MASTERARBEIT / MASTER'S THESIS

Titel der Masterarbeit / Title of the Master's Thesis

Aspects of the Robinson-Schensted-Knuth Correspondence in the Representation Theory of non-archimedean General Linear Groups

verfasst von / submitted by

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angestrebter akademischer Grad / in partial fulfilment of the requirements for the degree of

Master of Science (MSc)

Wien, 2022 / Vienna, 2022

Studienkennzahl lt. Studienblatt / degree programme code as it appears on the student record sheet:	UA 066 821
Studienrichtung lt. Studienblatt / degree programme as it appears on the student record sheet:	Masterstudium Mathematik
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Abstract

We give an introduction to so called RSK-standard modules that were introduced by Gurevich and Lapid in [GL21] and provide a link between the Robinson-Schensted-Knuth (RSK) correspondence in the representation theory of general linear groups over local non-archimedean fields F. To be as self-contained as possible we develop the classification of irreducible representations of $\mathrm{GL}_n(F)$ following Bernstein-Zelevinsky [BZ77], [Ze80] and give a comparison between the RSK algorithm developed in [GL21] with more standard descriptions of the RSK-correspondence. Finally, we report some open problems from [GL21].

Zusammenfassung

In dieser Arbeit geben wir eine Einführung in sogenannte RSK-standard Moduln welche von Gurevich und Lapid in [GL21] eingeführt wurden und eine Beziehung der Robinson-Schensted-Knuth (RSK) Korrespondenz mit der Darstellungstheorie der allgemeinen linearen Gruppen über lokalen nicht-archimedischen Körpern F darstellen. Um so selbstständig wie möglich zu sein, entwickeln wir nach Bernstein-Zelevinsky [BZ77], [Ze80] die Klassifikation von irreduziblen Darstellungen von $\mathrm{GL}_n(F)$ und geben einen Vergleich zwischen dem RSK Algorithmus, welcher in [GL21] entwickelt wurde, und Standardbeschreibungen der RSK Korrespondenz. Letztlich geben wir auch einige offene Fragen aus [GL21] wieder.

Acknowledgements

Firstly and most importantly I want to thank my advisor for this thesis, Alberto Minguez. He introduced me to the subject of p-adic representation theory and in particular to Zelevinsky's classification and gave me interesting recent research and conjectures to work on. Throughout the ups and downs of this experience, he has patiently supported me and was a great help in the process of writing this thesis, both mathematically and personally.

I also want to thank Anton Mellit for many interesting suggestions he made during my work on this subject and for his financial support through his ERC grant.

My thanks go out to Ilse Fischer and her research group for discussing the combinatorial aspects of the RSK construction that we present here and who were very valuable for my understanding of the link to the classical RSK correspondence.

Last but not least, I want to thank all my friends, family and colleagues, who have supported me mathematically and socially throughout this work. Special thanks go to Florian Fürnsinn, Florian Lang, Stephan Schneider and Fabian Schuh for their frequent help with often very silly mathematical questions and to Emma Schwentner for her support with proofreading this thesis and much more.

Contents

Introduction

In this work, we give an introduction to a connection between the representation theory of p-adic groups and the Robinson-Schensted-Knuth (RSK) correspondence that was recently developed by Gurevich and Lapid in [GL21]. Our focus will lie mostly in providing a thorough exposition of the backgrounds needed from the representation theory of $\mathrm{GL}_n(F)$, for a local non-archimedean field F, and from combinatorics.

Overview

Chapter 1 gives a quick overview of the necessary concepts from representation theory. We assume the reader is familiar with the basics of the representation theory of groups and has some familiarity with p-adic groups. We introduce the concept of smooth representation, which will be the only kind of representations that we deal with in this work. At the center of this chapter lie the operations of parabolic induction and its adjoint, parabolic restriction. These operations allow us to combine representations $\pi_1, \ldots, \pi_k$ of several $\mathrm{GL}_{n_1}(F), \ldots, \mathrm{GL}_{n_k}(F)$ into a representation $\pi_1 \times \cdots \times \pi_k$ of $\mathrm{GL}_{n_1+\cdots+n_k}(F)$, called the product of the π_i, which will be the fundamental tool in studying the representation theory of all $\mathrm{GL}_n(F)$ at the same time.
A special role in this theory is played by the so-called *cuspidal* representations. These are representations of $\mathrm{GL}_n(F)$ that can not be constructed using parabolic induction. Hence, when trying to study the representations of $\mathrm{GL}_n(F)$ using parabolic induction, these representations naturally serve as a kind of building block for this study.
Finally we introduce a fundamental computational result, the Geometric Lemma of Bernstein-Zelevinsky that enables the study of restriction-induction operations.

In chapter 2 we turn towards classifying the irreducible representations of the $\mathrm{GL}_n(F)$ using parabolic induction and cuspidal representations. This was developed by Bernstein and Zelevinsky in [BZ77] and [Ze80] and we largely follow their approach. Essentially, they showed that the set Irr of isomorphism classes of all smooth irreducible representations of all $\mathrm{GL}_n(F)$ can be described combinatorially by so-called multisegments. A segment is a pair (Δ, ρ) of a subset $\Delta \subset \mathbb{Z}$ of the form $\Delta = \{a, a+1, \ldots, b\}$, $a \leq b$ and an irreducible cuspidal representation ρ. To each such segment we can associate, using parabolic induction, an irreducible representation $Z(\Delta, \rho)$. A multisegment $\mathfrak{m}$ is a (finite) multiset of segments, to which we can also associate an irreducible representation $Z(\mathfrak{m})$. In this way, we gain a complete classification of Irr.
After this, we discuss some questions arising from this classification, particularly how the irreducible subquotients of a product $Z(\Delta_1) \times \cdots \times Z(\Delta_n)$ can be described through multisegments. Then we turn to investigating a special class of multisegments, so called ladders.

The following chapter 3 is devoted entirely to the Robinson-Schensted-Knuth correspondence. We begin by abstracting the concept multisegments a small step by dropping the cuspidal representations, and thus only working with mutlisets of intervals in $\mathbb{Z}$. In this setting, we report the version of the RSK algorithm from [GL21]. This algorithm takes in a multisegment $\mathfrak{m}$ and produces a sequence $(\mathfrak{l}_1, \dots, \mathfrak{l}_k)$ of ladder multisegments with some properties. We then link this construction to the well-known RSK correspondence between matrices and pairs of tableaux. For this we first recall the standard insertion and row-bumping algorithm of Schensted. Following this, we reformulate this into the so-called matrix-ball construction, an extension of Viennot's geometric realization of the Robinson-Schensted correspondence. Looking at this construction, a clear analogy to the algorithm from [GL21] arises and we explicate this relationship precisely.

The final two chapters focus on the RSK-standard modules introduced in [GL21]. From the RSK construction we can produce a sequence of ladder multisegments $(\mathfrak{l}_1, \dots, \mathfrak{l}_k)$ and to each of these ladders we can associate an irreducible representation $Z(\mathfrak{l}_i)$. Using parabolic induction, we can thus associate to each multisegment $\mathfrak{m}$ a product $\Lambda(\mathfrak{m}) = Z(\mathfrak{l}_k) \times \cdots \times Z(\mathfrak{l}_1)$ of ladders. These $\Lambda(\mathfrak{m})$ are called RSK-standard modules. A remarkable property of these modules is that they retain much of the representation theoretic information of the original multisegment $\mathfrak{m}$: indeed, in [Gu21] it is shown that for all $\mathfrak{m}$, the RSK-standard module $\Lambda(\mathfrak{m})$ has a unique irreducible submodule which equals $Z(\mathfrak{m})$. Expanding our concepts of multisegments to allow also for 'dummy segments' of the form $[a+1, a]$, we introduce also enhanced RSK-standard modules. These allow us to produce many familiar representations as special cases of RSK-standard modules. In particular, the Zelevinsky and Langlands-standard modules can both be described in this way.
We end with some conjectures of [GL21] that remain open and that were the main motivation for writing this thesis.

Notations

Some notational conventions should be fixed before we begin. We will always work over a local non-archimedean field F that we fix throughout this article. Everything we will do does not depend on the characteristic of F. In fact we could work in even greater generality, by considering some division algebra D over F and the groups $\mathrm{GL}_n(D)$ (see [LM16] for more details on this approach). We will denoted by $|.|_F$ the absolute value of F and by $|.|$ the character $\mathrm{GL}_n(F) \to \mathbb{C}^\times$, $g \mapsto |\det(g)|_F$.

Definition 0.1. A multiset A (on X) is a function $A : X \to \mathbb{N}$ from some set X to the non-negative integers $\mathbb{N} = \{0, 1, \dots\}$ such that for all but finitely many $x \in X$ we have $A(x) = 0$.
We identify the set of multisets on a set X with the free monoid generated by X. In other words, if $A : X \to \mathbb{N}$ is a multiset such that $A(x_1) = n_1, \dots, A(x_k) = n_k$ and all other $A(x) = 0$, then we usually write A as a formal sum $A = n_1 x_1 + \cdots + n_k x_k = \underbrace{x_1 + \cdots + x_k}_{n_k \text{ times}} + \cdots + \underbrace{x_k + \cdots + x_k}_{n_k \text{ times}}$.

We will sometimes abuse the set-theoretic notation for multisets to write things like '$\{x_1, \dots, x_n\}$ considered as a multiset' and really mean the multiset $x_1 + \cdots + x_n$. This

should not cause any confusion.
Given a positive integer n a *partition* of n is a tuple $\lambda = (\lambda_1, \dots, \lambda_k)$ of non-negative integers with $\lambda_1 + \cdots + \lambda_k = n$. A partition $\mu = (\mu_1, \dots, \mu_k)$ of n is called a *subpartition* of λ, if there are indices $1 = i_1 < i_2 < \cdots < i_{k+1} = n + 1$ such that for all $j = 1, \dots, k$ we have

$$\lambda_j = \sum_{i=i_j}^{i_{j+1}-1} \mu_i.$$

In other words, blocks of μ partition the λ_i. If μ is a subpartition λ we denote this by $\mu \leq \lambda$ and call μ a *proper* subpartition, if $\mu < \lambda$. We call λ a *proper* partition of n, if $\lambda < (n)$.

1 Background from Representation Theory

In this section we want to introduce some basic properties of the representation theory of p-adic groups. We will usually state all definitions and results only for the groups $G_n := \mathrm{GL}_n(F)$. Most results hold in much larger generality however, for this more general setting see for example [BH06].

Definition 1.1. Let $n \in \mathbb{N}$ and V be a complex vector space, possibly of infinite dimension. A representation $\pi : \mathrm{GL}_n(F) \to \mathrm{GL}(V)$ is called *smooth*, if for all $v \in V$ there is a compact, open subgroup $K \leq \mathrm{GL}_n(F)$ that stabilizes v: $\pi(k)v = v$ for all $k \in K$.

From this point on we will consistently use the term *representation* to mean complex *smooth* representation unless explicitly stated otherwise. The category of these smooth representations of G_n will be denoted $\mathrm{Rep}(G_n)$, where a morphism f between representations $(\pi_1, V_1), (\pi_2, V_2)$ of G_n is a linear map $f \colon V_1 \to V_2$ such that $f \circ \pi_1(g) = \pi_2(g) \circ f$ for every $g \in G_n$. The set of these will be denoted by $\mathrm{Hom}_{G_n}(\pi_1, \pi_2)$. We will denote by $\mathrm{R}(G_n)$ the category of smooth representations of finite length of G_n and by $\mathrm{Irr}(G_n)$ the set of equivalence classes of smooth irreducible representations.
The natural concepts of sub- and quotient representations carry over verbatim to the smooth case: if (π, V) is a smooth representation of G_n and U is a G_n-stable subspace of V then the action of G_n restricted to U defines a *smooth* representation of G_n. Likewise, given such a stable subspace U also the quotient V/U with the obvious G_n-action defines a *smooth* representation of G_n.

Definition 1.2. Let $\pi : G_n \to \mathrm{GL}(V)$ be a smooth, complex representation of finite length. Then there is a filtration $0 = \pi_0 \subsetneq \pi_1 \subsetneq \cdots \subsetneq \pi_n = \pi$ such that all quotients π_i/π_{i-1} for $i = 1, \ldots, n$ are irreducible. The multiset $\mathrm{JH}(\pi)$ of these quotients π_i/π_{i-1} is uniquely determined by π and the representations π_i/π_{i-1} are called Jordan-Hölder factors (or composition factors or irreducible subquotients) of π.

Similarly, concepts such as unitary representations, contragredient representations carry over to smooth representations in exactly the same way.
While smooth representations of G_n will usually be infinite dimensional, many of them possess a useful finiteness condition.

Definition 1.3. A representation $\rho \colon G_n \to \mathrm{GL}(V)$ is called *admissible* if for all open compact subgroups $K \leq G_n$ the set of K-fixed vectors V^K is finite dimensional.

The main theorem on admissible representations is that they include all the irreducible representations:

Proposition 1.4 (cf. [BZ77] Theorem 3.25)**.** *If $\rho \colon G_n \to \mathrm{GL}(V)$ is an irreducible representation of G_n then it is admissible.*

As a corollary of this, we see that all representations of finite length are admissible. Thus in this article we will exclusively deal with admissible representations. We let Irr_n denote the set of irreducible representations of G_n and $\mathrm{Irr} := \bigsqcup_{n \geq 0} \mathrm{Irr}_n$.
Since representations of G_n tend not to be semisimple, we also introduce the concepts of socle and cosocle.

Definition 1.5. Let $\pi \in \mathrm{Rep}(G_n)$. Then we define the *socle* $\mathrm{soc}(\pi)$ as the largest semisimple subrepresentation of π. Equivalently, $\mathrm{soc}(\pi)$ is simply the sum of all irreducible subrepresentations of π. Dually, we define the *cosocle* $\cos(\pi)$ as the largest semisimple quotient of π.
We call π *socle* (resp. *cosocle*) *irreducible*, abbreviated SI (resp. CI), if $\mathrm{soc}(\pi)$ (resp. $\cos(\pi)$) is irreducible and appears with multiplicity 1 in $\mathrm{JH}(\pi)$, the multiset of composition factors of π.

Definition 1.6. Let G be a locally compact group (such as $\mathrm{GL}_n(F)$) with a left Haar measure μ. Then we have a continuous homomorphism $\delta_G \colon G \to \mathbb{R}^+$ such that for all $g \in G$ and measurable $A \subset G$ we have:

$$\mu(Ag) = \delta_G(g^{-1})\mu(A).$$

We call δ_G the *modulus* or *modular function* of G. It is independent of the choice of Haar measure.

1.1 Induction and Jacquet Modules

The most important technical tool for our study of the representation theory of $\mathrm{GL}_n(F)$ will be *parabolic induction* and its adjoint functor *parabolic restriction* (also known as Jacquet functors). The classification of Zelevinsky relies primarily on building up *all* irreducible representations from just a few building blocks using parabolic induction.

Given a partition $\alpha = (n_1, \dots, n_k)$ of a positive integer n we denote by G_α the subgroup of G_n given by block-diagonal matrices with blocks of size $n_1, \dots, n_k$ and by P_α the block upper triangular matrices of this shape. Finally, let U_α denote the upper unitriangular matrices of the shape, i.e. the subgroup of P_α consisting of matrices with the identity along the diagonal.

$$P_\alpha = \left\{ \begin{pmatrix} A_1 & \star & \dots & \star \\ 0 & A_2 & \dots & \star \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & A_n \end{pmatrix} : A_i \in \mathrm{GL}_{n_i} \right\}, G_\alpha = \left\{ \begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & A_n \end{pmatrix} : A_i \in \mathrm{GL}_{n_i} \right\},$$

$$U_\alpha = \left\{ \begin{pmatrix} I_{n_1} & \star & \dots & \star \\ 0 & I_{n_2} & \dots & \star \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & I_{n_k} \end{pmatrix} \right\}, \text{ where } I_n \text{ is the identity } n \times n \text{ matrix.}$$

The P_α are parabolic subgroups of G_n with Levi factors G_α and unipotent radicals U_α. We call a partition β a subpartition of α, if G_β is a subgroup of G_α. This is equivalent to saying that all blocks of the partition β are contained in the blocks of the partition α.

Definition 1.7. Let $\alpha = (n_1, \dots, n_k)$ be a partition of n and let (π, V) be a representation of G_α. Then let $X(V)$ denote the set of functions $f \colon G_n \to V$ such that:

1. $f(umg) = \delta_{P_\alpha}^{1/2}(m)\pi(m)f(g)$, for all $u \in U_\alpha$, $m \in G_\alpha$, $g \in G$;
2. there is an open, compact subgroup $K \leq G_n$ (depending on f) such that $f(gk) = f(g)$ for all $g \in G_n$, $k \in K$.

This is a complex vector space and we define an action Π of G_n on it in the natural way:

$$(\Pi(g)f)(h) := f(hg), \quad g, h \in G_n.$$

This defines a smooth representation $(\Pi, X(V))$ of G_n which we will henceforth denote by $i_\alpha(\pi)$, the (normalized) parabolically induced representation of π.

In fact i_α extends naturally to a functor

$$i_\alpha \colon \mathrm{Rep}(G_\alpha) \to \mathrm{Rep}(G_n),$$

acting on objects in the way just described. Given a G_α-morphism $f \colon (\pi_1, V_1) \to (\pi_2, V_2)$ between representations of G_α we get a G_n-morphism

$$\begin{aligned} i_\alpha(f) \colon i_\alpha(\pi_1) &\to i_\alpha(\pi_2) \\ \varphi &\mapsto f \circ \varphi, \end{aligned}$$

and this defines the functor i_α.
Finally we can define more generally for all subpartitions β of α an induction functor

$$i_{\alpha,\beta} \colon \mathrm{Rep}(G_\beta) \to \mathrm{Rep}(G_\alpha)$$

by using the fact that the blocks of α are by definition partitioned by the blocks of β and using

$$\mathrm{Rep}(G_\alpha) = \mathrm{Rep}(G_{n_1}) \otimes \cdots \otimes \mathrm{Rep}(G_{n_k}). \tag{1}$$

This last identification of categories is crucial for the study of the representation theory of the groups $\mathrm{GL}_n(F)$. Specifically, given a partition $\alpha = (n_1, \dots, n_k)$ of the positive integer n and representations $\pi_i \in \mathrm{Rep}(G_{n_i})$ for $i = 1, \dots, k$ we can combine these into a representation $\pi_1 \otimes \cdots \otimes \pi_k$ of G_α following (1). Applying the induction functor we have just defined, one obtains a representation $\pi_1 \times \cdots \times \pi_k := i_\alpha(\pi_1 \otimes \cdots \otimes \pi_k)$ of G_n,

which we will simply refer to as the product of the π_i.
We have thus defined, for a partition $\alpha = (n_1, \dots, n_k)$ of n, a functor

$$\begin{aligned} \mathrm{Rep}(G_{n_1}) \times \cdots \times \mathrm{Rep}(G_{n_k}) &\to \mathrm{Rep}(G_n) \\ (\pi_1, \dots, \pi_k) &\mapsto \pi_1 \times \cdots \times \pi_k, \end{aligned}$$

that allows us to build up representations of G_n from representations of smaller G_m groups. The classification of irreducible representations we will discuss in this section tells us that with this operation we can in fact get essentially all irreducible representations of G_n.

To study the structure of such products we also need the left adjoint of the induction functor i_α.

Definition 1.8. Let $\pi : G_n \to \mathrm{GL}(V)$ be a representation of G_n and $\alpha = (n_1, \dots, n_k)$ a partition of n. Then we let $V(U_\alpha)$ be the subspace of V spanned by vectors of the form $\pi(u)v - v$ for all $u \in U_\alpha$, $v \in V$ and define $r_\alpha(V) := V/V(U_\alpha)$. We can define a representation $r_\alpha(\pi)$ of G_α on this space in a natural way by setting:

$$(r_\alpha(\pi)(x))(v + V(U_\alpha)) := \delta_{P_\alpha}^{-1/2}(x)\pi(x)v + V(U_\alpha).$$

One checks that this indeed defines a smooth representation of G_α which we call the *parabolic restriction* or the *Jacquet module* of π (w.r.t. α). In the natural way we get a functor

$$r_\alpha : \mathrm{Rep}(G_n) \to \mathrm{Rep}(G_\alpha).$$

Similar to the induction functor, we can also apply this construction to individual blocks to get functors

$$r_{\beta,\alpha} : \mathrm{Rep}(G_\alpha) \to \mathrm{Rep}(G_\beta),$$

for a subpartition β of α.

We collect the basic properties of these functors in the following proposition.

Proposition 1.9. *Let n be a positive integer and β, α be partitions of n, where β is a subpartition of α.*

1. *The functors $i_{\alpha,\beta}, r_{\beta,\alpha}$ are exact.*
2. *The functors $i_{\alpha,\beta}, r_{\beta,\alpha}$ preserve the properties of finite length and admissibility.*
3. *The functor $r_{\beta,\alpha}$ is left adjoint to $i_{\alpha,\beta}$. In other words for representations $\pi \in \mathrm{Rep}(G_\alpha), \sigma \in \mathrm{Rep}(G_\beta)$ we have a natural isomorphism*

$$\mathrm{Hom}_{G_\alpha}(\pi, i_{\alpha,\beta}(\sigma)) \cong \mathrm{Hom}_{G_\beta}(r_{\beta,\alpha}(\pi), \sigma).$$

4. *The functors $i_{\alpha,\beta}, r_{\beta,\alpha}$ commute with tensor products in the following sense. Let $\alpha = (n_1, \dots, n_k)$ and decompose $\beta = (\beta_1, \dots, \beta_k)$ where each β_i is a partition of n_i. Then for $\pi_i \in \mathrm{Rep}(G_{n_i})$ and $\sigma_i \in \mathrm{Rep}(G_{\beta_i})$ we have*

$$i_{\alpha,\beta}(\sigma_1 \otimes \cdots \otimes \sigma_k) = i_{\beta_1}(\sigma_1) \otimes \cdots \otimes i_{\beta_k}(\sigma_k),$$
$$r_{\beta,\alpha}(\pi_1 \otimes \cdots \otimes \pi_k) = r_{\beta_1}(\pi_1) \otimes \cdots \otimes r_{\beta_k}(\pi_k).$$

Proof. [Ze80] Proposition 1.1, [Be92] Theorem 20, [Re10] Theorem VI.9.6. and [Au04] Proposition 4.20. □

Part 3 of the proposition is usually called *Frobenius reciprocity* and is of great practical importance. As a consequence of the properties described here we prove a useful Lemma.

Lemma 1.10. *Let $\pi_i \in \mathrm{Irr}(G_{n_i})$ for $i = 1, \dots, k$ and $\alpha = (n_1, \dots, n_k)$. Assume that $\pi_1 \otimes \cdots \otimes \pi_k$ appears with multiplicity 1 in $\mathrm{JH}(r_\alpha(\pi_1 \times \cdots \times \pi_k))$, then $\pi_1 \times \cdots \times \pi_k$ is SI. In other words, it has a unique irreducible subrepresentation that appears with multiplicity 1 in $\mathrm{JH}(\pi_1 \times \cdots \times \pi_k)$.*

Proof. The proof is a straightforward application of Frobenius reciprocity. Assume that there are two distinct irreducible subrepresentations σ_1, σ_2 of $\pi_1 \times \cdots \times \pi_k$. Then by Frobenius reciprocity we deduce

$$2 \leq \dim_{\mathbb{C}}(\mathrm{Hom}(\sigma_1 \oplus \sigma_2, \pi_1 \times \cdots \times \pi_k)) = \dim_{\mathbb{C}}(\mathrm{Hom}(r_\alpha(\sigma_1 \oplus \sigma_2), \pi_1 \otimes \cdots \otimes \pi_k)).$$

But by exactness of the Jacquet functors, we have $\mathrm{JH}(r_\alpha(\sigma_1 \oplus \sigma_2)) \leq \mathrm{JH}(r_\alpha(\pi_1 \times \cdots \times \pi_k))$ and therefore $\pi_1 \otimes \cdots \otimes \pi_k$ would appear with multiplicity ≥ 2 in $\mathrm{JH}(r_\alpha(\pi_1 \times \cdots \times \pi_k))$; a contradiction! □

1.2 Grothendieck group

In this section we briefly introduce the Grothendieck group of smooth representations of finite length $\mathscr{R}_n$ of G_n. Denote by R_n the set of isomorphism classes of smooth representations of finite length of G_n and let $\mathbb{Z}^{\oplus R_n}$ be the free abelian group generated by R_n, where we denote the basis element corresponding to a representation π as (π), and consider its subgroup generated by all elements

$$(\pi_2) - (\pi_1) - (\pi_3)$$

for which there is a short exact sequence

$$0 \to \pi_1 \to \pi_2 \to \pi_3 \to 0.$$

Definition 1.11. The quotient $\mathscr{R}_n$ of $\mathbb{Z}^{\oplus R_n}$ by the subgroup just described is called the *Grothendieck group* of $\mathrm{R}(G_n)$, we denote the image of a representation π in the Grothendieck group as $[\pi]$. We also define the group $\mathscr{R} := \bigoplus_{n \geq 0} \mathscr{R}_n$.

It is easy to describe the basic structure of $\mathscr{R}_n$. By the way it is defined, it is clear that for all representations $\pi \in \mathrm{Rep}^{f.l.}(G_n)$ we have a decomposition in $\mathscr{R}_n$ as

$$[\pi] = \sum_{\sigma \in \mathrm{JH}(\pi)} [\sigma],$$

where $\mathrm{JH}(\pi)$ denotes the multiset of irreducible subquotients of π counted with multiplicities (equivalently, the multiset of quotients in a Jordan-Hölder sequence of π). This decomposition also easily implies that $\mathscr{R}_n$ is simply the free abelian group generated by $\mathrm{Irr}(G_n)$.
In the same way we can of course also define the Grothendieck groups for other categories of representations to get $\mathscr{R}_\alpha$ for all partitions α and by the exactness of the functors $r_{\beta,\alpha}, i_{\alpha,\beta}$ we obtain for all subpartitions β of α also homomorphisms

$$\begin{aligned} i_{\alpha,\beta} &\colon \mathscr{R}_\beta \to \mathscr{R}_\alpha \\ r_{\beta,\alpha} &\colon \mathscr{R}_\alpha \to \mathscr{R}_\beta. \end{aligned}$$

We define an ordering on the Grothendieck group by setting

$$[\pi_1] \leq [\pi_2] \text{ if and only if } [\pi_2] - [\pi_1] \in \mathscr{R}^+,$$

where $\mathscr{R}^+$ denotes the submonoid of $\mathscr{R}$ consisting of all elements that only have non-negative coefficients when developed in the basis Irr.

Definition 1.12. We define a map $m\colon \mathscr{R} \times \mathscr{R} \to \mathscr{R}$, by taking two irreducible representations $\pi_1 \in \mathrm{Irr}_n, \pi_2 \in \mathrm{Irr}_m$ and setting $m(\pi_1, \pi_2) := [\pi_1 \times \pi_2]$. Since Irr gives a basis of $\mathscr{R}$ this gives a well-defined map by extending linearly.

With this multiplication, $\mathscr{R}$ clearly becomes a graded ring with 1 (the trivial representation of the trivial group G_0).

Proposition 1.13. *[cf. [Ze80] Theorem 1.9.] The ring $\mathscr{R}$ is commutative. In other words, if π_1, π_2 are representations of some G_n, G_m, then $\pi_1 \times \pi_2$ and $\pi_2 \times \pi_1$ have the same Jordan-Hölder factors.*

1.3 Cuspidal representations

We have already briefly mentioned that the basic goal of this section is to classify all irreducible representations of the G_n using parabolic induction. For this we need certain *building blocks* which we combine through induction to gain all irreducible representations. These are the *cuspidal* representations which we will define now.

Definition 1.14. Let α be a partition of n. A representation $\rho \in \mathrm{Rep}(G_\alpha)$ is called *cuspidal*, if for all *proper* subpartitions β of α we have $r_{\beta,\alpha}(\rho) = 0$.

We denote by $\mathcal{C}(G_\alpha)$ the set of irreducible, cuspidal representations of G_α and set $\mathcal{C} := \bigcup_{n\geq 0} \mathcal{C}(G_n)$.
From the compatibility of the Jacquet functors with tensor products(Proposition 1.9 4.) that a representation $\rho = \rho_1 \otimes \cdots \otimes \rho_k$ of G_α, $\alpha = (n_1, \ldots, n_k)$ is cuspidal if and only if all ρ_i are.

Proposition 1.15. *Let $\pi \in \mathrm{Rep}(G_n)$. Then*

1. *π decomposes into $\pi = \pi_c \oplus \pi_c^\perp$ with π_c cuspidal and $\pi_c^\perp$ having no non-zero cuspidal subquotients,*
2. *If π has an irreducible cuspidal subquotient ρ, then ρ appears both as a subrepresentation and as a quotient of π,*
3. *If α is a proper partition of n and $\pi = i_\alpha(\sigma)$ for some $\sigma \in \mathrm{Rep}(G_\alpha)$, then $\pi_c = 0$.*

Proof. We only prove part 3., for proofs of the other points, see [BZ77] Theorem 4.17, Proposition 3.30 respectively.
For point 3., since π_c is a cuspidal submodule of π by defintion, we deduce from Frobenius reciprocity:

$$\mathrm{Hom}_{G_n}(\pi_c, \underbrace{\pi}_{=i_\alpha(\sigma)}) = \mathrm{Hom}_{G_\alpha}(\underbrace{r_\alpha(\pi_c)}_{=0}, \sigma) = 0.$$

Therefore we see $\pi_c = 0$. □

The last point of the proposition illustrates the role that cuspidal representations will play as building blocks of the classification.

Proposition 1.16 ([Ze80] Proposition 1.10.)**.** *Let $\pi \in \mathrm{Irr}(G_n)$. Then there is a partition $\alpha = (n_1, \ldots, n_k)$ of n and a cuspidal representation $\rho = \rho_1 \otimes \cdots \otimes \rho_k$ of G_α such that $[\pi] \leq [i_\alpha(\rho)] = [\rho_1 \times \cdots \times \rho_k]$, i.e. π is an irreducible subquotient of $\rho_1 \times \cdots \times \rho_k$. Furthermore the ρ_i can be ordered in such a way that π is a submodule of $\rho_1 \times \cdots \times \rho_k$. The multiset $\rho_1 + \cdots + \rho_k$ only depends on π and is called the support of π, denoted* $\mathrm{supp}(\pi)$.

This ensures that our strategy of constructing all irreducible representations of G_n from cuspidal representations using parabolic induction works in principle. A fundamental result for this is the following theorem, which tells us how the product of two cuspidal representations behaves.

Theorem 1.17 ([Ze80] Proposition 1.11.)**.** *Let $\rho \in \mathcal{C}$ be a cuspidal representation. Then for any cuspidal representation $\rho' \in \mathcal{C}$ we have*

$$\rho \times \rho' \text{ is irreducible if and only if } \rho' \ncong \rho|.|^{\pm 1}.$$

This theorem is the main motivation for the definitions of segments and multisegments we will see shortly. Essentially it tells us that when trying to combine cuspidal representations using parabolic induction, we only really need to study what happens when we form products $\rho|.|^a \times \rho|.|^{a+1} \times \cdots \times \rho|.|^b$. It is then natural to study in greater detail sets of the form $\{\rho|.|^a, \dots, \rho|.|^b\}$, which will be called *segments*. We will continue the study of this in section 2.1, before that we include an important technical tool: the Geometric Lemma.

1.4 Geometric Lemma

We recall the combinatorial formulation of the Geometric Lemma of Bernstein-Zelevinsky. It is of fundamental importance to the computations in all of the following.

Let $\alpha = (n_1, \dots, n_k)$, $\beta = (m_1, \dots, m_l)$ be partitions of the same integer $n \geq 1$. Then we can consider the functor $F_{\beta,\alpha} := r_\beta \circ i_\alpha : \mathrm{Rep}(G_\alpha) \to \mathrm{Rep}(G_\beta)$. Fix now a $\pi = \pi_1 \otimes \cdots \otimes \pi_k \in \mathrm{Rep}(G_\alpha)$. The goal of the Geometric Lemma is to describe $[F_{\beta,\alpha}(\pi)]$.

Let $\mathbb{N}^{\alpha,\beta}$ denote the set of $k \times l$ matrices B with entries from $\mathbb{N}$ such that

$$\sum_{j=1}^{l} b_{ij} = n_i, \quad i \in \{1, \dots, k\},$$
$$\sum_{i=1}^{k} b_{ij} = m_j, \quad j \in \{1, \dots, l\}.$$

Fixing such a matrix B, the definition ensures that we get partitions $\alpha_i := (b_{i1}, \dots, b_{il})$ of n_i for each i and can thus form the Jaquet modules $r_{\alpha_i}(\pi_i)$. Let $\{\sigma_i^{(1)}, \dots, \sigma_i^{(r_i)}\}$ denote the multiset $\mathrm{JH}(r_{\alpha_i}(\pi_i))$ and write each $\sigma_i^{(k)}$ as

$$\sigma_i^{(s)} = \sigma_{i1}^{(s)} \otimes \cdots \otimes \sigma_{il}^{(s)}, \quad \sigma_{ij}^{(s)} \in \mathrm{Irr}(G_{b_{ij}}).$$

Then for any $j = 1, \dots, l$ and any sequence $S = (s_1, \dots, s_k)$ with $1 \leq s_i \leq r_i$ we can form representations

$$\Sigma_j^{B,S} := \sigma_{1j}^{s_1} \times \cdots \times \sigma_{kj}^{s_k} \in \mathrm{Rep}(G_{m_j}).$$

Theorem 1.18 (Geometric Lemma)**.** *With notation as above, we have*

$$[r_\beta(\pi_1 \times \cdots \times \pi_k)] = \sum_{B,S} [\Sigma_1^{B,S} \otimes \cdots \otimes \Sigma_l^{B,S}].$$

Proof. A proof of the original, more geometric version, of the claim can be found in [BZ77] Theorem 5.2. The combinatorial form of the Geomteric Lemma we use here can be found in [Ze80] Section 1.6. □

To illustrate the utility of the Geometric Lemma, we prove some results that are important for the classification.

Lemma 1.19. *Let $\pi_i \in \mathrm{Rep}(G_{n_i})$ and $\sigma_i := \mathrm{soc}(\pi_i)$ for $i = 1, 2$. Suppose the σ_i are irreducible and $\sigma_1 \otimes \sigma_2$ appears with multiplicity 1 in $[r_{(n_1,n_2)}(\pi_1 \times \pi_2)]$, then $\pi_1 \times \pi_2$ is SI.*

Proof. We begin by showing that for any submodule ρ of $\pi_1 \times \pi_2$, $\sigma_1 \otimes \sigma_2$ is a subquotient of $r_{(n_1,n_2)}(\rho)$. By Frobenius reciprocity we have

$$\mathrm{Hom}_{G_{n_1+n_2}}(\rho, \pi_1 \times \pi_2) = \mathrm{Hom}_{G_{(n_1,n_2)}}(r_{(n_1,n_2)}(\rho), \pi_1 \otimes \pi_2),$$

and therefore the inclusion $\rho \hookrightarrow \pi_1 \times \pi_2$ gives us a non-trivial $G_{(n_1,n_2)}$-morphism $f\colon r_{(n_1,n_2)}(\rho) \to \pi_1 \otimes \pi_2$. If we can show $\mathrm{Im}(f) \supset \sigma_1 \otimes \sigma_2$ then we have shown our first claim.
Since $f \neq 0$ there is necessarily some irreducible $\omega_1 \otimes \omega_2 \subset \mathrm{Im}(f) \subset \pi_1 \otimes \pi_2$. Since $\sigma_i = \mathrm{soc}(\pi_i)$ for $i = 1, 2$ are both irreducible we further have $\omega_i = \sigma_i$ since there are no other irreducible subrepresentations of π_i. The map f thus shows $\sigma_1 \otimes \sigma_2$ to be a subquotient of ρ.
Suppose now that we had two distinct irreducible subrepresentations ρ, ρ' of $\pi_1 \times \pi_2$. By what we have just shown, we thus deduce

$$2[\sigma_1 \otimes \sigma_2] \leq [r_{(n_1,n_2)}(\rho + \rho')] \leq [r_{(n_1,n_2)}(\pi_1 \times \pi_2)],$$

contradicting our assumption. Thus $\mathrm{soc}(\pi_1 \times \pi_2)$ is irreducible. Since $\sigma_1 \otimes \sigma_2$ is obviously a subquotient of $r_{(n_1,n_2)}(\mathrm{soc}(\pi_1 \times \pi_2))$ (exactness of Jacquet functor) we see from the same argument as above that $\mathrm{soc}(\pi_1 \times \pi_2)$ can only appear with multiplicity 1 in $\pi_1 \times \pi_2$. This finishes the proof. □

Proposition 1.20. *If $\rho_i \in \mathcal{C}_{n_i}$, $i = 1, \ldots, k$ are pairwise inequivalent cuspidal representations, then $\rho_1 \times \cdots \times \rho_k$ is SI.*

Proof. We prove the theorem by induction on k, starting with $k = 2$. For this, we only need to check the conditions of Lemma 1.19 using the Geometric Lemma. Indeed we can explicitly compute $[r_{(n_1,n_2)}(\rho_1 \times \rho_2)]$. Since the ρ_i are cuspidal, any proper partition α_i of n_i will produce $r_{\alpha_i}(\rho_i) = 0$ and thus the only matrices B that can contribute to the sum in Theorem 1.18 is the diagonal matrix $\mathrm{diag}(n_1, n_2)$ which shows $[r_{(n_1,n_2)}(\rho_1 \times \rho_2)] = [\rho_1 \otimes \rho_2]$ and thus the conditions of Lemma 1.19 are met, proving the case $k = 2$.
Suppose now that the result holds for all $r < k$. We will again use Lemma 1.19 to deduce the result. By assumption $\pi := \rho_1 \times \cdots \times \rho_{k-1}$ is socle irreducible, say $\sigma := \mathrm{soc}(\pi) \in \mathrm{Irr}$, and ρ_k is irreducible itself and hence SI. We thus only need to compute how often $\pi \otimes \rho_k$ appears in $r_{(n_1+\cdots+n_{k-1},n_k)}(\rho_1 \times \cdots \times \rho_k)$. For ease of notation, set $m := n_1 + \cdots + n_{k-1}$. We again appeal to the Geometric Lemma for this computation. Let $B \in \mathbb{N}^{(n_1,\ldots,n_k),(m,n_k)}$. The last row $\alpha_k = (b_{k1}, b_{k2})$ of B is a partition of n_k and since ρ_k is cuspidal, $r_{\alpha_k}(\rho_k) = 0$ unless the partition is trivial. So one possibility for B is $b_{k1} = 0$ and $b_{k2} = n_k$ which necessitates $b_{i1} = n_i$ and $b_{i2} = 0$ for all $i < k$ and thus contributes $[\pi \otimes \rho_k]$ to $[r_{(n_1+\cdots+n_{k-1},n_k)}(\rho_1 \times \cdots \times \rho_k)]$. By induction hypothesis, π is SI with socle σ and so

this choice of B contributes $[\sigma \otimes \rho_k]$ with multiplicity 1.

We thus need to show that the other choices of B don't contribute $[\sigma \otimes \rho_k]$ to the Jacquet module. Necessarily if B is not as before we have $b_{k1} = n_k$ and $b_{k2} = 0$. Since each ρ_i is cuspidal, the only matrices that can give a non-zero contribution in the Jacquet module have i-th row $(n_i, 0)$ or $(0, n_i)$ for all i. Since $\sum_{i=1}^{k} b_{i2} = n_k$ we thus have a sequence $1 \leq i_1 < \cdots < i_l \leq k-1$ such that $n_k = n_{i_1} + \cdots + n_{i_l}$ and the i_j are exactly those indices i where the i-th row of B is equal to $(0, n_i)$. Therefore such a B gives a contribution of $[\pi_{i_1,\dots,i_l} \otimes (\rho_{i_1} \times \cdots \times \rho_{i_l})]$, where $\pi_{i_1,\dots,i_l}$ is the product of the ρ_i with indices *not equal* to any i_j. Since the ρ_i are inequivalent by assumption, we see that ρ_k cannot appear as a composition factor of $\rho_{i_1} \times \cdots \times \rho_{i_k}$ (essentially by the Geometric Lemma) and so these choices of B never contribute $[\sigma \otimes \rho_k]$.

We have now shown that $[\sigma \otimes \rho_k]$ appears with multiplicity 1 in $[r_{(n_1+\cdots+n_{k-1}, n_k)}(\rho_1 \times \cdots \times \rho_k)]$ and so by Lemma 1.19 we deduce that $\rho_1 \times \cdots \times \rho_k$ is SI. $\square$

2 Zelevinsky's Classification

The aim of this section is to report Zelevinsky's classification of all irreducible representations of all G_n through multisegments and cuspidal representations. We follow the original strategy of [BZ77] and [Ze80].

2.1 Segments and Multisegments

As discussed at the end of section 1.3 we will now investigate sets of the form $\Delta = \{\rho|.|^a, \dots, \rho|.|^b\}$ for some cuspidal ρ and integers $a \leq b$ and associate irreducible representations $Z(\Delta)$ to them.

Definition 2.1. A *segment* Δ is a set of the form $\{\rho|.|^a, \dots, \rho|.|^b\}$ for a cuspidal representation $\rho \in \mathcal{C}$ and integers $a \leq b$. We will usually denote such a segment by $[a, b]_\rho$. We denote the set of all segments by Seg. We also adopt the convention that $[a, b]_\rho = \emptyset$ if $b < a$.

We will need a whole slew of notation connected to segments which we will introduce right away. So let $\Delta = [a, b]_\rho$ be a segment, then we will adopt the following notations for the rest of this article:

$$\begin{aligned}
l(\Delta) &= b - a + 1, \\
\deg(\Delta) &= l(\Delta)n \text{ where } \rho \in \mathcal{C}(G_n)\ , \\
b(\Delta) &= \rho|.|^a, \\
e(\Delta) &= \rho|.|^b \\
\Delta^{\pm} &= [a, b \pm 1]_\rho, \\
{}^{\pm}\Delta &= [a \pm 1, b]_\rho, \\
\overleftarrow{\Delta} &= [a - 1, b - 1]_\rho.
\end{aligned}$$

Definition 2.2. Let $\Delta = [a, b]_\rho$, $\Delta' = [c, d]_{\rho'}$ be segments. We call them *linked*, if $\Delta \cup \Delta'$ is also a segment but neither $\Delta \subset \Delta'$ nor $\Delta' \subset \Delta$ hold. Otherwise we say that they are *unlinked*.
In case Δ, Δ' are linked, we say that Δ *precedes* Δ', and write $\Delta \prec \Delta'$, if there is $j > 0$ such that $b(\Delta)|.|^j = b(\Delta')$.

As a consequence of Proposition 1.20 we can associate an irreducible representation to each segment Δ as follows.

Definition 2.3. Let $\Delta = [a, b]_\rho$ be a segment. Then we define

$$Z(\Delta) := \mathrm{soc}(\rho|.|^a \times \dots \times \rho|.|^b).$$

By Proposition 1.20 this is an irreducible representation (of $G_{\deg(\Delta)}$ to be precise).

For many computations and proofs it is essential to understand the Jacquet modules of a segment representation $Z(\Delta)$, we will now compute these.

Proposition 2.4. *Let $\rho \in \mathcal{C}(G_n)$ and $\Delta = [a,b]_\rho$. Then $Z(\Delta) \in \mathrm{Irr}(G_m)$ where $m = nl(\Delta)$ and*

if $0 \leq k \leq m$ is not divisible by n, then $r_{(k,m-k)}(Z(\Delta)) = 0$,

if $k = nr$ for some $r \in \{0, \dots, l(\Delta)\}$, then $r_{(k,m-k)}(Z(\Delta)) = Z([a, a+r-1]) \otimes Z([a+r, b])$.

Proof. Fix some $k \leq m$ and let $\alpha = \underbrace{(n, \dots, n)}_{l(\Delta) \text{ times}}$ and $\beta = (k, m-k)$. Then we can compute $r_\beta(i_\alpha(\rho|.|^a \otimes \dots \otimes \rho|.|^b))$ using the Geometric Lemma. If $k \nmid n$ then α is not a subpartition of β and it is then easy to see that in the description of $[r_\beta(i_\alpha(\rho|.|^a \otimes \dots \otimes \rho|.|^b))]$ that the Geometric Lemma gives us, we always have to take Jacquet modules $r_{\beta_i}(\rho|.|^i)$ with a *proper* partition β_i of n for some i and thus all contributions are 0.
Therefore $r_{(k,m-k)}(i_\alpha(\rho|.|^a \otimes \dots \otimes \rho|.|^b)) = 0$ and since $r_\beta(Z(\Delta))$ is a submodule of this, we are done.
If $k = nr$ we first note that α is a subpartition of β and $r_\alpha(Z(\Delta)) = \rho|.|^a \otimes \dots \otimes \rho|.|^b$. To see this last bit, note that by Frobenius reciprocity we have a non-zero map $r_\alpha(Z(\Delta)) \hookrightarrow \rho|.|^a \otimes \dots \otimes \rho|.|^b$ and since the later representation is irreducible we get the equality. Since $r_\alpha = r_{\alpha,\beta} \circ r_\beta$ we therefore see that $r_{\alpha,\beta}(r_\beta(Z(\Delta)))$ is irreducible, we want to show that even $r_\beta(Z(\Delta))$ is already irreducible. For this, it is enough to show that for any irreducible submodule π of $r_\beta(Z(\Delta))$ we have $r_{\alpha,\beta}(\pi) \neq 0$.
Since $Z(\Delta) = \mathrm{soc}(\rho|.|^a \times \dots \rho|.|^b)$ and by exactness of Jacquet functors, π is a submodule of $r_\beta(\rho|.|^a \times \dots \times \rho|.|^b)$. $\pi \neq 0$ implies that there must be some subpartition γ of β such that $r_{\gamma,\beta}(\pi)$ is cuspidal and non-zero, since the set of subpartitions of β for which the Jacquet module of π is non-zero must have some minimal elements. Let σ be some irreducible subquotient of $r_{\gamma,\beta}(\pi)$. By exactness of the Jacquet functor, this is again cuspidal and it appears as a subquotient of $r_\gamma(\rho|.|^a \times \dots \times \rho|.|^b)$. But by the Geometric Lemma it is easy to see that the only way that $r_\gamma(\rho|.|^a \times \dots \times \rho|.|^b)$ can have a cuspidal irreducible subrepresentation is that $\gamma = (n, \dots, n) = \alpha$ and so $r_{\alpha,\beta}(\pi)$ is non-zero.
We can thus conclude that $r_\beta(Z(\Delta))$ is irreducible and so write it as $\omega_1 \otimes \omega_2$ for some $\omega_1 \in \mathrm{Irr}(G_k)$ and $\omega_2 \in \mathrm{Irr}(G_{m-k})$. Further, we have

$$\rho|.|^a \otimes \dots \otimes \rho|.|^b = r_\alpha(Z(\Delta)) = r_{\alpha,\beta}(\omega_1 \otimes \omega_2) = r_{(n,\dots,n)}(\omega_1) \otimes r_{(n,\dots,n)}(\omega_2),$$

and therefore deduce

$$r_{(n,\dots,n)}(\omega_1) = \rho|.|^a \otimes \dots \otimes \rho|.|^{a+r-1}$$
$$r_{(n,\dots,n)}(\omega_2) = \rho|.|^{a+r} \otimes \dots \otimes \rho|.|^b.$$

It is not hard to see (cf. [Ze80], Section 3.1.) that this in fact implies $\omega_1 = Z([a, a+r-1]_\rho)$ and $\omega_2 = Z([a+r, b]_\rho)$ which completes the proof. □

Segments already give us a large class of irreducible representations that we can get from cuspidals using parabolic induction. Unfortunately, these are not all irreducible representations and we now need to study how we can combine several segment representations using parabolic induction to get even more irreducibles. This leads fairly naturally to the concept of multisegment, which we will introduce now.

Definition 2.5. A multisegment $\mathfrak{m}$ is a (finite) multiset of segments. We denote by $\mathfrak{M}$ the set of multisegments.

As with segments, we need to introduce some notation concerning multisegments before we proceed. Given a multisegment $\mathfrak{m} = \Delta_1 + \cdots + \Delta_n$, we set $\deg(\mathfrak{m}) = \deg(\Delta_1) + \cdots + \deg(\Delta_n)$ and $\mathrm{supp}(\mathfrak{m}) = \bigcup_i \Delta_i \subset \mathcal{C}$.

Definition 2.6. Given a cuspidal $\rho \in \mathcal{C}$ we define

$$\begin{aligned} \mathbb{Z}_\rho &:= \{\rho|.|^b \mid b \in \mathbb{Z}\}, \\ \mathrm{Seg}_\rho &:= \{\Delta \in \mathrm{Seg} \mid \Delta \subset \mathbb{Z}_\rho\}, \\ \mathfrak{M}_\rho &:= \{\mathfrak{m} \in \mathfrak{M} \mid \mathrm{supp}(\mathfrak{m}) \subset \mathbb{Z}_\rho\}. \end{aligned}$$

Given a multisegment $\mathfrak{m} = \sum \Delta_i$ we introduce some further terminology.

1. We say that $(\Delta_1, \ldots, \Delta_n)$ is an *arranged form* of $\mathfrak{m}$ if Δ_i does not precede Δ_j whenever $i < j$. Clearly, each multisegment $\mathfrak{m}$ possesses such an arranged form.

2. We say that $\mathfrak{m}$ is *totally unlinked*, if the segments Δ_i are pairwise unlinked.

Proposition 2.7. *[cf. [Ze80] Theorem 4.2.] Let $\mathfrak{m} = \Delta_1 + \cdots + \Delta_n$ be a totally unlinked multisegment, then $\zeta(\mathfrak{m}) := Z(\Delta_1) \times \cdots \times Z(\Delta_n)$ is irreducible.*

We can now associate an irreducible representation $Z(\mathfrak{m})$ to each multisegment $\mathfrak{m}$.

Proposition 2.8. *If $(\Delta_1, \ldots, \Delta_n)$ is an arranged form of a multisegment $\mathfrak{m}$, the representation $\zeta(\mathfrak{m}) := Z(\Delta_1) \times \cdots \times Z(\Delta_n)$ is SI. We note that, due to Proposition 2.7, $\zeta(\mathfrak{m})$ doesn't depend on the choice of arranged form up to isomorphism. We denote by $Z(\mathfrak{m}) := \mathrm{soc}(\zeta(\mathfrak{m})) \in \mathrm{Irr}$ its socle.*

Proof. Without loss of generality, we can assume that the Δ_i all belong to Seg_ρ for some fixed ρ and that the Δ_i are ordered with decreasing beginnings. Then we get indices $1 \leq n_1 < \cdots < n_k = n$ such that

$$b(\Delta_1) = \cdots = b(\Delta_{n_1}) > \cdots > b(\Delta_{n_{k-1}+1}) = \cdots = b(\Delta_n),$$

For $1 \leq i \leq k$, define multisegments (with $n_0 = 0$):

$$\mathfrak{m}_i := \sum_{j=n_{i-1}+1}^{n_i} \Delta_j.$$

Since all the segments in $\mathfrak{m}_i$ for a fixed i have the same ends, each $\mathfrak{m}_i$ is totally unlinked and the previous proposition guarantees that all $Z(\mathfrak{m}_i) = Z(\Delta_{n_{i-1}+1}) \times \ldots Z(\Delta_{n_i})$ are irreducible representations and $\zeta(\mathfrak{m}) = Z(\mathfrak{m}_1) \times \cdots \times Z(\mathfrak{m}_k)$. A careful application of the Geomteric Lemma (cf. [Mi09] Chapter 2) shows that $Z(\mathfrak{m}_1) \otimes \cdots \otimes Z(\mathfrak{m}_k)$ appears with multiplicity 1 in $r_{(\deg(\mathfrak{m}_1),\ldots,\deg(\mathfrak{m}_k))}(\zeta(\mathfrak{m}))$. By Lemma 1.10 this implies that $\zeta(\mathfrak{m})$ is SI, which finishes the proof. $\square$

Proposition 2.9. *[cf. [Ze80] Proposition 8.5.] Let $\mathfrak{m} = \sum_i \Delta_i, \mathfrak{n} = \sum_j \Delta'_j$ be multisegments, such that no two segments Δ_i, Δ'_j are linked. Then $Z(\mathfrak{m}) \times Z(\mathfrak{n})$ is irreducible and hence $Z(\mathfrak{m}) \times Z(\mathfrak{n}) \cong Z(\mathfrak{m}+\mathfrak{n}) \cong Z(\mathfrak{n}) \times Z(\mathfrak{m})$.*

Remark 2.10. As a particular consequence of the proposition, we can often work with multisegments in $\mathfrak{M}_\rho$ for some fixed $\rho \in \mathcal{C}$. Indeed, given a multisegment Given a multisegment $\mathfrak{m}$, we can decompose it uniquely into a sum $\mathfrak{m} = \mathfrak{m}_1 + \cdots + \mathfrak{m}_k$ such that each $\mathfrak{m}_i \in \mathfrak{M}_{\rho_i}$ with $\mathbb{Z}_{\rho_i} \cap \mathbb{Z}_{\rho_j} = \emptyset$ for $i \neq j$. By the proposition, it follows that

$$Z(\mathfrak{m}) = Z(\mathfrak{m}_1) \times \cdots \times Z(\mathfrak{m}_k).$$

Conversely, taking $\mathfrak{m} \in \mathfrak{M}_\rho$, $\mathfrak{n} \in \mathfrak{M}_{\rho'}$ with disjoint $\mathfrak{M}_\rho, \mathfrak{M}_{\rho'}$ the product $Z(\mathfrak{m}) \times Z(\mathfrak{n})$ is always irreducible and isomorphic to $Z(\mathfrak{m}+\mathfrak{n})$. This means that it often is enough to work only with multisegments over a fixed $\rho \in \mathcal{C}$ since the above discussion shows that combining these results is often very simple. When fixing a $\rho \in \mathcal{C}$ and working only in $\mathfrak{M}_\rho$ it is often more convenient to drop the ρ from all notations. A segment $[a,b]_\rho \in \mathrm{Seg}_\rho$ will then just be written as $[a,b]$ and sometimes identified with the interval $\{a, a+1, \ldots, b\} \subset \mathbb{Z}$. We will use this identification extensively in chapters 3 and 4.

2.2 Langlands standard modules

So far we have associated to a multisegment $\mathfrak{m}$ an irreducible representation $Z(\mathfrak{m}) = \mathrm{soc}(\zeta(\mathfrak{m}))$. We will occasionally refer to $\zeta(\mathfrak{m})$ as the *Zelevinsky standard module* associated to $\mathfrak{m}$.
Instead of taking socles everywhere in the construction of $Z(\mathfrak{m})$ we could also take cosocles to obtain a different irreducible representation $L(\mathfrak{m})$, which we quickly explain in this section. All proofs will be omitted since they work exactly analogously to what we have done for Z so far.

Definition 2.11. Given a segment $\Delta = [a,b]_\rho$ we define the representation

$$L(\Delta) = \cos(\rho|.|^a \times \cdots \times \rho|.|^b).$$

This is an irreducible representation appearing with multiplicity 1 in $[\rho|.|^a \times \cdots \times \rho|.|^b]$.

Also like before, we can associate an irreducible $L(\mathfrak{m})$ to any multisegment $\mathfrak{m}$.

Proposition 2.12. *Let $(\Delta_1, \ldots, \Delta_n)$ be an arranged form of a multisegment $\mathfrak{m}$. Then*

$$\lambda(\mathfrak{m}) := L(\Delta_1) \times \cdots \times L(\Delta_n),$$

doesn't depend on the choice of arranged form up to isomorphism and is CI, i.e. $\cos(\lambda(\mathfrak{m}))$ is irreducible and appears with multiplicity 1 in $[L(\Delta_1) \times \cdots \times L(\Delta_n)]$.

Definition 2.13. Given a multisegment $\mathfrak{m}$, we define

$$L(\mathfrak{m}) := \cos(\lambda(\mathfrak{m})).$$

This is an irreducible representation of $G_{\deg(\mathfrak{m})}$.

In all that follows, we will mostly use the association $\mathfrak{m} \mapsto Z(\mathfrak{m})$ but want to note here that we could always replace Z by L and things would work exactly analogously.

2.3 Classification

The previous discussion culminates in the following classification theorem of irreducible representations of G_n.

Theorem 2.14. *The maps $Z, L : \mathfrak{M} \to \mathrm{Irr}$ are bijections.*

Proof. For the proof concerning Z see [Ze80]. The proof concerning L works essentially the same way, since all the statements one needs also hold for L. $\square$

An interesting question that arises is the following. Given a multisegment $\mathfrak{m}$ we associate to it an irreducible representation $Z(\mathfrak{m})$. By the previous result, there must also be a unique multisegment $\mathfrak{m}^{\#}$ such that $Z(\mathfrak{m}) = L(\mathfrak{m}^{\#})$. It is not at all obvious that this map $\mathfrak{m} \mapsto \mathfrak{m}^{\#}$ is actually an *involution*, i.e. $(\mathfrak{m}^{\#})^{\#} = \mathfrak{m}$. In particular this implies also $L(\mathfrak{m}) = Z(\mathfrak{m}^{\#})$. This operation is usually referred to as the *Zelevinsky* or *Moeglin-Waldspurger* involution. For more details on it, especially the algorithmic description of the map $\mathfrak{m} \mapsto \mathfrak{m}^{\#}$ see [MW86] and [Ze80] Section 9.

2.4 Decomposition of induced representations

In this section we want to give a complete description of the subquotients of a representation of the form $Z(\Delta_1) \times \cdots \times Z(\Delta_k)$ for a multisegment $\mathfrak{m} = \sum_i \Delta_i$. Note that although the representation $Z(\Delta_1) \times \cdots \times Z(\Delta_k)$ depends on the ordering of the Δ_i, this dependence is only up to semi-simplification: in the Grothendieck Group $\mathscr{R}$ we have $[Z(\Delta_1) \times \cdots \times Z(\Delta_k)] = [\zeta(\mathfrak{m})]$ for any order of the Δ_i. We have already seen that the socle of $\zeta(\mathfrak{m})$ is irreducible and we can describe all irreducible representations of G_n in this way. It is now a natural question to ask how the other irreducible subquotients of these representations can be described in terms of multisegments. For this, we start with the simplest case of two segments $\Delta_1 + \Delta_2$.

Proposition 2.15. *Given segments Δ_1, Δ_2 we have:*

1. *if Δ_1, Δ_2 are unlinked, the representation $Z(\Delta_1) \times Z(\Delta_2)$ is irreducible and thus equals $Z(\Delta_1 + \Delta_2)$,*

2. *if Δ_1, Δ_2 are linked, the representation $Z(\Delta_1) \times Z(\Delta_2)$ is reducible of length 2 and we have*

$$[Z(\Delta_1) \times Z(\Delta_2)] = [Z(\Delta_1 + \Delta_2)] + [Z(\Delta_1 \cap \Delta_2 + \Delta_1 \cup \Delta_2)].$$

Proof. 1. We have already proven this as a special case of Proposition 2.7.

2. One first proves this in the case that the two segments are juxtaposed, i.e. $\Delta_1 \cap \Delta_2 = \emptyset$. For this, see [Ze80] Prop. 5.3. which makes use of Zelevinskys study of representations of multiplicity-free support which we have not reported here.
For the general case, we do the proof by induction on $l = l(\Delta_1) + l(\Delta_2)$. We assume that Δ_2 precedes Δ_1, since by Proposition 1.13 the ordering of the Δ_i

doesn't influence the claim.
For $l=2$ we can assume without loss of generality that $\Delta_2=\rho$ and $\Delta_1=\rho|.|$ so $Z(\Delta_1)\times Z(\Delta_2)=\rho|.|\times\rho$. We then already know $\mathrm{soc}(Z(\Delta_1)\times Z(\Delta_2))=Z([0,0]_\rho+[1,1]_\rho)$ which appears with multiplicity 1 in $[Z(\Delta_1)\times Z(\Delta_2)]$. But we also know $\cos(Z(\Delta_1)\times Z(\Delta_2))=\cos(\rho|.|\times\rho)=\mathrm{soc}(\rho\times\rho|.|)=Z([0,1]_\rho)$ and this appears with multiplicity 1 as well. This shows our claim for $l=2$.
Now assume $l>2$ and Δ_1,Δ_2 are not juxtaposed. Let $\alpha=(\deg(\Delta_1),\deg(\Delta_2)),\beta=(\deg(\Delta_1\cup\Delta_2),\deg(\Delta_1\cap\Delta_2))$ and $\deg\rho=m$. By Frobenius Reciprocity we have

$$\begin{aligned}&\mathrm{Hom}(Z(\Delta_1)\times Z(\Delta_2),Z(\Delta_1\cup\Delta_2)\times Z(\Delta_1\cap\Delta_2))\\&\qquad=\mathrm{Hom}(r_\beta(i_\alpha(Z(\Delta_1)\otimes Z(\Delta_2))),Z(\Delta_1\cup\Delta_2)\otimes Z(\Delta_1\cap\Delta_2)).\end{aligned}$$

using the Geometric Lemma, it is easy to see that $r_\beta(i_\alpha(Z(\Delta_1)\otimes Z(\Delta_2)))$ has a quotient isomorphic to $Z(\Delta_1)\times Z(\Delta_2\setminus(\Delta_1\cap\Delta_2))\otimes Z(\Delta_1\cap\Delta_2)$ and we note that $\Delta_2\setminus(\Delta_1\cap\Delta_2)$ and Δ_1 are juxtaposed segments and hence we know that $Z(\Delta_1)\times Z(\Delta_2\setminus(\Delta_1\cap\Delta_2))$ has the unique quotient $Z(\Delta_1\cup(\Delta_2\setminus(\Delta_1\cap\Delta_2)))=Z(\Delta_1\cup\Delta_2)$. This shows in total, that $Z(\Delta_1\cup\Delta_2)\times Z(\Delta_1\cap\Delta_2)$ is a quotient of $Z(\Delta_1)\times Z(\Delta_2)$, proving a part of our claim. Furthermore by proposition 2.8 we have $Z(\Delta_1+\Delta_2)=\mathrm{soc}(Z(\Delta_1)\times Z(\Delta_2))$ and this representation appears with multiplicity 1 in $[Z(\Delta_1)\times Z(\Delta_2)]$. We have therefore shown $[Z(\Delta_1)\times Z(\Delta_2)]\geq[Z(\Delta_1+\Delta_2)]+[Z(\Delta_1\cup\Delta_2)\times Z(\Delta_1\cup\Delta_2)]$.
So far, we have not made use of our induction setup. To show $[Z(\Delta_1)\times Z(\Delta_2)]\leq[Z(\Delta_1+\Delta_2)]+[Z(\Delta_1\cup\Delta_2)\times Z(\Delta_1\cup\Delta_2)]$ we will use a strategy of 'splitting off' the beginnings of segments using Jacquet functors and the Geometric Lemma which will allow us to prove the result by induction.
We will prove

$$[Z(\Delta_1+\Delta_2)]\geq[Z(\Delta_1)\times Z(\Delta_2)]-[Z(\Delta_1\cup\Delta_2)\times Z(\Delta_1\cup\Delta_2)],$$

which is equivalent to our claim. Recall from the proof of Proposition 2.4 that for any irreducible subquotient ω of $Z(\Delta_1)\times Z(\Delta_2)$ we have $r_{(m,(l-1)m)}(\omega)\neq 0$. Because of this, we reduce further to proving

$$[r(Z(\Delta_1+\Delta_2))]\geq[r(Z(\Delta_1)\times Z(\Delta_2))]-[r(Z(\Delta_1\cup\Delta_2)\times Z(\Delta_1\cup\Delta_2))],\qquad(2)$$

where we have set $r:=r_{(m,(l-1)m)}$ for purposes of legibility. Using the Geometric Lemma we compute

$$\begin{aligned}r_{(m,(l-1)m)}(Z(\Delta_1)\times Z(\Delta_2))&=b(\Delta_1)\otimes Z(^-\Delta_1)\times Z(\Delta_2)\\&\quad+b(\Delta_2)\otimes Z(\Delta_1)\times Z(^-\Delta_2).\end{aligned}$$

We therefore deduce that the right hand side of (2) is equal to $A+B$ where:

$$\begin{aligned}A&=[b(\Delta_1)\otimes Z(^-\Delta_1)\times Z(\Delta_2)]-[b(\Delta_1)\otimes Z(^-\Delta_1\cap\Delta_2)\times Z(^-\Delta_1\cup\Delta_2)]\\B&=[b(\Delta_2)\otimes Z(\Delta_1)\times Z(^-\Delta_2)]-[b(\Delta_2)\otimes Z(\Delta_1\cap^-\Delta_2)\times Z(\Delta_1\cup^-\Delta_2)].\end{aligned}$$

Note that if Δ_1 or Δ_2 have length 1, we have $A = 0$ or $B = 0$ and so we will assume $l(\Delta_i) > 1$ from now on. It is obvious from the definitions that $Z(\Delta_1 + \Delta_2)$ is a subrepresentation of $b(\Delta_1) \times Z(^-\Delta_1) \times Z(\Delta_2)$ and we know that $Z(^-\Delta_1) \times Z(\Delta_2)$ has the unique irreducible subrepresentation $Z(^-\Delta_1 + \Delta_2)$, since $^-\Delta_1$ does not precede Δ_2, so in total we see

$$Z(\Delta_1 + \Delta_2) \hookrightarrow b(\Delta_1) \times Z(^-\Delta_1 + \Delta_2).$$

Frobenius reciprocity then immediately implies

$$[r(Z(\Delta_1 + \Delta_2))] \geq [b(\Delta_1) \otimes Z(^-\Delta_1 + \Delta_2)].$$

Now since Δ_1 and Δ_2 are not juxtaposed and Δ_2 precedes Δ_1, the segments $^-\Delta_1$ and Δ_2 must also be linked. Therefore we can deduce by induction that

$$[Z(^-\Delta_1 + \Delta_2)] = [Z(^-\Delta_1) \times Z(\Delta_2)] - [Z(^-\Delta_1 \cup \Delta_2) \times Z(^-\Delta_1 \cap \Delta_2)],$$

and so

$$\begin{aligned}[b(\Delta_1) \otimes Z(^-\Delta_1 + \Delta_2)] = {} & [b(\Delta_1) \otimes Z(^-\Delta_1) \times Z(\Delta_2)] \\ & - [b(\Delta_1) \otimes Z(^-\Delta_1 \cap \Delta_2) \times Z(^-\Delta_1 \cup \Delta_2)] = A.\end{aligned}$$

We have therefore proven $[r(Z(\Delta_1 + \Delta_2))] \geq A$.
Since Δ_2 precedes Δ_1 and $l(\Delta_2) > 1$, we see that $b(\Delta_2)$ and Δ_1 are unlinked and so we see

$$Z(\Delta_1 + \Delta_2) \hookrightarrow Z(\Delta_1) \times b(\Delta_2) \times Z(^-\Delta_2) \cong b(\Delta_2) \times Z(\Delta_1) \times Z(^-\Delta_2).$$

Furthermore, we know that $\mathrm{soc}(Z(\Delta_1) \times Z(^-\Delta_2)) = Z(\Delta_1 +^- \Delta_2)$ and hence

$$Z(\Delta_1 + \Delta_2) \hookrightarrow b(\Delta_2) \times Z(\Delta_1 +^- \Delta_2).$$

Again, using Frobenius reciprocity and induction, we deduce also $[r(Z(\Delta_1+\Delta_2))] \geq B$ and therefore $[r(Z(\Delta_1 + \Delta_2))] \geq A + B$; this finishes the proof.

□

This answers the question completely for the case of two segments. It also leads naturally to the following definition.

Definition 2.16. Given linked segments Δ, Δ' we call the pair $(\Delta^\cup = \Delta \cup \Delta', \Delta^\cap = \Delta \cap \Delta')$ their *union-intersection.* Given a multisegment $\mathfrak{m} = \sum_i \Delta_i$ we call the operation of replacing a pair $\Delta_i + \Delta_j$ of *linked* segments of $\mathfrak{m}$ by their union-intersection $\Delta_i \cup \Delta_j + \Delta_i \cap \Delta_j$ an *elementary operation.* We define a partial order on $\mathfrak{M}$ by letting $\mathfrak{n} \leq \mathfrak{m}$ if we can reach $\mathfrak{n}$ from $\mathfrak{m}$ by a sequence of elementary operations.

Definition 2.17. Given multisegments $\mathfrak{m} = \Delta_1 + \cdots + \Delta_n$ and $\mathfrak{n}$ we say that $\mathfrak{n}$ is *subordinate* to $\mathfrak{m}$, denoted $\mathfrak{n} \dashv \mathfrak{m}$, if we can obtain $\mathfrak{n}$ from $\mathfrak{m}$ by replacing some of the Δ_i with Δ_i^-. We also denote $\mathfrak{m}^- = \Delta_1^- + \cdots + \Delta_n^-$.

Note that this indeed defines a partial ordering on $\mathfrak{M}$ since the number of pairs of linked segments is strictly decreased by an elementary operation. It is also easy to see that the number of multisegments $\mathfrak{n}$ that are smaller than a given $\mathfrak{m}$ in this ordering is always finite. The main result of this section can now be stated as follows.

Theorem 2.18. *Given a multisegment $\mathfrak{m} = \Delta_1 + \cdots + \Delta_n$ we have the decomposition*

$$[Z(\Delta_1) \times \cdots \times Z(\Delta_n)] = \sum_{\mathfrak{n} \leq \mathfrak{m}} c_{\mathfrak{m},\mathfrak{n}} [Z(\mathfrak{n})],$$

where $c_{\mathfrak{n},\mathfrak{m}} \neq 0 \Leftrightarrow \mathfrak{n} \leq \mathfrak{m}$, and $c_{\mathfrak{m},\mathfrak{m}} = 1$.

The determination of the $c_{\mathfrak{m},\mathfrak{n}}$ is more difficult than what we prove here and involves the Kazhdan-Lusztig polynomials of the symmetric group $S_{\deg(\mathfrak{m})}$ with respect to the Bruhat order. (cf. [Gu20] section 3.3 and the references given there).

Proof. We prove the theorem in several steps: First we show $c_{\mathfrak{m},\mathfrak{m}} = 1$, then we show $c_{\mathfrak{m},\mathfrak{n}} > 0$ if $\mathfrak{n} \leq \mathfrak{m}$ and finally we prove also the converse, if $c_{\mathfrak{m},\mathfrak{n}} > 0$ then $\mathfrak{n} \leq \mathfrak{m}$.
We have already proven that for an arranged form $(\Delta_1, \ldots, \Delta_n)$ of a multisegment $\mathfrak{m}$ the representation $\zeta(\mathfrak{m}) = Z(\Delta_1) \times \ldots Z(\Delta_n)$ is SI with socle $Z(\mathfrak{m})$. Since the Grothendieck group $\mathscr{R}$ is commutative, this means that $c_{\mathfrak{m},\mathfrak{m}} = 1$ since $[Z(\Delta_1) \times \cdots \times Z(\Delta_n)]$ does not depend on the order of the Δ_i and so one can assume without loss of generality that $(\Delta_1, \ldots, \Delta_n)$ is an arranged form.
Assume now that we have multisegments $\mathfrak{m}, \mathfrak{n}, \mathfrak{u}$ with $\mathfrak{n} \leq \mathfrak{m}$. Then we claim $c_{\mathfrak{u},\mathfrak{n}} \geq c_{\mathfrak{u},\mathfrak{m}}$. Indeed by an obvious induction it is enough to show this in the case that we can obtain $\mathfrak{n}$ from $\mathfrak{m}$ by a single elementary operation $(\Delta, \Delta') \mapsto (\Delta \cup \Delta', \Delta \cap \Delta')$ of two linked segments Δ, Δ' appearing in $\mathfrak{m}$. From Proposition 2.15 we know that $Z(\Delta \cup \Delta') \times Z(\Delta \cap \Delta')$ is a composition factor of $Z(\Delta) \times Z(\Delta')$ and this immediately implies the claim.
What we have just shown in particular implies that for $\mathfrak{n} \leq \mathfrak{m}$ we have

$$c_{\mathfrak{m},\mathfrak{n}} \geq c_{\mathfrak{m},\mathfrak{m}} = 1.$$

The only thing we now have to still prove is that if $c_{\mathfrak{m},\mathfrak{n}} > 0$ then we must have $\mathfrak{n} \leq \mathfrak{m}$. For this we follow the outline of [Ze80] 7.4. Let $\mathfrak{m} = \Delta_1 + \cdots + \Delta_n$, we prove the statement by induction on $\deg \mathfrak{m}$. If $\deg \mathfrak{m} = 1$ then there is nothing to proof. As explained in [Ze80] 7.4. from $[Z(\mathfrak{n})] \leq [\zeta(\mathfrak{m})]$ we deduce that there exists a $\mathfrak{m}' \dashv \mathfrak{m}$ with $[Z(\mathfrak{n}^-)] \leq [\zeta(\mathfrak{m}')]$ and so by induction we have $\mathfrak{n}^- \leq \mathfrak{m}'$. So we are no reduced to the purely combinatorial claim that $\mathfrak{m}' \dashv \mathfrak{m}$ and $\mathfrak{n}^- \leq \mathfrak{m}'$ imply $\mathfrak{n} \leq \mathfrak{m}$ if $\mathrm{supp}(\mathfrak{n}) = \mathrm{supp}(\mathfrak{m})$. We prove this in two steps which we isolate as lemmas.

Lemma 2.19. *Given multisegments $\mathfrak{m}, \mathfrak{m}', \mathfrak{n}'$ such that $\mathfrak{m}' \dashv \mathfrak{m}$ and $\mathfrak{n}' \leq \mathfrak{m}'$ there exists $\mathfrak{n}$ such that $\mathfrak{n}' \dashv \mathfrak{n}$ and $\mathfrak{n} \leq \mathfrak{m}$.*

Proof of Lemma. Without loss of generality there is a $k \in \{0, \dots, n\}$ such that we can write:

$$\begin{aligned}
\mathfrak{m} &= \Delta_1 + \dots + \Delta_n \\
\mathfrak{m}' &= \Delta_1^- + \dots + \Delta_k^- + \Delta_{k+1} + \dots + \Delta_n \\
\mathfrak{n}' &= \Delta_1' + \dots + \Delta_m'.
\end{aligned}$$

By an obvious induction, we can assume that $\mathfrak{n}'$ is obtained from $\mathfrak{m}'$ by a single elementary operation. We now have to distinguish some cases.
If $\mathfrak{n}'$ is obtained from $\mathfrak{m}'$ by replacing $\Delta_i + \Delta_j$ $(i, j > k)$ with $\Delta_i \cup \Delta_j + \Delta_i \cap \Delta_j$, then we define $\mathfrak{n}$ by replacing $\Delta_i \cup \Delta_j + \Delta_i \cap \Delta_j$ appearing in $\mathfrak{n}$ by $\Delta_i + \Delta_j$. This $\mathfrak{n}$ then satisfies all conditions.
If $\mathfrak{n}'$ is obtained from $\mathfrak{m}'$ by replacing $\Delta_i^- + \Delta_j$ $(i \le k < j)$ by their union-intersection and Δ_i^- precedes Δ_j, then we compute:

$$\Delta_i^- \cap \Delta_j = (\Delta_i \cap \Delta_j) \setminus \{e(\Delta_i)\} = (\Delta_i \cap \Delta_j)^-, \quad \Delta_i^- \cup \Delta_j = \Delta_i \cup \Delta_j.$$

So we see that

$$\begin{aligned}
\mathfrak{n}' = \Delta_1^- + \dots + \Delta_{i-1}^- + (\Delta_i \cap \Delta_j)^- + \Delta_{i+1}^- + \dots + \Delta_k^- \\
+ \Delta_{k+1} + \dots + \Delta_{j-1} + \Delta_i \cup \Delta_j + \Delta_{j+1} + \dots + \Delta_n.
\end{aligned}$$

We define $\mathfrak{n}$ by replacing $(\Delta_i \cap \Delta_j)^-$ with $\Delta_i \cap \Delta_j$ and all other Δ_l^- with Δ_l in $\mathfrak{n}'$. Clearly $\mathfrak{n}' \dashv \mathfrak{n}$. There are now two cases, if $e(\Delta_i) \neq e(\Delta_j)$ then Δ_i also precedes Δ_j and so $\mathfrak{n} \le \mathfrak{m}$. If $e(\Delta_i) = e(\Delta_j)$ then $\Delta_i \supset \Delta_j$ and so $\Delta_i \cap \Delta_j = \Delta_j$, $\Delta_i = \Delta_i \cup \Delta_j$ and therefore $\mathfrak{n} = \mathfrak{m} \le \mathfrak{m}$.
The other cases, i.e. if Δ_i^- succeeds Δ_j or if we obtain $\mathfrak{n}'$ by replacing Δ_i^-, Δ_j^- by their union-intersection, work analogously to what we have shown now. ☐

Lemma 2.20. *Given multisegments* $\mathfrak{m}, \mathfrak{n}$ *with* $\operatorname{supp}(\mathfrak{m}) = \operatorname{supp}(\mathfrak{n})$ *and* $\mathfrak{n}^- \dashv \mathfrak{m}$*, we have* $\mathfrak{n} \le \mathfrak{m}$*.*

Proof of Lemma. From the definition of $\dashv$ we see that there must be a k such that

$$\begin{aligned}
\mathfrak{m} &= \Delta_1 + \dots + \Delta_n \\
\mathfrak{n} &= \Delta_1' + \dots + \Delta_m' \\
\mathfrak{n}^- &= (\Delta_1')^- + \dots + (\Delta_m')^- = \Delta_1^- + \dots + \Delta_k^- + \Delta_{k+1} + \dots + \Delta_n.
\end{aligned}$$

Since $\operatorname{supp}(\mathfrak{m}) = \operatorname{supp}(\mathfrak{n})$ we see that, as multisets, $\{e(\Delta_i') : i = 1, \dots, m\} = \{e(\Delta_i) : i = 1, \dots, k\}$. Hence in particular $m = k$ and $\mathfrak{n} = \Delta_1' + \dots + \Delta_k'$. A careful but straightforward consideration of the equality $(\Delta_1')^- + \dots + (\Delta_m')^- = \Delta_1^- + \dots + \Delta_k^- + \Delta_{k+1} + \dots + \Delta_n$ combined with $\operatorname{supp}(\mathfrak{m}) = \operatorname{supp}(\mathfrak{n})$, taking account of which segments might be empty in this sum, leads to the identity

$$\begin{aligned}
\mathfrak{m} = \Delta_1' + \dots + \Delta_i' + (\Delta_{i+1}')^- + \dots + (\Delta_k')^- \\
+ [e(\Delta_{i+1}'), e(\Delta_{i+1}')] + \dots + [e(\Delta_k'), e(\Delta_k')],
\end{aligned}$$

for some $1 \leq i \leq k$. It is clear that for $j = i+1, \ldots, k$ the segments $(\Delta'_j)^-, e(\Delta'_j)$ are linked and their union-intersection is simply Δ'_j and therefore it follows that $\mathfrak{n} \leq \mathfrak{m}$. $\square$

Returning to the proof of Theorem 2.18 we combine the two lemmas to find that in fact $\mathfrak{n} \leq \mathfrak{m}$ follows whenever $[Z(\mathfrak{n})] \leq [\zeta(\mathfrak{m})]$ holds and this finishes the proof. $\square$

A particular consequence of the theorem we have just proven is that the representations $[\zeta(\mathfrak{m})] = [Z(\Delta_1) \times \cdots \times Z(\Delta_n)]$ for all $\mathfrak{m} \in \mathfrak{M}$ actually form a *basis* of the Grothendieck group $\mathscr{R}$. This follows immediately from the theorem since by definition the irreducible representations $[Z(\mathfrak{m})]$ form a basis of $\mathscr{R}$ and the above result shows that the 'matrix' of coefficients one obtains when expanding all the $[\zeta(\mathfrak{m})]$ in this basis is *unitriangular* and hence invertible over $\mathbb{Z}$.

Corollary 2.21. *The ring $\mathscr{R}$ is isomorphic to $\mathbb{Z}[\Delta\colon \Delta \in \mathrm{Seg}]$ the ring of polynomials in Seg over $\mathbb{Z}$.*

Proof. As noted, we can write every element x of $\mathscr{R}$ in a unique way as a linear combination $x = \sum_{i=1}^{n} [\zeta(\mathfrak{m}_i)]$ for multisegments $\mathfrak{m}_i \in \mathfrak{M}$. But for a multisegment $\mathfrak{m} = \Delta_1 + \cdots + \Delta_k$ we have by definition of the multiplication in $\mathscr{R}$

$$[\zeta(\mathfrak{m})] = [Z(\Delta_1) \times \cdots \times Z(\Delta_k)] = [Z(\Delta_1)] \ldots [Z(\Delta_k)].$$

Thus we have shown that each element of $\mathscr{R}$ can be written in a unique way as a linear combination of monomials $[Z(\Delta_1)] \ldots [Z(\Delta_k)]$ which leads in the obvious way to an isomorphism $\mathscr{R} \to \mathbb{Z}[\Delta\colon \Delta \in \mathrm{Seg}]$. $\square$

2.5 Properties of Ladder Multisegments

In this section we want to introduce *ladder multisegments* and study some of their properties. For ease of notation and because it does not make a significant difference (remark 2.10), we fix a cuspidal $\rho \in \mathcal{C}$ and will tacitly understand all segment and multisegments appearing to be relative to this ρ, dropping the subscript ρ in segments. In particular, given a segment $\Delta = [a,b]_\rho$ we will interpret $b(\Delta) = a$, $e(\Delta) = b$ just as integers, since ρ is fixed.
We begin with a kind of multisegment that is at the centre of this article. First we need a small notational convention. If Δ, Δ' are two segments, we write $\Delta \ll \Delta'$, if $b(\Delta) < b(\Delta')$ and $e(\Delta) < e(\Delta')$ (where fixing ρ allows us to interpret $b(\Delta)$, etc. simply as integers).

Definition 2.22. Let $\mathfrak{l} = \Delta_1 + \cdots + \Delta_n$ be a multisegment. We call $\mathfrak{l}$ a *ladder* if there is an ordering $i_1, \ldots, i_n$ of $\{1, \ldots, n\}$ such that $\Delta_{i_1} \gg \Delta_2 \gg \cdots \gg \Delta_{i_n}$. Further, we call $\mathfrak{l}$ *Speh* if we have $\Delta_{i+1} = \overleftarrow{\Delta}_i$ for all $i = 1, \ldots, n-1$.

Unless we explicitly state otherwise, all ladders $\mathfrak{l} = \Delta_1 + \cdots + \Delta_n$ appearing in the following will be ordered in such a way that $\Delta_1 \gg \cdots \gg \Delta_n$. If we want to emphasize this, we say that the ladder $\mathfrak{l}$ is *canonically ordered*.

Definition 2.23. Let $\mathfrak{m} = \Delta_1 + \cdots + \Delta_n$ be a multisegment and without loss of generality assume that the Δ_i are ordered such that $b(\Delta_1) \leq b(\Delta_2) \leq \cdots \leq b(\Delta_n)$. Then we can find indices $i_1 < \cdots < i_l$ such that:

$$b(\Delta_1) = \cdots = b(\Delta_{i_1}) < b(\Delta_{i_1+1}) = \cdots = b(\Delta_{i_2}) < \cdots < b(\Delta_{i_l+1}) = \cdots = b(\Delta_n)$$

We define for each $k = 0, \dots, l$ the multisegment

$$\mathfrak{m}_k := \sum_{i=i_k+1}^{i_{k+1}} \Delta_i,$$

and finally the *indicator representation* $Z(\mathfrak{m})_\otimes$ of $\mathfrak{m}$ as

$$Z(\mathfrak{m})_\otimes := Z(\mathfrak{m}_0) \otimes \cdots \otimes Z(\mathfrak{m}_l).$$

Remark 2.24. The indicator representation defined here is very similar to the construction used in the proof of proposition 2.8. The only real difference is in the order of the blocks: here we chose to order the $\mathfrak{m}_i$ with *ascending* beginnings, in the proof of proposition 2.8 we ordered the ends *decreasingly*. The point there was to obtain arranged forms which can not be ensured with the order that we have now defined for the indicator representation.

Notation 2.25. Given a multisegment $\mathfrak{m} = \sum_{i \in I} \Delta_i$ we let $\{b(\Delta_i) : i \in I\} = \{\lambda_1, \dots, \lambda_n\}$, $\{e(\Delta_i) : i \in I\} = \{\mu_1, \dots, \mu_n\}$ (as multisets) where we have ordered the λ, μ such that $\lambda_1 \leq \cdots \leq \lambda_n$ and $\mu_1 \leq \cdots \leq \mu_n$. There is therefore a permutation $\sigma \in S_n$ (not necessarily unique) such that

$$\mathfrak{m} = \sum_{i=1}^{n} [\lambda_i, \mu_{\sigma(i)}].$$

Furthermore, we can parameterize *all* multisegment whose beginnings and ends are exactly given by the λ_i, μ_i as multisets using permutations $x \in S_n$. Indeed, for $x \in S_n$ such that $\lambda_i \leq \mu_{x(i)} + 1$ (we allow here empty segments of the form $[i, i-1]$ which we think of as producing the trivial representation 1 of the trivial group G_0) we get a multisegment

$$\mathfrak{m}(x) := \sum_{i=1}^{n} [\lambda_i, \mu_{x(i)}].$$

Strictly speaking, we should not consider this to be defined by a permutation $x \in S_n$ but rather by a double coset in $S_\lambda \backslash S_n / S_\mu$ that takes account of possible multiplicities in λ, μ. We denote by $S_{\lambda,\mu}$ the subset of $S_\lambda \backslash S_n / S_\mu$ consisting of double cosets $S_\lambda x S_\mu$ where $\lambda_i \leq \mu_{x(i)} + 1$ for $i = 1 \dots, n$.

Definition 2.26. We say that a permutation $x \in S_n$ is 321-avoiding, if there is no triple $(i, j, k) \in \{1, \dots, n\}$ with $i < j < k$ and $x(i) > x(j) > x(k)$. Given subgroups $G, H \leq S_n$ we call a double coset $GxH \in G \backslash S_n / H$ 321-avoiding, if all representatives of GxH in S_n are 321-avoiding.

The following two theorems allow us to compute algorithmically the irreducible subquotients of a product $Z(\mathfrak{l}_1) \times Z(\mathfrak{l}_2)$ of two ladder representations.

Theorem 2.27 (cf. [Gu20] Corollary 5.4.)**.** *Given two ladders $\mathfrak{l}_1, \mathfrak{l}_2$ and σ a representation with $[\sigma] \leq [Z(\mathfrak{l}_1) \times Z(\mathfrak{l}_2)]$, then σ appears with multiplicity 1 in $[Z(\mathfrak{l}_1) \times Z(\mathfrak{l}_2)]$.*

Theorem 2.28 (cf. [Gu21] Theorem 4.1.)**.** *Let $\mathfrak{l}_1, \mathfrak{l}_2$ be ladders, set $\mathfrak{m} := \mathfrak{l}_1 + \mathfrak{l}_2$ and denote by $\lambda = \{\lambda_1, \dots, \lambda_n\}$ resp. $\mu = \{\mu_1, \dots, \mu_n\}$ the multisets of beginnings resp. endings of the segments of $\mathfrak{m}$ ordered increasingly. Then for a permutation $\sigma \in S_{\lambda,\mu}$ the following are equivalent:*

1. *σ is 321-avoiding and $Z(\mathfrak{m}(\sigma))_{\otimes}$ occurs as a subquotient of the Jacquet module $r_{\alpha_\sigma}(Z(\mathfrak{l}_1) \times Z(\mathfrak{l}_2))$*
2. *$Z(\mathfrak{m}(\sigma))$ appears as a subquotient of $Z(\mathfrak{l}_1) \times Z(\mathfrak{l}_2)$.*

Here α_σ is the partition which satisfies $Z(\mathfrak{m}(\sigma)) \in \mathrm{Irr}(G_{\alpha_\sigma})$.

These theorems give a complete description of the composition factors of the product of two ladder representations. In the case that there are no multiplicities in the beginnings of the $\mathfrak{m} = \mathfrak{l}_1 + \mathfrak{l}_2$ we can give a completely combinatorial condition that is easy to check and implement in computer assisted searching.
We begin by slightly reformulating a result of Kret and Lapid on the Jacquet modules of ladder representations.

Proposition 2.29 (cf. [KL12] Corollary 2.2.)**.** *Let $\mathfrak{l} = \Delta_1 + \cdots + \Delta_n$ be a ladder multisegment with $\Delta_1 \gg \Delta_2 \gg \cdots \gg \Delta_n$ and $\Delta_i = [a_i, b_i]$. Further, let $\alpha = (n_1, \dots, n_k)$ be a partition of $\deg(\mathfrak{l})$. Then the Jacquet module $r_\alpha(Z(\mathfrak{l}))$ is*

$$\bigoplus_C Z([c_1^1, c_1^2 - 1] + \cdots + [c_n^1, c_n^2 - 1]) \otimes \cdots \otimes Z([c_1^k, c_1^{k+1} - 1] + \cdots + [c_n^k, c_n^{k+1} - 1]).$$

Here the sum is over all matrices $C = (c_i^j)_{i=1,\dots,n}^{j=1,\dots,k+1}$ such that

(i) For all $1 \leq i \leq n$ we have $a_i = c_i^1 \leq c_i^2 \leq \cdots \leq c_i^{k+1} = b_i + 1$,

(ii) For every $1 \leq j \leq k$ and all $1 \leq i_1 < i_2 \leq n$ we have $c_{i_1}^{j+1} < c_{i_2}^j$,

(iii) For all $1 \leq j \leq k$ we have $\sum_{i=1}^n (c_i^{j+1} - c_i^j) = \deg(\rho) n_j$

Proof. The statement is essentially [KL12] Corollary 2.2. but replacing the colorings with matrices. Following the notation of [KL12], we set $f(j,i) = l$ if and only if $j \in [c_i^l, c_i^{l+1} - 1]$ for all $1 \leq i \leq n$, $1 \leq l \leq k$ and $j \in \Delta_i$. This gives a correspondence between the colorings f of [KL12] and the matrices C in our statement. □

We remark that the conditions on the matrices in the proposition guarantee that all the $[c_1^j, c_1^{j+1} - 1] + \cdots + [c_n^j, c_n^j + 1 - 1]$ appearing in $r_\alpha(\mathfrak{l})$ are also ladders. Jacquet modules of ladders are therefore also made up of ladders.

With this in hand we can employ the Geometric Lemma to make the condition that $Z(\mathfrak{m}(\sigma))_{\otimes}$ occur as a subquotient of the Jacquet module $r_{\alpha(\sigma)}(Z(\mathfrak{l}_1)\times Z(\mathfrak{l}_2))$ from theorem 2.28 completely explicit. We emulate [Gu21] Proposition 3.4. but prove a slightly more general statement. For clarity, we isolate a small technical argument as a lemma.

Lemma 2.30. *Let $\Delta = [a,b]$ be a segment and let $\mathfrak{l}_1, \mathfrak{l}_2$ be ladders. Then we have $[Z(\Delta)] \leq [Z(\mathfrak{l}_1)\times Z(\mathfrak{l}_2)]$ if and only if both $\mathfrak{l}_i$, $i=1,2$ are totally unlinked and* $\mathrm{supp}(\mathfrak{l}_1)\sqcup \mathrm{supp}(\mathfrak{l}_2) = [a,b]$.

Proof. We first show that if $\mathfrak{l}_1, \mathfrak{l}_2$ are each totally unlinked and $\mathrm{supp}(\mathfrak{l}_1)\sqcup\mathrm{supp}(\mathfrak{l}_2) = [a,b]$ then $[Z(\Delta)] \leq [Z(\mathfrak{l}_1)\times Z(\mathfrak{l}_2)]$ holds.
Since $\mathfrak{l}_1, \mathfrak{l}_2$ are each totally unlinked we find by 2.7 that

$$Z(\mathfrak{l}_1) = Z(\Delta_1)\times\cdots\times Z(\Delta_n), \quad Z(\mathfrak{l}_2) = Z(\Delta_1')\times\cdots\times Z(\Delta_m'),$$

where $\mathfrak{l}_1 = \Delta_1+\cdots+\Delta_n$, $\mathfrak{l}_2 = \Delta_1'+\cdots+\Delta_m'$. But then it follows that

$$Z(\mathfrak{l}_1)\times Z(\mathfrak{l}_2) = Z(\Delta_1)\times\cdots\times Z(\Delta_n)\times Z(\Delta_1')\times\cdots\times Z(\Delta_m'),$$

and since $\mathrm{supp}(\mathfrak{l}_1)\sqcup\mathrm{supp}(\mathfrak{l}_2) = [a,b]$ we see that $[a,b]\leq \Delta_1+\cdots+\Delta_n+\Delta_1'+\cdots+\Delta_m'$ in the union-intersection order (Definition 2.16) and so Theorem 2.18 implies

$$[Z([a,b])] \leq [Z(\mathfrak{l}_1)\times Z(\mathfrak{l}_2)].$$

We will prove the converse statement by induction on $l([a,b])$. If $a=b$ then $[\rho|.|^a] = [Z([a,a])] \leq [Z(\mathfrak{l}_1)\times Z(\mathfrak{l}_2)]$ means that $\mathrm{supp}(Z(\mathfrak{l}_1))\cup\mathrm{supp}(Z(\mathfrak{l}_2)) = \{\rho|.|^a\}$ and this implies that $\mathfrak{l}_1 = [a,a]$ and $\mathfrak{l}_2 = 0$ or vice-versa. In either case the claim of the lemma holds. Now assume we have proven the statement for all segments of smaller length than $l([a,b])$ and set $d := \deg\rho$. We note that the condition $\mathrm{supp}(\mathfrak{l}_1)\sqcup\mathrm{supp}(\mathfrak{l}_2) = [a,b]$. is clear and we can assume w.l.o.g. that $\rho|.|^a\in\mathrm{supp}(\mathfrak{l}_1)$. From exactness of the Jacquet functors we obtain

$$[r_{(d,(b-a)d)}(Z([a,b]))] \leq [r_{(d,(b-a)d)}(Z(\mathfrak{l}_1)\times Z(\mathfrak{l}_2))].$$

From Proposition 2.4 we obtain $r_{(d,(b-a)d)}(Z([a,b])) = \rho|.|^a\otimes Z([a+1,b])$. Furthermore, we can use the Geometric Lemma to decompose

$$r_{(d,(b-a)d)}(Z(\mathfrak{l}_1)\times Z(\mathfrak{l}_2)) = \rho|.|^a\otimes(Z(^-\mathfrak{l}_1)\times Z(\mathfrak{l}_2)) + \sum_i \sigma_i\otimes\pi_i, \tag{3}$$

where $^-\mathfrak{l}_1$ is the multisegment obtained from $\mathfrak{l}_1$ by deleting $\rho|.|^a$ and σ_i, ρ_i are representations such that no σ_i allows $\rho|.|^a$ as an irreducible subquotient. For more details on this type of construction, see [AM20]. The only way that (3) can hold is therefore that we have

$$[Z([a+1,b])] \leq [Z(^-\mathfrak{l}_1)\times Z(\mathfrak{l}_2)]$$

and by induction we see that $^-\mathfrak{l}_1, \mathfrak{l}_2$ are each totally unlinked. This implies that also $\mathfrak{l}_1$ is totally unlinked and this finishes the proof. ∎

Proposition 2.31 (cf. [Gu21] Proposition 3.4.). *Let $\mathfrak{l}_1, \mathfrak{l}_2$ be ladders and assume we can write $\mathfrak{m} := \mathfrak{l}_1 + \mathfrak{l}_2$ as $\mathfrak{m} = \sum_{i=1}^{n} [\lambda_i, \mu_{x(i)}]$ for a permutation $x \in S_n$ and $\lambda_1 < \cdots < \lambda_n$, $\mu_1 \leq \cdots \leq \mu_m$. Furthermore we let $J_1, J_2 \subset \{1, \ldots, n\}$ be such that $\mathfrak{l}_j = \sum_{i \in J_j} [\lambda_i, \mu_{x(i)}]$ for $j = 1, 2$. Then a permutation $\sigma \in S_{\lambda,\mu}$ satisfies the conditions 1. of Theorem 2.28 if and only if it is 321-avoiding and we can find a matrix $C = (c_i^j)_{i=1,\ldots,n}^{j=1,\ldots,n+1} \in \mathbb{Z}^{n\times(n+1)}$ with the following properties:*

(i) For every $i = 1, \ldots, n$ we have $\lambda_i = c_i^1 \leq \cdots \leq c_i^{n+1} = \mu_{x(i)} + 1$.

(ii) If $i_1 < i_2$ with either $i_1, i_2 \in J_1$ or $i_1, i_2 \in J_2$ we have for all $j = 1, \ldots, n$: $c_{i_1}^{j+1} < c_{i_2}^{j}$.

(iii) For every $j = 1, \ldots, n$ we have a disjoint union

$$\bigsqcup_{i=1}^{n} [c_i^j, c_i^{j+1} - 1] = [\lambda_{n+1-j}, \mu_{\sigma(n+1-j)}]$$

Proof. Assume that $[Z(\mathfrak{m}(\sigma))_\otimes] \leq [r_{\alpha_\sigma}(Z(\mathfrak{l}_1) \times Z(\mathfrak{l}_2))]$ and write $Z(\mathfrak{m}(\sigma)) = \tau_n \otimes \cdots \otimes \tau_1$ where $\tau_j = Z([\lambda_{n+1-j}, \mu_{\sigma(n+1-j)}])$. Note that some of the τ_j may be trivial if $\lambda_{n+1-j}+1 = \mu_{\sigma(n+1-j)}$.
From Proposition 2.29 and the Geometric Lemma, we know that $[Z(\mathfrak{m}(\sigma))_\otimes] \leq [r_{\alpha_\sigma}(Z(\mathfrak{l}_1) \times Z(\mathfrak{l}_2))]$ is equivalent to the existence of a matrix $C = (c_i^j) \in \mathbb{Z}^{n\times(n+1)}$ such that $Z(\mathfrak{m}(\sigma))_\otimes$ is a subquotient of $(\pi_1^n \times \pi_2^n) \otimes \cdots \otimes (\pi_1^1 \times \pi_2^1)$, where for $1 \leq j \leq n$:

$$\pi_1^j = Z\left(\sum_{i \in J_1} [c_i^j, c_i^{j+1} - 1]\right), \quad \pi_1^j = Z\left(\sum_{i \in J_2} [c_i^j, c_i^{j+1} - 1]\right).$$

Proposition 2.29 also gives us conditions that the matrix C must satisfy:

1. For all $i = 1, \ldots, n$ we have $\lambda_i = c_i^1 \leq \cdots \leq c_i^{n+1} = \mu_{x(i)} + 1$,

2. For all $i_1 < i_2$ such that either $i_1, i_2 \in J_1$ or $i_1, i_2 \in J_2$ and all $j = 1, \ldots, n$ we have $c_{i_1}^j < c_{i_2}^j$.

In particular, all the π_i^j for $i = 1, 2,\ j = 1, \ldots, n$ are ladder representations. We thus only need to show that for i_1, i_2, j as immediately above we have $c_{i_1}^{j+1} < c_{i_2}^j$ and that $\bigsqcup [c_i^j, c_i^{j+1} - 1] = [\lambda_{n+1-j}, \mu_{\sigma(n+1-j)}]$. Fix now $j \in \{1, \ldots, n\}$. Then from Lemma 2.30 we see that $\tau_j = Z([\lambda_{n+1-j}, \mu_{\sigma(n+1-j)}])$ being a subquotient of

$$Z\left(\sum_{i \in J_1} [c_i^j, c_i^{j+1} - 1]\right) \times Z\left(\sum_{i \in J_2} [c_i^j, c_i^{j+1} - 1]\right),$$

is equivalent to π_1^j, π_2^j each being totally unlinked and

$$\bigsqcup_{i=1}^{n} [c_i^j, c_i^{j+1} - 1] = [\lambda_{n+1-j}, \mu_{\sigma(n+1-j)}].$$

But these two conditions combined also imply $c_{i_1}^{j+1} < c_{i_2}^{j}$ if $i_1 < i_2$ and $i_1, i_2 \in J_1$ or $i_1, i_2 \in J_2$ since otherwise the representations π_1^j, π_2^j could not be totally unlinked. Thus we have checked all the conditions *(i)-(iii)* if $[Z(\mathfrak{m}(\sigma))_\otimes] \leq [r_{\alpha_\sigma}(Z(\mathfrak{l}_1) \times Z(\mathfrak{l}_2))]$. Now we prove also the converse.
So assume now that we have a permutation σ and a matrix C satisfying the conditions of the proposition. Then we define

$$\pi_1^j = Z\left(\sum_{i \in J_1} [c_i^j, c_i^{j+1} - 1]\right), \quad \pi_1^j = Z\left(\sum_{i \in J_2} [c_i^j, c_i^{j+1} - 1]\right).$$

From Proposition 2.29 and the Geometric Lemma, we then see that it is enough to prove $[Z([\lambda_{n+1-j}, \mu_{\sigma(n+1-j)}])] \leq [\pi_1^j \times \pi_2^j]$ which by Lemma 2.30 is equivalent to π_1^j, π_2^j each being totally unlinked and $\mathrm{supp}(\pi_1^j) \sqcup \mathrm{supp}(\pi_2^j) = [\lambda_{n+1-j}, \mu_{\sigma(n+1-j)}]$. But this is guaranteed by the conditions *(i)-(iii)* imposed on the matrix C. This finishes the proof. $\square$

3 RSK Correspondence

We give a formulation of the RSK correspondence following [GL21]. To make the exposition clearer, we will restrict ourselves to working only with multisegments relative to some fixed $\rho \in \mathcal{C}$ and thus view segments simply as subsets of $\mathbb{Z}$. By remark 2.10 this is no serious restriction of generality. Our goal in this section will be to associate to each multisegment $\mathfrak{m}$ a sequence $(\mathfrak{l}_1, \ldots, \mathfrak{l}_k)$ of ladder multisegments in a particular way.

Definition 3.1. Let $\mathfrak{m} = \sum_{i \in I} \Delta_i$ be a multisegment. Then we define the *depth* $\mathfrak{d}\colon I \to \mathbb{N}$ in $\mathfrak{m}$ as follows:

$$\mathfrak{d}(i) := \max\{k \colon \exists i_1, \ldots, i_k \in I \text{ s.t. } \Delta_i \ll \Delta_{i_1} \ll \cdots \ll \Delta_{i_k}\}.$$

In other words, the depth of a segment Δ_i in $\mathfrak{m}$ is the length of the longest ascending chain w.r.t. $\ll$ starting at Δ_i. Note that if $\mathfrak{d}(i) = \mathfrak{d}(j)$ we must have either $\Delta_i \subseteq \Delta_j$ or $\Delta_j \subseteq \Delta_i$. We further set $\mathfrak{d}(\mathfrak{m}) = \max_{i \in I} \mathfrak{d}(i)$.

Definition 3.2. Given two ladders $\mathfrak{l}_1 = \sum_{k=1}^{n} \Delta_k . \mathfrak{l}_2 = \sum_{k=1}^{m} \Delta'_k$ both canonically ordered, we say that $\mathfrak{l}_2$ is *dominant* w.r.t. $\mathfrak{l}_1$ if $m \geq n$ and for all $k = 1, \ldots, n$ we have $\Delta_i \ll \overleftarrow{\Delta'_i}$. We also say then that the pair $(\mathfrak{l}_2, \mathfrak{l}_1)$ is dominant.
If $(\mathfrak{l}_2, \mathfrak{l}_1)$ is dominant and for all i, j such that $\Delta_i \ll \Delta'_j$ and j is maximal w.r.t. this property (implying $j \geq i$ by dominance), we have $e(\Delta_r) \geq b(\Delta'_{j-i+r})$ for all $r = 1, \ldots, i$, then we say that the pair $(\mathfrak{l}_1, \mathfrak{l}_2)$ is *permissible*.
We denote by $\mathcal{L}$ the set of tuples $(\mathfrak{l}_1, \ldots, \mathfrak{l}_n)$ such that for $1 \leq i < j \leq n$ the pair $(\mathfrak{l}_i, \mathfrak{l}_j)$ is a permissible pair of ladders.

3.1 Knuth Algorithm

Having set up the objects involved, we now give a description of the algorithm that turns a multisegment $\mathfrak{m}$ into a sequence of ladders in $\mathcal{L}$. It is an adaption of the algorithm described in [Kn70] section 4 and we follow the description in [GL21] section 2.2.2.
Let $\mathfrak{m} = \sum_{i \in I} \Delta_i$ be a multisegment with $\mathfrak{d}(I) = \{0, \ldots, d\}$. For each $k = 0, \ldots, d$ enumerate $\mathfrak{d}^{-1}(k) = \{i_1^k, \ldots, i_{l_k}^k\}$ such that $\Delta_{i_1^k} \supseteq \cdots \supseteq \Delta_{i_{l_k}^k}$, and set $j_k := i_{l_k}^k$.
Let σ be the permutation of I with cycle decomposition $\sigma = \prod_{k=0}^{d} (i_1^k, \ldots, i_{l_k}^k)$ and define for all $i \in I$

$$\Delta'_i := [b(\Delta_i), e(\Delta_{\sigma(i)})].$$

Setting $I^\sharp := \{j_0, \ldots, j_d\}$ and $I' := I \setminus I^\sharp$ we define multisegments

$$\mathfrak{l}(\mathfrak{m}) := \sum_{i \in I^\sharp} \Delta'_i, \quad \mathfrak{m}' := \sum_{i \in I'} \Delta'_i$$

and set $\mathcal{K}(\mathfrak{m}) := (\mathfrak{l}(\mathfrak{m}), \mathfrak{m}')$. Inductively we construct $\mathrm{RSK}(0) = \emptyset$, $\mathrm{RSK}(\mathfrak{m}) = (\mathfrak{l}(\mathfrak{m}), \mathrm{RSK}(\mathfrak{m}'))$. We claim that this defines a map $\mathrm{RSK}\colon \mathfrak{M} \to \mathcal{L}$. We prove this in two steps. First, let $\mathfrak{A}$ denote the set of pairs $(\mathfrak{l}, \mathfrak{m})$ of ladders and multisegments that are *permissible*, meaning that for each submultisegment $\mathfrak{l}'$ of $\mathfrak{m}$ that is a ladder, the pair $(\mathfrak{l}, \mathfrak{l}')$ is permissible.

Proposition 3.3. *The map $\mathcal{K}$ defines a bijection $\mathcal{K}\colon \mathfrak{M}\setminus\{0\}\to\mathfrak{A}$.*

Before we prove this, we cite a technical lemma without proof.

Lemma 3.4 (cf. [GL21] Corollary 2.3.)**.** *With the notation as above, for $i\in I^\sharp$ we have $\mathfrak{d}_{\mathfrak{m}'}(i)\leq\mathfrak{d}_{\mathfrak{m}}(i)$.*

Proof of Proposition 3.3. Fix a multisegment $\mathfrak{m}\neq 0$ and keep the notation of the previous discussion. First we show that $\mathfrak{l}(\mathfrak{m})$ is indeed a ladder. Let $I^\sharp=\{j_0,\dots,j_d\}$ with $\mathfrak{d}_{\mathfrak{m}}(\Delta_{j_k})=k$ for $k=0,\dots,d$ and so $\mathfrak{l}(\mathfrak{m})=\sum_{k=0}^d\Delta'_{j_k}$. For $i\in I^\sharp$ we have by construction

$$b(\Delta_i')=\max_{j\in I\colon\mathfrak{d}(i)=\mathfrak{d}(j)}b(\Delta_j),\quad e(\Delta_i')=\max_{j\in I\colon\mathfrak{d}(j)=\mathfrak{d}(i)}e(\Delta_j).\tag{4}$$

Taking $1\leq k\leq d$ there must be a segment Δ_j with $\mathfrak{d}(j)=k-1$ and $\Delta_{j_k}\ll\Delta_j$. This implies

$$b(\Delta'_{j_k})=b(\Delta_{j_k})<b(\Delta_j)\leq b(\Delta'_{j_{k-1}}).$$

In similar fashion one shows $e(\Delta'_{j_k})<e(\Delta'_{j_{k-1}})$ and therefore we have all together shown $\Delta'_{j_k}\ll\Delta'_{j_{k-1}}$ and so $\mathfrak{l}(\mathfrak{m})$ is a ladder.
Next we prove that $\mathcal{K}(\mathfrak{m})=(\mathfrak{l}(\mathfrak{m}),\mathfrak{m}')$ is permissible. By (4), for any $j\in I$ and $i\in I^\sharp$ such that $\mathfrak{d}(j)\geq\mathfrak{d}(i)$ we have $\overleftarrow{\Delta}_j\ll\Delta_i'$ and so in particular $\overleftarrow{\Delta'_j}\ll\Delta'_i$ for all such j,i. Together with Lemma (3.4) this shows that $(\mathfrak{l}(\mathfrak{m}),\mathfrak{m}')$ is dominant.
For any $i,j\in I$ we have $\Delta_i\subseteq\Delta_j$ or $\Delta_j\subseteq\Delta_i$ if $\mathfrak{d}(i)=\mathfrak{d}(j)$ and therefore we have $b(\Delta_i)\leq e(\Delta_j)$ whenever $\mathfrak{d}(i)=\mathfrak{d}(j)$. Again from (4) and the fact that $\mathfrak{l}(\mathfrak{m})$ is a ladder, we thus get $e(\Delta'_j)\geq b(\Delta'_i)$ for all $i\in I^\sharp$, $j\in I$ with $\mathfrak{d}(j)\leq\mathfrak{d}(i)$. Appealing again to Lemma 3.4 it follows that $(\mathfrak{l}(\mathfrak{m}),\mathfrak{m}')=\mathcal{K}(\mathfrak{m})$ is permissible. The map $\mathcal{K}:\mathfrak{M}\to\mathfrak{A}$ is therefore well defined.
To show that this map is a bijection, we construct an inverse $\mathcal{K}^*$.
So take a permissible pair $(\mathfrak{l},\mathfrak{m})\in\mathfrak{A}$ with $\mathfrak{l}=\sum_{j=1}^m\Delta_j$ with $\Delta_{j+1}\ll\Delta_j$ for $j=1,\dots,m-1$ and $\mathfrak{m}=\sum_{i\in I}\Delta_i$. We set $J:=\{1,\dots,m\}$ and the index sets I,J are disjoint.
We define maps (depending on $\mathfrak{l}$ and $\mathfrak{m}$) $f,g:I\to J$ for $i\in I$:

$$\begin{aligned}g(i)&:=\max\{j\in J\colon\overleftarrow{\Delta}_i\ll\Delta_j\},\\ f(i)&:=\min\{g(i_k)-k\colon\Delta_i\gg\Delta_{i_1}\gg\cdots\gg\Delta_{i_k},\,i_1,\dots,i_k\in I\}.\end{aligned}$$

Equivalently we could define f recursively by $f(i):=\min(g(i),\{f(j)-1\colon j\in I,\,\Delta_j\ll\Delta_i\})$. The permissibility of $(\mathfrak{l},\mathfrak{m})$ ensures that f,g are well-defined. For $j\in J$ we set $Y_j=f^{-1}(j)$ and write $Y_j=\{i_1^{(j)},\dots,i_{k_j}^{(j)}\}$ (where we could have $k_j=0$ and $Y_j=\emptyset$) such that $\Delta_{i^{(j)}_{r+1}}\subseteq\Delta_{i^{(j)}_r}$ for $r=1,\dots,k_j-1$. We can do this, since if two segments Δ_i,Δ_l satisfy neither $\Delta_i\subseteq\Delta_l$ not $\Delta_i\subseteq\Delta_l$ we must have either $\Delta_i\ll\Delta_l$ or $\Delta_l\ll\Delta_i$ and from the definition of f it follows that $f(i)\neq f(l)$. The definition of f and the permissibility of $(\mathfrak{l},\mathfrak{m})$ then further imply that for all $r=1,\dots,k_j$ we have $b(\Delta_{r^{(j)}})\leq b(\Delta_j)\leq e(\Delta_{r^{(j)}})$

and $e(\Delta_j) \geq e(\Delta_{i_1^{(j)}})$.
We let σ be the permutation of $I \cup J$ with cycle decomposition $\prod_{j \in J}(i_1^{(j)}, \dots, i_{k_j}^{(j)}, j)$. For any $i \in I \cup J$ we define

$$\Delta_i^* := [b(\Delta_{\sigma(i)}), e(\Delta_i)].$$

Note that we now have for all $j \in J$: $\Delta^*_{i_1^{(j)}} \supseteq \Delta^*_{i_2^{(j)}} \supseteq \cdots \supseteq \Delta^*_{i_{k_j}^{(j)}} \supseteq \Delta_j^*$.
Finally, we define $\mathcal{K}^*(\mathfrak{l}, \mathfrak{m}) := \sum_{i \in I \cup J} \Delta_i^*$. One can then check that the segments of equal depth are given precisely by the sets $\{i_1^{(j)}, \dots, i_{k_j}^{(j)}, j\}$ and from the definition of Δ^* it is clear that this operation is indeed inverse to $\mathcal{K}$. This finishes the proof and gives an algorithmic description of $\mathcal{K}^{-1} = \mathcal{K}^*$. $\square$

For one step in our construction we therefore have a complete understanding of what can happen. For the whole map RSK our understanding of the image is not as complete, however we do have the following result

Proposition 3.5. *The map* RSK: $\mathfrak{M} \to \mathcal{L}$ *is well-defined. This amounts to saying that, given a multisegment* $\mathfrak{m}$ *with* $\mathrm{RSK}(\mathfrak{m}) = (\mathfrak{l}_1, \dots, \mathfrak{l}_k)$, *all pairs* $(\mathfrak{l}_i, \mathfrak{l}_j)$ *with* $i < j$ *are permissible.*

Proof. Let $(\mathfrak{l}, \mathfrak{m}) \in \mathfrak{A}$ be a permissible pair of a ladder $\mathfrak{l} = \Delta_0 + \cdots + \Delta_n$ in canonical order and a multisegment $\mathfrak{m} = \sum_{i \in I} \Delta_i$ with $\{0, \dots, n\} \cap I = \emptyset$. Then it is enough to prove that $(\mathfrak{l}, \mathfrak{l}(\mathfrak{m})), (\mathfrak{l}, \mathfrak{m}')$ are also permissible.
We begin with $(\mathfrak{l}, \mathfrak{l}(\mathfrak{m}))$. Let $i \in I^\sharp$ and $j \in \{0, \dots, n\}$ maximal with the property $\overleftarrow{\Delta'_i} \ll \Delta_j$. By construction of $\mathcal{K}$ it is then clear that $\overleftarrow{\Delta_i}, \overleftarrow{\Delta_{\sigma(i)}} \ll \Delta_j$. If $j \neq n$ then by the maximality of j we cannot have $\overleftarrow{\Delta_i} \ll \Delta_{j+1}$ and $\overleftarrow{\Delta_{\sigma(i)}} \ll \Delta_{j+1}$ at the same time (since otherwise $\overleftarrow{\Delta'_i} \ll \Delta_{j+1}$). Let $i' \in \{i, \sigma(i)\}$ be an index with $\overleftarrow{\Delta_{i'}} \not\ll \Delta_{j+1}$. If $j = n$ we just let i' be i.
In either case, there is a sequence $k_0, \dots, k_s = i'$ with $s = \mathfrak{d}(i)$ such that $\Delta_{k_s} \ll \cdots \ll \Delta_{k_0}$ and therefore $\mathfrak{d}(k_r) = r$ for all $r = 0, \dots, s$. Since $(\mathfrak{l}, \mathfrak{m})$ is permissible, for any $r \leq s$ we have $e(\Delta_{k_r}) \geq b(\Delta_{j-s+r})$ which implies $e(\Delta'_k) \geq b(\Delta_{j-s+r})$ for $k \in I^\sharp$ with $\mathfrak{d}(k) = r$. This shows permissibility of $(\mathfrak{l}, \mathfrak{l}(\mathfrak{m}))$.
Now we turn to proving permissibility of $(\mathfrak{l}, \mathfrak{m}')$. Consider a subladder $\mathfrak{l}_1$ of $\mathfrak{m}'$, i.e. pick indices $i_0, \dots, i_k \in I'$ with $\Delta'_{i_k} \ll \cdots \ll \Delta'_{i_0}$ and set $\mathfrak{l}_1 = \Delta'_{i_0} + \cdots + \Delta'_{i_k}$. We claim that we can find segments $\Delta'_{j_k} \ll \cdots \ll \Delta'_{j_0}$ with $j_0, \dots, j_k \in I'$ such that $\mathfrak{d}_{\mathfrak{m}}(j_r) = \mathfrak{d}_{\mathfrak{m}}(i_r)$ and $\Delta_{j_r} \subseteq \Delta'_{i_r}$ for all $r = 0, \dots, k$ and $e(\Delta_{j_k}) = e(\Delta'_{i_k})$.
We construct the j_r by descending induction. For j_k we can pick $\sigma(i_k)$ (with σ as in the definition of $\mathcal{K}$): $\mathfrak{d}_{\mathfrak{m}}(\sigma(i)) = \mathfrak{d}_{\mathfrak{m}}(i)$ and $\Delta_{\sigma(i_k)} \subseteq \Delta'_{i_k}$ and $e(\Delta'_{i_k}) = e(\Delta_{\sigma(i_k)})$ hold by definition of σ and $\mathfrak{m}'$. Suppose we have constructed j_r for $r > 0$. If $\Delta_{j_r} \ll \Delta_{\sigma(i_{r-1})}$ then we set $j_{r-1} := \sigma(i_{r-1})$. Otherwise, since $e(\Delta_{\sigma(i_{r-1})}) = e(\Delta'_{i_{r-1}})$ and $e(\Delta_{j_r}) \leq e(\Delta'_{i_r}) < e(\Delta'_{i_{r-1}})$, we must have $\Delta_{j_r} \subseteq \Delta_{\sigma(i_{r-1})}$. In this case pick j_{r-1} with $\mathfrak{d}_{\mathfrak{m}}(j_{r-1}) = \mathfrak{d}_{\mathfrak{m}}(i_{r-1})$ and $\Delta_{j_r} \ll \Delta_{j_{r-1}}$ which must necessarily fulfill $\Delta_{j_{r-1}} \subseteq \Delta_{\sigma(i_{r-1})} \subseteq \Delta'_{i_{r-1}}$ as desired. This shows that we can construct $j_0, \dots, j_k$ with the required properties.

Now let $l \in \{0, \ldots, n\}$ be maximal with the property $\Delta_{j_k} \ll \Delta_l$. By the permissibility of $(\mathfrak{l}, \mathfrak{m})$ we then have $e(\Delta_{j_r}) \geq b(\Delta_{j+r-k})$ for all $r = 0, \ldots, k$. By the conditions imposed on the j_k we also find $\Delta'_{i_k} \ll \Delta_j$ and so if $j' \in \{0, \ldots, n\}$ is maximal with $\Delta'_{i_k} \ll \Delta_{j'}$ then $j' \geq j$. But since $\mathfrak{l} = \Delta_0 + \cdots + \Delta_n$ is a ladder, this implies $e(\Delta'_{i_r}) \geq e(\Delta_{j_r}) \geq b(\Delta_{j+r-k}) \geq b(\Delta_{j'+r-k})$. This shows permissibility of $(\mathfrak{l}, \mathfrak{l}_1)$ and hence of $(\mathfrak{l}, \mathfrak{m}')$. $\square$

The previous propositions thus show that we have found a bijective correspondence $\mathcal{K} : \mathfrak{M} \setminus \{0\} \to \mathfrak{A}$ and an injective construction $\mathrm{RSK} : \mathfrak{M} \to \mathcal{L}$. We caution the reader that the map RSK is not surjective onto $\mathcal{L}$.

Example 3.6 (cf. [GL21] Remark 2.5.)**.** Consider the triple of ladders $L = ([3,3] + [1,2], [2,3], [1,2])$ which one checks to be permissible. We claim that L is not in the image of the map RSK. Indeed we have $\mathcal{K}([1,3] + [2,2]) = ([2,3], [1,2])$ and so the only way that L could be in the image of RSK is that there is some multisegment $\mathfrak{m}$ with $\mathcal{K}(\mathfrak{m}) = ([3,3] + [1,2], [1,3] + [2,2])$. But $([3,3] + [1,2], [1,3] + [2,2])$ is not permissible and hence cannot be $\mathcal{K}(\mathfrak{m})$ for any multisegment $\mathfrak{m}$. Thus $L \notin \mathrm{RSK}(\mathfrak{M})$.

3.2 Young Tableaux

Having introduced the basic algorithm for transforming a multisegment into a permissible sequence of ladders, we now want to cast this description into a form that is more convenient from a combinatorial viewpoint.
We start by introducing some terminology. We may think of a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of an integer n as a *Young diagram*, i.e. a diagram of empty boxes, with the number of boxes in row i being equal to λ_i. We always order partitions such that the λ_i are decreasing from now on. For example we view the partition $(3,2,2,1)$ of 8 as a Young diagram as depicted below. Given such a Young diagram $\lambda = (\lambda_1, \ldots, \lambda_k)$, we can fill the

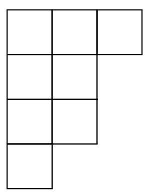


Figure 1: The partition $(3,2,2,1)$ viewed as a Young diagram

boxes with integers $a_{i,j}$ as depicted in figure 2. We will call a Young diagram λ filled with integers a *tableau* and call λ the *shape* $\mathrm{sh}(T)$ of T and $n = \lambda_1 + \cdots + \lambda_k$ the *size* of the tableau. Now take a tuple $L = (\mathfrak{l}_1, \ldots, \mathfrak{l}_k) \in \mathcal{L}$ with $\mathfrak{l}_i = \Delta_{i,1} + \cdots + \Delta_{i,n_i}$. For $i = 1, \ldots, k$ and $j = 1, \ldots, n_i$ we set $a_{i,j} := b(\Delta_{i,j})$ and $b_{i,j} := e(\Delta_{i,j})$ and $\lambda := (n_1, \ldots, n_k)$. Then we can form two tableaux $P(L), Q(L)$ of shape λ using the numbers $a_{i,j}, b_{i,j}$ similar to the situation depicted in figure 2. In other words, the tableau $P(L)$ consists of the beginnings of the $\Delta_{i,j}$ and the tableau $Q(L)$ of the endings of them. The i-th row in these tableaux is filled precisely with the beginnings resp. endings of $\mathfrak{l}_i$. Since this is a ladder in canonical order, we thus see that the entries of each row in both $P(L)$ and $Q(L)$

$a_{1,1}$	$a_{1,2}$	$a_{1,3}$
$a_{2,1}$	$a_{2,2}$	
$a_{3,1}$	$a_{3,2}$	
$a_{4,1}$		

Figure 2: A tableau of shape $(3, 2, 2, 1)$.

are strictly decreasing. Furthermore, since the tuple $(\mathfrak{l}_1, \dots, \mathfrak{l}_k)$ is permissible, each pair $(\mathfrak{l}_i, \mathfrak{l}_{i+1})$ for $i = 1, \dots, k-1$ is in particular dominant. Hence we have $n_1 \geq n_2 \geq \cdots \geq n_k$ and for each $i \in \{1, \dots, k\}$, $1 \leq j \leq n_{i+1}$

$$a_{i,j} \geq a_{i+1,j}, \quad b_{i,j} \geq b_{i+1,j}.$$

This means that the columns of the tableaux $P(L), Q(L)$ are weakly decreasing. We have thus associated to each tuple $L = (\mathfrak{l}_1, \dots, \mathfrak{l}_k) \in \mathcal{L}$ a pair $(P(L), Q(L))$ of tableaux of equal shape such that all rows of $P(L)$ resp. $Q(L)$ are strictly decreasing and all columns of $P(L)$ resp. $Q(L)$ are weakly decreasing. Furthermore, since all $\Delta_{i,j}$ are segments, we also have $a_{i,j} \leq b_{i,j}$, so each entry of $P(L)$ is less than or equal to the entry of $Q(L)$ in the same position.

Definition 3.7. Let T be a tableau of shape $\lambda = (n_1, \dots, n_k)$ with entries $a_{i,j}$. We call T a *inverted semi-standard Young tableau* (ISSYT) if all rows of T are strictly decreasing and all columns are weakly decreasing. In other words, T is ISSYT if

$$a_{i,1} > a_{i,2} > \cdots > a_{i,n_i}, \text{ for } i = 1, \dots, k,$$
$$a_{1,j} \geq a_{2,j} \geq \cdots \geq a_{\mu_j,j}, \text{ for } j = 1, \dots, n_1.$$

Here $\mu_j := \max\{i : n_i \geq j\}$ is the length of the j-th column in the Young diagram λ. We denote by $\mathcal{T}_{iss}^{\lambda}$ the set of ISSYT of shape λ, by $\mathcal{T}_{iss}^{n}$ the set of ISSYT of size n and by $\mathcal{T}_{iss}$ the set of all ISSYT. Furthermore we denote by $\mathcal{T}$ the set of pairs (P, Q) of ISSYT of equal shape such that $P_{i,j} \leq Q_{i,j}$ for all i, j.
Similarly, we define other types of tableaux. If the rows of a tableau T are weakly increasing and the columns strictly increasing we call T a *semi-standard Young tableau.* If the each entry of a semi-standard Young tableau T occurs at most once we call it a *standard Young tableau.* We define sets $\mathcal{T}_{ss}^{\lambda}, \mathcal{T}_{s}^{\lambda}$,etc. analogously to the case of inverted semi-standard Young tableaux.

We can thus view the algorithm described in section 3.1 as associating to each multisegment $\mathfrak{m}$ a unique pair $\mathrm{RSK}(\mathfrak{m}) = (P(\mathfrak{m}), Q(\mathfrak{m}))$ of inverted semi-standard Young tableaux of equal shape with $P_{i,j} \leq Q_{i,j}$ for all i, j. We will not distinguish in notation between the maps $\mathrm{RSK}\colon \mathfrak{M} \to \mathcal{L}$ and $\mathrm{RSK}\colon \mathfrak{M} \to \mathcal{T}$ that we have now defined since there is no risk of confusion.

In the following section we want to discuss how the algorithm we have dubbed 'RSK' in this article relates to more standard descriptions of the RSK correspondence from combinatorics.

3.3 Standard Descriptions of RSK

The aim of this section is to link the algorithm we have discussed above to well-known descriptions of the Robinson-Schensted-Knuth correspondence, beginning with the classical row bumping algorithm. Our main source for this is [Fu97].

3.3.1 Row Bumping

We will explain the operation of row bumping, which goes back to works of Schensted in [Sc61] to describe a bijection (usually called the Robinson-Schensted correspondence, since it was independently discovered by both) between permutations in S_n and pairs of standard Young tableaux with entries $\{1, \dots, n\}$ of equal shape. Later this was generalized by Knuth in [Kn70] to a correspondence between multisets of pairs of positive integers and pairs of semi-standard Young tableaux of equal shape.

We first describe the Robinson-Schensted correspondence through row bumping. Let $\sigma = \begin{pmatrix} 1 & 2 & \dots & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(n) \end{pmatrix}$ be a permutation and set P_0 to be the empty tableau. We now want to successively insert the numbers $\sigma(1), \dots, \sigma(n)$ into the tableau P_0 to create a standard Young tableau.
The first number $\sigma(1)$ is simply put into P_0 to make the tableau P_1 which has one box filled with $\sigma(1)$. Having constructed P_i, $i > 0$ already, we have to insert $\sigma(i+1)$ into it. Let $x = \sigma(i+1)$. If x is greater than all the entries in the first row of P_i then we simply append x to the first row of P_i and thus define P_{i+1}. Otherwise, we look for the smallest entry in the first row of P_i that is larger than x, call it y, and replace y with x in the first row. Then we repeat the same process with y and the second row of P_i, again either appending it to the second row (in particular, creating a second row if it was empty before) or replacing an element with y. We continue this process until it terminates and in this way we added the number $x = \sigma(i+1)$ to P_i. Inductively we can thus define $P := P_n$ and in each step it is clear that the P_i are standard Young tableaux, thus we have described a way of turning a permutation σ into a standard Young tableau P.
However this association is not injective, several different permutations may well produce the same P. Indeed going through the algorithm we see that the permutations $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}$ produce the same tableau $\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} \end{array}$. To fix this, we will associate a second tableau Q to σ aswell, and the pair (P, Q) will indeed determine σ uniquely. We define Q inductively together with P: we start with Q_0 being empty and in each step from P_i to P_{i+1} we add a box to Q_i in the same position that a box has

been added to P_i and fill it with $i+1$, the number of steps it took to create this box. The *recording tableau* $Q := Q_n$ thus remembers when the boxes of P have been created. It is clear that Q has the same shape as P.

Definition 3.8. The Robinson-Schensted construction

$$\mathrm{RS} : S_n \to \{(P, Q) \mid P, Q \in \mathcal{T}_s^n,\ \mathrm{sh}(P) = \mathrm{sh}(Q)\}$$

is the above described map from permutations to pairs of standard Young tableaux of equal shape and size n.

Example 3.9. To make the above construction more clear to the reader, we include here an example going through all the steps. Let $\sigma = (3, 6, 4, 7, 8, 1, 2, 5)$ be a permutation (not in cycle notation, rather $1 \mapsto 3$, $2 \mapsto 6$, etc.). Then below we draw all the pairs of tableaux $(P_1, Q_1), \ldots, (P_n, Q_n) = \mathrm{RS}(\sigma)$ that the above algorithm produces in figure 3.

$$\begin{array}{|c|}\hline 3\\\hline\end{array}\ \begin{array}{|c|}\hline 1\\\hline\end{array}\qquad \begin{array}{|c|c|}\hline 3&6\\\hline\end{array}\ \begin{array}{|c|c|}\hline 1&2\\\hline\end{array}\qquad \begin{array}{|c|c|}\hline 3&4\\\hline 6\\\cline{1-1}\end{array}\ \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$$

$$\begin{array}{|c|c|c|}\hline 3&4&8\\\hline 6\\\cline{1-1}\end{array}\ \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3\\\cline{1-1}\end{array}\qquad \begin{array}{|c|c|c|}\hline 3&4&7\\\hline 6&8\\\cline{1-2}\end{array}\ \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array}$$

$$\begin{array}{|c|c|c|}\hline 1&4&7\\\hline 3&8\\\cline{1-2} 6\\\cline{1-1}\end{array}\ \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2} 6\\\cline{1-1}\end{array}\qquad \begin{array}{|c|c|c|}\hline 1&2&7\\\hline 3&4\\\cline{1-2} 6&8\\\cline{1-2}\end{array}\ \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2} 6&7\\\cline{1-2}\end{array}$$

$$\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&7\\\hline 6&8\\\cline{1-2}\end{array}\ \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5&8\\\hline 6&7\\\cline{1-2}\end{array}$$

Figure 3: Computational example of the RS-algorithm applied to $\sigma = (3, 6, 4, 7, 8, 1, 2, 5)$.

Now we want to see also that this construction is bijective. For this we just think about how to reverse a single step. So take a pair of standard Young tableaux of equal shape (P, Q). Think of how Q was defined: the largest entry n of Q is in the box that was created in the last step of the RS construction. Thus, in the last step of the construction, we added the entry y sitting in this box of P to a tableau P' using row bumping. Our goal is to determine P'. But this is straightforward: take the entry y that we want to 'bump out' of the tableau P. If y is in the first row of P, then it is necessarily the last entry of this row and we just delete it to get the tableau P'. If y is in another row, we look at the row immediately above it and replace the last entry of this row that is strictly less than y. Then y replaces this element z and we repeat the process with z. This continues until an entry x is bumped out of the top row, leaving us with a standard Young tableau P'. From the description it is clear that the insertion and row-bumping algorithm we have used to describe the RS construction produces exactly the tableau P we have started with when inserting x into P'. This shows the following result.

Theorem 3.10 (cf. [Sc61] Lemma 3)**.** *The map* RS *is a bijection between permutations of* $\{1, \ldots, n\}$ *and pairs of standard Young tableaux of equal shape and size* n *with entries from* $\{1, \ldots, n\}$.

Example 3.11. To illustrate also the reverse step we use the example of $\sigma = (3, 6, 4, 7, 8, 1, 2, 5)$ from before. There we saw that $\mathrm{RS}(\sigma) = (P, Q)$ with

$P =$

1	2	5
3	4	7
6	8	

$Q =$

1	2	4
3	5	8
6	7	

Then from Q we know that in the last step the box where the 7 is in P was added to it. So the reverse bumping tells us to put the 7 in the first row of P, where it bumps out the left-most entry that is less than 7, i.e. 5. The 5 is bumped out and so we know that in the last step, a 5 was added to P, which was indeed the case. Recursively, this allows us to recover σ by continuing this process and filling in σ from the back.

$P' =$

1	2	7
3	4	
6	8	

, $Q' =$

1	2	4
3	5	
6	7	

, $\sigma = (\star, 5)$

Remark 3.12. We remark that it is not essential for the construction to begin with a permutation σ. If we take any totally ordered set X, we call a *two-line array* over X any $2 \times n$ matrix $\begin{pmatrix} u_1 & u_2 & \ldots & u_n \\ v_1 & v_2 & \ldots & v_n \end{pmatrix}$ with $u_i, v_i \in X$ that is lexicographically ordered, i.e. for all $i = 1, \ldots, n_1$ we have $u_i < u_{i+1}$ or $u_i = u_{i+1}$ and $v_i < v_{i+1}$.Note that here we do not require the u_i, v_i to be distinct. Then we inductively define P_i starting with an empty tableau and successively inserting the $v_1, \ldots, v_n$ as described before. Also as before, the tableaux Q_i are defined inductively by adding a box to Q_{i-1} where one was created in the step $P_{i-1} \to P_i$ and filling this box with u_i.

Just as before, one can check that this defines a bijection between two-line arrays over X and pairs of *semi*-standard Young tableaux of equal shape with entries from X (cf. [Fu97] Chapter 4.). This is the Robinson-Schensted-Knuth (RSK) correspondence and was introduced as an extension of the Robinson-Schnested contruction by Knuth in [Kn70].

3.3.2 Matrix-Ball Construction

In this paragraph we want to bridge the gap to the algorithm described in section 3.1 by describing a different way of realizing the classical RSK-corresondence defined by row bumping. This is called the Matrix-Ball construction in [Fu97] and has its roots in Viennot's geometric realization of the Robinson-Schensted correspondence in [Vi77]. Our description will follow [Fu97] section 4.2. and we refer to this source for proofs of all the claims made.
We begin by first explaining a simple correspondence between two-line arrays of positive integers and matrices over $\mathbb{N}$, since working with matrices will be more convenient for explaining this particular realization of the RSK-correspondence. Given such a two-line array $\alpha = \begin{pmatrix} i_1 & i_2 & \dots & i_n \\ j_1 & j_2 & \dots & j_n \end{pmatrix}$ of pairs of positive integers from the set $\{1, \dots, n\} \times \{1, \dots, m\}$ then for a given pair $(i, j) \in \{1, \dots, n\} \times \{1, \dots, m\}$ we denote by $c_{i,j}$ the number of times that the column $\begin{pmatrix} i \\ j \end{pmatrix}$ appears in the array α. The numbers $c_{i,j}$ then fit together into an $n \times m$ matrix with entries from $\{1, \dots, k\}$. It is clear that this operation defines a bijection.
The matrix-ball construction will now give a correspondence between matrices over $\mathbb{N}$ and pairs of semi-standard Young tableaux of equal shape. While explaining the algorithm, we will illustrate the steps in a specific example along the way.
We start with a matrix $A \in \mathbb{N}^{n \times m}$ and visualize this matrix as a grid of boxes where we fill the box in position (i, j) with $a_{i,j}$ balls. In figure 4 we illustrate this for $A = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix}$.
In each box we order the balls diagonally from top-left corner to bottom-right corner. We say that a ball in this configuration is *northwest* of another if it is in the box as the other but above it or if the box is to the left and above (weakly) of the other. We then label the balls recursively: if there is no ball northwest of one then we label this ball with 1. We label a ball with k if $k - 1$ is the biggest number occurring in a ball northwest of this one. In this way we obtain a configuration of labelled balls that we denote by $A^{(1)}$, see figure 5.
Let k be the largest label that a ball has in $A^{(1)}$. Then we think of the balls with label k as being lined up on a string as follows: We enter $A^{(1)}$ to attach the string to the ball with label k that is in the left-most column among these. Then we continue the string along the row that this ball is in until we find ourselves in a box in the same column as a ball with label k. We make a turn upwards to attach the string to this ball with label k and continue in this way through $A^{(1)}$ until the last ball with label k is attached to the

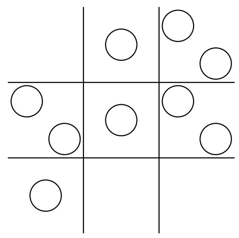


Figure 4: The matrix A and its representation as a grid of balls

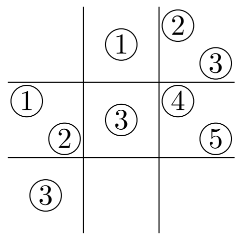


Figure 5: A configuration of labelled balls $A^{(1)}$

string. Finally we go out of $A^{(1)}$ straight rightward along the row in which the last ball is in. See figure 6. We do this string construction for all the labels $1, \ldots, k$ appearing in $A^{(1)}$. Then we note down the numbers $\left(p_1^{(1)}, \ldots, p_k^{(1)}\right)$ of the columns from left to right in which the strings for labels $1, \ldots, k$ entered and in the same way the numbers $\left(q_1^{(1)}, \ldots, q_k^{(1)}\right)$ of the rows from top to bottom in which the strings exited. These will be the first rows of the tableaux P and Q. In the example we are working out in the figures, this means that the first rows of P resp. Q are $(1,1,1,3,3)$ resp. $(1,1,1,2,2)$. Take a label $l \in \{1, \ldots, k\}$ and look at the string connecting the balls of label l. If there is only one ball with label l then we simply delete this ball. If there are multiple balls with label l, there are 'corners' in the string (as the dotted circles in figure 6 illustrate) where there is no ball. Then we create a new unlabelled ball in each of these corners and delete all the original balls with label l. Going through all the labels this leaves us with a new configuration of unlabelled balls. We label them again according to the same

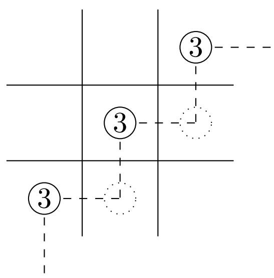


Figure 6: Illustrating a 'string' for the label 3 in the configuration form figure 5

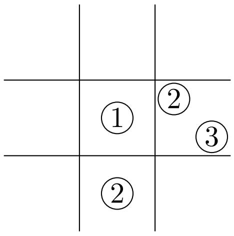


Figure 7: The configuration $A^{(2)}$ obtained from $A^{(1)}$ in figure 5.

rule that we used for the balls coming from the matrix A to obtain a configuration of labelled balls $A^{(2)}$, see figure 7. In the same way as before we get rows for P and Q from $A^{(2)}$ and we repeat the whole process until we reach a point where $A^{(n)}$ is empty. The tableaux P and Q we then have at this point are semi-standard Young tableaux and the following holds.

Proposition 3.13 (cf. [Fu97] Section 4.2. Proposition 1)**.** *The above description defines a bijection* $\mathrm{RSK}_c : \mathrm{Matr}(\mathbb{N}) \to \{(P,Q) \mid P,Q \in \mathcal{T}_{ss}, \mathrm{sh}(P) = \mathrm{sh}(Q)\}$ *between matrices over* $\mathbb{N}$ *and pairs of semi-standard Young tableaux of equal shape. Under the identification between matrices and two-line arrays of positive integers this coincides with the previous row-bumping description of the RSK-correspondence.*

Example 3.14. Starting with the matrix $A = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix}$ the algorithm described above gives first the pair of rows $(1,1,1,3,3)$, $(1,1,1,2,2)$, then $(2,2,3)$, $(2,2,2)$ and finally (3), (3). In total we thus associate through this construction with the matrix A the pair of semi-standard Young tableaux

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 3 & 3 \\ \hline 2 & 2 & 3 \\ \cline{1-3} 3 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 \\ \hline 2 & 2 & 2 \\ \cline{1-3} 3 \\ \cline{1-1} \end{array}$$

And indeed this is the same result we would get through insertion and row bumping.

A very elegant use of the matrix ball implementation of the RSK correspondence is the symmetry theorem:

Proposition 3.15 (cf. [Fu97] Section 4.1.)**.** *If an array* $\begin{pmatrix} u_1 & u_2 & \dots & u_n \\ v_1 & v_2 & \dots & v_n \end{pmatrix}$ *corresponds to the pair of tableaux* (P,Q) *then* $\begin{pmatrix} v_1 & v_2 & \dots & v_n \\ u_1 & u_2 & \dots & u_n \end{pmatrix}$ *(brought into lexicographic order) corresponds to the pair of tableaux* (Q,P)

Proof. This is clear from the above discussion: if $\begin{pmatrix} u_1 & u_2 & \dots & u_n \\ v_1 & v_2 & \dots & v_n \end{pmatrix}$ corresponds to the matrix A then $\begin{pmatrix} v_1 & v_2 & \dots & v_n \\ u_1 & u_2 & \dots & u_n \end{pmatrix}$ corresponds to A^t, the transpose of A. But since transposing simply switches rows and columns, the matrix-ball construction will produce $\mathrm{RSK}_c(A^t) = (Q, P)$ if $\mathrm{RSK}_c(A) = (P, Q)$. $\square$

As a particular consequence of this symmetry theorem, we note that $\mathrm{RSK}_c(A) = (P, P)$ if and only if the matrix A is symmetric. Thus the RSK-correspondence gives a bijection between symmetric matrices and semi-standard Young tableaux.
Another use of the matrix-ball construction is computational. Since in each step in this construction we create a row of P, Q that is not changed afterwards, this implementation can be much faster in practice than the row-bumping since we don't need to redraw the tableaux in each step adding a single box.
Now we turn again to our RSK correspondence from section 3.1.
Note that in this matrix-ball construction, if a ball of $A^{(2)}$ is created by a corner between two balls in $A^{(1)}$ then the new ball sits in the same row as one of the balls and in the same column as another ball. This is very similar to the way we create new segments in the construction from section 3.1 and in fact this construction for multisegments can be described using the matrix ball construction, finally linking this multisegment RSK to the classical one. A reference for this approach is [Fu97] Appendix A.4.
Instead of ordering the balls northwest as we did in our original discussion, we now order them SouthEast. So in each box we order the balls diagonally from top-left to bottom-right again and we say that one ball is SouthEast of another if the other is in a box that is to the left and below (strictly!) of the one. A ball that has no other one SouthEast of it will get the label 0 and any other ball gets the label k if $k-1$ is the maximal label of the balls SouthEast of it. Note that balls in the same box now have the same labels as well. In this way we get a configuration $\widehat{A^{(1)}}$ of labelled balls. We have chosen to begin the labelling with 0 here to keep the analogy with the depth function $\mathfrak{d}$ for multisegments from section 3.1.
Then we again draw strings along the balls of equal label. The balls with label k will be connected as follows: take the ball of label k in the bottom-most row and left-most among these and enter $\widehat{A^{(1)}}$ with the string along this row until we hit the ball with label k. Then we turn upward and follow this column until we reach a row in which there is a ball with label k. We turn toward this ball and follow the row until we reach it. If we enter a box with multiple balls of label k in this way, then we simply connect them in an arbitrary way with the string and exit the box upward along the same column. Continuing this process we eventually hit the ball of label k in the top-most row among them, from this we turn upward and follow the column until we exit $\widehat{A^{(1)}}$.
We then read out the rows in which the strings enter $\widehat{A^{(1)}}$ and let this be the first row of $\widehat{P}$. We also read out the columns in which the strings exit $\widehat{A^{(1)}}$ and let this be the first row of $\widehat{Q}$.
The rule for creating new balls is now superficially slightly different from the previous

description but is in fact the same. The earlier rule for creating new balls could also be described as follows: follow along the string of one label k and for each pair of successive balls along this path, make a new ball in the same row as the first ball and the same column as the second ball. Since in our earlier discussion the balls of equal label could never share the same rows or columns this always coincided with a corner of the string. Here we can now have balls of equal label sharing a column or row so the 'corners' don't capture the same information anymore. Instead we phrase the rule for creating new balls just like described above. In particular, if there are multiple balls in one box, say l balls, then this just amounts to creating l-1 balls in this box.
Having created these balls, delete all the original ones and then label the new balls according to the same SouthEast rule as before. This yields a new configuration $\widehat{A^{(2)}}$ of labelled balls and we can repeat the same string construction. Following this construction until it terminates we obtain two tableaux $(\widehat{P}, \widehat{Q})$ of equal shape. It is clear from the construction that the following holds:

Theorem 3.16 (cf. [Fu97] Appendix A.4. Theorem 1). *Given a multisegment $\mathfrak{m}$ we assume without loss of generality that all segments Δ appearing in $\mathfrak{m}$ are in $\mathbb{N}_+$. Identifying each segment Δ with the pair $(b(\Delta), e(\Delta)) \in \mathbb{N}_+^2$ and further identifying the two-line array of positive integers obtained in this way from $\mathfrak{m}$ with a matrix $A(\mathfrak{m})$ over $\mathbb{N}$, the pair of inverted semi-standard Young tableaux $(P, Q) = \mathrm{RSK}(\mathfrak{m})$ obtained using the algorithm from section 3.1 coincides with the pair $(\widehat{P}, \widehat{Q})$ obtained from the variant of the matrix-ball construction described above.*

To convince ourselves of this we go through both algorithms in an example.

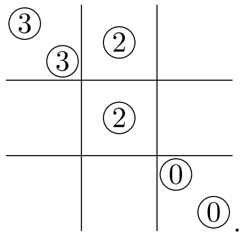
Example 3.17. Consider the multisegment $\mathfrak{m} = [1,1]+[1,1]+[1,2]+[2,2]+[3,3]+[3,3]$. This corresponds to the matrix $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.
We first compute the first step $\mathcal{K}(\mathfrak{m})$ of the RSK-algorithm described in section 3.1. The depths of the segments in $\mathfrak{m}$ are (if we index them with $1, \ldots, 6$ in the order that they appear in the sum defining $\mathfrak{m}$):

$$\mathfrak{d}(1) = 2,\ \mathfrak{d}(2) = 2,\ \mathfrak{d}(3) = 1,\ \mathfrak{d}(4) = 1,\ \mathfrak{d}(5) = 0,\ \mathfrak{d}(6) = 0.$$

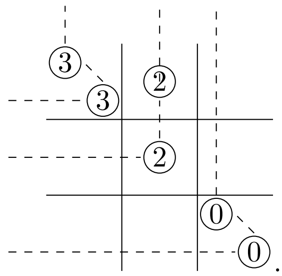
The permutation σ described in section 3.1 thus has the cycle decomposition $\sigma = (1,2)(3,4)(5,6)$ and $I^{\#} = \{6,4,2\}$. Thus we obtain

$$\mathcal{K}(\mathfrak{m}) = (\mathfrak{l}(\mathfrak{m}), \mathfrak{m}') = ([3,3] + [2,2] + [1,1], [1,1] + [1,2] + [3,3])$$

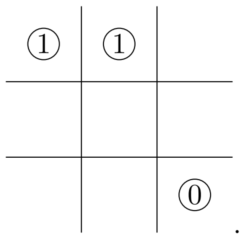
Now we turn to describing the first step in the above SouthEast matrix-ball construction for A. We begin by noting that the configuration $\widehat{A^{(1)}}$ is given by



If we add all the strings as described before to $\widehat{A^{(1)}}$ we obtain



And so we read off the first row of $\widehat{P}$ as $(3,2,1)$ and the first row of $\widehat{Q}$ as $(3,2,1)$. Furthermore, the new configuration $\widehat{A^{(2)}}$ is



Returning to the multisegment $\mathfrak{m}$, we have computed $\mathfrak{m}' = [1,1] + [1,2] + [3,3]$ and here we already notice that $\widehat{A^{(2)}}$ is exactly the configuration corresponding to $\mathfrak{m}'$. Thus we do not show how in the following steps the two constructions parallel each other as well and simply note that in both cases the result is the pair of inverted semi-standard Young tableaux

3	2	1
3	1	
1		

3	2	1
3	2	
1		

Remark 3.18. We have discussed in example 3.6 that the map RSK : $\mathfrak{M} \to \mathcal{L}$ is not surjective. With theorem 3.16 in mind, to find the image of this map, it is enough to find the image of the set of *upper-triangular* matrices under the SouthEast matrix ball construction. Indeed, since we always have $b(\Delta) \leq e(\Delta)$ for a segment Δ, a multisegment corresponds to an upper-triangular matrix.

3.3.3 Applications of RSK

One of the motivating problems for developing the Robinson-Schensted construction was finding longest increasing subsequences in some word $w = x_1 \dots x_n$ over a totally ordered alphabet, i.e. the maximal k such that one can find $i_1 < \cdots < i_k$ with $x_{i_1} \leq \cdots \leq x_{i_k}$. More generally we can ask for the number $L(w, k)$ of the largest number we can represent as the sum of the lengths of k *disjoint* increasing subsequenes of w.
The Robinsion-Schnested construction then gives a very simple way of determining the numbers $L(w, k)$. Indeed, let (P, Q) be the pair of tableaux corresponding to the word w, i.e. to the two-line array $\begin{pmatrix} 1 & 2 & \dots & n \\ x_1 & x_2 & \dots & x_n \end{pmatrix}$. Then $L(w, k)$ is the total number of boxes in the first k rows of P (equivalently Q). For more details on this see [Sc61] and [Fu97] where these concepts are discussed further and also a corresponding description for *decreasing* subsequences is given.
We next turn to a more algebraic application of the RSK-correspondence. Given a semi-standard Young tableau T with entries from $\{1, \dots, n\}$, we define a monomial $x^T = x_1^{r_1(T)} \dots x_n^{r_n(T)}$ where $r_i(T)$ denotes the number of times that the number i appears in the tableau T.

Definition 3.19. Given a partition λ with at most n rows we define the *Schur polynomial* s_λ of λ by

$$s_\lambda(x_1, \dots, x_n) = \sum_T x^T,$$

where the sum is over all semi-standard Young tableaux of shape λ filled with numbers from $\{1, \dots, n\}$

It is a non-obvious fact that the Schur polynomials are in fact *symmetric* in the variables x_i (cf. [Fu97] Section 4.3.). Many well-known symmetric polynomials are in fact Schur polynomials. For example, the elementary symmetric polynomials $e_k(x_1, \dots, x_n)$ defined for $1 \leq k \leq n$ by

$$e_k(x_1, \dots, x_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} x_{i_1} x_{i_2} \dots x_{i_k}.$$

Then we have $e_k = s_{u_k}$ for the partition $u_k = (1, \dots, 1)$ of k. In fact the Schur polynomials even form a basis of the vector space of the symmetric polynomials (cf. [Fu97] Section 6.1.).
The RSK-correspondence between matrices $A \in \mathbb{N}^{n \times n}$ and bi-tableaux (P, Q) of equal size then allows us to give a very elegant proof the Cauchy-Schur identity.

Theorem 3.20. *The following identity holds in the polynomial ring $\mathbb{Z}[x_1, \dots, x_n, y_1, \dots, y_m]$ for $n, m \in \mathbb{N}$:*

$$\prod_{i=1}^{n} \prod_{j=1}^{m} \frac{1}{1 - x_i y_j} = \sum_\lambda s_\lambda(x_1, \dots, x_n) s_\lambda(y_1, \dots, y_n),$$

where the sum is over all partitions λ.

Proof. On the left side, we sum over all $m \times n$ matrices $A = (a_{i,j})$ where each matrix contributes the product of all monomials $(x_j y_i)^{a_{i,j}}$. But if A corresponds to a pair of semi-standard Young tableaux (P, Q) under the RSK correspondence, then this product is exactly $x^P y^Q$. This proves the claim by recalling the definition 3.19 of the Schur polynomials. $\square$

Another prominent application of standard Young tableaux and by extension the RSK correspondence is in the representation theory of the symmetric groups S_n. For details on this see [Fu97] Chapter 7.

4 RSK-Standard Modules

We keep the conventions of the previous chapter regarding our notation for multisegments.
In the previous section, we have explained an algorithm that turns a multisegment $\mathfrak{m}$ into a permissible tuple of ladders $(\mathfrak{l}_1, \dots, \mathfrak{l}_k)$. Returning to representation theory, this allows us to associate a representation to $\mathfrak{m}$ based on the tuple ladders $(\mathfrak{l}_1, \dots, \mathfrak{l}_k)$.

Definition 4.1. Given a multisegment $\mathfrak{m}$ with $\mathrm{RSK}(\mathfrak{m}) = (\mathfrak{l}_1, \dots, \mathfrak{l}_k)$ we define the representation

$$\Lambda(\mathfrak{m}) := Z(\mathfrak{l}_k) \times \cdots \times Z(\mathfrak{l}_1),$$

called the *RSK-Standard* module associated to $\mathfrak{m}$.

The main result of [Gu22a] is the following remarkable statement.

Theorem 4.2 (cf. [Gu22a] Corollary 7.11.)**.** *Let $\mathfrak{m}$ be a multisegment. Then $\Lambda(\mathfrak{m})$ is SI with* $\mathrm{soc}(\Lambda(\mathfrak{m})) = Z(\mathfrak{m})$.

Proof. The proof requires techniques from the theory of quiver Hecke Algebras that we did not discuss. It can be found in [Gu22a] Theorem 7.9. □

Thus, the representation theoretic information that the multisegment $\mathfrak{m}$ describes is still retained through the RSK construction in some sense.
The theorem shows a certain 'minimality' property of $\Lambda(\mathfrak{m})$: if $\mathfrak{m} = \sum_{i \in I} \Delta_i$ and we set $\omega(\mathfrak{m}) := \max\{k : \exists i_1, \dots, i_k \in I \text{ pairwise distinct with } \Delta_{i_{r+1}} \subseteq \Delta_{i_r}\}$ then one can show (cf. [Kn70] Chapter 4.) that $\mathrm{RSK}(\mathfrak{m}) = (\mathfrak{l}_1, \dots, \mathfrak{l}_{\omega(\mathfrak{m})})$ and it was shown in [Gu20] (Proposition 4.7, Corollary 4.9) that if we have any sequence of ladders $(l_1, \dots, l_k)$ such that $Z(\mathfrak{m})$ is an irreducible subquotient of $Z(l_1) \times \cdots \times Z(l_k)$ we have $k \geq \omega(\mathfrak{m})$. Therefore $\Lambda(\mathfrak{m})$ is a product of a minimal number of ladder representations that admits $Z(\mathfrak{m})$ as an irreducible subquotient.
In the next section we expand our concept of multisegment to allow also for 'segments' of the form $\delta_a := [a+1, a]$ which appear naturally while doing union-intersection operations of juxtaposed segments. With this new concept in hand, we can describe how the RSK construction actually enables us to 'interpolate' between the two classifications $Z(\mathfrak{m}), L(\mathfrak{m})$ that we have discussed in the previous chapter.

4.1 Enhanced Mutlisegments

Definition 4.3. For $a \in \mathbb{Z}$ let $\delta_a := (a+1, a) \in \mathbb{Z} \times \mathbb{Z}$. Identifying segments with pairs of integers, we also view this δ_a as a 'segment' $[a+1, a]$ and will refer to it as a *dummy segments.* We call

$$\widehat{\mathrm{Seg}} := \{\delta_a \mid a \in \mathbb{Z}\} \cup \mathrm{Seg}$$

the set of *enhanced* segments.
In the same way we define an enhanced multisegment $\widehat{\mathfrak{m}}$ to be a multiset of enhanced segments. Note that this can also be seen as a pair $\widehat{\mathfrak{m}} = (\mathfrak{m}, \delta(\widehat{\mathfrak{m}}))$ where $\mathfrak{m}$ is the mutlisegment consisting of all the standard segments in $\widehat{\mathfrak{m}}$ and $\delta(\widehat{\mathfrak{m}}) = \delta_{a_1} + \cdots + \delta_{a_n}$ the multiset of dummy segments appearing in $\widehat{\mathfrak{m}}$. The set of these enhanced multisegments will be denoted $\widehat{\mathfrak{M}}$.

Remark 4.4. We note that enhanced multisegments occur naturally even when studying 'regular' multisegments. For example, given two segments Δ, Δ' that are *juxtaposed*, i.e. $\Delta \cap \Delta' = \emptyset$ and Δ, Δ' are linked, say Δ precedes Δ'. Then from Proposition 2.15 we know that $[Z(\Delta) \times Z(\Delta')] = [Z(\Delta + \Delta')] + [Z(\Delta \cup \Delta')]$. Instead of this, we could also note that $\Delta \cap \Delta' = \delta_{e(\Delta)}$ is a perfectly valid way of interpreting the 'empty' segment $\Delta \cap \Delta'$ as a dummy segment and thus we can describe the semi-simplification of $Z(\Delta) \times Z(\Delta')$ through the enhanced multisegments $\Delta + \Delta'$ and $\Delta \cup \Delta' + \delta_{e(\Delta)}$.

We define enhanced ladders in the obvious way and also the depth function in definition 3.1 translates naturally to the enhanced setting. Given a multiset of dummy segments $\delta = \delta_{a_1} + \cdots + \delta_{a_n}$ we let $\operatorname{supp}(\delta) = \{a_1, \ldots, a_n\}$ and for each $a \in \mathbb{Z}$ we let $n_\delta(a)$ be the multiplicity with which δ_a appears in δ. We denote by $\mathfrak{D}$ the set of multisets of dummy segments.

Now we can extend the map RSM: $\mathfrak{M} \to \mathcal{L}$ to enhanced multisegments using the exact same algorithm as described in the previous chapter. Defining analogously the permissibility of enhanced ladders $(\widehat{\mathfrak{l}_1}, \widehat{\mathfrak{l}_2})$ we define the set $\widehat{\mathcal{L}}$ of permissible tuples of enhanced ladders. Then everything working exactly as in the previous chapter, we obtain a bijection $\widehat{\mathcal{K}}\colon \widehat{\mathfrak{M}} \setminus \{0\} \to \widehat{\mathfrak{A}}$ and an injective map RSK : $\widehat{\mathfrak{M}} \to \widehat{\mathcal{L}}$. Note that this map is not surjective. This can be seen by considering the tuple $\widehat{L} = ([5,4]+[3,3]+[1,2], [2,3], [1,2]) \in \widehat{\mathcal{L}}$ (see example 3.6).

4.2 Enhanced RSK-Standard Modules

Turning back now to the representation theoretic side, the natural way to associate a representation to a dummy multisegment δ_a is simply 1, the trivial representation of the trivial group G_0. Accordingly, for an enhanced multisegment $\widehat{\mathfrak{m}} = (\mathfrak{m}, \delta)$ we simply associate the representation $Z(\widehat{\mathfrak{m}}) = Z(\mathfrak{m})$ corresponding to the multisegment $\mathfrak{m}$. So the introduction of dummy segments doesn't have an immediate impact on the representation theoretic side. However if δ is non-zero then $\mathrm{RSK}(\mathfrak{m}) = (\mathfrak{l}_1, \ldots, \mathfrak{l}_k)$ and $\mathrm{RSK}(\widehat{\mathfrak{m}}) = (\widehat{\mathfrak{l}_1}, \ldots, \widehat{\mathfrak{l}_m})$ clearly define different pairs of tableaux and accordingly the associated modules $\Lambda(\mathfrak{m})$ and

$$\Lambda(\widehat{\mathfrak{m}}) := Z(\widehat{\mathfrak{l}_m}) \times \cdots \times Z(\widehat{\mathfrak{l}_1}),$$

need not be the same representations. This motivates the following definition.

Definition 4.5. Given a multisegment $\mathfrak{m}$ and $\delta = \delta_{a_1} + \cdots + \delta_{a_n} \in \mathfrak{D}$ we define:

$$\Lambda_\delta(\mathfrak{m}) := Z(\widehat{\mathfrak{l}_k}) \times \cdots \times Z(\widehat{\mathfrak{l}_1}),$$

where $\mathrm{RSK}(\mathfrak{m}, \delta) = (\widehat{\mathfrak{l}_1}, \ldots, \widehat{\mathfrak{l}_k})$. We call $\Lambda_\delta(\mathfrak{m})$ the *RSK-standard module* associated to $(\mathfrak{m}, \delta)$.

Proposition 4.6 (cf. [Gu22b] Theorem 6.5.)**.** *Let $\mathfrak{m}$ be a multisegment and $\delta \in \mathfrak{D}$. Then $\Lambda_\delta(\mathfrak{m})$ is SI with socle $Z(\mathfrak{m})$.*

We now explain how different choices of δ lead to the standard modules of Zelevinsky or Langlands. In this sense, the RSK-standard modules 'interpolate' between these two classifications.

Theorem 4.7. *Let $\mathfrak{m} = \sum_{i \in I} \Delta_i \in \mathfrak{M}$ be a multisegment and δ a multiset of dummy segments.*

1. *Suppose that the following hold.*
   a) *Whenever $e(\Delta_h) < e(\Delta_i)$ we have*
   $$n_\delta(e(\Delta_i)) \geq n_\delta(e(\Delta_j)) + \#\{k \colon e(\Delta_k) = e(\Delta_j)\}$$
   b) *For all $i \in I$ and $t < e(\Delta_i)$ we have $n_\delta(t) \leq n_\delta(e(\Delta_i))$.*
   Then $\Lambda_\delta(\mathfrak{m}) \cong \zeta(\mathfrak{m})$.
2. *Suppose that $n_\delta(t) \geq n_\delta(t+1) + \#I$, for all $\min \mathfrak{m} \leq t < \max \mathfrak{m}$.*
   Then $\Lambda_\delta(\mathfrak{m}) \cong \lambda(\mathfrak{m}^\#)^t$, where $(.)^t$ denotes the contragredient of a representation.

Proof. The proof the of theorem is, as the conditions already suggest, largely technical in nature and works by induction on the number of segments $\#I$ of $\mathfrak{m}$. We begin by proving 1. Set $\widehat{\mathfrak{m}} := (\mathfrak{m}, \delta)$ and let $k \in I$ such that $e(\Delta_k)$ is minimal among all $e(\Delta_i)$, $i \in I$ and $b(\Delta_k)$ is maximal among these segments.
Note that the conditions guarantee $n_\delta(e(\Delta_i)) > 0$ for all $i \in I$ and that $e(\Delta_j) < e(\Delta_i)$ implies $n_\delta(e(\Delta_j)) < n_\delta(e(\Delta_j))$ (note the strict inequality here). Just following along the definitions, this easily implies that $\mathfrak{d}_{\widehat{\mathfrak{m}}}(i)$ depends only on $e(\Delta_i)$, $\mathfrak{l}(\mathfrak{m}, \delta) = \sum_{a \in \mathrm{supp}\,\delta} \delta_a$ and $\widehat{\mathfrak{m}}' = \widehat{m} - \mathfrak{l}(\mathfrak{m}, \delta)$. So for an enhanced multisegment like the one we are dealing with here, the first step $\mathcal{K}(\mathfrak{m}, \delta)$ in the RSK-algorithm is of a very simple nature. Accordingly, we can repeat this process until no dummy segments $\delta_{e(\Delta_k)}$ remain. This has not made any contribution to $\Lambda_\delta(\mathfrak{m})$ yet since the enhanced ladders we have 'split-off' from $\widehat{\mathfrak{m}}$ have so far only consisted of dummy segments.
We assume now that $n_\delta(e(\delta_k)) = 0$ for the minimal $e(\Delta_k)$ but for all other $e(\Delta_i)$, $i \in I$ we still have $n_\delta(e(\Delta_i)) > 0$. By condition $b)$ this also implies $n_\delta(t) = 0$ for all $t \leq e(\Delta_k)$. As before, one computes $\mathfrak{l}(\widehat{m}) = \Delta_k + \sum_{a \in \mathrm{supp}(\delta)} \delta_a$ and $\widehat{m}' = \widehat{m} - \mathfrak{l}(\widehat{m})$.
Now the actual non-enhanced multisegment underlying $\widehat{m}'$ is $\mathfrak{m} - \Delta_k$ and thus of smaller length than $\mathfrak{m}$. Furthermore, the enhanced multisegment $\widehat{\mathfrak{m}}'$ still satisfies conditions $a)$

and b) and therefore we find by induction that $\Lambda(\widehat{m}') = \zeta(\mathfrak{m} - \Delta_k)$ and by definition $\mathrm{RSK}(\mathfrak{m}, \delta) = (\mathfrak{l}(\widehat{m}), \mathrm{RSK}(\widehat{m}'))$. This shows

$$\Lambda_\delta(\mathfrak{m}) = \zeta(\mathfrak{m} - \Delta_k) \times Z(\Delta_k)$$

and since we chose Δ_k in a way that ensures it cannot succeed any segment Δ_i, $i \in I$ of $\mathfrak{m}$ we deduce that $\zeta(\mathfrak{m} - \Delta_k) \times Z(\Delta_k) = \zeta(\mathfrak{m})$, which proves 1.
The proof of claim 2 works along similarly straightforward lines. Since we have refrained from discussing the algorithmic description of the Moeglin-Waldspurger involution $\mathfrak{m} \mapsto \mathfrak{m}^{\#}$ and its interaction with the RSK-algorithm, we will not prove this part of the theorem here. For details see [GL21] Lemma 5.5. □

5 Open Problems

We now turn to the conjectures that were the main motivation for this article. They deal with the Jordan-Hölder factors of the representations $\Lambda(\mathfrak{m})$. The goal would be to find a description like in Theorem 2.18. The main point there is that we have a partial order $\leq$ defined on the set of multisegments such that $Z(\mathfrak{m})$ is an irreducible subquotient of $\zeta(\mathfrak{n})$ if and only if $\mathfrak{n} \leq \mathfrak{m}$. We want to now describe an ordering on the set of permissible tuples of ladders $\mathcal{L}$ (i.e. inverted semi-standard Young tableaux) that is a candidate for an ordering that would satisfy an analogous result.

Definition 5.1. Given two partitions $\alpha = (n_1, \ldots, n_k)$, $\beta = (m_1, \ldots, m_l)$ of the same integer n we let $\alpha \leq \beta$ if $k \leq l$ and for all $1 \leq i \leq k$:

$$\sum_{j=1}^{i} n_j \geq \sum_{j=1}^{i} m_j.$$

The partial order $\leq$ we have just defined is called the *reverse dominance order* on the partitions of n.

Given some inverted semi-standard Young tableau Y we denote by $\mathrm{sh}(Y)$ the *shape* of Y, i.e. the partition defined by the lengths of the rows of Y. Furthermore, given any integer r, we let $Y_{\geq r}$ be the inverted semi-standard Young tableau obtained form Y by only considering the entries that are $\geq r$ and omitting the rest.

Definition 5.2. Given inverted semi-standard Young tableaux X, Y whose entries agree as multisets, we define $X \leq Y$ if and only if:

$$\mathrm{sh}(X_{\geq r}) \leq \mathrm{sh}(Y_{\geq r}), \text{ for all } r \in \mathbb{Z}.$$

Finally, we define also an ordering for pairs (X, Y), (P, Q) of inverted semi-standard Young tableaux by taking the product ordering:

$$(X, Y) \leq (P, Q) \iff X \leq P \text{ and } Y \leq Q.$$

This clearly defines a partial ordering on the set of pairs inverted semi-standard Young tableaux of equal shape. Using the RSK construction, we can thus obtain a partial ordering $\leq_{\mathrm{RSK}}$ on the set $\mathfrak{M}$ of multisegments by setting:

$$\mathfrak{m} \leq_{\mathrm{RSK}} \mathfrak{n} \iff \mathrm{RSK}(\mathfrak{m}) \leq \mathrm{RSK}(\mathfrak{n}).$$

Conjecture 5.3 (Gurevich-Lapid [GL21] Conjecture 6.3.)**.** *Let $\mathfrak{m}$ be a multisegment, then we have*

$$[\Lambda(\mathfrak{m})] = \sum_{\mathfrak{n} \leq_{\mathrm{RSK}} \mathfrak{m}} l_{\mathfrak{m},\mathfrak{n}} [Z(\mathfrak{n})],$$

with integers $l_{\mathfrak{m},\mathfrak{n}} \geq 0$ such that $l_{\mathfrak{m},\mathfrak{m}} = 1$.

Remark 5.4. The case of $k = 2$, i.e. $\Lambda(\mathfrak{m}) = Z(\mathfrak{l}_2) \times Z(\mathfrak{l}_1)$ is the product of two ladders of this conjecture is almost purely combinatorial. From Theorems 2.28, 2.27 we have a complete description of the semi-simplification $[\Lambda(\mathfrak{m})]$ that can be phrased in a purely combinatorial way (Proposition 2.31).

Just as theorem 2.18 does for segment representations, this result would in particular show that the $[\Lambda(\mathfrak{m})]$ form a basis of the Grothendieck group $\mathscr{R}$. A weaker form of conjecture 5.3 would be this statement.

Conjecture 5.5 (cf. [GL21] Conjecture 6.4.)**.** *The classes* $[\Lambda(\mathfrak{m})]$ *for* $\mathfrak{m} \in \mathfrak{M}$ *form a* $\mathbb{Z}$*-basis of* $\mathscr{R}$*.*

We finish with some open questions that we do not have precise conjectures for.
In theorem 4.7 we have seen how enhanced multisegments can be used to produce both Zelevinsky and Langlands-standard modules as special cases of (enhanced) RSK-standard modules. It is then a natural question ask, given a multisegment $\mathfrak{m} \in \mathfrak{M}$, how the set $\{\Lambda_\delta(\mathfrak{m}) : \delta \in \mathfrak{D}\}$ can be described. This question is further investigated in [Gu22b] through the lens of *derived* RSK-modules (cf. [Gu22b] §6).

Lastly, there is a purely combinatorial question as well. We have noted in example 3.6 that the map RSK : $\mathfrak{M} \to \mathcal{L}$ is not surjective. By remark 3.18, finding the image of this map is equivalent to finding the image of the upper-triangular matrices under the SouthEast matrix ball construction. Similarly, one could investigate what tuples of enhanced ladders in $\widehat{\mathcal{L}}$ actually appear as RSK$(\widehat{m})$ for enhanced multisegments $\widehat{m}$

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