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Abstract

Unlike the well-known spin angular momentum of light, manifested as a specific polarization of a light beam, its complement known as orbital angular momentum (OAM) stayed mostly unnoticed for decades and it was not until the early 90s that its many interesting properties began to be studied in its own right. Since then the OAM of light has been used to spin colloidal particles, measure the angular velocity of rotating objects, and to send information via classical as well as quantum communication channels, to name a few. Driven by curiosity in both the fundamental questions and the applications, the understanding of the spatial structure of light beams and thus also OAM is nowadays much deeper than ever before. Apart from other notable application areas, certainly a very interesting one is the use of OAM of particles of light, photons, for the processing and transmission of information. This is the focus of the present text.

This thesis discusses two major topics. The first one is the manipulation of the quantum state of a single photon with an emphasis on the OAM degree of freedom. The second topic is the efficient and modular generation of entanglement in OAM. The former is addressed theoretically, whereas for the latter an experiment is reported that demonstrates the feasibility of a specific method of entanglement generation called entanglement by path identity. As orbital angular momentum is not a so well-known physical quantity, it is briefly introduced in the first part of the thesis, where some of the most important properties are mentioned that are relevant in the rest of the text.

In the second part of the thesis, the OAM of a single photon is used to represent an elementary quantum carrier of information, a qudit, and different techniques are analyzed that allow one to apply various classes of unitary transformations to it. Most importantly, several universal schemes for the implementation of general unitaries acting on OAM are discussed. A specific interferometric method called the path-assisted approach is described in detail and exemplified by high-dimensional Pauli operators. Apart from that, an implementation scheme of the Fourier transform of OAM of single photons is presented that uses only conventional optical elements and whose scaling is considerably better than the scaling of its counterpart that employs the path degree of freedom.

The third part of the thesis discusses a proof-of-principle experiment based on entanglement by path identity. Pairs of photons are generated whose quantum state is a three-dimensional entangled state in OAM. We demonstrate that the concept of entanglement by path identity is feasible and offers some advantages over alternative techniques. Among other things, we show that the local phases as well as local magnitudes of coefficients in the entangled state can be set at will. This allows one to produce a large class of states, which class is limited by the setup's geometry rather than intrinsic physical constraints such as energy conservation in non-linear processes. The thesis ends with concluding remarks about future prospects of manipulation and entanglement generation in OAM.

Kurzfassung

Im Gegensatz zum bekannten Spin-Drehimpuls des Lichts, der sich als spezifische Polarisierung eines Lichtstrahls manifestiert, blieb sein Komplement, der Bahndrehimpuls (OAM), jahrzehntelang weitgehend unbeachtet, und erst in den frühen 90er Jahren begann man, seine vielen interessanten Eigenschaften eigenständig zu untersuchen. Seitdem wird der OAM des Lichts genutzt, um kolloidale Teilchen zu drehen, die Winkelgeschwindigkeit rotierender Objekte zu messen und Informationen über klassische und Quantenkommunikationskanäle zu senden, um nur einige Beispiele zu nennen. Angetrieben von der Neugier auf die grundlegenden Fragen und die Anwendungen ist das Verständnis der räumlichen Struktur von Lichtstrahlen und damit auch des OAM heute viel tiefer als je zuvor. Neben anderen bemerkenswerten Anwendungsgebieten ist sicherlich ein sehr interessantes die Verwendung des OAM von Lichtteilchen, von so genannten Photonen, für die Verarbeitung und Übertragung von Informationen. Dies ist der Schwerpunkt des vorliegenden Textes.

In dieser Arbeit werden zwei Hauptthemen behandelt. Das erste ist die Manipulation von Quantenzuständen eines einzelnen Photons mit Schwerpunkt auf dem OAM-Freiheitsgrad. Das zweite Thema ist die effiziente und modulare Erzeugung von Verschränkung in OAM. Ersteres wird theoretisch behandelt, während für letzteres ein Experiment erläutert wird, das die Machbarkeit einer speziellen Methode der Verschränkungserzeugung, der Verschränkung durch Pfadidentität, demonstriert. Da der Bahndrehimpuls eine nicht so bekannte physikalische Größe ist, wird er im ersten Teil der Arbeit kurz beschrieben, wobei der Fokus auf einige der wichtigsten Eigenschaften gelegt wird, die für die Diskussion im restlichen Text relevant sind.

Im zweiten Teil der Arbeit wird der OAM eines einzelnen Photons verwendet, um einen elementaren Quanteninformationsträger, ein Qudit, darzustellen, und es werden verschiedene Techniken analysiert, die es uns erlauben, verschiedene Klassen von unitären Transformationen darauf anzuwenden. Vor allem werden mehrere universelle Schemata für die Implementierung allgemeiner unitären Transformationen, die auf OAM wirken, diskutiert. Eine spezielle interferometrische Methode, der so genannte pfadunterstützte Ansatz, wird ausführlich beschrieben und durch einige explizite Beispiele veranschaulicht. Außerdem wird ein Implementierungsschema der Fourier-Transformation von OAM einzelner Photonen vorgestellt, das nur konventionelle optische Elemente verwendet und dessen Skalierung erheblich besser ist als die seines Gegenstücks, das den Pfadfreiheitsgrad verwendet.

Im dritten Teil der Arbeit wird ein Proof-of-Principle-Experiment diskutiert, das auf Verschränkung durch Pfadidentität beruht. Es werden Paare von Photonen erzeugt, deren Quantenzustand ein dreidimensionaler verschränkter Zustand in OAM ist. Wir zeigen, dass das Konzept der Verschränkung durch Pfadidentität machbar ist und einige Vorteile gegenüber alternativen Techniken bietet. Unter anderem zeigen wir, dass sowohl die lokalen Phasen als auch die lokalen Beträge der Koeffizienten im verschränkten Zustand beliebig eingestellt werden können. Dies ermöglicht es uns, eine große Klasse von Zuständen zu erzeugen, die durch die Geometrie des Aufbaus und nicht durch intrinsische physikalische

Beschränkungen wie die Energieerhaltung bei nichtlinearen Prozessen begrenzt ist. Die Arbeit endet mit abschließenden Bemerkungen zu den Zukunftsaussichten der Manipulation und Verschränkungserzeugung in OAM.

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1 Introduction

Quantum physics seems to an untrained eye even nowadays as a sort of elusive dark magic. The intuition about the world one gains through everyday experience is based on phenomena that are usually too macroscopic for their underlying quantum nature to be directly accessible. As a result, the language that humans have developed over the generations is often not well-suited for the description of genuine quantum effects and one is sometimes misled by the improper use of seemingly clear words. This can be manifested by questions such as “How come a particle can be in two distinct places at the same time?” while one should rather talk in such situations about a quantum superposition of two different positional states of a particle. Yet, these and similar questions keep laymen in awe and it would only be fair to say that even the experts struggle to answer these questions in a satisfactory way. The last century has seen considerable progress in our understanding as well as harnessing quantum phenomena, while even larger progress can be expected in the future. This thesis is meant to make a contribution to this topic, to the fascinating fields of quantum foundations, quantum computation, and quantum technologies. It is of course impossible to even scratch the surface of all the major concepts studied in the context of quantum theory. In this text, we, therefore, focus on a specific topic: the manipulation of high-dimensional quantum states and the generation of high-dimensional entanglement. While we study the former on a theoretical level, for the latter we construct a proof-of-principle experiment.

Many quantum phenomena can be demonstrated with the help of fairly simple quantum systems that are associated with a two-dimensional state space. These two-dimensional quantum systems, or *qubits*, have been the focus of intensive research from the early days of quantum physics. Arguably the most famous example of such a system is a particle with a spin of $1/2$, such as an electron. It was effectively this system that allowed Otto Stern and Walther Gerlach in 1922 [1] to demonstrate experimentally the quantization of angular momentum. It was the same system that David Bohm used in his book [2] to simplify the argument of Einstein, Podolsky, and Rosen about entangled states. A qubit represents an elementary information carrier in quantum computation and is often described as a quantum generalization of a bit known from the classical computation. From the information-theoretic perspective, qubits are universal in that they provide a sufficient resource for carrying out any quantum algorithm and there is no need to employ quantum systems with larger than two-dimensional state space. Having said that, there are indeed tasks that profit from the use of high-dimensional generalizations of qubits, known as *qudits*. Even though there are some features that can be reproduced only by qubits [3], the higher-dimensional quantum systems offer improvements in many different areas of quantum communication and computation. The quantum cryptographic protocols, for instance, display more robust operation and increased noise resistance when qudits are used for the distribution of a secret key [4, 5]. Likewise, the tests of local realism provide more favorable bounds for higher-dimensional systems [6, 7, 8, 9]. In a communication protocol where a fixed set of symbols is to be transmitted, the natural choice for the information

carrier is the one that can encode directly each symbol in a separate mode and qudits thus appear as a natural choice. These facts motivate our study of the high-dimensional systems, presented in this text.

There are many different physical systems that can play the role of a qudit and even more that can represent a qubit, the polarization of photons being one of the prominent ones. The polarization is closely connected to the spin of photons, sometimes referred to as the intrinsic angular momentum. There is, however, another component of the photon's angular momentum that is known as the *orbital angular momentum* or OAM for short. Unlike polarization, the OAM of photons amounts to the spatial structure of the wave function assigned to a photon and is associated with a state space of a countably infinite dimension. As such, it can serve as a physical representation of high-dimensional qudits and it is indeed utilized this way throughout the text. Many of the results presented below are nevertheless independent of this specific choice and the same results are under some assumptions valid for other degrees of freedom as well. Let us note that in general there is only one type of angular momentum, the total angular momentum, with no separately identifiable components. Yet in many relevant scenarios, light forms a narrow beam propagating along a straight line and under such circumstances, the total angular momentum of each photon in the beam approximately splits into the two aforementioned components [10]. Also, the concept of a wave function of a photon is a rather controversial topic [11]. For the purposes of the present text we use the term “photon's wave function” as a synonym for a function that allows for the prediction of the form of single-photon interference patterns in the spatial domain and serves thus as a convenient mathematical tool. This way we leave aside not only the actual meaning of a wave function and its connection to the physical system it describes but also the problem of its very existence, resulting from the fact that non-relativistic wave functions cannot reproduce all the properties associated with photons [11]. To introduce the reader to the topic, the next chapter is devoted solely to the orbital angular momentum itself and its applications are discussed later.

The title of this thesis — “Entanglement, interference, and manipulation of systems in high dimensions” — contains three important terms closely related to quantum physics. In the chapters to come, we discuss each of these terms separately in a rather reversed order. We consider first single photons and the manipulation of their quantum state, specifically the OAM part of their state, with the help of conventional optical elements. A theoretical framework is presented that employs these elements to construct arbitrary unitary operations that act on such OAM states. Several examples of unitaries that can be implemented that way are worked out and their action on pure quantum states is studied in detail. These pure quantum states are quantum superpositions of multiple OAM modes. When such a quantum superposition is subjected to an appropriate set of measurements, the interference pattern emerges in the measurement results. Such interference can be observed both for states of single photons and for multi-photon states. This observation serves us as a bridge from single-partite systems to multi-partite ones. The latter display a richer structure when compared to the former since their states can exhibit a wide range of correlations including quantum entanglement. We study the generation of photons in high-dimensional entangled states of their OAM in the last chapter of the thesis. The proof-of-principle experiment is reported that successfully generates with reasonable fidelity different classes of entangled states in two and three dimensions. We conclude the thesis by listing some future prospects of the results presented in the text.

2 Orbital angular momentum

The entire text of this thesis revolves around a physical quantity termed *orbital angular momentum*. The word “revolve” is particularly fitting here as the orbital angular momentum has a lot in common with the revolving of an electromagnetic field about a given axis. Even though the primary focus of this thesis is the use of orbital angular momentum in the quantum domain, many of its features can be readily demonstrated in the classical regime. For this reason, the concept of angular momentum is recalled first as it is used in classical physics, followed by the separation of the total angular momentum into its components and only then does the discussion turn quantum and non-classical phenomena are discussed.

2.1 Angular momentum

In classical physics, angular momentum is a physical quantity that captures the intuitive idea of the strength of rotation of a massive object. A figure skater on an ice rink spins the more about her vertical axis the closer she keeps her arms to her body. When she stretches her arms, the angular speed of her rotation gets smaller, when she retracts her arms, the speed increases. This observation can be formally described as the conservation of angular momentum. In the simple case of a point particle that is characterized by its mass m , velocity $\mathbf{v}$, and position $\mathbf{r}$ with respect to some predefined coordinate system, see Fig. 2.1(a), the angular momentum of the particle is given by

$$\mathbf{J} = \mathbf{r} \times \mathbf{p}, \quad (2.1)$$

where $\mathbf{p} = m\mathbf{v}$ is the particle’s momentum. For a massive body with finite dimensions, the above expression represents the angular momentum of an infinitesimal element of the body at position $\mathbf{r}$ with mass $m = \rho(\mathbf{r})dV$, where dV is the volume of the infinitesimal element and $\rho(\mathbf{r})$ is the mass density at the given position. To obtain the total angular momentum of the body, one has to integrate over all infinitesimal elements.

Angular momentum is a property not restricted to massive objects only. An electromagnetic field may also possess non-zero linear as well as angular momentum. As the field varies in space and time, it is necessary to consider first the densities of linear and angular momentum, denoted by $\mathbf{p}$ and $\mathbf{j}$, respectively, which are given by classical formulas [10]

$$\mathbf{p} = \epsilon_0 \mathbf{E} \times \mathbf{B}, \quad (2.2)$$

$$\mathbf{j} = \epsilon_0 \mathbf{r} \times (\mathbf{E} \times \mathbf{B}), \quad (2.3)$$

where $\mathbf{E}$ and $\mathbf{B}$ denote the electric and magnetic field, and where ϵ_0 is the vacuum permittivity. The total linear momentum $\mathbf{P}$ and the total angular momentum $\mathbf{J}$ are calculated by integrating these formulas over space. It turns out that the latter contains terms that correspond to the polarization of electromagnetic radiation and terms that

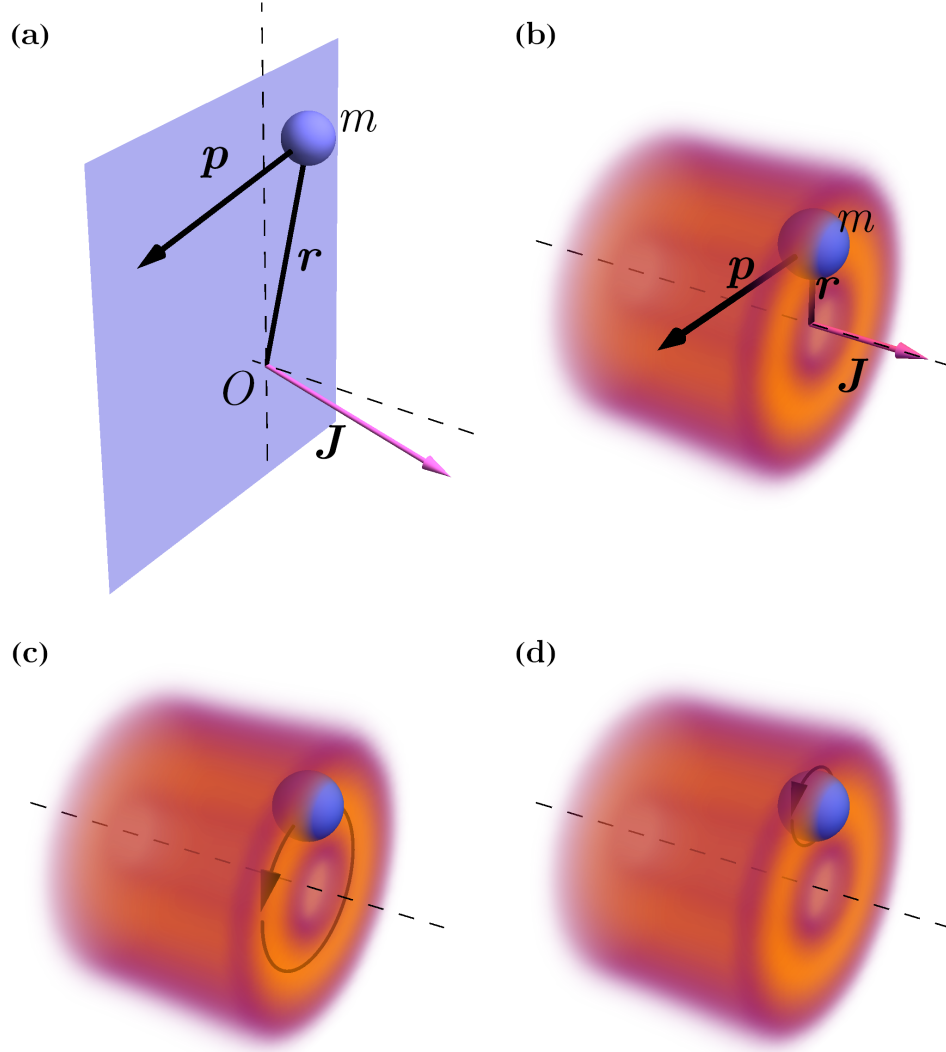


Figure 2.1: Angular momentum in classical physics. (a) The angular momentum of a point particle with respect to point O is given by $\mathbf{J} = \mathbf{r} \times \mathbf{p}$, where $\mathbf{r}$ is the position vector and $\mathbf{p}$ is the linear momentum of the point particle. (b) Not only point particles but also beams of light can have a specific value of angular momentum. When present in the beam, a point particle is propelled by the beam according to the local density of the angular momentum. (c) In the paraxial approximation, the total angular momentum of a Laguerre-Gaussian beam decomposes into the orbital and spin parts. The orbital angular momentum forces a particle to orbit around the beam axis. (d) On the contrary, the spin angular momentum, corresponding to the light polarization, causes the particle to spin around its own axis.

depend on the spatial profile of the field. These other terms are related to *orbital angular momentum* (OAM) [12]. The decomposition of the total angular momentum of a general electromagnetic field into its polarization and orbital part is problematic and has been a subject of thorough analysis by many authors [13, 14, 10, 15, 16]. One of the problems is that in general the individual components are not gauge invariant and thus considered unphysical. We avoid the complicated discussion of the general case by resorting to the special class of electromagnetic fields that roughly correspond to a collimated beam propagating along a specific direction. In such a case, the variation of the field along the axis of propagation, henceforth identified with the z -axis, is small compared to the variation in the transversal plane defined by the x - and y -axis. This regime is mathematically embodied in the so-called paraxial approximation. When such an approximation is applied to the wave equation that governs the evolution of the field, the equation reduces to the paraxial wave equation. The resulting solutions to the latter possess a well-defined polarization as well as orbital angular momentum. For such fields, one can write the total angular momentum $\mathbf{J}$ in the form [12, 17]

$$\mathbf{J} = \mathbf{S} + \mathbf{L}, \quad (2.4)$$

where $\mathbf{S}$ and $\mathbf{L}$ denotes polarization and orbital angular momentum, respectively. To illustrate the difference between the two components, one can consider the effect of a light beam on trapped particles, cf. Fig. 2.1(b). Whereas the OAM propels the particle around the axis of the beam [18], see Fig. 2.1(c), the circular polarization of the beam causes the particle to rotate around its own axis, as shown in Fig. 2.1(d). Even though the value of polarization is independent of the choice of the coordinate system, the same does not hold true for the orbital angular momentum. From now on, we fix the coordinate system such that its origin lies on the beam axis, which is identified with the z -axis, and will only focus on the z -component of the angular momentum that points in the direction of the beam propagation. For simplicity, we also consider only monochromatic beams in the following.

There are different classes of electromagnetic fields that can carry non-zero orbital angular momentum, the most prominent being the beams, whose phase winds around the beam axis to produce helical wavefronts. The electric field of such beams factorizes into the radial and azimuthal parts and its form can be expressed in cylindrical coordinates as

$$E(r, \phi, z) = A(r, z) \exp(im\phi), \quad (2.5)$$

where r is the radial coordinate, ϕ is the azimuthal coordinate, z is the axial position, m is a real number and $A(r, z)$ is the amplitude that satisfies the paraxial wave equation. In order for the field E to be smooth at $\phi = 2\pi \equiv 0 \pmod{2\pi}$, the number m has to be an integer as $E(r, 2\pi, z) = E(r, 0, z)$ is satisfied only when $\exp(2\pi mi) = 1$. By far the most utilized class of beams with helical wavefronts are the Laguerre-Gaussian modes, which are the subject of the next section. The existence of a well-defined OAM for such beams was pointed out first in the seminal paper by Allen and coworkers [12] and studied both theoretically and experimentally ever since. These authors noted that the helical profile of the beam wavefronts is to the orbital angular momentum what the polarization of the beam is to its spin. This way, they built on the work of Poynting [19], who suggested in 1909 that a circularly polarized light exerts a torque on neighboring particles. Since the angular momentum of a beam is an extensive quantity, one usually considers the angular momentum of a unit-length portion of the beam J_z and compares it with the total energy W of the same portion. Poynting's suggestion, confirmed experimentally in 1936 by Beth

[20], claims that the angular momentum of a circularly polarized monochromatic Gaussian beam is given by $J_z = \pm(1/\omega)W$, where the sign depends on whether the polarization is right- or left-handed and where ω is the angular frequency of the electromagnetic wave. Allen et al. [12] proposed an experiment to measure the torque of the beam of the form (2.5) on cylindrical lenses. They found that the total angular momentum of such a light beam (in the direction of propagation) is equal to

$$J_z = \frac{m + \sigma}{\omega} W, \quad (2.6)$$

where m is given in (2.5) and σ is the polarization term equal to $\sigma = \pm 1$ for circular polarization and $\sigma = 0$ for linear polarization. As a result, a helical beam has a non-zero value of angular momentum even if its polarization is linear. This additional contribution causes the illuminated object to rotate, as has also been tested experimentally [21, 22].

Let us mention for completeness that the Laguerre-Gaussian modes are not the only beams with non-zero orbital angular momentum. Another class of beams with helical wavefronts are higher-order Bessel beams [23], noteworthy for their self-healing properties, which also attracted the attention of researchers recently. The existence of non-zero OAM is nevertheless not restricted to the beams of the form (2.5). As shown in [24], an elliptic Gaussian beam propagated through a tilted cylindrical lens can acquire very high orbital angular momentum.

2.2 Laguerre-Gaussian modes

As already mentioned above, the prominent example of beams with helical wavefronts are the higher-order Laguerre-Gaussian beams. The purpose of this section is to discuss their form and basic properties in detail, to which end we start with the simplest thinkable beam of light, the collimated beam. For the sake of mathematical convenience we use the cylindrical coordinate system, already introduced above, where the axis of symmetry coincides with the direction of the beam propagation. Moreover, let us fix the intensity of the beam to one, such that it can later be interpreted as a wave function. Strictly speaking, there is not much to be said about a collimated beam of light since such a beam does not exist in the real world. It is only a handy approximation to the Gaussian beam, depicted in Fig. 2.2. The Gaussian beam, on the contrary, has many interesting properties, some of which we review below.

A Gaussian beam always exhibits some degree of divergence and there is always exactly one spot where the beam diameter takes on the minimum value, such that the beam in the course of its propagation gets focused down to this spot and after traversing the spot it diverges again. This narrowest spot is called the beam waist of the Gaussian beam and the radius of the beam at its waist is denoted by w_0 . This value and the wavelength λ are the only two parameters (apart from the direction, the intensity, and other degrees of freedom such as polarization) that fully characterize a Gaussian beam. The complex amplitude of such a beam in a spatial point described by cylindrical coordinates (r, ϕ, z) is given by formula [25, 26]

$$E_{0,0}(r, \phi, z) = \sqrt{\frac{2}{\pi}} \frac{w_0}{w(z)} \exp\left(-\frac{r^2}{w^2(z)}\right) \exp\left(-ikz - ik\frac{r^2}{2R(z)} + i\psi_{0,0}(z)\right), \quad (2.7)$$

where k is the wave number satisfying $k = 2\pi/\lambda$, and where the three functions $w(z)$, $R(z)$, and $\psi_{0,0}(z)$ are the beam radius, curvature, and Gouy phase of the beam in distance z from the beam waist, respectively. The beam radius is defined as the radial distance from the center, for which the intensity of the beam drops to $\exp(-2)$ when compared to the intensity in the center and reads [25]

$$w(z) = w_0 \sqrt{1 + \left(\frac{z}{z_R}\right)^2}, \quad (2.8)$$

where $z_R = \pi w_0^2/\lambda$ is the so-called Rayleigh range of the Gaussian beam. The curvature is determined by [25]

$$R(z) = z \left(1 + \left(\frac{z_R}{z}\right)^2\right), \quad (2.9)$$

and the Gouy phase reads $\psi_{0,0}(z) = \arctan(z/z_R)$. One can observe right away that the expression (2.7) does not depend on the azimuthal coordinate ϕ and the beam is therefore cylindrically symmetric. One can identify three parts of the expression (2.7). The first part, independent of r , is only the normalization factor that makes sure that the total intensity for each cross-section with fixed z is equal to one. The second part is a real-valued exponential factor that governs the radial intensity profile of the beam in each cross-section. Apart from the overall scaling, the beam intensity profile preserves its shape on propagation. Finally, the third part of the expression is a complex-valued exponential that depends both on the curvature of the beam at a given point and on the corresponding Gouy phase. This extra dependence results in the curved wavefronts of the Gaussian beam, as can be seen in Fig. 2.2. At the beam waist, the wavefront is completely flat as if it belonged to a plane wave. On the other hand, the further from the beam waist one goes, the more the wavefront resembles that of a spherical wave.

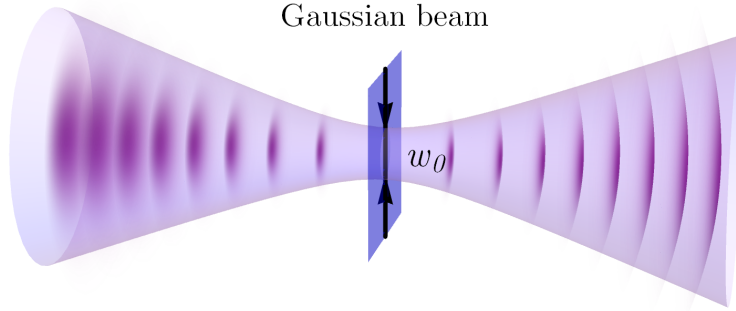


Figure 2.2: Gaussian beam is a physical realization of the abstract notion of a collimated beam. It is characterized by the wavelength λ and its radius at the beam waist w_0 . For convenience, a plane is drawn in the figure that intersects the beam at the beam waist. The wavefronts of the beam are curved and for a large enough distance from the beam waist they resemble spherical wavefronts.

So much for the Gaussian beam. Are there any other kinds of light beams that preserve their spatial profile under propagation? As a matter of fact, there are plenty of them. They

are in general referred to as *modes*. It also turns out that a given set of modes forms a basis in the sense that any spatial profile of an electromagnetic field, no matter how complex it is, can be represented as a linear combination of such modes. There are different families of modes, such as Hermite-Gaussian [26], Ince-Gaussian [27], or Laguerre-Gaussian, to name a few. It is the latter modes that are of primary focus in this thesis. The mathematical form of Laguerre-Gaussian modes can be ultimately derived from Maxwell equations [28, 25] when several additional constraints are in place. Namely, the equations are simplified using the paraxial approximation, the cylindrical symmetry is imposed from the outset and one considers free boundary conditions to model the free space propagation (as opposed to the propagation in an optical fiber, for instance). The derivation lies well beyond the scope of this text — for more information see Ref. [26] and references therein. The resulting set of modes is parameterized by two integer numbers l and p , where l can take on any negative, zero, or positive value, whereas p can only be non-negative. A Laguerre-Gaussian mode characterized by l and p is given by a somewhat complicated mathematical expression of the form [26]

$$E_{l,p}(r, \phi, z) = \sqrt{\frac{2p!}{\pi(p+|l|)!}} \frac{w_0}{w(z)} \left(\frac{r\sqrt{2}}{w(z)} \right)^{|l|} \exp\left(-\frac{r^2}{w^2(z)}\right) L_p^{|l|}\left(\frac{2r^2}{w^2(z)}\right) \times \exp\left(-ikz - ik\frac{r^2}{2R(z)} + i\psi_{l,p}(z)\right) \exp(-il\phi), \quad (2.10)$$

where the meaning of symbols k , $w(z)$, and $R(z)$ stays the same as for the case of the Gaussian beam, see Eqs. (2.8) and (2.9); $L_p^{|l|}$ is a generalized Laguerre polynomial; and where Gouy phase takes on a more general form $\psi_{l,p}(z) = (2p + |l| + 1) \arctan(z/z_R)$. The p parameter governs the radial structure of each mode. It is, however, of no interest to us in the rest of the thesis and we set it to $p = 0$. On the contrary, the l parameter is directly related to the orbital angular momentum and it will be of particular importance from now on. Many properties of the beam can be studied when one looks only at the beam at its waist, setting effectively $z = 0$. For this choice of parameters, the above complicated mathematical expression reduces to

$$E_{l,0}(r, \phi, 0) = \sqrt{\frac{2}{\pi|l|!}} \left(\frac{r\sqrt{2}}{w_0} \right)^{|l|} \exp\left(-\frac{r^2}{w_0^2}\right) \exp(-il\phi) \quad (2.11)$$

as for $p = 0$ the generalized Laguerre polynomial is identically equal to one, $L_0^{|l|}(x) = 1$. This formula can be further simplified by introducing a normalized radial distance $\rho = r/w_0$, for which

$$E_l(\rho, \phi) \equiv E_{l,0}(\rho w_0, \phi, 0) = \sqrt{I_l(\rho)} \exp(-il\phi), \quad (2.12)$$

where the intensity I_l of E_l is given by

$$I_l(\rho) = \frac{2^{|l|+1}}{\pi|l|!} \rho^{2|l|} \exp(-2\rho^2). \quad (2.13)$$

The expression (2.12) is (for each ρ and ϕ) a complex number in the polar form, where the first term determines the magnitude of E_l , whereas the second term determines the complex phase. As is easy to see, the intensity of the beam goes like $I_l(\rho) \sim \rho^{2|l|} \exp(-2\rho^2)$

as the radial coordinate ρ increases. This functional dependence has for any non-zero l a similar behavior: it starts at zero, rises until it reaches the maximum value for

$$\rho_{\max} = \sqrt{\frac{l}{2}} \quad (2.14)$$

and then it converges to zero again. Since the magnitude of E_l (2.12) is independent of angle ϕ , the resulting intensity profile resembles a doughnut, whose radius increases as the square root of parameter l and in whose center there is a phase singularity with the intensity null. The only exception to this doughnut-like property is when $l = 0$, in which case one recovers the familiar Gaussian beam, whose intensity maximum is precisely in the center of the beam. One can thus think of Laguerre-Gaussian modes as higher-order generalizations of the Gaussian mode. Moreover, the azimuthal dependence of the complex amplitude is restricted to the term $\exp(-il\phi)$. For $l = 0$ there is no dependence and the resulting wavefronts are flat. For $l = 1$ the phase linearly increases with ϕ from 0 to 2π . One can picture the resulting wavefront as a helix that winds around the beam axis. For $l = 2$ one obtains two helices and an analogous observation holds also for larger positive l . For negative l , the helix winds in the opposite sense. In Fig. 2.3, both the intensity and the phase of the lower-order Laguerre-Gaussian modes are shown explicitly.

2.3 Quantum regime

The discussion in the preceding section considered classical modes of light, where the corresponding expressions (2.10)–(2.13) were normalized, such that the total intensity in each cross-section along the axis of propagation was equal to one. At the same time though, the exact same expressions can be viewed as wave functions of a single photon, where the normalization represents the fact that the probability of a photon's detection somewhere in the transversal plane is equal to one and the photon is thus not lost on its way. In the following, we describe the scenario of previous sections from the quantum perspective, where angular momentum is represented by the corresponding quantum-mechanical operator.

First, let us take a quick look back to (2.6). This formula has a straightforward interpretation when recast into the language of photons. If we understand the beam as being composed of N photons, each of quantized energy $\hbar\omega$, where $\hbar$ is the reduced Planck constant, its energy reads $N\hbar\omega$ and the angular momentum of each photon $J_z^{(1)} = J_z/N$ reduces to

$$J_z^{(1)} = \hbar m + \hbar \sigma. \quad (2.15)$$

The angular momentum of each photon is thus quantized and decomposes into two terms. The second term corresponds to the spin, which is an intrinsic quantum property associated with the polarization of photons. It is well-studied and has been used extensively in many experiments in classical (as polarization) as well as quantum physics. The first term, on the contrary, corresponds to the orbital angular momentum and we re-derive its form using quantum-mechanical calculation in the following paragraph.

In quantum physics, the role of the position and momentum vectors $\mathbf{r}$ and $\mathbf{p}$ is played by the position and momentum operators $\hat{\mathbf{X}}$ and $\hat{\mathbf{P}}$, respectively, which act on the wave function of a given quantum system. As mentioned in the introduction, it is problematic to define a wave function for a photon. For this reason, we adopt the following approach that does not strive to present a rigorous derivation but rather presents a somewhat effective

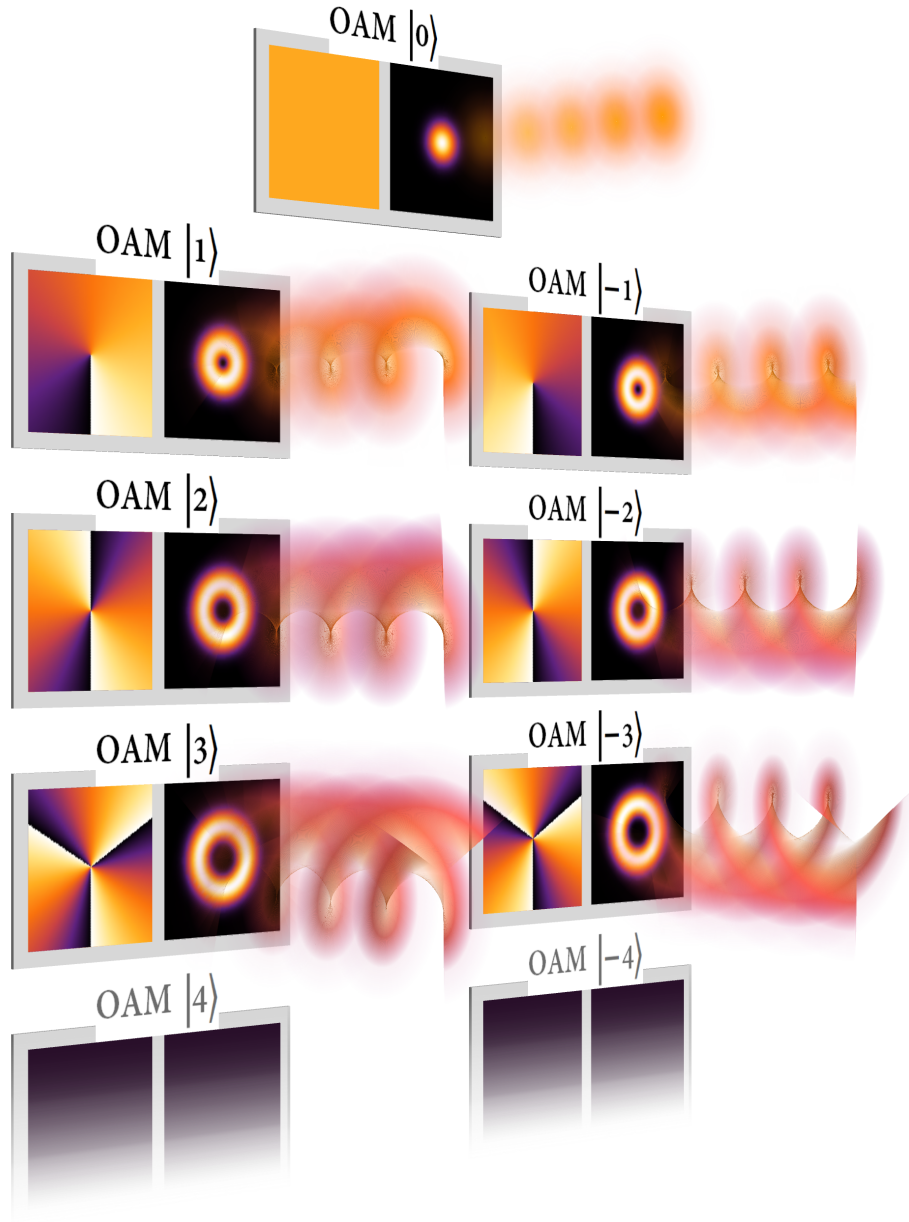


Figure 2.3: OAM modes. The fundamental mode, shown at the top, contains zero quanta of OAM and coincides with the Gaussian beam. The OAM values of higher-order modes can be both positive and negative with no upper or lower limit. For details see the text.

description and serves mainly as an introduction to the topics discussed later in the text. As far as the polarization of a photon is concerned, one can characterize a general mixed quantum state with a 2×2 density matrix. Since we are mostly interested not in the polarization but in the other component of the angular momentum — the orbital angular momentum — we consider a function $\psi(\mathbf{x})$ defined over the position space, for which the Born's rule applies and so $q = |\psi(\mathbf{x})|^2$ determines the probability density of detecting the photon at point $\mathbf{x} = (x_1, x_2, x_3)$. The individual Cartesian components of the position and momentum operators then act on such a function as [29]

$$\hat{X}_j \psi(\mathbf{x}) = x_j \psi(\mathbf{x}), \quad (2.16)$$

$$\hat{P}_j \psi(\mathbf{x}) = -i \hbar \partial_j \psi(\mathbf{x}). \quad (2.17)$$

When we plug these expressions into (2.1) and represent the cross product with the Levi-Civita symbol ϵ_{jkl} , we obtain the position representation of the quantum angular momentum operator $\hat{\mathbf{L}}$:

$$\hat{L}_j \psi(\mathbf{x}) = -i \hbar \epsilon_{jkl} x_k \partial_l \psi(\mathbf{x}). \quad (2.18)$$

In view of the upcoming discussion, it is advisable to express this formula in cylindrical coordinates, where $\mathbf{r} \equiv (r, \phi, z) = (\sqrt{x_1^2 + x_2^2}, \arctan(x_2/x_1), x_3)$. Also, we will focus only on the third component of the angular momentum operator, $\hat{L}_z$, which takes on in cylindrical coordinates a particularly simple form

$$\hat{L}_z \psi(\mathbf{r}) = -i \hbar \partial_\phi \psi(\mathbf{r}). \quad (2.19)$$

Up to a numerical prefactor, the angular momentum corresponds to the partial derivative of a wave function with respect to the azimuthal coordinate ϕ .

The possible values that one can get when measuring the orbital angular momentum and the corresponding post-measurement states are given by the eigenvalues and eigenvectors of operator $\hat{L}_z$. When we take an ansatz of the form (2.5), i.e., $\psi_m(r, \phi, z) = A_m(r, z) \exp(im\phi)$, it is a matter of elementary mathematics to show that for such a choice of the wave function the equation (2.19) reduces to

$$\hat{L}_z \psi_m(\mathbf{r}) = m \hbar \psi_m(\mathbf{r}). \quad (2.20)$$

One sees that ψ_m is an eigenfunction of $\hat{L}_z$ associated with eigenvalue $m\hbar$. Moreover, each ψ_m satisfies the normalization condition $\int_{\mathbb{R}^3} |\psi(r, \phi, z)|^2 r dr d\phi dz = 1$ and to ensure the continuity of the wave function, number m has to be an integer. This number, called the *topological charge* in literature, is otherwise unbounded and the orbital angular momentum thus, unlike spin, represents an infinite-dimensional discrete degree of freedom. The value of OAM is quantized with the increment equal to the reduced Planck constant $\hbar$, which we omit from now on and talk only about m quanta of OAM or alternatively about a value m of OAM. The eigenstates for different OAM values are mutually orthonormal and can be used to form a basis of OAM state space.

For eigenfunctions we henceforth take the Laguerre-Gaussian modes, which comply with the ansatz made above. We also adopt the bracket notation and instead of ψ_m work with $|m\rangle$, which is now a superposition of the form [30]

$$|m\rangle = \int_0^\infty d\rho \int_0^{2\pi} \rho d\phi \sqrt{I_m(\rho)} \exp(-im\phi) a^\dagger(\rho, \phi) |\Omega\rangle, \quad (2.21)$$

where $|\Omega\rangle$ is the vacuum state, $a^\dagger(\rho, \phi)$ is a photon creation operator associated with the point given by transversal coordinates (ρ, ϕ) , and where the formula (2.12) was used. Quantum states of the form (2.21) are notable for the fact that for large enough m , the corresponding wave function reaches macroscopic dimensions [31]. From Eq. (2.14) it is easy to derive that for $m \sim 10^5$ (when we assume physically achievable beam waist with $w_0 \sim 100 \mu m$) the mode has the size on the order of centimeters. Even though the number of OAM quanta m is unbounded, in applications one restricts oneself only to a finite set of modes with small m . Such a restriction allows for the physical representation of qudits in quantum information processing. In the next chapter, we use this particular implementation of qudits and study its potential for the construction of unitary gates in high dimensions.

3 Manipulation of OAM of single photons

Physical systems and the phenomena associated with their quantum behaviour can be studied from various points of view. Apart from the basic research that focuses on their fundamental aspects to better understand the quantum theory, and apart from the applied research that aims at finding potential applications for such phenomena and systems, there is the field of quantum information and quantum computation that can be argued to lie somewhere in between. On the one hand, this field questions the founding principles of quantum theory from the information-theoretic perspective, but on the other, it devises novel methods allowing to perform complicated computational tasks in a way significantly more efficient than classical computers can deliver.

Different quantum systems can be employed as carriers of information. Each system and degree of freedom has its pros and cons and it is the particular application area that decides which one is chosen in the end. Photons are usually the systems of choice when it comes to transmission of information over free-space links in applications such as quantum key distribution or quantum teleportation. A necessary prerequisite for utilizing the orbital angular momentum of photons as a full-fledged carrier of information is to identify its capabilities in terms of efficient generation, manipulation, and detection. From a somewhat abstract perspective, one has to be able to perform the following classes of operations on quantum systems for them to be useful:

1. Generation of the fundamental mode,
2. Performing local gates,
3. Performing entangling gates,
4. Detection in the computational basis.

In this chapter, we focus on the second point, that is the ability to perform local gates on qudits encoded in the orbital angular momentum of single photons. It turns out that any finite-dimensional unitary operation can be implemented experimentally that acts on the OAM of single photons. In the rest of the chapter, we review several different methods that can be used in this regard, mention their advantages and disadvantages and at the end of the chapter, specific examples of important operations are presented in greater detail.

Regarding the other three points mentioned above, one can replace the necessity to generate photons in a fundamental mode together with the ability to perform entangling gates with the requirement that photons can be generated in a well-chosen entangled state right away. As the measurement-based quantum computation architecture [32, 33] demonstrates the local measurements on a highly entangled state of many qubits are

sufficient for universal computation. From this perspective, chapter 4 is particularly of note as there the generation of customizable entangled states is discussed in detail. Considering the local measurements, any local measurement can be turned into the computational-basis measurement by a proper choice of local unitaries. The measurement of OAM modes in the computational basis is discussed in chapter 4 as well.

3.1 Available operations

As the first step in our program of demonstrating that any unitary can be implemented experimentally that acts on OAM of photons, we list the set of relevant elementary operations. There are multiple ways how the spatial profile of a photon's wave function can be manipulated. One of the most general transformations is delivered by spatial light modulators, devices akin to traditional LCD displays that can pixel by pixel adjust the local phase as well as the magnitude of the incoming light beam. Spatial light modulators can be used to impart complex phase profiles onto the incoming OAM modes, see Fig. 3.1, but their downside is their cost and the fact that they require an external control from a computer.

There is another class of elements, henceforth referred to as *conventional optical elements*, that are known from classical optics and display low level of design complexity. To these elements belong phase shifters, mirrors, traditional beam splitters, Dove prisms, and spiral phase plates. In the following, we briefly mention the action of these elements on OAM of a photon, see also Fig. 3.1. To that end, the following convention is adopted: Throughout the entire thesis we assume to work with qudits represented as OAM modes living in a d -dimensional Hilbert space, whose computational basis is formed by OAM eigenstates $\{|0\rangle_O, |1\rangle_O, \dots, |d-1\rangle_O\}$. (Other choices of eigenstates are of course possible, but in the following it suffices to consider this particular one.) We label the OAM eigenstates by subscript O to distinguish them from the path modes, which we denote by $\{|0\rangle_P, |1\rangle_P, \dots, |d-1\rangle_P\}$ with subscript P . A state of a photon that has k quanta of OAM and propagates along path p is thus denoted by $|k\rangle_O |p\rangle_P$. To capture the action of different optical elements, we sometimes assume that the OAM value of k is not constrained to lie between 0 and $d-1$. For example, to describe the action of a mirror, it is inevitable to take into account also negative values of k , as detailed below.

Mirror: A mirror reverses the sign of the incoming OAM beam, such that a photon with k quanta of OAM is reflected off the mirror having $-k$ quanta of OAM. That is, for an OAM mode k propagating along path p one gets:

$$\text{mirror : } |k\rangle_O |p\rangle_P \rightarrow -|-k\rangle_O |p\rangle_P. \quad (3.1)$$

As the prefactor of -1 can be removed by a phase shifter, discussed in the next paragraph, we will drop it and model a mirror as a mere reversal of the OAM value.

Phase shifter: The only action of a phase shifter is to modify the phase of the beam, such that an OAM mode $|k\rangle_O |p\rangle_P$ with k quanta of OAM and propagating along path p is transformed according to:

$$\text{phase shifter : } |k\rangle_O |p\rangle_P \rightarrow e^{i\varphi} |k\rangle_O |p\rangle_P, \quad (3.2)$$

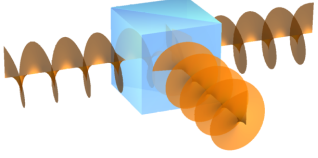
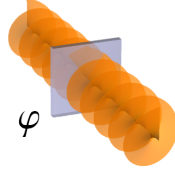
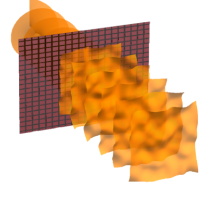
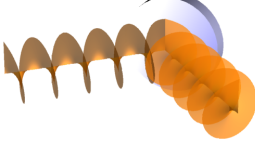
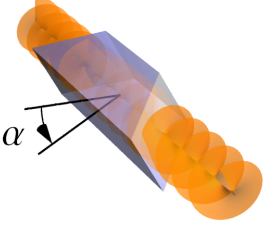
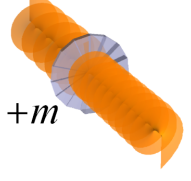
beam splitter  $ k\rangle \rightarrow \frac{ k_1\rangle + -k_2\rangle}{\sqrt{2}}$	phase shifter  $ k\rangle \rightarrow e^{i\varphi} k\rangle$	SLM  $ k\rangle \rightarrow U k\rangle$
mirror  $ k\rangle \rightarrow - -k\rangle$	Dove prism  $ k\rangle \rightarrow -e^{2i\alpha k} -k\rangle$	spiral phase plate  $+m$ $ k\rangle \rightarrow k+m\rangle$

Figure 3.1: Manipulation of a spatial profile of a photon's wave function. A spatial light modulator (SLM) allows for a very flexible modification of the phase of the beam, but in the text we focus on all the other elements, for convenience collectively referred to as conventional optical elements. For details see the text.

where φ is a real constant pertaining to a particular phase shifter and is independent of the OAM value of the beam.

Dove prism: This property is no longer true for a Dove prism. A Dove prism is a prism of a trapezoidal shape, whose operation is given by formula:

$$\text{Dove prism : } |k\rangle_O |p\rangle_P \rightarrow -e^{2\alpha k i} |-k\rangle_O |p\rangle_P. \quad (3.3)$$

Since the reversal of the OAM value as well as the extra phase of -1 can be corrected for by an extra mirror, from now on we will model the action of a Dove prism as an OAM-dependent phase shift with no change of the OAM value. This dependency will prove crucial later on.

Beam splitter: A symmetric beam splitter not only splits the propagation paths but also reverses the sign of the reflected beam, similarly to a mirror. For a beam splitter acting on two paths denoted by p and q we obtain the following transformation rules:

$$\text{beam splitter : } \begin{cases} |k\rangle_O |p\rangle_P \rightarrow \frac{1}{\sqrt{2}}(|k\rangle_O |p\rangle_P + |-k\rangle_O |q\rangle_P), \\ |k\rangle_O |q\rangle_P \rightarrow \frac{1}{\sqrt{2}}(|-k\rangle_O |p\rangle_P - |k\rangle_O |q\rangle_P). \end{cases} \quad (3.4)$$

Spiral phase plate: The last element we will need, and at the same time the only one that directly modifies the OAM value of the incoming photon, is a *spiral phase plate*, also

known as a *vortex plate*. Its action on the OAM mode $|k\rangle_O |p\rangle_P$ can be described by:

$$\text{spiral phase plate : } |k\rangle_O |p\rangle_P \rightarrow |k+m\rangle_O |p\rangle_P, \quad (3.5)$$

where m is a fixed integer value that is given by the shape of the spiral phase plate and which can be both positive and negative.

These are all the optical elements that we will need in the following discussion. It turns out that rather than working directly with these elements it is convenient in many cases to consider simple optical setups built out of these elements and use these setups instead as elementary building blocks for more complicated operations.

3.2 Arbitrary unitary in OAM of single photons

Several different approaches have been developed that allow for the construction of arbitrary unitaries that act on OAM of single photons. Loosely speaking, these approaches can be divided into two larger groups. The first group contains methods that are based on pixel-by-pixel phase modulation of the spatial profile of a photon's wave function. These techniques require additional optimization algorithms that generate the appropriate phase profiles that are subsequently imparted to the photon. The second group encompasses the methods that employ traditional optical elements and do not rely on optimization algorithms. Below we briefly discuss both groups of methods and list their advantages as well as possible disadvantages.

As just mentioned, the first group of methods makes use of complex computer-generated phase profiles. The simplest example of this group is the utilization of a spatial light modulator (SLM), mentioned in the previous section. This device can display on its screen nontrivial phase profiles that are subsequently imparted to the incoming beam, both in the classical and quantum regimes. By generalizing this idea, several methods have been developed that rely on the use of nontrivial phase profiles and free space propagation to implement broad classes of transformations. For instance, a sorter of OAM modes can be implemented with two specially designed optical lenses [34] or using scattering of light off heterogeneous medium that is controlled by an SLM [35]. A notable example of such techniques is the multi-plane light conversion (MPLC) [36, 37, 38], schematically presented in Fig. 3.2. This approach relies on the observation that any unitary transformation U of spatial modes can be decomposed into an alternating series of Fourier transforms $\mathcal{F}$ and phase planes $\mathcal{P}_k$ that impart an additional phase profile to the beam. That is:

$$U = \mathcal{F} \circ \mathcal{P}_n \circ \dots \circ \mathcal{F} \circ \mathcal{P}_3 \circ \mathcal{F} \circ \mathcal{P}_2 \circ \mathcal{F} \circ \mathcal{P}_1 \circ \mathcal{F}. \quad (3.6)$$

It may happen for a given unitary U that the number of phase planes would have to be infinite to precisely reconstruct U . However, when choosing n finite but large enough, numerical optimization algorithms can be used to find such phase profiles that approximate U with good accuracy. Let us emphasize that the Fourier transform as well as the multiplication by a phase profile are themselves unitary transformations and so is their product. The expression on the right-hand side of Eq. (3.6) is thus always some unitary — the nontrivial task is, however, how close this unitary is to our chosen U . The MPLC approach is very flexible and conceptually very simple, yet powerful. On the other hand,

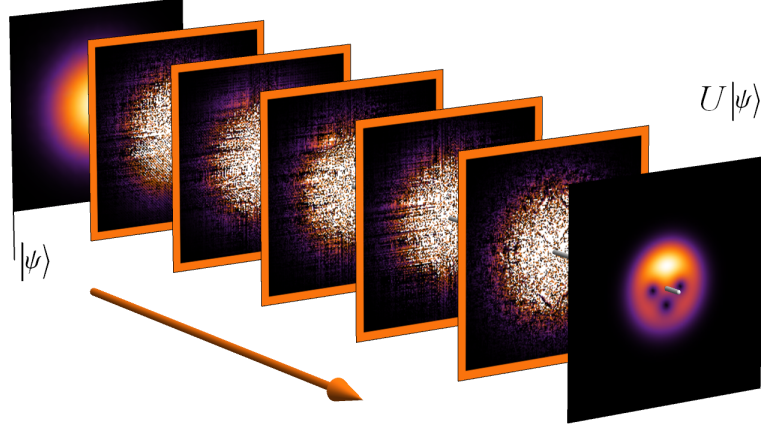


Figure 3.2: Computer-generated complex phase profiles can be used to apply an arbitrary unitary onto the OAM of the incoming beam. In the figure, a technique of multi-plane light conversion (MPLC) is shown, where the incoming beam in state $|\psi\rangle$ is transformed via propagation through a series of planes with complex phase profiles. When leaving the setup, the spatial form of the photon's wave function is approximately equal to $U|\psi\rangle$. The larger the number of planes, the more precisely the actual unitary evolution of the photon approximates the given unitary U . For details see the text.

sophisticated numeric algorithms have to be utilized in the design stage and precise tools are needed in the manufacturing stage. The equipment, such as SLMs, is also rather costly.

Let us turn our attention now to the second group of methods, which employ standard optical elements with no need to invoke computer optimization algorithms. One such method is based on a special kind of decomposition of a general unitary into a series of high-dimensional Pauli gates. Using the properties of Heisenberg-Weyl operators it can be shown that any finite-dimensional unitary can be expressed as a linear combination of X and Z gates. To be specific, consider an arbitrary unitary operator U acting on a d -dimensional space $\mathcal{H}$ with basis states $\{|m\rangle\}_{m=0}^{d-1}$. For this operator there exist complex coefficients α_{kl} , such that [39, 40]

$$U = \sum_{k=0}^{d-1} \sum_{l=0}^{d-1} \alpha_{kl} X^k Z^l, \quad (3.7)$$

where the X and Z gates are d -dimensional Pauli matrices, whose form as well as implementation are discussed in section 3.3.3.

This decomposition works in general, regardless of a specific way in which the qudits are implemented. When we focus on the case of qudits implemented with the OAM degree of freedom, we can identify three stages of the final experimental setup, as shown in Fig. 3.3. In the first part, the photon's wave function, propagating along a single path, is split into d^2 different paths by a d^2 -dimensional version of an asymmetric beam splitter. There is one path for each double of indices (k, l) . The splitting ratios of the beam splitter are given by the magnitudes of coefficients α_{kl} in Eq. (3.7). Likewise, the local phases of each outgoing beam are given by the phases of the corresponding coefficients. In the second stage, in

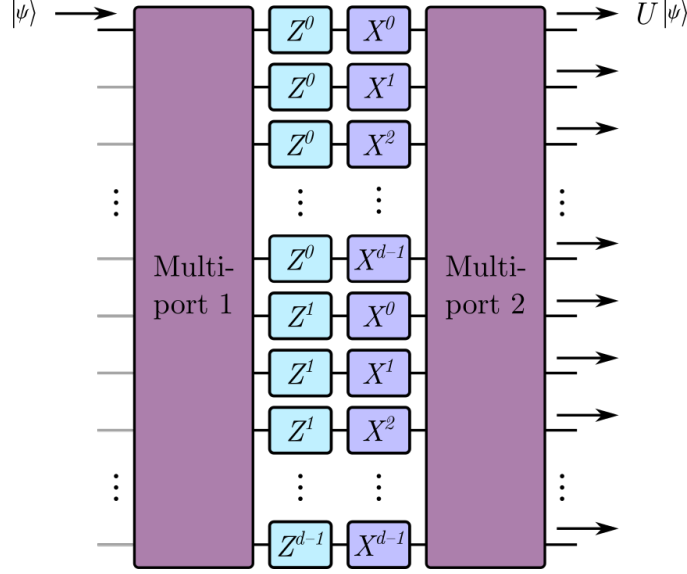


Figure 3.3: Any unitary can be decomposed into a linear combination of powers of X and Z gates. A schematic implementation is depicted that consists of a d^2 -dimensional asymmetric beam splitter, referred to as a multiport in the figure, followed by a series of elementary setups. Each of these setups implements a particular power of the Z and X gates (in this order). In the end, the wave-function terms from all the d^2 parts are recombined by another asymmetric beam splitter. Due to the unitarity of the beam splitter, the photons may leave the second multiport with non-zero probability from an undesirable port, which leads to the non-deterministic operation of the whole scheme.

the path indexed by (k, l) , there is a setup of the k -th power of the d -dimensional X gate and the l -th power of the d -dimensional Z gate, both acting on OAM of the propagating photon. In the third stage, all the beams are recombined again by another d^2 -dimensional beam splitter. This recombination comes at a price: there is a non-zero probability that a photon will be detected in one of $d^2 - 1$ undesirable paths. This unavoidable effect leads to the non-deterministic operation of the whole scheme.

The advantage of this proposal is that the experimental setup can be implemented using only conventional optical elements, such as beam splitters, phase shifters, spiral phase plates, Dove prisms and mirrors. The downside is, however, that the number of these elements scales unfavorably with the dimension. If we restrict ourselves only to a special class of dimensions of the form $d = 2^M$, we can easily estimate the number of elements. There are two asymmetric d^2 -dimensional beam splitters necessary, either of which can be implemented as an optical network of $O(d^4)$ classical beam splitters [41]. Then there are $O(d^3 \log_2(d))$ Dove prisms and $O(d^3 \log_2(d))$ beam splitters required for the implementation of the X and Z gates, see also section 3.3.3. Altogether, the whole scheme uses as many as $O(d^4)$ optical elements. In the following section, we discuss an alternative approach to that of Eq. (3.7) with more favorable scaling properties as well as fully deterministic operation.

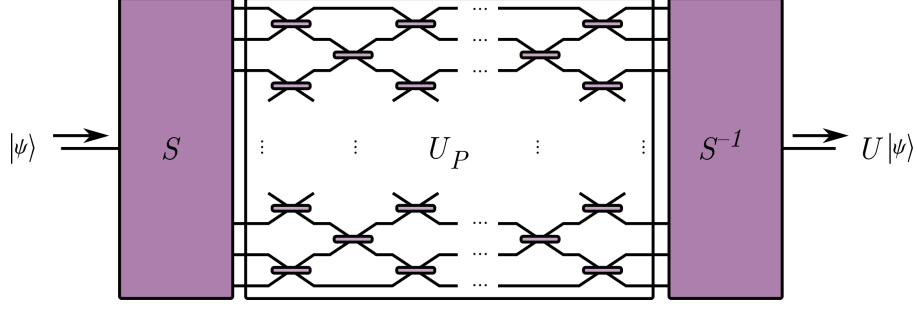


Figure 3.4: Path-assisted approach for implementing an arbitrary unitary in OAM of single photons. The OAM state enters the scheme from the left, where it is subjected to the action of a d -dimensional OAM sorter S , followed by the path-encoded version U_P of the unitary that one wishes to construct. All the transformed modes are rerouted back into a single output path by another OAM sorter, which is operated in reverse.

3.3 Path-assisted approach

An alternative technique for the implementation of an arbitrary unitary acting on OAM of single photons is the following method [42], which we from now on call the *path-assisted approach*. The following discussion is mostly based on, and builds upon the results of, our findings summarized in Refs. [42, 43]. One can transform the OAM degree of freedom of a photon into a spatial degree of freedom, apply the unitary encoded in path, and then transform path back into OAM. This approach is depicted schematically in Fig. 3.4 and can be summarized by formula

$$U = S^{-1} \cdot U_P \cdot S, \quad (3.8)$$

where U_P denotes the path-encoded unitary and S is an OAM sorter, which exchanges the path and OAM degrees of freedom. This way, we translated the problem of constructing arbitrary unitaries from OAM to the path, for which general implementation schemes are known. The above approach is conceptually very simple and one might even call it trivial. However obvious such an approach might be, it hasn't been reported in literature until Ref. [42], as far as the knowledge of the author goes. Not only is this approach conceptually simple but it also displays considerably better scaling properties than the approach based on the decomposition into X and Z gates and, unlike that approach, is deterministic.

The path-encoded unitary can be implemented using one of the available schemes, such as the Reck et al. scheme [41], or Clements et al. scheme [44]. Both schemes need only $O(d^2)$ beam splitters and $O(d^2)$ phase shifters for its implementation. As for the OAM sorter, many designs can be employed. Except those based on complex phase profiles [34, 37, 35, 38], there is also an implementation that requires only $O(d)$ conventional optical elements [45, 46, 47]. Altogether, this approach is lossless, deterministic, and requires only $O(d^2)$ elements.

A crucial role in this approach is played by the OAM sorter, which can be constructed from a set of elementary building blocks. Each building block is a simple passive interferometric setup, which we call an OAM *exchanger* from now on. The sorter is then an optical network of the form of a binary tree, where each node consists of an exchanger

of an appropriate order. We present the detailed description of exchangers, sorters, and path-encoded unitaries in their respective sections below.

3.3.1 Exchanger

The OAM exchanger is a simple interferometric setup, consisting of a Mach-Zehnder interferometer, supplemented by two Dove prisms and two spiral phase plates, see Fig. 3.5(a). It can be viewed as a generalization of both conventional beam splitters and polarizing beam splitters. The exact properties of a given exchanger are given by a parameter m , henceforth referred to as the *order* of the exchanger. The action of the exchanger of order m , denoted by symbol EX_m in the following, is governed by formulas:

$$\text{exchanger } \text{EX}_m : \begin{cases} |k\rangle_O |p\rangle_P \rightarrow e^{-i\beta_k} (\cos(\beta_k) |k\rangle_O |p\rangle_P + i \sin(\beta_k) |k-m\rangle_O |q\rangle_P), \\ |k\rangle_O |q\rangle_P \rightarrow e^{-i\beta_k} (\cos(\beta_k) |k+m\rangle_O |p\rangle_P + i \sin(\beta_k) |k\rangle_O |q\rangle_P), \end{cases} \quad (3.9)$$

where p and q are the paths on which the exchanger resides and where $\beta_k = \pi k / (2m)$. Despite the apparent complexity of the transformation formulas shown above, the exchangers have appealing algebraic properties that can be successfully employed in the construction of many complicated operations, as will be discussed below. As a matter of fact, a single exchanger of order m effectively works as $4m$ different types of (polarizing as well as non-polarizing) beam splitters.

By a closer inspection of formulas (3.9) one can see that the coefficients of the resulting states depend explicitly on the OAM mode value k . As a result, the exchanger works for some OAM modes as a symmetric beam splitter (BS), for some other modes it works as an asymmetric beam splitter, and for yet other modes it works as an OAM equivalent of a polarizing beam splitter (PBS). One can, among other things, identify two special regimes of operation of an OAM exchanger:

Exchanger as a polarizing beam splitter: For OAM modes whose value k is a multiple of the order m of the exchanger, the exchanger EX_m effectively works as a polarizing beam splitter. More specifically, when k is an even multiple of m , i.e., $k = 2lm$ for some $l \in \mathbb{Z}$, the transformation rules (3.9) reduce into a particularly simple form:

$$\text{even multiples : } \begin{cases} |k\rangle_O |p\rangle_P \rightarrow |k\rangle_O |p\rangle_P, \\ |k\rangle_O |q\rangle_P \rightarrow |k+m\rangle_O |p\rangle_P. \end{cases} \quad (3.10)$$

An analogous behavior is displayed for odd multiples of m , i.e., $k = (2l+1)m$ for some $l \in \mathbb{Z}$. In this case, the role of the two paths is basically swapped as:

$$\text{odd multiples : } \begin{cases} |k\rangle_O |p\rangle_P \rightarrow |k-m\rangle_O |q\rangle_P, \\ |k\rangle_O |q\rangle_P \rightarrow |k\rangle_O |q\rangle_P. \end{cases} \quad (3.11)$$

Even modes entering the exchanger along a given path p leave the exchanger unaffected, similarly to the H polarization in the case of traditional polarizing beam splitters, whereas odd modes entering the same path are rerouted into path q , similarly to the V polarization for PBSs, with simultaneous change of their OAM. In Figs. 3.5(b) and (c) the PBS-like property of exchangers of order 1 and 2, respectively, is shown explicitly.

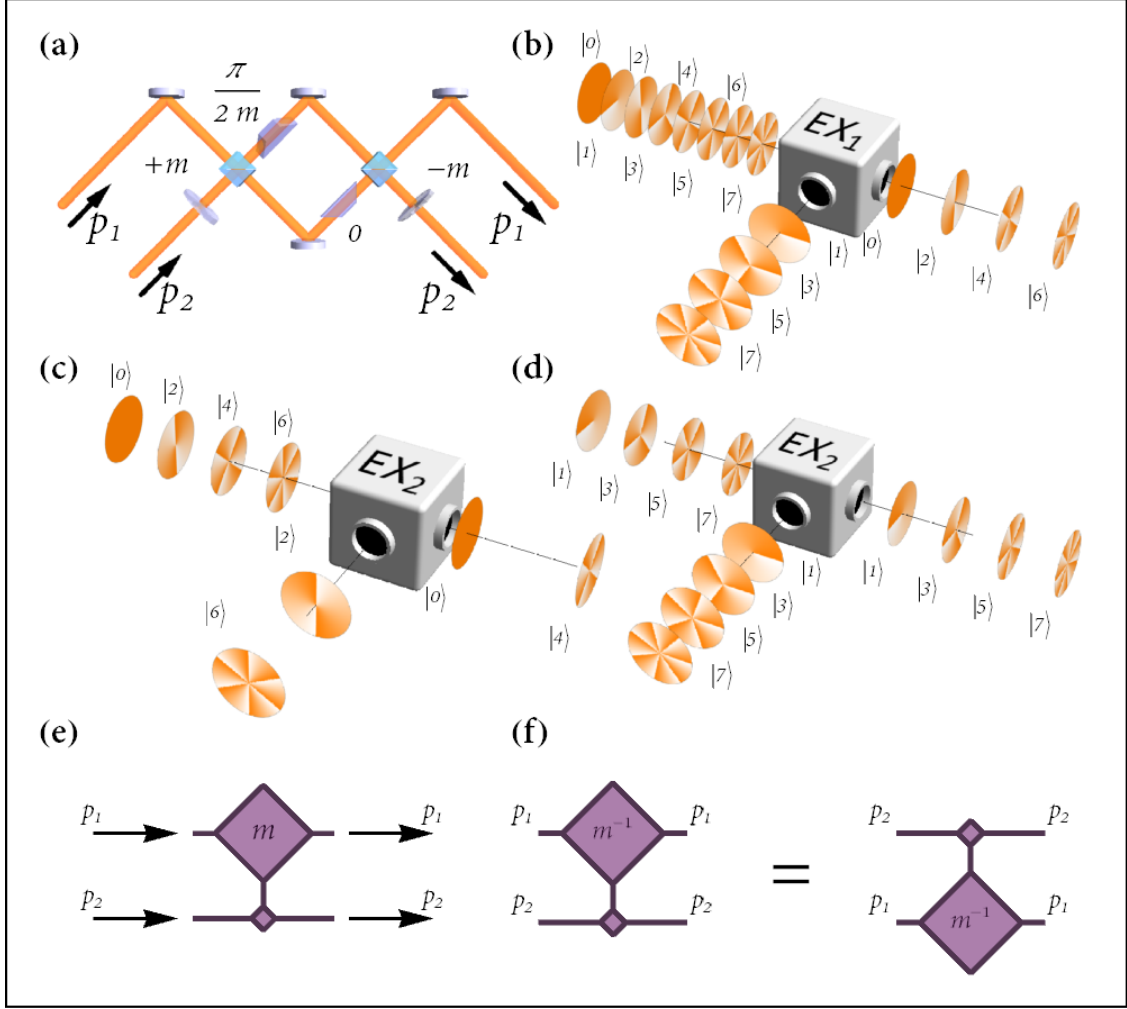


Figure 3.5: The OAM exchanger. (a) The exchanger is a passive two-input two-output device that consists of a Mach-Zehnder interferometer, in whose arms properly rotated Dove prisms are inserted, and two spiral phase plates that add and subtract m quanta of OAM, respectively. The exact angle $\alpha = \pi/(2m)$ of rotation of one of the Dove prisms is given by the order m of the exchanger and governs the sorting properties of the exchanger. (b) For instance, the exchanger EX_1 of order 1 sorts even and odd OAM modes into two different output ports. Note that for simplicity the modification of the OAM value due to spiral phase plates is not depicted. (c) When the same OAM modes are subjected to an exchanger of order 2, i.e., EX_2 , the even multiples of 2 leave via the upper port, and the odd multiples of 2 leave via the lower port. (d) Moreover, all the other OAM modes leave the exchanger in a balanced superposition of both ports. (e) In schematic diagrams, the exchanger of order m will be depicted as shown. (f) For the inverse of an exchanger, two slightly different symbols will be used.

Exchanger as a symmetric beam splitter: For OAM modes whose value k is an odd multiple of the order m divided by two, the exchanger EX_m effectively works as a symmetric beam splitter. When k is of the form $k = (4l + 1)m/2$ for some $l \in \mathbb{Z}$, the transformation rules (3.9) get simplified into:

$$(4l + 1)/2 \text{ multiples : } \begin{cases} |k\rangle_O |p\rangle_P & \rightarrow \frac{e^{-i\pi/4}}{\sqrt{2}}(|k\rangle_O |p\rangle_P + i|k - m\rangle_O |q\rangle_P), \\ |k\rangle_O |q\rangle_P & \rightarrow \frac{e^{-i\pi/4}}{\sqrt{2}}(|k + m\rangle_O |p\rangle_P + i|k\rangle_O |q\rangle_P). \end{cases} \quad (3.12)$$

One sees that the resulting modes are balanced superpositions of two spatial modes, akin to a symmetric beam splitter. Unlike the traditional beam splitter, in the present case one output mode possesses the same value of OAM as the input, whereas the value of the other mode is changed by m .

Again, an analogous behavior is obtained for k of the form $k = (4l + 3)m/2$ for some $l \in \mathbb{Z}$. Then the exchanger EX_m acts like:

$$(4l + 3)/2 \text{ multiples : } \begin{cases} |k\rangle_O |p\rangle_P & \rightarrow \frac{e^{-i\pi/4}}{\sqrt{2}}(i|k\rangle_O |p\rangle_P + |k - m\rangle_O |q\rangle_P), \\ |k\rangle_O |q\rangle_P & \rightarrow \frac{e^{-i\pi/4}}{\sqrt{2}}(i|k + m\rangle_O |p\rangle_P + |k\rangle_O |q\rangle_P). \end{cases} \quad (3.13)$$

By inspection of these formulas and relations (3.12) we see that the only difference from the previous case is a modified local phase between the two output modes. The BS-like behaviour of an exchanger of order 2 is shown explicitly in Fig. 3.5(d).

The exchangers of orders 1 and 2 can only exhibit the above two regimes. For exchangers of orders larger than 2 also the more general asymmetric-beam-splitter behaviour emerges. In the upcoming discussion it will become evident that the PBS-like property of exchangers can be utilized in the construction of an OAM sorter, whereas the BS-like property plays a crucial role in the construction of the SWAP operator that exchanges the path and OAM degrees of freedom of single photons. The action of the exchanger is periodic in that the OAM mode $|k\rangle_O$ is affected the exact same way as the mode $|k'\rangle_O$ with $k' = k + 4m$. It is thus sufficient to consider only modes with $0 \leq k < 4m$ to fully describe the operation of the exchanger of order m .

In the diagrams we will use a symbolic representation of an exchanger, as introduced in Fig. 3.5(e). The inverse of an exchanger can be obtained when we pass light in the exchanger backwards. Such a reversed setup is effectively identical to the setup from Fig. 3.5(a) with the only exception that the rotation angle of the upper Dove prism is negative, $\alpha = -\pi/(2m)$. We denote the inverse of an exchanger as shown in Fig. 3.5(f). Because some setups that we are going to discuss in the following attain a simpler form when individual propagation paths are permuted, we will also employ an upside-down version of the symbol for the inverted exchanger.

3.3.2 Sorter

As is evident from the preceding discussion on the path-assisted approach, the crucial role is played there by an OAM sorter. The OAM sorter is a device that reroutes individual OAM eigenstates into separate paths, such that an OAM eigenstate $|k\rangle_O$ entering the sorter S via the zeroth path $|0\rangle_P$ leaves the sorter via path $|k\rangle_P$. Various designs of the

sorter exist, we will focus on the interferometric approach in this section. The operation of the d -dimensional OAM sorter S_d is given by the formula:

$$S_d(|k\rangle_O |0\rangle_P) = |0\rangle_O |k\rangle_P. \quad (3.14)$$

One sees that as a side effect, the sorter also subtracts the OAM quanta of the incoming beam such that the beam leaves in the fundamental OAM mode $|0\rangle_O$. In the rest of this section, the structure of the OAM sorter is presented and explained. It turns out that the sorter can be constructed entirely out of exchangers of different orders. Moreover, due to the periodicity of exchangers, the sorter displays a useful property called *modulo property*, which is explained at the end of the section.

The sorter can be constructed in any dimension. Nevertheless, for convenience we focus on dimensions that are powers of two, i.e., $d = 2^M$ for some integer M . For other dimensions d one just constructs the sorter for the smallest dimension d' that is of the form $d' = 2^M$ and exceeds d . From the resulting sorter, all the exchangers corresponding to unused ports can be removed. The sorter for dimensions of the form $d = 2^M$ can be constructed recursively by splitting the task of sorting d modes into two tasks, each of which sorts $d/2$ modes. This splitting is provided by an exchanger of an appropriately chosen order. To be more specific, consider the action of an exchanger of order 1, EX_1 , on general OAM modes:

$$\text{EX}_1(|2l\rangle_O |0\rangle_P) = |2l\rangle_O |0\rangle_P, \quad (3.15)$$

$$\text{EX}_1(|2l+1\rangle_O |0\rangle_P) = |2l\rangle_O |1\rangle_P. \quad (3.16)$$

In other words, all even modes are passed on unaffected, whereas the odd modes are rerouted into the second path. As one can see, the OAM modes in both output arms lie in the same subspace. That is, we reduced the task of sorting d modes $\{|0\rangle_O, |1\rangle_O, |2\rangle_O, \dots, |d-1\rangle_O\}$ into the task of sorting $d/2$ modes $\{|0\rangle_O, |2\rangle_O, |4\rangle_O, \dots, |d/2\rangle_O\}$. Since to both output ports of EX_1 we append two identical $d/2$ -dimensional sorters, let us focus on the sorter connected to output path $|0\rangle_P$. We have to take into account at this point that the OAM values of the modes leaving the first exchanger are multiples of 2. As a result, in the next step we consider the action of an exchanger of order 2 on OAM modes of the form $|2k\rangle_O$, where k can be either an even or an odd integer. The action of EX_2 on such modes reads:

$$\text{EX}_2(|2(2l)\rangle_O |0\rangle_P) = |4l\rangle_O |0\rangle_P, \quad (3.17)$$

$$\text{EX}_2(|2(2l+1)\rangle_O |0\rangle_P) = |4l\rangle_O |2\rangle_P, \quad (3.18)$$

where $|2\rangle_P$ is the second output path of EX_2 . Analogously to the previous case, the OAM modes leaving the first exchanger along path $|0\rangle_P$ are rerouted by the second exchanger into two paths. This way, we reduced the task of sorting $d/2$ modes into the task of sorting $d/4$ modes. We could proceed along this line of thought until we reach the point when there are only two modes in each path to be sorted. From the above discussion it is easy to see that the order of the exchanger that sorts these two modes into different paths has to be equal to $d/2$ and the two outgoing modes have zero quanta of OAM, i.e., $|0\rangle_O$. The resulting structure of the OAM sorter has the form of a binary tree, where each node is represented by an OAM exchanger, see Fig. 3.6. From there it is also possible to derive the number of exchangers that constitute the d -dimensional sorter:

$$N_S^{(\text{EX})}(d) = 1 + 2 + 4 + \dots + d/2 = \sum_{j=0}^{M-1} 2^j = d - 1. \quad (3.19)$$

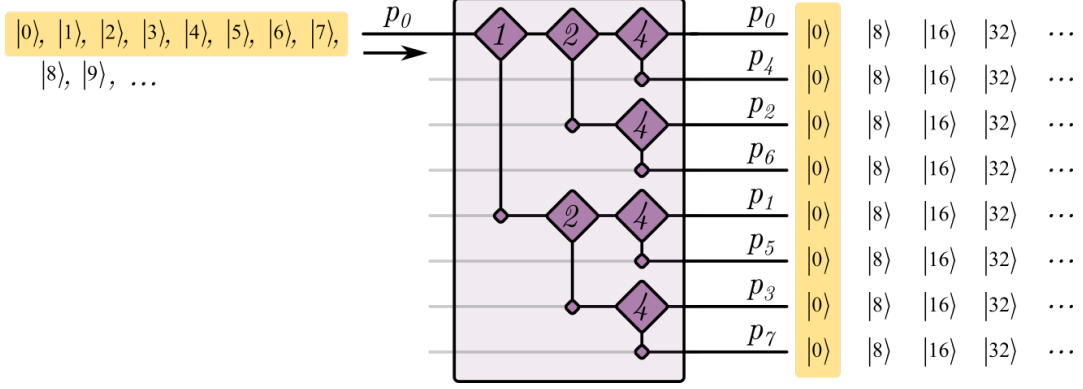


Figure 3.6: The OAM sorter is constructed as an optical network, whose structure is that of a binary tree, in which each node is an OAM exchanger. In the course of the propagation, the OAM modes lose their quanta of OAM such that in the end all modes $\{|0\rangle_O, \dots, |d-1\rangle_O\}$ leave the sorter in the fundamental mode $|0\rangle_P$. These modes are enclosed in a yellow box. Due to the periodicity of exchangers, the OAM sorter sorts also OAM modes that lie outside the intended subspace. For details see the text. In the figure, the specific example of a sorter for $d = 8$ is shown. Note that due to the recursive construction of the sorter, its output paths are not in the increasing order. An additional path permutation would be necessary to make them so.

In the discussion above, the two output paths of the first exchanger were labeled by $|0\rangle_P$ and $|1\rangle_P$. The paths of the second exchangers were labeled by $|0\rangle_P$ and $|2\rangle_P$ as the zeroth path can be reused. Similarly, we would label by $|1\rangle_P$ and $|3\rangle_P$ the two output paths of the third exchanger that is connected to the second output port of the first exchanger. If one follows this assignment of labels to individual output ports of exchangers that constitute the sorter, the final output ports are not labeled in an increasing order, as one would expect. In Fig. 3.6 this situation is explicitly shown for dimension $d = 8$. In order to comply with the formula (3.14), an additional path permutation has to be appended to the setup. This permutation takes path $|p\rangle_P$ into $|\pi(p)\rangle_P$, where π is for $d = 8$ given by

$$(0, 4, 2, 6, 1, 5, 3, 7) \xrightarrow{\pi} (0, 1, 2, 3, 4, 5, 6, 7). \quad (3.20)$$

Similar permutation can be found for any dimension and we come back to this topic in section 3.3.3, where the form of this permutation affects the structure of the X gate implementation.

There is one noteworthy property of the OAM sorter, when implemented as demonstrated above and when only dimensions of the form $d = 2^M$ are considered. It sorts not only the modes it is intended for, that is $\{|0\rangle_P, |1\rangle_P, |2\rangle_P, \dots, |d-1\rangle_P\}$, but also OAM modes with values that lie outside the desired subspace. This is easy to see from relations (3.15)–(3.18), where the value of l can be an arbitrary integer number. As a result, the OAM modes leaving a d -dimensional sorter are of the form $|dk\rangle$ for $k \in \mathbb{Z}$. For instance, modes $\{|d\rangle_O, \dots, |2d-1\rangle_O\}$ are rerouted properly and leave the sorter in mode $|d\rangle_O$, modes $\{|2d\rangle_O, \dots, |3d-1\rangle_O\}$ leave the sorter in mode $|2d\rangle_O$ etc. This behavior of the sorter is henceforth referred to as the *modulo property* and in Fig. 3.6 it is depicted for the special case of $d = 8$.

3.3.3 Pauli gates

The Pauli matrices constitute a very important set of matrices acting on two-dimensional quantum states. For instance, when supplemented with the identity matrix they form a basis of 2×2 Hermitian matrices, which has the consequence that any two-dimensional density matrix can be represented as a linear combination of Pauli matrices. This gives rise to the well-known Bloch representation. Similarly important role is played also by the high-dimensional generalizations of Pauli matrices. In the present context, the high-dimensional X and Z gates are crucial for constructing other quantum gates, when the approach of Eq. (3.7) is employed.

Let us discuss first, in a somewhat reversed manner, the form of the high-dimensional Z gate. Given the computational basis $\{|0\rangle, \dots, |d-1\rangle\}$ of an abstract d -dimensional Hilbert space $\mathcal{H}_d$, the d -dimensional Z gate acts on the basis states as:

$$Z|m\rangle = e^{\frac{2\pi i}{d}m}|m\rangle. \quad (3.21)$$

That is, it adds a state-dependent phase shift. When the Hilbert space corresponds to OAM, the Z gate as well as its powers is particularly easy to implement. Only a single properly rotated Dove prism suffices [43], cf. Eq. (3.3) and Fig. 3.1. Specifically, the k -th power of the d -dimensional Z gate is implemented with a Dove prism, whose rotation angle with respect to the vertical axis of the setup is set to $\alpha = k\pi/d$.

From now on we therefore focus on the X gate, whose implementation is less trivial. The d -dimensional X gate acts on the aforementioned basis states as follows:

$$X|m\rangle = |(m+1) \bmod d\rangle. \quad (3.22)$$

The X gate transforms each basis state $|m\rangle$ into its successor $|m+1\rangle$, with one exception. Specifically, the “last” state $|d-1\rangle$ is transformed back into the “first” state $|0\rangle$. In other words, this gate amounts to a cyclic permutation of the basis states with step equal to one. The k -th power of the X gate then in complete analogy corresponds to the cyclic permutation of the basis states with step equal to k , as is also evident from the formula:

$$X^k|m\rangle = |(m+k) \bmod d\rangle. \quad (3.23)$$

In the following, we present an implementation of X gates acting on OAM of single photons in arbitrarily high dimension. The implementation scheme makes use of only conventional optical elements, as defined in section 3.1, and is an example of the general path-assisted scheme (3.8). The construction of high-dimensional X gates in OAM of single photons was given for the first time in Ref. [43] and we present in section 3.3.3.1 its improved version. The integer powers of the X gate can be obtained by merely concatenating multiple setups for the X gate, which is however resource-demanding. In section 3.3.3.2 a streamlined design is presented that requires less resources.

It is instructive at this point to mention that the OAM degree of freedom lives naturally in an (countably) infinite-dimensional Hilbert space. In such a space, one can consider the computational (Hamel) basis composed of OAM eigenstates $\{\dots, |-2\rangle_O, |-1\rangle_O, |0\rangle_O, |1\rangle_O, |2\rangle_O, \dots\}$. It is then possible to consider a shift operator Q that adds a quantum of OAM to whatever basis state it encounters: $Q|m\rangle = |m+1\rangle$. Such a shift can be easily implemented with a spiral phase plate as defined in Eq. (3.5). When we work with finite-dimensional qudits, we have to artificially restrict the values of OAM carried by a photon only to a

certain range. The apparent close similarity between the X gate and the shift operation Q leads to the tendency of mistaking the latter for the former. The crucial difference is that the shift operator cannot act in a finite-dimensional space, it always brings a vector out of the intended subspace of OAM eigenstates after sufficiently many applications. Were there no need to transform the highest mode $|d-1\rangle$ back to the fundamental mode $|0\rangle$, a single spiral phase plate would suffice to implement the X gate. We will see below that in order to ensure the cyclicity, instead of a single spiral phase plate a whole optical network of elements has to be constructed.

3.3.3.1 X gate

In this section, we apply the path-assisted approach (3.8) to find an implementation of high-dimensional X gates. To that end, the path-encoded version U_P of the X gate has to be constructed first. As discussed in the preceding section, the X gate is a cyclic permutation of basis states, which are in the path encoding represented by propagation paths. The implementation of U_P is thus particularly simple: it is a mere permutation of paths, where no optical elements are required. In the center of Fig. 3.7(a) the path-encoded X gate in dimension $d = 16$ is depicted explicitly.

To finish the setup for the OAM-encoded X gate, it suffices to sandwich this path permutation between two d -dimensional sorters S and S^{-1} , the second of which is operated in reverse, cf. Eq. (3.8). At this point our job of constructing the X gate could come to an end. Nevertheless, we are going to show that the setup just described can be simplified considerably when one studies closely the structure of the path permutation in connection with the structure of the sorters. As follows from the previous section, both S and S^{-1} are networks of exchangers. It turns out that some of the exchangers can be removed from the setup of the X gate without any effect on the transformation.

As mentioned in section 3.3.2, for dimensions of the form $d = 2^M$, $M \in \mathbb{N}$, the sorter has a clean binary-tree structure consisting of exchangers of different orders. On top of that, one has to add a special path permutation to this structure to account for the proper ordering of output ports. This path permutation comes from the recursive construction of the sorter and its special example for $d = 8$ is shown in Eq. (3.20). In Fig. 3.7(a), this permutation is depicted graphically for $d = 16$. In the final scheme, the path permutation at the end of sorter S is composed with the path-encoded X gate from the left and analogously the permutation coming from the inverted sorter S^{-1} is composed with the X gate from the right. When simplified, the resulting path permutation follows a nested pattern, as one can see in Fig. 3.7(b). From the form of this pattern it becomes obvious that many exchangers that compose the sorter are immediately followed by inverted exchangers of the same order and acting on the same paths. All these exchangers are depicted in faded colors in Fig. 3.7(b). As their actions cancel each other out they can be removed without affecting the operation of the setup. This way, we arrive at the final simplified setup shown in Fig. 3.7(c), which has a highly symmetric structure and consists of $2 \log_2(d)$ exchangers, as is easy to observe.

From the above discussion and Fig. 3.7 it is straightforward to generalize the argument and construct the setup for the d -dimensional X gate, where d is an arbitrarily large dimension of the form $d = 2^M$ with $M \in \mathbb{N}$. The analysis for the dimensions of the general form is more involved and is omitted here for the sake of simplicity. For the complete discussion we refer an interested reader to Ref. [43].

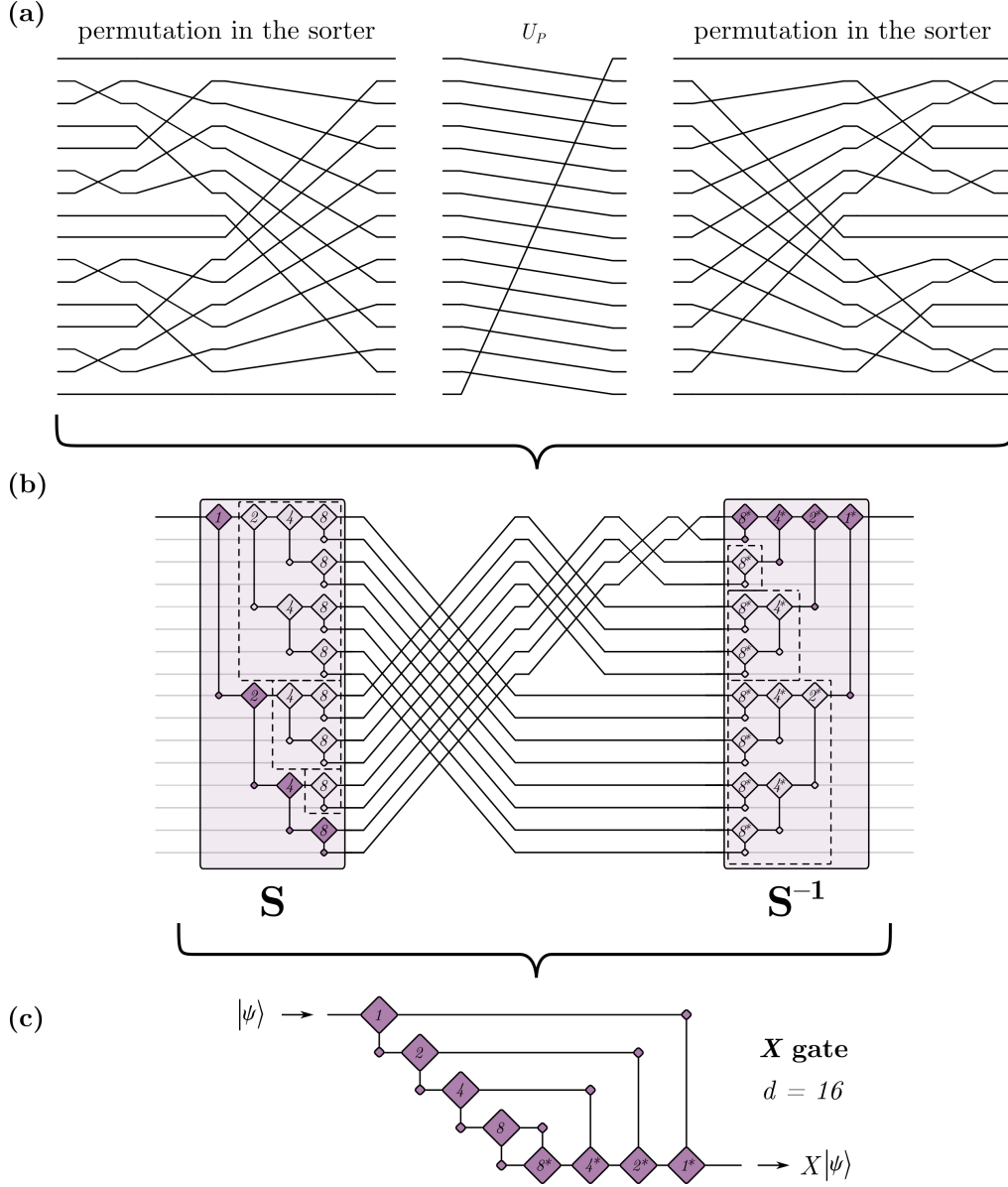


Figure 3.7: The d -dimensional X gate implemented via the path-assisted approach. (a) The path-encoded X gate, shown under U_P at the top, is a mere cyclic permutation of paths. This path permutation can be composed with the two path permutations on both sides that come from OAM sorters, see the text, to obtain the path permutation in (b). (b) The simplified path permutation has a nested structure and is sandwiched between the two sorters S and S^{-1} . Redundant exchangers are drawn in faded colors. (c) After the removal of redundant exchangers, the setup attains a remarkably simple form.

3.3.3.2 X^k gates

In the naive approach, the k -th power of the X gate can be constructed as a concatenation of k setups for the X gate, whose implementation is presented in the preceding section. For a dimension d such an approach requires $O(k \log_2(d))$ optical elements. In this section, we demonstrate a more efficient design, which allows us to build a setup for the X^k gate using only $O(k \log_2(d/k))$ elements [42].

As in the previous case, we start from the path-assisted scheme of Eq. (3.8) and then remove all the superfluous exchangers. The difference between the case of the X gate and its integer power is the form of the path permutation that is sandwiched between the two sorters. When one studies the structure of such a permutation, one can decide which exchangers can be removed without any effect on the operation of the whole setup. The detailed analysis is presented in Ref. [42]. Here we will focus only on the cases when both d and k are powers of two. That is, $d = 2^M$ and $k = 2^N$ for some $M, N \in \mathbb{N}$. In such a special case, the path permutation has a repetitive structure and it is easy to determine the final form of the setup for the X^k gate. Let us note that from the cyclicity of the X gate, its d -th power is equal to the identity operator, $X^d = \text{I}$. From there it follows that $X^{d-k} = X^{-k}$, which in turn implies that $X^{k'}$ gate with $k' = d - k$ can be implemented as the setup for the X^k gate that is operated backward. It is thus sufficient to study only powers that do not exceed the half of the dimension, i.e., $k \leq d/2$.

The path-encoded X^k gate can be constructed as a composition of k path permutations corresponding to the X gate, specifically the path permutation in Fig. 3.7(b) that already includes also the permutations coming from the sorters. This path permutation, which we denote by P_d , consists of a stack of path crossings. The first set of crossings exchanges the upper half of paths with the other half, the second crossing exchanges the first quarter with the second quarter etc. Let us discuss first the case of X^2 gate. It is not hard to observe that when two permutations P_d are composed, the resulting paths get divided into two independent groups. The first group, consisting of paths $p_0, \dots, p_{d/2-1}$ undergoes a path permutation that corresponds exactly to the $d/2$ -dimensional X gate. The second group comprises paths $p_{d/2}, \dots, p_d$ and is likewise subjected to permutation $P_{d/2}$ corresponding to the $d/2$ -dimensional X gate. This situation for $d = 16$ is depicted in Fig. 3.8(a). One can understand the resulting setup in the figure as two $d/2$ -dimensional X gates that are connected by EX_1 and EX_1^{-1} at the very beginning and the very end, respectively. We can apply at this point the argument from the previous section about removing redundant exchangers to simplify the final scheme, as shown on the right-hand side of Fig. 3.8(a).

The discussion just presented can be applied recursively to construct simplified setups for any other power k that is itself a power of two. For such k , the setup for X^k gate in dimension d effectively splits into k identical setups for X gate in dimension d/k . This series of k setups is sandwiched between two k -dimensional sorters. If we denote by $N_X^{(\text{EX})}(d, k)$ the number of exchangers necessary in the implementation of X^k gate in dimension d , where $k = 2^N$ for some $N \in \mathbb{N}$, we can write down directly:

$$N_X^{(\text{EX})}(d, k) = k N_X^{(\text{EX})}(d/k, 1) + 2 N_S^{(\text{EX})}(k), \quad (3.24)$$

where $N_S^{(\text{EX})}$ is given in Eq. (3.19). Explicitly this formula reads

$$N_X^{(\text{EX})}(d, k) = 2 \left(k \log_2 \left(\frac{d}{k} \right) + k - 1 \right). \quad (3.25)$$

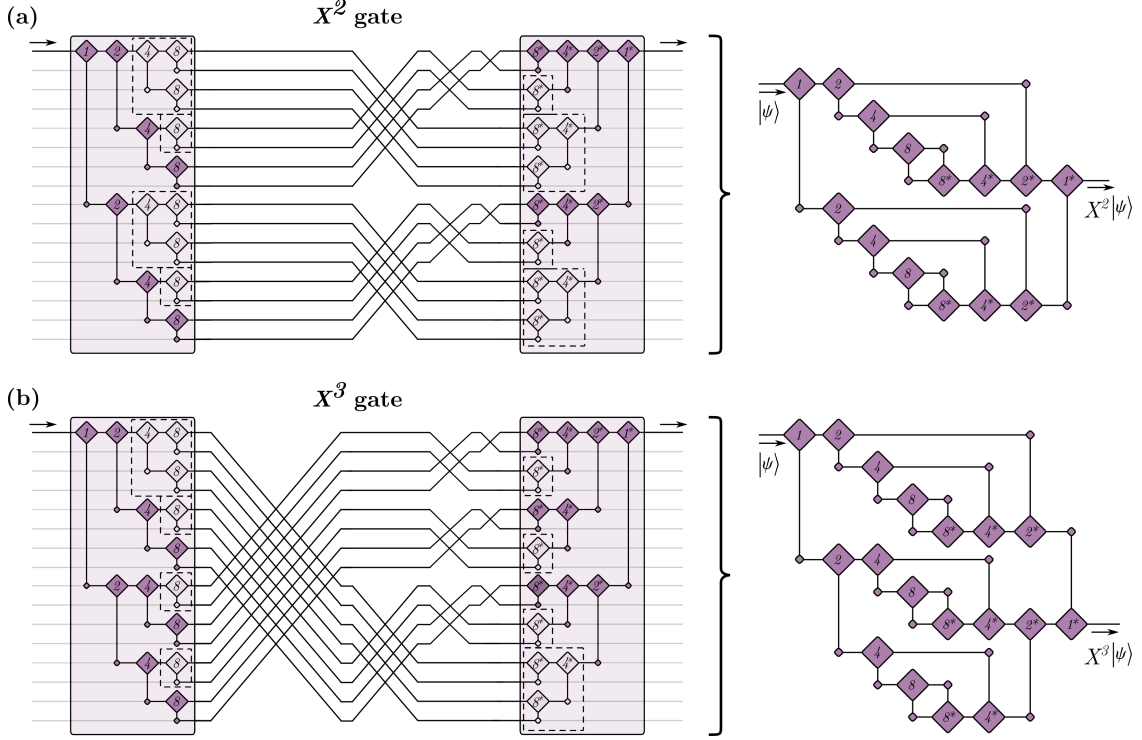


Figure 3.8: Examples of high-dimensional X^k gates. (a) The setup for d -dimensional X^2 gate effectively decomposes into two setups for X gate in dimension $d/2$. Similar property is displayed for any k that is a power of two. The general scheme on the left can be simplified into the form presented on the right. (b) For powers k that are not themselves powers of two, such as $k = 3$, the structure of the general setup is more complicated and the resulting network does not have a repetitive structure.

Note that as $d = 2^M$ and $k = 2^N$, the division $d/k = 2^{M-N}$ is again an integer number and so an X gate can indeed be constructed in dimension d/k , as required above. For $k = 1$ formula (3.25) reduces to $2 \log_2(d)$ and is consistent with our treatment of the X gate in section 3.3.3.1. As follows from Eq. (3.25), the number of exchangers that can be removed decreases as the power increases. In the extreme case of $k = d/2$ there are no exchangers left that can be removed. The number of exchangers in such a case is twice the number of exchangers that implement the d -dimensional sorter: $N_X^{(\text{EX})}(d, d/2) = 2(d - 1)$. If we implemented this case in the naive way by concatenating $d/2$ setups for the X gate, the number of exchangers would be equal to $d \log_2(d)$.

To get a glimpse of how the situation looks like for powers k that are not powers of two, in Fig. 3.8(b) a setup for the 16-dimensional X^3 is shown. It turns out that the layout of redundant exchangers is different in the sorter and the inverted sorter. No simple claims can be made in such a case about the precise structure of the simplified scheme. However, even in the general case it holds true that with the increasing power the number of redundant exchangers decreases and the scaling is still approximately $O(k \log_2(d/k))$ [42].

3.4 Single-photon control gates

In the previous section that analysed the path-assisted approach of implementing general unitaries in OAM, the role of OAM and path was rather asymmetric. The path modes serve there as a mere auxiliary degree of freedom that allows one to work eventually with OAM. There is, however, an important class of unitaries where both quantities are equally important — the single-photon controlled gates. In such gates, one quantity represents the control qudit, the other quantity the target qudit. We will focus in this section on the case when the control qudit is represented by OAM and briefly mention at the end the other case when the control is represented by path. It turns out that even these controlled gates, though not based on the path-assisted approach, can be implemented only using the conventional optical elements. The following discussion draws upon our article published in Ref. [42].

Consider a local unitary U that is to be applied to the path degree of freedom of a single photon. The quantum state of this same photon has also the OAM part. If one wants to apply a different power of the unitary U onto the path modes of the photon based on its OAM, one obtains a controlled unitary CU . Its action on a composite mode $|k\rangle_O |p\rangle_P$ reads

$$CU |k\rangle_O |p\rangle_P = |k\rangle_O (U^k |p\rangle_P), \quad (3.26)$$

where $0 \leq k < d$ and $0 \leq p < n$ for some integers d and n . In other words, when the mode in path p contains k quanta of OAM, the k -th power of unitary U is applied to the path mode. In the language of quantum circuits, the above transformation is represented by the circuit shown in Fig. 3.9(a).

Such a controlled unitary CU can be implemented in the following way. First we note that any unitary matrix is diagonalizable and so there exists a unitary matrix M and a diagonal matrix D such that

$$U = M^\dagger \cdot D \cdot M. \quad (3.27)$$

Due to unitarity, the eigenvalues of U are of the form $\lambda_j = \exp(i\varphi_j)$ for some real φ_j . These numbers form the diagonal of D . As can be shown easily, an integer power of U can be expressed as

$$U^k = M^\dagger \cdot D^k \cdot M, \quad (3.28)$$

where $k \in \mathbb{N}$. All the transformations U^k , M , $M^\dagger$, as well as D^k act on the path degree of freedom, the value of k is nevertheless given by a particular OAM mode.

So much for the theoretical derivation. When it comes to the implementation, the unitary U^k can be decomposed according to Eq. (3.28) into three modules. The first module consists of a unitary M that acts on the path modes of the photon. This can be implemented as a network of beam splitters and phase shifters, following for instance the Reck et al. architecture [41]. In the second module, the path mode $|p\rangle_P$ acquires an additional phase, such that $|p\rangle_P$ changes into $\lambda_p^k |p\rangle_P = \exp(i\varphi_p k) |p\rangle_P$. The third module is then the inverse $M^\dagger$ of the first module. If the power k was fixed, the second module could be implemented as a series of phase shifters where the phase shifter in path p adds the phase of $\varphi_p k$ to the beam. As mentioned above, this is not the case since the integer k depends on the OAM value of the traversing mode. There is however an easy fix to this problem — a single Dove prism works exactly in the desired way as it transforms an OAM mode $|k\rangle_O$ into $-\exp(i2\alpha k) |-k\rangle_O$ (3.3), which can be further modified by a reflection off

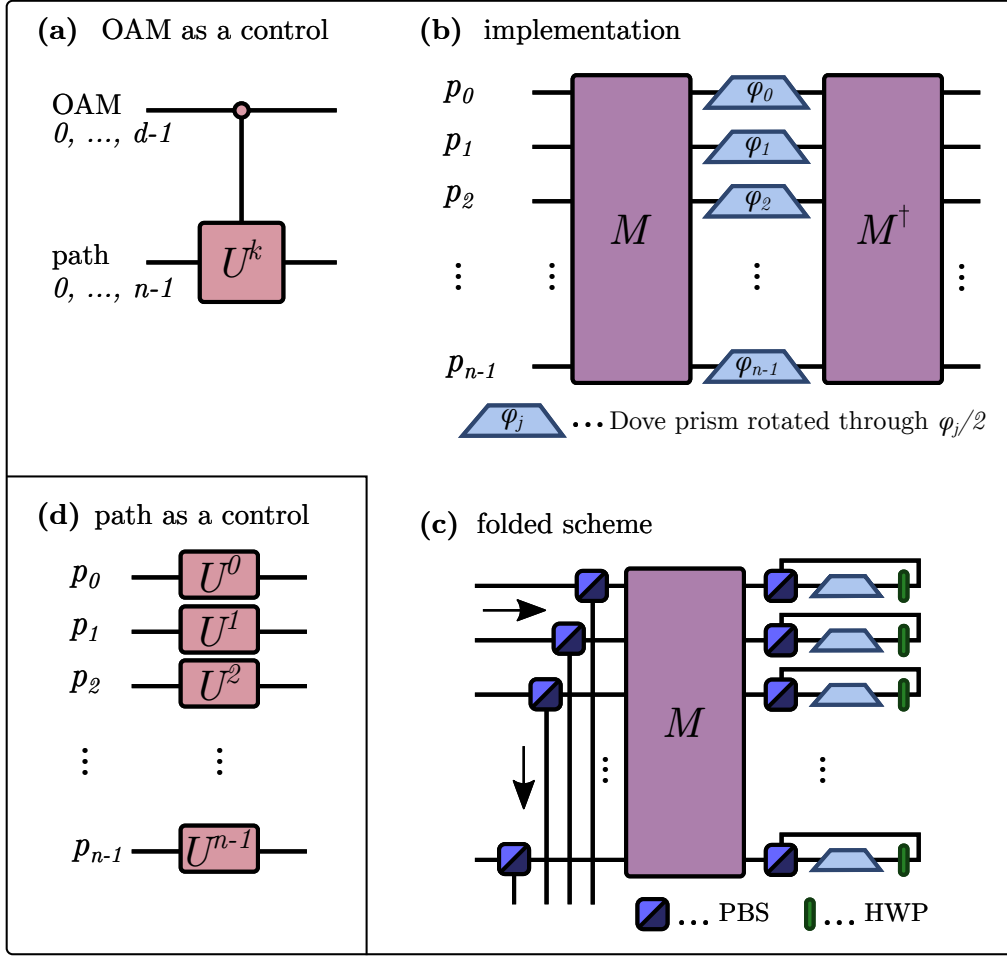


Figure 3.9: Single-photon controlled gates in OAM and path degree of freedom. (a) In the quantum-circuit picture, the OAM of a photon represents the upper horizontal line that corresponds to the control qudit. Similarly, the lower line corresponds to the target qudit and is represented by the path degree of freedom of the same photon. (b) The implementation consists of three modules. At first, a unitary transformation M is applied to the path. Then a series of properly rotated Dove prisms affects the OAM of propagating terms. Finally, the inverse operation $M^\dagger$ redistributes the individual terms into the final state. (c) The inverse transformation can be removed and polarization can be used as an auxiliary degree of freedom that controls whether the photon propagates forward or backward through the setup. For details see the text. (d) The role of OAM and path can be exchanged. The resulting setup then corresponds to one device in each path, where the devices implement different powers of the same unitary U that acts on OAM. Adapted from Ref. [42].

a mirror, see Eq. (3.1), into the form $\exp(i2\alpha k) |k\rangle_O$. When one inserts a Dove prism into each path and rotates the prism in path p through the angle of $\alpha = \varphi_p/2$, one obtains exactly the desirable behavior of the second module. To sum up, the controlled unitary CU (3.26) can be implemented in the way demonstrated in Fig. 3.9(b). In terms of a direct sum, the decomposition of CU into the three modules can be written as

$$CU = M^\dagger \cdot \left(\bigoplus_{j=0}^{n-1} \text{DP}_{\varphi_j/2} \right) \cdot M, \quad (3.29)$$

where $\text{DP}_{\varphi_j/2}$ is a Dove prism rotated through the angle of $\varphi_j/2$. As can be easily observed, $O(n)$ Dove prisms, $O(n^2)$ beam splitters, and $O(n^2)$ phase shifters are necessary for the construction of the resulting setup. Thanks to the symmetry of the setup, where the third module is just the inverse of the first one, a folded scheme can be considered that provides more favorable scaling of resources, see Fig. 3.9(c). The photon enters the folded setup in a specific polarization, say H , and traverses the first and second modules only to reach a series of half-wave plates that rotate the polarization into V . Polarizing beam splitters then reroute the photon back into the first module. After traversing the first module again, this time backwards, the photon leaves the setup in polarization V .

The general implementation scheme of Eq. (3.29) can be illustrated with several examples. For instance, when the unitary U is already diagonal, the transformation M is nothing but an identity operator and the controlled unitary CU is represented by a series of Dove prisms only. A special yet notable example of such an operator is the n -dimensional controlled Pauli Z operator, CZ , where the Dove prism in path p is rotated through $\pi p/n$. Another important example is the Leach interferometer [45], which functions as a parity sorter for OAM modes. In this case, the transformation M is a mere symmetric beam splitter and the two diagonal entries in D can be chosen as 1 and $\exp(i2\alpha)$, where the real number α determines the sorting properties of the interferometer. When $\alpha = \pi/2$, for instance, the input modes with even number of OAM quanta leave the interferometer via one port, whereas those with odd number of quanta leave via the other port. Yet another notable example is the high-dimensional CNOT gate. It corresponds to the case of U being a high-dimensional Pauli X gate. This gate satisfies the Fourier duality relation with the Z operator, such that $X = F^\dagger \cdot Z \cdot F$, where F is the Fourier transform in the appropriate dimension. Using the above notation, we have $U = X$, $M = F$, and $D = Z$. The setup for the high-dimensional CNOT was proposed in 2005 [48] and then rediscovered independently by two groups in 2016 [49, 50].

The above analysis is valid for the case when the control qudit is encoded in OAM. The other case, when the control is played by the path degree of freedom, can be mathematically formulated in an obvious way following the formula in Eq. (3.26). The corresponding setup can be schematically seen as a series of devices that act on OAM, where in each path there is a device representing a different power of the given unitary U , see Fig. 3.9(d). To implement such a setup one can recall the scheme in Eq. (3.29) and sandwich it between two swap operators. This way, the task of using the path as control is turned into the task of using OAM as control, which is already solved above. The swap operators exchange the role of OAM and path and are discussed in detail in the next section. Using such a solution provides improved scaling behavior over the naive approach of Fig. 3.9(d).

3.5 Parallelization

In the schemes discussed in section 3.3, a quantum state, originally propagating along a single path, is split by a sorter into a superposition of multiple terms, where each term propagates along a separate path. A natural question arises whether one can do better, whether a smaller number of paths would be enough. The smaller the number of paths, the more stable the entire setup is. Even though it is not clear as of now how to further simplify in a systematic way the setups presented in preceding sections, one can circumvent this problem and consider increasing the number of quantum systems instead of decreasing the number of paths. Suppose the task is to apply the same operation to multiple systems in parallel. If we apply the path-encoded approach presented above, we need as many setups as there are systems. Each setup splits the input path into multiple paths and then applies the path-encoded unitary. Is it possible to somehow merge all these setups into one? In the following, we will demonstrate that it is indeed possible. The key ingredient is to replace the OAM sorter with the OAM-path swap operator. This operator works in effect as a sorter for multiple systems at the same time. The discussion below provides a detailed description of the results that we presented in Ref. [42].

To be more specific, suppose we have n photons at our disposal, each being in a given d -dimensional OAM state and propagating along a separate path. Our task is to apply a specific unitary U individually to each photon. We can do so by first applying a swap operator with n input and d output paths to all the photons, which effectively exchanges the OAM and path degrees of freedom. The OAM at this point merely labels different photons, whereas the state to be acted upon by U is now stored in the path degree of freedom. As a next step, a single instance of the path-encoded version U_P of the unitary U is applied to all the paths. This operation leaves OAM intact. Finally, another swap operator is used to swap the role of OAM and path back such that at the end the unitary U is applied to OAM. This scheme, henceforth referred to as the *parallelized scheme* [42], is depicted in Fig. 3.10 and can be summarized by the following formula:

$$U^{\text{par}} = \text{SWAP}^{-1} \cdot U_P \cdot \text{SWAP}, \quad (3.30)$$

where SWAP stands for the OAM-path swap with n input and d output paths. Note that when $n = d$, the two-fold application of the SWAP leaves the input state intact, from where it follows that $\text{SWAP} = \text{SWAP}^{-1}$. The parallelized operation U^{par} acts on n photons and corresponds to the tensor product of n single-photon unitaries U :

$$U^{\text{par}} = U \otimes U \otimes \dots \otimes U. \quad (3.31)$$

It is not hard to realize that the exact same setup for U^{par} can also be used in a slightly different scenario when there is only *one* photon that propagates in a superposition of d different OAM eigenstates and n different paths. If the task now is to apply unitary U only to the photon's OAM and not to its propagation modes, we end up with the same physical setup as the one considered above. In this case, however, the setup effectively implements an operation of the form

$$U^{\text{par}} = U \otimes \text{I}, \quad (3.32)$$

where U acts on OAM and I is the identity operator that acts on the path degree of freedom.

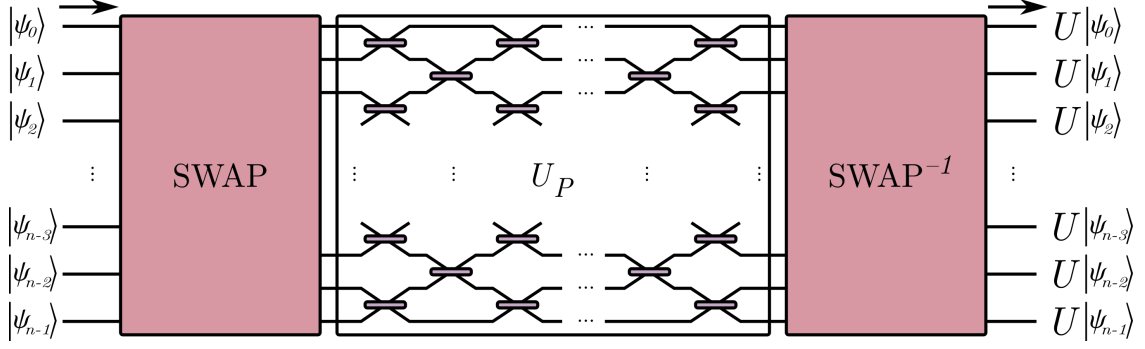


Figure 3.10: The parallelized scheme allows to apply the same unitary U to many systems that propagate along separate paths. The scheme can be understood as a generalization of the path-assisted scheme in that the use is made of the path-encoded version U_P of the unitary.

The parallelized scheme relies on the swap operator and a particular implementation of this operator greatly affects the scaling of the entire scheme. In the next section, we will demonstrate an efficient design of the swap that makes use of OAM exchangers. Later on, the parallelized versions of Pauli gates are discussed as specific examples of the proposed scheme.

3.5.1 SWAP operator

The swap operator is an important example of an operation in quantum information processing. When manipulating single photons, the swap exchanges the states of two degrees of freedom of a single photon. We will focus on the case of the path and OAM and present the corresponding implementation of the swap in terms of conventional optical elements. Formally speaking, the swap is characterized by the formula:

$$\text{SWAP}(|k\rangle_O |p\rangle_P) = |p\rangle_O |k\rangle_P, \quad (3.33)$$

where $|k\rangle_O$ and $|p\rangle_P$ form a basis of the OAM and path subspace, respectively. This transformation is well-defined only when the dimensions of both subspaces coincide. Nevertheless, we can understand the swap in a more general setting, where the dimensions of the two subspaces differ and are changed by the application of the swap. If there are d OAM modes and n input paths at the beginning, after the swap there are n OAM modes propagating along d output paths.

Let us discuss now the possible implementations of the swap operator. Consider a photon propagating in a superposition of d OAM modes along a superposition of n different paths. The naive approach how to swap the states of OAM and path would be to use a version of the path-assisted scheme, see Fig. 3.11. A d -dimensional OAM sorter would be inserted in each of the n paths to obtain nd paths, each containing only a single OAM mode. As a next step, the path-encoded version of the swap would be applied to these nd paths and at the end another series of d OAM sorters, this time n -dimensional ones, would be used to recombine the multitude of paths back into the expected d spatial modes. This naive approach requires approximately $O(nd)$ optical elements, which scales like $O(d^2)$ for $n \sim d$. In the following, we present a more efficient scheme that ingeniously makes

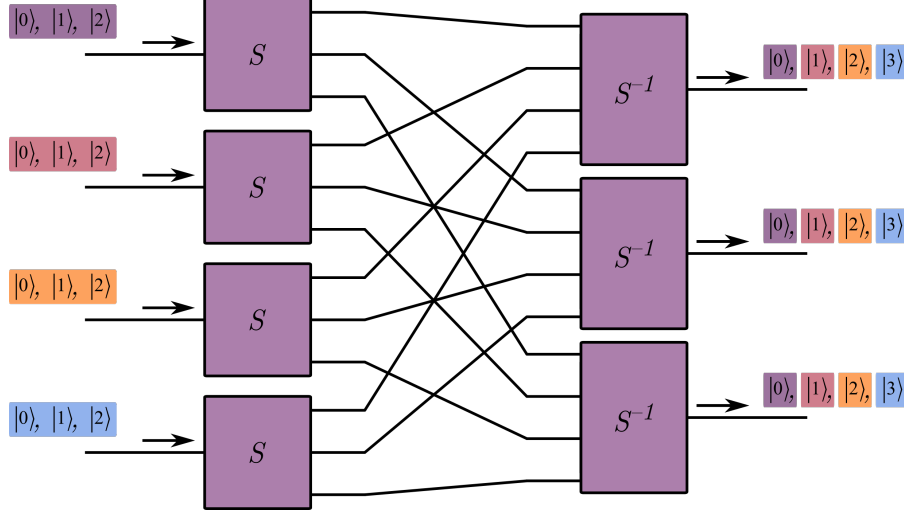


Figure 3.11: The naive implementation of the swap operator that exchanges the OAM and path degree of freedom of a single photon. First, a d -dimensional sorter is applied to each of n input paths. Second, the output paths are permuted according to the path-encoded version of the swap, and finally, a series of inverted sorters, each of dimension n , recombines all the modes into d output paths. Color coding is used to identify each set of input and corresponding output modes.

use of the BS-like property of OAM exchangers and requires only $O(d \log_2(d))$ elements. Similarly to previous sections, we consider primarily dimensions n and d that are powers of two. At first, we assume that the dimensions of both the OAM subspace and the path subspace are equal, $n = d$. Later we relax this constraint and discuss also the cases when the dimension d of the OAM subspace differs from the dimension n of the path subspace.

3.5.1.1 Holo beam splitter

As noted in the pertaining section, a single exchanger works as many beam splitters with different splitting ratios. A particular ratio depends not only on the order of the exchanger but also on the number of OAM quanta in the incoming mode. A special case of this property, namely the PBS-like behavior, was used in the construction of the OAM sorter in section 3.3.2. In this section, we use the whole range of splitting ratios to construct an efficient implementation of the swap operator. Exchangers alone however do not suffice and we have to introduce one more passive optical device that we from now on call a *holo beam splitter*. It consists of two spiral phase plates and an asymmetric beam splitter, see Fig. 3.12(a), while its operation is characterized by two parameters α and m that we collectively refer to as the *order* of the holo beam splitter, cf. Fig. 3.12(b). These two real numbers may be arbitrary but nonzero. The transformation rules of the holo beam splitter $\text{HBS}_{\alpha,m}$ of order (α, m) acting on paths p and q read

$$\text{holo beam splitter : } \begin{cases} |k\rangle_O |p\rangle_P \rightarrow e^{i\beta} (\cos(\beta) |k\rangle_O |p\rangle_P - i \sin(\beta) |k-m\rangle_O |q\rangle_P), \\ |k\rangle_O |q\rangle_P \rightarrow e^{i\beta} (\cos(\beta) |k\rangle_O |q\rangle_P - i \sin(\beta) |k+m\rangle_O |p\rangle_P), \end{cases} \quad (3.34)$$

where $\beta = -\pi\alpha/(2m)$. Note that the kets in the second line appear in a reversed order when compared to the definition of the exchanger (3.9). Otherwise the two definitions are very similar and one could make a discussion of the sorting properties of a holo beam splitter in an analogous way to the discussion in section 3.3.1. The main difference is that the splitting ratio of the holo beam splitter does not depend on the number of OAM quanta of the incoming beam.

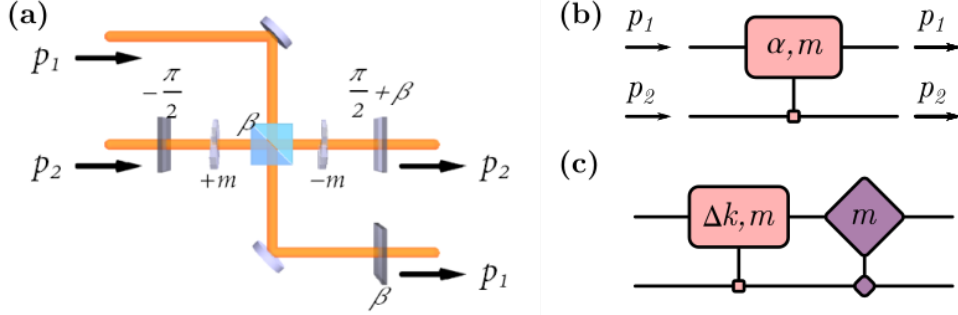


Figure 3.12: Holo beam splitters and their sorting properties. (a) A holo beam splitter is an interferometric device consisting of a single beam splitter with its splitting ratio set to $\beta = -\alpha\pi/(2m)$ and two spiral phase plates that add and subtract m quanta of OAM, respectively. Apart from that, three phase shifters are included that shift the phases as depicted in the figure. (b) The symbolic representation of a holo beam splitter of order (α, m) . (c) The sorting properties of an exchanger are modified when a holo beam splitter of a properly chosen order is prepended to it. For details see the text.

The importance of a holo beam splitter lies in the way it modifies the sorting properties of exchangers. Consider the simple setup in Fig. 3.12(c) where a holo beam splitter of order $(\Delta k, m)$ for some fixed integers m and Δk with $0 < \Delta k < m$ is followed by an exchanger of order m . From Eqs. (3.34) and (3.9) it is straightforward to determine how the OAM modes entering either of the two input paths are transformed by the setup. Specifically:

$$\begin{cases} |k\rangle_O |p\rangle_P \rightarrow e^{-i\beta_{k+\Delta k}} (\cos(\beta_{k+\Delta k}) |k\rangle_O |p\rangle_P + i \sin(\beta_{k+\Delta k}) |k-m\rangle_O |q\rangle_P), \\ |k\rangle_O |q\rangle_P \rightarrow e^{-i\beta_{k+\Delta k}} (\cos(\beta_{k+\Delta k}) |k+m\rangle_O |p\rangle_P + i \sin(\beta_{k+\Delta k}) |k\rangle_O |q\rangle_P), \end{cases} \quad (3.35)$$

where $\beta_{k+\Delta k} = \pi(k + \Delta k)/(2m)$. By comparison of these relations with Eq. (3.9) one sees that this simple setup works exactly like a single exchanger of order m with the crucial difference that the OAM modes are effectively shifted by Δk . The sorting properties of the exchanger, such as the BS-like and the PBS-like property, are thus preserved, but they are performed on modes that are shifted by an additive constant Δk . This fact is essential for the efficient construction of the swap, as presented below.

3.5.1.2 Construction of SWAP

The first thing we should realize when attempting to construct the swap operator is that the action of the sorter, defined by Eq. (3.14), is a special case of the action of the swap in Eq. (3.33) when $p = 0$. It is thus to be expected that also the setup for the swap will be a generalization of the setup for the sorter. The sorter is constructed as a binary

tree of exchangers of increasing orders, see Fig. 3.6. Even though the network for the d -dimensional sorter has strictly speaking d input as well as d output paths, for sorting only the first input path p_0 is used. In the following discussion, we will focus on all the other paths one by one and augment the setup for the sorter with additional elements, such that the final setup works as a proper swap. We describe our design below using the special example of the 8-dimensional swap, but identical ideas apply also in the general case. As we will see, by including more and more input paths, different regimes of the exchanger operation are invoked.

PBS-like behavior Let us start our discussion by studying the case when one wants to sort OAM modes entering the second input path. To that end, we replicate the binary-tree-like structure of the sorter and reuse its last column of exchangers, as demonstrated in Fig. 3.13(a). The OAM modes entering the second path propagate as if they were in a normal sorter, until they reach the last column of exchangers EX_4 , which they enter via the lower port. (Recall that in the sorter, only the upper input ports are used.) What these exchangers do is that they sort the OAM modes correctly but add 4 quanta of OAM to them, which corresponds to the PBS-like behavior of Eqs. (3.10) and (3.11). This side effect is actually what we need in order to implement Eq. (3.33), as will be clear later on.

BS-like behavior What if we replicate the structure of the OAM sorter in the third input port and inject into it OAM modes, as demonstrated in Fig. 3.13(b)? The modes are sorted by the first exchanger EX_1 as expected but they enter already the next column of exchangers EX_2 via their lower ports. As a result, the modes are sorted correctly but simultaneously acquire 2 quanta of OAM. This fact proves problematic when the modes enter the following column of exchangers EX_4 . The OAM values of modes entering EX_4 are multiples of 2 and so the exchangers do not work as polarizing beam splitters in the sense of Eqs. (3.10) and (3.11) any more. They rather act as symmetric beam splitters according to Eqs. (3.12) and (3.13). The OAM modes thus leave the last column of exchangers in a superposition of two paths (and two different OAM values).

A very similar situation occurs when the above procedure is applied to the fourth input path. This is the point where holo beam splitters come in. We can artificially create carefully designed superpositions of paths of the incoming OAM modes, such that these exactly cancel with the superpositions introduced by the exchangers. The net effect is that the modes are correctly rerouted and leave the network via a single output port. The artificial superposition is for the case of the third and fourth paths created by a single holo beam splitter whose order is set to $(2, 4)$, which can be deduced from Eq. (3.35). In Fig. 3.13(c) the propagation of the fundamental mode $|0\rangle_O$ is highlighted to demonstrate that very early on it is turned into a superposition that propagates along the third and fourth paths until it reaches the last exchanger EX_4 , where it gets recombined again and leaves as mode $|2\rangle_O$. The network of Fig. 3.13(c) can be understood as the simple setup in Fig. 3.12(c), where between the holo beam splitter and the exchanger a stack of additional exchangers is inserted. The role of this stack is only to reroute some OAM modes from the simple setup or into the setup, as necessary.

Asymmetric BS-like behavior When we study the propagation of OAM modes injected into the fifth, sixth, seventh, and eighth port of the network in Fig. 3.13(d), the situation

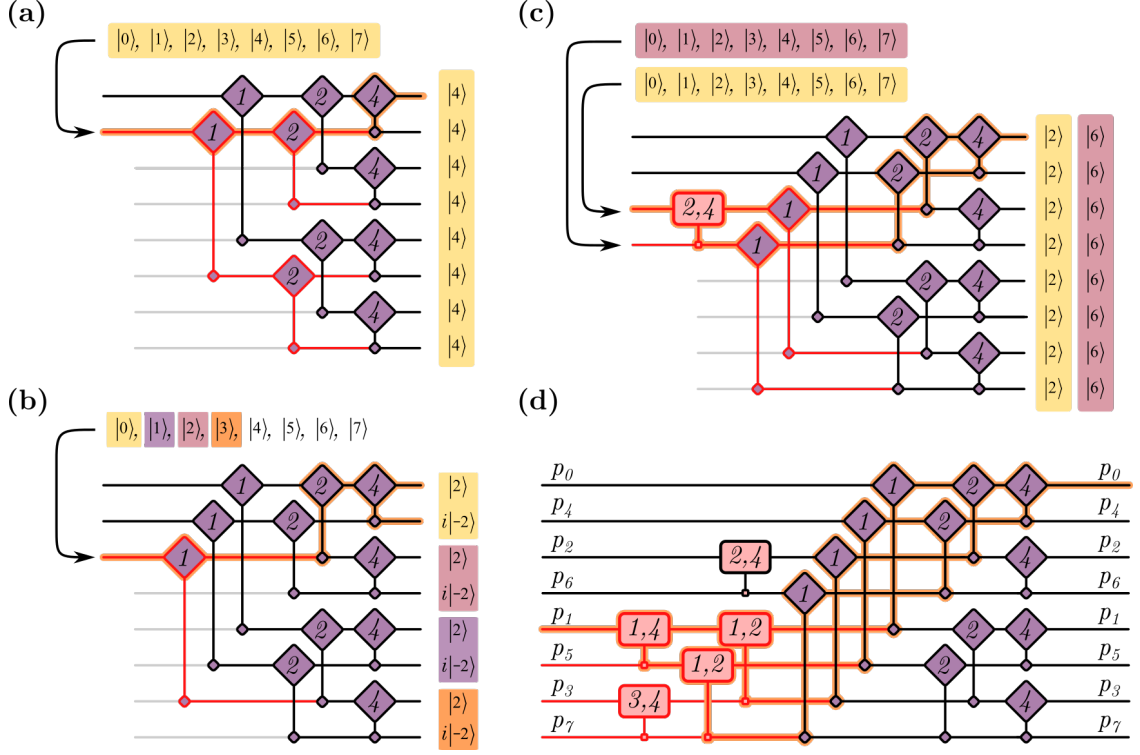


Figure 3.13: The construction of the OAM-path swap. (a) For OAM modes entering via the first path p_0 , the OAM sorter already works as a swap. When we build another sorter that shares with the first one the last column of exchangers, we obtain the optical network depicted in the figure. When injected into the second path, the OAM modes are rerouted correctly but leave the network in state $|4\rangle_O$. (b) When the same procedure is repeated also for the third input path, the OAM modes leave the network in a superposition of two output paths. A complementary behavior is observed also for the fourth input path. (c) The sorting property of the setup from Fig. 3.12(c) can be applied successfully to artificially introduce the superposition of incoming OAM modes, such that this superposition is undone by the action of the last column of exchangers, which originally caused the problem. (d) For the rest of input paths, the problems are caused not only by the last column but also by the second to last one. Similar reasoning to that of (c) can be applied to arrive at the final setup for the swap in dimension $d = 8$. Note the permuted order of the path indices resulting from the recursive structure of the sorter, cf. Fig. 3.6. The red outlines in each subfigure refer to the newly added components, black outlines correspond to the already placed components and orange thick outlines depict the propagation of chosen OAM modes.

becomes even more complicated. The modes enter already the first column of exchangers via their lower ports. It means that they leave these exchangers with an offset of 1 quantum of OAM, which results in the BS-like behavior of exchangers EX₂ in the second column. These introduce superposition of paths and such superpositions enter the third column of exchangers EX₄. These exchangers work for the incoming modes as asymmetric beam splitters with their splitting ratio equal to 25 : 75. As a result, each OAM mode injected into any of the four input paths leaves the network in a superposition of four output paths (and four OAM values). The above approach using the simple setup of Fig. 3.12(c) can be generalized for four paths and applied also in the present case. The idea remains the same: a superposition of four paths is introduced artificially by a network of holo beam splitters, such that it destructively interferes with superpositions introduced by exchangers in the course of propagation through the network. We end up with the setup of Fig. 3.13(d), which works correctly as the swap with 8 input and 8 output paths. The generalization of the above reasoning can be used to find setups for even larger dimensions and can be found in Ref. [51]. In the next section, instead of giving a full account of the construction of a general scheme we resort to presenting a summary of its main features.

3.5.1.3 General structure

In Fig. 3.14, the setups for the swap operator in several low dimensions are shown. In the simplest case of $d = 2$, a single exchanger of order 1 already works as a swap. However, already for $d = 4$ a holo beam splitter is necessary and the higher the dimension the larger number of holo beam splitters is required. Even though we have considered so far only the cases when the number of input paths n is equal to the number of output paths d , these two numbers may differ in general. To construct the swap for this general case it is enough to construct it first for the case with d' input and output paths where $d' = \max(n, d)$ and then use only first n input and first d output paths. The elements corresponding to unused ports can obviously be removed. An example of such a general case with $n = 8$ and $d = 4$ is shown in Fig. 3.14(c) when one removes exchangers enclosed in a dashed box. It is important to mention that additional path permutations have to be added in front and after the networks shown in Fig. 3.14 in order to comply with formula (3.33). These permutations are different for different n and d . Nevertheless, for clarity reasons these permutations are omitted in the figure and the details can be found in Ref. [51]. For the case of $n = 8$ and $d = 4$ in Fig. 3.14(c), the path permutations are shown explicitly in Fig. 3.20(a) in section 3.6.

As can be guessed from Fig. 3.14(c) (and proven as done in Ref. [51]) the optical network for the swap consists of two conceptually different parts. The first part is composed of a series of blocks henceforth referred to as H blocks, where each block is a complete network of holo beam splitters. The k -th block in the series acts on 2^k paths, such that the largest block is of size $\sqrt{d}$. This part merely introduces artificial superpositions of incoming OAM modes. The second part of the network comprises a single block, from now on called the E block, which is a network of exchangers. When the number n of input paths of the swap is equal to the number d of its output paths, the E block is a complete network, otherwise it is not. The E block is responsible for the proper sorting of the modes and its undesirable side effects are compensated for by the H blocks.

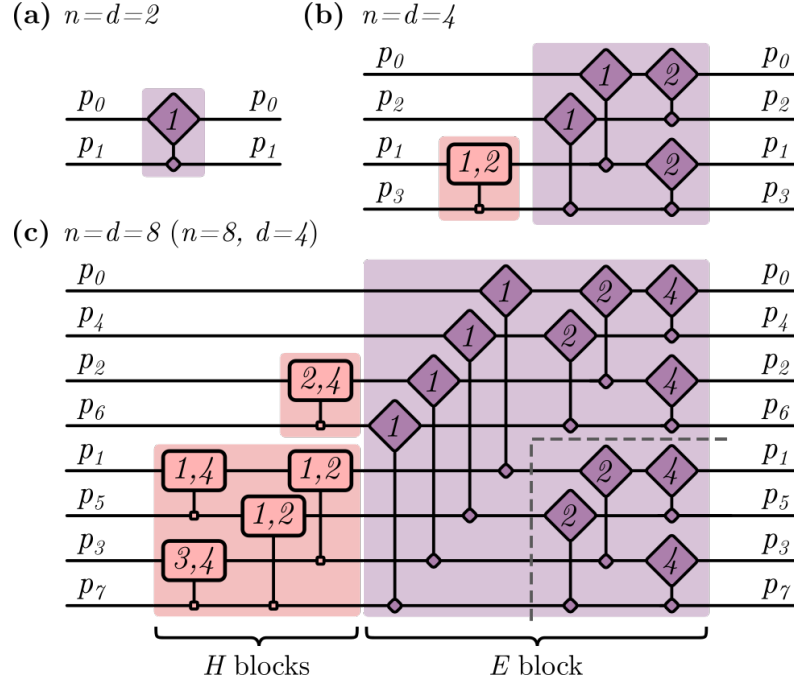


Figure 3.14: The structure of the OAM-path swap operator with n input and d output paths. (a) For $n = d = 2$, a single exchanger EX_1 works as a swap. (b) For $n = d = 4$, the situation gets more complicated as the complete network of exchangers does not suffice to work as a proper swap. A holo beam splitter $\text{HBS}_{1,2}$ has to be prepended to the network. (c) For $n = d = 8$, the general structure of a swap becomes visible and two conceptual parts of the swap can be identified: the E block, consisting of exchangers, and a series of H blocks, each of which consists of holo beam splitters. The swap can also be constructed for cases when $n \neq d$. In the figure, when the exchangers below the dashed line are removed, one obtains the swap with $n = 8$ and $d = 4$. For details see the text.

3.5.1.4 Number of elements

Before we move on and discuss parallelized Pauli operators, let us determine the number of optical elements that are necessary in the construction of the swap operator. At first, let us assume that $n = d = 2^M$ for some integer M . In that case, the E block is a complete network of exchangers. This network consists of $\log_2(d)$ columns, each of which contains $d/2$ exchangers. The whole block thus comprises $N_E^{(\text{EX})}(d) = d \log_2(d)/2$ exchangers of various orders. The other part of the swap is made out of a series of H blocks. Each H block is a complete network of holo beam splitters, whose structure is identical to that of the E block. The d' -dimensional H block thus contains $N_H^{(\text{HBS})}(d') = d' \log_2(d')/2$ holo beam splitters. In total, there is

$$\overline{N}_H^{(\text{HBS})}(n) = \sum_{j=0}^{M-1} N_H^{(\text{HBS})}(2^j) = \sum_{j=0}^{M-1} \frac{j}{2} 2^j = \frac{n}{2} \log_2(n) - n + 1 \quad (3.36)$$

holo beam splitters in the entire series of H blocks. Since both the holo beam splitters and the exchangers contain beam splitters as their components, we can count the beam splitters in the swap operator. Their total number reads $N_{\text{SWAP}}^{(\text{BS})}(n, d = n) = 2N_E^{(\text{EX})}(d) + \overline{N}_H^{(\text{HBS})}(n)$, where the factor of 2 comes from the fact that in each exchanger there are two beam splitters.

In the more general case when $n \neq d$, the E block is no longer a complete network of exchangers, whereas the H blocks remain unaffected. It is not difficult to derive that the number of exchangers in the E block with n input and d output paths, where $n \geq d$, is equal to $N_E^{(\text{EX})}(n, d) = d \log_2(d)/2 + n - d$. The total number of beam splitters required by the swap operator with n input paths and $d \leq n$ output paths is then given by

$$N_{\text{SWAP}}^{(\text{BS})}(n, d) = 2N_E^{(\text{EX})}(n, d) + \overline{N}_H^{(\text{HBS})}(n) \quad (3.37)$$

$$= \frac{n}{2} \log_2(n) + d \log_2(d) + n - 2d + 1. \quad (3.38)$$

The case with $n < d$ is more involved and we will not discuss it here. The resulting scaling in this case is $O((n/2) \log_2(n) + d \log_2(n))$ [42]. An analogous discussion could also be done for the other optical elements, but the resulting numbers would be very similar.

3.5.2 Parallelized Pauli gates

Having finished the description of the internal structure of the swap operator, we are in a position to mention some specific examples of the parallelized scheme, represented by Eq. (3.30). To follow the discussion of section 3.3.3.2, we recall again the high-dimensional Pauli gates and construct their parallelized versions. As discussed in section 3.3.3, the high-dimensional Z gate in OAM can be implemented with a single properly rotated Dove prism. The naive implementation of the parallelized scheme would be to use a series of Dove prisms. Obviously, in this case there is hardly a simpler solution than this naive approach. From now on we therefore focus on the X gate and its powers.

3.5.2.1 Parallelized X^k gates

The discussion in this section can largely follow the treatment for non-parallelized X gates and their powers, presented in sections 3.3.3.1 and 3.3.3.2, respectively. The only major difference is that now instead of removing exchangers from two sorters we want to remove them from two swap operators. An additional complication in the present case is that the structure of the swap depends not only on the dimension d of the OAM space but also on the number n of input paths. Based on this, there are two situations that have to be discussed separately, namely the cases with $n \geq d$ and those with $n < d$. The parallelization is usually employed when there is either a larger number of systems or a larger number of paths. Moreover, it is in general experimentally more challenging to generate with good quality the higher-order OAM modes. For these two reasons we focus on the case $n \geq d$ in the rest of the section. The analysis of the case $n < d$ is more involved and can be found in Ref. [42]. We also consider only those n that are powers of two, for simplicity.

As for the structure of the parallelized setup, the idea is analogous to that in the non-parallelized scheme in section 3.3.3 and is demonstrated in Fig. 3.15. The path-encoded X gate and its powers are mere path permutations that connect the first swap with the

second one. Thanks to the structure of the permutation, there are cases when an exchanger from the first swap is followed immediately by an inverted exchanger from the second swap that has the same order as the first exchanger and thus cancels it out. These exchangers can be removed and in Fig. 3.15 they are pictured in faded colors.

In general, the number of input paths n of the first swap may differ from the number d of its output paths that coincides with the dimension of the OAM subspace. Both swaps consist of $O(n \log_2(n))$ elements, provided that $n \geq d$. Consequently, the number of optical elements needed for the parallelized X^k gates will also not exceed $O(n \log_2(n))$. It is illustrative to compare this scaling with the brute-force implementation of the parallelization, where a separate setup for the d -dimensional X^k gate is placed in each of n paths. Such an approach requires $O(n k \log_2(d))$ elements if the consecutive application of the X gate is used to implement the X^k gate, or $O(n k \log_2(d/k))$ if the approach of section 3.3.3.2 is utilized. As the detailed calculation in Ref. [42] shows, when k is kept constant, then there is only a constant-factor improvement in the scaling. However, if we require that the power k scales with the dimension of the OAM subspace, then the improvement is well pronounced. For instance, if we need an operation that shifts OAM basis states exactly by the half of their number, i.e., $k = d/2$, then the brute-force parallelization requires $O(n d \log_2(d))$, or $O(n d)$, optical elements. The approach of this section needs $O(n \log_2(n))$ elements, which is significantly more efficient unless the number of paths n grows exponentially faster than the dimension d of the OAM subspace.

3.5.2.2 Number of elements

In the following, we present a detailed calculation of the exact number of beam splitters required for the construction of the parallelized X^k gate, which number we denote by $N_{X_{\text{par}}}^{(\text{BS})}(n, d, k)$. For simplicity, we focus only on the cases where the power k is itself a power of two and also $n \geq d$. For an overview of the general case, we refer the reader to Ref. [42]. We adopt a slightly different approach from that in section 3.3.3. There we analyzed the structure of the simplified path-assisted scheme and observed that the setup for an X^k gate is effectively identical to k setups for the X gate in dimension d/k , while these setups are connected by two k -dimensional sorters. From there it was straightforward to determine the number of exchangers necessary in the implementation, cf. Eq. (3.24). In this section, it is easier to go backward and rather analyze how many exchangers can be removed from the naive implementation. We denote this number by $R^{(\text{EX})}(n, d, k)$. It is easy to observe that

$$N_{X_{\text{par}}}^{(\text{BS})}(n, d, k) = 2N_{\text{SWAP}}^{(\text{BS})}(n, d) - 2R^{(\text{EX})}(n, d, k), \quad (3.39)$$

where $N_{\text{SWAP}}^{(\text{BS})}(n, d)$ is the number of beam splitters required by the swap operator, see Eq. (3.38), and where the factor of 2 in front of $R^{(\text{EX})}(n, d, k)$ comes from the fact that a single exchanger contains two beam splitters. The only elements we can remove from the naive scheme are the exchangers in the E blocks of both swaps. The crucial property of the E block for $n = d$ is that it is a *complete* network of exchangers — all the nodes of the network are occupied by an exchanger. Such a network consists of $\log_2(d)$ columns, each of which contains $d/2$ exchangers. In total, $N_E^{(\text{EX})}(d) = d \log_2(d)/2$ exchangers (or $d \log_2(d)$ beam splitters). If we consider only the X gate for $n = d$, see Fig. 3.15(a), we can observe that the removable exchangers can be grouped into a series of blocks of increasing sizes,

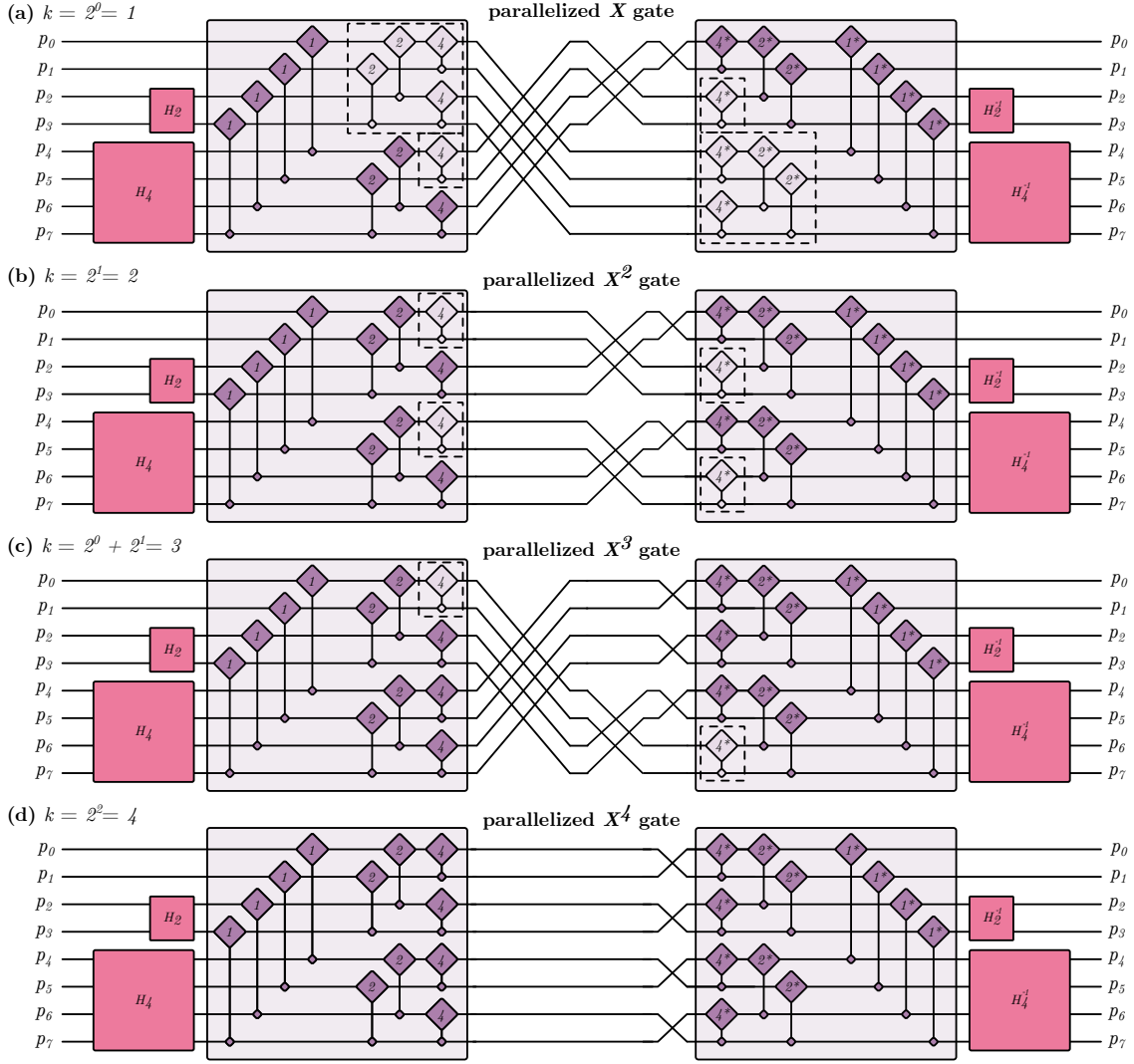


Figure 3.15: Parallelized X^k gates, exemplified for $n = d = 8$. The parallelized versions of the X gate and its powers can be obtained from the general scheme of Fig. 3.10 in a way analogous to that in section 3.3.3, where the non-parallelized X^k gates were discussed. (a) The X gate is constructed from two swap operators, from which complete subnetworks of idle exchangers are removed. (b) For powers k of the X gate, where k is a power of two, the setup effectively decomposes into k identical subsetups, each of which corresponds to d/k dimensional X gate. (c) For powers k that are themselves not a power of two, the resulting structure of the simplified setup is more complicated and is discussed in Ref. [42]. (d) When the power reaches the half of the dimension, i.e., $k = d/2$, there are no exchangers that can be removed. Reprinted from Ref. [42].

each of which is a complete network of exchangers. A minute of thought and a close look at Fig. 3.15(a) can convince one that the number of exchangers in all those blocks equals

$$R^{(\text{EX})}(n, d = n, 1) = 2 \sum_{j=0}^{M-1} N_E^{(\text{EX})}(2^j) = d \log_2(d) - 2d + 2, \quad (3.40)$$

where $n = d = 2^M$ and where the factor of 2 comes from the fact that we remove elements from both swaps. For $n > d$ the E block is no longer a complete network, but its part that has d output paths and that faces the path-encoded X^k gate is again indeed a complete subnetwork. (This situation is depicted in Fig. 3.20(a) in section 3.6.) The same reasoning as above thus applies and we end up with the number of removable exchangers that is identical to that in Eq. (3.40) except that now $n \neq d$. Putting all things together, the number of beam splitters $N_{X\text{par}}^{(\text{BS})}(n, d, 1)$ in Eq. (3.39) for the parallelized X gate reads explicitly

$$N_{X\text{par}}^{(\text{BS})}(n, d, 1) = n \log_2(n) + 2n - 2. \quad (3.41)$$

This number depends only on the number of input paths, not on the dimension of the OAM subspace (so long as $n \geq d$). For the powers of the X gate we can follow the reasoning of section 3.3.3 and observe that the setup is effectively a series of k smaller setups for the X gate in dimension d/k and so $R^{(\text{EX})}(n, d, k) = k R^{(\text{EX})}(d/k, d/k, 1)$. The number of beam splitters that constitute the parallelized X^k gate is thus given by

$$N_{X\text{par}}^{(\text{BS})}(n, d, k) = n \log_2(n) + 2n + 2 - 4k + 2d \log_2(k). \quad (3.42)$$

When $k = 1$, the above expression reduces to Eq. (3.41). Since we assume that $n \geq d$, it is evident from the above expressions that $N_{X\text{par}}^{(\text{BS})}(n, d, k)$ indeed scales as $O(n \log_2(n))$. This scaling is preserved even when we allow for general powers k that are not powers of two.

3.6 Fourier transform

Multiple approaches have been presented so far that aim to construct various classes of single-photon OAM gates. The path-assisted scheme, discussed at the beginning of this chapter, allows one to construct arbitrary unitaries with the help of conventional optical elements. When scaled up for more photons, the path-assisted approach gives rise to the parallelized scheme that enables a simultaneous application of the same unitary to multipartite systems. In addition to these, an alternative scheme can be used to construct single-photon OAM-controlled gates. We build on these results now and conclude this chapter with the general scheme for the construction of the Fourier transform.

The Fourier transform amounts to an exceptionally useful tool in modern mathematics, physics, as well as engineering. In the classical signal processing, for instance, it is used to retrieve the frequency information from time-series data. It is the underlying mechanism of popular compression algorithms such as MP3 for audio compression or JPEG for image compression. From the more fundamental point of view, the free space propagation can be modeled by the Fourier transform of the transversal profile of an electromagnetic beam. In the quantum domain, the Fourier transform embodies the duality relation between the position and momentum representation of the wave function.

The range of potential applications of OAM as a carrier of information would be thus far from complete if we had no means of applying the Fourier transform to it. In this section, we demonstrate how the Fourier transform can be applied to OAM while utilizing merely the conventional optical elements listed in section 3.1. Again, we restrict our analysis only to dimensions that are powers of two, i.e., $d = 2^M$ for some integer M . In the construction of the d -dimensional OAM Fourier transform we can utilize many results from the preceding sections. We apply a scheme akin to the path-assisted approach discussed in section 3.3 and then simplify the idea by utilizing the parallelization scheme introduced in section 3.5. Key elements of the resulting setups are OAM sorters, studied in section 3.3.2, CZ gates mentioned in section 3.4, and OAM-path swap operators, analyzed in section 3.5.1. The following discussion is mostly based on our results presented in Refs. [51, 52].

3.6.1 Theory

To describe the action of the Fourier transform, consider an abstract d -dimensional Hilbert space $\mathcal{H}$ spanned by basis states $\{|0\rangle, \dots, |d-1\rangle\}$. The d -dimensional Fourier transform acts on these states as

$$\mathcal{F}^{(d)}(|k\rangle) = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} e^{\frac{2\pi i}{d}jk} |j\rangle. \quad (3.43)$$

The action on all the other states from $\mathcal{H}$ is obtained by the linear extension of this formula to linear combinations of the basis states.

Before we delve into the details of the implementation, let us briefly comment on different kinds of Fourier transforms depending on the property acted upon. When applied to OAM, the Fourier transform takes individual OAM eigenstates and returns specific superpositions thereof with coefficients of equal magnitude. The example for the 4-dimensional Fourier transform acting on OAM modes $|0\rangle_O$, $|1\rangle_O$, $|2\rangle_O$, and $|3\rangle_O$ is shown in Fig. 3.16(a). The resulting states contain multiple phase singularities, but the beam waist of the photon's wave function remains the same. The Fourier transform of OAM modes, which form a special class of spatial transversal profiles of the wave function, has to be distinguished from the Fourier transform that represents the free space propagation of a beam or the action of a lens. This infinite-dimensional Fourier transform preserves the overall shape of the OAM mode but changes its beam waist, see Fig. 3.16(b).

To implement the Fourier transform on the abstract level, we can employ the recursive scheme of Cooley-Tuckey algorithm [53]. As follows from the algebraic structure of the formula in Eq. (3.43), the d -dimensional transform can be decomposed into a series of two lower-dimensional transforms provided that d is not a prime number, as is indeed the case in our discussion. Let us denote these two smaller dimensions by d_A and d_B , such that $d = d_A d_B$. One can consider the abstract space to be a tensor product of two spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, where $\dim \mathcal{H}_A = d_A$ and $\dim \mathcal{H}_B = d_B$. The recursive scheme relies on the application of the d_A -dimensional Fourier transform in space $\mathcal{H}_A$, followed by the d_B -dimensional Fourier transform in space $\mathcal{H}_B$ with some intermediate operations. Specifically, the recursive decomposition reads

$$\mathcal{F}^{(d)} = \mathcal{F}_B^{(d_B)} \cdot \overline{\text{SWAP}} \cdot CZ \cdot \mathcal{F}_A^{(d_A)}, \quad (3.44)$$

where individual operations are applied from right to left. The controlled Z gate CZ

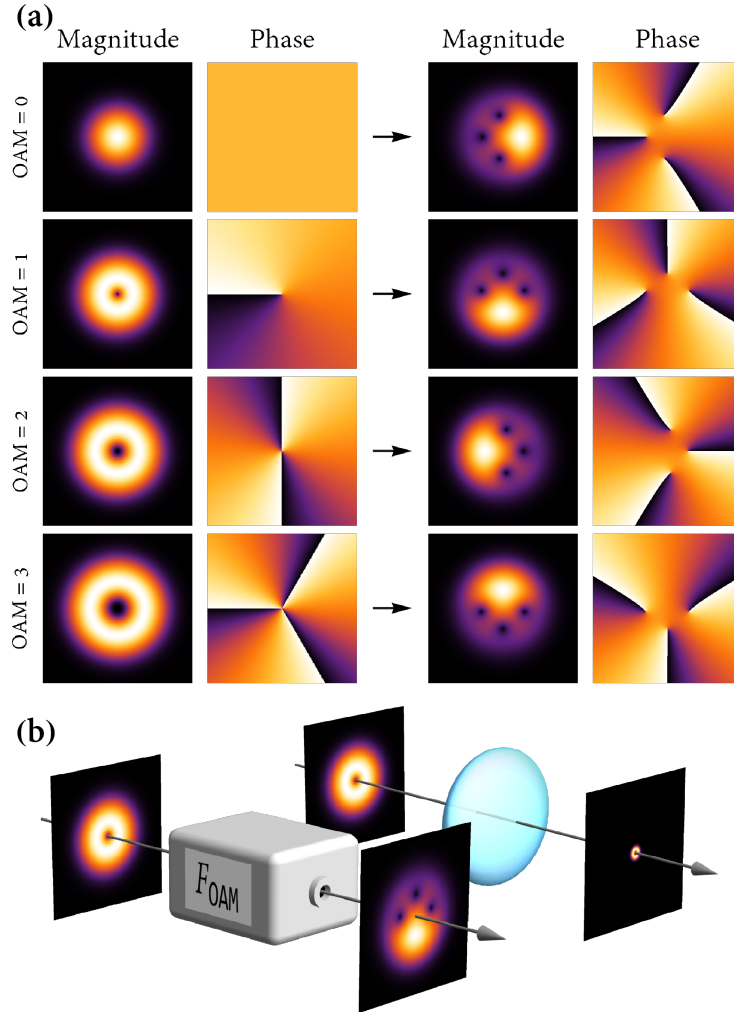


Figure 3.16: Fourier transform of OAM. (a) The transform takes OAM eigenstates with their characteristic central discontinuity into a linear combination of these modes, which has multiple discontinuities distributed around the center of the beam. (b) The Fourier transform of OAM is sometimes mistaken for the action of a single lens or free space propagation, both of which can be modeled by a Fourier transform. Whereas the free space propagation preserves the shape of OAM eigenstates, it is not so for the OAM Fourier transform. Reprinted from Ref. [52].

imparts a mode-dependent phase shift

$$CZ(|m\rangle_A |k\rangle_B) = (Z^k |m\rangle_A) |k\rangle_B = \omega^{mk} |m\rangle_A |k\rangle_B, \quad (3.45)$$

where $\omega = \exp(2\pi i/d)$, and $\overline{\text{SWAP}}$ stands for the mathematical swap operator, which swaps the two states as

$$\overline{\text{SWAP}}(|m\rangle_A |k\rangle_B) = |k\rangle_A |m\rangle_B. \quad (3.46)$$

The implementation of $\overline{\text{SWAP}}$, discussed in section 3.5.1, will be denoted simply by SWAP. The reason for this distinction is that the physical implementation of the swap has some limitations that have to be treated in special ways and that do not emerge for the mathematical swap.

3.6.2 OAM Fourier transform

If we identify spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ with the OAM and path degree of freedom, respectively, we obtain the scheme shown in Fig. 3.17(a), where individual gates in the quantum circuit are applied from left to right. The initial d_B -dimensional OAM sorter S reroutes the d different OAM modes into d_B different paths, each of which contains d_A different modes. This operation corresponds to splitting the abstract space $\mathcal{H}$ into the tensor product $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Then, the d_A -dimensional Fourier transform $\mathcal{F}_O^{(d_A)}$ is applied to OAM in each path, followed by the controlled Z gate. After that, the d_B -dimensional Fourier transform $\mathcal{F}_P^{(d_B)}$ is applied to all the path modes at once, which leaves the OAM degree of freedom unaffected. At the end, all the OAM modes in all the paths are rerouted by a swap and a reverted sorter S^{-1} back into a single output path. The path-encoded Fourier transform $\mathcal{F}_P^{(d_B)}$ can be implemented using the proposal of Ref. [54]. In order to implement the OAM Fourier transform $\mathcal{F}_O^{(d_A)}$, we apply the same scheme recursively until we reach the point when we are to implement the 2-dimensional transform, which is nothing else than a single beam splitter sandwiched between two 2-dimensional sorters, as shown below. The approach just presented was introduced for OAM in Ref. [51].

The bottleneck of this approach is the recursive step — the same lower-dimensional OAM Fourier transform has to be applied to each and every output port of the first sorter, which results in significant redundancy and the overall scaling $O(d)$ of resources [51]. We can, however, recall that this situation is exactly one of the two regimes that are addressed by the parallelized scheme. The series of d_A -dimensional OAM Fourier transforms can be replaced by two swaps and a d_A -dimensional path-encoded Fourier transform, as is done in Fig. 3.17(b). This modification entails the change in the dimension of the path space $\mathcal{H}_P$ from d_B to d_A , after which the d_A -dimensional transform is applied, and then back from d_A to d_B . As is evident from Fig. 3.17(b), the scheme can be decomposed into three modules. The first module, consisting of a sorter and a swap, reroutes OAM eigenstates into d_A different paths. The second module, henceforth referred to as the *transform proper*, performs the actual Fourier transform and has d_A input and d_B output paths. The third module consists of a swap and a reverted sorter and its purpose is to recombine into a single output path all the transformed OAM modes. The elementary building block of the recursive scheme is the 2-dimensional OAM Fourier transform. The structure of this block is different from that of the setup for higher dimensions. It comprises merely a 2-dimensional sorter, followed by the 2-dimensional path-encoded Fourier transform

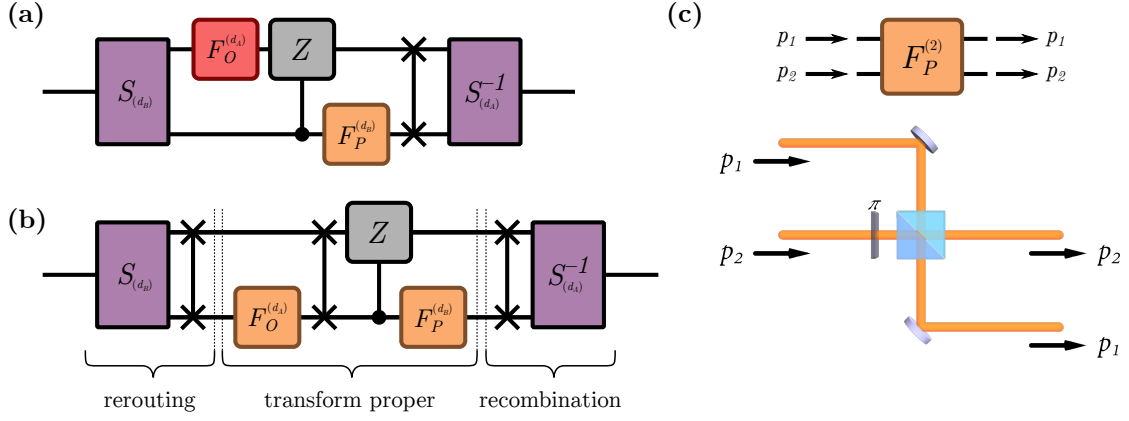


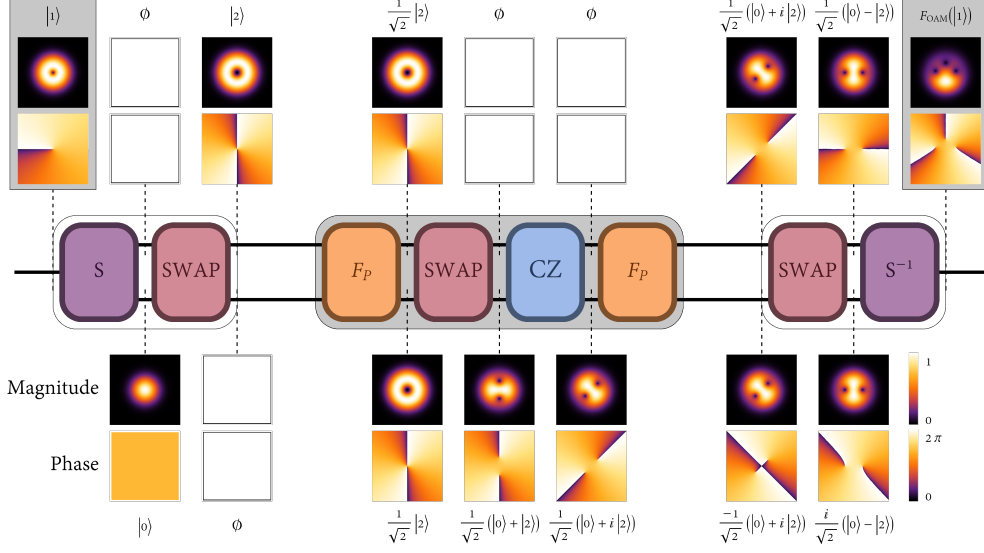
Figure 3.17: The conceptual scheme of the Fourier transform of OAM of single photons. (a) The d -dimensional OAM Fourier transform can be decomposed recursively into lower-dimensional transforms. In the quantum circuit picture, the upper wire represents the d_A -dimensional space of OAM and the lower wire stands for d_B different actual paths. The swap operator is represented by two crosses connected with a vertical line. (b) Using the parallelized scheme, the d_A -dimensional OAM Fourier transform can be implemented as the d_A -dimensional path-encoded Fourier transform. We can identify three modules, out of which the final setup is composed. For details see the text. (c) The elementary building block is the 2-dimensional path-encoded Fourier transform. It consists of a beam splitter supplemented by two mirrors and a phase shifter.

depicted in Fig. 3.17(c), and a reverted 2-dimensional sorter. The approach of Fig. 3.17(b) improves that of Ref. [51] and was analysed in detail in Ref. [52]. We will summarize the main results of the latter paper in the following.

3.6.3 Implementation

Since the circuit picture of Fig. 3.17 is rather abstract, in Fig. 3.18(a) the structure of the 4-dimensional OAM Fourier transform is shown explicitly. The original 4-dimensional OAM space is split by the first module into two subspaces. The OAM subspace is now only 2-dimensional and is spanned by eigenstates $|0\rangle_O$ and $|2\rangle_O$. The second subspace corresponds to the path degree of freedom and is also 2-dimensional. Its two basis vectors amount to two propagation paths that are visible in the figure. To illustrate the action of the transform on incoming modes, the propagation of OAM eigenstate $|1\rangle_O$ through the setup is demonstrated explicitly. Moreover, the action of the rerouting module can be easily summarized by the schematics in Fig. 3.18(b). The module reroutes first d_A OAM modes into the first output path, the second set of d_A modes into the second path etc. As a side effect, the OAM values of individual modes are changed in the process such that the OAM values in all the paths are multiples of d_A . This is a result of the modulo property of the sorter, discussed in section 3.3.2.

As follows from the conceptual scheme in Fig. 3.17 as well as from Fig. 3.18, there are four components of the setup that have to be discussed: the sorters, swaps, path-

(a) Fourier transform for $d = 4$


(b) Rerouting property of the first module

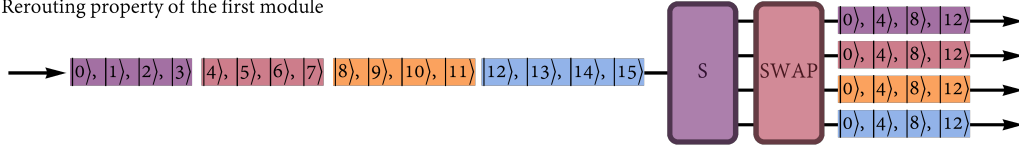


Figure 3.18: The setup of the OAM Fourier transform can be decomposed into three principal modules, each of which implements different stages of the transform. (a) The propagation of the $|1\rangle_O$ eigenstate through the setup of the 4-dimensional OAM Fourier transform is demonstrated explicitly. Both the magnitude and the phase of the beam are shown. The white squares denote the path segments through which no light propagates. (b) The initial rerouting of the OAM modes is performed by the first module. A color coding is used to identify sets of input and output modes that correspond to each other. The first d_A modes are rerouted into the first path, the second set of d_A modes is rerouted into the second path etc. Reprinted from Ref. [52].

encoded Fourier transforms, and the controlled Z gate. The sorters were analyzed in detail in section 3.3.2. Similarly, the structure of swaps was described in section 3.5.1. A high-dimensional path-encoded Fourier transform can be implemented by a variety of different general schemes [41, 44]. However, since we restrict ourselves only to dimensions of the form $d = 2^M$, the efficient design of Ref. [54] can be employed, which requires only $O(d \log_2(d))$ beam splitters and phase shifters for its construction. As we work with OAM as an additional degree of freedom, we also have to take into account the side effects that reflections of OAM modes off a beam splitter entail, namely the sign reversal, cf. Eq. (3.4). To compensate for this effect, each beam splitter in the path-encoded Fourier transform has to be supplemented by two mirrors, as demonstrated in Fig. 3.17(c). The last gate to be discussed is then the controlled Z gate. It is easy to see, nevertheless, that this gate can be implemented as a series of rotated Dove prisms, where in the k -th path there is a single Dove prism inserted that is rotated through angle $k\pi/(d_A d)$. The extra factor of d_A in the denominator accounts for the fact that the OAM modes that enter the CZ gate are of the form $|0\rangle_O, |d_A\rangle_O, |2d_A\rangle_O, \dots$ and their values are thus multiples of d_A . Altogether, the series of Dove prisms acts on such states as

$$|m d_A\rangle_O |k\rangle_P \rightarrow e^{2\frac{k\pi}{d_A d}(m d_A)i} |m d_A\rangle_O |k\rangle_P = e^{\frac{2\pi}{d} i m k} |m d_A\rangle_O |k\rangle_P, \quad (3.47)$$

which corresponds exactly to Eq. (3.45). We have thus all the ingredients we need to construct the setup of the high-dimensional Fourier transform acting on OAM of photons.

So far, we have not specified the integer values of dimensions d_A and d_B . The reason is that these values are not fixed, the only constraint they have to satisfy is $d = d_A d_B$. We can thus choose such values that optimize a given figure of merit. If we take the number of optical elements in the setup as our figure of merit, it can be shown, see Ref. [52] for details, that the optimal choice of these dimensions is given by

$$d_A = 2^{\lceil M/2 \rceil}, \quad d_B = 2^{\lfloor M/2 \rfloor}, \quad (3.48)$$

where $\lceil x \rceil$ and $\lfloor x \rfloor$ stands for the ceiling and floor function applied to x , respectively, and where M is such that $d = 2^M$. This formula simplifies into two distinct cases: those with M even and those with M odd. For the former case, the general formula simplifies into $d_A = d_B = \sqrt{d}$. The corresponding setup, depicted in Fig. 3.19(a), has a simple structure as the number of paths stays constant throughout the whole setup. For the latter case, shown in Fig. 3.19(b), one obtains $d_A = \sqrt{2d}$ and $d_B = \sqrt{d/2}$ and the number of propagation paths alternates between d_A and d_B .

To emphasize the fact that all the schematic setups can actually be in principle constructed in a lab, in Fig. 3.20 the setups of OAM Fourier transforms are depicted for several low dimensions. These setups are expressed in terms of exchangers, holo beam splitters, path-encoded Fourier transforms, and elementary operations. All these building blocks were discussed above where their implementation was demonstrated that uses only conventional optical elements. Furthermore, in Fig. 3.21 the setup of the 4-dimensional OAM Fourier transform is shown explicitly only in terms of conventional elements.

3.6.4 Modifications and alternatives

As demonstrated in Ref. [52], the polarization of photons can be used as an auxiliary degree of freedom to simplify the experimental setup for the Fourier transform. Specifically,

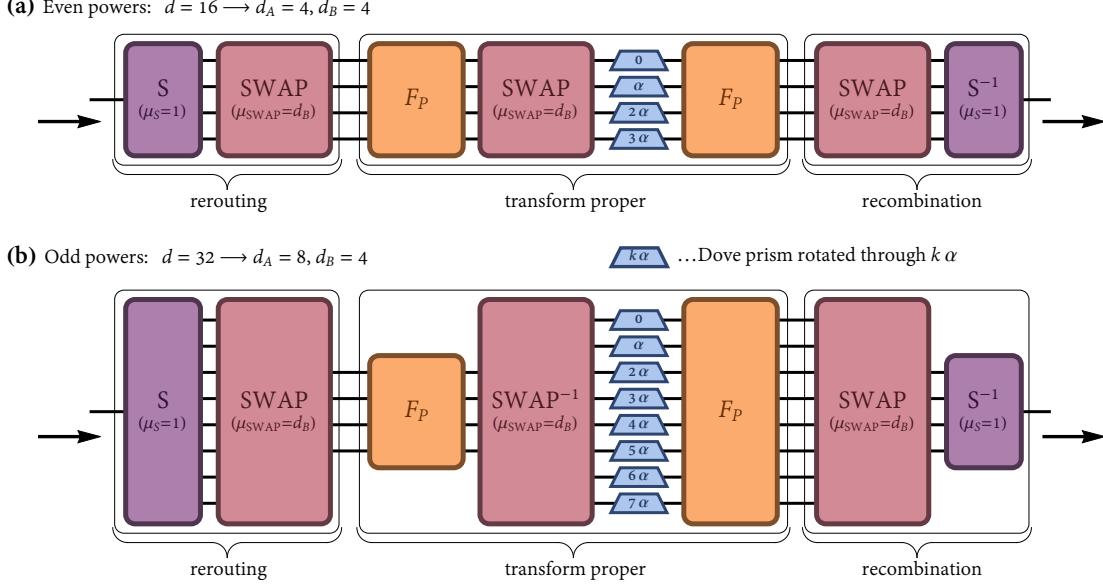


Figure 3.19: The general structure of the d -dimensional OAM Fourier transform. There are two classes of dimensions, for which the corresponding setups differ a little. (a) For dimensions of the form $d = 2^{2M}$ with $M \in \mathbb{N}$ the setup has a rectangular structure with the same number $\sqrt{d}$ of propagation paths all the way from the initial sorter to the final reverted sorter. (b) For dimensions of the form $d = 2^{2M+1}$ with $M \in \mathbb{N}$ the number of paths alternates between $\sqrt{d/2}$ and $\sqrt{2d}$. Reprinted from Ref. [52].

the last recombining module can be for some dimensions removed completely and the first module can be operated backward instead. The direction of propagation through this rerouting module is governed by the polarization. In Fig. 3.21, the explicit experimental setup is shown that implements the 4-dimensional OAM Fourier transform and uses polarization in this way. The photons enter the setup in the vertical (V) polarization and propagate the same way as in the original proposal until they reach the series of Dove prisms. At this point, a series of half-wave plates set to 45 degrees rotates the initial V polarization into the horizontal (H) polarization and the photon is injected back into the setup via a series of polarizing beam splitters. From this point onward, the photon propagates through the first part of the setup, but this time backward, until it reaches the initial polarizing beam splitter and leaves via its upper port.

The scheme for the implementation of the Fourier transform as discussed above is not, despite similarities, a special case of the path-assisted approach from section 3.3. This is due to the explicit use of the modulo property of OAM sorters. Using this scheme one can show that $O(\sqrt{d} \log_2(d))$ conventional optical elements are sufficient for the construction of the high-dimensional OAM Fourier transform in dimensions of the form $d = 2^M$ [52]. It is still an open question, however, how to modify the above setups for other dimensions and preserve appealing scaling properties. One could in principle use the genuine path-assisted approach also in this case, of course. The d -dimensional sorter can reroute all the incoming OAM modes into separate paths and then the d -dimensional path-encoded

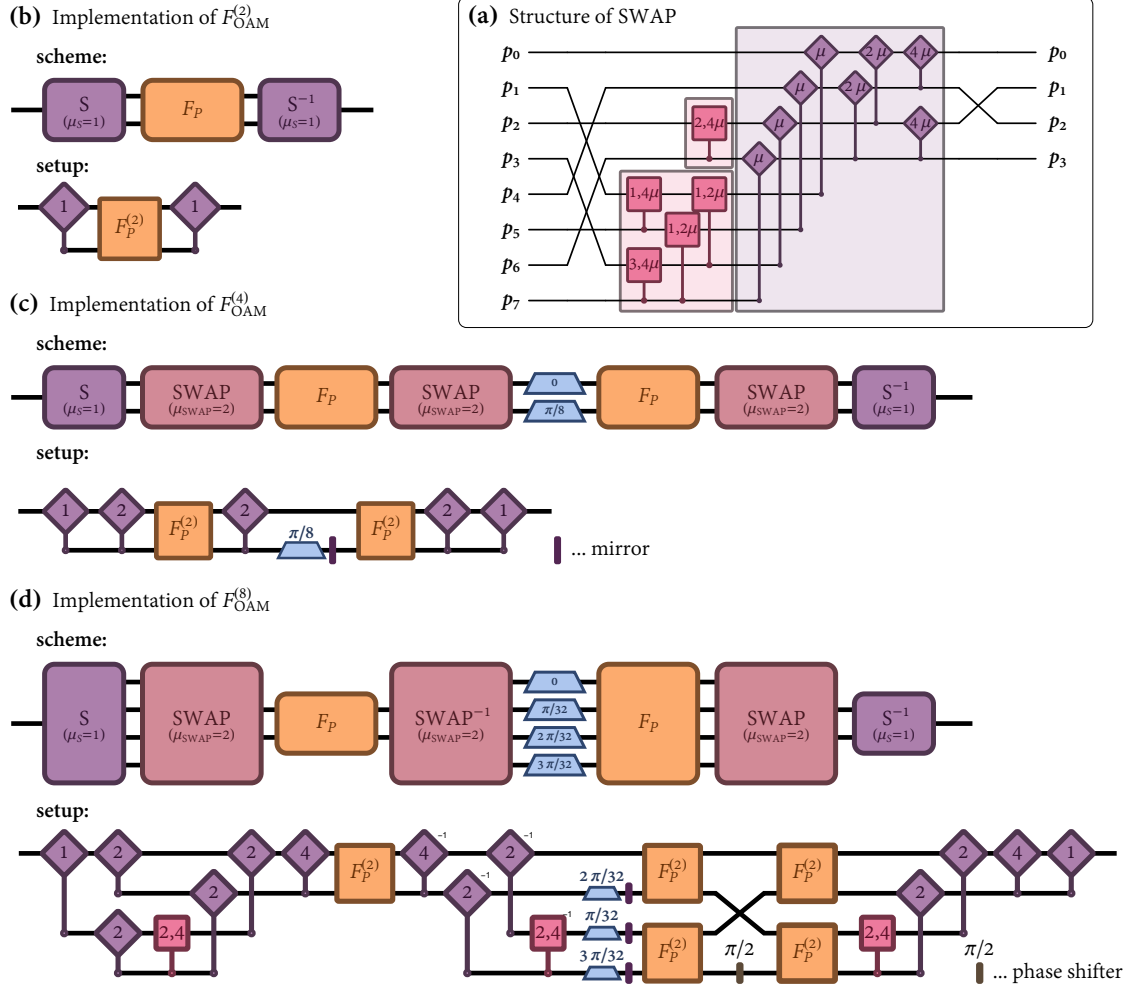


Figure 3.20: Setups of the OAM Fourier transform for several low dimensions of the form $d = 2^M$. (a) The swap operator with unequal number of input and output ports is required in some of the setups. In the figure, the specific case for $d = 8$ is shown explicitly. (b) The Fourier transform in $d = 2$ does not follow the general structure presented in the text as no swap or controlled Z gates are necessary. (c) For all the higher dimensions the general scheme applies. The Fourier transform in $d = 4$ is the simplest nontrivial example shown in Fig. 3.19(a). An extra mirror has to be added in a real physical setup in order to compensate for the sign-reversal due to Dove prisms. (d) The Fourier transform in $d = 8$ is the simplest nontrivial example of the general scheme shown in Fig. 3.19(b). The symbols for exchangers with -1 in superscript indicate the inverse exchangers. Reprinted from Ref. [52].

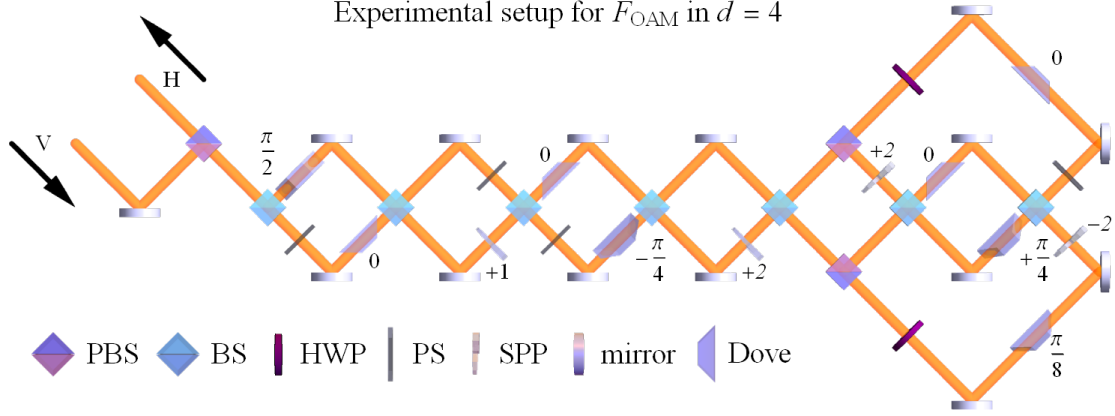


Figure 3.21: The explicit experimental setup for the 4-dimensional Fourier transform of OAM of single photons. The polarization is used as an auxiliary degree of freedom that controls the direction of propagation of a photon through the setup. The photon enters in V polarization from the left and leaves in H polarization. PBS — polarizing beam splitter, BS — symmetric beam splitter, HWP — half-wave plate set to 45° , PS — phase shifter that imparts π phase shift, SPP — spiral phase plate, also known as a vortex plate, that adds or subtracts a specific number of OAM quanta, which number is attached to individual plates in the figure, Dove — Dove prism rotated through the angle that is attached to each prism in the figure.

Fourier transform can be applied. The d -dimensional inverted sorter could then recombine all the modes into a single output path. Such an approach works for general dimension d since the universal schemes of Refs. [41, 44] can be used. The number of elements scales in such a case like $O(d^2)$, which is significantly worse than in our approach.

3.7 Discussion

Having presented an efficient implementation of the OAM Fourier transform, it is time to recapitulate this entire chapter. The primary focus of the chapter are the single-photon OAM gates in various regimes. At first, we demonstrate that any d -dimensional unitary operation acting on OAM of single photons can be constructed using only conventional optical elements whose number scales like $O(d^2)$. This implementation scheme, which we term the path-assisted approach, has better scaling properties than its alternative based on the decomposition of the unitary into powers of X and Z gates. We further show how any single-photon controlled gate can be implemented that uses the OAM and path of a photon as the control and target qudit, respectively. This construction also relies entirely on conventional optical elements. By generalization of the path-assisted approach one arrives at the idea of parallelization of local gates, where the same unitary is applied simultaneously and separately to multiple quantum systems. To illustrate the general schemes with specific examples, we construct a class of high-dimensional Pauli operators

using the path-assisted approach and present their parallelized versions as well. At the end, multiple ideas underlying both the path-assisted and parallelized scheme are employed to implement the high-dimensional Fourier transform of OAM of single photons.

The above approaches allow to apply the techniques of constructing arbitrary unitaries known for the path degree of freedom, such as the Reck et al. scheme [41], also to the orbital angular momentum. The approaches are conceptually very simple and make use of only conventional optical elements familiar to many people working with traditional optics. On the other hand, it is fair to say that these approaches are based on interferometers, where all the arms of each interferometer have to be kept stable in the real physical implementation to assure proper functioning of the corresponding OAM gate. The stability of the experimental setup is thus the major downside of the path-assisted and parallelization schemes. In contrast, the implementation based on the multi-plane light conversion (MPLC) described briefly in section 3.2 does not suffer so much from this problem and it has been applied in multiple scenarios with promising results [37, 38, 55]. Notably, the three-dimensional Fourier transform of OAM of single photons has been implemented this way [55]. On the other hand, our construction of the Fourier transform clearly demonstrates its algebraic structure and the role of individual modules, which can lead to further improvements and generalizations. This is something that complex phase profiles generated by optimization numerical algorithms in the MPLC approach can hardly deliver.

To conclude this chapter, there still remain many open questions related to the efficient implementation of general unitaries for the orbital angular momentum of photons. The results presented in this chapter are meant to provide an answer to at least some of them.

4 Entanglement generation in OAM of photons

Quantum entanglement represents one of the most intriguing features of quantum physics. This non-local effect went unnoticed for several decades after the establishment of quantum theory until it was discovered by Einstein and collaborators [56], who used it to support their claim that “the description of reality as given by a wave function is not complete.” Their observations were followed up by Schrödinger [57, 58, 59], who described this phenomenon as not “one but rather *the* characteristic trait of quantum mechanics.” Here we do not want to argue with Schrödinger about the distinctiveness of quantum entanglement, nor will we argue whether it is merely a natural consequence of the superposition principle for multipartite systems. Be it as it may, thanks to works of Bell [60] and others, the study of entanglement helped immensely in our understanding of quantum theory and shed light on features that have no classical counterparts. After several decades of intensive research, both theoretical and experimental, the field is still very active and there are many open questions about entanglement that wait to be satisfactorily answered. With the advent of quantum information science the phenomenon of entanglement has found concrete applications in processing of information, randomness generation, and cryptographic key distribution to name a few. As quantum technologies are gradually coming of age, an increasing number of originally purely theoretical proposals are being implemented and some of the most promising applications already left the laboratory environment and entered the commercial sphere and the real-world deployment.

In this chapter, we focus on the experimental generation of entanglement. More specifically, after a brief introduction into the concept of quantum entanglement we discuss the generation of higher-dimensional entangled quantum systems. The systems we choose are pairs of photons and we study their entanglement in OAM degree of freedom. There are multiple ways how such entanglement can be created and in the following we elaborate on one general method that was discovered recently [61] and whose first experimental demonstration we reported in Ref. [62].

4.1 Quantum entanglement in a nutshell

In the simplest terms, one can imagine quantum entanglement as a very strong correlation between measurement results that can be observed in quantum measurements, which correlation is too strong to be describable by classical physics. Unlike in classical physics, in quantum physics one can choose to measure a given property of some system in many different ways, many different measurement bases. It is the existence of strong correlations in measurements for many different bases that distinguishes quantum entanglement from classical correlations. Even though the entanglement can be characterized in terms of probabilities with which different results of joint measurements on distant parties occur, its traditional mathematical definition is given in terms of quantum states. Let us restrict

ourselves to the simplest possible scenario and consider two distant parties A and B and only pure quantum states. If these systems are independent, the first system is completely described by state $|\psi_A\rangle$ and similarly system B is fully described by state $|\psi_B\rangle$. The two systems can also be understood as a single composite system with its own state $|\psi_{AB}\rangle$. We say that a quantum state $|\psi_{AB}\rangle$ of the composite system is *entangled* if and only if there is no way how it can be decomposed into two independent states $|\psi_A\rangle$ and $|\psi_B\rangle$ of its subsystems. That is, the composite state cannot be expressed as a tensor product

$$|\psi_{AB}\rangle \neq |\psi_A\rangle \otimes |\psi_B\rangle \quad (4.1)$$

for any $|\psi_A\rangle$ and $|\psi_B\rangle$. The two systems are therefore not independent and state $|\psi_{AB}\rangle$ contains correlations between these systems. These correlations amount to the quantum entanglement. A textbook example of a state that cannot be brought into the tensor-product form is the bipartite state

$$|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}(|0,0\rangle + |1,1\rangle), \quad (4.2)$$

where either of the two systems is associated with a two-dimensional Hilbert space spanned by orthonormal states $|0\rangle$ and $|1\rangle$. The above definition (4.1) of entanglement is independent of the dimension of respective systems A and B and applies not only to the two-dimensional but also to the higher-dimensional scenario. It is the latter that will be the focus of our discussion in the rest of the chapter. Quantum states can be entangled to various extent. Some states exhibit no entanglement and there are also states that have maximal entanglement in a given dimension. Any maximally entangled state of two systems in dimension d can be brought by a suitable choice of local unitaries into the form [63]

$$|\psi_{AB}\rangle = \frac{1}{\sqrt{d}}(|0,0\rangle + |1,1\rangle + |2,2\rangle + \dots + |d-1,d-1\rangle), \quad (4.3)$$

where states $\{|j\rangle\}_{j=0}^{d-1}$ form an orthonormal basis of the single-system Hilbert space and where we tacitly assume that both systems A and B are associated with a Hilbert space of the same dimension d . At this point, it behooves us to mention that entangled states living in a d -dimensional space may not necessarily be entangled in d dimensions. The state $|\psi_{AB}\rangle = (1/\sqrt{2})(|0,0\rangle + |d-1,d-1\rangle)$ can serve us as an example. It lives in at least d -dimensional space but is obviously only two-dimensional when it comes to its entanglement, as only two states are required for its expression. In the rest of the thesis, every time we talk about d -dimensional entangled states we mean states that not only live in a d -dimensional space but also their entanglement is of this dimension. Let us restrict our discussion for the time being to the generation of states of the form in Eq. (4.3), in which a finite number d of basis states is present with equal probabilities [64, 65].

4.2 Generation of entanglement

Entanglement can be generated in various ways and the specific method depends on the physical system that is to be entangled as well as the dimension of the resulting entanglement. In the two-dimensional case, when $d = 2$, the maximally entangled state (4.3) can be represented by the polarization of two photons. Such entangled states have

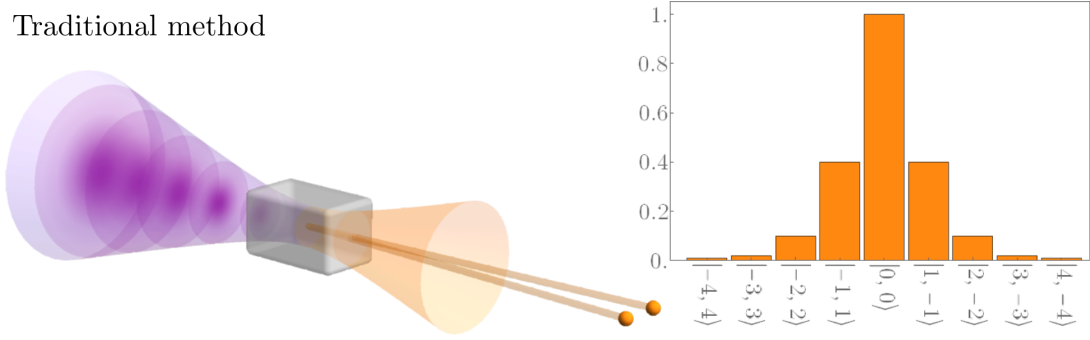
been traditionally generated using specifically manufactured non-linear crystals that are excited with sufficiently strong laser beams. The laser beam, referred to as the *pump beam*, gives rise to optical non-linear phenomena in the crystal, notably the *spontaneous parametric down-conversion* or SPDC for short. The rather long name of this process tells us all its important features: the process is not driven by an external agent, thus *spontaneous*. It is of a probabilistic nature and one does not know when exactly the photon pair is produced. The word *parametric* relates to the fact that the state of the medium after the process is the same as the one prior to the process [66] and the pump beam is, for all practical purposes, left undepleted. Also, the conservation law holds and the total energy before the process is equal to the total energy after it. Since at the beginning there was one photon from the pump beam that split into two photons, the two photons have to carry lower energies. Therefore, one talks about the *down-conversion*. With the name carefully explained, let us turn our attention to the quantum state of the down-converted photons. The detailed description of the SPDC process, its quantum dynamics as well as its semi-classical approximation lie well beyond the scope of this text and can be found in various sources, such as Refs. [67, 68, 69, 66].

It turns out that the exact same physical process can be used to generate higher-dimensional OAM-entangled photon pairs [70], see Fig. 4.1(a). In the SPDC process, the two photons are created in a superposition of different modes as a consequence of the fact that total polarization, total energy as well as other properties such as OAM have to be conserved in the down-conversion process. The resulting superposition represents an entangled state. When we omit all the other degrees of freedom, the quantum state of OAM of the two down-converted photons is in general of the form

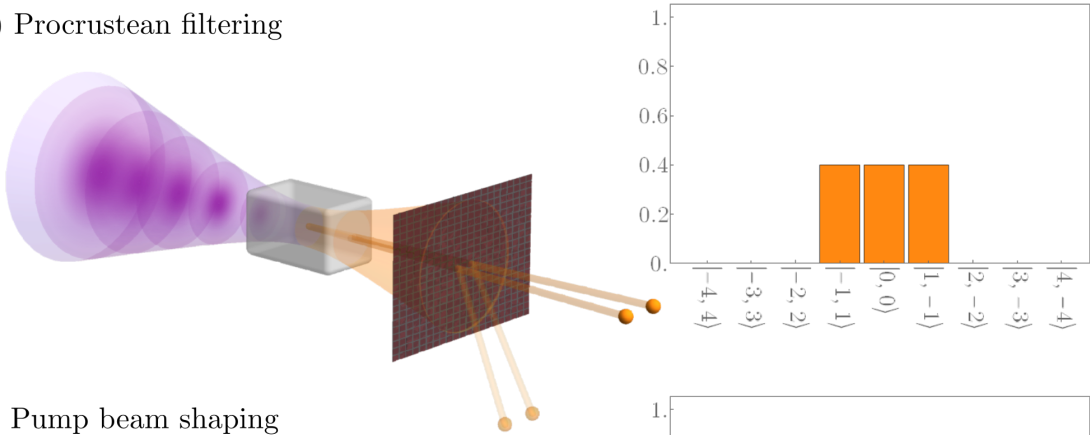
$$|\psi\rangle = \alpha_0 |0, 0\rangle + \alpha_1 |1, -1\rangle + \alpha_{-1} |-1, 1\rangle + \alpha_2 |2, -2\rangle + \alpha_{-2} |-2, 2\rangle + \dots, \quad (4.4)$$

where the dots stand for an infinite number of higher-order OAM modes and where the coefficients α_j satisfy the normalization condition $\sum_j |\alpha_j|^2 = 1$. For sufficiently large OAM values, the coefficients are virtually zero and one can approximate the state with a finite number of several lower-order modes. The SPDC process is a well-studied phenomenon that is relatively easy to exploit for entanglement generation. However, it features downsides also. The one that is of concern to us is that the distribution of OAM modes thus produced is not uniform [71, 72, 73, 74]. It is difficult if not impossible to create a state where the higher-order modes disappear completely and the remaining non-zero coefficients α_j are of equal magnitude. This condition is a necessary prerequisite for the generation of a maximally entangled high-dimensional state (4.3), for instance. Several techniques have been developed by different research groups to circumvent this problem. One of the post-processing techniques is called Procrustean filtering [75, 76] and relies on removing a part of the state (4.4) to make it maximally entangled, see Fig. 4.1(b). More specifically, the down-converted photons impinge on a spatial light modulator (SLM) on which a special hologram is displayed that reflects a certain portion of light residing in undesirable modes off the main propagation path and only those modes are collected that are left unaffected [72]. Another possibility is to use pre-processing techniques such as the pump beam shaping [77, 78], where a superposition of OAM modes is imprinted by holograms directly into the pump beam. Via the down-conversion process this one-photon superposition is translated into maximally entangled states of two photons, see Fig. 4.1(c). There is yet another technique, which is termed *entanglement by path identity* and which is described in detail in the following section.

(a) Traditional method



(b) Procrustean filtering



(c) Pump beam shaping

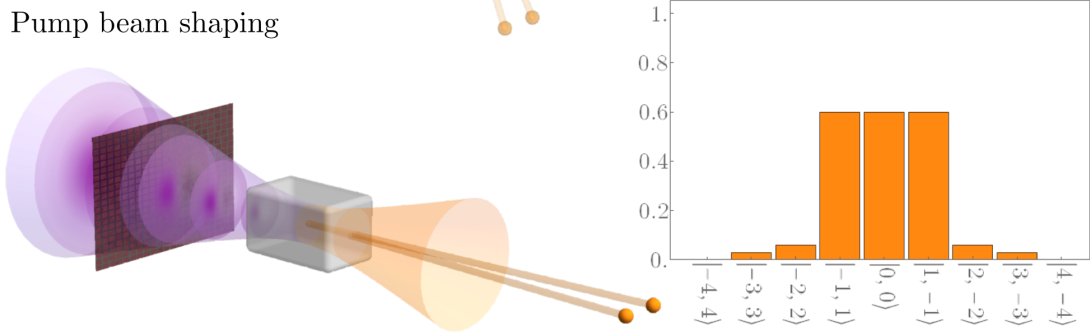


Figure 4.1: The generation of OAM entanglement using the SPDC process. (a) The traditional approach makes use of a single SPDC process in a non-linear crystal that gives rise to a non-maximally entangled states of two photons of the form given by Eq. (4.4). (b) By filtering out the non-uniform component of the state (4.4) by the so-called Procrustean filtering one can retrieve a maximally entangled state (4.3) with decreased efficiency. (c) An alternative method of the pump beam shaping imprints a superposition of OAM modes to the pump beam such that this superposition is turned into a superposition of two photons of the form (4.3) via SPDC.

4.3 Entanglement by path identity

The concept of entanglement by path identity is based on ideas of *induced coherence without induced emission* [79] and *frustrated down-conversion* [80]. When a photon is created in one of multiple photon sources and there exists no information as to which of them it is exactly, one can observe interference phenomena that can be interpreted in such a way that the photon was actually created in a coherent superposition of all the sources. It is the indistinguishability of multiple photon-generation processes that leads to the emergence of a coherent superposition. Each term in the superposition corresponds to a separate process and can be manipulated independently of the other terms. Thanks to this freedom and the use of post-selection, which can remove the undesirable extra terms, a broad class of quantum multidimensional as well as multipartite states can be produced. The structure of the resulting quantum state is governed by the geometry (or the topology, if you like) of the experimental setup.

The entanglement by path identity as a general framework was introduced in Ref. [61]. It rests on three principal ingredients, each of which is presented in detail in the following. The first ingredient is the non-deterministic source of photons, such as a non-linear crystal producing photons via SPDC. In stark contrast to the aforementioned methods though, we demand that the source does not produce any entanglement. For our purposes, the ideal state of the down-converted photons that we want the source to produce is the fundamental mode $|0,0\rangle$. In the real experiment this is nevertheless not always possible and so we work with the experimentally achievable approximation that has negligible higher-order contributions, as shown in Fig. 4.2(a).

The second ingredient amounts to a procedure that transforms the fundamental mode into a class of states necessary for the preparation of entanglement. The transformations we consider are those that can be implemented by spiral phase plates (SPPs). An SPP is a glass plate divided into circular segments such that it resembles a spiral staircase that adds or subtracts a given number of OAM quanta to (or from) the state while keeping all the other properties of photons unaffected. Its action on input states is given by Eq. (3.5) in chapter 3, at least approximately. It suffices to consider only the SPPs that add a single quantum of OAM, as any other integer value can be obtained by multiple applications of the same type of SPP, see Fig. 4.2(b).

The third and last ingredient that is essential for the entanglement generation via path identity is the indistinguishability of photon-generation processes. Let us delve into this topic a little more here. Indistinguishability is closely related to coherence and coherence is in turn a necessary prerequisite for the emergence of quantum superpositions. Let us take a look at Fig. 4.2(c) where a very simple experimental setup is depicted that consists of two non-linear crystals arranged in series. Both crystals are pumped by a single pump beam, shown on the left, and both produce with some finite probability a pair of photons in the fundamental mode. By detecting a pair of photons that leave the setup one cannot obtain any information as to which source produced them. This is a consequence of the setup's geometry where the paths along which photons propagate are made identical for both crystals. This lack of knowledge is a manifestation of the indistinguishability of the first and the second source so long as the photon generation is concerned. The photons leave the setup in the quantum superposition of being produced by the first source and being produced by the second source. The local phase φ in this superposition corresponds to the phase difference acquired by photons originating in either of the two crystals. By

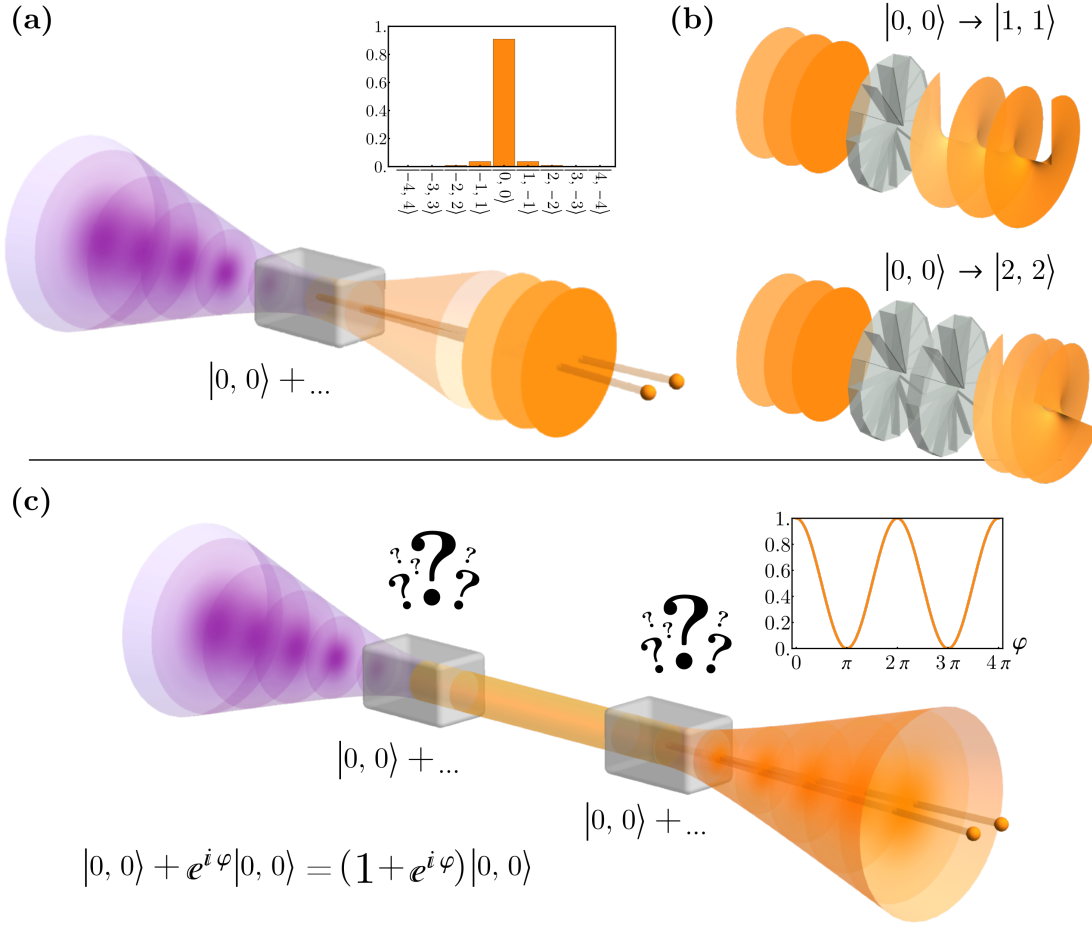


Figure 4.2: The approach of entanglement by path identity is based on three principal ingredients: (a) the sources that produce pairs of photons in the fundamental mode, (b) the ability to manipulate the OAM of single photons, and (c) the indistinguishability of photon-generation processes.

tuning this phase an interference pattern can be observed. But what kind of interference is that? What interferes destructively or constructively is the possibility that photons are produced by the two crystals. For some values of the local phase φ the production of photons by the setup is suppressed and for some other it is enhanced. The quantum state of each individual photon pair remains the same, $|0, 0\rangle$, but the rate of generated photons varies, see the inset in Fig. 4.2(c). This somewhat surprising behavior was experimentally confirmed in multiple experiments [79, 80, 81].

Let us emphasize that it is a misconception to think that both crystals produce photons independently and so sometimes there can be as many as four photons detected at a given instant. Since the SPDC process is relatively inefficient (the probability of a photon pair being produced is on the order of 10^{-7} [82]), the multi-photon emissions can be neglected in a usual case and so there is at most a single pair of photons present in the setup at any given moment. It is rather the setup as a whole that works as a source of photons. The setting of the local phase then affects the rate of photon-pair production.

We are now in a position to explain the underlying principle of entanglement by path identity. Even though the concept is much broader, here we restrict ourselves to a simple scenario of sources arranged in series. For ease of explanation, the figures in Fig. 4.3 serve as a visual guide through the upcoming discussion. Consider a generalization of the above setup, where three crystals are put in series. On top of that, two spiral phase plates are inserted into the spaces between the crystals. The resulting setup combines all three aforementioned ingredients. All the crystals are pumped by the same laser beam and all of them produce photon pairs with certain small probability. This probability is so small that at most a single crystal gives rise to a photon pair at any given time. Once a photon pair is detected after leaving the setup, there are three possibilities that could have happened and that are thanks to the geometry of the setup indistinguishable from each other:

1. The photon pair may have come from the last source, as shown in Fig. 4.3(a). If this is the case, the two photons reside in the fundamental mode $|0, 0\rangle$.
2. Another possibility is that the pair of photons originated in the middle crystal, in which case it must have traversed a spiral phase plate, see Fig. 4.3(b). The plate added a quantum of OAM to either photon and the pair thus left the setup in state $|1, 1\rangle$.
3. The last possibility, depicted in Fig. 4.3(c), is that the pair was produced by the first crystal and consequently passed through two spiral phase plates. In such a case, both photons carry two quanta of OAM and they leave the setup in state $|2, 2\rangle$.

Due to indistinguishability all these three possibilities are in a quantum superposition, cf. Fig. 4.3(d), and each pair of photons generated collectively by the three crystals leaves the setup in state

$$|\psi\rangle = \frac{1}{\sqrt{3}}(|0, 0\rangle + e^{i\varphi}|1, 1\rangle + e^{i\xi}|2, 2\rangle), \quad (4.5)$$

where the local phases φ and ξ are the phases between SPDC processes in the three crystals and can be tuned at will by properly adjusted phase shifters inserted into the setup. Not only the phases but also the magnitudes of individual terms in the above state can be tuned arbitrarily. If the pumping power is adjusted for each crystal separately, the magnitudes of each term are no longer equal to each other and the setup can generate any state of the form

$$|\psi\rangle = \alpha_0|0, 0\rangle + \alpha_1 e^{i\varphi}|1, 1\rangle + \alpha_2 e^{i\xi}|2, 2\rangle, \quad (4.6)$$

where $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{R}$ and $\alpha_0^2 + \alpha_1^2 + \alpha_2^2 = 1$. Moreover, if the *non*-collinear SPDC was utilized in all the crystals and the two down-converted photons could be manipulated separately, one could apply a different number of SPPs to either photon. That would result in a state with three terms, where not only phases and magnitudes of coefficients but also the values of OAM in each term can be chosen in an independent and arbitrary way. This observation demonstrates the power of the concept of entanglement by path identity.

In the above discussion we tacitly assumed that both photons in each down-converted pair have identical properties: identical polarization, frequency, momentum, etc. The only thing that we manipulated was the OAM. If that was not the case, the resulting state produced by the setup would contain three terms, where now each term would hold information about not only OAM but also all the other degrees of freedom. To retrieve the

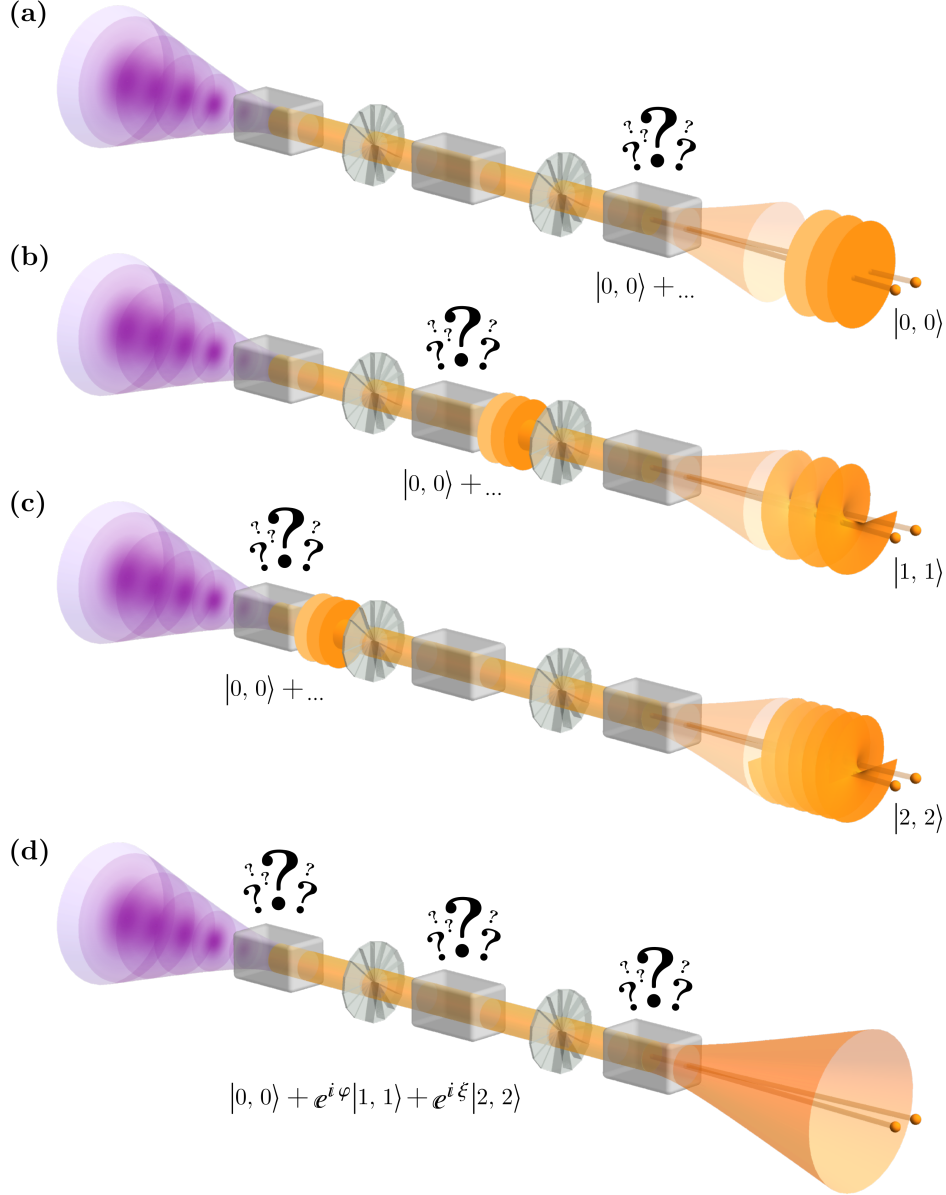


Figure 4.3: The principle of entanglement by path identity. Due to indistinguishability there is no way of knowing in which crystal the pair of photons originated. (a) If it was the last crystal, the two photons carry zero quanta of OAM. (b) If it was instead the middle crystal, the two photons, though created in state $|0, 0\rangle$, leave the setup in state $|1, 1\rangle$ due to an SPP inserted after the middle crystal. (c) Similarly, if the photons were produced by the first crystal, they propagate through two SPPs that impart two quanta of OAM to each. As a result, the photons leave in state $|2, 2\rangle$. (d) Taking into account all these possibilities, we conclude that the OAM state of photons is a coherent superposition $|\psi\rangle = (1/\sqrt{3})(|0, 0\rangle + e^{i\varphi}|1, 1\rangle + e^{i\xi}|2, 2\rangle)$. For details see the text.

state of OAM one would have to trace over all these additional quantities, resulting in a mixed OAM state. At the beginning of this section we argued that the indistinguishability of sources leads to quantum coherence, but in the setup of Fig. 4.3(d) one can actually distinguish the sources — the OAM of a photon carries the information about where it was generated. What do we mean then by indistinguishability? The point is that except for measurement of OAM eigenstates there is no other way how the three sources could be distinguished by subjecting the produced photons to some quantum measurement.

To summarize the discussion, we can see the entanglement by path identity as a kind of quantum state engineering where instead of relying on intrinsic properties of the SPDC process we build the entangled state from scratch ourselves. By appropriately designing the experimental setup we artificially make multiple processes indistinguishable. By adding a crystal into the setup we increase the dimension of entanglement by one. By inserting spiral phase plates we can modify the OAM of individual terms in the superposition. By pumping individual crystals separately we can tune the magnitudes of probability amplitudes and finally by changing the phase of photons between crystals we can modify the phases of the probability amplitudes. This approach is equally valid for photons as well as for other physical systems, it applies to the case of two as well as many sources, and is fully applicable for many different degrees of freedom. One can generate a large class of states artificially and engineer well-defined superpositions of various terms on demand so to say. The synthesis of the three ingredients in a unified framework of entanglement generation was done in Ref. [61], even though some earlier proposals rely on similar ideas: for instance, the cross-crystal scheme is the simplest example of the above approach, where two-particle states are entangled in polarization [83, 84, 85].

It is rather easy to generalize the above analysis for more crystals to increase the entanglement dimension. We have discussed the linear setup with two and three crystals, giving rise to two- and three-dimensional entanglement. Obviously, the more crystals are arranged in the setup, the larger the dimension of entanglement. In Fig. 4.4(a) the setup for the generation of four-dimensional entanglement is depicted and its generalization for even higher dimensions is straightforward. Instead of increasing the dimension of a bipartite entangled state one could think of increasing the number of photons. This is also possible but its proper explanation is much more involved. It turns out that by post-selection and properly designed pattern of path overlaps a broad class of multipartite entangled states can be generated. The detailed introduction into this topic lies beyond the scope of this work and can be found in Ref. [61]. Nevertheless, in Fig. 4.4(b) a taste of the more general approach is presented. The setup consisting of six non-linear crystals produces quadruples of photons. Out of all detection events only those are post-selected, for which there is exactly one photon in each of the four output paths. The propagation paths of individual photons generated by any of the crystals are rerouted such that only three possibilities are allowed by the post-selection. The first possibility is that the two leftmost crystals simultaneously generate a photon pair. In such a case, each down-converted photon traverses two SPPs and leaves the setup with two quanta of OAM, such that the state of all four photons reads $|2, 2, 2, 2\rangle$. The second possibility is that the two center crystals simultaneously fire. Each down-converted photon then propagates through only one SPP and the resulting state of four photons reads $|1, 1, 1, 1\rangle$. In the last allowed case the two rightmost crystals simultaneously produce a pair of photons such that they leave the setup in state $|0, 0, 0, 0\rangle$. Again, due to indistinguishability the photons end up in state $(1/\sqrt{3})(|0, 0, 0, 0\rangle + |1, 1, 1, 1\rangle + |2, 2, 2, 2\rangle)$.

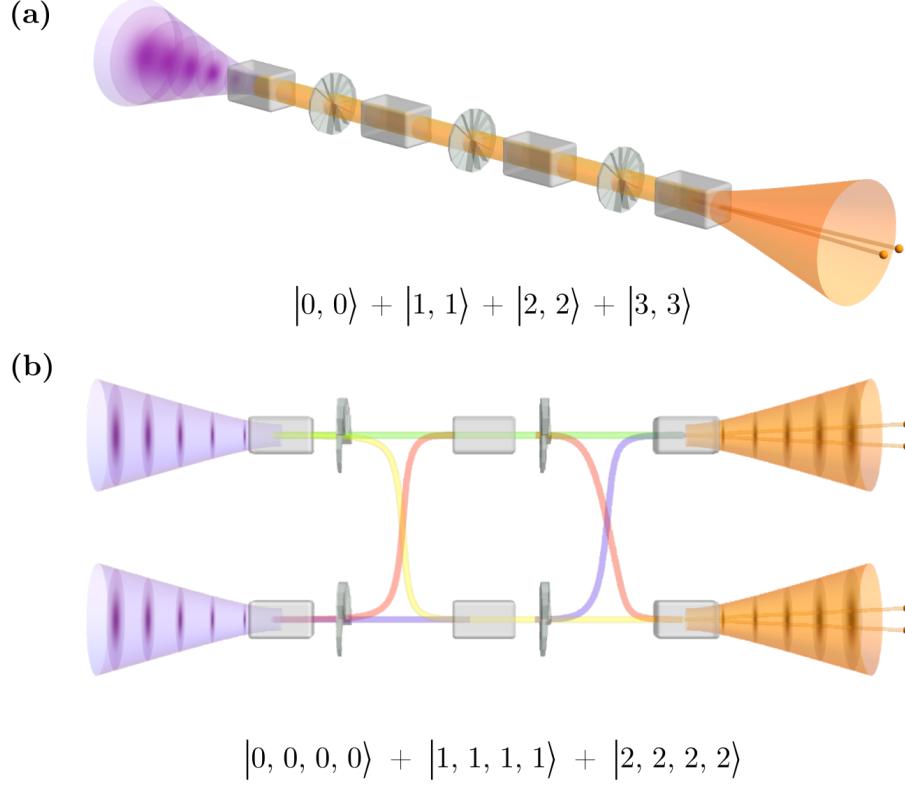


Figure 4.4: The generalization of the experimental arrangement. (a) The dimension of the entangled state can be increased by adding more crystals in series in an obvious way. (b) Generalizations for more than two photons are possible but more involved. The post-selection has to be used and the propagation paths of photons produced by different crystals have to be chosen carefully. The details can be found in Ref. [61]. In the figure, an example of the setup that creates four-photon three-dimensional entangled state is shown. Different propagation paths are marked by different colors. For instance, one of the photons generated by the crystal at the top left follows the green path, whereas the other photon traverses the yellow path. It turns out that the only cases when four photons are detected in different paths are those that give rise to states $|0, 0, 0, 0\rangle$, $|1, 1, 1, 1\rangle$, and $|2, 2, 2, 2\rangle$. Due to indistinguishability, they form a quantum superposition.

4.4 Experimental setup

Having introduced the concept, in this and following sections we turn our attention to its experimental realization. More specifically, we choose the setup in Fig. 4.3(d) with three crystals as the model and modify it to comply with the technical demands and limitations. The states we aim to generate are thus of the form of Eq. (4.6), where the two photons are entangled in the orbital angular momentum. In the rest of this chapter, we describe a proof-of-principle experiment that was performed in our group and whose results are published in Ref. [62]. The schematic plan of our experimental setup is depicted in Fig. 4.5, where individual stages of the setup are enclosed in colored boxes. This emphasizes the incremental manner in which the entanglement is built, so to speak, from the fundamental mode.

The core part of each stage is a non-linear crystal pumped by a laser. Typically, such a crystal generates via SPDC a pair of photons in an entangled state. In our case though, we want the exact opposite. We seek to be in a regime when the crystal produces a photon pair in a product state $|0, 0\rangle$ with zero quanta of OAM. In the real experimental implementation one can only approximate such a state and has to study also how large the higher-order OAM contributions are. Because of the crosstalk of the undesirable higher orders of OAM we choose in the actual setup to generate state

$$|\psi\rangle = \alpha_0 |0, 0\rangle + \alpha_1 e^{i\varphi_1} |2, 2\rangle + \alpha_2 e^{i\varphi_2} |-2, -2\rangle, \quad (4.7)$$

which can be transformed into state (4.6) by local unitaries applied to either photon and so the two states exhibit the same entanglement properties. As opposed to state (4.6), the technical limitations were more favorable for the generation of state (4.7). The details related to the SPDC source of photon pairs can be found below in section 4.4.1.

There are several differences between the conceptual scheme and the experimental implementation apart from those already mentioned. Unlike in the original proposal in Fig. 4.3, the spiral phase plates are inserted into the pump beam such that it is the laser photon that acquires higher-order OAM. Only due to SPDC is the OAM transferred to the down-converted photons. For details consult section 4.4.1. The second notable modification of the conceptual scheme is that the pump beam is guided along separate paths from those for the down-converted photons. This arrangement was chosen to make the setup customizable as the coherence, phase, as well as the power of each of the three branches of the pump beam can be adjusted independently. On the other hand, the resulting setup consists of two coupled Mach-Zehnder interferometers, which poses a problem with keeping the setup sufficiently stable on the time scales required for the experiment. In section 4.4.2 below, the role of coherence in our setup is discussed. The stabilization system of the interferometers can be at the same time used for a fine-tuned adjustment of the local phases, as is explained in the same section. A more detailed schematic of the experimental setup is shown in Fig. 4.6, where the stabilization system as well as the actual way of adjusting the magnitude, phase, and OAM of individual modes is shown explicitly.

At the end of the setup, two spatial light modulators in addition to single-mode fibers and single-photon detectors are used to detect the OAM state of the photon pairs. The experimental data are fed into the state-tomography procedures and post-processed by maximum likelihood estimation to obtain physically sound density matrices to represent our data. As our original goal was to produce entangled states, the entanglement of the

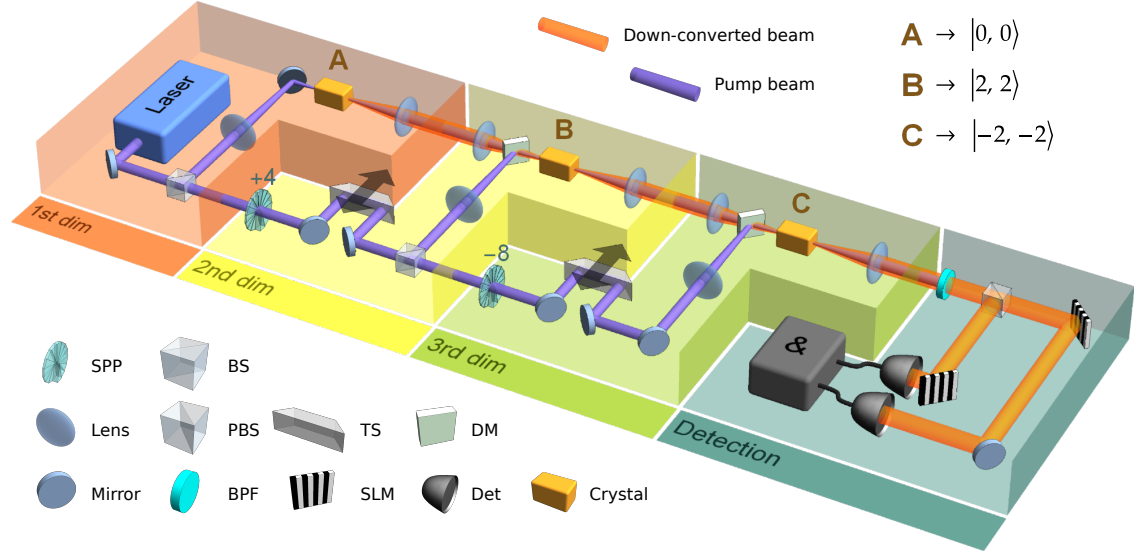


Figure 4.5: The experimental setup consists of three generation modules and one detection module. The pump beam is split into three beams of equal power by beam splitters (BSs) and focused by lenses into three non-linear crystals. Each generation module, enclosed in a colored box, adds one dimension to the resulting quantum state of two photons. The first module comprises a non-linear crystal that via SPDC gives rise to two photons in a separable state $|0, 0\rangle$. The second module contains not only a crystal but also a spiral phase plate (SPP) that adds 4 quanta of OAM to the pump, which translates via SPDC into the state $|2, 2\rangle$ of the two photons. A trombone system (TS) is used to adjust the local phase precisely. The third module resembles the second one, only the state created is $|-2, -2\rangle$. In the detection module, the two photons are separated by a polarizing beam splitter (PBS) and a state tomography is performed with the help of two spatial light modulators (SLMs) and single-photon detectors (Dets). The pump beam is diverted from the path of down-converted photons by dichroic mirrors (DMs) and a band-pass filter (BPF). Reprinted from Ref. [62].

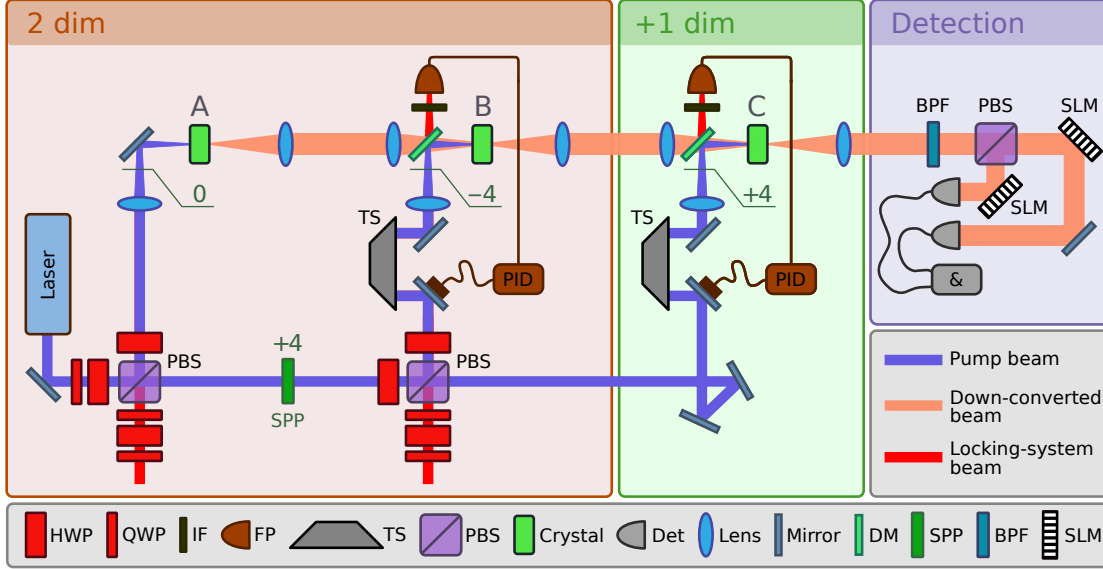


Figure 4.6: The detailed experimental setup. There are several differences between the schematic setup of Fig. 4.5 and the actual experimental implementation. First, the asymmetric beam splitters are constructed with the help of traditional polarizing beam splitters (PBSs), half-wave plates (HWPs), and quarter-wave plates (QWPs). The exact rotation of the polarization of the incoming beam effectively sets the splitting ratio, as explained in the text. The second difference is that an additional stabilization system is built along the experimental setup. This system consists of a separate laser diode, a pair of fast photodiodes (FDs), interference filters (IFs) and two PID modules. For details see the text. The last major difference is that instead of the third spiral phase plate (SPP), an additional mirror is inserted in the beam before it reaches the third crystal. This mirror reverts the sign of OAM and turns thus mode $|4\rangle$ into $|{-4}\rangle$, effectively working as an SPP that subtracts 8 quanta of OAM. The meaning of all the other acronyms is identical to those in Fig. 4.5. Reprinted from Ref. [62].

resulting states is assessed and fidelities of these state are estimated. This last stage of our experiment is described at length in sections 4.4.4 and 4.5.

In the following, all the major parts of our experimental setup are discussed in greater detail in separate sections. After that, the alignment procedure is described that puts all the parts together into a single experimental setup. For completeness, in Fig. 4.7 one can see a photo of the actual experimental setup as built in the laboratory.

4.4.1 Sources

The power for the experiment is delivered by a continuous-wave diode laser (Ondax LM-405-PLR-40-X1) with around 17 mW of the pumping power at the wavelength of 405 nm. The pump beam is split into three beams by two variable-splitting-ratio beam splitters such that the power with which individual non-linear crystals are pumped can be adjusted. Each non-linear crystal is a 10 mm long slab of a periodically poled potassium titanyl phosphate (ppKTP) with the poling period of $4.7 \mu\text{m}$ that is cut for the collinear type II SPDC. The temperature for each crystal is set to get degenerate down-conversion, where the central wavelength of both down-converted photons is 810 nm with FWHM roughly equal to 1 nm. As mentioned above, the quantum state of down-converted photons is in general entangled in many degrees of freedom including OAM. The amount of entanglement can be modified by tuning the pumping parameters of the crystal arrangement. For a crystal of a fixed length the amount can be changed by choosing the focusing and collecting lenses with different focal lengths [71]. In our setup, the focal lengths were chosen to be 25 cm to ensure that virtually only the fundamental mode is generated via SPDC. In Fig. 4.8 the experimental measurement is shown, where the quality of the quantum state is assessed in terms of the spiral spectrum of the crystal. In the ideal scenario, the only non-zero element in the figure would be the one in the center. In our experiment, the central peak is approximately twenty times larger than the next larger higher-order contribution. Nevertheless, even such a small contribution leads to a considerable crosstalk, where the higher-order modes from one crystal interfere with those from the other crystal. To avoid such an undesirable behavior, the OAM modes $|1\rangle$ and $|-1\rangle$ are not used in the setup and the next higher-order modes $|\pm 2\rangle$, which display small crosstalk, are used instead. For this reason the state generated by our setup is the one in Eq. (4.7) rather than (4.6).

The setup comprises three non-linear crystals. The first one emits photons in their fundamental mode. The other two crystals are meant to emit higher-order terms of the state in Eq. (4.7). These terms are prepared by a spiral phase plate (SPP) that adds a finite number of OAM quanta to the traversing photons. In the conceptual scheme, the photons are produced in the fundamental mode $|0, 0\rangle$ and only then are they subjected to the action of the SPP. This approach proved to be problematic in our setup since the SPP is very sensitive to precise alignment. For an SPP to work properly it has to be placed precisely into the center of the beam of down-converted photons. Even though we used the collinear type of SPDC, the two down-converted photons did not propagate along the exact same trajectory. The two trajectories rather subtended an angle of roughly 1° , which was already enough to ruin the operation of the SPP. For this reason, we decided to put the SPP into the pump beam before it reaches the crystal, which has several important consequences. The SPP was manufactured to impart 2 quanta of OAM to the beam of wavelength 810 nm. When placed into the pump beam, whose wavelength is 405 nm, it effectively adds twice as many quanta as it would to the down-converted beam. The first



Figure 4.7: The photo of the actual experimental setup. The blue pumping laser beam enters the setup at the far end of the optical table. The holders of the first and the second non-linear crystals labeled by C1 and C2, respectively, are visible in the upper half of the image. In the foreground, the trombone system for the coarse adjustment of the phase in the second interferometer is visible.

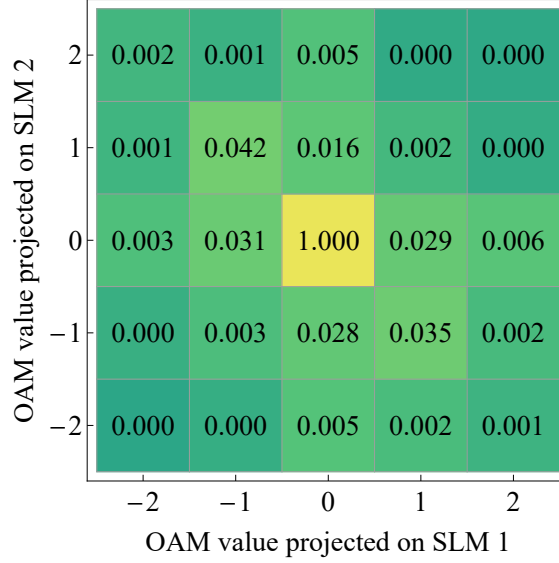


Figure 4.8: The spiral spectrum of the first non-linear crystal. The crystal was pumped by the fundamental mode and the resulting quantum state of down-converted photons was measured. Normalized contributions of different lower-order OAM modes are shown. In the ideal case, only the center element corresponding to the fundamental mode $|0, 0\rangle$ would be non-zero. In our case, the central peak is more than twenty times larger than the next largest value (the central peak amounts to roughly 54000 ± 230 counts per second, the next largest one to roughly 2280 ± 48 counts per second). For the second and third non-linear crystals very similar results were obtained. Reprinted from Ref. [62].

SPP thus adds 4 quanta of OAM to the pump such that each pump photon is in state $|4\rangle$. As the OAM is preserved in SPDC, the OAM of the pump is transformed into the OAM of the two down-converted photons, which are then in state $|2, 2\rangle$. For the third crystal one could use the same strategy with another SPP. There is, however, a simpler approach. One can recall that a mere mirror reverses the sign of OAM. Instead of another SPP one just inserts an additional mirror before the third crystal, which transforms the OAM state of each pump photon from $|4\rangle$ into $|-4\rangle$, effectively functioning as an SPP that subtracts 8 quanta of OAM. The two down-converted photons are then in state $|-2, -2\rangle$.

4.4.2 Interferometers

In the preceding section, we discussed the sources of photons as separate entities. In order for the setup to work we have to make sure that the sources are mutually coherent. The coherence is achieved by, first, setting the parameters of individual sources in such a way that the states of down-converted photons from each source are as similar as possible; and second, by a very careful alignment of down-converted beams from the three sources, such that interference emerges in the counting rate of photons collected by the detection system. The interference can be seen only when the same mode is produced by each crystal and so in the alignment stage, no spiral phase plates are inserted into the setup. All photons are thus in the fundamental mode the whole time. Moreover, it is easier to assess the

quality of interference when only two crystals are present and so for aligning, say, the first and the second crystal the pump beam for the third crystal is blocked. Similarly for the other combinations. This way, one can measure the pairwise interference where photon coincidence counts are monitored by the detection system while the local phase between the two involved crystals is varied. The interferometric visibility $\mathcal{V}$ of the resulting interference pattern serves as the figure of merit that is to be optimized. The visibility $\mathcal{V}$ is defined as

$$\mathcal{V} = \frac{\max C - \min C}{\max C + \min C}, \quad (4.8)$$

where C stands for the set of coincidence counts registered by detectors over a sufficiently long time span. To illustrate the quality of coherence in our setup, in Fig. 4.9 the interference pattern obtained for the interference of the first and the second crystal is shown, where the visibility equals $97.1 \pm 0.5\%$.

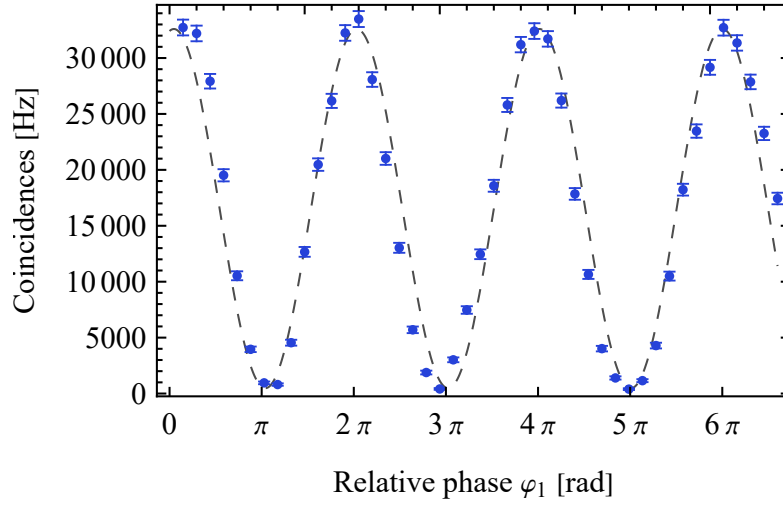


Figure 4.9: The coherence of the first and the second crystal measured in the fundamental mode $|0, 0\rangle$ when the relative phase φ_1 is varied. The dashed line represents the fit of the displayed data, whose visibility is equal to $97.1 \pm 0.5\%$. Adapted from Ref. [62].

As already mentioned above, the actual setup comprises two Mach-Zehnder interferometers that share a common arm. A traditional interferometer consists of two beam splitters (BSs) and two mirrors such that the incoming light splits at the first beam splitter into two arms that are recombined by the second beam splitter. In our case, the first BS has a variable splitting ratio that allows for adjusting the relative powers with which each crystal is pumped and thus the magnitude of the corresponding probability amplitudes. Such a beam splitter is implemented as a polarizing beam splitter (PBS) where the splitting ratio is controlled by the rotation of the polarization of the incoming light with the help of a half-wave plate (HWP), see also Fig. 4.6. The incoming laser light is horizontally polarized, the half-wave plate redistributes the power between the horizontal (H) and vertical (V) component and the PBS transmits the H component while reflecting the V component. This V component is rotated back to H by an additional HWP. The second beam splitter of the interferometer is effectively played in our case by a dichroic mirror. Into one arm

of the interferometer a non-linear crystal is placed and in the other a trombone system consisting of a movable linear stage with two mirrors is used for the coarse-grained phase adjustment.

This solution was chosen in order to control independently the arm of the pump beam and the down-converted photons. On the other hand, the interferometric arrangement puts stringent constraints on other aspects of the setup. The difference between the arms of an interferometer has an effect on the coherence of the down-converted photons. Consider for instance the first interferometer that connects the first and the second crystal. Let $L_{p,1}$ and $L_{p,2}$ be the distances traveled by the pump beam from the initial beam splitter of the interferometer to the two crystals, respectively. Furthermore, let L_{SPDC} be the distance that down-converted photons have to travel from one crystal to the other. It turns out [86, 87] that the following condition has to be satisfied in order to observe interference

$$|L_{p,2} - L_{p,1} - L_{\text{SPDC}}| \leq L_{\text{coh}}, \quad (4.9)$$

where L_{coh} is the coherence length of the pump laser, which in our case exceeds 2 cm. In other words, the optical path length difference between the two arms of the interferometer must be within the coherence length of the pump laser.

4.4.3 Stabilization system

Another downside of interferometers is that they require active stabilization. Otherwise, the random mechanical vibrations of the setup lead to the destruction of quantum coherence on time scales relevant for the experiment. For illustration, in Fig. 4.10 the orange curve demonstrates the rate and magnitude of fluctuations in coincidence counts when no stabilization is in effect. To compensate for this, the phase-lock system is built around the actual setup. The key part of this system is an auxiliary laser diode that emits a beam with wavelength of roughly 700 nm, which beam is split into two branches. Each branch is injected into one of the unused ports of variable-splitting-ratio beam splitters that constitute the two interferometers, see Fig. 4.6. A given branch then propagates through both arms of the respective interferometer and is recombined on a dichroic mirror before leaving the setup. The dichroic mirror at the end of the interferometer has the edge at around 700 nm and so it effectively works as a beam splitter for the auxiliary laser. The interference pattern observed in the unused output port of either dichroic mirror is monitored by a fast photodiode. The photodiode supplies its signal to an electronic unit responsible for the feedback generation. The parameters of the feedback loop are chosen such that the photodiode expects intensity of the auxiliary laser beam that lies roughly between the interference maximum and interference minimum. Even small perturbations from this middle-point intensity are then easy to detect. The setup thus contains two feedback loops, one for either interferometer. Either loop consists of the photodiode, a PID (proportional-integral-derivative) controller, and a piezoelectric element attached to a mirror in one arm of the given interferometer. The PID controller drives the piezoelectric element based on the signal from the photodiode.

The phase-lock system can also be used for the fine tuning of the mutual phase between two crystals. A series of quarter-, half-, and quarter-wave plates, henceforth denoted by QHQ, is used to prepare the auxiliary laser beam in polarization $(1/\sqrt{2})(|H\rangle + e^{i\omega}|V\rangle)$, where the local phase ω can be controlled by the experimenter. A polarizing beam splitter separates the H and V components of the auxiliary laser the same way as it does for

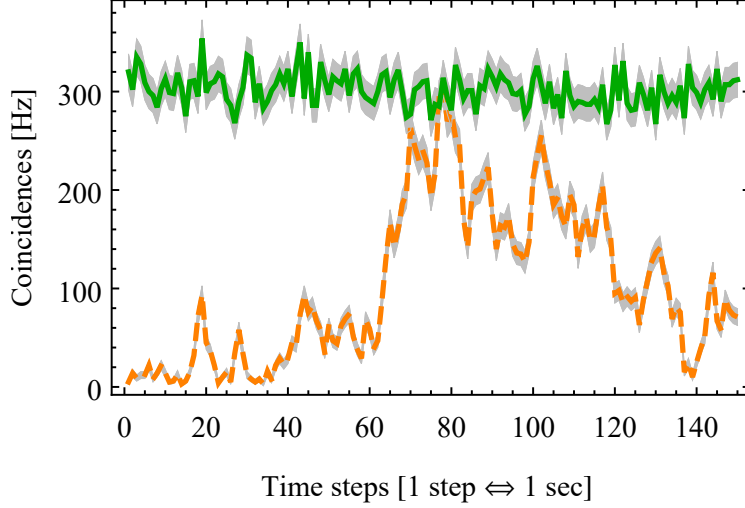


Figure 4.10: Phase stability of the first interferometer when both crystals produce photons in the fundamental mode. Without a phase-lock system the coincidence counts vary dramatically on time spans of tens of seconds (orange dashed curve). When the phase stabilization is in place, the fluctuations are mostly eradicated (green solid curve). The gray areas around both curves mark the one-standard-deviation bands around the measured values, as given by the Poissonian error propagation. Reprinted from Ref. [62].

the pump laser. The local phase ω between polarization components thus turns into a relative phase between the two arms of the interferometer and the phase-lock system thinks it is an additional phase caused by the environment that has to be corrected for. The feedback loop thus offsets the mirror attached to the piezoelectric crystal by an amount that depends on ω . This way, one can control the optical path length difference between the two arms of the interferometer. The setting of the QHQ series can be chosen such that the local phase ω is changed by a mere rotation of the center half-wave plate (HWP). In the supplemental material of Ref. [62] the derivation of the exact formula for the settings can be found. Here we only present the underlying idea. First, it can be shown that any elliptical polarization of the input laser light can be transformed into a linear polarization by a single quarter-wave plate (QWP). Second, this linear polarization can be turned into another linear polarization by a half-wave plate. Third, another QWP rotated through the angle of 45° transforms a linear polarization that points in the direction given by angle $\omega/2$ into an elliptical polarization, where both H and V component have equal magnitudes and their mutual phase is, up to a constant offset, equal to ω . Specifically, if the input polarization vector of the auxiliary laser reads $(a, be^{i\varphi})$ for $a, b, \varphi \in \mathbb{R}$ and $a^2 + b^2 = 1$, it can be transformed into the polarization $(1/\sqrt{2})(1, e^{i(\omega-\pi/2)})$ by the following operation

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{i(\omega-\pi/2)} \end{pmatrix} = \text{QWP}\left(\frac{\pi}{4}\right) \cdot \text{HWP}\left(\frac{\omega}{2} - \xi\right) \cdot \text{QWP}(\gamma) \cdot \begin{pmatrix} a \\ be^{i\varphi} \end{pmatrix}, \quad (4.10)$$

where γ is given by formula $\tan(2\gamma) = 2ab \cos(\varphi)/(a^2 - b^2)$ (for $a = b = 1/\sqrt{2}$ we set $\gamma = \pi/4$) and where ξ is the angle of linear polarization of light that leaves the first QWP, whose form is determined by the values of a , b , and φ , such that $\tan(\xi)^2 =$

$(b^2 + f \sin(2\varphi))/(a^2 - f \sin(2\varphi))$, where $f = a^2 b^2 / \sqrt{a^4 + b^4 + 2a^2 b^2 \cos(2\varphi)}$. By $\text{QWP}(\alpha)$ and $\text{HWP}(\alpha)$ we denote a QWP and a HWP rotated through α , respectively. These operations are given by matrices

$$\text{QWP}(\alpha) = \frac{e^{-i\frac{\pi}{4}}}{\sqrt{2}} \begin{pmatrix} i + \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & i - \cos(2\alpha) \end{pmatrix}, \quad \text{HWP}(\alpha) = \begin{pmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{pmatrix}. \quad (4.11)$$

4.4.4 Detection

The last stage of the experimental setup is used to detect the down-converted photons. At first, a band-pass filter (3 nm FWHM at 810 nm) gets rid of the pump beam such that only the down-converted beam reaches the detectors. Since we use type II SPDC, the two photons that make up each entangled pair have orthogonal polarizations. This allows us to deterministically separate the two photons by a polarizing beam splitter and guide both of them individually to their respective detectors. Before the photons reach the single-photon counting modules (Excelitas SPCM-CD 3321H), a measurement of OAM is performed by two SLMs (Hamamatsu LCOS-SLM X10468-2) and single-mode fibers that support light frequencies of around 810 nm, as described in detail below. The signals issued by both modules are collected by a coincidence logic and sent for further processing into a computer.

The OAM state of photons is measured sequentially mode by mode, where the information about each mode is used by a computer to reconstruct the state of the photons or calculate some important characteristics of the state. Each of the sequential measurements of a photon pair is performed by a simple detection setup that comprises two SLMs and two single-mode fibers, one for either photon. When a single photon impinges on the screen of an SLM, its wave functions acquires an additional spatial-dependent phase shift. Suppose we display a phase profile $e^{i\varphi(x,y)}$ on an SLM that is described by a real function $\varphi(x,y)$ that depends on the position in the plane perpendicular to the beam propagation. If the phase of the beam reads $e^{i\phi(x,y)}$ before the reflection of the beam off the SLM's screen, after the reflection the new phase is given by $e^{i(\varphi(x,y)+\phi(x,y))}$. Suppose the photon pair is in a superposition (4.4) and we want to know the magnitude of the coefficient α_k that corresponds to the OAM mode $|k, -k\rangle$. We do so by a projective measurement that projects the photons' state onto the basis state $|k, -k\rangle$ in the following way. The OAM mode $|k\rangle$ is associated with a phase profile $\phi(x,y) = k \arctan(y/x)$. It is thus sufficient to set $\varphi(x,y) = -k \arctan(y/x)$, such that the resulting wavefront reduces to the plane wave as $\varphi(x,y) + \phi(x,y) = 0$. The plane wave corresponds to the fundamental mode, which is efficiently coupled into a single-mode fiber, in contrast to the rest of the modes. The fiber then guides the photon into the detection module and from the number of detection events per a fixed time span one can subsequently estimate the magnitude of the coefficient α_k . As a result, the assembly of an SLM and a single-mode fiber works as a projective measurement when one sets the state onto which the photons' state is projected. To measure the local phases of individual terms in the photons' state, one can display well-chosen phase profiles on the SLM that correspond to various superpositions of OAM modes. It can be shown that all these projective measurements are sufficient for the reconstruction of the entire density matrix of the photons' OAM state [88].

4.4.5 Alignment procedure

In preceding sections, individual components of the experimental setup are discussed separately. Here, we describe the alignment procedure with the help of which the components are assembled together. The experimental setup has a modular structure and so also the alignment procedure is in the following described in a similar manner, one crystal after the other. At first, the pump beam generated by the diode laser is coupled into a single-mode fiber for spatial filtering and the ease of manipulation. This beam is subsequently coupled out, rerouted and then focused down to individual crystals. We proceed by describing the alignment of the first crystal in the next section.

4.4.5.1 First crystal

The alignment of the first crystal (as well as the other crystals) follows the established techniques that are used also in other experiments. The crystal is mounted in an oven whose temperature is adjusted and kept stable by an external controller. The oven itself is attached to a mechanical stage that allows for precise position and tilt adjustments. After the pump beam is focused down to the first crystal, the SPDC process gives rise to pairs of photons comprising the output beam. The output beam is collimated by a lens and the collimated beam is sent to a polarizing beam splitter, representing the first component of the detection module. There is large enough space left between the crystal and the beam splitter to accommodate the rest of the final setup. For either beam leaving the beam splitter there is a series of mirrors, an SLM, and a fiber coupler which the beam consecutively traverses. This way, the beam leaving the beam splitter is reflected off the SLM screen and coupled into a single-mode fiber.

The setup consisting of the crystal and the detection module is first coarsely aligned using a bright laser beam as it is considerably easier than working with single photons right away. The pump beam propagating forward through the setup is made collinear with the backward propagating temporary laser beam. A laser pointer connected to the output end of the fiber leading from the detection module can be used as the source of the temporary beam. Let us emphasize that during the alignment procedure the SLM screens have to be turned on and the pattern displayed on them is the fundamental OAM mode $|0\rangle$ superposed with a diffraction grating. The first diffraction order is the one that is to be coupled into the fiber. The reason for that is that the zeroth order contains an admixture of light that is not affected by the hologram projected on the SLM, thus contributing to an undesirable background. This is not an issue for diffracted orders. After the two laser beams are sufficiently overlapping, the laser pointer is replaced with a single-photon detector and only the forward propagating laser is used that generates single photons and that is filtered away with a band-pass filter before entering the fiber coupler.

The signal generated by the detector is fed into a coincidence-logic unit, whose output is in turn sent to a computer. A suite of LabVIEW programs was developed for the purposes of the experiment, one of which programs allows for the visualization of the detected photon counts. If the detected photons cannot all be attributed to the background, which can be checked by comparing the counts with and without the laser beam being blocked, there are down-converted photons reaching the detector. If this is the case, the alignment continues while the rate of the detected photons is maximized. Otherwise, the above procedure with the backward propagating laser beam is applied again and the entire process is repeated

until the down-converted photons are detected. This is performed for both detectors separately. Once the down-conversion is detected by both detectors, one maximizes not the individual single-photon counts but rather the coincidence counts where pairs of photons are detected simultaneously (with the coincidence window set to 3.5 ns). As the last step, the holograms on the SLMs have to be centered with respect to the beam. The centering is automated and performed separately for the vertical and horizontal directions. For the former a dedicated program projects on the screen a superposition of OAM modes $|1\rangle + |-1\rangle$ where the position of the pattern is moved over the screen while the photon counts are monitored. The minimum of the photon rate corresponds to the right position of the pattern. Analogously, for the vertical centering the superposition $|1\rangle + i|-1\rangle$ is used instead. The beam waist at the plane of the SLM screen can be determined in a similar manner. At the end, the first module as well as the detection module of the setup, depicted as the orange and the cyan box in Fig. 4.5, respectively, is ready.

4.4.5.2 Second crystal

The alignment of the other two crystals proceeds analogously. The additional difficulty there is that the states of photons produced by each crystal have to be as indistinguishable as possible. The second crystal is placed in line after the first one and aligned in the aforementioned way. Moreover, the collimating lens of the first crystal together with the focusing lens of the second crystal form a 4f system. The role of this system is to image the center plane of the first crystal to the center plane of the second crystal to make sure the spatial profiles of the down-converted beams originating in the two crystals are identical. The pump beam is split by a beam splitter into two arms, the first of which pumps the first crystal and the second pumps the second crystal. The two arms form a Mach-Zehnder interferometer of approximate size 60×40 cm, which has to be stabilized. Before that, it is useful to keep it unstable since the random fluctuations allow us to see the interference in the photon counts, provided that the two crystals are coherent with each other. There is an additional catch related to the group velocity mismatch of photons generated in the first crystal — it has to be compensated for in order to see the interference. For this reason, the second crystal is rotated by 90 degrees with respect to the first crystal. This way it functions not only as an SPDC source but also as a compensating crystal. If the spatial alignment is precise enough and the pump laser has large enough coherence length, the interference should be visible. The temperature of the two crystals can then be adjusted to increase the visibility of the interference pattern.

So far only the fundamental OAM mode was used. At this point, we place an SPP into the pump beam for the second crystal and center it with respect to this beam. Due to the SPP the rate of the down-converted photons generated by the second crystal is lower than that for the first crystal. The relative pump power for the two crystals then has to be adjusted accordingly. Once this is done, the setup is almost ready to generate two-dimensional entangled states in OAM. The last obstacle to cope with is the interferometric stability. For this reason, the stabilization system is built that makes use of an extra red laser beam. The beam is injected into the interferometer via the beam splitter that is otherwise used to split the pump beam. The two arms of the red light are then recombined on the dichroic mirror that reflects the pump beam into the second crystal. The red light leaving the interferometer is then collected by a fast photodiode, whose signal is fed to the feedback loop. The details of the stabilization system are given in section 4.4.3.

4.4.5.3 Third crystal

The alignment procedure for the third crystal is analogous to that for the second crystal. Only instead of an SPP one additional mirror is used, as shown in Fig. 4.6, and the crystal has the same orientation as the first one to compensate for the group velocity mismatch of photons originating in the second crystal. The only tricky part now is the simultaneous alignment of all three crystals such that the mutual coherence is retained. Also, one has to make sure that the red beams used to stabilize the first and the second interferometer do not interfere with each other and do not arrive at the wrong photodiode. The two feedback loops are independent even though the two interferometers share one arm and so changes in this arm affect both loops. The final setup acts as a source of three-dimensional entanglement that has to be kept stable for around 20 minutes. This is the time necessary to perform the full quantum state tomography in three dimensions. The raw data produced by each measurement are the photon counts collected per a time unit for different holograms projected on the two SLMs. The post-processing of these raw data is the subject of the next section.

4.5 Post-processing

At this point, the description of the hardware part of the experiment comes to an end and its software part just begins. The output of the measurement setup are photon counts collected by the coincidence logic. The counts are post-processed by a computer program in several stages before the resulting density matrix is returned. These stages are as follows:

1. Count renormalization — It turns out that the efficiency of the measurement process differs when different measurement bases are chosen for the same subspace. Even though the total number of counts for a fixed subspace should be the same irrespective of the measurement basis, it is not so in the actual setup. To counteract the possible distortion of reconstructed states due to this imbalance we renormalize experimentally obtained coincidence counts by their sum for each measurement basis of a given subspace.
2. State tomography — The main goal of the measurement process is to reconstruct the density matrix of emitted photons. There are multiple ways how this can be achieved. One of them is the linear-inversion method, where the measurement results are plugged in the formulas that describe the same measurement from the theoretical perspective and the density matrix is calculated from these formulas. Even though this seems as the natural method for the density matrix reconstruction, it suffers from one major problem. Due to experimental inaccuracies and unavoidable external effects, the acquired experimental data do not lead to matrices that represent a valid quantum state. Most importantly, the resulting matrices turn out not to be always positive semi-definite. The following method is thus used to circumvent this problem.
3. Maximum likelihood estimation — This approach can be seen as fitting a physically meaningful matrix to the experimentally obtained data. In an iterative algorithm a matrix is found that is positive semi-definite, has the unit trace, and gives predictions that are as close to the experimental results as possible.

4. Fidelity calculation and entanglement quantification — Once the density matrix of a given quantum state is reconstructed, we can easily calculate all the figures of merit such as fidelity or the dimension of entanglement carried by the state.
5. Monte Carlo simulation to obtain errors — As any experiment features some imperfections that affect the values of its various characteristics, it is important to estimate the errors associated with our measurements. We do so by running a simple Monte Carlo simulation that mimics the Poissonian distribution of errors.

Last four of these post-processing stages are described in detail below in their respective sections.

4.5.1 State tomography

In this section, we briefly summarize the working principle of a tomography technique that is sometimes referred to as the linear inversion. As follows from the quantum theory, when a physical quantity represented by an observable O is measured on a system that is in state described by a density matrix ρ , the mean value $\langle O \rangle_\rho$ of this quantity is given by

$$\langle O \rangle_\rho = \text{Tr}(O\rho), \quad (4.12)$$

where Tr stands for the trace of a matrix. This expression is a generalization of the Born rule. In the quantum state tomography, the density matrix ρ is not known and one designs a suitable set of observables such that their mean values provide sufficient information for the reconstruction of ρ . Since we consider only finite-dimensional state spaces, the reconstruction can proceed as follows. Formally speaking, a density matrix is an element of a d^2 -dimensional space $\mathcal{B}(\mathcal{H})$ of operators that act on vectors from a d -dimensional space $\mathcal{H}$. Let $\{|m\rangle\}_{m=0}^{d-1}$ be the computational basis of this space and $P_m = |m\rangle\langle m|$ be projectors onto individual basis vectors. Moreover, let $X_{m,k} = |m\rangle\langle k| + |k\rangle\langle m|$ and $Y_{m,k} = i|m\rangle\langle k| - i|k\rangle\langle m|$ for $m \neq k$. Operators $\{P_m, X_{m,k}, Y_{m,k}\}_{m,k=0}^{d-1}$ form a set of observables such that

$$\langle P_m \rangle_\rho = \rho_{m,m}, \quad (4.13)$$

$$\langle X_{m,k} \rangle_\rho = 2 \text{Re}(\rho_{m,k}), \quad (4.14)$$

$$\langle Y_{m,k} \rangle_\rho = 2 \text{Im}(\rho_{m,k}), \quad (4.15)$$

where $\rho_{m,k} = \langle m|\rho|k\rangle$ is an entry of the density matrix ρ in the m -th row and k -th column. As is easy to see from the above relations, the m -th diagonal entry of the matrix is determined by the measurement of projector P_m and the off-diagonal entry $\rho_{m,k}$ can be calculated as

$$\rho_{m,k} = \frac{1}{2}(\langle X_{m,k} \rangle_\rho + i \langle Y_{m,k} \rangle_\rho). \quad (4.16)$$

We have shown explicitly the set of observables whose measurement gives enough information for the reconstruction of the unknown state ρ . In the case of a bipartite state ρ , as is the case in our experiment, one also has to take into account that the above computational-basis states $\{|m\rangle\}_m$ are non-local. Our detection system, however, can perform only local projective measurements. It can be shown [88] that all the aforementioned observables can be decomposed into a linear combination of terms, each

of which can be measured locally. In the bipartite case, each computational basis state is described by two indices such that $|m\rangle = |q, r\rangle$ for some $0 \leq q < d$ and $0 \leq r < d$. The real part of a matrix element $\rho_{q,r;s,t}$ of the bipartite density matrix is determined by $\text{Re}(\rho_{q,r;s,t}) = \text{Tr}(\rho X_{q,r;s,t})/2$ in analogy to (4.14), where $X_{q,r;s,t} = |q, r\rangle \langle s, t| + |s, t\rangle \langle q, r|$, and similarly for the imaginary part. A straightforward calculation gives us

$$X_{q,r;s,t} = |q\rangle \langle s| \otimes |r\rangle \langle t| + |s\rangle \langle q| \otimes |t\rangle \langle r| \quad (4.17)$$

$$= \frac{1}{2}(X_{q,s} \otimes X_{r,t} - Y_{q,s} \otimes Y_{r,t}) \quad (4.18)$$

$$= \frac{1}{2} \left((P_{q,s}^{(+1)} - P_{q,s}^{(-1)}) \otimes (P_{r,t}^{(+1)} - P_{r,t}^{(-1)}) \right. \\ \left. - (P_{q,s}^{(+i)} - P_{q,s}^{(-i)}) \otimes (P_{r,t}^{(+i)} - P_{r,t}^{(-i)}) \right), \quad (4.19)$$

where $X_{q,s}, \dots, Y_{r,t}$ act now only on a single particle. The expression (4.19) comes from the eigendecomposition of these single-particle operators, as $X_{q,s} = P_{q,s}^{(+1)} - P_{q,s}^{(-1)}$ and $Y_{q,s} = P_{q,s}^{(+i)} - P_{q,s}^{(-i)}$, where $P_{q,s}^{(\alpha)} = |q, s, (\alpha)\rangle \langle q, s, (\alpha)|$ with $|q, s, (\alpha)\rangle = (|q\rangle + \alpha|s\rangle)/\sqrt{2}$. From Eq. (4.19) it follows that the real part of an element $\rho_{q,r;s,t}$ of the bipartite density matrix can be represented as a linear combination of terms $\text{Tr}(\rho P_{q,s}^{(\alpha)} \otimes P_{r,t}^{(\beta)})$. Each of these terms corresponds to a local measurement, where the first (second) SLM implements projection $P_{q,s}^{(\alpha)}$ ($P_{r,t}^{(\beta)}$). An analogous conclusion can be drawn also for the imaginary part of $\rho_{q,r;s,t}$. No conceptual problems therefore arise with the above mentioned technique when entangled states are to be reconstructed.

Despite the clarity of the reconstruction process described in the previous paragraph, it faces some problems when applied to real experimental data. Specifically, the resulting matrix may not be positive semi-definite and consequently it may predict negative probabilities for some measurements. This trouble results from experimental imperfections and external effects of the environment. For this reason, in the experiment a more robust technique of the state reconstruction was used, which technique is described in the following section.

4.5.2 Maximum likelihood estimation

As mentioned in the preceding section, some methods of quantum state reconstruction may suffer from bottlenecks related to the inconsistency of experimental data. In what follows, we describe an algorithm that reconstructs matrices that not only approximate the experimental data but also represent valid density operators.

In this approach, the density matrix is not calculated from Eqs. (4.13)–(4.19). The main objects of interest are instead the elementary projective measurements $\text{Tr}(\rho P_{q,s}^{(\alpha)} \otimes P_{r,t}^{(\beta)})$. For convenience, let us group all the parameters into a single multiindex $\vec{i} = (\alpha, \beta, q, s, r, t)$ and set

$$p_{\vec{i}}(\rho) \equiv \text{Tr}(\rho P_{q,s}^{(\alpha)} \otimes P_{r,t}^{(\beta)}). \quad (4.20)$$

In general, the mean value of a projective measurement amounts to the probability with which that particular projector is measured. In a real experiment, such probabilities are not accessible and the only thing that is actually measured are numbers of photons that arrive at a detector per a given period of time. By normalizing these counts by their total sum one obtains relative frequencies of individual outcomes. Let us denote these frequencies by $f_{\vec{i}}$. On the other hand, the probabilities $p_{\vec{i}}$ depend on the density matrix

ρ and one can thus understand the expression $p_{\vec{i}}(\rho)$ as a functional that takes density matrices as inputs and returns non-negative numbers as outputs. Our goal then is to find such a density matrix ρ , for which the resulting non-negative numbers $p_{\vec{i}}$ match as close as possible the relative frequencies $f_{\vec{i}}$ while still preserving the normalization property $\sum_{\vec{i}} p_{\vec{i}} = 1$. It turns out [89] that the search for such a density matrix amounts to the maximization of the following functional

$$\mathcal{L}(\rho) = \prod_{\vec{i}} p_{\vec{i}}(\rho)^{n_{\vec{i}}} \quad (4.21)$$

over the set of all density matrices of the appropriate dimension, where $n_{\vec{i}}$ is the number of coincidence counts detected for the measurement characterized by $\vec{i}$, such that $n_{\vec{i}} = N f_{\vec{i}}$ with N being the total number of counts.

The value of the functional from Eq. (4.21) for a particular density matrix ρ can be understood as a likelihood that the experimentally measured counts $\{f_{\vec{i}}\}_{\vec{i}}$ were produced by the density matrix ρ . Since our aim is to maximize this likelihood, the reconstruction method just described is called *maximum likelihood estimation* (or just MLE) in literature. Instead of the brute-force maximization, we implemented the iterative algorithm from Ref. [90] that calculates directly the density matrix for which (4.21) is maximum. The resulting density matrices can be found in section 4.6 below.

4.5.3 Figures of merit

Having determined the density matrix corresponding to the generated state, one might calculate various characteristics of the state such as fidelity, purity or von Neumann entropy. The experimentally obtained states ρ are in general mixed and the fidelity F of these states with their pure-state theoretical counterparts $|\psi\rangle$ is thus a very important figure of merit of our setup. All the states we focused on in the experiment were pure states, for which the general Uhlmann formula for fidelity [91, 92] reduces to

$$F(\rho, |\psi\rangle) = \langle \psi | \rho | \psi \rangle. \quad (4.22)$$

The specific values of fidelities for different states are to be found in section 4.6 below.

The density matrices themselves would be of no great use for us had we no way to certify that they indeed correspond to high-dimensional entanglement as was our intention from the very beginning. In our experiment, we employed a qualitative technique of Refs. [93, 94], which can certify the dimension of entanglement. This technique is based on the fact that the set of entangled states has in some sense a nested structure, where the low-dimensional entangled states form a subset of the set of higher-dimensional entangled states. Moreover, there are fidelity bounds that are satisfied by all d -dimensional entangled states but that can be exceeded by $(d+1)$ -dimensional states. These bounds divide the interval between zero and one into subintervals. For the experimentally obtained state ρ , one first takes its theoretical counterpart $|\psi\rangle$ and determines the fidelity bounds F_d corresponding to $|\psi\rangle$ for all possible dimensions d . To assess the entanglement dimension of state ρ one just compares its fidelity $F(\rho, |\psi\rangle)$ with all F_d 's. The smallest d for which $F(\rho, |\psi\rangle) > F_d$ gives us the lower bound on the dimension of entanglement exhibited by ρ . For example, for the three-dimensional maximally entangled state of the form (4.3), the fidelity bound is equal to $2/3$. It means that the experimentally obtained counterpart ρ of

this state is entangled in at least three dimensions whenever its fidelity exceeds $2/3$. Even though this method allows us to determine the lower bound only, it is sufficient for our purposes of the high-dimensional entanglement certification.

4.5.4 Error estimation

As we have just seen in the previous section, the values of fidelities for a given quantum state can be compared with precalculated fidelity bounds to determine the dimension of the entanglement displayed by the state. Such a comparison is meaningful only when the difference between the fidelity and a given bound is larger than the error associated with the measurement. For this and other reasons, the effect of errors in our experiment has to be considered. Apart from possible systematic errors, there are also intrinsically random errors that follow a certain probability distribution. The only type of signal that is measured by our apparatus and that is subjected to noise is the number of counts taken as a function of time for one specific measurement setting. The main source of randomness in our experimental signal is the shot noise, which can be modeled as a random variable governed by Poissonian distribution.

To quantify the effect of the shot noise we perform a simple Monte Carlo simulation, in which the measured counts serve as mean values and to which a random noise is added that is distributed according to Poissonian statistics with the standard deviation set to the square root of the mean value. As the first step, a collection of a large number M of data sets $\{D_j\}_{j=0}^{M-1}$ is generated, where each data set differs from the experimentally acquired data set of coincidence counts only in the artificially added shot noise. As the second step, to each data set D_j the MLE method is applied to reconstruct the corresponding state ρ_j , which is further used for the calculation of fidelity F_j . As the last step, the resulting collection of fidelities is averaged to obtain the mean fidelity $\bar{F} = (1/M) \sum_{j=0}^{M-1} F_j$ and it is this mean fidelity that is used in the entanglement certification described above. Obviously, not only for the fidelity but also for any other figure of merit the error estimation can be performed this way.

4.6 Experimental results

In the previous section, the somewhat detailed explanation of the physical setup as well as the post-processing techniques is given. The last thing that remains to be shown is the actual experimental data. Our experimental setup consists of three non-linear crystals and is thus capable of generating not only three-dimensional but also two-dimensional entanglement. When the power for the third crystal is rerouted to the first two crystals, two-dimensional maximally entangled states can be created. We choose to create two of the well-known Bell states

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0,0\rangle + |2,2\rangle), \quad (4.23)$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|0,0\rangle - |2,2\rangle), \quad (4.24)$$

where the OAM of the two photons effectively lies in a two-dimensional space spanned by vectors $|0\rangle$ and $|2\rangle$. In Fig. 4.11, the reconstructed density matrices for these two states are depicted explicitly. In part (a) we demonstrate the ability of our setup (and therefore the

ability of the concept of entanglement by path identity itself) to generate entangled states. In part (b) we further show that we can tune the local phase of individual terms that make up the quantum superposition. The other two Bell states $|\Psi^\pm\rangle = (1/\sqrt{2})(|0, 2\rangle \pm |2, 0\rangle)$ have different number of OAM quanta in each term and one would have to augment our setup with additional spiral phase plates to allow for the generation of these states. Even though we have not done it, there is no conceptual obstacle that would prevent the setup from generating also such states. In our proof-of-principle setup, we were able to generate states (4.23) and (4.24) with fidelity $90.4 \pm 0.5\%$ and $89.1 \pm 0.5\%$, respectively. To certify the two-dimensional entanglement of these states, their fidelities have to exceed the bound of $1/2$, see section 4.5.3. This is indeed the case and we can conclude that our source produces with reasonable fidelity two-dimensional entanglement. The photon count rates generated by the setup were for these two-dimensional states on the level of around 1400 Hz.

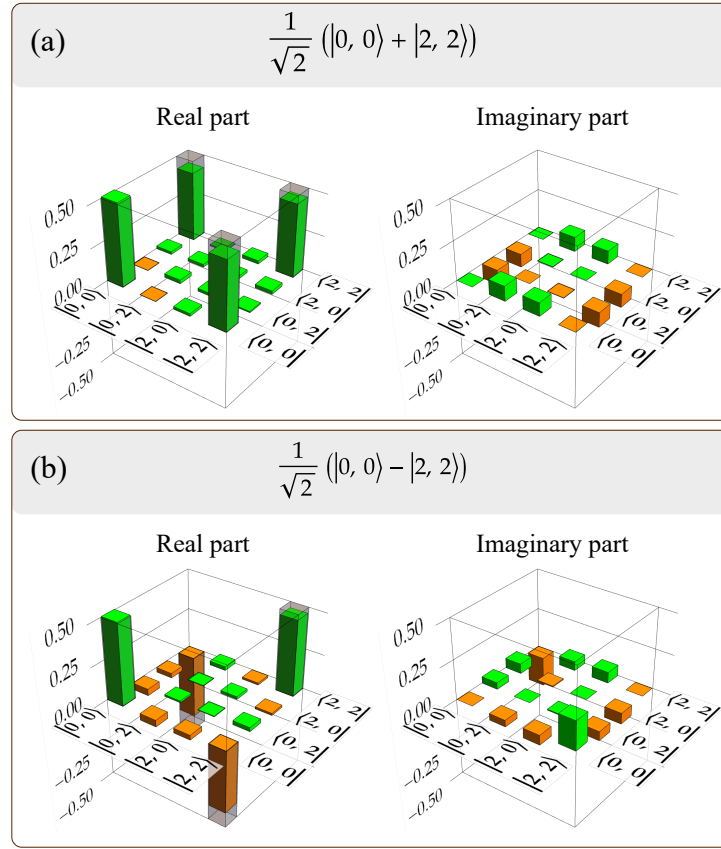


Figure 4.11: The experimentally obtained two-dimensional quantum states of two photons. Two Bell states are shown, which were created with fidelities (a) $90.4 \pm 0.5\%$, and (b) $89.1 \pm 0.5\%$, respectively. The green bars denote positive values, the orange bars the negative ones. Reprinted from Ref. [62].

The entanglement in two dimensions is well-studied and polarization-entangled pairs of photons generated via SPDC have been in use for decades. The higher-dimensional case is therefore more interesting. We have seen that the setup is indeed capable of producing entanglement and so we set the splitting ratios of the two beam splitters in the setup such

that all three crystals are pumped evenly. This allows us to generate bipartite states in three dimensions. Moreover, as explained above we can tune the local phase of individual probability amplitudes in the resulting quantum superpositions. To demonstrate this ability, we choose to produce the following three-dimensional entangled states of two photons:

$$|\psi_1\rangle = \frac{1}{\sqrt{3}}(|0,0\rangle + |2,2\rangle + |-2,-2\rangle), \quad (4.25)$$

$$|\psi_2\rangle = \frac{1}{\sqrt{3}}(|0,0\rangle + \omega |2,2\rangle + \omega^{-1} |-2,-2\rangle), \quad (4.26)$$

$$|\psi_3\rangle = \frac{1}{\sqrt{3}}(|0,0\rangle + \omega^{-1} |2,2\rangle + \omega |-2,-2\rangle), \quad (4.27)$$

$$|\psi_4\rangle = \frac{1}{\sqrt{3}}(|0,0\rangle - |2,2\rangle - |-2,-2\rangle), \quad (4.28)$$

where $\omega = \exp(2\pi i/3)$. All these states are maximally entangled in three dimensions and differ only in local phases. Moreover, states $|\psi_1\rangle$, $|\psi_2\rangle$, and $|\psi_3\rangle$ form an orthonormal set. The reconstructed density matrices for states $|\psi_1\rangle$ and $|\psi_4\rangle$ are explicitly shown in Fig. 4.12(a) and (b), respectively. The fidelity bound for the three-dimensional entanglement is for states (4.25)–(4.28) equal to $2/3$. The actual fidelities for all these states are shown in Tab. 4.1 and all of them exceed this bound by at least 26 standard deviations. We thus produce fairly good three-dimensional entanglement.

As noted above, we have the full control not only over the local phases but also over the magnitudes. By changing the splitting ratios of the beam splitters in our setup, we can redistribute the pumping power over the three crystals arbitrarily. To demonstrate also this feature of the concept of entanglement by path identity, we generated state

$$|\psi_5\rangle = \frac{1}{\sqrt{22}}(2|0,0\rangle + 3|2,2\rangle + 3|-2,-2\rangle), \quad (4.29)$$

which is very similar to the state that maximally violates a three-dimensional generalization of Bell inequalities [6, 95]. This state has the term $|0,0\rangle$ suppressed with respect to the other two and its density matrix is shown in Fig. 4.12(c). Even though state (4.29) cannot be maximally entangled, it still exhibits genuine three-dimensional entanglement. The corresponding fidelity bound is for this state equal to $9/11 \approx 0.82$ and we generated it with fidelity 0.848 ± 0.008 , cf. Tab. 4.1, which exceeds the bound by more than 3 standard deviations. The three-dimensional states were generated by our setup with the rates of around 1200 Hz.

Let us conclude this section by a brief discussion of possible experimental imperfections that led to the imperfect fidelities, as reported in Tab. 4.1. One can identify two main factors. The first one is the imperfect coherence of the SPDC processes, which amounts to approximately 5 % decrease in the fidelities for all the reported states, both two- and three-dimensional. Coherence is limited by the minor distinguishability of the SPDC sources caused by small differences in the spectral and polarization degrees of freedom of the down-converted photons. The second factor is state-dependent and arises from the imprecise adjustment of local phases, slight misalignment and the presence of higher-order OAM modes. The two factors together can explain the range 85 – 90% of fidelities for different three-dimensional states. None of these imperfections is however fundamental and can be overcome with a better experimental setup.

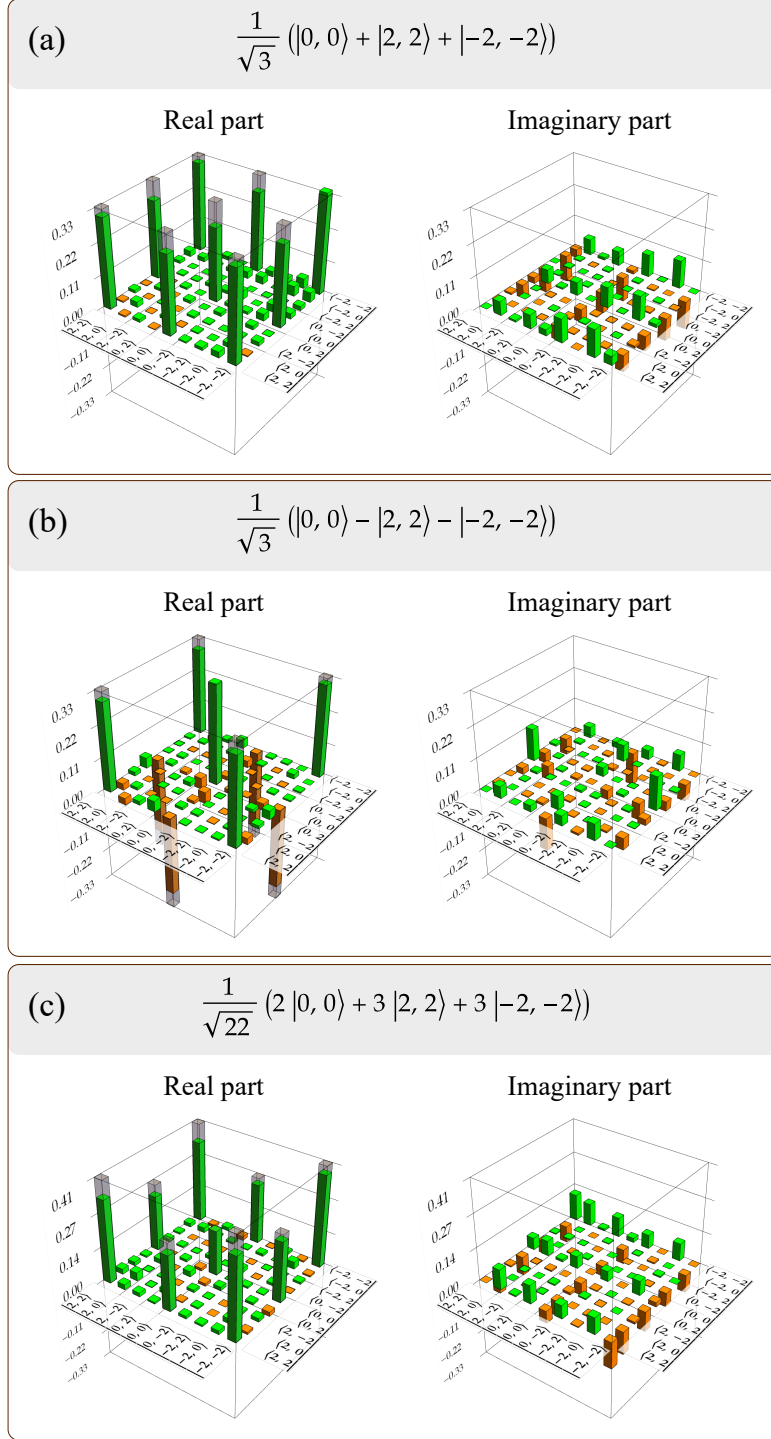
State	Fidelity
$ \psi_1\rangle = \frac{1}{\sqrt{3}}(0, 0\rangle + 2, 2\rangle + -2, -2\rangle)$	0.870 ± 0.005
$ \psi_2\rangle = \frac{1}{\sqrt{3}}(0, 0\rangle + \omega 2, 2\rangle + \omega^{-1} -2, -2\rangle)$	0.852 ± 0.007
$ \psi_3\rangle = \frac{1}{\sqrt{3}}(0, 0\rangle + \omega^{-1} 2, 2\rangle + \omega -2, -2\rangle)$	0.903 ± 0.006
$ \psi_4\rangle = \frac{1}{\sqrt{3}}(0, 0\rangle - 2, 2\rangle - -2, -2\rangle)$	0.890 ± 0.004
$ \psi_5\rangle = \frac{1}{\sqrt{22}}(2 0, 0\rangle + 3 2, 2\rangle + 3 -2, -2\rangle)$	0.848 ± 0.008

Table 4.1: The fidelities between three-dimensional entangled states $|\psi_i\rangle$ and their experimental realizations ρ_i . Reprinted from Ref. [62].

4.7 Discussion

As has been already touched upon above, the experimental setup can be augmented in a straightforward way to generate more than three-dimensional entanglement. It suffices to add an arbitrary number of additional modules, each of which consists of a non-linear crystal and a spiral phase plate to increase incrementally the entanglement dimension. Since the interferometric solution, used in the experiment reported above, requires active stabilization, it is advisable to implement the experimental setup in a way that is not based on interferometers and follows more closely the conceptual scheme. One possible avenue of possible improvements in this regard can be the use of wavelength-dependent phase shifters and q-plates [96] designed in such a way that the OAM is added only to the down-converted beam, while leaving the pump beam unaffected. The entire setup can also be scaled down and isolated from the environment. In the extreme case, the setup can in principle be implemented on a photonic chip. The integrated optics and photonic micro-chip manufacturing have been very successful technologies for path- and polarization-encoded qudits. Our source of OAM entanglement would also immensely benefit from the integrated implementation. On the other hand though, such a technology that can guide and manipulate OAM modes on a chip is still in its infancy [97].

As our experimental findings suggest, the very concept of entanglement by path identity is a promising and highly customizable method that allows for efficient generation of a vast class of both entangled and separable states. As such, it may find applications in the expanding field of quantum technologies such as entanglement-based quantum key distribution, quantum teleportation, dense coding, as well as in yet immature fields such as photonic quantum computation.



5 Conclusion

In this thesis, we consider a property of photons called orbital angular momentum (OAM) and discuss two aspects of its usability in quantum information processing — the manipulation of OAM of single photons and the generation of entangled multi-partite OAM states. To get everyone on the same page, in chapter 2 we briefly introduce the orbital angular momentum of photons and list some of its basic properties. After that, either of the two aspects is studied in detail in a dedicated chapter.

In chapter 3, we discuss how an arbitrary single-photon unitary operation acting on OAM can be implemented. It turns out that a universal scheme exists, referred to as the path-assisted approach, where only conventional optical elements are sufficient for the construction of any unitary and no specially designed elements have to be involved. We present several notable examples of operations implemented with the universal scheme, namely the high-dimensional Pauli operators. Complementary to the path-assisted approach is the scheme of implementing single-photon controlled gates where the control qudit is represented by OAM. The implementation of such gates is presented that differs from the path-assisted approach, yet relies also on conventional optical elements. Furthermore, we generalize the idea of the universal scheme and present the parallelized scheme that allows one to apply the same unitary to multiple systems efficiently and simultaneously. We exemplify this scheme by constructing the parallelized versions of high-dimensional Pauli operators. At the end of the chapter, we present an implementation of one of the most prominent unitary operations in modern mathematics, physics, as well as engineering — the Fourier transform. The high-dimensional OAM Fourier transform can be implemented with only conventional optical elements in a way that combines ideas of both the universal and the parallelized scheme. As a result, an optical setup is obtained whose scaling properties are more favorable than any of the previous implementations based on the path-encoded qudits. All of the aforementioned theoretical findings are, despite them being noticeably more efficient than previous approaches, still rather challenging when it comes to their implementation in a real experiment. On the other hand, they can serve as a stepping stone for further improvements that can seek real physical implementation. Multiple techniques have been developed that enable the universal manipulation of OAM states of photons and it will be interesting to see the range of applicability of each of these techniques.

The second aspect, the entanglement generation, is tackled in chapter 4, where a proof-of-principle experiment is reported that demonstrates the feasibility of a general scheme of entanglement generation termed entanglement by path identity. This concept relies on the indistinguishability of photon-generation processes. Instead of utilizing intrinsic physical laws such as the conservation of angular momentum, the entangled state is built incrementally in a modular source that consists of multiple non-linear crystals. Thanks to the geometry of the experimental setup, the indistinguishability of sources allows for well-controlled engineering of entangled quantum states in an arbitrary dimension. We experimentally demonstrate the generation of three-dimensional maximally as well non-

maximally entangled states of two photons. The relative indifference of resulting fidelities to whether the two- or three-dimensional states are created suggests that our approach is feasible also in even higher dimensions. We identify several issues that can be addressed in our approach to increase the fidelity of generated states as well as the stability of the experimental setup. The concept of entanglement by path identity is a framework that can be employed for the production of entangled states in many different degrees of freedom of various physical systems. It represents a promising platform that can find applications in the ever-increasing field of quantum computation and communication.

Quantum technologies are finally some one hundred years after the foundation of quantum theory coming of age and seek real-life deployment. It will be surely very exciting to see what new areas of research, both basic and applied, will benefit from this development. Quantum cryptographic devices are already commercially available, but these are rather simple applications of the quantum phenomena that have much more to offer. The large-scale harnessing of the full potential of features such as quantum superposition or quantum entanglement for massive data processing can bring to life yet unexpected insights into not only the processed data but also the quantum algorithms and quantum phenomena themselves. The manipulation of a single photon's wave function as well as customized entanglement generation are indispensable tools in reaching this goal.

6 Author's publications related to the presented work

Manipulation of OAM of single photons

The general framework of implementing arbitrary OAM unitaries using the path-assisted approach is described in the following article, where also the parallelization is discussed:

- Jaroslav Kysela. “Arbitrary unitaries in orbital angular momentum of single photons”. *EPJ Quantum Technology* 9(22), (2022). [doi:10.1140/epjqt/s40507-022-00142-1](https://doi.org/10.1140/epjqt/s40507-022-00142-1).

The implementation of the high-dimensional Pauli X gate in OAM is introduced in the following article:

- Xiaoqin Gao, Mario Krenn, Jaroslav Kysela, and Anton Zeilinger. “Arbitrary d -dimensional Pauli X gates of a flying qudit”. *Physical Review A* 99 (2019), 023825. [doi:10.1103/PhysRevA.99.023825](https://doi.org/10.1103/PhysRevA.99.023825).

The implementation of the high-dimensional Fourier transform of OAM of single photons is introduced and analyzed in the following two articles:

- Jaroslav Kysela. “High-dimensional quantum Fourier transform of twisted light”. *Physical Review A* 104 (2021), 012413. [doi:10.1103/PhysRevA.104.012413](https://doi.org/10.1103/PhysRevA.104.012413).
- Jaroslav Kysela, Xiaoqin Gao, and Borivoje Dakić. “Fourier transform of the orbital angular momentum of a single photon”. *Physical Review Applied* 14 (2020), 034036. [doi:10.1103/PhysRevApplied.14.034036](https://doi.org/10.1103/PhysRevApplied.14.034036).

Entanglement generation

The proof-of-principle experiment on the generation of entangled photons in OAM via path identity is reported in the following article:

- Jaroslav Kysela, Manuel Erhard, Armin Hochrainer, Mario Krenn, and Anton Zeilinger. “Path identity as a source of high-dimensional entanglement”. *Proceedings of the National Academy of Sciences* 117(42) (2020), 26118–26122. [doi:10.1073/pnas.2011405117](https://doi.org/10.1073/pnas.2011405117).

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