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„New types of suppression of KR-orders in causal set theory “

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Abstract

Causal set theory as a quantum gravity theory describes space-time geometry as partially ordered sets, that is, sets of elements with an order relation between them. Some of these sets can be viewed as space-time points on a manifold, linked causally. This minimal structure does not a priori exclude structures, that cannot approximate space-time geometries that resemble our universe. The set of all causal sets contains many classes of sets, which in the continuum limit are non-manifold like. The largest of such classes are the so-called Kleitman-Rothschild orders. Showing that the path integral of the causal sets suppresses non-physical sets, allows for the universe we observe to manifest out of causal sets. This path integral uses an action analogous to the Einstein-Hilbert-action but expressed in causal set formulation. Some suppression methods of these Kleitman-Rothschild orders have already been shown, as in [1]. In this thesis two additional approaches are studied. The first is by studying the variation of the set size. This approach will lead to a diverging path integral. However, possibilities to recover suppression are discussed. The second is the variation of links while the causal sets are coupled to an Ising interaction. This approach will lead to a suppression of the path integral, whose suppression condition will give a bound on the length scale of the spacing of space-time points in terms of the coupling constant of the Ising model.

Zusammenfassung

Causal Set Theorie ist eine Quantengravitationstheorie, welche Raumzeit als eine Menge ausgestattet mit einer partiellen Ordnung, den sogenannten kausale Sets, beschreibt. Manche dieser kausalen Sets können als Raumzeitpunkte gesehen werden, welche kausal verbunden sind. Diese minimale Struktur schließt a priori nicht Strukturen aus, welche nicht unser Universum beschreiben. Die Menge aller kausalen Sets beinhaltet viele solche Klassen, die keine Mannigfaltigkeit im Kontinuumslimes beschreiben. Die größte solcher Klassen sind die Kleitman-Rothschild Ordnungen. Wenn das Pfadintegral für solche nicht physikalischen Mengen unterdrückt ist, bedeutet das, dass das messbare Universum, sich aus den kausalen Sets entwickeln kann. Das Pfadintegral verwendet eine Wirkung, welche analog zu der Einstein-Hilbert Wirkung ist, jedoch ausgedrückt in der Formulierung der Causal Set Theorie. Eine mögliche Methode zu beweisen, dass Kleitman-Rothschild Ordnungen unterdrückt werden, wurde bereits gezeigt in [1]. In dieser Arbeit werden zwei weitere Methoden analysiert. Die erste ist die Variation von der Mengengröße, die zu einem divergentem Pfandintergal führt. Jedoch werden mögliche Wege zur Wiederherstellung der Unterdrückung besprochen. Die zweite ist die Variation der Ordnungsrelationen, wobei die kausalen Sets an ein Ising-Modell gekoppelt ist. Diese Methode führt zu einer Unterdrückung des Pfadintegrals, dessen Unterdrückungsbedingung eine Grenze für

die Längenskala der Abstände der Einbettung der Punkte des kausalen Sets ist. Diese Grenze ist in Abhängigkeit der Kopplungskonstante.

Contents

1 Introduction

Causal set theory is an approach to discretized space-time, postulating a granular structure for space-time. This could lead to a quantum theory, which, as pointed out in [2], intrinsically answers the questions: "Why does our universe have a Lorentzian signature?" and "Why does space-time have topological and differential structure?". In this context causal set theory will be introduced in chapter 2. Further, as a quantum-gravity theory it might give insight into the singularities appearing in a semi-classical approach to physics. A general overview of this theory was given in this review paper of causal set theory [3].

Causal set theory postulates a new fundamental objects of space-time, which are certain types of partially ordered sets. This new viewpoint leads to non-physical solutions which one must address if the quantum gravity theory is meant to hold. An analogy to this is the exclusion of non-orientable manifolds in general relativity. Similarly, as will be described in section 2, there are many such sets, that do not describe our universe as it is observed. It would be possible to axiomatically exclude these as "non-physical" but this seems arbitrary as it is an a priori choice. Using the causal set formulation one can introduce a path integral for the causal sets themselves, which might give a more physical reason for the suppression of non-physical causal sets [4]. This concept was already proven to work for 2-layer sets [5] and certain layered sets [1].

In the following sections this thesis explores the suppression of so called KR-orders, which are already shown to be suppressed by varying over the amount of links at a fixed set size in the paper [1]. In that paper the suppression was shown by a variation over different relation configurations of large layered orders, but that might not be the only way in that these are suppressed. In this thesis two further types are explored. In chapter 3, the variation over set size is considered and conditions for suppression are given. Further, in chapter 4 an interaction term in the action is added, in this case an Ising interaction between set elements. The aim is to analyze if such an interaction gives a different suppression.

2 Causal set theory

This chapter introduces the foundational notions of causal set theory and the elements relevant to the problem at hand. These include the definition of causal set theory and its mathematical and physical motivation as well as the path integral for the causal sets themselves. The latter derives from the causal set formulation of the Einstein-Hilbert action done by the introduction of a d'Alembertian on causal sets. Lastly, the class of causal sets, that will be analyzed in this thesis, the so-called Kleitman-Rothschild orders, are introduced.

Causal set theory as an approach to quantum gravity was first introduced in [2]. It describes our universe as a fundamentally discontinuous object resembling in approximation a manifold, which is the fundamental object in semi classical general relativity. The basis for this argumentation is the paper [6], which introduces the path topology on strongly causal manifolds. This approach was later refined in [7] to future and past distinguishing manifolds.

2.1 Path topology

This section will summarize the important results for this thesis form [6]. Let a path be a map $\gamma : I \to M$ from an interval I on the real numbers $\mathbb{R}$ to the manifold M. Further its image is called a curve. A path is called continuous if it is continuous with respect to the standard topologies on I and M. Furthermore, a path γ is called future directed and timelike if at every point $t_0 \in I$, γ is continuous and there exists both a connected neighborhood U of t_0 in I and an open neighborhood $\mathcal{U}$ of $p = \gamma(t_0)$ such that

$$t \in U \text{ and } t < t_0 \Rightarrow \gamma(t) \in J^-(p, \mathcal{U}) \tag{2.1}$$

and

$$t \in U \text{ and } t > t_0 \Rightarrow \gamma(t) \in J^+(p, \mathcal{U}). \tag{2.2}$$

Here $J^-(p, \mathcal{U})$ and $J^+(p, \mathcal{U})$ are the causal past and future of the point p in the neighborhood $\mathcal{U}$ with respect to the metric on M. One can further show, that a path is timelike if all sub intervals of I are timelike. Therefore, the notion of a Feynman path is introduced, which glues endpoints and starting points of timelike paths together. This is not necessarily smooth but continuous, as it is likely to develop zig-zags. This is the most general version of a timelike continuous curve and these are also the objects, which define the pathtopology.

Definition 1 (Pathtopology) *The pathtopology $\mathscr{P}$ of M is defined as the finest topology such that the induced topology on every timelike curve γ coincides with the induced topology from the standard topology $\mathcal{M}$ on M.*

This means that if there is a set $E \subset M$ which is $\mathscr{P}$-open for every timelike curve there is an $O \in \mathcal{M}$ with

$$E \cap \gamma = O \cap \gamma. \tag{2.3}$$

The most important part of this topology is that its causal, differential and conformal structure of M is retained. This was shown in this paper [6] for strongly causal space-times and in [7] extended for future and past distinguishing ones. One can show that homeomorphisms with respect to $\mathscr{P}$ are chronology preserving or reversing. Further a homeomorphism with respect to $\mathscr{M}$, which maps null geodesics to null geodesics, is a smooth diffeomorphism. Therefore, it is also a smooth diffeomorphism in $\mathscr{P}$. This means a $\mathscr{P}$-homeomorphism is a conformal diffeomorphism. This shows that almost all information of the manifold is preserved in this topology, except for the conformal scaling factor. Implying, that up to conformal rescaling one can derive the metric and topology from the causal relation on the paths.

2.2 Space-time as a causal set

In analogy to the Feynman paths, the original causal set paper [2] postulates that one can randomly dot the manifold M with space-time events and connect these causaly with respect to the metric, and then forget everything except the causal order derived from the light cone of the metric g. This, by similarity to the pathtopology, should retain the information of the conformal metric $g_{ab}/|\det g|^{1/n}$, the differential structure and the topology. However, the problem of the non-definiteness of the space-time volume arises. It is postulated that if the space-time points are embedded with some constant density then the volume is directly counting the elements of the causal set. This density is not yet specified but needs to be constant for this argumentation to hold. Additionally, it needs to be such that the mean distance between points is small enough, meaning of the order of Planck scale, as its effects are have not yet been possible to measure. This collection of points and their relationship is the causal set. This causal set is then the fundamental object of space-time and not the manifold. The manifold is simply a continuum approximation of the causal set. An interesting observation of this description of the universe is that the approximation manifold is always Lorentzian by the fact of the causal structure embedded into the causal set.

To describe the causal set in a physically meaningful way, partially ordered sets are used. For this the definition of partially ordered set is given:

Definition 2 (Partially ordered set) *A partially ordered set $(C, \prec)$ is a set of elements C and an order relation $\prec$ which satisfies at least the following two conditions for $x, y, z \in C$.*

- *transitive: $x \prec y \prec z \Rightarrow x \prec z$*
- *non-circular: $x \prec y \prec x \neq y$ is not a allowed configuration*

Sometimes, the condition of reflexivity ($x \prec x$) is added, however this thesis does not use this additional condition. Further, the causal set $(C, \prec)$ will be denoted only by C. To make this physically meaningful, one needs to make sure that the idea of volume counting makes sense, meaning that the causal set is finite between any two set elements. Thus there cannot be infinity many elements of the set between two points. This condition is called locally finite.

Definition 3 (Locally finite partially ordered sets) *A partially ordered set is called locally finite if every Alexandrov set $A(x,y)$, i.e. causal diamond, is finite, where the Alexandrov set is given by*

$$A(x,y) = \{z | x \prec z \prec y\}. \tag{2.4}$$

Following the argumentation above, the causal sets are defined as locally finite partially ordered sets.

Viewing the causal set C as fundamental leaves the question of whether one can determine which manifold M this causal set is describing. This means one wants to find an embedding map of the type $f : C \to M$ and then implement physical or reasonable restrictions on f. These are the following three properties:

- The embedding should respect the causal order, i.e. $f(x) \in J^-(f(y))$ if and only if $x < y$.
- Embedded points should be distributed uniformly but randomly with some density of the order of Planck scale.
- The variations in the manifold should be over longer distances than the spacing of the points as it would introduce structure not implied by the causal set.

Essential factors to take into account when defining such maps are existence and uniqueness. To answer the question posed by the first factor, "Does such an embedding exist for all causal sets?", the answer is almost never. What exactly this means is discussed in chapter 2.3. As for the second question, "If it exists is it unique?", it is conjectured to be essentially unique, meaning unique up to variations smaller than the spacing of the points, which is information not contained in the causal set. All causal sets, which do not have a embedding as described above, are called non-manifold like sets. A subset of these are exactly the ones considered in this thesis.

2.3 Kleitman-Rothschild sets

It turns out most sets are not manifold like and non-physical. The predominant ones, as the number of elements n in the sets increases, are the so-called Kleitman Rothschild (KR) sets as proven in [8]. As a subset of all possible causal sets they make up almost all causal sets for large number of elements n, specifically they are $(1+\mathcal{O}(1/n))$ of all sets. To define KR-sets one needs the definition of a layer in a causal set. The i'th layer of a causal set is the set of all points with no elements preciding it after deleting all elements in all layers L_m, where $m < i$. Following the definition of [1], KR orders with n elements are three layer sets with layers L_1, L_2, L_3 with following properties:

- The amount of elements in each layer is $|L_1| = n/4 + o(n) = |L_3|$ and $|L_2| = n/2 + o(n)$
- $x \prec y$, $|A(x,y)| = 0$ and $x \in L_j$ implies $y \in L_{j+1}$
- Every $x \in L_j$ is asymptotically, as $n \to \infty$, connected to half of the elements in both L_{j-1} and L_{j+1}
- $\forall x \in L_1$ and $y \in L_3$, $x \prec y$

In Figure 1 a schematic depiction of a KR-order is shown to illustrate the limits given by the definitions above. It shows a 20 element set with the three layers separated to visualize the relations, which uniformly go into one direction, either up or down.

It was shown in [9] that KR-orders are already dominant at set size of the order 100. If one assumes a mean distance of embedded space-time points to be in the order of the Planck scale the universe has multiple orders of magnitude higher set elements than 100.

These sets are clearly causal sets as the number of links is limited by the requirements on the set, but equally these are not embeddable into a manifold as they always consist of

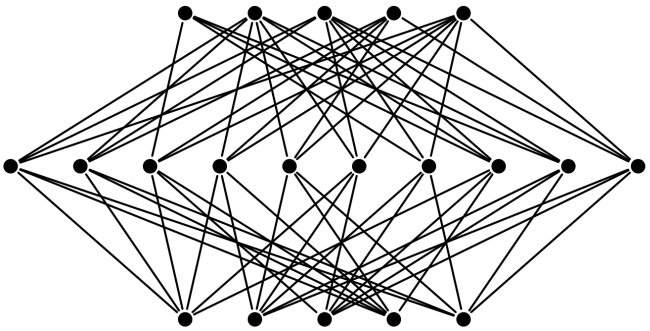


Figure 1: KR-order with $n = 20$

three layers. A priori one does not want to assume that these sets are excluded intrinsically, as this would again assume some manifold like structure to be the fundamental structure of the universe. A much more natural approach to this problem is that one can think of some mechanism on these causal sets that describes how the causal set of our universe has to behave. This is usually done by finding the action that describes the specific system. In the case of causal sets this is the Benincasa–Dowker–Glaser (BDG) action, which will be derived later and is analogous to the Einstein-Hilbbert action in the manifold picture. For this, one needs an expression of the Ricci scalar in terms of the causal set. Here, the Ricci scalar is a notion of a manifold not a causal set, therefore one needs to assume the existence of an embedding to derive this expression. Afterwards, one can assume the inverse argument, that the Ricci scalar is a result of the expression when applied to a causal set, which is embeddable.

2.4 The d'Alembertian operator

In the paper [10] a d'Alembetian operator in the context of preserving the Lorentz symmetry was introduced. A priori one cannot assume that the manifold generated by a causal set will be Lorentz invariant. As discussed above, the manifold is an approximation of the causal set, which looks like space-time points on a manifold, connected causally. For example, if these space-time points are spaced uniformly on a grid, it is clear that this grid will not be invariant under Lorentz boost, as it admits different properties when boosted in different directions. However, by having the space-time points distributed according to a

Poisson distribution, it was shown in [11] that this will on average persevere the Lorentz invariance of the manifold. To be more specific, Poisson distributed means the probability $P(m.V)$ that m elements are found in a volume V is given by:

$$P(m,V) = \frac{(\rho V)^m}{m!} e^{-\rho V}. \tag{2.5}$$

Here, ρ is the density of the embedding as required by the embedding map. The inverse process, deriving a causal set from a given manifold with such a distribution, is called *sprinkling*. Further, in causal set theory it is assumed that a space-time may correspond only to causal sets that look embedded like a sprinkling.

2.4.1 2D d'Alembertian

Looking at the massless wave equation $\Box\phi = 0$, one merely needs to find a representation of $\Box$ on the causal set induced by the manifold structure. If one assumes the $\Box$ operates linearly, then the operator should take the form of a matrix B_{xy}, where the indices reach over all elements of the causal set C. One can immediately restrict B_{xy} by causality by setting $B_{xy} = 0$ for incompatible elements, which means that neither $x \prec y$ or $x \succ y$ is true. The biggest issue with the derivation of the d'Alembertian is that due to the fact that one wants to preserve Lorentz symmetry, the notion of closest neighbor will mostly include non-local partners along the light cone.

Green's function approach The 2D free wave operator has a retarded Green's function, which is proportional to the step function $\Theta(t-x)$. This has a simple representation in C. As the Green's function is the inverse of the differential operator, one can define a matrix B^{-1} as a representation in causal set theory of the Green's function. In this case, the prescription that is suited best works as follows: picking a region of Minkowski space $\Omega \in \mathbb{M}^2$ and sprinkling N points $x_i, i = 1,\ldots,N$ into it. One can then apply this to some discretized scalar fields $\phi_i = \phi(x_i)$ and $\psi_i = \psi(x_i)$ on the causal set C created by the sprinkled in dots. Then, one builds a $N \times N$ matrix by transposing and rescaling the linking matrix of C, called C_{xy}, which is 1 for all $x \prec y$ and $|A(x,y)| = 0$, else it is 0. This is then the Green's function represented in the formulation of causal sets. This will then be symmetrized to be invertable and subsequently inverted, as the Green's function is the

inverse of the differential operator in the sense of distributions. Then one compares the continuum limit to the application of the matrix to these scalar fields.

$$B(\phi,\psi)=\sum_{ij}B_{ij}\phi_i\psi_j \to \int \mathrm{d}^2x\mathrm{d}^2y\phi(x)\Box\psi(y). \tag{2.6}$$

This holds true for test functions that vanish on first order and vary only weakly on the boundary of Ω. The reason behind this dependence on the boundary is, that one does not know what this expression exactly approximates to. For the above stated case one could not distinguish between, for example $\int \mathrm{d}^2x\mathrm{d}^2y\phi(x)\Box\psi(y)$ and $\int \mathrm{d}^2x\mathrm{d}^2y\nabla\phi(x)\nabla\psi(y)$. Further, this approach is not frame independent, even though the Green's function is. The reason for this is that due to Lorentz invariance a region needs to support both longer and and shorter light waves and a region can only support waves with longer wavelengths than the spacing of the grid. Lastly, this approach is dependent on the form of the retarded Green's function and this has not always a good representation in the causal set.

Causal link approach The B from the Green's function approach does give roughly as many positive as negative entries as well as the highest amplitude around the light cone, meaning it has the property of Lorentzian proper distance. This gives hope that such a cancellation/suppression of non-local links is taking place. One can make this a bit more precise: Looking at a minimal causal diamond in Figure 2 the combination $\phi(a)+\phi(d)-\phi(b)-\phi(c)$ will converge, under suitable normalization and slowly varying fields, to $-\Box\phi$ as the square shrinks. Boosting this will give a second rectangle in Figure

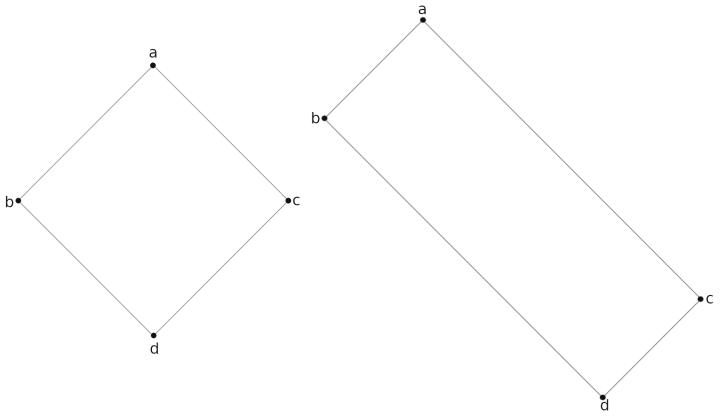


Figure 2: Minimal Alexandroff diamond. Recreated from a figure in [10]

2, but in this the longer legs will make sure that these boosted diamonds contribute less, as $\phi(a)-\phi(b)+\phi(d)-\phi(c)$ will be considerably smaller in this boosted frame. This might conserve the Lorentz invariance once all the boosts of the original square are considered. In one dimension, one can simply use a "mid point method" to describe the d'Alembert operator, in analogy to taking the distance between b and c to be zero, so $\phi(a)-2\phi(b)+\phi(d)$. One can plug in causally related vertices $a \prec b \prec d$, such that $|A(a,b)|=|A(b,d)|=0$. A similar speculative d'Alembert in 2D can be found by summing over the ancestors of some point x at which the d'Alembertian acts upon

$$B(\phi)=\frac{4}{l^2}\Big(-\frac{1}{2}\phi(x)+\Big(\sum_{L_1}-2\sum_{L_2}+\sum_{L_3})\Big)\phi(y)\Big). \tag{2.7}$$

L_i denotes the level of the ancestor, concretely a point y is in L_i of x if $|A(y,x)|=i-1$. Further l is the length-scale of the sprinkling. This is a retarded description of the wave operator. Explicitly in matrix description the B matrix for $x \succ y$ and $x=y$ is

$$\frac{l^2}{4}B_{xy}=\begin{cases}-1/2 & \text{for } x=y\\ 1,-2,1 & \text{depending on the level of the ancestor}\\ 0 & else\end{cases}. \tag{2.8}$$

Of course this is dependent on the sprinkling. Picking a fixed point x of the sprinkling, one can apply B to ϕ at x. This is more or less random as there are many sprinklings of the same manifold. However, the mean does converge to the $\Box$ operator as $l \to 0$.

$$\mathbb{E}\sum_y B_{xy}\phi_y \to \Box\phi(x) \tag{2.9}$$

Here, $\mathbb{E}$ describes the expectation with respect to the Poisson distribution. This is a much better description, it is intrinsically Lorentz invariant and non-local only at Planck scales. As the average over all sprinklings of B acts linearly one notices that $\mathbb{E}(B\phi)=(\mathbb{E}B)\phi =: \bar{B}\phi$. $\bar{B}$ is then the continuous limit integral kernel, which acts as $\int \bar{B}(x-y)\phi(y)\mathrm{d}^2y$. This kernel for the equation 2.7 has the form

$$\bar{B}(x-y)=\frac{4}{l^4}p(\xi)e^{-\xi}-\frac{2}{l^2}\delta^{(2)}(x-y) \tag{2.10}$$

where $p(\xi) = 1 - 2\xi + \frac{1}{2}\xi^2$, $\xi = v/l^2$ and $v = \frac{1}{2}||x-y||^2$. The norm given here is $||x-y|| = |(x-y)(x-y)|$, and therefore it is Lorentz invariant. More generally, one can write this with a general parameter K

$$\bar{B}(x-y) = 4K^2 p(\xi) e^{-\xi} - 2K\delta^{(2)}(x-y). \tag{2.11}$$

This helps in generalizing it higher dimensions. Additionally one can, if necessary, add a dampening function as it is not clear that fluctuations do not start growing out of proportion.

2.4.2 Higher dimensions

It is straight forward to expand this idea to higher dimensions by using more "support" vertices as in using 4 levels with coefficients $1, -3, 3, -1$ associated with the L_is for 4 dimensions. However, one can also start from an integral kernel, finding a suitable polynomial $p(\xi)$, where $\xi = Kv(x,y)$ and $v(x,y)$ is the volume associated to the Alexandrov set $A(x,y)$. Any such function can be expressed by a differential operator depending on K, $\mathcal{O}(\partial_K)$ acting on $\exp(-\xi)$. This is equivalent to

$$\int p(\xi)\exp(-\xi)dx = \int \mathcal{O}\exp(-\xi)dx = \mathcal{O}\int \exp(-\xi)dx \equiv \mathcal{O}J. \tag{2.12}$$

In 2D, one wants ϕ to be polynomial of x^μ of degree two or less, meaning that up to negligible parts the solution is just a linear combination of terms of the form $1/K^n$ and $\log K/K^n$. Further, the only monomials that contribute to the logarithmic part are the squared and the constant ones. All terms except the squared ones vanish for large K in the d'Alemabertian. Now, one needs to find a form of the operator where it removes the logarithms as these would lead to infrared dependencies and annihilates all terms $1/K^n$. Such an operator is given by

$$\mathcal{O} = \frac{1}{2}(H+1)(H+2) \quad \text{where} \quad H = K\frac{\partial}{\partial K}. \tag{2.13}$$

This satisfies these requirements and converts $\log K/K$ to $1/K$, which can be canceled by adding a delta function in complete analogy with the version from equation 2.11. This concept was generalized in the paper [12] in which a general form of the differential operator was given for even dimensions as

$$\mathcal{O}_{2n} = \frac{(H+2)(H+4)\cdots(H+2n+2)}{2^{n+1}(n+1)!}. \tag{2.14}$$

For odd dimensions, one can use the one from the dimensions mentioned above, so $\mathcal{O}_{2n+1} = \mathcal{O}_{2n}$. Lastly, in the paper [4] it was mentioned that a general form of the d'Alembertion operator can be given as

$$B^{(d)}\phi(x) = \frac{1}{l^2}\left(\alpha_d\phi(x) + \beta_d \sum_{i=1}^{n_d} \lambda_i^{(d)} \sum_{y\in L_i} \phi(y)\right) \tag{2.15}$$

where α_d, β_d, n_d and $\lambda_i^{(d)}$ are dimensional dependent constants.

2.5 Benincasa–Dowker–Glaser action

One need not or should not assume the structure of the causal set that describes our universe. A priori this structure is not defined. However, one can find an action that describes the dynamics of causal sets themselves. This action is in essence the Einstein-Hilbert action expressed in terms of causal sets, this was shown in [4]. The Einstein-Hilbert action is up to constants the integral over the Ricci scalar

$$S_{EH} = \int R\sqrt{-g}\mathrm{d}^d x. \tag{2.16}$$

It was shown in [13], that if the causal set can be embedded into a curved manifold with metric g as defined in chapter 2.2 with some density ρ, then the d'Alembertian operator defined in 2.15 has as a classical limit not only the box operator $\Box$ as is required, but also attains an contribution of the Ricci scalar.

$$\lim_{l\to 0} \bar{B}^{(d)}\phi(x) = \Box^{(d)}\phi - \frac{1}{2}R^{(d)}(x)\phi(x) \tag{2.17}$$

It is easy to see that the description of R in terms of causal sets can be found by setting $\phi(x)$ constant, say 1, as the box operator will equate to 0.

$$R^{(d)}(x) = -\frac{2}{l^2}\left(\alpha_d + \beta_d \sum_{i=1}^{n_d} \lambda_i^{(d)} N_i(x)\right) \tag{2.18}$$

Where $N_i(x)$ denotes the cardinality of the i'th layer toward the past of x. As was shown in [4]. plugging this into the Einstein-Hilbert action, one obtains an action for the causal sets themselves. The integral in this discrete case becomes the sum over all elements in the causal set, here implicitly assumed to be embedded into a manifold, meaning it is

equivalent to the sum over all space-time points. The resulting action is known as the Benincasa–Dowker–Glaser (BDG) action on a causal set C.

$$\frac{1}{\hbar}S^{(d)}(C) = \mu\left(n + \sum_{k=0}^{k_{max}} \lambda_k N_k\right). \tag{2.19}$$

Here, N_k is not dependent on an element of the causal set, but counts the number of pairs of elements such that the Alexandrov set of them has cardinality of k. Further, due to restructuring of the constants μ now contains information of the length scale of the sprinkling l. With an action, one can also define a path integral, which should in this case give the most probable structure of our causal set or rather universe. The partition function of this path integral is given as the sum over all causal sets of the exponentiated BDG-action. As the embedding will always be Lorentzian, the Lorentzian path integral is used and the it will sum over all causal sets.

$$Z = \sum_{C\in\mathcal{C}} \exp\left(\frac{i}{\hbar}S(C)\right) = \sum_{C\in\mathcal{C}} \exp\left(i\mu\left(n + \sum_{k=0}^{k_{max}} \lambda_k N_k\right)\right) \tag{2.20}$$

Here, $\mathcal{C}$ denotes the set of all causal sets.

This concludes the introduction into causal set theory. To recap, an intuition for the mathematical validity was given through the path topology, which led to the description of space-time through locally finite partially ordered sets, also called causal sets. The KR-orders were introduced as the dominant subset of all sets for large set size n, being $1 + \mathcal{O}(1/n)$ of all sets. Further these do not describe our universe as we can measure it, therefore are non-physical. Lastly, the path integral of the causal sets was motivated and introduced. It will be the main tool of the following analysis as this describes the way our universe behaves.

3 Variation over the set size

In this section, the path integral sum restricted to the KR-orders is analyzed. Firstly, the problem will be set up by arguing that for KR-orders only links, meaning direct relations of elements, will be of significance in the action of the causal sets, as well as brought to a general form in which the lemma of Van der Corput and the lemma of the non-stationary phase can apply. These lemmas will then be used to show that for the lemma of Van der Corput the path integral is not suppressed. Lastly, it will be argued that under the assumption of a cosmological constant that depends on the size of the universe, the lemma of the non-stationary phase might give a scenario under which the path integral can be suppressed. This argumentation will not be the most general solution of the problem.

3.1 General setup of the problem

As described in the paper of Kleitman and Rothschild [8], a KR-order is a set, which at large numbers of elements n in the set takes the following form: It is a 3-layer set, where the layers will be denoted by L_i with $i = 1,2,3$. The layers L_1 and L_3 will have a quarter of all the elements in them, while L_2 has half of them. Further, there is a restriction on how the sub sets are connected. For large n all elements of L_1 and L_3 are connected to $n/4$ elements of L_2, similarly every element in L_2 is connected to $n/8$ elements of L_1 and L_3 respective. The relations are such that the causal flow starts at the L_1 layer and ends in L_3. As was shown in the KR paper, these are the most dominant sets, in fact the larger the size of the set, the larger is the percentage of KR-orders appearing. The probability, to find these sets if picked at random out of the set of all partially ordered sets, approaches 1 as $n \to \infty$.

For any n large enough, the first thing one must consider in the partition function 2.20 are the N_k for $k > 0$. These are all of the relations one can find going from an element in L_1 to an element in L_3. Because KR-sets are three layer sets, k will equivalently denote the amount of different paths between two specific elements in L_1 and L_3. It is possible to find a upper bound for any N_k for $k > 0$ and it will be shown that this upper bound goes to zero as $n \to \infty$. The process, which leads to this upper bound, first suggested in [5], starts by picking a $x \in L_1$ and connect it to $n/4$ elements in L_2. Now, by picking an element $y \in L_3$, one can check how many different possibilities there are to connect this point in such a way that there are no connections, one connection, etc. For zero connections it is 1 possibility, for 1 connection one can take any one of the already connected elements of L_2 and connect it to that to see how many other possibilities there are to connect the remaining ones. This means there are $(n/4)^2$ possibilities. That means for each k one can

find the probability $P_{N_k}(n,k)$ that a pair of points of a set of size n has a contribution to N_k, by norming it with the norm κ.

$$P_{N_k}(n,k) \leq \frac{1}{\kappa}\binom{n/4}{k}\binom{n/4}{n/4-k} = \left(\sum_m^{n/4}\binom{n/4}{m}^2\right)^{-1}\binom{n/4}{k}^2 = \binom{n/2}{n/4}^{-1}\binom{n/4}{k}^2. \tag{3.1}$$

This resembles a bell-like curve with a moving peak, which is at $n/8$. The large n limit of this probability for a fixed k converges exponentially to 0.

$$P_{N_k}(n,k) \leq \binom{n/2}{n/4}^{-1}\binom{n/4}{k}^2 \to 2^{-n/2}\frac{(n/4)^k}{k!} \to 0 \quad \text{as} \quad n \to \infty. \tag{3.2}$$

As n becomes larger, the peak moves to higher values of k, meaning given a large enough set all N_k with $k > 0$, which appear in the partition function go to zero as the peak moves further to higher k's, which are not part of the action. This behavior is illustrated in Figure 3 where $P_{N_k}(n,k)$ is plotted for different set sizes. Using this the partition function on the

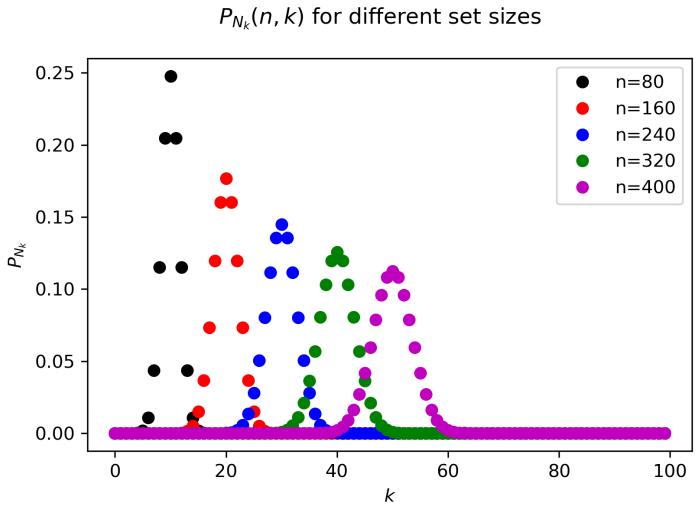


Figure 3: Probability of a pair $x \in L_1$ and $y \in L_3$ to have a contribution to N_k. This is the analytic solution from equation 3.1

set of all KR-orders, here called $\mathcal{K}$, can be written as

$$Z = \sum_{\mathcal{K}} \exp(i\mu[n + \lambda_0 N_0]). \tag{3.3}$$

It follows that the action of the KR-orders is dominated only by the amount of direct relations between elements of the set. These direct relations are called links and therefore the action is called the linking action.

Additionally, the behavior of the movement of the peak is studied by a numerical simulation. In this simulation KR-orders are randomly generated, counting the amount of point pairs $x \in L_1$ and $y \in L_3$ with k relations to each other and then averaged over the KR-orders created. To further elaborate, the KR-orders are generated by taking n elements and dividing them into two lists of $n/4$ elements and one with $n/2$ elements. Afterwards, the links between each element are established by assigning a link between them with a probability of 50%, which will lead to KR-orders most of the time. In Figure 4 the values of N_k for different set sizes, which were calculated with the simulation, are shown. This demonstrates that N_k's with small k tend to 0 as the set size grows, meaning that the argumentation for the analytic probability is accurate.

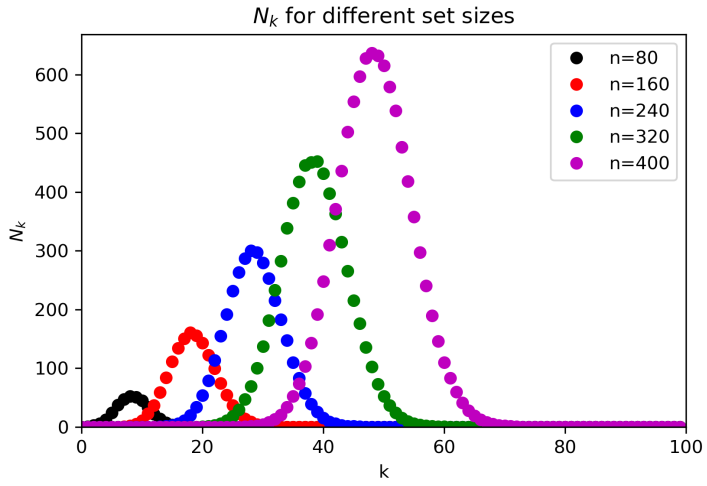


Figure 4: Amount of N_k for different set sizes obtained by numerical simulations.

An interesting result to see is that the peak is slightly lower then $n/8$. This is due to the fact that the above consideration does not include the restriction that points in L_2 have limited links to other layers as well as the condition that every element in L_1 and L_3 have to be related to each other.

N_0 on KR-sets are more or less fixed for large n. As described above, KR-orders have a fixed amount of links that is $n^2/8+o(n^2)$. Therefore,

$$Z=\sum_{\mathcal{K}}\exp\big(i\mu[n+\lambda_0 n^2/8]\big). \tag{3.4}$$

Because, the exponential is independent of the structure of the KR-order it follows, that the sum over one specific n is the same as counting the number of KR-orders with n elements, $\mathcal{K}_n$. Further, for large n the function in the exponential may be taken to be $i\mu\lambda_0\frac{n^2}{8}$. This gives the final form of the partition function to be analyzed:

$$Z=\sum_n|\mathcal{K}_n|\exp\left(i\mu\lambda_0\frac{n^2}{8}\right). \tag{3.5}$$

The quantity of $|\mathcal{K}_n|$ is given in the paper [8] and it was shown that $|\mathcal{K}_n|\approx 2^{n^2/4}$. As a first step one can arrange all terms to have the same exponential base, defining $a=4/\ln(2)$, $2^{n^2/4}=e^{n^2/a}$. This grows exponentially and is not suppressed, but one could assume a finite amount of elements and vary over some range I of n. In the limit for many n one can use an integral instead of a sum as an approximation.

$$Z=\int_{I(n)}\mathrm{d}n e^{n^2/a}\exp\left(i\mu\lambda_0\frac{n^2}{8}\right) \tag{3.6}$$

Making a change of variable and setting

$$x=\frac{\sqrt{\pi a}}{2}\mathrm{erfi}\left(\frac{n}{\sqrt{a}}\right), \tag{3.7}$$

the integral becomes

$$Z=\int_{x(I(n))}\exp\left(i\mu\lambda_0\frac{n^2}{8}\right)dx=\int_{x(I(n))}\exp\left(i\mu\lambda_0\frac{a\ \mathrm{erfi}^{-1}\left(\frac{2x}{\sqrt{a\pi}}\right)^2}{8}\right)dx. \tag{3.8}$$

This allows us to make use of the lemma of Van der Corput, which states the following:

Lemma 1 (Leamma of Van der Corput (1D)) *The lemma of Van der Corput gives a condition when an one dimensional integral on an interval $J \subseteq \mathbb{R}$ of the form*

$$I(\lambda) = \int_J e^{i\lambda\phi(x)} dx \tag{3.9}$$

is bounded. Where $\lambda \in \mathbb{R}$ and $\phi(x) : J \to \mathbb{R}$ is a smooth function and is called the phase. The phase further has to satisfy the following bound for the k-th derivative: $|\phi^{(k)}(x)| \geq 1$ for all $x \in J$. Then, for all $\lambda > 0$ and $k \leq 2$ the bound on the integral is given as

$$|I(\lambda)| < d_k \frac{1}{\lambda^k} \tag{3.10}$$

where d_k is a constant dependent on the degree of the derivative. For $k = 1$, an additional constraint has to be added for the bound to remain true, which is that the first derivative of the phase has to be monotone. [14]

To make analysis easier let

$$\phi(x) = \frac{a}{8} \text{erfi}^{-1}\left(\frac{2x}{\sqrt{a\pi}}\right)^2, \qquad \varepsilon = \mu\lambda_0. \tag{3.11}$$

Therefore,

$$Z = \int_{x(I(n))} \exp(i\varepsilon\phi(x)) dx. \tag{3.12}$$

A problem here arises due to the fact that ϕ is proportional to the inverse imaginary error function erfi^{-1}. The imaginary error function is defined by the relation $\text{erfi}(x) = -i\text{erf}(ix)$. The inverse of this function does not have an analytic description on all of $\mathbb{C}$. However, one can find an inverse of each function value along $\mathbb{R}$ and can at least analyze the behavior of the function, which is the only important part for the Van der Corput method. To start, one can plot or evaluate $\text{erfi}(x)$ at all points. In Figure 5 a interval of this function is depicted. Taking the inverse is then done numerically as also seen in the same figure.

In this figure it is not yet easily seen if the inverse may plateau at $x \to \infty$. However, the fact that the erfi function is bijective means it cannot plateau and does indeed diverge to infinity at $x \to \infty$. The numerical derivative is then taken 4 times to see how the derivative behaves at large x values. This is seen in Figure 6 and all of them go to to 0 as x goes to infinity, meaning the bound on the integral in the lemma of Van der Corput is infinity. We thus have to conclude that the KR-orders are not suppressed when varying over n.

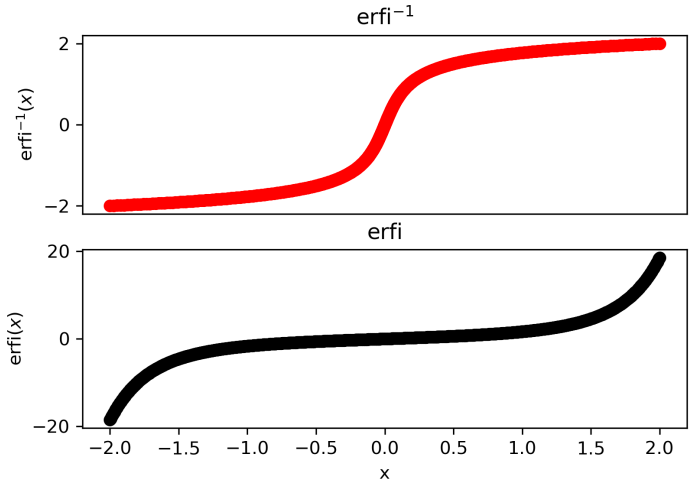


Figure 5: Schematic plot of erfi and its numerical inverse erfi^{-1}

3.2 Principle of non-stationary phase

A possible way to recover boundedness is to introduce a physically motivated functional measure in analogy to [15]. Meaning a function such that the measure of the path integral is not just dx but $b(x)$dx or more specifically $\tilde{b}(n)$dn. This is not the most general such function as the functional measure is a function of the causal sets and not of the set size. Further, as explained in [15] a non-trivial functional measure is equivalent to additional terms tn the Lagrangian as it can be incorporated into it. The path integral under the assumption that such a measure function exists becomes

$$I = \int_{x(n)} b(x) \exp(i\varepsilon\phi(x))dx. \tag{3.13}$$

If the function now satisfies the properties stated by the principle of non-stationary phase, which is introduced below, the integral is suppressed as long as $\varepsilon > 0$, where ε is an one dimensional parameter.

Lemma 2 (Principle of non-stationary phase) *Let b be a bounded and smooth, $C_b^\infty(\mathbb{R})$, function and ϕ a smooth function such that its derivative is non-zero on the support of b. Then the integral above is suppressed by $I(\lambda) = \mathcal{O}(\lambda^{-N})$ for all $N \geq 0$. [14]*

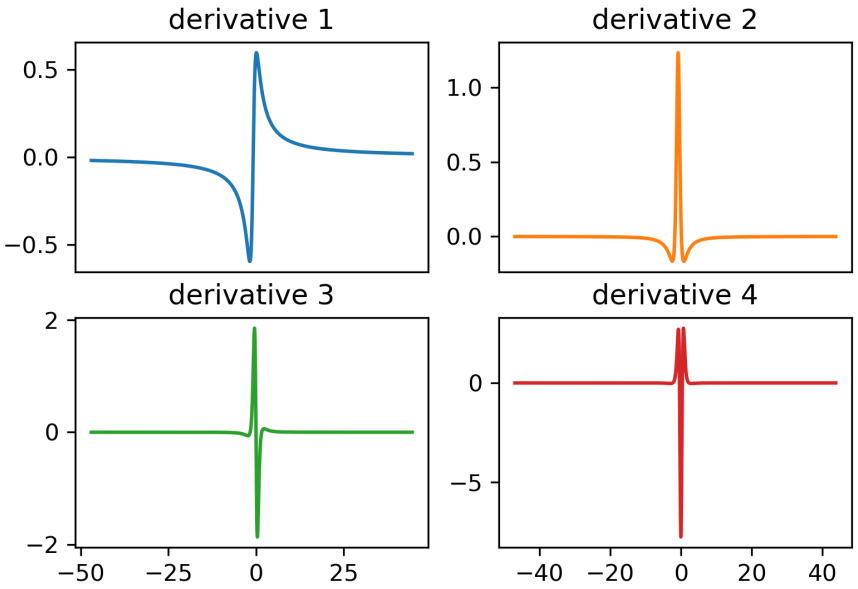


Figure 6: Plot of four derivatives of ϕ

This description is true for a lot of possible functions, but to pick one of them, one ought to resort to physical motivations. This is a choice and not the most general solution of this problem. A possible way to find such a function is to treat it as a weight to the free path integral, which takes into account the amount of sets with a certain size appearing over a given evolution of a causal set. Therefore the question asked is: "Given all 'time slices' of the evolution, what is the size of the causal set?". Using how many times the set has taken a certain size as a weight. To simplify the analysis through such a function, a limited but not fixed set size will be assumed. The biggest issue with a set sizes that could become larger and larger in such a scenario, is that it would leave the weight at 1 for all sets of such an evolution. This would mean the new Lagrangian that follows from the function b would be the free Lagrangian.

Here a formulation of how one can derive such a function from an evolution of set will be introduced, with this one can argue that KR-orders undergo a specific evolution, namely due to fluctuations of the cosmological constant as discussed in Chapter 3.3. However, before a specific example of such function is inspected more closely, a more general prescription can be formulated abstractly. Assume some specific universe set U and its

evolution, especially the function $E : \mathbb{N} \to \mathbb{N}$, which describes the amount of elements in the set U at a given time step t of the evolution. From this, one can define an countable infinite vector Y, such that for each time step $t \in \mathbb{N}$

$$Y_i = \begin{cases} 1 & i = E(t) \\ 0 & \text{else} \end{cases}. \tag{3.14}$$

Meaning this vector marks how many elements the set has at each time step. The b function is then the sum of all these Y for all universes. This prescription does not need to assume that the set is intrinsically embeddable. As a side note, this would not work with the sequential growth model [16], because b for all universes are a string of ones.

To motivate such an evolution the everpresent Λ will be introduced. It is postulated in [17] that causal sets that emit space-time like structure fluctuate in the cosmological constant Λ due to the Poission distribution of the embedding. Assuming that this fluctuation manifests itself as a fluctuation of the set size one can apply this prescription to all causal sets, particularly KR-orders.

3.3 Everpresent Λ

This explanation will follow the paper [17]. The derivation of this concept was done heuristically but applied to three models in the same paper, which shows that these models do coincide with some phenomenological observations. Due to the Lorentz invariance preserving nature of the embedding, the number of set elements in a given volume V is subject to Poisson fluctuations, which in Planck units are of the order of $\sqrt{V}$, $N \sim V \pm \sqrt{V}$. Further, the part of the Einstein-Hilbert-action that describes the variation the cosmological constant Λ is of the form:

$$-\int \Lambda \mathrm{d}V \tag{3.15}$$

This suggest that the cosmological constant Λ and the volume V are canonical conjugates, therefore the Heisenberg uncertainty principle suggests that

$$\Delta\Lambda \cdot \Delta V \sim 1. \tag{3.16}$$

Using further, due to the volume being proportional to the amount of elements in the causal set, $V = NV_{planck}$, the identification of

$$\Delta V \sim \Delta N \sim \sqrt{V} \sim \sqrt{N} \tag{3.17}$$

leads to a variation of Λ proportional to the square root of the size of the causal set

$$\Delta\Lambda \sim \frac{1}{\sqrt{N}}. \tag{3.18}$$

Therefore, one assumes that the varying cosmological constant is a result of the variation of the causal set size. The KR-orders are always limited to 3-layers, therefore one can assume that this variation of the set size actually only widens the layers almost equally. The last question left is what shape does the b function have exactly. First, one can assume an "equilibrium" state of the set, where the cosmological constant takes the average value 0. These fluctuations, depending on the model, seem to be erratic fluctuations as seen in [17]. Given enough time, one would find a bounded function as described above with the expectation value at the "rest" amount of elements of the KR-order. This distribution likely takes the form of a normal distribution, as it is a continuous limit of the Poisson distribution, but this is not a bounded function. However, given the discrete nature of the fluctuation there will probably be some cutoff size that is never crossed. On one side this is definitely 0, on the other it is some size $N_{\Lambda max}$. A possible solution to such a function b is:

$$b(n) = \begin{cases} \exp\left(-\frac{1}{1-(\frac{1}{\Delta N}(n-N))^2}\right), & n \in [N-\Delta N, N+\Delta N] \\ 0, & \text{else} \end{cases} \tag{3.19}$$

Here, N is the expectation value and ΔN gives the span of set sizes allowed by the evolution. Of course, one would need to make sure that b is 0 at $n = 0$. If this is true then it suppresses KR-orders with a size N but diverges in other set combinations. It is not clear to what modification to the Lagrangian this would correspond to but it shows one approach to recover boundedness of the integral.

In this chapter the approach for the suppression of KR-orders through set size variation was discussed. Firstly, the integral was brought into a more manageable form. Starting from this form the van der Corput lemma was used to argue that the KR-orders were not suppressed in this scheme. Then the principle of non-stationary phase was used to argue a possible way to recover boundedness of the path integral. For this the functional measure was assumed to be non-trivial. This combined with the principle of non-stationary phase

led to conditions on the measure function. Lastly, a physically motivated guess for such a function was presented through the use of the idea of a fluctuating cosmological constant.

4 The Ising model

In this section, the argumentation of the suppression through variation of links as shown in [1] will be reviewed as it builds the basis of the later suppression argumentation. In this analysis an interaction term will be introduced to the action of the causal set, and analogously shown to be suppressed. This interaction term has the form of an Ising model as introduced in [18]. It will be then shown that this is also suppressed. However, the bound will be weaker due to approximations done in deriving that limit.

4.1 Suppression of free KR-orders

In this section the results of the paper [1] on suppression of KR-orders are reviewed. The same approach to suppression is then used with the additional Ising interaction term. It is necessary to consider larger sets of causal sets to show suppression through variation of links. Form here on out, naturally labeled sets are considered. These are orders $c \in \Omega_n$ such that $\forall e_r, e_s \in c, e_r \prec e_s \Rightarrow r < s$, where Ω_n denotes the finite n-element orders. To make it easier to define certain sets of causal sets used in the analysis, the k-layer map is introduced.

Definition 4 (k-layer map) *A k-layer map assigns all elements e_n in a k-layer set its layer number*

$$\xi : \{e_1, e_2 \dots e_n\} \to \{1, \dots k\}. \tag{4.1}$$

This allows to formulate sets in terms of layer relations. The largest and simplest of these are the so called k-layer orders.

Definition 5 (k-layer orders) *A k-layer order c is a partially ordered set in, which to every element a k-layer map can be assigned, such that for all elements $e_r \in c$:*

- $e_r \prec e_s \Rightarrow \xi(e_r) < \xi(e_s)$
- $\xi(e_s) > \xi(e_r) + 1 \Rightarrow e_r \prec e_s$

holds. Further, the set of all k-layer orders is denoted by $\mathcal{D}_n^k$.

To show suppression, two new sets need to be defined, these are the k-quasi-layer (k-QL) orders and the k-pseudo-quasi-layer (k-PQL). These are sets where only adjacent layers are connected to each other.

Definition 6 (k-pseudo-quasi-layer) *A k-pseudo-quasi-layer order $c \in \Omega_n$ is a partially ordered set on which a k-layer map θ can be defined, such that for $e_r, e_s \in c$ if $e_r \prec e_s$ and $A(e_r, e_s) = \emptyset$ then $\theta(e_s) = \theta(e_r) + 1$. The set of all k-pseudo-quasi-layer orders are denoted by $\mathcal{P}_n^k$.*

A necessary definition for k-QL are irreducible causally disconnected subsets. These are called causally disconnected if they are subsets $\hat{c} \subset c$, such that there exist no relations between its elements and all other elements of the set not in $\hat{c}$, so its compliment. If this subset further does not contain any smaller non-trivial causally disconnected subsets, then it is called an irreducible sub set.

Definition 7 (k-quasi-layer) *A k-quasi-layer order $c \in \Omega_n$ are k-pseudo-quasi-layer orders with a k-layer map η, in which additionally for every causally disconnected subset $\hat{c}, \exists e_r \in \hat{c}$, such that $\eta(e_r) = 1$. These sets are denoted by $\mathcal{Q}_n^k$.*

As was proven in the original paper [1], every k-QL order can be uniquely assigned to a labeled order in Ω_n. Further, not every set in $\mathcal{Q}_n^k$ is in $\mathcal{D}_n^k$ and vice versa. An interesting observation is that KR-orders do in fact satisfy both the conditions for $\mathcal{Q}_n^k$ and $\mathcal{D}_n^k$, therefore they are a part of $\mathcal{Q}_n^k \cap \mathcal{D}_n^k$. The way to think of generating these sets is to build them from a n-element set in Ω_n and a k-layer map and then add in the relations of the elements such that these then satisfy the conditions given by the k-layer map afterwards. This will generate a subset of $\mathcal{Q}_n^k$ and $\mathcal{P}_n^k$ respectively. A notable problem is that the set $\mathcal{P}_n^k$ is contrary to $\mathcal{Q}_n^k$ not one-to-one in Ω_n. This follows from the weaker condition on k-PQL layers, as these can have different assignments of k-PQL, k-layer maps, as for example, minimal elements, which are ones that do not have any elements preceding them, can be assigned any layer label. This means the same element of Ω_n can be generated through different assignments of k-PQL orders. $\mathcal{P}_n^k$ is the set over which one wants to sum the partition function Z_n and are not in one-to-one correspondence with the Ω_n, therefore one needs a subset $\mathcal{P}_{\vec{q}n}^* \subset \mathcal{P}_n^k$, defined below, which removes these redundancies.

Let $\mathcal{V}_n^k$ be the set of all possible ways to assign PQLs over the set of n integers. Having a n element set, one can distribute the elements in the k different layers with the cardinalities given by some fraction of all elements $q_i n$ for $|\vec{q}| = 1$. Meaning, elements in $\mathcal{V}_n^k$ can be characterized by these so called filling fractions $\vec{q}$. Further, because there is no unique way to distribute these distinguishable elements into these layers the filling fraction has a multiplicity given by

$$m(\vec{q}) = \frac{n!}{(q_1 n)!\,(q_2 n)! \ldots (q_k n)!}. \tag{4.2}$$

This also gives the cardinality of the set $\mathcal{V}_n^k$ as

$$|\mathcal{V}_n^k| = \sum_{\vec{q}} m(\vec{q}) = k^n. \tag{4.3}$$

To make the set $\mathcal{V}_n^k$ unique, one restricts to so called naturally labeled PQL assignments $\mathcal{V}_n^*$. This means, the first PQ layer is labeled by the first $q_1 n$ indices, i.e. $\{e_1 \dots e_{q_1 n}\}$, the second one with the next $q_2 n$ indices $\{e_{q_1 n+1} \dots e_{q_1 n+q_2 n}\}$ and so on for all layers. The restricted set $\mathcal{P}^*_{\vec{q},n}$ is defined as the naturally labeled k-PQL orders on $\mathcal{V}_n^*$. From here on, one can add the links in these orders denoting k-PQL orders with pn^2 links by $\mathcal{P}^*_{\vec{q},p,n} \subset \mathcal{P}^*_{\vec{q},n}$ and analogously with k-QL orders with pn^2 links with, $\mathcal{Q}^k_{pn} \subset \mathcal{Q}^k_n$. There exists a injective map from $\mathcal{P}^*_{\vec{q},p,n}$ to $\mathcal{Q}^k_{p,n}$ such that $\mathcal{P}^*_{\vec{q},p,n}$ is a proper subset of $\mathcal{Q}^k_{p,n}$. This is achieved by taking every irreducible causally disconnected subset $\hat{c}_i$ of an order in $\mathcal{P}^*_{\vec{q},p,n}$ and lowering the layer map assigned so that the lowest layer number element has the layer assignment 1. Let the lowest assigned layer in each irreducible causally disconnected set be denoted by t_i, then the relation between layer maps is given by

$$\eta(e_r) = \theta(e_r) - t_i + 1 \quad \forall e_r \in \hat{c}_i. \tag{4.4}$$

From here, it is possible to define the partition function over the set $\mathcal{P}^*_{p,n}$, as it is now unique. As for the filling fraction, this can be chosen more arbitrarily, for example the choice $\vec{q} = (1/4, 1/2, 1/4)$ is the KR-order without higher order terms. Leaving $\vec{q}$ general, one can find the maximum amount of links in $\mathcal{P}^*_{p,n}$, where every element is connected to every element in the adjacent layer:

$$N_{max} = (q_1 q_2 + q_2 q_3 + \dots + q_{k-1} q - k) n^2 =: \alpha(\vec{q}) n^2. \tag{4.5}$$

Further, one can express the specific number of links in any given set by a fraction of the maximum number of links $N_0 = \tilde{p} N_{max} = pn^2$, where $0 < \tilde{p} < 1$ is called the linking fraction. For every given p, the cardinality of the set $\mathcal{P}^*_{\vec{q},p,n}$ is simply given by the binomial coefficient.

$$|\mathcal{P}^*_{\vec{q},p,n}| = \binom{N_{max}}{N_0} = \binom{\alpha(\vec{q}) n^2}{pn^2} \tag{4.6}$$

The logarithm is in leading order a scaling of the Dhar's entropy function $h(x) = -x \ln x - (1-x)\ln(1-x)$ [19].

$$\ln|\mathcal{P}^*_{\vec{q},p,n}| = \alpha(\vec{q}) n^2 h(\tilde{p}) + o(n^2) \tag{4.7}$$

As shown in the previous chapter, the BDG action reduces to the linking action in this case as well. Looking at the partition function restricted to $\mathcal{P}^*_{\vec{q},n}$ and parametrising it with the linking fraction, it reduces to

$$\begin{aligned} Z_n|_{\mathcal{P}^*_{\vec{q},n}} &= \int_0^1 \mathrm{d}\tilde{p}\,|\mathcal{P}^*_{\vec{q},p,n}|\exp\big(i\mu\lambda_0 N_0 + o(n^2)\big) \\ &= \int_0^1 \mathrm{d}\tilde{p}\exp\big(\alpha(\vec{q})n^2(i\mu\lambda_0\tilde{p} + h(\tilde{p})) + o(n^2)\big). \end{aligned} \tag{4.8}$$

This is just a rescaling of the partition function $Z_n|_{\mathcal{Q}^2_n}$ of the bi-layer orders shown to be suppressed in [5], and this suppression is independent of rescaling, therefore this is also suppressed with the same bounds.

$$\begin{aligned} \tan\left(\frac{\mu\lambda_0}{2}\right) &< -\sqrt{\frac{27}{4}e^{-1/2} - 1} \quad \text{for} \quad \mu\lambda_0 < 0 \\ \tan\left(\frac{\mu\lambda_0}{2}\right) &> \sqrt{\frac{27}{4}e^{-1/2} - 1} \quad \text{for} \quad \mu\lambda_0 > 0 \end{aligned} \tag{4.9}$$

The last issue to be solved is that $\mathcal{P}^*_{\vec{q},n}$ is overlapping for different $\vec{q}$, so this would only hold for "typical" KR-orders, but it can be extended to the restriction to the set $\mathcal{Q}^k_n$.

Because $\mathcal{P}^k_{p,n}$, the set of all k-PQL orders with pn^2 links, has different PQL labelings for one element in $\mathcal{Q}^k_{p,n}$, $\mathcal{Q}^k_{p,n} \subset \mathcal{P}^k_{p,n}$ holds. Further, from the argumentation above, $\mathcal{P}^*_{\vec{q},p,n} \subset \mathcal{Q}^k_{p,n}$ therefore

$$|\mathcal{P}^*_{\vec{q},p,n}| \leq |\mathcal{Q}^k_{p,n}| \leq |\mathcal{P}^k_{p,n}|. \tag{4.10}$$

From here on, one can vary over $\vec{q}$, for example for KR-order, one can vary over $\vec{q}_x = (1/4 - x, 1/2, 1/4 + x)$. Let q_0 be the configuration that maximizes $\alpha(\vec{q})$, then summing over the maximal value one finds

$$|\mathcal{P}^k_{p,n}| = \sum_{\vec{q}} m(\vec{q})|\mathcal{P}^*_{\vec{q},p,n}| \leq \sum_{\vec{q}} m(\vec{q})|\mathcal{P}^*_{\vec{q}_0,p,n}| = k^n|\mathcal{P}^*_{\vec{q}_0,p,n}|, \tag{4.11}$$

meaning

$$|\mathcal{P}^*_{\vec{q}_0,p,n}| \leq |\mathcal{Q}^k_{p,n}| \leq k^n|\mathcal{P}^*_{\vec{q}_0,p,n}|. \tag{4.12}$$

As the k^n factor only contributes to higher orders, the first orders are the same

$$\ln|\mathcal{Q}^k_{p,n}| = \ln|\mathcal{P}^*_{\vec{q}_0,p,n}| + o(n^2). \tag{4.13}$$

This shows, that the KR-orders are suppressed for the same configurations of $\mu\lambda_0$ as in [5].

4.2 Addition of the Ising term

The BDG action can be expanded by an interaction term as defined in [18] to read

$$S_I = \mu\left(n + \sum_{k=0}^{k_{max}} \lambda_k N_k\right) + \sum_{i,j} \beta s_i s_j L^{ij}. \tag{4.14}$$

Here, s is the spin configuration of elements in the causal set, which is a vector of length n with entries ± 1. β is the coupling constant and L^{ij} is the so-called linking matrix, which contains an 1 if the i'th element e_i is linked with the j'th, e_j, $L_{ij} = \delta_{i\prec j,|A(e_i,e_j)|=0}$. Further, the partition function now does not only sum over all causal sets, but also all spin configurations. Here the following notation for the sum in the partition function will be used:

$$\sum_s := \prod_{i=1}^{n} \sum_{s_i=\pm 1} := \sum_{s_1=\pm 1} \sum_{s_2=\pm 1} \cdots \sum_{s_n=\pm 1} \tag{4.15}$$

Looking specifically at the restriction to $\mathcal{P}^*_{\vec{q},n}$, it is clear that the linking matrix takes an off-diagonal block matrix form, as only adjacent layers are connected as well as this connection being unidirectional. Further, it is naturally labeled, meaning the linking matrix rows and column of lower levels are always at lower indices.

$$(L^{ij}) = \begin{pmatrix} 0 & A_1^{i_1 j_1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & A_2^{i_2 j_2} & 0 & \cdots & 0 \\ 0 & 0 & 0 & A_3^{i_3 j_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & A_{k-1}^{i_{k-1} j_{k-1}} \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{4.16}$$

The A_i matrices are the link matrices between the i'th and $i+1$'th layer, these have dimension $q_i n \times q_{i+1} n$.

4.3 3-layer sets

For now, this analysis is restricted to three layer sets. From here on out, the Einstein sum convention is used.

$$s_i s_j L^{ij} = s_{1,i_1} s_{2,j_1} A_1^{i_1 j_1} + s_{2,i_2} s_{3,j_2} A_2^{i_2 j_2} \tag{4.17}$$

where $s_{i,j}$ are the restriction of s_j to the i'th layer. One can define two new vectors to make the analysis easier, these are $v_1^j := s_{1,i} A_1^{ij}$ and $v_3^j := s_{3,i} A_2^{ji}$. Collecting them into one vector, gives:

$$s_i s_j L^{ij} = (v_1 + v_3)^j s_{2,j} := \tilde{v}^j s_{2,j}. \tag{4.18}$$

Plugging the interaction action into the partition function, the sum over s can be pulled through the product

$$Z_I|_{\mathcal{P}^*_{\vec{q},n}} = \sum_{\mathcal{P}^*_{\vec{q},n}} \exp\left(i\mu \left[n + \sum_{k=0}^{k_{max}} \lambda_k N_k \right] \right) \sum_s \exp\left(i\beta s_i s_j L^{ij} \right) \tag{4.19}$$

Only considering the interaction term, one can rewrite the sum over the whole s vector to an independent sum over the vectors s_1, s_2 and s_3.

$$\sum_s \exp\left(i\beta s_i s_j L^{ij} \right) = \sum_{s_1,s_2,s_3} \exp\Big(i\beta (s_{1,i_1} s_{2,j_1} A_1^{i_1 j_1} + s_{2,i_2} s_{3,j_2} A_2^{i_2 j_2}) \Big) = \sum_{s_1,s_2,s_3} \exp\left(i\beta s_{2,j} \tilde{v}^j \right). \tag{4.20}$$

The sum, implied by the sum convention, can now be interpreted as a product of exponentials

$$\sum_s \exp\left(i\beta s_i s_j L^{ij} \right) = \sum_{s_1,s_2,s_3} \prod_j \exp\Big(i\beta s_2^j \tilde{v}^j \Big). \tag{4.21}$$

To work on this further, one can exchange the product and the sum as follows for a set with k elements in the second layer:

$$\begin{aligned}\sum_{s_2}\prod_j \exp\left(i\beta s_2^j \tilde{v}^j\right) &= \sum_{s_2} \exp\left(i\beta s_2^1 \tilde{v}^1\right)\exp\left(i\beta s_2^2 \tilde{v}^2\right)\ldots\exp\left(i\beta s_2^k \tilde{v}^k\right)\\ &= \sum_{s_2^1}\sum_{s_2^2}\ldots\sum_{s_2^k} \exp\left(i\beta s_2^1 \tilde{v}^1\right)\exp\left(i\beta s_2^2 \tilde{v}^2\right)\ldots\exp\left(i\beta s_2^k\right)\\ &= \left(\sum_{s_2^1}\exp\left(i\beta s_2^1 \tilde{v}^1\right)\right)\left(\sum_{s_2^2}\exp\left(i\beta s_2^2 \tilde{v}^2\right)\right)\\ &\qquad\qquad \ldots\left(\sum_{s_2^k}\exp\left(i\beta s_2^k \tilde{v}^k\right)\right)\\ &= \prod_j\sum_{s_2}\exp\left(i\beta s_2^j \tilde{v}^j\right) \end{aligned} \tag{4.22}$$

For every $\tilde{v}^j$, there is a spin configuration $s_2^j = \pm 1$. Taking the sum of the s_2^j, one simply gets the representation of the cosines in terms of exponential, therefore the result is a product over cosines

$$\sum_s \exp\left(i\beta s_i s_j L^{ij}\right) = 2^{q_2 n}\sum_{s_1,s_3}\prod_j \cos\left(\beta\tilde{v}^j\right). \tag{4.23}$$

$\beta\tilde{v}^j$ is now a sum of all s_1^j and s_3^j with a prefactor of either β or 0, depending on L_{ij}. This can be expanded by using the trigonometric identity

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y). \tag{4.24}$$

For simplicity the sum terms inside the cosine will be, for generality, denoted with a_k^j. The decomposition of the cosine, if done iteratively for all sum terms, is a sum of products with even amounts of sines such that all permutations appear once. To specify, permutation in this case mean that for example for 3 elements, (a,b,c) both the term $\sin(a)\sin(b)\cos(c)$ and $\sin(a)\sin(c)\cos(b)$ would appear as a summand. Further all terms with $4m-2, m \in \mathbb{N}_+$ sines will have a negative sign. Let the sum of all combinations where $2l$ a_k^j's are in sines while the rest are in cosins be denoted by Υ_l, then:

$$\cos\left(\sum_k a_k^j\right) = \sum_{l=0}^{\lfloor\frac{(q_1+q_3)n}{2}\rfloor}(-1)^l\Upsilon_l \tag{4.25}$$

The floor appearing in the sum limit insures that for odd numbers of summands this expression still holds true. The product is then the multinomial of 4.25, but heuristically one can find a simpler expression for the sum over all s_1^j and s_3^j and the product. The sum will filter out all odd parity functions from the multinomial or will just lead to a factor of $2^{(q_1+q_3)n}$ in front of them due to symmetry. Depending on L^{ij} more of these terms will fall out as L^{ij} will contain 0's, which, if it appears in a sine, will set the whole term 0. One can write the expression of 4.23, disregarding subleading terms in n, as follows:

$$\sum_{s_1,s_3}\prod_j \cos\left(\beta\tilde{v}^j\right) = \sum_{j=0}^{\lfloor\frac{(q_1+q_3)n}{2}\rfloor q_2 n\tilde{p}} (-1)^j c_j f_{j,\tilde{p}} \cos(\beta)^{\alpha(\vec{q})n^2\tilde{p}-2j}\sin(\beta)^{2j}. \tag{4.26}$$

Here $c_j f_{j,\tilde{p}}$ describes the multiplicity of terms with $2j$ sines in the multinomial. These factors can be derived by combinatorics. All permutations with a certain amount of sines will contribute to the same sum term. A subtlety here is that the multiplicity depends non-trivially on the linking matrix. This follows from some sines being zero if there is no link to the middle layer for the given element. This dependency has been separated into the functions $f_{j,\tilde{p}}$. Therefore, c_j are the maximal multiplicity for each term. Because c_j is maximal it is the combinatorical case for $\tilde{p} = 1$, meaning its maximally linked.

Starting with the function $f_{j,\tilde{p}}$ these are dependent on the filling fractions $\tilde{p}$ over which one wants to integrate. Further these describe the percent of all summand terms that do not vanish due to a missing link in the linking matrix appearing in a sine. This is equivalent to asking how many permutations of a n long vector, filled with $\tilde{p}\alpha(\vec{q})n^2$ ones and $\alpha(\vec{q})n^2 - \tilde{p}\alpha(\vec{q})n^2$ zeros, have k ones in the first k entries. A norm is used such that the case for $\tilde{p} = 1$ is 1. The norm condition follows from the fact that if all elements are linked then all sines appear. This leads to the expression for $f_{j,\tilde{p}}$ being:

$$f_{j,\tilde{p}} = \frac{(\tilde{p}\alpha(\vec{q})n^2)!\,(\alpha(\vec{q})n^2 - 2j)!}{(\alpha(\vec{q})n^2)!\,(\tilde{p}\alpha(\vec{q})n^2 - 2j)!}, \tag{4.27}$$

which can be written as

$$f_{j,\tilde{p}} = \begin{cases} 0, & \tilde{p} < \frac{2j}{\alpha(\vec{q})n^2} \\ \frac{\Gamma(1-2j+\alpha(\vec{q})n^2)\Gamma(1+\alpha(\vec{q})n^2\tilde{p})}{\Gamma(1+\alpha(\vec{q})n^2)\Gamma(1-2j+\alpha(\vec{q})n^2\tilde{p})}, & \text{else} \end{cases} \tag{4.28}$$

to account for $\tilde{p}$ being continuous.

Finding c_j equates to counting all product combinations of terms in the expansions of $\cos\left(\sum_k a_k^m\right)$ for all m such that the resulting term has $2j$ sines and is symmetric in s. This

has a complicated closed form, so one would need to distinguish odd amount of elements in the second layer and even amounts. Further one would need to do the same for the outside layers. For simplicity, the symmetry condition on the sines is dropped. Then one can choose any amount less than the required sine amount and multiply them together. The leading combinations are the ones closest to the middle of Pascal's triangle, under the condition that all the $q_2 n$ multiplications are used

$$c_{lead,nonsym} = \binom{(q_1+q_3)n}{2j}^{q_2 n}. \tag{4.29}$$

To make an error analysis for the disregard of the symmetry conditions one can try to find all ways to combine terms such that the resulting terms are actually symmetric. The largest contribution should come from the combinations that uses the most amount of different sines. These should in leading order behave like

$$c_{lead,sym} = \binom{(q_1+q_3)n}{2j}^{\frac{q_2 n}{2}}. \tag{4.30}$$

This is smaller than the non-symmetric case by a factor of square root and quantifies the error of this approximation.

The idea to solve this would be to approximate all terms by the largest c_j, but this is not necessarily a good approximation as the sum contains negative terms. However, one can group them up to make the resulting terms have all the same sign. This will lead to an error, which should be subleading.

The first thing to note is that $\cos(\beta)^{\alpha(\vec{q})n^2\tilde{p}-2j}\sin(\beta)^{2j}$ has the same sign behavior for all j. Further c_j and $f_{j,\tilde{p}}$ are all positive. Therefore only the signs of the expansion of the cosinus matters. One can use the fact that if the $2j$ sines are symmetric in s the cosines will also contain a symmetric combinations of s. This means exchanging the sines with cosines will give a valid symmetric combination. Thus c_j for m sines is the same as for m cosines. Grouping the terms $j = m$ and $j = \alpha(\vec{q})n^2\tilde{p}/2 - m$ for $m \in [0, \alpha(\vec{q})n^2\tilde{p}/2]$ makes it possible to pull out the c_j in front of them as they are the same. These terms will have the same sign, meaning one can approximate the $f_{j,\tilde{p}}$ of the group by the larges one. Because $f_{j,\tilde{p}} > f_{j+1,\tilde{p}}$ for all $\tilde{p}$, one uses the $f_{j,\tilde{p}}$ of the smallest j.

This leaves half the terms with alternating signs, and such that for growing j, c_j becomes larger. Starting from the maximal c_j, c_{max}, term this might have a positive or negative sign but it is not of importance because the terms have alternating signs and the following argument will grouped them in such a way that the resulting terms all have the

sign of the largest term. Comparing it with the term c_{max-1} one sees that $c_{max} > c_{max-1}$ especially as $n \to \infty$. This holds for all further pairs, $c_m > c_{m-1}$. Furthermore, the f's will retain their behavior as $n \to \infty$, meaning only the c_j will contribute to the limit of the prefactors.

Because the $f_{j,\tilde{p}}$'s for lower j start growing earlier, there is a small interval of $\tilde{p}$ in which these pairs will have the sign of the term with the lower j. However, this interval is very small and gets smaller as $n \to \infty$. This makes the approximation of both terms by the $f_{j,\tilde{p}}$ with the larger j reasonable and should only contribute to a small error. Now doing this with all pairs starting from c_{max} down to the last, one attains a sum with all signs being the same for most $\tilde{p}$. Meaning that finally one can approximate all terms by the maximal c_j and $f_{j,\tilde{p}}$ by 1. With this one gets an approximate upper bound for the integral if the term for c_{max} is positive and and a lower bound for it being negative.

For large enough n the partition function 4.19 can be described by an integral over all possible linking fractions $\tilde{p}$ and with the argumentations above written as:

$$\begin{aligned} Z_I|_{\mathcal{P}^*_{\vec{q},n}} &= c_{max} \int_0^1 \mathrm{d}\tilde{p} \sum_{j=0}^{\lfloor \frac{(q_1+q_3)n}{2} \rfloor q_2 n\tilde{p}} (-1)^j \cos(\beta)^{\alpha(\vec{q})n^2\tilde{p}-2j} \sin(\beta)^{2j} \exp\big(\alpha(\vec{q})n^2(i\mu\lambda_0\tilde{p}+h(\tilde{p}))\big) \\ &= c_{max} \int_0^1 \mathrm{d}\tilde{p} \cos(\beta)^{\alpha(\vec{q})n^2\tilde{p}} \exp\big(\alpha(\vec{q})n^2(i\mu\lambda_0\tilde{p}+h(\tilde{p}))\big) \sum_{j=0}^{\lfloor \frac{(q_1+q_3)n}{2} \rfloor q_2 n\tilde{p}} (-1)^j \tan(\beta)^{2j} \end{aligned} \tag{4.31}$$

The sum is just a geometric series with a finite boundary, this is where the finiteness of the sum comes into play and where extra conditions on β would arise if the sum was infinite. Approximating for large n the upper bound as $\alpha(\vec{q})n^2/2$ the path sum is:

$$\begin{aligned} Z_I|_{\mathcal{P}^*_{\vec{q},n}} = c_{max} \int_0^1 \mathrm{d}\tilde{p} \cos(\beta)^{\alpha(\vec{q})n^2\tilde{p}+2} \\ \cdot \exp\big(\alpha(\vec{q})n^2(i\mu\lambda_0\tilde{p}+h(\tilde{p}))\big)\big((-1)^{\frac{\alpha(\vec{q})\tilde{p}n^2}{2}} \tan(\beta)^{\alpha(\vec{q})\tilde{p}n^2+2}+1\big) \end{aligned} \tag{4.32}$$

Lastly one wants to find c_{max}. If one ignores the restriction of the sine combinations being symmetric the amount of combinations with the most combinations being at $2j = \alpha(\vec{q})n^2/2$ is

$$c_{max} = \binom{(q_1+q_3)n}{\frac{(q_1+q_3)n}{2}}^{q_2 n} + \xi(n,\vec{q}). \tag{4.33}$$

The function ξ describes all combinations of sines such that they add up to $\alpha(\vec{q})n^2/2$ sines, which uses terms that have a multiplicity that is not the middle of Pascal's triangle. In leading order of n, c_{max} behaves as $2^{\alpha(\vec{q})n^2}$. Therefore the partition function up to constants and subleading terms becomes

$$Z_I|_{\mathcal{P}^*_{\vec{q},n}} = \int_0^1 \mathrm{d}\tilde{p}(\cos(\beta)^{\alpha(\vec{q})n^2\tilde{p}} + (-1)^{\frac{\alpha(\vec{q})\tilde{p}n^2}{2}} \sin(\beta)^{\alpha(\vec{q})n^2\tilde{p}}) \cdot \exp\big(\alpha(\vec{q})n^2(\ln(2) + i\mu\lambda_0\tilde{p} + h(\tilde{p}))\big). \quad (4.34)$$

A sum of exponentials are exactly suppressed when all of the exponents are negative as $n \to \infty$. This follows from the fact that exponentials are always positive numbers which diverge either to infinity or converge to 0. Even if the exponentials are subtracted from one another it will diverge as long as the exponents do not diverge at the same rate. The only special case is if all exponents are equivalent and there are alternating signs. This is still zero for diverging exponentials. This case might arise in this approximation but without the approximations of c_j these terms would have different prefactors, which would lead to a divergence if the exponentials are divergent. Now a case distinction is made. For $\cos(\beta) = 0$ this reduces only to a term with only sines. This is the case for $\tilde{p} = 1$. Then $h(\tilde{p}) = 0$ and it is not suppressed or at least not in this approximation. The second case is $\sin(\beta) = 0$ this corresponds to the free case as $\cos(\beta)$ is 1.

4.4 Suppression bound

In this section, the suppression condition for this new partition function 4.34 is calculated via the method of steepest decent. The calculations and figures in this and the following section were done with Mathematica. Firstly, β is a coupling constant, so one can assume it as constant and let $\tilde{\beta} \in [-1, 1]$. This new $\tilde{\beta}$ can be used to describe both integrals at the same time as both perfactors are in that range. This more explicitly shows, that the factor has a constant exponent base, which can be changed to the Euler's number.

$$\tilde{\beta}^{\tilde{p}\alpha(\vec{q})n^2} = \exp\Big(\tilde{p}\alpha(\vec{q})n^2 \ln\Big(\tilde{\beta}\Big)\Big) \quad (4.35)$$

With this the first integral takes the form of

$$Z_n|_{\mathcal{P}^*_{\vec{q},n}} = \int_0^1 \mathrm{d}\tilde{p} \exp\Big(\alpha(\vec{q})n^2\Big(\ln(2) + \Big(\ln\Big(\tilde{\beta}\Big) + i\mu\lambda_0\Big)\tilde{p} + h(\tilde{p})\Big)\Big) \quad (4.36)$$

and the second one has the form of

$$Z_n|_{\mathcal{P}^*_{\vec{q},n}} = \int_0^1 \mathrm{d}\tilde{p} \exp\Big(\alpha(\vec{q}) n^2 \Big(\ln(2) + \Big(\ln\Big(\tilde{\beta}\Big) + i(\mu\lambda_0 + \frac{\pi}{2}) \Big) \tilde{p} + h(\tilde{p}) \Big) \Big). \tag{4.37}$$

These are complex valued Laplace type integrals, which are of the form:

$$I = \int_a^b r(\tilde{p}) e^{\lambda g(\tilde{p})} \mathrm{d}\tilde{p} \tag{4.38}$$

For these types of integral the method of steepest decent [20] can be used to evaluate the integral as a whole but for the large λ limit one can more easily calculate the leading order contribution as follows. In this case $r(\tilde{p}) = 1$, further writing the function $g(\tilde{p})$ as a real part and an imaginary part with $\tilde{p} = x + iy$ where $x, y \in \mathbb{R}$ one attains

$$g(x,y) = u(x,y) + iv(x,y). \tag{4.39}$$

Subsequently one can use the Cauchy theorem to deform the integration contour γ from a line between $(a,0)$ to $(b,0)$ as long as one does not cross any singularities or branch cuts. Using this one can reshape the contour such that the imaginary part is constant, $v(x,y) = \text{const.}$, denoting this new contour as γ'. If $g(x,y)$ is an analytic function in the region of the contour this coincides with the direction of steepest decent. This follows from the Cauchy–Riemann equations.

$$\partial_x u(x,y) = \partial_y v(x,y) \text{ and } \partial_y u(x,y) = -\partial_x v(x,y) \;\Rightarrow\; \nabla u(x,y) \cdot \nabla v(x,y) = 0 \tag{4.40}$$

This contour does not have to be continuous or have the same constant of the imaginary part along the whole contour. Therefore, deforming the contour in such a way gives

$$I = \int_a^b e^{\lambda g(\tilde{p})} \mathrm{d}\tilde{p} = \int_\gamma e^{\lambda g(z)} \mathrm{d}z = e^{i\lambda v} \int_{\gamma'} e^{\lambda u(z)} \mathrm{d}z. \tag{4.41}$$

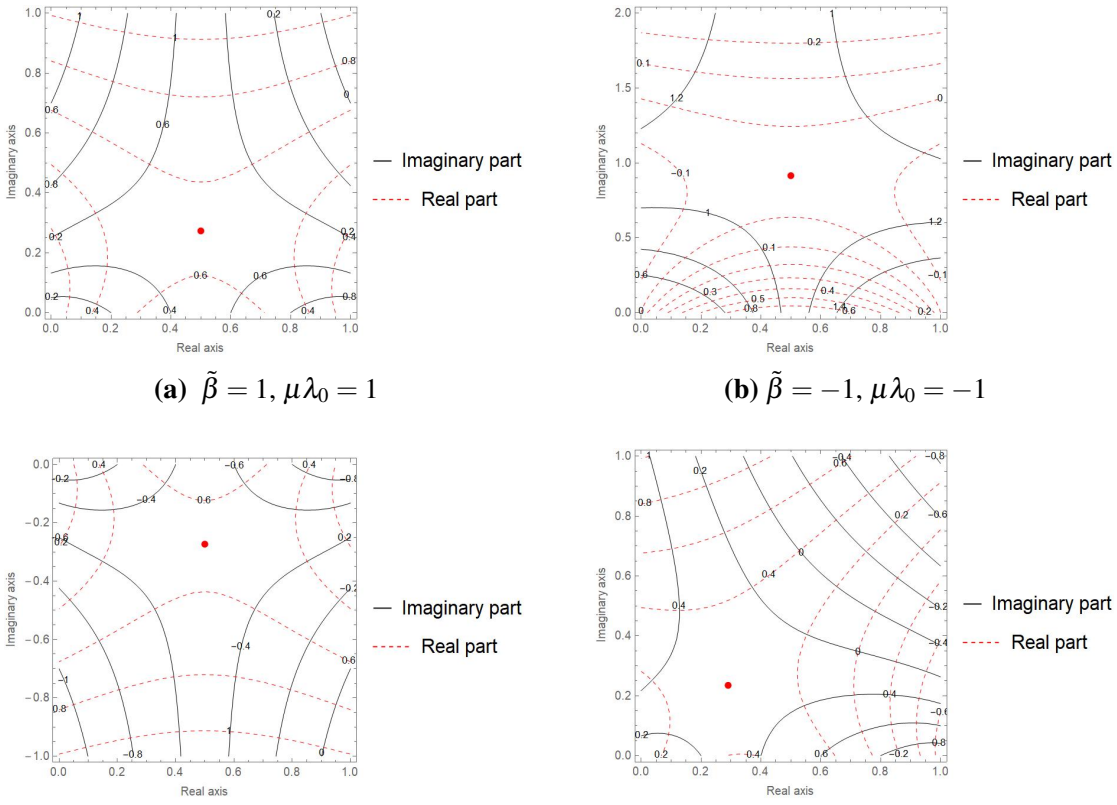
Up until now no approximation was made, still one can find the point in the complex plane that contributes the most towards the integral. This will be the maximum point of $u(x,y)$ along a contour of constant $v(x,y)$. Assume $u(z)$ attains its maximum value at z_0 and let the contour curve be parametrized with the parameter s then from the choice of constant imaginary part follows that $\partial_s v(s) = 0$ at z_0. Since this point is the maximum of $u(z)$, $\partial_s u(s) = 0$ at z_0. Therefore $\frac{\partial g(s)}{\partial s} = 0$ and subsequently due to analyticity this is not only

true along the contour but in all directions. Hence z_0 is a maximum or a saddle point of $u(z)$ around which one can Taylor expand. For the first case in equation 4.36

$$g(z) = (i\mu\lambda_0 + \ln\left(\tilde{\beta}\right))z - z\ln(z) - (1-z)\ln(1-z). \tag{4.42}$$

This function has a saddle point at

$$z_0 = 1 - \frac{1}{1+\tilde{\beta}e^{-i\mu\lambda_0}}. \tag{4.43}$$



(a) $\tilde{\beta} = 1, \mu\lambda_0 = 1$ **(b)** $\tilde{\beta} = -1, \mu\lambda_0 = -1$

(c) $\tilde{\beta} = 1, \mu\lambda_0 = -1$ **(d)** $\tilde{\beta} = 0.5, \mu\lambda_0 = 1$

Figure 7: Constant value contours of $u(x,y)$, in dashed lines, and $v(x,y)$, in solid lines, for different $\tilde{\beta}$ and $\mu\lambda_0$ and the saddle point are at the red point.

Picking a contour line such that it has the steepest decent through the saddle point depends on $\tilde{\beta}$ and $\mu\lambda_0$. To illustrate how $u(x,y)$ and $v(x,y)$ behave under certain choices of $\tilde{\beta}$ and $\mu\lambda_0$, it is plotted for different choices of these in Figure 7.

The leading order can be found by the value of $u(z)$ at z_0. In this case the leading term of the integral is given by:

$$Z_n|_{\mathcal{P}^*_{\vec{q},n}} \approx \exp\left(\alpha(\vec{q})n^2\left(\mathrm{Re}\left(\left(i\mu\lambda_0+\ln\left(\tilde{\beta}\right)\right)z - z\ln(z) - (1-z)\ln(1-z)\right)\right)\right)|_{z=z_0} \tag{4.44}$$

The second order will be given by a Gaussian type integral. To confirm now that this is exponentially suppressed one needs to check where the exponential is negative. The first issue that arises is that the real part of this function depends on the argument of the logarithm. Except for the term that contains the arguments 4.45 every occurrence of $\mu\lambda_0$ is in 2π periodic functions.

$$\mu\lambda_0 + \arg\left(\frac{1}{1+\tilde{\beta}e^{i\mu\lambda_0}}\right) - \arg\left(1-\frac{1}{1+\tilde{\beta}e^{i\mu\lambda_0}}\right) \tag{4.45}$$

This is proportional to the choice of branch that the logarithm takes on, as seen in Figure 8. Choosing the branch which is proportional to 0, one can find exactly the region of

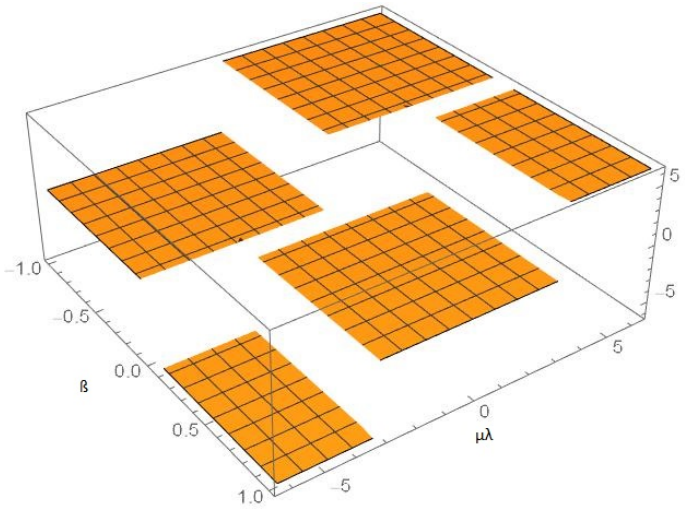
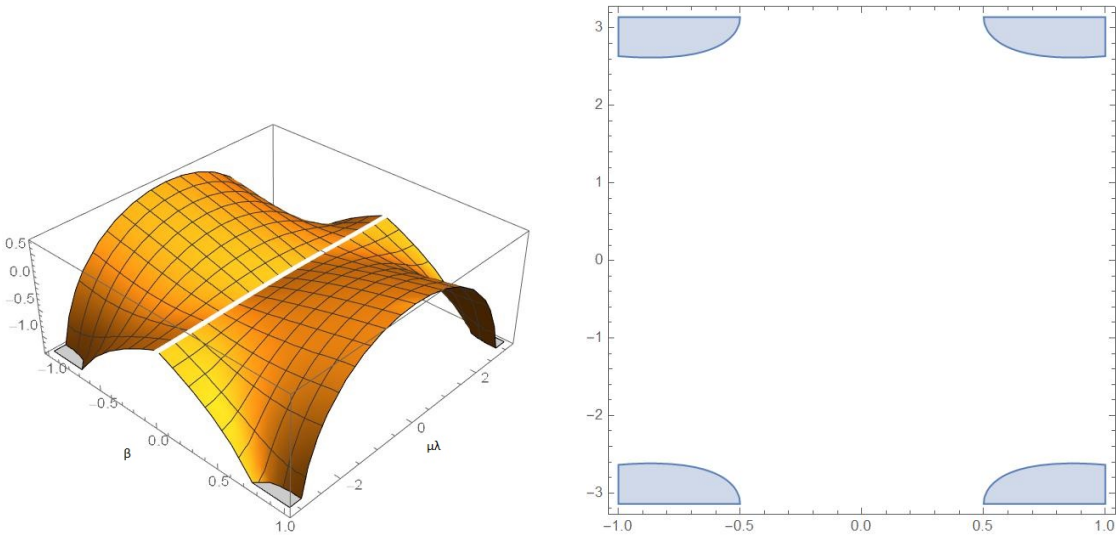


Figure 8: Plot of equation 4.45

$\mu\lambda_0$ and $\tilde{\beta}$ at which the exponent becomes negative. This needs a case distinction for positive and negative $\tilde{\beta}$. In Figure 9a $u(z)$ is plotted in terms of $\tilde{\beta} \in [-1,1]$, $\tilde{\beta} \neq 0$ and

$\mu\lambda_0 \in [-\pi/2, \pi/2]$ restricted to the chosen branch cut. Further in Figure 9b the region of $\mathrm{Re}(u(z_0)) \leq \ln(2)$ was plotted to show at which points the integral becomes suppressed. Note that this only holds for $\tilde{\beta} \neq 0$, however for $\tilde{\beta} = 0$ the integral 4.36 is 0.



(a) Plot of $u(z_0)$ **(b)** Region plot of $u(z_0) \leq \ln(2)$

Figure 9: Plots of the exponent of the leading order of the integral

For the second integral it has the exact same behavior, the difference is that $\tilde{\beta}$ has a different dependency on β. Further, the choice of branch is different, if chosen such that they are zero like before it has the exact same behavior. To fully find the bound one needs to find for each β which of the $\tilde{\beta}$ are larger. This means $\frac{1}{\sqrt{2}} < |\tilde{\beta}| < 1$ giving the bounds as seen in Figure 10.

4.5 Higher orders

4.5.1 Odd layer setting

In the setting of odd numbered layers, one can expand the argumentation above to z layers. Starting off with the interaction term

$$s_i s_j L^{ij} = \sum_k s_{k,i_k} s_{k+1,j_k} L^{i_k,j_k} \quad \text{where} \quad k \in 2\mathbb{R}+1 < z \tag{4.46}$$

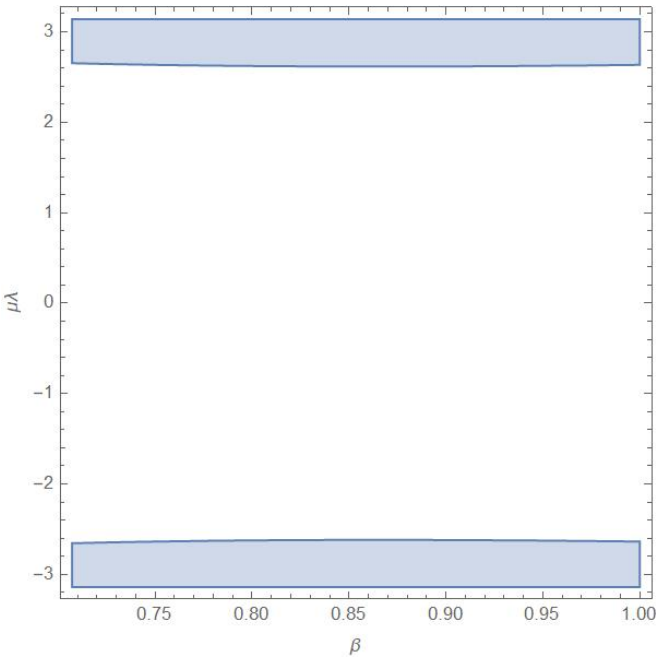


Figure 10: Plot of bounds for relevant coupling constants

one can group the layers in packs of three as was done in the three layer case.

$$s_i s_j L^{ij} = \sum_k (v_k + w_{k+2})^j s_{k+1,j} \quad \text{where} \quad k \bmod 2 = 1 \ \text{ and } \ k \leq z \tag{4.47}$$

Where $v_k^j = s_{k,i} A_k^{ij}$ and $w_k^j = s_{k,i} A_{k-1}^{ji}$. After plugging this into the path integral, the part that describes the interaction is

$$\begin{aligned}
\sum_s \exp\left(i\beta s_i s_j L^{ij}\right) &= \sum_{v,w} \prod_k \sum_{s_{k+1}} \exp\left(i\beta (v_k + w_{k+2})^j s_{k+1,j}\right) \\
&= \sum_{v,w} \prod_k \prod_j \sum_{s_{k+1}} \exp\left(i\beta (v_k + w_{k+2})^j s_{k+1}^j\right) \\
&= \sum_{v,w} \prod_k 2^{q_{k+1} n} \prod_j \cos\left(\beta (v_k + w_{k+2})^j\right).
\end{aligned} \tag{4.48}$$

From here on, the argumentation is the same as for the three layer set, however, further there is always a v and w corresponding to the same layer. Therefore, a function $\tilde{\Xi}(z, \beta, L^{ij})$

is introduced. This is the sum of all symmetric product combinations in s of all expansions of $\cos\big(\beta(v_k + w_{k+2})^j\big)$ for all admissible k's.

If one then sums over all s, one gets a function proportional to $\Xi(z,\beta,L^{ij})$, which describes all these combinations without the s but with an additional factor in front. Then the interaction term gives the factor

$$\sum_s \exp\big(\beta s_i s_j L^{ij}\big) \propto \Xi(z,\beta,L^{ij}). \tag{4.49}$$

4.5.2 Even layer setting

In this setting, one has a leftover term which cannot be grouped up by three layers, so for an z layer even order, this means

$$s_i s_j L^{ij} = s_{z-1,j_{z-1}} s_{z,j_z} A_z^{j_{z-1},j_z} + \sum_k (v_k + w_{k+2})^j s_{k+1,j}. \tag{4.50}$$

Following the argumentation form the odd layer sets one simply gets an additional cosine in the sum over s_z. Therefore, one needs to modify the function $\Xi(z,\beta,L^{ij})2^n$ as it needs to include the symmetricity of $\cos\Big(\beta s_{z-1,j_{z-1}} A_z^{j_{z-1},j_z}\Big)$. Let this new combinations be denoted by $\Xi_e(z,\beta,L^{ij})$. For completeness, the factor gained is

$$\sum_s \exp\big(\beta s_i s_j L^{ij}\big) \propto \Xi_e(z,\beta,L^{ij}). \tag{4.51}$$

This section shows the suppression of k-layer orders after introducing an additional Ising interaction term for all β. After introducing the suppression method as shown in [1] the suppression of free KR-orders were reviewed. The same approach was used for the extended action. This interaction term led to a prefactor consisting of a sum of products of cosines and sines depending on the linking matrix and a combinatorial factor. The combinatorial factor was argued to be approximable by the largest of these. From there the integral was evaluated to leading order using the method of steepest decent to find the suppression condition for the path integral.

5 Summary and Outlook

As seen in Chapter 3, the variation of the path integral with respect to the number of set elements, or equivalently space-time volumes, leads to a divergent path integral. Due to the fact that the path integral does not need to have a trivial measure for all causal sets, an idea discussed was to modify the measure of the integral to include the physical assumption of everpresent Λ. Assuming KR-orders with a certain given size, it was argued that due to uncertainty of a Poisson distribution, the cosmological constant fluctuates around a equilibrium value, which here is 0. This means, over time KR-orders with the given equilibrium size have the largest weight and ones that are closely adjacent are weighted less. Based on the paper of the everpresent Λ [17] the weights were assumed to be of the form of a normal distribution. This is not a bounded function. However, due to discreteness, a distribution similar to it was assumed. The resulting integral is then indeed suppressed. The two issues with this approach is that this solution requires the assumption of the fluctuating Λ and the assumption of a non-trivial measure function. The latter is equivalent to a modification to the Lagrangian. It was further shown, that there is a general method to find such a weight for a given set and its evolution, but this is neither unique nor necessarily finite. To resolve this problem a particular form of the evolution was assumed. But the assumption of this specific type of measure function itself is a choice, therefore this is an example to explore possible ways to recover boundedness and not a general solution.

In Chapter 4, the suppression of layered stets through an interacting action was analyzed. This was done in analogy to the paper [1], in which the suppression through variation of links was shown for the free case. In this thesis an Ising interaction term was added to the action. This additional interaction term led to a prefactor consisting of products of cosines and sines of the coupling constant. These prefactors further had a multiplicity depending on the amount of links a given set has, because if a link is missing that is associated with a sine this sine becomes 0. It was argued that this can be approximated by the largest prefactor. However, this approximation is rough as is introduces a more divergent exponent in the final path integral. A way to see that this suppression condition is too large is to analyze the free case $\beta = 0$. This should give back the result seen in [1] but is too large. The original result was

$$\mu\lambda_0 \approx \pm 2.108 \tag{5.1}$$

while in this case here it is found that this bound is at

$$\mu\lambda_0 \approx \pm 2.636. \tag{5.2}$$

If the same calculation as described in chapter 4.4 is done with $\cos(\beta) = 1$ from the beginning one finds

$$\mu\lambda_0 \approx \pm 2.094. \tag{5.3}$$

This traces back to the rough approximations as the prefactor of the term governing the free case, which is the term with no sines, has a combinatorial factor of 1. This was approximated by the maximal combinatorial factor, which led to the bound being lifted by a factor of $\ln(2)$. Further it was not taken into account that terms with higher multiplicity have lower factors of the trigonometric products. These is a result of interference between cosines and sines, which could counteract parts of the large multiplicity. With these approximations the sum was a geometric series, which was evaluated to give the final expression of the path sum. Both terms were evaluated to first order with the method of steepest decent giving suppression conditions on both terms. Both terms decay exponentially, therefore both have to satisfy their respective suppression conditions. Meaning that depending on the coupling constant one needs to find the largest value of either $\cos(\beta)$ or $\sin(\beta)$ to find the corresponding suppression condition. Note that these conditions contain information about the minimum length scale of the sprinkling as shown by equations 2.18 and 2.19.

The solution shows, that interacting terms do not need to destroy the suppression of non-manifold like sets. This means, there is hope for more complicated interacting terms. Interesting potentials to look for are the ϕ^4 and ϕ^3 potentials on the causal set, as these are common and semi-simple scalar potentials. Potentials which are not scalar are not yet definable in causal sets, as the notion of a tangent space is not yet defined in terms of the causal set. This is possible in the approximation of the set as a manifold, however an analysis of non-manifold like sets is not possible without such a description.

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