# DIPLOMARBEIT

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## Evolutionary Games and Imitation Dynamics

Verfasser

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Abstract. This work deals with imitation dynamics of the form $\dot{x}_i = x_i \sum_j x_j \psi((Ax)_i - (Ax)_j)$, where the switching rate is a function of the payoff differences. Our main tools in the process of analyzing are Poincare maps and Hamiltonian functions.

After a general introduction to imitation dynamics, we outline how to use imitation processes in bimatrix games. A special emphasis is given to the case of two players with two strategies. Non-degenerate games and several degenerate cases are analyzed. There is a topological equivalence between the replicator equation and all our imitation dynamics for the non-degenerate games but in the case of degenerate games the imitate the better dynamics is not topologically equivalent to the replicator equation.

In chapter 4 we analyze RSP games under imitation dynamics. Zero sum games and the case of cyclic symmetry are covered in full detail here. Open problems remain of how to extend the analysis to the general RSP game.

# Contents

CHAPTER 1

# Introduction

### From the Replicator Equation to Imitation Dynamics

The concept of evolution is older than most of us believe, maybe it existed ever since human beings were capable of thinking. The Taoists were well aware of the constant change in the world as well as Heraclit was with his famous "Panta Rhei". They all realized that we are living in a dynamic world, a world of constant change, but it took a while to realize how far reaching these primitive assumptions are.

From this merely natural and primitive idea everything came down to Darwin [**3**] who was fascinated by the diversity and wealth of species and set out to put the evolutionary thinking of Lamarck and his predecessors on a solid basis. Darwin was the one who proposed the first correct mechanism for evolutionary processes but he was not aware of Mendelian genetics and so his theory was partly rejected because of the diminishing variability in successive generations.

Nevertheless the theory of Darwin combined with Mendels genetic theory started to blossom in the unified theory of mathematical biology. Everyone in the field was concerned with fitness, reproductive success, mutation and selection, i.e. the evolution of populations under (sexual) reproduction.

Josef Hofbauer, Karl Sigmund and Peter Taylor [**5**] introduced and intensively explored the replicator equation, fundamental to evolutionary game theory. In its simplest form, the net growth rate of individuals using a strategy $R_i$ is given by

$$\dot{x}_i = x_i((Ax)_i - xAx)$$

where $x_i$ is the relative frequency of individually using strategy $R_i$, $A = a_{ij}$ is the $n \times n$ payoff matrix, $(Ax)_i$ is the payoff that a strategy

$R_i$ player receives and $xAx$ is the average payoff of the population. The basic interpretation was a biological one, payoff was interpreted as fitness, respectively reproductive success for each individual and the offspring of each individual inherits the strategy used by her single parent.

Thus so far we have considered information transfer in genetic terms (vertically, i.e. from the parent generation to the offspring) only but in human societies successful strategies are much more likely adopted through imitation (horizontally, we copy individuals, with successful strategies, of the same generation). Recently some researchers pointed out that our ability to imitate others is unique among animals and has set up for a new form of evolution. Our societies, the different cultures, the ability to acquire language are all based on our ability to imitate any form of behaviour. And human beings are masters in imitating others [**1**].

Nowak [**7**] thinks that language is the fundamental difference that sets us apart from all other animal species, but maybe there is a deeper underlying property of which only the human species is capable of imitation. We are so accustomed to imitation that we seem to have lost our sense of how complex this phenomenon is and how difficult it is for any other animal species to imitate at all. Although there are simple forms of imitation among animals, they are purely task specific, like a bird imitating a melody.

Maynard Smith and Szathmary [**6**] consider the evolution of language as one of the major transitions of life. One can easily say that it is the biggest invention of the last six hundred million years, comparable in significance only to the origin of life, the first bacteria, the first cells and the evolution of complex multicellularity. But why is imitation in any form and specially considering language so important to us [**1**]? Genetic information transfer is a rather slow process, while learning through imitation allows for a much quicker adaptation to new strategic concepts and gains in fitness. Dawkins proposed a so called memetic theory, in which ideas are changing and evolving. He calls a (cultural) information unit a meme.

So maybe the initial ability to imitate set up for a first step for the evolution of language. Selection might favor individuals who adapt quickly to new strategies, thus setting an environment in which good imitation skills, genetically encoded became better and better. It seems that language did not evolve as a by-product of a big brain, but that the cognitive demands for imitating and later on improving language might have provided a strong selection pressure for increased brain size.

The ability to imitate seems to be the most important aspect of human evolution and we will study it in some very simplistic models here.

CHAPTER 2

# Imitation Dynamics

We give a very general ansatz as in Hofbauer and Sigmund [**5**] how to model imitation processes. Consider a large population of individuals, called players, playing a symmetric $n \times n$ game with payoff matrix $A$, i.e. the players may choose from a set of $n$ different pure strategies $R_1$ to $R_n$. The frequency of players using strategy $R_i$ at time $t$ is given by $x_i(t)$. Thus the state of the population, for every moment $t$ is given by a point on the simplex $x(t) \in S_n$. The payoff $(Ax)_i$ of a player using strategy $R_i$ is given by $\sum a_{ij}x_j$ and the average payoff in the population is $xAx$.

To model the imitation process we randomly and independently choose two individuals from time to time. Player 1 has the opportunity to change his strategy and Player 2 is used as a model which may be imitated. Player 1 will adopt the strategy of Player 2 with a certain probability, depending on the payoffs of both players. A general ansatz is given by an input-output model of the form

$$\dot{x}_i = x_i \sum_j [f_{ji}(x) - f_{ij}(x)]x_j \tag{1}$$

where $f_{ij}$ denotes the switching rate from Player 1 using strategy $R_i$ to Player 2 using $R_j$. The product $x_i x_j$ just represents the independent and random sampling of a Player 1 using strategy $R_i$ and a Player 2 using $R_j$. The flow in a small time interval $\Delta t$ from strategy $R_i$ to strategy $R_j$ is thus given by $x_i x_j f_{ij} \Delta t$.

We now take a closer look at the form of the switching rate, assuming that it depends on the current payoffs of both players, we have

$$f_{ij}(x) = f((Ax)_i, (Ax)_j) \tag{2}$$

If the switching rate is a function of the payoff difference, i.e. $f(u,v) = \rho(u-v)$ with monotonically increasing $\rho$, with $\psi(u) = \rho(u) - \rho(-u)$ we finally arrive at the dynamics

$$\dot{x}_i = x_i \sum_j x_j \psi((Ax)_i - (Ax)_j) \tag{3}$$

with $\psi$ being an odd and strictly increasing function. A particular model of interest arises for $\rho(u) = u_+^\alpha$ which yields $\psi(u) = |u|^\alpha \operatorname{sgn} u$. Two cases are of special interest. For the limit $\alpha \to 0$ one obtains a dynamic called "imitate the better". Under this dynamics players only change their strategy to strategies with a current higher payoff, i.e. $f(u,v) = 0$ if $u < v$ and $f(u,v) = 1$ if $u > v$. This dynamics is discontinuous on $S_n$. For $\alpha = 1$ we have a proportional imitation rule $\psi(u) = u$. Players imitate strategies that perform better, with a probability proportional to the expected gain. This case yields the well analyzed replicator dynamics.

Other imitation dynamics will not be discussed in this paper.

CHAPTER 3

# Bimatrix Games

## 1. A General Framework for Asymmetric Games under Imitation Dynamics

Bimatrix games are used to model asymmetric conflicts in evolutionary game theory. We will use these games to describe two populations interacting with each other. An underlying assumption is that individuals have only pairwise encounters with individuals from the opposite population and that there is only a finite number of pure strategies available.

We have to distinguish players in position 1 and in position 2. We assume that in position 1 a player has $n$ strategies, while in position 2 a player has $m$ strategies. $A$ and $B$ represent the payoff matrices for 1 and 2. So a 1 player (i.e. a player in position 1) using strategy $i$ against a 2 player using $j$ receives payoff $a_{ij}$, while the 2 player receives $b_{ji}$.

Mixed strategies for the 1 player are denoted by $p \in S_n$ and $q \in S_m$ for the 2 player respectively, where $S_n$ is the $n$-dimensional simplex. The corresponding payoffs are $pAq$ and $qBp$. A Nash equilibrium is a pair of strategies $(p, q) \in S_n \times S_m$ where $p$ is a best reply to $q$ and $q$ a best reply to $p$.

We consider two different populations. Let $x \in S_n$ and $y \in S_m$ denote the frequencies of the strategies for the players in position 1 and 2 respectively. The differential equation of asymmetric games under imitation dynamics is given by

$$\dot{x}_i = \sum_j x_j \rho_{ji} - x_i \sum_j \rho_{ij} \tag{4}$$

$$\dot{y}_i = \sum_j y_j \tilde{\rho_{ji}} - y_i \sum_j \tilde{\rho_{ij}} \tag{5}$$

with $\rho_{ij} = x_j[(Ay)_j - (Ay)_i]_+^\alpha$ and $\tilde{\rho_{ij}} = y_j[(Bx)_j - (By)_i]_+^\alpha$.

## 2. The Case of Two Strategies per Player

We now take a look at the case $n = m = 2$.

We can still use the fact that like in the symmetric case adding a constant to every column of A and B does not change the dynamics on $S_n \times S_m$. The dynamics of these games only depends on the relative payoffs. So we prove the following proposition

Proposition 1. *Adding a constant to a column of the payoff matrix does not change the dynamics of the system.*

Proof. It suffices to prove that adding a constant to a column of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ does not change the equations. We add a constant $K$ to the first column of $A$ and get $\tilde{A} = \begin{bmatrix} a+K & b \\ c+K & d \end{bmatrix}$. The payoff for strategy 1 is given by $\tilde{a}_1 = (a+K)y_1 + by_2$ and $\tilde{a}_2 = (c+K)y_1 + dy_2$ for strategy 2. The payoff difference $\tilde{a}_1 - \tilde{a}_2 = (a + K - (c + K))y_1 + (b - d)y_2 = (a - c)y_1 + (b - d)y_2$ is equal to the payoff difference $a_1 - a_2$ of $A$. $\square$

So without loss of generality, we can set the diagonal terms equal to zero. Then the corresponding payoff matrices are

$$A = \begin{bmatrix} 0 & a_{12} \\ a_{21} & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & b_{12} \\ b_{21} & 0 \end{bmatrix} \tag{6}$$

Setting $x = x_1, x_2 = 1 - x, y = y_1$ and $y_2 = 1 - y$ we obtain a two dimensional system of differential equations on the square $Q = [0, 1] \times [0, 1] = S_2 \times S_2$. To further simplify notation we use the

Definition 1. $\{u\}^\alpha := |u|^\alpha \operatorname{sgn} u$

and get

$$\begin{aligned} \dot{x} &= x(1-x)\,\mathrm{sgn}\,[a_{12}-(a_{12}+a_{21})y]|a_{12}-(a_{12}+a_{21})y|^{\alpha} \\ &= x(1-x)\{a_{12}-(a_{12}+a_{21})y\}^{\alpha} \\ \dot{y} &= y(1-y)\,\mathrm{sgn}\,[b_{12}-(b_{12}+b_{21})x]|b_{12}-(b_{12}+b_{21})x|^{\alpha} \\ &= y(1-y)\{b_{12}-(b_{12}+b_{21})x\}^{\alpha} \end{aligned} \tag{7}$$

We further simplify this system of equations by multiplying with a so called Dulac function $B(x,y)$. This does not affect the orbits as it is only a change in velocity. We choose $B(x,y)$ to be

$$B(x,y) = \frac{1}{xy(1-x)(1-y)} \tag{8}$$

The modified system is

$$\begin{aligned} \dot{x} &= \frac{\{a_{12}-(a_{12}+a_{21})y\}^{\alpha}}{y(1-y)} = f(y) \\ \dot{y} &= \frac{\{b_{12}-(b_{12}+b_{21})x\}^{\alpha}}{x(1-x)} = g(x) \end{aligned} \tag{9}$$

We will encounter the expressions $a_{12}-(a_{12}+a_{21})y \geq 0 \iff y \leq \frac{a_{12}}{a_{12}+a_{21}}$frequently. To simplify notation we use

Definition 2. $\frac{a_{12}}{a_{12}+a_{21}} = c$ *and* $\frac{b_{12}}{b_{12}+b_{21}} = k$.

We explicitly mention here that $f(c) = g(k) = 0$. We state our first result about this system as a

Proposition 2. *For $a_{12}a_{21} \leq 0$ the sign of $\dot{x}$ does not change in intQ and so one strategy of player 1 dominates the other one. This is equally alike for player 2 if $b_{12}b_{21} \leq 0$. The phase portrait is given by figure 3.3.*

Proof. We rewrite $a_{12}-(a_{12}+a_{21})y$ as

$$a_{12}(1-y) - a_{21}y \tag{10}$$

. We have $(1-y) \geq 0$, $y \geq 0$ and $\mathrm{sgn}[a_{12}] = \mathrm{sgn}[-a_{21}]$, therefore the sign of $a_{12}-(a_{12}+a_{21})y$ does not change in *intQ*. $\square$

For $a_{12}a_{21} > 0$ and $b_{12}b_{21} > 0$ we have an interior equilibrium $E = (\frac{b_{12}}{b_{12}+b_{21}}, \frac{a_{12}}{a_{12}+a_{21}}) = (k, c)$. The dynamic outcome of the system depends on the signs of $a_{12}, a_{21}, b_{12}$ and $b_{21}$. We have to distinguish 4 cases, which we can group into two classes.

$$Class \quad 1 \quad \begin{cases} a_{12}, a_{21} > 0 & \text{and} \quad b_{12}, b_{21} > 0 \\ a_{12}, a_{21} < 0 & \text{and} \quad b_{12}, b_{21} < 0 \end{cases}$$

$$Class \quad 2 \quad \begin{cases} a_{12}, a_{21} > 0 & \text{and} \quad b_{12}, b_{21} < 0 \\ a_{12}, a_{21} < 0 & \text{and} \quad b_{12}, b_{21} > 0 \end{cases}$$

In Class 1 the signs of $a_{ij}$ and $b_{ij}$ are identical, whereas in Class 2 the signs of $a_{ij}$ are exactly opposite to the signs of $b_{ij}$. In both of these classes we have the system being split up into 4 quadrants, separated by the lines $y = c$ and $x = k$. In the case of $\alpha = 0$ the system (9) is discontinuous at the connection point, for $0 < \alpha < 1$ the system is continuous but not differentiable at the connection point and for $\alpha \geq 1$ the system is at least $C^1$. We start with some definitions

Definition 3. *We explicitly name the regions $\Delta_i$*

$$\Delta_1 = \{(x, y) \in \mathbb{R}^2 | 0 < x < k, 0 < y < c\}$$
$$\Delta_2 = \{(x, y) \in \mathbb{R}^2 | k < x < 1, 0 < y < c\}$$
$$\Delta_3 = \{(x, y) \in \mathbb{R}^2 | k < x < 1, c < y < 1\}$$
$$\Delta_4 = \{(x, y) \in \mathbb{R}^2 | 0 < x < k, c < y < 1\}$$

Definition 4. *And the corresponding half rays $\delta_i$ separating these regions*

$$\delta_1 = \{(x, y) \in \mathbb{R}^2 | 0 < x < k, y = c\}$$
$$\delta_2 = \{(x, y) \in \mathbb{R}^2 | x = k, 0 < y < c\}$$
$$\delta_3 = \{(x, y) \in \mathbb{R}^2 | k < x < 1, y = c\}$$
$$\delta_4 = \{(x, y) \in \mathbb{R}^2 | x = k, c < y < 1\}$$

*So the half ray $\delta_1$ is above the region $\Delta_1$ and the other half rays are orientated counter clockwise around $F$.*

Before we state our final theorem of this chapter we abbreviate the integrals of the vector field defined in

DEFINITION 5. *For $y \in (0,1)$ we define*

$$F(y) = \int_c^y f(s)\mathrm{d}s = \int_c^y \frac{\{a_{12} - (a_{12} + a_{21})s\}^\alpha}{s(1-s)}\mathrm{d}s \tag{11}$$

$$G(x) = -\int_k^x g(t)\mathrm{d}t = -\int_k^x \frac{\{b_{12} - (b_{12} + b_{21})t\}^\alpha}{t(1-t)}\mathrm{d}t \tag{12}$$

*Thus $F(y)$ is a continuous function $F : (0,1) \to \mathbb{R}$ with $F(c) = 0$. It is an injection if restricted to $(0,c]$ or $[c,1)$. The same properties are valid for $G(x)$ on the intervals $(0,k]$ and $[k,1)$, with $G(k) = 0$. We explicitly name the functions*

$$F^1 := F|_{(0,c]} \quad F^2 := F|_{[c,1)} \tag{13}$$

$$G^1 := G|_{(0,k]} \quad G^2 := G|_{[k,1)} \tag{14}$$

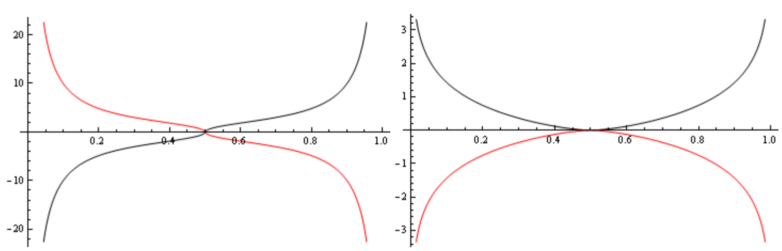


FIGURE 3.1. Sample plots of $f(y) : [0,1] \to \mathbb{R}$ and $F(y) : (0,1) \to (0,\infty)$ or $(-\infty,0)$ for Class 1 and 2

THEOREM 1. *For Class 1 the interior fixed point $E$ is a saddle as shown in figure 3.5 and almost all orbits converge to one or the other of two opposite corners of $Q$. For Class 2 all orbits in int$Q$ are periodic orbits surrounding the fixed point $F$, see figure 3.6. In both cases, the system is Hamiltonian in int$Q$, that means it can be written as*

$$\dot{x} = \frac{\partial H}{\partial y} \quad \dot{y} = -\frac{\partial H}{\partial x} \tag{15}$$

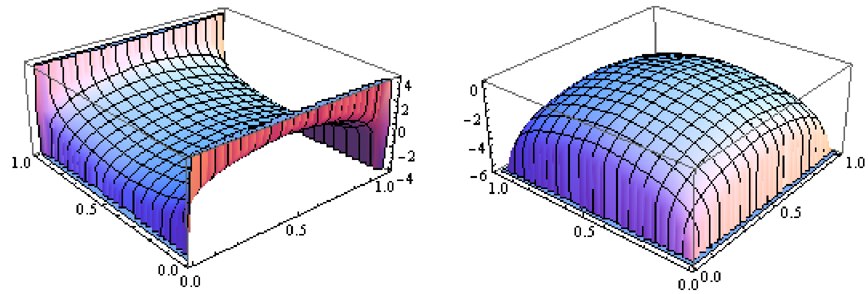


FIGURE 3.2. The Hamiltonians of *Class* 1 and *Class* 2

PROOF. For both systems we have piecewise defined Hamiltonians in the regions of $intQ$ with Hamiltonian

$$H(x,y) = \int f(y)\mathrm{d}y + \int g(x)\mathrm{d}x = F(y) + G(x) \tag{16}$$

We have to be careful because the Hamiltonian is not $C^\infty$. For $\alpha > 0$ the Hamiltonian is $C^1$ and for $\alpha = 0$ it is continuous. We split the proof into two pieces, starting with
Class 1: W.l.o.g. we choose $a_{12}, a_{21} > 0$ and $b_{12}, b_{21} > 0$. To prove the dynamic behaviour of Class 1, we use geometric concepts. In region $\Delta_1$ we have $a_1 - a_2 > 0 \iff y \leq c$ and $b_1 - b_2 > 0 \iff x \leq k$, i.e. the vector field is pointing up and to the right and exactly one orbit converges to the fixed point $F$. In region $\Delta_3$ the reverse condition applies which means that the vector field points down and to the left. Again exactly one orbit in $\Delta_3$ converges to the fixed point $F$. Whereas in region $\Delta_2$ and $\Delta_4$ the vector field points away from the fixed point $F$, thus we have shown that the fixed point $F$ is a saddle and almost all orbits converge to either $P(0,1)$ or $P(1,0)$.

It is easy to see that only one orbit in $\Delta_1$ and $\Delta_3$ respectively converges to the equilibrium $F$. As the system is Hamiltonian the value of the invariant of motion in region $\Delta_1$ at the fixed point is given by $H^{\Delta_1}(k,c)$ and there is only one solution curve that reaches the fixed point keeping this constant value.

For if we assume there are two orbits $(x, y_1(x))$ and $(x, y_2(x))$ with

$$H^{\Delta_1}(x, y_1(x)) = H^{\Delta_1}(k,c) = H^{\Delta_1}(x, y_2(x)) \tag{17}$$

then

$$F_1(x) + G_1(y_1(x)) = F_1(x) + G_1(y_2(x)) \tag{18}$$

which is equivalent to $G_1(y_1(x)) = G_1(y_2(x))$ and since $G(y)$ is a continuous bijection in $\Delta_1$ we have

$$G_1(y_1(x)) = G_1(y_2(x)) \iff y_1(x) = y_2(x) \tag{19}$$

which contradicts our assumption that we have two different orbits and there is only one orbit in $\Delta_1$ converging to the saddle point $F$.

The same reasoning is valid for $a_{12}, a_{21} < 0$ and $b_{12}, b_{21} < 0$ the phase portrait is just the time reversal of the former case, i.e. almost all orbits converge to either $P(0,0)$ or $P(1,1)$.

Class 2: We will again only prove the result w.l.o.g. for $a_{12}, a_{21} > 0$ and $b_{12}, b_{21} < 0$. Our Hamiltonian is defined piecewise for each of the 4 quadrants and we are looking for a piecewise, on these quadrants defined solution that gives us a closed orbit. The Hamiltonian for the regions is denoted by $H^{\Delta_i}(x,y)$. Starting with the Hamiltonian for region $\Delta_1$ we get

$$H^{\Delta_1}(x,y) = F^1(y) + G^2(x) \quad H^{\Delta_2}(x,y) = F^1(y) + G^1(x)$$
$$H^{\Delta_3}(x,y) = F^2(y) + G^1(x) \quad H^{\Delta_4}(x,y) = F^2(y) + G^2(x)$$

Since $\dot{H}^{\Delta_i}(x,y) = \frac{\partial H^{\Delta_i}}{\partial x}\dot{x} + \frac{\partial H^{\Delta_i}}{\partial y}\dot{y} = -\dot{y}\dot{x} + \dot{y}\dot{x} \equiv 0$, the Hamiltonian is an invariant of motion. We will now show that all orbits are closed using a Poincare return map. We start by determining the value of the constant of motion $H^{\Delta_1}(x,y)$ on a point $P_1(r,c)$ on the half ray $\delta_1$ for any given value $r = x \in [0,k]$, i.e. we calculate $H^{\Delta_1}(r,c)$.

$$H^{\Delta_1}_{\delta_1}(r,c) = F^1(c) + G^1(r)$$

We follow the solution until it reaches $\delta_2$. The value of the invariant of motion for a point $P_2(k,s)$ on $\delta_2$ is given by

$$H^{\Delta_1}_{\delta_2}(k,s) = F^1(s) + G^1(k) = H^{\Delta_1}_{\delta_1}(r,c)$$

which is equal to $H_{\delta_1}^{\Delta_1}(r,c)$ as the value of the solution is constant along every orbit. Solving this system for the free parameter F(s) gives

$$F^1(s) = F^1(c) + G^1(r) - G^1(k).$$

The connection point $(k,s)$ is the new starting point to determine the value of the Hamiltonian in region $\Delta_2$ on the half ray $\delta_2$. Using the same point $P_2(k,s)$ for $H_{\delta_2}^{\Delta_2}$ and following the solution until it reaches a point $P_3(t,c)$ on $\delta_3$ yields

$$\begin{aligned} H_{\delta_2}^{\Delta_2}(k,s) &= F^1(s) + G^2(k) = \\ &= F^1(c) + G^1(r) - G^1(k) + G^2(k) \\ H_{\delta_3}^{\Delta_2}(t,c) &= F^1(c) + G^2(t) \end{aligned}$$

We solve this equation for the free parameter $G^2(t)$ and get

$$G^2(t) = F^1(s) + G^2(k) - F^1(c) = G^1(r) - G^1(k) + G^2(k)$$

Repeating this procedure we get a point $P_4(k,u)$ on $\delta_4$

$$\begin{aligned} H_{\delta_3}^{\Delta_3}(t,c) &= F^2(c) + G^2(t) \\ H_{\delta_4}^{\Delta_3}(k,u) &= F^2(u) + G^2(k) \end{aligned}$$

Solving the system for $F^2(u)$ yields

$$F^2(u) = F^2(c) + G^2(t) - G^2(k) = F^2(c) + G^1(r) - G^1(k)$$

In the last step we move on from the point $P_4(k,u)$ on the line $\delta_4$ to a point $P_1(v,c)$ on our starting line $\delta_1$.

$$\begin{aligned} H_{\delta_4}^{\Delta_4}(k,u) &= F^2(u) + G^1(k) \\ H_{\delta_1}^{\Delta_4}(v,c) &= F^2(c) + G^1(v) \end{aligned}$$

Solving for the free parameter $G^1(v)$

$$G^1(v) = F^2(u) + G^1(k) - F^2(c) = G^1(r)$$

The value of $G^1(v)$ after following the solution around the fixed point is the same value we calculated for $G^1(r)$ at our starting point, which implies that we arrived at our starting point, i.e. $P_1(r,c) = P_1(v,c)$.

This proofs that every orbit for the Class 2 games is closed under this imitation dynamics. □

On the following four pages are the phase portraits of the dynamics discussed above. They are grouped into series of 4 portraits, always in the following order a) Imitate the Better $\alpha = 0$, b) $\alpha = 1/2$, c) Replicator Equation $\alpha = 1$ and d) $\alpha = 2$.

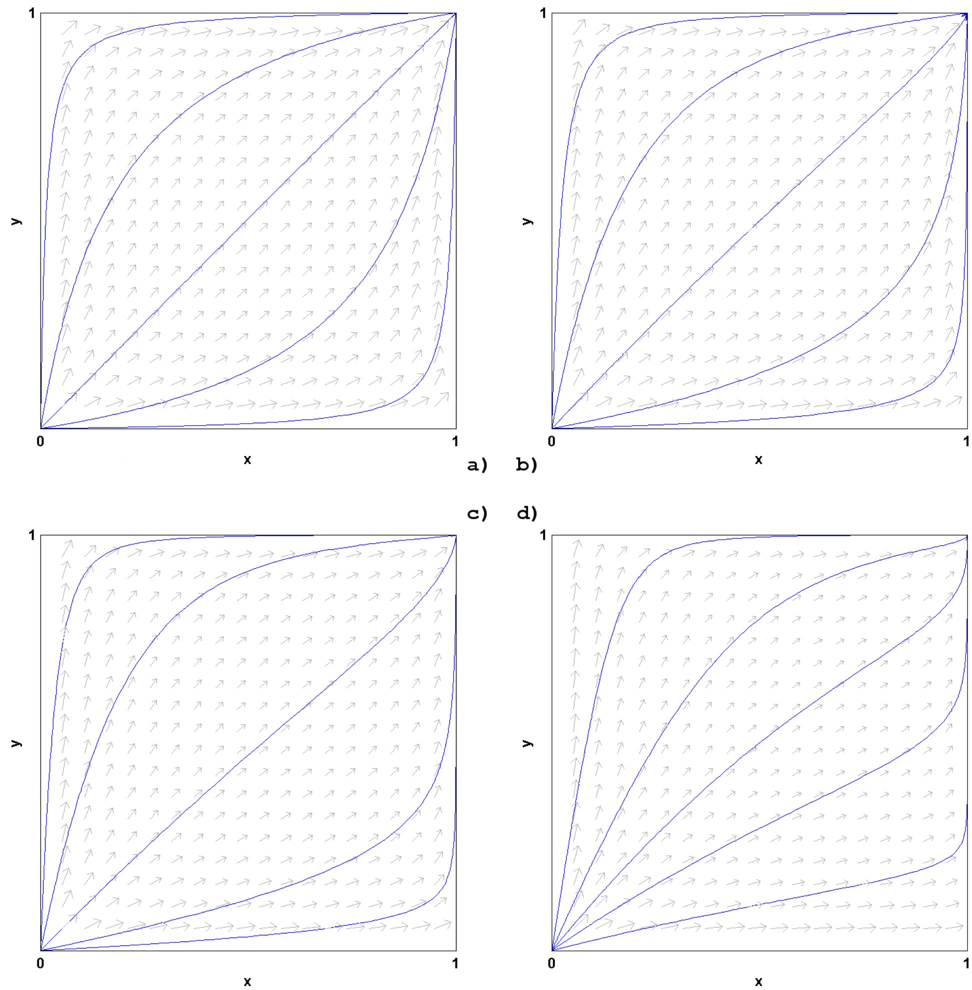


FIGURE 3.3. Direct: $a_{12} = 1$, $a_{21} = -\frac{1}{2}$, $b_{12} = 1$ and $b_{21} = -\frac{1}{3}$

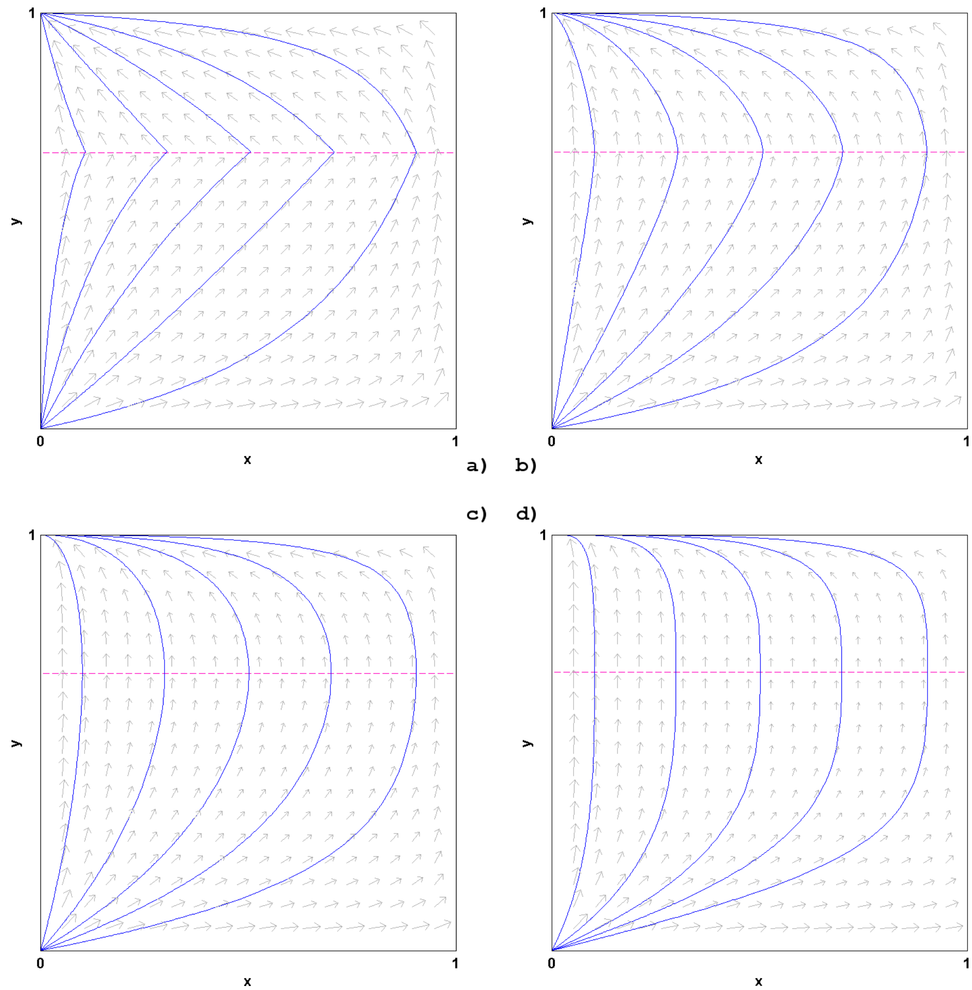


FIGURE 3.4. Indirect: $a_{12} = 1$, $a_{21} = \frac{1}{2}$, $b_{12} = 1$ and $b_{21} = -\frac{1}{3}$

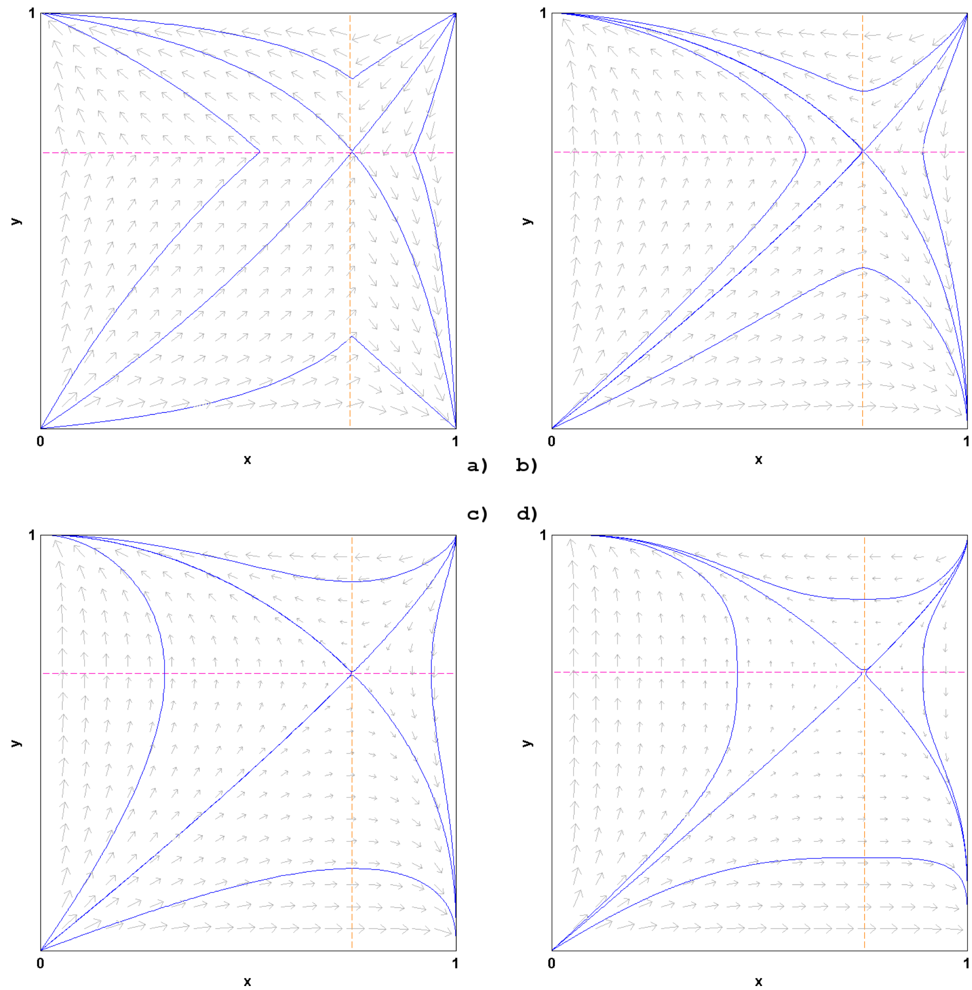


FIGURE 3.5. *Class* 1, $F$ is a saddle point with $a_{12} = 1$, $a_{21} = \frac{1}{2}$, $b_{12} = -1$ and $b_{21} = -\frac{1}{2}$

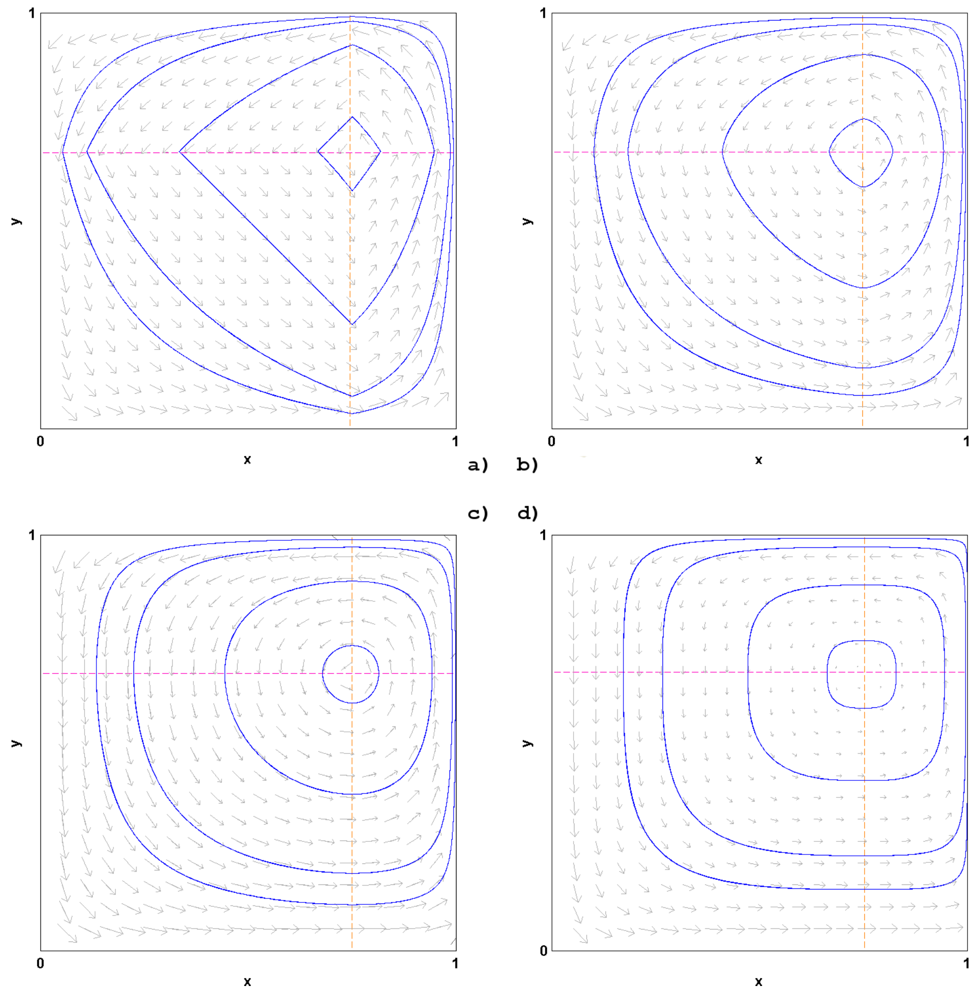


FIGURE 3.6. *Class* 2, closed orbits around $F$ with $a_{12} = 1$, $a_{21} = \frac{1}{2}$, $b_{12} = 1$ and $b_{21} = \frac{1}{3}$

## 3. Degenerate $2 \times 2$ Games

There are four cases of degenerate $2 \times 2$ bimatrix games with one degenerate strategy (see [**2**], [**8**]) and the main task of this part is to explain the difference in the dynamic outcome of the "imitate the better" dynamics, i.e. $\alpha = 0$ and the dynamics with $\alpha > 0$ for all the games. We will ignore games of higher degeneracy because their dynamic behaviour is less interesting. For Game 1 and 2 we have $b_{12} = 0$ and for Game 3 (Centipede) and Game 4 (Chain-Store) we have $b_{21} = 0$, these cases are represented by the modified equations

$$\begin{aligned} \dot{x} &= \frac{\{a_{12} - (a_{12} + a_{21})y\}^{\alpha}}{y(1-y)} \\ \dot{y} &= \frac{\{-b_{21}x\}^{\alpha}}{x(1-x)} \quad \text{or} \\ \dot{y} &= \frac{\{b_{12}(1-x)\}^{\alpha}}{x(1-x)} \end{aligned} \tag{20}$$

The first equation is valid for both systems and the second and third one describe the degeneration, where the second equation belongs to Game 1 and 2 and the third equation to Game 3 and 4.

We reintroduce the original forms of the equations because some properties of the system are easier to understand and analyze in these forms.

$$\begin{aligned} \dot{x} &= x(1-x)\{a_{12} - (a_{12} + a_{21})y\}^{\alpha} = f(x,y) \\ \dot{y} &= y(1-y)\{-b_{21}x\}^{\alpha} = g_1(x,y) \quad \text{or} \\ \dot{y} &= y(1-y)\{b_{12}(1-x)\}^{\alpha} = g_2(x,y) \end{aligned} \tag{21}$$

As we will frequently use Jacobians in the analysis of these games, $J_i$ represents the Jacobian of Game $i$. The partial derivatives of $f(x,y)$,

$g_1(x,y)$ and $g_2(x,y)$ are given by

$$\frac{\partial f(x,y)}{\partial x} = (1-2x)\{a_{12}-(a_{12}+a_{21})y\}^{\alpha} \tag{22}$$

$$\frac{\partial f(x,y)}{\partial y} = -(a_{12}+a_{21})x(1-x)\alpha|(a_{12}-(a_{12}+a_{21})y)|^{\alpha-1}$$

$$\frac{\partial g_1(x,y)}{\partial x} = -b_{21}y(1-y)\alpha\{-b_{21}x\}^{\alpha-1} \tag{23}$$

$$\frac{\partial g_1(x,y)}{\partial y} = (1-2y)\{-b_{21}x\}^{\alpha} \tag{24}$$

$$\frac{\partial g_2(x,y)}{\partial x} = -b_{12}y(1-y)\alpha\{b_{12}(1-x)\}^{\alpha-1} \tag{25}$$

$$\frac{\partial g_2(x,y)}{\partial y} = (1-2y)\{b_{12}(1-x)\}^{\alpha} \tag{26}$$

**Game 1.** We start with Game 1, which is given by $a_{12} > 0$, $a_{21} < 0$, $b_{21} < 0$ and $b_{12} = 0$. Because of Proposition 2 the sign of $\dot{x}$ and $\dot{y}$ does not change in $intQ$. For these parameter values we have $\{a_{12} - (a_{12} + a_{21})y\}^{\alpha} = (a_{12} - (a_{12} + a_{21})y)^{\alpha}$ and we arrive at the Jacobian for $(1,1)$ which is

$$J_1(1,1) = \begin{bmatrix} -(-a_{21})^{\alpha} & 0 \\ 0 & -(-b_{21})^{\alpha} \end{bmatrix}$$

and we have two negative eigenvalues which implies that $(1,1)$ is asymptotically stable.

**Game 2.** For Game 2, given by $a_{12} < 0$, $a_{21} > 0$, $b_{21} < 0$ and $b_{12} = 0$ the same reasoning applies and signs do not change in $intQ$. So we have $\{a_{12} - (a_{12} + a_{21})y\}^{\alpha} = -(-a_{12} + (a_{12} + a_{21})y)^{\alpha}$ and the corresponding Jacobian at a point $(0, \hat{y})$ on the y-axis is given by

$$J_2(0,\hat{y}) = \begin{bmatrix} -(-a_{12} + (a_{12}+a_{21})\hat{y})^{\alpha} & 0 \\ 0 & 0 \end{bmatrix}$$

We have one negative eigenvalue and one eigenvalue equal zero at each equilibrium point $(0, \hat{y})$. The Hamiltonian is $H(x, y) = F(y) + G(x)$

with bijections $F:(0,1)\to\mathbb{R}$ and $G:[0,1)\to[0,\infty)$:

$$F(y)=\int_d^y f(s)\mathrm{d}s=-\int_d^y\frac{(-a_{12}+(a_{12}+a_{21})s)^\alpha}{s(1-s)}\mathrm{d}s$$
$$G(x)=-\int_0^x g(t)\mathrm{d}t=-\int_0^x\frac{(-b_{21}t)^\alpha}{t(1-t)}\mathrm{d}t$$

with $d\in(0,1)$ arbitrary. $G(x)$ is continuous for $x=0$ because integrals of the form $\int_0^x\frac{t^\alpha}{t(1-t)}\mathrm{d}t$ are convergent for $\alpha>0$. Therefore we have $H(0,y)=F(y)+G(0)=F(y)+0=\mathit{constant}$ and we are able to solve $F(y)=C$ explicitly for every value of $C\in\mathbb{R}$. This implies that there is an orbit converging to every point on the degenerate edge $(0,y)$ and we have a continuum of stable equilibria there. The functions $F(y)$ and $G(x)$ for an arbitrary value of $\alpha$ are given by hypergeometric series and will not be explicitly mentioned here!

**Game 3, the Centipede Game.** For the Centipede Game we have $a_{12}<0$, $a_{21}<0$ and $b_{12}>0$. At the line $(x,c)$ parallel to the x-axis $\dot{x}$ changes sign for the Centipede and Chain-Store Game. For $x=1$ and $\hat{y}>c$, we have $\{a_{12}-(a_{12}+a_{21})y\}^\alpha=(a_{12}-(a_{12}+a_{21})y)^\alpha$ again and therefore the Jacobian at $(1,\hat{y})$ is

$$J_3(1,\hat{y})=\begin{bmatrix}-(a_{12}-(a_{12}+a_{21})\hat{y})^\alpha & 0\\ 0 & 0\end{bmatrix}$$

We have one negative eigenvalue and one eigenvalue 0. At $(1,c)$ the Hamiltonian has a global minimum on $(0,1]\times(0,1)$, this implies that no orbit reaches or leaves $(1,c)$ and it is a stable equilibrium too. For $\hat{y}<c$ the sign of the matrix element $j_{11}$ of $J_3(1,\hat{y})$ changes and we have one positive eigenvalue and one eigenvalue equal 0.

Proceeding in the same way as in Game 2 one can show that there is an orbit leading to every point $(1,\hat{y})$ with $\hat{y}>c$ and one has a continuum of stable equilibria.

**Game 4, the Chain-Store Game.** For the Chain-Store Game we have $a_{12}<0$, $a_{21}<0$ and $b_{12}<0$. The Jacobian for $(1,\hat{y}>c)$ is equal to the Jacobian $J_3(1,\hat{y})$ of the Centipede game. So we have one negative eigenvalue and one equal 0 and a continuum of stable equilibria like in the Centipede Game, whereas the point $(1, c)$ is not stable for the Chain-Store Game, because there is one orbit leading to and one orbit going away from $(1, c)$. They both have $H(x, y) = H(1, c)$.

For $(0, 0)$, with $\{b_{12}\}^\alpha = -(-b_{12})^\alpha$ we arrive at

$$J_4(0,0) = \begin{bmatrix} (-a_{12})^\alpha & 0 \\ 0 & -(-b_{12})^\alpha \end{bmatrix}$$

with two negative eigenvalues and hence asymptotic stability for $(0, 0)$.

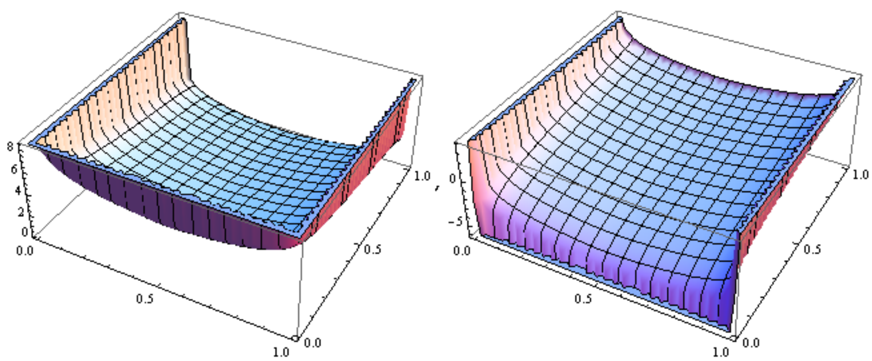


Figure 3.7. The Hamiltonians of the Centipede and Chain-Store Game

The difference in the phase portraits for $\alpha = 0$ and $\alpha > 0$ is due to the loss of degeneration in the limiting case $\alpha = 0$. The degenerated line becomes the two strategy single population replicator dynamics with no internal equilibrium.

**The Limiting Case $\alpha = 0$ in Degenerate Bimatrix Games.** The equations for $\alpha = 0$ become

$$\begin{aligned} \dot{x} &= x(1-x)\,\mathrm{sgn}[a_{12} - (a_{12} + a_{21})y] \\ \dot{y} &= y(1-y)\,\mathrm{sgn}[-b_{21}x] \quad \text{or} \\ \dot{y} &= y(1-y)\,\mathrm{sgn}[b_{12}(1-x)] \end{aligned} \tag{27}$$

In $intQ$ this is just the zero sum replicator dynamics. We will use the following matrices to describe the games

$$A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad B_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

For Game 1 this is just the zero sum game replicator dynamics with payoff matrices $A_1$ and $B_1$, thus the system for $x > 0$ is

$$\dot{x} = x(1-x)$$
$$\dot{y} = y(1-y)$$

but on the degenerate edge $x = 0$ we have $\operatorname{sgn}[-b_{21}x] = \operatorname{sgn}[0] = 0$ and $\dot{y} = y(1-y)\operatorname{sgn}[0] = 0$.
For Game 2 we have $\dot{x} = -x(1-x)$ and we arrive at the replicator equation with payoff matrices $A_2$ and $B_1$. For the behaviour on the degenerate edge see Game 1.
For the Centipede and Chain-Store Game we still have the line $(x, c)$ where $\dot{x}$ changes sign, i.e. for $y > c$ we have $\dot{x} = x(1-x)$ and therefore $A_1$ and for $y < c$ we have $\dot{x} = -x(1-x)$ with $A_2$

The games only differ in the sign of $\dot{y}$. For the Centipede Game we have $\dot{y} = y(1-y)$ and for the Chain-Store Game we arrive at $\dot{y} = -y(1-y)$. In the Centipede Game the dynamics for $y$ is equivalent to the replicator dynamics with $B_1$ and the Chain-Store corresponds to $B_2$. For both games the degenerate edge $x = 1$ has $\dot{y} = 0$ and we have a continuum of stable equilibria there. The signum function produces a discontinuity in the vector field at the degenerate edge. The behaviour is similar to that of the BNN or best response dynamics [**8**].

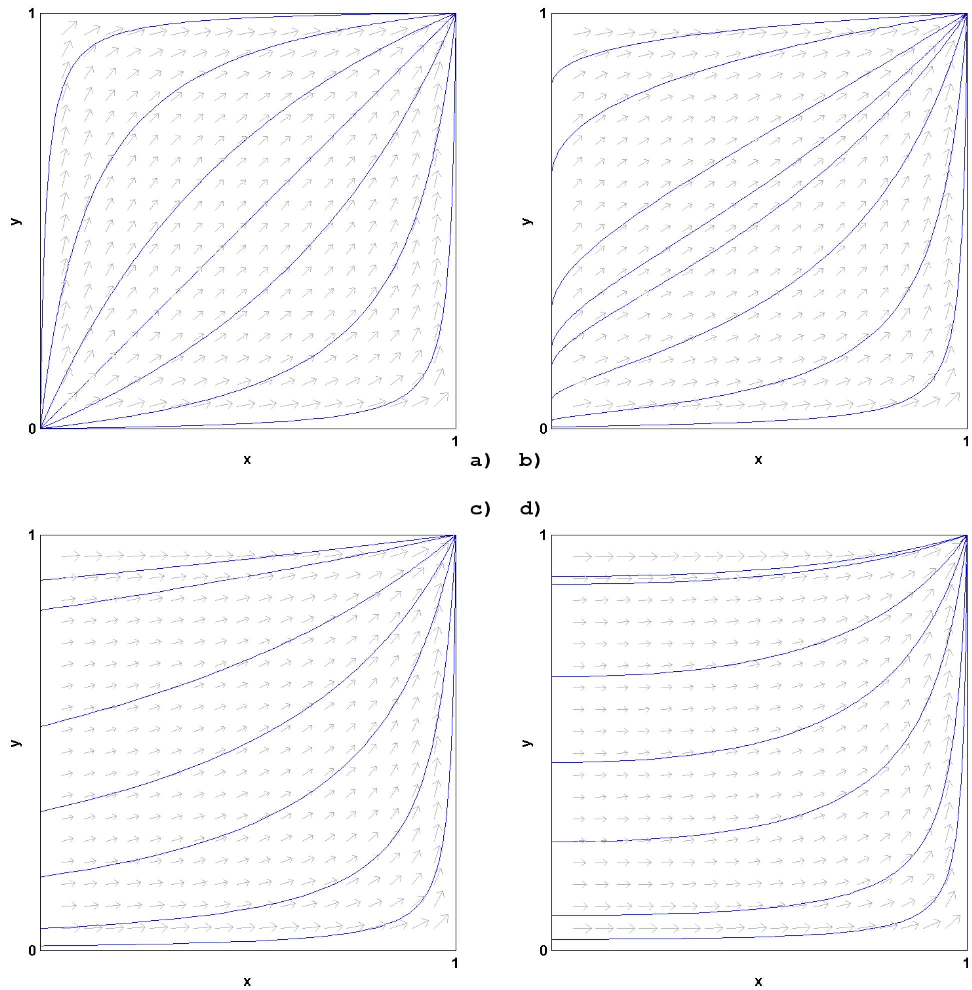


FIGURE 3.8. Game 1: $a_{12} = 1$, $a_{21} = -1$, $b_{12} = 0$ and $b_{21} = -1$

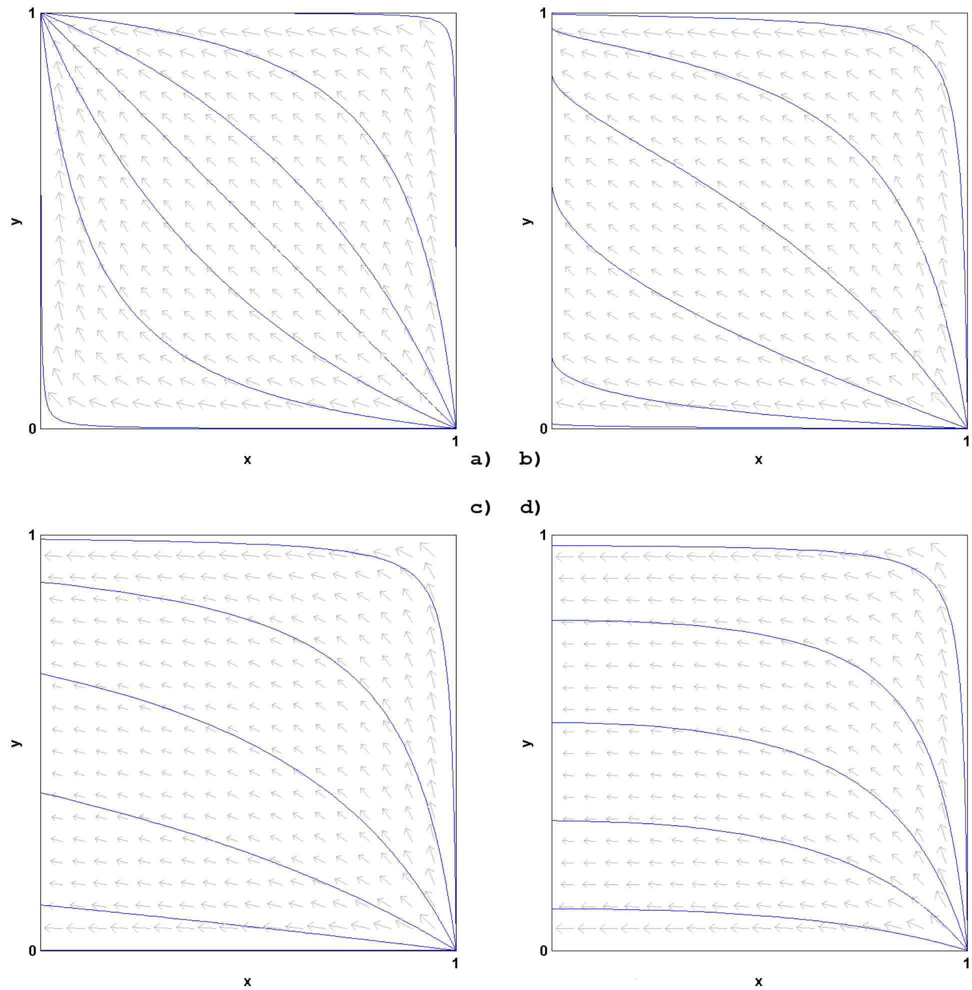


FIGURE 3.9. Game 2: $a_{12} = -1$, $a_{21} = 1$, $b_{12} = 0$ and $b_{21} = -1$

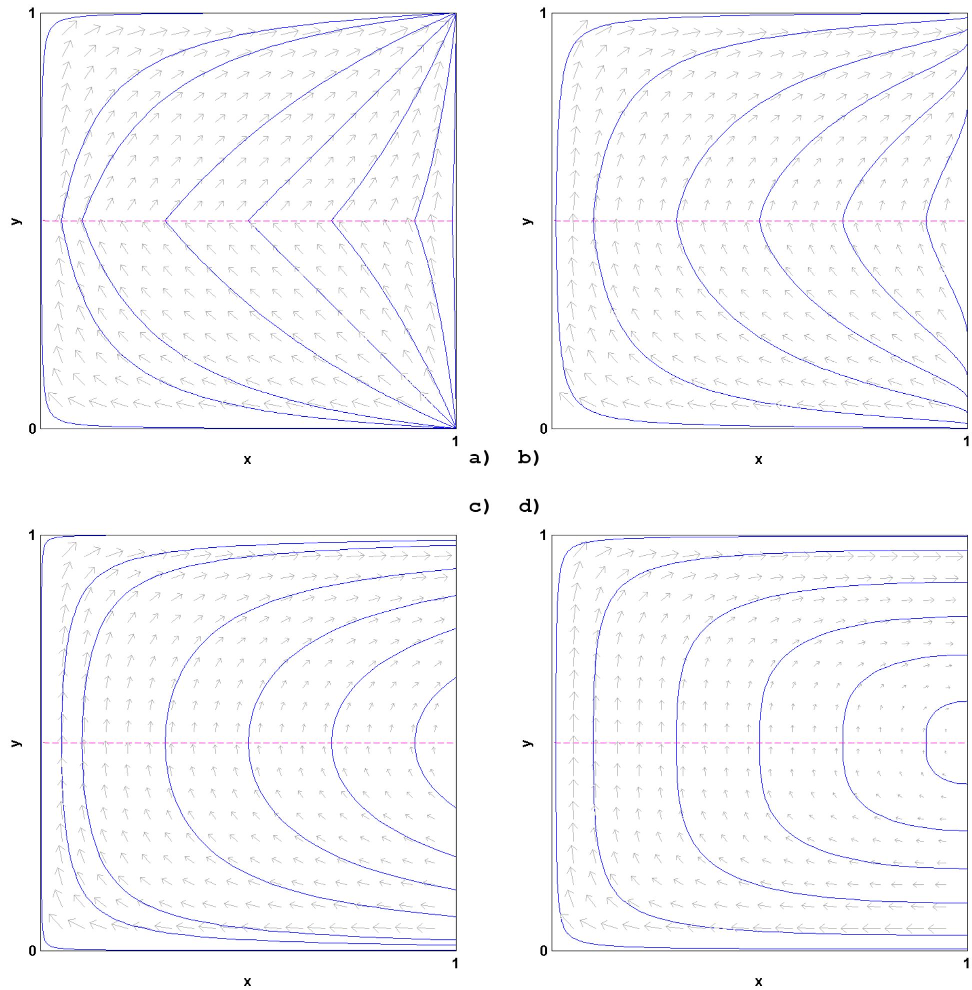


FIGURE 3.10. Centipede: $a_{12} = -1$, $a_{21} = -1$, $b_{12} = 1$ and $b_{21} = 0$

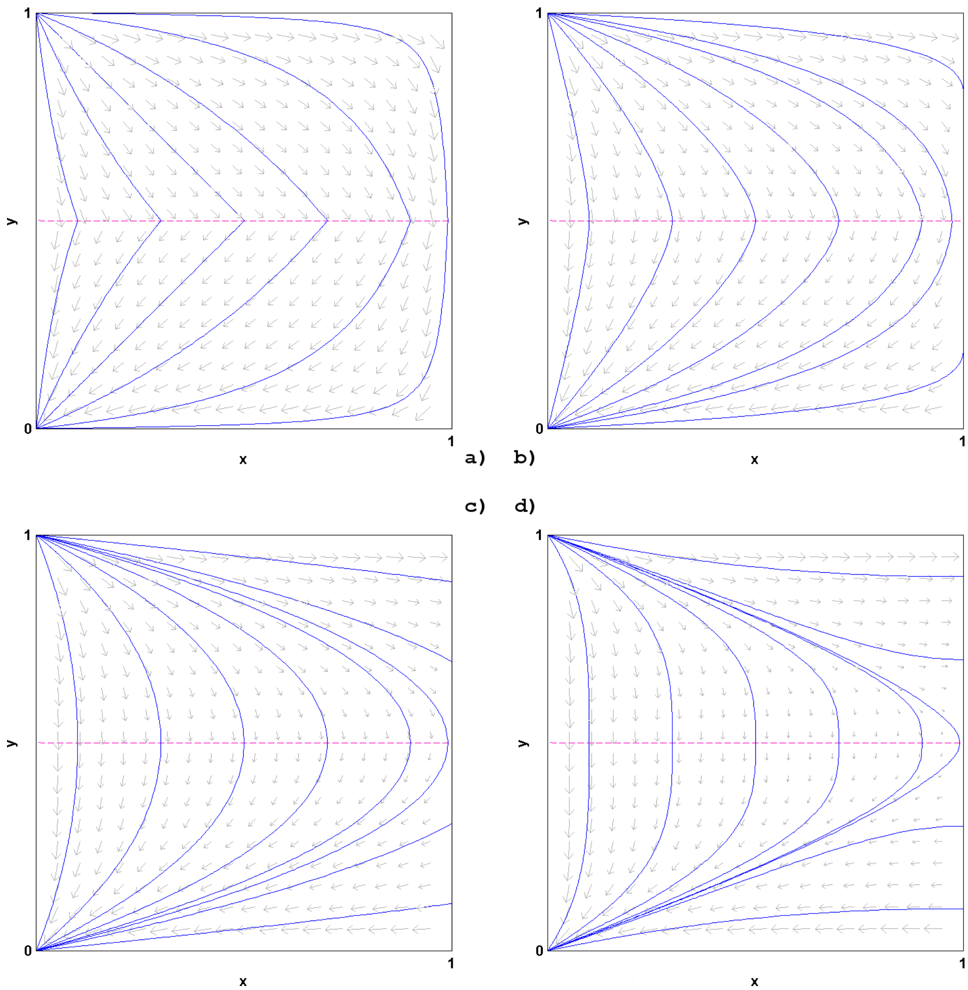


FIGURE 3.11. Chain-Store: $a_{12} = -1$, $a_{21} = -1$, $b_{12} = -1$ and $b_{21} = 0$

CHAPTER 4

# The Rock-Scissors-Paper Game

Generally a rock-scissors-paper game, short RSP game is characterized by having three different pure strategies. Each strategy is superior to one strategy and inferior to the remaining other one, i.e. strategy $R_1$ is beaten by $R_2$, which is beaten by $R_3$, which is finally beaten by $R_1$ again. In its normalized form this game is given by the payoff matrix

$$A = \begin{bmatrix} 0 & -a_2 & b_3 \\ b_1 & 0 & -a_3 \\ -a_1 & b_2 & 0 \end{bmatrix} \tag{28}$$

with $a_i, b_i > 0$. This game always admits an interior equilibrium $p$ which is given by

$$p = \frac{(b_2b_3 + b_2a_3 + a_3a_2, b_3b_1 + b_3a_1 + a_1a_3, b_1b_2 + b_1a_2 + a_1a_2)}{\Sigma}$$

with $\Sigma > 0$ such that $p \in S_3$. The only rest points on the boundary are the vertices. The behaviour of RSP under the replicator dynamics is well analyzed in Hofbauer and Sigmund [**5**]. The replicator dynamics is just a special case of the imitation dynamics we consider, as mentioned earlier in the text. We will show that some properties valid for the replicator dynamics are also valid for the imitation dynamics under consideration.

## 1. Zero Sum Games

In this section we will restrict our attention to zero sum games, where the gain of one player is the loss of the other player, i.e. $b_i = a_{i+1}$. We will show that these games admit closed orbits under imitation dynamics. The payoff matrix of a $3 \times 3$ zero sum game is

$$A = \begin{bmatrix} 0 & -a & b \\ a & 0 & -c \\ -b & c & 0 \end{bmatrix} \tag{29}$$

where $A$ is skew-symmetric and $a, b, c > 0$. Our system of differential equations looks like

$$\dot{x_i} = \sum_{j=1}^{3} x_j \rho_{ji} - x_i \sum_{j=1}^{3} \rho_{ij} \tag{30}$$

with $\rho_{ij} = x_j[(Ay)_j - (Ay)_i]_+^\alpha = x_j(a_j - a_i)_+^\alpha$. Explicitly written out this is

$$\begin{aligned} \dot{x_1} &= x_1(x_2\{a_1 - a_2\}^\alpha + x_3\{a_1 - a_3\}^\alpha) \\ \dot{x_2} &= x_2(x_1\{a_2 - a_1\}^\alpha + x_3\{a_2 - a_3\}^\alpha) \\ \dot{x_3} &= x_3(x_1\{a_3 - a_1\}^\alpha + x_2\{a_3 - a_2\}^\alpha) \end{aligned} \tag{31}$$

The payoffs are given by

$$a_1 = -ax_2 + bx_3 \quad a_2 = ax_1 - cx_3 \quad a_3 = -bx_1 + cx_2$$

differences are given by

$$\begin{aligned} a_1 - a_2 &= x_3(b + c) - a(x_1 + x_2) = \\ &= x_3(a + b + c) - a \\ a_2 - a_1 &= a - x_3(a + b + c) \end{aligned}$$

$$\begin{aligned} a_1 - a_3 &= b(x_1 + x_3) - x_2(a + c) = \\ &= b - x_2(a + b + c) \\ a_3 - a_1 &= x_2(a + b + c) - b \end{aligned}$$

$$
\begin{aligned}
a_2 - a_3 &= x_1(a+b) - c(x_2+x_3) = \\
&= x_1(a+b+c) - c \\
a_3 - a_2 &= c - x_1(a+b+c)
\end{aligned}
$$

After a change in velocity, with the Dulac function

$$B(x_1, x_2, x_3) = \frac{1}{x_1 x_2 x_3 (a+b+c)}$$

and setting $A = \frac{a}{a+b+c}$, $B = \frac{b}{a+b+c}$ and $C = \frac{c}{a+b+c}$ we finally arrive at

$$
\begin{aligned}
\dot{x_1} &= \frac{\{x_3 - A\}^\alpha}{x_3} - \frac{\{x_2 - B\}^\alpha}{x_2} = \\
&= h(x_3) - g(x_2) \\
\dot{x_2} &= \frac{\{x_1 - C\}^\alpha}{x_1} - \frac{\{x_3 - A\}^\alpha}{x_3} = \\
&= f(x_1) - h(x_3) \\
\dot{x_3} &= \frac{\{x_2 - B\}^\alpha}{x_2} - \frac{\{x_1 - C\}^\alpha}{x_1} = \\
&= g(x_2) - f(x_1)
\end{aligned}
\tag{32}
$$

We have three sign-functions invoked in this expression, which split up the simplex in six distinct regions. The change of direction in the vector field depends on the value of $\alpha$. The change for $\alpha = 0$ is discontinuous, getting to a continuous but not differentiable change for $0 < \alpha < 1$ and arriving at a continuously differentiable $C^1$ vector field for $\alpha \geq 1$. As in chapter 3 we state

Definition 6. *For $x_1, x_2, x_3 \in (0, 1]$ and $A + B + C = 1$ we have*

$$F(x_1) = \int_C^{x_1} f(s)\mathrm{d}s = \int_C^{x_1} \frac{\{s - C\}^\alpha}{s}\mathrm{d}s \tag{33}$$

$$G(x_2) = \int_B^{x_2} g(t)\mathrm{d}t = \int_B^{x_2} \frac{\{t - B\}^\alpha}{t}\mathrm{d}t \tag{34}$$

$$H(x_3) = \int_A^{x_3} h(u)\mathrm{d}u = \int_A^{x_3} \frac{\{u - A\}^\alpha}{u}\mathrm{d}u \tag{35}$$

*We have the continuous functions $F, G, H : (0, 1] \to \mathbb{R}$ with $F(C) = G(B) = H(A) = 0$, which is a global minimum for all three functions.*

*We also define restrictions of the functions to the domains*

$$\hat{F} := F|_{(0,C]} \quad \tilde{F} := F|_{[C,1]}$$
$$\hat{G} := G|_{(0,B]} \quad \tilde{G} := G|_{[B,1]}$$
$$\hat{H} := H|_{(0,A]} \quad \tilde{H} := H|_{[A,1]}$$

*where they are monotonic.*

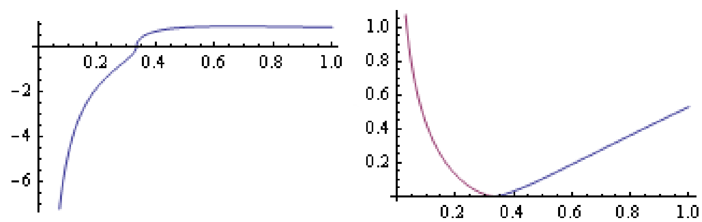


FIGURE 4.1. Sample plot of the function $f$ and its Integral $F$, with $C = 1/3$

The change $\dot{x}_i$ is independent of $x_i$ and the system is Hamiltonian in some sense, see [**4**].

$$V(x_1, x_2, x_3) = F(x_1) + G(x_2) + H(x_3)$$

defined on the interior of the simplex, respectively in each region (depending on $\alpha$) serves as a sort of Hamiltonian.

$$\begin{aligned}
\dot{V}(x_1, x_2, x_3) &= \frac{\partial V}{\partial x_1}\frac{\partial x_1}{\partial t} + \frac{\partial V}{\partial x_2}\frac{\partial x_2}{\partial t} + \frac{\partial V}{\partial x_3}\frac{\partial x_3}{\partial t} \\
&= f(x_1)\dot{x_1} + g(x_2)\dot{x_2} + h(x_3)\dot{x_3} = \\
&= f(x_1)(h(x_3) - g(x_2)) + \\
&+ g(x_2)(f(x_1) - h(x_3)) + \\
&+ h(x_3)(g(x_2) - f(x_1)) \equiv 0
\end{aligned}$$

As $F(C) = 0$, $G(B) = 0$ and $H(A) = 0$ have a global minimum in these points, the function $V(x_1, x_2, x_3)$ has a global minimum for $V(C, B, A)$.

Definition 7. *The lines*

$$l_1 = \{x \in S_3 : a_2(x) = a_3(x)\} = \{x \in S_3 : x_1 = C\}$$
$$l_2 = \{x \in S_3 : a_1(x) = a_3(x)\} = \{x \in S_3 : x_2 = B\}$$
$$l_3 = \{x \in S_3 : a_1(x) = a_2(x)\} = \{x \in S_3 : x_3 = A\}$$

*divide $S_3$ into six regions.*

Definition 8. *The regions are given by all permutations of the ordering of the payoffs*

$$A_1 = \{x \in S_3 : a_1 > a_2 > a_3\}$$
$$A_2 = \{x \in S_3 : a_2 > a_1 > a_3\}$$
$$A_3 = \{x \in S_3 : a_2 > a_3 > a_1\}$$
$$A_4 = \{x \in S_3 : a_3 > a_2 > a_1\}$$
$$A_5 = \{x \in S_3 : a_3 > a_1 > a_2\}$$
$$A_6 = \{x \in S_3 : a_1 > a_3 > a_2\}$$

Definition 9. *The Hamiltonians in these regions are given by*

$$V^{A_1}(x_1, x_2, x_3) = \hat{F}(x_1) + \tilde{G}(x_2) + \hat{H}(x_3)$$
$$V^{A_2}(x_1, x_2, x_3) = \hat{F}(x_1) + \tilde{G}(x_2) + \tilde{H}(x_3)$$
$$V^{A_3}(x_1, x_2, x_3) = \hat{F}(x_1) + \hat{G}(x_2) + \tilde{H}(x_3)$$
$$V^{A_4}(x_1, x_2, x_3) = \tilde{F}(x_1) + \hat{G}(x_2) + \tilde{H}(x_3)$$
$$V^{A_5}(x_1, x_2, x_3) = \tilde{F}(x_1) + \hat{G}(x_2) + \hat{H}(x_3)$$
$$V^{A_6}(x_1, x_2, x_3) = \tilde{F}(x_1) + \tilde{G}(x_2) + \hat{H}(x_3)$$

Theorem 2. *All orbits of the zero sum RSP game are closed orbits under the imitation dynamics.*

Proof. We proceed as in chapter 2 using a Poincare return map, i.e. we follow a solution around the interior fixed point $F$ to show that the orbit of every solution is closed. We start at the line $l_1 : x_1 = C$, which is a one dimensional manifold so we only need to specify one parameter $r$ to define a point on it, i.e. $P_1 = (C, 1-A-C-r, A+r) = (C, B-r, A+r)$. Starting at $P_1$ the solution has constant value in $A_1$ on the curve satisfying

$$V^{A_1}(x_1, x_2, x_3) = V_{l_1}^{A_1}(P_1) = \hat{F}(C) + \tilde{G}(B-r) + \hat{H}(A+r)$$

It reaches a point $P_3 = (C+s, 1-A-C-s, A) = (C+s, B-s, A)$ on the line $l_3 : x_3 = A$ depending on the new parameter $s$ only. The value of the invariant of motion their is

$$V_{l_3}^{A_1}(P_3) = \hat{F}(C+s) + \tilde{G}(B-s) + \hat{H}(A)$$

which still equals $V_{l_1}^{A_1}(C, B-r, A+r)$. We abbreviate $V_{l_i}(x_1, x_2, x_3) = V(P_i)$ and solve the system

$$\begin{aligned} V_{l_1}^{A_1}(P_1) &= \hat{F}(C) + \tilde{G}(B-r) + \hat{H}(A+r) \\ &= \hat{F}(C+s) + \tilde{G}(B-s) + \hat{H}(A) = V_{l_3}^{A_1}(P_3) \end{aligned}$$

with respect to the new parameter s which yields

$$\hat{F}(C+s) + \tilde{G}(B-s) = \hat{F}(C) + \tilde{G}(B-r) + \hat{H}(A+r) - \hat{H}(A)$$

Our next starting point for the solution curve in $A_2$ is therefor given by $P_3(C+s, B-s, A)$ and the value of the constant of motion is

$$V_{l_3}^{A_2}(P_3) = \hat{F}(C+s) + \tilde{G}(B-s) + \tilde{H}(A)$$

The solution reaches a point $P_2 = (C+t, B, A-t)$ on $l_2$ that satisfies

$$\begin{aligned} V_{l_2}^{A_2}(P_2) &= \hat{F}(C+s) + \tilde{G}(B-s) + \tilde{H}(A) \\ &= \hat{F}(C+t) + \tilde{G}(B) + \tilde{H}(A-t) = V_{l_3}^{A_2}(P_3) \end{aligned}$$

Solving this system for the new parameter $t$ yields

$$\hat{F}(C+t) + \tilde{H}(A-t) = \hat{F}(C) - \tilde{G}(B) + \tilde{H}(A) - \hat{H}(A) + \tilde{G}(B-r) + \hat{H}(A+r)$$

Starting on the line $l_2$ at the point $P_2$ the constant of motion in region $A_3$ is given by

$$V_{l_2}^{A_3}(P_2) = \hat{F}(C+t) + \hat{G}(B) + \tilde{H}(A-t)$$

Reaching a point $P_{1*} = (C, B+u, A-u)$ on $l_1$, yields the system

$$\begin{aligned} V_{l_2}^{A_3}(P_2) &= \hat{F}(C+t) + \hat{G}(B) + \tilde{H}(A-t) \\ &= \hat{F}(C) + \hat{G}(B+u) + \tilde{H}(A-u) = V_{l_1}^{A_3}(P_{1*}) \end{aligned}$$

Solving this system for the new parameter $u$ yields

$$\hat{G}(B+u)+\tilde{H}(A-u) = \hat{G}(B)-\tilde{G}(B)+\tilde{H}(A)-\hat{H}(A)+\tilde{G}(B-r)+\tilde{H}(A+r)$$

Going from $A_3$ to $A_4$, we start at $P_{1*}$, finish at $P_{3*} = (C-v, B+v, A)$ and get the system

$$\begin{aligned} V_{l_1}^{A_4}(P_{1*}) &= \tilde{F}(C) + \hat{G}(B+u) + \tilde{H}(A-u) = \\ &= \tilde{F}(C-v) + \hat{G}(B+v) + \tilde{H}(A) = V_{l_3}^{A_4}(P_{2*}) \end{aligned}$$

Solving this system for the new parameter $v$ yields

$$\tilde{F}(C-v)+\hat{G}(B+v) = \hat{F}(C)+\hat{G}(B)-\tilde{G}(B)-\hat{H}(A)+\tilde{G}(B-r)+\tilde{H}(A+r)$$

In $A_5$, we start at $P_{3*}$ on $l_3$, finish at $P_{2*} = (C-w, B, A+w)$ on $l_2$ and get the system

$$\begin{aligned} V_{l_3}^{A_5}(P_{3*}) &= \tilde{F}(C-v) + \hat{G}(B+v) + \hat{H}(A) \\ &= \tilde{F}(C-w) + \hat{G}(B) + \hat{H}(A+w) = V_{l_2}^{A_5}(P_{2*}) \end{aligned}$$

Solving this system for the new parameter $w$ yields

$$\tilde{F}(C-w) + \hat{H}(A+w) = \hat{F}(C) - \tilde{G}(B) + \tilde{G}(B-r) + \tilde{H}(A+r)$$

in the last region $A_6$ we start the line $l_2$ at the point $P_{2*}$ and finally arrive at our starting line $l_1$ at a $\bar{P}_1 = (C, B-\bar{r}, A+\bar{r})$

$$\begin{aligned} V_{l_2}^{A_6}(P_{2*}) &= \tilde{F}(C-w) + \tilde{G}(B) + \hat{H}(A+w) \\ &= \tilde{F}(C) + \tilde{G}(B-\bar{r}) + \hat{H}(A+\bar{r}) = V_{l_1}^{A_6}(\bar{P}_1) \end{aligned}$$

Solving this system for the new parameter $\bar{r}$ yields

$$\begin{aligned} \tilde{G}(B-\bar{r}) + \hat{H}(A+\bar{r}) &= \hat{F}(C) - \tilde{F}(C) + \tilde{G}(B-r) + \hat{H}(A+r) = \\ &= \tilde{G}(B-r) + \hat{H}(A+r) \end{aligned}$$

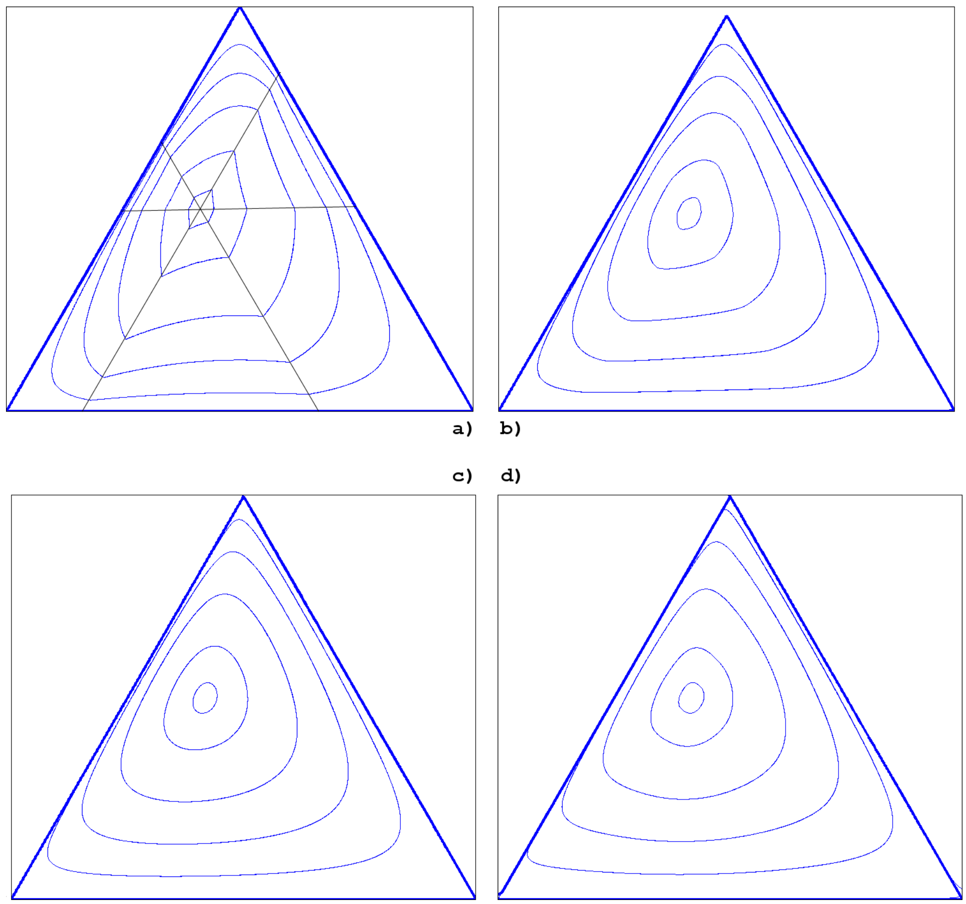


FIGURE 4.2. The Zero Sum Game with $A = \frac{1}{2}$, $B = \frac{1}{6}$ and $C = \frac{1}{3}$

The only possible and valid solution of this system is $\bar{r} = r$, since $\tilde{G}$ is a continuous bijection. So after following the solution around the interior fixed point we arrive at the point $\bar{P}(C, B - \bar{r}, A + \bar{r})$ plugging in $\bar{r} = r$ the Hamiltonians are equal and we arrive at the starting point $P(C, B - r, A + r)$. □

## 2. Imitate the Better, the Special Case $\alpha = 0$

We will now take a closer look at the special case $\alpha = 0$, but for general RSP games with payoff matrix (28) for we need some result about it later on in the text. In this case the system (31) looks like

$$\begin{aligned} \dot{x_1} &= x_1[x_2 \operatorname{sgn}[a_1 - a_2] - x_3 \operatorname{sgn}[a_1 - a_3]] \\ \dot{x_2} &= x_2[-x_1 \operatorname{sgn}[a_1 - a_2] + x_3 \operatorname{sgn}[a_2 - a_3]] \\ \dot{x_3} &= x_3[x_1 \operatorname{sgn}[a_1 - a_3] - x_2 \operatorname{sgn}[a_2 - a_3]] \end{aligned} \tag{36}$$

The vector fields in the regions $A_i$ are given explicitly by

Region $A_1$ with $a_1 > a_2 > a_3$

$$\begin{aligned} \dot{x_1} &= x_1(x_2 + x_3) \\ \dot{x_2} &= x_2(-x_1 + x_3) \\ \dot{x_3} &= x_3(-x_1 - x_2) \end{aligned} \tag{37}$$

Taking a look at the equation one immediately recognizes that this is just the replicator dynamics with payoff matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix} \tag{38}$$

So the connection is that smooth pieces of the orbits of the imitate the better dynamics are part of orbits of the replicator dynamics, but for a different game. There is no interior rest point for these games under the replicator equation. Nevertheless one gets the usual invariant of motion $V(x_1, x_2, x_3) = \prod x_i^{p_i}$ where $p = (p_1, p_2, p_3)$ is a fixed point of the replicator equation with $p \notin S_3$. So for region $A_1$ we have $p = (1, -1, 1)$ and a constant of motion in region $A_1$ is given by $V^{A_1}(x_1, x_2, x_3) = \frac{x_1 x_3}{x_2}$

Region $A_2$ with $a_2 > a_1 > a_3$

$$\begin{aligned} \dot{x_1} &= x_1(-x_2 + x_3) \\ \dot{x_2} &= x_2(x_1 + x_3) \qquad V^{A_2}(x_1, x_2, x_3) = \frac{x_2 x_3}{x_1} \\ \dot{x_3} &= x_3(-x_1 - x_2) \end{aligned} \tag{39}$$

Region $A_3$ with $a_2 > a_3 > a_1$

$$\begin{aligned} \dot{x_1} &= x_1(-x_2 - x_3) \\ \dot{x_2} &= x_2(x_1 + x_3) \qquad V^{A_3}(x_1, x_2, x_3) = \frac{x_1 x_2}{x_3} \\ \dot{x_3} &= x_3(x_1 - x_2) \end{aligned} \tag{40}$$

Region $A_4$ with $a_3 > a_2 > a_1$

$$\begin{aligned} \dot{x_1} &= x_1(-x_2 - x_3) \\ \dot{x_2} &= x_2(x_1 - x_3) \qquad V^{A_4}(x_1, x_2, x_3) = \frac{x_1 x_3}{x_2} \\ \dot{x_3} &= x_3(x_1 + x_2) \end{aligned} \tag{41}$$

Region $A_5$ with $a_3 > a_1 > a_2$

$$\begin{aligned} \dot{x_1} &= x_1(x_2 - x_3) \\ \dot{x_2} &= x_2(-x_1 - x_3) \qquad V^{A_5}(x_1, x_2, x_3) = \frac{x_2 x_3}{x_1} \\ \dot{x_3} &= x_3(x_1 + x_2) \end{aligned} \tag{42}$$

Region $A_6$ with $a_1 > a_3 > a_2$

$$\begin{aligned} \dot{x_1} &= x_1(x_2 + x_3) \\ \dot{x_2} &= x_2(-x_1 - x_3) \qquad V^{A_6}(x_1, x_2, x_3) = \frac{x_1 x_2}{x_3} \\ \dot{x_3} &= x_3(-x_1 + x_2) \end{aligned} \tag{43}$$

## 3. Cyclic Symmetry $a_i = a$ and $b_i = b$ $\forall i$ and $\alpha = 0$

In this special case the payoff matrix has the form

$$A = \begin{bmatrix} 0 & -a & b \\ b & 0 & -a \\ -a & b & 0 \end{bmatrix} \tag{44}$$

with $a, b > 0$. Actually the flow of the system only depends on the ratio of $a$ and $b$. So we set $a = bm$ and arrive at

$$A = \begin{bmatrix} 0 & -bm & b \\ b & 0 & -bm \\ -bm & b & 0 \end{bmatrix} = b \begin{bmatrix} 0 & -m & 1 \\ 1 & 0 & -m \\ -m & 1 & 0 \end{bmatrix} \tag{45}$$

THEOREM 3. *The interior rest point* $F = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ *of* (31) *with payoff matrix* (45) *and* $\alpha = 0$ *is a sink for* $b > a$ *(i.e.* $1 > m$*) and a source for* $b < a$ *(i.e.* $1 < m$*), for* $a = b$ *this is just the zero sum game* (36) *with* $a_i = b_i = 1$.

Before we start with the proof, we need a redefinition of the lines $l_i$.

DEFINITION 10. *The lines*

$$\begin{aligned} l_1 &= \{x \in S_3 : a_2(x) = a_3(x)\} = \{x \in S_3 : x_1 = \frac{x_2 + mx_3}{1+m}\} \\ l_2 &= \{x \in S_3 : a_1(x) = a_3(x)\} = \{x \in S_3 : x_2 = \frac{x_3 + mx_1}{1+m}\} \\ l_3 &= \{x \in S_3 : a_1(x) = a_2(x)\} = \{x \in S_3 : x_3 = \frac{x_1 + mx_2}{1+m}\} \end{aligned}$$

*divide* $S_3$ *into six regions.*

DEFINITION 11. *Points* $P_i$ *on the lines* $l_i$ *are ordered counter-clockwise around the equilibrium and are thus given by a parametrization of the lines*

$$\begin{aligned} P_1 &= (\frac{1}{3} - \frac{1-m}{1+2m}r, \frac{1}{3} - r, \frac{1}{3} + \frac{2+m}{1+2m}r\} \\ P_2 &= (\frac{1}{3} + \frac{2+m}{1+2m}s, \frac{1}{3} - \frac{1-m}{1+2m}, \frac{1}{3} - s\} \\ P_3 &= (\frac{1}{3} + \frac{1+2m}{2+m}t, \frac{1}{3} - t, \frac{1}{3} + \frac{1-m}{2+m}t\} \end{aligned}$$

*with* $a = bm$ *and* $r, s, t \in [0, 1/3]$

PROOF. The vector field of this game for $\alpha = 0$ is just the vector field of (36) under the imitate the better dynamics. We have the same constants of motion. We will prove this theorem by showing that every solution, starting on a point $P_i$ on $l_i$ is getting closer to or farer away from the equilibrium, depending on the values of $a$ and $b$,respectively on the value of $m$. The system is cyclically symmetric on the simplex, which implies that it suffices to analyze the flow from a point $P_1$ on $l_1$, to $P_3$ and finally arriving at a point $P_2$ on $l_2$. As earlier mentioned the

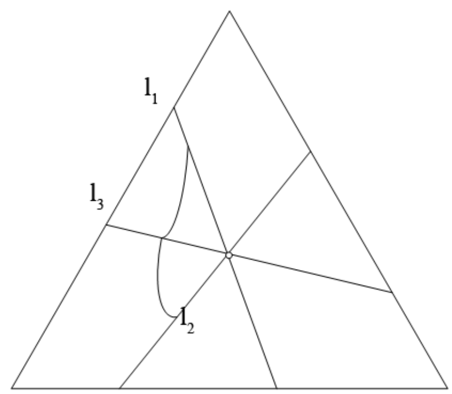


FIGURE 4.3. The simplex with a solution curve starting at the line $l_1$, going to $l_2$ and finally arriving at $l_3$

constants of motion are the same as for (36), we list only the two used in this proof i.e.

$$V^{A_1}(x_1, x_2, x_3) = \frac{x_1 x_3}{x_2}$$
$$V^{A_2}(x_1, x_2, x_3) = \frac{x_2 x_3}{x_1}$$

So the value of the constant of motion $V^{A_1}$ for a given point $P_1(r)$ on $l_1$ in region $A_1$ is given by

$$V^{A_1}(P_1(r)) = V^{A_1}(r) = \frac{(\frac{1}{3} - \frac{1-m}{1+2m}r)(\frac{1}{3} + \frac{2+m}{1+2m}r)}{\frac{1}{3} - r} \tag{46}$$

Following this procedure we calculate $V^{A_1}$ for a point $P_3(t)$ on $l_3$.

$$V^{A_1}(P_3(t)) = V^{A_1}(t) = \frac{(\frac{1}{3} + \frac{1+2m}{2+m}t)(\frac{1}{3} + \frac{1-m}{2+m}t)}{\frac{1}{3} - t} \tag{47}$$

When the solution reaches $A_2$ the value of the constant of motion changes according to the change of the region. So we take $V^{A_2}$ instead of $V^{A_1}$ and start the procedure again. We calculate $V^{A_2}$ for a point $P_3(t)$ on $l_3$, which yields

$$V^{A_2}(P_3(t)) = V^{A_2}(t) = \frac{(\frac{1}{3} - t)(\frac{1}{3} + \frac{1-m}{2+m}t)}{\frac{1}{3} + \frac{1+2m}{2+m}t} \tag{48}$$

and finally the solution arrives at the line $l_2$ at a point $P_2(s)$, where the constant of motion has the value

$$V^{A_2}(P_2(s)) = V^{A_2}(s) = \frac{(\frac{1}{3} - \frac{1-m}{1+2m}s)(\frac{1}{3} - s)}{\frac{1}{3} + \frac{2+m}{1+2m}s} \tag{49}$$

The next step is to find out at which point $P_3$ on $l_3$ a solution arrives when started from a point $P_1$ on $l_1$. Solving the quadratic system

$$V^{A_1}(r) = V^{A_1}(t) \tag{50}$$

$$\frac{(\frac{1}{3} - \frac{1-m}{1+2m}r)(\frac{1}{3} + \frac{2+m}{1+2m}r)}{\frac{1}{3} - r} = \frac{(\frac{1}{3} + \frac{1+2m}{2+m}t)(\frac{1}{3} + \frac{1-m}{2+m}t)}{\frac{1}{3} - t} \tag{51}$$

for the variable $t$ gives us the two solutions

$$t_{1,2}(r,m) = \frac{1}{6(-1+3r)(-1+m)(1+2m)^3}[-8 - 9r^2(-1+m)(2+m)^3 - 2m(5+2m)(4+m(5+2m)) \pm \\ \pm \sqrt{(2+m)^2(81r^4(-1+m)^2(2+m)^4 + 4(2+m)^2(1+2m)^4 - 24r(-1+m)(1+2m)^5 + +108(-1+m)^2(2+m)(r+2rm)^3 + 108(-1+m)(4+5m)(r+2rm)^2(1+m+m^2)}]$$

we can discard $t_2(r,m)$ because $t_2(r,m) \leq 0$ for all $r, m \geq 0$. We set $t_1(r,m) = t(r,m)$

We use the same procedure to calculate a function $s(t,m)$. Solving the system for the region $A_2$

$$V^{A_2}(t) = V^{A_2}(s) \tag{52}$$

$$\frac{(\frac{1}{3} - t)(\frac{1}{3} + \frac{1-m}{2+m}t)}{\frac{1}{3} + \frac{1+2m}{2+m}t} = \frac{(\frac{1}{3} - \frac{1-m}{1+2m}s)(\frac{1}{3} - s)}{\frac{1}{3} + \frac{2+m}{1+2m}s} \tag{53}$$

yields the two solutions

$$s_{1,2}(t,m) = \frac{1}{(6(-1+m)(2+m+t(3+6m)))}[-2(2+m)^2 - 9t^2(-2+m+m^2) \pm \\ \pm \sqrt{(2+m)^2(9t^2(-1+m) + 2(2+m))^2 - 12t(-2+3t(-1+m) - 4m)(-1+m)(1+2m)(2+m+t(3+6m))}]$$

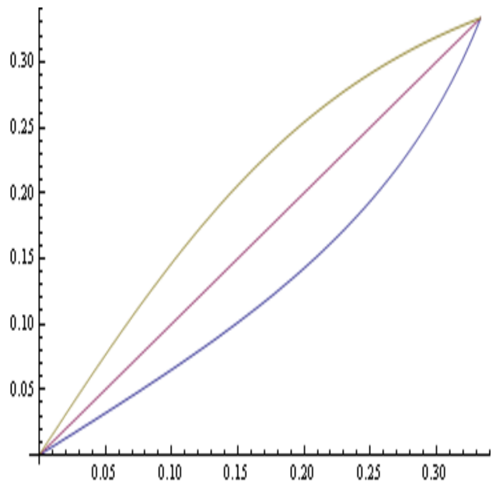


FIGURE 4.4. The function $s(r,m)$, with $m = 1/2, 1, 2$. For $0 < m < 1$ we have $s(r,m) < r$ and for $1 < m$ we have $s(r,m) > r$. For $m = 1$ we have $s(r,1) = r$

Once again we can discard the second solution as $s_2(t,m) \geq \frac{1}{3}$ for all $t, m \geq 0$. Setting $s_1(t,m) = s(t,m)$

So far we have seen where a solution starting at $l_1$ ends up at $l_3$ and what happens if we go from $l_3$ to $l_2$. Combining these two functions we get a map $s(t(r),m) = s(r,m)$ assigning to each value $r$, which represents a point on $l_1$ a value $s$ on $l_2$. The function $s : [0,1/3] \times (0,\infty) \to [0,1/3]$ is a rather clumsy object, but with the help of a computer algebra system (we used mathematica with Reduce[$s[r,m] < r \&\& r > 0 \&\& y > 0$,r,m,Reals]), one easily verifies that for $0 < m < 1$ the function $s(r,m) < r$ and the interior fixed point is globally attractive.

For $m = 1$ we arrive at (36) with $a_i = b_i = 1$ and conclude that all orbits are closed. To see this one only has to solve (51)and (53), which yields $r = t = s$.

For $1 < m$ the function $s(r,m) > r$ and therefore the interior fixed point is a source and every orbit converges to the boundary, which is a heteroclinic cycle. $\square$

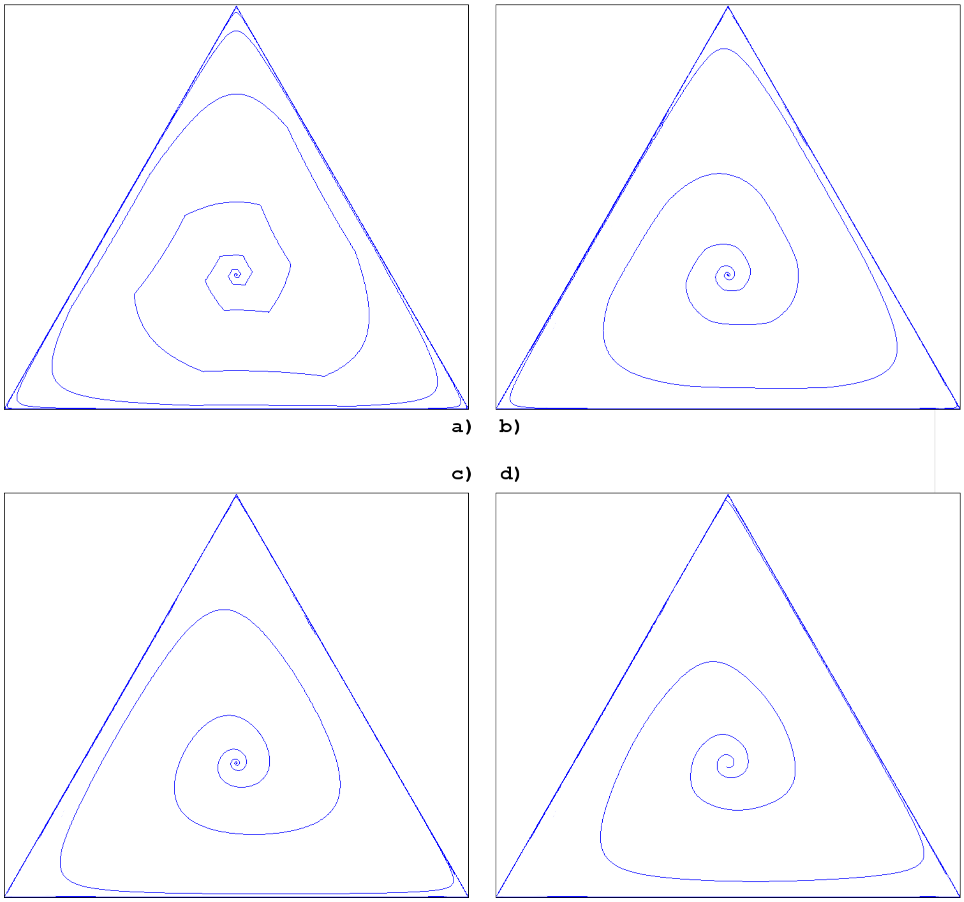


FIGURE 4.5. Cyclic Symmetry for $m = 2$

## Deutsche Zusammenfassung

Diese Arbeit beschäftigt sich mit Imitationsdynamiken der Form $\dot{x}_i = x_i \sum_j x_j \psi((Ax)_i - (Ax)_j)$, in denen die Wechselrate eine Funktion der Auszahlungsdifferenz ist. Für die Analyse der Dynamiken werden Poincare-Abbildungen und Hamiltonfunktionen verwendet.

Nach einer Einführung über Imitationsdynamiken beschreiben wir Bimatrixspiele unter diesen Dynamiken. Besonderes Augenmerk wird auf den Fall mit zwei Spielern und zwei Strategien gelegt. Es werden sowohl nicht-degenerierte, als auch einige degenerierte Fälle analysiert. Für nicht-degenerierte Spiele ist die Replikatorgleichung topologisch equivalent zu den betrachteten Imitationsdynamiken. Im Falle der degenerierten Spiele ist die ”imitate the better” Dynamik $\alpha = 0$ nicht mehr topologisch equivalent zur Replikatorgleichung und den Fällen $\alpha > 0$.

Das Kapitel 4 beschäftigt sich mit Schere-Stein-Papier Spielen unter den Imitationsdynamiken. Null-Summen-Spiele und der Fall zyklischer Symmetrie werden vollständig behandelt. Das dynamische Verhalten des allgemeinen Schere-Stein-Papier Spiels konnte nicht geklärt werden.

# List of Figures

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## Vita Academica

I was born on August 20, 1982 in Vienna, to Christoph and Elisabeth Rieder. I have 2 brothers, Clemens and Maximilian and a sister Viktoria. I made my Matura in 2001 at the Piaristen Gymnasium (humanistic branch) and after finishing the military service, I started studying physics at the Technical University of Vienna and mathematics at the University of Vienna.