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"On the open sea propagation of water waves in a flow with vorticity generated by a moving bed"

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Abriss

Wir betrachten Tsunamis ausgelöst durch ein Seebeben am offenen Meer. Wir nehmen an, dass die Strömung anfangs konstante Vortizität hat.
Im ersten Abschnitt dieser Arbeit werden die Gleichungen, die die Strömung beschreiben, eingeführt und dann auf dimensionslose Variablen skaliert. Diese dimensionslosen Gleichungen werden linearisiert und weiter umgeformt, bis man ein System von Gleichungen erhält, das nur mehr von der vertikalen Bewegung des Seebodens abhängt.
Im zweiten Abschnitt werden Methoden aus der Fourieranalysis benutzt, um einige Formeln für f, welches die vertikale Auslenkung an der Meeresoberfläche beschreibt, herzuleiten. Es stellt sich heraus, dass die Formeln bei nicht-verschwindender Vortizität komplizierter werden und man andere Methoden benutzen müsste.
Im letzten Abschnitt wird besprochen, wie man das Modell (potentiell) anwenden kann: Im Fall von Gewässern, wo die durchschnittliche Meerestiefe viel kleiner als die Wellenlänge der Tsunamiwelle ist, liefert das Modell mit verschwindender Vortizität gute Vorhersagen für das Verhalten einiger historischen Tsunamis. In anderen Situationen könnte das sogenannte *stationary-phase principle* Einsicht in das asymptotische Verhalten von f geben, allerdings nicht bei verschwindender Vortizität.

Abstract

We consider tsunamis triggered by a seaquake on the open sea. We assume that the flow is initially of constant vorticity.
In the first section of this thesis, we introduce the equations governing the flow, which are then scaled to dimensionless parameters. We then linearize these dimensionless equations and rewrite them, until we obtain a system of equations only depending on the vertical movement of the sea bed.
In the second section, we use Fourier methods to extract formulae for f, which describes the vertical displacement of the water's free surface. It turns out that the formulae get more complicated for non-vanishing vorticity and one would need to use different methods.
In the last section, we discuss how the model can be (potentially) applied: In the case of waters, where the average depth of the sea is much smaller than the wavelength of the tsunami wave, the model with vanishing vorticity gives good predictions for the behaviour of some historical tsunamis. In different situations, the so called *stationary-phase principle* might give insight into the asymptotic behaviour of f, however not for vanishing vorticity.

Contents

1 The governing equations

1.1 The physical equations

We consider an analogous setup as Constantin and Germain [4]:
We model the sea two-dimensional with unbounded X-direction. The sea is bounded above by the water's free surface $Y = d + F(X,T)$, where d is the average depth of the sea and bounded below by the seabed $Y = H(X,T)$. At time $T = 0$, we assume that $F(X,0) = 0$ and $H(X,0) = 0$. Then the seabed moves for a short time in some compact region $X \in [-L, L]$ and is flat after that again.
We are ultimately interested in the deformation of the free surface, i.e. in $F(X,T)$. To do this, we study a PDE. Consider (U,V) the velocity field of the flow in rectangular Cartesian coordinates (X,Y). We assume that the water is inviscid and that the resulting flow is with constant vorticity. Note that the presence of non-uniform currents makes the hypothesis of irrotational flow inappropriate and, since in a setting in which the waves are long compared with the water depth, the existence of a non-zero mean vorticity is important rather than its specific distribution (see Da Silva and Peregrine [5]), we consider the case of flows of constant vorticity. This is not merely a mathematical simplification: Wind-generated currents in shallow waters with nearly flat beds have been shown to be accurately described as flows with constant vorticity [7]. We have the incompressible Euler equations

$$\left.\begin{aligned} U_X + V_Y &= 0, \\ \rho(U_T + UU_X + VU_Y) &= -P_X, \\ \rho(V_T + UV_X + VV_Y) &= -P_Y - \rho g, \end{aligned}\right\} \tag{1.1}$$

in $H(X,T) < Y < d + F(X,T)$, where ρ is the constant density, g is the constant acceleration of gravity and P is the pressure. The effects of surface tension water are negligible for wavelengths greater than a few centimetres [1, 9], hence the major factor governing the wave motion is the balance between gravity and the inertia of the system. We get the kinematic boundary conditions

$$V = F_T + UF_X \quad \text{on the free surface } Y = d + F(X,T) \tag{1.2}$$

and

$$V = H_T + UH_X \quad \text{on the rigid bed } Y = H(X,T), \tag{1.3}$$

which ensure that particles on these boundaries are confined to them at all times. We also have the dynamic boundary condition

$$P - P_{\mathrm{atm}} = 0 \quad \text{on the free surface } Y = d + F(X,T), \tag{1.4}$$

where P_{atm} is the constant atmospheric pressure. This decouples the motion of the water from the motion of the air above it [8]. The system (1.1)-(1.4) is to be solved with the following initial conditions

$$F(X,0) = 0, \quad U(X,Y,0) = AY + B, \quad V(X,Y,0) = 0. \tag{1.5}$$

These initial conditions express the fact that we assume that at $T = 0$ there is only horizontal flow depending linearly on the depth. This is a generalization of [4], where it was assumed that the water is completely at rest at $T = 0$, which corresponds to $A = B = 0$. An incompressible inviscid flow that is initially with constant vorticity, remains so at later times, which implies that throughout the wave propagation the vorticity of the flow remains constant (see the discussion in Chapter 1 of Constantin [2]). By the initial condition (1.5) this implies

$$U_Y - V_X = A \tag{1.6}$$

for $H(X,T) < Y < d + F(X,T)$. 1.1 illustrates our setup in the case where $A, B > 0$.

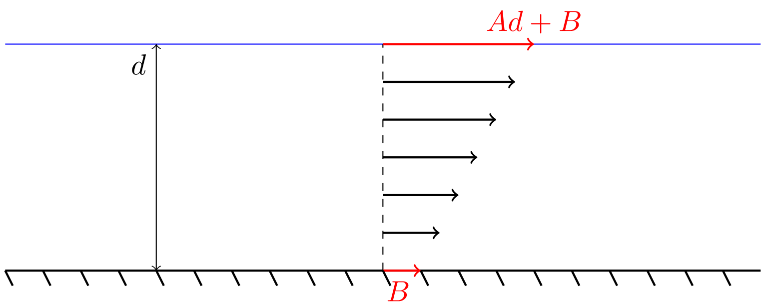

Figure 1: Pure current flow (in the absence of waves) at the initial time $T = 0$

1.2 Non-dimensionalization and linearization of the physical equations

The first step in the analysis of the system (1.1)-(1.6) will be to non-dimensionalize and scale the equations. First, it will be convenient to introduce the non-dimensional excess pressure p, relative to the hydrostatic pressure distribution by

$$P = P_{\text{atm}} + \rho g(d - Y) + \rho g d p. \tag{1.7}$$

To obtain meaningful scales, we introduce the average or typical wavelength λ of the wave, we use $\sqrt{gd}$ as a scale for the wave speed and $\lambda/\sqrt{gd}$ as a time scale. We use the change of variables

$$X = \lambda x, \quad Y = dy, \quad T = \frac{\lambda}{\sqrt{gd}} t, \quad U = \sqrt{gd} u, \quad V = \frac{d\sqrt{gd}}{\lambda} v \tag{1.8}$$

with

$$F = af, \quad H = ah, \tag{1.9}$$

where a is a typical, perhaps maximal, amplitude of the wave. Note that (1.9) should be interpreted as the fact that the variations of the wave and of the seabed are of comparable size.

We introduce the parameters

$$\varepsilon = \frac{a}{d}, \quad \delta = \frac{d}{\lambda}. \tag{1.10}$$

ε measures the relative size of the amplitude to the average water depth and δ measures the average water depth to the wavelength.

Now we get the non-dimensional equations

$$\left.\begin{array}{l} \begin{cases} u_x + v_y = 0 \\ u_t + u u_x + v u_y = -p_x \\ \delta^2 (v_t + u v_x + v v_y) = -p_y \\ u_y - \delta^2 v_x = \sqrt{\frac{d}{g}} A, \end{cases} \quad \text{in } \varepsilon h(x,t) < y < 1 + \varepsilon f(x,t) \\ p = \varepsilon f \qquad \text{on } y = 1 + \varepsilon f(x,t) \\ v = \varepsilon (f_t + u f_x) \quad \text{on } y = 1 + \varepsilon f(x,t) \\ v = \varepsilon (h_t + u h_x) \quad \text{on } y = \varepsilon h(x,t) \\ f(x,0) = 0, \quad u(x,y,0) = \sqrt{\dfrac{d}{g}} A y + \dfrac{B}{\sqrt{gd}}, \quad v(x,y,0) = 0 \end{array}\right\} \tag{1.11}$$

The magnitudes of ε and δ correspond to the different general types of water wave problem: The limits $\delta \to 0$ and $\delta \to \infty$ produce shallow water and deep water regime, respectively. On the other hand, $\varepsilon \to 0$ corresponds to regime of waves of small amplitude. We want to linearize the problem and in this case we will want to let $\varepsilon \to 0$ and keep δ fixed. But the system (1.11) shows that v and p are of order ε and hence so is u. In order to deal with this, we need to introduce a new scaling of the form $u = \varepsilon u_1$, $v = \varepsilon v_1$ and $p = \varepsilon p_1$, but then the fourth equation and the initial condition for u would become singular. Instead we use the following scaling:

$$u \mapsto \sqrt{\frac{d}{g}}Ay + \frac{B}{\sqrt{gd}} + \varepsilon u, \quad v \mapsto \varepsilon v, \quad p \mapsto \varepsilon p, \tag{1.12}$$

where we avoid a new notation. We then obtain the system

$$\left.\begin{aligned} u_x + v_y &= 0 \\ u_t + \left(\sqrt{\frac{d}{g}}Ay + \frac{B}{\sqrt{gd}} + \varepsilon u\right)u_x + v\left(\sqrt{\frac{d}{g}}A + \varepsilon u_y\right) &= -p_x \\ \delta^2\left(v_t + \left(\sqrt{\frac{d}{g}}Ay + \frac{B}{\sqrt{gd}} + \varepsilon u\right)v_x + \varepsilon v v_y\right) &= -p_y \\ u_y - \delta^2 v_x &= 0 \end{aligned}\right\} \tag{1.13}$$

in $\varepsilon h(x,t) < y < 1 + \varepsilon f(x,t)$ and

$$\left.\begin{aligned} &p = f \quad \text{on } y = 1 + \varepsilon f(x,t) \\ &v = f_t + \left(\sqrt{\frac{d}{g}}A(1+\varepsilon f) + \frac{B}{\sqrt{gd}} + \varepsilon u\right)f_x \quad \text{on } y = 1 + \varepsilon f(x,t) \\ &v = h_t + \left(\sqrt{\frac{d}{g}}A\varepsilon h + \frac{B}{\sqrt{gd}} + \varepsilon u\right)h_x \quad \text{on } y = \varepsilon h(x,t) \\ &f(x,0) = 0, \quad u(x,y,0) = 0, \quad v(x,y,0) = 0. \end{aligned}\right\} \tag{1.14}$$

We now linearize this system by taking $\varepsilon \to 0$ and obtain

$$\left.\begin{aligned}
&\begin{cases}
u_x + v_y = 0 \\
u_t + \left(\sqrt{\frac{d}{g}}Ay + \frac{B}{\sqrt{gd}}\right) u_x + \sqrt{\frac{d}{g}}Av = -p_x \\
\delta^2 \left(v_t + \left(\sqrt{\frac{d}{g}}Ay + \frac{B}{\sqrt{gd}}\right) v_x\right) = -p_y \\
u_y - \delta^2 v_x = 0
\end{cases} \qquad \text{in } 0 < y < 1 \\
&p = f \quad \text{on } y = 1 \\
&v = f_t + \left(\sqrt{\frac{d}{g}}A + \frac{B}{\sqrt{gd}}\right) f_x \quad \text{on } y = 1 \\
&v = h_t + \frac{B}{\sqrt{gd}} h_x \quad \text{on } y = 0 \\
&f(x,0) = 0, \quad u(x,y,0) = 0, \quad v(x,y,0) = 0.
\end{aligned}\right\} \tag{1.15}$$

By the first equation in (1.15), there exists a stream function ψ such that

$$u = \psi_y, \quad v = -\psi_x \quad \text{in } 0 < y < 1. \tag{1.16}$$

If we plug this in (1.15), we get

$$\left.\begin{aligned}
&\begin{cases}
\psi_{yy} + \delta^2 \psi_{xx} = 0 \\
\psi_{yt} + \left(\sqrt{\frac{d}{g}}Ay + \frac{B}{\sqrt{gd}}\right) \psi_{xy} - \sqrt{\frac{d}{g}}A\psi_x = -p_x \\
\delta^2 \left(\psi_{xt} + \left(\sqrt{\frac{d}{g}}Ay + \frac{B}{\sqrt{gd}}\right) \psi_{xx}\right) = p_y
\end{cases} \qquad \text{in } 0 < y < 1 \\
&p = f \quad \text{on } y = 1 \\
&\psi_x = -f_t - \left(\sqrt{\frac{d}{g}}A + \frac{B}{\sqrt{gd}}\right) f_x \quad \text{on } y = 1 \\
&\psi_x = -h_t - \frac{B}{\sqrt{gd}} h_x \quad \text{on } y = 0 \\
&f(x,0) = 0, \quad \psi_y(x,y,0) = 0, \quad \psi_x(x,y,0) = 0
\end{aligned}\right\} \tag{1.17}$$

Note that we want to eventually find f. But p and ψ are also unknown, only h is given. We will now derive equations only involving ψ and h.
We first differentiate the second equation in (1.17) with respect to x and t, respectively,

to obtain

$$\psi_{xyt} + \left(\sqrt{\frac{d}{g}}Ay + \frac{B}{\sqrt{gd}}\right)\psi_{xxy} - \sqrt{\frac{d}{g}}A\psi_{xx} = -p_{xx} \tag{1.18}$$

and

$$\psi_{ytt} + \left(\sqrt{\frac{d}{g}}Ay + \frac{B}{\sqrt{gd}}\right)\psi_{xyt} - \sqrt{\frac{d}{g}}A\psi_{xt} = -p_{xt}. \tag{1.19}$$

Furthermore, we can differentiate the fifth equation in (1.17) with respect to x to obtain

$$\psi_{xx} = -f_{xt} - \left(\sqrt{\frac{d}{g}}A + \frac{B}{\sqrt{gd}}\right)f_{xx} \quad \text{on } y = 1. \tag{1.20}$$

Observe that the fourth equation in (1.17) implies that we can exchange the derivatives of f in (1.20) by the corresponding derivatives of p. But then we can insert the equations (1.18) and (1.19) in (1.20) to obtain

$$\begin{aligned}\psi_{xx} = \psi_{ytt} &+ 2\left(\sqrt{\frac{d}{g}}A + \frac{B}{\sqrt{gd}}\right)\psi_{xyt} + \left(\sqrt{\frac{d}{g}}A + \frac{B}{\sqrt{gd}}\right)^2\psi_{xxy} \\ &- \sqrt{\frac{d}{g}}A\left(\sqrt{\frac{d}{g}}A + \frac{B}{\sqrt{gd}}\right)\psi_{xx} - \sqrt{\frac{d}{g}}A\psi_{xt}\end{aligned}$$

on $y = 1$.

We introduce the constant

$$C = \sqrt{\frac{d}{g}}A + \frac{B}{\sqrt{gd}}, \tag{1.21}$$

which is the non-dimensional horizontal velocity of the flow at the surface. This allows us to rewrite the previous equation as

$$\left(1 + \sqrt{\frac{d}{g}}AC\right)\psi_{xx} = \psi_{ytt} + 2C\psi_{xyt} + C^2\psi_{xxy} - \sqrt{\frac{d}{g}}A\psi_{xt}$$

on $y = 1$. By observing the quadratic expression on the right-hand side and introducing the differential operator

$$S = C\partial_x + \partial_t, \tag{1.22}$$

we can rewrite this again as

$$\left(1+\sqrt{\frac{d}{g}}AC\right)\psi_{xx}=S^2\psi_y-\sqrt{\frac{d}{g}}A\psi_{xt}.$$

So we obtain

$$\left.\begin{aligned}\psi_{yy}+\delta^2\psi_{xx}=0,\quad \text{in } 0<y<1\\ \left(1+\sqrt{\frac{d}{g}}AC\right)\psi_{xx}=S^2\psi_y-\sqrt{\frac{d}{g}}A\psi_{xt},\quad \text{on } y=1\\ \psi_x=-h_t-\frac{B}{\sqrt{gd}}h_x,\quad \text{on } y=0\\ \psi_x(x,y,0)=0,\quad \psi_y(x,y,0)=0,\end{aligned}\right\}\tag{1.23}$$

which only involves ψ and h. We can then finally recover f by the following: We know by the fourth and fifth equation in (1.17) that

$$f_x=p_x=-\psi_{yt}+C\psi_{xy}-\sqrt{\frac{d}{g}}A\psi_x$$

on $y=1$ and hence

$$f_t=-\psi_x-Cf_x=C\psi_{yt}-C^2\psi_{xy}+\left(\sqrt{\frac{d}{g}}AC-1\right)\psi_x\tag{1.24}$$

on $y=1$. Together with the initial condition

$$f(x,0)=0$$

we can recover f by

$$f(x,t)=\int_0^t f_t(x,\tau)d\tau.$$

We can reconstruct p similarly.

2 General solution formulae for linear waves

We will derive solution formulae for (1.23) using Fourier methods.

2.1 Definitions

We will consider the space and space-time Fourier transform, with notations

$$\hat{\varphi}(\xi, y, t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi(x, y, t) e^{-ix\xi} dx$$

and

$$\tilde{\varphi}(\xi, y, \omega) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(x, y, t) e^{-i(x\xi + t\omega)} dx dt.$$

The Fourier transform is well defined and invertible on the Schwartz class (with respect to the x and t variable), with the inversion formulae

$$\varphi(x, y, t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{\varphi}(\xi, y, t) e^{ix\xi} d\xi$$
$$\varphi(x, y, t) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \tilde{\varphi}(\xi, y, \omega) e^{i(x\xi + t\omega)} d\xi d\omega.$$

An important feature of the Fourier transform for PDEs is that it transforms derivatives to multiplication with polynomials:

$$\widetilde{\varphi_x} = i\xi\tilde{\varphi}, \quad \widetilde{\varphi_t} = i\omega\tilde{\varphi}$$

and, by exchanging order of integration and differentiation,

$$\widetilde{\varphi_y} = (\tilde{\varphi})_y.$$

Similarly

$$\widehat{\varphi_x} = i\xi\hat{\varphi}, \quad \widehat{\varphi_y} = \hat{\varphi}_y, \quad \widehat{\varphi_t} = \hat{\varphi}_t.$$

We also define Fourier multipliers: Let $m : \mathbb{R} \to \mathbb{C}$ be some function. We define $m(D)$ by

$$\left(m(D)\varphi\right)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} m(\xi)\hat{\varphi}(\xi) e^{ix\xi} d\xi,$$

or equivalently

$$\left(\widehat{m(D)\varphi}\right)(\xi) = m(\xi)\hat{\varphi}(\xi),$$

where $D = -i\partial_x$. This is well-defined pointwise if φ is Schwartz and m is smooth with derivatives of at most polynomial growth. Note that $m(D)$ maps real-valued functions to real-valued functions whenever the condition

$$m(-\xi) = \overline{m(\xi)} \tag{2.1}$$

is satisfied. In particular, this is satisfied if m is real-valued and even.

2.2 Fourier Transform of the equations

We can apply the space-time Fourier transform to the first three equations of system (1.23) to obtain

$$\left.\begin{aligned} \tilde{\psi}_{yy} - \delta^2\xi^2\tilde{\psi} = 0, \quad \text{in } 0 < y < 1 \\ \left(1 + \sqrt{\frac{d}{g}}AC\right)\xi^2\tilde{\psi} = Q^2\tilde{\psi}_y - \sqrt{\frac{d}{g}}A\xi\omega\tilde{\psi}, \quad \text{on } y = 1 \\ \xi\tilde{\psi} = -\omega\tilde{h} - \frac{B}{\sqrt{gd}}\xi\tilde{h}, \quad \text{on } y = 0 \end{aligned}\right\}, \tag{2.2}$$

where

$$Q(\xi, \omega) = C\xi + \omega \tag{2.3}$$

with the relation

$$\widetilde{S\varphi} = iQ\tilde{\varphi}. \tag{2.4}$$

(2.2) is just a linear ODE of second order in y for fixed ξ and ω, where we can, somewhat explicitly, write down the solution:

$$\tilde{\psi}(\xi, y, \omega) = D_1(\xi, \omega)e^{\delta\xi y} + D_2(\xi, \omega)e^{-\delta\xi y}, \tag{2.5}$$

where D_1, D_2 are determined by the boundary conditions. We can then reconstruct $\tilde{f}$ by applying the Fourier transform on the first equation in (1.24) to obtain

$$\omega\tilde{f}(\xi, \omega) = -\xi\tilde{\psi}(\xi, 1, \omega) - C\xi\tilde{f}(\xi, \omega)$$

and hence

$$\tilde{f}(\xi,\omega) = \frac{-\xi}{C\xi+\omega}\tilde{\psi}(\xi,1,\omega) = \frac{-\xi}{Q(\xi,\omega)}\tilde{\psi}(\xi,1,\omega). \tag{2.6}$$

We ignore the potential singularity for the time being.

2.3 Calculations

We know that the functions D_1 and D_2 are uniquely determined. Once we have formulae for them, we get a formula for $\tilde{\psi}$ by inserting in (2.5). Then we can insert this formula in (2.6) to obtain an expression for $\tilde{f}$ depending on $\tilde{h}$.

To this end, we will do the calculations first in special cases.

2.3.1 A = 0

In this case, we have

$$C = \frac{B}{\sqrt{gd}}$$

and

$$Q(\xi,\omega) = C\xi+\omega = \frac{B}{\sqrt{gd}}\xi+\omega.$$

Our system (2.2) is in this case

$$\left.\begin{aligned} \tilde{\psi}_{yy} - \delta^2\xi^2\tilde{\psi} = 0, &\quad \text{in } 0<y<1 \\ \xi^2\tilde{\psi} = Q^2\tilde{\psi}_y, &\quad \text{on } y=1 \\ \xi\tilde{\psi} = -Q\tilde{h}, &\quad \text{on } y=0 \end{aligned}\right\}.$$

By inserting the formula (2.5) for $y=0$ and $y=1$ in the above, we obtain

$$\left.\begin{aligned} \xi(D_1(\xi,\omega)+D_2(\xi,\omega)) = -Q(\xi,\omega)\tilde{h}(\xi,\omega) \\ \xi(D_1(\xi,\omega)e^{\delta\xi}+D_2(\xi,\omega)e^{-\delta\xi}) = Q(\xi,\omega)^2\delta(D_1(\xi,\omega)e^{\delta\xi}-D_2(\xi,\omega)e^{-\delta\xi}) \end{aligned}\right\}.$$

This is just a linear equation, which we can rewrite as

$$M\begin{pmatrix} D_1(\xi,\omega) \\ D_2(\xi,\omega)\end{pmatrix} = \begin{pmatrix} -Q(\xi,\omega)\tilde{h}(\xi,\omega) \\ 0\end{pmatrix},$$

where

$$M = \begin{pmatrix} \xi & \xi \\ e^{\delta\xi}(\xi - \delta Q(\xi,\omega)^2) & e^{-\delta\xi}(\xi + \delta Q(\xi,\omega)^2) \end{pmatrix}.$$

Whenever this matrix is invertible, we get

$$D_1(\xi,\omega) = \frac{-e^{-\delta\xi}(\xi + \delta Q(\xi,\omega)^2)Q(\xi,\omega)}{\det(M)}\tilde{h}(\xi,\omega)$$

and

$$D_2(\xi,\omega) = \frac{e^{\delta\xi}(\xi - \delta Q(\xi,\omega)^2)Q(\xi,\omega)}{\det(M)}\tilde{h}(\xi,\omega),$$

where

$$\begin{aligned}\det(M) &= \xi(e^{-\delta\xi}(\xi + \delta Q(\xi,\omega)^2) - e^{\delta\xi}(\xi - \delta Q(\xi,\omega)^2)) \\ &= 2\xi(\delta Q(\xi,\omega)^2\cosh(\delta\xi) - \xi\sinh(\delta\xi)).\end{aligned}$$

By (2.6) we have

$$\begin{aligned}\tilde{f}(\xi,\omega) &= \frac{-\xi}{Q(\xi,\omega)}\left(D_1(\xi,\omega)e^{\delta\xi} + D_2(\xi,\omega)e^{-\delta\xi}\right) \\ &= \frac{-\xi}{Q(\xi,\omega)}\frac{-e^{-\delta\xi}(\xi + \delta Q(\xi,\omega)^2)Q(\xi,\omega)}{\det(M)}\tilde{h}(\xi,\omega)e^{\delta\xi} \\ &+ \frac{-\xi}{Q(\xi,\omega)}\frac{e^{\delta\xi}(\xi - \delta Q(\xi,\omega)^2)Q(\xi,\omega)}{\det(M)}\tilde{h}(\xi,\omega)e^{-\delta\xi} \\ &= \frac{\xi}{\det(M)}\left(\xi + \delta Q(\xi,\omega)^2 - (\xi - \delta Q(\xi,\omega))\right)\tilde{h}(\xi,\omega) \\ &= \frac{2\xi\delta Q(\xi,\omega)^2}{2\xi(\delta Q(\xi,\omega)^2\cosh(\delta\xi) - \xi\sinh(\delta\xi))}\tilde{h}(\xi,\omega) \\ &= \frac{\delta Q(\xi,\omega)^2}{\delta Q(\xi,\omega)^2\cosh(\delta\xi) - \xi\sinh(\delta\xi)}\tilde{h}(\xi,\omega).\end{aligned}$$

It will be convenient to write

$$\tilde{f}(\xi,\omega) = \frac{\delta^2 Q(\xi,\omega)^2}{\delta^2 Q(\xi,\omega)^2\cosh(\delta\xi) - \delta\xi\sinh(\delta\xi)}\tilde{h}(\xi,\omega). \tag{2.7}$$

In our calculation, we needed $\det(M) \neq 0$ (or, equivalently, that the denominator in (2.7) is $\neq 0$), so one would need to check that this works out. We will briefly discuss this here, but later, cf. (2.13), we will have obtained a formula with a potential singularity, with which we can deal directly. We are looking for roots of

$$\delta^2 Q(\xi,\omega)^2 \cosh(\delta\xi) - \delta\xi \sinh(\delta\xi) = 0$$

or, equivalently, of

$$Q(\delta\xi,\delta\omega)^2 - \delta\xi \tanh(\delta\xi) = 0.$$

Here, the behaviour of the roots depends on whether $B = 0$ or $B \neq 0$.
For the case $B = 0$, first note that the function $s \mapsto s\tanh(s)$ is even and maps $[0,\infty)$ bijectively to $[0,\infty)$. Since Q does not depend on ξ, Q is just a constant depending on ω. Furthermore, it is strictly positive for $\omega \neq 0$. In this case there are two roots $\xi_\pm(\omega)$ with $\xi_-(\omega) = -\xi_+(\omega)$, where $\xi_+(\omega) > 0$. In the limit $\omega \to 0$, the two roots coalesce to the single root $\xi_\pm(0) = 0$. Since $\tanh(s) \asymp 1$ for $s \to \infty$, we have $\xi_\pm(\omega) \asymp \pm\delta\omega^2$ as $\omega \to \infty$.
Now, the case $B \neq 0$: For any fixed ω, the function $s \mapsto Q(s,\delta\omega) = (Cs + \delta\omega)^2$ is a quadratic polynomial. One can check that we again get exactly two roots $\xi_\pm(\omega)$ if $\omega \neq 0$ and exactly one root if $\omega = 0$. As before the two roots coalesce in the limit to the single root $\xi_\pm(0) = 0$. On the other hand, since the roots of the polynomial $(Cs + \delta\omega)^2 - s$ are given by

$$\frac{1 - 2C\delta\omega \pm \sqrt{1 - 4C\delta\omega}}{2C^2}$$

we conclude that $\xi_\pm(\omega) \asymp -\frac{\omega}{C}$ as $\omega \to \infty$.

Next, we can rewrite (2.7) as

$$\left(Q(\xi,\omega)^2 - \frac{\xi}{\delta}\tanh(\delta\xi)\right)\tilde{f}(\xi,\omega) = \frac{Q(\xi,\omega)^2}{\cosh(\delta\xi)}\tilde{h}(\xi,\omega).$$

After applying the inverse Fourier transform we obtain

$$\left(S^2 + \frac{D}{\delta}\tanh(\delta D)\right) f = \frac{S^2}{\cosh(\delta D)} h. \tag{2.8}$$

We will extract a formula for f whenever it solves this type of an equation: Let $\tau : \mathbb{R} \to \mathbb{R}$ be an even function such that $\tau(\xi) > 0$ for all $\xi \neq 0$. Note that in particular

(2.1) is satisfied. We try to derive a formula for the solution f of

$$\left.\begin{aligned}\left(S^2+\tau(D)\right)f=\theta\\ \lim_{t\to-\infty} f=0\end{aligned}\right\}, \tag{2.9}$$

where θ is a given forcing term. We will assume f and θ to be in $\mathcal{S}(\mathbb{R}^2,\mathbb{R})$ and interpret the limit in (2.9) in the Schwartz class. We can decompose the operator $S^2+\tau(D)$ into

$$S^2+\tau(D)=(S+i\sqrt{\tau(D)})(S-i\sqrt{\tau(D)}), \tag{2.10}$$

since S and $i\sqrt{\tau(D)}$ commute. If we now set

$$f_1=(S+i\sqrt{\tau(D)})f$$

then we see by (2.10) and the fact that we can commute the two factors that

$$(S-i\sqrt{\tau(D)})f_1=\theta.$$

We can now filter f_1 by the group $e^{it\sqrt{\tau(D)}}$: Let $f_2=e^{-it\sqrt{\tau(D)}}f_1$. A calculation shows that

$$Sf_2=(C\partial_x+\partial_t)f_2=e^{-it\sqrt{\tau(D)}}\theta.$$

This is an inhomogeneous transport equation, with solution

$$f_2(x,t)=\int_{-\infty}^{t} e^{-is\sqrt{\tau(D)}}\theta(x+C(s-t),s)ds.$$

This gives, by using the fact that Fourier multipliers commute with integrals,

$$f_1(x,t)=\int_{-\infty}^{t} e^{i(t-s)\sqrt{\tau(D)}}\theta(x+C(s-t),s)ds$$

and since f (and hence also $\sqrt{\tau(D)}f$ by (2.1)) is real-valued

$$\left(\sqrt{\tau(D)}f\right)(x,t)=\operatorname{Im}\left(\int_{-\infty}^{t} e^{i(t-s)\sqrt{\tau(D)}}\theta(x+C(s-t),s)ds\right).$$

If we write out the Fourier multipliers explicitly, we get

$$f(x,t)=\frac{1}{\sqrt{2\pi}}\,\mathrm{Im}\left(\int_{-\infty}^{t}\int_{\mathbb{R}}\frac{e^{i(t-s)\sqrt{\tau(\xi)}}}{\sqrt{\tau(\xi)}}e^{ix\xi}e^{iC(s-t)\xi}\hat{\theta}(\xi,s)d\xi ds\right). \tag{2.11}$$

We will deal with the possible singularity in the for us relevant case: We have $\tau(\xi)=\frac{\xi}{\delta}\tanh(\delta\xi)$ and $\theta=\frac{S^2}{\cosh(\delta D)}h$. We see that τ is even and $\tau(\xi)>0$ for $\xi\neq 0$. We have furthermore that τ is smooth, with derivatives of at most polynomial growth at infinity. For the potential singularity, it will be relevant that

$$\lim_{\xi\to 0}\frac{\tau(\xi)}{\xi^2}=1,$$

or, equivalently

$$\lim_{\xi\to 0}\frac{|\xi|}{\sqrt{\tau(\xi)}}=1. \tag{2.12}$$

Now observe that the function $\frac{1}{\sqrt{\tau(D)}}\theta$ is real-valued and hence we can subtract $\left(\frac{1}{\sqrt{\tau(D)}}\theta\right)(0,\cdot)$ in (2.11) to obtain

$$f(x,t)=\frac{1}{\sqrt{2\pi}}\,\mathrm{Im}\left(\int_{-\infty}^{t}\int_{\mathbb{R}}\frac{e^{i(t-s)\sqrt{\tau(\xi)}}e^{ix\xi}e^{iC(s-t)\xi}-1}{\sqrt{\tau(\xi)}}\hat{\theta}(\xi,s)d\xi ds\right). \tag{2.13}$$

Clearly, $\hat{\theta}(\xi,s)$ remains bounded around 0. For the quotient, note that we can rewrite it as follows:

$$\left|\frac{e^{i(t-s)\sqrt{\tau(\xi)}}e^{ix\xi}e^{iC(s-t)\xi}-1}{\sqrt{\tau(\xi)}}\right|=\left|\frac{e^{i(t-s)\sqrt{\tau(\xi)}}e^{ix\xi}e^{iC(s-t)\xi}-1}{\xi}\right|\left|\frac{\xi}{\sqrt{\tau(\xi)}}\right|$$

The second factor converges to 1 by (2.12). For the first factor, note that the function

$$\xi\mapsto e^{i(t-s)\sqrt{\tau(\xi)}}e^{ix\xi}e^{iC(s-t)\xi}$$

is left and right differentiable in 0 and hence the first factor remains bounded as $\xi\to 0$ for any s. We conclude that (2.13) holds (at least pointwise).

2.3.2 B = 0

In this case we have

$$C = \sqrt{\frac{d}{g}}A$$

and

$$Q(\xi, \omega) = C\xi + \omega = \sqrt{\frac{d}{g}}A\xi + \omega.$$

System (2.2) is then

$$\left.\begin{aligned} \tilde{\psi}_{yy} - \delta^2\xi^2\tilde{\psi} = 0, \quad \text{in } 0 < y < 1 \\ \left(1 + C^2\right)\xi^2\tilde{\psi} = Q^2\tilde{\psi}_y - C\xi\omega\tilde{\psi}, \quad \text{on } y = 1 \\ \xi\tilde{\psi} = -\omega\tilde{h}, \quad \text{on } y = 0 \end{aligned}\right\}.$$

We again insert (2.5) for $y = 0$ and $y = 1$ to obtain

$$\xi(D_1(\xi, \omega) + D_2(\xi, \omega)) = -\omega\tilde{h}(\xi, \omega)$$

and

$$\begin{aligned} &(\xi + CQ(\xi, \omega))\left(D_1(\xi, \omega)e^{\delta\xi} + D_2(\xi, \omega)e^{-\delta\xi}\right) \\ &\quad = Q(\xi, \omega)^2\delta(D_1(\xi, \omega)e^{\delta\xi} - D_2(\xi, \omega)e^{-\delta\xi}). \end{aligned}$$

Again, we can write this as

$$M\begin{pmatrix} D_1(\xi, \omega) \\ D_2(\xi, \omega) \end{pmatrix} = \begin{pmatrix} -\omega\tilde{h}(\xi, \omega) \\ 0 \end{pmatrix},$$

where M is given by

$$\begin{pmatrix} \xi & \xi \\ e^{\delta\xi}(\xi + CQ(\xi, \omega) - \delta Q(\xi, \omega)^2) & e^{-\delta\xi}(\xi + CQ(\xi, \omega) + \delta Q(\xi, \omega)^2) \end{pmatrix}.$$

Whenever this matrix is invertible, we have

$$D_1(\xi,\omega) = \frac{-e^{-\delta\xi}(\xi + CQ(\xi,\omega) + \delta Q(\xi,\omega)^2)\omega}{\det(M)}\tilde{h}(\xi,\omega)$$

and

$$D_2(\xi,\omega) = \frac{e^{\delta\xi}(\xi + CQ(\xi,\omega) - \delta Q(\xi,\omega)^2)\omega}{\det(M)}\tilde{h}(\xi,\omega),$$

where $\det(M)$ is given by

$$\begin{aligned}\xi(e^{-\delta\xi}(\xi + CQ(\xi,\omega) + \delta Q(\xi,\omega)^2) - e^{\delta\xi}(\xi + CQ(\xi,\omega) - \delta Q(\xi,\omega)^2))\\ = 2\xi(\delta Q(\xi,\omega)^2\cosh(\delta\xi) - (\xi + CQ(\xi,\omega))\sinh(\delta\xi)).\end{aligned}$$

We calculate

$$\tilde{\psi}(\xi,1,\omega) = D_1(\xi,\omega)e^{\delta\xi} + D_2(\xi,\omega)e^{-\delta\xi} = \frac{-2\delta\omega Q(\xi,\omega)^2}{\det(M)}\tilde{h}(\xi,\omega)$$

and hence get

$$\begin{aligned}\tilde{f}(\xi,\omega) &= \frac{-\xi}{Q(\xi,\omega)}\tilde{\psi}(\xi,1,\omega) = \frac{-\xi}{Q(\xi,\omega)}\frac{-2\delta\omega Q(\xi,\omega)^2}{\det(M)}\tilde{h}(\xi,\omega)\\ &= \frac{2\delta\xi\omega Q(\xi,\omega)}{\det(M)}\tilde{h}(\xi,\omega)\\ &= \frac{2\delta\xi\omega Q(\xi,\omega)}{2\xi(\delta Q(\xi,\omega)^2\cosh(\delta\xi) - (\xi + CQ(\xi,\omega))\sinh(\delta\xi))}\tilde{h}(\xi,\omega)\\ &= \frac{\delta\omega Q(\xi,\omega)}{\delta Q(\xi,\omega)^2\cosh(\delta\xi) - (\xi + CQ(\xi,\omega))\sinh(\delta\xi)}\tilde{h}(\xi,\omega).\end{aligned}$$

We will write this as

$$\tilde{f}(\xi,\omega) = \frac{\delta^2\omega Q(\xi,\omega)}{\delta^2 Q(\xi,\omega)^2\cosh(\delta\xi) - (\delta\xi + C\delta Q(\xi,\omega))\sinh(\delta\xi)}\tilde{h}(\xi,\omega). \tag{2.14}$$

We can write this as

$$\left(S^2 + \frac{D - iCS}{\delta}\tanh(\delta D)\right) f = \frac{S\partial_t}{\cosh(\delta D)}h. \tag{2.15}$$

For a rigorous justification of (2.15), one would need to consider the roots of the denominator in (2.14).

Ideally, we would like to employ the same strategy as in the case $A = 0$. However, the operator

$$S^2 + \frac{D - iCS}{\delta} \tanh(\delta D) = S^2 + (1 - C^2)\frac{D}{\delta} \tanh(\delta D) - \frac{iC}{\delta}\partial_t \tanh(\delta D)$$

does not admit an analogous decomposition as in (2.10), since there is no apparent way to obtain a root for

$$(1 - C^2)\frac{D}{\delta} \tanh(\delta D) - \frac{iC}{\delta}\partial_t \tanh(\delta D).$$

More precisely, we do not know whether this operator is positive, which could guarantee the existence of a root (cf. for example Werner [11]).

2.3.3 General Case

We expect long formulae in the following. We again insert (2.5) in (2.2) to obtain

$$\xi \left(D_1(\xi, \omega) + D_2(\xi, \omega)\right) = -\left(\omega + \frac{B}{\sqrt{gd}}\xi\right) \tilde{h}(\xi, \omega)$$

and

$$\begin{aligned}\left(\xi + \sqrt{\frac{d}{g}}AQ(\xi, \omega)\right) &\left(D_1(\xi, \omega)e^{\delta\xi} + D_2(\xi, \omega)e^{-\delta\xi}\right) \\ &= \delta Q(\xi, \omega)^2 \left(D_1(\xi, \omega)e^{\delta\xi} - D_2(\xi, \omega)e^{-\delta\xi}\right).\end{aligned}$$

Again, we can write this as

$$M \begin{pmatrix} D_1(\xi, \omega) \\ D_2(\xi, \omega) \end{pmatrix} = \begin{pmatrix} -\left(\omega + \frac{B}{\sqrt{gd}}\xi\right) \tilde{h}(\xi, \omega) \\ 0 \end{pmatrix},$$

where M is given by

$$\begin{pmatrix} \xi & \xi \\ M_{2,1} & M_{2,2} \end{pmatrix}$$

and

$$\begin{aligned}
M_{2,1} &= e^{\delta\xi}\left(\xi + \sqrt{\frac{d}{g}}AQ(\xi,\omega) - \delta Q(\xi,\omega)^2\right) \\
M_{2,2} &= e^{-\delta\xi}\left(\xi + \sqrt{\frac{d}{g}}AQ(\xi,\omega) + \delta Q(\xi,\omega)^2\right).
\end{aligned}$$

Whenever this matrix is invertible, we have

$$D_1(\xi,\omega) = \frac{-e^{-\delta\xi}\left(\xi + \sqrt{\frac{d}{g}}AQ(\xi,\omega) + \delta Q(\xi,\omega)^2\right)\left(\omega + \frac{B}{\sqrt{gd}}\xi\right)}{\det(M)}\tilde{h}(\xi,\omega)$$

and

$$D_2(\xi,\omega) = \frac{e^{\delta\xi}\left(\xi + \sqrt{\frac{d}{g}}AQ(\xi,\omega) - \delta Q(\xi,\omega)^2\right)\left(\omega + \frac{B}{\sqrt{gd}}\xi\right)}{\det(M)}\tilde{h}(\xi,\omega),$$

where $\det(M)$ is given by

$$\xi(M_{2,2} - M_{2,1}) = 2\xi\left(\delta Q(\xi,\omega)^2\cosh(\delta\xi) - \left(\xi + \sqrt{\frac{d}{g}}AQ(\xi,\omega)\right)\sinh(\delta\xi)\right).$$

We calculate

$$\begin{aligned}
\tilde{\psi}(\xi,1,\omega) &= D_1(\xi,\omega)e^{\delta\xi} + D_2(\xi,\omega)e^{-\delta\xi} \\
&= \frac{-2\delta Q(\xi,\omega)^2\left(\omega + \frac{B}{\sqrt{gd}}\xi\right)}{\det(M)}\tilde{h}(\xi,\omega)
\end{aligned}$$

and hence get by (2.6)

$$\begin{aligned}
\tilde{f}(\xi,\omega) &= \frac{-\xi}{Q(\xi,\omega)}\tilde{\psi}(\xi,1,\omega) = \frac{2\delta\xi Q(\xi,\omega)^2\left(\omega + \frac{B}{\sqrt{gd}}\xi\right)}{\det(M)Q(\xi,\omega)}\tilde{h}(\xi,\omega) \\
&= \frac{\delta Q(\xi,\omega)\left(\omega + \frac{B}{\sqrt{gd}}\xi\right)}{\left(\delta Q(\xi,\omega)^2\cosh(\delta\xi) - \left(\xi + \sqrt{\frac{d}{g}}AQ(\xi,\omega)\right)\sinh(\delta\xi)\right)}\tilde{h}(\xi,\omega).
\end{aligned}$$

We will write this as

$$\tilde{f}(\xi,\omega) = \frac{\delta^2 Q(\xi,\omega)\left(\omega + \frac{B}{\sqrt{gd}}\xi\right)\tilde{h}(\xi,\omega)}{\left(\delta^2 Q(\xi,\omega)^2\cosh(\delta\xi) - \left(\delta\xi + \sqrt{\frac{d}{g}}A\delta Q(\xi,\omega)\right)\sinh(\delta\xi)\right)}. \tag{2.16}$$

We can rewrite this as

$$\begin{aligned}\left(Q(\xi,\omega)^2 - \frac{\xi + \sqrt{\frac{d}{g}}AQ(\xi,\omega)}{\delta}\tanh(\delta\xi)\right)\tilde{f}(\xi,\omega) \\ = \frac{Q(\xi,\omega)\left(\omega + \frac{B}{\sqrt{gd}}\xi\right)}{\cosh(\delta\xi)}\tilde{h}(\xi,\omega)\end{aligned}$$

and as

$$\left(S^2 + \frac{D - i\sqrt{\frac{d}{g}}AS}{\delta}\tanh(\delta D)\right)f = \frac{S\left(\partial_t + i\frac{B}{\sqrt{gd}}D\right)}{\cosh(\delta D)}h. \tag{2.17}$$

Again, one would need to consider the roots of the denominator in (2.16) in order to justify (2.17). Also, we run into the same problem as in the case $B = 0$: the operator on the left in (2.17) does not admit an (apparent) analogous decomposition as in (2.10).

3 Behaviour in different regimes

We want to see for which regimes this model gives useful predictions. We will consider on the one hand the case where δ is small, which justifies considering the model as $\delta \to 0$. On the other hand, we will discuss the case where δ is finite and non-vanishing. The case where δ is large (corresponding to the formal limit $\delta \to \infty$) is not relevant for us since deep-water tsunamis are rare.

3.1 The shallow water regime

We first use the model in the case $A = 0$: As $\delta \to 0$, (2.8) becomes

$$(S^2 - \partial_x^2)f = S^2 h. \tag{3.1}$$

The initial conditions are on the one hand $f(x,0) = 0$ from (1.15). On the other hand, this gives $f_x(x,0) = 0$ and inserting this in the sixth relation in (1.15) for $t = 0$ yields

$f_t(x,0) = 0$. By using Duhamel's principle (similar to [6, Chapter 2.4.2]), we get the formula

$$f(x,t) = \frac{1}{2}\int_0^t \int_{x-(t-s)(C+1)}^{x-(t-s)(C-1)} S^2 h(r,s)drds. \tag{3.2}$$

We will assume that h can be separated in the following way: $h(x,t) = a(t)b(x)$ for $a \in C^2(\mathbb{R},[0,\infty))$, $b \in C^2(\mathbb{R},\mathbb{R})$ such that $a(t) = 0$ for $t \leq 0$ and $a(t) = 1$ for $t > t_0$ (where t_0 represents the duration of the earthquake), and with $b(x) = 0$ for $x \notin (-L, L)$ modelling a localized tsunami source. Inserting this into (3.2) yields the formula

$$\begin{aligned} f(x,t) &= a(t)b(x) \\ &\quad + \frac{1}{2}\int_0^t a(s)\left[b'(x-(t-s)(C-1)) - b'(x-(t-s)(C+1))\right] ds, \end{aligned}$$

which is a lot simpler. In the limiting case $t_0 \searrow 0$, a becomes the Heaviside step function (modelling an instantaneous upward thrust of the seabed near to the earthquake's epicentre) and the formula further simplifies to

$$f(x,t) = \frac{C^2}{C^2-1}b(x) + \frac{b(x-t(1+C))}{2(1+C)} + \frac{b(x+t(1-C))}{2(1-C)}, \tag{3.3}$$

for $x \in \mathbb{R}$ and $t > 0$. We can draw some insightful conclusions:

- at each instant $t > 0$ after initiation, the generated wave is localized as $f(x,t) = 0$ for $|x| \geq L + t(1+C)$;
- the surface wave consists of one stationary part (which can be disregarded, since $C \ll 1$) and two travelling waves, one moving to the right and the other moving to the left;
- the wave travelling to the left moves with non-dimensionalized speed $1 - C$ (corresponding to $\sqrt{gd} - B$ in the physical variables) and the wave travelling to the right moves with non-dimensionalized speed $1 + C$ (corresponding to $\sqrt{gd} + B$ in the physical variables);
- the shapes of the waves travelling to the left and to the right remain unchanged and are precisely that of the bed deformation at a scale of $\frac{1}{2(1-C)}$ and $\frac{1}{2(1+C)}$, respectively.

This behaviour is illustrated in Figure 2 and Figure 3. On the one hand, the wave travelling to the right is faster for bigger C, but the maximal amplitude decreases. On

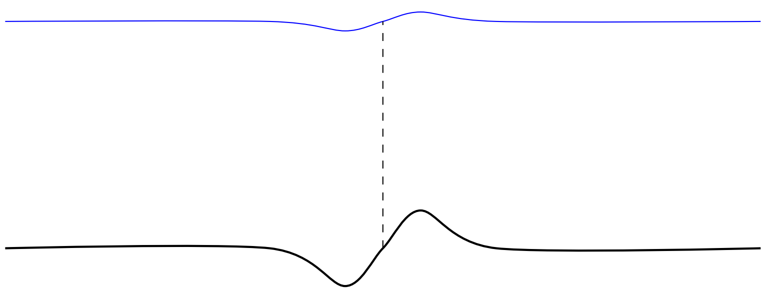

Figure 2: Model shortly after the instantaneous upward and downward thrust

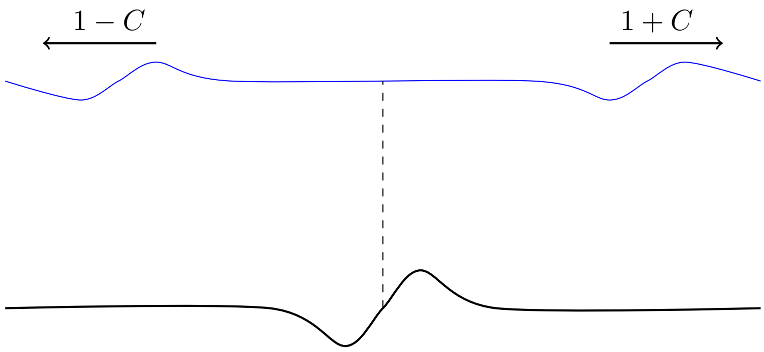

Figure 3: Expected long-term behaviour

the other hand, the wave travelling to the left becomes slower for bigger C, but the maximal amplitude increases.

For $B = 0$ and hence $C = 0$, this model gives good predictions for the maximal amplitude and speed of the propagating wave for the two largest historical tsunamis: the December 2004 and the May 1960 tsunamis, see [4]. Since $B \ll \sqrt{gd}$, i.e. $C \ll 1$, the model for $B \neq 0$ will give a very similar prediction, which might be slightly more precise, if one were in a situation, where B is known almost exactly, say due to a current.

One could try to apply the model in the case $A \neq 0$ and let $\delta \to 0$ in (2.15) and (2.17). This would give the equations

$$\left(S^2 - (\partial_x + CS)\partial_x\right) f = S\partial_t h$$

and

$$\left(S^2 - \left(\partial_x + \sqrt{\frac{d}{g}}AS\right)\partial_x\right) f = S\left(\partial_t + \frac{B}{\sqrt{gd}}\partial_x\right) h,$$

respectively. These equations would be quite tedious to analyze compared to (3.1), where the corresponding homogeneous equation can be solved rather easily, which is not expected here. Even then, we still made quite a few assumptions on h to obtain (3.3), so we will not analyze this further.

3.2 The finite, non-vanishing regime

Here we cannot expect to be able to write down f in any explicit fashion. However, we will discuss a method to analyze the asymptotic behaviour of f in (2.13). Ultimately, we will see that this approach does not work, since the method only works for too large t, see (3.10).

We cannot do the same for $B = 0$ or the general case, since we lack a formula which would need to correspond to (2.11) or (2.13). If such a formula were available, stationary-phase analysis might then be appropriate.

3.2.1 Principle of stationary-phase

We want to analyse the asymptotic behaviour of (2.13). For this we try to use the stationary-phase principle. Recall that we set $\tau(\xi) = \frac{\xi}{\delta}\tanh(\delta\xi)$ and $\theta = \frac{S^2}{\cosh(\delta D)}h$. We first rewrite (2.13) as

$$\begin{aligned} f(x,t) &= \frac{1}{\sqrt{2\pi}} \operatorname{Im}\left(\int_{-\infty}^{t}\int_{\mathbb{R}} \frac{e^{i(t-s)\sqrt{\tau(\xi)}}e^{ix\xi}e^{iC(s-t)\xi} - 1}{\sqrt{\tau(\xi)}}\hat{\theta}(\xi,s)d\xi ds\right) \\ &= \frac{1}{\sqrt{2\pi}} \operatorname{Im}\left(\int_{-\infty}^{t}\int_{\mathbb{R}} \frac{e^{-is\left(\sqrt{\tau(\xi)}-C\xi\right)}e^{it\left(\sqrt{\tau(\xi)}+(\mathcal{X}-C)\xi\right)} - 1}{\sqrt{\tau(\xi)}}\hat{\theta}(\xi,s)d\xi ds\right), \end{aligned}$$

where $\mathcal{X} = \frac{x}{t}$. The derivative of the phase factor $\sqrt{\tau(\xi)} + (\mathcal{X} - C)\xi$ is given by

$$[\sqrt{\tau}]'(\xi) + \mathcal{X} - C$$

and one can check that

$$\partial_\xi \sqrt{\tau(\xi)} = \frac{\sinh(\delta\xi)\cosh(\delta\xi) + \delta\xi}{2\delta\cosh^2(\delta\xi)}\sqrt{\frac{\delta\cosh(\delta\xi)}{\xi\sinh(\delta\xi)}} \tag{3.4}$$

and

$$\partial_\xi^2 \sqrt{\tau(\xi)} = -\frac{1}{\tau^{\frac{3}{2}}(\xi)}\frac{(\sinh(\delta\xi)\cosh(\delta\xi) - \delta\xi)^2 + 4\delta^2\xi^2\sinh^2(\delta\xi)}{4\delta^2\cosh^4(\delta\xi)} < 0. \tag{3.5}$$

One computes the limits

$$
\begin{aligned}
&\lim_{\xi\searrow 0}\partial_\xi\sqrt{\tau(\xi)}=1,\\
&\lim_{\xi\nearrow 0}\partial_\xi\sqrt{\tau(\xi)}=-1,\\
&\lim_{\xi\to\infty}\partial_\xi\sqrt{\tau(\xi)}=0,\\
&\lim_{\xi\to-\infty}\partial_\xi\sqrt{\tau(\xi)}=0
\end{aligned}
$$

and concludes that $\xi\mapsto\partial_\xi\sqrt{\tau(\xi)}$ is strictly decreasing on $(0,\infty)$ from the asymptotic value 1 towards the asymptotic value 0 and on $(-\infty,0)$ from the asymptotic value 0 to the asymptotic value -1. So we conclude that

$$[\sqrt{\tau}]'(\xi_0)+\mathcal{X}-C=0$$

is possible if $\mathcal{X}-C\in(-1,0)\cup(0,1)$ for exactly one ξ_0 for given $\mathcal{X}$. So as long as $\mathcal{X}$ is in the corresponding range, we can write $\xi_0=\xi_0(\mathcal{X})$. The stationary-phase principle (cf. for example [10]) gives

$$
\begin{aligned}
f(x,t)\sim&\frac{1}{\sqrt{t\tau(\xi_0)|[\sqrt{\tau}]''(\xi_0)|}}\cdot\\
&\operatorname{Im}\left(\int_{-\infty}^{t}e^{-is\left(\sqrt{\tau(\xi_0)}-C\xi_0\right)}\hat{\theta}(\xi_0,s)e^{it\left(\sqrt{\tau(\xi_0)}+(\mathcal{X}-C)\xi_0+\sigma\frac{\pi}{4})\right)}ds\right)
\end{aligned}
$$

for large t, where σ is the sign of $[\sqrt{\tau}]''(\xi_0)$, which is just -1 by (3.5). We see that for large t this is a Fourier transform (with respect to time) and hence get

$$
\begin{aligned}
f(x,t)\sim&\frac{\sqrt{2\pi}}{\sqrt{t\tau(\xi_0)|[\sqrt{\tau}]''(\xi_0)|}}\cdot\\
&\operatorname{Im}\left(\tilde{\theta}(\xi_0,\sqrt{\tau(\xi_0)}-C\xi_0)e^{it\left(\sqrt{\tau(\xi_0)}+(\mathcal{X}-C)\xi_0-\frac{\pi}{4})\right)}\right).
\end{aligned}
$$

We have

$$
\begin{aligned}
\tilde{\theta}(\xi_0,\sqrt{\tau(\xi_0)}-C\xi_0)&=-\frac{Q(\xi_0,\sqrt{\tau(\xi_0)}-C\xi_0)^2}{\cosh(\delta\xi_0)}\tilde{h}(\xi_0,\sqrt{\tau(\xi_0)}-C\xi_0)\\
&=-\frac{\tau(\xi_0)}{\cosh(\delta\xi_0)}\tilde{h}(\xi_0,\sqrt{\tau(\xi_0)}-C\xi_0)
\end{aligned}
$$

and hence get

$$
\begin{aligned}
f(x,t) \sim & \frac{-\sqrt{2\pi\tau(\xi_0)}}{\cosh(\delta\xi_0)\sqrt{t|[\sqrt{\tau}]''(\xi_0)|}} \cdot \\
& \operatorname{Im}\left(e^{it\left(\sqrt{\tau(\xi_0)}+(\mathcal{X}-C)\xi_0-\frac{\pi}{4}\right)}\tilde{h}(\xi_0, \sqrt{\tau(\xi_0)}-C\xi_0)\right). \qquad (3.6)
\end{aligned}
$$

Finally, we try to extract the asymptotics with respect to δ: We fix a $\xi > 0$. A look at (3.4) gives that $\delta \mapsto \partial_\xi\sqrt{\tau(\xi)}$ is C^∞ and clearly even in δ. Since we have $\lim_{\delta\searrow 0}\partial_\xi\sqrt{\tau(\xi)} = 1$, the Taylor expansion with respect to δ around 0 has the form

$$
\partial_\xi\sqrt{\tau(\xi)} = 1 + \alpha(\xi)\delta^2 + O(\delta^3),
$$

for some coefficient $\alpha(\xi)$. One could find this coefficient through tedious calculations, but one can instead notice that $2\alpha(\xi)$ is the coefficient of δ^2 of the expansion of $\left(\partial_\xi\sqrt{\tau(\xi)}\right)^2$. We have

$$
\begin{aligned}
\left(\partial_\xi\sqrt{\tau(\xi)}\right)^2 &= \frac{\sinh(\delta\xi)}{4\delta\xi\cosh(\delta\xi)} + \frac{1}{2\cosh^2(\delta\xi)} + \frac{\delta\xi}{4\sinh(\delta\xi)\cosh^3(\delta\xi)} \\
&= 1 - \delta^2\xi^2 + O(\delta^3)
\end{aligned}
$$

and hence $\partial_\xi\sqrt{\tau(\xi)} = 1 - \frac{1}{2}\delta^2\xi^2 + O(\delta^3)$. We conclude that for $\mathcal{X} - C \in (-1, 0)$ the stationary-phase point $\xi_0 > 0$ with $[\sqrt{\tau}]'(\xi_0) + \mathcal{X} - C = 0$ satisfies $\delta\xi_0 = O(1)$. Inserting this into (3.5) yields

$$
|\partial_\xi^2\sqrt{\tau(\xi_0)}| = O(\delta). \qquad (3.7)
$$

3.2.2 Inserting typical physical parameters

We want to see whether stationary-phase analysis, using the formula

$$
\begin{aligned}
f(x,t) \sim & \frac{-\sqrt{2\pi\tau(\xi_0)}}{\cosh(\delta\xi_0)\sqrt{t|[\sqrt{\tau}]''(\xi_0)|}} \cdot \\
& \operatorname{Im}\left(e^{it\left(\sqrt{\tau(\xi_0)}+(\mathcal{X}-C)\xi_0-\frac{\pi}{4}\right)}\tilde{h}(\xi_0, \sqrt{\tau(\xi_0)}-C\xi_0)\right),
\end{aligned}
$$

could be justified here. Since the stationary-phase principle applies for large t we insert typical values of the physical parameters for tsunamis propagating at open sea (see [3]):

$$a = 1\text{m}, \quad d = 4\text{km}, \quad \lambda = 200\text{km}$$

This leads to $\varepsilon = 0.00025$ and $\delta = 0.02$, which point to linear wave approximation. Applying the stationary-phase principle is justified if

$$t|\sqrt{\tau}''(\xi)| \gg 1. \tag{3.8}$$

We have by (3.7), $|\sqrt{\tau}''(\xi)| = O(\delta)$, which translates to

$$t \gg \frac{1}{\delta^2}. \tag{3.9}$$

In physical variables this means

$$T \gg \frac{\lambda}{\sqrt{gd}\delta^2} \approx 2.5 \times 10^6 \text{ s} \approx 700 \text{ h}, \tag{3.10}$$

which takes way too long to be applicable. If one were to relax (3.9) to a condition of the form

$$t \geq \frac{1}{\delta},$$

one would arrive at

$$T \geq \frac{\lambda}{\sqrt{gd}\delta} \approx 5 \times 10^4 \text{ s} \approx 14 \text{ h},$$

which would barely be applicable in the case, where the epicentre of the seaquake is far off the shore. Nevertheless, relaxing (3.9) is quite dubious.

We conclude that using stationary-phase analysis is not justified in the case $A = 0$.

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