sigma\widetilde{\mathbb{N}}} x_n := l.$$

As usual, a hypersequence (not) having a hyperlimit is said to be (non-)convergent. We can also similarly say that $(x_n)_n : {}^\sigma\widetilde{\mathbb{N}} \longrightarrow {}^\rho\widetilde{\mathbb{R}}$ is divergent to $+\infty$ $(-\infty)$ if

$$\forall q \in \mathbb{N}\, \exists M \in {}^\sigma\widetilde{\mathbb{N}}\, \forall n \in {}^\sigma\widetilde{\mathbb{N}}_{\geq M} : \ x_n > \mathrm{d}\rho^{-q} \quad (x < -\mathrm{d}\rho^{-q}).$$

Example 27 (i) If $\sigma \leq \rho^R$ for some $R \in \mathbb{R}_{>0}$, we have ${}^\rho\lim_{n\in{}^\sigma\widetilde{\mathbb{N}}} \frac{1}{n} = 0$. In fact, $\frac{1}{n} < \mathrm{d}\rho^q$ holds e.g. if $n > \left[\mathrm{int}\left(\rho_\varepsilon^{-q}\right) + 1\right] \in {}^\sigma\widetilde{\mathbb{N}}$ because $\rho_\varepsilon^{-q} \leq \sigma_\varepsilon^{-q/R}$ for ε small.

(ii) Let ρ be a gauge and set $\sigma_\varepsilon := \exp\left(-\rho_\varepsilon^{-\frac{1}{\rho_\varepsilon}}\right)$, so that σ is also a gauge. We have

$$ {}^\rho\lim_{n\in{}^\sigma\widetilde{\mathbb{N}}} \frac{1}{\log n} = 0 \in {}^\rho\widetilde{\mathbb{R}} \text{ whereas } \nexists\, {}^\rho\lim_{n\in{}^\rho\widetilde{\mathbb{N}}} \frac{1}{\log n}$$

In fact, if $n > 1$, we have $0 < \frac{1}{\log n} < \mathrm{d}\rho^q$ if and only if $\log n > \mathrm{d}\rho^{-q}$, i.e. $n > e^{\mathrm{d}\rho^{-q}}$(in ${}^\rho\widetilde{\mathbb{R}}$). We can thus take $M := \left[\mathrm{int}\left(e^{\rho_\varepsilon^{-q}}\right) + 1\right] \in {}^\sigma\widetilde{\mathbb{N}}$ because $e^{\rho_\varepsilon^{-q}} < \exp\left(\rho_\varepsilon^{-\frac{1}{\rho_\varepsilon}}\right) = \sigma_\varepsilon^{-1}$ for ε small. Vice versa, by contradiction, if $\exists\, {}^\rho\lim_{n\in{}^\rho\widetilde{\mathbb{N}}} \frac{1}{\log n} =: l \in {}^\rho\widetilde{\mathbb{R}}$, then by the definition of hyperlimit from ${}^\rho\widetilde{\mathbb{N}}$ to ${}^\rho\widetilde{\mathbb{R}}$, we would get the existence of $M \in {}^\rho\widetilde{\mathbb{N}}$ such that

$$\forall n \in {}^\rho\widetilde{\mathbb{N}} : n \geq M \ \Rightarrow \ \frac{1}{\log n} - \mathrm{d}\rho < l < \frac{1}{\log n} + \mathrm{d}\rho \tag{5.1}$$

We have to explore two possibilities: if l is not invertible, then $l_{\varepsilon_k} = 0$ for some sequence $(\varepsilon_k) \downarrow 0$ and some representative $[l_\varepsilon] = l$. Therefore from 25, we get

$$\frac{1}{\log M_{\varepsilon_k}} < l_{\varepsilon_k} + \rho_{\varepsilon_k} = \rho_{\varepsilon_k}$$

hence $M_{\varepsilon_k} > e^{-\frac{1}{\rho_{\varepsilon_k}}}$ $\forall k \in \mathbb{N}$, in contradiction with $M \in {}^\rho\widetilde{\mathbb{R}}$. If l is invertible, then $\mathrm{d}\rho^p < |l|$ for some $p \in \mathbb{N}$. Setting $q := \min\{p \in \mathbb{N} \mid \mathrm{d}\rho^p < |l|\} + 1$, we get that $l_{\bar{\varepsilon}_k} < \rho_{\bar{\varepsilon}_k}^q$ for some sequence $(\bar{\varepsilon}_k)_k \downarrow 0$. Therefore

$$\frac{1}{\log M_{\bar{\varepsilon}_k}} < l_{\bar{\varepsilon}_k} + \rho_{\bar{\varepsilon}_k} \leq |l_{\bar{\varepsilon}_k}| + \rho_{\bar{\varepsilon}_k} < \rho_{\bar{\varepsilon}_k}^q + \rho_{\bar{\varepsilon}_k}$$

and hence $M_{\bar{\varepsilon}_k} > \exp\left(\frac{1}{\rho_{\bar{\varepsilon}_k}^q + \rho_{\varepsilon_k}}\right)$ for all $k \in \mathbb{N}$, which is in contradiction with $M \in {}^\rho\widetilde{\mathbb{R}}$ because $q \geq 1$.

Analogously, we can prove that ${}^{\rho}\lim_{n\in{}^{\sigma}\widetilde{\mathbb{N}}}\frac{1}{\log(\log n)}=0$ if $\sigma=[\sigma_\epsilon]=\left[e^{-e^{\rho_\epsilon^{-\frac{1}{\rho_\epsilon}}}}\right]$ whereas $\nexists\,{}^{\rho}\lim_{n\in{}^{\rho}\widetilde{\mathbb{N}}}\frac{1}{\log(\log n)}$ (and similarly using $\log(\log(\ldots\overset{k}{\ldots}(\log n)\ldots)$).

(iii) Set $x_n:=\mathrm{d}\rho^{-n}$ if $n\in\mathbb{N}$, and $x_n:=\frac{1}{n}$ if $n\in{}^{\rho}\widetilde{\mathbb{N}}\setminus\mathbb{N}$, then $\{x_n\mid n\in{}^{\rho}\widetilde{\mathbb{N}}\}$ is unbounded in ${}^{\rho}\widetilde{\mathbb{R}}$ even if ${}^{\rho}\lim_{n\in{}^{\rho}\widetilde{\mathbb{N}}}x_n=0$. Similarly, if $x_n:=\mathrm{d}\rho^n$ if $n\in\mathbb{N}$ and $x_n:=\sin(n)$ otherwise, then $\lim_{\substack{n\to+\infty\\ n\in\mathbb{N}}}x_n=0$ whereas $\nexists\,{}^{\rho}\lim_{n\in{}^{\rho}\widetilde{\mathbb{N}}}x_n$. In general, we can hence only state that convergent hypersequence are eventually bounded:

$$\exists\,{}^{\rho}\lim_{n\in{}^{\sigma}\widetilde{\mathbb{N}}}x_n\ \Rightarrow\ \exists M\in{}^{\rho}\widetilde{\mathbb{R}}\,\exists N\in{}^{\sigma}\widetilde{\mathbb{N}}\,\forall n\in{}^{\sigma}\widetilde{\mathbb{N}}_{\ge N}:\ |x_n|\le M.$$

(iv) If $k<_{\mathrm{s}}1$ and $k>_{\mathrm{s}}1$, then ${}^{\rho}\lim_{n\in{}^{\rho}\widetilde{\mathbb{N}}}k^n=_{\mathrm{s}}0$ and ${}^{\rho}\lim_{n\in{}^{\rho}\widetilde{\mathbb{N}}}k^n=_{\mathrm{s}}+\infty$, hence $\nexists^{\rho}\lim_{n\in{}^{\rho}\widetilde{\mathbb{N}}}k^n$.

(v) Since for $n\in\mathbb{N}$ we have $(1-\mathrm{d}\rho)^n=1-n\mathrm{d}\rho+O_n(\mathrm{d}\rho^2)$, it is not hard to prove that $((1-\mathrm{d}\rho)^n)_{n\in\mathbb{N}}$ is not a Cauchy sequence. Therefore, $\nexists\lim_{n\in\mathbb{N}}(1-\mathrm{d}\rho)^n$, whereas ${}^{\rho}\lim_{n\in{}^{\rho}\widetilde{\mathbb{N}}}(1-\mathrm{d}\rho)^n=0$.

A sufficient condition to extend an ordinary sequence $(a_n)_{n\in\mathbb{N}}:\mathbb{N}\longrightarrow{}^{\rho}\widetilde{\mathbb{R}}$ of ρ-generalized numbers to the whole ${}^{\sigma}\widetilde{\mathbb{N}}$ is

$$\forall n\in{}^{\sigma}\widetilde{\mathbb{N}}:\ \left(a_{\mathrm{ni}(n)_\varepsilon}\right)\in\mathbb{R}_\rho. \tag{5.2}$$

In fact, in this way $a_n:=\left[a_{\mathrm{ni}(n)_\varepsilon}\right]\in{}^{\rho}\widetilde{\mathbb{R}}$ for all $n\in{}^{\sigma}\widetilde{\mathbb{N}}$, is well-defined because of Lemma 16; on the other hand, we have defined an extension of the old sequence $(a_n)_{n\in\mathbb{N}}$ because if $n\in\mathbb{N}$, then $\mathrm{ni}(n)_\varepsilon=n$ for ε small and hence $a_n=[a_n]$. For example, the sequence of infinities $a_n=\frac{1}{n}+\mathrm{d}\rho^{-1}$ for all $n\in\mathbb{N}$ can be extended to any ${}^{\sigma}\widetilde{\mathbb{N}}$, whereas $a_n=\mathrm{d}\sigma^{-n}$ can be extended as $a:{}^{\sigma}\widetilde{\mathbb{N}}\longrightarrow{}^{\rho}\widetilde{\mathbb{R}}$ only for some gauges ρ, e.g. if the gauges satisfy

$$\exists N\in\mathbb{N}\,\forall n\in\mathbb{N}\,\forall^0\varepsilon:\ \sigma_\varepsilon^n\ge\rho_\varepsilon^N, \tag{5.3}$$

(e.g. $\sigma_\varepsilon=\varepsilon$ and $\rho_\varepsilon=\varepsilon^{1/\varepsilon}$).

The following result allows us to obtain hyperlimits by proceeding ε-wise

Theorem 28 *Let $(a_{n,\varepsilon})_{n,\varepsilon}:\mathbb{N}\times I\longrightarrow\mathbb{R}$. Assume that for all ε*

$$\exists\lim_{n\to+\infty}a_{n,\varepsilon}=:l_\varepsilon, \tag{5.4}$$

and that $l:=[l_\varepsilon]\in{}^{\rho}\widetilde{\mathbb{R}}$. Then there exists a gauge σ (not necessarily a monotonic one) such that

(i) *There exists $M\in{}^{\sigma}\widetilde{\mathbb{N}}$ and a hypersequence $(a_n)_n:{}^{\sigma}\widetilde{\mathbb{N}}\longrightarrow{}^{\rho}\widetilde{\mathbb{R}}$ such that $a_n=[a_{ni(n)_\varepsilon,\varepsilon}]\in{}^{\rho}\widetilde{\mathbb{R}}$ for all $n\in{}^{\sigma}\widetilde{\mathbb{N}}_{\ge M}$;*

(ii) *$l={}^{\rho}\lim_{n\in{}^{\sigma}\widetilde{\mathbb{N}}}a_n$.*

Proof From (5.4), we have

$$\forall\varepsilon\,\forall q\,\exists M_{\varepsilon q}\in\mathbb{N}_{>0}\,\forall n\geq M_{\varepsilon q}:\ \rho_\varepsilon^q-l_\varepsilon<a_{n,\varepsilon}<\rho_\varepsilon^q+l_\varepsilon. \tag{5.5}$$

Without loss of generality, we can assume to have recursively chosen $M_{\varepsilon q}$ so that

$$M_{\varepsilon q}\leq M_{\varepsilon,q+1}\quad\forall\varepsilon\,\forall q. \tag{5.6}$$

Set $\bar{M}_\varepsilon:=M_{\varepsilon,\lceil\frac{1}{\varepsilon}\rceil}>0$; since $\forall q\in\mathbb{N}\,\forall^0\varepsilon:\ q\leq\lceil\frac{1}{\varepsilon}\rceil$, (5.6) implies

$$\forall q\in\mathbb{N}\,\forall^0\varepsilon:\ \bar{M}_\varepsilon\geq M_{\varepsilon q}. \tag{5.7}$$

If the net $(\bar{M}_\varepsilon)$ is ρ-moderate, set $\sigma:=\rho$, otherwise set $\sigma_\varepsilon:=\min\left(\rho_\varepsilon,\bar{M}_\varepsilon^{-1}\right)\in(0,1]$. Thereby, the net $\sigma_\varepsilon\to0$ as $\varepsilon\to0^+$ (note that not necessarily σ is non-decreasing, e.g. if $\lim_{\varepsilon\to\frac{1}{k}}\bar{M}_\varepsilon=+\infty$ for all $k\in\mathbb{N}_{>0}$ and $\bar{M}_\varepsilon\geq\rho_\varepsilon^{-1}$), i.e. it is a gauge. Now set $\bar{M}:=[\bar{M}_\varepsilon]\in{}^\sigma\widetilde{\mathbb{N}}$ because our definition of σ yields $\bar{M}_\varepsilon\leq\sigma_\varepsilon^{-1}$, $M_q:=[M_{\varepsilon q}]\in{}^\sigma\widetilde{\mathbb{N}}$ because of (5.7), and

$$a_n:=\begin{cases}[a_{\mathrm{ni}(n)_\varepsilon,\varepsilon}] & \text{if } n\geq M_1 \text{ in } {}^\sigma\widetilde{\mathbb{N}}\\ 1 & \text{otherwise}\end{cases}\qquad\forall n\in{}^\sigma\widetilde{\mathbb{N}}. \tag{5.8}$$

We have to prove that this well-defines a hypersequence $(a_n)_n:{}^\sigma\widetilde{\mathbb{N}}\longrightarrow{}^\rho\widetilde{\mathbb{R}}$. First of all, the sequence is well-defined with respect to the equality in ${}^\sigma\widetilde{\mathbb{N}}$ because of Lemma 16. Moreover, setting $q=1$ in (5.5), we get $\rho_\varepsilon-l_\varepsilon<a_{n,\varepsilon}<\rho_\varepsilon+l_\varepsilon$ for all ε and for all $n\geq M_{\varepsilon1}$. If $n\geq M_1$ in ${}^\sigma\widetilde{\mathbb{N}}$, then $\mathrm{ni}(n)_\varepsilon\geq M_{\varepsilon1}$ for ε small, and hence $\rho_\varepsilon-l_\varepsilon<a_{\mathrm{ni}(n)_\varepsilon,\varepsilon}<\rho_\varepsilon+l_\varepsilon$. This shows that $a_n\in{}^\rho\widetilde{\mathbb{R}}$ because we assumed that $l=[l_\varepsilon]\in{}^\rho\widetilde{\mathbb{R}}$. Finally, (5.5) and (5.6) yield that if $n\geq M_q$ then $n\geq M_1$ and hence $|a_n-l|<\mathrm{d}\rho^q$. ◻

From the proof it also follows, more generally, that if $(M_{\varepsilon q})_{\varepsilon,q}$ satisfies (5.5) and if

$$\exists(q_\varepsilon)\to+\infty:\ \left(M_{\varepsilon,q_\varepsilon}\right)\in\mathbb{R}_\rho,$$

then we can repeat the proof with q_ε instead of $\lceil\frac{1}{\varepsilon}\rceil$ and setting $\sigma:=\rho$.

5.2 Operations with hyperlimits and inequalities

Thanks to Definition 9 of sharp topology and our notation for $x<y$ (and of the consequent Lemma 2), some results about hyperlimits can be proved by trivially generalizing classical proofs. For example, if $(x_n)_{n\in{}^\sigma\widetilde{\mathbb{N}}}$ and $(y_n)_{n\in{}^\sigma\widetilde{\mathbb{N}}}$ are two convergent hypersequences then their sum $(x_n+y_n)_{n\in{}^\sigma\widetilde{\mathbb{N}}}$, product $(x_n\cdot y_n)_{n\in{}^\sigma\widetilde{\mathbb{N}}}$ and quotient $\left(\frac{x_n}{y_n}\right)_{n\in{}^\sigma\widetilde{\mathbb{N}}}$ (the last one being defined only when y_n is invertible for all $n\in{}^\sigma\widetilde{\mathbb{N}}$) are convergent hypersequences and the corresponding hyperlimits are sum, product and quotient of the corresponding hyperlimits.

The following results generalize the classical relations between limits and inequalities.

Theorem 29 *Let $x, y, z : {}^\sigma\widetilde{\mathbb{N}} \longrightarrow {}^\rho\widetilde{\mathbb{R}}$ be hypersequences, then we have:*

(i) *If ${}^\rho\lim_{n\in{}^\sigma\widetilde{\mathbb{N}}} x_n < {}^\rho\lim_{n\in{}^\sigma\widetilde{\mathbb{N}}} y_n$, then $\exists M \in {}^\sigma\widetilde{\mathbb{N}}$ such that $x_n < y_n$ for all $n \ge M$, $n \in {}^\sigma\widetilde{\mathbb{N}}$.*
(ii) *If $x_n \le y_n \le z_n$ for all $n \in {}^\sigma\widetilde{\mathbb{N}}$ and ${}^\rho\lim_{n\in{}^\sigma\widetilde{\mathbb{N}}} x_n = {}^\rho\lim_{n\in{}^\sigma\widetilde{\mathbb{N}}} z_n =: l$, then $\exists\, {}^\rho\lim_{n\in{}^\sigma\widetilde{\mathbb{N}}} y_n = l$,*

Proof (i) follows from Lemma 2 and the Definition 25 of hyperlimit. For (ii), the proof is analogous to the classical one. In fact, since ${}^\rho\lim_{n\in{}^\sigma\widetilde{\mathbb{N}}} x_n = {}^\rho\lim_{n\in{}^\sigma\widetilde{\mathbb{N}}} z_n =: l$ given $q \in \mathbb{N}$, there exist $M', M'' \in {}^\sigma\widetilde{\mathbb{N}}$ such that $l - \mathrm{d}\rho^q < x_n$ and $z_n < l + \mathrm{d}\rho^q$ for all $n > M'$, $n > M''$, $n \in {}^\sigma\widetilde{\mathbb{N}}$, then for $n > M := M' \vee M''$, we have $l - \mathrm{d}\rho^q < x_n \le y_n \le z_n < l + \mathrm{d}\rho^q$. □

Theorem 30 *Assume that C is a sharply closed subset of ${}^\rho\widetilde{\mathbb{R}}$, that $\exists\, {}^\rho\lim_{n\in{}^\sigma\widetilde{\mathbb{N}}} x_n =: l$ and that x_n eventually lies in C, i.e. $\exists N \in {}^\sigma\widetilde{\mathbb{N}}\, \forall n \in {}^\sigma\widetilde{\mathbb{N}}_{\ge N} : \ x_n \in C$. Then also $l \in C$. In particular, if $(y_n)_n$ is another hypersequence such that $\exists\, {}^\rho\lim_{n\in{}^\sigma\widetilde{\mathbb{N}}} y_n =: k$, then $\exists N \in {}^\sigma\widetilde{\mathbb{N}}\, \forall n \in {}^\sigma\widetilde{\mathbb{N}}_{\ge N} : \ x_n \ge y_n$ implies $l \ge k$.*

Proof A reformulation of the usual proof applies. In fact, let us suppose that $l \in {}^\rho\widetilde{\mathbb{R}} \setminus C$. Since ${}^\rho\widetilde{\mathbb{R}} \setminus C$ is sharply open, there is an $\eta > 0$, for which $B_\eta(l) \subseteq {}^\rho\widetilde{\mathbb{R}} \setminus C$. Let $\bar{n} \in {}^\sigma\widetilde{\mathbb{N}}_{\ge N}$ be such that $|x_n - l| < \eta$ when $n > \bar{n}$. Then we have $x_n \in C$ and $x_n \in B_\eta(l) \subseteq {}^\rho\widetilde{\mathbb{R}} \setminus C$, a contradiction. □

The following result applies to all generalized smooth functions (and hence to all Colombeau generalized functions, see e.g. [11,12]; see also [1] for a more general class of functions) because of their continuity in the sharp topology.

Theorem 31 *Suppose that $f : U \longrightarrow {}^\rho\widetilde{\mathbb{R}}$. Then f is sharply continuous function at $x = c$ if and only if it is hyper-sequentially continuous, i.e. for any hypersequence $(x_n)_n$ in U converging to c, the hypersequence $(f(x_n))_n$ converges to $f(c)$, i.e. $f\left({}^\rho\lim_{n\in{}^\sigma\widetilde{\mathbb{N}}} x_n\right) = {}^\rho\lim_{n\in{}^\sigma\widetilde{\mathbb{N}}} f(x_n)$.*

Proof We only prove that the hyper-sequential continuity is a sufficient condition, because the other implication is a trivial generalization of the classical one. By contradiction, assume that for some $Q \in \mathbb{N}$

$$\forall n \in \mathbb{N}\, \exists x_n \in U : \ |x_n - c| < \mathrm{d}\rho^n, \ |f(x_n) - f(c)| >_{\mathrm{s}} \mathrm{d}\rho^Q. \tag{5.9}$$

For $n \in \mathbb{N}$ set $\omega_n := n$ and for $n \in {}^\rho\widetilde{\mathbb{N}} \setminus \mathbb{N}$ set $\omega_n := \min\left\{N \in \mathbb{N} \mid n \le \mathrm{d}\rho^{-N}\right\}$ and $x_n := x_{\omega_n}$. Then for all $n \in {}^\rho\widetilde{\mathbb{N}}$, from (5.9) we get $|x_n - c| < \mathrm{d}\rho^{\omega_n} \to 0$ because $\omega_n \to +\infty$ as $n \to +\infty$ in $n \in {}^\rho\widetilde{\mathbb{N}}$. Therefore, $(x_n)_n$ is an hypersequence of U that converges to c, which yields $f(x_n) \to f(c)$, in contradiction with (5.9). □

Example 32 Let $\sigma \leq \rho^R$ for some $R \in \mathbb{R}_{>0}$. The following inequalities hold for all generalized numbers because they also hold for all real numbers:

$$\begin{aligned} &\ln(x) \leq x \\ &e\left(\frac{n}{e}\right)^n \leq n! \leq en\left(\frac{n}{e}\right)^n. \end{aligned} \tag{5.10}$$

From the first one it follows $0 \leq \frac{\ln(n)}{n} = \frac{2\ln\sqrt{n}}{n} \leq \frac{2\sqrt{n}}{n}$, so that ${}^{\rho}\lim_{n\in^{\sigma}\widetilde{\mathbb{N}}} \frac{\ln(n)}{n} := 0$ from Theorem 29 and ${}^{\rho}\lim_{n\in^{\sigma}\widetilde{\mathbb{N}}} n^{1/n} = 1$ from Theorem 31 and hence ${}^{\rho}\lim_{n\in^{\sigma}\widetilde{\mathbb{N}}} (n!)^{1/n} = +\infty$ by (5.10). Similarly, we have ${}^{\rho}\lim_{n\in^{\sigma}\widetilde{\mathbb{N}}} \left(1+\frac{1}{n}\right)^n = e$ because $n\log\left(1+\frac{1}{n}\right) = 1 - \frac{1}{2n} + O\left(\frac{1}{n^2}\right) \to 1$ and because of Theorem 31.

A little more involved proof concerns L'Hôpital rule for generalized smooth functions. For the sake of completeness, here we only recall the equivalent definition:

Definition 33 Let $X \subseteq {}^{\rho}\widetilde{\mathbb{R}}^n$ and $Y \subseteq {}^{\rho}\widetilde{\mathbb{R}}^d$. We say that $f : X \longrightarrow Y$ is a *generalized smooth function* (GSF) if

(i) $f : X \longrightarrow Y$ is a set-theoretical function.
(ii) There exists a net $(f_\varepsilon) \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^d)^{(0,1]}$ such that for all $[x_\varepsilon] \in X$:
   (a) $f(x) = [f_\varepsilon(x_\varepsilon)]$
   (b) $\forall \alpha \in \mathbb{N}^n : (\partial^\alpha f_\varepsilon(x_\varepsilon))$ is $\rho-$moderate.

For generalized smooth functions lots of results hold: closure with respect to composition, embedding of Schwartz's distributions, differential calculus, one-dimensional integral calculus using primitives, classical theorems (intermediate value, mean value, Taylor, extreme value, inverse and implicit function), multidimensional integration, Banach fixed point theorem, a Picard-Lindelöf theorem for both ODE and PDE, several results of calculus of variations, etc.

In particular, we have the following (see also [9] for the particular case of Colombeau generalized functions):

Theorem 34 *Let $U \subseteq {}^{\rho}\widetilde{\mathbb{R}}$ be a sharply open set and let $f : U \longrightarrow {}^{\rho}\widetilde{\mathbb{R}}$ be a GSF defined by the net of smooth functions $f_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$. Then*

(i) *There exists an open neighbourhood T of $U \times \{0\}$ and a GSF $R_f : T \to {}^{\rho}\widetilde{\mathbb{R}}$, called the* generalized incremental ratio *of f, such that*

$$f(x+h) = f(x) + h \cdot R_f(x,h) \qquad \forall (x,h) \in T. \tag{5.11}$$

   Moreover $R_f(x,0) = \left[f'_\varepsilon(x_\varepsilon)\right] = f'(x)$ is another GSF and we can hence recursively define $f^{(k)}(x)$.
(ii) *Any two generalized incremental ratios of f coincide on the intersection of their domains.*

(iii) *More generally, for all $k \in \mathbb{N}_{>0}$ there exists an open neighbourhood T of $U \times \{0\}$ and a GSF $R_f^k : T \to {}^\rho\widetilde{\mathbb{R}}$, called k-*th order Taylor ratio*of f, such that*

$$f(x+h) = \sum_{j=0}^{k-1} \frac{f^{(j)}(x)}{j!} h^j + R_f^k(x,h) \cdot h^k \qquad \forall (x,h) \in T. \tag{5.12}$$

Any two ratios of f of the same order coincide on the intersection of their domains.

We can now prove the following generalization of one of L'Hôpital rule:

Theorem 35 *Let $U \subseteq {}^\rho\widetilde{\mathbb{R}}$ be a sharply open set $(x_n)_n$, $(y_n)_n : {}^\sigma\widetilde{\mathbb{N}} \longrightarrow U$ be hypersequences converging to $l \in U$ and $m \in U$ respectively and such that*

$$^\rho\lim_{n\in{}^\sigma\widetilde{\mathbb{N}}} \frac{x_n - l}{y_n - m} =: C \in {}^\rho\widetilde{\mathbb{R}}.$$

Let $k \in \mathbb{N}_{>0}$ and $f, g : U \longrightarrow {}^\rho\widetilde{\mathbb{R}}$ be GSF such that for all $n \in {}^\sigma\widetilde{\mathbb{N}}$ and all $j = 0, \ldots, k-1$

$$\begin{aligned} &g^{(j)}(y_n) \in {}^\rho\widetilde{\mathbb{R}}^* \\ &f^{(j)}(l) = g^{(j)}(m) = 0 \\ &g^{(k)}(m) \in {}^\rho\widetilde{\mathbb{R}}^* \end{aligned} \tag{5.13}$$

Then for all $j = 0, \ldots, k-1$

$$\exists\, {}^\rho\lim_{n\in{}^\sigma\widetilde{\mathbb{N}}} \frac{f^{(j)}(x_n)}{g^{(j)}(y_n)} = C^k \cdot {}^\rho\lim_{n\in{}^\sigma\widetilde{\mathbb{N}}} \frac{f^{(k)}(x_n)}{g^{(k)}(y_n)}.$$

Proof Using (5.12) and (5.13), we can write

$$\begin{aligned} \frac{f(x_n)}{g(y_n)} &= \frac{\sum_{j=0}^{k-1} \frac{f^{(j)}(l)}{j!}(x_n - l)^j + (x_n - l)^k R_f^k(l, x_n - l)}{\sum_{j=0}^{k-1} \frac{g^{(j)}(m)}{j!}(y_n - m)^j + (y_n - m)^k R_g^k(m, y_n - m)} \\ &= \left(\frac{x_n - l}{y_n - m}\right)^k \cdot \frac{R_f^k(l, x_n - l)}{R_g^k(m, y_n - m)}. \end{aligned}$$

Since R_f^k and R_g^k are GSF, they are sharply continuous. Therefore, the right hand side of the previous equality tends to $C^k \cdot \frac{R_f^k(l,0)}{R_g^k(m,0)} = C^k \cdot \frac{f^{(k)}(l)}{g^{(k)}(m)}$. At the same limit converges the quotient $C^k \frac{f^{(k)}(x_n)}{g^{(k)}(y_n)}$ because $f^{(k)}$ and $g^{(k)}$ are also GSF and hence they are sharply continuous. The claim for $j = 1, \ldots, k-1$ follows by applying the conclusion for $j = 0$ with $f^{(j)}$ and $g^{(j)}$ instead of f and g. □

Note that for $x_n = y_n$, $l = m$, we have $C = 1$ and we get the usual L'Hôpital rule (formulated using hypersequences). Note that a similar theorem can also be proved without hypersequences and using the same Taylor expansion argument as in the previous proof.

5.3 Cauchy criterion and monotonic hypersequences.

In this section, we deal with classical criteria implying the existence of a hyperlimit.

Definition 36 We say that $(x_n)_{n\in{}^{\sigma}\widetilde{\mathbb{N}}}$ is a *Cauchy hypersequence* if

$$\forall q \in \mathbb{N}\, \exists M \in {}^{\sigma}\widetilde{\mathbb{N}}\, \forall n, m \in {}^{\sigma}\widetilde{\mathbb{N}}_{\ge M} : \ |x_n - x_m| < \mathrm{d}\rho^q .$$

Theorem 37 *A hypersequence converges if and only if it is a Cauchy hypersequence*

Proof To prove that the Cauchy criterion is a necessary condition it suffices to consider the inequalities:

$$|x_n - x_m| \le |x_n - l| + |x_m - l| \le \mathrm{d}\rho^{q+1} + \mathrm{d}\rho^{q+1} < \mathrm{d}\rho^q$$

Vice versa, assume that

$$\forall q \in \mathbb{N}\, \exists M_q \in {}^{\sigma}\widetilde{\mathbb{N}}\, \forall n, m \in {}^{\sigma}\widetilde{\mathbb{N}}_{\ge M_q} : \ |x_n - x_m| < \mathrm{d}\rho^q . \tag{5.14}$$

The idea is to use Cauchy completeness of ${}^{\rho}\widetilde{\mathbb{R}}$. In fact, set $h_1 := M_1$ and $h_{q+1} := M_{q+1} \vee h_q$. We claim that $(x_{h_q})_{q\in\mathbb{N}}$ is a standard Cauchy sequence converging to the same limit of $(x_n)_{n\in{}^{\sigma}\widetilde{\mathbb{N}}}$. From (5.14) it follows that $(x_{h_q})_{q\in\mathbb{N}}$ is a standard Cauchy sequence (in the sharp topology). Therefore, there exists $\bar{x} \in {}^{\rho}\widetilde{\mathbb{R}}$ such that $\lim_{q\to+\infty} x_{h_q} = \bar{x}$. Now, fix $q \in \mathbb{N}$ and pick any $m \ge q + 1$ such that

$$|x_{h_m} - \bar{x}| < \mathrm{d}\rho^{q+1}. \tag{5.15}$$

Then for all $N \ge M_{q+1}$ we have:

$$|x_N - \bar{x}| \le |x_N - x_{h_m}| + |x_{h_m} - \bar{x}| < 2\mathrm{d}\rho^{q+1} < \mathrm{d}\rho^q$$

because $h_m \ge h_{q+1} \ge M_{q+1}$ so that we can apply (5.14) and (5.15). □

Theorem 38 *A hypersequence converges if and only if*

$$\forall q \in \mathbb{N}\, \exists M \in {}^{\sigma}\widetilde{\mathbb{N}}\, \forall n, m \in {}^{\sigma}\widetilde{\mathbb{N}}_{\ge M} : m \ge n \ \Rightarrow \ |x_n - x_m| < \mathrm{d}\rho^q .$$

Proof It suffices to apply the inequality $|x_n - x_m| \le |x_n - x_{n\vee m}| + |x_{n\vee m} - x_m|$. □

The second classical criterion for the existence of a hyperlimit is related to the notion of monotonic hypersequence. The existence of several chains in ${}^{\sigma}\widetilde{\mathbb{N}}$ does not allow to arrive at any $M \in {}^{\sigma}\widetilde{\mathbb{N}}$ starting from any other lower $N \in {}^{\sigma}\widetilde{\mathbb{N}}$ and using the successor operation only a finite number of times. For this reason, the following is the most natural notion of monotonic hypersequence:

Definition 39 We say that $(x_n)_{N \in {}^{\sigma}\widetilde{\mathbb{N}}}$ is a *non-decreasing* (or *increasing*) hypersequence if

$$\forall n, m \in {}^{\sigma}\widetilde{\mathbb{N}} : \; n \geq m \;\Rightarrow\; x_n \geq x_m .$$

Similarly, we can define the notion of non-increasing (decreasing) hypersequence.

Theorem 40 *Let* $(x_n)_n : {}^{\sigma}\widetilde{\mathbb{N}} \longrightarrow {}^{\rho}\widetilde{\mathbb{R}}$ *be a non-decreasing hypersequence. Then*

$$\exists \, {}^{\rho}\lim_{n \in {}^{\sigma}\widetilde{\mathbb{N}}} x_n \iff \exists \sup \left\{ x_n \mid n \in {}^{\sigma}\widetilde{\mathbb{N}} \right\},$$

and in that case they are equal.

Proof Assume that $(x_n)_{n \in {}^{\sigma}\widetilde{\mathbb{N}}}$ converges to l and set $S := \{x_n \mid n \in {}^{\sigma}\widetilde{\mathbb{N}}\}$, we will show that $l = \sup(S)$. Now, using Definition 25, we have that $\forall n \in {}^{\sigma}\widetilde{\mathbb{N}}_{\geq N} : \; x_n < l + \mathrm{d}\rho^q$ for some $N \in {}^{\sigma}\widetilde{\mathbb{N}}$. But from Definition 39 $\forall n \in {}^{\sigma}\widetilde{\mathbb{N}} : \; x_n \leq x_{n \vee N} < l + \mathrm{d}\rho^q$. Therefore $x_n \leq l + \mathrm{d}\rho^q$ for all $n \in {}^{\sigma}\widetilde{\mathbb{N}}$, and the conclusion $x_n \leq l$ follows since $q \in \mathbb{N}$ is arbitrary. Finally, from Definition 25 of hyperlimit, for all $q \in \mathbb{N}$ we have the existence of $L \in {}^{\sigma}\widetilde{\mathbb{N}}$ such that $l - \mathrm{d}\rho^q < x_L \in S$ which completes the necessity part of the proof. Now, assume that $\exists \sup(S) =: l$. We have to prove that ${}^{\rho}\lim_{n \in {}^{\sigma}\widetilde{\mathbb{N}}} x_n = l$. In fact, using Rem. (i), we get

$$\forall q \in \mathbb{N} \, \exists x_N \in S : \; l - \mathrm{d}\rho^q < x_N ,$$

and $x_N \leq x_n \leq l < l + \mathrm{d}\rho^q$ for all $n \in {}^{\sigma}\widetilde{\mathbb{N}}_{\geq N}$ by Definition 39 of monotonicity. That is, $|l - x_n| = x_n - l < \mathrm{d}\rho^q$. □

Example 41 The hypersequence $x_n := \mathrm{d}\rho^{1/n}$ is non-decreasing. Assume that $(x_n)_n$ converges to l and that $\sigma \leq \rho^R$ for some $R \in \mathbb{R}_{>0}$. Since $x_n \geq \mathrm{d}\rho$, by Theorem 30, we get $l \geq \mathrm{d}\rho$. Therefore, applying the logarithm and the exponential functions, from Theorem 31 we obtain that $l = 1$ because from $\sigma \leq \rho^R$ it follows that ${}^{\rho}\lim_{n \in {}^{\sigma}\widetilde{\mathbb{N}}} \frac{\log(\mathrm{d}\rho)}{n} = 0$. But this is impossible since $1 \approx 1 - \mathrm{d}\rho \not\leq \mathrm{d}\rho^{1/n}$. Thereby, $\nexists \sup \left\{ \mathrm{d}\rho^{1/n} \mid n \in {}^{\sigma}\widetilde{\mathbb{N}}_{>0} \right\}$.

6 Limit superior and inferior

We have two possibilities to define the notions of limit superior and inferior in a non-Archimedean setting such as ${}^{\rho}\widetilde{\mathbb{R}}$: the first one is to assume that both $\alpha_m := \sup\{x_n \mid n \in {}^{\sigma}\widetilde{\mathbb{N}}_{\geq m}\}$ and $\inf\{\alpha_m \mid m \in {}^{\sigma}\widetilde{\mathbb{N}}\}$ exist (the former for all $m \in {}^{\sigma}\widetilde{\mathbb{N}}$); the second possibility is to use inequalities to avoid the use of supremum and infimum. In fact, in the real case we have $\iota \le \sup_{n\ge m} x_n \le \iota + \varepsilon$ if and only if

$$\forall n \ge m : \ x_n \le \iota + \varepsilon$$
$$\forall \varepsilon \, \exists \bar{n} \ge m : \ \iota - \varepsilon \le x_{\bar{n}}.$$

Definition 42 Let $(x_n)_n : {}^{\sigma}\widetilde{\mathbb{N}} \longrightarrow {}^{\rho}\widetilde{\mathbb{R}}$ be an hypersequence, then we say that $\iota \in {}^{\rho}\widetilde{\mathbb{R}}$ is the *limit superior* of $(x_n)_n$ if

(i) $\forall q \in \mathbb{N} \, \exists N \in {}^{\sigma}\widetilde{\mathbb{N}} \, \forall n \ge N : \ x_n \le \iota + \mathrm{d}\rho^q$;
(ii) $\forall q \in \mathbb{N} \, \forall N \in {}^{\sigma}\widetilde{\mathbb{N}} \, \exists \bar{n} \ge N : \ \iota - \mathrm{d}\rho^q \le x_{\bar{n}}$.

Similarly, we say that $\sigma \in {}^{\rho}\widetilde{\mathbb{R}}$ is the *limit inferior* of $(x_n)_n$ if

(iii) $\forall q \in \mathbb{N} \, \exists N \in {}^{\sigma}\widetilde{\mathbb{N}} \, \forall n \ge N : \ x_n \ge \sigma - \mathrm{d}\rho^q$;
(iv) $\forall q \in \mathbb{N} \, \forall N \in {}^{\sigma}\widetilde{\mathbb{N}} \, \exists \bar{n} \ge N : \ \sigma + \mathrm{d}\rho^q \ge x_{\bar{n}}$.

We have the following results (clearly, dual results hold for the limit inferior):

Theorem 43 *Let* $(x_n)_n, (y_n)_n : {}^{\sigma}\widetilde{\mathbb{N}} \longrightarrow {}^{\rho}\widetilde{\mathbb{R}}$ *be hypersequences, then*

(i) *There exists at most one limit superior and at most one limit inferior. They are denoted with* ${}^{\rho}\limsup_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} x_n$ *and* ${}^{\rho}\liminf_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} x_n$.
(ii) *If* $\exists \sup \left\{x_n \mid n \in {}^{\sigma}\widetilde{\mathbb{N}}_{\ge m}\right\} =: \alpha_m$ *for all* $m \in {}^{\sigma}\widetilde{\mathbb{N}}$, *then* $\exists \, {}^{\rho}\limsup_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} x_n$ *if and only if* $\exists \inf \left\{\alpha_m \mid m \in {}^{\sigma}\widetilde{\mathbb{N}}\right\}$, *and in that case*

$$ {}^{\rho}\limsup_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} x_n = {}^{\rho}\lim_{m\in{}^{\sigma}\widetilde{\mathbb{N}}} \alpha_m = \inf\left\{\alpha_m \mid m \in {}^{\sigma}\widetilde{\mathbb{N}}\right\}.$$

(iii) ${}^{\rho}\limsup_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} (-x_n) = -{}^{\rho}\liminf_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} x_n$ *in the sense that if one of them exists, then also the other one exists and in that case they are equal.*
(iv) $\exists {}^{\rho}\lim_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} x_n$ *if and only if* $\exists {}^{\rho}\limsup_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} x_n = {}^{\rho}\liminf_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} x_n$.
(v) *If* $\exists {}^{\rho}\limsup_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} x_n$, ${}^{\rho}\limsup_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} y_n$, ${}^{\rho}\limsup_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} (x_n + y_n)$, *then*

$$ {}^{\rho}\limsup_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} (x_n + y_n) \le {}^{\rho}\limsup_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} x_n + {}^{\rho}\limsup_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} y_n.$$

In particular, if $\forall N \in {}^{\sigma}\widetilde{\mathbb{N}} \, \forall \bar{n}, \hat{n} \ge N \, \exists n \ge N : \ x_{\bar{n}} + y_{\hat{n}} \le x_n + y_n$, *then the existence of the single limit superiors implies the existence of the limit superior of the sum.*
(vi) *If* $x_n, y_n \ge 0$ *for all* $n \in {}^{\sigma}\widetilde{\mathbb{N}}$ *and if* $\exists {}^{\rho}\limsup_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} x_n$, ${}^{\rho}\limsup_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} y_n$, ${}^{\rho}\limsup_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} (x_n \cdot y_n)$, *then*

$$ {}^{\rho}\limsup_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} (x_n \cdot y_n) \le {}^{\rho}\limsup_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} x_n \cdot {}^{\rho}\limsup_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} y_n.$$

In particular, if $\forall N \in {}^{\sigma}\widetilde{\mathbb{N}} \, \forall \bar{n}, \hat{n} \ge N \, \exists n \ge N : \ x_{\bar{n}} \cdot y_{\hat{n}} \le x_n \cdot y_n$, *then the existence of the single limit superiors implies the existence of the limit superior of the product.*
(vii) *If* $\exists \, {}^{\rho}\limsup_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} x_n =: \iota$, *then there exists a sequence* $(\bar{n}_q)_{q\in\mathbb{N}}$ *of* ${}^{\sigma}\widetilde{\mathbb{N}}$ *such that*

(a) $\bar{n}_{q+1} > \bar{n}_q$ *for all* $q \in \mathbb{N}$;
(b) $\lim_{q\to+\infty} \bar{n}_q = +\infty$ *in* ${}^{\sigma}\widetilde{\mathbb{R}}$;
(c) $\exists \lim_{q\to+\infty} x_{\bar{n}_q} = \iota$.

(viii) *Assume to have a sequence* $(\bar{n}_q)_{q\in\mathbb{N}}$ *satisfying the previous conditions (a), (b), (c) and*

$$\forall n \in {}^{\sigma}\widetilde{\mathbb{N}}\, \exists p \in \mathbb{N} : \ \bar{n}_p \ge n, \ x_n \le x_{\bar{n}_p}. \tag{6.1}$$

Then $\exists\, {}^{\rho}\limsup_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} x_n =: \iota$.

Proof (i): Let ι_1, ι_2 be both limit superior of $(x_n)_n$. Based on Lemma 6.(iii), without loss of generality we can assume that $\iota_1 <_{\mathrm{s}} \iota_2$. According to Lemma 2, there exists $m \in \mathbb{N}$ such that $\iota_1 + \mathrm{d}\rho^m <_{\mathrm{s}} \iota_2$. Take q_1, q_2 large enough so that $\mathrm{d}\rho^{q_1} + \mathrm{d}\rho^{q_2} < \mathrm{d}\rho^m$. Using the last two inequalities, we obtain

$$\iota_1 + \mathrm{d}\rho^{q_1} <_{\mathrm{s}} \iota_2 - \mathrm{d}\rho^{q_2}. \tag{6.2}$$

Using Definition 42.(i), we can find $N_1 \in {}^{\sigma}\widetilde{\mathbb{N}}$ such that

$$\forall n \in {}^{\sigma}\widetilde{\mathbb{N}}_{\ge N_1} : \ x_n \le \iota_1 + \mathrm{d}\rho^{q_1}. \tag{6.3}$$

Using Definition 42.(ii) with $q = q_1$ and $N = N_1$, we get

$$\exists \bar{n} \in {}^{\sigma}\widetilde{\mathbb{N}}_{\ge N_1} : \ \iota_2 - \mathrm{d}\rho^{q_2} \le x_{\bar{n}}. \tag{6.4}$$

We now use (6.2), (6.4) and (6.3) for $n = \bar{n}$ and we obtain $x_{\bar{n}} \le \iota_1 + \mathrm{d}\rho^{q_1} <_{\mathrm{s}} \iota_2 - \mathrm{d}\rho^{q_2} \le x_{\bar{n}}$, which is a contradiction.

(ii): Lemma 24. (iii) implies that $(\alpha_m)_m$ is non-increasing. Therefore, we have ${}^{\rho}\lim_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} \alpha_m = \inf \left\{\alpha_m \mid m \in {}^{\sigma}\widetilde{\mathbb{N}}\right\}$ if these terms exist from Theorem 40. But Cor. ?? and Definition 42.(i) imply $\alpha_m \le \iota + \mathrm{d}\rho^q$. Finally, Definition 42.(ii) yields $\iota - \mathrm{d}\rho^q \le x_{\bar{n}} \le \alpha_m$, which proves that $\exists\, {}^{\rho}\lim_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} \alpha_m = {}^{\rho}\limsup_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} x_m = \iota$.

(iii): Directly from Definition 42.

(iv): Assume that hyperlimit superior and inferior exist and are equal to l. From Definition 42.(i) and Definition 42.(iii) we get $l - \mathrm{d}\rho^q \le x_n \le l + \mathrm{d}\rho^q$ for all $n \ge N$. Vice versa, assume that the hyperlimit exists and equals l, so that $l - \mathrm{d}\rho^q \le x_n \le l + \mathrm{d}\rho^q$ for all $n \ge N$. Then both Definition 42.(i) and Definition 42.(iii) trivially hold. Finally, Definition 42.(ii) and Definition 42.(iv) hold taking e.g. $\bar{n} = N$.

(v): Setting

$$\begin{aligned}
\iota &:= {}^{\rho}\limsup_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} x_n \\
j &:= {}^{\rho}\limsup_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} y_n \\
l &:= {}^{\rho}\limsup_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} (x_n + y_n),
\end{aligned}$$

from Definition 42 we get $l - \mathrm{d}\rho^q \le x_{\bar{n}} + y_{\bar{n}} \le \iota + j + 2\mathrm{d}\rho^q$, which implies $l \le \iota + j$ for $q \to +\infty$. Adding Definition 42.(ii) we obtain $\iota + j - 2\mathrm{d}\rho^q \le x_{\bar{n}} + y_{\hat{n}}$ for some $\bar{n}, \hat{n} \ge N \in {}^\sigma\widetilde{\mathbb{N}}$. Therefore, if $x_{\bar{n}} + y_{\hat{n}} \le x_n + y_n$ for some $n \ge N$, this yields the second claim. Similarly, one can prove (vi).

(vii): From Definition 42.(i), choose an $N_q = N$ for each $q \in \mathbb{N}$, i.e.

$$\forall q \in \mathbb{N}\, \exists N_q \in {}^\sigma\widetilde{\mathbb{N}}\, \forall n \ge N_q : \ x_n \le \iota + \mathrm{d}\rho^q. \tag{6.5}$$

Applying Definition 42.(ii) with $q > 0$ and $N = N_q \vee \left(\bar{n}_{q-1} + 1\right) \vee \left[\mathrm{int}(\sigma_\varepsilon^{-q})\right] \in {}^\sigma\widetilde{\mathbb{N}}$, we get the existence of $\bar{n}_q \ge N_q$ such that both (a) and (b) hold and $\iota - \mathrm{d}\rho^q \le x_{\bar{n}_q}$. Thereby, from (6.5) we also get (c).

(viii): Write (c) as

$$\forall q \in \mathbb{N}\, \exists Q_q \in \mathbb{N}\, \forall p \in \mathbb{N}_{\ge Q_q} : \ \iota - \mathrm{d}\rho^p \le x_{\bar{n}_p} \le \iota + \mathrm{d}\rho^p. \tag{6.6}$$

Set $N := \bar{n}_{Q_q} \in {}^\sigma\widetilde{\mathbb{N}}$. For $n \ge N$, from (6.1) we get the existence of $p \in \mathbb{N}$ such that $\bar{n}_p \ge n$ and $x_n \le x_{\bar{n}_p}$. Thereby, $\bar{n}_p \ge \bar{n}_{Q_q}$ and hence $p \ge Q_q$ because of (a) and thus $x_n \le x_{\bar{n}_p} \le \iota + \mathrm{d}\rho^q$. Finally, condition (ii) of Definition 42 follows from (6.6) and (b). □

It remains an open problem to show an example that proves as necessary the assumption of Theorem 43.(ii), i.e. that that the previous definition of limit superior and inferior is strictly more general than the simple transposition of the classical one.

Example 44 (i) Directly from Definition 42, we have that

$${}^\rho\limsup_{n \in {}^\sigma\widetilde{\mathbb{N}}} (-1)^n = 1, \ {}^\rho\liminf_{n \in {}^\sigma\widetilde{\mathbb{N}}} (-1)^n = -1$$

(ii) Let $\mu \in {}^\rho\widetilde{\mathbb{R}}$ be such that $\mu|_L = 1$ and $\mu|_{L^c} = -1$, where $L, L^c \subseteq_0 I$. Then $\mu^n \le 1$ and $1 - \mathrm{d}\rho^q \le \mu^{\bar{n}}$ if $\mathrm{ni}(\bar{n})_\varepsilon$ is even for all ε small. Therefore ${}^\rho\limsup_{n \in {}^\sigma\widetilde{\mathbb{N}}} \mu^n = 1$, $\sup_{n \ge m} \mu^n = 1$, whereas $\nexists\, {}^\rho\lim_{n \in {}^\sigma\widetilde{\mathbb{N}}} \mu^n$.

(iii) From (vii) and (viii) of Theorem43 it follows that for an increasing hypersequence $(x_n)_n$, $\exists\, {}^\rho\limsup_{n \in {}^\sigma\widetilde{\mathbb{N}}} x_n$ if and only if $\exists\, {}^\rho\lim_{n \in {}^\sigma\widetilde{\mathbb{N}}} x_n$. Therefore, example 41 implies that $\nexists\, {}^\rho\limsup_{n \in {}^\sigma\widetilde{\mathbb{N}}} \mathrm{d}\rho^{1/n}$.

7 Conclusions

In this work we showed how to deal with several deficiencies of the ring of Robinson-Colombeau generalized numbers ${}^\rho\widetilde{\mathbb{R}}$: trichotomy law for the order relations $\le$ and $<$, existence of supremum and infimum and limits of sequences with a topology generated by infinitesimal radii. In each case, we obtain a faithful generalization of the classical case of real numbers. We think that some of the ideas we presented in this article can inspire similar works in other non-Archimedean settings such as (constructive) nonstandard analysis, p-adic analysis, the Levi-Civita field, surreal numbers, etc.

Clearly, the notions introduced here open the possibility to extend classical proofs in dealing with series, analytic generalized functions, sigma-additivity in integration of generalized functions, non-Archimedean functional analysis, just to mention a few.

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Chapter 6

Hyperseries in the non-Archimedean ring of Colombeau generalized numbers

By D. Tiwari, P. Giordano

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Hyperseries in the non-Archimedean ring of Colombeau generalized numbers

Diksha Tiwari[1] · **Paolo Giordano**[1]

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Abstract

This article is the natural continuation of the paper: Mukhammadiev et al. *Supremum, infimum and hyperlimits of Colombeau generalized numbers* in this journal. Since the ring ${}^{\rho}\widetilde{\mathbb{R}}$ of Robinson-Colombeau is non-Archimedean and Cauchy complete, a classical series $\sum_{n=0}^{+\infty} a_n$ of generalized numbers $a_n \in {}^{\rho}\widetilde{\mathbb{R}}$ is convergent *if* and only if $a_n \to 0$ in the sharp topology. Therefore, this property does not permit us to generalize several classical results, mainly in the study of analytic generalized functions (as well as, e.g., in the study of sigma-additivity in integration of generalized functions). Introducing the notion of hyperseries, we solve this problem recovering classical examples of analytic functions as well as several classical results.

Keywords Colombeau generalized numbers · Non-archimedean rings · Generalized functions

Mathematics Subject Classification 46F-XX · 46F30 · 26E30

1 Introduction

In this article, the study of supremum, infimum and hyperlimits of Colombeau generalized numbers (CGN) we carried out in [13] is applied to the introduction of a corresponding notion of hyperseries. In [13], we recalled that $(x_n)_{n\in\mathbb{N}} \in \widetilde{\mathbb{R}}^{\mathbb{N}}$ is a

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Paolo Giordano
paolo.giordano@univie.ac.at

Diksha Tiwari
diksha.tiwari@univie.ac.at

1 University of Vienna, Vienna, Austria

Cauchy sequence if and only if $\lim_{n\to+\infty}|x_{n+1}-x_n|=0$ (in the sharp topology). As a consequence, a series of CGN

$$\sum_{n\in\mathbb{N}} a_n \text{ converges} \iff a_n \to 0 \text{ (in the sharp topology).} \tag{1.1}$$

Once again, this is a well-known property of every ultrametric space, cf., e.g., [11]. The point of view of the present work is that in a non-Archimedean ring such as ${}^{\rho}\widetilde{\mathbb{R}}$, the notion of hyperseries $\sum_{n\in{}^{\rho}\widetilde{\mathbb{N}}} a_n$, i.e. where we sum over the set of hyperfinite natural numbers ${}^{\rho}\widetilde{\mathbb{N}}$, yields results which are more closely related to the classical ones, e.g. in studying analytic functions, sigma additivity and limit theorems for integral calculus, or in possible generalization of the Cauchy-Kowalevski theorem to generalized smooth functions (GSF; see e.g. [7]).

Considering the theory of analytic CGF as developed in [17] for the real case and in [20] for the complex one, it is worth to mention that several properties have been proved in both cases: closure with respect to composition, integration over homotopic paths, Cauchy integral theorem, existence of analytic representatives u_ε, a real analytic CGF is identically zero if it is zero on a set of positive Lebesgue measure, etc. (cf. [14,17,20] and references therein). On the other hand, even if in [20] it is also proved that each complex analytic CGF can be written as a Taylor series, necessarily this result holds only in an infinitesimal neighborhood of each point. The impossibility to extend this property to a finite neighborhood is due to (1.1) and is hence closely related to the approach we follow in the present article.

We refer to [13] for notions such as the ring of Robinson-Colombeau, subpoints, hypernatural numbers, supremum, infimum and hyperlimits. See also [12] for a more general approach to asymptotic gauges. In the present paper, we focus only on examples and properties related to hyperseries, postponing those about analytic functions, integral calculus and the Cauchy-Kowalevski theorem to subsequent works. Once again, the ideas presented in the present article, which needs only [13] as prior knowledge, can surely be useful to explore similar ideas in other non-Archimedean Cauchy complete settings, such as [2,3,11,19].

2 Hyperseries and their basic properties

2.1 Definition of hyperfinite sums and hyperseries

In order to define these notions, the main idea, like in the Archimedean case $\mathbb{R}$, is to reduce the notion of hyperseries to that of hyperlimit. Since to take an hyperlimit we have to consider two gauges ρ, σ, it is hence natural to consider the same assumption for hyperseries. However, in a non-Archimedean setting, the aforesaid main idea implies that the main problem is not in defining hyperseries itself, but already in defining what an hyperfinite sum is. In fact, if $(a_i)_{i\in{}^{\sigma}\widetilde{\mathbb{N}}}$ is a family of generalized numbers of ${}^{\rho}\widetilde{\mathbb{R}}$ indexed in ${}^{\sigma}\widetilde{\mathbb{N}}$, it is not even clear what $\sum_{i=0}^{4} a_i$, means because there are infinite $i\in{}^{\sigma}\widetilde{\mathbb{N}}$ such that $0\le i\le 4$. In general, in a sum of the form $\sum_{i=0}^{N} a_i$ with $N\in{}^{\sigma}\widetilde{\mathbb{N}}$, the number of addends depends on how small is the gauge σ and hence how large can be $N \le \mathrm{d}^{-R}$ (for some $R \in \mathbb{N}$). The problem is simplified if we consider only *ordinary* sequences $(a_n)_{n\in\mathbb{N}}$ of CGN in ${}^\rho\widetilde{\mathbb{R}}$ and an ε-wise definition of hyperfinite sum. In order to accomplish this goal, we first need to extend the sequence

$$\left(N \in \mathbb{N} \mapsto \sum_{n=0}^{N} a_n \in {}^\rho\widetilde{\mathbb{R}}\right) : \mathbb{N} \longrightarrow {}^\rho\widetilde{\mathbb{R}} \tag{2.1}$$

of partial sums with summands $a_n \in {}^\rho\widetilde{\mathbb{R}}$, $n \in \mathbb{N}$, to the entire set ${}^\sigma\widetilde{\mathbb{N}}$ of hyperfinite numbers. This problem is not so easy to solve: in fact, the sequence of representatives of zero: $a_{n\varepsilon} = 0$ if $\varepsilon \le \frac{1}{n}$ and $a_{n\varepsilon} = (1-\varepsilon)^{-n}$ otherwise, where $n \in \mathbb{N}_{>0}$, satisfies

$$\sum_{n=0}^{N_\varepsilon} a_{n\varepsilon} = \sum_{n=\left\lceil \frac{1}{\varepsilon}\right\rceil}^{N_\varepsilon} \left(\frac{1}{1-\varepsilon}\right)^n \quad \forall \varepsilon \in (0,1)_{\mathbb{R}}, \tag{2.2}$$

but if $N_\varepsilon \to +\infty$ this sum diverges to $+\infty$ because $\frac{1}{1-\varepsilon} > 1$; moreover, for suitable N_ε, the net in (2.2) is of the order of $\left(\frac{1}{1-\varepsilon}\right)^{N_\varepsilon}$, which in general is not ρ-moderate. To solve this first problem, we have at least two possibilities: the first one is to consider a Robinson-Colombeau ring defined by the index set $\mathbb{N} \times I$ and ordered by $(n,\varepsilon) \le (m,e)$ if and only if $\varepsilon \le e$. In this solution, moderate representatives are nets $(a_{n\varepsilon})_{n,\varepsilon} \in \mathbb{R}^{\mathbb{N}\times I}$ satisfying the uniformly moderate condition

$$\exists Q \in \mathbb{N}\, \forall^0\varepsilon\, \forall n \in \mathbb{N} : \; |a_{n\varepsilon}| \le \rho_\varepsilon^{-Q}. \tag{2.3}$$

Negligible nets are $(a_{n\varepsilon})_{n,\varepsilon} \in \mathbb{R}^{\mathbb{N}\times I}$ such that

$$\forall q \in \mathbb{N}\, \forall^0\varepsilon\, \forall n \in \mathbb{N} : \; |a_{n\varepsilon}| \le \rho_\varepsilon^{q}. \tag{2.4}$$

Note that, with respect to the aforementioned directed order relation, for any property $\mathcal{P}$, we have

$$(\exists (n_0,\varepsilon_0) \in \mathbb{N} \times I\, \forall (n,\varepsilon) \le (n_0,\varepsilon_0) : \; \mathcal{P}\{n,\varepsilon\}) \iff \forall^0\varepsilon\, \forall n \in \mathbb{N} : \; \mathcal{P}\{n,\varepsilon\}.$$

The main problem with this solution is that it works to define hyperfinite sums only if

$$\exists Q_{\sigma,\rho} \in \mathbb{N}\, \forall^0\varepsilon : \; \sigma_\varepsilon \ge \rho_\varepsilon^{Q_{\sigma,\rho}}. \tag{2.5}$$

Assume, indeed, that (2.3) holds for all $\varepsilon \le \varepsilon_0$, so that if $N = [N_\varepsilon] \in {}^\sigma\widetilde{\mathbb{N}}$, $N_\varepsilon \in \mathbb{N}$, we have

$$\left|\sum_{n=0}^{N_\varepsilon} a_{n\varepsilon}\right| \le N_\varepsilon \cdot \rho_\varepsilon^{-Q} \le \sigma_\varepsilon^{-R} \cdot \rho_\varepsilon^{-Q} \le \rho_\varepsilon^{-RQ_{\sigma,\rho}-Q}, \tag{2.6}$$

where we assumed that $N_\varepsilon \le \sigma_\varepsilon^{-R}$ for these $\varepsilon \le \varepsilon_0$. This is intuitively natural since the Robinson-Colombeau ring defined by the aforementioned order relation depends only on the gauge ρ, whereas the moderateness in (2.6) depends on the product $N_\varepsilon \cdot \rho_\varepsilon^{-Q} \le \sigma_\varepsilon^{-R} \cdot \rho_\varepsilon^{-Q}$ between how many addends we are taking (that depends on σ) and the uniform moderateness property (2.3).

This first solution has three drawbacks: The first one, as explained above, is that in this ring we can consider hyperfinite sum only if the relation (2.5) between the two considered gauges holds. The second one is that we cannot consider divergent hyperseries such as e.g. ${}^\rho\sum_{n\in{}^\sigma\widetilde{\mathbb{N}}} n$ because $n \le \rho_\varepsilon^{-Q}$ do not hold for ε small and uniformly for all $n \in \mathbb{N}$. The third one is that we would like to apply [13, Thm. 28] to prove the convergence of hyperseries by starting from the corresponding converging series of ε-representatives; however, [13, Thm. 28] do not allow us to get the limitation (2.5) (and later, we will see that in general the limitation (2.5) is impossible to achieve: see just after the proof of Theorem 12).

The second possibility to extend (2.1) to ${}^\sigma\widetilde{\mathbb{N}}$, a possibility that depends on two gauges but works for any σ and ρ, is to say that hyperseries can be computed only for representatives $(a_{n\varepsilon})_{n\varepsilon}$ which are *moderate over hypersums*, i.e. to ask

$$\forall N \in {}^\sigma\widetilde{\mathbb{N}} : \left(\sum_{n=0}^{\mathrm{ni}(N)_\varepsilon} a_{n\varepsilon}\right) \in \mathbb{R}_\rho. \tag{2.7}$$

For example, if $k = [k_\varepsilon] \in (0, 1) \subseteq {}^\rho\widetilde{\mathbb{R}}$, then $\left(k_\varepsilon^n\right)_{n,\varepsilon}$ is moderate over hypersums (see also Example 8 below); but also the aforementioned $a_{n\varepsilon} = n$ for all $n \in \mathbb{N}$ and $\varepsilon \in I = (0, 1]_{\mathbb{R}}$ is clearly of the same type if $\mathbb{R}_\sigma \subseteq \mathbb{R}_\rho$.

However, if $a_n = [a_{n\varepsilon}] = [\bar{a}_{n\varepsilon}]$ for all $n \in \mathbb{N}$ are two sequences which are moderate over hypersums, does the equality $\left[\sum_{n=0}^{\mathrm{ni}(N)_\varepsilon} a_{n\varepsilon}\right] = \left[\sum_{n=0}^{\mathrm{ni}(N)_\varepsilon} \bar{a}_{n\varepsilon}\right]$ hold? The answer is negative: let $a_{n\varepsilon}{:=}0$ if $\varepsilon < \frac{1}{n+1}$ and $a_{n\varepsilon}{:=}1$ otherwise, then $[a_{n\varepsilon}] = 0$ are representatives of zero, but the corresponding series is not zero:

$$\forall N = [N_\varepsilon] \in {}^\rho\widetilde{\mathbb{N}} : \sum_{n=0}^{N_\varepsilon} a_{n\varepsilon} = \sum_{n=\left\lceil\frac{1}{\varepsilon}\right\rceil-1}^{N_\varepsilon} 1 = N_\varepsilon - \left\lceil\frac{1}{\varepsilon}\right\rceil + 2. \tag{2.8}$$

Note that both examples (2.2) and (2.8) show that in dealing with hypersums, also the values $a_{n\varepsilon}$ for "ε large" may play a role. We can then proceed like for (2.7) by saying that $(a_{n\varepsilon})_{n,\varepsilon} \sim_{\sigma\rho} (\bar{a}_{n\varepsilon})_{n,\varepsilon}$ if

$$\forall M, N \in {}^\sigma\widetilde{\mathbb{N}} : \left(\sum_{n=\mathrm{ni}(N)_\varepsilon}^{\mathrm{ni}(M)_\varepsilon} (a_{n\varepsilon} - \bar{a}_{n\varepsilon})\right) \sim_\rho 0. \tag{2.9}$$

The idea of this second solution is hence to consider hyperseries only for representatives which are moderate over hypersums modulo the equivalence relation (2.9).

In the following definition, we will consider both solutions:

Definition 1 The quotient set $\left(\mathbb{R}^{\mathbb{N}\times I}\right)_{\sigma\rho} / \sim_{\sigma\rho} =: {}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$ of the set $\left(\mathbb{R}^{\mathbb{N}\times I}\right)_{\sigma\rho}$ of nets which are σ, ρ*-moderate over hypersums*

$$(a_{n\varepsilon})_{n,\varepsilon} \in \left(\mathbb{R}^{N\times I}\right)_{\sigma\rho} \ :\Longleftrightarrow \ \forall N \in {}^{\sigma}\widetilde{\mathbb{N}} : \left(\sum_{n=0}^{\mathrm{ni}(N)_\varepsilon} a_{n\varepsilon}\right) \in \mathbb{R}_{\rho},$$

by σ, ρ*-negligible nets*

$$(a_{n\varepsilon})_{n,\varepsilon} \sim_{\sigma\rho} (\bar{a}_{n\varepsilon})_{n,\varepsilon} \ :\Longleftrightarrow \ \forall M, N \in {}^{\sigma}\widetilde{\mathbb{N}} : \left(\sum_{n=\mathrm{ni}(N)_\varepsilon}^{\mathrm{ni}(M)_\varepsilon} (a_{n\varepsilon} - \bar{a}_{n\varepsilon})\right) \sim_{\rho} 0,$$

is called the *space of sequences for hyperseries*. Nets of $\mathbb{R}^{\mathbb{N}\times I}$ are denoted as $(a_{n\varepsilon})_{n,\varepsilon}$ or simply as $(a_{n\varepsilon})$; equivalence classes of ${}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$ are denoted as $(a_n)_n = [(a_{n\varepsilon})_{n,\varepsilon}] = [a_{n\varepsilon}] \in {}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$.

The ring of Robinson-Colombeau defined by the index set $\mathbb{N} \times I$ ordered by

$$(n, \varepsilon) \le (m, e) \ :\Longleftrightarrow \ \varepsilon \le e, \tag{2.10}$$

is denoted as ${}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{u}}$. We recall that ${}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{u}}$ is the quotient space of moderate nets $(a_{n\varepsilon})_{n,\varepsilon}$ (i.e. (2.3) holds) modulo negligible nets (i.e. (2.4) holds) with respect to the directed order relation (2.10).

The letter 'u' in ${}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{u}}$ recalls that in this case we are considering *uniformly* moderate (and *uniformly* negligible) sequences. The letter 's' in ${}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$ recalls that this is the space of *sequences* for hyperseries. We recall that ${}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{u}}$ is a ring with respect to pointwise multiplication $(a_n)_n \cdot (b_n)_n = \left[(a_{n\varepsilon} \cdot b_{n\varepsilon})_{n\varepsilon}\right]$. When we want to distinguish equivalence classes in these two quotient sets, we use the notations

$$(a_n)_n = [a_{n\varepsilon}]_{\mathrm{s}} \in {}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}, \quad \{a_n\}_n = [a_{n\varepsilon}]_{\mathrm{u}} \in {}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{u}}.$$

Note explicitly that, on the contrary with respect to ${}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$, the ring ${}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{u}}$ depends on only one gauge ρ.

We already observed the importance to consider suitable relations between the two gauges σ and ρ:

Definition 2 Let σ, ρ be two gauges, then we define

$$\sigma \ge \rho^* \ :\Longleftrightarrow \ \exists Q_{\sigma,\rho} \in \mathbb{N}\, \forall^0 \varepsilon : \ \sigma_\varepsilon \ge \rho_\varepsilon^{Q_{\sigma,\rho}}.$$
$$\sigma \le \rho^* \ :\Longleftrightarrow \ \exists Q_{\sigma,\rho} \in \mathbb{N}_{>0}\, \forall^0 \varepsilon : \ \sigma_\varepsilon \le \rho_\varepsilon^{Q_{\sigma,\rho}}.$$

(Note that if $\sigma \ge \rho^*$, then $Q_{\sigma,\rho} > 0$ since $\sigma_\varepsilon \to 0$).

It is easy to prove that $(-) \le (-)^*$ is a reflexive, transitive and antisymmetric relation, in the sense that $\sigma \le \rho^*$ and $\rho \le \sigma^*$ imply $\sigma_\varepsilon = \rho_\varepsilon$ for ε small, whereas $(-) \ge (-)^*$

is only a partial order. Clearly, $\sigma \geq \rho^*$ is equivalent to the inclusion of σ-moderate nets $\mathbb{R}_\sigma \subseteq \mathbb{R}_\rho$, whereas $\sigma \leq \rho^*$ is equivalent to $\mathbb{R}_\sigma \supseteq \mathbb{R}_\rho$.

It is well-known that there is no natural product between two ordinary series in $\mathbb{R}$ and involving summations with only one index set in $\mathbb{N}$ (see e.g. [4] and Sect. 4 below). This is the main motivation to consider only a structure of ${}^\rho\widetilde{\mathbb{R}}$-module on ${}^\rho_\sigma\widetilde{\mathbb{R}}_{\mathrm{s}}$ and, later, the natural Cauchy product between hyperseries:

Theorem 3 *${}^\rho_\sigma\widetilde{\mathbb{R}}_{\mathrm{s}}$ is a quotient ${}^\rho\widetilde{\mathbb{R}}$-module.*

Proof The closure of the set of σ, ρ-moderate nets over hypersums, i.e. $(\mathbb{R}^{\mathbb{N}\times I})_{\sigma\rho} = \left\{(a_{n\varepsilon}) \mid \forall N \in {}^\sigma\widetilde{\mathbb{N}} : \left(\sum_{n=0}^{\mathrm{ni}(N)_\varepsilon} a_{n\varepsilon}\right) \in \mathbb{R}_\rho\right\}$ with respect to pointwise sum $(a_{n\varepsilon} + b_{n\varepsilon})$ and product $(r_\varepsilon) \cdot (a_{n,\varepsilon}) = (r_\varepsilon \cdot a_{n,\varepsilon})$ by $(r_\varepsilon) \in \mathbb{R}_\rho$ follows from similar properties of the ring $\mathbb{R}_\rho$. Similarly to the case of ${}^\rho\widetilde{\mathbb{R}}$, we can finally prove that the equivalence relation (2.9) is a congruence with respect to these operations. □

We now prove that if $(a_n)_n \in {}^\rho_\sigma\widetilde{\mathbb{R}}_{\mathrm{s}}$, then hyperfinite sums are well-defined:

Theorem 4 *Let $(a_n)_n = [a_{n\varepsilon}] \in {}^\rho_\sigma\widetilde{\mathbb{R}}_{\mathrm{s}}$ and let σ, ρ be two arbitrary gauges, then the map*

$$(M, N) \in {}^\sigma\widetilde{\mathbb{N}}^2 \mapsto \left[\sum_{n=\mathrm{ni}(N)_\varepsilon}^{\mathrm{ni}(M)_\varepsilon} a_{n\varepsilon}\right] \in {}^\rho\widetilde{\mathbb{R}}$$

is well-defined.

Proof ρ-moderateness directly follows from $(a_n)_n \in {}^\rho_\sigma\widetilde{\mathbb{R}}_{\mathrm{s}}$; in fact:

$$\left|\sum_{n=\mathrm{ni}(N)_\varepsilon}^{\mathrm{ni}(M)_\varepsilon} a_{n\varepsilon}\right| = \left|\sum_{n=0}^{\mathrm{ni}(M)_\varepsilon} a_{n\varepsilon} - \sum_{n=0}^{\mathrm{ni}(N)_\varepsilon - 1} a_{n\varepsilon}\right| \leq \rho_\varepsilon^{-Q_1} + \rho_\varepsilon^{-Q_2}. \tag{2.11}$$

Now, assume that $(a_n)_n = [\bar{a}_{n\varepsilon}]$ is another representative. For $q \in \mathbb{N}$, condition (2.9) yields

$$\left|\sum_{n=\mathrm{ni}(N)_\varepsilon}^{\mathrm{ni}(M)_\varepsilon} a_{n\varepsilon} - \sum_{n=\mathrm{ni}(N)_\varepsilon}^{\mathrm{ni}(M)_\varepsilon} \bar{a}_{n\varepsilon}\right| \leq \rho_\varepsilon^q$$

for ε small. □

We can finally define hyperfinite sums and hyperseries.

Definition 5 Let $(a_n)_n = [a_{n\varepsilon}] \in {}^\rho_\sigma\widetilde{\mathbb{R}}_{\mathrm{s}}$ and let σ and ρ be two arbitrary gauges then the term

$$\sum_{n=N}^{M} a_n := \left[\sum_{n=\mathrm{ni}(N)_\varepsilon}^{\mathrm{ni}(M)_\varepsilon} a_{n\varepsilon}\right] \in {}^\rho\widetilde{\mathbb{R}} \quad \forall N, M \in {}^\sigma\widetilde{\mathbb{N}} \tag{2.12}$$

is called σ, ρ-*hypersum* of $(a_n)_n$ (*hypersum* for brevity).

Moreover, we say that *s is the ρ-sum of hyperseries with terms $(a_n)_{n\in\mathbb{N}}$ as $n \in {}^{\sigma}\widetilde{\mathbb{N}}$* if s is the hyperlimit of the hypersequence $N \in {}^{\sigma}\widetilde{\mathbb{N}} \mapsto \sum_{n=0}^{N} a_n \in {}^{\rho}\widetilde{\mathbb{R}}$. In this case, we write

$$s = {}^{\rho}\lim_{N\in{}^{\sigma}\widetilde{\mathbb{N}}} \sum_{n=0}^{N} a_n =: {}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n.$$

In case the use of the gauge ρ is clear from the context, we simply say that *s is the sum of the hyperseries with terms $(a_n)_{n\in\mathbb{N}}$ as $n \in {}^{\sigma}\widetilde{\mathbb{N}}$*.

As usual, we also say that the hyperseries ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n$ is *convergent* if

$$\exists\, {}^{\rho}\lim_{N\in{}^{\sigma}\widetilde{\mathbb{N}}} \sum_{n=0}^{N} a_n \in {}^{\rho}\widetilde{\mathbb{R}}.$$

Whereas, we say that a hyperseries ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n$ *does not converge* if ${}^{\rho}\lim_{\sigma\in\widetilde{\mathbb{N}}} N \sum_{n=0}^{N} a_n$ does not exist in ${}^{\rho}\widetilde{\mathbb{R}}$. More specifically, if ${}^{\rho}\lim_{N\in{}^{\sigma}\widetilde{\mathbb{N}}} \sum_{n=0}^{N} a_n = +\infty\ (-\infty)$, we say that ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n$ *diverges to* $+\infty$ $(-\infty)$.

For the sake of brevity, when dealing with hyperseries or with hypersums, we always *implicitly assume* that σ, ρ are two gauges and that $(a_n)_n \in {}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$.

Remark 6 (i) Note that we are using an abuse of notations, since the term $\sum_{n=N}^{M} a_n$ actually depends on the two considered gauges ρ, σ.

(ii) Explicitly, $s = {}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n$ means

$$\forall q \in \mathbb{N}\, \exists M \in {}^{\sigma}\widetilde{\mathbb{N}}\, \forall N \in {}^{\sigma}\widetilde{\mathbb{N}}_{\geq M} : \left| \sum_{n=0}^{N} a_n - s \right| < \mathrm{d}^{q}. \tag{2.13}$$

(iii) Also note that if $N, M \in \mathbb{N}$, then $\sum_{n=N}^{M} a_n = a_N + \ldots + a_M \in {}^{\rho}\widetilde{\mathbb{R}}$ is the usual finite sum. This is related to our motivating discussion about the definition of hyperfinite series at the beginning of Sect. 2.

2.2 Relations between ${}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$ and ${}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{u}}$

A first consequence of the condition $\sigma \geq \rho^*$ is that if the net $(a_{n\varepsilon})$ is uniformly moderate, then it is also moderate over hypersums. Similarly, we can argue for the equality, so that we have a natural map ${}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{u}} \longrightarrow {}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$:

Lemma 7 *We have the following properties:*

(i) If $N = [N_\varepsilon] \in {}^{\sigma}\widetilde{\mathbb{N}}$, with $N_\varepsilon \in \mathbb{N}$, and $(a_n)_n = [a_{n\varepsilon}] \in {}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$, then $a_N := [a_{N_\varepsilon,\varepsilon}] \in {}^{\rho}\widetilde{\mathbb{R}}$ is well defined. That is, any sequence $(a_n)_n \in {}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$ can be extended from $\mathbb{N}$ to the entire set ${}^{\sigma}\widetilde{\mathbb{N}}$ of hyperfinite numbers.

Moreover, if we assume that $\sigma \geq \rho^*$, then we have:

(ii) *Both* ${}^{\rho}\widetilde{\mathbb{R}} \subseteq {}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{u}}$ *and* ${}^{\rho}\widetilde{\mathbb{R}} \subseteq {}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$ *via the embedding* $[x_\varepsilon] \mapsto [(x_\varepsilon)_{n,\varepsilon}]$.

(iii) *The mapping*

$$\lambda : [a_{n\varepsilon}]_{\mathrm{u}} \in {}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{u}} \mapsto [a_{n\varepsilon}]_{\mathrm{s}} \in {}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$$

is a well-defined ${}^{\rho}\widetilde{\mathbb{R}}$*-linear map.*

(iv) *Let us define a hypersum operator*

$$\sum_{n=N}^{M} : {}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{u}} \longrightarrow {}^{\rho}\widetilde{\mathbb{R}} \quad \text{as} \quad \sum_{n=N}^{M} [a_{n\varepsilon}]_{\mathrm{u}} := \left[\sum_{n=\mathrm{ni}(N)_\varepsilon}^{\mathrm{ni}(M)_\varepsilon} a_{n\varepsilon} \right]$$

for all $M, N \in {}^{\sigma}\widetilde{\mathbb{N}}$. *Then* $\sum_{n=N}^{M} [a_{n\varepsilon}]_{\mathrm{u}} = \sum_{n=N}^{M} \lambda\left([a_{n\varepsilon}]_{\mathrm{u}}\right) \in {}^{\rho}\widetilde{\mathbb{R}}$. *Therefore, the character (convergent or divergent) of the corresponding hyperseries is identical in the two spaces* ${}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{u}}$ *and* ${}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$.

Proof (i) In fact, if $(N_\varepsilon) \in \mathbb{N}_\sigma$, then from $(a_n)_n \in {}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$ we get the existence of $Q_i \in \mathbb{N}$ such that

$$\left| a_{N_\varepsilon,\varepsilon} \right| = \left| \sum_{n=0}^{N_\varepsilon} a_{n\varepsilon} - \sum_{n=0}^{N_\varepsilon - 1} a_{n\varepsilon} \right| \le \rho_\varepsilon^{-Q_1} + \rho_\varepsilon^{-Q_2}.$$

Finally, if $(a_n)_n = [a_{n\varepsilon}] = [\bar{a}_{n\varepsilon}] \in {}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$, then directly from (2.9) with $M = N$ we get $[(a_{\mathrm{ni}(N)_\varepsilon,\varepsilon})_{n,\varepsilon}] = [(\bar{a}_{\mathrm{ni}(N)_\varepsilon,\varepsilon})_{n,\varepsilon}]$ (note that it is to have this result that we defined (2.9) using $\sum_{\mathrm{ni}(N)_\varepsilon}^{\mathrm{ni}(M)_\varepsilon}$ instead of $\sum_{n=0}^{\mathrm{ni}(N)_\varepsilon}$ like in (2.7)). Therefore, $a_N := [a_{N_\varepsilon,\varepsilon}] \in {}^{\rho}\widetilde{\mathbb{R}}$ is well-defined. In particular, this applies with $N = n \in \mathbb{N}$, so that any equivalence class $(a_n)_n = [(a_{n\varepsilon})_{n,\varepsilon}] \in {}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$ also defines an ordinary sequence $(a_n)_{n\in\mathbb{N}} = ([a_{n\varepsilon}])_{n\in\mathbb{N}}$ of ${}^{\rho}\widetilde{\mathbb{R}}$. On the other hand, let us explicitly note that if $(a_n)_{n\in\mathbb{N}} = (\bar{a}_n)_{n\in\mathbb{N}}$, i.e. if $a_n = \bar{a}_n$ for all $n \in \mathbb{N}$, then not necessarily (2.9) holds, i.e. we can have $(a_n)_n \ne (\bar{a}_n)_n$ as elements of the quotient module ${}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$.

Claim (ii) is left to the reader.

(iii) Assume that inequality in (2.3) holds for $\varepsilon \le \varepsilon_0$, i.e.

$$\forall \varepsilon \le \varepsilon_0 \, \forall n \in \mathbb{N} : \ |a_{n\varepsilon}| \le \rho_\varepsilon^{-Q}. \tag{2.14}$$

Without loss of generality, we can also assume that $\mathrm{ni}(N)_\varepsilon + 1 \le \sigma_\varepsilon^{-R}$ and $\sigma_\varepsilon \ge \rho_\varepsilon^{-Q_{\sigma,\rho}}$. Then, for each $\varepsilon \le \varepsilon_0$, using (2.14) we have $\left| \sum_{n=0}^{\mathrm{ni}(N)_\varepsilon} a_{n\varepsilon} \right| \le (\mathrm{ni}(N)_\varepsilon + 1) \cdot \rho_\varepsilon^{-Q} \le \sigma_\varepsilon^{-R} \cdot \rho_\varepsilon^{-Q} \le \rho_\varepsilon^{-R \cdot Q_{\sigma,\rho} - Q}$. Moreover, if $[a_{n\varepsilon}]_{\mathrm{u}} = 0$ then, for all $q \in \mathbb{N}$

$$\left| \sum_{n=0}^{\mathrm{ni}(N)_\varepsilon} a_{n\varepsilon} \right| \le (\mathrm{ni}(N)_\varepsilon + 1) \cdot \rho_\varepsilon^{q}$$

for ε small. Since $\text{ni}(N)_\varepsilon$ is σ-moderate and $\sigma \geq \rho^*$, this proves that the linear map λ is well-defined.

(iv) This follows directly from Definition 5. □

It remains an open problem the study of injectivity of the map λ.

We could say that $\sigma \geq \rho^*$ and $[a_{n\varepsilon}]_\text{u} \in {}^\rho\widetilde{\mathbb{R}}_\text{u}$ are sufficient conditions to get $(a_n)_n = [a_{n\varepsilon}]_\text{s} \in {}^\rho_\sigma\widetilde{\mathbb{R}}_\text{s}$ and hence to start talking about hyperfinite sums $\sum_{n=N}^M a_n$ and hyperseries ${}^\rho\sum_{n\in{}^\sigma\widetilde{\mathbb{N}}} a_n$. On the other hand, if $\sigma \geq \rho^*$ is false, we can still consider the space ${}^\rho_\sigma\widetilde{\mathbb{R}}_\text{s}$ and hence talk about hyperseries, but the corresponding space ${}^\rho_\sigma\widetilde{\mathbb{R}}_\text{s}$ lacks elements such as $(1)_n$ because of the subsequent Lemma 10. As we will see in the following example 8 (3) and (6), the space ${}^\rho_\sigma\widetilde{\mathbb{R}}_\text{s}$ still contains sequences corresponding to interesting converging hyperseries.

The following examples of convergent hyperseries justify our definition of hyperseries by recovering classical examples such such geometric and exponential hyperseries. We recall that this is not possible using classical series in a non-Archimedean setting.

Example 8 (1) Let $N = [N_\varepsilon] \in {}^\sigma\widetilde{\mathbb{N}}$, where $N_\varepsilon \in \mathbb{N}$ for all ε, then $\sum_{n=N}^N a_n = \left[a_{N_\varepsilon,\varepsilon}\right] = a_N \in {}^\rho\widetilde{\mathbb{R}}$. We recall that $a_N = \left[a_{N_\varepsilon,\varepsilon}\right]$ is the extension of $(a_n)_n \in {}^\rho_\sigma\widetilde{\mathbb{R}}_\text{s}$ to $N \in {}^\sigma\widetilde{\mathbb{N}}$ (see (i) of Lemma 7).

(2) For all $k \in {}^\rho\widetilde{\mathbb{R}}$, $0 < k < 1$, we have (note that $\sigma = \rho$)

$$ {}^\rho\sum_{n\in{}^\rho\widetilde{\mathbb{N}}} k^n = \frac{1}{1-k}. \tag{2.15}$$

We first note that $k^n \leq 1$, so that $\{k^n\}_n = \left[k_\varepsilon^n\right]_\text{u} \in {}^\rho\widetilde{\mathbb{R}}_\text{u}$. Now,

$$ {}^\rho\sum_{n\in{}^\rho\widetilde{\mathbb{N}}} k^n = {}^\rho\lim_{N\in{}^\rho\widetilde{\mathbb{N}}} \sum_{n=0}^N k^n = {}^\rho\lim_{N\in{}^\rho\widetilde{\mathbb{N}}} \frac{1-k^{N+1}}{1-k}.$$

But $k^{N+1} < \mathrm{d}^q$ if and only if $(N+1)\log k < q\log \mathrm{d}$. Since $0 < k < 1$, we have $\log k < 0$ and we obtain $N > q\frac{\log \mathrm{d}}{\log k} - 1$. It suffices to take $M_\varepsilon := \text{int}\left(q\frac{\log \rho_\varepsilon}{\log k_\varepsilon}\right)$ in the definition of hyperseries.

(3) More generally, if $k \in {}^\rho\widetilde{\mathbb{R}}$, $0 < k < 1$, we can evaluate

$$\left[\left|\sum_{n=0}^{N_\varepsilon} k_\varepsilon^n\right|\right] = \left[\left|\frac{1-k_\varepsilon^{N_\varepsilon+1}}{1-k_\varepsilon}\right|\right] \leq \frac{2}{1-k} \in {}^\rho\widetilde{\mathbb{R}}.$$

This shows that $(k^n)_n = [k_\varepsilon^n] \in {}^\rho_\sigma\widetilde{\mathbb{R}}_\text{s}$ for all gauges σ. If we assume $\sigma \leq \rho^*$ then $M_\varepsilon := \text{int}\left(q\frac{\log \rho_\varepsilon}{\log k_\varepsilon}\right) \in \mathbb{N}_\sigma$ and hence, proceeding as above, we can prove that ${}^\rho\sum_{n\in{}^\sigma\widetilde{\mathbb{N}}} k^n = \frac{1}{1-k}$.

(4) Let $k \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ be such that $k >_\text{s} 1$ (see [13] for the relations $>_\text{s}$ and $=_\text{s}$ and, more generally, for the language of subpoints), then the hyperseries ${}^\rho\sum_{n\in{}^\rho\widetilde{\mathbb{N}}} k^n$

is not convergent. In fact, by contradiction, in the opposite case we would have $\sum_{n=0}^{N} k^n \in (l-1, l+1)$ for some $l \in {}^{\rho}\widetilde{\mathbb{R}}$ for all N sufficiently large, but this is impossible because for all fixed $K \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$ and for N sufficiently large, $\sum_{n=0}^{N} k^n = {}^{\rho}\lim_{N\in{}^{\rho}\widetilde{\mathbb{N}}} \frac{1-k^{N+1}}{1-k} >_{\mathrm{s}} K$ because ${}^{\rho}\lim_{N\in{}^{\rho}\widetilde{\mathbb{N}}} k^{N+1} =_{\mathrm{s}} +\infty$.

(5) For all $x \in {}^{\rho}\widetilde{\mathbb{R}}$ finite, we have ${}^{\rho}\sum_{n\in{}^{\rho}\widetilde{\mathbb{N}}} \frac{x^n}{n!} = e^x$. We have $|x| < M \in \mathbb{R}_{>0}$ because x is finite, and hence $|x_\varepsilon| \le M$ for all $\varepsilon \le \varepsilon_0$. Thereby $\frac{x_\varepsilon^n}{n!} \le \frac{|x_\varepsilon|^n}{n!} \le e^M$ for all $n \in \mathbb{N}$ and thus $\left\{\frac{x^n}{n!}\right\}_n = \left[\frac{x_\varepsilon^n}{n!}\right]_{\mathrm{u}} \in {}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{u}}$. For all $N = [N_\varepsilon] \in {}^{\sigma}\widetilde{\mathbb{N}}$, $N_\varepsilon \in \mathbb{N}$, and all ε, we have

$$\sum_{n=0}^{N_\varepsilon} \frac{x_\varepsilon^n}{n!} = e^{x_\varepsilon} - \sum_{n=N_\varepsilon+1}^{+\infty} \frac{x_\varepsilon^n}{n!}. \tag{2.16}$$

Now, take $N \in {}^{\sigma}\widetilde{\mathbb{N}}$ such that $\frac{M}{N+1} < \frac{1}{2}$, so that we can assume $N_\varepsilon + 1 > 2M$ for all ε. We have $\left|\sum_{n=N_\varepsilon+1}^{+\infty} \frac{x_\varepsilon^n}{n!}\right| \le \sum_{n>N_\varepsilon} \frac{M^n}{n!}$, and for all $n \ge N_\varepsilon$ we have (by induction)

$$\frac{M^{n+1}}{(n+1)!} < \frac{1}{2^{n+1}}.$$

Therefore $\left|\sum_{n=N_\varepsilon+1}^{+\infty} \frac{x_\varepsilon^n}{n!}\right| \le \sum_{n>N_\varepsilon} \frac{1}{2^n}$ and hence ${}^{\rho}\lim_{N\in{}^{\rho}\widetilde{\mathbb{N}}} \sum_{n=N+1}^{+\infty} \frac{x^n}{n!} = 0$ by (2.15). This and (2.16) yields the conclusion.

(6) In the same assumptions of the previous example, we have $\left[\left|\sum_{n=0}^{N_\varepsilon} \frac{x_\varepsilon^n}{n!}\right|\right] \le e^M$ and hence $\left(\frac{x^n}{n!}\right)_n = \left[\frac{x_\varepsilon^n}{n!}\right] \in {}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$ for all gauges σ. If $\sigma \le \rho^*$, proceeding as above, we can prove that ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} \frac{x^n}{n!} = e^x$.

Finally, the following lemma shows that a sharply bounded sequence of ${}^{\rho}\widetilde{\mathbb{R}}$ always defines a sequence for hyperseries, i.e. an element of ${}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$.

Lemma 9 *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of ${}^{\rho}\widetilde{\mathbb{R}}$. If $(a_n)_{n\in\mathbb{N}}$ is sharply bounded:*

$$\exists M \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}\, \forall n \in \mathbb{N} : \ |a_n| \le M, \tag{2.17}$$

then there exists a sequence $(a_{n\varepsilon})_{n\in\mathbb{N}}$ of $\mathbb{R}_\rho$ such that

(i) $a_n = [a_{n\varepsilon}] \in {}^{\rho}\widetilde{\mathbb{R}}$ *for all* $n \in \mathbb{N}$;
(ii) $[a_{n\varepsilon}]_{\mathrm{u}} \in {}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{u}}$;
(iii) *If* $\sigma \ge \rho^*$, *then* $[a_{n\varepsilon}]_{\mathrm{s}} \in {}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$.

Proof Let $M = [M_\varepsilon]$ be any representative of the bound satisfying (2.17), so that $M_\varepsilon \le \rho_\varepsilon^{-Q}$ for $\varepsilon \le \varepsilon_0$ and for some $Q \in \mathbb{N}$. From (2.17), for each $n \in \mathbb{N}$ we get the existence of a representative $a_n = [\bar{a}_{n\varepsilon}]$ such that $|\bar{a}_{n\varepsilon}| \le M_\varepsilon$ for $\varepsilon \le \varepsilon_{0n} \le \varepsilon_0$. It suffices to define $a_{n\varepsilon} := \bar{a}_{n\varepsilon}$ if $\varepsilon \le \varepsilon_{0n}$ and $a_{n\varepsilon} := M_\varepsilon$ otherwise to have $\forall \varepsilon \le \varepsilon_0\, \forall n \in \mathbb{N} : \ |a_{n\varepsilon}| \le \rho_\varepsilon^{-Q}$, so that $[a_{n\varepsilon}]_{\mathrm{u}} \in {}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{u}}$. If $\sigma \ge \rho^*$, we can then apply Lemma 7. $\square$

However, let us note that, generally speaking, changing representatives of a_n as in the previous proof, we also get a different value of the corresponding hyperseries, as proved by example (2.8).

2.3 Divergent hyperseries

By analyzing when the constant net (1) is moderate over hypersums, we discover the relation (2.5) between the gauges σ and ρ:

Lemma 10 *The constant net* $(1) \in (\mathbb{R}^{\mathbb{N}\times I})_{\sigma\rho}$, *i.e. it is* σ, ρ*-moderate over hypersums, if and only if* $\sigma \geq \rho^*$.

Proof If $(1) \in (\mathbb{R}^{\mathbb{N}\times I})_{\sigma\rho}$, we set $N_\varepsilon := \text{int}(\sigma_\varepsilon^{-1}) + 1$, so that we get $\sigma_\varepsilon^{-1} \leq \sum_{n=0}^{N_\varepsilon} 1 = N_\varepsilon \leq \rho_\varepsilon^{-Q_{\sigma,\rho}}$ for some $Q_{\sigma,\rho} \in \mathbb{N}$, i.e. $\sigma \geq \rho^*$. Vice versa, if $\sigma_\varepsilon \geq \rho_\varepsilon^{Q_{\sigma,\rho}}$ for all $\varepsilon \leq \varepsilon_0$, then for those ε, we have $\left|\sum_{n=0}^{\text{ni}(N)_\varepsilon} 1\right| = \text{ni}(N)_\varepsilon \leq \sigma_\varepsilon^{-R} \leq \rho_\varepsilon^{-R\cdot Q_{\sigma,\rho}}$ for some $R \in \mathbb{N}$, because $N \in {}^\sigma\widetilde{\mathbb{N}}$. $\square$

One could argue that we are mainly interested in converging hyperseries and hence it is not worth considering the constant net (1). On the other hand, we would like to argue in the following way: the hypersums $N \in {}^\sigma\widetilde{\mathbb{N}} \mapsto \sum_{n=0}^{N} 1 \in {}^\rho\widetilde{\mathbb{R}}$ can be considered, but they do not converge because $1 \not\to 0$. As we will see in Lemma 15, this argumentation is possible only if $\sigma \geq \rho^*$ because of the previous Lemma 10.

Let $\sigma \geq \rho^*$ and $\omega \in {}^\rho\widetilde{\mathbb{N}}$ be an infinite number. If $(a_n)_n \in {}^\rho_\sigma\widetilde{\mathbb{R}}_\text{s}$, we can also think at $\sum_{n=0}^{\omega} a_n$ as another way to compute an infinite summation of the numbers a_n. In other words, the following examples can be considered as related to calculation of divergent series. They strongly motivate and clarify our definition of hyperfinite sum.

Example 11 Assuming $\sigma \geq \rho^*$ and working with the module ${}^\rho_\sigma\widetilde{\mathbb{R}}_\text{s}$, we have:

(5) $\sum_{n=1}^{\omega} 1 = \omega \in {}^\rho\widetilde{\mathbb{R}} \setminus \mathbb{R}$.

(6) $\sum_{n=1}^{\omega} n = \left[\sum_{n=1}^{\text{ni}(\omega)_\varepsilon} n_\varepsilon\right] = \left[\frac{\text{ni}(\omega)_\varepsilon(\text{ni}(\omega)_\varepsilon+1)}{2}\right] = \frac{\omega(\omega+1)}{2}$.

(7) $\sum_{n=1}^{\omega}(2n-1) = 2\sum_{n=1}^{\omega} n - \sum_{n=1}^{\omega} 1 = \omega^2$ because we know from Theorem 3 that ${}^\rho_\sigma\widetilde{\mathbb{R}}_\text{s}$ is an ${}^\rho\widetilde{\mathbb{R}}$-module.

(8) $\sum_{n=1}^{\omega}(a + (n-1)d) = \omega a + \frac{\omega^2 d}{2} - \frac{\omega d}{2}$.

(9) Using ε-wise calculations, we also have $\sum_{n=0}^{\omega}(-1)^n = \frac{1}{2}(-1)^{\omega+1} + \frac{1}{2}$. Note that the final result is a finite generalized number of ${}^\rho\widetilde{\mathbb{R}}$, but it does not converge for $\omega \to +\infty$, $\omega \in {}^\rho\widetilde{\mathbb{N}}$.

10) The net $(2^n)_{n,\varepsilon}$ is not ρ-moderate over hypersums, in fact if $\omega = \left[\text{int}(\rho_\varepsilon^{-1})\right]$, then $\sum_{n=0}^{\omega_\varepsilon} 2^{n-1} = 2^{\omega_\varepsilon} - 1$, which is not ρ-moderate. Another possibility to consider the function 2^ω is to take another gauge $\mu \leq \rho$ and the subring of ${}^\mu\widetilde{\mathbb{R}}$ defined by

$$\widetilde{\mathbb{R}} := \{x \in {}^\mu\widetilde{\mathbb{R}} \mid \exists N \in \mathbb{N} : \ |x| \leq \text{d}^{-N}\},$$

where only here we have set $\text{d} := [\rho_\varepsilon]_{\sim_\mu} \in {}^\mu\widetilde{\mathbb{R}}$. If we have

$$\forall N \in \mathbb{N}\, \exists M \in \mathbb{N} : \ \text{d}^{-N} \leq -M \log \text{d}, \tag{2.18}$$

then $2^{(-)} : [x_\varepsilon] \in {}^{\mu}_{\rho}\widetilde{\mathbb{R}} \mapsto \left[2^{x_\varepsilon}\right] \in {}^{\mu}\widetilde{\mathbb{R}}$ is well defined. For example, if $\mu_\varepsilon := \exp\left(-\rho_\varepsilon^{1/\varepsilon}\right)$, then $\mu \le \rho$ and (2.18) holds for $M = 1$. Note that the natural ring morphism $[x_\varepsilon]_{\sim_\sigma} \in {}^{\mu}_{\rho}\widetilde{\mathbb{R}} \mapsto [x_\varepsilon]_{\sim_\rho} \in {}^{\rho}\widetilde{\mathbb{R}}$ is surjective but generally not injective. Now, if $\omega \in {}^{\mu}_{\rho}\widetilde{\mathbb{R}} \cap {}^{\sigma}\widetilde{\mathbb{N}}$, then $\sum_{n=0}^{\omega} 2^{n-1} = 2^\omega - 1 \in {}^{\mu}\widetilde{\mathbb{R}} \setminus {}^{\mu}_{\rho}\widetilde{\mathbb{R}}$.

(11) Once again, proceeding by ε-wise calculations, we also have the binomial formula: For all $a, b \in {}^{\rho}\widetilde{\mathbb{R}}$ and $n \in {}^{\rho}\widetilde{\mathbb{N}}$, if $(a+b)^n \in {}^{\rho}\widetilde{\mathbb{R}}$, then

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

where $n! := [ni(n)_\varepsilon!]$ and $\binom{n}{k} := \frac{n!}{k!(n-k)!}$.

2.4 ε-wise convergence and hyperseries

The following result allows us to obtain hyperseries by considering ε-wise convergence of its summands. Its proof is clearly very similar to that of [13, Theorem 28], but with a special attention to the condition $(a_n)_n \in {}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$ that we need beforehand to talk about hyperseries.

Theorem 12 *Let $a : \mathbb{N} \longrightarrow {}^{\rho}\widetilde{\mathbb{R}}$ and $a_n = [a_{n\varepsilon}]$ for all $n \in \mathbb{N}$. Let $q_\varepsilon, M_\varepsilon \in \mathbb{N}_{>0}$ be such that $(q_\varepsilon) \to +\infty$ as $\varepsilon \to 0^+$, and*

$$\forall \varepsilon\, \forall N \ge M_\varepsilon : \ s_\varepsilon - \rho_\varepsilon^{q_\varepsilon} < \sum_{n=0}^{N} a_{n\varepsilon} < s_\varepsilon + \rho_\varepsilon^{q_\varepsilon} \tag{2.19}$$

(note that this implies that the standard series $\sum_{n=0}^{+\infty} a_{n\varepsilon}$ converges to s_ε). Finally, let μ be another gauge. If $s = [s_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}}$, then setting $\sigma_\varepsilon := \min(\mu_\varepsilon, M_\varepsilon^{-1})$, we have:

(i) $\sigma \le \mu$ is a gauge (not necessarily a monotonic one);
(ii) $M := [M_\varepsilon] \in {}^{\sigma}\widetilde{\mathbb{N}}$;
(iii) $s_M := \left[\sum_{n=0}^{M_\varepsilon - 1} a_{n\varepsilon}\right] \in {}^{\rho}\widetilde{\mathbb{R}}$;
(iv) $(a_{n+M})_n := \left[\left(a_{n+M_\varepsilon,\varepsilon}\right)_{n,\varepsilon}\right]_{\mathrm{s}} \in {}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$;
(v) $s = s_M + {}^{\rho}\sum_{n \in {}^{\sigma}\widetilde{\mathbb{N}}} a_{n+M}$.

Proof The net $\sigma_\varepsilon = \min(\mu_\varepsilon, M_\varepsilon^{-1}) \to 0^+$ as $\varepsilon \to 0^+$ because $\mu_\varepsilon \to 0^+$, i.e. it is a gauge (note that not necessarily σ is non-decreasing, e.g. if $\lim_{\varepsilon \to \frac{1}{k}} M_\varepsilon = +\infty$ for all $k \in \mathbb{N}_{>0}$ and $M_\varepsilon \ge \mu_\varepsilon^{-1}$). We have $M := [M_\varepsilon] \in {}^{\sigma}\widetilde{\mathbb{N}}$ because our definition of σ yields $M_\varepsilon \le \sigma_\varepsilon^{-1}$. Moreover, since $(q_\varepsilon) \to +\infty$, for all ε condition (2.19) yields $s_\varepsilon - 1 \le s_\varepsilon - \rho_\varepsilon^{q_\varepsilon} < \sum_{n=0}^{N} a_{n\varepsilon} < s_\varepsilon + \rho_\varepsilon^{q_\varepsilon} \le s_\varepsilon + 1$ for all $N \ge M_\varepsilon$. For $N = M$ this gives $s_M = \left[\sum_{n=0}^{M_\varepsilon} a_{n\varepsilon}\right] \in {}^{\rho}\widetilde{\mathbb{R}}$, because we assumed that $s = [s_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}}$. If $N \in {}^{\sigma}\widetilde{\mathbb{N}}$, then $\mathrm{ni}(N)_\varepsilon + M_\varepsilon \ge M_\varepsilon$ for all ε, and hence $s_\varepsilon - 1 < \sum_{n=0}^{M_\varepsilon - 1} a_{n\varepsilon} + \sum_{n=M_\varepsilon}^{\mathrm{ni}(N)_\varepsilon + M_\varepsilon} a_{n\varepsilon} =$

$s_{M\varepsilon} + \sum_{n=0}^{\text{ni}(N)_\varepsilon} a_{n+M_\varepsilon,\varepsilon} < s_\varepsilon + 1$. This shows that $\left[\sum_{n=0}^{\text{ni}(N)_\varepsilon} a_{n+M_\varepsilon,\varepsilon}\right] \in {}^\rho\widetilde{\mathbb{R}}$, because $s_M, s \in {}^\rho\widetilde{\mathbb{R}}$, i.e. $(a_{n+M})_n \in {}^\rho_\sigma\widetilde{\mathbb{R}}_{\text{s}}$. Similarly, if $q \in \mathbb{N}$, then $q < q_\varepsilon$ for ε sufficiently small and, proceeding as above, we can prove that $|\sum_{n=0}^{N} a_{n+M} - s - s_M| < \text{d}^q$. □

Note that, if (M_ε) is not ρ-moderate and we set $\sigma_\varepsilon := \min\left(\rho_\varepsilon, M_\varepsilon^{-1}\right) \in (0, 1]$, then

$$\forall Q \in \mathbb{N}\, \exists L \subseteq_0 I\, \forall \varepsilon \in L : \ \sigma_\varepsilon^{-1} > \bar{M}_\varepsilon > \rho_\varepsilon^{-Q}$$

and this shows that $\sigma \geq \rho^*$, i.e. the fundamental condition to have moderateness of hyperfinite sums in the ring ${}^\rho\widetilde{\mathbb{R}}_{\text{u}}$ (see (2.6)), *always necessarily does not hold*. On the other hand, if (M_ε) is ρ-moderate, then $\mathbb{R}_\sigma = \mathbb{R}_\mu$; in particular, we can simply take $\sigma = \mu = \rho$.

Assuming absolute convergence, we can obtain a stronger and clearer result:

Theorem 13 *Let $a : \mathbb{N} \longrightarrow {}^\rho\widetilde{\mathbb{R}}$ and $a_n = [a_{n\varepsilon}]$ for all $n \in \mathbb{N}$. Assume that the standard series $\sum_{n=0}^{+\infty} a_{n\varepsilon}$ converges absolutely to $\bar{s}_\varepsilon$ and $[\bar{s}_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}$. Finally, let μ be another gauge and set $s_\varepsilon := \sum_{n=0}^{+\infty} a_{n\varepsilon}$, then there exists a gauge $\sigma \leq \mu$ such that:*

(i) $(a_n)_n = [a_{n\varepsilon}]_{\text{s}} \in {}^\rho_\sigma\widetilde{\mathbb{R}}_{\text{s}}$;
(ii) $s = {}^\rho\sum_{n\in{}^\sigma\widetilde{\mathbb{N}}} a_n$.

Proof Set $q_\varepsilon := \left\lceil \frac{1}{\varepsilon} \right\rceil$, so that the absolute convergence of $\sum_{n=0}^{+\infty} a_{n\varepsilon}$ yields

$$\forall \varepsilon\, \exists M_\varepsilon \in \mathbb{N}_{>0}\, \forall N \geq M_\varepsilon : \ \bar{s}_\varepsilon - \rho_\varepsilon^{q_\varepsilon} < \sum_{n=0}^{N} |a_{n\varepsilon}| < \bar{s}_\varepsilon + \rho_\varepsilon^{q_\varepsilon} \tag{2.20}$$

and we can thereby apply the previous Theorem 12 obtaining $\bar{s}_M := \left[\sum_{n=0}^{M_\varepsilon-1} |a_{n\varepsilon}|\right]$, ${}^\rho\sum_{n\in{}^\sigma\widetilde{\mathbb{N}}} |a_{n+M}| \in {}^\rho\widetilde{\mathbb{R}}$. Therefore, for an arbitrary $N \in {}^\sigma\widetilde{\mathbb{N}}$, we have:

$$\begin{aligned}\left[\left|\sum_{n=0}^{\text{ni}(N)_\varepsilon} a_{n\varepsilon}\right|\right] &\leq \left[\sum_{n=0}^{\text{ni}(N)_\varepsilon} |a_{n\varepsilon}|\right] \leq \left[\sum_{n=0}^{M_\varepsilon+\text{ni}(N)_\varepsilon} |a_{n\varepsilon}|\right] \\ &= \bar{s}_M + {}^\rho\sum_{n\in{}^\sigma\widetilde{\mathbb{N}}} |a_{n+M}| \in {}^\rho\widetilde{\mathbb{R}},\end{aligned}$$

and this shows that $(a_n)_n \in {}^\rho_\sigma\widetilde{\mathbb{R}}_{\text{s}}$. Now, proceeding as above we can prove that $s = {}^\rho\sum_{n\in{}^\sigma\widetilde{\mathbb{N}}} a_n$. □

2.5 Basic properties of hyperfinite sums and hyperseries

We now study some basic properties of hyperfinite sums (2.12).

Lemma 14 *Let σ, ρ be arbitrary gauges, $(a_n)_n \in {}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$, and $M, N \in {}^{\sigma}\widetilde{\mathbb{N}}$, then*

$$\sum_{n=0}^{N+M} a_n - \sum_{n=0}^{N} a_n = \sum_{n=N+1}^{N+M} a_n. \tag{2.21}$$

Proof For simplicity, if $N = [N_\varepsilon]$, $M = [M_\varepsilon]$ with $N_\varepsilon, M_\varepsilon \in \mathbb{N}$ for all ε, then $\sum_{n=0}^{N} a_n = \left[\sum_{n=0}^{N_\varepsilon} a_{n\varepsilon}\right] \in {}^{\rho}\widetilde{\mathbb{R}}$ and

$$\begin{aligned}\sum_{n=0}^{N+M} a_n &= \left[\sum_{n=0}^{N_\varepsilon+M_\varepsilon} a_{n\varepsilon}\right]\\ &= \left[\sum_{n=0}^{N_\varepsilon} a_{n\varepsilon} + \sum_{N_\varepsilon+1}^{N_\varepsilon+M_\varepsilon} a_{n\varepsilon}\right]\\ &= \sum_{n=0}^{N} a_n + \sum_{n=N+1}^{N+M} a_n.\end{aligned}$$

□

Lemma 15 *If ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n$ is convergent and $M \in {}^{\sigma}\widetilde{\mathbb{N}}$, then*

$${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n = {}^{\rho}\lim_{N\in{}^{\sigma}\widetilde{\mathbb{N}}} \sum_{n=0}^{N+M} a_n.$$

Therefore, from (2.21), we also have

$${}^{\rho}\lim_{N\in{}^{\sigma}\widetilde{\mathbb{N}}} \sum_{n=N}^{N+M} a_n = 0.$$

In particular, ${}^{\rho}\lim_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n = 0$.

Proof Let $s := {}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n$. Directly from the definition of convergent hyperseries (2.13), we have

$$\forall q \in \mathbb{N}\, \exists K \in {}^{\sigma}\widetilde{\mathbb{N}}\, \forall N \in {}^{\sigma}\widetilde{\mathbb{N}}_{\geq K} : \left|\sum_{n=0}^{N} a_n - s\right| < \mathrm{d}^{q}. \tag{2.22}$$

In particular, if $N \in {}^{\sigma}\widetilde{\mathbb{N}}_{\geq K}$, then also $N + M \in {}^{\sigma}\widetilde{\mathbb{N}}_{\geq K}$, which is our conclusion. □

Directly from Lemma 14, we also have:

Corollary 16 *Let* ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n$ *be a convergent hyperseries. Then adding a hyperfinite number of terms have no effect on the convergence of the hyperseries, that is*

$$\sum_{n=0}^{K} a_n + {}^{\rho}\lim_{N\in{}^{\sigma}\widetilde{\mathbb{N}}} \sum_{n=K+1}^{N} a_n = {}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n \quad \forall K \in {}^{\sigma}\widetilde{\mathbb{N}}.$$

Mimicking the classical theory, we can also say that the hyperseries ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n$ is *Cesàro hypersummable* if $\exists\, {}^{\rho}\lim_{N\in{}^{\sigma}\widetilde{\mathbb{N}}} \frac{1}{N+1}\sum_{n=0}^{N}\sum_{k=0}^{n} a_k$ (this hyperlimit is clearly called *Cesàro hypersum*). For example, proceeding ε-wise, we have that the hyperseries ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} (-1)^n$ has Cesàro hypersum equal to $\frac{1}{2}$. Trivially generalizing the usual proof, we can show that if ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n$ converges to s, then it is also Cesàro hypersummable with the same hypersum s.

3 Hyperseries convergence tests

3.1 *p*-hyperseries

To deal with p-hyperseries, i.e. hyperseries of the form

$${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}_{\geq 1}} \frac{1}{n^p},$$

we always assume that:

(i) $p \in {}^{\rho}\widetilde{\mathbb{R}}$ is a *finite* generalized number such that $p > 0$ or $p < 0$; we recall that if p is an infinite number (at least on a subpoint), the operation n^p is in general not well-defined since it leads to a non ρ-moderate net (see also Example 11.10)).

(ii) The gauges σ and ρ are chosen so that $\left(\frac{1}{n^p}\right)_n \in {}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$. A *sufficient* condition for this is that $\sigma \geq \rho^*$, i.e. that $\mathbb{R}_\sigma \subseteq \mathbb{R}_\rho$. In fact, since p is finite, we have $\left(\sum_{n=1}^{N_\varepsilon} n^{-p_\varepsilon}\right) \leq N_\varepsilon N_\varepsilon^{-p_\varepsilon} \in \mathbb{R}_\sigma \subseteq \mathbb{R}_\rho$ whenever $N_\varepsilon \in \mathbb{N}$, $(N_\varepsilon) \in \mathbb{N}_\sigma$.

Note explicitly that, from the assumption $\left(\frac{1}{n^p}\right)_n \in {}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$, we hence have

$$\sum_{n=1}^{N} \frac{1}{n^p} = \left[\sum_{n=1}^{N_\varepsilon} \frac{1}{n^{p_\varepsilon}}\right] \in {}^{\rho}\widetilde{\mathbb{R}} \tag{3.1}$$

even if $N \in {}^{\sigma}\widetilde{\mathbb{N}} \subseteq {}^{\sigma}\widetilde{\mathbb{R}}$, and ${}^{\sigma}\widetilde{\mathbb{R}}$ is a different quotient ring with respect to ${}^{\rho}\widetilde{\mathbb{R}}$ (e.g., in general we do *not* have ${}^{\sigma}\widetilde{\mathbb{N}} \subseteq {}^{\rho}\widetilde{\mathbb{N}}$). In other words, it is not correct to say that the left hand side of (3.1) is the sum for $1 \leq n \leq N \in {}^{\sigma}\widetilde{\mathbb{N}}$ of terms $\frac{1}{n^p} \in {}^{\sigma}\widetilde{\mathbb{R}}$, but instead it is a way to use $(N_\varepsilon) \in \mathbb{N}_\sigma$ to compute a generalized number of ${}^{\rho}\widetilde{\mathbb{R}}$.

If $p < 0$, the general term $\frac{1}{n^p} \not\to 0$ because $n^{-p} \leq \mathrm{d}^q$ only if $n < \mathrm{d}^{-q/p}$. Therefore, Lemma 15 yields that the p-series diverges. We can hence consider only the case $p > 0$. Let us note that Theorem 4, which is clearly based on our Definition 1 of ${}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$, allows us to consider partial hypersums of the form $\sum_{n=1}^{N} \frac{1}{n^p}$ even if $p < 0$. In other words, we can correctly argue that the p-hyperseries does not converge if $p < 0$ not because the partial hypersums cannot be computed, but because the general term does not converge to zero. The situation would be completely different if we restrict our attention only to sequences $(a_n)_n$ satisfying the (more natural) uniformly moderate condition (2.3). Similar remarks can be formulated for the calculus of divergent hyperseries in Example 8, 6)-10).

To prove the following results, we will follow some ideas of [10].

Theorem 17 *Let $p \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$ and let $S_N(p)$ be the N-th partial sum of the p-hyperseries, where $N \in {}^{\sigma}\widetilde{\mathbb{N}}_{\geq 1}$, then*

$$1 - \frac{1}{2^p} + \frac{2}{2^p} S_N(p) < S_{2N}(p) < 1 + \frac{2}{2^p} S_N(p). \tag{3.2}$$

Proof Let $N = [N_\varepsilon] \in {}^{\sigma}\widetilde{\mathbb{N}}_{\geq 1}$, with $N_\varepsilon \in \mathbb{N}_{\geq 1}$ for all ε, and $p = [p_\varepsilon]$, where $p_\varepsilon > 0$ for all ε. Since

$$\begin{aligned} S_{2N}(p) = \left[\sum_{n=1}^{2N_\varepsilon} \frac{1}{n^{p_\varepsilon}}\right] &= \left[1 + \frac{1}{2^{p_\varepsilon}} + \frac{1}{3^{p_\varepsilon}} + \cdots + \frac{1}{(2N_\varepsilon))^{p_\varepsilon}}\right] \\ &= 1 + \left[\frac{1}{2^{p_\varepsilon}} + \frac{1}{4^{p_\varepsilon}} + \cdots + \frac{1}{((2N_\varepsilon))^{p_\varepsilon}}\right] \\ &\quad + \left[\frac{1}{3^{p_\varepsilon}} + \frac{1}{5^{p_\varepsilon}} + \cdots + \frac{1}{((2N_\varepsilon - 1))^{p_\varepsilon}}\right]. \end{aligned}$$

But

$$1 + \left[\frac{1}{2^{p_\varepsilon}} + \frac{1}{4^{p_\varepsilon}} + \cdots + \frac{1}{((2N_\varepsilon))^{p_\varepsilon}}\right] = 1 + \frac{1}{2^p} S_N(p) \tag{3.3}$$

and, since $p_\varepsilon > 0$, we also have

$$\begin{aligned} \left[\frac{1}{3^{p_\varepsilon}} + \frac{1}{5^{p_\varepsilon}} + \cdots + \frac{1}{((2N_\varepsilon - 1))^{p_\varepsilon}}\right] &> \left[\frac{1}{4^{p_\varepsilon}} + \frac{1}{6^{p_\varepsilon}} + \cdots + \frac{1}{((2N_\varepsilon))^{p_\varepsilon}}\right] \\ &= \frac{1}{2^p} S_N(p) - \frac{1}{2^p}. \end{aligned} \tag{3.4}$$

Summing (3.3) and (3.4) yields $S_{2N}(p) > 1 - \frac{1}{2^p} + \frac{2}{2^p} S_N(p)$. Finally, from (3.3) and

$$\begin{aligned} \left[\frac{1}{3^{p_\varepsilon}} + \frac{1}{5^{p_\varepsilon}} + \cdots + \frac{1}{((2N_\varepsilon - 1))^{p_\varepsilon}}\right] &< \left[\frac{1}{2^{p_\varepsilon}} + \frac{1}{4^{p_\varepsilon}} + \cdots + \frac{1}{((2N_\varepsilon))^{p_\varepsilon}}\right] = \\ &= \frac{1}{2^p} S_N(p) \end{aligned}$$

we get $S_{2N}(p) < 1 + \frac{1}{2^p} S_N(p) + \frac{1}{2^p} S_N(p)$. □

Theorem 18 *The p-hyperseries is divergent when $0 < p \le_s 1$. When $p > 1$, there exist some gauge $\tau \le \rho$ such that $\left(\frac{1}{n^p}\right)_n \in {}^\rho\widetilde{\mathbb{R}}_\tau$ and the p-series is convergent with respect to the gauges τ, ρ (to the usual real value $\zeta(p)$ if $p \in \mathbb{R}$), and in this case*

$$\frac{2^p-1}{2^p-2} \le {}^\rho\sum_{n\in{}^\tau\widetilde{\mathbb{N}}_{\ge 1}} \frac{1}{n^p} = \sum_{n=1}^{+\infty} \frac{1}{n^p} \le \frac{2^p}{2^p-2}. \tag{3.5}$$

Proof If $0 < p \le_s 1$, there exist $J \subseteq_0 I$ such that $p|_J \le 1$. Assume that the $p-$hyperseries is convergent and set ${}^\rho\lim_{n\in{}^\sigma\widetilde{\mathbb{N}}} S_N(p) =: S(p)$. Taking $N \to \infty$ in the first inequality in (3.2), we have

$$1 - \frac{1}{2^p} + \frac{2}{2^p}S(p) = \frac{2^p-1}{2^p} + \frac{2}{2^p}S(p) \le S(p).$$

Since for $\varepsilon \in J$ sufficiently small we have $0 < p_\varepsilon < 1$ for some $[p_\varepsilon] = p$, we obtain

$$0 < \frac{2^{p_\varepsilon}-1}{2^{p_\varepsilon}} \le \frac{2^{p_\varepsilon}-2}{2^{p_\varepsilon}} S(p) \le 0.$$

This shows that the $p-$hyperseries is divergent when $p \le_s 1$. Now let $p > 1$, so that we can assume $p_\varepsilon > 1$ for all ε. From (3.2), we have

$$S_{N_\varepsilon}(p_\varepsilon) < S_{2N_\varepsilon}(p_\varepsilon) < 1 + \frac{2}{2^{p_\varepsilon}} S_{N_\varepsilon}(p_\varepsilon),$$

where $S_{N_\varepsilon}(p_\varepsilon) = \sum_{n=1}^{N_\varepsilon} \frac{1}{n^{p_\varepsilon}} \in \mathbb{R}$. Thereby

$$\begin{aligned} 0 &< \left(1 - \frac{2}{2^{p_\varepsilon}}\right) S_{N_\varepsilon}(p_\varepsilon) < 1 \\ S_{N_\varepsilon}(p_\varepsilon) &< \frac{2^{p_\varepsilon}}{2^{p_\varepsilon}-2}. \end{aligned}$$

So $S_{N_\varepsilon}(p)$ is bounded and increasing, and applying Theorem 12, we get the existence of a gauge τ such that $\left(\frac{1}{n^p}\right)_n \in {}^\rho\widetilde{\mathbb{R}}_\tau$ and ${}^\rho\sum_{n\in{}^\tau\widetilde{\mathbb{N}}_{\ge 1}} \frac{1}{n^p}$ converges to $\sum_{n=1}^{+\infty} \frac{1}{n^p} = \zeta(p)$. □

Later, in Corollary 34 and using the integral test Thm. 33, we will prove the convergence of p-hyperseries for arbitrary gauges σ, ρ (even if we will not get the estimates (3.5)).

3.2 Absolute convergence

Theorem 19 *If ${}^\rho\sum_{n\in{}^\sigma\widetilde{\mathbb{N}}} |a_n|$ converges then also ${}^\rho\sum_{n\in{}^\sigma\widetilde{\mathbb{N}}} a_n$ converges.*

Proof Cauchy criterion [13, Thm. 37], yields

$$\forall q \in N\, \exists M \in {}^{\sigma}\widetilde{\mathbb{N}}\, \forall N_1, N_2 \in {}^{\sigma}\widetilde{\mathbb{N}}_{\geq M} : \left| \sum_{n=0}^{N_1} |a_n| - \sum_{n=0}^{N_2} |a_n| \right| < \mathrm{d}.$$

The conclusion hence follows from the inequality

$$\left| \sum_{n=0}^{N_1} a_n - \sum_{n=0}^{N_2} a_n \right| < \left| \sum_{n=0}^{N_1} |a_n| - \sum_{n=0}^{N_2} |a_n| \right|,$$

and from Cauchy criterion for hypersequences, i.e. [13, Thm. 37]. □

3.3 Direct and limit comparison tests

In the direct comparison test, we need to assume a relation of the form $a_n \leq b_n$ for all $n \in \mathbb{N}$ between general terms of two hyperseries. If $a_n = [a_{n\varepsilon}]$, $b_n = [b_{n\varepsilon}] \in {}^{\rho}\widetilde{\mathbb{R}}$ are two representatives, then this inequality would yield

$$\exists [z_{n\varepsilon}] = 0 \in {}^{\rho}\widetilde{\mathbb{R}}\, \exists \varepsilon_{0n}\, \forall \varepsilon \leq \varepsilon_{0n} : \ a_{n\varepsilon} \leq b_{n\varepsilon} + z_{n\varepsilon}.$$

The dependence of ε_{0n} from $n \in \mathbb{N}$ does not allow to prove, e.g., that $\sum_{n=0}^{N} a_n \leq \sum_{n=0}^{N} b_n$ (e.g. as in (2.8), we can consider $b_{n\varepsilon}$:=0 and $a_{n\varepsilon}$:=0 if $\varepsilon < \frac{1}{n+1}$ but $a_{n\varepsilon}$:=1 otherwise, then $a_n = [a_{n\varepsilon}] = 0 = b_n$, but $\sum_{n=0}^{N} a_n = N - \left[\left\lceil \frac{1}{\varepsilon} \right\rceil\right] + 2$).

Moreover, for convergence tests we also need to perform pointwise (in $n \in \mathbb{N}$) operations of the form $\left(\frac{a_n}{b_n}\right)_n$, $\left(\left|\frac{a_{n+k}}{a_n}\right|\right)_n$ or $\left(|a_n|^{1/n}\right)_n$. This kind of pointwise operations can be easily considered if we restrict us to sequences $\{a_n\}_n = [a_{n\varepsilon}]_{\mathrm{u}} \in {}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{u}}$ and we assume that $\sigma \geq \rho^*$. Anyway, this is a particular sufficient condition, and we can more generally state some convergence tests using the more general space ${}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$. For these reasons, we define

Definition 20 Let $(a_n)_n, (b_n)_n \in {}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$, then we say that $(a_n)_n \leq (b_n)_n$ if

$$\forall N, M \in {}^{\sigma}\widetilde{\mathbb{N}} : \sum_{n=N}^{M} a_n \leq \sum_{n=N}^{M} b_n \text{ in } {}^{\rho}\widetilde{\mathbb{R}}. \tag{3.6}$$

Note explicitly that if $M <_L N$ on $L \subseteq_0 I$, then $\sum_{n=N}^{M} a_n =_L \sum_{n=N}^{M} b_n =_L 0$. Therefore, (3.6) is equivalent to

$$\forall N, M \in {}^{\sigma}\widetilde{\mathbb{N}} : \ M \geq N \ \Rightarrow \sum_{n=N}^{M} a_n \leq \sum_{n=N}^{M} b_n \text{ in } {}^{\rho}\widetilde{\mathbb{R}}.$$

On the other hand, we recall that the Robinson-Colombeau ring ${}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{u}}$ is already ordered by $\{a_n\}_n = [a_{n\varepsilon}]_{\mathrm{u}} \le [b_{n\varepsilon}]_{\mathrm{u}} = \{b_n\}_n$ if

$$\exists\, [z_{n\varepsilon}]_{\mathrm{u}} = 0 \in {}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{u}}\, \forall^0 \varepsilon\, \forall n \in \mathbb{N} :\ a_{n\varepsilon} \le b_{n\varepsilon} + z_{n\varepsilon}. \tag{3.7}$$

We also recall that ${}^{\rho}\widetilde{\mathbb{R}} \subseteq {}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$ through the embedding $x \mapsto [(x_\varepsilon)_{n\varepsilon}]$. Thereby, a relation of the form $x \le [a_{n\varepsilon}]_{\mathrm{u}}$ yields

$$\forall^0 \varepsilon\, \forall n \in \mathbb{N} :\ x_\varepsilon \le a_{n\varepsilon}$$

for some representative $x = [x_\varepsilon]$. Finally, [13, Lem. 2] for the ring ${}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{u}}$ implies that $\{a_n\}_n < \{b_n\}_n$ if and only if

$$\exists q \in \mathbb{R}_{>0}\, \forall^0 \varepsilon\, \forall n \in \mathbb{N} :\ b_{n\varepsilon} - a_{n\varepsilon} \ge \mathrm{d}^q.$$

Theorem 21 *We have the following properties:*

(i) $({}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}, \le)$ *is an ordered* ${}^{\rho}\widetilde{\mathbb{R}}$*-module.*
(ii) If $(a_n)_n \ge 0$*, then* $N \in {}^{\sigma}\widetilde{\mathbb{N}} \mapsto \sum_{n=0}^{N} a_n \in {}^{\rho}\widetilde{\mathbb{R}}$ *is increasing.*
(iii) If $\sigma \ge \rho^*$*, and* $[a_{n\varepsilon}]_{\mathrm{u}} \le [b_{n\varepsilon}]_{\mathrm{u}}$ *in* ${}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{u}}$*, then* $[a_{n\varepsilon}]_{\mathrm{s}} \le [b_{n\varepsilon}]_{\mathrm{s}}$ *in* ${}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$*.*

Proof (i): The relation $\le$ on ${}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$ is clearly reflexive and transitive. If $(a_n)_n \le (b_n)_n \le (a_n)_n$, then for all $N, M \in {}^{\sigma}\widetilde{\mathbb{N}}$ we have $\sum_{n=N}^{M} a_n = \sum_{n=N}^{M} b_n$ in ${}^{\rho}\widetilde{\mathbb{R}}$. From Definition 5 of hypersum, this implies (2.9), i.e. that $(a_n)_n = (b_n)_n$ in ${}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$.

(ii): Let $N, M \in {}^{\sigma}\widetilde{\mathbb{N}}$ with $N \le M$, then $\sum_{n=0}^{M} a_n = \sum_{n=0}^{N} a_n + \sum_{n=N+1}^{M} a_n$ by Lemma 14, and $\sum_{n=N+1}^{M} a_n \ge 0$ because $(a_n)_n \ge 0$ in ${}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$.

(iii): Assume that the inequality in (3.7) holds for all $\varepsilon \le \varepsilon_0$ and for all $n \in \mathbb{N}$. Then, if $N = [N_\varepsilon]$, $M = [M_\varepsilon]$, $N_\varepsilon, M_\varepsilon \in \mathbb{N}$, we have $\sum_{n=N_\varepsilon}^{M_\varepsilon} a_{n\varepsilon} \le \sum_{n=N_\varepsilon}^{M\varepsilon} b_{n\varepsilon} + \sum_{n=N_\varepsilon}^{M\varepsilon} z_{n\varepsilon}$ for all $\varepsilon \le \varepsilon_0$. Since $[z_{n\varepsilon}]_{\mathrm{u}} = 0 \in {}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{u}}$, for each $q \in \mathbb{N}$, for ε small and for all $n \in \mathbb{N}$, we have $|z_{n\varepsilon}| \le \rho_\varepsilon^q$ and hence $\left|\sum_{n=N_\varepsilon}^{M_\varepsilon} z_{n\varepsilon}\right| \le |M_\varepsilon - N_\varepsilon|\, \rho_\varepsilon^q$. Thereby, from the assumption $\sigma \ge \rho^*$, it follows that $\left(\sum_{n=N_\varepsilon}^{M_\varepsilon} z_{n\varepsilon}\right) \sim_\rho 0$, which proves the conclusion. □

The direct comparison test for hyperseries can now be stated as follows.

Theorem 22 *Let σ and ρ be arbitrary gauges, let* ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n$ *and* ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} b_n$ *be hyperseries with* $(a_n)_n, (b_n)_n \ge 0$ *and such that*

$$\exists N \in \mathbb{N} :\ (a_{n+N})_n \le (b_{n+N})_n. \tag{3.8}$$

Then we have:

(i) If ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} b_n$ *is convergent then so is* ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n$*.*
(ii) If ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n$ *is divergent to* $+\infty$*, then so is* ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} b_n$*.*

(iii) *If* $\sigma \ge \rho^*$ *and* $\{a_n\}_n:=[a_{n\varepsilon}]_{\mathrm{u}}$, $\{b_n\}_n = [b_{n\varepsilon}]_{\mathrm{u}} \in {}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{u}}$, *then we have the same conclusions if we assume that*

$$\exists N \in \mathbb{N} : \ \{a_{n+N}\}_n \le \{b_{n+N}\}_n \ \mathit{in}\ {}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{u}}.$$

Proof We first note that (3.8) can be simply written as $(\alpha_n)_n \le (\beta_n)_n$, where $\alpha_n:=a_{n+N}$ and $\beta_n:=b_{n+N}$. Thereby, Corollary 16 implies that, without loss of generality, we can assume $N = 0$. To prove (i), let us consider the partial sums $A_N:=\sum_{i=0}^N a_i$ and $B_N:=\sum_{i=0}^N b_i$, $N \in {}^{\sigma}\widetilde{\mathbb{N}}$. Since ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} b_n$ is convergent, we have $\exists^{\rho}\lim_{N\in{}^{\sigma}\widetilde{\mathbb{N}}} B_N =: B \in {}^{\rho}\widetilde{\mathbb{R}}$. The assumption $(a_n)_n \le (b_n)_n$ implies $A_N \le B_N$. The hypersequences $(A_N)_{N\in{}^{\sigma}\widetilde{\mathbb{N}}}$, $(B_N)_{N\in{}^{\sigma}\widetilde{\mathbb{N}}}$ are increasing because $(a_n)_n$, $(b_n)_n \ge 0$ (Theorem 21.(ii)) and hence $(B - B_N)_{N\in{}^{\sigma}\widetilde{\mathbb{N}}}$ decreases to zero because of the convergence assumption. Now, for all $N, M \in {}^{\sigma}\widetilde{\mathbb{N}}$, $M \ge N$, we have

$$A_N \le A_M = \sum_{n=0}^{M} a_n = \sum_{n=0}^{N} a_n + \sum_{n=N+1}^{M} a_n \le A_N + \sum_{n=N+1}^{M} b_n = A_N + (B - B_N). \tag{3.9}$$

Thereby, given $m, n \ge N$, applying (3.9) with m, n instead of M, we get that both A_n, A_m belong to the interval $[A_N, A_N + (B - B_N)]$, whose length $B - B_N$ decreases to zero as $N \in {}^{\sigma}\widetilde{\mathbb{N}}$ goes to infinity. This shows that $(A_n)_{n\in{}^{\sigma}\widetilde{\mathbb{N}}}$ is a Cauchy hypersequence, and therefore ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n$ converges.

The proof of (ii) follows directly from the inequality $A_N \le B_N$ for each $N \in {}^{\sigma}\widetilde{\mathbb{N}}$. The proof of (iii) follows from Lemma 21.(iii) and from Lemma 7.(iv). □

Note that the two cases (i) and (ii) do not cover the case where the non-decreasing hypersums $N \mapsto \sum_{n=0}^N a_n$ are bounded but anyway do not converge because it does not exists the supremum of their values (see [13]).

Example 23 Let $x \in {}^{\rho}\widetilde{\mathbb{R}}_{\ge 0}$ be a finite number so that $x \le M \in \mathbb{N}_{>0}$ for some M. Assume that $x_\varepsilon \le M$ for all $\varepsilon \le \varepsilon_0$. For these ε we have $\frac{n+x_\varepsilon}{n^3} \le \frac{2}{n^2}$ for all $n \in \mathbb{N}_{\ge M}$. This shows that $\left\{\frac{n+M+x}{(n+M)^3}\right\}_n \le \left\{\frac{2}{(n+M)^2}\right\}_n$ and we can hence apply Theorem 22.

The limit comparison test is the next

Theorem 24 *Let σ and ρ be arbitrary gauges, let ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n$ and ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} b_n$ be hyperseries, with $(a_n)_n \ge 0$. Assume that*

$$\exists m, M \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}\, \exists N \in \mathbb{N} : \ (m b_{n+N})_n \le (a_{n+N})_n \le (M b_{n+N})_n\,. \tag{3.10}$$

Then we have:

(i) *Either both hyperseries ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n$, ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} b_n$ converge or diverge to $+\infty$.*

(ii) *If $\sigma \ge \rho^*$, $\{a_n\}_n = [a_{n\varepsilon}]_{\mathrm{u}}$, $\{b_n\}_n = [b_{n\varepsilon}]_{\mathrm{u}} \in {}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{u}}$, $[b_{n\varepsilon}]_{\mathrm{u}} > 0$ and*

$$\exists m, M \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}\, \exists N \in \mathbb{N} : \ m \le \left\{\frac{a_{n+N}}{b_{n+N}}\right\}_n \le M,$$

then the same conclusion as in (i) holds.

Proof As in the previous proof, without loss of generality, we can assume $N = 0$. Now, if ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} b_n$ diverges to $+\infty$, then so does ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} mb_n$ because $m > 0$. Since $(mb_n)_n < (a_n)_n$, by the direct comparison test also the hyperseries ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n$ diverges to $+\infty$. Likewise, if the hyperseries ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} b_n$ converges then so does ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} Mb_n$. Since $(a_n)_n < (Mb_n)_n$, by the direct comparison test also the hyperseries ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n$ converges. Property (ii) can be proved as in the previous theorem. □

Note that if $M, a_n, b_n \in \mathbb{R}$, then the condition $\left\{\frac{a_n}{b_n}\right\}_n \le M$ is equivalent to $\limsup_{n\to+\infty} \frac{a_n}{b_n} \le M$. Analogously, $m \le \left\{\frac{a_n}{b_n}\right\}_n$ is equivalent to $\liminf_{n\to+\infty} \frac{a_n}{b_n} \ge m$. This shows that our formulation of the limit comparison test faithfully generalizes the classical version.

Both the root and the ratio tests are better formulated in ${}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{u}}$, because their proofs require term-by-term operations (such as e.g. that $|a_n|^{1/n} \le L$ implies $|a_n| \le L^n$), so that the order relation defined in Definition (20) seems too weak for these aims; see also the comment below the proof of the root test.

3.4 Root test

Theorem 25 *Let $\sigma \ge \rho^*$ and let $\{|a_n|\}_n \in {}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{u}}$ (so that also $\left\{|a_n|^{1/n}\right\}_n \in {}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{u}}$). Assume that*

$$\exists L \in {}^{\rho}\widetilde{\mathbb{R}} : \left\{|a_n|^{1/n}\right\}_n \le L. \tag{3.11}$$

Then

(i) If $L < 1$, then the hyperseries ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n$ converges absolutely.
(ii) If $J \subseteq_0 I$ and $L >_J 1$, then $(|a_n||_J)_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} \to +\infty$ and hence the hyperseries ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n|_J$ diverges.
(iii) If $L \not=_{\mathrm{s}} 1$, then the hyperseries ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n$ converges if and only if $L < 1$.

Proof (i): Our assumption $\left\{|a_n|^{1/n}\right\}_n \le L$ entails $\{|a_n|\}_n < (L^n)_n$. Since ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}_{\ge 0}} L^n$ is convergent because $0 \le L < 1$, by the direct comparison test, the series ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} |a_n|$ is also convergent.

(ii): Now, assume that $L_\varepsilon > 1$ for $\varepsilon \in J \subseteq_0 I$ and let us work directly in the ring ${}^{\rho}\widetilde{\mathbb{R}}|_J$. Proceeding as above, we have $\{|a_n|\}_n \ge_J (L^n)_n$, and thereby the last claim follows because of $L >_J 1$.

(iii): Assume that $L \not=_{\mathrm{s}} 1$, that ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n$ converges but $L >_{\mathrm{s}} 1$, then (ii) would yield that ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n|_J$ diverges for some $J \subseteq_0 I$ and this contradicts the convergence assumption. Therefore, we have $L \not=_{\mathrm{s}} 1$ and $L \not>_{\mathrm{s}} 1$, and hence [13, Lem. 6.(v)] gives $L \not\ge_{\mathrm{s}} 1$. Thereby, [13, Lem. 5.(ii)] finally implies $L < 1$. □

Actually, we can also simply reformulate the root test in ${}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$ by trivially asking that $(|a_n|)_n \le (L^n)_n$, but that would simply recall a non-meaningful consequence of the direct comparison test Theorem 22.

3.5 Ratio test

Proceeding as in the previous proof, i.e. by generalizing the classical proof for series of real numbers, we also have the following

Theorem 26 *Let $\sigma \ge \rho^*$ and let $\{a_n\}_n \in {}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{u}}$, with $\{|a_n|\}_n > 0$. Assume that for some $k \in \mathbb{N}$ we have*

$$\exists L \in {}^{\rho}\widetilde{\mathbb{R}} : \left\{ \frac{a_{n+k}}{a_n} \right\}_n \le L.$$

Then

(i) If $L < 1$, the hyperseries ${}^{\rho}\sum_{n \in {}^{\sigma}\widetilde{\mathbb{N}}} a_n$ converges.
(ii) If $J \subseteq_0 I$, $L >_J 1$, then $(|a_n| \,|_J)_{n \in {}^{\sigma}\widetilde{\mathbb{N}}} \to +\infty$ and hence the hyperseries ${}^{\rho}\sum_{n \in {}^{\sigma}\widetilde{\mathbb{N}}} a_n|_J$ diverges.
(iii) If $L \not\backsimeq 1$, then the hyperseries ${}^{\rho}\sum_{n \in {}^{\sigma}\widetilde{\mathbb{N}}} a_n$ converges if and only if $L < 1$.

As in the classical case, the convergent p-hyperseries ${}^{\rho}\sum_{n \in {}^{\sigma}\widetilde{\mathbb{N}}} \frac{1}{n^2}$ and the divergent hyperseries ${}^{\rho}\sum_{n \in {}^{\sigma}\widetilde{\mathbb{N}}} 1^n$ shows that the ratio and the root tests fails if $L = 1$.

Example 27 If $x \in {}^{\rho}\widetilde{\mathbb{R}}$ is a finite invertible number, then using the ratio test and proceeding as in Example 23, we can prove that ${}^{\rho}\sum_{n \in {}^{\sigma}\widetilde{\mathbb{N}}} \frac{x^n}{n!}$ converges if $\sigma \ge \rho^*$.

3.6 Alternating series test

Also in the proof of this test we need a term-by-term comparison (see (3.14) below).

Theorem 28 *Let $\sigma \ge \rho^*$ and let $\{a_n\}_n \in {}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{u}}$. Assume that we have*

$$\{a_{n+1}\}_n \le \{a_n\}_n \tag{3.12}$$

$${}^{\rho}\lim_{n \in {}^{\sigma}\widetilde{\mathbb{N}}} a_n = 0. \tag{3.13}$$

Then the alternating hyperseries ${}^{\rho}\sum_{n \in {}^{\sigma}\widetilde{\mathbb{N}}} (-1)^n a_n$ converges.

Proof By (3.12), assume that

$$a_{n+1,\varepsilon} \le a_{n\varepsilon} \quad \forall n \in \mathbb{N}\, \forall \varepsilon \le \varepsilon_0 \tag{3.14}$$

holds for all $n \in \mathbb{N}$ and all $\varepsilon \le \varepsilon_0$. Take $N = [N_\varepsilon]$, $M = [M_\varepsilon] \in {}^{\sigma}\widetilde{\mathbb{N}}$, where for $\varepsilon \le \varepsilon_0$ we have $N_\varepsilon, M_\varepsilon \in \mathbb{N}$. If M_ε is odd and $M_\varepsilon \le N_\varepsilon$, we can estimate the difference

$S_{N_\varepsilon} - S_{M_\varepsilon}$ as:

$$
\begin{aligned}
S_{N_\varepsilon} - S_{M_\varepsilon} &= \sum_{n=0}^{N_\varepsilon}(-1)^n a_n - \sum_{n=0}^{M_\varepsilon}(-1)^n a_k = \sum_{n=M_\varepsilon+1}^{N_\varepsilon}(-1)^n a_n \\
&= a_{M_\varepsilon+1} - a_{M_\varepsilon+2} + a_{M_\varepsilon+3} + \cdots + a_{N_\varepsilon} \\
&= a_{M_\varepsilon+1} - (a_{M_\varepsilon+2} - a_{M_\varepsilon+3}) - (a_{M_\varepsilon+4} - a_{M_\varepsilon+5}) - \cdots - a_{N_\varepsilon} \\
&\le a_{M_\varepsilon+1} \le a_{M_\varepsilon},
\end{aligned}
\tag{3.15}
$$

where we used (3.14). If M_ε is even and $M_\varepsilon \le N_\varepsilon$, a similar argument shows that $S_{N_\varepsilon} - S_{M_\varepsilon} \ge -a_{M_\varepsilon}$. If $M_\varepsilon > N_\varepsilon$, it suffices to revert the role of M_ε and N_ε in (3.15). This proves that $\min(-a_M, -a_N) \le S_N - S_M \le \max(a_M, a_N)$. The final claim now follows from (3.13) and Cauchy criterion [13, Thm. 44]. □

Example 29 Using the previous Theorem 28, we can prove, e.g., that the hyperseries ${}^\rho\sum_{n\in{}^\sigma\widetilde{\mathbb{N}}} \frac{(-1)^n}{n}$ is convergent.

3.7 Integral test

The possibility to prove the integral test for hyperseries is constrained by the existence of a notion of generalized function that can be defined on an unbounded interval, e.g. of the form $[0, +\infty) \subseteq {}^\rho\widetilde{\mathbb{R}}$. This is not possible for an arbitrary Colombeau generalized function, which are defined only on finite (i.e. compactly supported) points, i.e. on domains of the form $\widetilde{\Omega}_c$ (see e.g. [9]). Moreover, we would also need a notion of improper integral extended on $[0, +\infty)$. This notion clearly needs to have good relations with the notion of hyperlimit and with the definite integral over the interval $[0, N]$, where $N \in {}^\rho\widetilde{\mathbb{N}}$ (see Definition 32 below). This further underscores drawbacks of Colombeau's theory, where this notion of integral (if N is an infinite number) is not defined, see e.g. [1].

The theory of generalized smooth functions overcomes these difficulties, see e.g. [7,8]. Here, we only recall the equivalent definition. In the following, σ, ρ always denote arbitrary gauges.

Definition 30 Let $X \subseteq {}^\sigma\widetilde{\mathbb{R}}^n$ and $Y \subseteq {}^\sigma\widetilde{\mathbb{R}}^d$. We say that $f : X \longrightarrow Y$ is a ρ-moderate *generalized smooth function* (GSF), and we write $f \in {}^\rho_\sigma\mathcal{GC}^\infty(X, Y)$, if

(i) $f : X \longrightarrow Y$ is a set-theoretical function.
(ii) There exists a net $(f_\varepsilon) \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^d)^{(0,1]}$ such that for all $[x_\varepsilon] \in X$:

   (a) $f(x) = [f_\varepsilon(x_\varepsilon)]$
   (b) $\forall \alpha \in \mathbb{N}^n : (\partial^\alpha f_\varepsilon(x_\varepsilon))$ is $\rho -$ moderate.

For generalized smooth functions lots of results hold: closure with respect to composition, embedding of Schwartz's distributions, differential calculus, one-dimensional integral calculus using primitives, classical theorems (intermediate value, mean value, Taylor, extreme value, inverse and implicit function), multidimensional integration, Banach fixed point theorem, a Picard-Lindelöf theorem for both ODE and PDE, several results of calculus of variations, etc.

In particular, we have the following

Theorem 31 *Let a, b, $c \in {}^{\rho}\widetilde{\mathbb{R}}$, with $a < b$ and $c \in [a, b] \subseteq U$. Let $f \in {}^{\rho}_{\sigma}\mathcal{GC}^{\infty}([a, b], {}^{\rho}\widetilde{\mathbb{R}})$ be a generalized smooth function. Then, there exists one and only one generalized smooth function $F \in {}^{\rho}_{\sigma}\mathcal{GC}^{\infty}([a, b], {}^{\rho}\widetilde{\mathbb{R}})$ such that $F(c) = 0$ and $F'(x) = f(x)$ for all $x \in [a, b]$. Moreover, if f is defined by the net $f_{\varepsilon} \in \mathcal{C}^{\infty}(\mathbb{R}, \mathbb{R})$ and $c = [c_{\varepsilon}]$, then*

$$F(x) = \left[\int_{c_{\varepsilon}}^{x_{\varepsilon}} f_{\varepsilon}(s)\, d\right] \tag{3.16}$$

for all $x = [x_{\varepsilon}] \in [a, b]$.

We can hence define $\int_c^x f(t)\,\mathrm{d}{:=}F(x)$ for all $x \in [a, b]$ (note explicitly that a, b can also be infinite generalized numbers). For this notion of integral we have all the usual elementary property, monotonicity and integration by substitution included.

For the integral test for hyperseries, we finally need the following

Definition 32 Let $f \in {}^{\rho}_{\sigma}\mathcal{GC}^{\infty}([0, +\infty), {}^{\rho}\widetilde{\mathbb{R}})$, then we say

$$\begin{aligned}
\exists \int_0^{+\infty} f(x)\,\mathrm{d} \quad &:\Longleftrightarrow \quad \exists\, {}^{\rho}\lim_{N\in{}^{\sigma}\widetilde{\mathbb{N}}} \int_0^N f(x)\,\mathrm{d} =: \int_0^{+\infty} f(x)\,\mathrm{d} \in {}^{\rho}\widetilde{\mathbb{R}} \\
\int_0^{+\infty} f(x)\,\mathrm{d} = \pm\infty \quad &:\Longleftrightarrow \quad {}^{\rho}\lim_{N\in{}^{\sigma}\widetilde{\mathbb{N}}} \int_0^N f(x)\,\mathrm{d} = \pm\infty.
\end{aligned}$$

Theorem 33 *Let $f = [f_{\varepsilon}(-)] \in {}^{\rho}_{\sigma}\mathcal{GC}^{\infty}([0, +\infty), {}^{\rho}\widetilde{\mathbb{R}})$ be a GSF such that*

$$\forall^0\varepsilon\, \forall n \in \mathbb{N}\, \forall x \in [n, n+1)_{\mathbb{R}} : \ f_{\varepsilon}(x) \le f_{\varepsilon}(n) \tag{3.17}$$

$$\forall^0\varepsilon\, \forall n \in \mathbb{N}_{>0}\, \forall x \in [n-1, n)_{\mathbb{R}} : \ f_{\varepsilon}(n) \le f_{\varepsilon}(x) \tag{3.18}$$

then $(f(n))_n = [f_{\varepsilon}(n)]_{\mathrm{s}} \in {}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$ and

(i) The hyperseries ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} f(n)$ converges if $\exists \int_0^{+\infty} f(x)\, d$ and ${}^{\rho}\lim_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} f(n) = 0$.
(ii) The hyperseries ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} f(n)$ diverges to $+\infty$ if $\int_0^{+\infty} f(x)\, d = +\infty$.

Proof From (3.16), for all N, $M \in {}^{\rho}\widetilde{\mathbb{N}}$ we have

$$\int_N^{M+1} f(x)\,\mathrm{d} = \left[\int_{N_{\varepsilon}}^{M_{\varepsilon}+1} f_{\varepsilon}(x) dx\right].$$

Without loss of generality, we can assume $N_\varepsilon, M_\varepsilon \in \mathbb{N}$ for all ε. Thereby

$$\int_N^{M+1} f(x)dx = \left[\sum_{n=N_\varepsilon}^{M_\varepsilon} \int_n^{n+1} f_\varepsilon(x)\,\mathrm{d}\right] \le$$
$$= \left[\sum_{n=N_\varepsilon}^{M_\varepsilon} \int_n^{n+1} f_\varepsilon(n)\,\mathrm{d}\right] = \sum_{n=N}^{M} f(n),$$

where we used Definition 5 of hypersum and (3.17). On the other hand, since $N \in {}^\sigma\widetilde{\mathbb{N}} \subseteq [0, +\infty) \subseteq {}^\sigma\widetilde{\mathbb{R}}$, using (3.18) we have

$$\sum_{n=N}^{M} f(n) = f(N) + \sum_{n=N+1}^{M} \int_{n-1}^{n} f(n)\,\mathrm{d} = f(N) + \left[\sum_{n=N_\varepsilon+1}^{M_\varepsilon} \int_{n-1}^{n} f_\varepsilon(n)\,\mathrm{d}\right]$$
$$\le f(N) + \left[\sum_{n=N_\varepsilon+1}^{M_\varepsilon} \int_{n-1}^{n} f_\varepsilon(x)\,\mathrm{d}\right] = f(N) + \int_N^M f(x)\,\mathrm{d},$$

where we used again Definition 5 of hypersum and (3.16). Combining these two inequalities yields

$$\int_N^{M+1} f(x)\,\mathrm{d} \le \sum_{n=N}^{M} f(n) \le f(N) + \int_N^M f(x)\,\mathrm{d}. \tag{3.19}$$

These inequalities show that $(f(n))_n \in {}^\rho_\sigma\widetilde{\mathbb{R}}_s$. Now, we consider that $\sum_{n=0}^{M} f(n) - \sum_{n=0}^{N} f(n) = \sum_{n=N}^{M} f(n)$ for $M \ge N$, and

$$\int_N^M f(x)\,\mathrm{d} = -\int_0^N f(x)\,\mathrm{d} + \int_0^M f(x)\,\mathrm{d}.$$

Therefore, if $\exists \int_0^{+\infty} f(x)\,\mathrm{d}$ and ${}^\rho\lim_{n\in{}^\sigma\widetilde{\mathbb{N}}} f(n) = 0$, letting $N, M \to +\infty$ in (3.19) proves that our hyperseries satisfies the Cauchy criterion. If $\int_0^{+\infty} f(x)\,\mathrm{d} = +\infty$, then setting $N = 0$ in the first inequality of (3.19) and $M \to +\infty$ we get the second conclusion. □

Note that, generalizing by contradiction the classical proof, we only have

$$\exists \int_0^{+\infty} f(x)\,\mathrm{d} \text{ and } \exists^\rho \lim_{n\in{}^\sigma\widetilde{\mathbb{N}}} |f(n)| =: L \quad\Rightarrow\quad L \text{ is not invertible}$$

and not $L = 0$ like in classical case.

As usual, from the integral test we can also deduce the p-series test:

Corollary 34 *If $p \in {}^\rho\widetilde{\mathbb{R}}_{>1}$, then the p-hyperseries converges.*

Proof Since $p=[p_\varepsilon]>1$, we can assume that $p_\varepsilon>1$ for all $\varepsilon\le\varepsilon_0$. For these ε, if $x\in[n,n+1)_{\mathbb{R}}$, we also have $\frac{1}{x^{p_\varepsilon}}\le\frac{1}{n^{p_\varepsilon}}$, which proves (3.17). Similarly, we can prove (3.18). Note that the function $f(x)=\frac{1}{x^p}$ for all $x\in[1,+\infty)\subseteq{}^\sigma\widetilde{\mathbb{R}}$ is ρ-moderate because $x\ge1$ and $p>1$, so that $|f(x)|\le1$. Finally, like in the classical case, we have $\int_1^{+\infty}\frac{\mathrm{d}}{x^p}=\frac{1}{1-p}{}^\rho\lim_{N\in{}^\rho\widetilde{\mathbb{N}}}\left(\frac{1}{N^{p-1}}-1\right)=\frac{1}{p-1}$ because $p>1$. □

Note, however, that our proofs in Sect. 3.1 are independent from the notion of generalized smooth function.

4 Cauchy product of hyperseries

We recall that, on the basis of Theorem 3, the operations of sum and product by a number in ${}^\rho\widetilde{\mathbb{R}}$ are the sole operations defined in ${}^\rho_\sigma\widetilde{\mathbb{R}}_{\mathrm{s}}$. We now consider the classical Cauchy product:

Definition 35 The *Cauchy product of two sequences for hypersums* $(a_n)_n,(b_n)_n\in{}^\rho_\sigma\widetilde{\mathbb{R}}_{\mathrm{s}}$ is defined as

$$(a_n)_n\star(b_n)_n:=\left(\sum_{k=0}^n a_kb_{n-k}\right)_n.$$

The *Cauchy product of two hyperseries* is the hyperlimit of the hypersums with general term $(a_n)_n\star(b_n)_n$, assuming that it exists:

$$\left({}^\rho\sum_{n\in{}^\sigma\widetilde{\mathbb{N}}}a_n\right)*\left({}^\rho\sum_{n\in{}^\sigma\widetilde{\mathbb{N}}}b_n\right):={}^\rho\sum_{n\in{}^\sigma\widetilde{\mathbb{N}}}(a_n)_n\star(b_n)_n\,.$$

A similar *product of sequences* can be defined in ${}^\rho\widetilde{\mathbb{R}}_{\mathrm{u}}$ with

$$\{a_n\}_n\star\{b_n\}_n:=\left\{\sum_{k=0}^n a_kb_{n-k}\right\}_n.$$

Theorem 36 *The module ${}^\rho_\sigma\widetilde{\mathbb{R}}_{\mathrm{s}}$ is a ring with respect to the Cauchy product of sequences. The ring ${}^\rho\widetilde{\mathbb{R}}_{\mathrm{u}}$ is also a ring with respect to the Cauchy product of sequences.*

Proof Let $(a_n)_n,(b_n)_n\in{}^\rho\widetilde{\mathbb{R}}_{\mathrm{u}}$, $(a_{n\varepsilon})_{n,\varepsilon}\sim_{\sigma\rho}(\bar a_{n\varepsilon})_{n,\varepsilon}$, $(b_{n\varepsilon})_{n,\varepsilon}\sim_{\sigma\rho}(\bar b_{n\varepsilon})_{n,\varepsilon}$. We first prove that the Cauchy product $(a_n)_n\star(b_n)_n$ is well-defined in ${}^\rho_\sigma\widetilde{\mathbb{R}}_{\mathrm{s}}$. For simplicity, for $n,m\in{}^\sigma\widetilde{\mathbb{N}}$, set $A_{nm}:=\sum_{k=n}^m a_k$, $B_{nm}:=\sum_{k=n}^m b_k$ and similar notations $\bar A_{nm}$, $\bar B_{nm}$ using the other representatives. Then

$$\sum_{k=n}^m a_kb_{n-k}-\sum_{k=n}^m\bar a_k\bar b_{n-k}=\left(A_{nm}-\bar A_{nm}\right)B_{nm}+\bar A_{nm}\left(B_{nm}-\bar B_{nm}\right).$$

Now, $A_{nm} - \bar{A}_{nm}$ and $B_{nm} - \bar{B}_{nm}$ are ρ-negligible by Definition (2.9) of $\sim_{\sigma\rho}$, and B_{nm}, $\bar{A}_{nm}$ are ρ-moderate because of Lemma 4. This proves that the Cauchy product $\star$ in ${}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$ is well-defined. Similarly, we can proceed with ${}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{u}}$. The ring properties now follow from the same properties of the ring ${}^{\rho}\widetilde{\mathbb{R}}$. □

It is well-known that for each $n \in \mathbb{N}$, we have $(a_n)_n \star (b_n)_n = \left(\sum_{i=0}^{n} a_i\right) \cdot \left(\sum_{j=0}^{n} b_j\right) = \sum_{i=0}^{n}\sum_{j=0}^{n} a_i b_j =: c_n$, so $(c_n)_n \in {}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$ by Theorem 36. Moreover, if ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n$ and ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} b_n$ converge, taking the hyperlimit of the product of the two partial sums we obtain

$$\begin{aligned}\left({}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n\right) \cdot \left({}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} b_n\right) &= {}^{\rho}\lim_{N\in{}^{\sigma}\widetilde{\mathbb{N}}} c_N \\ &= {}^{\rho}\lim_{N\in{}^{\sigma}\widetilde{\mathbb{N}}} \sum_{i=0}^{N}\sum_{j=0}^{N} a_i b_j\end{aligned}$$

but, in general, this is different from

$$\begin{aligned}{}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} c_n &= {}^{\rho}\lim_{N\in{}^{\sigma}\widetilde{\mathbb{N}}} \sum_{n=0}^{N} c_n = {}^{\rho}\lim_{N\in{}^{\sigma}\widetilde{\mathbb{N}}} \sum_{n=0}^{N}\sum_{i=0}^{n}\sum_{j=0}^{i} a_i b_j \\ &= {}^{\rho}\lim_{N\in{}^{\sigma}\widetilde{\mathbb{N}}} \sum_{n=0}^{N}\sum_{k=0}^{n} a_k b_{n-k} = \left({}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n\right) * \left({}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} b_n\right).\end{aligned}$$

In other words, if the Cauchy product of these hyperseries exists, in general we have

$$\left({}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n\right) * \left({}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} b_n\right) \neq \left({}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n\right) \cdot \left({}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} b_n\right).$$

The classical Mertens' theorem also holds for hyperseries:

Theorem 37 *Assume that the hyperseries ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n$ converges to A and ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} b_n$ converges to B. Assume that the former hyperseries converges absolutely. Then their Cauchy product converges to AB.*

We first need the following

Lemma 38 *If $(b_n)_n$ converges to B, then*

$$\exists K \in {}^{\rho}\widetilde{\mathbb{R}}\, \forall N \in {}^{\sigma}\widetilde{\mathbb{N}} : \left|\sum_{k=0}^{N} b_k - B\right| \leq K. \tag{4.1}$$

Proof Set $B_N := \sum_{k=0}^{N} b_k \in {}^{\rho}\widetilde{\mathbb{R}}$ for all $N \in {}^{\sigma}\widetilde{\mathbb{N}}$. The assumption of convergence yields

$$\exists N_1 \in {}^{\sigma}\widetilde{\mathbb{N}}\, \forall M \in {}^{\sigma}\widetilde{\mathbb{N}}_{\geq N_1} : \ |B_M - B| < 1. \tag{4.2}$$

Set $K = 1 + \sum_{k=0}^{N_1} |b_k| \in {}^{\rho}\widetilde{\mathbb{R}}$ and let $N \in {}^{\sigma}\widetilde{\mathbb{N}}$. We use dichotomy law as in [13, Lem. 7.(ii)], i.e. we consider $L \subseteq_0 I$ and two cases $N \geq_L N_1$ or $N \leq_L N_1$. We claim that $\left|B_{N_\varepsilon} - B_\varepsilon\right| \leq K_\varepsilon$ for all $\varepsilon \in L$ sufficiently small, where $N_\varepsilon := \mathrm{ni}(N)_\varepsilon$, $N_{1\varepsilon} := \mathrm{ni}(N_1)_\varepsilon$, $(b_n)_n = [b_{n\varepsilon}]_{\mathrm{s}} \in {}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$, $B_{N_\varepsilon} := \sum_{k=0}^{N_\varepsilon} b_{k\varepsilon}$, $B = [B_\varepsilon]$ and $K_\varepsilon := 1 + \sum_{k=0}^{N_{1\varepsilon}} |b_{k\varepsilon}|$ (hence $K = [K_\varepsilon]$). We also note that, in general, $P, Q \in {}^{\sigma}\widetilde{\mathbb{N}}$, $P \leq Q$ implies $\mathrm{ni}(P)_\varepsilon \leq \mathrm{ni}(Q)_\varepsilon$ for ε small. In the first case $N \geq_L N_1$, we get $N_\varepsilon \geq N_{1\varepsilon}$ for ε small. Set $M_\varepsilon := N_\varepsilon$ if $\varepsilon \in L$ and $M_\varepsilon := N_{1\varepsilon}$ otherwise, so that $M \in {}^{\sigma}\widetilde{\mathbb{N}}_{\geq N_1}$ and from (4.2) we obtain $\left|B_{M_\varepsilon} - B_\varepsilon\right| < 1$ for ε small and, in particular

$$\forall^0 \varepsilon \in L : \ \left|B_{N_\varepsilon} - B_\varepsilon\right| < 1 \leq K_\varepsilon.$$

We claim the same conclusion also in the second case $N \leq_L N_1$, i.e.

$$\forall^0 \varepsilon \in L : \ N_\varepsilon \leq N_{1\varepsilon}. \tag{4.3}$$

In fact, we have $B_N = B_{N_1 \vee N} - \sum_{k=N+1}^{N_1 \vee N} b_k$ and hence

$$\begin{aligned} |B_N - B| &\leq \left|B_{N_1 \vee N} - B\right| + \sum_{k=N+1}^{N_1 \vee N} |b_k| < 1 \\ &\quad + \sum_{k=0}^{N_1 \vee N} |b_k| \forall^0 \varepsilon : \ \left|B_{N_\varepsilon} - B_\varepsilon\right| \leq 1 + \sum_{k=0}^{N_{1\varepsilon} \vee N_\varepsilon} |b_{k\varepsilon}|. \end{aligned} \tag{4.4}$$

Inequalities (4.3) and (4.4) prove the claim also in the second case. Therefore, the dichotomy law yields $|B_N - B| \leq K$ and K does not depend on N. □

Proof of Theorem 37 Define $A_n := \sum_{i=0}^{n} a_i$, $B_n := \sum_{i=0}^{n} b_i$ and $C_n := \sum_{i=0}^{n} c_i$ for $n \in {}^{\sigma}\widetilde{\mathbb{N}}$, and with $c_i := \sum_{k=0}^{i} a_k b_{i-k}$ for $i \in \mathbb{N}$. Then $C_n = \sum_{i=0}^{n} a_{n-i} B_i$ because the same equality holds ε-wise for any representatives, and hence $C_n = \sum_{i=0}^{n} a_{n-i}(B_i - B) + A_n B$. The idea is to estimate

$$\begin{aligned} |C_n - AB| &= \left|\sum_{i=0}^{n} a_{n-i}(B_i - B) + (A_n - A)B\right| \\ &\leq \sum_{i=0}^{N-1} |a_{n-i}||B_i - B| + \sum_{i=N}^{n} |a_{n-i}||B_i - B| + |A_n - A||B|, \end{aligned} \tag{4.5}$$

and to use our assumption (4.1) for the first summand. We start from the second summand in (4.5): Since ${}^{\rho}\sum_{n \in {}^{\sigma}\widetilde{\mathbb{N}}} a_n$ converges absolutely to some $\bar{A}$ and B_n converges to B, for all $q \in \mathbb{N}$ we can find $N \in {}^{\sigma}\widetilde{\mathbb{N}}_{>0}$ such that

$$|B_n - B| \leq \frac{\mathrm{d}^q}{\bar{A}+1}$$

for all $n \geq N$. Therefore, for the same $n \geq N$

$$\sum_{i=N}^{n} |a_{n-i}||B_i - B| \leq \mathrm{d}^q.$$

First summand in (4.5): Since ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n$ converges, $(a_n)_{n\in{}^{\sigma}\widetilde{\mathbb{N}}}$ must converge to zero. Hence for some $M \in {}^{\sigma}\widetilde{\mathbb{N}}$ and for all $n \geq M$ and because of Lemma 38, we have

$$|a_n| \leq \frac{\mathrm{d}^q}{N(K+1)} \leq \frac{\mathrm{d}^q}{N\,(|B_i - B|+1)} \quad \forall i \leq N-1 \tag{4.6}$$

In particular, if $n \geq M + N$ and $i \leq N - 1$, then $n - i \geq n - N + 1 > M$ and we can apply (4.6) with $n - i$ instead of n obtaining

$$\sum_{i=0}^{N-1} |a_{n-i}||B_i - B| \leq \frac{\mathrm{d}^q N}{N}.$$

Finally, since A_n converges to A, for some $L \in {}^{\sigma}\widetilde{\mathbb{N}}$ and for all $n \geq L$, we have

$$|A_n - A| \leq \frac{\mathrm{d}^q}{|B|+1}.$$

Then, for all $n \geq \max(L, M+N)$ using the aforementioned estimates, (4.5) yields $|C_n - AB| \leq 3\mathrm{d}^q$. □

5 Conclusions

One of the main goals of the present paper is to show that when dealing with non-Archimedean Cauchy complete rings, it can be worth to consider summations over infinite natural numbers instead of classical series indexed by $n \in \mathbb{N}$. As we proved for the Robinson-Colombeau ring ${}^{\rho}\widetilde{\mathbb{R}}$, this allows us to extend numerous classical results which do not hold using classical series in a non-Archimedean framework. We tried to motivate in a clear way why we have to consider two gauges σ and ρ and, for each result, what relationships we have to consider among them. To help the reader to summarize the condition on the gauges, we can use the following table

Therefore, a possible summary could be: for meaningful examples of convergent series, for all convergence tests, one can assume $\sigma \geq \rho^*$ and use both ${}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{u}}$ or ${}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$. For ε-wise convergence, to have larger domains (see Example 11.10)), one can assume

${}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$	σ, ρ arbitrary
${}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{u}}$	$\sigma \geq \rho^*$
Divergent hypersums	${}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$ and $\sigma \geq \rho^*$
Geometric hyperseries	$\sigma \geq \rho^*$ or $\sigma \leq \rho^*$
Taylor series exponential	$\sigma \geq \rho^*$ or $\sigma \leq \rho^*$
ε-wise convergence	$\exists\sigma \leq \rho^*$ or $\sigma = \rho$
p-series	σ, ρ arbitrary
Direct comparison test	σ, ρ arbitrary
Limit comparison test	σ, ρ arbitrary
Root test	${}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{u}}$ and $\sigma \geq \rho^*$
Ratio test	${}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{u}}$ and $\sigma \geq \rho^*$
Alternating series test	${}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{u}}$ and $\sigma \geq \rho^*$
Integral test	σ, ρ arbitrary

$\sigma \leq \rho^*$ and hence use ${}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$. For divergent hypersums, one must assume $\sigma \geq \rho^*$ and work in ${}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$.

Since non-Archimedean rings are not Dedekind-complete, we have been forced to substitute the least-upper-bound property using the weaker Cauchy completeness. We think that several of the proofs we presented here can be generalized to other non-Archimedean Cauchy complete settings.

Clearly, the present work opens the actual possibility to start the study of hyper-power series, (real or complex) hyper-analytic generalized smooth functions and sigma additivity of multidimensional integration of generalized smooth functions (see [8]).

We also finally mention that the embedding of Schwartz's distributions can be realized by regularization with an *entire* Colombeau mollifier (see e.g. [9]). We can hence conjecture that this embedding yields a hyper-analytic generalized smooth functions (but note that the hyper-power series of classical non-analytic smooth functions with a flat point would converge to zero under any invertible infinitesimal). This conjecture would open the possibility to extend the Cauchy-Kowalevski theorem to all Schwartz's distributions.

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Chapter 7

Hyper-power series and generalized real analytic functions

By D. Tiwari, A. Mukhammadiev, P. Giordano

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HYPER-POWER SERIES AND GENERALIZED REAL ANALYTIC FUNCTIONS

DIKSHA TIWARI, AKBARALI MUKHAMMADIEV, AND PAOLO GIORDANO

ABSTRACT. This article is a natural continuation of the paper Tiwari, D., Giordano, P., *Hyperseries in the non-Archimedean ring of Colombeau generalized numbers* in this journal. We study one variable hyper-power series by analyzing the notion of radius of convergence and proving classical results such as algebraic operations, composition and reciprocal of hyper-power series. We then define and study one variable generalized real analytic functions, considering their derivation, integration, a suitable formulation of the identity theorem and the characterization by local uniform upper bounds of derivatives. On the contrary with respect to the classical use of series in the theory of Colombeau real analytic functions, we can recover several classical examples in a non-infinitesimal set of convergence. The notion of generalized real analytic function reveals to be less rigid both with respect to the classical one and to Colombeau theory, e.g. including classical non-analytic smooth functions with flat points and several distributions such as the Dirac delta. On the other hand, each Colombeau real analytic function is also a generalized real analytic function.

1. INTRODUCTION

In this article, the study of hyperseries in the non-Archimedean ring of Colombeau generalized numbers (CGN), as carried out in [29], is applied to the corresponding notion of hyper-power series. As we will see, this yields results which are more closely related to classical ones, such as, e.g. the equality ${}^{\rho}\sum_{n\in{}^{\rho}\widetilde{\mathbb{N}}}\frac{x^n}{n!}=e^x$ that holds for all $x\in{}^{\rho}\widetilde{\mathbb{R}}$ where the exponential is moderate, i.e. if $|x|\le\log\left(\mathrm{d}\rho^{-R}\right)$ for some $R\in\mathbb{N}$. On the other hand, we will see that classical smooth but non-analytic functions, e.g. smooth functions with flat points, and Schwartz distributions like the Dirac delta, are now included in the related notion of *generalized real analytic function* (GRAF). This implies that necessarily we cannot have a trivial generalization of the identity theorem (see e.g. [22, Cor. 1.2.6, 1.2.7]) but, on the contrary, only a suitable sufficient condition (see Thm. 40 below). The notion of generalized real analytic function hence reveals to be less rigid than the classical concept, by including a large family of non-trivial generalized functions (e.g. Dirac delta δ,

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Heaviside function H, but also powers δ^k, $k \in \mathbb{N}$, and compositions $\delta \circ \delta$, $\delta^k \circ H^h$, $H^h \circ \delta^k$, etc., for $h, k \in \mathbb{N}$.

Conversely, GRAF preserve a lot of classical results: they can be thought of as infinitely long polynomials $f(x) = {}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n(x-c)^n$, with uniquely determined coefficient $a_n = \frac{f^{(n)}(c)}{n!}$, they can be added, multiplied, composed, differentiated, integrated term by term, are closed with respect to inverse function, etc. This lays the foundation for a potential interesting generalization of the Cauchy-Kowalevski theorem which is able to include many non-analytic (but generalized real analytic) generalized functions.

Concerning the theory of analytic Colombeau generalized functions, as developed in [26] for the real case and in [5, 6, 1, 7, 24, 20, 30] for the complex one, it is worth to mention that several properties have been proved in both cases: closure with respect to composition, integration over homotopic paths, Cauchy integral theorem, existence of analytic representatives, identity theorem on a set of positive Lebesgue measure, etc. (cf. [26, 30] and references therein). On the other hand, even if in [30] it is also proved that each complex analytic Colombeau generalized functions can be written as a Taylor series, necessarily this result holds only in an infinitesimal neighborhood of each point. The impossibility to extend this property to a finite neighborhood is a general drawback of the use of ordinary series in a (Cauchy complete) non-Archimedean framework instead of hyperseries, as explained in details in [29].

We refer to [23] for basic notions such as the ring of Robinson-Colombeau, subpoints, hypernatural numbers, supremum, infimum and hyperlimits, and [29] for the notion of hyperseries as well as their notations and properties. Once again, the ideas presented in the present article can be useful to explore similar ideas in other non-Archimedean settings, such as [3, 17, 18, 4, 19, 21, 28].

2. HYPER-POWER SERIES AND ITS BASIC PROPERTIES

2.1. **Definition of hyper-power series.** In the entire paper, ρ and σ are two arbitrary gauges; only when it will be needed, we will assume a relation between them, such as $\sigma \le \rho^*$ or $\sigma \ge \rho^*$ (see [29]).

A power series of *real numbers* is usually simply defined as "a series of the form $\sum_{n\in\mathbb{N}} a_n(x-c)^n$, where $x, c, a_n \in \mathbb{R}$ for all $n \in \mathbb{N}$". Actually, this (informal) definition does not state explicitly whether a non-convergent series is included or not. However, unlike the real case where finite sums can always be considered, this does not hold for hyperfinite sums in the ring ${}^{\rho}\widetilde{\mathbb{R}}$, see [29]. For this reason, we consider the ${}^{\rho}\widetilde{\mathbb{R}}$-module ${}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$ of sequences for hyperseries exactly as the space where we can consider hyperfinite sums regardless of convergence. This is the idea to define the space ${}^{\rho}_{\sigma}\widetilde{\mathbb{R}}[\![x-c]\!]$ of formal hyper-power series (HPS):

Definition 1. Let $x, c \in {}^{\rho}\widetilde{\mathbb{R}}$. We say $(b_n)_n \in {}^{\rho}_{\sigma}\widetilde{\mathbb{R}}[\![x-c]\!]$ if and only if there exist $(a_{n\varepsilon})_{n,\varepsilon} \in \mathbb{R}^{\mathbb{N}\times I}$ and representatives $[x_\varepsilon] = x$, $[c_\varepsilon] = c$ such that

$$(b_n)_n = [a_{n\varepsilon} \cdot (x_\varepsilon - c_\varepsilon)^n]_{\mathrm{s}} \in {}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}. \tag{2.1}$$

For the notation $[-]_{\mathrm{s}}$, see [29, Def. 1]. Elements of ${}^{\rho}_{\sigma}\widetilde{\mathbb{R}}[\![x-c]\!]$ are called *formal* HPS because here we are not considering their convergence. In other words, a formal HPS is a hyper series (i.e. an equivalence class $(b_n)_n \in {}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$ in the space of sequences for hyperseries) of the form $[a_{n\varepsilon} \cdot (x_\varepsilon - c_\varepsilon)^n]_{\mathrm{s}}$.

Remark 2.

(i) We explicitly note that $x-c$ is not an indeterminate, like in the case of formal power series $\mathbb{R}[\![x]\!]$, but a generalized number of ${}^{\rho}\widetilde{\mathbb{R}}$. For example, in Lem. 10 below, we will prove that if $x-c=y-d$, then ${}^{\rho}_{\sigma}\widetilde{\mathbb{R}}[\![x-c]\!]={}^{\rho}_{\sigma}\widetilde{\mathbb{R}}[\![y-d]\!]$.

(ii) On the contrary with respect to the case of real numbers, being a formal HPS, i.e. an element of ${}^{\rho}_{\sigma}\widetilde{\mathbb{R}}[\![x-c]\!]$, depends on the interplay of the two gauges ρ and σ: take e.g. $a_n=\frac{1}{n^2}$ and $x-c=2$, so that for all $N\in{}^{\sigma}\widetilde{\mathbb{N}}$ we have $\sum_{n=1}^{N}a_n(x-c)^n\ge\sum_{n=0}^{N}\frac{1}{n}\sim\log(N)$. Therefore, taking e.g. $\sigma_\varepsilon=\exp\left(-\exp\left(\frac{1}{\rho_\varepsilon}\right)\right)$ and $N_\varepsilon:=\text{int}\,(\sigma_\varepsilon)$, we have that $(\log N_\varepsilon)\notin\mathbb{R}_\rho$ and hence we cannot even consider hyperfinite sums of this form. Informally stated, for this gauge σ, we have that ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}}\frac{1}{n^2}2^n$ is *not* a formal HPS, i.e. even before considering its convergence or not, we cannot compute σ-hyperfinite sums and get a number in ${}^{\rho}\widetilde{\mathbb{R}}$.

(iii) In [29], we proved that if x is finite, then $\left[\frac{x^n}{n!}\right]_{\text{s}}\in{}^{\rho}_{\sigma}\widetilde{\mathbb{R}}[\![x]\!]$ is a formal HPS for all gauges ρ, σ. In Sec. 14, we will prove that $\left[\frac{\left(\text{d}\rho^{-1}\right)^n}{n!}\right]_{\text{s}}\notin{}^{\rho}_{\rho}\widetilde{\mathbb{R}}[\![\text{d}\rho^{-1}]\!]$; on the other hand, we will also see that if $x\le\log\left(\text{d}\rho^{-N}\right)$ and $\text{d}\sigma^{Q}\le\text{d}\rho^{N}$ for some $Q\in\mathbb{N}$, then $\left[\frac{x^n}{n!}\right]_{\text{s}}\in{}^{\rho}_{\sigma}\widetilde{\mathbb{R}}[\![x]\!]$ is a formal HPS.

The previous Def. 1 sets immediate problems concerning independence of representatives: every time we start from $[a_{n\varepsilon}]\in{}^{\rho}\widetilde{\mathbb{R}}$, for all $n\in\mathbb{N}$, $[x_\varepsilon]=x$, $[c_\varepsilon]=c$ and we have that $\left[a_{n\varepsilon}\cdot(x_\varepsilon-c_\varepsilon)^n\right]_{\text{s}}\in{}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\text{s}}$, we can consider whether the corresponding formal HPS ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}}[a_{n\varepsilon}]\cdot([x_\varepsilon]-[c_\varepsilon])^n$ converges or not. On the other hand, we also have to prove that it is well-defined, i.e. that taking different representatives $[\bar{a}_{n\varepsilon}]=[a_{n\varepsilon}]$, $[\bar{x}_\varepsilon]=x$, $[\bar{c}_\varepsilon]=c$, we have $\left[a_{n\varepsilon}\cdot(x_\varepsilon-c_\varepsilon)^n\right]_{\text{s}}=\left[\bar{a}_{n\varepsilon}\cdot(\bar{x}_\varepsilon-\bar{c}_\varepsilon)^n\right]_{\text{s}}$. However, from [29, Sec. 2] it follows that we can have $x-c=1$ and $[a_{n\varepsilon}]=[\bar{a}_{n\varepsilon}]=0$ for all $n\in\mathbb{N}$, but

$$\left[\sum_{n=0}^{N_\varepsilon}a_{n\varepsilon}(x_\varepsilon-c_\varepsilon)^n\right]\ne\left[\sum_{n=0}^{N_\varepsilon}\bar{a}_{n\varepsilon}(x_\varepsilon-c_\varepsilon)^n\right].$$

This means that $(b_n)_n:=\left[a_{n\varepsilon}(x_\varepsilon-c_\varepsilon)^n\right]_{\text{s}}$ and $(\bar{b}_n)_n:=\left[\bar{a}_{n\varepsilon}(x_\varepsilon-c_\varepsilon)^n\right]_{\text{s}}$ yield two different formal HPS (see [29, Thm. 4]) and hence, in general, the operation

$$((a_{n\varepsilon})_{n,\varepsilon},(x_\varepsilon),(c_\varepsilon))\in{}^{\rho}\widetilde{\mathbb{R}}^{\mathbb{N}}\times{}^{\rho}\widetilde{\mathbb{R}}^{2}\mapsto(b_n)_n:=\left[a_{n\varepsilon}(x_\varepsilon-c_\varepsilon)^n\right]_{\text{s}}\in{}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\text{s}}$$

is not well-defined.

The problem can also be addressed differently: what notion of equality do we have to set on a suitable subring of $\mathbb{R}^{\mathbb{N}\times I}$ so as to have independence on representatives? This notion of equality naturally emerges in proving that the following definition of radius of convergence is well-defined (see Lem. 4). What subring we need to consider arises from the idea to include $\left(\frac{\delta_\varepsilon^{(n)}(0)}{n!}\right)_{n,\varepsilon}$ in it, where $\delta=[\delta_\varepsilon(-)]$ is a suitable embedding of Dirac's delta function (see Example 5.(v)).

2.2. **Radius of convergence.** The idea to define the radius of convergence corresponding to coefficients $(a_{n\varepsilon})_{n,\varepsilon}\in\mathbb{R}^{\mathbb{N}\times I}$ is that it does not matter if

$$\left(\limsup_{n\to+\infty}|a_{n\varepsilon}|^{1/n}\right)^{-1}\in\mathbb{R}\cup\{+\infty\}$$

yields a non ρ-moderate net (for example for $\varepsilon \in L \subseteq_0 I$) because this case would intuitively identify a radius of convergence larger than any infinite number in ${}^{\rho}\widetilde{\mathbb{R}}$:

Definition 3.

(i) Let $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ be the extended real number system with the usual (partially defined) operations but where we define $\infty - \infty = \infty + (-\infty) = -\infty + \infty = -\infty - (-\infty) = 0$. We set ${}^{\rho}\overline{\mathbb{R}} := \overline{\mathbb{R}}^I / \sim_\rho$, where for arbitrary (x_ε), $(y_\varepsilon) \in \overline{\mathbb{R}}^I$, as usual we define

$$(x_\varepsilon) \sim_\rho (y_\varepsilon) \quad :\iff \quad \forall q \in \mathbb{N}\, \forall^0 \varepsilon : \ |x_\varepsilon - y_\varepsilon| \le \rho_\varepsilon^q.$$

Note that, e.g., $(\infty) \sim_\rho (\infty)$ because of our definition of $\infty - \infty$. In ${}^{\rho}\overline{\mathbb{R}}$, we can also consider the standard order relation

$$x \le y \quad :\iff \quad \exists [x_\varepsilon] = x, [y_\varepsilon] = y\, \forall^0 \varepsilon : \ x_\varepsilon \le y_\varepsilon.$$

Note that $\left({}^{\rho}\overline{\mathbb{R}}, +, \le\right)$ is an ordered group but, since we are considering arbitrary nets $\overline{\mathbb{R}}^I$, the set ${}^{\rho}\overline{\mathbb{R}}$ is not a ring: e.g. $+\infty \cdot 0$ is still undefined and $+\infty \cdot [z_\varepsilon] = [+\infty]$ for all $(z_\varepsilon) \in \mathbb{R}^I_{>0}$.

(ii) Moreover, we denote by ${}^{\rho}\widetilde{\mathbb{R}}_{\text{c}} := \left(\mathbb{R}^{\mathbb{N}\times I}\right)_\rho / \simeq_\rho$ the quotient ring of *coefficients for HPS*, where

$$(a_{n\varepsilon})_{n,\varepsilon} \in \left(\mathbb{R}^{\mathbb{N}\times I}\right)_\rho :\iff \exists Q, R \in \mathbb{N}\, \forall^0 \varepsilon\, \forall n \in \mathbb{N} : \ |a_{n\varepsilon}| \le \rho_\varepsilon^{-nQ-R} \tag{2.2}$$

is the ring of *weakly ρ-moderate nets*, and

$$(a_{n\varepsilon})_{n,\varepsilon} \simeq_\rho (\bar{a}_{n\varepsilon})_{n,\varepsilon} :\iff \forall q, r \in \mathbb{N}\, \forall^0 \varepsilon\, \forall n \in \mathbb{N} : \ |a_{n\varepsilon} - \bar{a}_{n\varepsilon}| \le \rho_\varepsilon^{nq+r}, \tag{2.3}$$

in this case, we say that these two nets are *strongly ρ-equivalent*. Equivalence classes of ${}^{\rho}\widetilde{\mathbb{R}}_{\text{c}}$ are denoted by $(a_n)_{\text{c}} := [a_{n\varepsilon}]_{\text{c}} \in {}^{\rho}\widetilde{\mathbb{R}}_{\text{c}}$.

(iii) Finally, if $(a_n)_{\text{c}} = [a_{n\varepsilon}]_{\text{c}} \in {}^{\rho}\widetilde{\mathbb{R}}_{\text{c}}$, then we set $\text{rad}\,(a_n)_{\text{c}\varepsilon} := r_\varepsilon$, and $\text{rad}\,(a_n)_{\text{c}} =: [r_\varepsilon] \in {}^{\rho}\overline{\mathbb{R}}$, where

$$r_\varepsilon := \left(\limsup_{n\to+\infty} |a_{n\varepsilon}|^{1/n}\right)^{-1} \in \mathbb{R} \cup \{+\infty\}. \tag{2.4}$$

In the following lemma, we prove that $\text{rad}\,(a_n)_{\text{c}}$ is well-defined:

Lemma 4. *Let $(a_n)_{\text{c}} = [a_{n\varepsilon}]_{\text{c}} = [\bar{a}_{n\varepsilon}]_{\text{c}} \in {}^{\rho}\widetilde{\mathbb{R}}_{\text{c}}$. Define r_ε as in (2.4) and similarly define $\bar{r}_\varepsilon$ using $\bar{a}_{n\varepsilon}$. Then $(r_\varepsilon) \sim_\rho (\bar{r}_\varepsilon)$, and hence $[r_\varepsilon] = [\bar{r}_\varepsilon]$ in ${}^{\rho}\overline{\mathbb{R}}$.*

Proof. For all $\varepsilon \in I$ and all $n \in \mathbb{N}_{>0}$, we have $|\bar{a}_{n\varepsilon}|^{1/n} \le (|\bar{a}_{n\varepsilon} - a_{n\varepsilon}| + |a_{n\varepsilon}|)^{1/n}$. The binomial formula yields $(x + y) \le \left(x^{1/n} + y^{1/n}\right)^n$ for all x, $y \in \mathbb{R}_{\ge 0}$, so that $|\bar{a}_{n\varepsilon}|^{1/n} \le |\bar{a}_{n\varepsilon} - a_{n\varepsilon}|^{1/n} + |a_{n\varepsilon}|^{1/n}$. Setting $r = 0$ in (2.3), for all $q \in \mathbb{N}$ and for ε small we have

$$\forall n \in \mathbb{N} : \ |a_{n\varepsilon} - \bar{a}_{n\varepsilon}| \le \rho_\varepsilon^{nq}.$$

Therefore, for the same ε we get $|\bar{a}_{n\varepsilon}|^{1/n} \le \rho_\varepsilon^q + |a_{n\varepsilon}|^{1/n}$. Taking the limit superior we obtain $\limsup_{n\to+\infty} |\bar{a}_{n\varepsilon}|^{1/n} \le \rho_\varepsilon^q + \limsup_{n\to+\infty} |a_{n\varepsilon}|^{1/n}$. Inverting the role of $(a_{n\varepsilon})_{n,\varepsilon}$ and $(\bar{a}_{n\varepsilon})_{n,\varepsilon}$ we finally obtain

$$\forall^0 \varepsilon : \ -\rho_\varepsilon^q \le \limsup_{n\to+\infty} |a_{n\varepsilon}|^{1/n} - \limsup_{n\to+\infty} |\bar{a}_{n\varepsilon}|^{1/n} \le \rho_\varepsilon^q,$$

which proves the claim. $\square$

Remark 5.

(i) If $(a_n)_c = [a_{n\varepsilon}]_c \in {}^\rho\widetilde{\mathbb{R}}_c$, then for each fixed $n \in \mathbb{N}$, we have that $[(a_{n\varepsilon})_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}$, i.e. the net $(a_{n\varepsilon})_\varepsilon$ is ρ-moderate. This is the main motivation to consider the exponent "$-R$" in (2.2) (recall that in our notation $0 \in \mathbb{N}$): without the term "$-R$", the only possibility to have $(a_n)_c \in {}^\rho\overline{\mathbb{R}}$ is that $|a_0| \le 1$, which is an unnecessary limitation. Similarly, we can motivate why we are considering the quantifier "$\forall n \in \mathbb{N}$" in the same formula (instead of, e.g., "$\exists N \in \mathbb{N}\,\forall n \in \mathbb{N}_{\ge N}$"). The proof of the next Lem. 10 will motivate why in (2.2) we consider the uniform property "$\forall^0\varepsilon\,\forall n \in \mathbb{N}$" and not "$\forall n \in \mathbb{N}\,\forall^0\varepsilon$".

(ii) Note that ${}^\rho\widetilde{\mathbb{R}} \subseteq {}^\rho\overline{\mathbb{R}}$ because the notion of equality $\sim_\rho$ in the two quotient sets is the same and because if (x_ε) is ρ-moderate and $(x_\varepsilon) \sim_\rho (y_\varepsilon)$, then also (y_ε) is ρ-moderate.

(iii) Condition (2.2) of being weakly ρ-moderate represents a constrain on what coefficients a_n we can consider in a hyperseries. For example, if $(a_n)_{n\in\mathbb{N}}$ is a sequence of real numbers satisfying $|a_n| \le p(n)$, where $p \in \mathbb{R}[x]$ is a polynomial, then $p(n) \le \rho_\varepsilon^{-nQ}$ for all ε sufficiently small and for all $n \in \mathbb{N}$ if $Q \ge \max\left(1, \max\left\{-\frac{\log n}{p(n)\log\rho_\varepsilon} \mid n < N_1\right\}\right)$, where $\frac{\log n}{P(n)} \le 1$ for all $n \ge N_1$ and $-\frac{1}{\log\rho_\varepsilon} \le 1$. Hence $(a_n)_{n,\varepsilon} \in \left(\mathbb{R}^{\mathbb{N}\times I}\right)_\rho$ is weakly ρ-moderate. On the contrary, we cannot have $n^n \le \rho_\varepsilon^{-nQ-R} = \rho_\varepsilon^{-R}\left(\frac{1}{\rho_\varepsilon^Q}\right)^n$ for all $n \in \mathbb{N}$. Similarly $(n!)_{n\in\mathbb{N}}$ is not weakly ρ-moderate and hence our theory does not apply to a "hyperseries" of the form ${}^\rho\sum_{n\in{}^\sigma\widetilde{\mathbb{N}}} n!\cdot x^n$. On the other hand, in Lem. 7.(i) we will show that, as a consequence of considering only weakly moderate coefficients, the radius of convergence of our hyperseries is always strictly positive.

(iv) Let $a_{n\varepsilon} = \rho_\varepsilon^{\frac{n+1}{\varepsilon}}$, so that $[a_{n\varepsilon}]_c = 0$. The corresponding radius of convergence is $r_\varepsilon = \lim_{n\to+\infty}|a_{n\varepsilon}|^{1/n} = \rho_\varepsilon^{1/\varepsilon}$ which is not ρ-moderate. In general, if $\operatorname{rad}(a_n)_c = [r_\varepsilon] =: r \in {}^\rho\overline{\mathbb{R}}$, we can have different behavior on different sub-points, e.g. $r|_{L_1} = +\infty$, $r|_{L_2} \in {}^\rho\widetilde{\mathbb{R}}$, $r|_{L_3}$ non ρ-moderate, etc., where $L_i \subseteq_0 I$. This behavior is studied in Lem. 7 below.

(v) Let $\mu := \mathcal{F}^{-1}(\beta) \in \mathcal{S}(\mathbb{R})$ be a Colombeau mollifier defined as the inverse Fourier transform of a smooth, supported in $[-1,1]_\mathbb{R}$, even bump function $0 \le \beta \le 1$ which identically equals 1 in a neighborhood of 0 (see e.g. [15]). Let $i_\mathbb{R}^b$ be the embedding of Schwartz distributions into generalized smooth functions (GSF) defined by μ and by the infinite number $b \in {}^\rho\widetilde{\mathbb{R}}$ (see e.g. [13]). The Schwartz's Paley-Wiener theorem implies that μ is an entire function and we know that if $\mathrm{d}\rho^{-Q} \ge b = [b_\varepsilon] \ge \mathrm{d}\rho^{-R}$, for some $Q,\ R \in \mathbb{R}_{>0}$, then the embedding of Dirac delta $\delta := \iota_\mathbb{R}^b(\varphi \mapsto \varphi(0)) \in {}^\rho\mathcal{GC}^\infty({}^\rho\widetilde{\mathbb{R}}, {}^\rho\widetilde{\mathbb{R}})$ is defined by the net $\delta_\varepsilon(x) = b_\varepsilon\mu(b_\varepsilon x)$ (see e.g. [13]). For $n \in \mathbb{N}$, we have $\mu^{(n)}(0) = \frac{1}{2\pi}\int\beta(x)(ix)^n\,\mathrm{d}x = 0$ if n is odd and $\left|\mu^{(n)}(0)\right| \le \frac{1}{2\pi}\left[\frac{x^{n+1}}{n+1}\right]_{-1}^1 \le 1$ if n if even. Thereby $\left|\frac{\delta_\varepsilon^{(n)}(0)}{n!}\right| = \left|\frac{\mu^{(n)}(0)}{n!}b_\varepsilon^{n+1}\right| \le \frac{1}{n!}\rho_\varepsilon^{-nQ-Q} \le \rho_\varepsilon^{-nQ-Q}$. This inequality shows that $\left(\frac{\delta^{(n)}(0)}{n!}\right)_c \in {}^\rho\widetilde{\mathbb{R}}_c$ and motivates our definition of weakly ρ-moderate nets. The corresponding radius of convergence is $r_\varepsilon^{-1} =$

$\limsup_{n\to+\infty}\left|\frac{\delta_\varepsilon^{(n)}(0)}{n!}\right|^{1/n} = \limsup_{n\to+\infty} b_\varepsilon^{1+1/n}\left|\frac{\mu^{(n)}(0)}{n!}\right|^{1/n} = b_\varepsilon\cdot 0 = 0$, i.e. $\mathrm{rad}\left(\frac{\delta^{(n)}(0)}{n!}\right)_{\mathrm{c}} = +\infty$.

(vi) Let $(a_n)_{\mathrm{c}} = [a_{n\varepsilon}]_{\mathrm{c}} \in {}^\rho\widetilde{\mathbb{R}}_{\mathrm{c}}$, and assume that for all ε there exists $r_\varepsilon := \left(\lim_{n\to+\infty}|a_{n\varepsilon}|^{1/n}\right)^{-1}$ such that $r := [r_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}$. Then from [23, Thm. 28], for some gauge $\sigma \le \rho$ we have ${}^\rho\lim_{n\in{}^\sigma\widetilde{\mathbb{N}}}|a_n|^{1/n} = \frac{1}{r}$ and $r = \mathrm{rad}\,(a_n)_{\mathrm{c}} \in {}^\rho\widetilde{\mathbb{R}}$. In Cor. 17, we will see the relationship between our definition of radius of convergence and the least upper bound of all the radii where the HPS converges.

In the following lemma, we show that ${}^\rho\widetilde{\mathbb{R}}_{\mathrm{c}}$ is a ring:

Lemma 6. *With pointwise operations, ${}^\rho\widetilde{\mathbb{R}}_{\mathrm{c}}$ is a quotient ring.*

Proof. Actually, the result follows from [14, Thm. 3.6] because the set

$$\mathcal{B} := \left\{\left(\rho_\varepsilon^{-nQ-R}\right)_{n,\varepsilon} \in \mathbb{R}^{\mathbb{N}\times I} \mid Q,\ R\in\mathbb{N}\right\}$$

is an asymptotic gauge with respect to the order $(n,\varepsilon) \le (\bar{n},\bar{\varepsilon})$ if and only if $\varepsilon \le \bar{\varepsilon}$. However, an independent proof follows the well-known lines of the corresponding proof for the ring ${}^\rho\widetilde{\mathbb{R}}$, and depends on the following properties of $\mathcal{B}$:

(a) $\forall p,q\in\mathcal{B}\,\exists r,s\in\mathcal{B}:\ p+q\le r,\ p\cdot q\le s$;
(b) $\forall p\in\mathcal{B}\,\exists r,s\in\mathcal{B}:\ r^{-1}+s^{-1}\le p^{-1}$;
(c) $\forall p,q,r\in\mathcal{B}\,\exists u,v\in\mathcal{B}:\ u^{-1}\cdot q+v^{-1}\cdot r\le p^{-1}$,

where $p=(p_{n\varepsilon})_{n,\varepsilon}\le(q_{n\varepsilon})_{n,\varepsilon}=q$ means $\forall^0\varepsilon\,\forall n\in\mathbb{N}:\ p_{n\varepsilon}\le q_{n\varepsilon}$. □

The following lemma represents a useful tool to deal with the radius of convergence. It essentially states that the radius of convergence equals $+\infty$ on some subpoint, or it is moderate on some subpoint or it is greater than any power $\mathrm{d}\rho^{-P}$.

Theorem 7. *Let $(a_n)_{\mathrm{c}} \in {}^\rho\widetilde{\mathbb{R}}_{\mathrm{c}}$ and $r=[r_\varepsilon]=\mathrm{rad}\,(a_n)_{\mathrm{c}}\in{}^\rho\overline{\mathbb{R}}$, then we have*

(i) $r>0$.
(ii) $r<+\infty$ *or* $r=_{\mathrm{s}}+\infty$.
(iii) *If* $r<+\infty$, *then the following alternatives hold*
  (a) $\forall P\in\mathbb{N}:\ r>\mathrm{d}\rho^{-P}$ *or*
  (b) *setting*

$$\begin{aligned}\left[r\le\rho^{-P}\right] &:= \left\{\varepsilon \mid r_\varepsilon\le\rho_\varepsilon^{-P}\right\} =: L_P\\ P_m &:= \min\left\{P\in\mathbb{N} \mid \left[r\le\rho^{-P}\right]\subseteq_0 I\right\}\end{aligned}\tag{2.5}$$

  we have
  (b.1) $I=\bigcup_{P\in\mathbb{N}}\left[r\le\rho^{-P}\right]$;
  (b.2) $\forall P\ge P_m:\ \left[r\le\rho^{-P}\right]\subseteq_0 I,\ r\le_{L_P}\mathrm{d}\rho^{-P}$;
  (b.3) $\forall P<P_m:\ \mathrm{d}\rho^{-P}\le r$;
  (b.4) *If* $P_m=0$ *and* $L_0^c\subseteq_0 I$, *then* $1\le_{L_0^c} r$; *if* $L_0^c\not\subseteq_0 I$, *then* $r\le 1$.
(iv) *Assume that for all* $L\subseteq_0 I$, *the following implication holds*

$$\left(\exists Q\in\mathbb{N}:\ r\le_L\mathrm{d}\rho^{-Q}\right)\ \text{or}\ \left(\forall Q\in\mathbb{N}:\ r>_L\mathrm{d}\rho^{-Q}\right)\ \Rightarrow\ \forall^0\varepsilon\in L:\ \mathcal{P}\left\{r_\varepsilon\right\}.\tag{2.6}$$

  Then $\forall^0\varepsilon:\ \mathcal{P}\{r_\varepsilon\}$, *i.e. the property* $\mathcal{P}\{r_\varepsilon\}$ *holds for all sufficiently small* ε.
(v) *If* $q\in{}^\rho\widetilde{\mathbb{R}}$ *and* $q<r$, *then* $\exists s\in{}^\rho\widetilde{\mathbb{R}}:\ q<s\le r$.

Proof. (i): Assume that $|a_{n\varepsilon}| \le \rho_\varepsilon^{-nQ-R}$ for all $\varepsilon \le \varepsilon_0$ and for all $n \in \mathbb{N}$. Then $\limsup_{n\to+\infty} |a_{n\varepsilon}|^{1/n} \le \lim_{n\to+\infty} \rho_\varepsilon^{-Q-\frac{R}{n}} = \rho_\varepsilon^{-Q}$, i.e. $r_\varepsilon \ge \rho_\varepsilon^Q$.

(ii): Set $L := \{\varepsilon \mid r_\varepsilon = +\infty\}$. If $L \subseteq_0 I$, then $r =_L +\infty$. Otherwise $(0, \varepsilon_0] \cap L = \emptyset$ for some ε_0, i.e. $r_\varepsilon < +\infty$ for all $\varepsilon \le \varepsilon_0$.

(iii): Since we assume that $r < +\infty$, without loss of generality we can take $r_\varepsilon < +\infty$ for all ε. We also assume that (a) is false, i.e. $r \le_M \mathrm{d}\rho^{-\bar{P}}$ for some $\bar{P} \in \mathbb{N}$ and some $M \subseteq_0 I$. We first prove (b.1): take $\varepsilon \in \bigcap_{P\in\mathbb{N}} \left[r > \rho^{-P}\right]$, then $r_\varepsilon > \rho_\varepsilon^{-P}$ for all $P \in \mathbb{N}$, so that $r_\varepsilon = +\infty$ for $P \to +\infty$, and this is not possible. We also note that $\left[r \le \rho^{-P}\right] \subseteq \left[r \le \rho^{-Q}\right]$ for all $Q \ge P$. From $M \subseteq_0 I$ and $r \le_M \mathrm{d}\rho^{-\bar{P}}$, we have $(0, \varepsilon_0] \cap M \subseteq \left[r \le \rho^{-(\bar{P}+1)}\right] \subseteq_0 I$, and hence definition (2.5) yields $P_\mathrm{m} \in \mathbb{N}$ and also proves (b.2). For all $P \in \mathbb{N}_{<P_\mathrm{m}}$, we hence have $\left[r \le \rho^{-P}\right] \not\subseteq_0 I$, i.e. $(0, \varepsilon_P] \subseteq \left[r > \rho^{-P}\right]$ for some ε_P. This implies $\mathrm{d}\rho^{-P} \le r$ and proves (b.3). Finally, if $P_\mathrm{m} = 0$ and $L^c_{P_\mathrm{m}} = L^c_0 \subseteq_0 I$, then $1 \le_{L^c_0} r$ because $L^c_0 = [r > 1]$. If $L^c_0 \not\subseteq_0 I$, then $(0, \varepsilon_0] \subseteq L_0$ for some ε_0, i.e. $r \le 1$.

(iv): By contradiction, assume that $\neg\mathcal{P}\{r_\varepsilon\}$ for all $\varepsilon \in L$ and for some $L \subseteq_0 I$. As usual, we assume that all the results we proved for ${}^\rho\overline{\mathbb{R}}$ can also be similarly proved for the restriction ${}^\rho\overline{\mathbb{R}}|_L$. From (ii) for ${}^\rho\overline{\mathbb{R}}|_L$, we have $r <_L +\infty$ or $r =_K +\infty$ for some $K \subseteq_0 L$. The second case implies $r >_L \mathrm{d}\rho^{-Q}$ for all $Q \in \mathbb{N}$. Since $K \subseteq_0 I$, we can apply the second alternative in the implication (2.6) to get $\forall^0 \varepsilon \in K : \mathcal{P}\{r_\varepsilon\}$, which gives a contradiction because $K \subseteq L$. We can hence consider the first case $r <_L +\infty$ and apply the subcase (a), i.e. $r >_L \mathrm{d}\rho^{-P}$ for all $P \in \mathbb{N}$, and we hence proceed as above applying the second alternative of the implication (2.6). In the remaining subcase, we can use (b.2) (with L instead of I). This yields $L_{P_\mathrm{m}} \subseteq_0 L$ and $r \le_{L_{P_\mathrm{m}}} \mathrm{d}\rho^{-P_\mathrm{m}}$. Since $L_{P_\mathrm{m}} \subseteq_0 I$, we can apply the first alternative in the implication (2.6) to get once again a contradiction.

(v): Assume that $r > q$ and take $s := \min(q+1, r) \in {}^\rho\widetilde{\mathbb{R}}_{>0}$. □

Explicitly note the meaning of Lem. 7.(iv): on an arbitrary subpoint $r|_L$ of the radius of convergence $r = \mathrm{rad}\,(a_n)_\mathrm{c}$, we have to consider only two cases: either $r|_L$ is ρ-moderate or it is greater than any power $\mathrm{d}\rho^{-Q}$ (the latter case including also the case $r|_L = +\infty$); if in both cases we are able to prove the property $\mathcal{P}\{r_\varepsilon\}$ for $\varepsilon \in L$ sufficiently small, then this property holds for *all* ε sufficiently small.

2.3. **Set of convergence.** Even if the radius of convergence of the exponential hyperseries is $\mathrm{rad}\left(\frac{1}{n!}\right)_\mathrm{c} = +\infty$, we have that $e^x = {}^\rho\sum_{n\in{}^\sigma\widetilde{\mathbb{N}}} \frac{x^n}{n!} \in {}^\rho\widetilde{\mathbb{R}}$ implies $|x| \le \log\left(\mathrm{d}\rho^{-R}\right)$ for some $R \in \mathbb{N}$: in other words, the constraint to get a ρ-moderate number implies that even if ${}^\rho\sum_{n\in{}^\sigma\widetilde{\mathbb{N}}} \frac{x^n}{n!}$ converges at x, the exponential HPS does *not* converge in the interval $[x, \mathrm{rad}\left(\frac{1}{n!}\right)_\mathrm{c}) = [x, +\infty) \subseteq {}^\rho\widetilde{\mathbb{R}}$.

Moreover, in all our examples, if the HPS ${}^\rho\sum_{n\in{}^\sigma\widetilde{\mathbb{N}}} a_n(x-c)^n \in {}^\rho\widetilde{\mathbb{R}}$ converges, then it converges exactly to $\left[\sum_{n=0}^{+\infty} a_{n\varepsilon}(x_\varepsilon - c_\varepsilon)^n\right] \in {}^\rho\widetilde{\mathbb{R}}$. The following definition of *set of convergence* closely recalls the definition of GSF:

Definition 8. Let $(a_n)_\mathrm{c} \in {}^\rho\widetilde{\mathbb{R}}_\mathrm{c}$ and $c \in {}^\rho\widetilde{\mathbb{R}}$. The set of convergence

$$ {}^\rho_\sigma\mathrm{conv}\left((a_n)_\mathrm{c}, c\right) $$

is the set of all $x \in {}^\rho\widetilde{\mathbb{R}}$ satisfying

(i) $|x - c| < \mathrm{rad}\,(a_n)_\mathrm{c}$, and such that there exist representatives $[x_\varepsilon] = x$, $[a_{n\varepsilon}]_{\text{c}} = (a_n)_{\text{c}}$ and $[c_\varepsilon] = c$ satisfying the following conditions:

(ii) $[a_{n\varepsilon} \cdot (x_\varepsilon - c_\varepsilon)^n]_{\text{s}} \in {}^\rho_\sigma\widetilde{\mathbb{R}}[\![x - c]\!]$, i.e. we have a formal HPS;

(iii) ${}^\rho\sum_{n\in{}^\sigma\widetilde{\mathbb{N}}} a_n(x-c)^n = \left[\sum_{n=0}^{+\infty} a_{n\varepsilon}(x_\varepsilon - c_\varepsilon)^n\right] \in {}^\rho\widetilde{\mathbb{R}}$;

(iv) For all representatives $[\bar{x}_\varepsilon] = x$ and all $k \in \mathbb{N}_{>0}$, the k-th derivative net is ρ-moderate:

$$\left(\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(\sum_{n=0}^{+\infty} a_{n\varepsilon}(x - c_\varepsilon)^n\right)_{x=\bar{x}_\varepsilon}\right) \in \mathbb{R}_\rho.$$

Note that condition (ii) is necessary because in (iii) we use a HPS; on the other hand, conditions (iii) and (iv) state that the function

$$x \in {}^\rho_\sigma\text{conv}\left((a_n)_{\text{c}}, c\right) \mapsto {}^\rho\sum_{n\in{}^\sigma\widetilde{\mathbb{N}}} a_n(x-c)^n \in {}^\rho\widetilde{\mathbb{R}}$$

is a GSF defined by the net of smooth functions $\left(\sum_{n=0}^{+\infty} a_{n\varepsilon}(x_\varepsilon - c_\varepsilon)^n\right)$. As for GSF, see [13, Thm. 16], condition (iv) will be useful to prove that we have independence from representatives of x in all the derivatives. In Cor. 25, we will see that under very general assumptions and if $\sigma \le \rho^*$, condition (iv) can be omitted.

In Sec. 14 we will show that $\log\left(\mathrm{d}\rho^{-1}\right) \in {}^\rho_\sigma\text{conv}\left(\left(\frac{1}{n!}\right)_{\text{c}}, 0\right)$ (the set of convergence of the exponential HPS at the origin), but $\mathrm{d}\rho^{-1} \notin \text{conv}\left(\left(\frac{1}{n!}\right)_n^{\text{c}}, 0\right)$. We immediately note that $x \in \text{conv}\left((a_n)_n^{\text{c}}, c\right)$ if and only if $x - c \in \text{conv}\left((a_n)_n^{\text{c}}, 0\right)$, and because of this property without loss of generality we will frequently assume $c = 0$.

We also note that condition (iii) states that the hyperseries ${}^\rho\sum_{n\in{}^\sigma\widetilde{\mathbb{N}}} a_n x^n$ converges, and it does exactly to the generalized number $\left[\sum_{n=0}^{+\infty} a_{n\varepsilon}x_\varepsilon^n\right]$. It is hence natural to wonder whether it is possible that it converges to some different quantity. This is the problem of the relation between hyperlimit and ε-wise limit:

$${}^\rho\lim_{N\in{}^\sigma\widetilde{\mathbb{N}}}\left[\sum_{n=0}^{\text{ni}(N)_\varepsilon} a_{n\varepsilon}x_\varepsilon^n\right], \quad \lim_{N\to+\infty}\sum_{n=0}^{N} a_{n\varepsilon}x_\varepsilon^n,$$

which has been already addressed in [29, Thm. 12, Thm. 13]. Intuitively speaking, if the gauge (σ_ε) is not sufficiently small, and hence the infinite nets $(\sigma_\varepsilon^{-N})$ are not sufficiently large, it can happen that $\text{ni}(N)_\varepsilon \to +\infty$ as $\varepsilon \to 0$ only very slowly, whereas the ε-wise limit could require $N \to +\infty$ at a greater speed to converge. This can be stated more precisely in the following way: Let $[a_{n\varepsilon} \cdot x_\varepsilon^n]_{\text{s}} \in {}^\rho_\sigma\widetilde{\mathbb{R}}[\![x]\!]$ be a formal HPS and assume that $\sum_{n=0}^{+\infty} a_{n\varepsilon}x_\varepsilon^n < +\infty$ for ε small. Then, for all $q \in \mathbb{N}$ and for all ε small, we can find $N_\varepsilon^q \in \mathbb{N}$ such that

$$\left|\sum_{n=0}^{N_\varepsilon^q} a_{n\varepsilon}x_\varepsilon^n - \sum_{n=0}^{+\infty} a_{n\varepsilon}x_\varepsilon^n\right| \le \rho_\varepsilon^q \quad \forall n \in \mathbb{N}_{\ge N_\varepsilon^q}.$$

However, only if $\left(\sum_{n=0}^{+\infty} a_{n\varepsilon}x_\varepsilon^n\right) \in \mathbb{R}_\rho$ and $(N_\varepsilon^q) \in \mathbb{R}_\sigma$, i.e. $[N_\varepsilon^q] \in {}^\sigma\widetilde{\mathbb{N}}$, then this also implies ${}^\rho\sum_{n\in{}^\sigma\widetilde{\mathbb{N}}} a_n x^n = \left[\sum_{n=0}^{+\infty} a_{n\varepsilon}x_\varepsilon^n\right]$.

As expected, for HPS the set of convergence is never a singleton:

Theorem 9. *Let* $(a_n)_c \in {}^\rho\widetilde{\mathbb{R}}_c$ *and* $c \in {}^\rho\widetilde{\mathbb{R}}$. *Then*

$$\exists q \in \mathbb{N} : \ (c - \mathrm{d}\rho^q, c + \mathrm{d}\rho^q) \subseteq {}^\rho_\sigma\mathrm{conv}\left((a_n)_c, c\right). \tag{2.7}$$

Proof. From Thm. 7.(i), we have $r := \mathrm{rad}\,(a_n)_c \geq \mathrm{d}\rho^{q_1}$ for some $q_1 \in \mathbb{N}$. We also have $|a_{n\varepsilon}| \leq \rho_\varepsilon^{-nQ-R}$ from (2.2). Assume that $|x - c| < \mathrm{d}\rho^q$: we want to find $q \in \mathbb{N}_{\geq q_1}$ so that $x \in {}^\rho_\sigma\mathrm{conv}\left((a_n)_c, c\right)$. To prove property Def. 8.(ii), for N_ε, $M_\varepsilon \in \mathbb{N}$ and for ε small, we estimate

$$\left|\sum_{n=N_\varepsilon}^{M_\varepsilon} a_{n\varepsilon}(x_\varepsilon - c_\varepsilon)^n\right| \leq \sum_{n=N_\varepsilon}^{M_\varepsilon} \rho_\varepsilon^{-nQ-R}\rho_\varepsilon^{nq} = \rho_\varepsilon^{-R}\sum_{n=N_\varepsilon}^{M_\varepsilon} \rho_\varepsilon^{-nQ+nq}.$$

Therefore, taking $q = \max(1 + Q, q_1)$, we get

$$\left|\sum_{n=N_\varepsilon}^{M_\varepsilon} a_{n\varepsilon}(x_\varepsilon - c_\varepsilon)^n\right| \leq \rho_\varepsilon^{-R}\sum_{n=N_\varepsilon}^{M_\varepsilon} \rho_\varepsilon^{n} \leq \frac{\rho_\varepsilon^{-R}}{1-\rho_\varepsilon},$$

and this proves Def. 8.(ii). Similarly, we have

$$\begin{aligned}\left|\sum_{n=0}^{M_\varepsilon} a_{n\varepsilon}(x_\varepsilon - c_\varepsilon)^n - \sum_{n=0}^{+\infty} a_{n\varepsilon}(x_\varepsilon - c_\varepsilon)^n\right| &\leq \sum_{n=M_\varepsilon+1}^{+\infty} \rho_\varepsilon^n \\ &\leq \frac{\rho_\varepsilon^{M_\varepsilon+1}}{1-\rho_\varepsilon}.\end{aligned}$$

Since ${}^\rho\lim_{M\in{}^\sigma\widetilde{\mathbb{N}}} \mathrm{d}\rho^{M+1} = 0$, this proves Def. 8.(iii). Finally, for all $k \in \mathbb{N}_{>0}$ and all representatives $[\bar{x}_\varepsilon] = x$, we have

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(\sum_{n=0}^{+\infty} a_{n\varepsilon}(x - c_\varepsilon)^n\right)_{x=\bar{x}_\varepsilon} = \sum_{n=k}^{+\infty} a_{n\varepsilon}(\bar{x}_\varepsilon - c_\varepsilon)^{n-k}\prod_{j=0}^{k-1}(n-j) \tag{2.8}$$

$$= k!\sum_{n=k}^{+\infty} a_{n\varepsilon}(\bar{x}_\varepsilon - c_\varepsilon)^{n-k}\binom{n}{k}, \tag{2.9}$$

and hence

$$\begin{aligned}\left|\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(\sum_{n=0}^{+\infty} a_{n\varepsilon}(\bar{x}_\varepsilon - c_\varepsilon)^n\right)\right| &\leq \sum_{n=k}^{+\infty} \rho_\varepsilon^{-nQ-R}\rho_\varepsilon^{(n-k)q}\prod_{j=0}^{k-1}(n-j) \\ &= \rho_\varepsilon^{-R-kQ}\sum_{n=k}^{+\infty}\rho_\varepsilon^{(q-Q)(n-k)}\prod_{j=0}^{k-1}(n-j) \\ &= \rho_\varepsilon^{-R-kQ}\frac{k!}{(1-\rho_\varepsilon^{q-Q})^{k+1}} \in \mathbb{R}_\rho.\end{aligned}$$

In the last step we used $q \geq Q+1$ and the binomial series $\sum_{n=k}^{+\infty} y^{n-k}\prod_{j=0}^{k-1}(n-j) = k!\sum_{n=k}^{+\infty}\binom{n}{k}y^{n-k} = \frac{k!}{(1-y)^{k+1}}$ for $|y| < 1$. □

We can now prove independence from representatives both in Def. 8 and in Def. 1:

Lemma 10. *Let* $(a_n)_c = [a_{n\varepsilon}]_c = [\bar{a}_{n\varepsilon}]_c \in {}^\rho\widetilde{\mathbb{R}}_c$, $x = [x_\varepsilon] = [\bar{x}_\varepsilon]$, $c = [c_\varepsilon] = [\bar{c}_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}$. *Assume that* $x \in {}^\rho_\sigma\mathrm{conv}\left((a_n)_c, c\right)$. *Then*

(i) The nets $(a_{n\varepsilon})_{n,\varepsilon}$, (x_ε) *and* (c_ε) *also satisfy all the conditions of Def. 8 of set of convergence.*

(ii) $\left[a_{n\varepsilon}\cdot(x_\varepsilon-c_\varepsilon)^n\right]_{\mathrm{s}}=\left[\bar{a}_{n\varepsilon}\cdot(\bar{x}_\varepsilon-\bar{c}_\varepsilon)^n\right]_{\mathrm{s}}$, *where the equality is in* ${}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$.

Proof. (i): Since we have similar steps for several claims, let $N_\varepsilon\in\mathbb{N}$ and $M_\varepsilon\in\mathbb{N}\cup\{+\infty\}$, so that a term of the form $\sum_{n=N_\varepsilon}^{M_\varepsilon}b_{n\varepsilon}$ represents both the ordinary series $\sum_{n=0}^{+\infty}b_{n\varepsilon}$ or the finite sum $\sum_{n=N_\varepsilon}^{M_\varepsilon}b_{n\varepsilon}$. From Def. 8 of set of convergence, we get the existence of representatives $[\hat{x}_\varepsilon]=x\in{}^{\rho}\widetilde{\mathbb{R}}$, $[\hat{a}_{n\varepsilon}]_{\mathrm{c}}=(a_n)_n^{\mathrm{c}}$ and $[\hat{c}_\varepsilon]=c$ satisfying Def. 8. Set $\hat{y}_\varepsilon:=\hat{x}_\varepsilon-\hat{c}_\varepsilon$, $\hat{y}:=[\hat{y}_\varepsilon]$. Let $r:=[r_\varepsilon]:=\mathrm{rad}\,(a_n)_{\mathrm{c}}$ be the radius of convergence. From Lem. 7.(v), take $s\in{}^{\rho}\widetilde{\mathbb{R}}$ satisfying $|\hat{y}|<s\le r$ and a representative $[s_\varepsilon]=s$ such that $|\hat{y}_\varepsilon|<s_\varepsilon\le r_\varepsilon$ for all ε small. Set $z_{n\varepsilon}:=a_{n\varepsilon}-\hat{a}_{n\varepsilon}$ and $\hat{z}_\varepsilon:=y_\varepsilon-\hat{y}_\varepsilon$. For all $k\in\mathbb{N}$, we have

$$\begin{aligned}\sum_{n=N_\varepsilon}^{M_\varepsilon}a_{n\varepsilon}y_\varepsilon^{n-k}\binom{n}{k}&=\sum_{n=N_\varepsilon}^{M_\varepsilon}\left(\hat{a}_{n\varepsilon}+z_{n\varepsilon}\right)\left(\hat{y}_\varepsilon+\hat{z}_\varepsilon\right)^{n-k}\binom{n}{k}\\&=\sum_{n=N_\varepsilon}^{M_\varepsilon}\hat{a}_{n\varepsilon}\left(\hat{y}_\varepsilon+\hat{z}_\varepsilon\right)^{n-k}\binom{n}{k}+\sum_{n=N_\varepsilon}^{M_\varepsilon}z_{n\varepsilon}\left(\hat{y}_\varepsilon+\hat{z}_\varepsilon\right)^{n-k}\binom{n}{k}.\end{aligned}\tag{2.10}$$

Since $\hat{z}=0$, we also have $|\hat{y}_\varepsilon|+|\hat{z}_\varepsilon|<s_\varepsilon\le r_\varepsilon$ for all ε small. For the same ε, assume that $|z_{n\varepsilon}|\le\rho_\varepsilon^{np+q}$ for fixed arbitrary p, $q\in\mathbb{N}$. We first consider the second summand in (2.10):

$$\begin{aligned}\left|\sum_{n=N_\varepsilon}^{M_\varepsilon}z_{n\varepsilon}\left(\hat{y}_\varepsilon+\hat{z}_\varepsilon\right)^{n-k}\binom{n}{k}\right|&\le\sum_{n=N_\varepsilon}^{M_\varepsilon}\rho_\varepsilon^{np+q}s_\varepsilon^{n-k}\binom{n}{k}=\rho_\varepsilon^{q+kp}\sum_{n=N_\varepsilon}^{M_\varepsilon}\left(\rho_\varepsilon^p s_\varepsilon\right)^{n-k}\binom{n}{k}\\&\le\rho_\varepsilon^{q+kp}\sum_{n=k}^{+\infty}\left(\rho_\varepsilon^p s_\varepsilon\right)^{n-k}\binom{n}{k}-\rho_\varepsilon^{q+kp}\sum_{n=k}^{N_\varepsilon-1}\left(\rho_\varepsilon^p s_\varepsilon\right)^{n-k}\binom{n}{k}\\&\le2\rho_\varepsilon^{q+kp}\sum_{n=k}^{+\infty}\left(\rho_\varepsilon^p s_\varepsilon\right)^{n-k}\binom{n}{k}.\end{aligned}$$

Since $s\in{}^{\rho}\widetilde{\mathbb{R}}$, we can take $p\in\mathbb{N}$ sufficiently large so that $\rho_\varepsilon^p s_\varepsilon<1$. This implies

$$\left|\sum_{n=N_\varepsilon}^{M_\varepsilon}z_{n\varepsilon}\left(\hat{y}_\varepsilon+\hat{z}_\varepsilon\right)^{n-k}\binom{n}{k}\right|\le\frac{2\rho_\varepsilon^{q+kp}}{\left(1-\rho_\varepsilon^p s_\varepsilon\right)^{k+1}}.$$

Thereby, for $q\to+\infty$, this summand defines a negligible net. For the first summand of (2.10), we can use the mean value theorem to get

$$\begin{aligned}&\left|\sum_{n=N_\varepsilon}^{M_\varepsilon}\hat{a}_{n\varepsilon}\left(\hat{y}_\varepsilon+\hat{z}_\varepsilon\right)^{n-k}\binom{n}{k}-\sum_{n=N_\varepsilon}^{M_\varepsilon}\hat{a}_{n\varepsilon}\hat{y}_\varepsilon^{n-k}\binom{n}{k}\right|\\&\quad\le\left|\sum_{n=N_\varepsilon}^{M_\varepsilon}\hat{a}_{n\varepsilon}(n-k)\xi_\varepsilon^{n-k-1}\binom{n}{k}\hat{z}_\varepsilon\right|=|\hat{z}_\varepsilon|\left|\sum_{n=N_\varepsilon}^{M_\varepsilon}\hat{a}_{n\varepsilon}(n-k)\xi_\varepsilon^{n-k-1}\binom{n}{k}\right|\end{aligned}\tag{2.11}$$

for some $\xi_\varepsilon\in[\hat{y}_\varepsilon,\hat{y}_\varepsilon+\hat{z}_\varepsilon]\cup[\hat{y}_\varepsilon+\hat{z}_\varepsilon,\hat{y}_\varepsilon]$. Thereby, the right hand side of (2.11) is negligible because of Def. 8.(iv).

We can hence state that for all $k \in \mathbb{N}$

$$\left(\sum_{n=N_\varepsilon}^{M_\varepsilon} a_{n\varepsilon}\left(y_\varepsilon+z_\varepsilon\right)^{n-k}\binom{n}{k}\right) \sim_\rho \left(\sum_{n=N_\varepsilon}^{M_\varepsilon} \hat{a}_{n\varepsilon}\hat{y}_\varepsilon^{n-k}\binom{n}{k}\right). \tag{2.12}$$

In the case $M_\varepsilon < +\infty$ for all ε and $k=0$, this proves that $[a_{n\varepsilon}\cdot y_\varepsilon^n]_{\mathrm{s}} \in {}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$ because $(\hat{a}_{n\varepsilon})_{n,\varepsilon}$ and $(\hat{y}_\varepsilon)$ satisfy Def. 8.(ii). In the case $M_\varepsilon = +\infty$ and $N_\varepsilon = 0 = k$, it proves the moderateness of $\left(\sum_{n=0}^{+\infty} a_{n\varepsilon}y_\varepsilon^n\right)$ too, i.e. the implicit moderateness requirement of Def. 8.(iii). Finally, for $k>0$, from property (2.12) we have that Def. 8.(iv) holds for $(a_{n\varepsilon})$ and (y_ε) because of (2.8). We can apply (2.12) with $k=0$ to $[\bar{a}_{n\varepsilon}]_{\mathrm{c}} = (a_n)_{\mathrm{c}}$, $[\bar{x}_\varepsilon]=x$, $[\bar{c}_\varepsilon]=c$, $\bar{y}_\varepsilon := \bar{x}_\varepsilon - \bar{c}_\varepsilon$ and with $M_\varepsilon < +\infty$, to get

$$\left(\sum_{n=N_\varepsilon}^{M_\varepsilon} a_{n\varepsilon}y_\varepsilon^n\right) \sim_\rho \left(\sum_{n=N_\varepsilon}^{M_\varepsilon} \hat{a}_{n\varepsilon}\hat{y}_\varepsilon^n\right) \sim_\rho \left(\sum_{n=N_\varepsilon}^{M_\varepsilon} \bar{a}_{n\varepsilon}\bar{y}_\varepsilon^n\right).$$

This proves claim (ii) and hence also Def. 8.(iii) because ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}}[\hat{a}_{n\varepsilon}]\cdot[\hat{y}_\varepsilon]^n$ converges to $\left[\sum_{n=0}^{+\infty}\hat{a}_{n\varepsilon}\hat{y}_\varepsilon^n\right] = \left[\sum_{n=0}^{+\infty} a_{n\varepsilon}y_\varepsilon^n\right] = \left[\sum_{n=0}^{+\infty}\bar{a}_{n\varepsilon}\bar{y}_\varepsilon^n\right] \in {}^{\rho}\widetilde{\mathbb{R}}$ from (2.12). □

2.4. **Examples.** We start studying geometric hyperseries, which in general are convergent HPS if $\sigma \le \rho^*$:

Example 11 (Geometric hyperseries)**.** Assume that $x \in (-1,1) \subseteq {}^{\rho}\widetilde{\mathbb{R}}$. We have:

$$\left[\left|\sum_{n=0}^{N_\varepsilon} x_\varepsilon^n\right|\right] \le \left[\left|\frac{1-|x_\varepsilon^{N_\varepsilon+1}|}{1-x_\varepsilon}\right|\right] \le \frac{2}{1-x} \in {}^{\rho}\widetilde{\mathbb{R}}. \tag{2.13}$$

This shows that $(x^n)_n = [x_\varepsilon^n] \in {}^{\rho}_{\sigma}\widetilde{\mathbb{R}}_{\mathrm{s}}$ for all gauges ρ, σ. Hence by Def. 1, $[x_\varepsilon^n]_{\mathrm{s}} \in {}^{\rho}_{\sigma}\widetilde{\mathbb{R}}[\![x]\!]$, i.e. the geometric series is a formal hyper-series. Since coefficients $a_{n\varepsilon}=1$, we have $[a_{n\varepsilon}]_{\mathrm{c}} \in {}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{c}}$ (see Def. 3.(ii)). Now, by Def. 3.(iii), $\mathrm{rad}\,(1)_{\mathrm{c}} = 1$. From Def. 8.(i), we have ${}^{\rho}_{\sigma}\mathrm{conv}\left((1)_{\mathrm{c}},0\right) \subseteq (-1,1)$. Now, take $x=[x_\varepsilon] \in (-1,1)$, with $-1 < x_\varepsilon < 1$ for all ε. From [29, Example 8], if $\sigma \le \rho^*$ (i.e. if $\exists Q \in \mathbb{R}_{>0}\,\forall^0\varepsilon : \sigma_\varepsilon \le \rho_\varepsilon^Q$), we have Def. 8.(iii). Finally, if $[\bar{x}_\varepsilon]=x$ is another representative and $k \in \mathbb{N}_{>0}$, then $-1 < \bar{x}_\varepsilon < 1$ for ε small, and from (2.8) we get $\sum_{n=k}^{+\infty} k!\binom{n}{k}\bar{x}_\varepsilon^{n-k} = \frac{k!}{(1-\bar{x}_\varepsilon)^{k+1}} \in \mathbb{R}_\rho$ because $1-x>0$ is invertible.

Note explicitly that $\sigma \le \rho^*$ is a sufficient condition ensuring the convergence of *any* geometric hyperseries with $|x|<1$. However, we already used (see e.g. Thm. 9) the convergence of the geometric hyperseries ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}}\mathrm{d}\rho^n = \frac{1}{1-\mathrm{d}\rho}$ for all gauges ρ, σ. More generally, exactly as proved in [29, Example 8], it is easy to see that ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} x^n = \frac{1}{1-x}$ if $\sigma_\varepsilon \le \left(\frac{\log x_\varepsilon}{\log \rho_\varepsilon}\right)^Q$ for ε small and some $Q \in \mathbb{R}_{>0}$.

Example 12 (A smooth function with a flat point)**.** Consider the GSF corresponding to the ordinary smooth function

$$f(x) := \begin{cases} e^{-1/x} & \text{if } x \in \mathbb{R}_{>0} \\ 0 & \text{otherwise} \end{cases}.$$

It is not hard to prove that $|f(x)| \le |x|^q$ for all $x \approx 0$ and all $q \in \mathbb{N}$. Thereby, $f(x)=0$ for all x such that $|x| \le \mathrm{d}\rho^r$ for some $r \in \mathbb{R}_{>0}$. Therefore, we trivially have $f(x) = {}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} 0\cdot x^n$ only for all x in this infinitesimal neighborhood of 0.

On the other hand, ${}^{\rho}_{\sigma}\text{conv}\left((0)_{\text{c}}, 0\right) = {}^{\rho}\widetilde{\mathbb{R}}$. Moreover, $\text{rad}\left(\frac{f^{(n)}(c)}{n!}\right)_{\text{c}} = +\infty$ and ${}^{\rho}_{\sigma}\text{conv}\left(\left(\frac{f^{(n)}(c)}{n!}\right)_{\text{c}}, c\right) = {}^{\rho}\widetilde{\mathbb{R}}$ for all $c \in {}^{\rho}\widetilde{\mathbb{R}}$ such that $|c| \gg 0$, i.e. satisfying $|c| \geq r$ for some $r \in \mathbb{R}_{>0}$, but $f(x) = {}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} \frac{f^{(n)}(c)}{n!} \cdot (x-c)^n$ only for all $x \in {}^{\rho}\widetilde{\mathbb{R}}$ such that $|x| \gg 0$, which is a strict subset of ${}^{\rho}_{\sigma}\text{conv}\left(\left(\frac{f^{(n)}(c)}{n!}\right)_{\text{c}}, c\right) = {}^{\rho}\widetilde{\mathbb{R}}$. The GSF f is therefore a candidate to be a GRAF, but not an entire GRAF.

Example 13 (A nowhere analytic smooth function)**.** A classical example of an infinitely differentiable function which is not analytic at any point is $F(x) = \sum_{k\in 2^{\mathbb{N}}} e^{-\sqrt{k}} \cos(kx)$, where $2^{\mathbb{N}} := \{2^n \mid n \in \mathbb{N}\}$. Since for all $x = \pi\frac{p}{q}$, with $p \in \mathbb{N}$ and $q \in 2^{\mathbb{N}}$ and for all $n \in 2^{\mathbb{N}}$, $n \geq 4$, $n > q$, we have $F^{(n)}(x) \geq e^{-2n}(4n^2)^n + O(q^n)$ as $n \to +\infty$, we have that $\left(\frac{F^{(n)}(x)}{n!}\right)_{n,\varepsilon} \notin \left(\mathbb{R}^{\mathbb{N}\times I}\right)_{\rho}$, i.e. they are *not* coefficients for a HPS. Note that this also proves that not even all smooth functions can be embedded as GRAF.

Example 14 (Exponential)**.** We clearly have $\left(\frac{1}{n!}\right)_{\text{c}} \in {}^{\rho}\widetilde{\mathbb{R}}_{\text{c}}$ and $\text{rad}\left(\frac{1}{n!}\right)_{\text{c}} = +\infty$, i.e. we have coefficients for an HPS with infinite radius of convergence. Set

$$C := \left\{x \in {}^{\rho}\widetilde{\mathbb{R}} \mid \exists K \in \mathbb{N} : \ |x| < -K \log \mathrm{d}\rho\right\}.$$

For all $x = [x_\varepsilon] \in C$ and all N_ε, $M_\varepsilon \in \mathbb{N}$, we have $\left|\sum_{n=N_\varepsilon}^{M_\varepsilon} \frac{x_\varepsilon^n}{n!}\right| \leq e^{|x_\varepsilon|} \leq \rho_\varepsilon^{-K}$ for ε small, and this shows that $\left[\frac{x^n}{n!}\right]_{\text{s}} \in {}^{\rho}_{\sigma}\widetilde{\mathbb{R}}[\![x]\!]$, i.e. for all $x \in C$, we have a formal HPS. We finally want to prove that $C = {}^{\rho}_{\sigma}\text{conv}\left(\left(\frac{1}{n!}\right)_{\text{c}}, 0\right)$ if $\sigma \leq \rho^*$. The inclusion $\supseteq$ follows directly from Def. 8.(iii). If $x = [x_\varepsilon] \in C$, then condition Def. 8.(iv) holds because the k-th derivative HPS $\left(k! \sum_{n=k}^{+\infty} \binom{n}{k} \cdot \frac{x_\varepsilon^{n-k}}{n!}\right) = (e^{x_\varepsilon}) \in \mathbb{R}_\rho$. To prove Def. 8.(iii), assume that $|x_\varepsilon| < -K \log \rho_\varepsilon =: M_\varepsilon$ for all ε and set $M := [M_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}}$. Take $N = [N_\varepsilon] \in {}^{\sigma}\widetilde{\mathbb{N}}$ such that $\frac{M}{N+1} < \frac{1}{2}$, so that, exactly as in [29, Example 8], we can prove that $\frac{M^{n+1}}{(n+1)!} < \frac{1}{2^{n+1}}$ and hence $\left|\sum_{n=N_\varepsilon+1}^{+\infty} \frac{x_\varepsilon^n}{n!}\right| \leq \sum_{n\geq N_\varepsilon} \frac{1}{2^n} \to 0$ as $N \to +\infty$, $N \in {}^{\sigma}\widetilde{\mathbb{N}}$, if $\sigma \leq \rho^*$. Similarly, we can consider trigonometric functions whose set of convergence is the whole of ${}^{\rho}\widetilde{\mathbb{R}}$.

Example 15 (Dirac delta)**.** In Rem. 5.(v), we already proved that $\left(\frac{\delta^{(n)}(0)}{n!}\right)_{\text{c}} \in {}^{\rho}\widetilde{\mathbb{R}}_{\text{c}}$ and $\text{rad}\left(\frac{\delta^{(n)}(0)}{n!}\right)_{\text{c}} = +\infty$. For all $x = [x_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}}$ and all N_ε, $M_\varepsilon \in \mathbb{N}$, we have $\left|\sum_{n=N_\varepsilon}^{M_\varepsilon} \frac{\delta_\varepsilon^{(n)}(0)}{n!} x_\varepsilon^n\right| \leq b_\varepsilon \cdot \sum_{n=0}^{+\infty} \frac{|\mu^{(n)}(0)|}{n!} |b_\varepsilon x_\varepsilon|^n$. But $|\mu^{(n)}(0)| = i^n \mu^{(n)}(0)$ because $\mu^{(n)}(0) = 0$ if n is odd, so that $\left|\sum_{n=N_\varepsilon}^{M_\varepsilon} \frac{\delta_\varepsilon^{(n)}(0)}{n!} x_\varepsilon^n\right| \leq b_\varepsilon \cdot \sum_{n=0}^{+\infty} \frac{\mu^{(n)}(0)}{n!} |i b_\varepsilon x_\varepsilon|^n = b_\varepsilon \mu(i|b_\varepsilon x_\varepsilon|) \in \mathbb{R}_\rho$, and this proves that $\left(\frac{\delta^{(n)}(0)}{n!}\right)_{\text{c}} \in {}^{\rho}_{\sigma}\widetilde{\mathbb{R}}[\![x]\!]$, i.e. we always have a formal HPS. Condition Def. 8.(iv) follows because derivatives $\delta^{(k)}(x) \in {}^{\rho}\widetilde{\mathbb{R}}$ are always moderate. It remains to prove Def. 8.(iii) for all $x = [x_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}}$ to show that ${}^{\rho}_{\sigma}\text{conv}\left(\left(\frac{\delta^{(n)}(0)}{n!}\right)_{\text{c}}, 0\right) = {}^{\rho}\widetilde{\mathbb{R}}$:

$$\sum_{n=0}^{N} \frac{\delta^{(n)}(0)}{n!} x^n = \left[b_\varepsilon \sum_{n=0}^{N_\varepsilon} \frac{\mu^{(n)}(0)}{n!} b_\varepsilon^n x_\varepsilon^n\right] = \delta(x) - b\left[\mu^{(N_\varepsilon+1)}(\bar{x}_\varepsilon) \frac{x_\varepsilon^{N_\varepsilon+1}}{(N_\varepsilon+1)!}\right]$$

where the existence of $\bar{x}_\varepsilon \in [0, x_\varepsilon] \cup [x_\varepsilon, 0]$ is derived from Taylor's formula. Since $|\mu^{(k)}(y)| \le \frac{1}{2\pi} \int \beta(x)|x|^k \, \mathrm{d}x =: C \in \mathbb{R}_{>0}$ for all $k \in \mathbb{N}$ and all $y \in \mathbb{R}$, we obtain

$$\left| \sum_{n=0}^{N} \frac{\delta^{(n)}(0)}{n!} x^n - \delta(x) \right| \le bC \left[\frac{|x_\varepsilon|^{N_\varepsilon+1}}{(N_\varepsilon+1)!} \right].$$

Using Stirling's approximation, we have $\frac{|x_\varepsilon|^{N_\varepsilon+1}}{(N_\varepsilon+1)!} \le 2 \left(\frac{|x_\varepsilon| e}{N_\varepsilon} \right)^N \le \rho_\varepsilon^{N_\varepsilon}$ for all $N \in {}^\sigma\widetilde{\mathbb{N}}$ such that $N > |x| e \mathrm{d}\rho^{-1}$, which is always possible if $\sigma \le \rho^*$. Since ${}^\rho\lim_{N \in {}^\sigma\widetilde{\mathbb{N}}} \mathrm{d}\rho^N = 0$, this proves the claim.

A different way to include a large class of examples is to use the characterization Thm. 37 by factorial growth of derivatives of GRAF.

When we say that a HPS ${}^\rho\sum_{n \in {}^\sigma\widetilde{\mathbb{N}}} a_n (x-c)^n$ is *convergent*, we already assume that its coefficients are correctly chosen and that the point x is in the set of convergence, as stated in the following

Definition 16. We say that ${}^\rho\sum_{n \in {}^\sigma\widetilde{\mathbb{N}}} a_n (x-c)^n$ is a *convergent* HPS if

(i) $(a_n)_c \in {}^\rho\widetilde{\mathbb{R}}_c$ are coefficients for HPS.
(ii) $x \in {}^\rho_\sigma\mathrm{conv}\left((a_n)_c, c\right)$.

In all the previous examples, we recognized that dealing with HPS is more involved than working with ordinary series, where we only have to check that the final result is a convergent series "of the form" $\sum_{n=0}^{\infty} a_n (x-c)^n$. On the contrary, for HPS we have to control the following steps:

1) We have to check that the net $(a_{n\varepsilon})_{n,\varepsilon}$ defines coefficients for HPS (Def. 3.(ii)), i.e. that
$$\forall^0 \varepsilon \, \forall n \in \mathbb{N} : \ |a_{n\varepsilon}| \le \rho_\varepsilon^{-nQ-R}$$
for some Q, $R \in \mathbb{N}_{>0}$. This allows us to talk about the radius of convergence $\mathrm{rad}\,(a_n)_c$ and of the set of convergence ${}^\rho_\sigma\mathrm{conv}\left((a_n)_c, c\right)$ (Def. 8). Because of Thm. 9, this set is always non-trivial
$$(c - \mathrm{d}\rho^q, c + \mathrm{d}\rho^q) \subseteq {}^\rho_\sigma\mathrm{conv}\left((a_n)_c, c\right) \subseteq (c - \mathrm{rad}\,(a_n)_c, c + \mathrm{rad}\,(a_n)_c), \tag{2.14}$$
but in general *is not an interval*, like the case of the exponential function clearly shows. This step already allows us to say that the HPS ${}^\rho\sum_{n \in {}^\sigma\widetilde{\mathbb{N}}} a_n (x-c)^n$ is convergent, i.e. Def. 16, if $x \in {}^\rho_\sigma\mathrm{conv}\left((a_n)_c, c\right)$.
2) At this point, we can study the set of convergence, e.g. to arrive at an explicit form $C = {}^\rho_\sigma\mathrm{conv}\left((a_n)_c, c\right) \subseteq (c - \mathrm{rad}\,(a_n)_c, c + \mathrm{rad}\,(a_n)_c)$. This depends mainly on three conditions:
   a) For all $x \in C$, we must have a formal HPS (Def. 1) because this allows us to talk of any hyperfinite sum $\sum_{n=N}^{M} a_n (x-c)^n$ for M, $N \in {}^\sigma\widetilde{\mathbb{N}}$. Here, the main step is to prove that the net $\left(\sum_{n=N_\varepsilon}^{M_\varepsilon} a_{n\varepsilon} (x_\varepsilon - c_\varepsilon)^n \right) \in \mathbb{R}_\rho$.
   b) For all $x \in C$, we have to check Def. 8.(iii), i.e. the equality:
   $${}^\rho\sum_{n \in {}^\sigma\widetilde{\mathbb{N}}} a_n (x-c)^n = \left[\sum_{n=0}^{+\infty} a_{n\varepsilon} (x_\varepsilon - c_\varepsilon)^n \right] \in {}^\rho\widetilde{\mathbb{R}}. \tag{2.15}$$
   c) Finally, we have to prove that for all representatives $x = [\bar{x}_\varepsilon] \in C$, all the derivatives $\frac{\mathrm{d}^k}{\mathrm{d}x^k} \left(\sum_{n=0}^{+\infty} a_{n\varepsilon} (\bar{x}_\varepsilon - c_\varepsilon)^n \right)$ are ρ-moderate.

d) After the previous three steps, we get $C \subseteq {}^{\rho}_{\sigma}\text{conv}\left((a_n)_{\text{c}}, c\right)$, and hence it remains to prove the opposite inclusion.

See Cor. 25 for sufficiently general conditions under which only (2.15) suffices to prove that x lies in the set of convergence.

Note explicitly that we *never* formally defined what is a HPS: we have *formal HPS* (Def. 1), the notion of *coefficients for HPS* (Def. 3.(ii)), which always have a strictly positive *radius of convergence* (Def. 3.(iii)) and a non trivial *set of convergence* (Def. 8 and Thm. 9), and finally *convergent HPS* (Def. 16).

2.5. **Topological properties of the set of convergence.** The first consequence of our definition of convergent HPS Def. 16 and radius of convergence Def. 3, is the following

Lemma 17. *Let ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n(x-c)^n$ be a convergent HPS. If the following least upper bound exists*

$$\text{lub}\left\{|\bar{x}-c| \mid {}^{\rho}\textstyle\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n(\bar{x}-c)^n \textit{ is a convergent HPS}\right\} =: r \in {}^{\rho}\widetilde{\mathbb{R}}, \tag{2.16}$$

then $r \le \text{rad}\,(a_n)_{\text{c}}$.

Proof. In fact, if ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n(\bar{x}-c)^n$ is a convergent HPS, then ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n(\bar{x}-c)^n = \left[\sum_{n=0}^{+\infty} a_{n\varepsilon}(\bar{x}_\varepsilon - c_\varepsilon)^n\right]$, and hence $|\bar{x}_\varepsilon - c_\varepsilon| \le (\limsup_n |a_{n\varepsilon}|^{1/n})^{-1}$ for all ε small, i.e. $|\bar{x}-c| \le \text{rad}\,(a_n)_{\text{c}}$. □

Note that from Example 14, we have that the least upper bound of

$$\left\{x \in {}^{\rho}\widetilde{\mathbb{R}} \mid {}^{\rho}\textstyle\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} \frac{x^n}{n!} \text{ is a convergent HPS}\right\} \tag{2.17}$$

does not exist in ${}^{\rho}\widetilde{\mathbb{R}}$, whereas Def. 3 yields the value $\text{rad}\left(\frac{1}{n!}\right)_{\text{c}} = +\infty$. Therefore, Def. 3 allows us to consider the exponential HPS even if the supremum of (2.17) does not exist. It remains an open problem whether $r = \text{rad}\,(a_n)_{\text{c}}$, at least if the least upper bound (2.16), or the corresponding sharp supremum, exists.

We now study absolute convergence of HPS, and sharply boundedness of the summands of a HPS. We first show that the hypersequence $\left(a_n(x-c)^n\right)_{n\in{}^{\sigma}\widetilde{\mathbb{N}}}$ of the terms of a HPS is sharply bounded:

Lemma 18. *Let $x, c \in {}^{\rho}\widetilde{\mathbb{R}}$. If ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n(x-c)^n$ is a convergent HPS, then*

$$\exists K \in {}^{\rho}\widetilde{\mathbb{R}}\,\forall n \in {}^{\sigma}\widetilde{\mathbb{N}}:\ |a_n(x-c)^n| < K. \tag{2.18}$$

Proof. We recall that because of the definition of formal HPS (Def. 1) and [29, Lem. 7] the term $a_n(x-c)^n \in {}^{\rho}\widetilde{\mathbb{R}}$ is well-defined for all $n \in {}^{\sigma}\widetilde{\mathbb{N}}$. Set $\bar{x} := x - c$, i.e. without loss of generality we can assume $c = 0$. Since ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n\bar{x}^n$ converges, from [29, Lem. 15] we have

$$\exists N \in {}^{\sigma}\widetilde{\mathbb{N}}\,\forall n \in {}^{\sigma}\widetilde{\mathbb{N}}_{\ge N}:\ |a_n\bar{x}^n| < 1. \tag{2.19}$$

Let us consider an arbitrary $n \in {}^{\sigma}\widetilde{\mathbb{N}}$. From [23, Lem. 13], we have either $n \ge N$ or $n <_L N$ for some $L \subseteq_0 I$. In the latter case, $|a_n\bar{x}^n| \le_L s := \sum_{n=0}^{N-1} |a_n\bar{x}^n| < \max(s+1, 1) =: K$. From [23, Lem. 7.(iii)] and from (2.19), the claim follows. □

The previous proof is essentially the generalization in our setting of the classical one, see e.g. [22]. However, property (2.18) does not allow us to apply the direct comparison test [29, Thm. 22]. Indeed, let us imagine that we only prove $|a_n x^n| < Kh^n$, with $h < 1$, for all $n \in {}^{\sigma}\widetilde{\mathbb{N}}$ and with K coming from (2.18); as we already explained in [29, Sec. 3.3], this would imply

$$\forall n \in \mathbb{N}\, \exists \varepsilon_{0n}\, \forall \varepsilon \le \varepsilon_{0n} :\ |a_{n\varepsilon} x_\varepsilon^n| \le K_\varepsilon h_\varepsilon^n,$$

and the dependence of ε_{0n} from $n \in \mathbb{N}$ is a problem in estimating inequalities of the form $\sum_{n=0}^{N_\varepsilon} |a_{n\varepsilon} x_\varepsilon^n| \le K_\varepsilon \sum_{n=0}^{N_\varepsilon} h_\varepsilon^n$, see [29]. A solution of this problem is to consider a uniform property of $n \in \mathbb{N}$ with respect to ε:

Definition 19. Let $(a_n)_{\mathrm{c}} \in {}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{c}}$ and $x, c \in {}^{\rho}\widetilde{\mathbb{R}}$, then we say that $(a_n(x-c)^n)_{n\in\mathbb{N}}$ *is eventually* ${}^{\rho}\widetilde{\mathbb{R}}$*-bounded in* ${}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{c}}$, if there exist representatives $(a_n)_{\mathrm{c}} = [a_{n\varepsilon}]_{\mathrm{c}}$, $[x_\varepsilon] = x$, $[c_\varepsilon] = c$ such that

$$\exists [R_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}}\, \exists N \in \mathbb{N}\, \forall^0 \varepsilon\, \forall n \in \mathbb{N}_{\ge N} :\ |a_{n\varepsilon}(x_\varepsilon - c_\varepsilon)^n| < R_\varepsilon. \tag{2.20}$$

Remark 20.

(i) The adverb *eventually* clearly refers to the validity of the uniform inequality in (2.20) only for n sufficiently large.

(ii) If for ε small, the series $\sum_{n=0}^{+\infty} |a_{n\varepsilon}(x_\varepsilon - c_\varepsilon)^n| =: R_\varepsilon$ of absolute values terms converges to a ρ-moderate net, then (2.20) holds for $N = 0$. This includes Example 11 of geometric hyperseries, Example 12 of a function with a flat point if both x, c are finite, and Example 14 of the exponential hyperseries at $c = 0$ if x is finite.

(iii) In Example 15 of Dirac delta at $c = 0$, if $|bx| \le 1$ (therefore, x is an infinitesimal number) we have $\left|\frac{\delta_\varepsilon^{(n)}(0)}{n!} x_\varepsilon^n\right| = \left|\frac{\mu^{(n)}(0)}{n!} b_\varepsilon^{n+1} x_\varepsilon^n\right| \le b_\varepsilon$ for all $n \in \mathbb{N}$ such that $\left|\frac{\mu^{(n)}(0)}{n!}\right| \le 1$. Therefore, $\left(\frac{\delta^{(n)}(0)}{n!} x^n\right)_{n\in\mathbb{N}}$ is eventually ${}^{\rho}\widetilde{\mathbb{R}}$-bounded in ${}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{c}}$ if $|bx| \le 1$. If $x \gg 0$, i.e. $x \ge s \in \mathbb{R}_{>0}$, then $\left|\frac{\delta_\varepsilon^{(n)}(0)}{n!} x_\varepsilon^n\right| = \left|\frac{\mu^{(n)}(0)}{n!} b_\varepsilon^{n+1} x_\varepsilon^n\right| \ge \left|\frac{\mu^{(n)}(0)}{n!}\right| s^n b_\varepsilon^{n+1}$ and hence condition (2.20) does not hold for any $[R_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}}$ because $b \ge \mathrm{d}\rho^{-a}$ for some $a \in \mathbb{R}_{>0}$ (see e.g. [13, Sec. 3.0.2]).

The last example also shows that property (2.20) does not hold for all points $x \in {}^{\rho}_{\sigma}\mathrm{conv}\left((a_n)_{\mathrm{c}}, c\right)$. However, it always holds for any c if x is sufficiently near to c:

Lemma 21. *Let* $(a_n)_{\mathrm{c}} \in {}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{c}}$ *and* $c \in {}^{\rho}\widetilde{\mathbb{R}}$*, then there exists* $\sigma \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$ *such that for all* $x \in B_\sigma(c)$*, the sequence of summands* $(a_n(x-c)^n)_{n\in\mathbb{N}}$ *is eventually* ${}^{\rho}\widetilde{\mathbb{R}}$*-bounded in* ${}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{c}}$.

Proof. Using the same notation as above, since $(a_n)_{\mathrm{c}} \in {}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{c}}$, we have $\forall^0 \varepsilon\, \forall n \in \mathbb{N} : |a_{n\varepsilon}| \le \rho_\varepsilon^{-nQ-R}$. Therefore, for $\sigma := \mathrm{d}\rho^Q$, we have $|a_{n\varepsilon}(x_\varepsilon - c_\varepsilon)^n| \le \rho_\varepsilon^{-nQ-R}\rho_\varepsilon^{nQ} = \rho_\varepsilon^{-R}$. □

The following result is a stronger version of the previous Lem. 18, and allows us to apply the dominated convergence test:

Lemma 22. *Let $(a_n)_\mathrm{c} \in {}^\rho\widetilde{\mathbb{R}}_\mathrm{c}$, x, $c \in {}^\rho\widetilde{\mathbb{R}}$, and assume that $(a_n(x-c)^n)_{n\in\mathbb{N}}$ is eventually ${}^\rho\widetilde{\mathbb{R}}$-bounded in ${}^\rho\widetilde{\mathbb{R}}_\mathrm{c}$, then*

$$\exists K \in {}^\rho\widetilde{\mathbb{R}} : \ ((a_n(x-c)^n))_\mathrm{c} < K \ \text{in } {}^\rho\widetilde{\mathbb{R}}_\mathrm{c}, \tag{2.21}$$

i.e. for all representatives $(a_n)_\mathrm{c} = [a_{n\varepsilon}]_\mathrm{c}$, $[x_\varepsilon] = x$, $[c_\varepsilon] = c$, $[K_\varepsilon] = K$, we have

$$\forall^0\varepsilon\, \forall n \in \mathbb{N} : \ |a_{n\varepsilon}(x_\varepsilon - c_\varepsilon)^n| < K_\varepsilon. \tag{2.22}$$

Since ${}^\rho\widetilde{\mathbb{R}} \subseteq {}^\rho\widetilde{\mathbb{R}}_\mathrm{c}$ by Rem. 5.(ii), property (2.21) also shows that Def. 19 does not depend on the representatives involved.

Proof. It suffices to set $K := R \vee \max_{n\le N} a_n$, where $R \in {}^\rho\widetilde{\mathbb{R}}$ and $N \in \mathbb{N}$ come from (2.20). $\square$

Even if the case of the exponential HPS (see Example 14) shows that in general the set of convergence is not an interval, it has very similar properties, at least if the gauge σ is sufficiently small:

Theorem 23. *Let $\sigma \le \rho^*$ and ${}^\rho\sum_{n\in{}^\sigma\widetilde{\mathbb{N}}} a_n(\bar{x}-c)^n$ be a convergent HPS whose sequence of summands $(a_n(\bar{x}-c)^n)_{n\in\mathbb{N}}$ is eventually ${}^\rho\widetilde{\mathbb{R}}$-bounded in ${}^\rho\widetilde{\mathbb{R}}_\mathrm{c}$. Then for all $x \in B_{|\bar{x}-c|}(0)$ we have:*

(i) *The HPS converges absolutely at x, and hence uniformly on every functionally compact $K \Subset_\mathrm{f} \overline{B}_{|\bar{x}-c|}(c)$;*
(ii) *$(a_n(x-c)^n)_{n\in\mathbb{N}}$ is eventually ${}^\rho\widetilde{\mathbb{R}}$-bounded in ${}^\rho\widetilde{\mathbb{R}}_\mathrm{c}$.*
(iii) *If $|\hat{x}-c| = |\bar{x}-c|$, then not necessarily ${}^\rho\sum_{n\in{}^\sigma\widetilde{\mathbb{N}}} a_n(\hat{x}-c)^n$ converges.*

If ${}^\rho\sum_{n\in{}^\sigma\widetilde{\mathbb{N}}} a_n(x-c)^n = \left[\sum_{n=0}^\infty a_{n\varepsilon}(x_\varepsilon - c_\varepsilon)^n\right] \in {}^\rho\widetilde{\mathbb{R}}$, then:

(iv) *$x \in {}^\rho_\sigma\mathrm{conv}\left((a_n)_\mathrm{c}, c\right)$;*
(v) *x is a sharply interior point, i.e. $B_s(x) \subseteq {}^\rho_\sigma\mathrm{conv}\left((a_n)_\mathrm{c}, c\right)$ for some $s \in {}^\rho\widetilde{\mathbb{R}}_{>0}$;*
(vi) *${}^\rho_\sigma\mathrm{conv}\left((a_n)_\mathrm{c}, c\right)$ is ${}^\rho\widetilde{\mathbb{R}}$-convex, i.e. if also $y \in {}^\rho_\sigma\mathrm{conv}\left((a_n)_\mathrm{c}, c\right)$, then $\forall t \in [0,1]$: $y + t(\bar{x}-y) \in {}^\rho_\sigma\mathrm{conv}\left((a_n)_\mathrm{c}, c\right)$;*
(vii) *The set of convergence ${}^\rho_\sigma\mathrm{conv}\left((a_n)_\mathrm{c}, c\right)$ is* strongly connected, *i.e. it is not possible to write it as union of two non empty strongly disjoint sets, i.e. such that*
  (a) *A, $B \subseteq {}^\rho\widetilde{\mathbb{R}}$, $A \ne \emptyset \ne B$,*
  (b) *$\exists \sup(A)$, $\exists \inf(B)$, $\sup(A) \le \inf(B)$,*
  (c) *${}^\rho_\sigma\mathrm{conv}\left((a_n)_\mathrm{c}, c\right) = A \cup B$,*
  (d) *$\exists m \in \mathbb{N} : \ B_{\mathrm{d}\rho^m}(A) \cap B_{\mathrm{d}\rho^m}(B) = \emptyset$.*

Proof. Without loss of generality we can assume $c = 0$. From [23, Lem. 5.(ii)], we have either $\bar{x} =_L 0$ or $|\bar{x}| > 0$ for some $L \subseteq_0 I$. The first case is actually impossible because $0 \le |x| < |\bar{x}| =_L 0$. We can hence work only in the latter case $|\bar{x}| > 0$. From Lem. 22, we have $\forall^0\varepsilon\, \forall n \in \mathbb{N} : \ |a_{n\varepsilon}\bar{x}_\varepsilon^n| \le K_\varepsilon$. Setting $h := \left|\frac{x}{\bar{x}}\right|$, we have $h < 1$ because $|x| \in B_{|\bar{x}|}(0)$, and

$$\forall^0\varepsilon\, \forall n \in \mathbb{N} : \ |a_{n\varepsilon}x_\varepsilon^n| = |a_{n\varepsilon}\bar{x}_\varepsilon^n| \,.\, \left|\frac{x_\varepsilon}{\bar{x}_\varepsilon}\right|^n < K_\varepsilon h_\varepsilon^n. \tag{2.23}$$

Thereby, $\sum_{n=N}^M |a_n x^n| \le \sum_{n=N}^M K h^n$ for all N, $M \in {}^\sigma\widetilde{\mathbb{N}}$. By the direct comparison test [29, Thm. 22], the HPS ${}^\rho\sum_{n\in{}^\sigma\widetilde{\mathbb{N}}} a_n x^n$ converges absolutely because ${}^\rho\sum_{n\in{}^\sigma\widetilde{\mathbb{N}}} K h^n$ converges since $\sigma \le \rho^*$ and $h < 1$. Finally, [13, Thm. 74] yields that pointwise convergence implies uniform convergence on functionally compact sets. This proves (i).

(ii): From (2.23) it follows that $\sum_{n=0}^{+\infty} |a_{n\varepsilon} x_\varepsilon^n| =: R_\varepsilon$ converges and is ρ-moderate. This implies condition (2.20).

For (iii), it suffices to consider that ${}^\rho\sum_{n\in{}^\rho\widetilde{\mathbb{N}}} \frac{(-1)^n}{n}$ converges (see [29, Sec. 3.6]) whereas ${}^\rho\sum_{n\in{}^\rho\widetilde{\mathbb{N}}} \frac{1}{n}$ does not by [29, Thm. 18]. Note however, that for $x = 1$, we have $|x| = \mathrm{rad}\left(\frac{1}{n}\right)_{\mathrm{c}}$ so that condition Def. 8.(i) does not hold.

(iv): From the assumptions, $x \in B_{|\bar{x}-c|}(0)$, $|\bar{x} - c| < \mathrm{rad}\,(a_n)_{\mathrm{c}}$, and hence Def. 8.(i) and Def. 8.(iii) follow. Note that Def. 8.(ii) can be proved as above from (2.23). Finally, if $[\hat{x}_\varepsilon] = x$ and $k \in \mathbb{N}_{>0}$, we have

$$\begin{aligned}\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(\sum_{n=0}^{+\infty} a_{n\varepsilon}\hat{x}_\varepsilon^n\right) &\le \sum_{n=k}^{+\infty} |a_{n\varepsilon}| k! \binom{n}{k} \left|\frac{\hat{x}_\varepsilon}{\bar{x}_\varepsilon}\right|^{n-k} |\bar{x}_\varepsilon|^{n-k} \\ &\le K_\varepsilon |\bar{x}_\varepsilon|^{-k} \sum_{n=k}^{+\infty} k! \binom{n}{k} \left|\frac{\hat{x}_\varepsilon}{\bar{x}_\varepsilon}\right|^{n-k} \in \mathbb{R}_\rho, \end{aligned} \tag{2.24}$$

where we used Lem. 22, and hence Def. 8.(iv) also holds.

(v): For $s := |\bar{x}| - |x| > 0$ and $\hat{x} \in B_s(x)$, we have $|\hat{x}| \le |\hat{x} - x| + |x| < s + |x| = |\bar{x}|$, and hence $\hat{x} \in {}^\rho_\sigma\mathrm{conv}\left((a_n)_{\mathrm{c}}, c\right)$ from (iv).

(vi): Setting $\hat{x} := y + t(\bar{x} - y)$, we have $y \le \hat{x} \le \bar{x}$. We can use trichotomy law [23, Lem. 7.(iii)] to distinguish the cases $y =_L 0$ or $y >_L 0$ or $y <_L 0$ for $L \subseteq_0 I$. The latter has to be subdivided into the sub-cases $\hat{x} >_M 0$ or $\hat{x} =_M 0$ or $\hat{x} <_M 0$ with $M \subseteq_0 L$, i.e. using [23, Lem. 7.(iii)] for the ring ${}^\rho\widetilde{\mathbb{R}}|_L$. Finally, the latter of these sub-cases has to be further subdivided into $\hat{x} >_K y$ or $\hat{x} <_K y$ or $\hat{x} =_K y$ with $K \subseteq_0 M$. In all these cases (clearly, some of these subcases cannot hold simultaneously) we can prove Def. 8 in the corresponding co-final set.

(vii): By contradiction, if $a \in A$ and $b \in B$, then $x := \frac{1}{2}(\sup(A) + \inf(B))$ lies in the segment $[a, b] \subseteq {}^\rho_\sigma\mathrm{conv}\left((a_n)_{\mathrm{c}}, c\right)$ by (vi). But property $B_{\mathrm{d}_{\rho^m}}(A) \cap B_{\mathrm{d}_{\rho^m}}(B)$ implies that $\sup(A) < \inf(B)$ and hence $x \notin A \cup B = {}^\rho_\sigma\mathrm{conv}\left((a_n)_{\mathrm{c}}, c\right)$. □

In spite of Thm. 23.(v), it remains open the problem whether the set of convergence is always a sharply open set or not. Using the previous theorem, this problem depends, for each point x in the set of convergence, on the existence of a point $\bar{x}$ satisfying assumptions of Thm. 23. However, in the case $x = 1 \in {}^\rho_\sigma\mathrm{conv}\left(\left(\frac{\delta^{(n)}(0)}{n!}\right)_{\mathrm{c}}, 0\right)$, Rem. 20.(iii) shows that $\left(\frac{\delta^{(n)}(0)}{n!} x^n\right)_{n\in\mathbb{N}}$ is not eventually ${}^\rho\widetilde{\mathbb{R}}$-bounded in ${}^\rho\widetilde{\mathbb{R}}_{\mathrm{c}}$, so that such a point $\bar{x}$ in this case does not exist and Thm. 23 cannot be applied.

Corollary 24. *Let $\sigma \le \rho^*$ and let R be the set of all the numbers of the form $s = |\bar{x} - c|$ for some $\bar{x} \in {}^\rho\widetilde{\mathbb{R}}$ satisfying:*

(i) ${}^\rho\sum_{n\in{}^\sigma\widetilde{\mathbb{N}}} a_n(\bar{x} - c)^n$ *is a convergent HPS,*

(ii) $(a_n(\bar{x} - c)^n)_{n\in\mathbb{N}}$ *is eventually* ${}^\rho\widetilde{\mathbb{R}}$*-bounded in* ${}^\rho\widetilde{\mathbb{R}}_{\mathrm{c}}$.

If $\exists \sup R =: r \in {}^\rho\widetilde{\mathbb{R}}$*, then* $B_r(c) \subseteq {}^\rho_\sigma\mathrm{conv}\left((a_n)_{\mathrm{c}}, c\right)$*, the HPS* ${}^\rho\sum_{n\in{}^\sigma\widetilde{\mathbb{N}}} a_n(x - c)^n$ *converges absolutely for all* $x \in B_r(c)$ *and uniformly on every functionally compact* $K \Subset_{\mathrm{f}} \overline{B}_r(c)$.

Proof. Without loss of generality, we assume $c = 0$, and let $x \in B_r(c)$. Since $|x| < r$, by the definition of sharp supremum, (see [23]) there exist $s = |\bar{x}|$ such that $|x| < |\bar{x}| \le r$ and such that (i) and (ii) hold. The conclusions then follow by Thm. 23. $\square$

Property Thm. 23.(iv) can also be written as a characterization of the set of convergence:

Corollary 25. *Let $\sigma \le \rho^*$, $(a_n)_{\mathrm{c}} = [a_{n\varepsilon}]_{\mathrm{c}} \in {}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{c}}$, $c = [c_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}}$ such that ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n(\bar{x}-c)^n$ is a convergent HPS whose sequence of summands $(a_n(\bar{x}-c)^n)_{n\in\mathbb{N}}$ is eventually ${}^{\rho}\widetilde{\mathbb{R}}$-bounded in ${}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{c}}$. If $x \in B_{|\bar{x}-c|}(0)$, then $x = [x_\varepsilon] \in {}^{\rho}_{\sigma}\mathrm{conv}\left((a_n)_{\mathrm{c}}, c\right)$ if and only if*

$$ {}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n(x-c)^n = \left[\sum_{n=0}^{\infty} a_{n\varepsilon}(x_\varepsilon - c_\varepsilon)^n\right] \in {}^{\rho}\widetilde{\mathbb{R}}. $$

2.6. **Algebraic properties of hyper-power series.** In this section, we extend to HPS the classical results concerning algebraic operations and composition of power series.

Theorem 26. *Assume that ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n\,(x-c)^n$ and ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} b_n\,(x-c)^n$ are two convergent HPS, then:*

(i) *For all $r \in {}^{\rho}\widetilde{\mathbb{R}}$, the product $r \cdot {}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n\,(x-c)^n$ is a convergent HPS with* $\mathrm{rad}\,(ra_n)_{\mathrm{c}} \ge \mathrm{rad}\,(a_n)_{\mathrm{c}}$, *and*

$$ r \cdot {}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n\,(x-c)^n = {}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} ra_n\,(x-c)^n. \tag{2.25} $$

(ii) *The sum of these HPS is a convergent HPS with*

$$ \mathrm{rad}\,(a_n+b_n)_{\mathrm{c}} \ge \min(\mathrm{rad}\,(a_n)_{\mathrm{c}}, \mathrm{rad}\,(b_n)_{\mathrm{c}}), $$

and

$$ {}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} (a_n+b_n)\,(x-c)^n = {}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n\,(x-c)^n + {}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} b_n\,(x-c)^n. \tag{2.26} $$

(iii) *For all $\bar{x} \in B_{|x-c|}(c)$, the product of these HPS converges to their Cauchy product:*

$$ \left({}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n\,(\bar{x}-c)^n\right) \cdot \left({}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} b_n\,(\bar{x}-c)^n\right) = {}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} \sum_{k=0}^{n} a_k b_{n-k}\,(\bar{x}-c)^n, \tag{2.27} $$

which is still a convergent HPS with radius of convergence greater or equal to $\min(\mathrm{rad}\,(a_n)_{\mathrm{c}}, \mathrm{rad}\,(b_n)_{\mathrm{c}})$.

(iv) *Let $[a_{n\varepsilon}]_c = (a_n)_{\mathrm{c}}$ and $[b_{n\varepsilon}]_c = (b_n)_{\mathrm{c}}$ be representatives of the coefficients of the given HPS. Assume that $b_0 = [b_{0\varepsilon}] \in {}^{\rho}\widetilde{\mathbb{R}}$ is invertible, and recursively define (for ε small) $d_{0\varepsilon} := \frac{a_{0\varepsilon}}{b_{0\varepsilon}}$,*

$$ d_{n\varepsilon} := \frac{1}{b_{0\varepsilon}}\left(a_{n\varepsilon} - \sum_{l=1}^{n} b_{l\varepsilon} d_{n-l,\varepsilon}\right) \qquad \forall n \in \mathbb{N}_{>0}. \tag{2.28} $$

Then coefficients $(d_n)_{\mathrm{c}} \in {}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{c}}$ define a convergent HPS with radius of convergence greater or equal to $\min(\mathrm{rad}\,(a_n)_{\mathrm{c}}, \mathrm{rad}\,(b_n)_{\mathrm{c}})$ *such that for all $\bar{x} \in$*

$B_{|x-c|}(c)$, *if* ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} b_n\left(\bar{x}-c\right)^n$ *is invertible, then*

$$\frac{{}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n\left(\bar{x}-c\right)^n}{{}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} b_n\left(\bar{x}-c\right)^n}={}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} d_n\left(\bar{x}-c\right)^n. \tag{2.29}$$

Proof. Equalities (2.25) and (2.26) follow directly from analogous properties of convergent hyperlimits, i.e. [23, Sec. 5.2]. All the inequalities concerning the radius of convergence can be proved in the same way from analogous results of the classical theory, because of Def. .(iii)3. For example, from Def. 8.(iii) we have that both the ordinary series $\sum_{n=0}^{+\infty} a_{n\varepsilon}(x_\varepsilon-c_\varepsilon)^n$ and $\sum_{n=0}^{+\infty} b_{n\varepsilon}(x_\varepsilon-c_\varepsilon)^n$ converge. Thereby, their sum $\sum_{n=0}^{+\infty}\left(a_{n\varepsilon}+b_{n\varepsilon}\right)\left(x_\varepsilon-c_\varepsilon\right)^n$ converges with radius $\operatorname{rad}\left(a_n+b_n\right)_{\mathrm{c}\varepsilon}\ge\min\left(\operatorname{rad}\left(a_n\right)_{\mathrm{c}\varepsilon},\operatorname{rad}\left(b_n\right)_{\mathrm{c}\varepsilon}\right)$. To prove (2.27) (assuming that $\bar{x}$ lies in the convergence set of the product HPS, see below), from Lem. 23 we have that both the series converge absolutely because $\bar{x}\in B_{|x-c|}(c)$. We can hence apply the generalization of Mertens' theorem to hyperseries (see [29, Thm. 37]).

To complete the proof of (iii), we start by showing that the representatives of the product $\left(\sum_{k=0}^{n} a_{k\varepsilon}b_{n-k,\varepsilon}\right)_{n,\varepsilon}$ defines coefficients for an HPS. Let $(a_n)_{\mathrm{c}}=\left[a_{n\varepsilon}\right]_{\mathrm{c}}$, $(b_n)_{\mathrm{c}}=\left[b_{n\varepsilon}\right]_{\mathrm{c}}\in{}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{c}}$, so that:

$$\exists Q_1,R_1\in\mathbb{N}\,\forall^0\varepsilon\,\forall n\in\mathbb{N}:\ \left|a_{n\varepsilon}\right|\le\rho_\varepsilon^{-nQ_1-R_1}. \tag{2.30}$$

$$\exists Q_2,R_2\in\mathbb{N}\,\forall^0\varepsilon\,\forall n\in\mathbb{N}:\ \left|b_{n\varepsilon}\right|\le\rho_\varepsilon^{-nQ_2-R_2}. \tag{2.31}$$

Without loss of generality we can assume $Q_2>Q_1$. We have

$$\begin{aligned}\left|\sum_{k=0}^{n} a_{k\varepsilon}b_{n-k,\varepsilon}\right| &\le\sum_{k=0}^{n}\left|a_{k\varepsilon}\right|\left|b_{n-k,\varepsilon}\right|\\ &\le\sum_{k=0}^{n}\rho_\varepsilon^{-kQ_1-R_1}.\rho_\varepsilon^{-(n-k)Q_2-R_2}\\ &\le\sum_{k=0}^{n}\rho_\varepsilon^{-kQ_1+kQ_2-nQ_2-R_1-R_2}.\end{aligned} \tag{2.32}$$

We have $\rho_\varepsilon^{Q_2-Q_1}<1$ because $Q_2>Q_1$, and hence

$$\left|\sum_{k=0}^{n} a_{k\varepsilon}b_{n-k,\varepsilon}\right|\le\frac{\rho_\varepsilon^{-nQ_2-R_1-R_2}}{1-\rho_\varepsilon^{-Q_1+Q_2}}\le\rho_\varepsilon^{-nQ-R},$$

where $R:=R_1+R_2$ and for a suitable $Q\in\mathbb{N}$ (that can be chosen uniformly with respect to $n\in\mathbb{N}$). Thereby, the product HPS has well-defined coefficients and hence a suitable set of convergence.

Now, we want to show that $\bar{x}$ lies in this set of convergence. Since Def. 8.(i) clearly holds and Def. 8.(iii) follows from Mertens' Theorem (both [29, Thm. 37] and the classical version), it remains to prove that we actually have a formal HPS (Def. 8.(ii)) and moderateness of derivatives (Def. 8.(iv)). The latter follows by the general Leibniz rule for the k-th derivative of a product. For the former one, without loss of generality we can assume $c=0$; let (M_ε), $(N_\varepsilon)\in\mathbb{N}_\sigma$, then for suitable $(\bar{M}_\varepsilon)$, $\left(\hat{M}_\varepsilon\right)\in\mathbb{N}_\sigma$ and $(\bar{N}_\varepsilon)$, $\left(\hat{N}_\varepsilon\right)\in\mathbb{N}_\sigma$ such that $M_\varepsilon=\bar{M}_\varepsilon+\hat{M}_\varepsilon$ and

$N_\varepsilon = \bar{N}_\varepsilon + \hat{N}_\varepsilon$, we have

$$\left(\sum_{n=N_\varepsilon}^{M_\varepsilon}\sum_{k=0}^{n} a_{n\varepsilon}b_{n-k,\varepsilon}\hat{x}_\varepsilon^n\right) = \left(\sum_{n=\bar{N}_\varepsilon}^{\bar{M}_\varepsilon} a_{n\varepsilon}\hat{x}_\varepsilon^n\right)\cdot\left(\sum_{n=\hat{N}_\varepsilon}^{\hat{M}_\varepsilon} b_{n,\varepsilon}\hat{x}_\varepsilon^n\right), \tag{2.33}$$

and thereby Def. 8.(ii) follows.

(iv): To prove that $(d_n)_{\mathrm{c}} \in {}^\rho\widetilde{\mathbb{R}}_{\mathrm{c}}$, without loss of generality, we can assume in (2.30) and (2.31) that $Q_1 = Q_2 =: \hat{Q} > R_1 = R_2 =: \hat{R}$ and $\hat{Q} > 0$. By induction on $n \in \mathbb{N}$, we want to prove that

$$\forall^0\varepsilon\,\forall n \in \mathbb{N}:\ |d_{n\varepsilon}| \le \rho_\varepsilon^{-n\hat{Q}-\hat{Q}}. \tag{2.34}$$

For $n = 0$, we have $|d_{0\varepsilon}| = \left|\frac{a_{0\varepsilon}}{b_{0\varepsilon}}\right| \le \rho_\varepsilon^{-\hat{R}+\hat{R}} \le \rho_\varepsilon^{-\hat{Q}}$ for all ε because $\hat{Q} > 0$. For the inductive step, we assume (2.34) and use the recursive definition (2.28):

$$\begin{aligned}|d_{n+1,\varepsilon}| &\le \left|\frac{a_{n+1,\varepsilon}}{b_{0\varepsilon}}\right| + \left|\frac{\sum_{l=1}^{n+1} b_{l\varepsilon}d_{n-l,\varepsilon}}{b_{0\varepsilon}}\right|\\ &\le \rho_\varepsilon^{-(n+1)\hat{Q}-\hat{R}}\cdot\rho_\varepsilon^{\hat{R}} + \sum_{l=1}^{n+1}\rho_\varepsilon^{-l\hat{Q}-\hat{R}}\cdot\rho_\varepsilon^{-(n-l)\hat{Q}-\hat{Q}}\cdot\rho_\varepsilon^{\hat{R}}\\ &= \rho_\varepsilon^{-n\hat{Q}-\hat{Q}} + \rho_\varepsilon^{-n\hat{Q}-\hat{Q}} \le 2\rho_\varepsilon^{-n\hat{Q}-\hat{Q}}.\end{aligned}$$

We have $2\rho_\varepsilon^{-n\hat{Q}-\hat{Q}} \le \rho_\varepsilon^{-(n+1)\hat{Q}-\hat{Q}}$ if and only if $2 \le \rho_\varepsilon^{-\hat{Q}}$, which holds for ε small (independently by n).

Finally, equality (2.29) can be proved as we did above for the product because $\bar{x} \in B_{|x-c|}(c)$ and

$$^\rho\!\!\sum_{n\in{}^\sigma\widetilde{\mathbb{N}}} a_n\,(\bar{x}-c)^{\,n} = {}^\rho\!\!\sum_{n\in{}^\sigma\widetilde{\mathbb{N}}} d_n\,(\bar{x}-c)^{\,n}.{}^\rho\!\!\sum_{n\in{}^\sigma\widetilde{\mathbb{N}}} b_n\,(\bar{x}-c)^{\,n}. \tag{2.35}$$

From this equality, it also follows Def. 16.(ii) because the product of an invertible non-moderate net (on a co-final set) by a moderate net cannot yield a moderate net. Finally, as above, moderateness of derivatives follows from Mertens' theorem and the k-th derivative of the quotient. □

The following theorem concerns the composition of HPS:

Theorem 27. *Let $(a_n)_{\mathrm{c}} = [a_{n\varepsilon}]_{\mathrm{c}}$, $(b_n)_{\mathrm{c}} = [b_{n\varepsilon}]_{\mathrm{c}} \in {}^\rho\widetilde{\mathbb{R}}_{\mathrm{c}}$ be coefficients for HPS. Set*

$$\begin{aligned}f(y) &:= {}^\rho\!\!\sum_{n\in{}^\sigma\widetilde{\mathbb{N}}} a_n(y-b_0)^n \quad \forall y \in {}^\rho_\sigma\mathrm{conv}\left((a_n)_{\mathrm{c}}, b_0\right)\\ g(x) &:= {}^\rho\!\!\sum_{n\in{}^\sigma\widetilde{\mathbb{N}}} b_n(x-c)^n \quad \forall x \in {}^\rho_\sigma\mathrm{conv}\left((b_n)_{\mathrm{c}}, c\right).\end{aligned}$$

Set

$$\begin{aligned}c_{0\varepsilon} &:= a_{0\varepsilon}\\ c_{n\varepsilon} &:= \sum_{k=0}^{+\infty} a_{k\varepsilon}\sum_{m_1+\ldots+m_k=n} b_{m_1\varepsilon}\cdot\ldots\cdot b_{m_k\varepsilon} \qquad \forall n \in \mathbb{N}_{>0}.\end{aligned}$$

If $x \in {}^{\rho}_{\sigma}\mathrm{conv}\left(\left(b_n\right)_{\mathrm{c}}, c\right)$ *and* $g(x) \in {}^{\rho}_{\sigma}\mathrm{conv}\left(\left(a_n\right)_{\mathrm{c}}, b_0\right)$*, then*

$$f(g(x)) = {}^{\rho}\sum_{n \in {}^{\sigma}\widetilde{\mathbb{N}}} c_n (x-c)^n$$

is a convergent HPS.

Proof. Since $[a_{n\varepsilon}]_c$, $[b_{n\varepsilon}]_{\mathrm{c}} \in {}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{c}}$, we can assume that both (2.2) and (2.3) hold with $\hat{Q} = Q_1 = Q_2 > 0$ and $\hat{R} = R_1 = R_2 > 0$. We have

$$\begin{aligned}
\left|\sum_{k=0}^{n} a_{k\varepsilon} \sum_{m_1+\ldots+m_k=n} b_{m_1\varepsilon} \cdot \ldots \cdot b_{m_k\varepsilon}\right| &\le \sum_{k=0}^{n} |a_{k\varepsilon}| \sum_{m_1+\ldots+m_k=n} |b_{m_1\varepsilon}| \cdot \ldots \cdot |b_{m_k\varepsilon}| \\
&\le \sum_{k=0}^{n} \rho_\varepsilon^{-k\hat{Q}-\hat{R}} \sum_{m_1+\ldots+m_k=n} \rho_\varepsilon^{-m_1\hat{Q}-\hat{R}} \cdot \ldots \cdot \rho_\varepsilon^{-m_k\hat{Q}-\hat{R}} \\
&= \sum_{k=0}^{n} \rho_\varepsilon^{-k\hat{Q}-\hat{R}} \sum_{m_1+\ldots+m_k=n} \rho_\varepsilon^{-n\hat{Q}-k\hat{R}} \\
&= \rho_\varepsilon^{-\hat{R}} + \sum_{k=1}^{n} \rho_\varepsilon^{-k\hat{Q}-\hat{R}} \sum_{m_1+\ldots+m_k=n} \rho_\varepsilon^{-n\hat{Q}-k\hat{R}} \\
&= \rho_\varepsilon^{-\hat{R}} + \sum_{k=1}^{n} \rho_\varepsilon^{-k\hat{Q}-\hat{R}-n\hat{Q}-k\hat{R}} \binom{n+k-1}{k-1} \\
&\le \rho_\varepsilon^{-\hat{R}} + 2^{2n} \rho_\varepsilon^{-\hat{R}-n\hat{Q}} \cdot \frac{1-\rho_\varepsilon^{-(n+1)(\hat{Q}+\hat{R})}}{1-\rho_\varepsilon^{-\hat{Q}-\hat{R}}} =: [*].
\end{aligned}$$

For ε small, we have $\frac{4}{\rho_{\varepsilon^{-1}}} \le 1$, hence $\frac{2^{2n}}{\rho_\varepsilon^{-n}} \le 1$ for the same ε and for all $n \in \mathbb{N}$. Now, take ε small so that also $\frac{1}{1-\rho_\varepsilon^{-\hat{Q}-\hat{R}}} \le 1$, and $\frac{1}{\rho_\varepsilon^{-1}} \le \frac{1}{3}$. We hence have

$$[*] \le \rho_\varepsilon^{-\hat{R}} + \rho_\varepsilon^{-n\hat{Q}-\hat{R}-n} + \rho_\varepsilon^{-2n\hat{Q}-n\hat{R}-2\hat{R}-n}.$$

Since

$$\begin{aligned}
\frac{\rho_\varepsilon^{-\hat{R}}}{\rho_\varepsilon^{-n(2\hat{Q}+\hat{R}+1)-2\hat{R}-1}} &\le \frac{1}{\rho_\varepsilon^{-1}} \le \frac{1}{3} \\
\frac{\rho_\varepsilon^{-n\hat{Q}-\hat{R}-n}}{\rho_\varepsilon^{-n(2\hat{Q}+\hat{R}+1)-2\hat{R}-1}} &\le \frac{1}{\rho_\varepsilon^{-1}} \le \frac{1}{3} \\
\frac{\rho_\varepsilon^{-2n\hat{Q}-n\hat{R}-2\hat{R}-n}}{\rho_\varepsilon^{-n(2\hat{Q}+\hat{R}+1)-2\hat{R}-1}} &\le \frac{1}{\rho_\varepsilon^{-1}} \le \frac{1}{3},
\end{aligned}$$

we finally get

$$\forall^0\varepsilon\, \forall n \in \mathbb{N}: \ |\sum_{k=0}^{n} a_{k\varepsilon} \sum_{m_1+\ldots+m_k=n} b_{m_1\varepsilon} \cdot \ldots \cdot b_{m_k\varepsilon}| \le \rho_\varepsilon^{-n(2\hat{Q}+\hat{R}+1)-2\hat{R}-1},$$

which proves that $(c_{n\varepsilon})_{n,\varepsilon}$ defines coefficients for an HPS.

To prove that $x \in {}^{\rho}_{\sigma}\mathrm{conv}\left(\left(c_n\right)_{\mathrm{c}}, c\right)$, we can proceed as follows: Def. 8.(i) can be proved like in the classical case; Def. 8.(ii) is a consequence of composition of polynomials if $M_\varepsilon < +\infty$ or it can be proved proceeding like in the case of composition of GSF if $M_\varepsilon = +\infty$: Def. 8.(iii) and Def. 8.(iv) can be proved like for GSF (see [13] and Thm. 28 below). $\square$

3. Generalized real analytic functions and their calculus

A direct consequence of Def. 8 of set of convergence is the following

Theorem 28. *Let $[a_{n\varepsilon}]_{\mathrm{c}} = (a_n)_{\mathrm{c}} \in {}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{c}}$ and $c = [c_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}}$. Set*

$$f(x) := {}^{\rho}\!\!\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n(x-c)^n = \left[\sum_{n=0}^{\infty} a_{n\varepsilon}(x_\varepsilon - c_\varepsilon)^n\right] =: [v_\varepsilon(x_\varepsilon)]$$

for all $x = [x_\varepsilon] \in {}^{\rho}_{\sigma}\mathrm{conv}\left((a_n)_{\mathrm{c}}, c\right)$. Then $f \in {}^{\rho}\mathcal{GC}^{\infty}\left({}^{\rho}_{\sigma}\mathrm{conv}\left((a_n)_{\mathrm{c}}, c\right), {}^{\rho}\widetilde{\mathbb{R}}\right)$ is a GSF defined by (v_ε).

Before defining the notion of GRAF, we need to prove that the derived HPS has the same set of convergence of the original HPS:

Theorem 29. *Assume $\sigma \le \rho^*$, $(a_n)_{\mathrm{c}} \in {}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{c}}$ and $c \in {}^{\rho}\widetilde{\mathbb{R}}$. Then the set of convergence of the derived series ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}_{>0}} na_n(x-c)^{n-1} = {}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} (n+1)a_{n+1}(x-c)^n$ is the same as the set of convergence of the original HPS ${}^{\rho}\sum_{n\in{}^{\rho}\widetilde{\mathbb{N}}} a_n(x-c)^n$. Thereby, recursively, all the derivatives have the same set of convergence of the original HPS and define a GSF.*

Proof. By Def. 3.(iii) of radius of convergence and the classical theory, we have

$$\begin{aligned}\mathrm{rad}\,(a_n)_{\mathrm{c}\varepsilon} &= \left(\limsup_{n\to+\infty} |a_{n\varepsilon}|^{1/n}\right)^{-1} = \left(\limsup_{n\to+\infty} |(n+1)a_{n+1,\varepsilon}|^{1/n+1}\right)^{-1}\\ &= \mathrm{rad}\,((n+1)a_{n+1})_{\mathrm{c}\varepsilon}\,,\end{aligned}$$

so Def. 8.(i) for the original HPS and the derived one are equivalent. From the condition $[a_{n\varepsilon}\cdot(x_\varepsilon - c_\varepsilon)^n]_{\mathrm{s}} \in {}^{\rho}_{\sigma}\widetilde{\mathbb{R}}[\![x-c]\!]$ and $\sigma \le \rho^*$, in the usual way it follows that $[(n+1)a_{n+1,\varepsilon}\cdot(x_\varepsilon - c_\varepsilon)^n]_{\mathrm{s}} \in {}^{\rho}_{\sigma}\widetilde{\mathbb{R}}[\![x-c]\!]$. Vice versa, from $(n+1)\,|a_{n+1,\varepsilon}| \ge |a_{n+1,\varepsilon}|$ the opposite implication follows. The condition Def. 8.(iv) about moderateness of derivatives for the original HPS clearly implies the analogue condition for the derived one. For the opposite inclusion, we can distinguish the case $x =_{\mathrm{s}} c$ or $|x-c| > 0$, the former one being trivial. We have

$$\begin{aligned}\left|\sum_{n=1}^{+\infty} a_{n\varepsilon} n(x_\varepsilon - c_\varepsilon)^{n-1}\right| &= |x_\varepsilon - c_\varepsilon|^{-1}\left|\sum_{n=1}^{+\infty} a_{n\varepsilon} n(x_\varepsilon - c_\varepsilon)^{n}\right|\\ &\ge |x_\varepsilon - c_\varepsilon|^{-1}\left|\sum_{n=0}^{+\infty} a_{n\varepsilon}(x_\varepsilon - c_\varepsilon)^{n}\right|,\end{aligned}$$

so that also the net $\left(\sum_{n=0}^{+\infty} a_{n\varepsilon}(x_\varepsilon - c_\varepsilon)^n\right) \in \mathbb{R}_\rho$ if the derivative is moderate. $\square$

Thm. 28 motivates the following definition:

Definition 30. Let $\sigma \le \rho^*$ and U be a sharply open set of ${}^{\rho}\widetilde{\mathbb{R}}$, then we say that f *is a GRAF on U* (with respect to ρ, σ), and we write $f \in {}^{\rho}_{\sigma}\mathcal{GC}^{\omega}(U, {}^{\rho}\widetilde{\mathbb{R}})$ if $f : U \longrightarrow {}^{\rho}\widetilde{\mathbb{R}}$ and for all $c \in U$ we can find $s \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$, $(a_n)_c \in {}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{c}}$ such that

(i) $(c-s, c+s) \subseteq U \cap {}^{\rho}_{\sigma}\mathrm{conv}\left((a_n)_{\mathrm{c}}, c\right)$,

(ii) $f(x) = {}^\rho\sum_{n\in{}^\sigma\widetilde{\mathbb{N}}} a_n\,(x-c)^n$ for all $x \in (c-s, c+s)$.

Moreover, we say that $f : {}^\rho\widetilde{\mathbb{R}} \to {}^\rho\widetilde{\mathbb{R}}$ *is an entire function* (with respect to ρ, σ) if we can find $c \in {}^\rho\widetilde{\mathbb{R}}$ and $(a_n)_c \in {}^\rho\widetilde{\mathbb{R}}_c$ such that

(iii) ${}^\rho\widetilde{\mathbb{R}} = {}^\rho_\sigma\mathrm{conv}\left((a_n)_c, c\right)$,

(iv) $f(x) = {}^\rho\sum_{n\in{}^\sigma\widetilde{\mathbb{N}}} a_n(x-c)^n$ for all $x \in {}^\rho\widetilde{\mathbb{R}}$.

We also say that f *is entire at* c if (iii) and (iv) hold.

Example 31.

(a) Clearly, if $(a_n)_c \in {}^\rho\widetilde{\mathbb{R}}_c$, $c \in {}^\rho\widetilde{\mathbb{R}}$, and we set $f(x) = {}^\rho\sum_{n\in{}^\sigma\widetilde{\mathbb{N}}} a_n\,(x-c)^n$, then f is a GRAF on the interior points of the set of convergence ${}^\rho_\sigma\mathrm{conv}\left((a_n)_c, c\right)$. Vice versa, if $f \in {}^\rho_\sigma\mathcal{GC}^\omega(U, {}^\rho\widetilde{\mathbb{R}})$, then U is contained in the union of all the sharp interior sets $\mathrm{int}\left({}^\rho_\sigma\mathrm{conv}\left((a_n)_c, c\right)\right)$, because of condition (i).

(b) Example 15 shows that Dirac δ is entire at 0 but it is not at any $c \in {}^\rho\widetilde{\mathbb{R}}$ such that $|c| \ge s$ for some $s \in \mathbb{R}_{>0}$.

(c) Example 12 of a function f with a flat point shows that f is a GRAF, but if $c = 0$, then $s \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ satisfying condition (i) is infinitesimal, whereas if $c \gg 0$, then $s \gg 0$ is finite, and these two types of set of convergence are always disjoint.

Corollary 32. *Let $\sigma \le \rho^*$, $U \subseteq {}^\rho\widetilde{\mathbb{R}}$ be a sharply open set and $f \in {}^\rho_\sigma\mathcal{GC}^\omega(U, {}^\rho\widetilde{\mathbb{R}})$, then also $f' \in {}^\rho_\sigma\mathcal{GC}^\omega(U, {}^\rho\widetilde{\mathbb{R}})$ and it can be computed with the derived HPS i.e. $f'(x) = {}^\rho\sum_{n\in{}^\sigma\widetilde{\mathbb{N}}} (n+1)a_{n+1}(x-c)^n$.*

Because of our definition Def. 8 of set of convergence, several classical results can be simply translated in our setting considering the real analytic function that defines a given GRAF.

Theorem 33. *Let $\sigma \le \rho^*$, $(a_n)_c \in {}^\rho\widetilde{\mathbb{R}}_c$, $c \in {}^\rho\widetilde{\mathbb{R}}$, and set $f(x) = {}^\rho\sum_{n\in{}^\sigma\widetilde{\mathbb{N}}} a_n(x-c)^n$ for all interior points $x \in {}^\rho_\sigma\mathrm{conv}\left((a_n)_c, c\right)$, then $a_k = \frac{f^{(k)}(c)}{k!}$ for all $k \in \mathbb{N}$.*

Proof. From Cor. 32, we have $f^{(k)}(x) = \left[\sum_{n=k}^\infty a_{n\varepsilon}k!\binom{n}{k}(x_\varepsilon - c_\varepsilon)^{n-k}\right]$ for all the interior points $x \in {}^\rho_\sigma\mathrm{conv}\left((a_n)_c, c\right)$. For $x = c$ (which is always a sharply interior point because of Thm. 9) this yields the conclusion. □

Corollary 34. *Let $\sigma \le \rho^*$, U be a sharply open set of ${}^\rho\widetilde{\mathbb{R}}$, and $f \in {}^\rho_\sigma\mathcal{GC}^\omega(U, {}^\rho\widetilde{\mathbb{R}})$. Then for all $c \in U$ the Taylor coefficients $\left(\frac{f^{(n)}(c)}{n!}\right)_c \in {}^\rho\widetilde{\mathbb{R}}_c$.*

The definition of 1-dimensional integral of GSF by using primitives, allows us to get a simple proof of the term by term integration of GRAF:

Theorem 35. *In the assumptions of the previous theorem, set*

$$F(x) := {}^\rho\sum_{n\in{}^\sigma\widetilde{\mathbb{N}}} \frac{a_n(x-c)^{n+1}}{n+1}$$

for all the interior points $x \in {}^\rho_\sigma\mathrm{conv}\left((a_n)_c, c\right)$. Then $F(x) = \int_c^x f(x)\,\mathrm{d}x$ and F is a GRAF on the interior points of ${}^\rho_\sigma\mathrm{conv}\left((a_n)_c, c\right)$.

Proof. The proof that ${}^\rho\sum_{n\in{}^\sigma\widetilde{\mathbb{N}}} \frac{a_n(x-c)^{n+1}}{n+1}$ is a convergent HPS with the same set of convergence of f can be done as in Thm. 29, and hence F is a GRAF on the interior points of ${}^{\rho}_{\sigma}\mathrm{conv}\left((a_n)_{\mathrm{c}}, c\right)$. The remaining part of the proof is straightforward by using Cor. 32, so that $F'(x) = f(x)$ and $F(c) = 0$. and using [13, Thm. 42, Def. 43]. $\square$

We close this section by first noting that, differently with respect to the classical theory, if $f(x) = {}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n(x-c)^n$ for all $x \in {}^{\rho}_{\sigma}\mathrm{conv}\left((a_n)_{\mathrm{c}}, c\right)$, and we take another point $\bar{c} \in {}^{\rho}_{\sigma}\mathrm{conv}\left((a_n)_{\mathrm{c}}, c\right)$, we do not have that $(\bar{c}-\mathrm{rad}\,(a_n)_{\mathrm{c}}+|c-\bar{c}|, \bar{c}+\mathrm{rad}\,(a_n)_{\mathrm{c}}-|c-\bar{c}|) \subseteq {}^{\rho}_{\sigma}\mathrm{conv}\left((a_n)_{\mathrm{c}}, c\right)$; in fact for $c = \bar{c}$ this would yield the false equality $(c-\mathrm{rad}\,(a_n)_{\mathrm{c}}, c+\mathrm{rad}\,(a_n)_{\mathrm{c}}) = {}^{\rho}_{\sigma}\mathrm{conv}\left((a_n)_{\mathrm{c}}, c\right)$. On the other hand, in the following result we show that ${}^{\rho}_{\sigma}\mathrm{conv}\left(\left(\frac{f^{(n)}(\bar{c})}{n!}\right)_{\mathrm{c}}, \bar{c}\right) \subseteq {}^{\rho}_{\sigma}\mathrm{conv}\left(\left(\frac{f^{(n)}(c)}{n!}\right)_{\mathrm{c}}, c\right)$:

Theorem 36. *In the assumptions of Thm. 33, if* $\bar{c} \in {}^{\rho}_{\sigma}\mathrm{conv}\left(\left(\frac{f^{(n)}(c)}{n!}\right)_{\mathrm{c}}, c\right)$, *then* ${}^{\rho}_{\sigma}\mathrm{conv}\left(\left(\frac{f^{(n)}(\bar{c})}{n!}\right)_{\mathrm{c}}, \bar{c}\right) \subseteq {}^{\rho}_{\sigma}\mathrm{conv}\left(\left(\frac{f^{(n)}(c)}{n!}\right)_{\mathrm{c}}, c\right)$.

Proof. In fact, since $\bar{c} \in {}^{\rho}_{\sigma}\mathrm{conv}\left(\left(\frac{f^{(n)}(c)}{n!}\right)_{\mathrm{c}}, c\right)$, we have

$$f^{(n)}(\bar{c}) = {}^{\rho}\sum_{m\in{}^{\sigma}\widetilde{\mathbb{N}}_{\geq n}} \frac{f^{(m)}(c)}{m!} n! \binom{m}{n} (x-c)^{m-n}.$$

Thereby, if $x \in {}^{\rho}_{\sigma}\mathrm{conv}\left(\left(\frac{f^{(n)}(\bar{c})}{n!}\right)_{\mathrm{c}}, \bar{c}\right)$, we have

$$\begin{aligned} f(x) &= {}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} \frac{f^{(n)}(\bar{c})}{n!}(x-\bar{c})^n \\ &= {}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} \frac{(x-\bar{c})^n}{n!} \cdot {}^{\rho}\sum_{m\in{}^{\sigma}\widetilde{\mathbb{N}}_{\geq n}} \frac{f^{(m)}(c)}{m!} n! \binom{m}{n} (x-c)^{m-n} \\ &= \left[\sum_{n=0}^{+\infty} \frac{(x_\varepsilon-\bar{c}_\varepsilon)^n}{n!} \sum_{m\geq n}^{+\infty} \frac{f_\varepsilon^{(m)}(c_\varepsilon)}{m!} n! \binom{m}{n} (x_\varepsilon-c_\varepsilon)^{m-n}\right]. \end{aligned}$$

Therefore, the usual proof, see e.g. [22], yields

$$\sum_{n=0}^{+\infty} \frac{(x_\varepsilon-\bar{c}_\varepsilon)^n}{n!} \sum_{m\geq n}^{+\infty} \frac{f_\varepsilon^{(m)}(c_\varepsilon)}{m!} n! \binom{m}{n} (x_\varepsilon-c_\varepsilon)^{m-n} = \sum_{n=0}^{+\infty} \frac{f_\varepsilon^{(n)}(c)}{n!}(x_\varepsilon-c_\varepsilon)^n$$

and hence $f(x) = {}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} \frac{f^{(n)}(c)}{n!}(x-c)^n = \left[\sum_{n=0}^{+\infty} \frac{f_\varepsilon^{(n)}(c)}{n!}(x_\varepsilon-c_\varepsilon)^n\right]$, which implies the conclusion. $\square$

4. Characterization of generalized real analytic functions, inversion and identity principle

The classical characterization of real analytic functions by the growth rate of the derivatives establishes a difference between GRAF and Colombeau real analytic functions:

Theorem 37. *Let* $\sigma \leq \rho^*$, U *be a sharply open set of* ${}^{\rho}\widetilde{\mathbb{R}}$, *and* $f \in {}^{\rho}\mathcal{GC}^{\infty}(U, {}^{\rho}\widetilde{\mathbb{R}})$ *be a GSF defined by the net* (f_ε). *Then* $f \in {}^{\rho}_{\sigma}\mathcal{GC}^{\omega}(U, {}^{\rho}\widetilde{\mathbb{R}})$ *if and only if for each* $c \in U$

there exist $s=[s_\varepsilon]$, $C=[C_\varepsilon]$, $R=[R_\varepsilon]\in{}^{\rho}\widetilde{\mathbb{R}}_{>0}$ *such that* $B_s(c)\subseteq U$ *and*

$$\forall[x_\varepsilon]\in B_s(c)\,\forall^0\varepsilon\,\forall n\in\mathbb{N}:\ \left|f_\varepsilon^{(n)}(x_\varepsilon)\right|\le C_\varepsilon\frac{n!}{R_\varepsilon^n}. \tag{4.1}$$

Proof. We prove that condition (4.1) is necessary. For $c\in U$, we have $f(x)={}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}}\frac{f^{(n)}(c)}{n!}(x-c)^n$ for all $x\in(c-\bar{s},c+\bar{s})$ for some $\bar{s}>0$ from Def. 30 and Thm. 33. We first note that condition (4.1) can also be formulated as an inequality in ${}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{c}}$ and as such it does not depend on the representatives involved. Therefore, from Thm. 28 and Thm. 33, without loss of generality, we can assume that the given net (f_ε) is of real analytic functions satisfying $f_\varepsilon(x)=\sum_{n=0}^{+\infty}\frac{f_\varepsilon^{(n)}(c_\varepsilon)}{n!}(x-c_\varepsilon)^n$ for all $x\in(c_\varepsilon-\mathrm{rad}\,(a_n)_{\mathrm{c}\varepsilon},c_\varepsilon+\mathrm{rad}\,(a_n)_{\mathrm{c}\varepsilon})$. From Lem. 21, locally the Taylor summands $\left(\frac{f^{(n)}(c)}{n!}(\bar{x}-c)^n\right)_{n\in\mathbb{N}}$ are eventually ${}^{\rho}\widetilde{\mathbb{R}}$-bounded in ${}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{c}}$ if $\bar{x}$ is sufficiently near to $c=[c_\varepsilon]$, i.e. there exists $\sigma\in{}^{\rho}\widetilde{\mathbb{R}}_{>0}$ such that for each $\bar{x}=[\bar{x}_\varepsilon]\in B_\sigma(c)$ we have

$$\forall^0\varepsilon\,\forall j\in\mathbb{N}:\ \left|\frac{f_\varepsilon^{(j)}(c_\varepsilon)}{j!}(\bar{x}_\varepsilon-c_\varepsilon)^j\right|\le K_\varepsilon, \tag{4.2}$$

for some $K=[K_\varepsilon]\in{}^{\rho}\widetilde{\mathbb{R}}$. Set $s:=\frac{1}{2}\min(\sigma,\bar{s})\in\mathbb{R}_{>0}$ and $S:=|\bar{x}-c|$, where $\bar{x}$ is any point such that $s<|\bar{x}-c|<\sigma$, so that $0<\frac{s}{S}<1$ and from (4.2) we obtain

$$\forall^0\varepsilon\,\forall j\in\mathbb{N}:\ \left|f_\varepsilon^{(j)}(c_\varepsilon)\right|\le K_\varepsilon\frac{j!}{S_\varepsilon^j}. \tag{4.3}$$

For each $[x_\varepsilon]\in B_s(c)$, we have

$$f_\varepsilon^{(n)}(x_\varepsilon)=\sum_{j=n}^{+\infty}\frac{f_\varepsilon^{(j)}(c_\varepsilon)}{j!}n!\binom{j}{n}(x_\varepsilon-c_\varepsilon)^{j-n},$$

and hence from (4.3):

$$\begin{aligned}\left|\frac{f_\varepsilon^{(n)}(x_\varepsilon)}{n!}\right|&\le\sum_{j=n}^{+\infty}K_\varepsilon\binom{j}{n}\frac{|x_\varepsilon-c_\varepsilon|^{j-n}}{S_\varepsilon^j}\\&\le\frac{K_\varepsilon}{S_\varepsilon^n}\sum_{j=n}^{\infty}\binom{j}{n}\left(\frac{s_\varepsilon}{S_\varepsilon}\right)^{j-n}\\&=\frac{K_\varepsilon}{S_\varepsilon^n}\cdot\frac{1}{\left(1-\frac{s_\varepsilon}{S_\varepsilon}\right)^{n+1}}=\frac{K_\varepsilon}{\left(1-\frac{s_\varepsilon}{S_\varepsilon}\right)}\cdot\frac{1}{\left(S_\varepsilon\left(1-\frac{s_\varepsilon}{S_\varepsilon}\right)\right)^n},\end{aligned}$$

which is our claim for $C:=\frac{K}{1-\frac{s}{S}}$ and $R:=S\left(1-\frac{s}{S}\right)$. Note that, differently with respect to the case of Colombeau real analytic functions [26], not necessarily the constant $\frac{1}{R}$ is finite, e.g. if $s\approx S$.

We now prove that the condition is sufficient. Let $c=[c_\varepsilon]\in U$ and $s=[s_\varepsilon]$, $C=[C_\varepsilon]$, $R=[R_\varepsilon]\in{}^{\rho}\widetilde{\mathbb{R}}_{>0}$ be the constants satisfying (4.1). Set $\bar{s}:=\frac{1}{2}\min\left(s,R,\mathrm{rad}\left(\frac{f^{(n)}(c)}{n!}\right)_{\mathrm{c}}\right)$ and take $x\in B_{\bar{s}}(c)$. We first prove the equality $f(x)={}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}}\frac{f^{(n)}(c)}{n!}(x-c)^n$. Let $N=[N_\varepsilon]\in{}^{\sigma}\widetilde{\mathbb{N}}$, with $N_\varepsilon\in\mathbb{N}$. For all ε, from Taylor's formula for the smooth f_ε, we have

$$\left|\sum_{n=0}^{N}\frac{f^{(n)}(c)}{n!}(x-c)^n-f(x)\right|=\left[\left|\frac{f_\varepsilon^{(N_\varepsilon+1)}(\xi_\varepsilon)}{(N_\varepsilon+1)!}(x_\varepsilon-c_\varepsilon)^{N_\varepsilon+1}\right|\right]$$

for some $t_\varepsilon\in[0,1]_{\mathbb{R}}$ and for $\xi_\varepsilon:=(1-t_\varepsilon)c_\varepsilon+t_\varepsilon x_\varepsilon$. Since $|\xi_\varepsilon-c_\varepsilon|=t_\varepsilon|x_\varepsilon-c_\varepsilon|<\bar{s}_\varepsilon<s_\varepsilon$, we can apply (4.1) and get $\forall^0\varepsilon\,\forall n\in\mathbb{N}:\ \left|\frac{f_\varepsilon^{(n)}(\xi_\varepsilon)}{n!}\right|\le\frac{C_\varepsilon}{R_\varepsilon^n}$. Thereby, for these small ε and for $n=N_\varepsilon+1$ we obtain

$$\left|\sum_{n=0}^{N}\frac{f^{(n)}(c)}{n!}(x-c)^n-f(x)\right|\le C\left(\frac{\bar{s}}{R}\right)^{N+1},$$

and hence the claim follows by ${}^{\rho}\lim_{n\in{}^{\sigma}\widetilde{\mathbb{N}}}\left(\frac{\bar{s}}{R}\right)^{N+1}=0$.

Now, we prove that ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}}\frac{f^{(n)}(c)}{n!}(x-c)^n=\left[\sum_{n=0}^{+\infty}\frac{f_\varepsilon^{(n)}(c_\varepsilon)}{n!}(x_\varepsilon-c_\varepsilon)^n\right]$. In fact, once again from (4.1) we have

$$\begin{aligned}\left|\sum_{n=0}^{N}\frac{f^{(n)}(c)}{n!}(x-c)^n-\left[\sum_{n=0}^{+\infty}\frac{f_\varepsilon^{(n)}(c_\varepsilon)}{n!}(x_\varepsilon-c_\varepsilon)^n\right]\right|&=\left[\left|\sum_{n=N_\varepsilon+1}^{+\infty}\frac{f_\varepsilon^{(n)}(c_\varepsilon)}{n!}(x_\varepsilon-c_\varepsilon)^n\right|\right]\\&\le\left[\sum_{n=N_\varepsilon+1}^{+\infty}\frac{C_\varepsilon}{R_\varepsilon^n}|x_\varepsilon-c_\varepsilon|^n\right]\\&\le C\cdot{}^{\rho}\!\!\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}_{\ge N+1}}\left(\frac{\bar{s}}{R}\right)^n\to0\end{aligned}$$

because $\bar{s}<R$ and hence ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}}\left(\frac{\bar{s}}{R}\right)^n$ converges. Finally, take $\bar{x}\in B_{\bar{s}}(c)$ such that $|x-c|<|\bar{x}-c|$. As above, we can prove that ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}}\frac{f^{(n)}(c)}{n!}(\bar{x}-c)^n$ converges; moreover from (4.1) we also have $\forall^0\varepsilon\,\forall n\in\mathbb{N}:\ \left|\frac{f_\varepsilon^{(n)}(c_\varepsilon)}{n!}(\bar{x}_\varepsilon-c_\varepsilon)^n\right|\le C_\varepsilon\left(\frac{\bar{s}_\varepsilon}{R_\varepsilon}\right)^n\le\frac{C_\varepsilon}{1-\frac{\bar{s}_\varepsilon}{R_\varepsilon}}$. This proves that $\left(\frac{f^{(n)}(c)}{n!}(\bar{x}-c)^n\right)_{n\in\mathbb{N}}$ is eventually ${}^{\rho}\widetilde{\mathbb{R}}$-bounded in ${}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{c}}$ and hence $x\in{}^{\rho}_{\sigma}\mathrm{conv}\left(\left(\frac{f^{(n)}(c)}{n!}\right)_{\mathrm{c}},c\right)$ by Cor. 25. $\square$

As we have already noted in this proof, differently with respect to the definition of Colombeau real analytic function [26], we have that, generally speaking, $\frac{1}{R}\in{}^{\rho}\widetilde{\mathbb{R}}$ is not finite. For example, for $f=\delta$ at $c=0$, we have $\left|\frac{\delta_\varepsilon^{(n)}(x_\varepsilon)}{n!}\right|=\left|\frac{\mu^{(n)}(x_\varepsilon)}{n!}b_\varepsilon^{n+1}\right|=\left|\frac{\mu^{(n)}(x_\varepsilon)}{n!}b_\varepsilon\right|\frac{1}{\left(b_\varepsilon^{-1}\right)^n}\le\frac{\bar{C}b_\varepsilon}{(b_\varepsilon^{-1})^n}$, where $\left|\mu^{(n)}(x_\varepsilon)\right|\le\int\beta=:\bar{C}$ and hence $\frac{1}{R}=b$ which is an infinite number. Thereby, in the particular case when $\frac{1}{R}$ is finite, f is a Colombeau real analytic function in a neighborhood of c. Vice versa, any Colombeau real analytic function and any ordinary real analytic function are GRAF.

This characterization also yields the closure of GRAF with respect to inversion. We first recall that the local inverse function theorem holds for GSF, see [10]. Therefore, if $f\in{}^{\rho}_{\sigma}\mathcal{GC}^{\omega}(U,{}^{\rho}\widetilde{\mathbb{R}})\subseteq{}^{\rho}\mathcal{GC}^{\infty}(U,{}^{\rho}\widetilde{\mathbb{R}})$ and at the point $x_0\in U$ the derivative $f'(x_0)$ is invertible, we can find open neighborhoods of $x_0\in X\subseteq U$ and of $y_0:=f(x_0)\in Y$ such that $f|_X:X\to Y$ is invertible, $(f|_X)^{-1}\in{}^{\rho}\mathcal{GC}^{\infty}(Y,X)$ and $f'(x)$ is invertible for all $x\in X$.

Theorem 38. *If $\sigma \le \rho^*$ and we use notations and assumptions introduced above, then* $(f|_X)^{-1} \in {}^{\rho}_{\sigma}\mathcal{GC}^{\omega}(Y, X)$.

Proof. For simplicity, set $g := (f|_X)^{-1}$ and $h(x) := \frac{1}{f'(x)}$ for all $x \in X$, so that $g'(y) = h[g(y)]$ for all $y \in Y$. From Cor. 32 and Thm. 26, we know that h is a GRAF. Therefore, Thm. 37 yields $\forall^0\varepsilon\, \forall n \in \mathbb{N} : \left|h^{(j)}_\varepsilon(x_\varepsilon)\right| \le C_\varepsilon \frac{j!}{R^j_\varepsilon}$ for all $[x_\varepsilon] \in B_s(x_0)$ and suitable constants s, C, $R \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$. For $[y_\varepsilon] \in f(B_s(x_0))$ (note that this is an open neighborhood of y_0 because f is an open map) and these ε, formula (1.15) of [22, Thm.1.5.3] yields $\left|g^{(j)}_\varepsilon(y_\varepsilon)\right| \le j!(-1)^{j-1}\binom{1/2}{j}\frac{(2C_\varepsilon)^j}{R^{j-1}_\varepsilon}$ for all $j \in \mathbb{N}_{>0}$, and hence $g \in {}^{\rho}_{\sigma}\mathcal{GC}^{\omega}(U, {}^{\rho}\widetilde{\mathbb{R}})$ once again by Thm. 37. □

Since δ is a GRAF, in general the identity principle does not hold for GRAF. From our point of view this is a feature of GRAF because it allows to include as GRAF a large class of interesting generalized functions and hence pave the way to a more general related Cauchy-Kowalevski theorem. The following theorem clearly shows that the identity principle does not hold in our framework exactly because we are in a non-Archimedean setting: every interval is not connected in the sharp topology because the set of all the infinitesimals is a clopen set, see e.g. [9].

Theorem 39. *Let $U \subseteq {}^{\rho}\widetilde{\mathbb{R}}$ be an open set and f, $g \in {}^{\rho}_{\sigma}\mathcal{GC}^{\omega}(U, {}^{\rho}\widetilde{\mathbb{R}})$. Then the set*

$$\mathcal{O} := \operatorname{int}\{x \in U \mid f(x) = g(x)\}$$

is clopen in the sharp topology.

Proof. For simplicity, considering $f - g$, without loss of generality we can assume $g = 0$. We only have to show that $\mathcal{O}$ is closed in U. Assume that c is in the closure of $\mathcal{O}$ in U, i.e.

$$c \in U,\ \forall r \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}\, \exists \bar{c} \in B_r(c) \cap \mathcal{O}. \tag{4.4}$$

We have to prove that $c \in \mathcal{O}$. We first note that for each $\bar{c} \in \mathcal{O}$, we have $B_p(\bar{c}) \subseteq \mathcal{O}$ for some $p \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$ and hence

$$f(\bar{x}) = 0 \quad \forall \bar{x} \in B_p(\bar{c}). \tag{4.5}$$

Now, fix $n \in \mathbb{N}$ in order to prove that $f^{(n)}(c) = 0$. From (4.4), for all $r \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$ we can find $\bar{c}_r \in B_r(c) \cap \mathcal{O}$ such that $f^{(n)}(\bar{c}_r) = 0$ from (4.5). From sharp continuity of $f^{(n)}$, we have $f^{(n)}(c) = \lim_{r\to 0^+} f^{(n)}(\bar{c}_r) = 0$. Since $f \in {}^{\rho}_{\sigma}\mathcal{GC}^{\omega}(U, {}^{\rho}\widetilde{\mathbb{R}})$ and $c \in U$, we can hence find $\sigma > 0$ such that $f(x) = {}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} \frac{f^{(n)}(c)}{n!}(x-c)^n = 0$ for all $x \in B_\sigma(c)$, i.e. $c \in \mathcal{O}$. □

For example, if $f = \delta$ and $g = 0$, the set

$$\operatorname{int}\left\{x \in {}^{\rho}\widetilde{\mathbb{R}} \mid \delta(x) = 0\right\} \supseteq \left\{x \in {}^{\rho}\widetilde{\mathbb{R}} \mid |x| \gg 0\right\}$$

is clopen. Thereby, also ${}^{\rho}\widetilde{\mathbb{R}} \setminus \operatorname{int}\left\{x \in {}^{\rho}\widetilde{\mathbb{R}} \mid \delta(x) = 0\right\}$ is clopen, and we have

$$\begin{aligned}\left\{x \in {}^{\rho}\widetilde{\mathbb{R}} \mid \delta(x) \neq 0\right\} &\subseteq {}^{\rho}\widetilde{\mathbb{R}} \setminus \operatorname{int}\left\{x \in {}^{\rho}\widetilde{\mathbb{R}} \mid \delta(x) = 0\right\} \\ &\subseteq \left\{x \in {}^{\rho}\widetilde{\mathbb{R}} \mid \forall r \in \mathbb{R}_{>0} :\ |x| \le_{\mathrm{s}} r\right\}.\end{aligned}$$

If we assume that all the derivatives of f are finite and the neighborhoods of Def. 30 are also finite, then we can repeat the previous proof considering only standard points $c \in \mathbb{R}$ and radii $r \in \mathbb{R}_{>0}$, obtaining the following sufficient condition:

Theorem 40. *Let $U \subseteq {}^{\rho}\widetilde{\mathbb{R}}$ be an open set such that $U \cap \mathbb{R}$ is connected. Let f, $g \in {}^{\rho}_{\sigma}\mathcal{GC}^{\omega}(U, {}^{\rho}\widetilde{\mathbb{R}})$ be such that $f|_{V\cap\mathbb{R}} = g|_{V\cap\mathbb{R}}$ for some nonempty subset $V \subseteq U$ such that $V \cap \mathbb{R}$ is open in the Fermat topology, i.e.*

$$\forall x \in V \cap \mathbb{R}\, \exists r \in \mathbb{R}_{>0} : \ B_r(x) \subseteq V \cap \mathbb{R}.$$

Finally, assume that all the following quantities are finite:

(i) The neighborhood length s in Def. 30 is finite for each $c \in U \cap \mathbb{R}$,

(ii) $\forall x \in U\, \forall n \in \mathbb{N} : \ f^{(n)}(x)$ and $g^{(n)}(x)$ are finite.

Then $f|_{U\cap\mathbb{R}} = g|_{U\cap\mathbb{R}}$.

Proof. The proof proceeds exactly as in Thm. 39 but considering

$$\mathcal{O} := \mathrm{int}_{\mathrm{F}} \left\{ x \in U \cap \mathbb{R} \mid f(x) = g(x) \right\},$$

where $\mathrm{int}_{\mathrm{F}}$ is the interior in the Fermat topology (i.e. the topology generated by the balls $B_r(c)$ for $c \in {}^{\rho}\widetilde{\mathbb{R}}$ and $r \in \mathbb{R}_{>0}$, see [9]). We have to note that assumption (ii) implies that all $f^{(n)}$ are continuous in this topology (see [12]). □

For example, if $f \in \mathcal{C}^{\omega}(\mathbb{R})$ is an ordinary real analytic function and K, $h \in {}^{\rho}\widetilde{\mathbb{R}}$ are finite numbers, the GRAF $x \in \mathrm{int}(\mathrm{c}({}^{\rho}\widetilde{\mathbb{R}})) \mapsto Kf(hx) \in {}^{\rho}\widetilde{\mathbb{R}}$, where $\mathrm{c}({}^{\rho}\widetilde{\mathbb{R}})$ is the set of compactly supported points, satisfies the assumptions of the last theorem.

5. Conclusions

Sometimes, e.g. in the study of PDE, the class of real analytic functions is described as a too rigid set of solutions. In spite of their good properties with respect to algebraic operations, composition, differentiation, integration, inversion, etc., this rigidity is essentially well represented by the identity principle that necessarily excludes e.g. solitons with compact support or interesting generalized functions. Thanks to Thm. 39, we can state that this rigidity is due to the banishing of non-Archimedean numbers from mathematical analysis. The use of hyperseries allows one to recover all these features including also interesting non trivial generalized functions and compactly supported functions. This paves the way for an interesting generalization of the Cauchy-Kowalevski theorem for GRAF that we intend to develop in a subsequent work. Its proof can be approached by trying a generalization of the classical method of majorants, or using the Picard-Lindelöf theorem for PDE with GSF and then using characterization Thm. 37 to show that the GSF solution is actually a GRAF.

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Faculty of Mathematics, University of Vienna, Austria
Email address: `diksha.tiwari@univie.ac.at`

Faculty of Mathematics, University of Vienna, Austria
Email address: `akbarali.mukhammadiev@univie.ac.at`

Faculty of Mathematics, University of Vienna, Austria
Email address: `paolo.giordano@univie.ac.at`

Chapter 8

Appendix

For the sake of completeness, this appendix presents some basic results of GSF theory. This part of the dissertation is taken from [42] and, to underscore that it has not been developed by the candidate D. Tiwary, we present here only the statements; the proofs can be found in [42].

8.1 Generalized smooth functions and their calculus

Using the ring ${}^{\rho}\widetilde{\mathbb{R}}$, it is easy to consider a Gaussian with an infinitesimal standard deviation. If we denote this probability density by $f(x,\sigma)$, and if we set $\sigma = [\sigma_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$, where $\sigma \approx 0$, we obtain the net of smooth functions $(f(-,\sigma_\varepsilon))_{\varepsilon\in I}$. This is the basic idea we are going to develop in the following

Definition 1. Let (Ω_ε) be a net of open subsets of $\mathbb{R}^n$. Let $X \subseteq {}^{\rho}\widetilde{\mathbb{R}}^n$ and $Y \subseteq {}^{\rho}\widetilde{\mathbb{R}}^d$ be arbitrary subsets of generalized points. Then we say that

$$f : X \longrightarrow Y \text{ is a } \textit{generalized smooth function}$$

if there exists a net $f_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon, \mathbb{R}^d)$ defining the map $f : X \longrightarrow Y$ in the sense that

1. $X \subseteq \langle \Omega_\varepsilon \rangle$,
2. $f([x_\varepsilon]) = [f_\varepsilon(x_\varepsilon)] \in Y$ for all $x = [x_\varepsilon] \in X$,
3. $(\partial^\alpha f_\varepsilon(x_\varepsilon)) \in \mathbb{R}^d_\rho$ for all $x = [x_\varepsilon] \in X$ and all $\alpha \in \mathbb{N}^n$.

The space of generalized smooth functions (GSF) from X to Y is denoted by ${}^{\rho}\mathcal{GC}^\infty(X,Y)$.

Let us note explicitly that this definition states minimal logical conditions to obtain a set-theoretical map from X into Y and defined by a net of smooth functions of which we can take arbitrary derivatives still remaining in the space of ρ-moderate nets. In particular, the following Thm. 2 states that the equality $f([x_\varepsilon]) = [f_\varepsilon(x_\varepsilon)]$ is meaningful, i.e. that we have independence from the representatives for all derivatives $[x_\varepsilon] \in X \mapsto [\partial^\alpha f_\varepsilon(x_\varepsilon)] \in {}^\rho\widetilde{\mathbb{R}}^d$, $\alpha \in \mathbb{N}^n$.

Theorem 2. *Let $X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ and $Y \subseteq {}^\rho\widetilde{\mathbb{R}}^d$ be arbitrary subsets of generalized points. Let $f_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon, \mathbb{R}^d)$ be a net of smooth functions that defines a generalized smooth map of the type $X \longrightarrow Y$, then*

1. $\forall \alpha \in \mathbb{N}^n \, \forall (x_\varepsilon), (x'_\varepsilon) \in \mathbb{R}^n_\rho : \; [x_\varepsilon] = [x'_\varepsilon] \in X \;\Rightarrow\; (\partial^\alpha f_\varepsilon(x_\varepsilon)) \sim_\rho (\partial^\alpha f_\varepsilon(x'_\varepsilon))$.
2. *Each $f \in {}^\rho\mathcal{GC}^\infty(X, Y)$ is continuous with respect to the sharp topologies induced on X, Y.*
3. *$f : X \longrightarrow Y$ is a GSF if and only if there exists a net $v_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^d)$ defining a generalized smooth map of type $X \longrightarrow Y$ such that $f = [v_\varepsilon(-)]|_X$.*
4. *GSF are closed with respect to composition, i.e. subsets $S \subseteq {}^\rho\widetilde{\mathbb{R}}^s$ with the trace of the sharp topology, and GSF as arrows form a subcategory of the category of topological spaces. We will call this category ${}^\rho\mathcal{GC}^\infty$, the* category of GSF. *Therefore, with pointwise sum and product, any space ${}^\rho\mathcal{GC}^\infty(X, {}^\rho\widetilde{\mathbb{R}})$ is an algebra.*

The differential calculus for GSF can be introduced by showing existence and uniqueness of another GSF serving as incremental ratio (sometimes this is called *derivative á la Carathéodory*, see e.g. [58]).

Theorem 3 (Fermat-Reyes theorem for GSF)**.** *Let $U \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ be a sharply open set, let $v = [v_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}^n$, and let $f \in {}^\rho\mathcal{GC}^\infty(U, {}^\rho\widetilde{\mathbb{R}})$ be a GSF generated by the net of smooth functions $f_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon, \mathbb{R})$. Then*

1. *There exists a sharp neighborhood T of $U \times \{0\}$ and a generalized smooth map $r \in {}^\rho\mathcal{GC}^\infty(T, {}^\rho\widetilde{\mathbb{R}})$, called the* generalized incremental ratio *of f* along *v, such that*
$$\forall (x, h) \in T : \; f(x + hv) = f(x) + h \cdot r(x, h).$$
2. *Any two generalized incremental ratios coincide on a sharp neighborhood of $U \times \{0\}$, so that we can use the notation $f[x; h] := r(x, h)$ if (x, h) are sufficiently small.*
3. *We have $f[x; 0] = \left[\frac{\partial f_\varepsilon}{\partial v_\varepsilon}(x_\varepsilon)\right]$ for every $x \in U$ and we can thus define $Df(x) \cdot v := \frac{\partial f}{\partial v}(x) := f[x; 0]$, so that $\frac{\partial f}{\partial v} \in {}^\rho\mathcal{GC}^\infty(U, {}^\rho\widetilde{\mathbb{R}})$.*

Note that this result permits us to consider the partial derivative of f with respect to an arbitrary generalized vector $v \in {}^\rho\widetilde{\mathbb{R}}^n$ which can be, e.g., infinitesimal or infinite. Using recursively this result, we can also define subsequent differentials $D^j f(x)$ as $j-$multilinear maps, and we set $D^j f(x) \cdot h^j :=$

$D^j f(x)(h, \overset{j}{\ldots}, h)$. The set of all the $j-$multilinear maps $\left({}^\rho\widetilde{\mathbb{R}}^n\right)^j \longrightarrow {}^\rho\widetilde{\mathbb{R}}^d$ over the ring ${}^\rho\widetilde{\mathbb{R}}$ will be denoted by $L^j({}^\rho\widetilde{\mathbb{R}}^n, {}^\rho\widetilde{\mathbb{R}}^d)$. For $A = [A_\varepsilon(-)] \in L^j({}^\rho\widetilde{\mathbb{R}}^n, {}^\rho\widetilde{\mathbb{R}}^d)$, we set $\|A\| := [|A_\varepsilon|]$, the generalized number defined by the operator norms of the multilinear maps $A_\varepsilon \in L^j(\mathbb{R}^n, \mathbb{R}^d)$.

The following result follows from the analogous properties for the nets of smooth functions defining f and g.

Theorem 4. *Let $U \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ be an open subset in the sharp topology, let $v \in {}^\rho\widetilde{\mathbb{R}}^n$ and $f,\, g : U \longrightarrow {}^\rho\widetilde{\mathbb{R}}$ be generalized smooth maps. Then*

1. $\frac{\partial(f+g)}{\partial v} = \frac{\partial f}{\partial v} + \frac{\partial g}{\partial v}$
2. $\frac{\partial(r\cdot f)}{\partial v} = r \cdot \frac{\partial f}{\partial v} \quad \forall r \in {}^\rho\widetilde{\mathbb{R}}$
3. $\frac{\partial(f\cdot g)}{\partial v} = \frac{\partial f}{\partial v} \cdot g + f \cdot \frac{\partial g}{\partial v}$
4. *For each $x \in U$, the map $\mathrm{d}f(x).v := \frac{\partial f}{\partial v}(x) \in {}^\rho\widetilde{\mathbb{R}}$ is ${}^\rho\widetilde{\mathbb{R}}$-linear in $v \in {}^\rho\widetilde{\mathbb{R}}^n$.*

Using the Fermat-Reyes theorem, it is also possible to give intrinsic proofs (i.e. without using nets of smooth functions that define a given GSF), as exemplified in the following

Theorem 5. *Let $U \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ and $V \subseteq {}^\rho\widetilde{\mathbb{R}}^d$ be open subsets in the sharp topology and $g \in {}^\rho\mathcal{GC}^\infty(V, U)$, $f \in {}^\rho\mathcal{GC}^\infty(U, {}^\rho\widetilde{\mathbb{R}})$ be generalized smooth maps. Then for all $x \in V$ and all $v \in {}^\rho\widetilde{\mathbb{R}}^d$*

$$\frac{\partial\,(f\circ g)}{\partial v}(x) = \mathrm{d}f\,(g(x))\,.\frac{\partial g}{\partial v}(x)$$
$$\mathrm{d}(f\circ g)\,(x) = \mathrm{d}f\,(g(x)) \circ \mathrm{d}g(x).$$

8.2 Integral calculus using primitives

Lemma 6. *Let $a,\, b \in {}^\rho\widetilde{\mathbb{R}}$ be such that $a < b$, then the interior $int([a,b])$ in the sharp topology is dense in $[a,b]$.*

Theorem 7. *Let $X \subseteq {}^\rho\widetilde{\mathbb{R}}$ and let $f \in {}^\rho\mathcal{GC}^\infty(X, {}^\rho\widetilde{\mathbb{R}})$ be a generalized smooth function. Let $a,\, b \in {}^\rho\widetilde{\mathbb{R}}$, with $a < b$, such that $(a,b) \subseteq X$. If $f'(x) = 0$ for all $x \in \mathrm{int}(a,b)$, then f is constant on (a,b). An analogous statement holds if we take any other type of interval (closed or half closed) instead of (a,b).*

Theorem 8. *Let $a,\, b \in {}^\rho\widetilde{\mathbb{R}}$ with $a < b$, and $f \in {}^\rho\mathcal{GC}^\infty([a,b], {}^\rho\widetilde{\mathbb{R}})$ be a generalized smooth function. Then for all $x \in [a,b]$, the following limit exists in the sharp topology*

$$\lim_{\substack{y\to x\\ y\in\mathrm{int}([a,b])}} f^{(k)}(y) =: f^{(k)}(x).$$

Moreover, if the net $f_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon, \mathbb{R})$ defines f and $x = [x_\varepsilon]$, then $f^{(k)}(x) = [f_\varepsilon^{(k)}(x_\varepsilon)]$ and hence $f^{(k)} \in {}^\rho\mathcal{GC}^\infty([a,b], {}^\rho\widetilde{\mathbb{R}})$.

Theorem 9. *Let a, b, $c \in {}^{\rho}\widetilde{\mathbb{R}}$, with $a < b$ and $c \in [a,b]$. Let $f \in {}^{\rho}\mathcal{GC}^{\infty}([a,b], {}^{\rho}\widetilde{\mathbb{R}})$ be a generalized smooth function. Then, there exists one and only one generalized smooth function $F \in {}^{\rho}\mathcal{GC}^{\infty}([a,b], {}^{\rho}\widetilde{\mathbb{R}})$ such that $F(c) = 0$ and $F'(x) = f(x)$ for all $x \in [a,b]$. Moreover, if f is defined by the net $f_\varepsilon \in \mathcal{C}^{\infty}(\mathbb{R}, \mathbb{R})$ and $c = [c_\varepsilon]$, then $F(x) = \left[\int_{c_\varepsilon}^{x_\varepsilon} f_\varepsilon(s)\,\mathrm{d}s\right]$ for all $x = [x_\varepsilon] \in [a,b]$.*

Definition 10. Under the assumptions of Theorem 9, we denote by $\int_c^{(-)} f := \int_c^{(-)} f(s)\,\mathrm{d}s \in {}^{\rho\rho}\mathcal{GC}^{\infty}([a,b], {}^{\rho}\widetilde{\mathbb{R}})$ the unique generalized smooth function such that:

1. $\int_c^c f = 0$
2. $\left(\int_c^{(-)} f\right)'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_c^x f(s)\,\mathrm{d}s = f(x)$ for all $x \in [a,b]$.

Theorem 11. *Let $f \in {}^{\rho\rho}\mathcal{GC}^{\infty}(X, {}^{\rho}\widetilde{\mathbb{R}})$ and $g \in {}^{\rho}\mathcal{GC}^{\infty}(Y, {}^{\rho}\widetilde{\mathbb{R}})$ be generalized smooth functions defined on arbitrary domains in ${}^{\rho}\widetilde{\mathbb{R}}$. Let a, $b \in {}^{\rho}\widetilde{\mathbb{R}}$ with $a < b$ and $[a,b] \subseteq X \cap Y$, then*

1. $\int_a^b (f+g) = \int_a^b f + \int_a^b g$
2. $\int_a^b \lambda f = \lambda \int_a^b f \quad \forall \lambda \in {}^{\rho}\widetilde{\mathbb{R}}$
3. $\int_a^b f = \int_a^c f + \int_c^b f$ *for all* $c \in [a,b]$
4. $\int_a^b f = -\int_b^a f$
5. $\int_a^b f' = f(b) - f(a)$
6. $\int_a^b f' \cdot g = [f \cdot g]_a^b - \int_a^b f \cdot g'$
7. *If* $f(x) \le g(x)$ *for all* $x \in [a,b]$, *then* $\int_a^b f \le \int_a^b g$.

Theorem 12. *Let $f \in {}^{\rho}\mathcal{GC}^{\infty}(T, {}^{\rho}\widetilde{\mathbb{R}})$ and $\varphi \in {}^{\rho}\mathcal{GC}^{\infty}(S, T)$ be generalized smooth functions defined on arbitrary domains in ${}^{\rho}\widetilde{\mathbb{R}}$. Let a, $b \in {}^{\rho}\widetilde{\mathbb{R}}$, with $a < b$, such that $[a,b] \subseteq S$, $\varphi(a) < \varphi(b)$ and $[\varphi(a), \varphi(b)] \subseteq T$. Finally, assume that $\varphi([a,b]) \subseteq [\varphi(a), \varphi(b)]$. Then*

$$\int_{\varphi(a)}^{\varphi(b)} f(t)\,\mathrm{d}t = \int_a^b f\left[\varphi(s)\right] \cdot \varphi'(s)\,\mathrm{d}s.$$

8.3 Multidimensional integration

In this section we want to introduce integration of GSF over functionally compact sets with respect to an arbitrary Borel measure μ.

The possibility to achieve results mirroring classical limit theorems for this notion of integral is closely linked to the introduction of the notion of hyperlimit, i.e. of limits of sequences of generalized numbers $a = (a_n)_{n\in\mathbb{N}} : {}^{\rho}\widetilde{\mathbb{N}} \longrightarrow {}^{\sigma}\widetilde{\mathbb{R}}$, where σ and ρ are two gauges and $n \to +\infty$ along generalized natural numbers, i.e. for

$$n \in {}^{\rho}\widetilde{\mathbb{N}} := \left\{ [n_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}} \mid n_\varepsilon \in \mathbb{N} \ \forall\varepsilon \right\}.$$

Mimicking nonstandard analysis, the numbers $n \in {}^{\rho}\widetilde{\mathbb{N}}$ are called *hypernatural* numbers. To glimpse the necessity of studying ${}^{\rho}\widetilde{\mathbb{N}}$, it suffices to note that $\frac{1}{n} < \mathrm{d}\rho^q$ is always false for $n \in \mathbb{N}$ but it can be satisfied for suitable $n \in {}^{\rho}\widetilde{\mathbb{N}}$. Therefore, if $\lim_{n\to+\infty} a_n = 0$ in the classical sense, i.e. for $n \in \mathbb{N}$ and with respect to the sharp topology, then necessarily a_n is infinitesimal for $n \in \mathbb{N}$ sufficiently large. This represents a severe limitation for this notion of limit. It is also clear from the fact that ${}^{\rho}\widetilde{\mathbb{R}}$ with the sharp topology is an ultra-pseudometric space, see e.g. [82], and hence a series in ${}^{\rho}\widetilde{\mathbb{R}}$ converges in the sharp topology if and only if its general term $a_n \to 0$ as $n \to +\infty$, $n \in \mathbb{N}$, in the sharp topology, see [55].

8.3.1 Integration over functionally compact sets

In section 8.2, we already defined a notion of integral over intervals using the notion of primitive. This notion does not help if we want to define the integral $\int_D f$ of a GSF f over a domain $D \subseteq {}^{\rho}\widetilde{\mathbb{R}}^n$ which is more general than an interval. In this case, it is natural to try an ε-wise definition of the type $\int_D f\,\mathrm{d}\mu := \left[\int_{D_\varepsilon} f_\varepsilon\,\mathrm{d}\mu\right] \in {}^{\rho}\widetilde{\mathbb{R}}$, where the net (f_ε) defines the GSF f and the net (D_ε) determines, in some way, the subset $D \subseteq {}^{\rho}\widetilde{\mathbb{R}}^n$, e.g. $D = [D_\varepsilon]$ in case of internal sets. In pursuing this idea, it is important to recall that the internal set (interval) $[0,1] = [[0,1]_{\mathbb{R}}]$ can also be defined by a net of finite sets. Indeed, if $\mathrm{int}(-)$ is the integer part function, and we set

$$\begin{aligned} N_\varepsilon &:= \mathrm{int}\left(\rho_\varepsilon^{-1/\varepsilon}\right) \\ K_\varepsilon &:= \{\rho_\varepsilon^{1/\varepsilon}, 2\rho_\varepsilon^{1/\varepsilon}, \ldots, N_\varepsilon\rho_\varepsilon^{1/\varepsilon}\} \end{aligned} \tag{8.3.1}$$

then the Hausdorff distance $d_{\mathrm{H}}([0,1]_{\mathbb{R}}, K_\varepsilon) = \rho_\varepsilon^{1/\varepsilon}$ and hence $[0,1] = [K_\varepsilon]$ (see also [94, 41]). Consequently, if λ is the Lebesgue measure on $\mathbb{R}$, we have that the generalized number $[\lambda([0,1]_{\mathbb{R}})] = 1$, whereas $[\lambda(K_\varepsilon)] = 0$ and, in general, $\left[\int_{[0,1]_{\mathbb{R}}} f_\varepsilon\,\mathrm{d}\lambda\right] \neq \left[\int_{K_\varepsilon} f_\varepsilon\,\mathrm{d}\lambda\right] = 0$. Therefore, even the definition of integral over an interval cannot be easily accomplished by proceeding ε-wise, i.e. on defining nets.

If we try to understand when such an ε-wise definition can be accomplished, it turns out that we have to consider an enlargement $\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(K_\varepsilon)$ and then take $m \to +\infty$. This is indeed quite natural if one keeps in mind that $[K_\varepsilon] = [L_\varepsilon]$ if and only if the Hausdorff distance $d_{\mathrm{H}}(K_\varepsilon, L_\varepsilon)$ defines a negligible nets, (see [94, 41]). In the following, we say that (K_ε) is a *representative of* $K \Subset_{\mathrm{f}} {}^{\rho}\widetilde{\mathbb{R}}^n$ if $K = [K_\varepsilon]$, (K_ε) is sharply bounded, and $K_\varepsilon \Subset \mathbb{R}^n$ for all ε.

Definition 13. Let μ be a Borel measure on $\mathbb{R}^n$ and let K be a functionally compact subset of ${}^\rho\widetilde{\mathbb{R}}^n$. Then we call K μ-measurable if the limit

$$\mu(K) := \lim_{\substack{m\to\infty\\ m\in\mathbb{N}}} [\mu(\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(K_\varepsilon))] \tag{8.3.2}$$

exists for some representative (K_ε) of K. The limit is taken in the sharp topology on ${}^\rho\widetilde{\mathbb{R}}$, and $\overline{B^{\mathrm{E}}}_r(A) := \{x \in \mathbb{R}^n : d(x, A) \le r\}$.

In the following result, we will prove that this definition satisfies our requirements. We will occasionally integrate generalized functions more general than GSF:

Definition 14. Let $K \Subset_{\mathrm{f}} {}^\rho\widetilde{\mathbb{R}}^n$. Let (Ω_ε) be a net of open subsets of $\mathbb{R}^n$, and (f_ε) be a net of continuous maps $f_\varepsilon\colon \Omega_\varepsilon \longrightarrow \mathbb{R}$. Then we say that

$$(f_\varepsilon) \textit{ defines a generalized integrable map} : K \longrightarrow {}^\rho\widetilde{\mathbb{R}}$$

if

1. $K \subseteq \langle\Omega_\varepsilon\rangle$ and $[f_\varepsilon(x_\varepsilon)] \in {}^\rho\widetilde{\mathbb{R}}$ for all $[x_\varepsilon] \in K$.
2. $\forall (x_\varepsilon), (x'_\varepsilon) \in \mathbb{R}^n_\rho :\ [x_\varepsilon] = [x'_\varepsilon] \in K \ \Rightarrow\ (f_\varepsilon(x_\varepsilon)) \sim_\rho (f_\varepsilon(x'_\varepsilon))$.

If $f \in \mathbf{Set}(K, {}^\rho\widetilde{\mathbb{R}})$ is such that

$$\forall [x_\varepsilon] \in K :\ f([x_\varepsilon]) = [f_\varepsilon(x_\varepsilon)] \tag{8.3.3}$$

we say that $f : K \longrightarrow {}^\rho\widetilde{\mathbb{R}}$ is a *generalized integrable function*.
We will again say that f *is defined by the net* (f_ε) or that the net (f_ε) *represents* f. The set of all these generalized integrable functions will be denoted by ${}^\rho\mathcal{GI}(K, {}^\rho\widetilde{\mathbb{R}})$.

E.g., if $f = [f_\varepsilon(-)]|_K \in {}^\rho\mathcal{GC}^\infty(K, {}^\rho\widetilde{\mathbb{R}})$, then both f and $|f| = [|f_\varepsilon(-)|]|_K$ are integrable on K.

8.4 Some classical theorems for generalized smooth functions

Corollary 15. *Let $f \in {}^\rho\mathcal{GC}^\infty(X, {}^\rho\widetilde{\mathbb{R}})$ be a generalized smooth function defined on the subset $X \subseteq {}^\rho\widetilde{\mathbb{R}}$. Let a, $b \in {}^\rho\widetilde{\mathbb{R}}$, with $a < b$, such that $[a, b] \subseteq X$. Assume that $f(a) < f(b)$. Then*

$$\forall y \in {}^\rho\widetilde{\mathbb{R}} :\ f(a) \le y \le f(b) \ \Rightarrow\ \exists c \in [a, b] :\ y = f(c).$$

Theorem 16. *Let $f \in {}^\rho\mathcal{GC}^\infty(X, {}^\rho\widetilde{\mathbb{R}}^d)$ be a generalized smooth function defined in the sharply open set $X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$. Let a, $b \in {}^\rho\widetilde{\mathbb{R}}^n$ such that $[a, b] \subseteq X$. Then*

1. *If* $n = d = 1$, *then* $\exists c \in [a,b] : \ f(b) - f(a) = (b-a) \cdot f'(c)$.

2. *If* $n = d = 1$, *then* $\exists c \in [a,b] : \ \int_a^b f(t)\,\mathrm{d}t = (b-a) \cdot f(c)$.

3. *If* $d = 1$, *then* $\exists c \in [a,b] : \ f(b) - f(a) = \nabla f(c) \cdot (b-a)$.

4. *Let* $h := b - a$, *then* $f(a+h) - f(a) = \int_0^1 \mathrm{d}f(a + t \cdot h).h\,\mathrm{d}t$.

Lemma 17. *Let* $\emptyset \neq K = [K_\varepsilon] \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ *be an internal set generated by a sharply bounded net* (K_ε) *of compact sets* $K_\varepsilon \Subset \mathbb{R}^n$ *Assume that* $\alpha : K \longrightarrow {}^\rho\widetilde{\mathbb{R}}$ *is a well-defined map given by* $\alpha(x) = [\alpha_\varepsilon(x_\varepsilon)]$ *for all* $x \in K$, *where* $\alpha_\varepsilon : K_\varepsilon \longrightarrow \mathbb{R}$ *are continuous maps (e.g.* $\alpha(x) = |x|$*). Then*

$$\exists m, M \in K \, \forall x \in K : \ \alpha(m) \leq \alpha(x) \leq \alpha(M).$$

Corollary 18. *Let* $f \in {}^\rho\mathcal{GC}^\infty(X, {}^\rho\widetilde{\mathbb{R}})$ *be a generalized smooth function defined in the subset* $X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$. *Let* $\emptyset \neq K = [K_\varepsilon] \subseteq X$ *be an internal set generated by a sharply bounded net* (K_ε) *of compact sets* $K_\varepsilon \Subset \mathbb{R}^n$, *then*

$$\exists m, M \in K \, \forall x \in K : \ f(m) \leq f(x) \leq f(M). \tag{8.4.1}$$

Definition 19. A subset K of ${}^\rho\widetilde{\mathbb{R}}^n$ is called *functionally compact*, denoted by $K \Subset_\mathrm{f} {}^\rho\widetilde{\mathbb{R}}^n$, if there exists a net (K_ε) such that

1. $K = [K_\varepsilon] \subseteq {}^\rho\widetilde{\mathbb{R}}^n$
2. (K_ε) is sharply bounded
3. $\forall \varepsilon \in I : \ K_\varepsilon \Subset \mathbb{R}^n$

If, in addition, $K \subseteq U \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ then we write $K \Subset_\mathrm{f} U$. Finally, we write $[K_\varepsilon] \Subset_\mathrm{f} U$ if 2, 3 and $[K_\varepsilon] \subseteq U$ hold.

Theorem 20. *Let* $f \in {}^\rho\mathcal{GC}^\infty(U, {}^\rho\widetilde{\mathbb{R}})$ *be a generalized smooth function defined in the sharply open set* $U \subseteq {}^\rho\widetilde{\mathbb{R}}^d$. *Let* $a, b \in {}^\rho\widetilde{\mathbb{R}}^d$ *such that the line segment* $[a,b] \subseteq U$, *and set* $h := b - a$. *Then, for all* $n \in \mathbb{N}$ *we have*

1. $\exists \xi \in [a,b] : \ f(a+h) = \sum_{j=0}^n \frac{\mathrm{d}^j f(a)}{j!} \cdot h^j + \frac{\mathrm{d}^{n+1} f(\xi)}{(n+1)!} \cdot h^{n+1}$.

2. $f(a+h) = \sum_{j=0}^n \frac{\mathrm{d}^j f(a)}{j!} \cdot h^j + \frac{1}{n!} \cdot \int_0^1 (1-t)^n \, \mathrm{d}^{n+1} f(a + th) \cdot h^{n+1} \,\mathrm{d}t$.

Moreover, there exists some $R \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ *such that*

$$\forall k \in B_R(0) \, \exists \xi \in [a, a+k] : \ f(a+k) = \sum_{j=0}^n \frac{\mathrm{d}^j f(a)}{j!} \cdot k^j + \frac{\mathrm{d}^{n+1} f(\xi)}{(n+1)!} \cdot k^{n+1} \tag{8.4.2}$$

$$\frac{\mathrm{d}^{n+1} f(\xi)}{(n+1)!} \cdot k^{n+1} = \frac{1}{n!} \cdot \int_0^1 (1-t)^n \, \mathrm{d}^{n+1} f(a + tk) \cdot k^{n+1} \,\mathrm{d}t \approx 0. \tag{8.4.3}$$

Definition 21. 1. Let $U \subseteq {}^\rho\widetilde{\mathbb{R}}$ be a sharp neighborhood of 0 and P, $Q : U \longrightarrow {}^\rho\widetilde{\mathbb{R}}$ be maps defined on U. Then we say that

$$P(u) = o(Q(u)) \quad \text{as } u \to 0$$

if there exists a function $R : U \longrightarrow {}^\rho\widetilde{\mathbb{R}}$ such that

$$\forall u \in U : \ P(u) = R(u) \cdot Q(u) \quad \text{and} \quad \lim_{u \to 0} R(u) = 0,$$

where the limit is taken in the sharp topology.

2. Let x, $y \in {}^\rho\widetilde{\mathbb{R}}$ and k, $j \in \mathbb{R}_{>0}$, then we write $x =_j y$ if there exist representatives (x_ε), (y_ε) of x, y, respectively, such that

$$|x_\varepsilon - y_\varepsilon| = O(\rho_\varepsilon^{\frac{1}{j}}). \tag{8.4.4}$$

We will read $x =_j y$ as *x is equal to y up to $j+1$-th order infinitesimals*. Finally, if $k \in \mathbb{N}_{>0}$, we set $D_{kj} := \left\{ x \in {}^\rho\widetilde{\mathbb{R}} \mid x^{k+1} =_j 0 \right\}$, which is called the *set of k-th order infinitesimals for the equality $=_j$*, and

$$D_{\infty j} := \left\{ x \in {}^\rho\widetilde{\mathbb{R}} \mid \exists k \in \mathbb{N}_{>0} : \ x^{k+1} =_j 0 \right\}$$

which is called the *set of infinitesimals for the equality $=_j$*.

Theorem 22. *Let $f \in {}^\rho\mathcal{GC}^\infty(U, {}^\rho\widetilde{\mathbb{R}})$ be a generalized smooth function defined in the sharply open set $U \subseteq {}^\rho\widetilde{\mathbb{R}}$. Let x, $\delta \in {}^\rho\widetilde{\mathbb{R}}$, with $\delta > 0$ and $[x - \delta, x + \delta] \subseteq U$. Let k, l, $j \in \mathbb{R}_{>0}$. Then*

1. $\forall n \in \mathbb{N} : \ f(x+u) = \sum_{r=0}^{n} \frac{f^{(r)}(x)}{r!} u^r + o(u^n)$ *as* $u \to 0$.
2. *The definition of $x =_j y$ does not depend on the representatives of x, y.*
3. *$=_j$ is an equivalence relation on ${}^\rho\widetilde{\mathbb{R}}$.*
4. *If $x =_j y$ and $l \geq j$, then $x =_l y$.*
5. *If $\forall^0 j \in \mathbb{R}_{>0} : \ x =_j y$, then $x = y$.*
6. *If $x =_j y$ and $z =_j w$ then $x + z =_j y + w$. If x and z are finite, then $x \cdot z =_j y \cdot w$.*
7. $\forall h \in D_{kj} : \ h \approx 0$.
8. $D_{mj} \subseteq D_{kj} \subseteq D_{\infty j}$ *if* $m \leq k$.
9. *D_{kj} is a subring of ${}^\rho\widetilde{\mathbb{R}}$. For all $h \in D_{kj}$ and all finite $x \in {}^\rho\widetilde{\mathbb{R}}$, we have $x \cdot h \in D_{kj}$.*

10. *Let $n \in \mathbb{N}_{>0}$ and assume that j and f satisfy*

$$\forall z \in {}^{\rho}\widetilde{\mathbb{R}}\, \forall \xi \in [x-\delta, x+\delta] : \; z =_j 0 \quad \Longrightarrow \quad z \cdot f^{(n+1)}(\xi) =_j 0. \tag{8.4.5}$$

Then, we have

$$\forall u \in D_{nj} : \; f(x+u) =_j \sum_{r=0}^{n} \frac{f^{(r)}(x)}{r!} u^r.$$

11. *For all $n \in \mathbb{N}_{>0}$ there exist e such that $e \leq j$, and $\forall u \in D_{ne} : \; f(x+u) =_j \sum_{r=0}^{n} \frac{f^{(r)}(x)}{r!} u^r$.*

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