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"Norm-controlled inversion in ℓ^1:
A support-dependent theorem and applications"

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Abstract

The aim of this thesis is to investigate the problem of norm-controlled inversion in $\ell^1(\mathbb{Z}, *)$. In the late nineties, El-Fallah, Nikolski and Zarrabi showed that the norm of some $a \in \ell^1(\mathbb{Z}, *)$ and a lower bound on its Fourier series are not enough to control the norm of a^{-1}, contrary to the weighted case where it is indeed possible to do so assuming that the weight grows sufficiently fast. They provide a lower bound for the norm of the inverses which implies a blowup if the condition on the Fourier series is too weak.

We generalize this lower bound by also taking the support of the involved sequences into account. This new estimate will be applied to some weighted setting which yields a lower bound for the growth of the inverses that does not follow from the arguments of El-Fallah, Nikolski and Zarrabi.

Moreover, we transfer their result to the twisted convolution algebra, a structure being very prominent in the field of time-frequency analysis. An application of this result to the theory of Gabor analysis yields that in general the M^1-norm of a dual window cannot be controlled by the M^1-norm of the window itself and the corresponding Riesz sequence or frame bounds.

Contents

Introduction

The problem of norm-controlled inversion is formulated in the context of Banach algebras: Which information do we need about an element a in some Banach algebra $\mathcal{A}$ to get an upper bound for the norm of its inverse? It can easily be seen that the bare norm of a is not enough so some further information on the element is needed. El-Fallah, Nikolski and Zarrabi investigated this problem in [1], mainly in the setting of Beurling-Sobolev algebras $\ell^p(\mathbb{Z}, w)$. They showed that if one additionally imposes a lower bound on the Fourier series induced by the sequence a (which, more abstractly, corresponds to a lower bound on the spectrum of a), then norm-control is possible if the weight grows sufficiently fast. However, in the unweighted case this method fails: If the lower bound on the Fourier series is smaller than or equal to $\frac{1}{2}$, then controlling the norm of the inverses is not possible. This is even more surprising in view of a famous result by Wiener (Theorem 2.1.13) which says that any lower bound on the Fourier series implies invertibility of the underlying sequence a in the (unweighted) Banach algebra $\ell^1(\mathbb{Z})$.

In this thesis, we first recall the proof of the fact that norm-controlled inversion is in general not possible in $\ell^1(\mathbb{Z})$. We then provide a more general lower estimate for the inverses which takes the support of the involved sequences into account. A corresponding upper estimate for the inverses is also given. These two estimates differ in their asymptotical behaviour and the exact growth of the inverses is still unknown to us. We also apply these results to the weighted setting to obtain a corresponding lower bound.

Going back to the original problem, one can also view the lower bound on the Fourier series as an upper bound on the inverse Fourier series, which allows us to formulate the problem in the more general setting of one Banach algebra embedded into another. Then the question is whether one can bound the norm of the inverse in the smaller Banach algebra using the norm of the inverse in the larger Banach algebra. Using this generalization, we transfer all the results from the ordinary convolution to the twisted convolution, a structure very prominent in the field of time-frequency analysis. This, in turn, yields a corresponding lower estimate in the context of Gabor frames and Gabor Riesz sequences, which in particular shows that in general it is not possible to bound the M^1-norm of the dual window using only the M^1-norm of the window itself and the frame or Riesz sequence bounds.

In Chapter 1 we introduce some notational conventions and the necessary concepts which will be used in the subsequent chapters.

Chapter 2 discusses the problem of norm-controlled inversion. After an introduction to the topic, we mention the concrete results given in [1], which are the starting points for our considerations (Theorems 2.1.12 and 2.1.14). Then we recall the proof of the result which says that, in general, norm-controlled inversion in $\ell^1(\mathbb{Z}, *)$ fails (Theorem 2.1.14). Based on that, we give an improved version of the original result by also taking the support of the involved sequences into account (Corollary 2.3.9). To complete the picture, we also provide a corresponding upper estimate (Proposition 2.3.10). As a byproduct, we derive a lower estimate for the weighted case (Theorem 2.4.3).

Chapter 3 is devoted to the theory of time-frequency analysis, which is fundamental to the developments in the subsequent chapters. We first introduce some important operators, then discuss the symplectic group, which serves as a technical tool in Section 4.3, and finally, we give a brief overview of modulation spaces, which are needed in Section 5.3.

Chapter 4 concerns the twisted convolution algebra, a Banach algebra directly connected to time-frequency analysis. After the initial definition, we discuss several embeddings of this Banach algebra into operator algebras (Sections 4.2.1 and 4.2.2). Finally, we transfer the results from Chapter 2 to the twisted convolution algebra (Theorems 4.3.12, 4.3.13 and 4.3.14 and Corollary 4.3.15).

In Chapter 5 we first give a brief introduction into frame theory and Gabor analysis and then apply our findings from Chapter 4 to the theory of Gabor analysis (Theorems 5.3.9 and 5.3.11).

We mention that Section 2.3, Theorems 2.4.3, 4.3.14, 5.3.9 and 5.3.11 and Corollary 4.3.15 are original contributions.

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1 Preliminaries

In this chapter we introduce the necessary notation, language and concepts which are used throughout the thesis.

1.1 Notational conventions

The natural numbers $\mathbb{N}$ by definition include 0. Whenever d occurs without a definition it is a natural number greater than 0 and represents the dimension of the space we consider. The term "countable" always refers to an infinite set, i.e. a set I is countable if and only if there exists a bijection $\pi\colon \mathbb{N} \to I$.

Furthermore, we define the following sets of matrices:

- $\mathrm{M}(d,\mathbb{R})$ is the set of all $d \times d$-matrices with real entries
- $\mathrm{GL}(d,\mathbb{R})$ is the set of all invertible $d \times d$-matrices with real entries.
- $\mathrm{Sym}(d,\mathbb{R})$ is the set of all symmetric $d \times d$-matrices with real entries.

For $A, B \in \mathrm{M}(d,\mathbb{R})$ their matrix product is denoted by AB where

$$(AB)_{ij} = \sum_{k=1}^{d} A_{ik}B_{kj}.$$

Moreover, the transpose of A is denoted by A^T and $A^{-T} := (A^{-1})^T = (A^T)^{-1}$ whenever A is invertible.

For $a, b \in \mathbb{R}^d$ their inner product or dot product is denoted by

$$a \cdot b = \sum_{k=1}^{d} a_k b_k.$$

For $p \in [1,\infty)$ $L^p(\mathbb{R}^d)$ refers to the well-known Banach space of equivalence classes of measurable functions $f\colon \mathbb{R}^d \to \mathbb{C}$ which coincide almost everywhere and whose p-th moment exists, i.e.

$$\int_{\mathbb{R}^d} |f(x)|^p dx < \infty.$$

$L^\infty(\mathbb{R}^d)$ refers to the well-known Banach space of equivalence classes of measurable functions $f\colon \mathbb{R}^d \to \mathbb{C}$ which coincide almost everywhere and are uniformly bounded almost everywhere.

In the context of inequalities we sometimes use the notation

$$a \gtrsim b :\Leftrightarrow \exists C > 0 : a \geq Cb.$$

Whenever this will be used, the left- and right-hand side will depend on some parameter whereas the implied constant will be independent of this parameter. We will always mention on which parameters the implied constant can depend.

Finally, whenever we talk about a space of sequences over some index set I, we denote the corresponding unit vectors by u_i, where $i \in I$. Concretely,

$$(u_i)_j = \begin{cases} 1, & j = i \\ 0, & j \neq i \end{cases} .$$

1.2 Banach algebras

Banach algebras are the framework in which the problem of norm-controlled inversion is formulated. In the following, we provide the main definitions and some properties. The main references for this section is Allan's book on Banach spaces and algebras [2] and chapter 1 of Folland's book [3].

Definition 1.2.1. *A Banach algebra is an associative algebra $\mathcal{A}$ over $\mathbb{R}$ or $\mathbb{C}$ endowed with a norm $\|\cdot\|$ such that*

- *$\mathcal{A}$ is complete with respect to $\|\cdot\|$.*
- *For all $a, b \in \mathcal{A}$ holds: $\|ab\| \leq \|a\|\|b\|$.*

Definition 1.2.2. *Let $\mathcal{A}$ be a Banach algebra.*

(i) $\mathcal{A}$ is unital if it has a multiplicative unit.

(ii) $\mathcal{A}$ is commutative if the multiplication is commutative.

Definition 1.2.3. *An involutive Banach algebra is a Banach algebra $\mathcal{A}$ over $\mathbb{C}$ with a mapping ${}^*\colon \mathcal{A} \to \mathcal{A}$ (called involution) such that for all $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{C}$ holds:*

- $(a^*)^* = a$
- $(\lambda a + b)^* = \overline{\lambda} a^* + b^*$
- $(ab)^* = b^* a^*$
- $\|a^*\| = \|a\|$

Remark 1.2.4. Some authors define involutive Banach algebras without the property $\|a^*\| = \|a\|$ (e.g. [4, p. 237]). However, in most cases (including all examples we encounter) the involution is isometric.

Involutive Banach algebras are the right framework to generalize concepts like self-adjointness and unitarity which are already known for bounded operators on Hilbert spaces.

Definition 1.2.5. *Let $\mathcal{A}$ be an involutive Banach algebra and $a \in \mathcal{A}$.*

(i) a is self-adjoint if $a^* = a$.

(ii) Let $\mathcal{A}$ be additionally unital. a is unitary if $a^* = a^{-1}$.

The following theorem yields an example of a Banach algebra which also serves as the "basis" for other structures we investigate later. The proof consists of some elementary verifications which will be left to the reader.

Theorem 1.2.6. *For $a, b \in \ell^1(\mathbb{Z})$ we define the convolution*

$$(a * b)_k = \sum_{l \in \mathbb{Z}} a_l b_{k-l}$$

and the involution

$$(a^*)_k = \overline{a_{-k}}.$$

Then $\ell^1(\mathbb{Z}, *, ^*)$ is a commutative, unital and involutive Banach algebra with unit given by $e \in \ell^1(\mathbb{Z})$ where

$$e_k = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases} .$$

The convolution fits also nicely into the picture for other ℓ^p-spaces:

Theorem 1.2.7 (Theorem 13.9, [5])**.** *Let $p \in [1, \infty]$, $a \in \ell^1(\mathbb{Z})$ and $b \in \ell^p(\mathbb{Z})$. Then $a * b \in \ell^p(\mathbb{Z})$ and $\|a * b\|_p \leq \|a\|_1 \|b\|_p$.*

Next, we define a special type of involutive Banach algebras.

Definition 1.2.8. *A C*-algebra is an involutive Banach algebra $\mathcal{A}$ which additionally satisfies $\|a^* a\| = \|a\|^2$ for all $a \in \mathcal{A}$.*

The following definitions and lemmata introduce the prototypical examples of C*-algebras.

Definition 1.2.9. *Let X be a non-empty Hausdorff space. Then $C_0(X)$ is the Banach space of all continuous functions $f \colon X \to \mathbb{C}$ that vanish at infinity.*

Lemma 1.2.10. *Let X be a non-empty Hausdorff space. Pointwise multiplication turns $C_0(X)$ into a Banach algebra. If X is compact, then $C_0(X) = C(X)$ and the Banach algebra is unital. Moreover, the operation*

$$\begin{aligned} ^* \colon C_0(X) &\longrightarrow C_0(X) \\ f &\longmapsto f^* = (x \mapsto \overline{f(x)}) \end{aligned}$$

turns $C_0(X)$ into a C-algebra.*

Proof. Obviously, pointwise multiplication of functions is compatible with the linear structure of $C_0(X)$ and the inequality $\|fg\|_\infty \leq \|f\|_\infty \|g\|_\infty$ is also evident, so $C_0(X)$ forms a Banach algebra. Moreover, if X is compact, then the constant function 1 is an element of $C_0(X) = C(X)$ and it clearly serves as the identity with respect to the pointwise product. Thus, we are left to show that the given involution satisfies all needed properties to turn $C_0(X)$ into a C*-algebra. But all these properties are trivial observations in this context. □

Definition 1.2.11. *Let $\mathcal{H}$ be a Hilbert space. Then $\mathcal{B}(\mathcal{H})$ is the Banach space of all bounded, linear operators on $\mathcal{H}$.*

Lemma 1.2.12. *Let $\mathcal{H}$ be a Hilbert space. The operation of composition turns $\mathcal{B}(\mathcal{H})$ into a unital Banach algebra. Furthermore, the action of taking the adjoint A^* of some $A \in \mathcal{B}(\mathcal{H})$ turns $\mathcal{B}(\mathcal{H})$ into a unital C*-algebra.*

Proof. It is clear that the composition of operators is compatible with the linear structure of $\mathcal{B}(\mathcal{H})$. For $A, B \in \mathcal{B}(\mathcal{H})$ and $f \in \mathcal{H}$ we have

$$\|ABf\|_{\mathcal{H}} \leq \|A\|_{\mathcal{B}(\mathcal{H})} \|Bf\|_{\mathcal{H}} \leq \|A\|_{\mathcal{B}(\mathcal{H})} \|B\|_{\mathcal{B}(\mathcal{H})} \|f\|_{\mathcal{H}}$$

and thus

$$\|AB\|_{\mathcal{B}(\mathcal{H})} \leq \|A\|_{\mathcal{B}(\mathcal{H})} \|B\|_{\mathcal{B}(\mathcal{H})}.$$

Furthermore, the identity operator $\mathrm{Id}_{\mathcal{H}} \in \mathcal{B}(\mathcal{H})$ clearly serves as the identity with respect to composition of operators. Consequently, composition turns $\mathcal{B}(\mathcal{H})$ into a unital Banach algebra. Finally, all properties of a C*-algebra concerning the involution are well-known facts about adjoints (see e.g. [4, p. 31 f.]). Thus, $\mathcal{B}(\mathcal{H})$ carries a unital C*-algebra structure. □

Like in the case for bounded operators, unitary elements in unital C*-algebras automatically have norm 1.

Lemma 1.2.13 (p. 269, [2])**.** *Let $\mathcal{A}$ be a unital C*-algebra with unit e and $a \in \mathcal{A}$ unitary. Then $\|a\| = 1$.*

In order to deal with norm-controlled inversion, we need to work with unital Banach algebras. There we have a notion of invertibility (with respect to the multiplication) and the following lemmas concern topological properties of the set of invertible elements. In the following, $a^0 := e$ for all $a \in \mathcal{A}$ if $\mathcal{A}$ is a unital Banach algebra.

Lemma 1.2.14 (Lemma 1.3, [3])**.** *Let $\mathcal{A}$ be a unital Banach algebra with unit e. If $a \in \mathcal{A}$ satisfies $\|a\| < 1$, then $e - a$ is invertible and*

$$(e-a)^{-1} = \sum_{k=0}^{\infty} a^k.$$

In other words, the last lemma shows that we can find a neighbourhood of e which contains only invertible elements. This can be translated to any invertible element.

Lemma 1.2.15 (Theorem 1.4, [3]). *Let $\mathcal{A}$ be a unital Banach algebra with unit e. Then the set of invertible elements is open. More precisely, if $a \in \mathcal{A}$ is invertible and $b \in \mathcal{A}$ such that*

$$\|a - b\| < \frac{1}{2} \left\|a^{-1}\right\|^{-1}$$

then b is also invertible and

$$\left\|a^{-1} - b^{-1}\right\| \leq 2 \left\|a^{-1}\right\|^2 \|a - b\|.$$

Finally, we will also introduce the concept of exponential of an element in a Banach algebra.

Definition 1.2.16. *Let $\mathcal{A}$ be a unital Banach algebra with unit e. For $a \in \mathcal{A}$ we define*

$$\exp(a) := \sum_{k=0}^{\infty} \frac{a^k}{k!}.$$

The exponential is well-defined due to absolute convergence of the series. It also satisfies its usual functional equation, but only if the involved elements commute.

Lemma 1.2.17 (Proposition 4.98, [2]). *Let $\mathcal{A}$ be a unital Banach algebra with unit e. For $a, b \in \mathcal{A}$ with $ab = ba$ we have $\exp(a + b) = \exp(a)\exp(b)$. Consequently, $\exp(a)$ is always invertible and $\exp(a)^{-1} = \exp(-a)$.*

We finish this part with another lemma about unitary elements.

Lemma 1.2.18. *Let $\mathcal{A}$ be a unital, involutive Banach algebra and $a \in \mathcal{A}$ be self-adjoint. Then $\exp(ia)$ is unitary.*

Proof. From Lemma 1.2.17 we know that $\exp(ia)^{-1} = \exp(-ia)$, so it suffices to show that $\exp(ia)^* = \exp(-ia)$:

$$\exp(ia)^* = \left(\sum_{k=0}^{\infty} \frac{(ia)^k}{k!}\right)^* = \sum_{k=0}^{\infty} \frac{(-i)^k a^k}{k!} = \exp(-ia),$$

where the interchange of limit and involution is justified by the fact that the involution is by definition an antilinear isometry and therefore continuous. $\square$

1.3 Fourier series and convolution

In this part we introduce the Fourier series operator and discuss its relation to convolution. We keep things condensed and focused on our needs here. For a broader introduction into these topics we refer to [6].

Definition 1.3.1. *The Fourier series operator is the linear operator*

$$\begin{aligned}\mathcal{F}\colon \ell^1(\mathbb{Z}) &\longrightarrow C(\mathbb{T})\\ (a_k) &\longmapsto \sum_{k\in\mathbb{Z}} a_k e^{2\pi i k\cdot}\end{aligned}$$

We note that the continuity of the induced Fourier series is just a consequence of the dominated convergence theorem. Moreover, the domain of this operator can be extended to $\ell^2(\mathbb{Z})$ and yields an isometry in the corresponding setting.

Theorem 1.3.2 (Plancherel)**.** *The Fourier series operator extends to an isometric operator $\mathcal{F}\colon \ell^2(\mathbb{Z}) \to L^2(\mathbb{T})$.*

Proof. This follows from standard techniques of functional analysis because the collection $\{e^{2\pi i k\cdot}\}_{k\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{T})$ (see e.g. [6, p. 176 f.]). □

One of the most fundamental properties of the Fourier series operator is that it turns convolutions into pointwise multiplications.

Theorem 1.3.3. *Let $a, b \in \ell^1(\mathbb{Z})$. Then*

$$\mathcal{F}(a * b)(t) = \mathcal{F}(a)(t)\mathcal{F}(b)(t) \tag{1.1}$$

for all $t \in \mathbb{T}$. Moreover, this holds in a similar manner for the extended version of $\mathcal{F}$: For $a \in \ell^1(\mathbb{Z})$ and $b \in \ell^2(\mathbb{Z})$ we have

$$\mathcal{F}(a * b) = \mathcal{F}(a)\mathcal{F}(b).$$

Proof. (1.1) is a straightforward calculation which can be found in e.g. [5, Theorem 13.10 (b)]. Concerning the second part, we choose $a \in \ell^1(\mathbb{Z})$ and $b \in \ell^2(\mathbb{Z})$. Moreover, for $N \in \mathbb{N}$ we define the following cut-off versions of b:

$$b_k^N := \begin{cases} b_k, & |k| \leq N \\ 0, & \text{otherwise} \end{cases}$$

Then clearly $b^N \in \ell^1(\mathbb{Z})$ and $\|b^N - b\|_2 \xrightarrow{n\to\infty} 0$. Consequently, the first part yields

$$\mathcal{F}(a * b^N) = \mathcal{F}(a)\mathcal{F}(b^N)$$

for all $N \in \mathbb{N}$. Taking the limit $N \to \infty$ the left-hand side converges to $\mathcal{F}(a * b)$ in $L^2(\mathbb{T})$ because

$$\begin{aligned}\|\mathcal{F}(a * b^N) - \mathcal{F}(a * b)\|_{L^2(\mathbb{T})} = \|\mathcal{F}(a * (b^N - b))\|_{L^2(\mathbb{T})} &\overset{\textit{Theorem } 1.3.2}{=} \|a * (b^N - b)\|_2 \\ &\leq \|a\|_1 \|b^N - b\|_2 \xrightarrow{N\to\infty} 0,\end{aligned}$$

where we used Theorem 1.2.7 in the last inequality. On the other hand, the right-hand side converges to $\mathcal{F}(a)\mathcal{F}(b)$ in $L^2(\mathbb{T})$ because

$$\begin{aligned}\|\mathcal{F}(a)\mathcal{F}(b^N) - \mathcal{F}(a)\mathcal{F}(b)\|_{L^2(\mathbb{T})} &\leq \|\mathcal{F}(a)\|_\infty \|\mathcal{F}(b^N - b)\|_{L^2(\mathbb{T})} \\ &= \|\mathcal{F}(a)\|_\infty \| \|b^N - b\|_2 \xrightarrow{N\to\infty} 0,\end{aligned}$$

where we used Theorem 1.3.2 in the last inequality. □

The preceding theorem yields a connection between Fourier series and convolution. In the following we are going to explore the consequences of this relation.

Corollary 1.3.4. *$\mathcal{F} \colon \ell^1(\mathbb{Z}) \to C(\mathbb{T})$ is a homomorphism between unital, involutive Banach algebras.*

Proof. Compatibility with the multiplications follows from Theorem 1.3.3. Moreover,

$$\mathcal{F}(u_0)(t) = e^{2\pi i 0 t} = 1$$

for all $t \in \mathbb{T}$, so $\mathcal{F}$ respects the corresponding units. Finally, for $a \in \ell^1(\mathbb{Z})$ and $t \in \mathbb{T}$ holds:

$$\mathcal{F}(a^*)(t) = \sum_{k\in\mathbb{Z}} \overline{a_{-k}} e^{2\pi i k t} \overset{l=-k}{=} \sum_{l\in\mathbb{Z}} \overline{a_l} e^{-2\pi i l t} = \overline{\sum_{l\in\mathbb{Z}} a_l e^{2\pi i l t}} = (\mathcal{F}(a))^*(t).$$ □

Definition 1.3.5. *Let $a \in \ell^1(\mathbb{Z})$. We define the operator*

$$\begin{aligned}\mathrm{F_a} \colon \ell^2(\mathbb{Z}) &\longrightarrow \ell^2(\mathbb{Z}) \\ b &\longmapsto a * b\end{aligned}$$

which is well-defined by Theorem 1.2.7.

Due to Theorem 1.3.3 we can easily deduce the norm of the operator F_a as it is isometrically equivalent to a multiplication operator via the Fourier series operator. To this end, we need a basic statement about multiplication operators.

Lemma 1.3.6 (Problem 64, [7])**.** *Let (X, μ) be a σ-finite measure space and $a \colon X \to \mathbb{C}$ be a bounded, measurable function. Let $A \colon L^2(X) \to L^2(X)$ be the multiplication operator acting by multiplication of a, i.e. A is of the form*

$$(Af)(x) = a(x)f(x)$$

for all $x \in X$. Then

$$\|A\|_{\mathcal{B}(L^2(X))} = \|a\|_\infty.$$

Using the preceding theorem we can readily calculate the norm of F_a, which hints again at the connection between Fourier series and convolution.

Lemma 1.3.7. *Let* $a \in \ell^1(\mathbb{Z})$*. Then*

$$\|F_a\|_{\mathcal{B}(\ell^2(\mathbb{Z}))} = \|\sum_{k\in\mathbb{Z}} a_k e^{2\pi i k \cdot}\|_\infty.$$

Proof. We already know from Theorem 1.3.3 that

$$\mathcal{F}(a * b) = \mathcal{F}(a)\mathcal{F}(b)$$

for $a \in \ell^1(\mathbb{Z})$ and $b \in \ell^2(\mathbb{Z})$. Consequently, using Theorem 1.3.2, we get

$$\begin{aligned} \|F_a\|_{\mathcal{B}(\ell^2(\mathbb{Z}))} &= \left\|\mathcal{F} F_a \mathcal{F}^{-1}\right\|_{\mathcal{B}(L^2(\mathbb{T}))} = \left\|\mathcal{F}(a * \mathcal{F}^{-1}(\cdot))\right\|_{\mathcal{B}(L^2(\mathbb{T}))} \\ &= \|\mathcal{F}(a)\cdot\|_{\mathcal{B}(L^2(\mathbb{T}))} \overset{Lemma\ 1.3.6}{=} \|\mathcal{F}(a)\|_\infty . \end{aligned}$$

□

2 The problem of norm-controlled inversion

This chapter is devoted to the problem of norm-controlled inversion. After an introduction to the topic, we recall some results from [1]. Then we give several original contributions into this direction.

2.1 What norm-controlled inversion is

In this chapter we introduce the problem of norm-controlled inversion. We start with a small motivation: Let $\mathcal{A}$ be a unital Banach algebra with unit e and let $a \in \mathcal{A}$ be invertible. Then the simple calculation

$$\|e\| = \|aa^{-1}\| \leq \|a\|\|a^{-1}\|$$

yields the lower estimate

$$\frac{\|e\|}{\|a\|} \leq \|a^{-1}\|$$

for $\|a^{-1}\|$. What about an upper estimate? The lower bound for $\|a^{-1}\|$ suggests that the only possibility to bound $\|a^{-1}\|$ is to restrict to elements $a \in \mathcal{A}$ with large norm. But there is no hope, this is far from being achievable as the following theorem shows:

Theorem 2.1.1. *Let $\mathcal{A}$ be a unital Banach algebra with unit e containing some nonzero element which is not invertible (i.e. $\mathcal{A}$ is not a division algebra). Then for any $M > 0$ holds:*

$$\sup\left\{\|a^{-1}\| : \|a\| \geq M\right\} = \infty.$$

Proof. First, notice that if $(a_n) \in \mathcal{A}^{\mathbb{N}}$ satisfies that $\|a_n\| \geq M$ for all $n \in \mathbb{N}$ and $\|a_n^{-1}\| \xrightarrow{n\to\infty} \infty$, then clearly $\left\|\frac{N}{M}a_n\right\| \geq N$ and

$$\left\|\left(\frac{N}{M}a_n\right)^{-1}\right\| = \frac{M}{N}\left\|a_n^{-1}\right\| \xrightarrow{n\to\infty} \infty.$$

So it suffices to show the statement just for any specific $M > 0$. Now let $a \in \mathcal{A} \setminus \{0\}$ be not invertible and consider the curve

$$\begin{aligned} \mathrm{c}\colon [0,1] &\longrightarrow \mathcal{A} \\ t &\longmapsto (1-t)e + ta \end{aligned}$$

As $c(0)$ is invertible and $c(1)$ is not,

$$s := \inf\{t \in [0,1] : c(t) \text{ is not invertible}\}$$

is well-defined and $b := c(s)$ is not invertible by Lemma 1.2.15. Moreover, $b \neq 0$ because otherwise a would be a multiple of e, contradicting the fact that a is not invertible. Thus, we can find a sequence $(b_n) \in \mathcal{A}^{\mathbb{N}}$ such that b_n is invertible for all $n \in \mathbb{N}$, $b_n \xrightarrow{n\to\infty} b$ and without loss of generality $\|b_n\| \geq \|b\|/2$ for all $n \in \mathbb{N}$.
Let us assume now that we can find $C > 0$ such that $\|b_n^{-1}\| \leq C$ for all $n \in \mathbb{N}$. Then

$$\|e - b_n^{-1}b\| = \|b_n^{-1}b_n - b_n^{-1}b\| \leq \|b_n^{-1}\|\|b_n - b\| \leq C\|b_n - b\| \xrightarrow{n\to\infty} 0$$

and therefore by Lemma 1.2.15 and the fact that e is invertible we get that $b_n^{-1}b$ is invertible for n sufficiently large. Consequently, $b = b_n(b_n^{-1}b)$ must be invertible as a product of invertible elements and we have a contradiction. Hence, we must have

$$\sup\left\{\|b_n^{-1}\| : n \in \mathbb{N}\right\} = \infty$$

and in particular,

$$\sup\left\{\|a^{-1}\| : \|a\| \geq \|b\|/2\right\} = \infty.$$

□

Remark 2.1.2. The assumption that $\mathcal{A}$ is not a division algebra from the last theorem is not very restrictive: Every Banach algebra which is a division algebra is isomorphic to $\mathbb{R}$, $\mathbb{C}$ or the quaternions $\mathbb{H}$ (see e.g. [10, Theorem 5.5]).

The last theorem shows that we need more information about a than just its norm in order to have a chance to control $\|a^{-1}\|$. An approach to how one could tackle this problem was given by El-Fallah, Nikolski and Zarrabi in [1]. In the following, we summarize the initial motivation for their considerations. We start by briefly recalling the relevant language from Gelfand theory. A comprehensive introduction into this topic can be found in [11].

Definition 2.1.3. *Let $\mathcal{A}$ be a commutative Banach algebra with dual space $\mathcal{A}'$. A character of $\mathcal{A}$ is an element $\phi \in \mathcal{A}' \setminus \{0\}$ which is also multiplicative, i.e. $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in \mathcal{A}$. The set of all characters of $\mathcal{A}$ will be denoted by $\Delta(\mathcal{A})$.*

Remark 2.1.4. In other words, a character $\phi : \mathcal{A} \to \mathbb{C}$ is a continuous, nonzero algebra homomorphism. Moreover, it can be shown that the continuity requirement in this case is superfluous: Every algebra homomorphism from a commutative Banach algebra into $\mathbb{C}$ is automatically continuous ([11, Corollary 2.1.10]).

As $\Delta(\mathcal{A}) \subset \mathcal{A}'$, we can endow $\Delta(\mathcal{A})$ with the subspace topology induced by the weak-*-topology on $\mathcal{A}'$. The resulting topological space is called the maximal ideal space (also known as the structure space or the Gelfand space).

Remark 2.1.5. The name "maximal ideal space" originates from the fact that there is a one-to-one correspondence between characters and maximal ideals of $\mathcal{A}$ via the identification $\phi \in \Delta(\mathcal{A}) \mapsto \ker(\phi)$ (see [11, Corollary 2.1.8]).

It can be shown that the maximal ideal space $\Delta(\mathcal{A})$ is a locally compact Hausdorff space and if $\mathcal{A}$ is unital, then $\Delta(\mathcal{A})$ is compact ([11, Theorem 2.2.3]). Consequently, it makes sense to talk about the Banach algebra $C_0(\Delta(\mathcal{A}))$ of continuous functions from $\Delta(\mathcal{A})$ into $\mathbb{C}$ vanishing at infinity with the usual supremum norm (see also Lemma 1.2.10). Next we define, generalizing the concept from bounded linear operators, the spectrum of an element in a Banach algebra.

Definition 2.1.6. *Let $\mathcal{A}$ be a unital Banach algebra with identity e. The resolvent set of $a \in \mathcal{A}$ is $\rho(a) = \{\lambda \in \mathbb{C} : a - \lambda e \text{ is invertible}\}$. The spectrum of a is $\sigma(a) = \mathbb{C} \setminus \rho(a)$.*

Remark 2.1.7. From the last definition it follows readily that $a \in \mathcal{A}$ is invertible if and only if $0 \notin \sigma(a)$.

Finally, the next theorem defines the Gelfand transform which is the starting point for the considerations in [1].

Theorem 2.1.8 (p. 54 f., [11])**.** *Let $\mathcal{A}$ be a commutative Banach algebra with corresponding maximal ideal space $\Delta(A)$. Then the Gelfand transform*

$$\begin{aligned} \mathcal{A} &\longrightarrow C_0(\Delta(\mathcal{A})) \\ a &\longmapsto \hat{a} = (\phi \mapsto \phi(a)) \end{aligned}$$

is a continuous algebra homomorphism. Moreover, if $\mathcal{A}$ is unital, then $\hat{a}(\Delta(A)) = \sigma(a)$.

Remark 2.1.9. To provide a connection to the preceding sections one should remark at this point that the Gelfand transform on $\ell^1(\mathbb{Z})$ is actually the Fourier series operator (see Definition 1.3.1), using a suitable identification of the set of characters with the torus $\mathbb{T}$.

Now using the last theorem we are in the following situation: Starting with a commutative unital Banach algebra $\mathcal{A}$, the Gelfand transform yields an embedding (i.e. a continuous Algebra homomorphism mapping the unit to the unit) of $\mathcal{A}$ into the Banach algebra $C_0(\Delta(\mathcal{A}))$. For such an embedding it is always true that if $a \in \mathcal{A}$ is invertible, then also the corresponding element $\hat{a} \in C_0(\Delta(\mathcal{A}))$ is invertible. What is special here is that also the other direction is true: If for some $a \in \mathcal{A}$ the corresponding $\hat{a}$ is invertible in $C_0(\Delta(\mathcal{A}))$, then $0 \not\ni \hat{a}(\Delta(\mathcal{A})) = \sigma(a)$ by Theorem 2.1.8 and thus a is invertible in $\mathcal{A}$. This led to the idea of using similar embeddings of $\mathcal{A}$ into spaces of the form $C_0(X)$ (via transforms $a \mapsto \hat{a}$), where X is a topological Hausdorff space, to investigate the problem of norm-controlled inversion. Concretely, can we control $\|a^{-1}\|$ using $\|a\|$ and $\inf_{x \in X} |\hat{a}(x)|$? This approach was used to show that norm-controlled inversion is possible in a variety of so-called Beurling-Sobolev algebras. We are mostly interested in the case of weighted ℓ^1.

Definition 2.1.10. *Let $w \colon \mathbb{Z} \to (0, \infty)$. Then the Beurling algebra $l^1(\mathbb{Z}, w)$ is the set*

$$\{(x_k) \in \mathbb{C}^{\mathbb{N}} : \|x\|_{1,w} := \sum_{k \in \mathbb{Z}} |x_k w_k| < \infty\}$$

together with the convolution (see Theorem 1.2.6) whenever this forms a Banach algebra.

Remark 2.1.11. The conditions on w needed for $l^1(\mathbb{Z}, w)$ to be a Banach algebra can be found in [1, Section 3.2.2].

Theorem 2.1.12 (Section 3.8.4, [1]). *Let*

$$w_k := \begin{cases} |k|^\alpha, & k \neq 0 \\ 1, & k = 0 \end{cases}$$

and denote by $\hat{a}$ the Fourier series (see Definition 1.3.1) of some $a \in \ell^1(\mathbb{Z})$. Then for the following choices of α, $l^1(\mathbb{Z}, w)$ is a Beurling algebra and norm-controlled inversion is possible with the given bound:

(i) $\alpha \geq 1$:

$$\sup\left\{\|a^{-1}\|_{1,w} : a \in \ell^1(\mathbb{Z}, w) \text{ with } \|a\|_{1,w} \leq 1, \inf_{t \in \mathbb{T}} |\hat{a}(t)| \geq \delta\right\} \leq \frac{1}{\delta} + \frac{C_\alpha}{\delta^{4\alpha+2}}.$$

(ii) $1/2 < \alpha < 1$:

$$\sup\left\{\|a^{-1}\|_{1,w} : a \in \ell^1(\mathbb{Z}, w) \text{ with } \|a\|_{1,w} \leq 1, \inf_{t \in \mathbb{T}} |\hat{a}(t)| \geq \delta\right\} \leq \frac{1}{\delta} + \frac{C_\alpha}{\delta^{2+2(1+\beta)(2-\alpha)}},$$

where $\beta = 1/(2\alpha - 1)$.

However, in the unweighted case norm-control is not possible and this is the starting point for this thesis. We first recall a qualitative (and actually positive) statement in this direction which is due to Wiener.

Theorem 2.1.13 (p. 14, Lemma IIe, [12]). *Let $a \in \ell^1(\mathbb{Z})$ with corresponding Fourier series $\hat{a}$. If $\hat{a}(t) \neq 0$ for all $t \in \mathbb{T}$ then $1/\hat{a} = \hat{b}$ for some $b \in \ell^1(\mathbb{Z})$, i.e. the inverse is also a Fourier series.*

Nevertheless, the quantitative version of the above statement (i.e. norm-controlled inversion in $\ell^1(\mathbb{Z})$) fails.

Theorem 2.1.14 (Theorem 1.4.3, [1]). *Let $\hat{a}$ denote the Fourier series of $a \in \ell^1(\mathbb{Z})$. Then for $\frac{1}{2} < \delta < 1$ holds:*

$$\sup\left\{\|a^{-1}\|_1 : a \in \ell^1(\mathbb{Z}) \text{ with } \|a\|_1 \leq 1, \inf_{t \in \mathbb{T}} |\hat{a}(t)| \geq \delta\right\} \geq \frac{1}{2\delta - 1}.$$

In particular, no norm control is possible for $\delta \leq 1/2$.

We will transfer this counterexample to a similar Banach algebra in Chapter 4 and apply the result to Gabor analysis in Section 5.3. However, before we can do that we need to generalize the situation a little bit more. Due to the structure of $C_0(X)$, it is clear that, in the context from before, $\inf_{x \in X} |\hat{a}(x)| = \|\hat{a}^{-1}\|_\infty^{-1}$. Consequently, we can rephrase the problem from above in the following way: Can we control $\|a^{-1}\|$ using $\|a\|$ and $\|\hat{a}^{-1}\|$? This new formulation does not require the Banach algebra $\mathcal{A}$ to be embedded in a space of the form $C_0(X)$ but rather in any other Banach algebra. The following definitions are taken from [13] and [14].

Definition 2.1.15. *Let $\mathcal{A}, \mathcal{B}$ be unital Banach algebras and $\iota \colon \mathcal{A} \to \mathcal{B}$ be a continuous algebra homomorphism mapping the identity of $\mathcal{A}$ to the identity of $\mathcal{B}$.*

(i) $\mathcal{A}$ is inverse-closed in $\mathcal{B}$ if $a \in \mathcal{A}$ and $\iota(a)$ invertible in $\mathcal{B}$ implies that a is invertible in $\mathcal{A}$.

(ii) $\mathcal{A}$ admits norm-controlled inversion in $\mathcal{B}$ if $\mathcal{A}$ is inverse-closed in $\mathcal{B}$ and there exists a function $f \colon (0,\infty)^2 \to (0,\infty)$ such that for any invertible $a \in \mathcal{A}$ holds:

$$\|a^{-1}\| \leq f\left(\|a\|, \|\iota(a)^{-1}\|\right)$$

Remark 2.1.16. The condition that the embedding $\iota \colon \mathcal{A} \to \mathcal{B}$ from the definition above maps the identity of $\mathcal{A}$ to the identity of $\mathcal{B}$ is crucial (and also satisfied by the Gelfand transform) because it ensures that for an invertible element $a \in \mathcal{A}$ the corresponding element $\iota(a)$ is invertible in $\mathcal{B}$.
Without going into details we refer to the following example: Let $\mathcal{A}$ be a unital Banach algebra and $\mathcal{A}_e$ its unitization (see e.g. [11, p. 5 f.]). Then $\mathcal{A}$ can be embedded naturally into $\mathcal{A}_e$ via a continuous Algebra homomorphism ι but this embedding does not map the unit of $\mathcal{A}$ to the unit of $\mathcal{A}_e$. Moreover, $\iota(a)$ is not invertible in $\mathcal{A}_e$ for any $a \in \mathcal{A}$. Consequently, $\mathcal{A}$ would trivially be inverse-closed in $\mathcal{A}_e$. However, this is just due to the fact that the embedding does not respect the structure of $\mathcal{A}$.

2.2 A lack of norm-controlled inversion

Before we go into the proof of Theorem 2.1.14 we start with some preliminary facts about convolution.

Lemma 2.2.1. *Let $a, b \in \ell^1(\mathbb{Z})$ with real, nonnegative entries, i.e. $a_k, b_k \geq 0$ for all $k \in \mathbb{Z}$. Then $\|a * b\| = \|a\|\|b\|$.*

Proof.

$$\begin{aligned} \|a * b\| &= \sum_{k\in\mathbb{Z}} |(a*b)_k| = \sum_{k\in\mathbb{Z}} \left| \sum_{l\in\mathbb{Z}} a_l b_{k-l} \right| = \sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} a_l b_{k-l} \\ &= \sum_{l\in\mathbb{Z}} a_l \sum_{k\in\mathbb{Z}} b_{k-l} \overset{n=k-l}{=} \sum_{l\in\mathbb{Z}} a_l \sum_{n\in\mathbb{Z}} b_n = \|a\|\|b\|, \end{aligned}$$

where the interchange of the summations is justified due to the absolute convergence. □

Lemma 2.2.2. *Let $a, b \in \ell^1(\mathbb{Z}, *)$. Then* $\operatorname{supp}(a * b) \subseteq \operatorname{supp}(a) + \operatorname{supp}(b)$, *where $+$ denotes the sumset. Consequently,* $\operatorname{sup}(a_1 * \cdots * a_n) \subseteq \operatorname{supp}(a_1) + \cdots + \operatorname{sup}(a_n)$.

Proof. Let $n \in \operatorname{supp}(a * b)$. Then

$$0 \neq (a * b)_n = \sum_{k\in\mathbb{Z}} a_k b_{n-k}.$$

Clearly, at least one summand in the series on the right has to be nonzero. We choose one of these and denote the corresponding index by $\tilde{k}$. Hence, $0 \neq a_{\tilde{k}} b_{n-\tilde{k}}$ and so clearly $a_{\tilde{k}}, b_{n-\tilde{k}} \neq 0$. Consequently, $n = \tilde{k} + (n - \tilde{k})$, where $\tilde{k} \in \operatorname{supp}(a)$ and $(n-\tilde{k}) \in \operatorname{supp}(b)$. □

Now we follow the arguments of El-Fallah, Nikolski and Zarrabi in [1, p. 907 ff.] very closely. As the following uses mainly Banach algebra language, we will drop $*$ from the notation and write ab for $a * b$, as usual in this context. We start with a theorem yielding a sufficient condition for impossibility of norm-controlled inversion.

Lemma 2.2.3. *Let $\mathcal{A}$ be a unital Banach algebra, X a compact Hausdorff space and $\iota : \mathcal{A} \to C(X)$ a homomorphism of unital Banach algebras. If for any*

$$0 < \theta < 1, p \in \mathbb{N} \text{ and } N > 0 \tag{2.1}$$

we can find $b \in \mathcal{A}$ such that

(i) $\|b\| \geq N$

(ii) $\|\iota(b)\|_\infty = 1$

(iii) $\|\sum_{k=0}^{p} \lambda^k b^k\| \geq \theta \sum_{k=0}^{p} \lambda^k \|b\|^k \quad \forall 0 < \lambda < 1$

then for $1 < M < 2$ holds:

$$\sup\left\{\|a^{-1}\| : a \in \mathcal{A} \text{ with } \|a\| \leq 1, \|\iota(a)^{-1}\|_\infty \leq M\right\} \geq \frac{M}{2-M}.$$

Proof. We define an element a which depends on b (which again implicitly depends on some choice of θ, N and p) without explicitly denoting the dependence. The idea is to use a limiting argument in the end to obtain the needed lower estimate. Denoting the identity of $\mathcal{A}$ as e, we define

$$a := \frac{1}{1+\mu}\left(e - \mu \frac{b}{\|b\|}\right)$$

for some $0 < \mu < 1$. a clearly satisfies

$$\|a\| = \frac{1}{1+\mu}\left\|e - \mu \frac{b}{\|b\|}\right\| \leq \frac{1}{1+\mu}(1+\mu) = 1.$$

Moreover, for $x \in X$ we have

$$\begin{aligned} |\iota(a)(x)| &= \left|\frac{1}{1+\mu}\left(\iota(e)(x) - \mu \frac{\iota(b)(x)}{\|b\|}\right)\right| \geq \frac{1}{1+\mu}\left(|\iota(e)(x)| - \mu \frac{|\iota(b)(x)|}{\|b\|}\right) \\ &\geq \frac{1}{1+\mu}\left(1 - \frac{\mu}{N}\right) \geq \frac{1}{M}, \end{aligned} \tag{2.2}$$

where the last inequality holds as long as we assume that $\frac{1}{1+\mu} > \frac{1}{M}$, meaning

$$\mu < M - 1, \tag{2.3}$$

and then choose N large enough accordingly. This implies that $\iota(a)$ is invertible and $\|\iota(a)^{-1}\|_\infty \leq M$. As $\|\mu\frac{b}{\|b\|}\| < 1$, it follows from Lemma 1.2.14 that a is invertible with inverse given by

$$a^{-1} = (1+\mu)\sum_{k=0}^{\infty} \mu^k \frac{b^k}{\|b\|^k}.$$

We estimate the norm of a^{-1}:

$$\begin{aligned}
\|a^{-1}\| &= (1+\mu)\left\| \sum_{k=0}^{\infty} \mu^k \frac{b^k}{\|b\|^k} \right\| \geq (1+\mu)\left(\left\| \sum_{k=0}^{p} \mu^k \frac{b^k}{\|b\|^k} \right\| - \left\| \sum_{k=p+1}^{\infty} \mu^k \frac{b^k}{\|b\|^k} \right\| \right) \\
&\overset{(iii)}{\geq} (1+\mu)\left(\theta \sum_{k=0}^{p} \mu^k - \sum_{k=p+1}^{\infty} \mu^k \right) = (1+\mu)\left(\theta \frac{1-\mu^{p+1}}{1-\mu} - \frac{\mu^{p+1}}{1-\mu} \right) \\
&= \frac{1+\mu}{1-\mu}(\theta - (\theta+1)\mu^{p+1}).
\end{aligned}$$

This converges to $\frac{M}{2-M}$ if we first let $p \to \infty$, then $\theta \to 1$ and finally $\mu \to M-1$. Consequently, we get

$$\sup\left\{ \|a^{-1}\| : a \in \mathcal{A} \text{ with } \|a\| \leq 1, \|\iota(a)^{-1}\|_\infty \leq M \right\} \geq \frac{M}{2-M}. \qquad \square$$

In the following, we will show that $\ell^1(\mathbb{Z}, *)$ satisfies the assumptions from Lemma 2.2.3. To this end, we define some special elements which we will use later on.

Definition 2.2.4. *For $j \in \mathbb{N}$ we define $c_j \in \ell^1(\mathbb{Z}, *)$ via*

$$(c_j)_k = \begin{cases} \frac{1}{2}, & k = \pm 3^j \\ 0, & \textit{otherwise} \end{cases}$$

We collect some basic properties of these elements.

Lemma 2.2.5. *For any two index sets $I \neq J \subseteq \{1, \ldots, n\}$ the products $\Pi_{j\in J} c_j$ and $\Pi_{j\in I} c_j$ have disjoint supports.*

Proof. Using Lemma 2.2.2 we can immediately infer that every $k \in \operatorname{supp}\left(\prod_{j\in J} c_j\right)$ has the form $k = \sum_{j\in J} \alpha_j 3^j$ for some $\alpha_j \in \{1, -1\}$. Consequently, in order to prove the claim, we need to consider

$$\sum_{j\in J} \alpha_j 3^j \in \operatorname{supp}\left(\prod_{j\in J} c_j \right)$$

and

$$\sum_{j\in I} \beta_j 3^j \in \operatorname{supp}\left(\prod_{j\in I} c_j \right)$$

satisfying

$$\sum_{j\in J}\alpha_j 3^j = \sum_{j\in I}\beta_j 3^j,$$

where $\alpha_j, \beta_j \in \{1,-1\}$, and show that $I = J$. This follows in particular if we show that

$$\sum_{j=1}^{n}\alpha_j 3^j = \sum_{j=1}^{n}\beta_j 3^j,$$

where $\alpha_j, \beta_j \in \{1,0,-1\}$, implies that $\alpha_j = \beta_j$ for $1 \leq j \leq n$. To this end, we assume the above equality and add $\sum_{j=1}^{n} 3^j$ on both sides. This yields

$$\sum_{j=1}^{n}(\alpha_j + 1)3^j = \sum_{j=1}^{n}(\beta_j + 1)3^j,$$

which is a triadic expansion. Consequently, we immediately get $\alpha_j = \beta_j$ for all $1 \leq j \leq n$, which finishes the proof. □

In order to construct the elements we need for Lemma 2.2.3, we will use imaginary exponentials. As we need to consider linear combinations of these elements (Lemma 2.2.3, property (iii)), the following auxiliary lemma will be of important use. It gives an exact value for the norm of a linear combination of the first-order Taylor polynomial of imaginary exponentials.

Lemma 2.2.6. *For all $n, p \in \mathbb{N}$, $\mu_k, f_k \geq 0$ $(0 \leq k \leq p)$:*

$$\left\| \sum_{k=0}^{p}\mu_k \prod_{j=1}^{n}(e + i f_k c_j)\right\| = \sum_{k=0}^{p}\mu_k(1+f_k)^n,$$

where e denotes the identity of $\ell^1(\mathbb{Z},*)$.

Proof. We expand the product:

$$\begin{aligned}\left\| \sum_{k=0}^{p}\mu_k \prod_{j=1}^{n}(e + i f_k c_j)\right\| &= \left\| \sum_{k=0}^{p}\mu_k \sum_{m=0}^{n} i^m f_k^m \sum_{\substack{J\subseteq\{1,\cdots,n\}\\ |J|=m}} \prod_{j\in J} c_j\right\| \\ &= \left\| \sum_{m=0}^{n} \sum_{\substack{J\subseteq\{1,\cdots,n\}\\ |J|=m}} \left(i^m \sum_{k=0}^{p}\mu_k f_k^m\right) \prod_{j\in J} c_j\right\|.\end{aligned}$$

Then using Lemma 2.2.5 we get

$$\left\| \sum_{m=0}^{n} \sum_{\substack{J\subseteq\{1,\cdots,n\}\\ |J|=m}} \left(i^m \sum_{k=0}^{p}\mu_k f_k^m\right) \prod_{j\in J} c_j\right\| = \sum_{m=0}^{n} \sum_{\substack{J\subseteq\{1,\cdots,n\}\\ |J|=m}} \left| i^m \sum_{k=0}^{p}\mu_k f_k^m\right| \left\| \prod_{j\in J} c_j\right\|.$$

Moreover, Lemma 2.2.1 yields

$$\sum_{m=0}^{n} \sum_{\substack{J \subseteq \{1,\cdots,n\} \\ |J|=m}} \left| i^m \sum_{k=0}^{p} \mu_k f_k^m \right| \left\| \prod_{j \in J} c_j \right\| = \sum_{m=0}^{n} \sum_{\substack{J \subseteq \{1,\cdots,n\} \\ |J|=m}} \sum_{k=0}^{p} \mu_k f_k^m \prod_{j \in J} \|c_j\| .$$

Finally, $\|c_j\| = 1$ for all $1 \leq j \leq n$ implies

$$\begin{aligned} \sum_{m=0}^{n} \sum_{\substack{J \subseteq \{1,\cdots,n\} \\ |J|=m}} \sum_{k=0}^{p} \mu_k f_k^m \prod_{j \in J} \|c_j\| &= \sum_{m=0}^{n} \sum_{\substack{J \subseteq \{1,\cdots,n\} \\ |J|=m}} \sum_{k=0}^{p} \mu_k f_k^m \\ &= \sum_{k=0}^{p} \mu_k \sum_{m=0}^{n} \binom{n}{m} f_k^m = \sum_{k=0}^{p} \mu_k (1+f_k)^n. \end{aligned}$$ □

The preceding lemma yields the key estimate for the main construction.

Lemma 2.2.7. *For all $p \in \mathbb{N}$, $r > 0$, $0 < \theta < 1$ and $\mu_k \geq 0$ $(0 \leq k \leq p)$ we can find $a \in \ell^1(\mathbb{Z}, *)$ such that*

(i) a is self-adjoint

(ii) $\|a\| = 1$

(iii) $\| \sum_{k=0}^{p} \mu_k \exp(ikra) \| \geq \theta \sum_{k=0}^{p} \mu_k e^{kr}$

Proof. We define

$$a_n := \frac{1}{n} \sum_{j=1}^{n} c_j.$$

Then clearly $\|a_n\| = 1$ and a_n is self-adjoint for all $n \in \mathbb{N}$. We will show that also the third condition is satisfied if we choose n large enough.

$$\begin{aligned} \exp(ikra_n) = \prod_{j=1}^{n} \exp(ikrc_j/n) &= \prod_{j=1}^{n} \left(e + \frac{ikrc_j}{n} + r_{j,k} \right) \\ &= \prod_{j=1}^{n} \left(e + \frac{ikrc_j}{n} \right) + s_k \end{aligned}$$

where $r_{j,k}$ and s_k are defined as the corresponding remainder terms which will be estimated in the following: Using $\|c_j\| = 1$ we get

$$\|r_{j,k}\| = \left\| \sum_{l=2}^{\infty} \frac{1}{l!} \frac{(ikr)^l}{n^l} (c_j)^l \right\| \leq \sum_{l=2}^{\infty} \frac{1}{l!} \frac{(kr)^l}{n^l} = e^{\frac{kr}{n}} - \left(1 + \frac{kr}{n} \right). \tag{2.4}$$

Concerning s_k, we first note that

$$\left\| e + \frac{ikrc_j}{n} \right\| \leq 1 + \frac{kr}{n}. \tag{2.5}$$

Every summand of s_k contains at least one factor of the form $r_{j,k}$ and we will estimate the summands by the number of factors they have of the form $r_{j,k}$ using (2.4) and (2.5):

$$\begin{aligned}\|s_k\| &\le \sum_{l=1}^{n}\binom{n}{l}\left(1+\frac{kr}{n}\right)^{n-l}\left(e^{\frac{kr}{n}}-\left(1+\frac{kr}{n}\right)\right)^{l} \\ &= \left(1+\frac{kr}{n}+e^{\frac{kr}{n}}-\left(1+\frac{kr}{n}\right)\right)^{n}-\left(1+\frac{kr}{n}\right)^{n}=e^{kr}-\left(1+\frac{kr}{n}\right)^{n}.\end{aligned} \tag{2.6}$$

Now we are ready to prove the third property:

$$\left\|\sum_{k=0}^{p}\mu_k\exp(ikra_n)\right\| \ge \left\|\sum_{k=0}^{p}\mu_k\prod_{j=1}^{n}\left(e+\frac{ikrc_j}{n}\right)\right\|-\sum_{k=0}^{p}\mu_k\|s_k\|.$$

Lemma 2.2.6 and (2.6) yield

$$\begin{aligned}&\left\|\sum_{k=0}^{p}\mu_k\prod_{j=1}^{n}\left(e+\frac{ikrc_j}{n}\right)\right\|-\sum_{k=0}^{p}\mu_k\|s_k\| \\ &\ge \sum_{k=0}^{p}\mu_k\left(1+\frac{kr}{n}\right)^{n}-\sum_{k=0}^{p}\mu_k\left(e^{kr}-\left(1+\frac{kr}{n}\right)^{n}\right) \\ &= \sum_{k=0}^{p}\mu_k\left(2\left(1+\frac{kr}{n}\right)^{n}-e^{kr}\right).\end{aligned}$$

As $\left(1+\frac{kr}{n}\right)^n \xrightarrow{n\to\infty} e^{kr}$ we can choose n so large that

$$\sum_{k=0}^{p}\mu_k\left(2\left(1+\frac{kr}{n}\right)^{n}-e^{kr}\right) \ge \sum_{k=0}^{p}\mu_k\left(2\cdot\frac{1+\theta}{2}e^{kr}-e^{kr}\right)=\theta\sum_{k=0}^{p}\mu_k e^{kr}, \tag{2.7}$$

which concludes the proof. $\square$

Now we just need to put together all we have in order to prove that the assumptions from Lemma 2.2.3 are satisfied.

Theorem 2.2.8. *Let X be a compact Hausdorff space and $\iota:\ell^1(\mathbb{Z},*)\to C(X)$ a homomorphism of unital, involutive Banach algebras. For all $0<\theta<1$, $p\in\mathbb{N}$ and $N>0$ we can find $b\in\ell^1(\mathbb{Z},*)$ such that*

(i) $\|b\|\ge N$

(ii) $\|\iota(b)\|_\infty = 1$

(iii) $\|\sum_{k=0}^{p}\lambda^k b^k\| \ge \theta\sum_{k=0}^{p}\lambda^k\|b\|^k \quad \forall 0<\lambda<1$

Proof. Fix θ and p. For these values and $r>0$ to be determined later, we define a as in Lemma 2.2.7 and consider

$$b \coloneqq \exp(ira).$$

As

$$r > 0, \tag{2.8}$$

we can use property (iii) from Lemma 2.2.7, which yields

$$\|\exp(ira)\| \geq \theta e^r \geq N \tag{2.9}$$

for r large enough. Moreover, as a is self-adjoint, we know that $\iota(a)$ is self-adjoint and then it follows from Lemma 1.2.18 that $\iota(b) = \exp(ir\iota(a))$ is unitary and thus in particular $\|\iota(b)\|_\infty = 1$ according to Lemma 1.2.13. Finally, again by property (iii) from Lemma 2.2.7:

$$\begin{aligned}\left\|\sum_{k=0}^{p} \lambda^k b^k\right\| = \left\|\sum_{k=0}^{p} \lambda^k \exp(ikra)\right\| &\geq \theta \sum_{k=0}^{p} \lambda^k e^{kr} \geq \theta \sum_{k=0}^{p} \lambda^k \|\exp(ira)\|^k \\ &= \theta \sum_{k=0}^{p} \lambda^k \|b\|^k.\end{aligned}$$

□

Now we are ready to formulate the main result.

Theorem 2.2.9. *Let X be a compact Hausdorff space and $\iota : \ell^1(\mathbb{Z}, *) \to C(X)$ a homomorphism of involutive Banach algebras mapping the unit to the unit. Then for $1 < M < 2$ holds:*

$$\sup\left\{\|a^{-1}\| : a \in \ell^1(\mathbb{Z}, *) \text{ with } \|a\| \leq 1, \|\iota(a)^{-1}\|_\infty \leq M\right\} \geq \frac{M}{2-M}.$$

Proof. From Theorem 2.2.8 we know that the assumptions from Lemma 2.2.3 are satisfied and application of the latter yields the claim. □

As a special case, we apply the preceding theorem to the Fourier series operator $\mathcal{F}$ (see Theorem 2.1.14).

Proof of Theorem 2.1.14. This follows immediately from Theorem 2.2.9 due to Corollary 1.3.4. □

2.3 A support-dependent version

The goal of this section is to give an extension of Theorem 2.1.14, where we also take the support of the sequence into account. We start with a collection of basic inequalities which will be used in this part.

Lemma 2.3.1. *The following inequalities hold:*

(i) $1 + x \leq e^x$ *for* $x \in \mathbb{R}$
(ii) $(1 + x)^y \leq e^{xy}$ *for* $x > -1$ *and* $y > 0$
(iii) $\left(1 + \frac{x}{n}\right)^n \leq e^x$ *for* $n \in \mathbb{N} \setminus \{0\}$ *and* $x > 0$
(iv) $\log(x) \leq y \left(x^{\frac{1}{y}} - 1\right)$ *for* $x, y > 0$
(v) $\left(1 + \frac{x}{n}\right)^n \geq e^x \left(1 - \frac{x^2}{n}\right)$ *for* $n \in \mathbb{N} \setminus \{0\}$ *and* $0 \leq x \leq n$
(vi) $\sqrt{2\pi n} \left(\frac{n}{e}\right)^n < n!$ *for* $n \in \mathbb{N}$

Proof. All these (and many more) inequalities can be found in Bullen's book [15]:

(i) page 84 (c);
(ii) follows from (i) by taking the y-th power;
(iii) follows from (i) inserting x/n and then taking the n-th power;
(iv) inserting $\log(x^{\frac{1}{y}})$ into (i) yields

$$1 + \log\left(x^{\frac{1}{y}}\right) \leq x^{\frac{1}{y}} \iff \frac{1}{y}\log(x) \leq x^{\frac{1}{y}} - 1$$
$$\iff \log(x) \leq y\left(x^{\frac{1}{y}} - 1\right);$$

(v) on page 86 (c);
(vi) follows from the version of Stirling's formula on page 285.

□

We also give an estimate of the tail of the exponential series.

Lemma 2.3.2. *Let* $x > 0$ *and* $K \in \mathbb{N}$*. Then*

$$\sum_{k=K}^{\infty} \frac{x^k}{k!} \leq e^x \frac{x^K}{K!}.$$

Proof. Notice that

$$\sum_{k=K}^{\infty} \frac{x^k}{k!} = \frac{x^K}{K!} \sum_{k=K}^{\infty} \frac{x^{k-K} K!}{k!} \leq \frac{x^K}{K!} \sum_{k=K}^{\infty} \frac{x^{k-K}}{(k-K)!} = \frac{x^K}{K!} e^x.$$ □

Now we move towards our main result. The following results in this section are original contributions. Our goal is to use the construction from Section 2.2, which gives a lower bound for the growth of the inverses, and resolve all the dependencies on the support of the involved sequence in order to get a lower estimate which also takes the support into account. To this end, we first recall the set of elements for which an explicit lower bound for their inverses is given.

Definition 2.3.3. *Let $n \in \mathbb{N} \setminus \{0\}$ and $1 < M \leq 2$. Using the elements c_l from Definition 2.2.4 and denoting the unit of $\ell^1(\mathbb{Z}, *)$ as e, $\mathcal{D}_{n,\delta}$ is the set of all*

$$\frac{1}{1+\mu}\left(e - \mu \frac{\exp\left(\frac{ir}{n}\sum_{j=1}^{n} c_j\right)}{\left\|\exp\left(\frac{ir}{n}\sum_{j=1}^{n} c_j\right)\right\|}\right)$$

such that

(i) $0 < \mu < M - 1$ *(see (2.3));*

(ii) $0 < \theta < 1$ *(see (2.1));*

(iii) $p \in \mathbb{N}$ *(see (2.1));*

(iv) $N, r > 0$ *(see (2.1) and (2.8));*

(v) $\frac{1}{1+\mu}\left(1 - \frac{\mu}{N}\right) \geq \frac{1}{M}$ *(see (2.2));*

(vi) $\left(1 + \frac{kr}{n}\right)^n \geq e^{kr\frac{1+\theta}{2}}$ *for all* $k \in \{0, \ldots, p\}$ *(see (2.7));*

(vii) $\theta e^r \geq N$ *(see (2.9)).*

Moreover, we define

$$D_{n,M} \coloneqq \sup\left\{\left\|a^{-1}\right\| : a \in \mathcal{D}_{n,M}\right\}.$$

First, we reduce the set of conditions to one which is easier to handle.

Lemma 2.3.4. *Let $\mu, \theta, p, N, r \in \mathbb{R}$, $n \in \mathbb{N}$ and $1 < M \leq 2$. The set of conditions*

(i) $0 < \mu < M - 1$

(ii) $0 < \theta < 1$

(iii) $p \in \mathbb{N}$

(iv) $N, r > 0$

(v) $\frac{1}{1+\mu}\left(1 - \frac{\mu}{N}\right) \geq \frac{1}{M}$

(vi) $\left(1 + \frac{kr}{n}\right)^n \geq e^{kr\frac{1+\theta}{2}}$ *for all* $k \in \{0, \ldots, p\}$

(vii) $\theta e^r \geq N$.

is implied by the set of conditions

(a) $0 < \mu < M - 1$

(b) $0 < \theta < 1$

(c) $p \in \mathbb{N}$

(d) $N = \frac{2M\mu}{M-1-\mu}$

(e) $r = \log\left(\frac{N}{\theta}\right)$

(f) $r > 0$

(g) $\frac{p^2r^2}{n} \leq \frac{1-\theta}{2}$

Proof. The first three conditions are equal on both sides. $N > 0$ from (iv) follows from

(d) and (a). $r > 0$ from (iv) is exactly (f). Regarding (v), we note that

$$\begin{aligned}\frac{1}{1+\mu}\left(1-\frac{2\mu}{N}\right) \geq \frac{1}{M} \iff 1-\frac{2\mu}{N} \geq \frac{1+\mu}{M} \iff \frac{M-1-\mu}{M} \geq \frac{2\mu}{N} \\ \iff N \geq \frac{2M\mu}{M-1-\mu},\end{aligned} \tag{2.10}$$

due to (a). Consequently, (v) is satisfied as the rightmost inequality follows from (d). Concerning (vi), we first note that due to (g) and (b):

$$p^2r^2 \leq \frac{(1-\theta)n}{2} < n.$$

Now if $p^2r^2 > 1$, then $pr < p^2r^2 < n$ and if $p^2r^2 \leq 1$, then $pr \leq 1 \leq n$. So in all cases, $pr \leq n$ (and so in particular $kr \leq n$ for all $k \in \{0, \ldots, p\}$). Consequently, we can use (v) from Lemma 2.3.1 together with the following equivalent formulation of (g),

$$\frac{p^2r^2}{n} \leq \frac{1-\theta}{2} \iff 1-\frac{p^2r^2}{n} \geq \frac{1+\theta}{2}, \tag{2.11}$$

to show that (vi) is satisfied:

$$\left(1+\frac{kr}{n}\right)^n \geq e^{kr}\left(1-\frac{k^2r^2}{n}\right) \geq e^{kr}\left(1-\frac{p^2r^2}{n}\right) \geq e^{kr}\frac{1+\theta}{2},$$

for all $k \in \{0, \ldots, p\}$. Finally, (vii) follows immediately from (e). □

The first theorem yields a lower estimate of $D_{n,M}$ in terms of the parameters n and M.

Theorem 2.3.5. *Let $1/4 < b < c < 1/2$. Then*

$$D_{n,M} \geq \frac{1+Mn^b}{1+(2-M)n^b}\left(1-\frac{1}{n^{\frac{1-2c}{4}}} - 2\exp\left(-\frac{n^{c-b}}{2}\right)\right)$$

for $1 < M \leq 2$ and

$$n > \left(\frac{16b}{1-2c}\right)^{\frac{8}{1-2c}}.$$

Proof. Let $n \in \mathbb{N} \setminus \{0\}$ and $1 < M \leq 2$. From the proof of Lemma 2.2.3 it follows that taking an element of $\mathcal{D}_{n,M}$ (depending on μ, θ, p, N and r), the norm of its inverse is larger than

$$\frac{1+\mu}{1-\mu}\left(\theta-(\theta+1)\mu^{p+1}\right). \tag{2.12}$$

We want to choose the element in $\mathcal{D}_{n,M}$ in such a way that the latter expression is large. To this end, we choose the variables μ, θ and p in such a way that they satisfy the reduced set of conditions from Lemma 2.3.4, depending on n and M. So for $b < c < 1/2$ we define

$$p := \lfloor n^c \rfloor, \tag{2.13}$$

$$\mu := (M-1)\frac{n^b}{1+n^b} = (M-1)\left(1-\frac{1}{1+n^b}\right) \tag{2.14}$$

and

$$\theta := 1-\frac{1}{n^{\frac{1-2c}{4}}} \iff \frac{1-\theta}{2} = \frac{1}{2n^{\frac{1-2c}{4}}}.$$

It is evident that these definitions satisfy (a), (b) and (c). Applying (d) and (e), (f) and (g) turn into

$$\log\left(\frac{2M\mu}{(M-1-\mu)\theta}\right) > 0 \tag{2.15}$$

and

$$\frac{p^2\log\left(\frac{2M\mu}{(M-1-\mu)\theta}\right)^2}{n} \leq \frac{1-\theta}{2}. \tag{2.16}$$

Concerning (2.15), we note that

$$\frac{2M\mu}{(M-1-\mu)\theta} = \frac{2M(M-1)\frac{n^b}{1+n^b}}{(M-1)\frac{1}{1+n^b}\theta} = \frac{2Mn^b}{\theta} > 1, \tag{2.17}$$

which implies (2.15). Now the only inequality left to show is (2.16). To this end, we use (2.17) to simplify the left-hand side:

$$\frac{p^2\log\left(\frac{2M\mu}{(M-1-\mu)\theta}\right)^2}{n} = \frac{\lfloor n^c\rfloor^2\log\left(\frac{2Mn^b}{\theta}\right)^2}{n}.$$

Assuming

$$n > 2^{\frac{4}{1-2c}}$$

yields

$$\frac{\lfloor n^c\rfloor^2\log\left(\frac{2Mn^b}{\theta}\right)^2}{n} \leq \frac{\log\left(8n^b\right)^2}{n^{1-2c}}.$$

Using (iv) of Lemma 2.3.1 with $y = \frac{4b}{1-2c}$ yields

$$\frac{\log\left(8n^b\right)^2}{n^{1-2c}} \leq \frac{(4b)^2\left(8^{\frac{1-2c}{4b}}n^{\frac{1-2c}{4}}-1\right)^2}{(1-2c)^2n^{1-2c}} \leq \frac{(4b)^2 8^{\frac{1-2c}{2b}}n^{\frac{1-2c}{2}}}{(1-2c)^2n^{1-2c}} \leq \frac{(4b)^2 8}{(1-2c)^2}\frac{1}{n^{\frac{1-2c}{2}}}. \tag{2.18}$$

Assuming

$$n > \left(\frac{16b}{1-2c}\right)^{\frac{8}{1-2c}}$$

yields

$$\frac{(4b)^2 8}{(1-2c)^2}\frac{1}{n^{\frac{1-2c}{2}}} \leq \frac{1}{2n^{\frac{1-2c}{4}}} = \frac{1-\theta}{2},$$

proving (2.16).

As p, θ and μ satisfy all needed conditions, we get a lower bound for M_n by inserting the corresponding values into (2.12). We estimate the first factor of (2.12) using (2.14):

$$\frac{1+\mu}{1-\mu} = \frac{1+(M-1)\frac{n^b}{1+n^b}}{1-(M-1)\frac{n^b}{1+n^b}} = \frac{\frac{1+n^b+(M-1)n^b}{1+n^b}}{\frac{1+n^b-(M-1)n^b}{1+n^b}} = \frac{1+n^b+(M-1)n^b}{1+n^b-(M-1)n^b} = \frac{1+Mn^b}{1+(2-M)n^b}. \tag{2.19}$$

Concerning the term μ^{p+1}, we first rewrite μ

$$\mu = (M-1)\left(1-\frac{1}{1+n^b}\right) \leq 1-\frac{1}{1+n^b}$$

and then we estimate

$$\mu^{p+1} \leq \left(1-\frac{1}{1+n^b}\right)^{\lfloor n^c\rfloor+1} \leq \left(1-\frac{1}{1+n^b}\right)^{n^c}.$$

(ii) from Lemma 2.3.1 yields

$$\leq \exp\left(-\frac{n^c}{1+n^b}\right) = \exp\left(-\frac{n^{c-b}}{\frac{1}{n^b}+1}\right) \leq \exp\left(-\frac{n^{c-b}}{2}\right). \tag{2.20}$$

Finally, we put all estimates together to get the claimed result:

$$M_n \geq \frac{1+\mu}{1-\mu}\left(\theta-(\theta+1)\mu^{p+1}\right)$$

and using the estimates $\theta < 1$, (2.20) and (2.19), we get

$$\frac{1+\mu}{1-\mu}\left(\theta-(\theta+1)\mu^{p+1}\right) \geq \frac{1+Mn^b}{1+(2-M)n^b}\left(1-\frac{1}{n^{\frac{1-2c}{4}}}-2\exp\left(-\frac{n^{c-b}}{2}\right)\right)$$

for all $1/4 < b < c < 1/2$ and

$$n > \left(\frac{16b}{1-2c}\right)^{\frac{8}{1-2c}}.$$

□

We will use the last theorem to prove the following result which is in the spirit of Theorem 2.1.14 but we additionally take the support of the corresponding sequence into account this time.

Theorem 2.3.6. *Let $\mathcal{F}$ be the Fourier series operator (see Definition 1.3.1). Then for $1 < M \leq 2$, $1/4 < b < c < 1/2$ and*

$$n > \left(\frac{16b}{1-2c}\right)^{\frac{8}{1-2c}}:$$

$$\begin{aligned}
\sup\Big\{ &\left\|a^{-1}\right\| : a \in \ell^1(\mathbb{Z},*) \text{ with } \|a\| \leq 1, \left\|\mathcal{F}(a)^{-1}\right\| \leq M, \\
&\qquad \operatorname{supp}(a) \subseteq \left[-\lceil 3be^{b+1}\log(8^{\frac{1}{b}}n)\rceil 3^n, \lceil 3be^{b+1}\log(8^{\frac{1}{b}}n)\rceil 3^n\right] \Big\} \\
\geq &\frac{1+Mn^b}{1+(2-M)n^b}\left(1 - \frac{1}{n^{\frac{1-2c}{4}}} - 2\exp\left(-\frac{n^{c-b}}{2}\right)\right)\left(1 - \left(3be^{b+1}\log(8^{\frac{1}{b}}n)+1\right)^{-\frac{1}{2}}\right).
\end{aligned} \tag{2.21}$$

Proof. The idea of this proof is to use the lower estimate and sequence of Lemma 2.2.3 and cut it off to obtain a sequence of the desired support. To this end, we first recall the needed definitions from Theorem 2.3.5, Lemma 2.2.7 and Lemma 2.2.3: Let $n \in \mathbb{N}$, $1 < M \leq 2$ and $1/4 < b < c < 1/2$:

$$\begin{aligned}
p_n &:= \lfloor n^c \rfloor, \\
\theta_n &:= 1 - \frac{1}{n^{\frac{1-2c}{4}}}, \\
\mu_n &:= (M-1)\left(1 - \frac{1}{1+n^b}\right) \\
d_n &:= \frac{1}{n}\sum_{j=1}^{n} c_j, \\
r_n &:= \log\left(\frac{2Mn^b}{\theta}\right), \\
b_n &:= \exp\left(ir_n d_n\right), \\
a_n &:= \frac{1}{1+\mu_n}\left(e - \mu_n \frac{b_n}{\|b_n\|}\right),
\end{aligned} \tag{2.22}$$

where the elements c_j are defined in Definition 2.2.4. We also recall that

$$\|\mathcal{F}(b_n)\|_\infty = 1 \tag{2.23}$$

due to it being a unitary element (see Theorem 2.2.8). Now Theorem 2.3.5 already gives a lower estimate for $\left\|a_n^{-1}\right\|$:

$$\left\|a_n^{-1}\right\| \geq \frac{1+Mn^b}{1+(2-M)n^b}\left(1 - \frac{1}{n^{\frac{1-2c}{4}}} - 2\exp\left(-\frac{n^{c-b}}{2}\right)\right)$$

for

$$n > \left(\frac{16b}{1-2c}\right)^{\frac{8}{1-2c}}.$$

In order to get an estimate for finitely-supported sequences we define a cut-off version of a_n. For $K \in \mathbb{N}$ we define

$$a_{n,K} := \frac{1}{1+\mu_n}\left(e - \mu_n \frac{\sum_{k=0}^{K} \frac{(ir_n d_n)^k}{k!}}{\|b_n\|}\right).$$

By definition of c_j we have $\operatorname{supp}(c_j) \subseteq [-3^j, 3^j]$ and so $\operatorname{supp}(d_n) \subseteq [-3^n, 3^n]$. Together with Lemma 2.2.2 this yields $\operatorname{supp}\left(d_n^k\right) \subseteq [-k3^n, k3^n]$ and thus

$$\operatorname{supp}\left(a_{n,K}\right) \subseteq [-K3^n, K3^n]. \tag{2.24}$$

Next, we estimate the difference

$$\begin{aligned} \|a_n - a_{n,K}\| &= \frac{\mu_n}{(1+\mu_n)\|b_n\|}\left\| \sum_{k=K+1}^{\infty} \frac{(ir_n d_n)^k}{k!} \right\| \\ &\le \frac{\mu_n}{(1+\mu_n)\|b_n\|} \sum_{k=K+1}^{\infty} \frac{r_n^k}{k!} \overset{Lemma\ 2.3.2}{\le} \frac{\mu_n}{(1+\mu_n)\|b_n\|} e^{r_n} \frac{r_n^{K+1}}{(K+1)!}. \end{aligned} \tag{2.25}$$

Inserting the definition of r and using that $\|b_n\| \geq \theta_n e^{r_n}$ (see Lemma 2.2.7 (iii)) we get

$$\frac{\mu_n}{(1+\mu_n)\|b_n\|} e^{r_n} \frac{r_n^{K+1}}{(K+1)!} \le \frac{1}{(1+\mu_n)\theta_n} \frac{\log\left(\frac{2Mn^b}{\theta}\right)^{K+1}}{(K+1)!} \le \frac{2\log\left(8n^b\right)^{K+1}}{(K+1)!} \tag{2.26}$$

and then Stirling's approximation (Lemma 2.3.1 (v)) yields

$$\frac{2\log\left(8n^b\right)^{K+1}}{(K+1)!} \le \frac{2}{\sqrt{2\pi(K+1)}} \frac{e^{K+1}\log\left(8n^b\right)^{K+1}}{(K+1)^{K+1}} = \frac{2}{\sqrt{2\pi(K+1)}} \left(\frac{eb\log(8^{\frac{1}{b}}n)}{K+1}\right)^{K+1}. \tag{2.27}$$

Now we use Lemma 1.2.15 to estimate the difference of the inverses. We first note that

$$\left\|a_n^{-1}\right\| = (1+\mu_n)\left\| \sum_{k=0}^{\infty} \mu_n^k \frac{b_n^k}{\|b_n\|^k} \right\| \le (1+\mu_n)\sum_{k=0}^{\infty} \mu_n^k = \frac{1+\mu_n}{1-\mu_n}. \tag{2.28}$$

Using this we get

$$\begin{aligned} \left\|a_n^{-1} - a_{n,K}^{-1}\right\| &\overset{Lemma\ 1.2.15}{\le} 2\left\|a_n^{-1}\right\|^2 \|a_n - a_{n,K}\| \\ &\overset{(2.27),(2.28)}{\le} 2\left\|a_n^{-1}\right\| \frac{1+\mu_n}{1-\mu_n} \frac{2}{\sqrt{2\pi(K+1)}} \left(\frac{eb\log(8^{\frac{1}{b}}n)}{K+1}\right)^{K+1}, \end{aligned} \tag{2.29}$$

where we postpone the verification of the assumptions of Lemma 1.2.15 to the point where we actually define K. Consequently, we can estimate

$$\begin{aligned}\left\|a_{n,K}^{-1}\right\| &\geq \left\|a_n^{-1}\right\| - \left\|a_n^{-1} - a_{n,K}^{-1}\right\| \\ &\overset{(2.29)}{\geq} \left\|a_n^{-1}\right\| - 2\left\|a_n^{-1}\right\| \frac{1+\mu_n}{1-\mu_n} \frac{2}{\sqrt{2\pi(K+1)}} \left(\frac{eb\log(8^{\frac{1}{b}}n)}{K+1}\right)^{K+1} \\ &\geq \left\|a_n^{-1}\right\| \left(1 - \frac{1}{1-\mu_n} \frac{8}{\sqrt{2\pi(K+1)}} \left(\frac{eb\log(8^{\frac{1}{b}}n)}{K+1}\right)^{K+1}\right).\end{aligned} \tag{2.30}$$

In order to further estimate this, we first note that

$$\begin{aligned}\frac{1}{1-\mu_n} &= \frac{1}{1-(M-1)\frac{n^b}{1+n^b}} = \frac{1+n^b}{1+n^b-(M-1)n^b} = \frac{1+n^b}{1+(2-M)n^b} \\ &\leq 2n^b = 2\big(e^b\big)^{\log(n)}.\end{aligned} \tag{2.31}$$

Using this we can bound the term inside the parenthesis in (2.30) assuming that $\log(n) \leq K(n)$:

$$\frac{1}{1-\mu_n} \frac{8}{\sqrt{2\pi(K+1)}} \left(\frac{eb\log(8^{\frac{1}{b}}n)}{K+1}\right)^{K+1} \leq \frac{16}{\sqrt{2\pi(K+1)}} \left(\frac{b\log(8^{\frac{1}{b}}n)e^{b+1}}{K+1}\right)^{K+1}. \tag{2.32}$$

Now we specify $K := \lceil 3be^{b+1}\log(8^{\frac{1}{b}}n)\rceil$ (which obviously satisfies $\log(n) \leq K(n)$) and insert this into the expression above:

$$\begin{aligned}&\frac{16}{\sqrt{2\pi(K+1)}} \left(\frac{b\log(8^{\frac{1}{b}}n)e^{b+1}}{K+1}\right)^{K+1} \\ &= \frac{16}{\sqrt{2\pi(\lceil 3be^{b+1}\log(8^{\frac{1}{b}}n)\rceil + 1)}} \left(\frac{b\log(8^{\frac{1}{b}}n)e^{b+1}}{\lceil 3be^{b+1}\log(8^{\frac{1}{b}}n)\rceil + 1}\right)^{\lceil 3be^{b+1}\log(8^{\frac{1}{b}}n)\rceil+1} \\ &\leq \frac{8}{\sqrt{\lceil 3be^{b+1}\log(8^{\frac{1}{b}}n)\rceil + 1}} \left(\frac{1}{3}\right)^{\lceil 3be^{b+1}\log(8^{\frac{1}{b}}n)\rceil+1} \\ &\overset{\lceil 3be^{b+1}\log(8^{\frac{1}{b}}n)\rceil \geq 1}{\leq} \big(\lceil 3be^{b+1}\log(8^{\frac{1}{b}}n)\rceil + 1\big)^{-\frac{1}{2}} \leq \big(3be^{b+1}\log(8^{\frac{1}{b}}n) + 1\big)^{-\frac{1}{2}}.\end{aligned} \tag{2.33}$$

Combining this with (2.30) yields the desired lower estimate

$$
\begin{aligned}
\left\|a_{n,K}^{-1}\right\| &\geq \left\|a_n^{-1}\right\| \left(1 - \left(3be^{b+1}\log(8^{\frac{1}{b}}n)+1\right)^{-\frac{1}{2}}\right) \\
&\geq \left(\frac{1+Mn^b}{8C+(2-M)n^b}\right)\left(1-2\left(\frac{1}{4n^{\frac{1-2c}{2}}}+\exp\left(-2n^{c-b}\right)\right)\right)\cdot \\
&\quad \left(1 - \left(3be^{b+1}\log(8^{\frac{1}{b}}n)+1\right)^{-\frac{1}{2}}\right).
\end{aligned}
$$

We also note at this point that the assumptions of Lemma 1.2.15 (which we applied in (2.29)) are satisfied:

$$
\begin{aligned}
\frac{4}{1-\mu_n}\left\|a_n - a_{n,K}\right\| &\overset{(2.27)}{\leq} \frac{1}{1-\mu_n}\frac{8}{\sqrt{2\pi(K+1)}}\left(\frac{eb\log(8^{\frac{1}{b}}n)}{K+1}\right)^{K+1} \\
&\overset{(2.32)\text{ and }(2.33)}{\leq} \left(3be^{b+1}\log(8^{\frac{1}{b}}n)+1\right)^{-\frac{1}{2}} \leq 1
\end{aligned}
$$

and thus

$$
\left\|a_n - a_{n,K}\right\| \leq \frac{1-\mu_n}{4} \leq \frac{1}{2}\frac{1-\mu_n}{1+\mu_n} \overset{(2.28)}{\leq} \frac{1}{2}\left\|a_n^{-1}\right\|^{-1}.
$$

Finally,

$$
\left\|a_{n,K}\right\| \leq \left\|a_n\right\| \leq 1
$$

and (2.24) yields

$$
\operatorname{supp}(a_{n,K}) \subseteq [-K3^n, K3^n] = [-\lceil 3be^{b+1}\log(8^{\frac{1}{b}}n)\rceil 3^n, \lceil 3be^{b+1}\log(8^{\frac{1}{b}}n)\rceil 3^n].
$$

Moreover,

$$
\left\|\mathcal{F}\Big(\sum_{k=K+1}^{\infty}\frac{(ir_nd_n)^k}{k!}\Big)\right\|_\infty \leq \left\|\sum_{k=K+1}^{\infty}\frac{(ir_nd_n)^k}{k!}\right\|
$$

and then analogously to the calculation in (2.25) we get

$$
\left\|\sum_{k=K+1}^{\infty}\frac{(ir_nd_n)^k}{k!}\right\| \leq e^{r_n}\frac{r_n^{K+1}}{(K+1)!} \overset{(2.22)}{=} \frac{2Mn^b}{\theta_n}\frac{r_n^{K+1}}{(K+1)!}.
$$

Now the same calculations as in (2.26) and (2.27) yield

$$
\begin{aligned}
\frac{2Mn^b}{\theta_n}\frac{r_n^{K+1}}{(K+1)!} &\leq \frac{8n^b}{\sqrt{2\pi(K+1)}}\left(\frac{eb\log(8^{\frac{1}{b}}n)}{K+1}\right)^{K+1} \\
&\leq \frac{8}{\sqrt{2\pi(K+1)}}\left(\frac{b\log(8^{\frac{1}{b}}n)e^{b+1}}{K+1}\right)^{K+1}
\end{aligned}
$$

and applying the arguments from (2.32) and gives

$$\frac{8}{\sqrt{2\pi(K+1)}}\left(\frac{b\log(8^{\frac{1}{b}}n)e^{b+1}}{K+1}\right)^{K+1} \leq \frac{8}{\sqrt{3be^{b+1}\log(n)+1}} \leq 1. \tag{2.34}$$

Consequently, for all $x \in \mathbb{T}$ we have

$$\begin{aligned}
|\mathcal{F}(a_{n,K})(x)| &= \left|\mathcal{F}\left(\frac{1}{1+\mu_n}\left(e-\mu_n\frac{\sum_{k=0}^{K}\frac{(ir_nd_n)^k}{k!}}{\|b_n\|}\right)\right)(x)\right| \\
&\geq \frac{1}{1+\mu_n}\left(1-\mu_n\frac{\|\mathcal{F}\left(\sum_{k=0}^{K}\frac{(ir_nd_n)^k}{k!}\right)\|_\infty}{\|b_n\|}\right) \\
&\geq \frac{1}{1+\mu_n}\left(1-\mu_n\frac{\|\mathcal{F}\left(\sum_{k=K+1}^{\infty}\frac{(ir_nd_n)^k}{k!}\right)\|_\infty+\|\mathcal{F}\left(\sum_{k=0}^{\infty}\frac{(ir_nd_n)^k}{k!}\right)\|_\infty}{\|b_n\|}\right) \\
&\overset{(2.34)\text{ and }(2.23)}{\geq} \frac{1}{1+\mu_n}\left(1-\mu_n\frac{2}{\|b_n\|}\right) \overset{(2.10)}{\geq} \frac{1}{M}
\end{aligned}$$

and thus $\|\mathcal{F}(a_{n,K})^{-1}\|_\infty \leq M$, which completes the argument. $\square$

In the following, we set the free parameters b and c from the last theorem.

Corollary 2.3.7. *Let $\mathcal{F}$ be the Fourier series operator (see Definition 1.3.1). Then for $1 < M \leq 2$ and $n > e^{e^5}$:*

$$\begin{aligned}
&\sup\left\{\left\|a^{-1}\right\| : a \in \ell^1(\mathbb{Z},*) \text{ with } \|a\| \leq 1, \left\|\mathcal{F}(a)^{-1}\right\| \leq M \right. \\
&\qquad\qquad \left. \text{and } \operatorname{supp}(a) \subseteq [-7\log(4096n)3^n, 7\log(4096n)3^n]\right\} \\
&\geq \frac{1+M\frac{n^{\frac{1}{2}}}{\log(n)^5}}{1+(2-M)\frac{n^{\frac{1}{2}}}{\log(n)^5}}\left(1-\frac{2}{\log(n)^2}\right)\left(1-\left(2\log(64n)+1\right)^{-\frac{1}{2}}\right)
\end{aligned}$$

Proof. We define

$$c := \frac{1-\frac{8\log(\log(n))}{\log(n)}}{2} \tag{2.35}$$

and

$$b := c - \frac{\log(\log(n))}{\log(n)} = \frac{1-\frac{8\log(\log(n))}{\log(n)}}{2} - \frac{\log(\log(n))}{\log(n)} = \frac{1}{2} - \frac{5\log(\log(n))}{\log(n)}.$$

Then we have

$$\log\left(\left(\frac{16b}{1-2c}\right)^{\frac{8}{1-2c}}\right) = \frac{8}{1-2c}\log\left(\frac{16b}{1-2c}\right).$$

Using that $b < 1/2$ yields

$$\frac{8}{1-2c}\log\left(\frac{16b}{1-2c}\right) < \frac{8}{1-2c}\log\left(\frac{8}{1-2c}\right)$$

and then (2.35) gives

$$\frac{8}{1-2c}\log\left(\frac{8}{1-2c}\right) = \frac{\log(n)}{\log(\log(n))}\log\left(\frac{\log(n)}{\log(\log(n))}\right) < \log(n),$$

where the last inequality follows from the fact that

$$\log\left(\frac{\log(n)}{\log(\log(n))}\right) < \log(\log(n)),$$

for $n > e^e$. Consequently,

$$\left(\frac{16b}{1-2c}\right)^{\frac{8}{1-2c}} < n.$$

Moreover, it is evident that $b < c < 1/2$ for $n > e^e$ and choosing

$$n > e^{e^5}$$

also establishes $1/4 < b$. Thus we can apply Theorem 2.3.6 to our choice of b and c. The claim then follows after simplifying the support condition using

$$3be^{b+1}\log(8^{\frac{1}{b}}n) < 7\log(4096n)$$

and adjusting the given lower bound. To this end, we first note that

$$n^{\frac{\log(\log(n))}{\log(n)}} = e^{\log(n)\frac{\log(\log(n))}{\log(n)}} = e^{\log(\log(n))} = \log(n)$$

and thus

$$n^b = \frac{n^{\frac{1}{2}}}{n^{\frac{5\log(\log(n))}{\log(n)}}} = \frac{n^{\frac{1}{2}}}{\log(n)^5}.$$

Consequently, we get

$$\begin{aligned}
&\frac{1+Mn^b}{1+(2-M)n^b}\left(1-\frac{1}{n^{\frac{1-2c}{4}}}-2\exp\left(-\frac{n^{c-b}}{2}\right)\right)\left(1-\left(3be^{b+1}\log(8^{\frac{1}{b}}n)+1\right)^{-\frac{1}{2}}\right)\\
&= \frac{1+M\frac{n^{\frac{1}{2}}}{\log(n)^5}}{1+(2-M)\frac{n^{\frac{1}{2}}}{\log(n)^5}}\left(1-\frac{1}{\log(n)^2}-2\exp\left(-\frac{\log(n)}{2}\right)\right)\cdot\\
&\quad\left(1-\left(3be^{b+1}\log(8^{\frac{1}{b}}n)+1\right)^{-\frac{1}{2}}\right).
\end{aligned}$$

Using $1/4 < b < 1/2$ in the last term yields

$$\frac{1 + M\frac{n^{\frac{1}{2}}}{\log(n)^5}}{1 + (2-M)\frac{n^{\frac{1}{2}}}{\log(n)^5}} \left(1 - \frac{1}{\log(n)^2} - 2\exp\left(-\frac{\log(n)}{2}\right)\right)\left(1 - \left(3be^{b+1}\log(8^{\frac{1}{b}}n) + 1\right)^{-\frac{1}{2}}\right)$$
$$\geq \frac{1 + M\frac{n^{\frac{1}{2}}}{\log(n)^5}}{1 + (2-M)\frac{n^{\frac{1}{2}}}{\log(n)^5}} \left(1 - \frac{1}{\log(n)^2} - \frac{2}{n^{\frac{1}{2}}}\right)\left(1 - (2\log(64n) + 1)^{-\frac{1}{2}}\right).$$

Finally, as

$$\frac{2}{n^{\frac{1}{2}}} < \frac{1}{\log(n)^2}$$

for $n > e^{e^5}$, we arrive at the claimed lower bound:

$$\frac{1 + M\frac{n^{\frac{1}{2}}}{\log(n)^5}}{1 + (2-M)\frac{n^{\frac{1}{2}}}{\log(n)^5}} \left(1 - \frac{1}{\log(n)^2} - \frac{2}{n^{\frac{1}{2}}}\right)\left(1 - \left(2\log(64n) + 1\right)^{-\frac{1}{2}}\right)$$
$$\geq \frac{1 + M\frac{n^{\frac{1}{2}}}{\log(n)^5}}{1 + (2-M)\frac{n^{\frac{1}{2}}}{\log(n)^5}} \left(1 - \frac{2}{\log(n)^2}\right)\left(1 - \left(2\log(64n) + 1\right)^{-\frac{1}{2}}\right).$$

□

Remark 2.3.8.

- If we let $n \to \infty$ in the preceding theorem we recover the original result which does not take the support of the sequence into account (see Theorem 2.1.14).
- We note that, concerning the expression $\frac{n^{\frac{1}{2}}}{\log(n)^5}$, which determines the asymptotical growth in the case $M = 2$, the denominator $\log(n)^5$ is not chosen to be optimal. It can be improved by altering the choice of b and c in the last theorem at the expense of the admissible range for n.

After inverting the support condition and estimating some constants, we arrive at the following corollary.

Corollary 2.3.9. *Let $\mathcal{F}$ be the Fourier series operator (see Definition 1.3.1). Then for $1 < M \leq 2$ and $N \geq 3^{2e^{e^5}}$:*

$$\sup\left\{ \left\|a^{-1}\right\| : a \in \ell^1(\mathbb{Z}, *) \text{ with } \|a\| \leq 1, \left\|\mathcal{F}(a)^{-1}\right\| \leq M \text{ and } \operatorname{supp}(a) \subseteq [-N, N] \right\}$$
$$\geq \frac{9}{10} \frac{1 + M\frac{\log(N)^{\frac{1}{2}}}{\log(\log(N))^6}}{1 + (2-M)\frac{\log(N)^{\frac{1}{2}}}{\log(\log(N))^6}}.$$

Proof. Our first goal is to invert the support condition from the previous theorem. To this end, we note that the lower bound (2.21) is increasing with respect to n. Consequently, the inequality

$$\lceil 7\log(4096n)\rceil 3^n \le 3^{2n},$$

which is satisfied due to

$$n > e^{e^5},$$

together with the fact that

$$N = 3^{2n} \iff n = \frac{\log_3(N)}{2}$$

yield the inequality with inverted support condition. For $1 < M \le 2$ and $N \ge 3^{2e^{e^5}}$:

$$\begin{aligned}
&\sup\left\{ \left\|a^{-1}\right\| : a \in \ell^1(\mathbb{Z},*) \text{ with } \|a\| \le 1, \left\|\mathcal{F}(a)^{-1}\right\| \le M \text{ and } \operatorname{supp}(a) \subseteq [-N,N] \right\}\\
&\ge \frac{1 + M\frac{\left\lfloor \frac{\log_3(N)}{2}\right\rfloor^{\frac{1}{2}}}{\log\left(\left\lfloor \frac{\log_3(N)}{2}\right\rfloor\right)^5}}{1 + (2-M)\frac{\left\lfloor \frac{\log_3(N)}{2}\right\rfloor^{\frac{1}{2}}}{\log\left(\left\lfloor \frac{\log_3(N)}{2}\right\rfloor\right)^5}} \left(1 - 2\log\left(\left\lfloor \frac{\log_3(N)}{2}\right\rfloor\right)^{-2}\right)\cdot\\
&\quad \left(1 - \left(2\log\left(64\left\lfloor \frac{\log_3(N)}{2}\right\rfloor\right) + 1\right)^{-\frac{1}{2}}\right).
\end{aligned}$$

The floor function is needed because the original parameter n has to be a positive integer. Moreover, using the inequalities

$$\left(1 - \left(2\log\left(64\left\lfloor \frac{\log_3(N)}{2}\right\rfloor\right) + 1\right)^{-\frac{1}{2}}\right) \ge \frac{19}{20}$$

and

$$\left(1 - 2\log\left(\left\lfloor \frac{\log_3(N)}{2}\right\rfloor\right)^{-2}\right) \ge \frac{180}{190}$$

for all $N \ge 3^{2e^{e^5}}$, we deduce the simplified version:

$$\begin{aligned}
&\sup\left\{ \left\|a^{-1}\right\| : a \in \ell^1(\mathbb{Z},*) \text{ with } \|a\| \le 1, \left\|\mathcal{F}(a)^{-1}\right\| \le M \text{ and } \operatorname{supp}(a) \subseteq [-N,N] \right\}\\
&\ge \frac{9}{10}\frac{1 + M\frac{\left\lfloor \frac{\log_3(N)}{2}\right\rfloor^{\frac{1}{2}}}{\log\left(\left\lfloor \frac{\log_3(N)}{2}\right\rfloor\right)^5}}{1 + (2-M)\frac{\left\lfloor \frac{\log_3(N)}{2}\right\rfloor^{\frac{1}{2}}}{\log\left(\left\lfloor \frac{\log_3(N)}{2}\right\rfloor\right)^5}}.
\end{aligned}$$

Finally, we note that the function

$$f(x) := \frac{1 + ax}{1 + bx}$$

is increasing with respect to x whenever $a \geq b$, as

$$f'(x) = \frac{(1+bx)a - (1+ax)b}{(1+bx)^2} = \frac{a-b}{(1+bx)^2} \geq 0 \iff a \geq b. \tag{2.36}$$

As $M \geq 2 - M$, we can apply this to our last lower bound. To this end, we first collect the estimates

$$\left\lfloor \frac{\log_3(N)}{2} \right\rfloor \leq \log(N) \tag{2.37}$$

and

$$\begin{aligned} \left\lfloor \frac{\log_3(N)}{2} \right\rfloor^{\frac{1}{2}} &= \left\lfloor \frac{\log(N)}{2\log(3)} \right\rfloor^{\frac{1}{2}} \geq \left(\frac{\log(N)}{4\log(3)} \right)^{\frac{1}{2}} = \left(\frac{1}{4\log(3)} \right)^{\frac{1}{2}} \log(N)^{\frac{1}{2}} \\ &\geq \frac{1}{\log(\log(N))} \log(N)^{\frac{1}{2}}, \end{aligned} \tag{2.38}$$

where the last inequality follows because $N \geq 3^{2e^{e^5}}$. Combining (2.37) and (2.38) yields

$$\frac{\left\lfloor \frac{\log_3(N)}{2} \right\rfloor^{\frac{1}{2}}}{\log\left(\left\lfloor \frac{\log_3(N)}{2} \right\rfloor \right)^5} \geq \frac{\log(N)^{\frac{1}{2}}}{\log(\log(N))^6}$$

and due to (2.36) this proves the claim, as

$$\frac{9}{10} \frac{1 + M \frac{\left\lfloor \frac{\log_3(N)}{2} \right\rfloor^{\frac{1}{2}}}{\log\left(\left\lfloor \frac{\log_3(N)}{2} \right\rfloor \right)^5}}{1 + (2-M) \frac{\left\lfloor \frac{\log_3(N)}{2} \right\rfloor^{\frac{1}{2}}}{\log\left(\left\lfloor \frac{\log_3(N)}{2} \right\rfloor \right)^5}} \geq \frac{9}{10} \frac{1 + M \frac{\log(N)^{\frac{1}{2}}}{\log(\log(N))^6}}{1 + (2-M) \frac{\log(N)^{\frac{1}{2}}}{\log(\log(N))^6}}. \qquad \square$$

We finish this chapter by giving an upper estimate in the case $M = 2$, where the blow-up occurs, using the results from Theorem 2.1.12.

Proposition 2.3.10. *Let $\mathcal{F}$ be the Fourier series operator (see Definition 1.3.1). Then*

$$\sup \left\{ \left\| a^{-1} \right\| : a \in \ell^1(\mathbb{Z}, *) \text{ with } \|a\| \leq 1, \left\| \mathcal{F}(a)^{-1} \right\| \leq 2 \text{ and } \operatorname{supp}(a) \subseteq [-N, N] \right\} \lesssim N^7.$$

Proof. Let $a \in \ell^1(\mathbb{Z}, *)$ with $\|a\| \leq 1$, $\left\| \mathcal{F}(a)^{-1} \right\| \leq 2$ and $\operatorname{supp}(a) \subseteq [-N, N]$. We consider the weight

$$w_k := \begin{cases} |k|, & k \neq 0 \\ 1, & k = 0 \end{cases}$$

with corresponding norm

$$\|a\|_{1,w} := \sum_{k \in \mathbb{Z}} |a_k w_k|$$

and define

$$b := \frac{1}{N}a.$$

As the weight w is increasing with respect to the absolute value of its argument and $\operatorname{supp}(a) \subseteq [-N, N]$, we get

$$\|b\|_{1,w} = \frac{1}{N}\|a\|_{1,w} \leq \frac{1}{N}\|a\|_1 N \leq \frac{1}{N}N = 1.$$

Moreover,

$$\left\|\mathcal{F}(b)^{-1}\right\| = N\left\|\mathcal{F}(a)^{-1}\right\| \leq 2N.$$

Consequently, Theorem 2.1.12 yields a constant $C > 0$ such that

$$\left\|a^{-1}\right\| = \frac{1}{N}\left\|b^{-1}\right\| \leq \frac{1}{N}\left\|b^{-1}\right\|_{1,w} \leq \frac{1}{N}(2N + C(2N)^6) \lesssim N^5. \qquad \square$$

Remark 2.3.11. The upper bound from the last theorem grows significantly faster than the corresponding lower bound given in Corollary 2.3.9. The exact asymptotic behavior is unknown to us.

2.4 Application to the weighted case

We now want to briefly look into the case of weighted ℓ^1 and apply our findings from the last section there. We start with a preliminary lemma about the polylogarithm.

Lemma 2.4.1. *Let $0 < x_0 < 1$ and $s > -1$. Then*

$$\sum_{k=1}^{\infty} x^k k^s \geq C_{x_0,s}(1-x)^{-1-s}$$

for all $x_0 \leq x < 1$, where the constant $C_{x_0,s}$ is independent of x.

Proof. Both $\sum_{k=1}^{\infty} x^k k^s$ and $(1-x)^{-1-s}$ are well-defined and do not vanish in $[x_0, 1)$. Moreover,

$$\lim_{x\to 1^-}(1-x)^{1+s}\sum_{k=1}^{\infty} x^k k^s = \Gamma(1+s),$$

where Γ is the gamma function, which can be found in [16, p. 30, (12) and (14)]. Consequently, the claim follows immediately. $\square$

We continue by recalling a result concerning $\ell^1(\mathbb{Z}, w)$ which is outlined in the last part of the article [1] by El-Fallah, Nikolski and Zarrabi. We also provide the proof as we want to refer to it afterwards.

Theorem 2.4.2 (Section 3.8.6, [1]). *Let $\mathcal{F}$ be the Fourier series operator (see Definition 1.3.1) and w be the weight*

$$w_k := \begin{cases} |k|^\alpha, & k \neq 0 \\ 1, & k = 0, \end{cases} \tag{2.39}$$

where $\alpha > 0$. Then for $M > 2$,

$$\sup\left\{\|a^{-1}\|_{1,w} : a \in \ell^1(\mathbb{Z}, w) \text{ with } \|a\|_{1,w} \leq 1, \left\|\mathcal{F}(a)^{-1}\right\|_\infty \leq M\right\} \geq C_\alpha M^{1+\alpha},$$

where $C_\alpha > 0$ is a constant depending only on α.

Proof. Let $M > 2$ and $0 < \lambda < 1$ whose exact value will be specified later. Denoting the k-th unit vector by $u_k \in \ell^1(\mathbb{Z})$ we define

$$a := \frac{1}{1+\lambda}(u_0 - \lambda u_1).$$

Then

$$\|a\|_{1,w} = \frac{1}{1+\lambda}(1+\lambda) = 1.$$

Moreover,

$$|\mathcal{F}(a)(t)| = \frac{1}{1+\lambda}|1 - \lambda e^{2\pi i t}| \geq \frac{1-\lambda}{1+\lambda}$$

for all $t \in \mathbb{T}$ and thus $\left\|\mathcal{F}(a)^{-1}\right\|_\infty \leq \frac{1+\lambda}{1-\lambda}$. Thus, $\left\|\mathcal{F}(a)^{-1}\right\|_\infty \leq M$ can be ensured by choosing

$$\frac{1+\lambda}{1-\lambda} = M \iff \lambda = \frac{M-1}{M+1}. \tag{2.40}$$

Furthermore, using Lemma 1.2.14 we immediately get that a is invertible and

$$a^{-1} = (1+\lambda)\sum_{k=0}^{\infty} \lambda^k u_1^k.$$

At this point we note that $u_1^k = u_k$ and so we can compute the norm of a^{-1} directly. In the following calculation, the implied constant corresponding to the symbol $\gtrsim$ depends only on α.

$$\begin{aligned} \|a^{-1}\|_{1,w} &= (1+\lambda)\sum_{k=0}^{\infty}\lambda^k w_k \overset{(2.40)}{=} \frac{2M}{M+1}\sum_{k=0}^{\infty}\left(\frac{M-1}{M+1}\right)^k w_k \\ &\overset{(2.39)}{=} \frac{2M}{M+1}\left(1 + \sum_{k=1}^{\infty}\left(\frac{M-1}{M+1}\right)^k k^\alpha\right) \geq \sum_{k=1}^{\infty}\left(\frac{M-1}{M+1}\right)^k k^\alpha \\ &\overset{\textit{Lemma 2.4.1}}{\gtrsim} \left(\frac{2}{M+1}\right)^{-1-\alpha} \gtrsim M^{1+\alpha}. \end{aligned} \tag{2.41}$$

□

For a slightly different setting, we can apply Corollary 2.3.9 to get the following result.

Theorem 2.4.3. *Let $\mathcal{F}$ be the Fourier series operator (see Definition 1.3.1) and w be the weight*

$$w_k := \begin{cases} |k|^\alpha, & k \neq 0 \\ 1, & k = 0, \end{cases}$$

where $\alpha > 0$. Then for $M > 2 \cdot 9^\alpha$

$$\sup\left\{\left\|a^{-1}\right\|_1 : a \in \ell^1(\mathbb{Z}, w) \text{ with } \|a\|_{1,w} \leq 1, \left\|\mathcal{F}(a)^{-1}\right\|_\infty \leq M\right\} \geq C_\alpha M \frac{\log(M)^{\frac{1}{2}}}{\log(\log(M))^6},$$

where $C_\alpha > 0$ is a constant depending only on α.

Remark 2.4.4. Note that if we apply the methods of Theorem 2.4.2 to this setting, we get something weaker:

$$\sup\left\{\left\|a^{-1}\right\|_1 : a \in \ell^1(\mathbb{Z}, w) \text{ with } \|a\|_{1,w} \leq 1, \left\|\mathcal{F}(a)^{-1}\right\|_\infty \leq M\right\} \geq C_\alpha M.$$

Since Theorem 2.4.3 provides a better lower bound for the weaker unweighted 1-norm, it would be interesting to see if it actually implies Theorem 2.4.2. Regrettably, this still remains unknown to us.

Proof of Theorem 2.4.3. We first note that whenever $\gtrsim$ appears in this proof its implied constant depends only on α. For $N \geq 3^{2e^{e^5}}$ Corollary 2.3.9 yields $b \in \ell^1(\mathbb{Z})$ with $\|b\|_1 \leq 1$, $\left\|\mathcal{F}(b)^{-1}\right\|_\infty \leq 2$, $\operatorname{supp}(b) \subseteq [-N, N]$ and

$$\|b^{-1}\|_1 \geq \frac{9}{10}\left(1 + 2\frac{\log(N)^{\frac{1}{2}}}{\log(\log(N))^6}\right).$$

As the weight w is increasing with respect to the absolute value of its argument and $\operatorname{supp}(b) \subseteq [-N, N]$, we get

$$\|b\|_{1,w} \leq \|b\|_1 w(N) \leq N^\alpha.$$

Thus, $a := \frac{1}{N^\alpha} b$ satisfies $\|a\|_{1,w} \leq 1$, $\left\|\mathcal{F}(a)^{-1}\right\|_\infty \leq 2N^\alpha =: \tilde{M}$ and

$$\|a^{-1}\|_1 \geq N^\alpha \frac{9}{10}\left(1 + 2\frac{\log(N)^{\frac{1}{2}}}{\log(\log(N))^6}\right) \gtrsim N^\alpha 2 \frac{\log(N)^{\frac{1}{2}}}{\log(\log(N))^6}. \tag{2.42}$$

After noting that

$$2N^\alpha = \tilde{M} \iff N = \left(\frac{\tilde{M}}{2}\right)^{\frac{1}{\alpha}},$$

we can express the right-hand side of (2.42) in terms of $\tilde{M}$:

$$N^\alpha 2 \frac{\log(N)^{\frac{1}{2}}}{\log(\log(N))^6} = \tilde{M} \frac{\log\left(\left(\frac{\tilde{M}}{2}\right)^{\frac{1}{\alpha}}\right)^{\frac{1}{2}}}{\log\left(\log\left(\left(\frac{\tilde{M}}{2}\right)^{\frac{1}{\alpha}}\right)\right)^6} \gtrsim \tilde{M} \frac{\log(\tilde{M})^{\frac{1}{2}}}{\log(\log(\tilde{M}))^6}.$$

This shows the claim for $M \in \{2N^\alpha : N \geq 3^{2e^{e^5}}\}$. For general $M > 2 \cdot 3^{2\alpha e^{e^5}}$, we define the closest $\tilde{M}$ which is smaller and falls into the above category:

$$\tilde{M} := 2 \left\lfloor \left(\frac{M}{2}\right)^{\frac{1}{\alpha}} \right\rfloor^\alpha.$$

Then we conclude by noting that

$$\begin{aligned}
&\sup\left\{\left\|a^{-1}\right\|_1 : a \in \ell^1(\mathbb{Z}, w) \text{ with } \|a\|_{1,w} \leq 1, \left\|\mathcal{F}(a)^{-1}\right\|_\infty \leq M\right\} \\
&\geq \sup\left\{\left\|a^{-1}\right\|_1 : a \in \ell^1(\mathbb{Z}, w) \text{ with } \|a\|_{1,w} \leq 1, \left\|\mathcal{F}(a)^{-1}\right\|_\infty \leq \tilde{M}\right\} \\
&\gtrsim \tilde{M} \frac{\log(\tilde{M})^{\frac{1}{2}}}{\log(\log(\tilde{M}))^6} \gtrsim M \frac{\log(M)^{\frac{1}{2}}}{\log(\log(M))^6},
\end{aligned}$$

where the last estimate follows from

$$\left(\frac{\tilde{M}}{M}\right)^{\frac{1}{\alpha}} = \left(\frac{2\left\lfloor\left(\frac{M}{2}\right)^{\frac{1}{\alpha}}\right\rfloor^\alpha}{M}\right)^{\frac{1}{\alpha}} = \frac{\left\lfloor\left(\frac{M}{2}\right)^{\frac{1}{\alpha}}\right\rfloor}{\left(\frac{M}{2}\right)^{\frac{1}{\alpha}}} \xrightarrow{M \to \infty} 1$$

and the fact that the lowest possible $\tilde{M}$ (which is $2 \cdot 3^{2\alpha e^{e^5}}$) still satisfies

$$\log\left(2 \cdot 3^{2\alpha e^{e^5}}\right) > 0. \qquad \square$$

3 Time-frequency analysis

The following section introduces some concepts of time-frequency analysis which are fundamental for the further development. This is far from giving an exhaustive overview of the field but rather tailored to our needs. For a more extensive introduction into this topic we refer to the book of Gröchenig [17].

3.1 Fundamental operations

We start by defining and collecting properties about some transformations playing an essential role in the theory of time-frequency analysis. We start with the most prominent one.

Definition 3.1.1. *The Fourier transform is the linear operator*

$$\begin{aligned} \mathcal{F}\colon L^1(\mathbb{R}^d) &\longrightarrow L^\infty(\mathbb{R}^d) \\ f &\longmapsto \hat{f} = \left(\xi \mapsto \int_{\mathbb{R}^d} f(t) e^{-2\pi i t\cdot\xi} dt\right) \end{aligned}$$

Although the Fourier transform is naturally defined for $L^1(\mathbb{R}^d)$-functions, it can be transferred to the space $L^2(\mathbb{R}^d)$ by a density argument. This Fourier transform then fits nicely into the Hilbert space structure of $L^2(\mathbb{R}^d)$.

Theorem 3.1.2 (Theorem 1.1.2, [17])**.** *The Fourier transform, restricted to the space $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, extends to a unitary operator on $L^2(\mathbb{R}^d)$.*

Remark 3.1.3. For another proof of Plancherel's theorem (and an extensive introduction into the topic of Fourier analysis) which is not motivated by time-frequency analysis we refer to the excellent book of Constantin [6].

The next (quite simple) operators are at the heart of time-frequency analysis. They are the main motivation for the definition of the twisted convolution (see Section 4.1) and also play a vital role in the theory of gabor frames (see Section 5.2).

Definition 3.1.4. *Let $x, \xi \in \mathbb{R}^d$.*

(i) The translation operator is

$$\begin{aligned} T_x\colon L^2(\mathbb{R}^d) &\longrightarrow L^2(\mathbb{R}^d) \\ f &\longmapsto f(\cdot - x) \end{aligned}$$

(ii) The modulation operator is

$$\begin{aligned} M_\xi \colon L^2(\mathbb{R}^d) &\longrightarrow L^2(\mathbb{R}^d) \\ f &\longmapsto e^{2\pi i \xi \cdot} f \end{aligned}$$

(iii) The time-frequency shift operator is

$$\begin{aligned} \pi(x,\xi) \colon L^2(\mathbb{R}^d) &\longrightarrow L^2(\mathbb{R}^d) \\ f &\longmapsto M_\xi T_x f \end{aligned}$$

(iv) The symmetric time-frequency shift operator is

$$\begin{aligned} \rho(x,\xi) \colon L^2(\mathbb{R}^d) &\longrightarrow L^2(\mathbb{R}^d) \\ f &\longmapsto M_{\frac{\xi}{2}} T_x M_{\frac{\xi}{2}} f \end{aligned}$$

We collect some properties of these operations.

Lemma 3.1.5. *Let $x, y, \xi, \omega \in \mathbb{R}^d$. Then $T_x T_y = T_{x+y}$ and $M_\xi M_\omega = M_{\xi+\omega}$. Moreover, $T_x^{-1} = T_{-x} = T_x^*$ and $M_\xi^{-1} = M_{-\xi} = M_\xi^*$. In particular, all the operators T_x, M_ξ, $\pi(x,\xi)$ and $\rho(x,\xi)$ are unitary.*

Proof. For $f \in L^2(\mathbb{R}^d)$ and $t \in \mathbb{R}^d$ clearly holds:

$$T_x T_y f(t) = T_y f(t-x) = f(t-x-y) = T_{x+y} f(t)$$

and

$$M_\xi M_\omega f(t) = e^{2\pi i \xi \cdot t} M_\omega f(t) = e^{2\pi i \xi \cdot t} e^{2\pi i \omega \cdot t} f(t) = e^{2\pi i (\xi+\omega)\cdot t} f(t) = M_{\xi+\omega} f(t).$$

Next we also choose $g \in L^2(\mathbb{R}^d)$ and calculate

$$\langle T_x f, g \rangle = \int_{\mathbb{R}^d} f(t-x)\overline{g(t)}dt \overset{s=t-x}{=} \int_{\mathbb{R}^d} f(s)\overline{g(s+x)}ds = \langle f, T_{-x} g \rangle$$

and

$$\langle M_\xi f, g \rangle = \int_{\mathbb{R}^d} e^{2\pi i \xi \cdot t} f(t)\overline{g(t)}dt = \int_{\mathbb{R}^d} f(t)\overline{e^{2\pi i (-\xi)\cdot t} g(t)}dt = \langle f, M_{-\xi} g \rangle.$$

Now the first observation together with the trivial fact that $T_0 = M_0 = \mathrm{Id}$ implies that T_x and M_ξ are unitary. Consequently, $\pi(x,\xi)$ and $\rho(x,\xi)$ are unitary since they are a composition of unitary operators. □

The following lemma shows that translation and modulation can be seen as two sides of the same medal although they may look quite different. Concretely, a modulation is just a translation on the Fourier domain and vice versa.

Lemma 3.1.6 (p. 8, [17])**.** *Let $x \in \mathbb{R}^d$. Then $\mathcal{F} T_x = M_{-x} \mathcal{F}$ and $\mathcal{F} M_x = T_x \mathcal{F}$.*

Now we investigate the crucial fact that the composition of time-frequency shifts is almost a time-frequency shift. At this point it is customary to directly consider the time-frequency plane $\mathbb{R}^{2d}$ instead of always taking $x, \xi \in \mathbb{R}^d$. So in the following, whenever we take an element $\lambda \in \mathbb{R}^{2d}$ in the time-frequency plane its time component will be denoted by $\lambda_1 \in \mathbb{R}^d$ and its frequency component by $\lambda_2 \in \mathbb{R}^d$, so $\lambda = (\lambda_1, \lambda_2)$.

Lemma 3.1.7. *Both types of time-frequency shifts are closed under composition up to a phase factor. We have the following identities for $\lambda, \mu \in \mathbb{R}^{2d}$:*

(i) $T_{\lambda_1} M_{\lambda_2} = e^{-2\pi i \lambda_1 \cdot \lambda_2} M_{\lambda_2} T_{\lambda_1}$

(ii) $\pi(\lambda)\pi(\mu) = e^{-2\pi i \lambda_1 \cdot \mu_2} \pi(\lambda + \mu)$

(iii) $\pi(\lambda)^{-1} = e^{-2\pi i \lambda_1 \cdot \lambda_2} \pi(-\lambda)$

(iv) $\rho(\lambda) = e^{-2\pi i \lambda_1 \cdot \frac{\lambda_2}{2}} \pi(\lambda)$

(v) $\rho(\lambda)\rho(\mu) = e^{-2\pi i \frac{\lambda_1 \cdot \mu_2 - \lambda_2 \cdot \mu_1}{2}} \rho(\lambda + \mu)$

(vi) $\rho(\lambda)^{-1} = \rho(-\lambda)$

Proof.

(i) For $f \in L^2(\mathbb{R}^d)$ we have

$$T_{\lambda_1} M_{\lambda_2} f(t) = e^{2\pi i \lambda_2 \cdot (t - \lambda_1)} f(t - \lambda_1) = e^{-2\pi i \lambda_2 \cdot \lambda_1} M_{\lambda_2} T_{\lambda_1} f(t)$$

and thus $T_{\lambda_1} M_{\lambda_2} = e^{-2\pi i \lambda_1 \cdot \lambda_2} M_{\lambda_2} T_{\lambda_1}$.

(ii)

$$\begin{aligned} \pi(\lambda)\pi(\mu) &= M_{\lambda_2} T_{\lambda_1} M_{\mu_2} T_{\mu_1} = e^{-2\pi i \lambda_1 \cdot \mu_2} M_{\lambda_2} M_{\mu_2} T_{\lambda_1} T_{\mu_1} \\ &= e^{-2\pi i \lambda_1 \cdot \mu_2} \pi(\lambda + \mu). \end{aligned}$$

(iii) Using part (ii) we get

$$\pi(\lambda)\pi(-\lambda) = e^{2\pi i \lambda_1 \cdot \lambda_2} \, \mathrm{Id} = \pi(-\lambda)\pi(\lambda)$$

and thus $\pi(\lambda)^{-1} = e^{-2\pi i \lambda_1 \cdot \lambda_2} \pi(-\lambda)$.

(iv)

$$\rho(\lambda) = M_{\frac{\lambda_2}{2}} T_{\lambda_1} M_{\frac{\lambda_2}{2}} = e^{-2\pi i \lambda_1 \cdot \frac{\lambda_2}{2}} \pi(\lambda).$$

(v)

$$\begin{aligned} \rho(\lambda)\rho(\mu) &= M_{\frac{\lambda_2}{2}} T_{\lambda_1} M_{\frac{\lambda_2}{2}} M_{\frac{\mu_2}{2}} T_{\mu_1} M_{\frac{\mu_2}{2}} \\ &= e^{-2\pi i \lambda_1 \cdot \frac{\mu_2}{2}} e^{2\pi i \mu_1 \cdot \frac{\lambda_2}{2}} M_{\frac{\lambda_2}{2}} M_{\frac{\mu_2}{2}} T_{\lambda_1} T_{\mu_1} M_{\frac{\lambda_2}{2}} M_{\frac{\mu_2}{2}} \\ &= e^{-2\pi i \frac{\lambda_1 \cdot \mu_2 - \lambda_2 \cdot \mu_1}{2}} \rho(\lambda + \mu). \end{aligned}$$

(vi) Using part (v) we get
$$\rho(\lambda)\rho(-\lambda) = \mathrm{Id} = \rho(-\lambda)\rho(\lambda)$$
and thus $\rho(\lambda)^{-1} = \rho(-\lambda)$.

□

Having discussed the most basic building blocks of time-frequency analysis, we can now turn to transformations which, as opposed to the Fourier transform, try to capture both time and frequency properties of a function at the same time.

Definition 3.1.8. *Let $g \in L^2(\mathbb{R}^d)$. The Short-time Fourier transform (STFT) with window g is the operator*

$$\begin{aligned} V_g \colon L^2(\mathbb{R}^d) &\longrightarrow L^\infty(\mathbb{R}^{2d}) \\ f &\longmapsto (\lambda \mapsto \langle f, \pi(\lambda)g\rangle) \end{aligned}$$

The STFT is maybe the most important transform in time-frequency analysis. It is used in the development of the theory as well as in practical applications like audio signal processing (see e.g. [18], [19]). In our case, we mainly need the STFT to define the modulation spaces in Section 3.3.

The second transform we discuss is the Wigner transform which was originally introduced in the field of quantum mechanics by Eugene Wigner ([20]). Since then it became a vital part of time-frequency analysis (see e.g. [21] or [22]). The Wigner transform will play an important part in the development of Section 4.3.

Definition 3.1.9. *The cross-Wigner transform is the operator*

$$\begin{aligned} W \colon L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) &\longrightarrow L^\infty(\mathbb{R}^{2d}) \\ (f,g) &\longmapsto \left((x,\omega) \mapsto \int_{\mathbb{R}^d} f\left(x + \frac{t}{2}\right) \overline{g\left(x - \frac{t}{2}\right)} e^{-2\pi i \omega \cdot t} dt \right) \end{aligned}$$

and the Wigner transform is the operator

$$\begin{aligned} W \colon L^2(\mathbb{R}^d) &\longrightarrow L^\infty(\mathbb{R}^{2d}) \\ f &\longmapsto W(f,f) \end{aligned}$$

We first summarize the action of the Wigner transform on time-frequency shifted functions.

Lemma 3.1.10. *Let $f \in L^2(\mathbb{R}^d)$ and $\lambda \in \mathbb{R}^{2d}$. Then*

$$W(\pi(\lambda)f)(x,\omega) = Wf(x - \lambda_1, \omega - \lambda_2)$$

and

$$W(\rho(\lambda)f)(x,\omega) = Wf(x - \lambda_1, \omega - \lambda_2).$$

Proof. The proof of the first equality can be found in [17, Proposition 4.3.2 (c)]. The second equality then follows immediately from Lemma 3.1.7 (iv) and the fact that for any $c \in \mathbb{C}$:

$$W(cf) = |c|^2 Wf.$$

□

Finally, we finish this part with the following reconstruction results.

Theorem 3.1.11 (p. 66, [17])**.** *Let $f, g \in L^2(\mathbb{R}^d)$. Then $Wf = Wg \iff f = cg$ for some $c \in \mathbb{C}$.*

Theorem 3.1.12. *Let $A, B \in \mathcal{B}(L^2(\mathbb{R}^d))$ be invertible. Then $W \circ A = W \circ B$ implies that $A = cB$ for some $c \in \mathbb{C}$ with $|c| = 1$.*

Proof. From Theorem 3.1.11 we know that for any $f \in L^2(\mathbb{R}^d)$ holds

$$Af = c(f)Bf$$

where $c(f) \in \mathbb{C}$ with $|c(f)| = 1$. Now let $f, g \in L^2(\mathbb{R}^d)$ be linearly independent. Then

$$c(f+g)Bf + c(f+g)Bg = c(f+g)B(f+g) = A(f+g) = Af + Ag = c(f)Bf + c(g)Bg$$

and therefore we have

$$(c(f) - c(f+g))Bf + (c(g) - c(f+g))Bg = 0.$$

As B is invertible we know that Bf and Bg are also linearly independent and thus $c(f) = c(f+g) = c(g)$. For linearly dependent $f, g \in L^2(\mathbb{R}^d)$ we can apply the argument from above twice using an intermediate $h \in L^2(\mathbb{R}^d)$ being linearly independent of f and g. Thus, $c(f) = c(g)$ for all $f, g \in L^2(\mathbb{R}^d) \setminus \{0\}$. Consequently, $A = cB$ for some $c \in \mathbb{C}$ with $|c| = 1$. □

3.2 The symplectic group

The symplectic group occurs quite frequently in the field of time-frequency analysis (see e.g. the book by de Gosson [23]), mainly because it is closely related to the composition of time-frequency shifts. It will be important for us in Section 4.3 where one of the fundamental results about the symplectic group in time-frequency analysis (Theorem 4.3.3) is a crucial ingredient.

In order to define the symplectic group we recall one of the identities of Lemma 3.1.7: The symmetric time-frequency shifts (see Definition 3.1.4) satisfy

$$\rho(\lambda)\rho(\mu) = e^{-2\pi i \frac{\lambda_1 \cdot \mu_2 - \lambda_2 \cdot \mu_1}{2}} \rho(\lambda + \mu). \tag{3.1}$$

Definition 3.2.1. *Let $\lambda, \mu \in \mathbb{R}^{2d}$. Then we denote the occurring phase-factor in (3.1) by*

$$\sigma(\lambda, \mu) = \frac{\lambda_1 \cdot \mu_2 - \lambda_2 \cdot \mu_1}{2}.$$

Lemma 3.2.2. *σ is an antisymmetric bilinear form. In particular, for $\lambda, \mu \in \mathbb{R}^{2d}$ we have: $\sigma(\lambda - \mu, \mu) = \sigma(\lambda, \mu)$*

Now we can define the symplectic group: These are all matrices under which σ is invariant. However, one usually uses the bilinear form 2σ to define the symplectic group and that is also what we will do now.

Definition 3.2.3.

(i) The standard symplectic form is $\tilde{\sigma} := 2\sigma$.

(ii) The symplectic group is

$$\mathrm{Sp}(2d, \mathbb{R}) := \{S \in \mathrm{M}(2d, \mathbb{R}) : \forall \lambda, \mu \in \mathbb{R}^{2d} : \tilde{\sigma}(S\lambda, S\mu) = \tilde{\sigma}(\lambda, \mu)\}.$$

(iii) The standard symplectic matrix is

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

where $I \in M(d, \mathbb{R})$ is the identity matrix.

First, we give some basic results and reformulations.

Lemma 3.2.4 ([24], Proposition 4.1)**.**

(i) J is the matrix corresponding to the bilinear form $\tilde{\sigma}$ and

$$S \in \mathrm{Sp}(2d, \mathbb{R}) \iff S^T J S = J.$$

(ii) $\mathrm{Sp}(2d, \mathbb{R})$ *is indeed a group under matrix multiplication.*

Lemma 3.2.5 ([24], Proposition 4.1)**.** $\mathrm{Sp}(2d, \mathbb{R})$ *is closed under transposition, i.e.*

$$S \in \mathrm{Sp}(2d, \mathbb{R}) \iff S^T \in \mathrm{Sp}(2d, \mathbb{R}).$$

Next, we discuss some special examples of symplectic matrices.

Definition 3.2.6. *Let $I \in \mathrm{M}(d, \mathbb{R})$ be the identity matrix. We define the sets*

$$\mathrm{U}(2d, \mathbb{R}) := \left\{ \begin{pmatrix} I & P \\ 0 & I \end{pmatrix} : P \in \mathrm{Sym}(d, \mathbb{R}) \right\}$$
$$\mathrm{V}(2d, \mathbb{R}) := \left\{ \begin{pmatrix} I & 0 \\ Q & I \end{pmatrix} : Q \in \mathrm{Sym}(d, \mathbb{R}) \right\}$$
$$\mathrm{D}(2d, \mathbb{R}) := \left\{ \begin{pmatrix} L & 0 \\ 0 & L^{-T} \end{pmatrix} : L \in \mathrm{GL}(d, \mathbb{R}) \right\}.$$

Lemma 3.2.7 ([24], Proposition 4.9). $\mathrm{U}(2d,\mathbb{R}), \mathrm{V}(2d,\mathbb{R})$ *and* $\mathrm{D}(2d,\mathbb{R})$ *are subgroups of* $\mathrm{Sp}(2d,\mathbb{R})$.

Remark 3.2.8. The importance of the subgroups from Lemma 3.2.7 lies in the fact that they give rise to generators of the symplectic group: Both

$$\{J\} \cup \mathrm{U}(2d,\mathbb{R}) \cup \mathrm{D}(2d,\mathbb{R})$$

and

$$\{J\} \cup \mathrm{V}(2d,\mathbb{R}) \cup \mathrm{D}(2d,\mathbb{R})$$

generate $\mathrm{Sp}(2d,\mathbb{R})$ (see e.g. [24, Prop 4.10]). This in turn is used in the development of the theory of metaplectic operators which is used in many areas of mathematics and physics (see e.g. [25] and [26]), one of which is also time-frequency analysis (see e.g. [17] or [24]).

3.3 Modulation spaces

Modulation spaces were introduced by Feichtinger in the early 1980s, but already appeared implicitly in the work of Bargmann (see [27]). Since then they became an essential part of time-frequency analysis (see e.g. the research article [28] by Feichtinger). In order to define them, we first need some background in distribution theory. However, we cover these things very superficially here and refer to [29] and [30] for a broader introduction.

Definition 3.3.1. $\alpha \in \mathbb{N}^d$ *is called a multi-index. We introduce the following notation:*

$$\begin{aligned} x^\alpha &:= x_1^{\alpha_1} \cdots x_d^{\alpha_d} \\ D^\alpha &:= \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d} \\ |\alpha| &:= \sum_{k=1}^{d} \alpha_k \end{aligned}$$

Definition 3.3.2. *The Schwartz space* $\mathcal{S}(\mathbb{R}^d)$ *is the set of all* $f \in C^\infty(\mathbb{R}^d, \mathbb{C})$ *such that*

$$\left\| x \mapsto x^\alpha D^\beta f(x) \right\|_\infty < \infty$$

for all multi-indices $\alpha \in \mathbb{N}^n$, $\beta \in \mathbb{N}^m$ *and* $n, m \in \mathbb{N}$, *together with the usual addition and scalar multiplication of functions.*

It follows immediately from the definition that $C_C^\infty(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$. An important example for a function in $\mathcal{S}(\mathbb{R}^d)$ which does not have compact support is the Gaussian

$$g(x) = e^{-\|x\|^2}.$$

Based on the Schwartz space we can define the space of tempered distributions.

Definition 3.3.3. *The space of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$ is the space of all linear functionals $\Lambda\colon \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$ which satisfy that there exists $N \in \mathbb{N}$ and $C > 0$ such that for all $f \in \mathcal{S}(\mathbb{R}^d)$:*

$$|\Lambda(f)| \leq C \sum_{|\alpha|\leq N, |\beta|\leq N} \|x \mapsto x^\alpha D^\beta f(x)\|_\infty.$$

Remark 3.3.4. It is worth mentioning that the family

$$\left\{ f \mapsto \left\| x \mapsto x^\alpha D^\beta f(x) \right\|_\infty \right\}_{\alpha,\beta}$$

of seminorms actually turns $\mathcal{S}(\mathbb{R}^d)$ into a Fréchet-space and $\mathcal{S}'(\mathbb{R}^d)$ is the corresponding dual space.

First, we show that we can view all L^p-functions (and thus in particular all Schwartz functions) as elements of $\mathcal{S}'$:

Lemma 3.3.5. *Let $p \in [1, \infty]$ and $f \in L^p(\mathbb{R}^d)$. Then f induces an element $\Lambda_f \in \mathcal{S}'(\mathbb{R}^d)$ via*

$$\begin{aligned} \Lambda_{\mathrm{f}}\colon \mathcal{S}(\mathbb{R}^d) &\longrightarrow \mathbb{C} \\ g &\longmapsto \langle g, f\rangle \end{aligned}$$

Proof. It is clear that Λ_f is linear. Moreover, we get for $g \in \mathcal{S}(\mathbb{R}^d)$ and $N \in \mathbb{N}$ that

$$\begin{aligned} |\Lambda_f(g)| = |\langle g, f\rangle| &= \left| \left\langle \left(1 + \|\cdot\|^2\right)^N g, \left(1 + \|\cdot\|^2\right)^{-N} f \right\rangle \right| \\ &\overset{\text{Hölder}}{\leq} \left\| \left(1 + \|\cdot\|^2\right)^{-N} f \right\|_1 \left\| \left(1 + \|\cdot\|^2\right)^N g \right\|_\infty \\ &\overset{\text{Hölder}}{\leq} \left\| \left(1 + \|\cdot\|^2\right)^{-N} \right\|_q \|f\|_p \left\| \left(1 + \|\cdot\|^2\right)^N g \right\|_\infty, \end{aligned}$$

where $q \in [1, \infty]$ is such that $\frac{1}{p} + \frac{1}{q} = 1$. Choosing N large enough the first term on the right-hand side becomes finite and thus

$$\left\| \left(1 + \|\cdot\|^2\right)^{-N} \right\|_q \|f\|_p \left\| \left(1 + \|\cdot\|^2\right)^N g \right\|_\infty \leq C \left\| \left(1 + \|\cdot\|^2\right)^N g \right\|_\infty$$

for some $C > 0$. As the term involving g is of the right form after expanding the power and applying the triangle inequality we conclude that Λ_f is a tempered distribution. □

Next, we want to extend the short-time Fourier transform from Section 3.1 to $\mathcal{S}'(\mathbb{R}^d)$.

Lemma 3.3.6. *Let $\phi \in \mathcal{S}(\mathbb{R}^d)$. Then the short-time Fourier transform with respect to the window ϕ extends to $\mathcal{S}'(\mathbb{R}^d)$ via*

$$\begin{aligned} \widetilde{V_\phi}\colon \mathcal{S}'(\mathbb{R}^d) &\longrightarrow C(\mathbb{R}^{2d}) \\ \Lambda &\longmapsto \left(\lambda \mapsto \overline{\Lambda(\pi(\lambda)\phi)}\right) \end{aligned}$$

Proof. As
$$\pi(\lambda)\phi \in \mathcal{S}(\mathbb{R}^d) \iff \phi \in \mathcal{S}(\mathbb{R}^d)$$
the expression on the right-hand side is well-defined. The continuity of $\widetilde{V_\phi}(\Lambda)$ follows from a simple calculation which can for example be found in [17, Corollary 11.2.2]. Thus, the only thing left to check is that $\widetilde{V_\phi}$ really extends V_ϕ in the sense that it is compatible with V_ϕ and the inclusion of $L^2(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$ from Lemma 3.3.5. To this end, let $f \in L^2(\mathbb{R}^d)$ and $\lambda \in \mathbb{R}^{2d}$. Then
$$V_\phi(f)(\lambda) = \langle f, \pi(\lambda)\phi\rangle = \overline{\langle \pi(\lambda)\phi, f\rangle} = \overline{\Lambda_f(\pi(\lambda)\phi)} = \widetilde{V_\phi}(\Lambda_f)(\lambda)$$
and thus $\widetilde{V_\phi}(\Lambda_f) = V_\phi(f)$. □

After these technical preparations we are ready to define the modulation spaces.

Definition 3.3.7. *Let $\phi \in \mathcal{S}(\mathbb{R}^d)$ and $p \in [1,\infty]$. The modulation space $M^p(\mathbb{R}^d)$ is defined via*
$$M^p(\mathbb{R}^d) := \{f \in \mathcal{S}'(\mathbb{R}^d) : \|V_\phi f\|_p < \infty\}.$$
For $f \in M^p(\mathbb{R}^d)$ we further define
$$\|f\|_{M^p} := \|V_\phi f\|_p.$$

Remark 3.3.8. Although the modulation spaces depend on the chosen window function ϕ, we did not denote this dependence explicitly. There is a reason for that: One can show that the definition of the modulation spaces is actually independent of the chosen window (as long as it is a Schwartz function) meaning that choosing different windows yields the same subsets of $\mathcal{S}'(\mathbb{R}^d)$ and equivalent norms (see [17, Proposition 11.3.2 c]). However, we do not want to go into this and so we just implicitly assume a choice of some window if we talk about modulation spaces. Whenever there is some dependence on the chosen window we will denote it explicitly.

We finish this discussion by collecting some results about modulation spaces.

Theorem 3.3.9. *Let $p, q \in [1,\infty]$ with $p < q$ and $\lambda \in \mathbb{R}^{2d}$.*

(i) $M^p(\mathbb{R}^d)$ is a Banach space ([17, Theorem 11.3.5]).

(ii) We have the following inclusion relation:
$$M^p(\mathbb{R}^d) \subset M^q(\mathbb{R}^d)$$
([17, Theorem 12.2.2]).

(iii) $M^2(\mathbb{R}^d) = L^2(\mathbb{R}^d)$ via the inclusion from Lemma 3.3.5 ([17, Proposition 11.3.1 b]).

(iv) $M^p(\mathbb{R}^d)$ is invariant under $\pi(\lambda)$ and
$$\|\pi(\lambda)f\|_{M^p} = \|f\|_{M^p}$$
([17, Proposition 11.3.5 b]).

There is of course much more to say about modulation spaces. They fit very well into the theory of Gabor analysis and are ubiquitous in time-frequency analysis in general (see e.g. chapter 11 and 12 of Gröchenig's book [17] for more on modulation spaces). In particular, $M^1(\mathbb{R}^d)$ turned out to be a nice space for windows in Gabor analysis (see Section 5.3) which is the main reason why we introduced the modulation spaces.

4 The twisted convolution algebra

In this section we introduce a noncommutative structure on $\ell^1(\mathbb{Z}^d)$ that occurs frequently in time-frequency analysis (see e.g. [31] or [32]), namely the twisted convolution. Furthermore, we discuss embeddings of this algebra into two operator algebras. Finally, we show that both settings do not admit norm-controlled inversion.

4.1 Definition of twisted convolution

This subsection defines the twisted convolution and all the necessary constructions associated to it and collects some basic properties of them. The material covered here can be found in Gröchenig's article [31] (but note that our twisted convolution is defined differently, see Definition 4.1.7).

Definition 4.1.1. *A lattice is a set of the form $\Lambda = A\mathbb{Z}^d$ where $A \in \mathrm{GL}(d, \mathbb{R})$.*

We start by recalling one of the identities of Lemma 3.1.7: The symmetric time-frequency shifts (see Definition 3.1.4) satisfy

$$\rho(\lambda)\rho(\mu) = e^{-2\pi i \frac{\lambda_1 \cdot \mu_2 - \lambda_2 \cdot \mu_1}{2}} \rho(\lambda + \mu) = e^{-2\pi i \sigma(\lambda, \mu)} \rho(\lambda + \mu).$$

This motivates the definition of a noncommutative version of convolution on $\ell^1(\Lambda)$:

Definition 4.1.2. *Let $\Lambda \subset \mathbb{R}^{2d}$ be a lattice and a, b be two sequences on Λ. Then the twisted convolution of a and b is another sequence on Λ defined via*

$$(a \natural b)_\lambda = \sum_{\mu \in \Lambda} a_\mu b_{\lambda - \mu} e^{-2\pi i \sigma(\mu, \lambda - \mu)} = \sum_{\mu \in \Lambda} a_\mu b_{\lambda - \mu} e^{-2\pi i \frac{\mu_1 \cdot \lambda_2 - \mu_2 \cdot \lambda_1}{2}},$$

for all $\lambda \in \Lambda$, whenever this is well-defined.

We can directly transfer Young's inequality (Theorem 1.2.7) to the twisted convolution.

Theorem 4.1.3. *Let $p \in [1, \infty]$, $a \in \ell^1(\Lambda)$ and $b \in \ell^p(\Lambda)$. Then $a \natural b \in \ell^p(\Lambda)$ and $\|a \natural b\|_p \leq \|a\|_1 \|b\|_p$.*

Proof. For $\tilde{p} \in [1, \infty]$ and $c \in \ell^{\tilde{p}}(\Lambda)$ we define $|c| \in \ell^{\tilde{p}}(\Lambda)$ via $|c|_\lambda = |c_\lambda|$. Clearly, $\|c\|_{\tilde{p}} = \||c|\|_{\tilde{p}}$. Using this we get

$$|(a \natural b)_\lambda| \leq \sum_{\mu \in \Lambda} |a_\mu||b_{\lambda - \mu}| = (|a| * |b|)_\lambda.$$

Therefore, Young's inequality (Theorem 1.2.7) yields

$$\|a \natural b\|_p \le \| |a| * |b| \|_p \le \| |a| \|_1 \| |b| \|_p = \|a\|_1 \|b\|_p . \qquad \square$$

Remark 4.1.4. The twisted convolution actually depends crucially on the underlying lattice. For example, if we consider the lattice $\Lambda = (2\mathbb{Z})^{2d}$, then twisted convolution reduces to ordinary convolution. Nevertheless, we write $\natural$ instead of $\natural_\Lambda$ because we do not want to clutter up the notation. In any case, we will never consider two different lattices at the same time so there should be no confusion and the twisted convolution always refers to the current lattice we consider.

Taking a suitable involution turns $\ell^1(\Lambda, \natural)$ into an involutive Banach algebra.

Definition 4.1.5. *Let $\Lambda \subset \mathbb{R}^{2d}$ be a lattice and $a \in \ell^1(\Lambda)$. Then $a^* \in \ell^1(\Lambda)$ is defined via $(a^*)_\lambda = \overline{a_{-\lambda}}$.*

Theorem 4.1.6. *Let $\Lambda \subset \mathbb{R}^{2d}$ be a lattice. Then $\ell^1(\Lambda, \natural, ^*)$ is a unital, involutive Banach algebra.*

Proof. It is well-known that $\ell^1(\Lambda)$ is a Banach space and it is also immediate that $\ell^1(\Lambda, \natural)$ is an algebra. In order to prove associativity let $a, b, c \in \ell^1(\Lambda, \natural)$:

$$\begin{aligned}\sum_{\mu\in\Lambda}\sum_{\nu\in\Lambda}|a_\nu b_{\mu-\nu} c_{\lambda-\mu}| &\le \sum_{\nu\in\Lambda}|a_\nu|\sum_{\mu\in\Lambda}|b_{\mu-\nu}|\|c\|_\infty \overset{\tilde{\mu}=\mu-\nu}{=} \sum_{\nu\in\Lambda}|a_\nu|\sum_{\tilde{\mu}\in\Lambda}|b_{\tilde{\mu}}|\|c\|_\infty \\ &= \|a\|_1\|b\|_1\|c\|_\infty.\end{aligned}$$

Thus, the interchange of summation in the following calculation is justified:

$$\begin{aligned}((a \natural b) \natural c)_\lambda &= \sum_{\mu\in\Lambda}(a \natural b)_\mu c_{\lambda-\mu} e^{-2\pi i\sigma(\mu,\lambda-\mu)} \\ &= \sum_{\mu\in\Lambda}\sum_{\nu\in\Lambda} a_\nu b_{\mu-\nu} e^{-2\pi i\sigma(\nu,\mu-\nu)} c_{\lambda-\mu} e^{-2\pi i\sigma(\mu,\lambda-\mu)} \\ &\overset{\tilde{\mu}=\mu-\nu}{=} \sum_{\nu\in\Lambda} a_\nu \sum_{\tilde{\mu}\in\Lambda} b_{\tilde{\mu}} e^{-2\pi i\sigma(\nu,\tilde{\mu})} c_{\lambda-\nu-\tilde{\mu}} e^{-2\pi i\sigma(\tilde{\mu}+\nu,\lambda-\nu-\tilde{\mu})}.\end{aligned}$$

Lemma 3.2.2 yields

$$\begin{aligned}&\sum_{\nu\in\Lambda} a_\nu \sum_{\tilde{\mu}\in\Lambda} b_{\tilde{\mu}} e^{-2\pi i\sigma(\nu,\tilde{\mu})} c_{\lambda-\nu-\tilde{\mu}} e^{-2\pi i\sigma(\tilde{\mu}+\nu,\lambda-\nu-\tilde{\mu})} \\ &= \sum_{\nu\in\Lambda} a_\nu \sum_{\tilde{\mu}\in\Lambda} b_{\tilde{\mu}} e^{-2\pi i\sigma(\nu,\tilde{\mu})} c_{\lambda-\nu-\tilde{\mu}} e^{-2\pi i\sigma(\tilde{\mu},\lambda-\nu-\tilde{\mu})} e^{-2\pi i\sigma(\nu,\lambda-\nu-\tilde{\mu})} \\ &= \sum_{\nu\in\Lambda} a_\nu \sum_{\tilde{\mu}\in\Lambda} b_{\tilde{\mu}} e^{-2\pi i\sigma(\nu,\lambda-\nu)} c_{\lambda-\nu-\tilde{\mu}} e^{-2\pi i\sigma(\tilde{\mu},\lambda-\nu-\tilde{\mu})} \\ &= \sum_{\nu\in\Lambda} a_\nu (b \natural c)_{\lambda-\nu} e^{-2\pi i\sigma(\nu,\lambda-\nu)} = (a \natural (b \natural c))_\lambda,\end{aligned}$$

which proves associativity. Moreover, from Theorem 4.1.3 follows that $\|a \natural b\|_1 \leq \|a\|_1 \|b\|_1$, so $\ell^1(\Lambda, \natural)$ is a Banach algebra. Considering the involution *, the only nontrivial fact we need to show is that $(a \natural b)^* = b^* \natural a^*$. Using Lemma 3.2.2 a few times, we get

$$\begin{aligned}
(b^* \natural a^*)_\lambda &= \sum_{\mu\in\Lambda} (b^*)_\mu (a^*)_{\lambda-\mu} e^{-2\pi i \sigma(\mu,\lambda-\mu)} = \overline{\sum_{\mu\in\Lambda} a_{\mu-\lambda} b_{-\mu} e^{-2\pi i\sigma(\lambda-\mu,\mu)}} \\
&\overset{\tilde{\mu}=\mu-\lambda}{=} \overline{\sum_{\tilde{\mu}\in\Lambda} a_{\tilde{\mu}} b_{-\lambda-\tilde{\mu}} e^{-2\pi i\sigma(-\tilde{\mu},\lambda+\tilde{\mu})}} = \overline{\sum_{\tilde{\mu}\in\Lambda} a_{\tilde{\mu}} b_{-\lambda-\tilde{\mu}} e^{-2\pi i\sigma(\tilde{\mu},-\lambda-\tilde{\mu})}} \\
&= \overline{(a \natural b)_{-\lambda}} = ((a \natural b)^*)_\lambda \, .
\end{aligned}$$
□

In the article [31] the twisted convolution is defined using the ordinary time-frequency shifts $\pi(\lambda)$ (see Definition 3.1.4) which satisfy

$$\pi(\lambda)\pi(\mu) = e^{-2\pi i \lambda_1 \cdot \mu_2} \pi(\lambda + \mu).$$

Definition 4.1.7. *Let $\Lambda \subset \mathbb{R}^{2d}$ be a lattice and $a, b \in \ell^1(\Lambda)$. Then we define the asymmetric variant of the twisted convolution of a and b via $a \,\widetilde{\natural}\, b \in \ell^1(\Lambda)$ where*

$$(a \,\widetilde{\natural}\, b)_\lambda = \sum_{\mu\in\Lambda} a_\mu b_{\lambda-\mu} e^{-2\pi i \mu_1 \cdot (\lambda-\mu)_2}.$$

Moreover, $a^{\widetilde{*}} \in \ell^1(\Lambda)$ is defined via

$$\left(a^{\widetilde{*}}\right)_\lambda = \overline{a_{-\lambda}} e^{-2\pi i \lambda_1 \cdot \lambda_2}.$$

This variant of the twisted convolution will be important in Section 5.3. The twisted convolution we defined at first has a symmetric phase factor and thus some calculations are more transparent. However, in the end there is no difference between these two structures as we see in the following lemma.

Lemma 4.1.8. *Let $\Lambda \subset \mathbb{R}^{2d}$ be a lattice. Then $\ell^1(\Lambda, \widetilde{\natural}, ^{\widetilde{*}})$ is a unital, involutive Banach algebra. Moreover, the following map is an isometric isomorphism between involutive Banach algebras:*

$$\begin{aligned}
\mathrm{i} \colon \ell^1(\Lambda, \natural, ^*) &\longrightarrow \ell^1\left(\Lambda, \widetilde{\natural}, ^{\widetilde{*}}\right) \\
(a_\lambda)_{\lambda\in\Lambda} &\longmapsto \left(a_\lambda e^{-2\pi i \frac{\lambda_1 \cdot \lambda_2}{2}}\right)_{\lambda\in\Lambda}
\end{aligned}$$

Proof. The first part is similar to the corresponding statement for the symmetric version of the twisted convolution (Theorem 4.1.6). The second part is a direct computation which we skip. □

Corollary 4.1.9. *Let $\Lambda \subset \mathbb{R}^{2d}$ be a lattice, $p \in [1, \infty]$, $a \in \ell^1(\Lambda)$ and $b \in \ell^p(\Lambda)$. Then $a \,\widetilde{\natural}\, b \in \ell^p(\Lambda)$ and $\|a \,\widetilde{\natural}\, b\|_p \leq \|a\|_1 \|b\|_p$.*

Proof. This follows immediately from Lemma 4.1.8 after noting that the isomorphism defined there leaves all ℓ^p-norms invariant. □

4.2 Embeddings into two operator algebras

In the following, we will embed $\ell^1(\Lambda, \natural)$ into the two operator algebras $\mathcal{B}(\ell^2(\Lambda))$ and $\mathcal{B}(L^2(\mathbb{R}^d))$.

4.2.1 Embeddings into $\mathcal{B}(\ell^2(\Lambda))$

Definition 4.2.1. *Let $\Lambda \subset \mathbb{R}^{2d}$ be a lattice. We consider the operator*

$$\begin{aligned} \mathrm{L}\colon \ell^1(\Lambda, \natural, ^*) &\longrightarrow \mathcal{B}(\ell^2(\Lambda)) \\ a &\longmapsto L_a = (b \mapsto a \natural b), \end{aligned}$$

whenever it is well-defined.

Lemma 4.2.2. *Let $\Lambda \subset \mathbb{R}^{2d}$ be a lattice. Then the operator L is indeed well-defined and a homomorphism between involutive Banach algebras.*

Proof. Due to Theorem 4.1.3 the mapping is well-defined and it is clear that the mapping is compatible with the linear structure. Moreover, for $a_1, a_2 \in \ell^1(\Lambda)$ and $b \in \ell^2(\Lambda)$ we have

$$L_{a_1} L_{a_2} b = L_{a_1}(a_2 \natural b) = a_1 \natural (a_2 \natural b) = (a_1 \natural a_2) \natural b = L_{a_1 \natural a_2} b$$

and so $L_{a_1} L_{a_2} = L_{a_1 \natural a_2}$. Concerning continuity, Theorem 4.1.3 yields

$$\|L_a\| = \sup_{\|b\|_2 = 1} \|a \natural b\|_2 \leq \sup_{\|b\|_2=1} \|a\|_1 \|b\|_2 = \|a\|_1$$

and so $\|L\| \leq 1$. To show compatibility with the involution, we calculate the adjoint of L_a. For $b, \tilde{b} \in \ell^2(\Lambda)$ we have (using Lemma 3.2.2 a few times)

$$\begin{aligned} \langle L_a b, \tilde{b} \rangle &= \sum_{\lambda \in \Lambda} (a \natural b)_\lambda \overline{\tilde{b}_\lambda} = \sum_{\lambda \in \Lambda} \sum_{\mu \in \Lambda} a_\mu b_{\lambda - \mu} e^{-2\pi i \sigma(\mu, \lambda - \mu)} \overline{\tilde{b}_\lambda} \\ &\overset{\tilde{\mu} = \lambda - \mu}{=} \sum_{\tilde{\mu} \in \Lambda} b_{\tilde{\mu}} \sum_{\lambda \in \Lambda} a_{\lambda - \tilde{\mu}} \overline{\tilde{b}_\lambda} e^{-2\pi i \sigma(\lambda - \tilde{\mu}, \tilde{\mu})} \overset{\tilde{\lambda} = \tilde{\mu} - \lambda}{=} \sum_{\tilde{\mu} \in \Lambda} b_{\tilde{\mu}} \sum_{\tilde{\lambda} \in \Lambda} a_{-\tilde{\lambda}} \overline{\tilde{b}_{\tilde{\mu} - \tilde{\lambda}}} e^{-2\pi i \sigma(-\tilde{\lambda}, \tilde{\mu})} \\ &= \sum_{\tilde{\mu} \in \Lambda} b_{\tilde{\mu}} \overline{\sum_{\tilde{\lambda} \in \Lambda} \overline{a_{-\tilde{\lambda}}} \tilde{b}_{\tilde{\mu} - \tilde{\lambda}} e^{-2\pi i \sigma(\tilde{\lambda}, \tilde{\mu})}} = \sum_{\tilde{\mu} \in \Lambda} b_{\tilde{\mu}} \overline{\sum_{\tilde{\lambda} \in \Lambda} (a^*)_{\tilde{\lambda}} \tilde{b}_{\tilde{\mu} - \tilde{\lambda}} e^{-2\pi i \sigma(\tilde{\lambda}, \tilde{\mu} - \tilde{\lambda})}} \\ &= \langle b, L_{a^*} \tilde{b} \rangle, \end{aligned}$$

where the interchange of the summations is justified due to absolute convergence:

$$\sum_{\lambda \in \Lambda} \sum_{\mu \in \Lambda} |a_\mu b_{\lambda - \mu} \tilde{b}_\lambda| = \langle |a| * |b|, |\tilde{b}| \rangle \leq \|a\|_1 \|b\|_2 \|\tilde{b}\|_2$$

by Young's inequality (Theorem 1.2.7) and Cauchy-Schwarz. Consequently, $(L_a)^* = L_{a^*}$ and thus L is compatible with the involution. We conclude by noting that L obviously maps the unit of $\ell^1(\Lambda, \natural)$ to the unit of $\mathcal{B}(\ell^2(\Lambda))$. $\square$

Lemma 4.2.2 shows that the embedding L fits into the framework of Definition 2.1.15. The next theorem (which is provided without proof) yields that this embedding is actually inverse-closed.

Theorem 4.2.3 (Theorem 2.14, [32])**.** *Let $a \in \ell^1(\Lambda)$ and L_a be invertible on $\ell^2(\Lambda)$. Then $L_a^{-1} = L_b$ for some $b \in \ell^1(\Lambda)$.*

Remark 4.2.4. Alternative proofs of Theorem 4.2.3 can be found in [33] and [34].

We finish this chapter by defining an operator similar to L which will be important in Chapter 5 and Section 5.3.

Definition 4.2.5. *Let $\Lambda \subset \mathbb{R}^{2d}$ be a lattice. We consider the operator*

$$\begin{aligned}\tilde{R}\colon \ell^1(\Lambda, \widetilde{\natural}, \tilde{}^{*}) &\longrightarrow \mathcal{B}(\ell^2(\Lambda))\\ a &\longmapsto \tilde{R}_a = (b \mapsto b \,\widetilde{\natural}\, a),\end{aligned}$$

whenever it is well-defined.

The following lemma is in the spirt of Lemma 4.2.2. As the proof is also very similar, we omit it.

Lemma 4.2.6. *Let $\Lambda \subset \mathbb{R}^{2d}$ be a lattice. Then the operator $\tilde{R}$ is indeed well-defined and an antihomomorphism between involutive Banach algebras, i.e. $\tilde{R}_{a \widetilde{\natural} b} = \tilde{R}_b \tilde{R}_a$ for all $a, b \in \ell^1(\Lambda, \widetilde{\natural}, \tilde{}^{*})$.*

Finally, we show that invertibility of $\tilde{R}_a$ yields invertibility of a.

Lemma 4.2.7. *Let $\Lambda \subset \mathbb{R}^{2d}$ be a lattice and $a \in \ell^0(\Lambda)$ such that $\tilde{R}_a$ is invertible as an operator on $\ell^2(\Lambda)$. Then a is invertible with respect to $\widetilde{\natural}$.*

Proof. As $\tilde{R}_a$ is invertible, we have

$$\mathrm{Id} = \tilde{R}_a \tilde{R}_a^{-1}.$$

Applying the unit u_0 to both sides yields

$$u_0 = \tilde{R}_a \tilde{R}_a^{-1} u_0 = \left(\tilde{R}_a^{-1} u_0\right) \widetilde{\natural}\, a,$$

which yields that a is invertible with inverse given by $\tilde{R}_a^{-1} u_0$. ☐

4.2.2 Embeddings into $\mathcal{B}(L^2(\mathbb{R}^d))$

Definition 4.2.8. *Let $\Lambda \subset \mathbb{R}^{2d}$ be a lattice and recall the time-frequency shifts ρ from Definition 3.1.4. We consider the operator*

$$\begin{aligned} \rho\colon \ell^1(\Lambda, \natural, ^*)) &\longrightarrow \mathcal{B}(L^2(\mathbb{R}^d)) \\ a &\longmapsto \sum_{\lambda\in\Lambda} a_\lambda \rho(\lambda), \end{aligned}$$

whenever it is well-defined.

Lemma 4.2.9. *Let $\Lambda \subset \mathbb{R}^{2d}$ be a lattice. Then the operator ρ is indeed well-defined and a homomorphism between involutive Banach algebras.*

Proof. $\rho(a) \in \mathcal{B}(L^2(\mathbb{R}^d))$ as $\mathcal{B}(L^2(\mathbb{R}^d))$ is a Banach space and the sum converges absolutely:

$$\sum_{\lambda\in\Lambda} \|a_\lambda \rho(\lambda)\| = \sum_{\lambda\in\Lambda} |a_\lambda| \, \|\rho(\lambda)\| = \sum_{\lambda\in\Lambda} |a_\lambda| = \|a\| < \infty.$$

Moreover, it is clear that the identification is compatible with the linear structure. Concerning the multiplication, we explicitly calculate

$$\begin{aligned} \rho(a)\rho(b) &= \left(\sum_{\lambda\in\Lambda} a_\lambda \rho(\lambda)\right)\left(\sum_{\mu\in\Lambda} b_\mu \rho(\mu)\right) = \sum_{\lambda\in\Lambda}\sum_{\mu\in\Lambda} a_\lambda b_\mu \rho(\lambda)\rho(\mu) \\ &= \sum_{\lambda\in\Lambda}\sum_{\mu\in\Lambda} a_\lambda b_\mu e^{-2\pi i\sigma(\lambda,\mu)} \rho(\lambda+\mu) \\ &\overset{\nu=\lambda+\mu}{=} \sum_{\nu\in\Lambda}\sum_{\lambda\in\Lambda} a_\lambda b_{\nu-\lambda} e^{-2\pi i\sigma(\lambda,\nu-\lambda)} \rho(\nu) \\ &= \sum_{\nu\in\Lambda} (a \natural b)_\nu \rho(\nu) = \rho(a\natural b), \end{aligned}$$

where the interchange of the sums is justified due to absolute convergence:

$$\begin{aligned} \sum_{\lambda\in\Lambda}\sum_{\mu\in\Lambda} \left\| a_\lambda b_\mu e^{-2\pi i\sigma(\lambda,\mu)} \rho(\lambda+\mu) \right\| &= \sum_{\lambda\in\Lambda}\sum_{\mu\in\Lambda} |a_\lambda b_\mu| \\ &= \sum_{\lambda\in\Lambda} |a_\lambda| \sum_{\mu\in\Lambda} |b_\mu| = \|a\|\|b\| < \infty. \end{aligned}$$

Furthermore,

$$\rho(a^*) = \sum_{\lambda\in\Lambda} \overline{a_{-\lambda}} \rho(\lambda) = \sum_{\lambda\in\Lambda} \overline{a_\lambda} \rho(-\lambda) = \left(\sum_{\lambda\in\Lambda} a_\lambda \rho(\lambda)\right)^* = \rho(a)^*.$$

To conclude, we note that ρ maps the unit of $\ell^1(\Lambda, \natural)$ to the unit of $\mathcal{B}(L^2(\mathbb{R}^d))$, as $\rho(0) = \mathrm{Id}_{L^2(\mathbb{R}^d)}$. $\square$

As in Section 4.2.1, this embedding is actually inverse-closed.

Theorem 4.2.10 (Theorem 3.1, [32]). *Let $a \in \ell^1(\Lambda)$ and $\rho(a)$ be invertible on $L^2(\mathbb{R}^d)$. Then $\rho(a)^{-1} = \rho(b)$ for some $b \in \ell^1(\Lambda)$.*

We finish this chapter by briefly defining the corresponding operator for asymmetric time-frequency shifts.

Definition 4.2.11. *Let $\Lambda \subset \mathbb{R}^{2d}$ be a lattice and recall the time-frequency shifts π from Definition 3.1.4. We consider the operator*

$$\begin{aligned} \pi\colon \ell^1(\Lambda, \widetilde{\natural}, \widetilde{^*})) &\longrightarrow \mathcal{B}(L^2(\mathbb{R}^d)) \\ a &\longmapsto \sum_{\lambda\in\Lambda} a_\lambda \pi(\lambda), \end{aligned}$$

whenever it is well-defined.

Lemma 4.2.12. *Let $\Lambda \subset \mathbb{R}^{2d}$ be a lattice. Then the operator π is indeed well-defined and a homomorphism between involutive Banach algebras.*

Proof. Recalling the isomorphism i from Lemma 4.1.8 the claim follows immediately because $\pi = \rho \circ i^{-1}$:

$$(\rho \circ i^{-1})(a) = \sum_{\lambda\in\Lambda} e^{\pi i \lambda_1 \cdot \lambda_2} a_\lambda \rho(\lambda) \overset{Lemma\ 3.1.7}{=} \sum_{\lambda\in\Lambda} a_\lambda \pi(\lambda) = \pi(a)$$

for all $a \in \ell^1(\Lambda, \widetilde{\natural})$. $\square$

4.3 A lack of norm-controlled inversion

Now we are going to carry the results from Chapter 2 over to the twisted convolution. The important observation is that we can find a subalgebra of $\ell^1(\Lambda, \natural)$ which is isomorphic to $\ell^1(\mathbb{Z}, *)$.

Lemma 4.3.1. *Let $\Lambda \subset \mathbb{R}^{2d}$ be a lattice and $\tilde{\lambda} \in \Lambda \setminus \{0\}$. Then the map*

$$\begin{aligned} \mathrm{i}\colon \ell^1(\mathbb{Z}, *) &\longrightarrow \ell^1(\Lambda, \natural) \\ (a_k) &\longmapsto (b_\lambda) \ \textit{where}\ b_\lambda = \begin{cases} a_k, & \lambda = k\tilde{\lambda} \\ 0, & \textit{otherwise} \end{cases} \end{aligned}$$

is an isometric homomorphism between Banach algebras.

Proof. It is clear that for $a \in \ell^1(\mathbb{Z}, *)$ we have $\|a\| = \|i(a)\|$ and that i preserves the linear structure. So it suffices to show that i also preserves the multiplication: For $\lambda \in \mathbb{Z}\tilde{\lambda}$, say $\lambda = l\tilde{\lambda}$, we have:

$$\begin{aligned}(i(a) \natural i(b))_{l\tilde{\lambda}} &= \sum_{\mu\in\Lambda} i(a)_\mu i(b)_{l\tilde{\lambda}-\mu} e^{-2\pi i l \frac{\mu_1\tilde{\lambda}_2-\mu_2\tilde{\lambda}_1}{2}} \\ &= \sum_{k\in\mathbb{Z}} i(a)_{k\tilde{\lambda}} i(b)_{l\tilde{\lambda}-k\tilde{\lambda}} e^{-2\pi i k l \frac{\tilde{\lambda}_1\tilde{\lambda}_2-\tilde{\lambda}_2\tilde{\lambda}_1}{2}} = \sum_{k\in\mathbb{Z}} a_k b_{l-k} = (a*b)_l = (i(a*b))_{l\tilde{\lambda}}.\end{aligned}$$

For $\lambda \notin \mathbb{Z}\tilde{\lambda}$, we get

$$\begin{aligned}(i(a) \natural i(b))_{\lambda} &= \sum_{\mu\in\Lambda} i(a)_\mu i(b)_{\lambda-\mu} e^{-2\pi i \frac{\mu_1\lambda_2-\mu_2\lambda_1}{2}} \\ &= \sum_{k\in\mathbb{Z}} i(a)_{k\tilde{\lambda}} i(b)_{\lambda-k\tilde{\lambda}} e^{-2\pi i k \frac{\tilde{\lambda}_1\lambda_2-\tilde{\lambda}_2\lambda_1}{2}} = 0 = (i(a*b))_\lambda\end{aligned}$$

because $i(b)_{\lambda-k\tilde{\lambda}} = 0$ for all $k \in \mathbb{Z}$ as $\lambda \notin \mathbb{Z}\tilde{\lambda}$. □

Remark 4.3.2. Of course the embedding i from Lemma 4.3.1 depends on the chosen $\tilde{\lambda}$. However, most results we prove about i will be true for any $\tilde{\lambda} \neq 0$ so we just implicitly assume that we have chosen $\tilde{\lambda}$ whenever we use i in these cases. If the results do depend on $\tilde{\lambda}$, we will mention it explicitly.

The idea is now to use the result that $\ell^1(\mathbb{Z}, *)$ does not admit norm-controlled inversion and transfer it via the embedding i from Lemma 4.3.1 to $\ell^1(\Lambda, \natural)$. The only thing left to show at this point is that i is also compatible with the embedding of $\ell^1(\mathbb{Z}, *)$ into $C(\mathbb{T})$ via Fourier series (see Theorem 2.1.14) and the embedding of $\ell^1(\Lambda, \natural)$ into $\mathcal{B}(L^2(\mathbb{R}^d))$ or $\mathcal{B}(\ell^2(\Lambda))$ via ρ or L (see Section 4.2), respectively. Concerning the first embedding ρ, we need to show that

$$\left\| \sum_{k\in\mathbb{Z}} a_k \rho(k\tilde{\lambda}) \right\|_{\mathcal{B}(L^2)} = \left\| \sum_{k\in\mathbb{Z}} a_k e^{2\pi i k \cdot} \right\|_\infty . \tag{4.1}$$

We investigate some special cases for specific choices of $\tilde{\lambda}$ to get some motivation on how to tackle this problem: If $\tilde{\lambda}$ is of the form $(0, \omega)$ for $\omega \in \mathbb{R}^d$, then

$$\sum_{k\in\mathbb{Z}} a_k \rho(k\tilde{\lambda}) = \left(f \mapsto \sum_{k\in\mathbb{Z}} a_k e^{2\pi i k \omega \cdot} f \right).$$

This is a multiplication operator and thus (see Lemma 1.3.6)

$$\left\| \sum_{k\in\mathbb{Z}} a_k \rho(k\tilde{\lambda}) \right\|_{\mathcal{B}(L^2)} = \left\| \sum_{k\in\mathbb{Z}} a_k e^{2\pi i k \omega \cdot} \right\|_\infty = \left\| \sum_{k\in\mathbb{Z}} a_k e^{2\pi i k \cdot} \right\|_\infty .$$

Therefore, if $\tilde{\lambda}$ contains only a frequency component, then we are done. On the other hand, if $\tilde{\lambda}$ is of the form $(x, 0)$ for $x \in \mathbb{R}^d$, then using the fact that the Fourier transform is unitary (Theorem 3.1.2) and that it converts translations into modulations (Lemma 3.1.6) we get

$$\begin{aligned}\Big\| \sum_{k\in\mathbb{Z}} a_k \rho(k\tilde{\lambda}) \Big\|_{\mathcal{B}(L^2)} &= \Big\| \sum_{k\in\mathbb{Z}} a_k T_{kx} \Big\|_{\mathcal{B}(L^2)} = \Big\| \mathcal{F}\Big(\sum_{k\in\mathbb{Z}} a_k T_{kx} \Big) \Big\|_{\mathcal{B}(L^2)} \\ &= \Big\| \sum_{k\in\mathbb{Z}} a_k M_{-kx} \Big\|_{\mathcal{B}(L^2)} = \Big\| \sum_{k\in\mathbb{Z}} a_k e^{2\pi i k \cdot} \Big\|_{\infty},\end{aligned}$$

where the last inequality follows from the calculation before. So the Fourier transform turned the translation into a modulation and then we are again in the first case and we are done. Looking at the coordinates of $\tilde{\lambda}$ we can use the picture that the Fourier transform corresponds to a rotation by $-\frac{\pi}{2}$ in the time-frequency plane, thus it converts the symmetric time-frequency shift $(x, 0)$ into $(0, -x)$. Consequently, in the case of a general $\tilde{\lambda}$ we need to find a unitary operator which maps $\tilde{\lambda}$ to an element of the form $(0, \omega)$ for some $\omega \in \mathbb{R}^d$. At the very heart of the argument we use the following theorem from the theory of metaplectic operators (see also Remark 3.2.8) which we supply without a proof.

Theorem 4.3.3 (Prop 4.28, [24]). *For any $S \in \mathrm{Sp}(2d, \mathbb{R})$ there exists a unitary operator*

$$\mathcal{F}_S : L^2(\mathbb{R}^{2d}) \to L^2(\mathbb{R}^{2d})$$

such that

$$W(\mathcal{F}_S(f))(x, \omega) = W(f)(S(x, \omega)),$$

where W is the Wigner transform from Definition 3.1.9.

Remark 4.3.4. The reference for Theorem 4.3.3 actually shows that for $S \in \mathrm{Sp}(2d, \mathbb{R})$ there exists a unitary operator

$$\mu(S) : L^2(\mathbb{R}^{2d}) \to L^2(\mathbb{R}^{2d})$$

such that

$$W(\mu(S)(f))(x, \omega) = W(f)(S^T(x, \omega)),$$

but together with Lemma 3.2.5 this immediately implies the claim from Theorem 4.3.3.

The identification of S with $\mathcal{F}_S$ given by Theorem 4.3.3 is almost compatible with taking inverses.

Lemma 4.3.5. *Let $S \in \mathrm{Sp}(2d, \mathbb{R})$. Using the notation of Theorem 4.3.3, we have ${\mathcal{F}_S}^{-1} = c\mathcal{F}_{S^{-1}}$ for some $c \in \mathbb{C}$ with $|c| = 1$.*

Proof. Let $x, \omega \in \mathbb{R}^d$. Then Theorem 4.3.3 yields

$$W(\mathcal{F}_{S^{-1}}\mathcal{F}_S f)(x,\omega) = W(\mathcal{F}_S f)(S^{-1}(x,\omega)) = W(f)(SS^{-1}(x,\omega)) = W(f)(x,\omega).$$

Consequently, Theorem 3.1.12 yields

$$\mathcal{F}_{S^{-1}}\mathcal{F}_S = c\,\mathrm{Id}$$

for some $c \in \mathbb{C}$ with $|c| = 1$. Thus,

$$\mathcal{F}_{S^{-1}} = c\mathcal{F}_S{}^{-1}.$$ □

Using Theorem 4.3.3 the interpretation of unitary operators as linear maps on the time-frequency plane becomes very concrete: If we assume that $S^{-1}\tilde{\lambda} = (0, \tilde{\omega})$ for some $\tilde{\omega} \in \mathbb{R}^d$, we get, using Lemma 3.1.10, Theorem 4.3.3 and Lemma 4.3.5,

$$\begin{aligned} W((\mathcal{F}_S \circ \rho(\tilde{\lambda}) \circ \mathcal{F}_S^{-1})(f))(x,\omega) &= W((\rho(\tilde{\lambda}) \circ \mathcal{F}_S^{-1})(f))(S(x,\omega)) \\ &= W(\mathcal{F}_S^{-1}(f))(S(x,\omega) - (\tilde{\lambda}_1, \tilde{\lambda}_2)) \\ &= c_1 W(f)((x,\omega) - S^{-1}(\tilde{\lambda}_1, \tilde{\lambda}_2)) \\ &= c_1 W(f)((x,\omega) - (0,\tilde{\omega})) = W(c_1 M_{\tilde{\omega}} f)(x,\omega), \end{aligned}$$

where $c_1 \in \mathbb{C}$ with $|c_1| = 1$ comes from the application of Lemma 4.3.5. The reconstruction result for the Wigner transform, Theorem 3.1.12, then yields

$$\mathcal{F}_S \circ \rho(\tilde{\lambda}) \circ \mathcal{F}_S^{-1} = cM_{\tilde{\omega}}$$

for some $c \in \mathbb{C}$ with $|c| = 1$, which is basically what we wanted.

We are now going to carry out these arguments in detail: First, we need to be able to map an arbitrary point of the time-frequency plane to a point without time component by a symplectic matrix.

Lemma 4.3.6. *For any $x, \omega \in \mathbb{R}^d$ with $\omega \neq 0$ we can find $S \in \mathrm{Sp}(2d, \mathbb{R})$ such that $S(x,\omega) = (0,\omega)$.*

Proof. We show that there exists $S \in \mathrm{U}(2d, \mathbb{R}) \subseteq \mathrm{Sp}(2d, \mathbb{R})$ satisfying our claim (see Definition 3.2.6 and Lemma 3.2.7). To this end, let

$$S := \begin{pmatrix} I & P \\ 0 & I \end{pmatrix}$$

where $P \in \mathrm{Sym}(d, \mathbb{R})$. Then the condition $S(x,\omega) = (0,\omega)$ is equivalent to the condition $P\omega = -x$. To show the latter, we first note that as $\omega \neq 0$ we can find $1 \leq k \leq d$ such that $\omega_k \neq 0$. Then we define P in the following way: If $\omega_i \neq 0$, then $P_{ii} := -\frac{x_i}{\omega_i}$ and if $\omega_i = 0$, then $P_{ik} = P_{ki} = -\frac{x_i}{\omega_k}$. Otherwise, the entries of P are defined to be 0. Then

$$(P\omega)_i = \sum_{j=1}^{d} P_{ji}\omega_j = P_{ii}\omega_i + \sum_{\substack{j=1 \\ j\neq i}}^{d} P_{ji}\omega_j = -x_i + \sum_{\substack{j=1 \\ j\neq i, P_{ji}\neq 0}}^{d} P_{ji}\omega_j = -x_i$$

Now for $1 \le i \le d$ such that $\omega_i \neq 0$ we have

$$(P\omega)_i = \sum_{j=1}^{d} P_{ji}\omega_j = P_{ii}\omega_i + \sum_{\substack{j=1\\ j\neq i}}^{d} P_{ji}\omega_j = -x_i + \sum_{\substack{j=1\\ j\neq i, P_{ji}\neq 0}}^{d} P_{ji}\omega_j = -x_i$$

where the last equality follows because if $P_{ji} \neq 0$, then $\omega_j = 0$ by definition of P. On the other hand, for $1 \le i \le d$ such that $\omega_i = 0$ we have

$$(P\omega)_i = \sum_{j=1}^{d} P_{ji}\omega_j = P_{ki}\omega_k = -x_i$$

where the second-but-last equality follows from the fact that only $P_{ki} \neq 0$ in the sum as an offdiagonal nonzero entry of P must have row or column index k and $i \neq k$. Consequently, $P\omega = -x$ and the claim follows from the remark at the beginning. □

The next theorem already supplies us with the needed unitary operator. The proof is mainly putting together all the things we have.

Theorem 4.3.7. *Let $\lambda \in \mathbb{R}^{2d}$. Then there exists a unitary operator*

$$\mathcal{F}_\lambda \colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$$

and $c \in \mathbb{C}$ with $|c| = 1$ such that

$$\mathcal{F}_\lambda \circ \rho(\lambda) \circ \mathcal{F}_\lambda^{-1} = cM_\xi$$

for some $\xi \in \mathbb{R}^d$. More concretely, if $\lambda_2 \neq 0$ then $\xi = \lambda_2$ and otherwise $\xi = -\lambda_1$.

Proof. For $S \in \mathrm{Sp}(2d, \mathbb{R})$ we denote the corresponding unitary operator from Theorem 4.3.3 by $\mathcal{F}_S$. Then, as we already calculated,

$$\begin{aligned} W((\mathcal{F}_S \circ \rho(\lambda) \circ \mathcal{F}_S^{-1})(f))(x,\omega) &= W((\rho(\lambda) \circ \mathcal{F}_S^{-1})(f))(S(x,\omega)) \\ &= W(\mathcal{F}_S^{-1}(f))(S(x,\omega) - (\lambda_1, \lambda_2)) \\ &= c_1 W(f)((x,\omega) - S^{-1}(\lambda_1,\lambda_2)), \end{aligned}$$

where $c_1 \in \mathbb{C}$ with $|c| = 1$. Now we distinguish two cases:

- $\lambda_2 = 0$: We choose $S^{-1} = J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, which is symplectic.
- $\lambda_2 \neq 0$: We choose S^{-1} according to Lemma 4.3.6 such that $S^{-1}(\lambda_1, \lambda_2) = (0, \lambda_2)$.

In both cases we get

$$W(f)((x,\omega) - S^{-1}(\lambda_1,\lambda_2)) = W(f)((x,\omega) - (0,\xi)) = W(M_\xi(f))(x,\omega)$$

for some $\xi \in \mathbb{R}^d$, which in total means that

$$W \circ \mathcal{F}_S \circ \rho(\lambda) \circ \mathcal{F}_S^{-1} = W \circ c_1 M_\xi$$

and then Theorem 3.1.12 yields

$$\mathcal{F}_S \circ \rho(\lambda) \circ \mathcal{F}_S^{-1} = c M_\xi$$

for $c \in \mathbb{C}$ with $|c| = 1$. □

Finally, we are ready to show the compatibility condition (4.1):

Theorem 4.3.8. *Let $(a_k) \in \ell^1(\mathbb{Z}, *)$ and i be the embedding from Lemma 4.3.1. Then*

$$\|\rho(i(a))\|_{\mathcal{B}(L^2)} = \left\| \sum_{\lambda \in \Lambda} (i(a))_\lambda \rho(\lambda) \right\|_{\mathcal{B}(L^2)} = \left\| \sum_{k \in \mathbb{Z}} a_k e^{2\pi i k \cdot} \right\|_\infty = \|\hat{a}\|_\infty .$$

Proof. We denote the unitary operator corresponding to $\tilde{\lambda}$ (being implicitly defined via i, see Remark 4.3.2), which we get from Theorem 4.3.7, by $\mathcal{F}_{\tilde{\lambda}}$. Then, using Lemma 3.1.7,

$$\left\| \sum_{\lambda \in \Lambda} (i(a))_\lambda \rho(\lambda) \right\|_{\mathcal{B}(L^2)} = \left\| \sum_{k \in \mathbb{Z}} a_k \rho(k\tilde{\lambda}) \right\|_{\mathcal{B}(L^2)} = \left\| \sum_{k \in \mathbb{Z}} a_k \rho(\tilde{\lambda})^k \right\|_{\mathcal{B}(L^2)} . \tag{4.2}$$

Using Theorem 4.3.7 and then unitarity of $\mathcal{F}_{\tilde{\lambda}}^{-1}$ and $\mathcal{F}_{\tilde{\lambda}}$ we get

$$\begin{aligned} \left\| \sum_{k \in \mathbb{Z}} a_k \rho(\tilde{\lambda})^k \right\|_{\mathcal{B}(L^2)} &= \left\| \sum_{k \in \mathbb{Z}} a_k \mathcal{F}_{\tilde{\lambda}}^{-1} c^k M_\xi^k \mathcal{F}_{\tilde{\lambda}} \right\|_{\mathcal{B}(L^2)} = \left\| \sum_{k \in \mathbb{Z}} a_k c^k M_\xi^k \right\|_{\mathcal{B}(L^2)} \\ &= \left\| \sum_{k \in \mathbb{Z}} a_k c^k e^{2\pi i k \xi \cdot} \right\|_\infty , \end{aligned} \tag{4.3}$$

where the last equality follows from the fact that the operator

$$\sum_{k \in \mathbb{Z}} a_k c^k M_\xi^k$$

is a multiplication operator acting by multiplication of a bounded, continuous function and Lemma 1.3.6. Now taking $s \in \mathbb{R}$ such that $c = e^{2\pi i s}$ we get

$$\sum_{k \in \mathbb{Z}} a_k c^k M_\xi^k = \sum_{k \in \mathbb{Z}} a_k e^{2\pi i k (s + \xi \cdot)} \tag{4.4}$$

and clearly the function on the right-hand side has the same image as

$$\sum_{k \in \mathbb{Z}} a_k e^{2\pi i k \cdot}.$$

Consequently, the result follows from combining (4.2), (4.3) and (4.4). □

Concerning the second embedding $L\colon \ell^1(\Lambda, \natural, ^*) \to \mathcal{B}(\ell^2(\Lambda))$ from Definition 4.2.1, we use a similar strategy. As before, we embed $\ell^1(\mathbb{Z}, *)$ into $\ell^1(\Lambda, \natural)$ via the mapping i from Lemma 4.3.1 and again, the only thing left to show is that for $a \in \ell^1(\mathbb{Z})$ we have

$$\left\|L_{i(a)}\right\|_{\mathcal{B}(L^2)} = \left\| \sum_{k\in\mathbb{Z}} a_k e^{2\pi i k \cdot} \right\|_{\infty}. \tag{4.5}$$

We are going to show (4.5) by using the corresponding result for the embedding ρ we just obtained. We first note that the upper inequality from (4.5) can be shown directly from what we already have.

Lemma 4.3.9. *Let $\Lambda \subset \mathbb{R}^{2d}$ be a lattice and $a \in \ell^1(\mathbb{Z})$. Then*

$$\left\|L_{i(a)}\right\|_{\mathcal{B}(l^2(\Lambda))} \geq \left\| \sum_{k\in\mathbb{Z}} a_k e^{2\pi i k \cdot} \right\|_{\infty}.$$

Proof. This is just a consequence of Lemma 4.3.1 and Lemma 1.3.7:

$$\begin{aligned}
\left\| \sum_{k\in\mathbb{Z}} a_k e^{2\pi i k \cdot} \right\|_{\infty} &\overset{\text{Lemma 1.3.7}}{=} \sup_{\|b\|=1} \|F_a(b)\|_2 = \sup_{\|b\|=1} \|a * b\|_2 \\
&\overset{\text{Lemma 4.3.1}}{=} \sup_{\|b\|=1} \|i(a) \natural\, i(b)\|_2 = \sup_{\|b\|=1} \left\|L_{i(a)}(i(b))\right\|_2 \\
&\overset{\text{Lemma 4.3.1}}{\leq} \sup_{\substack{c\in\ell^2(\Lambda) \\ \|c\|=1}} \left\|L_{i(a)}(b)\right\|_2 = \left\|L_{i(a)}\right\|_{\mathcal{B}(\ell^2(\Lambda))}.
\end{aligned}$$

□

Finally, we need another theorem from the theory of C*-algebras.

Theorem 4.3.10 (Theorem 1.3.2, [35])**.** *Let $\mathcal{A}, \mathcal{B}$ be C*-algebras and $\phi\colon \mathcal{A} \to \mathcal{B}$ be an injective homomorphism between C*-algebras. Then ϕ is automatically isometric, meaning $\|\phi(a)\| = \|a\|$ for all $a \in \mathcal{A}$.*

Using the last theorem we are ready to prove the equality (4.5). The following proof is closely related to [32, p. 13 f.], where Gröchenig and Leinert proved this statement as an auxiliary result.

Theorem 4.3.11. *Let $\Lambda \subset \mathbb{R}^{2d}$ be a lattice and $a \in \ell^1(\mathbb{Z})$. Then*

$$\left\|L_{i(a)}\right\|_{\mathcal{B}(l^2(\Lambda))} = \|\rho(i(a))\|_{\mathcal{B}(L^2(\mathbb{R}^d))} = \left\| \sum_{k\in\mathbb{Z}} a_k e^{2\pi i k \cdot} \right\|_{\infty}.$$

Proof. We already showed the second equality in Theorem 4.3.8 so it suffices to show the first one. To this end, we consider

$$C^*\left(\{L_{i(a)} : a \in \ell^1(\mathbb{Z})\}\right)$$

and

$$C^*\left(\{\rho(i(a)) : a \in \ell^1(\mathbb{Z})\}\right)$$

which are the C*-algebras generated by the corresponding sets. As both generating sets are already algebras closed under the involution, they form dense subsets of the generated C*-algebras. Now we consider the following mapping:

$$\begin{aligned} \tilde{\phi}\colon \{L_{i(a)} : a \in \ell^1(\mathbb{Z})\} &\longrightarrow \{\rho(i(a)) : a \in \ell^1(\mathbb{Z})\} \\ L_{i(a)} &\longmapsto \rho(i(a)) \end{aligned}$$

$\tilde{\phi}$ is clearly well-defined and compatible with the algebra and involutive structure (compare with Lemma 4.2.2 and Lemma 4.2.9). Moreover,

$$\left\|\tilde{\phi}\left(L_{i(a)}\right)\right\| = \|\rho(i(a))\| = \left\|\sum_{k\in\mathbb{Z}} a_k e^{2\pi i k\cdot}\right\|_\infty \overset{\text{Lemma 4.3.9}}{\leq} \|L_{i(a)}\|$$

which shows that $\tilde{\phi}$ is bounded. Consequently, $\tilde{\phi}$ extends to a homomorphism

$$\phi\colon C^*\left(\{L_{i(a)} : a \in \ell^1(\mathbb{Z})\}\right) \to C^*\left(\{\rho(i(a)) : a \in \ell^1(\mathbb{Z})\}\right)$$

between C*-algebras. Our goal is to invoke Theorem 4.3.10 so we are just left to show that ϕ is injective. To this end, we assume that $\phi(L) = 0$ for some $L \in C^*\left(\{L_{i(a)} : a \in \ell^1(\mathbb{Z})\}\right)$. By density we can find $(a^{(n)}) \in \ell^1(\mathbb{Z})^{\mathbb{N}}$ such that

$$\left\|L - L_{i\left(a^{(n)}\right)}\right\| \xrightarrow{n\to\infty} 0$$

and then our assumption $\phi(L) = 0$ translates to

$$0 \xleftarrow{n\to\infty} \left\|\rho\left(i\left(a^{(n)}\right)\right)\right\| \overset{\text{Theorem 4.3.8}}{=} \left\|\mathcal{F}\left(a^{(n)}\right)\right\|.$$

Next, we use Plancherel's theorem (Theorem 1.3.2) to show that the ℓ^2-norm of $a^{(n)}$ also tends to 0 for $n \to \infty$:

$$\left\|a^{(n)}\right\|_2 = \left\|\mathcal{F}\left(a^{(n)}\right)\right\|_{L^2(\mathbb{T})} \leq \left\|\mathcal{F}\left(a^{(n)}\right)\right\|_\infty \xrightarrow{n\to\infty} 0.$$

Now we consider for some fixed $\mu \in \Lambda$ the unit mass $u^\mu \in \ell^2(\Lambda)$ defined via

$$u^\mu_\lambda = \begin{cases} 1, & \lambda = \mu \\ 0, & \text{otherwise} \end{cases}$$

and apply it to $L_{i(a^{(n)})}$:

$$\begin{aligned}\left\|L_{i(a^{(n)})}(u^\mu)\right\|_2^2 &= \sum_{\lambda\in\Lambda}\left|\sum_{k\in\mathbb{Z}} a_k^{(n)} u^\mu_{\lambda-k\tilde\lambda} e^{-2\pi i\sigma(k\tilde\lambda,\lambda-k\tilde\lambda)}\right|^2 \\ &= \sum_{l\in\mathbb{Z}}\left|\sum_{k\in\mathbb{Z}} a_k^{(n)} u^\mu_{\mu+l\tilde\lambda-k\tilde\lambda} e^{-2\pi i\sigma(k\tilde\lambda,\mu+l\tilde\lambda-k\tilde\lambda)}\right|^2 = \sum_{l\in\mathbb{Z}}\left|a_l^{(n)} e^{-2\pi i\sigma(l\tilde\lambda,\mu)}\right|^2 \\ &= \left\|a^{(n)}\right\|_2^2 \xrightarrow{n\to\infty} 0.\end{aligned}$$

As

$$\left\|L - L_{i\left(a^{(n)}\right)}\right\| \xrightarrow{n\to\infty} 0,$$

this implies that $L(u^\mu) = 0$ for all $\mu \in \Lambda$. Consequently, $L = 0$ and this shows injectivity. Therefore, Theorem 4.3.10 yields that ϕ is actually an isometry and this means exactly that

$$\left\|L_{i(a)}\right\|_{\mathcal{B}(l^2(\Lambda))} = \|\rho(i(a))\|_{\mathcal{B}(L^2(\mathbb{R}^d))},$$

which was to be shown. □

Having prepared everything carefully, we are ready to transfer the results from Chapter 2 to the twisted convolution.

Theorem 4.3.12. *Let $\lambda \subset \mathbb{R}^{2d}$ be a lattice and ι be either the embedding ρ from Definition 4.2.8 or the embedding L from Definition 4.2.1. Then for $1 < M < 2$ holds:*

$$\sup\left\{\|a^{-1}\| : a \in \ell^1(\Lambda, \natural) \text{ with } \|a\| \le 1, \|\iota(a)^{-1}\| \le M\right\} \ge \frac{M}{2-M}.$$

Proof. This follows immediately from embedding $\ell^1(\mathbb{Z}, *)$ into $\ell^1(\Lambda, \natural)$ via i and the corresponding result for $\ell^1(\mathbb{Z}, *)$ (Theorem 2.1.14) due to Lemma 4.3.1, Theorem 4.3.8 and Theorem 4.3.11. □

Theorem 4.3.13. *Let $\Lambda \subset \mathbb{R}^{2d}$ be a lattice. Then for $1 < M < 2$ holds:*

$$\sup\left\{\|a^{-1}\| : a \in \ell^1(\Lambda, \widetilde{\natural}) \text{ with } \|a\| \le 1, \left\|\tilde R_a^{-1}\right\| \le M\right\} \ge \frac{M}{2-M}$$

where $\tilde R_a$ is the operator from Definition 4.2.5.

Proof. We first show that the claimed result holds for the operator

$$b \mapsto b \natural a$$

on $\ell^2(\Lambda)$. In order to do this, we define the matrix

$$M = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

where $I \in M(d,\mathbb{R})$ is the identity matrix and consider the lattice $\hat{\Lambda} := M\Lambda$. To every sequence c on Λ we assign a sequence $\hat{c}$ on $\hat{\Lambda}$ via $\hat{c}_{M\lambda} = c_\lambda$ and we investigate the operator $L_{\hat{a}}$, which acts on sequences on the lattice $\hat{\Lambda}$. For $\hat{b} \in \ell^2(\hat{\Lambda}, \natural)$ and $\lambda \in \Lambda$ this operators satisfies

$$
\begin{aligned}
(\hat{L}_{\hat{a}}\hat{b})_{M\lambda} = (\hat{a} \natural \hat{b})_{M\lambda} &= \sum_{\hat{\mu}\in\hat{\Lambda}} \hat{a}_{\hat{\mu}} \hat{b}_{M\lambda-\hat{\mu}} e^{-2\pi i \frac{\hat{\mu}_1\cdot(M\lambda)_2 - \hat{\mu}_2\cdot(M\lambda)_1}{2}} \\
&\overset{\mu=M^{-1}\hat{\mu}}{=} \sum_{\mu\in\Lambda} \hat{a}_{M\mu} \hat{b}_{M\lambda-M\mu} e^{-2\pi i \frac{(M\mu)_1\cdot(M\lambda)_2 - (M\mu)_2\cdot(M\lambda)_1}{2}} \\
&= \sum_{\mu\in\Lambda} a_\mu b_{\lambda-\mu} e^{-2\pi i \frac{\mu_2\cdot\lambda_1 - \mu_1\cdot\lambda_2}{2}} \\
&\overset{\nu=\lambda-\mu}{=} \sum_{\nu\in\Lambda} b_\nu a_{\lambda-\nu} e^{-2\pi i \frac{(\lambda-\nu)_2\cdot\lambda_1 - (\lambda-\nu)_1\cdot\lambda_2}{2}} = \sum_{\nu\in\Lambda} b_\nu a_{\lambda-\nu} e^{-2\pi i \frac{\nu_1\cdot\lambda_2 - \nu_2\cdot\lambda_1}{2}} = (b \natural a)_\lambda.
\end{aligned}
$$

Thus, the identification from above yields $\|a\|_1 = \|\hat{a}\|_1$, $(\hat{a})^{-1} = \widehat{(a^{-1})}$ and

$$\|\cdot \underset{\Lambda}{\natural} a\|_{\mathcal{B}(\ell^2(\Lambda))} = \|\hat{a} \underset{\hat{\Lambda}}{\natural} \cdot\|_{\mathcal{B}(\ell^2(\hat{\Lambda}))}.$$

Consequently, we can transfer Theorem 4.3.12 to the operator which acts by convolution from the right. Now the only thing left to show is that we can replace the twisted convolution by its asymmetric variant. To this end, let i be the isomorphism from Lemma 4.1.8. Then for $a \in \ell^1(\Lambda, \natural)$ we clearly have $\|a\| = \|i(a)\|$, $\|a^{-1}\| = \|i(a)^{-1}\|$ and

$$\sup_{\|i(b)\|=1} \|i(b) \,\widetilde{\natural}\, i(a)\| = \sup_{\|i(b)\|=1} \|i(b \natural a)\| = \sup_{\|i(b)\|=1} \|b \natural a\| = \sup_{\|b\|=1} \|b \natural a\|.$$

So we can transfer the result also to the asymmetric variant via the identification given by i and this finishes the proof. □

The following two results are original contributions.

Theorem 4.3.14. *Let $\Lambda \subset \mathbb{R}^{2d}$ be a lattice, i.e. $\Lambda = A\mathbb{Z}^{2d}$ for some $A \in \mathrm{GL}(2d,\mathbb{R})$ and ι be either the embedding ρ from Definition 4.2.8 or the embedding L from Definition 4.2.1. Then for $1 < M \le 2$ and $N \ge 3^{2e^{e^5}}$:*

$$
\begin{aligned}
&\sup\left\{ \left\|a^{-1}\right\| : a \in \ell^1(\Lambda,\natural) \textit{ with } \|a\| \le 1, \left\|\iota(a)^{-1}\right\| \le M \textit{ and } \operatorname{supp}(a) \subseteq A\,[-N,N]^{2d} \right\} \\
&\ge \frac{9}{10} \frac{1 + M^{\frac{\log(N)^{\frac{1}{2}}}{\log(\log(N))^6}}}{1 + (2-M)^{\frac{\log(N)^{\frac{1}{2}}}{\log(\log(N))^6}}}.
\end{aligned}
$$

Proof. This follows immediately from embedding $\ell^1(\mathbb{Z},*)$ into $\ell^1(\Lambda,\natural)$ via i, where we explicitly choose $\tilde{\lambda} := Au_1$, where u_1 is the first unit vector, (see Lemma 4.3.1), and Corollary 2.3.9 due to Lemma 4.3.1, Theorem 4.3.8 and Theorem 4.3.11. □

Corollary 4.3.15. *Let* $\Lambda \subset \mathbb{R}^{2d}$ *be a lattice, i.e.* $\Lambda = A\mathbb{Z}^{2d}$ *for some* $A \in \mathrm{GL}(2d, \mathbb{R})$ *and* $\tilde{R}$ *be the embedding from Definition 4.2.5. Then for* $1 < M \leq 2$ *and* $N \geq 3^{2e^{e^5}}$ *:*

$$\sup \left\{ \left\| a^{-1} \right\| : a \in \ell^1(\Lambda, \natural) \text{ with } \|a\| \leq 1, \left\| \tilde{R}_a^{-1} \right\| \leq M \text{ and } \operatorname{supp}(a) \subseteq A\,[-N, N]^{2d} \right\}$$
$$\geq \frac{9}{10} \frac{1 + M \frac{\log(N)^{\frac{1}{2}}}{\log(\log(N))^6}}{1 + (2 - M) \frac{\log(N)^{\frac{1}{2}}}{\log(\log(N))^6}}.$$

Proof. This follows from Theorem 4.3.14 and the proof of Theorem 4.3.13 because the identification given there, which transfers the corresponding result from the operator L_a to the operator $\tilde{R}_a$, preserves the support of the involved sequences. $\square$

5 Gabor analysis

Gabor analysis was proposed by Dennis Gabor in his paper [36]. It uses the methods of frame theory to reconstruct functions. The frame in this case consists of a family of time-frequency shifts of a fixed function g (the so-called window). The reconstruction is by means of the dual window $\tilde{g}$ (see Definition 5.2.7) corresponding to g. In this chapter we first give a quick introduction into frame theory and Gabor analysis and then apply the results from Section 4.3 to show that the norm of the dual window $\tilde{g}$ cannot be controlled using the norm of the window g itself and the corresponding frame or Riesz sequence bounds (see Definition 5.1.9 and Definition 5.1.18).

5.1 Frame theory

In this section we give an introduction into frame theory in order to define the necessary concepts and language we need later. For a comprehensive introduction into the field of frames and Riesz sequences we refer to the book by Christensen [37]. We supply most proofs in this section because in contrast to the given reference we want to focus on the duality between these concepts.

5.1.1 Frames

The term "frame" was introduced by Duffin and Schaeffer in their paper [38], initially in the context of Fourier series. We motivate the concept by considering a separable Hilbert space $\mathcal{H}$ with orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$. There it is well-known that for any $f \in \mathcal{H}$ we have

$$f = \sum_{k=0}^{\infty} \langle f, e_k \rangle e_k. \tag{5.1}$$

In other words, we can reconstruct f from the measurements $(\langle f, e_k \rangle)_{k\in\mathbb{N}}$. In frame theory we try to get reconstruction formulas for f like the one before without having to assume that $\{e_k\}_{k\in\mathbb{N}}$ is an orthonormal basis. In this case it is often natural to index the measurements not over $\mathbb{N}$ but over other sets (as we will see for example in Section 5.2) which have no natural ordering. To avoid artificial orderings, we work with the concept of unconditional convergence.

Definition 5.1.1. *Let I be a countable index set, $\mathcal{B}$ a Banach space, $(a_i) \in \mathcal{B}^I$ and $a \in \mathcal{B}$. $\sum_{i\in I} a_i$ converges unconditionally to a if for any bijection $\pi\colon \mathbb{N} \to I : \sum_{k=0}^{\infty} a_{\pi(k)} = a$.*

In the following, whenever an infinite series over some unordered index set appears, we automatically assume that it converges unconditionally without mentioning it explicitly.

Motivated by the formula $f = \sum_{k=0}^{\infty} \langle f, e_k \rangle e_k$, we split the whole process into two parts and define the corresponding operators: One which calculates the coefficients for a given f and one which constructs an element in the Hilbert space from a given sequence:

Definition 5.1.2. *Let I be a countable index set, $\mathcal{H}$ a Hilbert space and $(e_i)_{i\in I} \in \mathcal{H}^I$. The analysis (or coefficient) operator C is*

$$\begin{aligned} C\colon \mathcal{H} &\longrightarrow \ell^2(I) \\ f &\longmapsto (\langle f, e_i \rangle)_{i\in I} \end{aligned}$$

and the synthesis operator D is

$$\begin{aligned} D\colon \ell^2(I) &\longrightarrow \mathcal{H} \\ (c_i) &\longmapsto \sum_{i\in I} c_i e_i \end{aligned}$$

whenever they are well-defined.

Next, we look into conditions under which these operators are well-defined and list some properties of them.

Theorem 5.1.3 (Lemma 3.2.1, [37]). *Let I be a countable index set, $\mathcal{H}$ a Hilbert space and $(e_i)_{i\in I} \in \mathcal{H}^I$ such that for any $(c_i)_{i\in I} \in \ell^2(I)$ the sum $\sum_{i\in I} c_i e_i$ converges unconditionally. Then $C\colon \mathcal{H} \longrightarrow \ell^2(I)$ and $D\colon \ell^2(I) \longrightarrow \mathcal{H}$ are well-defined, bounded and linear operators, which are adjoint to each other, so $D^* = C$. In particular, we have the estimate*

$$\sum_{i\in I} |\langle f, e_i \rangle|^2 \leq \|D\|^2 \|f\|^2.$$

The condition in the last theorem that $\sum_{i\in I} c_i e_i$ converges unconditionally for any $(c_i) \in \ell^2(I)$ is of course a little bit artificial. Luckily, this is actually equivalent to the analysis operator being bounded.

Definition 5.1.4. *Let I be a countable index set and $\mathcal{H}$ a Hilbert space. $(e_i) \in \mathcal{H}^I$ is called a Bessel sequence if there exists $B > 0$ such that*

$$\sum_{i\in I} |\langle f, e_i \rangle|^2 \leq B \|f\|^2$$

for all $f \in \mathcal{H}$.

Lemma 5.1.5 (Corollaries 3.2.4 and 3.2.5, [37]). *Let I be a countable index set and $\mathcal{H}$ a Hilbert space. $(e_i) \in \mathcal{H}^I$ is a Bessel sequence if and only if $\sum_{i\in I} c_i e_i$ converges unconditionally for any $(c_i) \in \ell^2(I)$.*

Now we return to our original goal: Motivated by the reconstruction formula for orthonormal bases (5.1) we consider the composition of the coefficient and synthesis operator:

Definition 5.1.6. *Let I be a countable index set, $\mathcal{H}$ a Hilbert space and $(e_i) \in \mathcal{H}^I$. Then the frame operator S is*

$$\begin{aligned} S\colon \mathcal{H} &\longrightarrow \mathcal{H} \\ f &\longmapsto \sum_{i\in I} \langle f, e_i\rangle e_i, \end{aligned}$$

whenever this is well-defined.

We first give an exact criterion for when the frame operator is well-defined and collect some properties of it.

Lemma 5.1.7. *Let I be a countable index set, $\mathcal{H}$ a Hilbert space and $(e_i) \in \mathcal{H}^I$. Then the frame operator S corresponding to (e_i) is well-defined if and only if (e_i) is a Bessel sequence. Moreover, under these conditions $S = DC$, where C and D are the analysis and synthesis operator, respectively, and it is a positive operator satisfying*

$$\langle Sf, f\rangle = \sum_{i\in I} |\langle f, e_i\rangle|^2$$

for all $f \in \mathcal{H}$.

Proof. $\Longrightarrow$: Assuming that S is well-defined as a bounded operator on $\mathcal{H}$, we first note that for $f \in \mathcal{H}$

$$\langle Sf, f\rangle = \Big\langle \sum_{i\in I} \langle f, e_i\rangle e_i, f\Big\rangle = \sum_{i\in I} \langle f, e_i\rangle\langle e_i, f\rangle = \sum_{i\in I} |\langle f, e_i\rangle|^2. \tag{5.2}$$

Consequently,

$$\|S\|\|f\|^2 \geq \langle Sf, f\rangle = \sum_{i\in I} |\langle f, e_i\rangle|^2,$$

which readily shows that (e_i) is a Bessel sequence.

$\Longleftarrow$: As (e_i) is a Bessel sequence, it follows from Theorem 5.1.3 and Lemma 5.1.5 that C and D are well-defined and then it is evident that $S = DC$. □

In the case of an orthonormal basis the reconstruction formula (5.1) yields that the corresponding frame operator is just the identity operator. If we substitute orthonormal bases by a more general concept this will of course be no longer the case but this is not a problem. Our goal is reconstruction and as long as the frame operator is invertible this is (at least theoretically) possible. So we need to find a condition for $(e_i)_{i\in I}$ such that the corresponding frame operator is invertible. Clearly, being a Bessel sequence is not sufficient:

Example 5.1.8. Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $(e_n)_{n\in\mathbb{N}}$. If we remove e_0 from $(e_n)_{n\in\mathbb{N}}$ the resulting sequence will still be Bessel because for $f \in \mathcal{H}$ we get

$$\sum_{n=1}^{\infty} |\langle f, e_i\rangle|^2 \leq \sum_{n=0}^{\infty} |\langle f, e_i\rangle|^2 = \|f\|^2.$$

But obviously the frame operator corresponding to the reduced sequence is not injective because due to orthogonality we have

$$Se_0 = \sum_{n=1}^{\infty} \langle e_0, e_n\rangle e_n = 0.$$

In order for the frame operator to be invertible, we for sure need that the analysis operator C is injective. This motivates the following definition.

Definition 5.1.9. *Let I be a countable index set and $\mathcal{H}$ a Hilbert space. $(e_i) \in \mathcal{H}^I$ is called a frame if there exist $A, B > 0$ such that*

$$A\|f\|^2 \leq \sum_{i\in I} |\langle f, e_i\rangle|^2 \leq B\|f\|^2 \tag{5.3}$$

for all $f \in \mathcal{H}$. The optimal frame bounds are the lowest possible B and the highest possible A which satisfy (5.3). A frame is called tight if the bounds coincide, i.e. $A = B$.

We start by giving some examples for frames and their properties.

Example 5.1.10. Let $(e_n) \in \mathcal{H}^{\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$. Then

$$\|f\|^2 = \sum_{n\in\mathbb{N}} |\langle f, e_n\rangle|^2$$

for all $f \in \mathcal{H}$ and so (e_n) is a tight frame with optimal frame bounds $A = B = 1$. If we now consider $(e_n)_{n\in\mathbb{N}, k\in\{0,1\}}$, i.e. the same elements but every element is contained twice in the sequence, then this still forms a frame as

$$2\|f\|^2 = \sum_{n\in\mathbb{N}, k\in\{0,1\}} |\langle f, e_n\rangle|^2.$$

This example shows that frames can be overcomplete. On the other hand, if we consider $(e_n) \in \mathcal{H}^{\mathbb{N}}$ which is not a generating system of $\mathcal{H}$, i.e.

$$\overline{\operatorname{span}\{e_n\}_{n\in\mathbb{N}}} \neq \mathcal{H}$$

then this can never be a frame: In this case we can find

$$0 \neq f \in \{e_n\}_{n\in\mathbb{N}}^{\perp}$$

and then clearly

$$\sum_{n\in\mathbb{N}} |\langle f, e_n\rangle|^2 = 0$$

and so no lower frame bound can exist.

Next, we show that the concept of frame is the one we searched for: The corresponding frame operator is invertible. Moreover, the inverse gives rise to another frame and the optimal frame bounds can be expressed in terms of the norm of the frame operator and its inverse. In order to prove this, we need the existence of a square root of positive operators.

Theorem 5.1.11 (Korollar VII.1.16., [39]). *Let $\mathcal{H}$ be a Hilbert space and $S\colon \mathcal{H} \to \mathcal{H}$ a positive operator. Then there exists a positive $T\colon \mathcal{H} \to \mathcal{H}$ such that $T^2 = S$.*

Theorem 5.1.12. *Let $\mathcal{H}$ be a Hilbert space and $S\colon \mathcal{H} \to \mathcal{H}$ a self-adjoint operator satisfying*

$$A\|f\|^2 \leq \langle Sf, f\rangle \leq B\|f\|^2. \tag{5.4}$$

Then S is invertible on $\mathcal{H}$ and S^{-1} satisfies

$$\frac{1}{B}\|f\|^2 \leq \langle S^{-1}f, f\rangle \leq \frac{1}{A}\|f\|^2. \tag{5.5}$$

Moreover, the optimal upper bound in (5.4) is $\|S\|$ and the optimal lower bound is $\|S^{-1}\|^{-1}$.

Proof. Let $f \in \mathcal{H}$ such that $Sf = 0$. Then by (5.4) $A\|f\|^2 = 0$ and so $f = 0$. Therefore, S is injective. Furthermore,

$$\ker(S) = \operatorname{im}(S)^{\perp}$$

due to the self-adjointness of S and thus by injectivity $\operatorname{im}(S)$ is dense in $\mathcal{H}$. Closedness of $\operatorname{im}(S)$ follows directly from the lower estimate in (5.4): Let $(f_n) \in \mathcal{H}^{\mathbb{N}}$ such that (Sf_n) is Cauchy. Then

$$A\|f_n - f_m\|^2 \leq \langle S(f_n - f_m), f_n - f_m\rangle \leq \|S(f_n - f_m)\|\|f_n - f_m\|$$

and so

$$A\|f_n - f_m\| \leq \|S(f_n - f_m)\|$$

which shows that (f_n) is Cauchy and thus converges to some $f \in \mathcal{H}$. Then by continuity of S, $Sf_n \xrightarrow{n\to\infty} Sf$ and therefore $\operatorname{im}(S)$ is closed. All in all, $S^{-1}\colon \mathcal{H} \to \mathcal{H}$ exists and boundedness of S^{-1} follows again from the lower inequality of (5.4) applied to $S^{-1}f$:

$$A\|S^{-1}f\|^2 \leq \langle SS^{-1}f, S^{-1}f\rangle \leq \|f\|\|S^{-1}f\|$$

implies that

$$\|S^{-1}f\| \leq \frac{1}{A}\|f\|.$$

Concerning the frame inequality (5.5), we first note that the upper bound follows immediately:

$$\langle S^{-1}f, f\rangle \leq \|S^{-1}f\|\|f\| \leq \frac{1}{A}\|f\|^2.$$

For the lower inequality we invoke Theorem 5.1.11 which yields a positive operator $T\colon \mathcal{H} \to \mathcal{H}$ such that $T^2 = S$. Then self-adjointness of T and (5.4) yield

$$\begin{aligned}\|f\|^2 &= \langle SS^{-1}f, SS^{-1}f\rangle = \langle T^2S^{-1}f, T^2S^{-1}f\rangle = \langle STS^{-1}f, TS^{-1}f\rangle \\ &\leq B\|TS^{-1}f\|^2 = B\langle TS^{-1}f, TS^{-1}f\rangle = B\langle S^{-1}f, f\rangle.\end{aligned}$$

Consequently,

$$\frac{1}{B}\|f\|^2 \leq \langle S^{-1}f, f\rangle.$$

In order to show the last claim regarding the optimal bounds, we first note that

$$\|S\| = \sup_{\|f\|=1} \langle Sf, f\rangle$$

as S is a self-adjoint operator (see e.g. [39, Satz V.5.7]). Thus, we can consider the upper inequality from (5.4),

$$\langle Sf, f\rangle \leq B\|f\|^2,$$

and taking the supremum over all $f \in \mathcal{H}$ with $\|f\| = 1$ on both sides yields

$$\|S\| \leq B.$$

Therefore, every upper bound of (5.4) is at least as large as $\|S\|$. On the other hand,

$$\langle Sf, f\rangle \leq \|Sf\|\|f\| \leq \|S\|\|f\|^2$$

and so $\|S\|$ is actually a possible bound. Consequently, $\|S\|$ has to be the optimal upper bound. As S^{-1} is also a self-adjoint operator the same argument yields that the optimal upper bound in (5.5) is $\|S^{-1}\|$. Then we conclude using our first observation: We know that for each pair of possible bounds $A, B > 0$ for S we get a pair $\frac{1}{B}, \frac{1}{A}$ of bounds for the corresponding inequality for S^{-1} and from this it follows immediately that if we choose the optimal bounds for one operator we automatically get the optimal bounds for the inverse operator. Therefore, the optimal lower bound in (5.4) is exactly the inverse of the optimal upper bound in (5.5), which is $\|S^{-1}\|^{-1}$, as claimed. □

It is not a coincidence that S^{-1} satisfies the same type of inequality as the frame operator S. Indeed, S^{-1} is also a frame operator as the following theorem shows.

Corollary 5.1.13. *Let I be a countable index set, $\mathcal{H}$ a Hilbert space, $(e_i) \in \mathcal{H}^I$ a frame and S its corresponding frame operator. Then $(S^{-1}e_i)$ is also a frame and its corresponding frame operator is S^{-1}. Furthermore, we have the following reconstruction formulas for $f \in \mathcal{H}$:*

$$f = \sum_{i\in I}\langle f, S^{-1}e_i\rangle e_i = \sum_{i\in I}\langle f, e_i\rangle S^{-1}e_i.$$

Proof. Let $A, B > 0$ be frame bounds for (e_i). Then for $f \in \mathcal{H}$ we have

$$\langle S^{-1}f, f\rangle = \langle S^{-1}f, SS^{-1}f\rangle \overset{(5.2)}{=} \sum_{i\in I} |\langle S^{-1}f, e_i\rangle|^2 = \sum_{i\in I} |\langle f, S^{-1}e_i\rangle|^2.$$

Consequently, Theorem 5.1.12 yields that

$$\frac{1}{B}\|f\|^2 \le \sum_{i\in I} |\langle f, S^{-1}e_i\rangle|^2 \le \frac{1}{A}\|f\|^2$$

and so $(S^{-1}e_i)$ constitutes a frame. Moreover, as

$$\begin{aligned} S^{-1}f = S^{-1}SS^{-1}f = S^{-1}\left(\sum_{i\in I}\langle S^{-1}f, e_i\rangle e_i\right) &= \sum_{i\in I}\langle S^{-1}f, e_i\rangle S^{-1}e_i \\ &= \sum_{i\in I}\langle f, S^{-1}e_i\rangle S^{-1}e_i, \end{aligned}$$

it follows that S^{-1} is the frame operator corresponding to $(S^{-1}e_i)$. Finally,

$$f = SS^{-1}f = \sum_{i\in I}\langle S^{-1}f, e_i\rangle e_i = \sum_{i\in I}\langle f, S^{-1}e_i\rangle e_i$$

and

$$f = S^{-1}Sf = S^{-1}\left(\sum_{i\in I}\langle f, e_i\rangle e_i\right) = \sum_{i\in I}\langle f, e_i\rangle S^{-1}e_i.$$

□

Definition 5.1.14. *Let I be a countable index set, $\mathcal{H}$ a Hilbert space, $(e_i) \in \mathcal{H}^I$ a frame and S its corresponding frame operator. Then $(S^{-1}e_i)$ is called the canonical dual frame to (e_i).*

Remark 5.1.15. The term "dual frame" from Definition 5.1.14 originates from the concept of dual basis from linear algebra: For a basis $\{v_1, \cdots, v_n\}$ of some finite-dimensional vector space V we consider the corresponding dual basis $\{v_1^*, \cdots, v_n^*\} \subset V^*$ defined by $v_j^*(v_k) = \delta_{jk}$, where δ is the Kronecker delta. This yields the expansion

$$v = \sum_{i=1}^{n} v_i^*(v) v_i \tag{5.6}$$

for any $v \in V$. In the context of a Hilbert space $\mathcal{H}$ we know that $\mathcal{H} \cong \mathcal{H}^*$ via

$$v \mapsto \langle \cdot, v\rangle.$$

Consequently, in this case the expansion (5.6) reduces to

$$v = \sum_{i=1}^{n} \langle v, w_i\rangle v_i$$

for some specific $w_i \in \mathcal{H}$ which are independent of $v \in \mathcal{H}$. This is exactly the type of expansion we also get if we use a frame and its canonical dual (see Corollary 5.1.13), which motivated the name dual frame. However, note that despite the similarities a frame and its canonical dual frame do not form a pair of dual bases as a frame does not even have to be a basis itself (it could be overcomplete, see Example 5.1.10).

5.1.2 Riesz sequences

The whole theory about the frame operator was motivated by the problem of reconstructing functions. If we put this motivation aside and just view the frame operator as a mathematical object it would be equally reasonable to consider the operator which consists of first taking the synthesis operator and then the analysis operator (i.e. invert the order of the operators). This yields a similar but in some way complementary theory.

Definition 5.1.16. *Let I be a countable index set, $\mathcal{H}$ a Hilbert space and $(e_i) \in \mathcal{H}^I$. Then the Gramian operator G is*

$$(c_i) \longmapsto \Big(\langle \sum_{j \in I} c_j e_j, e_i \rangle \Big),$$

whenever this is well-defined.

We again start with an exact criterion for when the Gramian operator is well-defined and collect some properties of it.

Lemma 5.1.17. *Let I be a countable index set, $\mathcal{H}$ a Hilbert space and $(e_i) \in \mathcal{H}^I$. Then the Gramian operator G corresponding to (e_i) is well-defined if and only if (e_i) is a Bessel sequence. Moreover, under these conditions $G = CD$, where C and D are the analysis and synthesis operator, respectively, and it is a positive operator satisfying*

$$\langle Gc, c \rangle = \big\| \sum_{i \in I} c_i e_i \big\|^2$$

for all $c \in \ell^2(I)$.

Proof. $\Longrightarrow$: As G is assumed to be well-defined and bounded, we have, in particular, that $\sum_{j \in I} c_j e_j$ converges unconditionally for any $c \in \ell^2(I)$. Thus, Lemma 5.1.5 implies that (e_i) is a Bessel sequence.

$\Longleftarrow$: As (e_i) is a Bessel sequence, it follows from Theorem 5.1.3 and Lemma 5.1.5 that C and D are well-defined and then it is evident that $G = CD$.

Concerning the last claim, we note that for $c \in \ell^2(I)$:

$$\langle Gc, c \rangle = \langle D^* Dc, c \rangle = \langle Dc, Dc \rangle = \|Dc\|^2 = \big\| \sum_{i \in I} c_i e_i \big\|^2. \tag{5.7}$$

□

As in the case for frames, we need an additional condition on the Bessel sequence (e_i) to guarantee that the corresponding Gramian operator is invertible. This condition is quite similar to the frame condition (5.3).

Definition 5.1.18. *Let I be a countable index set and $\mathcal{H}$ a Hilbert space. $(e_i) \in \mathcal{H}^I$ is called a Riesz sequence if there exist $A, B > 0$ such that*

$$A\|c\|^2 \leq \Big\|\sum_{i\in I} c_i e_i\Big\|^2 \leq B\|c\|^2 \tag{5.8}$$

for all $c \in \ell^2(I)$. The optimal bounds are the lowest possible B and the highest possible A which satisfy (5.8). A Riesz sequence is called tight if the bounds coincide, i.e. $A = B$.

Again, we start with some examples for Riesz sequences which already hint at the duality between the concepts.

Example 5.1.19. Let $(e_n) \in \mathcal{H}^{\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$. Then

$$\Big\|\sum_{n\in\mathbb{N}} c_n e_n\Big\|^2 = \sum_{n\in\mathbb{N}} \|c_n e_n\|^2 = \sum_{n\in\mathbb{N}} |c_n|^2 = \|c\|^2$$

for all $c \in \ell^2(\mathbb{N})$ and so (e_n) is a tight Riesz sequence with optimal bounds $A = B = 1$. If we now consider $(e_n)_{n=1}^{\infty}$, i.e. the same sequence without the first element, then this still forms a Riesz sequence as still

$$\Big\|\sum_{n=1}^{\infty} c_n e_n\Big\|^2 = \|c\|^2$$

for all $c \in \ell^2(\mathbb{N} \setminus \{0\})$ by the same reasoning as above. This example shows that Riesz sequences do not have to generate $\mathcal{H}$. On the other hand, if we consider $(e_n) \in \mathcal{H}^N$ which is overcomplete, i.e.

$$\sum_{n\in\mathbb{N}} c_n e_n = 0$$

for some $0 \neq c \in \ell^2(\mathbb{N})$, then this can never be a Riesz sequence because

$$\Big\|\sum_{n\in\mathbb{N}} c_n e_n\Big\|^2 = 0$$

for this specific choice of c and so no lower bound can exist. This is exactly contrary to the situation for frames (see Example 5.1.10).

We can now formulate results about the Gramian operator and its inverse completely parallel to the ones concerning the frame operator. At this point we can reuse the theory we developed in Section 5.1.1 which makes the proofs quite short.

Theorem 5.1.20. *Let I be a countable index set, $\mathcal{H}$ a Hilbert space and $(e_i) \in \mathcal{H}^I$ a Riesz sequence with bounds $A, B > 0$, i.e.*

$$A\|c\|^2 \leq \langle Gc, c\rangle \leq B\|c\|^2 \tag{5.9}$$

for all $c \in \ell^2(I)$, with corresponding Gramian G. Then G is invertible on $\ell^2(I)$ with G^{-1} satisfying

$$\frac{1}{B}\|c\|^2 \leq \langle G^{-1}c, c\rangle \leq \frac{1}{A}\|c\|^2 \tag{5.10}$$

for all $c \in \ell^2(I)$. Moreover, the optimal upper bound in (5.9) is $\|G\|$ and the optimal lower bound is $\|G^{-1}\|^{-1}$.

Proof. This follows immediately from Lemma 5.1.17 and Theorem 5.1.12. □

There is also a reconstruction result in the spirit of Corollary 5.1.13. In the following, we denote the i-th unit vector of $\ell^2(I)$ by u_i.

Theorem 5.1.21. *Let I be a countable index set, $\mathcal{H}$ a Hilbert space, $(e_i) \in \mathcal{H}^I$ a Riesz sequence, G its corresponding Gramian operator and D its corresponding synthesis operator. Then $(DG^{-1}u_i)$ is also a Riesz sequence and its corresponding Gramian operator is G^{-1}. Furthermore, we have the biorthogonality condition*

$$\left\langle e_i, DG^{-1}u_j\right\rangle = \delta_{ij} \tag{5.11}$$

where $i, j \in I$ and δ is the Kronecker delta. Finally, we can reconstruct any $f \in \overline{\operatorname{span}\{e_i\}_{i\in I}}$ via:

$$f = \sum_{i\in I}\langle f, DG^{-1}u_i\rangle e_i = \sum_{i\in I}\langle f, e_i\rangle DG^{-1}u_i.$$

Proof. Let $A, B > 0$ be bounds for the Riesz sequence (e_i). Then Theorem 5.1.20 implies that

$$\frac{1}{B}\|c\|^2 \leq \langle G^{-1}c, c\rangle \leq \frac{1}{A}\|c\|^2 \tag{5.12}$$

for all $c \in \ell^2(I)$. Moreover, we have

$$\begin{aligned}\Big\|\sum_{i\in I} c_i DG^{-1}u_i\Big\|^2 &= \Big\langle \sum_{i\in I} c_i DG^{-1}u_i, \sum_{i\in I} c_i DG^{-1}u_i\Big\rangle = \left\langle DG^{-1}c, DG^{-1}c\right\rangle \\ &\overset{\text{Theorem 5.1.3}}{=} \left\langle G^{-1}c, CDG^{-1}c\right\rangle = \left\langle G^{-1}c, GG^{-1}c\right\rangle = \left\langle G^{-1}c, c\right\rangle.\end{aligned}$$

Together with (5.12) this yields

$$\frac{1}{B}\|c\|^2 \leq \left\|\sum_{i\in I} c_i DG^{-1}u_i\right\|^2 \leq \frac{1}{A}\|c\|^2$$

and thus $(DG^{-1}u_i)$ is a Riesz sequence. Denoting its corresponding Gramian by $\tilde{G}$, we get

$$\begin{aligned}(\tilde{G}c)_i &= \Big\langle \sum_{j\in I} c_j DG^{-1}u_j, DG^{-1}u_i \Big\rangle = \big\langle DG^{-1}c, DG^{-1}u_i \big\rangle \\ &\overset{Theorem\ 5.1.3}{=} \big\langle G^{-1}c, CDG^{-1}u_i \big\rangle = \big\langle G^{-1}c, GG^{-1}u_i \big\rangle = \big\langle G^{-1}c, u_i \big\rangle = \big(G^{-1}c\big)_i\end{aligned}$$

for all $i \in I$ and $c \in \ell^2(I)$. Consequently,

$$\tilde{G} = G^{-1}. \tag{5.13}$$

Concerning the biorthogonality condition (5.11) we calculate for $i, j \in I$:

$$\begin{aligned}\big\langle e_i, DG^{-1}u_j \big\rangle &= \big\langle Du_i, DG^{-1}u_j \big\rangle \overset{Theorem\ 5.1.3}{=} \big\langle u_i, CDG^{-1}u_j \big\rangle \\ &= \big\langle u_i, GG^{-1}u_j \big\rangle = \langle u_i, u_j \rangle = \delta_{ij}.\end{aligned}$$

Now the only thing left to show are the reconstruction formulas. To this end, we first show that

$$\overline{\operatorname{span}\{e_i\}_{i\in I}} = \overline{\operatorname{span}\{DG^{-1}u_i\}_{i\in I}}.$$

By construction,

$$DG^{-1}u_j \in \overline{\operatorname{span}\{e_i\}_{i\in I}}$$

for all $j \in I$ and thus

$$\overline{\operatorname{span}\{DG^{-1}u_i\}_{i\in I}} \subseteq \overline{\operatorname{span}\{e_i\}_{i\in I}}.$$

Concerning the other inclusion, we note that

$$\sum_{j\in I} (Gu_i)_j DG^{-1}u_j = DG^{-1}Gu_i = e_i,$$

where the first sum is well-defined as $Gu_i \in \ell^2(I)$ for all $i \in I$. Consequently,

$$\overline{\operatorname{span}\{e_i\}_{i\in I}} \subseteq \overline{\operatorname{span}\{DG^{-1}u_i\}_{i\in I}}.$$

Now in order to simplify the notation we define

$$g_i \coloneqq DG^{-1}u_i$$

for $i \in I$. Finally, for $f \in \overline{\operatorname{span}\{e_i\}_{i\in I}}$ and $i \in I$ we have

$$\Big\langle f - \sum_{j\in I} \langle f, g_j \rangle e_j, g_i \Big\rangle = \langle f, g_i \rangle - \sum_{j\in I} \langle f, g_j \rangle \langle e_j, g_i \rangle \overset{(5.11)}{=} \langle f, g_i \rangle - \langle f, g_i \rangle = 0$$

and as

$$\overline{\operatorname{span}\{e_i\}_{i\in I}} = \overline{\operatorname{span}\{g_i\}_{i\in I}}$$

this yields

$$f = \sum_{i \in I} \langle f, g_i \rangle e_i.$$

Analogously,

$$\Big\langle f - \sum_{j \in I} \langle f, e_j \rangle g_j, e_i \Big\rangle = \langle f, e_i \rangle - \sum_{j \in I} \langle f, e_j \rangle \langle g_j, e_i \rangle \overset{(5.11)}{=} \langle f, e_i \rangle - \langle f, e_i \rangle = 0$$

and thus

$$f = \sum_{i \in I} \langle f, e_i \rangle g_i. \qquad \square$$

Definition 5.1.22. *Let I be a countable index set, $\mathcal{H}$ a Hilbert space, $(e_i) \in \mathcal{H}^I$ a Riesz sequence, G its corresponding Gramian operator and D its corresponding synthesis operator. Then $(DG^{-1}u_i)$ is called the canonical dual Riesz sequence to (e_i).*

Remark 5.1.23. At this point we should again compare the last result with the corresponding result for frames: In the case of frames, the reconstruction formula holds for the whole Hilbert space, in the case of Riesz sequences, it only holds for the Hilbert space generated by the Riesz sequence (which is in general not the whole Hilbert space as we have seen for example in Example 5.1.19). On the other hand, a Riesz sequence and its canonical dual satisfy the biorthogonality conditions (5.11) (see also the discussion in Remark 5.1.15) which is not necessarily the case for frames.

Remark 5.1.24. In the picture of the preceding comparison between frames and Riesz bases, we want to look again at the case of orthonormal bases: Every orthonormal basis is a frame with corresponding bounds $A = B = 1$. However, there are examples of frames with bounds $A = B = 1$ where the frame elements are not orthogonal to each other (see [37, Example 5.1.4 (ii)]). On the other hand, it is also true that every orthonormal basis forms a Riesz sequence with bounds $A = B = 1$. Still, a Riesz sequence with bounds $A = B = 1$ does not have to be an orthonormal basis because there is no guarantee that it spans the space (see Example 5.1.19). But in this case, the elements form an orthonormal system, i.e. $\langle e_i, e_j \rangle = \delta_{ij}$ for $i, j \in I$. This follows immediately as (e_i) is the image of (u_i) (which is an orthonormal basis) under the synthesis operator D and in the case $A = B = 1$, D is an isometry.

A Riesz sequence satisfies a strong form of linear independence, which follows immediately from (5.8). Consequently, if a Riesz sequence is also a generating system, i.e. $\overline{\operatorname{span}\{e_i\}_{i \in I}} = \mathcal{H}$, it forms a (Schauder) basis. This motivates the following:

Definition 5.1.25. *Let I be a countable index set, $\mathcal{H}$ a Hilbert space and $(e_i) \in \mathcal{H}^I$ a Riesz sequence. If (e_i) is a generating system of $\mathcal{H}$, then (e_i) is called a Riesz basis.*

We close this chapter with the result that the invertibility of the Gramian operator implies being a Riesz sequence.

Lemma 5.1.26. *Let $\mathcal{H}$ be a Hilbert space and $S\colon \mathcal{H} \to \mathcal{H}$ a positive operator which is invertible on $\mathcal{H}$. Then for all $f \in \mathcal{H}$ holds:*

$$\left\|S^{-1}\right\|^{-1} \|f\|^2 \leq \langle Sf, f\rangle \leq \|S\|\|f\|^2. \tag{5.14}$$

Proof. The upper inequality follows immediately from

$$\langle Sf, f\rangle \leq \|Sf\|\|f\| \leq \|S\|\|f\|^2. \tag{5.15}$$

For the lower inequality we use the same arguments as in Theorem 5.1.12, replacing S^{-1} by S: Theorem 5.1.11 yields a positive operator $T\colon \mathcal{H} \to \mathcal{H}$ such that $T^2 = S^{-1}$. Then self-adjointness of T and (5.15) applied to S^{-1} yield

$$\begin{aligned}\|f\|^2 &= \langle S^{-1}Sf, S^{-1}Sf\rangle = \langle T^2Sf, T^2Sf\rangle = \langle S^{-1}TSf, TSf\rangle \\ &\leq \left\|S^{-1}\right\| \|Tf\|^2 = \left\|S^{-1}\right\| \langle TSf, TSf\rangle = \left\|S^{-1}\right\| \langle Sf, f\rangle.\end{aligned}$$

Consequently,

$$\left\|S^{-1}\right\|^{-1} \|f\|^2 \leq \langle Sf, f\rangle.$$

□

Theorem 5.1.27. *Let I be a countable index set, $\mathcal{H}$ a Hilbert space and $(e_i)_{i\in I} \in \mathcal{H}^I$ a Bessel sequence. If the Gramian operator G corresponding to (e_i) is invertible on $\mathcal{H}$, then (e_i) is a Riesz sequence.*

Proof. As G is positive due to Lemma 5.1.17, we can apply Lemma 5.1.26 and together with (5.7) this yields that (e_i) is a Riesz sequence. □

5.2 Gabor systems

We now turn to a specific class of frames and Riesz sequences. Namely those which consist of a function and some time-frequency shifts of it. This provides the connection between frame theory and Chapter 4.

Definition 5.2.1. *Let $\Lambda \subset \mathbb{R}^{2d}$ be a lattice and $g \in L^2(\mathbb{R}^d)$. Then*

$$\mathcal{G}(g, \Lambda) \coloneqq \{\pi(\lambda)g\}_{\lambda\in\Lambda}$$

is called the Gabor system associated to g.

Remark 5.2.2. One should remark at this point that there is also a theory of Gabor systems over any countable set $\Lambda \subset \mathbb{R}^{2d}$ (not necessarily a lattice) (see e.g. [40, Chapter 5]). Such Gabor systems are usually called irregular Gabor systems. However, we will only consider Gabor systems over lattices.

Definition 5.2.3. *Let $\Lambda \subset \mathbb{R}^{2d}$ be a lattice and $g \in L^2(\mathbb{R}^d)$.*

- *If $\mathcal{G}(g,\Lambda)$ is a Bessel sequence, we denote the corresponding operators C, D, S and G (see Section 5.1) by C_g, D_g, S_g and G_g.*
- *$\mathcal{G}(g,\Lambda)$ is called a Gabor frame if it is a frame.*
- *$\mathcal{G}(g,\Lambda)$ is called a Gabor Riesz sequence if it is a Riesz sequence.*

Remark 5.2.4. In the context of above, the analysis operator C_g can be expressed using the STFT (see Definition 3.1.8):

$$(C_g f)_\lambda = \langle f, \pi(\lambda)g\rangle = V_g f(\lambda)$$

for all $\lambda \in \Lambda$ and $f \in L^2(\mathbb{R}^d)$.

The first thing we want to show is that Gabor systems fit nicely into the context of frames and Riesz sequences: If we have a Gabor frame then the canonical dual frame is also a Gabor frame and the same happens for Riesz sequences. However, before we prove this we need an auxiliary result concerning the Gramian operator. This actually yields the first direct connection to Chapter 4.

Lemma 5.2.5. *Let $\Lambda \subset \mathbb{R}^{2d}$ be a lattice and $g \in L^2(\mathbb{R}^d)$. We consider the sequence a defined as*

$$a_\lambda = \langle g, \pi(\lambda)g\rangle.$$

Then $G_g = \tilde{R}_a$, where $\tilde{R}_a$ is the operator from Definition 4.2.5, as bounded operators on $l^2(\Lambda)$ as soon as one of them is well-defined. In particular, if G_g is invertible, then a is invertible.

Proof. Let $c \in \ell^2(\Lambda)$. Then

$$\begin{aligned}
(G_g c)_\lambda &= \Big\langle \sum_{\mu\in\Lambda} c_\mu \pi(\mu)g, \pi(\lambda)g\Big\rangle = \sum_{\mu\in\Lambda} c_\mu \langle \pi(\mu)g, \pi(\lambda)g\rangle \\
&= \sum_{\mu\in\Lambda} c_\mu \langle g, e^{-2\pi i\mu_1\cdot\mu_2}\pi(-\mu)\pi(\lambda)g\rangle = \sum_{\mu\in\Lambda} c_\mu \langle g, e^{-2\pi i\mu_1\cdot\mu_2}e^{2\pi i\mu_1\cdot\lambda_2}\pi(\lambda-\mu)g\rangle \\
&= \sum_{\mu\in\Lambda} c_\mu e^{-2\pi i\mu_1\cdot(\lambda_2-\mu_2)}\langle g, \pi(\lambda-\mu)g\rangle = \sum_{\mu\in\Lambda} c_\mu a_{\lambda-\mu} e^{-2\pi i\mu_1\cdot(\lambda_2-\mu_2)} = (c \,\widetilde{\natural}\, a)_\lambda \\
&= \left(\tilde{R}_a c\right)_\lambda,
\end{aligned}$$

which yields the first claim. The last claim then follows immediately from Lemma 4.2.7. □

Theorem 5.2.6. *Let $\Lambda \subset \mathbb{R}^{2d}$ be a lattice and $g \in L^2(\mathbb{R}^d)$.*

(i) If $\mathcal{G}(g,\Lambda)$ is a Gabor frame with corresponding frame operator S_g then its canonical dual frame is given by $\mathcal{G}(S_g^{-1}g,\Lambda)$.

(ii) If $\mathcal{G}(g,\Lambda)$ is a Gabor Riesz sequence with corresponding Gramian operator G_g and synthesis operator D_g then its canonical dual Riesz sequence is given by $\mathcal{G}(D_g G_g^{-1}u_0,\Lambda)$.

Proof.

(i) By definition the canonical dual frame is given by

$$\big(S_g^{-1}\pi(\lambda)g\big)_{\lambda\in\Lambda},$$

so if we can show that S_g^{-1} commutes with time-frequency shifts from the lattice, we are done. To this end, let $\tilde{\lambda}\in\Lambda$ and $f\in L^2(\mathbb{R}^d)$. Then

$$\begin{aligned}
\pi\big(\tilde{\lambda}\big)^{-1}S_g\pi\big(\tilde{\lambda}\big)f &= \pi\big(\tilde{\lambda}\big)^{-1}\sum_{\lambda\in\Lambda}\big\langle\pi\big(\tilde{\lambda}\big)f,\pi(\lambda)g\big\rangle\pi(\lambda)g\\
&= \sum_{\lambda\in\Lambda}\big\langle\pi\big(\tilde{\lambda}\big)f,\pi(\lambda)g\big\rangle\pi\big(\tilde{\lambda}\big)^{-1}\pi(\lambda)g\\
&\overset{Lemma\ 3.1.5}{=} \sum_{\lambda\in\Lambda}\big\langle f,\pi\big(\tilde{\lambda}\big)^{-1}\pi(\lambda)g\big\rangle\pi\big(\tilde{\lambda}\big)^{-1}\pi(\lambda)g\\
&\overset{Lemma\ 3.1.7}{=} \sum_{\lambda\in\Lambda}\big\langle f,e^{-2\pi i\tilde{\lambda}_1\cdot\tilde{\lambda}_2}\pi\big(-\tilde{\lambda}\big)\pi(\lambda)g\big\rangle e^{-2\pi i\tilde{\lambda}_1\cdot\tilde{\lambda}_2}\pi\big(-\tilde{\lambda}\big)\pi(\lambda)g\\
&= \sum_{\lambda\in\Lambda}\big\langle f,\pi\big(-\tilde{\lambda}\big)\pi(\lambda)g\big\rangle\pi\big(-\tilde{\lambda}\big)\pi(\lambda)g\\
&\overset{Lemma\ 3.1.7}{=} \sum_{\lambda\in\Lambda}\big\langle f,e^{2\pi i\tilde{\lambda}_1\cdot\lambda_2}\pi(\lambda-\tilde{\lambda})g\big\rangle e^{2\pi i\tilde{\lambda}_1\cdot\lambda_2}\pi(\lambda-\tilde{\lambda})g\\
&= \sum_{\lambda\in\Lambda}\big\langle f,\pi(\lambda-\tilde{\lambda})g\big\rangle\pi(\lambda-\tilde{\lambda})g\\
&\overset{\mu=\lambda-\tilde{\lambda}}{=} \sum_{\mu\in\Lambda}\langle f,\pi(\mu)g\rangle\,\pi(\mu)g = S_gf,
\end{aligned}$$

which shows that S_g commutes with time-frequency shifts from the lattice and so does S_g^{-1}.

(ii) By Definition 5.1.22 the canonical dual Riesz sequence is given by

$$\big(D_gG_g^{-1}u_\lambda\big)_\lambda = \Big(\sum_{\mu\in\Lambda}\big(G_g^{-1}u_\lambda\big)_\mu\,\pi(\mu)g\Big)_{\lambda\in\Lambda}.$$

We show that this sequence coincides with $\mathcal{G}(D_gG_g^{-1}u_0,\Lambda)$: For $\lambda\in\Lambda$ we have

$$\begin{aligned}
\pi(\lambda)D_gG_g^{-1}u_0 &= \pi(\lambda)\sum_{\mu\in\Lambda}\big(G_g^{-1}u_0\big)_\mu\,\pi(\mu)g = \sum_{\mu\in\Lambda}\big(G_g^{-1}u_0\big)_\mu\,\pi(\lambda)\pi(\mu)g\\
&= \sum_{\mu\in\Lambda}\big(G_g^{-1}u_0\big)_\mu\,e^{-2\pi i\lambda_1\cdot\mu_2}\pi(\lambda+\mu)g\\
&\overset{\tilde{\mu}=\lambda+\mu}{=} \sum_{\tilde{\mu}\in\Lambda}\big(G_g^{-1}u_0\big)_{\tilde{\mu}-\lambda}\,e^{-2\pi i\lambda_1\cdot(\tilde{\mu}-\lambda)_2}\pi(\tilde{\mu})g.
\end{aligned}$$

So if we can show that

$$(G_g^{-1}u_\lambda)_\mu = (G_g^{-1}u_0)_{\mu-\lambda}e^{-2\pi i\lambda_1\cdot(\mu-\lambda)_2}$$

for all $\mu \in \Lambda$, then

$$\pi(\lambda) D_g G_g^{-1} u_0 = \sum_{\mu \in \Lambda} (G_g^{-1} u_\lambda)_\mu \pi(\mu) g = D_g G_g^{-1} u_\lambda.$$

To this end, we use Lemma 5.2.5: As G_g is well-defined and invertible, a^{-1} exists and $G_g^{-1} = \tilde{R}_{a^{-1}}$, where a is the sequence from Lemma 5.2.5. Consequently,

$$\begin{aligned}(G_g^{-1} u_\lambda)_\mu &= (\tilde{R}_{a^{-1}} u_\lambda)_\mu = (u_\lambda \mathbin{\widetilde{\natural}} a^{-1})_\mu = a^{-1}_{\mu-\lambda} e^{-2\pi i \lambda_1 \cdot (\mu-\lambda)_2} \\ &= (\tilde{R}_{a^{-1}} u_0)_{\mu-\lambda} e^{-2\pi i \lambda_1 \cdot (\mu-\lambda)_2} = (G_g^{-1} u_0)_{\mu-\lambda} e^{-2\pi i \lambda_1 \cdot (\mu-\lambda)_2}. \qquad \square\end{aligned}$$

Definition 5.2.7. *Let $\Lambda \subset \mathbb{R}^{2d}$ be a lattice and $g \in L^2(\mathbb{R}^d)$ such that $\mathcal{G}(g, \Lambda)$ is a Gabor frame. Then $\tilde{g} := S_g^{-1} g$ is called the dual window corresponding to g. Analogously, if $\mathcal{G}(g, \Lambda)$ is a Gabor Riesz sequence, then $\tilde{g} := D_g G_g^{-1} u_0$ is called the dual window corresponding to g.*

In the context of Gabor systems, there is a duality between frames and Riesz sequences.

Definition 5.2.8. *Let $\Lambda \subset \mathbb{R}^{2d}$ be a lattice, i.e. $\Lambda = A\mathbb{Z}^{2d}$ for some $A \in \mathrm{GL}(2d, \mathbb{R})$.*

- *The volume of Λ is*
$$\mathrm{vol}(\Lambda) := |\det(A)|.$$
- *The density of Λ is*
$$\delta(\Lambda) := \mathrm{vol}(\Lambda)^{-1}.$$
- *The adjoint lattice Λ° of Λ is*
$$\Lambda^\circ := J A^{-T} \mathbb{Z}^{2d},$$
where J is the standard symplectic matrix (see Definition 3.2.3).

Theorem 5.2.9. *Let $\Lambda \subset \mathbb{R}^{2d}$ be a lattice and $g \in L^2(\mathbb{R}^d)$. Then $\mathcal{G}(g, \Lambda)$ is a frame with frame bounds A, B if and only if $\mathcal{G}(g, \Lambda^\circ)$ is a Riesz sequence with corresponding bounds* $\mathrm{vol}(\Lambda)A, \mathrm{vol}(\Lambda)B$.

There were several people involved in the development of the duality theory (e.g. Wexler and Raz [41] or Ron and Shen [42]). The version for general lattices as formulated above was first proven by Feichtinger and Kozek in [43].

5.3 Applications to Gabor analysis

In this section we apply the counterexamples concerning norm-controlled inversion we constructed in Chapter 4 to the theory of Gabor frames and Gabor Riesz sequences. The windows we consider here are taken from the space $M^1(\mathbb{R}^d)$, which is not a problem as

$$M^1(\mathbb{R}^d) \subset M^2(\mathbb{R}^d) = L^2(\mathbb{R}^d)$$

by Theorem 3.3.9 and so we can apply the theory from Chapter 5. Throughout this section we assume that the involved modulation spaces are defined using a fixed window $\phi \in \mathcal{S}(\mathbb{R}^d)$ and mention any dependence explicitly. We start by collecting some theorems which show that $M^1(\mathbb{R}^d)$ is a reasonable space for choosing windows in Gabor analysis. The first one shows that all the operators from Gabor analysis are automatically bounded if the window is in $M^1(\mathbb{R}^d)$.

Theorem 5.3.1 (Lemma 2.4, [31])**.** *Let $\Lambda \subset \mathbb{R}^{2d}$ be a lattice and $g \in M^1(\mathbb{R}^d)$. There exists $K_{\Lambda,\phi} > 0$ (independent of g) such that the analysis operator C_g and the synthesis operator D_g corresponding to $\mathcal{G}(g, \Lambda)$ satisfy*

$$\|C_g f\|_1 \leq K_{\Lambda,\phi} \|g\|_{M^1} \|f\|_{M^1}$$

and

$$\|D_g a\|_{M^1} \leq K_{\Lambda,\phi} \|g\|_{M^1} \|a\|_1$$

for all $f \in M^1(\mathbb{R}^d)$ and $a \in \ell^1(\Lambda)$. In particular, $C_g \colon M^1(\mathbb{R}^d) \to \ell^1(\Lambda)$ and $D_g \colon \ell^1(\Lambda) \to M^1(\mathbb{R}^d)$ are bounded.

The next theorem is part of a result by Gröchenig which gives several criteria for the invertibility of the Gabor frame operator for M^1-windows.

Theorem 5.3.2 (Lemma 3.1, [31])**.** *Let $\Lambda \subset \mathbb{R}^{2d}$ be a lattice and $g \in M^1(\mathbb{R}^d)$. Then the following are equivalent:*

- *$\mathcal{G}(g, \Lambda)$ is a frame*
- *S_g is invertible on $L^2(\mathbb{R}^d)$*
- *S_g is invertible on $M^1(\mathbb{R}^d)$*
- *$\mathcal{G}(g, \Lambda^\circ)$ is a Riesz sequence*
- *G_g is invertible on $\ell^2(\Lambda^\circ)$*
- *G_g is invertible on $\ell^1(\Lambda^\circ)$*

Moreover, we need an existence result for orthonormal Gabor systems. To this end, we refer to a recent existence result for frames.

Definition 5.3.3. *Let $\Lambda \subset \mathbb{R}^{2d}$ be a lattice, i.e. $\Lambda = A\mathbb{Z}^{2d}$ for some $A \in \mathrm{GL}(2d, \mathbb{R})$. Λ is called rational if*

$$\sigma(Au_i, Au_j) \in \mathbb{Q}$$

for all $i, j \in \{1, \ldots 2d\}$ with $i \neq j$ where σ is the symplectic form from Definition 3.2.1. Otherwise, Λ is called non-rational.

Theorem 5.3.4 (Theorem 5.4, [44])**.** *Let $\Lambda \subset \mathbb{R}^{2d}$ be a non-rational lattice with $\mathrm{vol}(\Lambda) < 1$. Then there exists $f \in M^1(\mathbb{R}^d)$ such that $\mathcal{G}(f, \Lambda)$ is a tight Gabor frame.*

Using this result, we can easily deduce an existence result for orthonormal Gabor systems.

Corollary 5.3.5. *Let $\Lambda \subset \mathbb{R}^{2d}$ be a non-rational lattice with* $\mathrm{vol}(\Lambda) > 1$. *Then there exists $f \in M^1(\mathbb{R}^d)$ such that $\mathcal{G}(f, \Lambda)$ is an orthonormal system.*

Proof. As

$$\mathrm{vol}(\Lambda^\circ) = \left|\det\left(JA^{-T}\right)\right| = \mathrm{vol}(\Lambda)^{-1}$$

we get $\mathrm{vol}(\Lambda^\circ) < 1$ and thus by Theorem 5.3.4 we can find $g \in M^1(\mathbb{R}^d)$ such that $\mathcal{G}(g, \Lambda^\circ)$ is a tight Gabor frame with some frame bound $A > 0$. Consequently, $\mathcal{G}(\mathrm{vol}(\Lambda)^{1/2}A^{-1/2}g, \Lambda^\circ)$ is a tight frame with frame bound $\mathrm{vol}(\Lambda)$. But then the duality theorem Theorem 5.2.9 yields that $\mathcal{G}(\mathrm{vol}(\Lambda)^{1/2}A^{-1/2}g, \Lambda)$ is a tight Riesz sequence with bound 1, which is an orthonormal system by Remark 5.1.24. □

Now we are ready for the final results. We show that in general it is not possible to control the M^1-norm of a dual window $\tilde{g}$ with the M^1-norm of the window g itself and the Riesz sequence or frame bounds. We start with the corresponding qualitative statement, which is indeed true.

Theorem 5.3.6. *Let $\Lambda \subset \mathbb{R}^{2d}$ be a lattice and $g \in M^1(\mathbb{R}^d)$ such that $\mathcal{G}(g, \Lambda)$ is a Gabor frame or a Gabor Riesz sequence. Then, in either case, the corresponding dual window $\tilde{g} \in M^1(\mathbb{R}^d)$.*

Proof. First, assume that $\mathcal{G}(g, \Lambda)$ is a frame. Then, due to Theorem 5.3.2, the corresponding frame operator S_g is invertible on $M^1(\mathbb{R}^d)$. Consequently, $\tilde{g} = S_g^{-1}g \in M^1(\mathbb{R}^d)$. Second, if $\mathcal{G}(g, \Lambda)$ is a Riesz sequence, then, due to Theorem 5.3.2, the corresponding Gramian operator G_g is invertible on $\ell^1(\Lambda)$. Moreover, Theorem 5.3.1 implies that the synthesis operator D_g maps from $\ell^1(\Lambda)$ to $M^1(\mathbb{R}^d)$. Thus, $\tilde{g} = D_g G_g^{-1} u_0 \in \mathrm{M}^1(\mathbb{R}^d)$. □

Remark 5.3.7. Theorem 5.3.6 was a famous conjecture which originally has been proven by Gröchenig and Leinert in [32, Theorem 4.2].

However, as already mentioned above, the quantitative statement is wrong. The following two results are original contributions.

Definition 5.3.8. *Let $\Lambda \subset \mathbb{R}^{2d}$ be a lattice and $A, B > 0$. $\mathcal{R}(A, B, \Lambda)$ denotes the set of all $g \in M^1(\mathbb{R}^d)$ such that $\mathcal{G}(g, \Lambda)$ is a Riesz sequence with (not necessarily optimal) bounds A and B.*

In the following two theorems we use the notation $\Lambda_N := A[-N, N]^{2d}$ whenever $\Lambda \subset \mathbb{R}^{2d}$ is a lattice with generating matrix $A \in \mathrm{GL}(2d, \mathbb{R})$, i.e. $\Lambda = A\mathbb{Z}^{2d}$, and $N \in \mathbb{N}$.

Theorem 5.3.9. *Let $\Lambda \subset \mathbb{R}^{2d}$ be a non-rational lattice with* $\mathrm{vol}(\Lambda) > 1$ *and generating matrix $A \in \mathrm{GL}(2d, \mathbb{R})$ and $f \in M^1(\mathbb{R}^d)$ be such that $\mathcal{G}(f, \Lambda)$ is an orthonormal system (see Corollary 5.3.5). Then for $\|V_\phi f\|_{L^1} < M \leq 2\|V_\phi f\|_{L^1}$ and $N \geq 3^{2e^{e^5}}$:*

$$\sup \Big\{ \|V_\phi \tilde{g}\|_{L^1} : \sum_{\lambda\in\Lambda} a_\lambda \pi(\lambda) f = g \in \mathcal{R}\Big(\frac{1}{M^2}, \frac{1}{\|V_\phi f\|_{L^1}^2}, \Lambda\Big), \\ \|V_\phi g\|_{L^1} \leq 1 \text{ and } \mathrm{supp}(a) \subseteq \Lambda_N \Big\}$$
$$\geq \frac{9}{10K_{\Lambda,\phi}} \left(\frac{1 + \frac{M}{\|V_\phi f\|_{L^1}} \frac{\log(N)^{\frac{1}{2}}}{\log(\log(N))^6}}{1 + \left(2 - \frac{M}{\|V_\phi f\|_{L^1}}\right) \frac{\log(N)^{\frac{1}{2}}}{\log(\log(N))^6}} \right),$$

where $K_{\Lambda,\phi} > 0$ is the constant we get from Theorem 5.3.1.

Proof. The central idea is to use Corollary 4.3.15. To this end, we need to identify an invertible $a \in \ell^1(\Lambda, \natural)$ satisfying $\|a\|_1 < 1$ with a window $h \in M^1(\mathbb{R}^d)$ constituting a Gabor Riesz sequence. We do this by defining

$$h := \pi(a)f = \sum_{\lambda\in\Lambda} a_\lambda \pi(\lambda) f.$$

In order to show that $\mathcal{G}(h, \Lambda)$ is a Riesz sequence we use Theorem 5.1.27 and Lemma 5.2.5 according to which it suffices to show that the operator $\tilde{R}_b$, where b is given by $b_\lambda = \langle h, \pi(\lambda) h\rangle$, is bounded and invertible on $\ell^2(\Lambda)$.

$$b_\lambda = \langle h, \pi(\lambda)h\rangle = \Big\langle \sum_\mu a_\mu \pi(\mu) f, \sum_\nu a_\nu \pi(\lambda)\pi(\nu) f\Big\rangle = \sum_\mu \sum_\nu a_\mu \overline{a_\nu} \langle \pi(\mu) f, \pi(\lambda)\pi(\nu) f\rangle$$
$$= \sum_\mu \sum_\nu a_\mu \overline{a_\nu} \big\langle \pi(\mu) f, e^{-2\pi i \lambda_1\cdot\nu_2} \pi(\lambda+\nu) f\big\rangle = \sum_\mu \sum_\nu a_\mu \overline{a_\nu} e^{2\pi i \lambda_1\cdot\nu_2} \langle \pi(\mu) f, \pi(\lambda+\nu) f\rangle .$$

Using orthonormality we get

$$\sum_\mu \sum_\nu a_\mu \overline{a_\nu} e^{2\pi i \lambda_1\cdot\nu_2} \langle \pi(\mu) f, \pi(\lambda+\nu) f\rangle = \sum_\mu a_\mu \overline{a_{\mu-\lambda}} e^{2\pi i \lambda_1\cdot(\mu-\lambda)_2}$$

and then the definition of the involution $\widetilde{*}$ (see Definition 4.1.7) yields

$$\sum_\mu a_\mu \overline{a_{\mu-\lambda}} e^{2\pi i \lambda_1\cdot(\mu-\lambda)_2} = \sum_\mu a_\mu \left(a^{\widetilde{*}}\right)_{\lambda-\mu} e^{2\pi i (\lambda-\mu)_1\cdot(\lambda-\mu)_2} e^{2\pi i \lambda_1\cdot(\mu-\lambda)_2}$$
$$= \sum_\mu a_\mu \left(a^{\widetilde{*}}\right)_{\lambda-\mu} e^{-2\pi i \mu_1\cdot(\lambda-\mu)_2} = (a \,\widetilde{\natural}\, a^{\widetilde{*}})_\lambda.$$

So $b = a \,\widetilde{\natural}\, a^{\widetilde{*}}$ and as $a \in \ell^1(\Lambda)$ is invertible it follows that $b \in \ell^1(\Lambda)$ and b is invertible as a product of invertible elements. Consequently, $\tilde{R}_b$ is bounded and invertible on $\ell^2(\Lambda)$ by Lemma 4.2.6 and thus $\mathcal{G}(h, \Lambda)$ is a Riesz sequence with corresponding Gramian

$$G_h = \tilde{R}_b. \tag{5.16}$$

Moreover, we know from Theorem 5.2.6 that the dual window $\tilde{h}$ is given by

$$\begin{aligned}
\tilde{h} &= D_h G_h^{-1} u_0 = \pi(G_h^{-1} u_0) h = \pi\left(\tilde{R}_{b^{-1}} u_0\right) h = \pi(b^{-1}) h \\
&= \pi\big((a^{\tilde{*}})^{-1} \natural\, a^{-1}\big)\pi(a) f \overset{Lemma\ 4.2.12}{=} \pi\left((a^{\tilde{*}})^{-1}\right)\pi(a)^{-1}\pi(a) f \\
&= \pi\left((a^{-1})^{\tilde{*}}\right) f = \sum_{\lambda\in\Lambda} \left(a^{-1}\right)^{\tilde{*}}_{\lambda} \pi(\lambda) f.
\end{aligned}$$

Using the bound $\|a\| < 1$, we get

$$\|h\|_{M^1} = \left\| \sum_{\lambda\in\Lambda} a_\lambda \pi(\lambda) f \right\|_{M^1} \leq \sum_{\lambda\in\Lambda} |a_\lambda|\, \|\pi(\lambda) f\|_{M^1} \overset{Theorem\ 3.3.9}{=} \|a\|_1 \|f\|_{M^1} \leq \|f\|_{M^1}.$$

Furthermore, orthonormality immediately implies that

$$(C_f \tilde{h})_\lambda = \langle \tilde{h}, \pi(\lambda) f\rangle = \sum_{\mu\in\Lambda} \left(a^{-1}\right)^{\tilde{*}}_{\mu} \langle \pi(\mu) f, \pi(\lambda) f\rangle = \left(a^{-1}\right)^{\tilde{*}}_{\lambda}$$

and hence Theorem 5.3.1 yields

$$\|a^{-1}\|_1 = \left\| \left(a^{-1}\right)^{\tilde{*}} \right\|_1 = \left\| C_f \tilde{h} \right\|_1 \leq K_{\Lambda,\phi} \|f\|_{M^1} \|\tilde{h}\|_{M^1}.$$

Thus,

$$\|\tilde{h}\|_{M^1} \geq \frac{\|a^{-1}\|_1}{K_{\Lambda,\phi}\|f\|_{M^1}}.$$

Finally, the Riesz sequence bounds are given by

$$\begin{aligned}
B_h &\overset{Theorem\ 5.1.20}{=} \|G_h\|_{\mathcal{B}(\ell^2)} \overset{(5.16)}{=} \left\|\tilde{R}_{a \natural a^{\tilde{*}}}\right\|_{\mathcal{B}(\ell^2)} \overset{Lemma\ 4.2.6}{=} \left\|\left(\tilde{R}_a\right)^* \tilde{R}_a\right\|_{\mathcal{B}(\ell^2)} \\
&= \left\|\tilde{R}_a\right\|^2_{\mathcal{B}(\ell^2)} \leq \|a\|_1^2 \leq 1
\end{aligned}$$

and analogously

$$A_h \overset{Theorem\ 5.1.20}{=} \|G_h^{-1}\|^{-1}_{\mathcal{B}(\ell^2)} = \left\|\left(\tilde{R}_a\right)^{-1}\right\|^{-2}_{\mathcal{B}(\ell^2)}.$$

Thus, a normalized version

$$g := \frac{1}{\|f\|_{M^1}} h$$

satisfies $\|g\|_{M^1} \leq 1$,

$$\|\tilde{g}\|_{M^1} \geq \frac{\|a^{-1}\|_1}{K_{\Lambda,\phi}}$$

and the corresponding Riesz-sequence bounds are given by

$$B_g \leq \frac{1}{\|f\|^2_{M^1}}$$

and

$$A_g = \big(\|(\tilde{R}_a)^{-1}\|_{\mathcal{B}(\ell^2)}\|f\|_{M^1}\big)^{-2}.$$

Consequently, an application of Corollary 4.3.15 immediately yields

$$\begin{aligned}\sup\Big\{\|V_\phi \tilde{g}\|_{L^1} : &\sum_{\lambda\in\Lambda} a_\lambda \pi(\lambda) f = g \in \mathcal{R}\Big(\frac{1}{\|V_\phi f\|_{L^1}^2 \tilde{M}^2}, \frac{1}{\|V_\phi f\|_{L^1}^2}, \Lambda\Big),\\ &\|V_\phi g\|_{L^1} \leq 1 \text{ and } \operatorname{supp}(a) \subseteq \Lambda_N\Big\}\\ \geq \frac{9}{10K_{\Lambda,\phi}}&\left(\frac{1+\tilde{M}\frac{\log(N)^{\frac12}}{\log(\log(N))^6}}{1+(2-\tilde{M})\frac{\log(N)^{\frac12}}{\log(\log(N))^6}}\right)\end{aligned}$$

for $1 < \tilde{M} \leq 2$ and $N \geq 3^{2e^{e^5}}$. The claimed result then follows from renormalizing $\tilde{M}$ via $M := \|V_\phi f\|_{L^1}\tilde{M}$. □

Using the duality theorem (Theorem 5.2.9) we immediately obtain the corresponding statement for frames.

Definition 5.3.10. *Let $\Lambda \subset \mathbb{R}^{2d}$ be a lattice and $A, B > 0$. $\mathcal{F}(A, B, \Lambda)$ denotes the set of all $g \in M^1(\mathbb{R}^d)$ such that $\mathcal{G}(g, \Lambda)$ is a frame with (not necessarily optimal) bounds A and B.*

Theorem 5.3.11. *Let $\Lambda \subset \mathbb{R}^{2d}$ be a non-rational lattice with $\operatorname{vol}(\Lambda) < 1$ and generating matrix $A \in \mathrm{GL}(2d, \mathbb{R})$ and $f \in M^1(\mathbb{R}^d)$ be such that $\mathcal{G}(f, \Lambda)$ is an orthonormal system (see Corollary 5.3.5). Then for $\sqrt{\operatorname{vol}(\Lambda)}\|V_\phi f\|_{L^1} < M \leq 2\sqrt{\operatorname{vol}(\Lambda)}\|V_\phi f\|_{L^1}$ and $N \geq 3^{2e^{e^5}}$:*

$$\begin{aligned}\sup\Big\{\|V_\phi \tilde{g}\|_{L^1} : &\sum_{\lambda\in\Lambda} a_\lambda \pi(\lambda) f = g \in \mathcal{F}\Big(\frac{1}{M^2}, \frac{1}{\operatorname{vol}(\Lambda)\|V_\phi f\|_{L^1}^2}, \Lambda\Big),\\ &\|V_\phi g\|_{L^1} \leq 1 \textit{ and } \operatorname{supp}(a) \subseteq \Lambda_N\Big\}\\ \geq \frac{9}{10K_{\Lambda,\phi}}&\left(\frac{1+\frac{M}{\sqrt{\operatorname{vol}(\Lambda)}\|V_\phi f\|_{L^1}}\frac{\log(N)^{\frac12}}{\log(\log(N))^6}}{1+\big(2-\frac{M}{\sqrt{\operatorname{vol}(\Lambda)}\|V_\phi f\|_{L^1}}\big)\frac{\log(N)^{\frac12}}{\log(\log(N))^6}}\right)\end{aligned}$$

where $K_{\Lambda,\phi} > 0$ is the constant we get from Theorem 5.3.1.

Proof. As $\operatorname{vol}(\Lambda) < 1$ we have $\operatorname{vol}(\Lambda^\circ) > 1$ and thus we can apply Theorem 5.3.9. Moreover, Theorem 5.2.9 yields that

$$g \in \mathcal{R}\Big(\frac{1}{\tilde{M}^2}, \frac{1}{\|V_\phi f\|_{L^1}^2}, \Lambda^\circ\Big) \iff g \in \mathcal{F}\Big(\frac{1}{\operatorname{vol}(\Lambda)\tilde{M}^2}, \frac{1}{\operatorname{vol}(\Lambda)\|V_\phi f\|_{L^1}^2}, \Lambda\Big).$$

These two statements immediately yield

$$\begin{aligned}
&\sup\Big\{\|V_\phi\tilde{g}\|_{L^1} : \sum_{\lambda\in\Lambda} a_\lambda \pi(\lambda) f = g \in \mathcal{F}\left(\frac{1}{\mathrm{vol}(\Lambda)\tilde{M}^2}, \frac{1}{\mathrm{vol}(\Lambda)\|V_\phi f\|_{L^1}^2}, \Lambda\right),\\
&\qquad\qquad \|V_\phi g\|_{L^1} \le 1 \text{ and } \mathrm{supp}(a) \subseteq \Lambda_N\Big\}\\
&\ge \frac{9}{10K_{\Lambda,\phi}}\left(\frac{1+\frac{\tilde{M}}{\|V_\phi f\|_{L^1}}\frac{\log(N)^{\frac{1}{2}}}{\log(\log(N))^6}}{1+\left(2-\frac{\tilde{M}}{\|V_\phi f\|_{L^1}}\right)\frac{\log(N)^{\frac{1}{2}}}{\log(\log(N))^6}}\right)
\end{aligned}$$

for $\|V_\phi f\|_{L^1} < \tilde{M} \le 2\|V_\phi f\|_{L^1}$ and $N \ge 3^{2e^{e^5}}$. The claim then follows by renormalizing $\tilde{M}$ via $M := \sqrt{\mathrm{vol}(\Lambda)}\tilde{M}$. □

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Zusammenfassung

Ziel dieser Arbeit ist es, das Problem von Inversion mit Normkontrolle in $\ell^1(\mathbb{Z}, *)$ zu beleuchten. In den späten Neunzigerjahren zeigten El-Fallah, Nikolski und Zarrabi, dass die Norm einer Folge $a \in \ell^1(\mathbb{Z}, *)$ und eine untere Abschätzung der dazugehörigen Fourierreihe nicht genug sind, um die Norm von a^{-1} zu kontrollieren. Selbiges funktioniert jedoch, wenn man die Norm mit einem genügend schnell wachsenden Gewicht versieht. Im ungewichteten Fall bewiesen El-Fallah, Nikolski und Zarrabi eine untere Schranke für das Wachstum der Inversen, welche beliebig groß wird, wenn die Anforderung an die zugehörige Fourierreihe zu schwach ist.

Wir verallgemeinern diese untere Schranke, indem wir auch den Träger der involvierten Folge berücksichtigen. Eine Anwendung dieser neuen Abschätzung liefert eine untere Schranke für das Wachstum der Inversen in einem gewichteten Fall, die nicht aus den Argumenten von El-Fallah, Nikolski and Zarrabi folgt.

Weiters transferieren wir die Resultate zur verdrehter Faltung, einer Struktur, die für das Gebiet der Zeit-Frequenz-Analyse von großer Bedeutung ist. Damit können wir unsere Resultate auch auf die Theorie der Gabor-Analyse anwenden. Wir zeigen dort, dass es im Allgemeinen nicht möglich ist, die M^1-Norm eines dualen Fensters mit der M^1-Norm des ursprünglichen Fensters und den Schranken des aufgespannten Frames beziehungsweise der aufgespannten Riesz-Folge zu kontrollieren.