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Counting Integral Points in Polytopes

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Zusammenfassung

Probleme, die sich durch das Zählen von ganzzahligen Punkten in Polytopen lösen lassen, treten in den unterschiedlichsten mathematischen Gebieten auf. Es ist das Anliegen der vorliegenden Arbeit, einige Methoden darzustellen, mit denen diesen Abzählproblemen begegnet werden kann.

Die ersten Untersuchungen, die sich dezidiert mit dem Abzählen von ganzzahligen Punkten in Polytopen auseinandersetzen, finden sich in Arbeiten von Eugène Ehrhart aus den 1960er Jahren. Ehrhart betrachtete Streckungen von rationalen Polytopen um einen positiven ganzzahligen Faktor. Dabei entdeckte er, dass die Anzahl der ganzzahligen Punkte in diesen Streckungen durch ein Quasi-Polynom im Streckungsfaktor berechnet werden kann. Während wir uns im ersten Kapitel mit einer kurzen Einführung in die Polytoptheorie begnügen, die als Grundlage für die weiteren Betrachtungen dienen soll, wenden wir uns im Anschluss eingehend der Ehrhart-Theorie zu und liefern eine Darstellung ihrer wichtigsten Resultate.

Unter einem allgemeineren Blickwinkel kann man die Anzahl der ganzzahligen Punkte in Polytopen betrachten, die durch unabhängige Parallelverschiebung der Randflächen eines Polytops entstehen. Mit Hilfe der Theorie von "vector partition functions" lassen sich die Resultate der Ehrhart-Theorie für diese Polytope verallgemeinern.

Vor dem Hintergrund der Arbeit von Wolfgang Dahmen und Charles A. Micchelli [6] die im Zuge ihrer Untersuchungen von "box splines" auf "vector partition functions" gestoßen sind, führen wir im dritten Kapitel einen elementaren Beweis für eine Verallgemeinerung von Ehrharts Satz. Während der Ansatz von Dahmen und Micchelli auf den Eigenschaften von splines beruht, fußt unser Beweis auf geometrischen Betrachtungen und macht sich die Eigenschaften von Polytopen zunutze. Mit diesen Mitteln können wir schließlich eine stärkere Version des Satzes aus [6] beweisen. Diese Variante wurde bereits von András Szenes und Michèle Vergne in [25] mit Hilfe von Methoden der Komplexen Analysis gezeigt.

Abstract

Problems that may be approached by counting integral points in polytopes occur in many mathematical fields. It is the objective of the present thesis to provide methods that enable us to face these problems.

The first attempts that explicitly dealt with the issue of counting integral points in polytopes were made by Eugène Ehrhart in the 1960s. Ehrhart concerned himself with integral dilations of rational polytopes. In doing so, he discovered that the number of integral points in these dilated polytopes may be computed by a quasi–polynomial in the dilation factor. While we content ourselves with a short introduction to the theory of polytopes in the first chapter we will subsequently engage in a more detailed study on Ehrhart Theory that will provide us with a description of its most important results.

From a more general point of view, it is convenient to examine the number of integral points in polytopes that result from independent parallel motions of the bounding hyperplanes of a given rational polytope. By means of the theory of vector partition functions, it is possible to generalize the results of Ehrhart Theory in order to apply them to this type of problem.

Based on an article of Wolfgang Dahmen and Charles A. Micchelli [6] who came across vector partition functions in the course of their investigation on box splines, we will establish a proof of a generalization of Ehrhart's Theorem in chapter three. While the approach of Dahmen and Micchelli relies on properties of splines, our proof is based on geometrical considerations and makes use of properties of polytopes.

Eventually, we may, in this wise, provide a proof of a more general version of the theorem given in [6]. This result has already been achieved by András Szenes and Michèle Vergne in [25] by means of complex analysis.

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Contents

Introduction

A polytope is a bounded convex subset of $\mathbb{R}^n$ enclosed by a finite number of hyperplanes. Two–dimensional convex polygons and three–dimensional convex polyhedra, for instance a cube or a pyramid, are examples of polytopes. More precisely, a rational polytope can be described as a set $\mathcal{P} = \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} \leq \boldsymbol{b}\}$, specified by an $s \times n$ integer matrix A and an integer vector $\boldsymbol{b} \in \mathbb{Z}^s$. The aim of this thesis is to investigate the number of integer points, i.e. points with integral coordinates, inside the polytopes which are obtained by varying the vector $\boldsymbol{b}$ in $\mathbb{Z}^s$ while the matrix A is fixed. That is to say, we study the counting function

$$\Phi_A(\boldsymbol{b}) = \#\{\boldsymbol{z} \in \mathbb{Z}^n : A\boldsymbol{z} \leq \boldsymbol{b}\}. \tag{0.1}$$

Problems that may be approached by counting integral points in polytopes occur in various mathematical areas. Coin problems in Number Theory [1], the designation of the number of contingency tables in Statistics [13] or the computation of Gröbner bases of toric ideals in Commutative Algebra [24] may be adduced as instances. Techniques for counting lattice points in polytopes are also useful for precise program analysis in Theoretical Computer Science [15], for counting integer flows in networks in Network Theory [2], or for solving integer programs in the area of Operations Research [17]. Moreover, many combinatorial structures can be counted as integral points in rational polytopes and may therefore be considered in a unified framework. Gelfand Patterns [20], Littlewood–Richardson coefficients [21], linear codes and combinatorial designs [14], magic squares [22] and alternating sign matrices are a few examples of such structures.

Astonishingly, the first and best known theory which explicitly deals with integral–point enumeration in variable rational polytopes goes back to the work of Eugène Ehrhart in the 1960s. Ehrhart concerned himself with integral dilations of rational polytopes. In other words, he studied the counting function (0.1) in the case that the vector $\boldsymbol{b}$ changes in a single integral factor, i.e. $\boldsymbol{b} = t\boldsymbol{v}$ for a fixed vector $\boldsymbol{v} \in \mathbb{Z}^s$ and arbitrary non–negative integers $t \in \mathbb{Z}_{\geq 0}$. Now, a function $f : \mathbb{Z} \to \mathbb{C}$ is called a quasi–polynomial if there exist a non–negative integer p and polynomials $f_0, f_1, \ldots, f_{p-1}$ such that $f(t) = f_i(t)$ if $t \equiv i \pmod{p}$. The minimal integer p with this property is said to be the period of the quasi–polynomial and the degree of f is defined as the maximum degree among the polynomials $f_0, f_1, \ldots, f_{p-1}$. Ehrhart's seminal theorem states that the number of integral points in the dilated polytopes $t\mathcal{P} = \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} \leq t\boldsymbol{v}\}$ may be computed by a quasi–polynomial in the dilation factor t, the so called Ehrhart quasi–polynomial of the polytope. The degree of this quasi–polynomial equals the dimension of the polytope while its period divides the least common multiple of the denominators of the coordinates of the vertices of the polytope. An intriguing and unexpected application of the Ehrhart quasi–polynomial is that it enables us to calculate the volume of a rational polytope discretely, by counting integral points. Another interesting fact is that the evaluation of the Ehrhart quasi–polynomial at negative integers yields the number of the integral points in the interior of the polytope dilated by the absolute value of this integer. This reciprocity relation had been conjectured and proved in several special cases by Eugène Ehrhart about a decade before I. G. Macdonald found a general proof in 1971 [18]. There are still many open questions related to Ehrhart quasi–polynomials and Ehrhart Theory is still an active source of research. A number of related open problems is presented in [3].

While the Ehrhart Theory is well known by now and there are several books and lecture notes on various topics that include chapters on this theory, it is relatively unknown that there exist similar results for the general case considering the vector $\boldsymbol{b}$ in (0.1) a variable in $\mathbb{Z}^s$. Geometrically speaking, this means we will not only consider the number of integral points in integral dilations of a rational polytope, but also the variation of the number of integral points when the bounding hyperplanes of a polytope are changed by independent parallel motions. Now, from another point of view, by introducing slack variables, one may equivalently to (0.1) consider a function that counts the number of ways in which a variable vector $\boldsymbol{b} \in \mathbb{Z}^s$ can be written as a non–negative integral linear combination of the column vectors of an $s \times d$ matrix A' of full rank $s \leq d$. By means of the theory of vector partition functions, one can prove that there exists a finite decomposition of the vector space $\mathbb{R}^s$ into regions, called the chambers of A, such that the function $\Phi_A(\boldsymbol{b})$, as defined in (0.1), agrees with a multivariate quasi–polynomial in $\boldsymbol{b} \in \mathbb{Z}^s$ on each of these chambers. The degree of these multivariate quasi–polynomials equals $d - s$ which is, at the same time, the dimension of the polytope $P_{A'}(\boldsymbol{b}) = \{\boldsymbol{x} \in \mathbb{R}^d : A'\boldsymbol{x} = \boldsymbol{b}\}$. Moreover, if we consider neighboring chambers, the corresponding quasi–polynomials coincide along a strip near their common boundary. This provides a generalization of Ehrhart's Theorem and it is the final objective of this thesis to give a basic proof of this theorem. This result and slightly weaker assertions have already been proved by several different methods and different approaches, c.f. [6],[23],[5],[25],[8]. Nevertheless, by now, it is rather hard to find literature on vector partition functions. Surprisingly, the only book we found that contains a survey on this nice counting problem is entitled "Box Spiles".

Eventually, some comments refering to the structure of the thesis at hand.

In order to keep this thesis self–contained, we start with a brief introduction to the basics of polytope theory in the first chapter. Then, we introduce the notation of quasi–polynomials and examine the relation between these functions and rational generating functions. Since the material of these sections is only preliminary, it may be skipped or can be consulted on demand.

Our second chapter provides a detailed exposition of the Ehrhart Theory. We carry out the proof of Ehrhart's Theorem for rational polytopes and give a basic proof of the Ehrhart–Macdonald Reciprocity that appears in [4]. Furthermore, we study the connection between the volumes of polytopes and coefficients of Ehrhart quasi–polynomials and discuss some facts about the period of Ehrhart quasi–polynomials. As an example for an application of Ehrhart Theory, we study the number of restricted partitions of an integer.

Finally, the third chapter is devoted to a generalization of Ehrhart's Theorem. By means of vector partition functions, we may not only study the number of integral points in integral dilations of a polytope, but also the number of such points in polytopes that result from independent parallel motions of the facets of a given polytope. Based on an article of Wolfgang Dahmen and Charles A. Micchelli [6], who came across vector partition functions in the course of their investigation on box splines, we will establish proof of a generalization of Ehrhart's Theorem to the setting of these polytopes. It is worth pointing out that the article [6] includes a result that can be considered as a generalization of Ehrhart–Macdonald Reciprocity. In this context, we provide an interpretation of the according theorem in terms of variable polytopes.

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Chapter 1

Preliminaries

1.1 Basic Notation

First of all we like to clarify some notations that we use in the thesis at hand.

For a non–negative integer $n \in \mathbb{Z}_{\geq 0}$, we denote the vector space of all column vectors of length n with real entries by $\mathbb{R}^n$. In the same manner, we denote by $\mathbb{Z}^n$ the subset of all vectors with integral entries in the vector space $\mathbb{R}^n$. Such vectors are called integral points or integer vectors.

Subsequently, bold letters shall denote column vectors in $\mathbb{R}^n$ and represent affine points, while italic letters stand for real numbers. Accordingly, we denote the i–th coordinate of a vector $\boldsymbol{x} \in \mathbb{R}^n$ by x_i, while $\boldsymbol{x}_i$ stands for the i–th element of an indexed family of vectors. When we need to refer to the i–th coordinate of the j–th element of an indexed family of vectors, we write $(\boldsymbol{x}_j)_i$. The i–th element of the standard basis of $\mathbb{R}^n$, i.e. the vector with a one in the i–th coordinate and zeros elsewhere, is denoted by $\boldsymbol{e}_i$. Moreover, the vector with all entries zero is denoted by $\mathbf{0}$. If each coordinate of a vector $\boldsymbol{x} \in \mathbb{R}^n$ is equal to or greater than the corresponding coordinate of the vector $\boldsymbol{y} \in \mathbb{R}^n$, i.e. $x_i \geq y_i$ for all $1 \leq i \leq n$, we write $\boldsymbol{x} \geq \boldsymbol{y}$. Accordingly, we denote by $\mathbb{R}^n_{\geq}$ the set of all vectors in $\mathbb{R}^n$ where all coordinates are given by non–negative reals.

Another basic notation that occurs frequently throughout the text is the greatest integer–function $\lfloor x \rfloor$, which denotes the greatest integer less than or equal to a real $x \in \mathbb{R}$, and the fractional part $\{x\} := x - \lfloor x \rfloor$. As usual, we denote by $|x|$ the absolute value of a real number x, and $\operatorname{sgn}(x)$ gives the sign of the real x.

We write $\bigcup M$ for the union of all elements of a set M. The set theoretic difference of two sets A and B, denoted $A \backslash B$, is the set of all elements which are members of A, but not members of B.

1.2 A Short Lecture on Polytopes

The aim of this section is to give a brief insight in the theory of polytopes, focusing on terms that form the bases of the upcoming chapters. As the headline suggests, the following outlines are mainly based on Günter Ziegler's book [26].

1.2.1 Finite–Dimensional Real Vector Spaces

We start with a review of some basic concepts of finite–dimensional real vector spaces, in which we will point out the analogy between linear, affine and convex subsets.

Linear Combinations and Linear Spaces

A *linear combination* of elements $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_d \in \mathbb{R}^n$ is a vector of the form $\lambda_1\boldsymbol{x}_1 + \ldots + \lambda_d\boldsymbol{x}_d$ for reals $\lambda_1, \ldots, \lambda_d \in \mathbb{R}$, and a family of vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_d$ is called *linearly independent* if and only if the only possible choice of reals $\lambda_1, \ldots, \lambda_d \in \mathbb{R}$ such that $\lambda_1\boldsymbol{x}_1 + \ldots + \lambda_d\boldsymbol{x}_d = \boldsymbol{0}$ is given by $\lambda_1 = \ldots = \lambda_d = 0$.

A non–empty subset $L \subseteq \mathbb{R}^n$ is called a *linear space* or *linear sub–space* of $\mathbb{R}^n$ if and only if any linear combination of vectors from L lies again in L.

Given a subset $M \subseteq \mathbb{R}^n$, the *linear hull* of M or, for short, $\operatorname{span}(M)$, is the set of all linear combinations of elements of M, and $\operatorname{span}(M)$ is the smallest linear sub–space of $\mathbb{R}^n$ containing the set M.

A *linear basis* of a linear sub–space $L \subseteq \mathbb{R}^n$ is a set of linearly independent elements $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k \in L$ such that $L = \operatorname{span}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k)$. It is a well known fact that k is then the maximum number of linearly independent elements in L. Thus, k is said to be the *dimension* of L, short $\dim(L)$.

Affine Combinations and Affine Spaces

An *affine combination* of elements $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_d \in \mathbb{R}^n$ is a vector $\lambda_1\boldsymbol{x}_1 + \ldots + \lambda_d\boldsymbol{x}_d$ where $\lambda_1, \ldots, \lambda_d$ stand for reals with the property that the sum $\sum_{i=1}^{d} \lambda_i = 1$.

A family of vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_d$ is called *affinely independent* if and only if the only reals $\lambda_1, \ldots, \lambda_d \in \mathbb{R}$ with $\sum_{i=1}^{d} \lambda_i = 0$ such that $\lambda_1\boldsymbol{x}_1 + \ldots + \lambda_d\boldsymbol{x}_d = \boldsymbol{0}$ are given by $\lambda_1 = \ldots = \lambda_d = 0$. Note that the affine independence of the vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_d$ is equivalent to the linear independence of the family $\boldsymbol{x}_1 - \boldsymbol{x}_i, \ldots, \boldsymbol{x}_{i-1} - \boldsymbol{x}_i, \boldsymbol{x}_{i+1} - \boldsymbol{x}_i, \ldots, \boldsymbol{x}_d - \boldsymbol{x}_i$, for any fixed index $1 \leq i \leq d$.

We call a non–empty subset $\mathcal{A} \subseteq \mathbb{R}^n$ an *affine space* or an *affine sub–space of* $\mathbb{R}^n$ if and only if any affine combination of elements of $\mathcal{A}$ is also contained in $\mathcal{A}$. It is easy to see that any affine sub–space is a translate of a linear sub–space, i.e. $\mathcal{A} = \boldsymbol{x} + L$ for some $\boldsymbol{x} \in \mathbb{R}^n$ and some linear sub–space $L \subseteq \mathbb{R}^n$. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are affine sub–spaces of $\mathbb{R}^n$ such that $\mathcal{A}_1 \subseteq \mathcal{A}_2$, then $\mathcal{A}_1$ is called an *affine sub–space* of $\mathcal{A}_2$.

Given a non–empty subset $M \subseteq \mathbb{R}^n$, the *affine hull*, $\mathrm{aff}(M)$, is the set of all affine combinations of elements of M, and as such the smallest affine space containing M.

An *affine basis* of an affine space $\mathcal{A} \subseteq \mathbb{R}^n$ is a family of affinely independent vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$ such that $\mathcal{A} = \mathrm{aff}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k)$. In this case it can be shown that k is the maximum number of affinely independent elements in $\mathcal{A}$ and the dimension of $\mathcal{A}$, denoted $\dim(\mathcal{A})$, is then defined as $k-1$. If an affine space $\mathcal{A} = \boldsymbol{x} + L$, for a linear sub–space of $L \subseteq \mathbb{R}^n$ and a vector $\boldsymbol{x} \in \mathbb{R}^n$, we obtain that $\dim(A) = \dim(L)$. Thus, if an affine space is actually a linear sub–space of $\mathbb{R}^n$, this definition agrees with the previous one. Subsequently, by the dimension of a subset $M \subseteq \mathbb{R}^n$ we mean the dimension of the affine sub–space which is spanned by M, i.e. $\dim(M) := \dim(\mathrm{aff}(M))$.

Furthermore, we call 0–dimensional affine spaces *points*, 1–dimensional affine spaces *lines* and $(n-1)$–dimensional affine sub–spaces of n–dimensional affine spaces are *hyperplanes*.

Convex Combinations and Convex Sets

A *convex combination* of vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_d \in \mathbb{R}^n$ is a vector of the form $\lambda_1 \boldsymbol{x}_1 + \ldots + \lambda_d \boldsymbol{x}_d$, where $\lambda_1, \ldots, \lambda_d$ stand for non–negative reals satisfying $\sum_{i=1}^d \lambda_i = 1$.

A subset $C \subseteq \mathbb{R}^n$ is called *convex* if any convex combination of vectors from C lies in C as well. Observe that convex combinations are just special cases of linear resp. affine combinations. Therefore, all linear sub–spaces and all affine sub–spaces of a finite dimensional vector space are convex.

Given a non–empty subset $M \subseteq \mathbb{R}^n$ the *convex hull*, denoted $\mathrm{conv}(M)$, is the set of all convex combinations of its elements, and it can be shown that this is the smallest convex set containing M. The convex hull of two points $\boldsymbol{a}$ and $\boldsymbol{b} \in \mathbb{R}^n$, i.e. $[a, b] := \mathrm{conv}(\boldsymbol{a}, \boldsymbol{b})$, is called the line segment connecting $\boldsymbol{a}$ and $\boldsymbol{b}$, or, the interval with endpoints $\boldsymbol{a}$ and $\boldsymbol{b}$. Moreover, it is easy to see that a set is convex if and only if it contains all the line segments connecting any pair of its elements.

Up to this point the theory of convex sets could be developed in complete analogy to the theory of linear and affine spaces. The concept of a basis has in general no analog for convex sets, since in general no finite set of elements of M suffices to generate $\mathrm{conv}(M)$. Therefore, we define the dimension of a convex set C, for short $\dim(C)$, as the dimension of the affine sub–space spanned by this set, i.e. $\dim(C) := \dim(\mathrm{aff}(C))$.

If $S = \mathrm{conv}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k)$ for affinely independent vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k \in \mathbb{R}^n$, then each element of the set S possesses a unique representation as convex combination of these vectors. In this sense the family $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$ is a "convex basis" for S, and such sets are called *simplices*.

A map from the vector space $\mathbb{R}^n$ into the vector space $\mathbb{R}^m$ given as $\boldsymbol{x} \mapsto A\boldsymbol{x} + \boldsymbol{b}$, where A denotes an $n \times m$ matrix and $\boldsymbol{b}$ a fixed vector in $\mathbb{R}^m$, is called an affine map. Two sets $P \subseteq \mathbb{R}^n$ and $Q \subseteq \mathbb{R}^m$ are called affinely isomorphic, denoted by $P \cong Q$, if there exists an affine map $f : \mathbb{R}^n \to \mathbb{R}^m$ that induces a bijection between the elements of P and Q. Observe that an affine map maps integral points to points with integral coordinates if and only if the matrix A as well as the vector $\boldsymbol{b}$ have solely integral entries.

Euclidean Spaces

A finite dimensional affine space whose underlying linear space is equipped with an inner product is called an Euclidean space. Most of the sets considered in this thesis are subsets of the n–dimensional real Euclidean space $\mathbb{R}^n$. In the sequel, we shall consider the standard inner product of the vector space $\mathbb{R}^n$, i.e. for all pairs of vectors $\boldsymbol{x}$ and $\boldsymbol{y} \in \mathbb{R}^n$, the inner product $\langle \boldsymbol{x}, \boldsymbol{y} \rangle := \sum_{i=1}^{n} x_i y_i$. An inner product exhibits positivity, $\langle \boldsymbol{x}, \boldsymbol{x} \rangle \geq 0$, symmetry, $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \langle \boldsymbol{y}, \boldsymbol{x} \rangle$, and bilinearity, $\langle \boldsymbol{x} + \lambda \boldsymbol{y}, \boldsymbol{z} \rangle = \langle \boldsymbol{x}, \boldsymbol{z} \rangle + \lambda \langle \boldsymbol{y}, \boldsymbol{z} \rangle$, for all vectors $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^n$ and all reals $\lambda \in \mathbb{R}$.

Observe that, denoting the row vector in $(\mathbb{R}^n)^*$ which has the same entries as the vector $\boldsymbol{x} \in \mathbb{R}^n$ by $\boldsymbol{x}^T$, we obtain that the inner product $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^T \boldsymbol{y}$, according to the rules of matrix multiplication.

Moreover, the *norm* induced by the standard inner product is given by $\|\boldsymbol{x}\| := \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle}$, while the corresponding metric $\mathrm{d}(\boldsymbol{x}, \boldsymbol{y}) := \sqrt{\langle \boldsymbol{x} - \boldsymbol{y}, \boldsymbol{x} - \boldsymbol{y} \rangle}$. The angle between two vectors $\boldsymbol{x}$ and $\boldsymbol{y} \in \mathbb{R}^n$ is defined as $\angle(\boldsymbol{x}, \boldsymbol{y}) := \arccos\left(\langle \boldsymbol{x}, \boldsymbol{y} \rangle / (\|\boldsymbol{x}\| \, \|\boldsymbol{y}\|)\right)$, and $\boldsymbol{x}$ and $\boldsymbol{y}$ are orthogonal if and only if the inner product $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0$.

Given a linear sub–space $L \subseteq \mathbb{R}^n$, the orthogonal complement, $L^\perp$, is defined as the linear space consisting of all vectors $\boldsymbol{x} \in \mathbb{R}^n$ for which the inner product $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0$, for all elements $\boldsymbol{y} \in L$. It is a well known fact that the dimension of the orthogonal complement of a d–dimensional linear sub–space of $\mathbb{R}^n$ is given by $n - d$.

Hyperplanes and Half–Spaces

The orthogonal complement of an $(n-1)$–dimensional linear sub–space $L \subseteq \mathbb{R}^n$ is given by a 1–dimensional linear sub–space, i.e. there exists a vector $\boldsymbol{a} \in \mathbb{R}^n$ such that $L^\perp = \mathrm{span}(\boldsymbol{a})$. Moreover, since $(L^\perp)^\perp = L$, we obtain that $L = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}, \boldsymbol{x} \rangle = 0\}$.

Now, recall that hyperplanes are $(n-1)$–dimensional affine sub–spaces of the vector space $\mathbb{R}^n$. Accordingly, for any hyperplane $\mathcal{H} \subseteq \mathbb{R}^n$ there exists an $(n-1)$–dimensional linear sub–space $L \subseteq \mathbb{R}^n$ and a vector $\boldsymbol{y} \in \mathcal{H}$ such that $\mathcal{H} = \boldsymbol{y} + L$. Therefore, using the same notation as above, we obtain that the hyperplane

$$\begin{aligned} \mathcal{H} &= \boldsymbol{y} + \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}, \boldsymbol{x} \rangle = 0\} = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}, \boldsymbol{x} - \boldsymbol{y} \rangle = 0\} \\ &= \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}, \boldsymbol{x} \rangle = c\} = \{\boldsymbol{x} \in \mathbb{R}^n : a_1 x_1 + \ldots + a_n x_n = c\}, \end{aligned} \tag{1.1}$$

where we set $c = \langle \boldsymbol{a}, \boldsymbol{y} \rangle$. On the other hand, any set of this form defines, obviously, a unique hyperplane in $\mathbb{R}^n$. Any hyperplane $\mathcal{H} \subseteq \mathbb{R}^n$ divides the vector space $\mathbb{R}^n$ into two *closed half–spaces*

$$\mathcal{H}^{\leq} := \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}, \boldsymbol{x} \rangle \leq c\} \quad \text{and} \quad \mathcal{H}^{\geq} := \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}, \boldsymbol{x} \rangle \geq c\}. \tag{1.2}$$

Note that the intersection of these half–spaces $\mathcal{H}^{\leq} \cap \mathcal{H}^{\geq} = \mathcal{H}$. Since the set $\{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}, \boldsymbol{x} \rangle = c'\} = \{\boldsymbol{x} \in \mathbb{R}^n : \langle -\boldsymbol{a}, \boldsymbol{x} \rangle = -c'\}$, any half–space can be described as $\mathcal{H}^{\leq}$ with an appropriate vector $\boldsymbol{a} \in \mathbb{R}^n$ and a suitable real $c \in \mathbb{R}$. If a hyperplane contains the origin we call it *linear* and the corresponding half–spaces are called *linear half–spaces*. Note that the intersection of finitely many half–spaces

$$\begin{aligned} \mathcal{H}_1^{\leq} &= \{\boldsymbol{x} \in \mathbb{R}^n : a_{1,1}x_1 + \ldots a_{1,n}x_n \leq b_1\} \\ &\vdots \\ \mathcal{H}_s^{\leq} &= \{\boldsymbol{x} \in \mathbb{R}^n : a_{s,1}x_1 + \ldots a_{s,n}x_n \leq b_s\}, \end{aligned} \tag{1.3}$$

agrees with the set $P = \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} \leq \boldsymbol{b}\}$, where A denotes the $s \times n$ matrix

$$A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \vdots & \vdots \\ a_{s,1} & \cdots & a_{s,n} \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_s \end{pmatrix}. \tag{1.4}$$

1.2.2 Polytopes and Polyhedral Sets

In the literature one may find several different definitions of polytopes, and not all of them yield exactly the same sets. Here, we demonstrate the equivalence of two common approaches to define these sets. In the sequel, it will be useful to bear in mind both definitions stated below. Moreover, we shall prove that any polytope, as to our definition, can be triangulated into finitely many simplices. This shows that polytopes as to our definition are also polytopes according to the definition given in Eugène Ehrhart's book [9].

Definition 1.1. A set which can be written as the intersection of finitely many closed half–spaces in $\mathbb{R}^n$ is called an *$\mathcal{H}$–polyhedron.* Put differently, a subset $P \subseteq \mathbb{R}^n$ is an $\mathcal{H}$–polyhedron if and only if there exists an $m \times n$ matrix A and a vector $\boldsymbol{b} \in \mathbb{R}^m$ such that

$$P = P(A, \boldsymbol{b}) := \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} \leq \boldsymbol{b}\}. \tag{1.5}$$

Definition 1.2. A polytope is a set which can be specified in one of the following ways.

- An $\mathcal{H}$–polyhedron which is bounded, i.e. does not contain a ray $\{\boldsymbol{x} + \lambda \boldsymbol{y} : \lambda \geq 0\}$, for a non–zero vector $\boldsymbol{y} \in \mathbb{R}^n$ and an element $\boldsymbol{x} \in \mathbb{R}^n$, is called an *$\mathcal{H}$–polytope.* We shall refer to this as the *hyperplane description* of a polytope.

- A *$\mathcal{V}$–polytope* is the convex hull of finitely many points $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m \in \mathbb{R}^n$, i.e.

$$\mathcal{P} := \mathrm{conv}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m) = \{\lambda_1 \boldsymbol{v}_1 + \ldots + \lambda_m \boldsymbol{v}_m : \lambda_i \in \mathbb{R}_{\geq 0}, \sum_{i=1}^{m} \lambda_i = 1\}, \tag{1.6}$$

  and we call this the *vertex description* of a polytope.

Note that the constraint that a polyhedron does not contain any ray is equivalent to the condition that it fits in a ball of finite radius, which is the definition of boundedness for general subsets of $\mathbb{R}^n$. A proof of this equivalence may be found in [11], Section 2.5.

A d–dimensional polytope or, for short, a *d–polytope*, is the convex hull of at least $d + 1$ vectors.

Definition 1.3. A d–polytope which is the convex hull of exactly $d + 1$ affinely independent vectors is called a *d–simplex.*

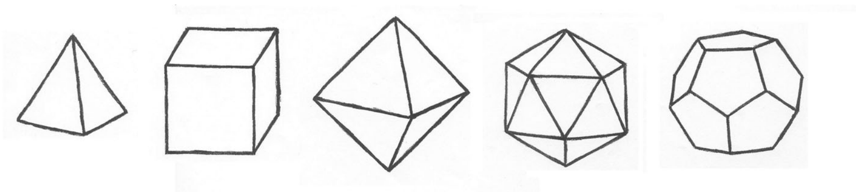


Figure 1.1: Some polytopes.

There is another class of polyhedral sets which will play an important role in our subsequent analysis. Again we consider two different approaches towards the definition of these sets that will turn out to be equivalent.

Definition 1.4. A polyhedral cone or, for short, a cone, is a subset of the vector space $\mathbb{R}^n$ which can be described as follows.

- If an $\mathcal{H}$–polyhedron is actually the intersection of finitely many *linear* half–spaces, we call it an *$\mathcal{H}$–cone*. That is, for any $\mathcal{H}$–cone $\mathcal{C} \subseteq \mathbb{R}^n$ there exists an $m \times n$ matrix A such that

$$\mathcal{C} = P(A, \mathbf{0}) = \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} \leq \mathbf{0}\}. \tag{1.7}$$

- A *$\mathcal{V}$–cone* is the non–negative hull of finitely many vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k \in \mathbb{R}^n$, i.e.

$$\mathcal{C} = \operatorname{cone}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k) := \{\lambda_1 \boldsymbol{v}_1 + \ldots + \lambda_k \boldsymbol{v}_k : \lambda_i \in \mathbb{R}_{\geq 0}\}. \tag{1.8}$$

Remark 1.5. According to the upper definition, polyhedral cones are not necessarily acute. In fact, this terminology includes infinite wedges as well as half–spaces.

Our next task is to accomplish the equivalence of the vertex and hyperplane description of polytopes. We shall give a constructive proof of this property, which uses several methods that will recur later on during the proof of Ehrhart's Theorem in the next section. Our main tool is a method called Fourier–Motzkin Elimination.

Definition 1.6. Given a subset $P \subseteq \mathbb{R}^n$, we define the *projection of P in direction $\boldsymbol{e}_k$* as

$$\operatorname{proj_k}(P) := \{\boldsymbol{x} - x_k \boldsymbol{e}_k : \boldsymbol{x} \in P\} = \{\boldsymbol{x} \in \mathbb{R}^n : x_k = 0, \exists \lambda \in \mathbb{R} : \boldsymbol{x} + \lambda \boldsymbol{e}_k \in \mathcal{P}\}. \tag{1.9}$$

The projection in direction $\boldsymbol{e}_k$ is always contained in the sub–space $\{\boldsymbol{x} \in \mathbb{R}^d : x_k = 0\}$.

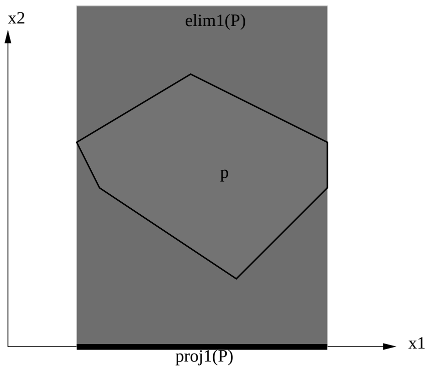
Moreover, the k^{th}*–elimination* of a set $P \subseteq \mathbb{R}^n$ is given by

$$\mathrm{elim_k}(P) := \{\boldsymbol{x} - \lambda \boldsymbol{e}_k : \boldsymbol{x} \in P, \lambda \in \mathbb{R}\}. \tag{1.10}$$

Remark 1.7. The k^{th}–elimination of P consists of all elements of the vector space $\mathbb{R}^n$ whose projection with respect to the k–th coordinate lies in $\mathrm{proj_k}(P)$, i.e.

$$\mathrm{elim_k}(P) \quad \cong \quad \mathrm{proj_k}(P) \times \mathbb{R} \quad \text{and} \tag{1.11}$$

$$\mathrm{proj_k}(P) \quad = \quad \mathrm{elim_k}(P) \cap \{\boldsymbol{x} \in \mathbb{R}^n : x_k = 0\}. \tag{1.12}$$



Theorem 1.8 (Fourier–Motzkin Elimination). *Let P be the $\mathcal{H}$–polyhedron $P(A, \boldsymbol{b})$, specified by an $m \times n$ matrix A and a vector $\boldsymbol{b} \in \mathbb{R}^m$. Then, for any fixed index $1 \leq k \leq n$ there exists an $m' \times n$ matrix $A^{\setminus k}$ and a vector $\boldsymbol{b}^{\setminus k} \in \mathbb{R}^{m'}$ such that*

$$\mathrm{elim_k}(P) = \{\boldsymbol{x} \in \mathbb{R}^n : A^{\setminus k}\boldsymbol{x} \leq \boldsymbol{b}^{\setminus k}\}. \tag{1.13}$$

In particular, we obtain that the k–th elimination of an $\mathcal{H}$–polyhedron is again an $\mathcal{H}$–polyhedron.

Proof. Denoting by $a_{i,k}$ the k–th entry of the i–th row of the matrix A, we obtain that a point $\boldsymbol{x} \in P$ if and only if it satisfies the inequalities

$$\begin{aligned} a_{1,1}x_1 + \ldots + a_{1,n}x_n &\leq b_1 \\ &\vdots \\ a_{m,1}x_1 + \ldots + a_{m,n}x_n &\leq b_m. \end{aligned} \tag{1.14}$$

We will now consider the coefficients $a_{i,k}$ of the variable x_k in this system of linear inequalities. By setting

$$\begin{aligned} A_> &:= \{i \in \{1,\dots,m\} : a_{i,k} > 0\}, \\ A_< &:= \{j \in \{1,\dots,m\} : a_{j,k} < 0\} \text{ and} \\ A_0 &:= \{l \in \{1,\dots,m\} : a_{l,k} = 0\} \end{aligned} \tag{1.15}$$

we obtain a partition of the index set $\{1,\dots,m\}$.

Given an index $i \in A_>$, a transformation of the corresponding inequality in (1.14) yields an upper bound for x_k, i.e.

$$x_k \le \frac{1}{a_{i,k}} \left(b_i - \langle \boldsymbol{a}_i, \boldsymbol{x}\rangle + a_{i,k}x_k\right), \tag{1.16}$$

where $\boldsymbol{a}_i$ denotes the i–th row vector of the matrix A. By the same manner we obtain a lower bound for the variable x_k for any index $j \in A_<$, namely

$$x_k \ge \frac{1}{a_{j,k}} \left(b_j - \langle \boldsymbol{a}_j, \boldsymbol{x}\rangle + a_{j,k}x_k\right). \tag{1.17}$$

Note that the right–hand sides of these inequalities do not depend on x_k and the condition that every lower bound on x_k is smaller than every upper bound is necessary for the solvability of the system of inequalities in (1.14).

Now, multiplying equation (1.16) by $a_{i,k} \cdot (-a_{j,k})$, and equation (1.17) by $a_{i,k} \cdot (-a_{j,k})$ yields that

$$a_{i,k}(-a_{j,k})x_k \quad \le \quad (-a_{j,k})\left(b_i - \langle \boldsymbol{a}_i, \boldsymbol{x}\rangle + a_{i,k}x_k\right) \text{ and} \tag{1.18}$$

$$a_{i,k}(-a_{j,k})x_k \quad \ge \quad (-a_{i,k})\left(b_j - \langle \boldsymbol{a}_j, \boldsymbol{x}\rangle + a_{j,k}x_k\right). \tag{1.19}$$

Combining these inequalities we obtain, after some basic transformations, that the condition "lower bound on x_k below upper bound" is equivalent to the "eliminating inequalities"

$$\langle a_{i,k}\boldsymbol{a}_j + (-a_{j,k})\boldsymbol{a}_i, \boldsymbol{x}\rangle \quad \le \quad a_{i,k}b_j + (-a_{j,k})b_i, \tag{1.20}$$

for $i \in A_<$ and $j \in A_<$. As these inequalities are positive combinations of the defining inequalities of the polyhedron P, they are valid for all elements $\boldsymbol{x} \in P$. Therefore, every element of the k–th elimination $\mathrm{elim_k}(P)$ satisfies all these inequalities. Conversely, if a vector $\boldsymbol{x} \in \mathbb{R}^n$ satisfies the eliminated inequalities (1.20) for all pairs $i \in A_<$ and $j \in A_<$, and in addition all inequalities of the original system (1.14) which do not affect x_k, it is possible to change the k–th coordinate of the vector $\boldsymbol{x}$ such that the resulting vector $\boldsymbol{x}'$ satisfies $A\boldsymbol{x}' \leq \boldsymbol{b}$, i.e. the point $\boldsymbol{x}$ is then contained in the set $\mathrm{elim_k}(P)$.

To formalize this idea, let $A^{\backslash k}$ denote the matrix consisting of the row vectors $\boldsymbol{a}_l$ for all $l \in A_0$ and the vectors $(a_{i,k}\boldsymbol{a}_j + (-a_{j,k})\boldsymbol{a}_i)$ for all pairs $(i,j) \in A_> \times A_<$. Accordingly, we denote the vector with coordinates b_l for $l \in A_0$ and $a_{i,k}b_j + (-a_{j,k})b_i$ for all pairs $(i,j) \in A_> \times A_<$ by $\boldsymbol{b}^{\backslash k}$. Then, according to our discussion above, we have that

$$\mathrm{elim_k}(P) = \{\boldsymbol{x} \in \mathbb{R}^n : A^{\backslash k}\boldsymbol{x} \leq b^{\backslash k}\}. \tag{1.21}$$

□

Corollary 1.9. *If $P \subseteq \mathbb{R}^n$ is an $\mathcal{H}$–polyhedron then so is the set $proj_k(P)$.*

Proof. Observe that $\mathrm{proj_k}(P) = \mathrm{elim_k}(P) \cap \{\boldsymbol{x} \in \mathbb{R}^n : x_k = 0\}$. □

Corollary 1.10. *The k–th elimination as well as the projection in direction $\boldsymbol{e}_k$ of an $\mathcal{H}$–cone are again $\mathcal{H}$–cones.*

Proof. Suppose that $\mathcal{C} = P(A, \boldsymbol{0}) \subseteq \mathbb{R}^n$ is a $\mathcal{H}$–cone, for an $m \times n$ matrix A. Since every $\mathcal{H}$–cone is at the same time an $\mathcal{H}$–polyhedron, there exists an $m' \times n$ matrix $A^{\backslash k}$ and a vector $\boldsymbol{b}^{\backslash k} \in \mathbb{R}^{m'}$ such that $\mathrm{elim_k}(\mathcal{C}) = \{\boldsymbol{x} \in \mathbb{R}^n : A^{\backslash k}\boldsymbol{x} \leq \boldsymbol{b}^{\backslash k}\}$. In consideration of the construction in the proof of Theorem 1.8, we obtain in the actual case that $\boldsymbol{b}^{\backslash k} = \boldsymbol{0}$, since the coordinates of the vector $\boldsymbol{b}^{\backslash k}$ are obtained as linear combinations of coordinates of $\boldsymbol{0} \in \mathbb{R}^m$. Thus, the set $\mathrm{elim_k}(\mathcal{C}) = \{\boldsymbol{x} \in \mathbb{R}^n : A^{\backslash k}\boldsymbol{x} \leq \boldsymbol{0}\}$ is an $\mathcal{H}$–cone and so is the set $\mathrm{proj_k}(\mathcal{C}) = \mathrm{elim_k}(\mathcal{C}) \cap \{\boldsymbol{x} \in \mathbb{R}^n : x_k = 0\}$. □

Theorem 1.11. *A subset $P \subseteq \mathbb{R}^n$ is a $\mathcal{V}$–polytope, i.e.*

$$P = \mathrm{conv}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k) \text{ with } \boldsymbol{v}_i \in \mathbb{R}^n \text{ for } 1 \leq i \leq k, \tag{1.22}$$

if and only if it is a bounded intersection of finitely many closed half–spaces of $\mathbb{R}^n$,

$$P = P(A, \boldsymbol{b}) \text{ for } A \in \mathbb{R}^{m \times n} \text{ and } \boldsymbol{b} \in \mathbb{R}^m, \tag{1.23}$$

or, in other words, an $\mathcal{H}$–polytope.

Theorem 1.12. *A subset $\mathcal{C} \subseteq \mathbb{R}^n$ is the non–negative hull of a finite set of elements of $\mathbb{R}^n$, that is a $\mathcal{V}$–cone, i.e.*

$$\mathcal{C} = \operatorname{cone}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n) \text{ for some } \boldsymbol{v}_i \in \mathbb{R}^n \text{ for } 1 \leq i \leq n, \tag{1.24}$$

if and only if

$$\mathcal{C} = P(A, \mathbf{0}) \text{ for some } A \in \mathbb{R}^{n \times m}, \tag{1.25}$$

or, in other words, an $\mathcal{H}$–cone.

In order to prove these two theorems we demonstrate, at first, that the equivalence of the two descriptions of polytopes can be deduced from Theorem 1.12. For this purpose we introduce a method called "coning over a polytope". Subsequently, we give proof of Theorem 1.12 by two lemmata.

Definition 1.13. Let $\mathcal{P} = \operatorname{conv}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k) \subseteq \mathbb{R}^n$ be a $\mathcal{V}$–polytope. The *cone over the polytope* is defined as

$$\operatorname{cone}(\mathcal{P}|1) := \{\lambda_1 \boldsymbol{w}_1 + \lambda_2 \boldsymbol{w}_2 + \ldots + \lambda_k \boldsymbol{w}_k : \lambda_i \in \mathbb{R}_{\geq 0}\}, \tag{1.26}$$

where $\boldsymbol{w}_i$ denotes the vector $(\boldsymbol{v}_i, 1) \in \mathbb{R}^{n+1}$, for all indices $1 \leq i \leq k$.

Lemma 1.14. *The equivalence of the vertex and hyperplane description of polytopes can be deduced from the equivalence of the vertex and hyperplane description of cones.*

Proof. Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a $\mathcal{V}$–polytope and observe that

$$P \cong \operatorname{cone}(\mathcal{P}|1) \cap \{\boldsymbol{x} \in \mathbb{R}^{n+1} : x_{n+1} = 1\}, \tag{1.27}$$

by omitting the $(n+1)$–st coordinate of each vector. Provided that Theorem 1.12 is valid, there is an $m \times (n+1)$ matrix A such that $\operatorname{cone}(\mathcal{P}|1) = \{\boldsymbol{x} \in \mathbb{R}^{n+1} : A\boldsymbol{x} \leq \mathbf{0}\}$. Let us denote the $(n+1)$–st column vector of the matrix A by $\boldsymbol{b}$ and the $m \times n$ matrix which is obtained by removing the vector $\boldsymbol{b}$ from the matrix A by A' . Then

$$\begin{aligned} \mathcal{P} &\cong \{\boldsymbol{x} \in \mathbb{R}^{n+1} : A\boldsymbol{x} \leq \mathbf{0}\} \cap \{\boldsymbol{x} \in \mathbb{R}^{n+1} : x_{n+1} = 1\} \\ &\cong \{\boldsymbol{x} \in \mathbb{R}^n : A'\boldsymbol{x} \leq -\boldsymbol{b}\}, \end{aligned} \tag{1.28}$$

which gives a hyperplane description of the polytope $\mathcal{P}$.

Conversely, suppose a non–empty polytope is given by $\mathcal{P} = \{\boldsymbol{x} \in \mathbb{R}^n : A'\boldsymbol{x} \leq -\boldsymbol{b}\}$ for an $m \times n$ matrix A' and a vector $\boldsymbol{b} \in \mathbb{R}^m$. Equivalently, we have that the polytope $\mathcal{P} = \{\boldsymbol{x} \in \mathbb{R}^n : A'\boldsymbol{x} + \boldsymbol{b} \leq \boldsymbol{0}\}$. If we denote the $m \times (n+1)$ matrix that is obtained by adding the vector $\boldsymbol{b}$ as $(n+1)$–st column to the matrix A' by A, we obtain that

$$\mathcal{P} \quad \cong \quad \{\boldsymbol{x} \in \mathbb{R}^{n+1} : A\boldsymbol{x} \leq \boldsymbol{0}, x_{n+1} \geq 0\} \cap \{\boldsymbol{x} \in \mathbb{R}^{n+1} : x_{n+1} = 1\}. \tag{1.29}$$

If the equivalence of the two descriptions of cones is valid, then there are vectors $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_k \in \mathbb{R}^{n+1}$ such that

$$\{\boldsymbol{x} \in \mathbb{R}^{n+1} : A\boldsymbol{x} \leq \boldsymbol{0}, x_{n+1} \geq 0\} = \operatorname{cone}(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_k) =: \mathcal{C}. \tag{1.30}$$

Since the intersection of this cone with the hyperplane $\{\boldsymbol{x} \in \mathbb{R}^{n+1} : x_{n+1} = 1\}$ is, by assumption, a non–empty polytope, and as such bounded, the coordinate $x_{n+1} \neq 0$ for all elements $\boldsymbol{x} \in \mathcal{C} \backslash \{\boldsymbol{0}\}$. Note that, otherwise, if there was a vector $\boldsymbol{y} \in \mathcal{C} \backslash \{\boldsymbol{0}\}$ with $y_{n+1} = 0$, then for any element $\boldsymbol{x} \in \mathcal{C}$ with $x_{n+1} = 1$ the ray $\{\boldsymbol{x} + \lambda \boldsymbol{y} : \lambda \geq 0\}$ would be contained in the intersection $\mathcal{C} \cap \{\boldsymbol{x} \in \mathbb{R}^{n+1} : x_{n+1} = 1\} = \mathcal{P}$ as well. Therefore, we have that the $(n+1)$–st coordinate of all vectors $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_k$ is non–zero. Moreover, if we set $\boldsymbol{w}'_i := (1/(\boldsymbol{w}_i)_{n+1})\ \boldsymbol{w}_i$, the $(n+1)$–st coordinate of the vectors $\boldsymbol{w}'_1, \ldots, \boldsymbol{w}'_k$ equals one and these vectors generate the same cone as the elements $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_k \in \mathbb{R}^{n+1}$. For $1 \leq i \leq k$ there exists a vector $\boldsymbol{v}_i \in \mathbb{R}^n$ such that $\boldsymbol{w}'_i = (\boldsymbol{v}_i, 1)^T$, and we obtain that

$$P \cong \operatorname{cone}(\boldsymbol{w}'_1, \ldots, \boldsymbol{w}'_n) \cap \{\boldsymbol{x} \in \mathbb{R}^{n+1} : x_{n+1} = 1\} \cong \operatorname{conv}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n). \tag{1.31}$$

□

The following lemma establishes the first direction of the proof of Theorem 1.12.

Lemma 1.15. *Every $\mathcal{V}$–cone is an $\mathcal{H}$–cone.*

Proof. Given $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_k \in \mathbb{R}^n$ we consider the $\mathcal{V}$–cone

$$\mathcal{K} = \operatorname{cone}(\boldsymbol{b}_1, \ldots \boldsymbol{b}_k) = \{y_1 \boldsymbol{b}_1 + \ldots + y_k \boldsymbol{b}_k : y_i \geq 0\} = \{B\boldsymbol{y} : \boldsymbol{y} \in \mathbb{R}^k_{\geq 0}\}. \tag{1.32}$$

Here B denotes the $n \times k$ matrix with column vectors $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_k$. Alternatively, we may describe the cone $\mathcal{K}$ as the projection of the set

$$\mathcal{K}' \quad = \quad \left\{ \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} \in \mathbb{R}^{n+k} : \boldsymbol{y} \geq \boldsymbol{0}, \boldsymbol{x} = B\boldsymbol{y} \right\} = \left\{ \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} \in \mathbb{R}^{n+k} : (\mathbb{I}_k, -B) \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} = \boldsymbol{0}, \boldsymbol{y} \geq \boldsymbol{0} \right\}$$

to the subspace $\left\{(\boldsymbol{x},\boldsymbol{y})^T \in \mathbb{R}^{n+k} : \boldsymbol{y} = 0\right\} \subseteq \mathbb{R}^{n+k}$. Here, $\mathbb{I}_n$ denotes the $n \times n$ identity matrix. Obviously, the set $\mathcal{K}'$ is an $\mathcal{H}$–cone. Thus, by Corollary 1.10, so is $\mathrm{proj_k}(\mathcal{K}')$. Therefore, projecting one component of the vector $\boldsymbol{y}$ at a time yields that the $\mathcal{V}$–cone $\mathcal{K}$ is an $\mathcal{H}$–cone. □

For the opposite direction of our proof, we need another lemma.

Lemma 1.16. *If $\mathcal{K} \subseteq \mathbb{R}^n$ is a $\mathcal{V}$–cone then so is the intersection $\mathcal{K} \cap \{\boldsymbol{x} \in \mathbb{R}^n : x_k = 0\}$.*

Proof. Suppose the $\mathcal{V}$–cone $\mathcal{K}$ is given by $\mathrm{cone}(\boldsymbol{b}_1, \ldots, \boldsymbol{b}_l)$ for some vectors $\boldsymbol{b}_i \in \mathbb{R}^n$. We construct a set B^k as follows. For all indices $1 \le j \le l$ such that the k–th coordinate of the vector $\boldsymbol{b}_j$, denoted by $(\boldsymbol{b}_j)_k$, equals zero, the vector $\boldsymbol{b}_j$ is an element of the set B^k. Moreover, for all pairs (i, j) of indices $1 \le i, j \le l$ such that $(\boldsymbol{b}_i)_k > 0$ and $(\boldsymbol{b}_j)_k < 0$ the set B^k includes the vector $(\boldsymbol{b}_i)_k\boldsymbol{b}_j - (\boldsymbol{b}_j)_k\boldsymbol{b}_i$. Thus, the k–th coordinate of all elements $\boldsymbol{x} \in \mathrm{cone}(B^k)$ equals zero, i.e.

$$\mathrm{cone}(B^k) \subseteq \mathcal{K} \cap \{\boldsymbol{x} \in \mathbb{R}^n : x_k = 0\}. \tag{1.33}$$

Conversely, suppose a vector $\boldsymbol{x}$ is contained in the set $\mathcal{K} \cap \{\boldsymbol{x} \in \mathbb{R}^n : x_k = 0\}$, i.e. $\boldsymbol{x} = \lambda_1\boldsymbol{b}_1 + \ldots + \lambda_l\boldsymbol{b}_l$ and $\lambda_1(\boldsymbol{b}_1)_k + \ldots + \lambda_l(\boldsymbol{b}_l)_k = 0$ for non–negative reals $\lambda_1, \ldots, \lambda_l \in \mathbb{R}$. We set

$$\Gamma := \sum_{i:(\boldsymbol{b}_i)_k>0} \lambda_i(\boldsymbol{b}_i)_k = -\sum_{j:(\boldsymbol{b}_j)_k<0} \lambda_j(\boldsymbol{b}_j)_k \ge 0. \tag{1.34}$$

Then we have that $\Gamma = 0$ if and only if $\lambda_i = 0$ for all $1 \le i \le l$ such that the coordinate $(\boldsymbol{b}_i)_k > 0$, and similarly $\lambda_j = 0$ for all $1 \le j \le l$ with $(\boldsymbol{b}_j)_k < 0$. Then the vector $\boldsymbol{x} = \sum_{m:(\boldsymbol{b}_m)_k=0} \lambda_m\boldsymbol{b}_m$ is contained in $\mathrm{cone}(B^k)$. Else, if the real Γ is positive, we obtain that

$$\begin{aligned}
\boldsymbol{x} &= \sum_{m:(\boldsymbol{b}_m)_k=0} \lambda_m\boldsymbol{b}_m + \frac{1}{\Gamma}\left(-\sum_{j:(\boldsymbol{b}_j)_k<0} \lambda_j(\boldsymbol{b}_j)_k\right)\left(\sum_{i:(\boldsymbol{b}_i)_k>0} \lambda_i\boldsymbol{b}_i\right) \\
&\quad + \frac{1}{\Gamma}\left(\sum_{i:(\boldsymbol{b}_i)_k>0} \lambda_i(\boldsymbol{b}_i)_k\right)\left(\sum_{j:(\boldsymbol{b}_j)_k<0} \lambda_j\boldsymbol{b}_j\right) \\
&= \sum_{l:(\boldsymbol{b}_l)_k=0} \lambda_l\boldsymbol{b}_l + \frac{1}{\Gamma}\sum_{i:(\boldsymbol{b}_i)_k>0, j:(\boldsymbol{b}_j)_k<0} \lambda_i\lambda_j((\boldsymbol{b}_i)_k\boldsymbol{b}_j - (\boldsymbol{b}_j)_k\boldsymbol{b}_i),
\end{aligned} \tag{1.35}$$

which is, according to our construction, contained in $\mathrm{cone}(B^k)$. □

We may now conclude the proof of the equivalence of the vertex and the hyperplane description of cones by the following lemma.

Lemma 1.17. *Every $\mathcal{H}$–cone is a $\mathcal{V}$–cone.*

Proof. Suppose an $\mathcal{H}$–cone $\mathcal{K}$ is given by $\{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} \leq \mathbf{0}\}$ with an $m \times n$ matrix A. Introducing an auxiliary variable $\boldsymbol{y} \in \mathbb{R}^m$ we obtain that

$$\mathcal{K} \cong \left\{ \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} \in \mathbb{R}^{n+m} : A\boldsymbol{x} \leq \boldsymbol{y} \right\} \cap \left\{ \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} \in \mathbb{R}^{n+m} : \boldsymbol{y} = \mathbf{0} \right\}. \tag{1.36}$$

Moreover, the set

$$\mathcal{K}' := \left\{ \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} \in \mathbb{R}^{n+m} : A\boldsymbol{x} \leq \boldsymbol{y} \right\} = \left\{ \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} \in \mathbb{R}^{n+m} : (A, -\mathbb{I}_m) \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} \leq \mathbf{0} \right\} \tag{1.37}$$

is an $\mathcal{H}$–cone. Here $\mathbb{I}_m$ denotes the $m \times m$ identity matrix. Given a vector $(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^{n+m}$, observe that

$$\begin{aligned} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} &= \begin{pmatrix} \boldsymbol{x} \\ A\boldsymbol{x} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ \boldsymbol{y} - A\boldsymbol{x} \end{pmatrix} = \sum_{j=1}^{d} x_j \begin{pmatrix} \boldsymbol{e}_j \\ A\boldsymbol{e}_j \end{pmatrix} + \sum_{k=1}^{m} (y_k - (A\boldsymbol{x})_k) \begin{pmatrix} \mathbf{0} \\ \boldsymbol{e}_k \end{pmatrix} \\ &= \sum_{j=1}^{d} |x_j| \operatorname{sgn}(\mathrm{x_j}) \begin{pmatrix} \boldsymbol{e}_j \\ A\boldsymbol{e}_j \end{pmatrix} + \sum_{k=1}^{m} (y_k - (A\boldsymbol{x})_k) \begin{pmatrix} \mathbf{0} \\ \boldsymbol{e}_k \end{pmatrix}. \end{aligned} \tag{1.38}$$

Note that a vector $(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^{n+m}$ is an element of the $\mathcal{H}$–cone $\mathcal{K}'$ if and only if $y_k - (A\boldsymbol{x})_k \geq 0$ for all $1 \leq k \leq m$. Hence, any vector $(\boldsymbol{x}, \boldsymbol{y})$ in the $\mathcal{H}$–cone $\mathcal{K}'$ is a non–negative linear combination of vectors $\pm(\boldsymbol{e}_j, A\boldsymbol{e}_j) \in \mathcal{K}'$ and $(\mathbf{0}, \boldsymbol{e}_k) \in \mathcal{K}'$, for $1 \leq j \leq n$ and $1 \leq k \leq m$. On the other hand, it is easy to check that any such vector is an element of $\mathcal{K}'$, c.f. (1.37). This shows that the $\mathcal{H}$–cone $\mathcal{K}'$ is at the same time a $\mathcal{V}$–cone, and it follows by Lemma 1.16 that the set $\mathcal{K}' \cap \{\boldsymbol{x} \in \mathbb{R}^n : x_k = 0\}$ is a $\mathcal{V}$–cone as well. Thus, by eliminating one coordinate of the vector $\boldsymbol{y}$ at a time, it is proved that the $\mathcal{H}$–cone $\mathcal{K}$ is at the same time a $\mathcal{V}$–cone. $\square$

1.2.3 Vertices, Faces and Facets

We will now discuss some basic notations of the theory of polytopes.

Definition 1.18. A hyperplane is called a *supporting hyperplane* of a polyhedron $P \subseteq \mathbb{R}^n$ if and only if P lies entirely within one of the two closed half–spaces of this hyperplane.

Definition 1.19. Suppose that $\mathcal{H}$ is a supporting hyperplane of a polyhedron $P \subseteq \mathbb{R}^n$. Then, the intersection

$$\mathcal{F} := P \cap \mathcal{H}, \tag{1.39}$$

is called a *face* of the polyhedron P.

A zero–dimensional face of a polyhedron is called a *vertex*, one–dimensional faces are referred to as *edges* and *facets* are $(d-1)$–dimensional faces of a d–polytope. We denote by $\mathrm{vert}(P)$ the set of the vertices of a polyhedron P.

Remark 1.20. Note that the hyperplanes occurring in the hyperplane description of a polyhedron are supporting hyperplanes. Therefore, the intersection of the polyhedron with one of these hyperplanes is always a face.

We shall now investigate some results in connection with the faces of polytopes. To this end, we start with a simple lemma which is well known in the in the theory of linear programming.

Lemma 1.21 (Farkas' Lemma). *Given an $m \times d$ matrix A and a vector $\boldsymbol{z} \in \mathbb{R}^m$, there exists* either *a vector $\boldsymbol{x} \in \mathbb{R}^d$ such that $A\boldsymbol{x} \leq \boldsymbol{z}$,* or *a row vector $\boldsymbol{c} \in (\mathbb{R}^m)^*$ with $\boldsymbol{c} \geq \boldsymbol{0}$ such that $\boldsymbol{c}A = \boldsymbol{0}$ and $\boldsymbol{c}\boldsymbol{z} < 0$,* but not both.

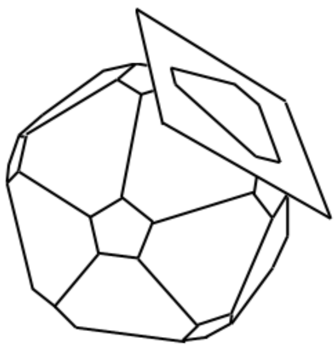


Figure 1.2: A face of a polytope.

Proof. At first, we observe that both conditions can not hold at the same time. Otherwise there would be a vector $\boldsymbol{x} \in \mathbb{R}^d$ and a row vector $\boldsymbol{c} \in (\mathbb{R}^m)^*$ such that

$$0 = \mathbf{0}\boldsymbol{x} = (\boldsymbol{c}A)\boldsymbol{x} = \boldsymbol{c}(A\boldsymbol{x}) \leq \boldsymbol{c}\boldsymbol{z} < 0. \tag{1.40}$$

Now, we define a polyhedron $\mathcal{P} := P(A, \boldsymbol{z}) = \{\boldsymbol{x} \in \mathbb{R}^d : A\boldsymbol{x} \leq \boldsymbol{z}\}$. Observe that this polyhedron is non–empty if and only if the cone,

$$\mathcal{Q} := P((-\boldsymbol{z}, A), \mathbf{0}) = \{\boldsymbol{x} \in \mathbb{R}^{d+1} : (-\boldsymbol{z}, A)\boldsymbol{x} \leq \mathbf{0}\}, \tag{1.41}$$

contains a vector $\boldsymbol{x}$ where the coordinate $x_1 > 0$. Now, we apply Fourier–Motzkin Elimination, Theorem 1.8, in order to construct the $\mathcal{H}-$ cone

$$(\mathrm{elim_2}(\mathrm{elim_3}(\dots(\mathrm{elim_{d+1}}(Q))))). \tag{1.42}$$

According to the proof of the Fourier–Motzkin Elimination, we obtain that the i–th elimination $\mathrm{elim_i}(P(D, \mathbf{0}))$ equals $P(D^{\setminus i}, \mathbf{0})$, where the left–hand side of each inequality in the eliminated system $P(D^{\setminus i}, \mathbf{0})$ is a positive combination of at most two rows of the matrix D. Thus, the matrix $D^{\setminus i}$ can be written as a product $C_i D$, where C_i denotes a matrix with solely non–negative entries of which at most two per row are non–zero. By repeated application of this procedure, we obtain that

$$\begin{aligned}(\mathrm{elim_2}(\mathrm{elim_3}(\dots(\mathrm{elim_{d+1}}(Q))))) &= P((-\boldsymbol{z}A)^{\setminus d+1 \setminus d \setminus \dots \setminus 2}, \mathbf{0}) \\ &= P(C_2 C_3 \dots C_{d+1}(-\boldsymbol{z}A), \mathbf{0}) \\ &= P(C(-\boldsymbol{z}A), \mathbf{0}),\end{aligned} \tag{1.43}$$

where C is a non–negative matrix. The inequalities in the system $C\ (-\boldsymbol{z}A)\ \boldsymbol{x} \leq 0$ are given as $\gamma_{i1}x_1 \leq 0$, since all variables different from x_1 have been eliminated.

Suppose that the set $\mathcal{P}$ is empty, i.e. there exists no vector $\boldsymbol{x} \in \mathbb{R}^d$ with $A\boldsymbol{x} \leq \boldsymbol{z}$. According to our previous thoughts, this is the case if and only if the cone Q lies within the half–space $\{\boldsymbol{x} \in \mathbb{R}^{d+1} : x_1 \leq 0\}$, and

$$(\mathrm{elim_2}(\mathrm{elim_3}(\dots(\mathrm{elim_{d+1}}(Q))))) \subseteq \{\boldsymbol{x} \in \mathbb{R}^{d+1} : x_1 \leq 0\}. \tag{1.44}$$

Thus, the system $C \cdot (-\boldsymbol{z}A)\boldsymbol{x} \leq 0$ has to contain an inequality $\gamma_{i1}x_1 \leq 0$ where the real

$\gamma_{i1} > 0$. Let $\boldsymbol{c}$ be the row vector of the matrix C that leads to this inequality. Then

$$\boldsymbol{c} \cdot (-\boldsymbol{z}, A) = (\gamma_{i0}, \mathbf{0}) \Rightarrow \boldsymbol{c} \cdot \boldsymbol{z} = -\gamma_{i0} < 0 \text{ and } \boldsymbol{c} \cdot A = \mathbf{0}. \tag{1.45}$$

□

Corollary 1.22. *Let A be an $m \times d$ matrix and $\boldsymbol{z} \in \mathbb{R}^m$ a vector. Then, there exists* either *a vector $\boldsymbol{x} \in \mathbb{R}^d_{\geq 0}$ such that $A\boldsymbol{x} = \boldsymbol{z}$,* or *a vector $\boldsymbol{c} \in \mathbb{R}^m$ with $A^T\boldsymbol{c} \geq \mathbf{0}$ and $\langle \boldsymbol{c}, \boldsymbol{z} \rangle < 0$,* but not both.

Proof. We assume there exists a vector $\boldsymbol{x} \in \mathbb{R}^d$ such that $A\boldsymbol{x} = \boldsymbol{z}$ and $\boldsymbol{x} \geq 0$. We can restate these conditions as $A\boldsymbol{x} \leq \boldsymbol{z}$, $A\boldsymbol{x} \geq \boldsymbol{z}$ and $-\boldsymbol{x} \leq \mathbf{0}$. Or, in a more compact notation, we obtain that

$$\begin{pmatrix} A \\ -A \\ -\mathbb{I}_d \end{pmatrix} \boldsymbol{x} \leq \begin{pmatrix} \boldsymbol{z} \\ -\boldsymbol{z} \\ \mathbf{0} \end{pmatrix}, \tag{1.46}$$

where the constructed matrix lies in $\mathbb{R}^{(2m+d)\times d}$.

Therefore, by Farkas' Lemma, Theorem 1.21, such a vector $\boldsymbol{x} \in \mathbb{R}^d$ exists if and only if there are no row vectors $\boldsymbol{c}_1 \in (\mathbb{R}^m_{\geq 0})^*$, $\boldsymbol{c}_2 \in (\mathbb{R}^m_{\geq 0})^*$ and $\mathbf{b} \in (\mathbb{R}^d_{\geq 0})^*$ such that

$$(\boldsymbol{c}_1, \boldsymbol{c}_2, \mathbf{b}) \begin{pmatrix} A \\ -A \\ -\mathbb{I}_d \end{pmatrix} = \mathbf{0} \text{ and } (\boldsymbol{c}_1, \boldsymbol{c}_2, \mathbf{b}) \begin{pmatrix} \boldsymbol{z} \\ -\boldsymbol{z} \\ \mathbf{0} \end{pmatrix} < \mathbf{0}. \tag{1.47}$$

Put differently, $(\boldsymbol{c}_1 - \boldsymbol{c}_2)\, A - \mathbf{b} = \mathbf{0}$ and $(\boldsymbol{c}_1 - \boldsymbol{c}_2)\, \boldsymbol{z} < 0$. Hence, if we set $\boldsymbol{c} = (\boldsymbol{c}_1 - \boldsymbol{c}_2)$, the assertion is equivalent to the condition that there is no row vector $\boldsymbol{c} \in (\mathbb{R}^m)^*$ and no row vector $\mathbf{b} \in (\mathbb{R}^d_{\geq 0})^*$ such that $\boldsymbol{c}A = \mathbf{b} \geq \mathbf{0}$ and $\boldsymbol{c}\boldsymbol{z} < 0$. The assertion follows since $\boldsymbol{c}A = A^T\boldsymbol{c}^T$ and $\boldsymbol{c}\boldsymbol{z} = \left\langle \boldsymbol{c}^T, \boldsymbol{z} \right\rangle$.

□

Theorem 1.23.

(i) Every polytope $\mathcal{P}$ is the convex hull of its vertices, i.e.

$$\mathcal{P} = \operatorname{conv}(\operatorname{vert}(\mathcal{P})).$$

(ii) If a polytope $\mathcal{P}$ can be written as the convex hull of a finite point set V, then this set contains all its vertices, i.e.

$$\mathcal{P} = \operatorname{conv}(V) \Rightarrow \operatorname{vert}(P) \subseteq V.$$

Proof. Let $V = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_d\}$ be a set of elements of $\mathbb{R}^n$ such that $\mathcal{P} = \operatorname{conv}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_d)$. Without loss of generality, we assume that none of the elements $\boldsymbol{v}_i \in V$ is a convex combination of the remaining ones. Expressed in a different manner, there is no proper subset $\widehat{V} \subsetneq V$ such that $\operatorname{conv}(\widehat{V}) = \operatorname{conv}(V)$.

The first claim of the theorem is valid if we can show that every element $\boldsymbol{v}_i \in V$ is a vertex of the polytope $\mathcal{P}$. Let us consider $\widehat{V}$ as the matrix whose column vectors are given by the elements of the set $V \backslash \{\boldsymbol{v}_i\}$. Since the number of elements of the set $V \backslash \{\boldsymbol{v}_i\}$ equals $d-1$, we obtain an $n \times (d-1)$–matrix, and a convex combination of vectors from $\widehat{V}$ can be displayed as matrix product $\widehat{V}\boldsymbol{t}$ for some vector $\boldsymbol{t} \in \mathbb{R}^{d-1}_{\geq \mathbf{0}}$ with $\|\boldsymbol{t}\|_1 = \sum_{k=0}^{n} t_i = 1$. If we denote the vector in $\mathbb{R}^{d-1}$ with all coordinates equal one by $\mathbb{1}$, we obtain that $\|\boldsymbol{t}\|_1 = \langle \mathbb{1}, \boldsymbol{t} \rangle = \mathbb{1}^T \boldsymbol{t}$. Hence, the assertion that the vector $\boldsymbol{v}_i$ is not contained in the convex hull $\operatorname{conv}(V \backslash \{\boldsymbol{v}_i\})$ is equivalent to the condition that there does not exist a vector $\boldsymbol{t} \in \mathbb{R}^{d-1}_{\geq \mathbf{0}}$ with

$$\begin{pmatrix} \mathbb{1}^T \\ \widehat{V} \end{pmatrix} \boldsymbol{t} = \begin{pmatrix} 1 \\ \boldsymbol{v}_i \end{pmatrix}. \tag{1.48}$$

Hence, by Corollary 1.22, there exists a vector $\boldsymbol{a} \in \mathbb{R}^{n+1}$ such that

$$\left(\mathbb{1}, \widehat{V}^T\right) \boldsymbol{a} \geq \mathbf{0} \text{ and } \left\langle \boldsymbol{a}, \begin{pmatrix} 1 \\ \boldsymbol{v}_i \end{pmatrix} \right\rangle < 0. \tag{1.49}$$

Now, if we write the vector $\boldsymbol{a}$ as $(\beta, -\boldsymbol{b})^T$, for a real β and an appropriate vector $\boldsymbol{b} \in \mathbb{R}^n$, (1.49) is equivalent to the condition that

$$\mathbb{1}\beta \geq \widehat{V}^T \boldsymbol{b} \quad \text{and} \quad \beta < \langle \boldsymbol{b}, \boldsymbol{v}_i \rangle. \tag{1.50}$$

Recall that the row vectors of the matrix $\widehat{V}^T$ are given by the elements ot the set $V \backslash \boldsymbol{v}_i$. Thus, according to the rules of matrix multiplication, we obtain that

$$\beta \geq \langle \boldsymbol{v}_j, \boldsymbol{b} \rangle \text{ for all } \boldsymbol{v}_j \in V \backslash \boldsymbol{v}_i \quad \text{ while } \beta < \langle \boldsymbol{b}, \boldsymbol{v}_i \rangle . \tag{1.51}$$

Therefore, the hyperplane $\mathcal{H} := \{\boldsymbol{x} \in \mathbb{R}^d : \langle \boldsymbol{x}, \boldsymbol{b} \rangle = \langle \boldsymbol{b}, \boldsymbol{v}_i \rangle\}$ is a supporting hyperplane of the polytope $\mathcal{P}$. Since the intersection $\mathcal{H} \cap \mathcal{P} = \boldsymbol{v}_i$, we conclude that the vector $\boldsymbol{v}_i$ is a vertex of the polytope $\mathcal{P}$.

Regarding the second claim of the theorem, we observe that a vertex can never be written as a convex combination of other points of the polytope. Note that, if $\boldsymbol{v}$ is a vertex of the polytope $\mathcal{P}$, there exists a hyperplane $\mathcal{H} := \{\boldsymbol{x} \in \mathbb{R}^d : \langle \boldsymbol{c}, \boldsymbol{x} \rangle = a\}$ such that $\langle \boldsymbol{c}, \boldsymbol{x} \rangle < a$ for all elements $\boldsymbol{x} \in \mathcal{P} \backslash \boldsymbol{v}$ while $\langle \boldsymbol{c}, \boldsymbol{v} \rangle = a$.

Now, suppose that $\boldsymbol{v} = \lambda_1 \mathbf{p}_1 + \ldots + \lambda_m \mathbf{p}_m$ for some points $\mathbf{p}_i \in \mathcal{P} \backslash \boldsymbol{v}$ and non–negative reals $\lambda_i \in \mathbb{R}$ with $\sum_{i=1}^m \lambda_i = 1$. Then

$$\langle \boldsymbol{c}, \boldsymbol{v} \rangle = \langle \boldsymbol{c}, \lambda_1 \mathbf{p}_1 + \ldots + \lambda_m \mathbf{p}_m \rangle = \sum_{i=1}^m \lambda_i \langle \boldsymbol{c}, \mathbf{p}_i \rangle < \sum_{i=1}^m \lambda_i a = a, \tag{1.52}$$

which contradicts the assumption that $\langle \boldsymbol{c}, \boldsymbol{v} \rangle = a$.

□

Since polytopes are bounded sets, there are always hyperplanes which do not intersect the polytope. Therefore, the empty set is a face of every polytope. Regarding the "degenerated" hyperplane $\{\boldsymbol{x} \in \mathbb{R}^d : \langle \mathbf{0}, \boldsymbol{x} \rangle = 0\} = \mathbb{R}^d$ we say that the polytope itself is one of its faces. Accordingly, we obtain the following theorem.

Theorem 1.24. *Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a polytope, $\mathcal{F} \subseteq \mathcal{P}$ a face and denote the vertex set of $\mathcal{P}$ by V, then the following assertions hold.*

(i) The face $\mathcal{F}$ is a polytope as well.

(ii) The vertex set of the polytope $\mathcal{F}$ is given by the intersection $\mathcal{F} \cap \operatorname{vert}(\mathcal{P})$.

(iii) The intersection of faces of $\mathcal{P}$ is again a face of $\mathcal{P}$.

(iv) The faces of $\mathcal{F}$ are exactly the faces of $\mathcal{P}$ which are contained in $\mathcal{F}$.

Proof. (i) According to the definition of a face, there exists a hyperplane

$$\mathcal{H}_{\mathcal{F}} = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{c}, \boldsymbol{x} \rangle = c_0\} \tag{1.53}$$

such that the polytope $\mathcal{P}$ is contained in the half–space $\{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{c}, \boldsymbol{x} \rangle \leq c_0\}$ and $\mathcal{F} = \mathcal{P} \cap \mathcal{H}_{\mathcal{F}}$. Due to the hyperplane description of a polytope, it is obvious that the face $\mathcal{F}$ is a polytope.

(ii) Suppose that $\mathrm{vert}(\mathcal{P}) = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$. Since, by Theorem 1.23 (i), any polytope is the convex hull of its vertices and any face is a subset of the polytope, we have that any element of the face $\mathcal{F}$ is a convex combination of elements of the set $\mathrm{vert}(\mathcal{P})$. Hence, for each element $\boldsymbol{x} \in \mathcal{F}$ there are non–negative reals $\lambda_1, \ldots, \lambda_k$ with $\sum_{i=1}^k \lambda_i = 1$ such that $\boldsymbol{x} = \sum_{i=1}^k \lambda_i \boldsymbol{v}_i$. Now, let $\mathcal{H}_{\mathcal{F}}$ be a supporting hyperplane of $\mathcal{P}$ such that $\mathcal{F} = \mathcal{P} \cap \mathcal{H}_{\mathcal{F}}$. Then, using the same notation as in the proof of item (i), we obtain that for all $\boldsymbol{x} \in \mathcal{F}$

$$c_0 = \langle \boldsymbol{c}, \boldsymbol{x} \rangle = \left\langle \boldsymbol{c}, \sum_{i=1}^k \lambda_i \boldsymbol{v}_i \right\rangle = \sum_{i=1}^k \lambda_i \langle \boldsymbol{c}, \boldsymbol{v}_i \rangle \leq \left(\sum_{i=1}^k \lambda_i \right) c_0 = c_0. \tag{1.54}$$

Thus, $\sum_{i=1}^k \lambda_i \langle \boldsymbol{c}, \boldsymbol{v}_i \rangle = \left(\sum_{i=1}^k \lambda_i \right) c_0$, or equivalently $\sum_{i=1}^k \lambda_i \left(\langle \boldsymbol{c}, \boldsymbol{v}_i \rangle - c_0 \right) = 0$. Therefore, and since $\langle \boldsymbol{c}, \boldsymbol{v}_i \rangle \leq c_0$ for all vertices $\boldsymbol{v}_i \in \mathrm{vert}(\mathcal{P})$, we obtain that

$$\left(\langle \boldsymbol{c}, \boldsymbol{v}_i \rangle - c_0 \right) \lambda_i = 0, \tag{1.55}$$

for all $1 \leq i \leq k$. As $\langle \boldsymbol{c}, \boldsymbol{v}_i \rangle < c_0$ for all vertices $\boldsymbol{v}_i \in \mathrm{vert}(\mathcal{P}) \backslash \mathcal{F}$, this implies $\lambda_i = 0$ in this cases. This establishes the inclusion $\mathcal{F} \subseteq \mathrm{conv}(\mathcal{F} \cap \mathrm{vert}(\mathcal{P}))$. Moreover, since $\mathcal{F}$ is a convex set, we obtain that $\mathrm{conv}(\mathcal{F} \cap \mathrm{vert}(\mathcal{P})) \subseteq \mathcal{F}$, and therefore we have that

$$\mathrm{conv}(\mathcal{F} \cap \mathrm{vert}(\mathcal{P})) = \mathcal{F}.$$

Now, item (ii) of Theorem 1.23 states that any finite point set, whose convex hull equals the polytope, contains all its vertices. Thus, we have proved the inclusion $\mathrm{vert}(\mathcal{F}) \subseteq (\mathcal{F} \cap \mathrm{vert}(\mathcal{P}))$. Conversely, any vertex of $\mathcal{P}$ which is contained in $\mathcal{F}$ is a vertex of $\mathcal{F}$. This is because any supporting hyperplane of the polytope $\mathcal{P}$ is also a supporting hyperplane of the polytope $\mathcal{F}$. We conclude that $\mathrm{vert}(\mathcal{F}) = \mathcal{F} \cap \mathrm{vert}(\mathcal{P})$.

(iii) Let $\mathcal{F}$ and $\mathcal{G}$ be faces of the polytope $\mathcal{P}$. Our aim is to show that the intersection $\mathcal{G} \cap \mathcal{F}$ is a face of the polytope $\mathcal{P}$ as well.

According to the definition of faces of polytopes, there are hyperplanes

$$\mathcal{H}_{\mathcal{F}} := \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}, \boldsymbol{x} \rangle = a_0\} \text{ and } \mathcal{H}_{\mathcal{G}} := \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{c}, \boldsymbol{x} \rangle = c_0\} \tag{1.56}$$

such that $\mathcal{P} \subseteq \mathcal{H}_{\mathcal{F}}^{\leq}$, $\mathcal{P} \subseteq \mathcal{H}_{\mathcal{G}}^{\leq}$, and $\mathcal{F} = \mathcal{P} \cap \mathcal{H}_{\mathcal{F}}$ while $\mathcal{G} = \mathcal{P} \cap \mathcal{H}_{\mathcal{G}}$. If we combine the two inequalities defining these half–spaces, we obtain that

$$\mathcal{P} \subseteq \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}, \boldsymbol{x} \rangle + \langle \boldsymbol{c}, \boldsymbol{x} \rangle \leq a_0 + c_0\} = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a} + \boldsymbol{c}, \boldsymbol{x} \rangle \leq a_0 + c_0\} \text{ and}$$
$$\mathcal{P} \cap \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}, \boldsymbol{x} \rangle + \langle \boldsymbol{c}, \boldsymbol{x} \rangle = a_0 + c_0\} = \mathcal{F} \cap \mathcal{G},$$

and consequently a supporting hyperplane of the polytope $\mathcal{P}$ such that the corresponding face is the intersection $\mathcal{F} \cap \mathcal{G}$.

(iv) For any face $\mathcal{G}$ of the polytope $\mathcal{P}$, which is contained in $\mathcal{F}$, a supporting hyperplane which defines $\mathcal{G}$ as a face of $\mathcal{P}$ is at the same time a supporting hyperplane of the polytope $\mathcal{F}$ which defines $\mathcal{G}$ as a face of $\mathcal{F}$. Therefore, any face of the polytope $\mathcal{P}$ which is a subset of $\mathcal{F}$ is also a face of this polytope.

Conversely, let $\mathcal{G} \subseteq \mathcal{F}$ be a face and suppose that $\mathcal{H}_1 := \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{c}, \boldsymbol{x} \rangle = c_0\}$ is a supporting hyperplane of the polytope $\mathcal{P}$ such that $\mathcal{P} \cap \mathcal{H}_1 = \mathcal{F}$ while $\mathcal{P}$ is contained in the half–space $\mathcal{H}_1^{\leq}$. Moreover, let $\mathcal{H}_2 = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{b}, \boldsymbol{x} \rangle = b_0\}$ be a supporting hyperplane of $\mathcal{F}$ such that $\mathcal{G} = \mathcal{F} \cap \mathcal{H}_2$ and $\mathcal{F} \subseteq H_2^{\leq}$. Note that $\mathcal{H}_2$ is not necessarily a supporting hyperplane of $\mathcal{P}$. We conclude our proof by constructing a supporting hyperplane which defines $\mathcal{G}$ as a face of $\mathcal{P}$ by use of the hyperplanes $\mathcal{H}_1$ and $\mathcal{H}_2$.

Combining the inequalities of the half–spaces $\mathcal{H}_1^{\leq}$ and $H_2^{\leq}$, we obtain that

$$\langle \boldsymbol{b} + \lambda \boldsymbol{c}, \boldsymbol{x} \rangle = \langle \boldsymbol{b}, \boldsymbol{x} \rangle + \lambda \langle \boldsymbol{c}, \boldsymbol{x} \rangle \leq b_0 + \lambda c_0, \tag{1.57}$$

for any $\boldsymbol{x} \in \mathcal{F}$ and any real λ. Hence, the face $\mathcal{F}$ is contained in the half–space $\{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{b} + \lambda \boldsymbol{c}, \boldsymbol{x} \rangle \leq b_0 + \lambda c_0\}$ for any real λ. Moreover, the equation $\langle \boldsymbol{b} + \lambda \boldsymbol{c}, \boldsymbol{x} \rangle = b_0 + \lambda c_0$ holds if and only if $\langle \boldsymbol{b}, \boldsymbol{x} \rangle = b_0$, i.e. for $\boldsymbol{x} \in \mathcal{G}$. Put differently, for any real $\lambda \in \mathbb{R}$

$$\mathcal{H}_\lambda := \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{b} + \lambda \boldsymbol{c}, \boldsymbol{x} \rangle = b_0 + \lambda c_0\} \tag{1.58}$$

is a supporting hyperplane of the polytope $\mathcal{F}$ and the corresponding face is given by $\mathcal{G}$.

Therefore, once we find a real λ such that the polytope $\mathcal{P}$ is contained in the half–space

$$\{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{b} + \lambda \boldsymbol{c}, \boldsymbol{x}\rangle \leq b_0 + \lambda c_0\}, \tag{1.59}$$

the hyperplane $\mathcal{H}_\lambda$ will be a supporting hyperplane of the polytope $\mathcal{P}$ which defines $\mathcal{G}$ as a face of $\mathcal{P}$. Now, recall that any polytope is the convex hull of its vertices. Thus, in order to proof that this inequality holds for all points of the polytope, it suffices to show the according property for any vertex. Given a vertex in $(\mathrm{vert}(\mathcal{P}) \cap \mathcal{F})$, equation (1.57) holds for all reals. For a vertex $\boldsymbol{v} \in \mathrm{vert}(\mathcal{P}) \backslash \mathcal{F}$, we have the following equivalence,

$$\begin{aligned} \langle \boldsymbol{b} + \lambda \boldsymbol{c}, \boldsymbol{v}\rangle \leq b_0 + \lambda c_0 &\Leftrightarrow \langle \boldsymbol{b}, \boldsymbol{v}\rangle + \lambda \langle \boldsymbol{c}, \boldsymbol{v}\rangle \leq b_0 + \lambda c_0 \\ &\Leftrightarrow \langle \boldsymbol{b}, \boldsymbol{v}\rangle - b_0 \leq \lambda \left(c_0 - \langle \boldsymbol{c}, \boldsymbol{v}\rangle\right) \\ &\Leftrightarrow -\frac{b_0 - \langle \boldsymbol{b}, \boldsymbol{v}\rangle}{c_0 - \langle \boldsymbol{c}, \boldsymbol{v}\rangle} \leq \lambda. \end{aligned} \tag{1.60}$$

Thus, with

$$\lambda > \max\nolimits_{\boldsymbol{v} \in \mathrm{vert}(\mathcal{P} \backslash \mathcal{F})} \left\{ -\frac{b_0 - \langle \boldsymbol{b}, \boldsymbol{v}\rangle}{c_0 - \langle \boldsymbol{c}, \boldsymbol{v}\rangle} \right\}, \tag{1.61}$$

equation (1.57) holds for all vertices $\boldsymbol{v} \in \mathrm{vert}(\mathcal{P})$, and $\mathcal{H}_\lambda$ gives a supporting hyperplane of the polytope $\mathcal{P}$ such that the corresponding face is $\mathcal{G}$. $\square$

Corollary 1.25. *The faces of simplices are simplices and therefore given by the convex hulls of subsets the vertex set.*

Proof. Let $\mathcal{F}$ be a face of a simplex Δ. According to item *(ii)* of Theorem 1.24, we have that $\mathrm{vert}(\mathcal{F}) \subseteq \mathrm{vert}(\Delta)$ and, by Theorem 1.23, $\mathcal{F} = \mathrm{conv}(\mathrm{vert}(\mathcal{F}))$. Since a subset of an affinely independent set is affinely independent, the face $\mathcal{F}$ is the convex hull of an affinely independent set, i.e. a simplex. $\square$

1.2.4 Triangulations of Polytopes

We have already mentioned that a concept analogous to the "basis" of a linear or affine space exists only for a very special class of convex sets, called simplices. Therefore we show that every polytope can be decomposed into finitely many simplices.

Definition 1.26. A *triangulation* of a d–polytope $\mathcal{P}$ is a finite set $T(\mathcal{P}) = \{\Delta_1, \dots, \Delta_n\}$ of $d-$simplices with the following properties.

(i) $\mathcal{P} = \bigcup T(\mathcal{P})$

(ii) $\Delta_i \cap \Delta_j$ is a common face of Δ_i and Δ_j, for any $\Delta_i, \Delta_j \in T(\mathcal{P})$.

We say that a polytope $\mathcal{P}$ can be *triangulated using no new vertices* if and only if there exists a triangulation $T(\mathcal{P})$ such that $\mathrm{vert}(\mathcal{P}) = \bigcup_{\Delta_i \in T(\mathcal{P})} \mathrm{vert}(\Delta_i)$.

Remark 1.27. Given a polytope $\mathcal{P}$, a simplex $\Delta \subseteq \mathcal{P}$ and a face $\mathcal{F}$ of $\mathcal{P}$, the intersection $\mathcal{F} \cap \Delta$ is a face of the simplex Δ and, as such, a simplex, by Corollary 1.25.

Lemma 1.28. *Suppose that $T(\mathcal{P})$ is a triangulation of a d–polytope $\mathcal{P}$, and let $\mathcal{F}$ be a facet of $\mathcal{P}$. Then the set*

$$T(\mathcal{F}) := \{\Delta \cap \mathcal{F} : \Delta \in T(\mathcal{P}), \dim(\Delta \cap \mathcal{F}) = d - 1\} \tag{1.62}$$

provides a triangulation of the polytope $\mathcal{F}$.

Proof. We have to show that $T(\mathcal{F})$ meets the two required properties of a triangulation.

First note that $\bigcup T(\mathcal{F}) \subseteq \mathcal{F}$, by definition. Now we will show that $\mathcal{F} \backslash \bigcup T(\mathcal{F}) = \emptyset$, by means of contradiction. To this end, observe that the set $\bigcup T(\mathcal{F})$ is a finite union of closed sets and therefore closed. Thus, for any element $\boldsymbol{x} \in \mathcal{F} \backslash \bigcup T(\mathcal{F})$ there exists an open neighborhood N of $\boldsymbol{x}$ in $\mathcal{F}$ with the property that $N \cap \bigcup T(\mathcal{F}) \neq \emptyset$. Now suppose there is an $\boldsymbol{x} \in \mathcal{F} \backslash \bigcup T(\mathcal{F})$ and assume, moreover, that the neighborhood $N \subseteq \mathcal{F}$ contains no points of $\bigcup T(\mathcal{F})$. This means that N consists only of points contained in some simplices of $T(\mathcal{P})$ that meet $\mathcal{F}$ in a set of dimension less than $d - 1$. Since there are only finitely many simplices in $T(\mathcal{P})$ this is a contradiction to $\dim(N) = d - 1$. Therefore, any open neighborhood of $\boldsymbol{x}$ in $\mathcal{F}$ has to contain elements of some simplex $\Delta \in \bigcup T(\mathcal{F})$. But this is also impossible since $\bigcup T(\mathcal{F})$ is closed. Therefore, such an $\boldsymbol{x}$ may not exist, i.e. $\mathcal{F} \backslash \bigcup T(\mathcal{F}) = \emptyset$. This shows that $\bigcup T(\mathcal{F})$ satisfies the first of the required properties of a triangulation.

It remains to prove that for any pair of simplices Δ_1 and $\Delta_2 \in T(\mathcal{F})$, the intersection $\Delta_1 \cap \Delta_2$ is a common face of Δ_1 and Δ_2.

By the definition of $T(\mathcal{F})$, there are simplices $S_1, S_2 \in T(\mathcal{P})$ such that $\Delta_1 = S_1 \cap \mathcal{F}$ and $\Delta_2 = S_2 \cap \mathcal{F}$, which implies that $\Delta_1 \cap \Delta_2 = S_1 \cap S_2 \cap \mathcal{F}$. Since $T(\mathcal{P})$ is a triangulation, $S_1 \cap S_2$ is a common face of S_1 and S_2. Now, $\mathcal{F}$ is a face of $\mathcal{P}$ and $S_1 \subseteq \mathcal{P}$. Thus, by Remark 1.27, the intersection $S_1 \cap \mathcal{F}$ is a face of S_1. The intersection of faces of a polytope is again a face, by Theorem 1.24 (*iii*). Therefore $(S_1 \cap S_2) \cap (S_1 \cap \mathcal{F}) = S_1 \cap S_2 \cap \mathcal{F}$ is a face of S_1. Since $\Delta_1 \subseteq S_1$, the intersection $(S_1 \cap S_2 \cap \mathcal{F}) \cap \Delta_1$ is a face of Δ_1. By definition $\Delta_1 = S_1 \cap \mathcal{F}$, and we obtain that $S_1 \cap S_2 \cap \mathcal{F} = \Delta_1 \cap \Delta_2$ is a face of the simplex Δ_1.

Analogously, $\Delta_1 \cap \Delta_2$ is a face of S_2. □

Definition 1.29. Let $\mathcal{Q} \subseteq \mathbb{R}^n$ be a polytope and $\boldsymbol{v}$ an element of $\mathbb{R}^n$. A facet $\mathcal{F}$ of $\mathcal{Q}$ is called *visible from* $\boldsymbol{v}$ if and only if for any point $\boldsymbol{x} \in \mathcal{F}$ the half–open line segment $(\boldsymbol{x}, \boldsymbol{v}]$ is disjoint from $\mathcal{Q}$.

Definition 1.30. For two points $\boldsymbol{x}$ and $\boldsymbol{y} \in \mathbb{R}^n$ the *line through* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ is defined as

$$\operatorname{aff}(\boldsymbol{x}, \mathbf{y}) = \{\lambda_1 \boldsymbol{x} + \lambda_2 \mathbf{y} : \lambda_1 + \lambda_2 = 1\}. \tag{1.63}$$

Theorem 1.31. *Every polytope can be triangulated into simplices using no new vertices.*

Proof. Our proof is by induction on the number of vertices of a d–polytope.

If a d–polytope $\mathcal{P}$ has $d+1$ vertices, it is a d–simplex and $T = \{\mathcal{P}\}$ provides a triangulation with the desired properties.

Suppose $\mathcal{P}$ is a d–polytope with at least $d+2$ vertices. Then, there exists a vertex $\boldsymbol{v}$ of $\mathcal{P}$ such that the polytope Q defined as the convex hull of the set $(\operatorname{vert}(\mathcal{P}) \backslash \boldsymbol{v})$ is still of dimension d. Therefore, by our induction hypothesis, the polytope Q can be triangulated into simplices using no new vertices. By Lemma 1.28, every triangulation $T(Q)$ induces a triangulation of a facet $\mathcal{F}$ of Q given by

$$T(\mathcal{F}) = \{\Delta \cap \mathcal{F} : \Delta \in T(Q), \dim(\Delta \cap \mathcal{F}) = d-1\}. \tag{1.64}$$

We denote by $\mathcal{F}_1, \ldots, \mathcal{F}_n$ the facets of Q which are visible from $\boldsymbol{v}$ and set

$$T := \{\operatorname{conv}(\boldsymbol{v}, \Delta) : \Delta \in T(\mathcal{F}_i) \text{ for some } 1 \le i \le n\}. \tag{1.65}$$

We claim that $(T \cup T(Q))$ provides a triangulation of the polytope $\mathcal{P}$.

First we prove that $\mathcal{P} = \bigcup(T \cup T(Q))$. To this end, we observe that $\mathcal{P} \supseteq \bigcup(T \cup T(Q))$, by definition. So it remains to show that $\mathcal{P} \subseteq \bigcup(T \cup T(Q))$. We consider a point $\boldsymbol{x}$ in $\mathcal{P}$. If $\boldsymbol{x} \in Q$, then $\boldsymbol{x} \in \bigcup T(Q) \subseteq \bigcup(T \cup T(Q))$. For $\boldsymbol{x} \in \mathcal{P} \backslash Q$, we consider the line through $\boldsymbol{v}$ and $\boldsymbol{x}$. Observe that for any point $\boldsymbol{x} \neq \boldsymbol{v}$ in the polytope $\mathcal{P} = \operatorname{conv}(\boldsymbol{v}, Q)$ there exists a point $\boldsymbol{y} \in Q$ such that $\mu \boldsymbol{v} + (1-\mu)\boldsymbol{y} = \boldsymbol{x}$, for some positive real μ. Hence, $\boldsymbol{y} = \frac{1}{1-\mu}\boldsymbol{x} - \frac{\mu}{1-\mu}\boldsymbol{v}$ lies on the line through $\boldsymbol{v}$ and $\boldsymbol{x}$, which shows that this line meets Q. So let $\boldsymbol{z}$ be the first point in Q one meets traveling along this line from $\boldsymbol{v}$ towards Q. Then, $[\boldsymbol{v}, \boldsymbol{z}) \cap Q = \emptyset$, and it can be shown that the point $\boldsymbol{z}$ is located on a facet $\mathcal{F}_i$ of Q that is visible from $\boldsymbol{v}$. Thus, the point $\boldsymbol{z}$ is contained in a simplex Δ in the triangulation $T(\mathcal{F}_i)$. Note that the point $\boldsymbol{x}$ is contained in the line segment $[\boldsymbol{v}, \boldsymbol{z}]$, and therefore it is also included in the set $\operatorname{conv}(\boldsymbol{v}, \Delta) \in T$.

It remains to show that for any simplices Δ_1 and $\Delta_2 \in (T \cup T(Q))$, the intersection $\Delta_1 \cap \Delta_2$ is a common face of Δ_1 and Δ_2. We distinguish three cases.

First, for Δ_1 and $\Delta_2 \in T(Q)$, we have that $\Delta_1 \cap \Delta_2$ is a face of both, Δ_1 and Δ_2, since $T(Q)$ is a triangulation.

If Δ_1 and $\Delta_2 \in T$, there are by definition simplices S_1 and S_2 in $T(Q)$ and facets $\mathcal{F}_1$ and $\mathcal{F}_2$ of the polytope Q which are visible from the vertex $\boldsymbol{v}$ such that the simplex $\Delta_1 = \operatorname{conv}(\boldsymbol{v}, S_1 \cap \mathcal{F}_1)$ and $\Delta_2 = \operatorname{conv}(\boldsymbol{v}, S_2 \cap \mathcal{F}_2)$. Now, $S_1 \cap S_2$ is a common face of S_1 and S_2. The intersection of faces is a face of a polytope, by Theorem 1.24 (*iii*). Therefore, $\mathcal{F}_1 \cap \mathcal{F}_2$ is a face of Q. Since S_1 and S_2 are contained in the polytope Q, we have that $\mathcal{F}_1 \cap \mathcal{F}_2 \cap S_1 \cap S_2$ is a common face of the simplices S_1 and S_2. Faces of a simplex are simplices given by the convex hulls of subsets of its vertex set, c.f. Corollary 1.25. Thus, the intersection $S_1 \cap S_2 \cap \mathcal{F}_1 \cap \mathcal{F}_2$ is the convex hull of common vertices of $S_1 \cap \mathcal{F}_1$ and $S_2 \cap \mathcal{F}_2$. By convexity,

$$\Delta_1 \cap \Delta_2 = \operatorname{conv}(\boldsymbol{v}, S_1 \cap \mathcal{F}_1) \cap \operatorname{conv}(\boldsymbol{v}, S_2 \cap \mathcal{F}_2) = \operatorname{conv}(\boldsymbol{v}, S_1 \cap S_2 \cap \mathcal{F}_1 \cap F_2). \quad (1.66)$$

Thus, we conclude that the intersection $\Delta_1 \cap \Delta_2$ is the convex hull of common vertices of Δ_1 and Δ_2, and therefore a common face of these simplices.

Finally, consider the case that $\Delta_1 \in T$ while $\Delta_2 \in T(Q)$. By definition of T, there exists a facet $\mathcal{F}$ of Q which is visible from $\boldsymbol{v}$ and a simplex $S \in T(\mathcal{F})$ such that $\Delta_1 = \operatorname{conv}(\boldsymbol{v}, S)$ and $\Delta_1 \cap Q = S$. As to the construction of $T(\mathcal{F})$ the set S is the face of a simplex $\Delta \in T(Q)$. Since $T(Q)$ is a triangulation, $\Delta \cap \Delta_2$ is a face of Δ and Δ_2.

Moreover, since Δ_2 is contained in the polytope Q, we have that

$$\Delta_1 \cap \Delta_2 = \Delta_1 \cap Q \cap \Delta_2 = S \cap \Delta_2 = S \cap \Delta \cap \Delta_2. \tag{1.67}$$

Thus, $S \cap (\Delta \cap \Delta_2)$ is the intersection of faces of Δ, and, as such, a face of this simplex. Hence, by (1.67), we obtain that $\Delta_1 \cap \Delta_2$ is a face of Δ. Note that $\Delta_1 \cap \Delta_2$ is contained in S, and because both sets are faces of the simplex Δ it follows that $\Delta_1 \cap \Delta_2$ is a face of S. By our construction, the vertices of S are a subset of the vertices of the simplex Δ_1, whence S is a face of Δ_1. Since faces of a face of a polytope are also faces of the polytope, we obtain that $\Delta_1 \cap \Delta_2$ is a face of Δ_1.

Because $\Delta \cap \Delta_2$ is a face of the simplices Δ_2 and Δ, this set is the convex hull of common vertices of these simplices. In the same way, S is the convex hull of a subset of vertices of Δ. Since all these sets are simplices, we get that $\Delta_1 \cap \Delta_2$ is the convex hull of $\mathrm{vert}(S) \cap \mathrm{vert}(\Delta) \cap \mathrm{vert}(\Delta_2)$. Hence, $\Delta_1 \cap \Delta_2$ is a face of Δ_2. $\square$

1.2.5 The Relative Interior of a Polytope

Lemma 1.32. *Let $\mathcal{P}$ be a polytope in a d–dimensional affine space. Then, the following conditions are equivalent for any element $\boldsymbol{y} \in \mathcal{P}$.*

(i) The point $\boldsymbol{y}$ is not contained in a face of the polytope $\mathcal{P}$ of dimension smaller than d.

(ii) If $\boldsymbol{y}$ lies on a hyperplane $\mathcal{H}$, then $\mathcal{H}$ is not a supporting hyperplane of $\mathcal{P}$.

(iii) The point $\boldsymbol{y}$ lies in the interior of the polytope $\mathcal{P}$ in the topological sense, i.e. there is an $\varepsilon > 0$ such that $\{\boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x} - \boldsymbol{y}\| \leq \varepsilon\} \subseteq \mathcal{P}$, resp. $\{\boldsymbol{y} + \boldsymbol{u} : \|\boldsymbol{u}\| \leq \varepsilon\} \subseteq \mathcal{P}$.

(iv) The point $\boldsymbol{y}$ can be represented as a sum $\boldsymbol{y} = \frac{1}{d+1} \sum_{i=1}^{d+1} \boldsymbol{x}_i$ of $d+1$ affinely independent elements $\boldsymbol{x}_1, \dots, \boldsymbol{x}_{d+1}$ of the polytope $\mathcal{P}$.

(v) The point $\boldsymbol{y}$ can be represented as convex combination $\boldsymbol{y} = \sum_{i=1}^{d+1} \lambda_i \boldsymbol{x}_i$ of $d+1$ affinely independent elements $\boldsymbol{x}_1, \dots, \boldsymbol{x}_{d+1}$ of the polytope $\mathcal{P}$, where all coefficients $\lambda_i \neq 0$ for all $1 \leq i \leq d+1$.

Proof. $(i) \Leftrightarrow (ii)$. This equivalence is an immediate consequence of the definition of faces of a polytope. Moreover, note that, if there exists a point $\boldsymbol{y}$ in the polytope $\mathcal{P}$, which meets the condition given in item (i), then $\mathcal{P}$ is a d–polytope.

$(iv) \Rightarrow (v)$. This follows since (iv) is a special case of (v).

$(v) \Rightarrow (i)$. Assume that the point $\boldsymbol{y}$ can be represented as a convex combination of some elements $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{d+1}$ of the polytope $\mathcal{P}$ and that there exists, at the same time, a supporting hyperplane $\mathcal{H}$ of $\mathcal{P}$, with $\dim(\mathcal{H} \cap \mathcal{P}) < d$, which contains $\boldsymbol{y}$.

Suppose this hyperplane is given by $\mathcal{H} = \{\boldsymbol{x} \in \mathbb{R}^d : \langle \boldsymbol{a}, \boldsymbol{x} \rangle = a_0\}$. Then

$$a_0 = \langle \boldsymbol{y}, \boldsymbol{a} \rangle = \sum_{i=1}^{d+1} \lambda_i \langle \boldsymbol{x}_i, \boldsymbol{a} \rangle \leq (\sum_{i=1}^{d+1} \lambda_i) a_0 = a_0. \tag{1.68}$$

Thus, the points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{d+1}$ lie on the hyperplane $\mathcal{H}$. Since the dimension of the face $\mathcal{H} \cap \mathcal{P}$ is by assumption less than d, the vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{d+1}$ are affinely dependent.

$(ii) \Rightarrow (iii)$. Consider the hyperplane description of the polytope, i.e. let A be an $m \times d$ matrix and $\boldsymbol{b}$ a vector in $\mathbb{R}^m$ such that $\mathcal{P} = \{\boldsymbol{x} \in \mathbb{R}^d : A\boldsymbol{x} \leq \boldsymbol{b}\}$. If the assertion of item (ii) is valid, then we have that $A\boldsymbol{y} < \boldsymbol{b}$. Therefore, there exists an $\varepsilon > 0$ such that all $\boldsymbol{u} \in \mathbb{R}^d$ with $\|\boldsymbol{u}\| < \varepsilon$ satisfy the inequality $A(\boldsymbol{y} + \boldsymbol{u}) \leq \boldsymbol{b}$.

$(iii) \Rightarrow (iv)$. Set $\alpha = \frac{\varepsilon}{\sqrt{d}} < \varepsilon$. Then, by item (iii), the points $(\boldsymbol{y} + \alpha \mathbf{e}_1), \ldots, (\boldsymbol{y} + \alpha \mathbf{e}_d)$, $(\boldsymbol{y} - \alpha(\mathbf{e}_1 + \ldots + \mathbf{e}_d))$ are contained in the polytope $\mathcal{P}$, are affinely independent and

$$\boldsymbol{y} = \frac{1}{d+1} \left(\boldsymbol{y} + \alpha \mathbf{e}_1 + \ldots + \boldsymbol{y} + \alpha \mathbf{e}_d + \boldsymbol{y} - \alpha(\mathbf{e}_1 + \ldots + \mathbf{e}_d) \right). \tag{1.69}$$

$\square$

Definition 1.33. Let $\mathcal{P}$ be a polytope in a d–dimensional affine space. A point $\boldsymbol{y} \in \mathcal{P}$ is called an *inner point* if it satisfies the conditions given in Lemma 1.32.

If the dimension of a polytope $\mathcal{P}$ in a d–dimensional affine space is less than d, then, according to our definition, the set of inner points of the polytope $\mathcal{P}$ is empty.

Definition 1.34. We define the relative interior of a polytope $\mathcal{P}$, denoted $\mathcal{P}^\circ$, as the set of inner points of $\mathcal{P}$ in $\mathrm{aff}(\mathcal{P})$.

Remark 1.35. Note that, if a polytope is non–empty, then its relative interior is non–empty. Furthermore, any polytope is the disjoint union of the relative interior of its faces, c.f. Lemma 1.32(i), i.e.

$$\mathcal{P} = \dot{\bigcup_{\substack{\mathcal{F} \subseteq \mathcal{P} \\ \text{Face}}}} \mathcal{F}^\circ. \tag{1.70}$$

Definition 1.36. The boundary of a polytope $\mathcal{P}$ is defined as the set $\partial\mathcal{P} := \mathcal{P} \backslash \mathcal{P}^\circ$.

1.3 Pointed Cones

Beside polytopes, another family of polyhedral sets will play a central role in the subsequent theories. The aim of this section is to give proofs for some properties of pointed cones, which are required during the next chapters.

In order to give an exact definition of pointed cones, we start with a few observations. First, recall that any cone can be described as the non–negative hull of a finite set of vectors. Therefore, a cone contains with any point $\boldsymbol{p}$ the half–line $\{t\boldsymbol{p} : t \in \mathbb{R}_{\geq 0}\}$, for which reason the sole point which can be a vertex of a cone is the origin.

Theorem 1.37. *A polyhedral cone $\mathcal{C} \subseteq \mathbb{R}^n$ has a vertex at the origin if one of the following equivalent conditions holds.*

(i) There exists a hyperplane whose intersection with the cone contains only the origin, while all the other points of the cone lie within one of the open half-spaces of $\mathbb{R}^n$ defined by this hyperplane.

(ii) Let $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\}$ be a set of generators of the cone $\mathcal{C}$ which does not contain the origin. Then $\boldsymbol{0} \notin \operatorname{conv}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m)$.

(iii) The cone $\mathcal{C}$ does not contains a line $\{\lambda\boldsymbol{x} : \lambda \in \mathbb{R}\}$ for any vector $\boldsymbol{x} \in \mathbb{R}^n \backslash \{\boldsymbol{0}\}$.

Proof.

$(i) \Rightarrow (ii)$. Let $\mathcal{H} = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}, \boldsymbol{x} \rangle = 0\}$ be the hyperplane with the property that $\mathcal{C}\backslash\{\boldsymbol{0}\} \subseteq \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}, \boldsymbol{x} \rangle > 0\}$. Since any cone includes its generators, it follows that the inner product $\langle \boldsymbol{a}, \boldsymbol{v}_i \rangle > 0$ for $1 \leq i \leq m$. Now suppose, contrary to the statement of item (ii), that the origin $\boldsymbol{0} \in \operatorname{conv}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m)$, i.e. there are non–negative reals $\lambda_1, \ldots, \lambda_m$ with $\sum_{i=1}^m \lambda_i = 1$ such that $\lambda_1 \boldsymbol{v}_1 + \ldots + \lambda_m \boldsymbol{v}_m = \boldsymbol{0}$. Together with our assumption, this would imply that

$$0 = \langle \boldsymbol{a}, \boldsymbol{0} \rangle = \langle \boldsymbol{a}, \lambda_1 \boldsymbol{v}_1 + \ldots + \lambda_m \boldsymbol{v}_m \rangle = \sum_{i=1}^{m} \lambda_i \langle \boldsymbol{a}, \boldsymbol{v}_i \rangle > 0, \tag{1.71}$$

which is impossible.

$(ii) \Rightarrow (i)$. We will now consider the polytope $\mathcal{P} = \operatorname{conv}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m)$. By the equivalence of the vertex and the hyperplane description of polytopes, there are finitely many closed half–spaces whose intersection is $\mathcal{P}$. Since we assume that the origin is not contained in this polytope, at least one of these closed half–spaces does not contain the origin. We denote the according half–space by $\mathcal{H}^{\geq} = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}, \boldsymbol{x} \rangle \geq b\}$, where the vector $\boldsymbol{a} \in \mathbb{R}^n$ and b stands for a real. Note that the condition that the origin is not located in the half–space $\mathcal{H}^{\geq}$ implies that $0 = \langle \boldsymbol{a}, \boldsymbol{0} \rangle < b$. Therefore, we observe that the inner product $\langle \boldsymbol{a}, \boldsymbol{v}_i \rangle$ is strictly positive for all generators $\boldsymbol{v}_i \in \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_m\}$. As all the elements of the cone are non–negative combinations of the generators, we obtain that the inner product $\langle \boldsymbol{a}, \boldsymbol{x} \rangle > 0$ for all points $\boldsymbol{x} \in \mathcal{C} \backslash \{\boldsymbol{0}\}$ while $\langle \boldsymbol{a}, \boldsymbol{0} \rangle = 0$. Accordingly, the hyperplane $\mathcal{H} = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}, \boldsymbol{x} \rangle = 0\}$ meets the desired properties.

$\neg(iii) \Rightarrow \neg(ii)$. If the cone $\mathcal{C}$ contains a line $\{\lambda \boldsymbol{x} : \lambda \in \mathbb{R}\}$, for some $\boldsymbol{x} \in \mathbb{R}^n \backslash \{\boldsymbol{0}\}$, then $\boldsymbol{x}$ as well as the point $-\boldsymbol{x}$ are elements of the cone and, as such, non–negative combinations of the generators. Therefore, there are non–negative reals $\lambda_1, \dots, \lambda_m$ with $\sum_{i=1}^m \lambda_i > 0$ and non–negative reals $\lambda'_1, \dots \lambda'_m$ satisfying $\sum_{i=1}^m \lambda'_i > 0$ such that

$$\boldsymbol{x} = \sum_{i=1}^{m} \lambda_i \boldsymbol{v}_i \quad \text{and} \quad -\boldsymbol{x} = \sum_{i=1}^{m} \lambda'_i \boldsymbol{v}_i. \tag{1.72}$$

Setting $\mu_1 = (\sum_{i=1}^m \lambda_i)^{-1}$ and $\mu_2 = (\sum_{i=1}^m \lambda'_i)^{-1}$, we obtain that the points $\mu_1 \boldsymbol{x}$ and $-\mu_2 \boldsymbol{x}$ are contained in the polytope $\operatorname{conv}(\boldsymbol{v}_1, \dots \boldsymbol{v}_m)$. Therefore, the origin

$$\boldsymbol{0} \in [\mu_1 \boldsymbol{x}, -\mu_2 \boldsymbol{x}] \subseteq \operatorname{conv}(\boldsymbol{v}_1, \dots \boldsymbol{v}_m). \tag{1.73}$$

$\neg(ii) \Rightarrow \neg(iii)$. Suppose the polytope $\operatorname{conv}(\boldsymbol{v}_1, \dots, \boldsymbol{v}_m)$ contains the origin. Since $\boldsymbol{0} \notin \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_m\}$, the origin has to be located on a line segment $[\boldsymbol{x}, \boldsymbol{y}]^\circ \subseteq \operatorname{conv}(\boldsymbol{v}_1, \dots, \boldsymbol{v}_m)$, for some points $\boldsymbol{x}$ and $\boldsymbol{y} \in \operatorname{conv}(\boldsymbol{v}_1, \dots, \boldsymbol{v}_m)$. This means that there exists a real $0 < \lambda < 1$ such that $\boldsymbol{0} = \lambda \boldsymbol{x} + (1 - \lambda) \boldsymbol{y}$, which implies that $-\boldsymbol{x} = (1 - \lambda) \boldsymbol{y} / \lambda$. Therefore, we obtain that the point $\boldsymbol{x}$ as well as $-\boldsymbol{x}$ are contained in $\operatorname{cone}(\boldsymbol{v}_1, \dots, \boldsymbol{v}_m)$, whence this cone contains the line $\{\lambda \boldsymbol{x} : \lambda \in \mathbb{R}\}$. □

Definition 1.38. A *pointed cone* $\mathcal{K} \subseteq \mathbb{R}^n$ is a set of the form

$$\mathcal{K} = \boldsymbol{v} + \mathcal{C}, \tag{1.74}$$

where $\mathcal{C}$ is a polyhedral cone with a vertex at the origin and $\boldsymbol{v} \in \mathbb{R}^n$ is a vector. Note that, according to our definition, pointed cones are not polyhedral–cones but translates of such sets.

If we recall the vertex description of a polyhedral–cone, we obtain that a pointed cone is a set of the form

$$\mathcal{K} = \{\boldsymbol{v} + \lambda_1 \boldsymbol{w}_1 + \ldots + \lambda_n \boldsymbol{w}_d : \lambda_1, \ldots, \lambda_d \geq 0\}, \tag{1.75}$$

for some vectors $\boldsymbol{v}, \boldsymbol{w}_1, \ldots, \boldsymbol{w}_d \in \mathbb{R}^n$ with the property that there exists a hyperplane $\mathcal{H} \subseteq \mathbb{R}^n$ such that $\mathcal{K} \cap \mathcal{H} = \boldsymbol{v}$, while the set $\mathcal{K}\backslash\{\boldsymbol{v}\}$ lies entirely in one of the open half–spaces of the vector space $\mathbb{R}^n$ defined by the hyperplane $\mathcal{H}$.

As to the hyperplane description, a pointed cone is given as the intersection of finitely many half–spaces where the corresponding hyperplanes meet in exactly one point.

Corollary 1.39. *The cone over a polytope is pointed with a vertex at the origin.*

Proof. Given a polytope $\mathcal{P} = \mathrm{conv}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m)$, the vectors $\{(\boldsymbol{v}_i, 1) : 1 \leq i \leq m\}$ generate $\mathrm{cone}(\mathcal{P}|1)$ and apparently the origin is not contained in $\mathrm{conv}((\boldsymbol{v}_1, 1), \ldots, (\boldsymbol{v}_m, 1))$. □

Definition 1.40. We call a pointed cone *simplicial* if and only if it possesses a set of linearly independent generators.

Proposition 1.41. *The cone over a simplex is simplicial.*

Proof. Let $\Delta \subset \mathbb{R}^n$ be a d–simplex, i.e. the convex hull of $d+1$ affinely independent points $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{d+1} \in \mathbb{R}^n$. In order to proof that $\mathrm{cone}(\Delta|1)$ is simplicial we have to show that the vectors $(\boldsymbol{v}_1, 1), \ldots, (\boldsymbol{v}_{d+1}, 1)$ are linearly independent in $\mathbb{R}^{n+1}$.

Therefore, observe that

$$\lambda_1 \begin{pmatrix} \boldsymbol{v}_1 \\ 1 \end{pmatrix} + \ldots + \lambda_{d+1} \begin{pmatrix} \boldsymbol{v}_{d+1} \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 0 \end{pmatrix} \tag{1.76}$$

if and only if $\sum_{i=1}^{d+1} \lambda_i = 0$ and $\sum_{i=1}^{d+1} \lambda_i \boldsymbol{v}_i = \mathbf{0}$. Since the vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{d+1}$ are, by assumption, affinely independent in $\mathbb{R}^n$, the sole solution of equation (1.76) is given by $\lambda_1 = \ldots = \lambda_{d+1} = 0$. □

Similar to the concept of the triangulation of polytopes, we introduce the following notation for pointed cones.

Definition 1.42. A family $\mathcal{T}$ of pointed simplicial d–cones is called a *triangulation of a pointed d–cone* $\mathcal{K}$ if and only if

(i) $\mathcal{K} = \bigcup \mathcal{T}$.

(ii) For any simplicial cones S_1 and $S_2 \in \mathcal{T}$, the intersection $S_1 \cap S_2$ is a face common to both, S_1 and S_2.

We say that a pointed cone can be triangulated using no new generators if and only if there exists a triangulation such that the generators of any simplicial cone in this triangulation are contained in the set of generators of the original cone.

We may deduce the following result from the according property of polytopes.

Theorem 1.43. *Any pointed cone can be triangulated into simplicial cones using no new generators.*

Proof. Since pointed cones are translates of polyhedral cones with an vertex at the origin, it is sufficient to prove the theorem for pointed polyhedral cones.

Given a d–dimensional pointed polyhedral cone $\mathcal{C}$, there exists a hyperplane $\mathcal{H}$ that intersects this cone solely at origin, c.f. Theorem 1.37. If we consider a translate of this hyperplane by some vector $\boldsymbol{x} \in \mathcal{C} \backslash \{\mathbf{0}\}$, the intersection,

$$\mathcal{P} = (\boldsymbol{x} + \mathcal{H}) \cap \mathcal{C}, \tag{1.77}$$

is a $(d-1)$–polytope whose vertices provide a set of generators of the cone $\mathcal{C}$. By Theorem 1.31, there exists a triangulation of the polytope $\mathcal{P}$ into simplices using no new vertices. The cone over each simplex of this triangulation is a simplicial cone whose faces are given by the cones over the faces of this simplex. Therefore, the cones over the simplices of the triangulation of the polytope $\mathcal{P}$ provide a triangulation of the pointed cone $\mathcal{C}$. □

Rational Pointed Cones

We will now study a particular class of pointed cones more closely. The aim of this section is to establish the fact that any rational pointed cone can be translated by a vector, such that there are no integral points on any of the faces of the pointed cone so obtained. This result will turn out to be quite useful at several points during the subsequent chapters.

Definition 1.44. We call a hyperplanes $\mathcal{H} \subseteq \mathbb{R}^n$ *rational* if and only if there exists an integral vector $\boldsymbol{a} \in \mathbb{Z}^n$ and an integer $b \in \mathbb{Z}$ such that

$$\mathcal{H} = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}, \boldsymbol{x} \rangle = b\}. \tag{1.78}$$

Lemma 1.45. *For any rational linear hyperplanes $\mathcal{H} \subseteq \mathbb{R}^n$ there exists a vector $\boldsymbol{v} \in \mathbb{R}^n$ such that all integral points are located on a translate of this hyperplanes by a integral multiple of this vector, i.e.*

$$\bigcup_{k \in \mathbb{Z}} (k\boldsymbol{v} + \mathcal{H}) \cap \mathbb{Z}^n = \mathbb{Z}^n. \tag{1.79}$$

Proof. As to the definition of rational hyperplanes, there exists an integral vector $\boldsymbol{a} \in \mathbb{Z}^n$ such that

$$\mathcal{H} = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}, \boldsymbol{x} \rangle = 0\}. \tag{1.80}$$

Accordingly, the orthogonal complement $\mathcal{H}^\perp = \{\lambda \boldsymbol{a} : \lambda \in \mathbb{R}\}$, and every vector $\boldsymbol{y} \in \mathbb{R}^n$ has a unique representation as

$$\boldsymbol{y} = \lambda \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|} + \boldsymbol{h} \tag{1.81}$$

with $\boldsymbol{h} \in \mathcal{H}$ and a real $\lambda \in \mathbb{R}$. Note that the real λ in (1.81) gives the distance between the vector $\boldsymbol{y}$ and the hyperplanes $\mathcal{H}$, which can be calculated, according to the Hessian normal form, as $\langle \boldsymbol{a}, \boldsymbol{y} \rangle / \|\boldsymbol{a}\|$. Furthermore, for any integral vector $\boldsymbol{y} \in \mathbb{Z}^n$, the inner product $\langle \boldsymbol{a}, \boldsymbol{y} \rangle =: k \in \mathbb{Z}$ and we set $\boldsymbol{a}/\|\boldsymbol{a}\|^2 =: \boldsymbol{v}$. Substituting this into (1.81), we obtain that $\boldsymbol{y} = k\boldsymbol{v} + \boldsymbol{h}$, and consequently,

$$\boldsymbol{y} \in \mathcal{H}_k := \{k\boldsymbol{v} + \boldsymbol{h} : \boldsymbol{h} \in \mathcal{H}, k \in \mathbb{Z}\}. \tag{1.82}$$

□

Lemma 1.46. *Given an integral vector $\boldsymbol{a} \in \mathbb{Z}^n$ and an integer $b \in \mathbb{Z}$, we consider the rational hyperplane $\mathcal{H}_b = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}, \boldsymbol{x} \rangle = b\}$. Then, there exists a vector $\boldsymbol{v} \in \mathbb{R}^n$ such that neither the open strip*

$$\bigcup_{\lambda \in (0,1)} (\lambda \boldsymbol{v} + \mathcal{H}_b) \ \textit{nor} \ \bigcup_{\lambda \in (0,1)} (-\lambda \boldsymbol{v} + \mathcal{H}_b) \tag{1.83}$$

contains any integral point. Moreover, this vector $\boldsymbol{v}$ can be chosen such that

$$\begin{aligned} &\bigcup_{\lambda \in (0,1)} (\lambda \boldsymbol{v} + \mathcal{H}_b) \subseteq \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}, \boldsymbol{x} \rangle > b\} \ \textit{and} \\ &\bigcup_{\lambda \in (0,1)} (-\lambda \boldsymbol{v} + \mathcal{H}_b) \subseteq \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}, \boldsymbol{x} \rangle < b\}. \end{aligned} \tag{1.84}$$

Proof. Consider the linear rational hyperplane $\mathcal{H} = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}, \boldsymbol{x} \rangle = 0\}$. Then, by Theorem 1.45, there exists a vector $\boldsymbol{v} \in \mathbb{R}^n$ such that each integral point is located on an affine hyperplanes $\mathcal{H}_k = k\boldsymbol{v} + \mathcal{H}$, for a suitable integer k. By the proof of Theorem 1.45, the vector $\boldsymbol{v} = \boldsymbol{a}/\|\boldsymbol{a}\|^2$ meets the required properties. Since the vector–space $\mathbb{R}^n$ equals the direct sum $\mathcal{H} \oplus \mathcal{H}^{\perp}$, we obtain that the hyperplanes

$$\mathcal{H}_b = \{b\boldsymbol{v} + \boldsymbol{h} : \boldsymbol{h} \in \mathcal{H}\}. \tag{1.85}$$

Therefore, we obtain, by Theorem 1.45, that

$$\bigcup_{\lambda \in (0,1)} (\lambda \boldsymbol{v} + \mathcal{H}_b) \cap \mathbb{Z}^n = \bigcup_{\lambda \in (0,1)} ((b+\lambda)\boldsymbol{v} + \mathcal{H}) \cap \mathbb{Z}^n = \emptyset, \tag{1.86}$$

for all reals $0 < \lambda < 1$, since then $(b + \lambda) \notin \mathbb{Z}$. The same argument holds for all reals $-1 < \lambda < 0$, and we obtain that $\bigcup_{\lambda \in (0,1)} (-\lambda \boldsymbol{v} + \mathcal{H}_b) \cap \mathbb{Z}^n = \emptyset$.

To verify (1.84), observe that any element $\boldsymbol{x} \in \bigcup_{\lambda \in (0,1)} (\lambda \boldsymbol{v} + \mathcal{H}_b)$ can be written as $\boldsymbol{x} = (b + \lambda)\boldsymbol{v} + \boldsymbol{h}$, for some $\boldsymbol{h} \in \mathcal{H}$ and $0 < \lambda < 1$. Thus,

$$\begin{aligned} \langle \boldsymbol{a}, \boldsymbol{x} \rangle = \langle \boldsymbol{a}, (b+\lambda)\boldsymbol{v} + \boldsymbol{h} \rangle &= \left\langle \boldsymbol{a}, (b+\lambda) \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|^2} + \boldsymbol{h} \right\rangle \\ &= \frac{(b+\lambda)}{\|\boldsymbol{a}\|^2} \langle \boldsymbol{a}, \boldsymbol{a} \rangle + \langle \boldsymbol{a}, \boldsymbol{h} \rangle = b + \lambda > b. \end{aligned} \tag{1.87}$$

The second inclusion follows along the same lines. □

Theorem 1.47. *Let $\mathcal{K} \subseteq \mathbb{R}^n$ be a rational pointed d–cone with a vertex at the origin. Then, there exists a vector $\boldsymbol{v} \in \mathbb{R}^n$ such that the points with integral coordinates in the shifted cone $\boldsymbol{v} + \mathcal{K}$ agree with the integer points of the relative interior of the cone $\mathcal{K}$, while the points with integral coordinates located in $-\boldsymbol{v} + \mathcal{K}$ coincide with the integral points in $\mathcal{K}$. Moreover, there are no integral points on any of the boundaries of these shifted cones.*

Proof. A rational pointed cone with a vertex at the origin is the intersection of finitely many rational linear half–spaces. Therefore, there exists a positive integer m and linear hyperplanes

$$\mathcal{H}_i = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle = 0\} \tag{1.88}$$

with $\boldsymbol{a}_i \in \mathbb{Z}^n$, for $1 \leq i \leq m$, such that the cone $\mathcal{K}$ is the intersection of the half–spaces $\mathcal{H}_i^{\geq} = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle \geq 0\}$, for $1 \leq i \leq m$. Now, by Lemma 1.46, for each of these rational hyperplanes there exists a vector $\boldsymbol{v}_i \in \mathbb{R}^n$ such that

$$\bigcup_{\lambda : |\lambda| \in (0,1)} (\lambda \boldsymbol{v}_i + \mathcal{H}_b) \cap \mathbb{Z}^n = \emptyset \tag{1.89}$$

while the open strip

$$\bigcup_{\lambda \in (0,1)} (\lambda \boldsymbol{v}_i + \mathcal{H}_b) \subseteq \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle > 0\}. \tag{1.90}$$

Recall that, according to the construction of the proof of Theorem 1.45, the vector $\boldsymbol{v}_i = \boldsymbol{a}_i / \|\boldsymbol{a}_i\|^2$ meets the required properties, for all $1 \leq i \leq m$. Now, our aim is to find a vector $\boldsymbol{v} \in \mathbb{R}^n$ such that the properties stated in (1.89) and (1.90) hold simultaneously for all indices $1 \leq i \leq m$. To this end, we choose a vector $\boldsymbol{s}$ from the interior of the cone, which is therefore an element of the open half–space $\{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle > 0\}$, for all $1 \leq i \leq m$. Since the vector–space $\mathbb{R}^n = \mathcal{H}_i \oplus \mathcal{H}_i^{\perp}$ where $\mathcal{H}_i^{\perp} = \operatorname{span}(\boldsymbol{v}_i)$, we may write $\boldsymbol{s} = \mu_i \boldsymbol{v}_i + \boldsymbol{h}_i$ for an element $\boldsymbol{h}_i \in \mathcal{H}_i$ and an appropriate real μ_i. Observe that

$$\begin{aligned} 0 < \langle \boldsymbol{a}_i, \boldsymbol{s} \rangle &= \langle \boldsymbol{a}_i, \mu_i \boldsymbol{v}_i + \boldsymbol{h} \rangle = \mu_i \langle \boldsymbol{a}_i, \boldsymbol{v}_i \rangle + \langle \boldsymbol{a}_i, \boldsymbol{h} \rangle \\ &= \mu_i \left\langle \boldsymbol{a}_i, \frac{\boldsymbol{a}_i}{\|\boldsymbol{a}_i\|^2} \right\rangle + \langle \boldsymbol{a}_i, \boldsymbol{h} \rangle = \mu_i, \end{aligned} \tag{1.91}$$

for all $1 \leq i \leq m$. Now, whenever $\mu_i \leq 1$, the shifted hyperplanes $\lambda \boldsymbol{s} + \mathcal{H}_i$ contain no integral points, for all reals $0 < |\lambda| < 1$, c.f. Lemma 1.46. In case that $\mu_i > 1$,

$$\bigcup_{\lambda\in(0,1)} (\lambda \frac{1}{\mu_i} \boldsymbol{s} + \mathcal{H}_i) \cap \mathbb{Z}^n = \emptyset. \tag{1.92}$$

Therefore, for the vector $\boldsymbol{v} := \lambda' \frac{1}{\mu}\ \boldsymbol{s}$, where $\mu = \max\{\mu_1, \ldots, \mu_m, 1\}$ and $0 < \lambda' < 1$, the assertions of Lemma 1.46 hold simultaneously for all $1 \leq i \leq m$, i.e.

$$(\lambda \boldsymbol{v} + \mathcal{H}_i) \cap \mathbb{Z}^n \quad = \quad \emptyset \qquad \text{for all } 0 < |\lambda| \leq 1, \tag{1.93}$$

$$\bigcup_{\lambda\in(0,1)} (\lambda \boldsymbol{v} + \mathcal{H}_i) \quad \subseteq \quad \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle > 0\} = \mathcal{H}_i^{>}, \tag{1.94}$$

$$\bigcup_{\lambda\in(0,1)} (-\lambda \boldsymbol{v} + \mathcal{H}_i) \quad \subseteq \quad \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle < 0\} = \mathcal{H}_i^{<}. \tag{1.95}$$

Hence, the translated cone

$$\boldsymbol{v} + \mathcal{K} = \bigcap_{i=1}^{m} (\boldsymbol{v} + \mathcal{H}_i^{\geq}) \subseteq \bigcap_{i=1}^{m} \mathcal{H}_i^{>} = \mathcal{K}^{\circ}. \tag{1.96}$$

Moreover, we obtain that the set difference,

$$\mathcal{K} \backslash (\boldsymbol{v} + \mathcal{K}) = \bigcup_{\lambda\in(0,1)} (\lambda \boldsymbol{v} + \partial \mathcal{K}), \tag{1.97}$$

contains no integral points. Thus, we have that

$$\mathcal{K}^{\circ} \cap \mathbb{Z}^n = (\boldsymbol{v} + \mathcal{K}) \cap \mathbb{Z}^n. \tag{1.98}$$

According to the inclusion

$$\partial(\boldsymbol{v} + \mathcal{K}) \subseteq \bigcup_{i=1}^{m} (\boldsymbol{v} + \mathcal{H}_i), \tag{1.99}$$

there are no integral points on the boundary of the shifted cone. Similarly, we may deduce that the cone $-\boldsymbol{v} + \mathcal{K}$ does not contain any integral point on its boundary. Moreover, since $\mathcal{H}_i^{\geq} \subseteq (-\boldsymbol{v} + \mathcal{H}_i^{\geq})$ for all $1 \leq i \leq m$, we have $\mathcal{K} \subseteq -\boldsymbol{v} + \mathcal{K}$, and because the difference $(-\boldsymbol{v} + \mathcal{K}) \backslash \mathcal{K} = \bigcup_{\lambda\in(0,1)} (\lambda \boldsymbol{v} + \partial \mathcal{K})$ does not contain any integral point, we conclude that $\mathcal{K} \cap \mathbb{Z}^n = (-\boldsymbol{v} + \mathcal{K}) \cap \mathbb{Z}^n$.

□

Remark 1.48. Let $\mathcal{K} \subseteq \mathbb{R}^n$ be a rational pointed d–cone a vertex at the origin and $\{\mathcal{K}_1, \ldots, \mathcal{K}_m\}$ a triangulation of the cone $\mathcal{K}$ into simplicial cones using no new generators. Then, there exists a vector $\boldsymbol{v} \in \mathbb{R}^n$ such that the shifted cone $\boldsymbol{v} + \mathcal{K}$ contains exactly the interior integral points of $\mathcal{K}$ while the integral points located in the pointed cone $-\boldsymbol{v}+\mathcal{K}$ are precisely the integral points of $\mathcal{K}$. Moreover, the vector $\boldsymbol{v}$ can be chosen such that there are no integral points on the boundaries of any of these shifted cones, i.e.

$$\begin{aligned}
\partial(\boldsymbol{v} + \mathcal{K}_j) \cap \mathbb{Z}^n &= \emptyset \qquad \text{for } 0 \leq j \leq m && (1.100)\\
\partial(-\boldsymbol{v} + \mathcal{K}_j) \cap \mathbb{Z}^n &= \emptyset \qquad \text{for } 0 \leq j \leq m && (1.101)\\
(\boldsymbol{v} + \mathcal{K}) \cap \mathbb{Z}^n &= \mathcal{K}^\circ \cap \mathbb{Z}^n && (1.102)\\
(-\boldsymbol{v} + \mathcal{K}) \cap \mathbb{Z}^n &= \mathcal{K} \cap \mathbb{Z}^n, && (1.103)
\end{aligned}$$

where we denoted the cone $\mathcal{K}$ by $\mathcal{K}_0$.

Proof. In fact, this theorem is no more than a corollary of the previous theorem. Since the set $\{\mathcal{K}_1, \ldots, \mathcal{K}_m\}$ is a triangulation of a rational pointed d–cone with no new generators, each of these cones is rational. Thus, any cone in this triangulation is the intersection of finitely many rational half–spaces. Therefore, we may choose a vector $\boldsymbol{s} \in \mathcal{K}^\circ$ which is, in addition, not contained in any bounding hyperplanes of the triangulation cones. Following the lines of the proof of Theorem 1.47, we may use this vector $\boldsymbol{s}$ in order to construct a vector $\boldsymbol{v} \in \mathbb{R}^n$ such that, furthermore, the translations of the bounding hyperplanes of the triangulation cones by the vectors $\boldsymbol{v}$ and $-\boldsymbol{v}$ include no integral points. $\square$

1.4 Quasi–Polynomials

We will now introduce a class of functions which is essential for theories on counting integer points in polytopes. Quasi–polynomials provide a generalization of polynomials in the sense that here the coefficients are replaced by periodic functions with integer period. Before we are able to state a definition of quasi–polynomials, we need a specification of the term periodic function.

Definition 1.49. We call a function $g : \mathbb{Z} \to \mathbb{C}$ *periodic* with *period* $p \in \mathbb{N}$ if and only if

$$g(t + p) = g(t) \quad \text{for all } t \in \mathbb{Z}. \tag{1.104}$$

Definition 1.50. A *quasi–polynomial* is a function $f : \mathbb{Z} \to \mathbb{C}$ of the form

$$f(t) = c_d(t)t^d + c_{d-1}(t)t^{d-1} + \ldots + c_0(t),$$

where $c_0(t), \ldots, c_d(t)$ denote periodic functions. If the function $c_d(t)$ is not identically zero, the *degree* of the quasi–polynomial is defined as d. A common period of the functions $c_0(t), \ldots, c_d(t)$ is called a *quasi–period*, the least integer with this property is said to be the *period* of the quasi–polynomial.

Equivalently, quasi–polynomials are frequently defined as follows.

Proposition 1.51. *A function $f : \mathbb{Z} \to \mathbb{C}$ is a* quasi–polynomial *if and only if there exist a non–negative integer p and polynomials $f_0, f_1, \ldots, f_{p-1}$ such that*

$$f(t) = f_i(t) \qquad \text{if } t \equiv i (\operatorname{mod} p). \tag{1.105}$$

Then, the integer p is a quasi–period *of the quasi–polynomial f and the minimal integer with this property is the* period *of the quasi–polynomial. The* degree *of f is the maximum degree among the polynomials $f_0, f_1, \ldots, f_{p-1}$.*

Proof. Note that a periodic function with period p is constant on each residue class modulo p. Conversely, the functions $c_k : \mathbb{Z} \to \mathbb{C}$, where $c_k(t)$ equals the coefficient of t^k of the polynomial f_i if $t \equiv i (\operatorname{mod} p)$ are periodic functions with period p. $\square$

There is another equivalent approach towards quasi–polynomials, based on the following observation.

Lemma 1.52. *Given two non–negative integers m and p, the sum over the m–th powers of the p–th roots of unity*

$$\sum_{k=0}^{p-1} e^{\frac{2i\pi km}{p}} = \begin{cases} p & \text{if } p|m \\ 0 & \text{else.} \end{cases} \tag{1.106}$$

Proof. Suppose that $p|m$ and let l denote the integer such that $m = l \cdot p$. Then,

$$e^{\frac{2i\pi km}{p}} = e^{\frac{2i\pi klp}{p}} = \left(e^{2i\pi}\right)^{k\cdot l} = 1 \Rightarrow \sum_{k=0}^{p-1} e^{\frac{2i\pi km}{p}} = \sum_{k=0}^{p-1} 1 = p. \tag{1.107}$$

If $p \not| m$, then $e^{\frac{2i\pi\cdot m}{p}} \neq 1$ and

$$\sum_{k=0}^{p-1} \left(e^{\frac{2i\pi m}{p}}\right)^k = \frac{\left(e^{\frac{2i\pi m}{p}}\right)^p - 1}{\left(e^{\frac{2i\pi m}{p}}\right) - 1} = 0. \tag{1.108}$$

□

Lemma 1.53. *A function $f : \mathbb{Z} \to \mathbb{C}$ is a quasi–polynomial of maximum degree d and quasi–period p if and only if there are polynomials $P_0, \dots, P_{p-1}$ of maximum degree d such that*

$$f(t) = \sum_{k=1}^{p-1} P_k(t) \left(e^{\frac{2i\pi k}{p}}\right)^t \qquad \text{for all } t \in \mathbb{Z}. \tag{1.109}$$

Proof. Let $f_0, f_1, \dots, f_{p-1}$ be polynomials such that $f(t) = f_i(t)$ if $t \equiv i (\operatorname{mod} p)$. Using the function introduced in Lemma 1.52, we obtain that

$$\begin{aligned} f(t) &= \frac{1}{p}\left(f_0(t)\left(\sum_{k=1}^{p-1} e^{\frac{2i\pi kt}{p}}\right) + f_1(t)\left(\sum_{k=1}^{p-1} e^{\frac{2i\pi k(t-1)}{p}}\right) + \dots \right. \\ &\qquad \left. \dots + f_{p-1}(t)\left(\sum_{k=1}^{p-1} e^{\frac{2i\pi k(t-(p-1))}{p}}\right)\right) \\ &= \sum_{k=1}^{p-1} \left(e^{\frac{2i\pi k}{p}}\right)^t \frac{1}{p}\left(f_0(t) + f_1(t)\left(e^{\frac{2i\pi k}{p}}\right)^{-1} + \dots + f_{p-1}(t)\left(e^{\frac{2i\pi k}{p}}\right)^{-(p-1)}\right) \\ &= \sum_{k=1}^{p-1} P_k(t)\left(e^{\frac{2i\pi k}{p}}\right)^t, \end{aligned}$$

where we set

$$P_k(t) := \frac{1}{p}\left(f_0(t) + f_1(t)\left(e^{\frac{2i\pi k}{p}}\right)^{-1} + \ldots + f_{p-1}(t)\left(e^{\frac{2i\pi k}{p}}\right)^{-(p-1)}\right). \tag{1.110}$$

Conversely, note that the function $t \mapsto (e^{\frac{2i\pi k}{p}})^t$ is periodic with period p. $\square$

1.5 Rational Generating Functions

Generating functions of quasi–polynomials belong to a simple class of functions, namely the rational functions, which we will briefly discuss in the following section. Our aim is to provide us with conditions which enable us to decide weather or not a rational term is the generating function of a quasi–polynomial, resp. of a polynomial. Moreover, we shall see how to determine the degree and the period of a quasi–polynomial from the closed form of the generating function. These considerations are essential for our proof of Ehrhart's Theorem and are also of interest with regard to the chapter on vector partition functions. Our subsequent explanations are based on [22], Chapter 4.

Definition 1.54. Let $f : \mathbb{N} \to \mathbb{C}$ be a function. The *generating function* of f is defined as the formal power series given by

$$F(z) := \sum_{t \geq 0} f(t) z^t. \tag{1.111}$$

The generating function of a function is called *rational* if and only there are polynomials $P(z), Q(z)$ such that $\sum_{t\geq 0} f(t)z^t = P(z)/Q(z)$.

In the next theorem we exhibit characteristics of rational generating functions.

Theorem 1.55. *Let $\alpha_1, \alpha_2, \ldots, \alpha_d$ be a sequence of complex numbers where $\alpha_d \neq 0$ and suppose that $f : \mathbb{N} \to \mathbb{C}$ is a function. Then, the following three conditions are equivalent.*

(i) The generating function of $f(n)$ is rational and given by

$$\sum_{n \geq 0} f(n) z^n = \frac{P(z)}{Q(z)}, \tag{1.112}$$

where $Q(z) = 1 + \alpha_1 z + \ldots + \alpha_d z^d$ and $P(z)$ a polynomial of degree less than d.

(ii) *For all non–negative integers $n \geq 0$ we have that*

$$f(n+d) + \alpha_1 f(n+d-1) + \ldots + \alpha_d f(n) = 0. \tag{1.113}$$

(iii) *Suppose $1 + \alpha_1 z + \ldots + \alpha_d z^d = \prod_{i=1}^{k}(1-\gamma_i z)^{d_i}$ for some non–negative integer k and pairwise distinct complex numbers γ_i, for $1 \leq i \leq k$. Then, for all $n \geq 0$, the function*

$$f(n) = \sum_{i=1}^{k} P_i(n)\gamma_i^n, \tag{1.114}$$

where $P_i(n)$ denotes a polynomial of degree less than d_i.

Proof. We will show that $(i) \Leftrightarrow (ii)$ and $(i) \Leftrightarrow (iii)$.

$(i) \Rightarrow (ii)$. Assume that $\sum_{n\geq 0} f(n)z^n = P(z)/Q(z)$, where the polynomial $Q(z) = 1 + \alpha_1 z + \ldots + \alpha_d z^d$ and $P(z)$ denotes a polynomial of degree less than d. Then, we have that

$$\begin{aligned}
Q(z)\sum_{n\geq 0} f(n)z^n &= \left(1 + \alpha_1 z + \ldots + \alpha_d z^d\right)\sum_{n\geq 0} f(n)z^n \\
&= \sum_{n\geq 0} f(n)z^n + \alpha_1 \sum_{n\geq 0} f(n)z^{n+1} + \ldots + \alpha_d \sum_{n\geq 0} f(n)z^{n+d} \\
&= \left(\sum_{k=0}^{d-1} f(k)z^k + \alpha_1 \sum_{k=0}^{d-2} f(k)z^{k+1} + \ldots + \alpha_{d-1} f(0)z^{d-1}\right) \\
&\quad + \sum_{n\geq 0} f(n+d)z^{n+d} + \alpha_1 \sum_{n\geq 0} f(n+d-1)z^{n+d} + \ldots + \alpha_d \sum_{n\geq 0} f(n)z^{n+d}.
\end{aligned}$$

Therefore, we obtain that

$$P(z) = \sum_{i=0}^{d-1}\left(\sum_{k=0}^{i} \alpha_k f(i-k)\right) z^i + \sum_{n\geq 0}\left(f(n+d) + \ldots + \alpha_d f(n)\right) z^{n+d}, \tag{1.115}$$

where we set $\alpha_0 = 1$. Since we assumed that the degree of the polynomial $P(z)$ is less than d, comparing coefficients on both sides of equation (1.115) yields that

$$f(n+d) + \alpha_1 f(n+d-1) + \ldots + \alpha_d f(n) = 0, \tag{1.116}$$

for all non–negative integers n.

$(ii) \Rightarrow (i)$ Suppose that

$$f(n+d) + \alpha_1 f(n+d-1) + \ldots + \alpha_d f(n) = 0, \tag{1.117}$$

for all integers $n \geq 0$. We have to show that the generating function of $f(n)$ is rational. Now, multiplying equation (1.117) by z^{n+d} and summation over all non–negative integers n yields that

$$\begin{aligned}
0 &= \sum_{n\geq 0} f(n+d)z^{n+d} + \alpha_1 \sum_{n\geq 0} f(n+d-1)z^{n+d} + \ldots + \alpha_d \sum_{n\geq 0} f(n)z^{n+d} \\
&= \sum_{l\geq d} f(l)z^l + \alpha_1 z \sum_{l\geq d-1} f(l)z^l + \ldots + \alpha_d z^d \sum_{l\geq 0} f(l)z^l \\
&= \left(\sum_{l\geq 0} f(l)z^l - \sum_{l=0}^{d-1} f(l)z^l\right) + \alpha_1 z \left(\sum_{l\geq 0} f(l)z^l - \sum_{l=0}^{d-2} f(l)z^l\right) + \ldots \\
&\qquad \ldots + \alpha_d z^d \sum_{l\geq 0} f(l)z^l.
\end{aligned}$$

Collecting the terms containing $\sum_{l\geq 0} f(l)z^l$ and subtracting them from both sides of the upper equation and multiplying by minus one, we obtain that

$$\begin{aligned}
&\sum_{l\geq 0} f(l)z^l \cdot \left(1 + \alpha_1 z + \cdots + \alpha_d z^d\right) = \ldots \\
&\qquad \ldots = \sum_{l=0}^{d-1} f(l)z^l + \alpha_1 z \sum_{l=0}^{d-2} f(l)z^l + \cdots + \alpha_{d-1} z^{d-1} f(0).
\end{aligned} \tag{1.118}$$

Observe that the right–hand side of the upper equation gives a polynomial of degree less than d, subsequently denoted by $P(z)$. Therefore, if we set $Q(z) := 1+\alpha_1 z+\cdots+\alpha_d z^d$, the generating function

$$\sum_{l\geq 0} f(l)z^l = \frac{P(z)}{Q(z)}. \tag{1.119}$$

$(i) \Rightarrow (iii)$ Again, we assume that

$$\sum_{n\geq 0} f(n)z^n = \frac{P(z)}{Q(z)}, \tag{1.120}$$

where $Q(z) = 1 + \alpha_1 z + \cdots + \alpha_d z^d$ and $P(z)$ is a polynomial of degree less than d.

Due to the fundamental theorem of algebra, there exists a positive integer k such that there are pairwise distinct complex numbers γ_i, for $1 \leq i \leq k$, with

$$1 + \alpha_1 z + \cdots + \alpha_d z^d = \prod_{i=1}^{k} (1 - \gamma_i z)^{d_i}. \tag{1.121}$$

Accordingly, by the partial fraction decomposition, there are polynomials $G_i(z)$ of degree less than d_i such that

$$\sum_{n\geq 0} f(n)z^n = \frac{P(z)}{Q(z)} = \frac{P(z)}{\prod\limits_{i=1}^{k} (1 - \gamma_i z)^{d_i}} = \sum_{i=1}^{k} \frac{G_i(z)}{(1 - \gamma_i z)^{d_i}}. \tag{1.122}$$

For any index $1 \leq i \leq k$, the function $G_i(z)/(1 - \gamma_i z)^{d_i}$ is a finite linear combination of terms of the form $z^j/(1 - \gamma_i z)^{d_i}$, for $j < d_i$. Now, observe that

$$\frac{z^j}{(1 - \gamma_i z)^{d_i}} \quad = \quad z^j \cdot \sum_{k\geq 0} \binom{d_i - 1 + k}{d_i - 1} \gamma_i^k z^k = \sum_{n\geq 0} \binom{d_i - 1 + n - j}{d_i - 1} \gamma_i^{-j} \gamma_i^n z^n.$$

Moreover, if we fix an integer j, the expression $\binom{d_i - 1 + n - j}{d_i - 1}\gamma_i^{-j}$ is a polynomial in n of degree $d_i - 1$. Since a linear combination of polynomials of degree $d_i - 1$ is again a polynomial of degree at most $d_i - 1$, the coefficient of z^n in $G_i(z)/(1 - \gamma_i z)^{d_i}$ equals γ_i^n times a polynomial in n of degree at most $d_i - 1$, denoted $P_i(n)$. Therefore, comparing coefficients in (1.122) implies that

$$f(n) = \sum_{i=1}^{k} P_i(n)\gamma_i^n, \tag{1.123}$$

which is the assertion of item (iii) in Theorem 1.55.

$(iii) \Rightarrow (i)$ Finally, we assume that

$$\sum_{n\geq 0} f(n)z^n = \sum_{n\geq 0}\sum_{i=1}^{k} P_i(n)\gamma_i^n z^n = \sum_{i=1}^{k}\sum_{n\geq 0} P_i(n)\gamma_i^n z^n, \tag{1.124}$$

where the γ_is stand for pairwise distinct complex numbers while $P_i(n)$ denotes a polynomial of degree less than d_i, for $1 \leq i \leq k$.

Put differently, for all indices $1 \leq i \leq k$, there are complex numbers $\alpha_{i,0}, \ldots, \alpha_{i,d_i-1}$ such that

$$P_i(n) = \alpha_{i,0} + \alpha_{i,1}n + \cdots + \alpha_{i,d_i-1}n^{d_i-1}. \tag{1.125}$$

Therefore, and since

$$\sum_{n\geq 0} n^l \gamma_i^n z^n = \left(z\frac{d}{dz}\right)^l \frac{1}{(1-\gamma_i z)} = \frac{g_l(z)}{(1-\gamma_i z)^{l+1}}, \tag{1.126}$$

for an appropriate polynomial $g_l(z)$ of degree at most l, we observe that

$$\begin{aligned}\sum_{n\geq 0} P_i(n)\gamma_i^n z^n &= \alpha_{i,0}\sum_{n\geq 0}\gamma_i^n z^n + \alpha_{i,1}\sum_{n\geq 0} n\gamma_i^n z^n + \cdots + \alpha_{i,d_i-1}\sum_{n\geq 0} n^{d_i-1}\gamma_i^n z^n \\ &= \frac{\alpha_{i,0}}{(1-\gamma_i z)} + \frac{\alpha_{i,1}\ g_1(z)}{(1-\gamma_i z)^2} + \cdots + \frac{\alpha_{i,d-1}\ g_{d_i-1}(z)}{(1-\gamma_i z)^{d_i}},\end{aligned}$$

where $g_i(z)$ stands for a polynomial of degree at most i, for $1 \leq i \leq d_i - 1$. Putting this sum over a common denominator, we get that

$$\sum_{n\geq 0} P_i(n)\gamma_i^n z^n = \frac{G_i(z)}{(1-\gamma_i z)^{d_i}}, \tag{1.127}$$

for some polynomial $G_i(z)$ of degree less than d_i. Finally we obtain that

$$\sum_{n\geq 0} f(n)z^n = \sum_{i=1}^{k}\frac{G_i(z)}{(1-\gamma_i z)^{d_i}} = \frac{P(z)}{\prod_{i=1}^{k}(1-\gamma_i z)^{d_i}}, \tag{1.128}$$

where $P(z)$ denotes a polynomial of degree less than $\sum\limits_{i=1}^{k} d_i$, as desired. $\square$

Corollary 1.56. *The fraction $P(x)/Q(x)$ in item (i) of Theorem 1.55 is written in lowest terms if and only if equation (1.113) in item (ii) gives the homogeneous linear recurrence with constant coefficients of least degree satisfied by the function $f(n)$. The latter is the case if and only if the degree of the polynomial $P_i(n)$ occurring in item (iii) equals $d_i - 1$, for all $1 \le i \le k$.*

The relation between quasi–polynomials and particular rational function is established by the following corollary.

Corollary 1.57. *Given a function $f : \mathbb{N} \to \mathbb{C}$ and a non–negative integer p, the following conditions are equivalent.*

(i) The function $f(n)$ is a quasi–polynomial with period p.

(ii) The generating function of the function $f(n)$ is given by

$$\sum_{n \ge 0} f(n) z^n = \frac{P(z)}{Q(z)}, \tag{1.129}$$

where $P(z)$ and $Q(z)$ are polynomials with $\deg(P(z)) < \deg(Q(z))$ and all roots of the polynomial $Q(z)$ are p–th roots of unity, provided that the rational function $P(z)/Q(z)$ has been reduced to lowest terms.

(iii) For all non–negative integers n we have that

$$f(n) = \sum_{i=1}^{k} P_i(n) \gamma_i^n, \tag{1.130}$$

where $P_i(n)$ is a polynomial and $\gamma_i^p = 1$, for all $1 \le i \le k$.

Proof. $(i) \Leftrightarrow (iii)$ is the subject of Lemma 1.53, while $(ii) \Leftrightarrow (iii)$ by Theorem 1.55. $\square$

Remark 1.58. Taking Corollary 1.56 into account, we obtain the following details. Provided that the rational expression $P(z)/Q(z)$ in item (ii) is written in lowest terms, the degree of the polynomial $P_i(n)$ in item (iii) is one less than the multiplicity of the root γ_i^{-1} of the polynomial $Q(z)$ in item (i). Accordingly, the degree of the quasi–polynomial $f(n)$ equals one less than the maximum multiplicity of a root of the polynomial $Q(z)$, c.f. Lemma 1.53.

As a special case of Corollary 1.57, we may now specify the relation between generating functions of polynomials and certain rational expressions.

Corollary 1.59. *Let $f : \mathbb{N} \to \mathbb{C}$ be a function and d a positive integer. If one of the following statements is valid, then so are the other two.*

(i) The generating function of $f(n)$ is rational and given as

$$\sum_{n\geq 0} f(n)z^n = \frac{P(z)}{(1-z)^{d+1}}, \tag{1.131}$$

where $P(z)$ denotes a polynomial of degree less than $d+1$.

(ii) For all integers $n > 0$ we have that

$$\sum_{i=0}^{d+1}(-1)^i \binom{d+1}{i} f(n+d+1-i) = 0. \tag{1.132}$$

(iii) The function $f(n)$ is a polynomial of degree not exceeding d, while the polynomial $f(n)$ has exactly degree d if and only if $P(1) \neq 0$.

Proof. By the binomial theorem, we have that

$$(1-z)^{d+1} = \sum_{i=0}^{d+1} \binom{d+1}{i} (-1)^i z^i \text{ and } \binom{d+1}{0} = \binom{d+1}{d+1} = 1 \tag{1.133}$$

for all non–negative integers $d \in \mathbb{Z}_{\geq 0}$. Therefore, the desired result follows by applying Theorem 1.55 to the sequence $\alpha_i = \binom{d+1}{i}(-1)^i$ for $1 \leq i \leq d+1$. Note that the polynomial $(1-z)^{d+1}$ has one root of multiplicity $(d+1)$ at $\gamma^{-1} = 1$ and the function $P(z)/(1-z)^{d+1}$ is written in lowest terms if and only if $P(1) \neq 0$. □

During the next chapter we require another lemma related to rational generating functions.

Lemma 1.60. *Let $p : \mathbb{Z} \to \mathbb{C}$ be a polynomial. The Laurent series, defined as*

$$R_p^+(z) := \sum_{t \geq 0} p(t) z^t \quad \text{and} \quad R_p^-(z) := \sum_{t<0} p(t) z^t \tag{1.134}$$

are rational functions which add up to zero, i.e. $R_p^+(z) + R_p^-(z) = 0$.

Proof. Note that

$$\sum_{t \geq 0} z^t + \sum_{t>0} \left(\frac{1}{z}\right)^t = \frac{1}{1-z} - \frac{z}{1-z} - 1 = 0. \tag{1.135}$$

Therefore, we conclude that

$$\begin{aligned} \sum_{t \geq 0} t^k z^t + \sum_{t \geq 0} (-t)^k \left(\frac{1}{z}\right)^t &= \left(z \frac{d}{dz}\right)^k \sum_{t \geq 0} z^t + \left(z \frac{d}{dz}\right)^k \sum_{t \geq 0} \left(\frac{1}{z}\right)^t \\ &= \left(z \frac{d}{dz}\right)^k \left(\frac{1}{1-z} - \frac{z}{1-z}\right) = \left(z \frac{d}{dz}\right)^k 1 = 0. \end{aligned} \tag{1.136}$$

Now, assume that the polynomial is given by $p(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_0$. If we insert this into our definition, we obtain that

$$\begin{aligned} R_p^+(z) &= a_n \sum_{t \geq 0} t^n z^t + a_{n-1} \sum_{t \geq 0} t^{n-1} z^t + \cdots + a_0 \sum_{t \geq 0} z^t, \\ R_p^-(z) &= a_n \sum_{t>0} (-t)^n \left(\frac{1}{z}\right)^t + a_{n-1} \sum_{t>0} (-t)^{n-1} \left(\frac{1}{z}\right)^t + \cdots + a_0 \sum_{t>0} \left(\frac{1}{z}\right)^t, \end{aligned}$$

and these two formal power series add up to zero, by Equation (1.136). Moreover, since $R_p^+(z)$ is the generating function of the polynomial $p(t)$ there are, by Corollary 1.59, polynomials $P(z)$ and $Q(z)$ such that $R_p^+(z) = P(z)/Q(z)$. Thus, the formal power series

$$R_p^-(z) = -R_p^+(z) = -P(z)/Q(z) \tag{1.137}$$

is a rational function as well.

□

Corollary 1.61. *Given a quasi–polynomial $q : \mathbb{Z} \to \mathbb{C}$, the Laurent series defined by*

$$R_q^+(z) := \sum_{t \geq 0} q(t) z^t \qquad \text{and} \qquad R_q^-(z) := \sum_{t<0} q(t) z^t \tag{1.138}$$

are rational functions which sum up to zero, i.e. $R_q^+(z) + R_q^-(z) = 0$.

Proof. By the definition of quasi–polynomials, there exist an integer k and polynomials $p_0, \ldots, p_{k-1}$ such that $q(t) = p_i(t)$ if $t \equiv i \pmod{k}$, for some $0 \leq i \leq k-1$.

Note that

$$\begin{aligned}
\sum_{t \geq 0} q(t) z^t &= \sum_{l \geq 0} p_0(kl) z^{kl} + \sum_{l \geq 0} p_1(kl+1) z^{kl+1} + \ldots + \sum_{l \geq 0} p_{k-1}(kl+k-1) z^{kl+k-1} \\
&= \sum_{l \geq 0} p_0(kl) \left(z^k\right)^l + z \sum_{l \geq 0} p_1(kl+1) \left(z^k\right)^l + \ldots \\
&\qquad \ldots + z^{k-1} \sum_{l \geq 0} p_{k-1}(kl+k-1) \left(z^k\right)^l .
\end{aligned}$$

Therefore, and since $p_j(kl+j)$ is a polynomial in l, the assertion follows by Lemma 1.60.

□

1.6 Lattices

The concept of lattices allows us to give a multivariate generalization of the definition of quasi–polynomials that we will need in the context of vector partition functions in Chapter 3. Here, we will also state some basic properties of lattices that we need in the following chapters.

Definition 1.62. A lattice Λ is a subgroup of $\mathbb{R}^s$ which can be generated from a basis of the vector space by forming all linear combinations with integral coefficients. Put differently, let $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_s\}$ be a set of linear independent vectors from $\mathbb{R}^s$. The lattice generated by this basis is defined as

$$\Lambda := \{n_1 \boldsymbol{a}_1 + \ldots + n_s \boldsymbol{a}_s : n_1, \ldots, n_s \in \mathbb{Z}\}. \tag{1.139}$$

The standard basis of the vector space $\mathbb{R}^s$ generates the standard lattice $\mathbb{Z}^s$.

Definition 1.63. We say that a lattice Λ_1 is a *sub–lattice* of a lattice Λ if and only if $\Lambda_1 \subseteq \Lambda$. Sub–lattices of the standard lattice $\mathbb{Z}^s$ are called *integral lattices.*

Given a sub–lattice $\Lambda_1 \subseteq \Lambda$ we can define an equivalence relation on Λ by setting

$$\boldsymbol{x}_1 \equiv \boldsymbol{x}_2 (\operatorname{mod} \Lambda_1) \quad \text{if and only if} \quad \boldsymbol{x}_1 - \boldsymbol{x}_2 \in \Lambda_1. \tag{1.140}$$

The equivalence classes of this relation are called the cosets of Λ modulo Λ_1 and we denote the set of these cosets by Λ/Λ_1.

Remark 1.64. For a given integer d, the set $\Lambda_d := \{\boldsymbol{z} \in \mathbb{Z}^s : d|z_i \text{ for all } 1 \leq i \leq s\}$ is an integral lattice which is generated by the basis $\{d\boldsymbol{e}_1, \ldots, d\boldsymbol{e}_s\}$.

Definition 1.65. Let Λ be a sub–lattice of $\mathbb{Z}^s$ generated by the basis $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_s\}$. The half–open parallelepiped defined as

$$\Pi_\Lambda := \{\lambda_1 \boldsymbol{a}_1 + \ldots + \lambda_s \boldsymbol{a}_s, 0 \leq \lambda_i < 1\}, \tag{1.141}$$

is called the *fundamental parallelepiped of the lattice.*

Definition 1.66. We define the determinant of a lattice $\Lambda \subseteq \mathbb{R}^s$, denoted $d(\Lambda)$, as the number of integral points in the fundamental paralellepiped Π_Λ.

Lemma 1.67. *Suppose the set* $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_s\}$ *is a basis of an integral lattice* Λ*. Then, any vector* $\boldsymbol{x} \in \mathbb{R}^s$ *has a unique representations as* $\boldsymbol{x} = \sum_{i=1}^{s} \lambda_i(\boldsymbol{x})\boldsymbol{a}_i$*, with* $\lambda_i(\boldsymbol{x}) \in \mathbb{R}$ *for* $1 \leq i \leq s$*. Two integral vectors* $\boldsymbol{x}_1$ *and* $\boldsymbol{x}_2$ *are contained in the same coset of* $\mathbb{Z}^s/\Lambda$ *if and only if* $\{\lambda_i(\boldsymbol{x}_1)\} = \{\lambda_i(\boldsymbol{x}_2)\}$*, for all* $1 \leq i \leq s$*.*

Proof. Since

$$\begin{aligned} \boldsymbol{x}_1 - \boldsymbol{x}_2 &= \sum_{i=1}^{s} \left(\lambda_i(\boldsymbol{x}_1) - \lambda_i(\boldsymbol{x}_2)\right) \boldsymbol{a}_i \\ &= \sum_{i=1}^{s} \left(\lfloor \lambda_i(\boldsymbol{x}_1) \rfloor + \{\lambda_i(\boldsymbol{x}_1)\} - \lfloor \lambda_i(\boldsymbol{x}_2) \rfloor - \{\lambda_i(\boldsymbol{x}_2)\}\right) \boldsymbol{a}_i, \end{aligned} \tag{1.142}$$

the vector $\boldsymbol{x}_1 - \boldsymbol{x}_2$ is an element of the lattice Λ if and only if $\{\lambda_i(\boldsymbol{x}_1)\} - \{\lambda_i(\boldsymbol{x}_2)\} = 0$ for all $1 \leq i \leq s$. $\square$

Corollary 1.68. *Let* Π_Λ *be the fundamental parallelepiped of an integral lattice* Λ*. Then the set* $\Pi_\Lambda \cap \mathbb{Z}^s$ *contains exactly one representant of each coset of* $\mathbb{Z}^s/\Lambda$*. Hence the number of cosets, or the index, of* $\mathbb{Z}^s/\Lambda$ *equals* $\boldsymbol{d}(\Lambda)$*.*

1.7 Multivariate Quasi-Polynomials

Definition 1.69. A multivariate *polynomial* is an expression of the form

$$p(\boldsymbol{x}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^s_{\geq 0}}^{\|\boldsymbol{n}\|_1\leq d} c_{\boldsymbol{n}} \boldsymbol{x}^{\boldsymbol{n}}$$

where $c_{\boldsymbol{n}}$ is a complex number for all $\boldsymbol{n} \in \mathbb{Z}^s_{\geq 0}$ with $\|\boldsymbol{n}\|_1 \leq d$, where $\boldsymbol{x}^{\boldsymbol{n}} := x_1^{n_1} \dots x_s^{n_s}$. The *degree* of a polynomial is defined as

$$\deg(p(\boldsymbol{x})) := \max_{\boldsymbol{n}\in\mathbb{Z}^s_{\geq 0}} \{\|\boldsymbol{n}\|_1 : c_{\boldsymbol{n}} \neq 0\}. \tag{1.143}$$

Here $\|\boldsymbol{n}\|_1 := |n_1| + \ldots + |n_s|$ denotes the 1–norm of a vector $\boldsymbol{n}$ in $\mathbb{R}^s$.

Definition 1.70. Let Λ be an integral lattice and suppose the set $\{\boldsymbol{z}_1, \ldots, \boldsymbol{z}_m\}$ is a complete set of representatives of the cosets of $\mathbb{Z}^s$ modulo Λ.

A function $q : \mathbb{Z}^s \to \mathbb{C}$ is called a quasi–polynomial with respect to the lattice Λ if and only if there exist polynomials $p_1(\boldsymbol{x}), \ldots, p_m(\boldsymbol{x})$ such that

$$q(\boldsymbol{x}) = p_i(\boldsymbol{x}) \quad \text{if } \boldsymbol{x} - \boldsymbol{z}_i \in \Lambda. \tag{1.144}$$

The maximum of the degrees of the polynomials $p_1(\boldsymbol{x}), \ldots, p_m(\boldsymbol{x})$ is called the *degree* of the quasi–polynomial.

Remark 1.71. A sub–lattice of $\mathbb{Z}$ is by definition a set of the form $\{ka : k \in \mathbb{Z}\}$ for some integer a. Hence, in the one–dimensional case, this definition agrees with the one given in Section 1.4.

Lemma 1.72. *Let $\Lambda_1 \subseteq \Lambda_2$ be sub–lattices of $\mathbb{Z}^s$. If $q : \mathbb{Z}^s \to \mathbb{R}$ is a quasi–polynomial with respect to the lattice Λ_1, then, q is a quasi–polynomial with respect to Λ_2 as well.*

Let q_1 be a quasi–polynomial with respect to the lattice Λ_1 and q_2 a quasi–polynomial with respect to the lattice Λ_2. Then $q_1 + q_2$ is a quasi–polynomial with respect to any lattice which is a sub–lattice of Λ_1 and Λ_2.

2 Chapter 2 Ehrhart Theory

The beginning of the general study of integral–point enumeration in variable polytopes goes back to the work of Eugène Ehrhart in the 1960's, perhaps with Pick's theorem as an earlier particular case. Ehrhart, who did most of his research working as teacher at a lycee in Strasbourg, France, and received his doctorate at the age of sixty[1], considered integral dilations of rational polytopes, leaving angles and proportions fixed. He discovered that the number of integral points in these dilated polytopes can be counted by a quasi–polynomial in the dilation factor. Another intriguing and unexpected application of Ehrhart's theory is that it enables us to calculate the volume rational polytopes by counting integral points.

Magic squares are perhaps the most famous example of combinatorial objects which may be counted using Ehrhart polynomials, c.f. [3], [22] and the references given there, but there are plenty more, for further examples see [14] or [16].

The theory of Ehrhart quasi–polynomials has a strong connection to Commutative Algebra, and much of the theory was developed initially from this point of view. For more information on this aspect see for example [24], [10].

Our aim is to give a survey of the most important properties of Ehrhart quasi–polynomials. We have arranged our explanations elementary, using basic notations and avoiding advanced language.

[1] To read more about Eugène Ehrhart see the tribute written by Philippe Clauss http://icps.u-strasbg.fr/∼ clauss/Ehrhart.html.

2.1 First Examples and the Main Theorem

The issue of the following chapter is to study the number of integral points in integral dilations of rational polytopes. In order to formulate the main aspects, we need to introduce some terms.

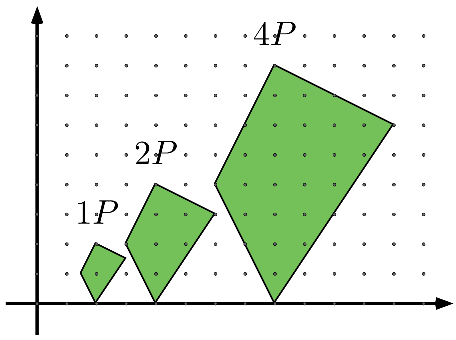


Figure 2.1: Integral dilations of a polytope.

Definition 2.1. Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a polytope. For a non–negative integer t the *t-th dilation* of the polytope is defined as

$$t\mathcal{P} := \{t\boldsymbol{x} : \boldsymbol{x} \in \mathcal{P}\}. \tag{2.1}$$

More general, the t-th dilation of a subset $S \subseteq \mathbb{R}^n$ is defined as $tS := \{t\boldsymbol{x} : \boldsymbol{x} \in S\}$.

Definition 2.2. A polytope is called *rational* if all its vertices have rational coordinates. The least common multiple of the denominators of the vertex coordinates is called the *denominator* of the polytope. In particular, if each vertex has integral coordinates the polytope is called *integral*.

Moreover, recall that any polytope $\mathcal{P} \subseteq \mathbb{R}^d$ can be described in hyperplane description as $\mathcal{P} = \{\boldsymbol{x} \in \mathbb{R}^d : A\boldsymbol{x} \leq \boldsymbol{b}\}$,where A denotes an $s \times d$–Matrix and $\boldsymbol{b} \in \mathbb{R}^s$ a vector. This representation is not unique, but it is easy to see that rational polytopes are exactly those for which we may find a representation where the matrix A as well as the vector $\boldsymbol{b}$ have only integral entries.

A cone is called *rational* if it has integral generators.

Definition 2.3. The *integer–point enumerator* of a polytope $\mathcal{P} \subseteq \mathbb{R}^n$ is a function which counts the integral points in the t–th dilation of the polytope in dependence of t, i.e.

$$\mathcal{L}_{\mathcal{P}}(t) \quad := \quad \#(t\mathcal{P} \cap \mathbb{Z}^n). \tag{2.2}$$

The generating function of the integer–point enumerator is called the *Ehrhart–series* of the polytope, subsequently denoted by

$$\mathrm{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{t \geq 1} \mathcal{L}_{\mathcal{P}}(t) z^t. \tag{2.3}$$

More generally, we define the integer–point enumerator of a finite set $S \subseteq \mathbb{R}^n$ as the counting function $\mathcal{L}_S(t) := \#(tS \cap \mathbb{Z}^n)$.

Remark 2.4. Note that $\mathcal{L}_S(0) = 1$ for any non–empty finite set $S \subseteq \mathbb{R}^n$.

Example 2.5. Polytopes in $\mathbb{R}$ are closed intervals. We consider an interval with integral boundary points $a < b$.

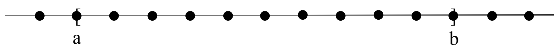


Figure 2.2: Integral points in the interval $[a, b]$.

The number of integral points located in such an interval, equals $b - a + 1$. Dilating the interval by a non–negative integral factor t, we obtain the closed interval $[ta, tb]$. Accordingly,

$$\mathcal{L}_{[a,b]}(t) = \#(t[a, b] \cap \mathbb{Z}) = t(b - a) + 1, \tag{2.4}$$

which is a polynomial in t of degree one. Observe that the leading term, $(b - a)$, agrees with the length, i.e. the volume of the interval, and that $\mathcal{L}_{[a,b]}(0) = 1$. Moreover, as to the interior integral points, we obtain that $\# ((a, b) \cap \mathbb{Z}) = (b - a) - 1$. This implies the reciprocity relation

$$\mathcal{L}_{[a,b]^\circ}(t) := \#(t(a, b) \cap \mathbb{Z}) = t(b - a) - 1 = (-1)\mathcal{L}_{[a,b]}(-t). \tag{2.5}$$

So the evaluation of the integer–point enumerator at negative integers gives -1 times the number of integral points in the interior of the polytope.

It is possible to generalize the properties established in the example given above to the setting of rational polytopes in general finite dimension. The objective of the next section will be to give proof of the following theorem.

Ehrhart's Theorem. Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a rational d–polytope, then the integer–point enumerator $\mathcal{L}_\mathcal{P}(t)$ is a quasi–polynomial in t of degree d whose period divides the denominator of the polytope $\mathcal{P}$. In honor of its discoverer, we call this function the *Ehrhart quasi–polynomial*, or the Ehrhart function, of the polytope.

In addition, we will give basic proofs of the following properties of Ehrhart quasi–polynomials.

- Given a rational d–polytope in $\mathbb{R}^d$, the leading coefficient of the Ehrhart quasi–polynomial equals the volume of the polytope.
- The evaluation of the Ehrhart quasi–polynomial at negative integers gives $(-1)^{\dim(\mathcal{P})}$ times the number of the integral points in the interior of the polytope dilated by the absolute value of this integer. This result is known as the Ehrhart–Macdonald Reciprocity.
- Regarding the period of the Ehrhart quasi–polynomial, we will find that the word *divides* in Ehrhart's Theorem can not be replaced by *equals*.

2.2 A Proof of Ehrhart's Theorem

Here, we carry out the proof of Ehrhart's Theorem for rational polytopes, as suggested in the exercise section of Chapter 3 of the book [3].

The essential idea of the proof is a method called coning over a polytope, which can be described as follows. Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a d–polytope. We lift the polytope into the vector space $\mathbb{R}^{n+1}$ by adding 1 as $(n+1)$–st coordinate to the points of the polytope. We consider the pointed cone which is generated by the vertices of the polytope so obtained. Then, for any positive integer t, the intersection

$$\operatorname{cone}(\mathcal{P}|1) \cap \{\boldsymbol{x} \in \mathbb{R}^{n+1} : x_{n+1} = t\} = t\mathcal{P} \times \{t\}. \tag{2.6}$$

Thus, this cone contains in a sense all positive integral dilations of the polytope. In fact, we have already used this method in order to prove the equivalence of the vertex and hyperplane description of polytopes, c.f. Section 1.2.2.

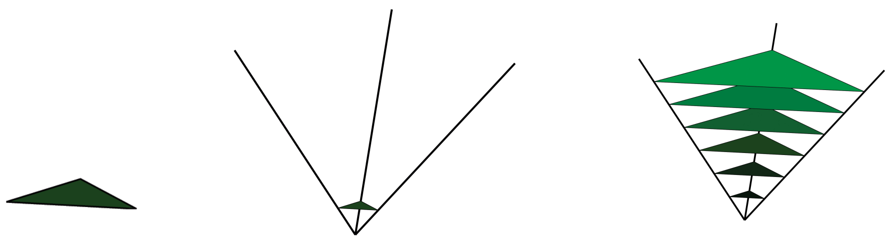


Figure 2.3: Coning over a 2−simplex.

It turns out that the integral points in rational simplicial cones have a nice description in terms of rational functions. As to the fact that any polytope can be triangulated into simplices using no new vertices, c.f. Theorem 1.31, and that cones over simplices are simplicial, c.f. Proposition 1.41, we may deduce Ehrhart's Theorem for general rational polytopes from the case of rational simplices.

To formalize these ideas, we introduce the following multivariate generating function, which lists the integral points in the set as a formal sum of monomials.

Definition 2.6. For an arbitrary set $S \subseteq \mathbb{R}^n$ the *integer–point transform* is defined as

$$\sigma_S(\boldsymbol{z}) = \sigma_S(z_1, \ldots, z_n) = \sum_{\boldsymbol{m} \in S \cap \mathbb{Z}^n} \boldsymbol{z}^{\boldsymbol{m}}. \tag{2.7}$$

Example 2.7. We consider the 1–dimensional cone $\mathcal{K} = [0, \infty)$. Then

$$\sigma_{\mathcal{K}}(z) = \sum_{m \in [0,\infty) \cap \mathbb{Z}} z^m = \sum_{m \geq 0} z^m = \frac{1}{1-z}. \tag{2.8}$$

Lemma 2.8. *If $S \subseteq \mathbb{R}^n$ is a bounded set, then the evaluation of the integer–point transform σ_S at the point $\boldsymbol{z} = (1, 1, \ldots, 1) \in \mathbb{R}^n$ counts the integral points in S.*

Proof. Observe that

$$\sigma_S(1, 1, \ldots, 1) = \sum_{\boldsymbol{m} \in S \cap \mathbb{Z}^n} (1, 1, \ldots, 1)^{\boldsymbol{m}} = \sum_{\boldsymbol{m} \in S \cap \mathbb{Z}^n} 1 = \#(S \cap \mathbb{Z}^n). \tag{2.9}$$

□

Remark 2.9. Given a polytope $\mathcal{P} \subseteq \mathbb{R}^n$ and a non–negative integer $t \in \mathbb{Z}_{\geq 0}$, we recover the t–th dilation $t\mathcal{P}$ as the intersection of $\mathrm{cone}(\mathcal{P}|1)$ with the hyperplane $\{\boldsymbol{x} \in \mathbb{R}^{n+1} : x_{n+1} = t\}$. Accordingly, the integer–point transform of the dilated polytope $t\mathcal{P}$ appears as the coefficient of z_{n+1}^t in the integer–point transform of the cone over $\mathcal{P}$. This relation between the integer–point transform of the cone over the polytope and the integral points contained in integral dilations of the polytope, yields that

$$\sigma_{\mathrm{cone}(\mathcal{P}|1)}(z_1, \ldots, z_n, z_{n+1}) = 1 + \sum_{t \geq 1} \sigma_{t\mathcal{P}}(z_1, \ldots, z_n) z_{n+1}^t. \tag{2.10}$$

We summarize the discussion above in the following theorem.

Theorem 2.10. *The integer–point transform of* $\mathrm{cone}(\mathcal{P}|1)$ *evaluated at* $(1, \ldots, 1, z)$ *yields the generating function of the integer–point enumerator of* $\mathcal{P}$*, i.e.*

$$\sigma_{\mathrm{cone}(\mathcal{P}|1)}(1, \ldots, 1, z) = 1 + \sum_{t \geq 1} \mathcal{L}_{\mathcal{P}}(t) z^t = Ehr_{\mathcal{P}}(z). \tag{2.11}$$

Now, basic geometrical considerations will lead to a nice form of the integer–point transform of rational simplicial cones. Recall that a rational d–cone $\mathcal{K} \subseteq \mathbb{R}^n$ is simplicial if it posesses exactly d linearly independent generators. In this case, any element of the cone posesses a unique representation as a non–negative linear combination of the generators. Therefore, we may tile the cone with translates of the half–open parallelepiped generated by these vectors. Since the cone is rational we may assume that the generators are given by integral vectors. Therefore, each integral point located in $\mathcal{K}$ lies in exactly one of these domains and has a unique representation as a translate of an integral point of the fundamental parallelepiped by a positive integral combination of the generators. We will specify these ideas during the proof of the next theorem.

Theorem 2.11. *Suppose that* $\mathcal{K} \subseteq \mathbb{R}^n$ *is a rational simplicial* d*–cone with integral generators* $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_d \in \mathbb{Z}^n$*. Then, for any vector* $\boldsymbol{v} \in \mathbb{R}^n$*, the integer–point transform of the shifted cone* $\boldsymbol{v} + \mathcal{K}$ *is given by the rational function*

$$\sigma_{\boldsymbol{v}+\mathcal{K}}(\boldsymbol{z}) = \frac{\sigma_{\boldsymbol{v}+\Pi_{\mathcal{K}}}(\boldsymbol{z})}{(1 - \boldsymbol{z}^{\boldsymbol{w}_1}) \ldots (1 - \boldsymbol{z}^{\boldsymbol{w}_d})}. \tag{2.12}$$

Here, $\Pi_{\mathcal{K}} := \{\lambda_1 \boldsymbol{w}_1 + \ldots + \lambda_d \boldsymbol{w}_d : 0 \leq \lambda_1, \ldots, \lambda_d < 1\}$*, denotes the fundamental parallelepiped of the cone* $\mathcal{K}$*.*

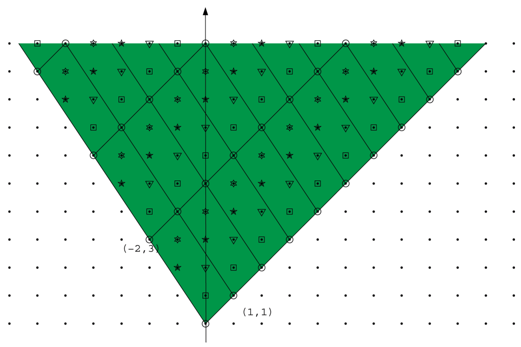


Figure 2.4: A simplicial cone generated by $(1,1)$ and $(-2,3)$.

Proof. Since we assumed that $\mathcal{K}$ is a simplicial cone, the generators $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_d$ are linearly independent. Therefore, each point $\boldsymbol{m} \in (\boldsymbol{v}+\mathcal{K}) \cap \mathbb{Z}^n$ has a unique representation as $\boldsymbol{m} = \boldsymbol{v} + \lambda_1 \boldsymbol{w}_1 + \ldots + \lambda_d \boldsymbol{w}_d$, for non–negative reals $\lambda_1, \ldots, \lambda_d$. Moreover, if we denote the real λ_i as $\lfloor \lambda_i \rfloor + \{\lambda_i\}$, we obtain that

$$\boldsymbol{m} = \boldsymbol{v} + \{\lambda_1\}\boldsymbol{w}_1 + \ldots + \{\lambda_d\}\boldsymbol{w}_d + \lfloor \lambda_1 \rfloor \boldsymbol{w}_1 + \ldots + \lfloor \lambda_d \rfloor \boldsymbol{w}_d. \tag{2.13}$$

Since $0 \leq \{\lambda_i\} < 1$, the point $\boldsymbol{p} := \boldsymbol{v} + \sum_{i=1}^{d} \{\lambda_i\} \boldsymbol{w}_i$ lies within $\boldsymbol{v} + \Pi_{\mathcal{K}}$, and because the vector $\sum_{i=1}^{d} \lfloor \lambda_i \rfloor \boldsymbol{w}_i$ as well as the vector $\boldsymbol{m}$ are integral, we conclude that $\boldsymbol{p} \in \mathbb{Z}^n$.

This shows that each point $\boldsymbol{m} \in (\boldsymbol{v} + \mathcal{K}) \cap \mathbb{Z}^n$ possesses a unique representation as

$$\boldsymbol{m} = \boldsymbol{p} + k_1 \boldsymbol{w}_1 + k_2 \boldsymbol{w}_2 + \ldots + k_d \boldsymbol{w}_d, \tag{2.14}$$

where $\boldsymbol{p}$ denotes an integral point in the parallelepiped $\boldsymbol{v} + \Pi_{\mathcal{K}}$ and $k_1, \ldots, k_d$ stand for non–negative integers. Conversely, each point of this form is an element of $(\boldsymbol{v}+\mathcal{K}) \cap \mathbb{Z}^d$. With this in mind, we will now consider the left hand side of Equation (2.12).

$$\begin{aligned}
&\frac{\sigma_{\boldsymbol{v}+\Pi_{\mathcal{K}}}(\boldsymbol{z})}{(1-\boldsymbol{z}^{\boldsymbol{w}_1})(1-\boldsymbol{z}^{\boldsymbol{w}_2})\ldots(1-\boldsymbol{z}^{\boldsymbol{w}_d})} = \\
&\quad = \left(\sum_{\boldsymbol{p} \in (\boldsymbol{v}+\Pi_{\mathcal{K}}) \cap \mathbb{Z}^n} \boldsymbol{z}^{\boldsymbol{p}} \right) \left(\sum_{k_1 \geq 0} \boldsymbol{z}^{\boldsymbol{w}_1 k_1} \right) \left(\sum_{k_2 \geq 0} \boldsymbol{z}^{\boldsymbol{w}_2 k_2} \right) \ldots \left(\sum_{k_d \geq 0} \boldsymbol{z}^{\boldsymbol{w}_d k_d} \right) \\
&\quad = \sum_{\substack{\boldsymbol{m} = \boldsymbol{p} + k_1 \boldsymbol{w}_1 + \ldots + k_d \boldsymbol{w}_d \\ \boldsymbol{p} \in (\boldsymbol{v}+\Pi_{\mathcal{K}}) \cap \mathbb{Z}^n; k_i \geq 0}} \boldsymbol{z}^{\boldsymbol{m}} = \sum_{\boldsymbol{m} \in (\boldsymbol{v}+\mathcal{K}) \cap \mathbb{Z}^n} \boldsymbol{z}^{\boldsymbol{m}} = \sigma_{\boldsymbol{v}+\mathcal{K}}(\boldsymbol{z}).
\end{aligned}$$

Note that this is exactly our claim. $\square$

Example 2.12. We calculate the integer–point transform of the rational cone displayed in Figure 2.4, i.e. $\mathcal{K} = \operatorname{cone}((1,1);(-2,3))$. According to the previous theorem, the crucial step of the computation of the integer–point transform of a simplicial cone is to specify the integer–point transform of the fundamental parallelepiped of the cone. In our example, Figure 2.4 displays that $\Pi_{\mathcal{K}} \cap \mathbb{Z}^2 = \{(0,0),(0,1),(-1,2),(-1,3)\}$. Hence,

$$\sigma_{\mathcal{K}}(\boldsymbol{z}) = \frac{z_1^0 z_2^0 + z_1^0 z_2^1 + z_1^{-1} z_2^2 + z_1^{-1} z_2^3}{(1-z_1 z_2)(1-z_1^{-2} z_2^3)} = \frac{1 + z_2^1 + z_1^{-1} z_2^2 + z_1^{-1} z_2^3}{(1-z_1 z_2)(1-z_1^{-2} z_2^3)}. \tag{2.15}$$

We are now able to prove the first instance of the main theorem of this chapter.

Theorem 2.13 (Ehrhart's Theorem for simplices). *Given a rational d–simplex $\Delta \subseteq \mathbb{R}^n$ with vertices $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{d+1}$ and denominator p, the integer–point enumerator $\mathcal{L}_\Delta(t)$ is a quasi–polynomial in t of degree d whose period divides p.*

Proof. By Theorem 2.10, the Ehrhart–series of a polytope is given by the integer–point transform of the cone over the polytope evaluated at $\boldsymbol{z} = (1, \dots, 1, z) \in \mathbb{R}^n$.

The cone over the simplex is the simplicial cone with generators $(\boldsymbol{v}_1, 1), \dots, (\boldsymbol{v}_{d+1}, 1)$, and since the denominator of the simplex Δ is given by the integer p, we obtain that $\{(p \cdot \boldsymbol{v}_1, p), \dots, (p \cdot \boldsymbol{v}_{d+1}, p)\}$ is a set of integral generators of $\operatorname{cone}(\Delta|1)$.
Accordingly, by Theorem 2.11, the integer–point transform

$$\sigma_{\operatorname{cone}(\Delta|1)}(\boldsymbol{z}) = \frac{\sigma_{\Pi_{\operatorname{cone}(\Delta|1)}}(\boldsymbol{z})}{(1-\boldsymbol{z}^{(p\cdot \boldsymbol{v}_1, p)}) \dots (1-\boldsymbol{z}^{(p\cdot \boldsymbol{v}_{d+1}, p)})}, \tag{2.16}$$

where

$$\Pi_{\operatorname{cone}(\Delta|1)} = \{\lambda_1(p \cdot \boldsymbol{v}_1, p) + \dots + \lambda_{d+1}(p \cdot \boldsymbol{v}_{d+1}, p) : 0 \le \lambda_1, \dots, \lambda_{d+1} < 1\} \tag{2.17}$$

denotes the fundamental parallelepiped of $\operatorname{cone}(\Delta|1)$.

The evaluation of the integer point transform $\sigma_{\Pi_{\operatorname{cone}(\Delta|1)}}(\boldsymbol{z})$ at $\boldsymbol{z} = (1, \dots, 1, z) \in \mathbb{R}^n$ yields a polynomial $P(z)$, where the coefficient of z^k counts the integral points in the half–open parallelepiped $\Pi_{\operatorname{cone}(\Delta|1)}$ with $(n+1)$–st coordinate k. This interpretation shows that the coefficients of the polynomial $P(z)$ are non–negative integers and the degree of $P(z)$ is given by the maximum value of the $(n+1)$–st coordinates of the integral points in $\Pi_{\operatorname{cone}(\Delta|1)}$. In particular, the degree of the polynomial $P(z)$ is smaller than $(d+1)p$, since the $(n+1)$–st coordinate of a point in $\Pi_{\operatorname{cone}(\Delta|1)}$ equals $\lambda_1 p + \dots + \lambda_{d+1} p$, for appropriate reals $0 \le \lambda_1, \dots, \lambda_{d+1} < 1$. Therefore, and by Theorem 2.10, we have that

$$1 + \sum_{t \geq 1} \mathcal{L}_\Delta(t) z^t = \sigma_{\text{cone}(\Delta|1)}(1, \ldots, 1, z) = \frac{P(z)}{(1 - z^p)^{d+1}}. \tag{2.18}$$

This shows that the Ehrhart–series of the simplex is rational. To decide whether or not the integer–point enumerator is a quasi–polynomial, we look over the conditions stated in Corollary 1.57, c.f. Section 1.5.

All zeros of the denominator polynomial in (2.18) are p–th roots of unity, and since the value $P(1)$ gives the number of integral points in the parallelepiped $\Pi_{\text{cone}(\Delta|1)}$, which contains at least the origin, we conclude that $P(1) \neq 0$. Therefore, at least the root one has multiplicity $d+1$ in the lowest term representation of the rational function in (2.18). Thus, by Corollary 1.57 and Remark 1.58, we conclude that the function $\mathcal{L}_\Delta(t)$ is a quasi–polynomial in t of degree d whose period divides p. □

Remark 2.14. Note that the leading coefficient of the Ehrhart quasi–polynomial $\mathcal{L}_\Delta(t)$ is given by a non–negative periodic function. One may deduce this property by the series expansion of the Ehrhart–series of the simplex, since the coefficient of z^k in the numerator polynomial $P(z)$ of this rational function counts the integral points in the half–open parallelepiped $\Pi_{\text{cone}(\Delta|1)}$ with $(n+1)$–st coordinate k. This observation implies that all coefficients of the polynomial $P(z)$ are given by non–negative integers.

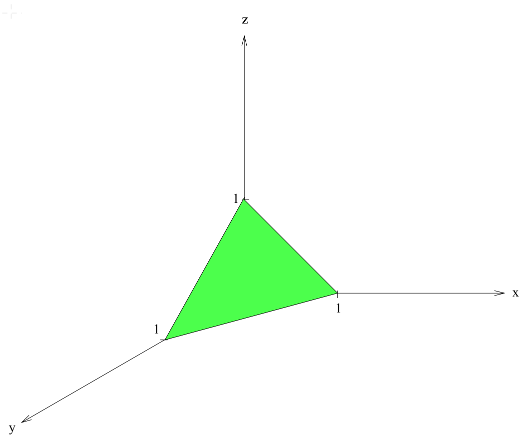


Figure 2.5: The standard 2-simplex.

Example 2.15. We consider the standard d–simplex

$$\Delta_d = \operatorname{conv}(\mathbf{e}_1, \ldots, \mathbf{e}_{d+1}) \subseteq \mathbb{R}^{d+1}. \tag{2.19}$$

Observe that the origin is the sole integral point lying within the fundamental parallelepiped of the cone over this simplex. Accordingly,

$$\operatorname{Ehr}_{\Delta_d} = \frac{1}{(1-z)^{d+1}} = \sum_{t \geq 0} \binom{d+t}{d} z^t. \tag{2.20}$$

Hence, the Ehrhart quasi–polynomial of the standard d–simplex $\mathcal{L}_{\Delta_d}(t) = \binom{d+t}{d}$, which is a polynomial in t of degree d.

Theorem 2.16 (Ehrhart's Theorem). *Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a rational d–polytope. The Ehrhart function, $\mathcal{L}_{\mathcal{P}}(t)$, is a quasi–polynomial in t of degree d whose period divides the denominator of the polytope $\mathcal{P}$.*

Proof. According to Theorem 1.31, any polytope can be triangulated in simplices using no new vertices, and the proper faces of a simplex are given by simplices of lower dimension, c.f. Corollary 1.25. Therefore, we may deduce Ehrhart's Theorem for general rational polytopes from the according result for rational simplices as follows.

The Ehrhart quasi–polynomial of a rational simplex in a triangulation of a rational d–polytope has degree d, and since all vertices of the simplices in a triangulation are given by vertices of the polytope, we conclude that the denominator of $\mathcal{P}$ divides the period of each of these quasi–polynomials. In order to calculate the Ehrhart quasi–polynomial of the original polytope one has to deal with over–counting at the faces where the simplices meet.[2] An important observation at this point is that the faces, where the simplices overlap, are given by simplices of dimension smaller than d. Therefore, the functions which have to be subtracted, or readded, are quasi–polynomials of degree smaller than d. Since the leading coefficient of the Ehrhart quasi–polynomial of a rational simplex is given by a non–negative periodic function, the result is again a quasi–polynomial of degree d whose period divides the denominator of the polytope. □

Corollary 2.17. *The Ehrhart quasi–polynomial of an integral d–polytope $\mathcal{P} \subseteq \mathbb{R}^n$ is a polynomial.*

[2] A formula for this may be found in [22] on page 224.

2.3 Further Properties of Ehrhart Quasi–Polynomials

In order to give basic proofs for some further properties of Ehrhart quasi–polynomials, we follow the outline given in [4]. It is the resourceful idea of this paper to make use of the fact that any rational pointed cone may be shifted by a vector such that the translated cone contains exactly the same integral points as the original cone while there are no integral points on any of the faces of the translated cone. We have already proved this result in Section 1.3.

Theorem 2.18. *Let $\mathcal{K}$ be a rational d–cone in $\mathbb{R}^n$ and $\{\mathcal{K}_1, \ldots, \mathcal{K}_m\}$ a triangulation of this cone into simplicial cones using no new generators. There exists a vector $\boldsymbol{v} \in \mathbb{R}^n$ such that*

$$\sigma_{\mathcal{K}}(\boldsymbol{z}) = \sum_{i=1}^{m} \sigma_{-\boldsymbol{v}+\mathcal{K}_j}(\boldsymbol{z}) \quad \textit{and} \quad \sigma_{\mathcal{K}^\circ}(\boldsymbol{z}) = \sum_{i=1}^{m} \sigma_{\boldsymbol{v}+\mathcal{K}_j}(\boldsymbol{z}). \tag{2.21}$$

Proof. By Remark 1.48, there exists a vector $\boldsymbol{v} \in \mathbb{R}^n$ such that

$$\begin{aligned}
\partial(\boldsymbol{v} + \mathcal{K}_j) \cap \mathbb{Z}^n &= \emptyset && \text{for } 0 \le j \le m && (2.22)\\
\partial(-\boldsymbol{v} + \mathcal{K}_j) \cap \mathbb{Z}^n &= \emptyset && \text{for } 0 \le j \le m && (2.23)\\
(\boldsymbol{v} + \mathcal{K}) \cap \mathbb{Z}^n &= \mathcal{K}^\circ \cap \mathbb{Z}^n && && (2.24)\\
(-\boldsymbol{v} + \mathcal{K}) \cap \mathbb{Z}^n &= \mathcal{K} \cap \mathbb{Z}^n, && && (2.25)
\end{aligned}$$

where we denoted the cone $\mathcal{K}$ by $\mathcal{K}_0$. For instance (2.24) and (2.22) mean that the translated cone $\boldsymbol{v} + \mathcal{K}$ contains all interior integral points of $\mathcal{K}$ while there are no integral points on the boundary of any of the shifted triangulating cones.

Now, the set $\{-\boldsymbol{v} + \mathcal{K}_1, \ldots, -\boldsymbol{v} + \mathcal{K}_m\}$ provides a triangulation of the cone $-\boldsymbol{v} + \mathcal{K}$, with the additional property that there are no integral points on the faces where these cones overlap. This implies that

$$\mathcal{K} \cap \mathbb{Z}^n = (-\boldsymbol{v} + \mathcal{K}) \cap \mathbb{Z}^n = \bigcup_{j=1}^{m} (-\boldsymbol{v} + \mathcal{K}_j) \cap \mathbb{Z}^n, \tag{2.26}$$

where the latter union is disjoint. Therefore, the integer–point transform

$$\sigma_{\mathcal{K}}(\boldsymbol{z}) = \sigma_{-\boldsymbol{v}+\mathcal{K}}(\boldsymbol{z}) = \sum_{i=1}^{m} \sigma_{-\boldsymbol{v}+\mathcal{K}_j}(\boldsymbol{z}). \tag{2.27}$$

Analogously, the set $\{\boldsymbol{v}+\mathcal{K}_1,\ldots,\boldsymbol{v}+\mathcal{K}_m\}$ gives a triangulation of the cone $\boldsymbol{v}+\mathcal{K}$ such that there are no integral points on any of the facets of the triangulating cones. Accordingly, we have that

$$\mathcal{K}^\circ\cap\mathbb{Z}^n=(\boldsymbol{v}+\mathcal{K})\cap\mathbb{Z}^n=\bigcup_{j=1}^m(\boldsymbol{v}+\mathcal{K}_j)\cap\mathbb{Z}^n, \tag{2.28}$$

which is again a union of disjoint sets. Translating these observations into terms of integer–point transforms, we obtain that

$$\sigma_{\mathcal{K}^\circ}(\boldsymbol{z})=\sigma_{\boldsymbol{v}+\mathcal{K}}(\boldsymbol{z})=\sum_{i=1}^m\sigma_{\boldsymbol{v}+\mathcal{K}_j}(\boldsymbol{z}). \tag{2.29}$$

□

Remark 2.19. Note that in the translated triangulations only one simplicial cone contains the origin. Thus, we may not use this method in our proof of Ehrhart's Theorem in order to avoid the over–counting at the faces where the simplices of the triangulations meet. However, Theorem 2.18 may be quite useful for the computation of the Ehrhart quasi–polynomial of a rational polytope. It is worth mentioning that this method is in fact used in [12] in order to improve Barvinok's Algorithm, the corresponding version of the algorithm is implemented in the computer program ”LattE macchiato”.

2.3.1 Ehrhart–Macdonald Reciprocity

While dilating a polytope by negative numbers does not make any sense, evaluating a quasi–polynomial is no problem. In fact, there is a nice combinatorial interpretation for the evaluation of Ehrhart quasi–polynomials of rational polytopes at negative integers.

Ehrhart–Macdonald Reciprocity had been conjectured and proved in several special cases by Eugène Ehrhart about a decade before I. G. Macdonald found a general proof in 1971 [18]. Our goal is to give a basic proof of the following theorem at the end of this section.

Theorem 2.20 (Ehrhart–Macdonald Reciprocity). *The evaluation of the Ehrhart quasi–polynomial at negative integers counts, up to a sign, the number of integral points in the relative interior of the dilated polytope, i.e.*

$$\mathcal{L}_{\mathcal{P}}(-t)=(-1)^{dim(\mathcal{P})}\mathcal{L}_{\mathcal{P}^\circ}(t). \tag{2.30}$$

Definition 2.21. The *Ehrhart–series of the interior* of a rational polytope $\mathcal{P}$,

$$\mathrm{Ehr}_{\mathcal{P}^\circ}(z) := \sum_{t\geq 1} L_{\mathcal{P}^\circ}(t) z^t. \tag{2.31}$$

Lemma 2.22. *Given an arbitrary subset $S \subseteq \mathbb{R}^n$, we set $-S := \{-\boldsymbol{x} : \boldsymbol{x} \in S\}$. The integer–point transform of the sets S and $-S$ are related by the equation*

$$\sigma_{-S}(z_1, \ldots, z_n) = \sigma_S\left(\frac{1}{z_1}, \ldots, \frac{1}{z_n}\right). \tag{2.32}$$

Proof. According to the definition of the integer–point transform, we have that

$$\sigma_{-S}(z_1, \ldots, z_n) \quad = \sum_{\boldsymbol{m}\in S\cap\mathbb{Z}^d} z_1^{-m_1} \ldots z_n^{-m_n} = \sigma_S\left(\frac{1}{z_1}, \ldots, \frac{1}{z_n}\right). \tag{2.33}$$

□

Lemma 2.23. *Let $\mathcal{K} \subseteq \mathbb{R}^n$ be a rational d–cone. Then, the integer–point transform of the translated cone $\sigma_{\boldsymbol{m}+\mathcal{K}}(\boldsymbol{z}) = \boldsymbol{z}^{\boldsymbol{m}}\sigma_{\mathcal{K}}(\boldsymbol{z})$, for any integral vector $\boldsymbol{m} \in \mathbb{Z}^n$.*

Proof. Observe that $\sigma_{\boldsymbol{m}+\mathcal{K}}(\boldsymbol{z}) = \sum_{\boldsymbol{s}\in(\boldsymbol{m}+\mathcal{K})\cap\mathbb{Z}^n} \boldsymbol{z}^{\boldsymbol{s}} = \sum_{\boldsymbol{k}\in\mathcal{K}\cap\mathbb{Z}^n} \boldsymbol{z}^{\boldsymbol{m}+\boldsymbol{k}} = \boldsymbol{z}^{\boldsymbol{m}}\sigma_{\mathcal{K}}(\boldsymbol{z})$. □

Lemma 2.24. *Let $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_d$ be linearly independent vectors in $\mathbb{R}^n$ and we denote by $\Pi^\circ = \{\lambda_1\boldsymbol{w}_1 + \ldots + \lambda_d\boldsymbol{w}_d : 0 < \lambda_1, \ldots, \lambda_d < 1\}$ the open parallelepiped generated by these vectors. Then, for any vector $\boldsymbol{v} \in \mathbb{R}^n$, we have that*

$$\boldsymbol{v} + \Pi^\circ = -(-\boldsymbol{v} + \Pi^\circ) + \boldsymbol{w}_1 + \boldsymbol{w}_2 + \ldots + \boldsymbol{w}_d. \tag{2.34}$$

Proof. As an immediate consequence of the definitions of the according sets, we have

$$\begin{aligned}
-(-\boldsymbol{v} + \Pi^\circ) + \boldsymbol{w}_1 + \ldots + \boldsymbol{w}_d = \\
&= \{\boldsymbol{v} - \lambda_1\boldsymbol{w}_1 - \ldots - \lambda_d\boldsymbol{w}_d + \boldsymbol{w}_1 + \ldots + \boldsymbol{w}_d : 0 < \lambda_i < 1\} \\
&= \{\boldsymbol{v} + \underbrace{(1-\lambda_1)}_{\mu_1}\boldsymbol{w}_1 + \underbrace{(1-\lambda_2)}_{\mu_2}\boldsymbol{w}_2 + \ldots + \underbrace{(1-\lambda_d)}_{\mu_d}\boldsymbol{w}_d : 0 < \lambda_i < 1\} \\
&= \{\boldsymbol{v} + \mu_1\boldsymbol{w}_1 + \mu_2\boldsymbol{w}_2 + \ldots + \mu_d\boldsymbol{w}_d : 0 < \mu_i < 1\} \\
&= \boldsymbol{v} + \Pi^\circ.
\end{aligned}$$

□

Lemma 2.25. *Let $\mathcal{K} \subseteq \mathbb{R}^n$ be a rational simplicial d–cone with integral generators $\boldsymbol{w}_1, \dots, \boldsymbol{w}_d$. For those vectors $\boldsymbol{v} \in \mathbb{R}^n$, which meet the condition that the boundaries of the shifted cones $\boldsymbol{v} + \mathcal{K}$ and $-\boldsymbol{v} + \mathcal{K}$ contain no integral points, the integer–point transform*

$$\sigma_{\boldsymbol{v}+\mathcal{K}}\left(\frac{1}{z_1}, \dots, \frac{1}{z_n}\right) = (-1)^d \sigma_{-\boldsymbol{v}+\mathcal{K}}(z_1, \dots, z_n). \tag{2.35}$$

Proof. Note that, if neither the cone $\boldsymbol{v} + \mathcal{K}$ nor $-\boldsymbol{v} + \mathcal{K}$ contain any integral point on their boundaries, the fundamental parallelepipeds of these cones may not include any integral points on their boundaries as well. That is, $\sigma_{\pm\boldsymbol{v}+\Pi}(\boldsymbol{z}) = \sigma_{\pm\boldsymbol{v}+\Pi^\circ}(\boldsymbol{z})$. Therefore, denoting $\frac{1}{\boldsymbol{z}} = \left(\frac{1}{z_1}, \dots, \frac{1}{z_n}\right)$, we obtain, by Theorem 2.11, that

$$\begin{aligned}
\sigma_{\boldsymbol{v}+\mathcal{K}}\left(\frac{1}{\boldsymbol{z}}\right) &= \frac{\sigma_{\boldsymbol{v}+\Pi^\circ}\left(\frac{1}{\boldsymbol{z}}\right)}{(1-\boldsymbol{z}^{-\boldsymbol{w}_1})(1-\boldsymbol{z}^{-\boldsymbol{w}_2})\dots(1-\boldsymbol{z}^{-\boldsymbol{w}_d})} \\
&= \frac{\sigma_{-\boldsymbol{v}+\Pi^\circ}(\boldsymbol{z})\, \boldsymbol{z}^{-\boldsymbol{w}_1} \boldsymbol{z}^{-\boldsymbol{w}_2} \dots \boldsymbol{z}^{-\boldsymbol{w}_d}}{(1-\boldsymbol{z}^{-\boldsymbol{w}_1})(1-\boldsymbol{z}^{-\boldsymbol{w}_2})\dots(1-\boldsymbol{z}^{-\boldsymbol{w}_d})} \\
&= \frac{\sigma_{-\boldsymbol{v}+\Pi^\circ}(\boldsymbol{z})}{(\boldsymbol{z}^{\boldsymbol{w}_1}-1)(\boldsymbol{z}^{\boldsymbol{w}_2}-1)\dots(\boldsymbol{z}^{\boldsymbol{w}_d}-1)} \\
&= (-1)^d \frac{\sigma_{-\boldsymbol{v}+\Pi}(\boldsymbol{z})}{(1-\boldsymbol{z}^{\boldsymbol{w}_1})(1-\boldsymbol{z}^{\boldsymbol{w}_2})\dots(1-\boldsymbol{z}^{\boldsymbol{w}_d})} \\
&= (-1)^d \sigma_{-\boldsymbol{v}+\mathcal{K}}(\boldsymbol{z}).
\end{aligned} \tag{2.36}$$

The second equality in this equation holds by Lemma 2.23 and Lemma 2.24. □

Theorem 2.26 (Stanley's Reciprocity). *Suppose that $\mathcal{K} \subseteq \mathbb{R}^n$ is a rational d-cone with a vertex at the origin. Then,*

$$\sigma_{\mathcal{K}}\left(\frac{1}{\boldsymbol{z}}\right) = (-1)^d \sigma_{\mathcal{K}^\circ}(\boldsymbol{z}). \tag{2.37}$$

Proof. Consider a triangulation of the cone $\mathcal{K}$ into rational simplicial cones $\mathcal{K}_1, \dots, \mathcal{K}_m$ using no new generators. According to Theorem 2.18, there exists a vector $\boldsymbol{v} \in \mathbb{R}^n$ with the property that the integer–point transform of the cone $\mathcal{K}$ equals the sum of the integer–point transforms of the triangulating cones shifted in direction $-\boldsymbol{v}$, i.e.

$$\sigma_{\mathcal{K}}(\boldsymbol{z}) = \sum_{i=1}^{m} \sigma_{-\boldsymbol{v}+\mathcal{K}_j}(\boldsymbol{z}), \text{ and } \sigma_{\mathcal{K}^\circ}(\boldsymbol{z}) = \sum_{i=1}^{m} \sigma_{\boldsymbol{v}+\mathcal{K}_j}(\boldsymbol{z}). \tag{2.38}$$

Recall that $\sigma_{\boldsymbol{v}+\mathcal{K}_i}(z_1,\ldots,z_n) = (-1)^d\sigma_{-\boldsymbol{v}+\mathcal{K}_i}\left(\frac{1}{z_1},\ldots,\frac{1}{z_n}\right)$, by Lemma 2.25. After all, we conclude that

$$\sigma_{\mathcal{K}}\left(\frac{1}{\boldsymbol{z}}\right) = \sum_{j=1}^{m}\sigma_{-\boldsymbol{v}+\mathcal{K}_j}\left(\frac{1}{\boldsymbol{z}}\right) = \sum_{j=1}^{m}(-1)^d\sigma_{\boldsymbol{v}+\mathcal{K}_j}(\boldsymbol{z}) = (-1)^d\sigma_{\mathcal{K}^\circ}(\boldsymbol{z}). \tag{2.39}$$

□

Lemma 2.27. *Given a rational polytope $\mathcal{P}\subseteq\mathbb{R}^n$, the evaluation of the Ehrhart–series*

$$\operatorname{Ehr}_{\mathcal{P}}\left(\frac{1}{z}\right) = (-1)^{dim(\mathcal{P})+1}\operatorname{Ehr}_{\mathcal{P}^\circ}(z). \tag{2.40}$$

Proof. In the same manner as we proved Theorem 2.10, we obtain that the Ehrhart–series of the interior of a polytope can be calculated by evaluating the integer–point transform of $\operatorname{cone}(\mathcal{P}|1)^\circ$ at $(1,\ldots,1,z)$. By Stanley's Reciprocity, we conclude that

$$\begin{aligned}\operatorname{Ehr}_{\mathcal{P}^\circ}(z) &= \sigma_{\operatorname{cone}(\mathcal{P}|1)^\circ}(1,\ldots,1,z) = (-1)^{d+1}\sigma_{\operatorname{cone}(\mathcal{P}|1)}\left(1,\ldots,1,\frac{1}{z}\right)\\ &= (-1)^{dim(\mathcal{P})+1}\operatorname{Ehr}_{\mathcal{P}}\left(\frac{1}{z}\right).\end{aligned} \tag{2.41}$$

□

Theorem 2.28 (Ehrhart–Macdonald Reciprocity). *Let $\mathcal{P}\subseteq\mathbb{R}^n$ be a rational d–polytope. The evaluation of the Ehrhart quasi–polynomial at negative integers*

$$\mathcal{L}_{\mathcal{P}}(-t) = (-1)^{dim(\mathcal{P})}\mathcal{L}_{\mathcal{P}^\circ}(t). \tag{2.42}$$

Proof. Since the Ehrhart–series of the polytope is the generating function of the integer–point enumerator, we obtain, by Lemma 2.27, that

$$\begin{aligned}\sum_{t\geq 1}L_{\mathcal{P}^\circ}(t)z^t = \operatorname{Ehr}_{\mathcal{P}^\circ(z)} &= (-1)^{d+1}\operatorname{Ehr}_{\mathcal{P}}\left(\frac{1}{z}\right) = (-1)^{d+1}\sum_{t\geq 0}L_{\mathcal{P}}(t)z^{-t}\\ &= (-1)^{d+1}\sum_{t\leq 0}L_{\mathcal{P}}(-t)z^t = (-1)^{d+1}\left(-\sum_{t\geq 1}L_{\mathcal{P}}(-t)z^t\right).\end{aligned}$$

Since the function $L_{\mathcal{P}}(t)$ is a quasi–polynomial, the latter equivalence follows by Corollary 1.61. Comparing the coefficients in this equation provides the reciprocity theorem. □

2.4 Continuous and Discrete Volume

The number of integral points in a set $S \subseteq \mathbb{R}^d$ is also referred to as the discrete volume. An interesting and rather unexpected application of Ehrhart theorie is that Ehrhart quasi–polynomials may be used in order to calculate the volume of a large class of rational polytopes discretely, simply by counting integral points. Here, we first give a short survey of the relationship between the volume $\mathrm{vol}(S)$ and the discrete volume $\#(S \cap \mathbb{Z}^d)$ of a subset of $\mathbb{R}^d$.

Suppose that S is a d–dimensional subset of $\mathbb{R}^d$. For each integral point i contained in the set $S \cap \mathbb{Z}^d$ we consider a d–dimensional hypercube with side length 1 and center i. These hypercubes overlap only on faces with zero volume and fill the set S with an error on the boundary. By shrinking the lattice by a positive integral factor t and using boxes with side length $1/t$ we will obtain a sucessively better approximations of $\mathrm{vol}(S)$ as the factor t grows. If the volume of the set S exists, it follows that

$$\mathrm{vol}(S) = \lim_{t\to\infty} \frac{1}{t^d}\ \#(S \cap \frac{1}{t}\mathbb{Z}^d). \tag{2.43}$$

In fact, this is the basic idea behind the concept of volume, if we, for example, consider the construction of the Lebesgue measure. Now, instead of shrinking the lattice we may equivalently think of dilating the set, i.e.

$$\#(S \cap \frac{1}{t}\mathbb{Z}^d) = \#(tS \cap \mathbb{Z}^d) \Rightarrow \mathrm{vol}(S) = \lim_{t\to\infty} \frac{1}{t^d}\#(tS \cap \mathbb{Z}^d). \tag{2.44}$$

Theorem 2.29. *Let $\mathcal{P}$ be a rational d–polytope in $\mathbb{R}^d$. Then, the leading coefficient of the Ehrhart quasi–polynomial of $\mathcal{P}$ is constant and agrees with the volume of the polytope.*

Proof. Ehrhart's Theorem states that the integer–point transform of a rational polytope is given by a quasi–polynomial in t of degree d, i.e.

$$\#(t\mathcal{P} \cap \mathbb{Z}^d) = \mathcal{L}_{\mathcal{P}}(t) = c_d(t)t^d + \ldots + c_0(t). \tag{2.45}$$

Since the volume of a polytope is well defined, our previous discussion implies that

$$\mathrm{vol}(\mathcal{P}) = \lim_{t\to\infty} \frac{\#(t\mathcal{P} \cap \mathbb{Z}^d)}{t^d} = \lim_{t\to\infty} \frac{c_d(t)t^d + \ldots + c_0(t)}{t^d} = c_d(t). \tag{2.46}$$

□

Now, we give an example of a set for which the continuous and the discrete volume coincide.

Lemma 2.30. *Suppose* $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_d \in \mathbb{Z}^d$ *are linearly independent integral vectors. Then, for the half–open parallelepiped* $\Pi = \{\lambda_1 \boldsymbol{w}_1 + \ldots + \lambda_d \boldsymbol{w}_d : 0 \leq \lambda_i < 1\}$ *and any positive integer* t *we have that*

$$\#(t\Pi \cap \mathbb{Z}^d) = t^d \cdot \mathrm{vol}(\Pi) = t^d |\det(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_d)|. \tag{2.47}$$

Proof. Since Π is a half–open parallelepiped, the t^{th} dilation of this set, $t\Pi$, can be tiled by t^d translates of Π. Thus, we obtain that

$$\mathcal{L}_\Pi(t) = \#(t\Pi \cap \mathbb{Z}^d) = \#(\Pi \cap \mathbb{Z}^d)t^d, \tag{2.48}$$

for all non–negative integers t. On the other hand, the function $t \mapsto \mathcal{L}_\Pi(t)$ is a quasi–polynomial in t with leading coefficient $\mathrm{vol}(\Pi)$, by Theorem 2.29, and $\mathrm{vol}(\Pi) = |\det(\boldsymbol{w}_1 \ldots \boldsymbol{w}_d)|$ by basic linear algebra. $\square$

We like to point out that an analog property regarding the leading coefficient of the Ehrhart quasi–polynomial does not hold for all rational d–polytopes in $\mathbb{R}^n$ with $n > d$. In order to clarify this aspect, we consider a simple example.

Example 2.31. Let $\mathcal{P}$ be the 1–polytope in $\mathbb{R}^2$ given by

$$\mathcal{P} = \mathrm{conv}\left(\begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} -1 \\ \frac{1}{2} \end{pmatrix}\right). \tag{2.49}$$

We calculate the Ehrhart quasi–polynomial of this simplex according to the proof of Theorem 2.13 and obtain that

$$\mathcal{L}_{\boldsymbol{p}}(t) = \begin{cases} \frac{1}{2}t + 1 & \text{if } t \text{ is even} \\ 0 & \text{if } t \text{ is odd.} \end{cases} \tag{2.50}$$

Hence, the leading coefficient of the Ehrhart quasi–polynomial of this polytope is not constant and does not agree with the volume or a relative volume of the polytope.

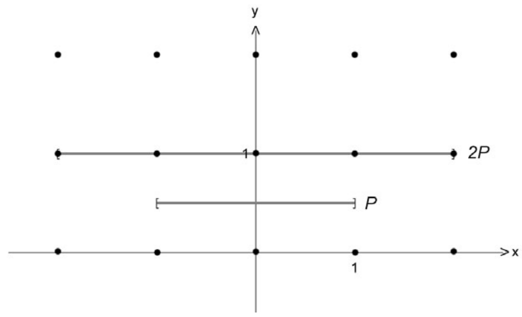


Figure 2.6: An 1–polytope in $\mathbb{R}^2$.

Anyway, for integral d–polytopes in a vector space $\mathbb{R}^n$ with $d < n$ we may define volume in a relative sense, so that we are able to show that the leading coefficient of the Ehrhart polynomials of these polytopes agrees with this relative volume.

2.4.1 The Relative Volume of Integral Polytopes

If the dimension of a set is not identical with the dimension of the surrounding space, its volume equals zero, according to the usual definition. Nevertheless, we may still consider the volume of a d–dimensional set $S \subseteq \mathbb{R}^n$ for $d < n$ relative to the d–dimensional affine space $\mathrm{aff}(S)$. This means that we choose an invertible affine map $\Phi : \mathrm{aff}(S) \to \mathbb{R}^d$ and define the relative volume of the set S as the volume of the image $\Phi(S)$ in $\mathbb{R}^d$. Now, if we want to maintain the relation between the number of integral points in a polytope and the volume, we have to choose an invertible affine map $\Phi : \mathrm{aff}(S) \to \mathbb{R}^d$ for which $\Phi(\mathrm{aff}(S) \cap \mathbb{Z}^n) = \mathbb{Z}^d$. It can be shown that such a map exists if and only if the d–dimensional affine space $\mathrm{aff}(S)$ contains $d+1$ affinely independent integral points, c.f. [22], Chapter 4.6. Since the vertex set of any integral d–polytope $\mathcal{P}$ contains $d+1$ affinely independent integral vectors, we may define the relative volume of integral d–polytopes as follows.

Definition 2.32. Let $\mathcal{P}$ be an integral d–polytope in $\mathbb{R}^n$ for $d \le n$. Then, there exists an invertible affine map $\Phi : \mathrm{aff}(\mathcal{P}) \to \mathbb{R}^d$ such that $\Phi(\mathrm{aff}(\mathcal{P}) \cap \mathbb{Z}^n) = \mathbb{Z}^d$. We define the *relative volume* of the polytope $\mathcal{P}$ as

$$\mathrm{vol}_{\mathcal{R}}(\mathcal{P}) := \mathrm{vol}(\Phi(\mathcal{P})). \tag{2.51}$$

Remark 2.33. It is easy to see that the relative volume is independ on the actual choice the map Φ and, thus, depends on $\mathcal{P}$ only, c.f. [22]. If $\mathcal{P}$ is an integral d–polytope in

$\mathbb{R}^d$, then $\mathrm{vol}_{\mathcal{R}}(\mathcal{P})$ agrees with the usual volume of $\mathcal{P}$, since, in this case, we may choose Φ as the identity map.

Corollary 2.34. *Let $\mathcal{P}$ be an integral d–polytope in $\mathbb{R}^n$. Then, the leading coefficient of the Ehrhart quasi–polynomial of $\mathcal{P}$ is constant and equals the relative volume of the polytope.*

Proof. Suppose that $\Phi : \mathrm{aff}(\mathcal{P}) \to \mathbb{R}^d$ is an invertible affine map with the property that $\Phi(\mathrm{aff}(\mathcal{P}) \cap \mathbb{Z}^n) = \mathbb{Z}^d$. Then, the image $\Phi(\mathcal{P})$ is an integral d–polytope in $\mathbb{R}^d$ and the number of integral points in $\Phi(\mathcal{P})$ agrees with the number of integral points in the polytope $\mathcal{P}$. Furthermore, it is easy to see that the number of integral points in the dilated polytope $t\mathcal{P}$ agrees with the number of integral points in $t\Phi(\mathcal{P})$, for all positive integers t. Thus, we obtain that $\mathcal{L}_{\mathcal{P}}(t) = \mathcal{L}_{\Phi(\mathcal{P})}(t)$, and our claim follows by Theorem 2.29. $\square$

Example 2.35. The Ehrhart polynomial of the standard d–simplex, $\Delta_d \subseteq \mathbb{R}^{d+1}$, is given by $\mathcal{L}_{\Delta_d}(t) = \binom{d+t}{d}$, c.f. Example 2.15. Since the leading coefficient of this polynomial equals $1/d!$ we conclude that the relative volume of the standard d–simplex is $1/d!$.

2.4.2 The Second Leading Coefficient of Ehrhart Polynomials

In case of integral polytopes there is also a nice interpretation of the second leading coefficient of the Ehrhart polynomial. We will see that this coefficient equals $1/2$ times the sum of the relative volumes of the facets of the polytope. Moreover, we will use this result in order to deduce Pick's Theorem from Ehrhart's results.

To shorten notation, we agree that the letter $\mathcal{F}$ denotes a face of a polytope $\mathcal{P}$ during the rest of this section and we set

$$F^k_{\mathcal{P}}(t) := \sum_{\substack{\mathcal{F} \subseteq \mathcal{P} \\ dim \mathcal{F} = k}} \mathcal{L}_{\mathcal{F}}(t), \tag{2.52}$$

i.e. $F^k_{\mathcal{P}}(t)$ denotes the sum of the Ehrhart polynomials of the k–dimensional faces of the polytope $\mathcal{P}$.

Lemma 2.36. *Let $\mathcal{P}$ be a rational d–polytope in $\mathbb{R}^n$ with Ehrhart quasi–polynomial $\mathcal{L}_{\mathcal{P}}(t)$. Regarding the number of integral points on the boundary of the polytope $\mathcal{P}$ we have that*

$$\mathcal{L}_{\partial_{\mathcal{P}}}(t) = \mathcal{L}_{\mathcal{P}}(t) - \mathcal{L}_{\mathcal{P}^\circ}(t) = \sum_{k=0}^{d-1} (-1)^k F^k_{\mathcal{P}}(-t). \tag{2.53}$$

Proof. In Section 1.2.5 we have seen that any polytope is the disjoint union of the relative interior of its faces, i.e.

$$\mathcal{P} = \dot{\bigcup_{\mathcal{F}\subseteq\mathcal{P}}} \mathcal{F}^\circ. \tag{2.54}$$

Thus, by Ehrhart–Macdonald Reciprocity, c.f. Theorem 2.28, we conclude that

$$\begin{aligned}\mathcal{L}_\mathcal{P}(t) &= \sum_{\mathcal{F}\subseteq\mathcal{P}} \mathcal{L}_{\mathcal{F}^\circ}(t) = \sum_{k=0}^{d} \sum_{\substack{\mathcal{F}\subseteq\mathcal{P}\\ \dim\mathcal{F}=k}} (-1)^k \mathcal{L}_\mathcal{F}(-t) \\ &= \sum_{k=0}^{d-1} (-1)^k F^k_\mathcal{P}(-t) + (-1)^d \mathcal{L}_\mathcal{P}(-t).\end{aligned} \tag{2.55}$$

The assertion of the lemma follows by subtracting $(-1)^d\mathcal{L}_\mathcal{P}(-t)$ from both sides of this equation. □

Theorem 2.37. *Suppose that $\mathcal{P}$ is an integral d–polytope with Ehrhart polynomial $\mathcal{L}_\mathcal{P}(t) = c_d t^d + c_{d-1}t^{d-1} + \ldots + c_0$. Then, the second leading coefficient of this polynomial equals $1/2$ times the sum of the relative volumes of the facets of $\mathcal{P}$, i.e.*

$$c_{d-1} = \frac{1}{2} \sum_{\substack{\mathcal{F}\subseteq\mathcal{P}\\ \dim\mathcal{F}=d-1}} \mathrm{vol}_\mathcal{R}(\mathcal{F}). \tag{2.56}$$

Proof. By Ehrhart–Macdonald Reciprocity,

$$\mathcal{L}_{\mathcal{P}^\circ}(t) \;=\; (-1)^d\mathcal{L}_\mathcal{P}(-t) = c_d t^d - c_{d-1}t^{d-1} + - \ldots (-1)^{d+1}c_1 t + (-1)^d c_0. \tag{2.57}$$

Therefore, by Lemma 2.36, we have that

$$\sum_{k=0}^{d-1}(-1)^k F^k_\mathcal{P}(-t) = \mathcal{L}_\mathcal{P}(t) - \mathcal{L}_{\mathcal{P}^\circ}(t) = 2c_{d-1}t^{d-1} + 2c_{d-3}t^{d-3} + \ldots + \tau, \tag{2.58}$$

where τ equals $2c_0$ if d is odd and $2c_1 t$ in the case that d is even.

Now, since proper faces of an integral d–polytope are integral polytopes of dimension smaller than d, only the Ehrhart polynomials of the facets of the polytope $\mathcal{P}$ contribute non–zero coefficients of t^{d-1} to the left hand side of this equation. Hence, this coefficient equals the sum of the leading coefficients of the Ehrhart polynomials of the facets of the polytope, which, by Corollary 2.34, agrees with the sum of the relative volumes of the facets of our polytope. □

Lemma 2.38. *Given an non–empty integral polygon $\mathcal{P}$, the Ehrhart polynomial*

$$\mathcal{L}_{\mathcal{P}}(t) = \mathrm{vol}(\mathcal{P})t^2 + \frac{1}{2}\left(\sum_{\substack{\mathcal{F}\subseteq\mathcal{P}\\ \dim\mathcal{F}=1}} \mathrm{vol}_{\mathcal{R}}(\mathcal{F})\right) t + 1. \tag{2.59}$$

Proof. Integral polygons are 2–dimensional integral polytopes. Accordingly, the Ehrhart polynomial of $\mathcal{P}$ has degree two and period one. By Theorem 2.29, the leading coefficient of the Ehrhart polynomial agrees with the volume of the polytope $\mathcal{P}$, while the second leading coefficient equals 1/2 times the sum of the relative volumes of the facets of $\mathcal{P}$, by Theorem 2.37. The constant term of an Ehrhart polynomial is one, for all non–empty polytopes. □

Corollary 2.39 (Pick's Theorem). *The volume of an integral polygon equals the number of integral points in the interior of the polygon plus* 1/2 *times the number of integral points on its boundary minus one, i.e.*

$$\mathrm{vol}(\mathcal{P}) = \#(\mathcal{P}^\circ \cap \mathbb{Z}^2) + \frac{1}{2}\#(\partial_{\mathcal{P}} \cap \mathbb{Z}^2) - 1 \tag{2.60}$$

Proof. The evaluation of the formula given in Lemma 2.38 at $t = 1$ gives

$$\#(\mathcal{P}^\circ \cap \mathbb{Z}^2) + \#(\partial_{\mathcal{P}} \cap \mathbb{Z}^2) = \#(\mathcal{P} \cap \mathbb{Z}^2) = \mathrm{vol}(\mathcal{P}) + \frac{1}{2}\left(\sum_{\mathcal{F}\subseteq\mathcal{P}} \mathrm{vol}_{\mathcal{R}}(\mathcal{F})\right) + 1. \tag{2.61}$$

The facets of an integral polygon are line segments with integral endpoints. In order to calculate their relative volume we have to consider closed intervals $[a, b] \subseteq \mathbb{R}$ for integers a, b with $a < b$. The volume of the interval $[a, b]$ is given by $(b - a)$ and the number of integral points $\#([a, b] \cap \mathbb{Z})$ equals $(b - a) + 1$, c.f. Example 2.5.

Since $\partial_{\mathcal{P}} = \mathcal{P}\backslash\mathcal{P}^\circ$ is a closed line segment, we obtain that

$$\sum_{\mathcal{F}\subseteq\mathcal{P}} \mathrm{vol}_{\mathcal{R}}(\mathcal{F}) = \#(\partial_{\mathcal{P}} \cap \mathbb{Z}^2). \tag{2.62}$$

□

2.5 The Period of Ehrhart Quasi–Polynomials

Ehrhart's Theorem indicates that the integer–point enumerator of a rational d–polytope is a quasi–polynomial whose period divides the least common multiple of the denominators of the vertex coordinates. Thus, we obtain in fact a polynomial, if the vertices of a polytope have integral coordinates. The question arises if it is possible to replace the word *divides* by *equals* in Ehrhart's Theorem. As we will see, this conclusion can be drawn in the case of $1-$dimensional rational polytopes in $\mathbb{R}$ but does not hold in higher dimensions.

As evidence for the latter claim we will give an instance of an infinite class of 2–dimensional triangles, where any positive integer occurs as denominator, even though the Ehrhart quasi–polynomials of all these triangles are polynomials, c.f. [19].

2.5.1 One Dimensional Rational Polytopes

Theorem 2.40. *The period of the Ehrhart quasi–polynomial of a rational* 1*–polytope in* $\mathbb{R}$ *is given by denominator of the polytope.*

Proof. In Example 2.5, we observed that the number of integral points in the closed interval $[a, b]$, is given by

$$\#([a, b] \cap \mathbb{Z}) = b - a + 1, \tag{2.63}$$

for all non–negative integers $a < b$. Therefore, we obtain that the Ehrhart quasi–polynomial of the polytope $[a, b]$ is given by $\mathcal{L}_{[a,b]}(t) = (b - a)t + 1$.

Rational 1–polytopes in $\mathbb{R}$ are closed intervals $[a/b, c/d]$ and we may assume, without loss of generality, that a, b, c, d are positive integers such that, $a/b < c/d$ and $\gcd(a, b) = \gcd(c, d) = 1$. Observe that the number of integral points in the interval $[a/b, c/d]$ agrees with the number of integral points in the integral interval $[\lceil a/b \rceil, \lfloor c/d \rfloor]$. Thus, by (2.63), we conclude that

$$\#\left(\left[\frac{a}{b}, \frac{c}{d}\right] \cap \mathbb{Z}\right) = \left\lfloor \frac{c}{d} \right\rfloor - \left\lceil \frac{a}{b} \right\rceil + 1. \tag{2.64}$$

We rephrase the identity above in terms of fractional parts and obtain that

$$\#\left(\left[\frac{a}{b}, \frac{c}{d}\right] \cap \mathbb{Z}\right) = \frac{c}{d} - \frac{a}{b} - \left\{\frac{c}{d}\right\} - \left\{-\frac{a}{b}\right\} + 1. \tag{2.65}$$

This implies that

$$\mathcal{L}_{\left[\frac{a}{b},\frac{c}{d}\right]}(t) = \left(\frac{c}{d} - \frac{a}{b}\right) t - \left\{\frac{tc}{d}\right\} - \left\{-\frac{ta}{b}\right\} + 1. \tag{2.66}$$

Since we assume that the integers c and d are relatively prime, $t \mapsto \left\{\frac{tc}{d}\right\}$ is a periodic function with period d. In the same way we obtain that the function $t \mapsto \left\{-\frac{ta}{b}\right\}$ is periodic in t with period b. Therefore, we conclude that the period of the Ehrhart quasi–polynomial $\mathcal{L}_{\left[\frac{a}{b},\frac{c}{d}\right]}(t)$ equals the lowest common multiple of the integers b and d. □

2.5.2 Minimal Periods of Ehrhart Quasi–Polynomials in Higher Dimensions

In contrast to the previous result, we may prove the following assertion in case of rational polytopes of dimension exceeding one.

Theorem 2.41. *For any integers $d \geq 2$ and $p \geq 1$ there exist rational d–polytopes with denominator p whose Ehrhart quasi–polynomials are polynomials.*

As a proof of this theorem we specify an infinite class of 2–polytopes where any integer occurs as denominator even though the Ehrhart functions of all these polytopes are polynomials. Since the Ehrhart quasi–polynomial of the cartesian product of two polytopes equals the product of the corresponding Ehrhart quasi–polynomials, it is possible, due to this class of polytopes, to construct polytopes of arbitrary dimension, whose Ehrhart functions are polynomials even though their denominator equals p.

Proposition 2.42. *Let $\mathcal{T}$ be the rational triangle with vertices $(0,0), (1,(p-1)/p)$ and $(p,0)$, for some integer $p \geq 2$. Then, the Ehrhart function of this polytope is given by the polynomial*

$$\mathcal{L}_{\mathcal{T}}(t) = \frac{p-1}{2}\, t^2 + \frac{p+1}{2}\, t + 1. \tag{2.67}$$

Proof. Since triangles are simplices, we may calculate the Ehrhart–series of this polytope according to the proof of Ehrhart's Theorem for simplices, Theorem 2.13. There, the crucial step was to specify the integer–point transform of the fundamental parallelepiped of the cone over this simplex, which is in our actual example given by

$$\Pi_{\mathrm{cone}(\mathcal{T}|1)} = \left\{ \lambda_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} p \\ p-1 \\ p \end{pmatrix} + \lambda_3 \begin{pmatrix} p \\ 0 \\ 1 \end{pmatrix} : 0 \leq \lambda_i < 1 \right\}. \tag{2.68}$$

$$\tag{2.69}$$

In order to specify the function $\sigma_{\Pi_{\mathrm{cone}(\mathcal{T}|1)}}(1, 1, \boldsymbol{z})$ we have to count the number of integral points $\boldsymbol{z} \in \Pi_{\mathrm{cone}(\mathcal{T}|1)}$ with $z_3 = l$, for all integers $0 \leq l \leq (p+1)$. Put differently, given an integer $0 \leq l \leq (p+1)$, our aim is to count the number of tuples (j, k) for which there exist reals $\lambda_1, \lambda_2, \lambda_3$ with $0 \leq \lambda_i < 1$ such that

$$(\lambda_2 + \lambda_3)\, p = j, \tag{2.70}$$

$$(p-1)\, \lambda_2 = k, \tag{2.71}$$

$$\lambda_1 + p\lambda_2 + \lambda_3 = l. \tag{2.72}$$

First of all, observe that equation (2.71) is equivalent to the condition that $\lambda_2 = \frac{k}{p-1}$. This implies that a suitable real $0 \leq \lambda_2 < 1$ may only exist for integers

$$k \in \{0, 1, \ldots, (p-2)\}. \tag{2.73}$$

Moreover, substituting the variable λ_2 by $k/(p-1)$ in equation (2.70) yields that $\lambda_3 = j/p - k/p - 1$. Hence, the condition that $0 \leq \lambda_3 < 1$ implies the estimation that $k + k/p - 1 \leq j < p + k + k/p - 1$. Because j, k and p denote integers and since the constraint that $0 \leq \lambda_2 < 1$ implies that $0 \leq k/p - 1 < 1$, we conclude that there may exist reals $0 \leq \lambda_2, \lambda_3 < 1$, such that the equations (2.71) and (2.70) hold at the same time, at the most for integers

$$\begin{gathered} j \in \{1+k, \ldots, p+k\} \text{ for } k \neq 0 \text{ and} \\ j \in \{0, \ldots, p-1\} \text{ for } k = 0. \end{gathered} \tag{2.74}$$

Finally, we study the third equation, (2.72). Here, we substitute the variable λ_2 by $k/p - 1$ and set $\lambda_3 = j/p - k/(p-1)$. Then, by applying some basic transformations, we obtain that $\lambda_1 = l - p\frac{k}{p-1} - (\frac{j}{p} - \frac{k}{p-1}) = l - k - \frac{j}{p}$. Taking into account the condition

$0 \le \lambda_1 < 1$, we obtain the estimation

$$k + \frac{j}{p} \le l < 1 + k + \frac{j}{p}. \tag{2.75}$$

If $k = 0$ then $0 \le j \le p - 1$ and the estimation $\frac{j}{p} \le l < 1 + \frac{j}{p}$ implies that

$$\begin{aligned} &l = 0 \text{ for } j = 0 \text{ and} \\ &l = 1 \text{ for } j > 0. \end{aligned} \tag{2.76}$$

In other words, there is one integral point with z_3–coordinate zero, the origin, and $p-1$ integral points with z_3–coordinate one, contained in the fundamental parallelepiped of the cone over our triangle.

Else, for $k \neq 0$, according to our previous thoughts, the integer j has to be an element of the set $\{k+1, k+2, \ldots, k+p\}$. Moreover, the estimations $0 \le \frac{j}{p} < 2$ and $\frac{j}{p} \le 1$ are valid for all $j \in \{k+1, \ldots, k+(p-k)\}$. Therefore, there are $p-k$ integers j, such that $l = k+1$, and k integers such that $l = k+2$, for a fixed integer $k \in \{1, \ldots, (p-2)\}$. Consequently, for $l \in \{1, 2, \ldots, p-1\}$ there are $p-1$ integral points in $\Pi_{\mathrm{cone}(\mathcal{T}|1)}$ with x_3–coordinate l and $p-2$ integral points with x_3–coordinate p. The origin is the only point with x_3-coordinate 0. Summarizing, we conclude that the Ehrhart–series of our triangle is given by

$$\begin{aligned} \mathrm{Ehr}_{\mathcal{T}}(z) &= \frac{1 + (p-1)(z + z^2 + \ldots + z^{p-1}) + (p-2)z^p}{(1-z)^2(1-z^p)} \\ &= \frac{(1 + (p-2)z)(1 + z + z^2 + \ldots + z^{p-1})}{(1-z)^3(1 + z + \ldots + z^{p-1})} \\ &= \frac{1 + (p-2)z}{(1-z)^3}. \end{aligned} \tag{2.77}$$

Eventually, we obtain that

$$\begin{aligned} \mathcal{L}_{\mathcal{T}}(t) &= \binom{t+2}{2} + (p-2)\binom{t+1}{2} = \frac{(t+2)(t+1)}{2} + (p-2)\frac{(t+1)t}{2} \\ &= \frac{t^2 + 2t + t + 2}{2} + \frac{(p-2)(t^2+t)}{2} = t^2\frac{1 + (p-2)}{2} + t\frac{t(3 + (p-2))}{2} + 1 \\ &= \frac{p-1}{2}t^2 + \frac{p+1}{2}t + 1. \end{aligned}$$

□

Based on the previous proof, we gain a necessary condition for the fact that the Ehrhart function of a rational simplex is actually a polynomial. This provides us with a method to prove that the period of the Ehrhart quasi–polynomial of certain rational simplices is strictly greater than one. To prove the subsequent Theorem, we need the following observation.

Lemma 2.43. *Let p be a positive integer and $g(z) = a_0 + a_1 z + \ldots + a_k z^k$ a polynomial of degree $k > p - 2$ with integral coefficients. If the integer p does not divide the sum of the coefficients of the polynomial $g(z)$, the polynomial $1 + z + z^2 + \ldots + z^{p-1}$ may not be a factor of this polynomial over $\mathbb{Z}$.*

Proof. On the contrary, suppose that $1 + z + z^2 + \ldots + z^{p-1}$ divides the polynomial $g(z)$ in $\mathbb{Z}[z]$, i.e. there exists a polynomial $q(z) = b_0 + b_1 z + \ldots + b_m$ with integral coefficients such that

$$\begin{aligned} g(z) &= (1 + z + z^2 + \ldots + z^{p-1})\; q(z) \\ &= q(z) + q(z)z + q(z)z^2 + \ldots + q(z)z^{p-1}. \end{aligned} \tag{2.78}$$

Summing up the coefficients on both sides of this equation yields that

$$\sum_{i=0}^{k} a_i = p \cdot \sum_{j=0}^{m} b_j \tag{2.79}$$

This shows that the integer p has to divide the sum of the coefficients of $g(z)$ in the case that the polynomial $1 + z + z^2 + \ldots + z^{p-1}$ is a factor of this polynomial. $\square$

Theorem 2.44. *Let $\Delta \subseteq \mathbb{R}^d$ be a $d-$simplex with vertices $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{d+1}$ and denote by p_i the minimal positive integer such that $p_i \boldsymbol{v}_i \in \mathbb{Z}^d$, for all indices $1 \le i \le d+1$. Then, if one of the integers*

$$p_i \nmid \det\left((p_1 \boldsymbol{v}_1, p_1), \ldots, (p_{d+1} \boldsymbol{v}_{d+1}, p_{d+1})\right), \tag{2.80}$$

for $0 \le i \le d+1$, we obtain that the period of the Ehrhart quasi–polynomial of the simplex is strictly greater than 1.

Proof. First of all, note that $\{(p_1\boldsymbol{v}_1, p_1), \ldots, (p_{d+1}\boldsymbol{v}_{d+1}, p_{d+1})\}$ provides a set of integral generators of the cone$(\Delta|1)$. Therefore, Equation (2.16), from the proof of Ehrhart's Theorem for simplices, implies that

$$\mathrm{Ehr}_\Delta(z) = \frac{\sigma_{\Pi_{\mathrm{cone}(\Delta|1)}}(1, 1, \ldots, z)}{(1 - z^{p_1}) \ldots (1 - z^{p_{d+1}})}. \tag{2.81}$$

Since the Ehrhart–series is the generating function of the Ehrhart quasi–polynomial, we obtain, by Corollary 1.59, that the Ehrhart quasi–polynomial of our simplex is a polynomial if and only if the denominator of the rational function $\mathrm{Ehr}_\Delta(z)$ equals $(1-z)^{d+1}$, if written in lowest terms. Since the polynomial

$$1 - z^{p_i} = (1 - z) \cdot (1 + z + z^2 + \ldots + z^{p_i - 1}), \tag{2.82}$$

this may only be the case if $(1 + z + z^2 + \ldots + z^{p_i-1}) \mid \sigma_{\Pi_{\mathrm{cone}(\Delta|1)}}(1, 1, \ldots, z)$, for all $0 \leq i \leq d+1$. A necessary condition for this property is, by Lemma 2.43, that all integers p_i, for $0 \leq i \leq d + 1$, divide the sum of the coefficients of the polynomial $\sigma_{\Pi_\Delta}(1, 1, \ldots, z)$. Note that we obtain this sum by evaluating the polynomial $\sigma_{\Pi_{\mathrm{cone}(\Delta|1)}}(1, 1, \ldots, 1, z)$ at $z = 1$. The combinatorial interpretation given in Theorem 2.11 states that the value $\sigma_{\Pi_{\mathrm{cone}(\Delta|1)}}(1, 1, \ldots, 1)$ corresponds with the number of integral points in the half–open parallelepiped

$$\Pi_{\mathrm{cone}(\Delta|1)} = \{\lambda_1(p_1\boldsymbol{v}_1, p_1) + \ldots + \lambda_{d+1}(p_{d+1}\boldsymbol{v}_{d+1}, p_{d+1}) : 0 \leq \lambda_1, \ldots, \lambda_{d+1} < 1\}. \tag{2.83}$$

By Lemma 2.30, this number equals the volume of the parallelepiped which is given by $\det((p_1\boldsymbol{v}_1, p_1), \ldots, (p_{d+1}\boldsymbol{v}_{d+1}, p_{d+1}))$. $\square$

2.6 Sylvester's Denumerants

An application of Ehrhart's Theorem which emerges from the hyperplane description of polytopes is that the number of restricted partitions [3] of a non–negative integer $t \in \mathbb{Z}_{\geq 0}$ can be counted by evaluating the Ehrhart quasi–polynomial of a simplex.

Given a sequence $A = (a_1, \ldots, a_d)$ of positive integers, we study the number of integral solutions $(x_1, \ldots, x_d) \in \mathbb{Z}^d_{\geq 0}$ of the linear diophantine equation

$$a_1 x_1 + a_2 x_2 + \ldots + a_d x_d = t, \tag{2.84}$$

i.e. the number of partitions of a positive integer t into given parts $a_1, \ldots, a_d$.

The English mathematician James Joseph Sylvester already investigated this counting problem in 1857 and named the function

$$\Phi_A(t) := \#\{(x_1, \ldots, x_d) \in \mathbb{Z}^d_{\geq 0} : a_1 x_1 + a_2 x_2 + \ldots + a_d x_d = t\} \tag{2.85}$$

the denumerant, c.f. [1].

These functions, for instance, occur in the context of Knapsack-problems within the field of combinatorial optimization and are also examined in conjunction with the Coin-Exchange-problem of Frobenius in number theory.

To establish the connection between denumerants and Ehrhart quasi–polynomials we consider the rational polytope given by

$$\mathcal{P} = \{(x_1, \ldots, x_d) \in \mathbb{R}^d_{\geq 0} : a_1 x_1 + a_2 x_2 + \ldots + a_d x_d = 1\}. \tag{2.86}$$

This represents the hyperplane description of a rational $d-1$ simplex in $\mathbb{R}^d$, whose vertices are given by the points where the hyperplane $\{\boldsymbol{x} \in \mathbb{R}^d : a_1 x_1 + \ldots + a_d x_d = 1\}$ intersects the coordinate axis, i.e. by the points $\{\frac{1}{a_1}\boldsymbol{e}_1, \ldots, \frac{1}{a_d}\boldsymbol{e}_d\}$. Given a positive integer $t \in \mathbb{Z}_{\geq 0}$, the t–th dilation of this simplex

$$t\mathcal{P} = \{(x_1, \ldots, x_d) \in \mathbb{R}^d_{\geq 0} : a_1 x_1 + a_2 x_2 + \ldots + a_d x_d = t\}. \tag{2.87}$$

Thus, the integral solutions of the diophantine equation (2.84) correspond with the integral points in the t–th dilation of the polytope $\mathcal{P}$. Moreover, the set of the integral

[3] A partition of a non–negative integer n is a finite, non–increasing sequence of positive integers $\lambda_1, \ldots, \lambda_r$ such that $\sum_{i=1}^r \lambda_i = n$

points in the interior of the dilated simplex $t\mathcal{P}$ is given as

$$(t\mathcal{P})^\circ \cap \mathbb{Z}^s \quad = \quad \{(x_1, \ldots, x_d) \in \mathbb{Z}^d_{>0} : a_1x_1 + a_2x_2 + \ldots + a_dx_d = t\}, \qquad (2.88)$$

and therefore corresponds with the partitions of the integer t into given parts $a_1, \ldots, a_d$, where each part actually occurs.

Summarizing, we gain the following result from Ehrhart's Theorem and Ehrhart–Macdonald Reciprocity.

Proposition 2.45. *Given a finite sequence of positive integers $A = (a_1, \ldots, a_d)$, the denumerant $\Phi_A(t)$ is a quasi–polynomial of degree $d-1$ whose period divides the lowest common multiple of the integers $a_1, \ldots, a_d$. Moreover, evaluating this quasi–polynomial at $-t$ gives $(-1)^{d-1}$ times the number of partitions of the integer t into given parts $(a_1, \ldots, a_d)$ where each part actually occurs.*

Now, we may obtain the same result by a different approach, using Theorem 1.55, which represents actually the one dimensional case of the method we will use in the next chapter in order to prove a multivariate generalization of Ehrhart's Theorem.

To this end, observe that

$$\begin{aligned} \frac{1}{(1-z^{a_i})\ldots(1-z^{a_d})} &= \sum_{x_1 \geq 0} z^{x_1a_1} \sum_{x_2 \geq 0} z^{x_2a_1} \ldots \sum_{x_d \geq 0} z^{x_da_1} \\ &= \sum_{x_1 \geq 0} \ldots \sum_{x_d \geq 0} z^{x_1a_1 + \ldots + x_da_d} = \sum_{t \geq 0} \Phi_A(t) z^t. \end{aligned} \qquad (2.89)$$

This shows that the generating function of the denumerant is rational. Moreover, all roots of the polynomial $(1-z^{a_i})\ldots(1-z^{a_d})$ are $lcm(a_1, \ldots, a_d)-th$ roots of unity with multiplicity no greater then d, and at least the root 1 occurs with multiplicity d. Therefore, by Corollary 1.57, it follows that the function $\Phi_A(t)$ is a quasi–polynomial of degree $d-1$ whose period divides the lowest common multiple of the integers $(a_1, \ldots, a_d)$.

Since we used here only the hyperplane description, while our proof of Ehrhart's Theorem was based on the vertex description of the polytope, the observation above constitutes a different approach to the problem of counting the integral points in the rational simplex (2.87) in dependence of the dilation factor $t \in \mathbb{Z}_{\geq 0}$.

Another interesting aspect regarding this approach arises when we consider item (ii) of the equivalent assertions stated in Theorem 1.55.

Denoting the sum of the elements of a sub–sequence $S = (a_{i_1}, \ldots, a_{i_1}) \subseteq (a_1, \ldots, a_d) = A$ by a_S and writing $A \backslash S$ for the sequence obtained by removing the terms of the sub–

sequence S from the sequence A, we obtain that the product

$$\prod_{i=1}^{d}(z^{a_i}-1)=\sum_{S\subseteq A}(-1)^{|A\setminus S|}z^{a_s}. \tag{2.90}$$

Accordingly, Theorem 1.55 (ii) implies that

$$\sum_{S\subseteq A}(-1)^{|A\setminus S|}\Phi_A(n+a_A-a_S)=0, \tag{2.91}$$

for all integers $n\geq 0$. This shows that the function Φ_A satisfies a homogeneous linear difference equation. Moreover, we may substitute the variable n in (2.91) by $b-a_A$, and gain that

$$\Phi_A(b)=\sum_{S\subseteq A, S\neq\emptyset}(-1)^{|A\setminus S|-1}\Phi_A(b-a_S), \tag{2.92}$$

for all integers $b>a_A$. This recurrence relation allows us to calculate the function value $\Phi_A(b)$ recursively from the according function values on the integral points in the interval $[b-a_A,b)$. Hence, all values of the function Φ_A can be calculated recursively from given initial values on the integral points in the interval $[0,a_A)$.

With regard to later considerations we observe that the interval $[0,a_A]=\{\sum_{i=1}^{d}\lambda_i a_i: 0\leq\lambda_i\leq 1\}=:B(A)$. Thus, given an integer $b\in\mathbb{Z}$, the integral points in the interval $[b-a_A,b)$ can be described as $(b+u-B(A))\cap\mathbb{Z}$ for some real $u\in(0,1)$. Although the statement may seem rather involved, it is precisely what we shall need later.

3 Chapter 3
Vector Partition Functions

We may gain a generalization of Ehrhart's Theorem by examining a multivariate generalization of the example analyzed in the previous section. In order to formulate the according result we need to elaborate the relation between rational polytopes and sets of non–negative solutions of systems of linear equations with integral coefficients.

3.1 Another Perspective on Polytopes

In the context of vector partition functions polytopes are often introduced as bounded sets

$$\mathcal{P} = \{\boldsymbol{x} \in \mathbb{R}^d_{\geq 0} : A\boldsymbol{x} = \boldsymbol{b}\}, \tag{3.1}$$

specified by an $s \times d$ matrix A and a vector $\boldsymbol{b} \in \mathbb{R}^s$. We will now briefly work out that any polytope is affinely isomorph to a set as described in (3.1).

The fact that (3.1) specifies a polytope is an immediate consequence of the hyperplane description of polytopes. On the other hand, we will now show that one may obtain any polytope as a translation of a projection of such a set. Moreover, we will be able to find such an affine map which maps integral points bijective to points with integral coordinates, for every rational polytope. To study the number of integral points contained in rational polytopes we may therefore restrict our considerations to sets as given in (3.1).

Example 3.1. We consider the polytope $\mathcal{P} \subseteq \mathbb{R}^2$ given by

$$\mathcal{P} \;=\; \{(x,y) \in \mathbb{R}^2 : 2x + 3y \leq 1, x \geq -1, y \geq -1\} \tag{3.2}$$

At first, we translate the polytope $\mathcal{P}$ by the integer vector $\boldsymbol{w} = (1,1)^T$ into the non–negative orthant $\mathbb{R}^2_{\geq 0}$ and obtain the equivalent polytope

$$\boldsymbol{w} + \mathcal{P} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2_{\geq 0} : \begin{pmatrix} 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leq 6 \right\}, \tag{3.3}$$

which contains the same number of integral points as $\mathcal{P}$. Furthermore, note that the inequality $2x+3y \leq 6$ is valid if and only if the real $z := 6-2x-3y$ is non–negative. If both, x and y are integers, then so is z. Hence, a point $(x,y) \in \mathbb{R}^2$ lies in the polytope $\boldsymbol{w}+\mathcal{P}$ if and only if there exists a non–negative real z such that $2x+3y+z = 6$ holds. Therefore, introducing an auxiliary variable z in (3.3) yields that

$$\mathcal{P} \cong \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3_{\geq 0} : \begin{pmatrix} 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 6 \right\}, \tag{3.4}$$

where the latter set contains the same number of integral points as the polytope $\mathcal{P}$. Conversely, we may retrieve the polytope $\boldsymbol{w}+\mathcal{P}$ by canceling the z–coordinates of the points in the polytope (3.4).

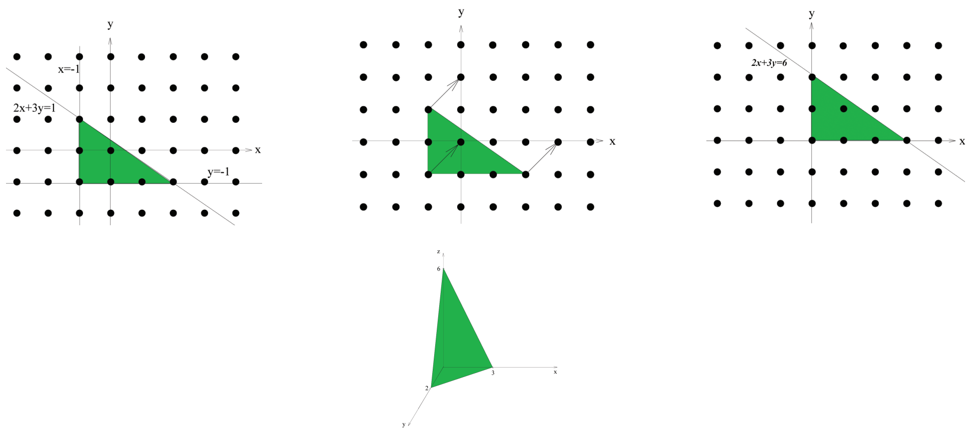


Figure 3.1: Introducing slack variables.

By the same method as illustrated in the upper example, we obtain a general result.

To this end, suppose that $\mathcal{P}$ is a polytope in $\mathbb{R}^n$. Since we are dealing with a bounded set, there exists an integral vector $\boldsymbol{v} \in \mathbb{Z}^n$ such that the translated polytope $\boldsymbol{v} + \mathcal{P}$ lies within the non–negative orthant $\mathbb{R}^n_{\geq 0}$. Consequently, as to the number of integral points, there is no loss of generality if we assume that the polytope $\mathcal{P}$ is located in $\mathbb{R}^n_{\geq 0}$. Thus, we consider polytopes given as

$$\mathcal{P} \quad = \quad \{\boldsymbol{x} \in \mathbb{R}^n_{\geq 0} : A\boldsymbol{x} \leq \boldsymbol{b}\}, \tag{3.5}$$

where A stands for an $s \times n$ matrix and $\boldsymbol{b}$ is a vector in $\mathbb{R}^s$. Denoting the j–th entry of the i–th row of the matrix A by $a_{i,j}$, we obtain that a point $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n_{\geq 0}$ lies within the polytope $\mathcal{P}$ if and only if the inequality

$$a_{i,1}x_1 + a_{i,2}x_2 + \ldots + a_{i,n}x_n \leq b_i, \tag{3.6}$$

holds for all indices $1 \leq i \leq s$. Subtracting b_i from both sides of the above equation leads to the conclusion that this inequality holds for a point $\boldsymbol{x} \in \mathbb{R}^n_{\geq 0}$ if and only if

$$x_{n+i} := b_i - a_{i,1}x_1 - a_{i,2}x_2 - \ldots - a_{i,n}x_n, \tag{3.7}$$

is non–negative, for all $1 \leq i \leq s$. Note that then $b_i = a_{i,1}x_1 + a_{i,2}x_2 + \ldots + a_{i,n}x_n + x_{n+i}$. Therefore, introducing an auxiliary slack–variable x_{n+i} for each of these inequalities gives that

$$\mathcal{P} \cong \{\boldsymbol{x} \in \mathbb{R}^{n+s}_{\geq 0} : A'\boldsymbol{x} = \boldsymbol{b}\}, \tag{3.8}$$

where $A' = (A \; \mathbb{I}_s)$. Conversely, we regain the polytope $\mathcal{P}$ by canceling the last s coordinates of the points of the set (3.8). Note that the corresponding affine maps are given by an inclusion map and a projection, respectively.

Given a rational polytope $\mathcal{P}$, we may moreover assume that the entries of the matrix A as well as the entries of the vector $\boldsymbol{b}$ in (3.5) are integers. It then follows by equation (3.7) that the slack variables $\{x_{n+i} : 0 \leq i \leq s\}$ are integers for any integral point $\boldsymbol{x} \in \mathcal{P} \cap \mathbb{Z}^n$. Therefore, in case of a rational polytope, our affine maps induce a bijection between the integral points in both sets. Thus,

$$\#(\mathcal{P} \cap \mathbb{Z}^n) = \# \left(\{\boldsymbol{x} \in \mathbb{R}^{n+s}_{\geq 0} : A'\boldsymbol{x} = \boldsymbol{b}\} \cap \mathbb{Z}^{n+s} \right). \tag{3.9}$$

This points out that as to the number of integral points in rational polytopes we may just as well count the integral points in the set on the right hand side of (3.9).

We proceed by stating a necessary and sufficient condition under which a polyhedron $P = \{\boldsymbol{x} \in \mathbb{R}^d_{\geq 0} : A\boldsymbol{x} = \boldsymbol{b}\}$, specified by an $s \times d$ matrix A and a vector$\boldsymbol{b} \in \mathbb{R}^s$, is actually a polytope.

Lemma 3.2. *The condition that the non–empty polyhedron $P = \{\boldsymbol{x} \in \mathbb{R}^d_{\geq 0} : A\boldsymbol{x} = \boldsymbol{b}\}$ is bounded is equivalent to the condition that the only solution of $A\boldsymbol{y} = \boldsymbol{0}$ contained in the non–negative orthant $\mathbb{R}^d_{\geq 0}$ is given by the zero vector.*

Proof. Recall that a polyhedron is not bounded if and only if it contains a half–line. Now suppose, contrary to our claim, that there exists a vector $\boldsymbol{y} \in \mathbb{R}^d_{\geq 0} \backslash \{\boldsymbol{0}\}$ satisfying $A\boldsymbol{y} = \boldsymbol{0}$. Then, the polyhedron contains with any point $\boldsymbol{x} \in P$ also the half–line $\{\boldsymbol{x} + \lambda \boldsymbol{y} : \lambda \in \mathbb{R}_{\geq 0}\}$. Conversely, assume there exists a point $\boldsymbol{x} \in P$ and a vector $\boldsymbol{y} \neq \boldsymbol{0}$ such that the half–line $\{\boldsymbol{x} + \lambda \boldsymbol{y} : \lambda \in \mathbb{R}_{\geq 0}\}$ lies within the polyhedron P. This implies that $\boldsymbol{y} \in \mathbb{R}^d_{\geq 0} \backslash \{\boldsymbol{0}\}$. Then we have that $A(\boldsymbol{x} + \lambda \boldsymbol{y}) = \boldsymbol{b}$ for all non–negative reals λ, which implies that $A\boldsymbol{y} = \boldsymbol{0}$. □

We summarize the discussion above in the following theorem.

Theorem 3.3. *Let $\mathcal{P}$ be a polytope in $\mathbb{R}^d$. Then there exists a vector $\boldsymbol{b} \in \mathbb{R}^s$ and a $(d + s) \times s$ matrix A, with the property that the only solution of $A\boldsymbol{y} = \boldsymbol{0}$ in the non–negative orthant $\mathbb{R}^d_{\geq 0}$ is given by the zero vector, such that*

$$\mathcal{P} \cong \{\boldsymbol{x} \in \mathbb{R}^{n+s}_{\geq 0} : A\boldsymbol{x} = \boldsymbol{b}\}, \tag{3.10}$$

and any such polyhedron is actually a polytope. The affine maps that establish (3.10) *are given by an inclusion map and a projection, respectively.*

Given a rational polytope $\mathcal{P} \subseteq \mathbb{R}^d$, we may moreover assume that the matrix A as well as the vector $\boldsymbol{b}$ have solely integral entries, in which case we have that

$$\#(\mathcal{P} \cap \mathbb{Z}^n) = \# \left(\{\boldsymbol{x} \in \mathbb{R}^{n+s}_{\geq 0} : A\boldsymbol{x} = \boldsymbol{b}\} \cap \mathbb{Z}^{n+s} \right). \tag{3.11}$$

By omitting redundant equations, we may assume, without loss of generality, that the matrix A has full rank.

Remark 3.4. In the sequel we refer to a finite sequence of vectors from $\mathbb{R}^s$ as a list and indicate by X the list of the column vectors of a matrix A. Moreover, by abuse of notation we denote by $X \backslash Y$ the list that is obtained by removing the elements of a sublist $Y \subseteq X$ from the list X.

Now, we may restate the condition that the only solution of $A\boldsymbol{y} = \mathbf{0}$ located in $\mathbb{R}^d_{\geq 0}$ is given by the zero vector, by requiring that $\mathbf{0} \notin \operatorname{conv}(X)$. Note that the convex hull $\operatorname{conv}(X) = \{A\boldsymbol{x} : \boldsymbol{x} \in \mathbb{R}^d_{\geq 0}, \|\boldsymbol{x}\|_1 = 1\}$. By Theorem 1.37, we may therefore equivalently demand that $\operatorname{cone}(A) := \operatorname{cone}(X)$ a pointed cone with a vertex at the origin. Thus, another way of stating Lemma 3.2 is to say that a polyhedron $P = \{\boldsymbol{x} \in \mathbb{R}^d_{\geq 0} : A\boldsymbol{x} = \boldsymbol{b}\}$ is bounded if and only if $\operatorname{cone}(X)$ is a pointed cone with vertex at the origin.

Regarding the theorem above, integral dilations of a polytope $\mathcal{P} = \{\boldsymbol{x} \in \mathbb{R}^d : A\boldsymbol{x} = \boldsymbol{b}\}$, as studied in the context of the Ehrhart Theory, are given as

$$t\mathcal{P} = \{\boldsymbol{x} \in \mathbb{R}^d : A\boldsymbol{x} = t\boldsymbol{b}\}. \tag{3.12}$$

In fact, this is the family of polytopes which can be represented with a fixed matrix A while the vector on the right hand side varies in a single integral dilation factor $t \in \mathbb{Z}_{\geq 0}$. An obvious question which arises at this point is weather we may obtain a result similar to Ehrhart's Theorem when we consider the vector $\boldsymbol{b}$ on the right hand side as a variable in $\mathbb{Z}^s$.

Convention. In the remainder of this section we assume A to be an $s \times d$ integer matrix of full rank $s \leq d$ which meets the condition that $\operatorname{cone}(A)$ is a pointed cone. Moreover, we assume that X denotes the list of the column vectors of the matrix A.

3.2 Variable Polytopes and Vector Partition Functions

Our aim is, geometrically spoken, to count the number of integral points in the family of rational polytopes that are obtained from a rational polytope by independent parallel motions of the facets, in dependence of the distance of the supporting hyperplanes to the origin. Taking our previous explanations into account, we consider the following objects.

Definition 3.5. For any vector $\boldsymbol{b} \in \mathbb{Z}^s$ we may define a polytope by setting

$$P_A(\boldsymbol{b}) := \{\boldsymbol{x} \in \mathbb{R}^d_{\geq 0} : A\boldsymbol{x} = \boldsymbol{b}\}. \tag{3.13}$$

We call $P_A(\boldsymbol{b})$ the *variable polytope* associated with the matrix A and the vector $\boldsymbol{b}$.

In the literature these polytopes appear under various names, for example as partition polytopes in [25]. We take the name variable polytope from [7], where one may also find a proof of the following elementary properties.

- If the vector $\boldsymbol{b}$ lies in the interior of cone(A), then $P_A(\boldsymbol{b})$ is a $d - s$ dimensional polytope in $\mathbb{R}^d$.
- The vertices of the polytope $P_A(\boldsymbol{b})$ are given by the non–empty sets $P_B(\boldsymbol{b})$, where B stands for an arbitrary $s \times s$ sub–matrix of A which has full rank.

Definition 3.6. We define the *vector partition function* associated with the matrix A as

$$\Phi_A : \mathbb{Z}^s \to \mathbb{N}, \quad \Phi_A(\boldsymbol{b}) := \#\{\boldsymbol{x} \in \mathbb{Z}^d_{\geq 0} : A\boldsymbol{x} = \boldsymbol{b}\}. \tag{3.14}$$

The function $\Phi_A(\boldsymbol{b})$ counts the number of integral points in the rational polytope $P_A(\boldsymbol{b})$ in dependence of a variable vector $\boldsymbol{b} \in \mathbb{Z}^s$. From another point of view, we may say that this function counts the number of ways in which a vector $\boldsymbol{b} \in \mathbb{Z}^s$ can be written as a non–negative integral linear combination of the column vectors of the matrix A and this is actually the reason for the name vector partition function. In this context it is sometimes convenient to consider lists rather than matrices, and we set

$$\Phi_X(\boldsymbol{b}) := \#\{\boldsymbol{x} \in \mathbb{Z}^d_{\geq 0} : x_1\boldsymbol{a}_1 + \ldots + x_d\boldsymbol{a}_d := \boldsymbol{b}\}. \tag{3.15}$$

Remark 3.7. Note that the values of the vector partition function are indeed well defined integers, since we are working under the assumption that cone(X) is a pointed cone. Vector partition functions appear incidentally as the multivariate discrete truncated power in the field of numerical analysis, cf.[6].

Before studying further properties of these functions, we examine a simple example.

Example 3.8. Let A be the 2×3 matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}. \tag{3.16}$$

The the cone generated by the list of the column vectors of the matrix A agrees with the positive orthant $\mathbb{R}^2_{\geq 0}$. Therefore, the function $\Phi_A(\boldsymbol{b})$ vanishes if one of the coordinates of the vector $\boldsymbol{b} \in \mathbb{Z}^2$ is smaller than zero. It remains to specify the function $\Phi_A(\boldsymbol{b})$ for vectors $\boldsymbol{b} \in \mathbb{Z}^2_{\geq 0}$. Observe that

$$b_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = (b_1 - a) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (b_2 - a) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + a \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{3.17}$$

for all integers $a \in \mathbb{Z}$. Now, the right side of equation (3.17) gives a representation of the vector $\boldsymbol{b} = (b_1, b_2)$ as a linear combination of the column vectors of the matrix A, where the coefficients are non–negative integers as long as $0 \leq a \leq \min\{b_1, b_2\}$. Since we obtain any such representation in this way, the vector partition function is given by $\Phi_A(\boldsymbol{b}) = b_1 + 1$ whenever $0 \leq b_1 \leq b_2$ and $\Phi_A(\boldsymbol{b}) = b_2 + 1$ if $0 \leq b_2 \leq b_1$, c.f. Figure 3.2.

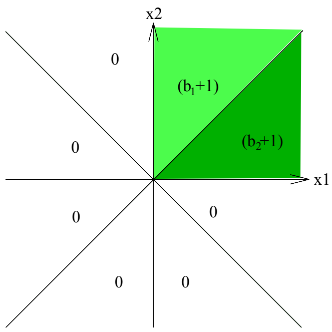


Figure 3.2

In the previous example we have observed that the vector partition function restricted to certain polyhedral cones agrees with (quasi–)polynomials. Moreover, the degrees of these polynomials are all one and equal the dimension of the variable polytope $P_A(\boldsymbol{b})$, for all integral vectors $\boldsymbol{b} \in \operatorname{cone}(A) \cap \mathbb{Z}^s$.

Our aim is to show that these observations are due to general properties of vector partition functions, which will provide a generalization of Ehrhart's Theorem to the setting of variable polytopes. The concepts needed to state the according theorem arise naturally from some properties of vector partition functions, which are immediate consequences of the definition.

Proposition 3.9. *The vector partition function $\Phi_X(\boldsymbol{b})$ vanishes at all points $\boldsymbol{b} \in \mathbb{Z}^s$ which do not belong to* $\operatorname{cone}(X) \cap \mathbb{Z}^s$.

Proposition 3.10. *Let $\boldsymbol{a}_i$ be an element of the list X. Then the vector partition function Φ_X satisfies*

$$\Phi_X(\boldsymbol{b}) - \Phi_X(\boldsymbol{b} - \boldsymbol{a}_i) = \Phi_{X\setminus\{\boldsymbol{a}_i\}}(\boldsymbol{b}), \tag{3.18}$$

at all points $\boldsymbol{b} \in \mathbb{Z}^s$.

Proof. Given a representation of the vector $\boldsymbol{b} \in \mathbb{Z}^s$ as a non–negative linear combination of the elements of the list X, we study the coefficient of the element $\boldsymbol{a}_i \in X$.

If this coefficient is strictly positive, we may obtain the same representation by adding $\boldsymbol{a}_i$ to a unique representation of the vector $\boldsymbol{b} - \boldsymbol{a}_i$ as a non–negative linear combination of the elements of the list X. Else, we have a representation of the vector $\boldsymbol{b}$ as a non–negative integral linear combination of the elements of the list $X\setminus\{\boldsymbol{a}_i\}$. This means that the vector partition function

$$\Phi_X(\boldsymbol{b}) = \Phi_X(\boldsymbol{b} - \boldsymbol{a}_i) + \Phi_{X\setminus\{\boldsymbol{a}_i\}}(\boldsymbol{b}). \tag{3.19}$$

Subtracting $\Phi_X(\boldsymbol{b} - \boldsymbol{a}_i)$ from both sides establishes (3.18). $\square$

In order to deduce further properties of vector partition functions from the previous results, we need the following terminology.

Definition 3.11. Given an integral vector $\boldsymbol{a} \in \mathbb{Z}^s$ we define the backwards difference operator $\nabla_{\boldsymbol{a}}$ by setting

$$\nabla_{\boldsymbol{a}} f(\boldsymbol{b}) := f(\boldsymbol{b}) - f(\boldsymbol{b} - \boldsymbol{a}), \tag{3.20}$$

for any function $f : \mathbb{Z}^s \to \mathbb{C}$, at all points $\boldsymbol{b} \in \mathbb{Z}^s$. Accordingly, we may define the difference operator associated with a list $Y = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_d)$ of integral vectors as

$$\nabla_Y f(\boldsymbol{b}) := \prod_{i=1}^{d} \nabla_{\boldsymbol{a}_i} f(\boldsymbol{b}) = \nabla_{\boldsymbol{a}_d}(\nabla_{\boldsymbol{a}_{d-1}}(\ldots(\nabla_{\boldsymbol{a}_1} f(\boldsymbol{b}))\ldots)). \tag{3.21}$$

Note that $\nabla_{\boldsymbol{a}}(\nabla_{\boldsymbol{c}} f(\boldsymbol{b})) = \nabla_{\boldsymbol{c}}(\nabla_{\boldsymbol{a}} f(\boldsymbol{b}))$ for all integral vectors $\boldsymbol{a}$ and $\boldsymbol{c} \in \mathbb{Z}^s$. Therefore, the operator ∇_Y does not depend on the order in which the elements of the list Y appear.

Using this notation we may now deduce the following result from Proposition 3.10.

Proposition 3.12. *Let $Y \subseteq X$ be a sublist. Then the vector partition function satisfies the multivariate homogeneous linear difference equation*

$$\nabla_Y \Phi_X(\boldsymbol{b}) = 0 \textit{ in all points } \boldsymbol{b} \in \mathbb{Z}^s \backslash \operatorname{cone}(X \backslash Y). \tag{3.22}$$

Proof. In the notation of backwards difference operators Proposition 3.10 states that $\nabla_{\boldsymbol{a}_i} \Phi_X \equiv \Phi_{X \backslash \{\boldsymbol{a}_i\}}$ for any element $\boldsymbol{a}_i \in X$. Repeated application of this result establishes that $\nabla_Y \Phi_X \equiv \Phi_{X \backslash Y}$. The assertion follows as the latter function vanishes outside $\operatorname{cone}(X \backslash Y)$, by Proposition 3.9. □

Definition 3.13. We call a minimal sublist $Y \subsetneq X$ with the property that the list $X \backslash Y$ does not span the vector space $\mathbb{R}^s$ a *cocircuit of the list X*. We denote the set of all cocircuits of the list X by $\mathcal{Y}(X)$.

Definition 3.14. We say that a point $\boldsymbol{x} \in \mathbb{R}^s$ is *strongly regular* with respect to the list X if and only if it is not contained in any cone of dimension less than s generated by a sublist of X. Using the notation of cocircuits, we can describe the set of regular points with respect to the list X as

$$\operatorname{reg}(X) := \mathbb{R}^s \backslash \bigcup_{Y \in \mathcal{Y}(X)} \operatorname{cone}(X \backslash Y). \tag{3.23}$$

The connected components of the set of strongly regular points are called the *chambers associated with the list X*, or short the chambers of X.

Remark 3.15. It can be shown that if the points $\boldsymbol{b}_1$ and $\boldsymbol{b}_2 \in \mathbb{Z}^s$ are contained in the same chamber associated with the list X, the variable polytopes $P_A(\boldsymbol{b}_1)$ and $P_A(\boldsymbol{b}_2)$ are combinatorial equivalent, i.e. there exists a bijection between the sets of faces of the polytopes $P_A(\boldsymbol{b}_1)$ and $P_A(\boldsymbol{b}_2)$, which preserves dimension and inclusion. For more details on this see [7].

Equivalently, one may define the chamber complex of X as the polyhedral subdivision of the $\operatorname{cone}(X)$, defined as the common refinement of the simplicial cones generated by bases of X, c.f. [23].

We illustrate these notations by a simple example.

Example 3.16. Consider the 3×6 matrix given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix},$$

where we denote the list of column vectors by $X = (\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4 \boldsymbol{a}_5, \boldsymbol{a}_6)$. The maximal sublists of X which do not span the vector space $\mathbb{R}^3$, i.e. the lists $X \backslash Y$ where Y is a cocircuit of the list X, are then given by $(\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_4)$, $(\boldsymbol{a}_1, \boldsymbol{a}_3)$, $(\boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_5)$, $(\boldsymbol{a}_1, \boldsymbol{a}_5, \boldsymbol{a}_6)$, $(\boldsymbol{a}_3, \boldsymbol{a}_4, \boldsymbol{a}_6)$, $(\boldsymbol{a}_4, \boldsymbol{a}_5)$ and $(\boldsymbol{a}_2, \boldsymbol{a}_6)$. We display the chambers of X in Figure 3.3, where the right picture represents a 2-dimensional slice of the chamber decomposition of cone(X).

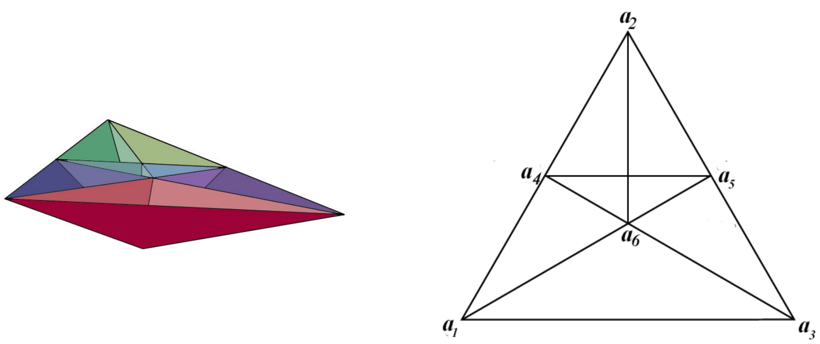


Figure 3.3: The chambers of cone(X)

Theorem 3.17. *Suppose that* $\boldsymbol{b} \in \mathbb{Z}^s$ *is a strongly regular point with respect to the list* X. *Then the vector partition function satisfies the homogeneous linear difference equation* $\nabla_Y \Phi_X(\boldsymbol{b}) = 0$ *for all cocircuits* $Y \in \mathcal{Y}(X)$.

The remainder of this chapter will be devoted to the proof of the following generalisation of Ehrhart's Theorem.

Theorem 3.18. *The restriction of the vector partition function* Φ_X *to a chamber of the list* X *agrees with a quasi-polynomial of total degree* $d - s$.

3.3 The space DM(X)

The previous theorem shows that the restriction of the vector partition function Φ_X to a chamber of X is a solution for certain difference equations. This suggests to study the following solution space of multivariate linear homogeneous difference equations.

Definition 3.19. We set

$$\mathrm{DM}(X) := \{f : \mathbb{Z}^s \to \mathbb{C} | \nabla_Y f \equiv 0, \text{ for all } Y \in \mathcal{Y}(X)\}. \tag{3.24}$$

Note that the condition that $\nabla_Y f \equiv 0$ for all cocircuits $Y \in \mathcal{Y}(X)$ implies also that $\nabla_{Y'} f \equiv 0$ for all sublists $Y' \subseteq X$ such that $\mathrm{span}(X \backslash Y') \neq \mathbb{R}^s$, since any such sublist has to contain a cocircuit.

The functions in the space $\mathrm{DM}(X)$ are quasi–polynomials which are uniquely determined by the function values on certain sets, cf. [8] or [6]. Here, we give another basic proof for these properties. To this end, we first observe that the difference equations that appear in the definition of the space $\mathrm{DM}(X)$ lead to certain recurrence relations. Geometric considerations regarding the points involved in these recurrence relations allow us then to specify sets with the property that arbitrary function values assigned to the elements of such a set lead to a unique continuation to a function in the space $\mathrm{DM}(X)$. The method we use to prove this property, will furthermore allow us to show that the restriction of the vector partition Φ_X to a chamber of X agrees with a unique function in the space $\mathrm{DM}(X)$. Altogether, these results will provide us with a proof of an even stronger version of Theorem 3.18.

3.3.1 Recurrence Relations and Multivariate Difference Equations

Our starting point is the following observation.

Lemma 3.20. *Let $f : \mathbb{Z}^s \to \mathbb{C}$ be a function and $Y = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)$ a list of integral vectors $\boldsymbol{a}_i \in \mathbb{Z}^s$. Then*

$$\nabla_Y f(\boldsymbol{b}) = \sum_{S \subseteq Y} (-1)^{|S|} f(\boldsymbol{b} - \boldsymbol{a}_S), \tag{3.25}$$

where the vector $\boldsymbol{a}_S$ denotes the sum of the elements of the sublist $S \subseteq Y$.

Proof. We argue by induction on the number of elements of the list Y. In the base case, i.e. for $Y = (\boldsymbol{a})$, the assertion agrees with the definition of the backwards difference operator. For the induction step, we assume that (3.25) holds for all lists having less than n elements and conclude that then

$$\begin{aligned}
\nabla_Y f(\boldsymbol{b}) &= \nabla_{\boldsymbol{a}}\left(\nabla_{Y\setminus\{\boldsymbol{a}\}} f(\boldsymbol{b})\right) = \nabla_{\boldsymbol{a}}\left(\sum_{S\subseteq Y\setminus\{\boldsymbol{a}\}} (-1)^{|S|} f(\boldsymbol{b}-\boldsymbol{a}_S)\right) \\
&= \sum_{S\subseteq Y\setminus\{\boldsymbol{a}\}} (-1)^{|S|} f(\boldsymbol{b}-\boldsymbol{a}_S) - \sum_{S\subseteq Y\setminus\{\boldsymbol{a}\}} (-1)^{|S|} f(\boldsymbol{b}-\boldsymbol{a}_S-\boldsymbol{a}) \\
&= \sum_{S\subseteq Y} (-1)^{|S|} f(\boldsymbol{b}-\boldsymbol{a}_S).
\end{aligned}$$

□

Now we may deduce the following result, which turns out to be quite useful in the context of the functions in the space DM(X).

Corollary 3.21. *Suppose that the function $f : \mathbb{Z}^s \to \mathbb{C}$ solves the difference equation $\nabla_Y f(\boldsymbol{b}) = 0$ at a point $\boldsymbol{b} \in \mathbb{Z}^s$, and let $S' \subseteq Y$ be a sublist. Then the function value $f(\boldsymbol{b}-\boldsymbol{a}_{S'})$ is uniquely determined by the values the function f takes at the points in the set $\{\boldsymbol{b}-\boldsymbol{a}_S : S \subseteq Y, S \neq S'\}$ and can be calculated as*

$$f(\boldsymbol{b}-\boldsymbol{a}_{S'}) = \sum_{S\subseteq Y, S\neq S'} (-1)^{|S|+|S'|+1} f(\boldsymbol{b}-\boldsymbol{a}_S). \tag{3.26}$$

Example 3.22. Suppose that $X = (2,3)$. Since the sole cocircuit of this list is given by the list itself, we obtain that $\mathrm{DM}(2,3) = \{f : \mathbb{Z} \to \mathbb{C} : \nabla_{(2,3)} f(b) = 0 \text{ for all } b \in \mathbb{Z}\}$. Now observe that

$$\nabla_{(2,3)} f(b) = f(b) - f(b-2) - f(b-3) + f(b-5), \tag{3.27}$$

and the equation $\nabla_{(2,3)} f(b) = 0$ implies that $f(b) = f(b-2) + f(b-3) - f(b-5)$. This shows that a function $f \in \mathrm{DM}(2,3)$ is uniquely determined by the function values on a single set $(a + [0,5)) \cap \mathbb{Z}$ for some integer $a \in \mathbb{Z}$, and an arbitrary choice of function values on such a set enables us to calculate recursively all values of a unique function in the space DM(X). With regard to the generalization to arbitrary dimensions, observe that we may describe the integral points in the half–open interval $[0,5)$ as well as

$$[0,5) \cap \mathbb{Z} = (u - B(2,3)) \cap \mathbb{Z} \tag{3.28}$$

where $B(2,3) := \{\lambda_1 2 + \lambda_2 3, 0 \leq \lambda_i \leq 1\} = [0,5]$ and $u \in (4,5)$.

As to the difference equations in the definition of the space $\mathrm{DM}(X)$, Lemma 3.20 suggests to study the set $\{\boldsymbol{b} - \boldsymbol{a}_S : S \subseteq Y\}$, where Y denotes a cocircuit of the list X and $\boldsymbol{b} \in \mathbb{Z}^s$ an integral vector. In this context, the following observation is crucial.

Proposition 3.23. *There is a bijective correspondence between cocircuits $Y \in \mathcal{Y}(X)$ and the linear hyperplanes spanned by sublists of X. Namely, if $H \subseteq \mathbb{R}^s$ is a hyperplane spanned by a sublist of X, the sublist $X \backslash H$ has to be a cocircuit. On the other hand, a sublist $Y \subseteq X$ is a cocircuit if and only if the elements of the list $X \backslash Y$ generate a hyperplane, subsequently denoted H_{Y^c}.*

Definition 3.24. The hyperplanes spanned by sublists of X are rational, since we assumed the elements of the list X to be integer vectors. Therefore, given a cocircuit $Y \in \mathcal{Y}(X)$ we may choose an integral vector $\boldsymbol{h}_{Y^c} \in \mathbb{Z}^s$ such that the corresponding hyperplane $H_{Y^c} = \{\boldsymbol{x} \in \mathbb{R}^s : \langle \boldsymbol{h}_{Y^c}, \boldsymbol{x} \rangle = 0\}$. We define the essential hyperplane arrangement related to the list X as

$$\mathcal{C}(X) := \{H_{Y^c} : Y \in \mathcal{Y}(X)\}. \tag{3.29}$$

Remark 3.25. The actual choice of the vectors $\boldsymbol{h}_{Y^c}$ is not important, we just need to specify these vectors to distinguish the half–spaces $H^{>}_{Y^c} := \{\boldsymbol{x} \in \mathbb{R}^s : \langle \boldsymbol{h}_{Y^c}, \boldsymbol{x} \rangle > 0\}$ and $H^{<}_{Y^c} := \{\boldsymbol{x} \in \mathbb{R}^s : \langle \boldsymbol{h}_{Y^c}, \boldsymbol{x} \rangle < 0\}$.

Definition 3.26. Given a cocircuit $Y \subseteq X$, we partition the elements of the list X into three sublists, according to their location relative to the hyperplane H_{Y^c}. That is, we define sublists

$$\begin{aligned} Y_+ &:= \{\boldsymbol{a} \in X : \langle \boldsymbol{h}_{Y^c}, \boldsymbol{a} \rangle > 0\}, \\ Y_- &:= \{\boldsymbol{a} \in X : \langle \boldsymbol{h}_{Y^c}, \boldsymbol{a} \rangle < 0\} \text{ and} \\ Y^c &:= H_{Y^c} \cap X = X \backslash Y. \end{aligned} \tag{3.30}$$

Thus, Y_+ denotes the sublist of the elements of the list X which lie on the same side of the hyperplane H_{Y^c} as the vector $\boldsymbol{h}_{Y^c}$, the sublist Y_- consists of the those list elements which are located in the opposite half–space, and the sublist Y^c includes all elements of the list X which lie on the hyperplane H_{Y^c}.

Accordingly we define three vectors related to a sublist $S \subseteq X$ and a cocircuit $Y \subseteq X$ by setting

- $\boldsymbol{a}_{S\cap Y_+} := \sum_{\boldsymbol{a}\in S\cap Y_+} \boldsymbol{a}$, with the property that $\langle \boldsymbol{a}_{S\cap Y_+}, h_{Y^c}\rangle \geq 0$,
- $\boldsymbol{a}_{S\cap Y_-} := \sum_{\boldsymbol{a}\in S\cap Y_-} \boldsymbol{a}$, satisfying $\langle \boldsymbol{a}_{S\cap Y_-}, h_{Y^c}\rangle \leq 0$, and
- $\boldsymbol{a}_{S\cap Y^c} := \sum_{\boldsymbol{a}\in S\cap Y^c} \boldsymbol{a}$, with $\langle \boldsymbol{a}_{S\cap Y^c}, h_{Y^c}\rangle = 0$.

For any sublist $S \subseteq X$ we have that $\boldsymbol{a}_S = \boldsymbol{a}_{S\cap Y_+} + \boldsymbol{a}_{S\cap Y_-} + \boldsymbol{a}_{S\cap Y^c}$.

Example 3.27. We consider the list $X = ((1,0),(0,1),(1,1),(-1,1))$. Then the sublist $Y = ((1,0),(0,1),(-1,1))$ is a cocircuit. The corresponding hyperplane is given by $H_{Y^c} = \operatorname{span}((1,1))$. Choosing the vector $\boldsymbol{h}_{Y^c} = (-2,2)$, we obtain that the sublist $Y_+ = ((0,1),(-1,1))$ while $Y_- = ((1,0))$. Moreover, for the sublist $S = ((-1,1),(1,0))$ the vector $\boldsymbol{a}_{S\cap Y_+} = (-1,1)$ and $\boldsymbol{a}_{S\cap Y_-} = (1,0)$. Therefore, $\boldsymbol{a}_S = (0,1)$.

Using this notation, we gain the following results regarding the location of the points that appear in our recurrence relations.

Lemma 3.28. *Let $Y \subseteq X$ be a cocircuit and $\boldsymbol{b} \in \mathbb{Z}^s$ an integral vector. For any sublist $S \subseteq Y$ we have that the vector $\boldsymbol{b} - \boldsymbol{a}_S$ lies in the closed strip between the hyperplanes $\boldsymbol{b} - \boldsymbol{a}_{Y_+} + H_{Y^c}$ and $\boldsymbol{b} - \boldsymbol{a}_{Y_-} + H_{Y^c}$, i.e.*

$$\{\boldsymbol{b} - \boldsymbol{a}_S : S \subseteq Y\} \subseteq (\boldsymbol{b} - \boldsymbol{a}_{Y_+} + H^{\geq}_{Y^c}) \cap (\boldsymbol{b} - \boldsymbol{a}_{Y_-} + H^{\leq}_{Y^c}). \tag{3.31}$$

Proof. Given a sublist $S \subseteq Y$ we have to show that the inner product $\langle \boldsymbol{b} - \boldsymbol{a}_{Y_+}, \boldsymbol{h}_{Y^c}\rangle \leq \langle \boldsymbol{b} - \boldsymbol{a}_S, \boldsymbol{h}_{Y^c}\rangle \leq \langle \boldsymbol{b} - \boldsymbol{a}_{Y_-}, \boldsymbol{h}_{Y^c}\rangle$. To this end observe that the inner product

$$\begin{aligned}\langle \boldsymbol{a}_S, \boldsymbol{h}_{Y^c}\rangle &= \langle \boldsymbol{a}_{S\cap Y_+} + \boldsymbol{a}_{S\cap Y_-} + \boldsymbol{a}_{S\cap Y^c}, \boldsymbol{h}_{Y^c}\rangle \\ &= \underbrace{\langle \boldsymbol{a}_{S\cap Y_+}, \boldsymbol{h}_{Y^c}\rangle}_{\geq 0} + \underbrace{\langle \boldsymbol{a}_{S\cap Y_-}, \boldsymbol{h}_{Y^c}\rangle}_{\leq 0} + \underbrace{\langle \boldsymbol{a}_{S\cap Y^c}, \boldsymbol{h}_{Y^c}\rangle}_{=0}.\end{aligned} \tag{3.32}$$

Thus, $\langle \boldsymbol{a}_{Y_+}, \boldsymbol{h}_{Y^c}\rangle \geq \langle \boldsymbol{a}_{S\cap Y_+}, \boldsymbol{h}_{Y^c}\rangle \geq \langle \boldsymbol{a}_S, \boldsymbol{h}_{Y^c}\rangle \geq \langle \boldsymbol{a}_{S\cap Y_-}, \boldsymbol{h}_{Y^c}\rangle \geq \langle \boldsymbol{a}_{Y_-}, \boldsymbol{h}_{Y^c}\rangle$, and we obtain our result by the bilinarity of the inner product. □

Definition 3.29. We call the set

$$B(X) := \{\sum_{\boldsymbol{a}\in X} \lambda_{\boldsymbol{a}} \boldsymbol{a} : 0 \le \lambda_{\boldsymbol{a}} \le 1\}, \tag{3.33}$$

the *zonotope* associated with the list X.

Remark 3.30. If the elements of the list X form a basis of $\mathbb{R}^s$, the zonotope $B(X)$ is actually a parallelepiped.

Our interest in these sets is based on the following property.

Lemma 3.31. *The zonotope $B(X)$ is a polytope with hyperplane description*

$$B(X) = \bigcap_{Y\in\mathcal{Y}(X)} \Big((\boldsymbol{a}_{Y_+} + H_{Y^c}^{\le}) \cap (\boldsymbol{a}_{Y_-} + H_{Y^c}^{\ge}) \Big). \tag{3.34}$$

Moreover, the facets of the polytope $B(X)$, that are parallel to the hyperplane H_{Y^c}, are given by the translated zonotopes $\boldsymbol{a}_{Y_+} + B(Y^c)$ and $\boldsymbol{a}_{Y_-} + B(Y^c)$, and all facets of this polytope are obtained in this way.

Proof. Given a point $\boldsymbol{x} \in B(X)$, there are, by definition, reals $0 \le \mu_1, \ldots, \mu_d \le 1$ such that $\boldsymbol{x} = \sum_{i=1}^d \mu_i \boldsymbol{a}_i$. Now, let $Y \subseteq X$ be a cocircuit. Then the inner product

$$\begin{aligned} \langle \boldsymbol{x}, \boldsymbol{h}_{Y^c} \rangle &= \sum_{\boldsymbol{a}_i\in Y_+} \mu_i \langle \boldsymbol{a}_i, \boldsymbol{h}_{Y^c}\rangle + \sum_{\boldsymbol{a}_j\in Y_-} \mu_j \langle \boldsymbol{a}_j, \boldsymbol{h}_{Y^c}\rangle + \sum_{\boldsymbol{a}_k\in Y^c} \mu_k \langle \boldsymbol{a}_k, \boldsymbol{h}_{Y^c}\rangle \\ &= \underbrace{\sum_{\boldsymbol{a}_i\in Y_+} \mu_i \langle \boldsymbol{a}_i, \boldsymbol{h}_{Y^c}\rangle}_{\ge 0} + \underbrace{\sum_{\boldsymbol{a}_j\in Y_-} \mu_j \langle \boldsymbol{a}_j, \boldsymbol{h}_{Y^c}\rangle}_{\le 0} + 0. \end{aligned} \tag{3.35}$$

Moreover, note that

$$\begin{aligned} 0 &\le \sum_{\boldsymbol{a}_i\in Y_+} \mu_i \langle \boldsymbol{a}_i, \boldsymbol{h}_{Y^c}\rangle \le \sum_{\boldsymbol{a}_i\in Y_+} \langle \boldsymbol{a}_i, \boldsymbol{h}_{Y^c}\rangle = \langle \boldsymbol{a}_{Y_+}, \boldsymbol{h}_{Y^c}\rangle \text{ and} \\ 0 &\ge \sum_{\boldsymbol{a}_i\in Y_-} \mu_i \langle \boldsymbol{a}_i, \boldsymbol{h}_{Y^c}\rangle \ge \sum_{\boldsymbol{a}_i\in Y_-} \langle \boldsymbol{a}_i, \boldsymbol{h}_{Y^c}\rangle = \langle \boldsymbol{a}_{Y_-}, \boldsymbol{h}_{Y^c}\rangle. \end{aligned} \tag{3.36}$$

Combining (3.35) and (3.36) we obtain the estimation

$$\langle \boldsymbol{a}_{Y_-}, \boldsymbol{h}_{Y^c}\rangle \le \sum_{\boldsymbol{a}_i\in Y_-} \mu_i \langle \boldsymbol{a}_i, \boldsymbol{h}_{Y^c}\rangle \le \langle \boldsymbol{x}, \boldsymbol{h}_{Y^c}\rangle \le \sum_{\boldsymbol{a}_i\in Y_+} \mu_i \langle \boldsymbol{a}_i, \boldsymbol{h}_{Y^c}\rangle \le \langle \boldsymbol{a}_{Y_+}, \boldsymbol{h}_{Y^c}\rangle, \tag{3.37}$$

for all points $\boldsymbol{x} \in B(X)$ and all cocircuits $Y \subseteq X$. Our assertion follows since, furthermore, the intersections with the hyperplanes

$$B(X) \cap (\boldsymbol{a}_{Y_+} + H_{Y^c}) = \big(\boldsymbol{a}_{Y_+} + B(Y^c)\big) \text{ and} \tag{3.38}$$

$$B(X) \cap (\boldsymbol{a}_{Y_-} + H_{Y^c}) = \big(\boldsymbol{a}_{Y_-} + B(Y^c)\big) \tag{3.39}$$

are non–empty sets of dimension $d-1$. To verify the assertion (3.38), first note that $\big(\boldsymbol{a}_{Y_+} + B(Y^c)\big) \subseteq \big(\boldsymbol{a}_{Y_+} + B(Y^c)\big)$, by definition. Moreover, suppose that the point $\boldsymbol{p} \in B(X) \cap (\boldsymbol{a}_{Y_+} + H_{Y^c})$. That is, there exist a vector $\boldsymbol{h} \in H_{Y^c}$ and $\boldsymbol{a}_1 \in B(Y_+), a_2 \in B(Y_-), \boldsymbol{a}_3 \in B(Y^c)$ respectively, such that

$$\boldsymbol{a}_{Y_+} + \boldsymbol{h} = \boldsymbol{p} = \boldsymbol{a}_1 + \boldsymbol{a}_2 + \boldsymbol{a}_3. \tag{3.40}$$

Subtracting $\boldsymbol{a}_{Y_+}$ from both sides of this equation yields that

$$\boldsymbol{h} = \boldsymbol{a}_1' + \boldsymbol{a}_2 + \boldsymbol{a}_3, \tag{3.41}$$

for some $\boldsymbol{a}_1' \in -B(Y_+)$. Now, we consider the inner product

$$\begin{aligned} \langle \boldsymbol{h}, \boldsymbol{h}_{Y^c} \rangle &= \langle \boldsymbol{a}_1', \boldsymbol{h}_{Y^c} \rangle + \langle \boldsymbol{a}_2, \boldsymbol{h}_{Y^c} \rangle + \langle \boldsymbol{a}_3, \boldsymbol{h}_{Y^c} \rangle \\ \Leftrightarrow 0 &= \underbrace{\langle \boldsymbol{a}_1', \boldsymbol{h}_{Y^c} \rangle}_{\leq 0} + \underbrace{\langle \boldsymbol{a}_2, \boldsymbol{h}_{Y^c} \rangle}_{\leq 0} \end{aligned} \tag{3.42}$$

This implies that $\boldsymbol{a}_1' = \boldsymbol{a}_2 = 0$ and we obtain that $\boldsymbol{h} = \boldsymbol{a}_3$. Hence, we have that $\boldsymbol{p} \in \boldsymbol{a}_{Y_-} + B(Y^c)$. Equation (3.39) follows along the same lines.

□

Corollary 3.32. *Let $\boldsymbol{b} \in \mathbb{Z}^s$ be an integral vector. Then*

$$\bigcap_{Y \in \mathcal{Y}(X)} \Big((\boldsymbol{b} - \boldsymbol{a}_{Y_+} + H^{\leq}_{Y^c}) \cap (\boldsymbol{b} - \boldsymbol{a}_{Y_-} + H^{\geq}_{Y^c}) \Big) = \boldsymbol{b} - B(X). \tag{3.43}$$

Let $\boldsymbol{b} \in \mathbb{Z}^s$ be an integral vector and $Y \subseteq X$ a cocircuit. Then observe that the translated zonotope $\boldsymbol{b} - B(X)$ contains all points in the set $\{\boldsymbol{b} - \boldsymbol{a}_S : S \subseteq Y\}$.

Thus, a function $f \in \mathrm{DM}(X)$ can not take arbitrary values on the integral points in the translated zonotope $\boldsymbol{b} - B(X)$, because, in general, at least the difference equations $\nabla_Y f(\boldsymbol{b}) = 0$ would not be valid, c.f Corollary 3.21. It turns out that we obtain a set with the property that an arbitrary choice of function values on the integer points in this set does not contradict any of the difference equations $\nabla_Y f(\boldsymbol{z}) = 0$, if we shift the zonotopes $\boldsymbol{b} - B(X)$ by a vector $\boldsymbol{u} \in \mathbb{R}^s$ such that the translated zonotope $\boldsymbol{u} + \boldsymbol{b} - B(X)$

- contains all interior integral points of the polytope $\boldsymbol{b} - B(X)$,
- includes for each cocircuit $Y \subseteq X$ at most the integral points of one of the facets of $\boldsymbol{b} - B(X)$ which is parallel to the hyperplane H_{Y^c},
- and there are no integral points on the boundary of the translated zonotope.

Example 3.33. We display the zonotope $-B(X)$ and the shifted set $\boldsymbol{u} - B(X)$ in the case when the list $X = ((1,0),(0,1),(1,1))$ and $\boldsymbol{u} = (1/4, 1/2)$ in Figure 3.4.

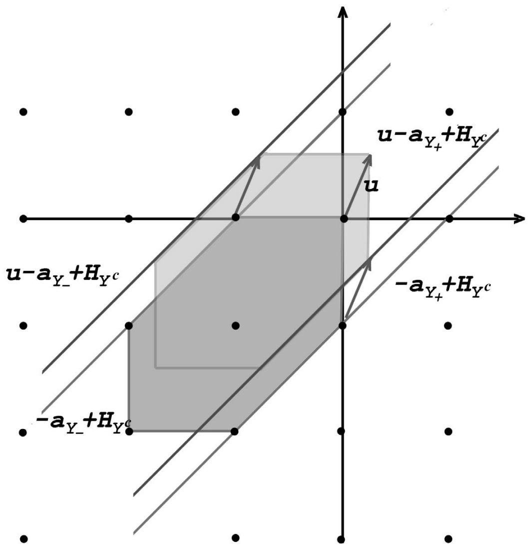


Figure 3.4: $-B(X)$ and $\boldsymbol{u} - B(X)$, $Y = (1,0),(0,1))$

To formalize the previous discussion, we introduce the following notion.

Definition 3.34. Consider the hyperplanes obtained by translating the elements of the essential hyperplane arrangement of the list X by integral vectors $\boldsymbol{z} \in \mathbb{Z}^s$.

We call a maximal connected subset

$$\mathfrak{c} \subseteq \mathbb{R}^s \setminus \bigcup_{\substack{\boldsymbol{z} \in \mathbb{Z}^s \\ H_{Y^c} \in \mathcal{C}(X)}} (\boldsymbol{z} + H_{Y^c}) \tag{3.44}$$

an X*–region.* Furthermore, we call an X–region $\mathfrak{c} \subseteq \mathbb{R}^s$ *initial* if and only if its closure, $\bar{\mathfrak{c}}$, contains the origin.

Remark 3.35. Note that the translated set $\mathfrak{c} - \boldsymbol{z}$ is again an X–region, for any X–region $\mathfrak{c} \subseteq \mathbb{R}^s$ and all integral vectors $\boldsymbol{z} \in \mathbb{Z}^s$.

Example 3.36. We display the X–regions of the list $X = ((1,0),(0,1),(1,1))$ in Figure 3.5. Initial X–regions are shades in gray.

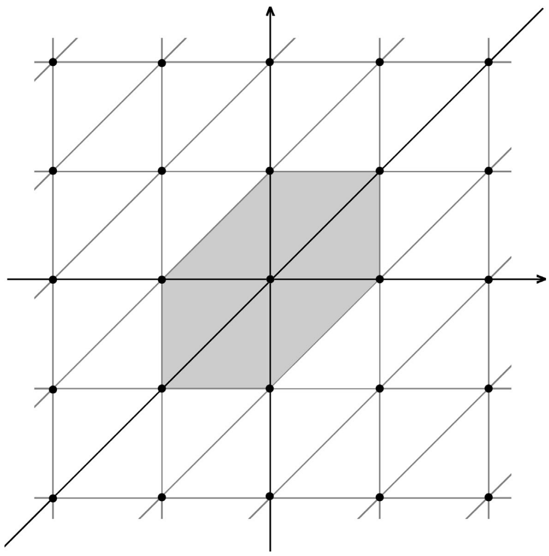


Figure 3.5: $X-$ regions

Definition 3.37. Given a vector $\boldsymbol{u} \in \mathbb{R}^s$ we denote the set of the integral points in the translated zonotope $\boldsymbol{u} - B(X)$ by

$$\delta(\boldsymbol{u}|X) := (\boldsymbol{u} - B(X)) \cap \mathbb{Z}^s = \{\boldsymbol{z} \in \mathbb{Z}^s : (\boldsymbol{u} - \boldsymbol{z}) \in B(X)\}. \tag{3.45}$$

Lemma 3.38. *Let $\mathfrak{c} \subseteq \mathbb{R}^s$ be an X–region. Then $\delta(\boldsymbol{u}_1|X) = \delta(\boldsymbol{u}_2|X)$ for all elements $\boldsymbol{u}_1$ and $\boldsymbol{u}_2 \in \mathfrak{c}$.*

Proof. Assuming the contrary, there exists a point $\boldsymbol{z} \in \delta(\boldsymbol{u}_1|X) \backslash \delta(\boldsymbol{u}_2|X)$, i.e. the point $\boldsymbol{u}_1 - \boldsymbol{z} \in B(X)$ while $\boldsymbol{u}_2 - \boldsymbol{z} \notin B(X)$. This implies that the points $\boldsymbol{u}_1 - \boldsymbol{z}$ and $\boldsymbol{u}_2 - \boldsymbol{z}$ lie on opposite sides of a supporting hyperplane of the zonotope $B(X)$. Since all these hyperplanes are integral translates of elements of the essential hyperplane arrangement of the list X, the points $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ cannot lie within the same X–region. $\square$

This justifies the following definition.

Definition 3.39. Given an X–region $\mathfrak{c} \subseteq \mathbb{R}^s$ we set $\delta(\mathfrak{c}|X) := \delta(\boldsymbol{u}|X)$, for $\boldsymbol{u} \in \mathfrak{c}$.

Now, we note some basic properties of these sets, which are more or less immediate consequences of the definition.

Lemma 3.40. *Let $\mathfrak{c} \subseteq \mathbb{R}^s$ be an X–region and $\boldsymbol{a} \in \mathbb{Z}^s$ an integral vector.*

- *For the X–region $\mathfrak{c} - \boldsymbol{a}$ the set $\delta(\mathfrak{c} - \boldsymbol{a}|X) = \delta(\mathfrak{c}|X) - \boldsymbol{a}$.*
- *The set $\delta(\mathfrak{c}|X)$ contains the origin if and only if the X–region $\mathfrak{c}$ lies in $B(X)$.*
- *An integral point $\boldsymbol{z} \in \mathbb{Z}^s$ is an element of the set $\delta(\mathfrak{c}|X)$ if and only if the translated X–region $\mathfrak{c} - \boldsymbol{z}$ lies within the zonotope $B(X)$.*

Proof. In order to verify the first assertion, note that the condition that a point $\boldsymbol{z} \in \mathbb{Z}^s$ lies in the set $\delta(\boldsymbol{u} - \boldsymbol{a}|X)$ is, by definition, that the point $\boldsymbol{u} - \boldsymbol{a} - \boldsymbol{z} \in B(X)$. This is at the same time the criterion that the point $\boldsymbol{a} + \boldsymbol{z} \in \delta(\boldsymbol{u}|X)$. In the same way we obtain that the origin $\mathbf{0} \in \delta(\mathfrak{c}|X)$ if and only if $\boldsymbol{u} - \mathbf{0} \in B(X)$ for all $\boldsymbol{u} \in \mathfrak{c}$. The third item is now no more than a corollary of the previous two. $\square$

3.3.2 Wall Crossing

In a next step we specify a method which enables us to calculate all values of a function in the space $\mathrm{DM}(X)$ recursively from the function values on a set $\delta(\mathfrak{c}|X)$ associated with an X–region $\mathfrak{c}$.

Definition 3.41. We call two X-regions $\mathfrak{c}$ and $\mathfrak{g} \subseteq \mathbb{R}^s$ *adjacent* if and only if their closures intersect in an $(s-1)$–dimensional face. Moreover, given adjacent X–regions $\mathfrak{c}$ and $\mathfrak{g}$ there exists a unique hyperplane $H_{Y^c} \in \mathcal{C}(X)$ and a unique integral vector $\boldsymbol{a} \in \mathbb{Z}^s$ such that $\mathfrak{f} := (\bar{\mathfrak{c}} \cap \bar{\mathfrak{g}})^\circ \subsetneq \boldsymbol{a} + H_{Y^c}$. Then, we call the X–regions $\mathfrak{c}$ and $\mathfrak{g}$ *Y–separated.*

Proposition 3.42. *Let $\mathfrak{c}$ and $\mathfrak{g} \subseteq \mathbb{R}^s$ be two adjacent, Y–separated X–regions. We assume, without loss of generality, that the X–region $\mathfrak{c}$ lies in the half-space $\boldsymbol{a} + H^{>}_{Y^c}$ while $\mathfrak{g}$ is contained in the opposite half-space $\boldsymbol{a} + H^{<}_{Y^c}$. Then we have the following result regarding the differences of the sets $\delta(\mathfrak{g}|X)$ and $\delta(\mathfrak{c}|X)$. If a point*

$$\begin{aligned} &\boldsymbol{z} \in \delta(\mathfrak{g}|X)\backslash\delta(\mathfrak{c}|X) \text{ then } \boldsymbol{z} + (\boldsymbol{a}_{S\cap Y_+} - \boldsymbol{a}_{S\cap Y_-}) \in \delta(\mathfrak{c}|X), \text{ and for} \\ &\boldsymbol{z} \in \delta(\mathfrak{c}|X)\backslash\delta(\mathfrak{g}|X) \text{ we have that } \boldsymbol{z} - (\boldsymbol{a}_{S\cap Y_+} - \boldsymbol{a}_{S\cap Y_-}) \in \delta(\mathfrak{g}|X). \end{aligned} \tag{3.46}$$

for all non–empty sublists $S \subseteq Y$.

Proof. We carry out the proof of the first statement, the second follows mostly along the same lines. To this end, note that $\delta(\mathfrak{h}|X) = \delta(\mathfrak{h} - \boldsymbol{z}|X) + \boldsymbol{z}$ for any X–region $\mathfrak{h} \subseteq \mathbb{R}^s$ and all integral vectors $\boldsymbol{z} \in \mathbb{Z}^s$, by Lemma 3.40. Thus, we obtain that

$$\boldsymbol{z} \in \delta(\mathfrak{g}|X)\backslash\delta(\mathfrak{c}|X) \Leftrightarrow \mathbf{0} \in \delta(\mathfrak{g} - \boldsymbol{z}|X)\backslash\delta(\mathfrak{c} - \boldsymbol{z}|X) =: \delta(\mathfrak{g}'|X)\backslash\delta(\mathfrak{c}'|X). \tag{3.47}$$

This equivalence implies that it is sufficient for our purpose to show that for all non–empty sublists $S \subseteq Y$ the point $(\boldsymbol{a}_{S\cap Y_+} - \boldsymbol{a}_{S\cap Y-})$ is an element of the set $\delta(\mathfrak{c}'|X)$. Now, recall that the zonotope

$$B(X) = \bigcap_{Y\in\mathcal{Y}(X)} \Big((\boldsymbol{a}_{Y_+} + H^{\leq}_{Y^c}) \cap (\boldsymbol{a}_{Y_-} + H^{\geq}_{Y^c})\Big). \tag{3.48}$$

By Lemma 3.40, the origin is an element of the set difference $\delta(\mathfrak{g}'|X)\backslash\delta(\mathfrak{c}'|X)$ if and only if the X–region $\mathfrak{g}'$ lies within the polytope $B(X)$ while the intersection $\mathfrak{c}' \cap B(X)$ is empty. Therefore, and by the definition of $\mathfrak{c}'$ and $\mathfrak{g}'$, we obtain that the X–regions $\mathfrak{g}' \subseteq \boldsymbol{a}_{Y_+} + H^{\leq}_{Y^c}$ while $\mathfrak{c}' \subseteq \boldsymbol{a}_{Y_+} + H^{\geq}_{Y^c}$. Thus, the set $\mathfrak{f}' := (\overline{\mathfrak{c}'} \cap \overline{\mathfrak{g}'})^\circ$ has to lie on the facet of the zonotope $B(X)$ given by $\boldsymbol{a}_{Y_+} + B(Y^c)$. This implies, for all $\boldsymbol{x} \in \mathfrak{f}'$, all $\varepsilon > 0$ and all non–empty sublist $S \subseteq Y$, that the point $\boldsymbol{x} + \varepsilon(\boldsymbol{a}_{S\cap Y_+} - \boldsymbol{a}_{S\cap Y_-})$ lies outside $B(X)$. We may furthermore choose $1 > \varepsilon > 0$ such that $\boldsymbol{y} := \boldsymbol{x} + \varepsilon(\boldsymbol{a}_{S\cap Y_+} - \boldsymbol{a}_{S\cap Y_-})$ is an element of the X–region $\mathfrak{c}'$. Now observe that

$$\begin{aligned}
\boldsymbol{y} - (\boldsymbol{a}_{S\cap Y_+} - \boldsymbol{a}_{S\cap Y_-}) &= \boldsymbol{x} - (1-\varepsilon)(\boldsymbol{a}_{S\cap Y_+} - \boldsymbol{a}_{S\cap Y_-}) \\
&= (\boldsymbol{x} - \boldsymbol{a}_{Y_+}) + \boldsymbol{a}_{Y_+} - \boldsymbol{a}_{S\cap Y_+} + \boldsymbol{a}_{S\cap Y_-} + \varepsilon(\boldsymbol{a}_{S\cap Y_+} - \boldsymbol{a}_{S\cap Y_-}) \\
&= (\boldsymbol{x} - \boldsymbol{a}_{Y_+}) + \boldsymbol{a}_{Y_+\backslash S} + (1-\varepsilon)\boldsymbol{a}_{S\cap Y_-} + \varepsilon\boldsymbol{a}_{S\cap Y_+} \\
&\in B(Y^c) + \boldsymbol{a}_{Y_+\backslash S} + (1-\varepsilon)\boldsymbol{a}_{S\cap Y_-} + \varepsilon\boldsymbol{a}_{S\cap Y_+} \subseteq B(X).
\end{aligned}$$

Since this is exactly the criterion that the point $(\boldsymbol{a}_{S\cap Y_+} - \boldsymbol{a}_{S\cap Y_-})$ lies in the set $\delta(\mathfrak{c}'|X)$, the proof is complete. ☐

Proposition 3.43. *Let $\mathfrak{c}$ and $\mathfrak{g} \subseteq \mathbb{R}^s$ be two adjacent, Y–separated X–regions and $\mathfrak{f} := (\overline{\mathfrak{c}} \cap \overline{\mathfrak{g}})^\circ$. Again we assume that the X–region $\mathfrak{c}$ lies in the half–space $\boldsymbol{a} + H^{\geq}_{Y^c}$ while $\mathfrak{g}$ is contained in the opposite half–space $\boldsymbol{a} + H^{\leq}_{Y^c}$. Then*

$$\begin{aligned}
&\delta(\mathfrak{g}|X)\backslash\delta(\mathfrak{c}|X) = \delta(\mathfrak{f}|Y^c) - \boldsymbol{a}_{Y_+} \subseteq \boldsymbol{a} + H_{Y^c} - \boldsymbol{a}_{Y_+} \text{ and} \\
&\delta(\mathfrak{c}|X)\backslash\delta(\mathfrak{g}|X) = \delta(\mathfrak{f}|Y^c) - \boldsymbol{a}_{Y_-} \subseteq \boldsymbol{a} + H_{Y^c} - \boldsymbol{a}_{Y_-}.
\end{aligned} \tag{3.49}$$

Proof. We start with a basic observation. Assume that the point $\boldsymbol{z} \in \mathbb{Z}^s$ lies in the union $\delta(\mathfrak{g}|X) \cup \delta(\mathfrak{c}|X)$. Then, as a consequence of Lemma 3.40, the origin has to be contained in the union $\delta(\mathfrak{g} - \boldsymbol{z}|X) \cup \delta(\mathfrak{c} - \boldsymbol{z}|X)$. Therefore, we obtain, by Lemma 3.40, that at least one of the X–regions $\mathfrak{c} - \boldsymbol{z}$ and $\mathfrak{g} - \boldsymbol{z}$ lies within the polytope $B(X)$. This implies, moreover, that $\mathfrak{f} - \boldsymbol{z} \subseteq B(X)$.

Now, the set $\mathfrak{f} - \boldsymbol{z}$ lies in the interior of $B(X)$ if and only if both X–regions, $\mathfrak{c} - \boldsymbol{z}$ as well as $\mathfrak{g} - \boldsymbol{z}$, are subsets of the zonotope $B(X)$. This can only be the case if the point $\boldsymbol{z}$ is an element of the intersection $\delta(\mathfrak{g}|X) \cap \delta(\mathfrak{c}|X)$, c.f. Lemma 3.40.

If $\boldsymbol{z} \in \delta(\mathfrak{g}|X) \backslash \delta(\mathfrak{c}|X)$, the X–region $\mathfrak{g} - \boldsymbol{z}$ lies within the zonotope $B(X)$ while the intersection $(\mathfrak{c} - \boldsymbol{z}) \cap B(X)$ is empty. Consequently, $(\overline{(\mathfrak{c} - \boldsymbol{z})} \cap \overline{(\mathfrak{g} - \boldsymbol{z})})^\circ = \mathfrak{f} - \boldsymbol{z}$ lies on a facet of the polytope $B(X)$. By our assumption on the positions of the X–regions $\mathfrak{c}$ and $\mathfrak{g}$, we obtain that the set $\mathfrak{f} - \boldsymbol{z}$ is then located on the facet $B(Y^c) + \boldsymbol{a}_{Y_+}$. Then, the set $\mathfrak{f} - \boldsymbol{z} - \boldsymbol{a}_{Y_+}$ lies in the zonotope $B(Y^c)$, which is exactly the criterion that the point $\boldsymbol{z} + \boldsymbol{a}_{Y_+}$ is an element of the set $\delta(\mathfrak{f}|Y^c)$. This means that $\boldsymbol{z} \in \delta(\mathfrak{f}|Y^c) - \boldsymbol{a}_{Y_+}$, i.e. we have the inclusion $\delta(\mathfrak{g}|X) \backslash \delta(\mathfrak{c}|X) \subseteq \delta(\mathfrak{f}|Y^c) - \boldsymbol{a}_{Y_+}$. On the other hand, suppose that $\boldsymbol{z} \in \delta(\mathfrak{f}|Y^c) - \boldsymbol{a}_{Y_+}$. That is,

$$\boldsymbol{f} - \boldsymbol{z} \in B(Y^c) + \boldsymbol{a}_{Y_+} \subseteq B(X) \subseteq H^{\leq}_{Y^c} + \boldsymbol{a}_{Y_+}, \tag{3.50}$$

for some $\boldsymbol{f} \in \mathfrak{f}$. Hence, we may choose a vector $\boldsymbol{u} \in \mathbb{R}^s$ with $\langle \boldsymbol{u}, \boldsymbol{h}_{Y^c} \rangle < 0$ such that the point $\boldsymbol{f} + \boldsymbol{u} - \boldsymbol{z} \in B(X)$ and $\boldsymbol{f} + \boldsymbol{u} \in \mathfrak{g}$, according to our assumption on the positions of the X–regions $\mathfrak{c}$ and $\mathfrak{g}$. Therefore, we conclude that $\boldsymbol{z} \in \delta(\mathfrak{g}|X)$.

Given a point $\boldsymbol{z} \in \delta(\mathfrak{c}|X) \backslash \delta(\mathfrak{g}|X)$, the X–region $\mathfrak{c} - \boldsymbol{z}$ lies within the zonotope $B(X)$ while $(\mathfrak{g} - \boldsymbol{z}) \cap B(X) = \emptyset$. Then, we have that the set $\mathfrak{f} - \boldsymbol{z}$ is located on the face $B(Y^c) + \boldsymbol{a}_{Y_-}$, which implies that $\boldsymbol{z} \in \delta(\mathfrak{f}|Y^c) - \boldsymbol{a}_{Y_-}$. □

Theorem 3.44 (Wall crossing formula). *Given two adjacent, Y–separated X–regions $\mathfrak{c}$ and $\mathfrak{g} \subseteq \mathbb{R}^s$, such that the X–region $\mathfrak{c}$ lies within the half–space $\boldsymbol{a} + H^{>}_{Y^c}$ while $\mathfrak{g}$ is located in the opposite half–space $\boldsymbol{a} + H^{<}_{Y^c}$. Moreover, suppose that a function $f : \mathbb{Z}^s \to \mathbb{C}$ solves the difference equation $\nabla_Y f(\boldsymbol{b}) = 0$ at all integral points $\boldsymbol{b} \in \delta(\mathfrak{f}|Y^c)$.*

Then we can uniquely calculate the values the function f takes on the points in the set $\delta(\mathfrak{g}|X)$ from the corresponding function values on the set $\delta(\mathfrak{c}|X)$, and vice versa.

Proof. Our proof relies on the previous two propositions. We specify how the function values on the set $\delta(\mathfrak{g}|X)$ can be calculated from the values the function f takes on $\delta(\mathfrak{c}|X)$. The proof of the corresponding assertion for the set $\delta(\mathfrak{c}|X)$ follows along the same lines and is omitted here. By Proposition 3.43, we have that

$$\delta(\mathfrak{g}|X)\backslash\delta(\mathfrak{c}|X) = \delta(\mathfrak{f}|Y^c) - \boldsymbol{a}_{Y_+}. \tag{3.51}$$

This shows that the point $\boldsymbol{z}+\boldsymbol{a}_{Y_+}$ lies in the set $\delta(\mathfrak{f}|Y^c)$, for all $\boldsymbol{z} \in \delta(\mathfrak{g}|X)\backslash\delta(\mathfrak{c}|X)$. Thus, we obtain, by our assumptions on the function f, that then

$$0 = \nabla_Y f(\boldsymbol{z}+\boldsymbol{a}_{Y_+}) = \sum_{S\subseteq Y}(-1)^{|S|}f(\boldsymbol{z}+\boldsymbol{a}_{Y_+}-\boldsymbol{a}_S), \tag{3.52}$$

which can be rephrased as

$$f(\boldsymbol{z}) = \sum_{S\subseteq Y, S\neq Y_+}(-1)^{|S|+|Y_+|+1}f(\boldsymbol{z}+\boldsymbol{a}_{Y_+}-\boldsymbol{a}_S). \tag{3.53}$$

Therefore, it remains to show that the point $\boldsymbol{z}+\boldsymbol{a}_{Y_+}-\boldsymbol{a}_S$ is an element of $\delta(\mathfrak{c}|X)$, for any sublist $S\subseteq Y$ with $S\neq Y_+$. Observe that

$$\boldsymbol{z}+\boldsymbol{a}_{Y_+}-\boldsymbol{a}_S = \boldsymbol{z}+\boldsymbol{a}_{Y_+}-\boldsymbol{a}_{S\cap Y_+}-\boldsymbol{a}_{S\cap Y_-} = \boldsymbol{z}+\boldsymbol{a}_{Y_+\backslash S}-\boldsymbol{a}_{S\cap Y_-}. \tag{3.54}$$

Recall that any sublist $S\subseteq Y$ equals the disjoint union $S=(S\cap Y_+)\cup(S\cap Y_-)$, and we set $S' := Y_+\backslash(S\cap Y_-)$, which is a sublist of Y. The sublist S' is non–empty as long as $S\neq Y_+$. Moreover, the vector $\boldsymbol{a}_{S'\cap Y_+} = \boldsymbol{a}_{Y_+\backslash S}$ and $\boldsymbol{a}_{S'\cap Y_-} = \boldsymbol{a}_{S\cap Y_-}$. Accordingly,

$$\boldsymbol{z}+\boldsymbol{a}_{Y_+}-\boldsymbol{a}_S = \boldsymbol{z}+\boldsymbol{a}_{S'\cap Y_+}-\boldsymbol{a}_{S'\cap Y_-}. \tag{3.55}$$

Therefore, by Proposition 3.42, the set $\{\boldsymbol{z}+\boldsymbol{a}_{Y_+}-\boldsymbol{a}_S : S\subseteq Y, S\neq Y_+\}$ is a subset of $\delta(\mathfrak{c}|X)$, for any element $\boldsymbol{z}\in\delta(\mathfrak{g}|X)\backslash\delta(\mathfrak{c}|X)$.

□

Theorem 3.45. *Let $\mathfrak{c}\subseteq\mathbb{R}^s$ be an X–region. Then any function $f\in \mathrm{DM}(X)$ is uniquely determined by the function values on the set $\delta(\mathfrak{c}|X)$.*

Proof. Given an integral point $\boldsymbol{z}\in\mathbb{Z}^s$, there exists at least one X–region $\mathfrak{g}\subseteq\mathbb{R}^s$ such that $\boldsymbol{z}\in\delta(\mathfrak{g}|X)$. Moreover, starting in $\mathfrak{c}\subseteq\mathbb{R}^s$ we can reach any X–region of $\mathbb{R}^s$ by crossing a finite number of hyperplanes $\boldsymbol{a}_i+H_{Y_i^c}$, where $\boldsymbol{a}_i\in\mathbb{Z}^s$ denotes an appropriate integral vector and $Y_i \subseteq X$ a cocircuit. Thus, there exists a finite sequence of X–regions, starting with $\mathfrak{g}_0 = \mathfrak{c}$ and ending with $\mathfrak{g}_d = \mathfrak{g}$, where consecutive sequence elements are adjacent X–regions. Note that a function $f \in \mathrm{DM}(X)$ solves the difference equation $\nabla_{Y_i} f(\boldsymbol{b}) = 0$ at all points $\boldsymbol{b} \in \mathbb{Z}^s$, for all cocircuits $Y_i \subseteq X$. Therefore, we can calculate the value $f(\boldsymbol{z})$ successively from the function values at the elements of $\delta(\mathfrak{c}|X)$ by the wall crossing formula, Theorem 3.44. □

3.3.3 The Functions in the Space DM(X)

We need to prove two more properties of the space $\mathrm{DM}(X)$. On the one hand, we will show that the functions in this space are given by quasi–polynomials, on the other hand we need to prove that any complex valued function on a set $\delta(\mathfrak{c}|X)$ possesses a unique continuation to a function in the space $\mathrm{DM}(X)$.

Recall that a quasi–polynomial agrees with a polynomial on the cosets of some sub–lattice of $\mathbb{Z}^s$, cf. Section 1.6. Here, we define a lattice which works just fine for our needs, but it is in general not the finest sub–lattice of $\mathbb{Z}^s$ for which one may prove the according property.

Definition 3.46. We define the *bases of the list* X as the elements of the set

$$\mathcal{B}(X) := \{Z \subseteq X : |Z| = s, \mathrm{span}(Z) = \mathbb{R}^s\}. \tag{3.56}$$

Furthermore, the *determinant of a basis* is the absolute value of the determinant of the matrix whose column vectors are given by the element of this sublist.

Definition 3.47. We define the basic lattice associated with the list X, subsequently denoted by $\hat{\Lambda}_X$, as the sub–lattice of $\mathbb{Z}^s$ consisting of all integer vectors whose coordinates, with respect to the standard basis of $\mathbb{R}^s$, are all divisible by the product of the determinants of all the bases of the list X.

Definition 3.48. If the elements of the list $Z = (\boldsymbol{a}_{i_1}, \dots, \boldsymbol{a}_{i_s})$ of integral vectors form a basis of $\mathbb{R}^s$, we denote by Λ_Z the sub–lattice of $\mathbb{Z}^s$ which is generated by this basis.

Proposition 3.49. *Suppose the integral vectors $\boldsymbol{a}_{i_1}, \dots, \boldsymbol{a}_{i_s}$ generate the lattice $\Lambda \subseteq \mathbb{Z}^s$. We denote by Z the integral $s \times s$ matrix with these vectors as its columns. The matrix Z is invertible, whence there exists an $s \times s$ integer matrix C such that $Z^{-1} = C/\det(Z)$. Denoting by $\boldsymbol{c}_1, \dots, \boldsymbol{c}_s$ the row vectors of this matrix C, we obtain that any vector*

$\boldsymbol{x} \in \mathbb{R}^s$ has a unique representation as

$$\boldsymbol{x} = \lambda_{1,Z}(\boldsymbol{x})\ \boldsymbol{a}_{i_1} + \lambda_{2,Z}(\boldsymbol{x})\ \boldsymbol{a}_{i_2} + \ldots + \lambda_{s,Z}(\boldsymbol{x})\ \boldsymbol{a}_{i_s}, \tag{3.57}$$

where the real $\lambda_{j,Z}(\boldsymbol{x}) = \langle \boldsymbol{c}_j, \boldsymbol{x} \rangle / \det(Z)$, for $1 \le j \le s$.

Using this notations, we may now state some properties of the basic lattice $\hat{\Lambda}_X$, which are essential for our proof of the quasi–polynomiality of the functions in $\mathrm{DM}(X)$.

Lemma 3.50. *The basic lattice $\hat{\Lambda}_X$ is a sub–lattice of all lattices generated by bases of the list X.*

Proof. Suppose that the sublist $Z = (\boldsymbol{a}_{i_1}, \ldots, \boldsymbol{a}_{i_s}) \subseteq X$ is a basis of X, and let $\boldsymbol{x}$ be an element of the basic lattice $\hat{\Lambda}_X$. Since $\det(Z)$ divides each coordinate of the vector $\boldsymbol{x}$, the reals $\lambda_{i,Z}(\boldsymbol{x})$, as introduced in Proposition 3.49, are integers, for $1 \le i \le s$. This shows that all elements $\boldsymbol{x} \in \hat{\Lambda}_X$ are contained in the lattice Λ_Z. □

Corollary 3.51. *Suppose that the sublist $Z = (\boldsymbol{a}_{i_1}, \ldots, \boldsymbol{a}_{i_s}) \subseteq X$ is a basis of X. Given an index $1 \le j \le s$, the function $\boldsymbol{x} \mapsto \{\lambda_{j,Z}(\boldsymbol{x})\}$ is constant on each coset of $\mathbb{Z}^s/\hat{\Lambda}_X$.*

Proof. Lemma 1.67 states that the function $\boldsymbol{x} \mapsto \{\lambda_{j,Z}(\boldsymbol{x})\}$ is constant on each coset of $\mathbb{Z}^s/\Lambda_Z$. Now, recall that two points $\boldsymbol{z}_1$ and $\boldsymbol{z}_2 \in \mathbb{Z}^s$ lie in the same coset of $\mathbb{Z}^s/\hat{\Lambda}_X$ if and only if the point $\boldsymbol{z}_1 - \boldsymbol{z}_2 \in \hat{\Lambda}_X$. Since $\hat{\Lambda}_X \subseteq \Lambda_Z$ we obtain that $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$ are in this case also contained in the same coset of $\mathbb{Z}^s/\Lambda_Z$. □

Lemma 3.52. *Let the sublist $Z = (\boldsymbol{a}_{i_1}, \ldots, \boldsymbol{a}_{i_s}) \subseteq X$ be a basis of X, suppose that $\boldsymbol{x} \in \hat{\Lambda}_X$ and let $\boldsymbol{a}_{i_j}$ be a fixed element of the list Z. Then, the integer vector $\lambda_{m,Z}(\boldsymbol{x})\boldsymbol{a}_{i_m}$, as introduced in Proposition 3.49, is an element the basic lattice related to the list $X \backslash \{\boldsymbol{a}_{i_j}\}$, for all $1 \le m \le s$.*

Proof. Our aim is to show is that each coordinate of the vector $\lambda_{m,Z}(\boldsymbol{x})\boldsymbol{a}_{i_m}$ is divisible by the product of the determinants of all bases of the list $X \backslash \{\boldsymbol{a}_{i_j}\}$. To this end, note that the sublist $Z = (\boldsymbol{a}_{i_1}, \ldots, \boldsymbol{a}_{i_s}) \in \mathcal{B}(X)$ can not be a basis of the list $X \backslash \{\boldsymbol{a}_{i_j}\}$. Therefore, we obtain that the coordinates of the vector $(1/\det(Z))\boldsymbol{x}$ are still divisible by the product of the determinants of all bases in $\mathcal{B}(X \backslash \{\boldsymbol{a}_{i_j}\})$ and conclude that this product divides the real $\lambda_{m,Z}(\boldsymbol{x}) = \langle \boldsymbol{c}_m, \boldsymbol{x} \rangle / \det(Z)$. □

Lemma 3.53. *Let* $\{\bar{\boldsymbol{z}}_1, \ldots, \bar{\boldsymbol{z}}_{d(j)}\}$ *be a complete set of coset representatives of* $\mathbb{Z}^s/\hat{\Lambda}_{X\setminus\{\boldsymbol{a}_j\}}$*. Then, for any vector* $\boldsymbol{a} \in \mathbb{Z}^s$*, the function* $i_{\boldsymbol{a}} : \mathbb{Z} \to \{1, \ldots, d(j)\}$

$$i_{\boldsymbol{a}}(k) := l \quad \textit{if} \quad k\boldsymbol{a} - \bar{\boldsymbol{z}}_l \in \hat{\Lambda}_{X\setminus\{\boldsymbol{a}_j\}} \tag{3.58}$$

is a periodic function in k *with period* $P_j := \prod_{Z \in \mathcal{B}(X\setminus\{\boldsymbol{a}_j\})} |\det(Z)|$.

Proof. First, note that the vectors $P_j\boldsymbol{e}_1, \ldots, P_j\boldsymbol{e}_s$ generate the lattice $\hat{\Lambda}_{X\setminus\{\boldsymbol{a}_j\}}$ and

$$\boldsymbol{a} = \frac{a_1}{P_j} P_j\boldsymbol{e}_1 + \ldots + \frac{a_s}{P_j} P_j\boldsymbol{e}_s. \tag{3.59}$$

Now suppose $k \equiv i \pmod{P_j}$. Then we have that $k\boldsymbol{a} - i\boldsymbol{a} = (k-i)\boldsymbol{a} \in \hat{\Lambda}_{X\setminus\{\boldsymbol{a}_j\}}$. □

To avoid case distinctions during the proof of our next theorem, we introduce the following generalization of the summation symbol.

Convention. In the sequel, we set $\sum_{i=m}^{n} b(i) := -\sum_{j=n+1}^{m-1} b(j)$ for $m > n$, while the sum $\sum_{i=m}^{m-1} b(i) := 0$.

Lemma 3.54. *Let* $f : \mathbb{Z}^s \to \mathbb{C}$ *be a function and suppose that* $\boldsymbol{a} \in \mathbb{Z}^s$*. As to our generalized summation rule we have that*

$$f(\boldsymbol{x}) = f(\boldsymbol{x} - k\boldsymbol{a}) + \sum_{j=1}^{k} \nabla_{\boldsymbol{a}} f(\boldsymbol{x} - k\boldsymbol{a} + j\boldsymbol{a}), \tag{3.60}$$

for all integers $k \in \mathbb{Z}$.

Proof. Given a positive integer k, repeated application of the backwards difference operator $\nabla_{\boldsymbol{a}}$ yields that

$$\begin{aligned} f(\boldsymbol{x}) &= f(\boldsymbol{x} - \boldsymbol{a}) + \nabla_{\boldsymbol{a}} f(\boldsymbol{x}) \\ &= f(\boldsymbol{x} - 2\boldsymbol{a}) + \nabla_{\boldsymbol{a}} f(\boldsymbol{x} - \boldsymbol{a}) + \nabla_{\boldsymbol{a}} f(\boldsymbol{x}) = \ldots \\ &= f(\boldsymbol{x} - k\boldsymbol{a}) + \sum_{i=0}^{k-1} \nabla_{\boldsymbol{a}} f(\boldsymbol{x} - i\boldsymbol{a}) \qquad (3.61) \\ &= f(\boldsymbol{x} - k\boldsymbol{a}) + \sum_{j=1}^{k} \nabla_{\boldsymbol{a}} f(\boldsymbol{x} - k\boldsymbol{a} + j\boldsymbol{a}). \end{aligned}$$

For an integer $k \le 0$ we obtain, by our extended summation rule, that

$$\begin{aligned} f(\boldsymbol{x}) &= f(\boldsymbol{x}+\boldsymbol{a}) - \nabla_{\boldsymbol{a}} f(\boldsymbol{x}+\boldsymbol{a}) = \dots \\ &= f(\boldsymbol{x}+|k|\,\boldsymbol{a}) - \sum_{j=0}^{|k|-1} \nabla_{\boldsymbol{a}} f(\boldsymbol{x}+|k|\,\boldsymbol{a} - j\boldsymbol{a}) \\ &= f(\boldsymbol{x}-k\boldsymbol{a}) - \sum_{m=k+1}^{0} \nabla_{\boldsymbol{a}} f(\boldsymbol{x}-k\boldsymbol{a}+m\boldsymbol{a}) \\ &= f(\boldsymbol{x}-k\boldsymbol{a}) + \sum_{j=1}^{k} \nabla_{\boldsymbol{a}} f(\boldsymbol{x}-k\boldsymbol{a}+j\boldsymbol{a}). \end{aligned} \tag{3.62}$$

□

Lemma 3.55. *Let k be an integer and l a non–negative integer. Then*

$$\sum_{k=1}^{r} \binom{k}{l} = \binom{r+1}{l+1} - \binom{1}{l+1}, \tag{3.63}$$

for all integers r.

Proof. This identity is a well known result for non–negative integers r. Therefore, it remains to verify the formula for negative integers according to our extended definition of the summation symbol. To this end, let $r < 0$ be an integer. Then

$$\sum_{k=1}^{r} \binom{k}{l} = -\sum_{k=r+1}^{0} \binom{k}{l} = -\left(\sum_{k=r+1}^{0} \binom{k+1}{l+1} - \binom{k}{l+1} \right) = \binom{r+1}{l+1} - \binom{1}{l+1}.$$

□

Lemma 3.56. *Let p be a polynomial–function in the variable $\boldsymbol{z} \in \mathbb{Z}^s$ of degree d and $\boldsymbol{u}, \boldsymbol{v}_1, \dots, \boldsymbol{v}_s$ given vectors from $\mathbb{Z}^s$. The function*

$$q : \mathbb{Z} \to \mathbb{C},\ k \mapsto p\left(\sum_{i=1}^{s} x_i \boldsymbol{v}_i + k\boldsymbol{u} \right) \tag{3.64}$$

is a polynomial–function in k of degree no greater than d, where the coefficient of k^m is given by a multivariate polynomial in $\boldsymbol{x} = (x_1, \dots, x_s)$ of degree no greater than $d - m$.

Proof. It is not difficult to verify this statement straight forward, by basic, but rather messy, computations. We, therefore, omit details. □

Theorem 3.57. *The functions in the space* $\mathrm{DM}(X)$ *are quasi–polynomials with respect to the basic lattice* $\hat{\Lambda}_X$ *of degree no greater than* $d - s$.

Proof. Our aim is to prove that any function $f \in \mathrm{DM}(X)$ agrees with a polynomial restricted to a coset $\overline{\boldsymbol{x}} + \hat{\Lambda}_X$ of $\mathbb{Z}^s/\hat{\Lambda}_X$. We proceed by induction on the number of elements of the list X. First there are a few general observations to make. Since the order of the elements of the list X is not of importance for the space $\mathrm{DM}(X)$, we may assume, without loss of generality, that the sublist $Z = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_s) \subseteq X$ forms a basis of $\mathbb{R}^s$. Thus, each vector $\boldsymbol{x} \in \mathbb{R}^s$ possesses a unique representation as

$$\boldsymbol{x} = \lambda_{1,Z}(\boldsymbol{x})\ \boldsymbol{a}_1 + \lambda_{2,Z}(\boldsymbol{x})\ \boldsymbol{a}_2 + \ldots + \lambda_{s,Z}(\boldsymbol{x})\ \boldsymbol{a}_s, \tag{3.65}$$

c.f. Proposition 3.49. By repeated application of Lemma 3.54, we obtain that

$$\begin{aligned} f(\boldsymbol{x}) &= f\left(\boldsymbol{x} - \lfloor \lambda_{1,Z}(\boldsymbol{x}) \rfloor\ \boldsymbol{a}_1\right) + \sum_{k=1}^{\lfloor \lambda_{1,Z}(\boldsymbol{x}) \rfloor} \nabla_{\boldsymbol{a}_1} f\left(\boldsymbol{x} - \lfloor \lambda_{1,Z}(\boldsymbol{x}) \rfloor\ \boldsymbol{a}_1 + k\boldsymbol{a}_1\right) \\ &= \ldots \\ &= f\left(\boldsymbol{x} - \sum_{i=1}^{s} \lfloor \lambda_{i,Z}(\boldsymbol{x}) \rfloor\ \boldsymbol{a}_i\right) \\ &\quad + \sum_{j=1}^{s} \sum_{k=1}^{\lfloor \lambda_{j,Z}(\boldsymbol{x}) \rfloor} \nabla_{\boldsymbol{a}_j} f\left(\boldsymbol{x} - \sum_{i=1}^{j} \lfloor \lambda_{i,Z}(\boldsymbol{x}) \rfloor\ \boldsymbol{a}_i + k\boldsymbol{a}_j\right), \end{aligned} \tag{3.66}$$

for any function $f : \mathbb{Z}^s \to \mathbb{C}$ and all points $\boldsymbol{x} \in \mathbb{Z}^s$. Now, observe that the function

$$\boldsymbol{x} \mapsto f\left(\boldsymbol{x} - \sum_{i=1}^{s} \lfloor \lambda_{i,Z}(\boldsymbol{x}) \rfloor\ \boldsymbol{a}_i\right) = f\left(\sum_{i=1}^{s} \{\lambda_{i,Z}(\boldsymbol{x})\}\ \boldsymbol{a}_i\right) \tag{3.67}$$

is constant on each coset of $\mathbb{Z}^s/\hat{\Lambda}_X$, since the map $\boldsymbol{x} \mapsto \{\lambda_{i,Z}(\boldsymbol{x})\}$ is constant on the cosets of $\mathbb{Z}^s/\hat{\Lambda}_X$, for all $1 \leq i \leq s$, by Corollary 3.51.

If $\mathrm{span}(X \backslash \{\boldsymbol{a}_j\}) \neq \mathbb{R}^s$, then $\nabla_{\boldsymbol{a}_j} f \equiv 0$ for all functions $f \in \mathrm{DM}(X)$, according to the definition of the space $\mathrm{DM}(X)$. Thus, we have that the sum of functions on the right hand side of (3.66) vanishes in the base case of our induction, i.e. for $d = s$. In this case equation (3.67) implies that the restriction of any function $f \in \mathrm{DM}(X)$ to a coset of $\mathbb{Z}^s/\hat{\Lambda}_X$ agrees with a polynomial of degree zero.

In general, i.e. for $d > s$, it remains to be shown that the sum of functions on the right hand side of (3.66) is a quasi–polynomial with respect to the lattice $\hat{\Lambda}_X$.

First note that the function $\nabla_{\boldsymbol{a}_j} f \in DM(X\backslash\{\boldsymbol{a}_j\})$ for all $f \in \mathrm{DM}(X)$. Therefore, we can apply our induction hypothesis to the function $\nabla_{\boldsymbol{a}_j} f$ and obtain that this function is a quasi–polynomial of degree $d-1-s$ with respect to the lattice $\hat{\Lambda}_{X\backslash\{\boldsymbol{a}_j\}}$. Hence, if $\{\overline{\boldsymbol{z}}_1^j, \ldots, \overline{\boldsymbol{z}}_{d(j)}^j\}$ is a complete set of coset representatives of $\mathbb{Z}^s/\hat{\Lambda}_{X\backslash\{\boldsymbol{a}_j\}}$, there are polynomials $p_1^j, \ldots, p_{d(j)}^j$ of degree no greater than $d-1-s$, such that the function

$$\nabla_{\boldsymbol{a}_j} f(\boldsymbol{z}) = p_i^j(\boldsymbol{z}), \quad \text{if } \boldsymbol{z} - \overline{\boldsymbol{z}}_i^j \in \hat{\Lambda}_{X\backslash\{\boldsymbol{a}_j\}}. \tag{3.68}$$

Thus, we examine the function

$$\begin{aligned} g_j(\boldsymbol{x}, k) &:= \boldsymbol{x} - \sum_{i=1}^{j} \lfloor \lambda_{i,Z}(\boldsymbol{x}) \rfloor \, \boldsymbol{a}_i + k\,\boldsymbol{a}_j \\ &= \sum_{i=1}^{j} \{\lambda_{i,Z}(\boldsymbol{x})\} \, \boldsymbol{a}_i + \sum_{m=j+1}^{s} \lambda_{m,Z}(\boldsymbol{x}) \boldsymbol{a}_m + k\boldsymbol{a}_j. \end{aligned} \tag{3.69}$$

occurring as argument of the functions in the sum on the right hand side of (3.66). Again, we make use of the fact that the function $\boldsymbol{x} \mapsto \{\lambda_{i,Z}(\boldsymbol{x})\}$ is constant on a coset of $\mathbb{Z}^s/\hat{\Lambda}_X$, for all $1 \le i \le s$, c.f. Corollary 3.51. Hence, for each coset $\overline{\boldsymbol{x}} + \hat{\Lambda}_X \in \mathbb{Z}^s/\hat{\Lambda}_X$ there exists a vector $\boldsymbol{u}_{\overline{\boldsymbol{x}}}^j \in \mathbb{Z}^s$ such that $\sum_{i=1}^{j} \boldsymbol{a}_i \{\lambda_{i,Z}(\boldsymbol{x})\} = \boldsymbol{u}_{\overline{\boldsymbol{x}}}^j$, for all $\boldsymbol{x} \in \overline{\boldsymbol{x}} + \hat{\Lambda}_X$. Moreover, given $\boldsymbol{x}_1$ and $\boldsymbol{x}_2 \in \overline{\boldsymbol{x}} + \hat{\Lambda}_X$ the difference

$$\sum_{m=j+1}^{s} \lambda_{m,Z}(\boldsymbol{x}_1) \boldsymbol{a}_m - \sum_{m=j+1}^{s} \lambda_{m,Z}(\boldsymbol{x}_2) \boldsymbol{a}_m \quad = \quad \sum_{m=j+1}^{s} \lambda_{m,Z}(\boldsymbol{x}_1 - \boldsymbol{x}_2) \boldsymbol{a}_m, \tag{3.70}$$

where $(\boldsymbol{x}_1 - \boldsymbol{x}_2) \in \hat{\Lambda}_X$. Thus, the vector $\sum_{m=j+1}^{s} \lambda_m(\boldsymbol{x}_1 - \boldsymbol{x}_2)\boldsymbol{a}_m$ is an element of the basic lattice $\hat{\Lambda}_{X\backslash\{\boldsymbol{a}_j\}}$, by Lemma 3.52. This shows that the coset of the vector $g_j(\boldsymbol{x}, k)$ in $\mathbb{Z}^s/\hat{\Lambda}_{X\backslash\{\boldsymbol{a}_j\}}$ varies only by the coset of the vector $k\boldsymbol{a}_j$ in $\mathbb{Z}^s/\hat{\Lambda}_{X\backslash\{\boldsymbol{a}_j\}}$, provided that the variable $\boldsymbol{x}$ is restricted to a coset $\overline{\boldsymbol{x}} + \hat{\Lambda}_X$. Hence, by Lemma 3.53, the function $i_j : \mathbb{Z} \to \{1, \ldots, d(j)\}$

$$i_j(k) := l, \quad \text{if } \; g_j(\boldsymbol{x}, k) - \overline{\boldsymbol{z}}_l^j \in \hat{\Lambda}_{X\backslash\{\boldsymbol{a}_j\}}, \tag{3.71}$$

is periodic with period $P_j = \prod_{Z \in \mathcal{B}(X\backslash\{\boldsymbol{a}_j\})} |\det(Z)|$, and we have that

$$\nabla_{\boldsymbol{a}_j} f(g_j(\boldsymbol{x}, k)) = p_{i_j(k)}^j (g_j(\boldsymbol{x}, k)). \tag{3.72}$$

Now, the functions $k \mapsto p_i\left(g_j(\boldsymbol{x}, k)\right)$, for $1 \leq i \leq d(j)$ are univariate polynomials in k of degree no greater than $d-1-s$, by Lemma 3.56. The coefficient of k^l in these polynomials is a multivariate polynomial in $\boldsymbol{x}$ of degree no greater than $d-1-s-l$. Consider these polynomials in k expanded in the basis $(\binom{\frac{k-n}{d(j)}}{l})_{l\geq 0}$. Summing over the integers in a equivalence class modulo P_j, we observe that

$$\begin{aligned}\sum_{\substack{1\leq k\leq \lfloor \lambda_{j,Z}(\boldsymbol{x})\rfloor \\ k\equiv n \pmod{P_j}}} \binom{\frac{k-n}{P_j}}{l} &= \sum_{\substack{1\leq k\leq \lfloor \lambda_{j,Z}(\boldsymbol{x})\rfloor \\ k\equiv n \pmod{P_j}}} \binom{\frac{k-n}{P_j}+1}{l+1} - \binom{\frac{k-n}{P_j}}{l+1} \\ &= \binom{\lfloor \frac{\lambda_{j,Z}(\boldsymbol{x})-n}{P_j} \rfloor}{l+1} - \binom{\lceil \frac{1-n}{P_j} \rceil}{l+1},\end{aligned} \tag{3.73}$$

by Lemma 3.55. Note that

$$\frac{\lambda_{j,Z}(\boldsymbol{x}) - \lambda_{j,Z}(\overline{\boldsymbol{x}})}{P_j} = \frac{\lambda_{j,Z}(\boldsymbol{x} - \overline{\boldsymbol{x}})}{P_j} = \frac{\langle \boldsymbol{c}_j, \boldsymbol{x} - \overline{\boldsymbol{x}}\rangle}{P_j \det(Z)} \tag{3.74}$$

is an integer, for all $\boldsymbol{x} \in \overline{\boldsymbol{x}} + \hat{\Lambda}_X$, since $\boldsymbol{x} - \overline{\boldsymbol{x}} \in \hat{\Lambda}_X$. Thus, the function

$$\boldsymbol{x} \mapsto \left\lfloor \frac{\lambda_{j,Z}(\boldsymbol{x}) - n}{P_j} \right\rfloor = \frac{\lambda_{j,Z}(\boldsymbol{x}) - \lambda_{j,Z}(\overline{\boldsymbol{x}})}{P_j} + \left\lfloor \frac{\lambda_{j,Z}(\overline{\boldsymbol{x}}) - n}{P_j} \right\rfloor \tag{3.75}$$

is a polynomial in $\boldsymbol{x}$ of degree one, on $\overline{\boldsymbol{x}} + \hat{\Lambda}_X$. Therefore, provided that the variable $\boldsymbol{x}$ is restricted to the coset $\overline{\boldsymbol{x}} + \hat{\Lambda}_X$, the expression in (3.73) is a polynomial in $\boldsymbol{x}$ of degree $l+1$. In summery, we conclude that, restricted to a coset $\overline{\boldsymbol{x}} + \hat{\Lambda}_X$, the function

$$\boldsymbol{x} \mapsto \sum_{k=1}^{\lfloor \lambda_j(\boldsymbol{x}) \rfloor} \nabla_{\boldsymbol{a}_j} f\left(\boldsymbol{x} - \sum_{i=1}^{j} \boldsymbol{a}_i \lfloor \lambda_{i,Z}(\boldsymbol{x}) \rfloor + k\boldsymbol{a}_j\right) \tag{3.76}$$

is again a polynomial in $\boldsymbol{x}$ of degree no greater than $d-s$ for all $1 \leq j \leq s$. Since the sum of polynomials of degree no greater than $d-s$ is again a polynomial of degree no greater than $d-s$, we conclude that the function

$$\boldsymbol{x} \mapsto \sum_{j=1}^{s} \sum_{k=1}^{\lfloor \lambda_j(\boldsymbol{x}) \rfloor} \nabla_{\boldsymbol{a}_j} f\left(\boldsymbol{x} - \sum_{i=1}^{j} \boldsymbol{a}_i \lfloor \lambda_{i,Z}(\boldsymbol{x}) \rfloor + k\boldsymbol{a}_j\right) \tag{3.77}$$

agrees with a polynomial of degree no greater than $d-s$, on each coset in $\mathbb{Z}^s \backslash \hat{\Lambda}_X$.

□

In addition, we may now state a condition on a list under which all functions in the space $\mathrm{DM}(X)$ are polynomials.

Definition 3.58. We call a list X *unimodular*, if and only if $|\det(Z)| = 1$ for all bases $Z \in \mathcal{B}(X)$.

Corollary 3.59. *Suppose the list X is unimodular, then the functions in the space* $\mathrm{DM}(X)$ *are actually polynomials.*

Proof. Note that the basic lattice $\hat{\Lambda}_X$ related to an unimodular list is $\mathbb{Z}^s$. □

Our next objective is to finally prove that there exists indeed a function $f \in \mathrm{DM}(X)$ for any given "initial" values assigned to the points in a set $\delta(\mathfrak{c}|X)$, for an arbitrary X–region $\mathfrak{c} \subseteq \mathbb{R}^s$. In order to achieve this goal, we need the following observations.

Lemma 3.60. *Let Z be a list of s linearly independent vectors in $\mathbb{Z}^s$ such that* $\mathrm{cone}(Z)$ *is a pointed cone. Then there exists an Z–region $\mathfrak{c}_* \subseteq \mathbb{R}^s$ such that*

$$\delta(\mathfrak{c}_*|Z) = \Pi_Z \cap \mathbb{Z}^s, \tag{3.78}$$

where Π_Z stands for the fundamental parallelepiped of the lattice Λ_Z. Consequently, the set $\delta(\mathfrak{c}_*|Z)$ contains exactly one representative of each coset of $\mathbb{Z}^s/\Lambda_Z$.

Proof. First recall that the set of the integral points located in the fundamental parallelepiped of the lattice Λ_Z, given by

$$\Pi_Z \cap \mathbb{Z}^s = \{\lambda_1 \boldsymbol{a}_1 + \ldots + \lambda_s \boldsymbol{a}_s : 0 \le \lambda_i < 1\} \cap \mathbb{Z}^s, \tag{3.79}$$

contains exactly one representative of each coset of $\mathbb{Z}^s/\Lambda_Z$, cf. Corollary 1.68. Furthermore, we have the inclusion that the set $\Pi_Z \subseteq B(Z) = \sum_{i=1}^s \boldsymbol{a}_i - B(Z)$.

Now, since the elements of the list Z form a basis of $\mathbb{R}^s$, the cocircuits of this list are given by the singleton sublists. Thus, the essential hyperplane arrangement associated with the list Z is given by

$$\mathcal{C}(Z) = \{H_{(\boldsymbol{a}_i)^c} : \boldsymbol{a}_i \in Z\}. \tag{3.80}$$

Without loss of generality, we suppose that the vector $\boldsymbol{h}_{(\boldsymbol{a}_i)^c} \in \mathbb{R}^s$ meets the condition that the inner product $\langle \boldsymbol{h}_{(\boldsymbol{a}_i)^c}, \boldsymbol{a}_i \rangle = 1$ and obtain that

$$B(Z) = \bigcap_{\boldsymbol{a}_i \in Z} \left((\boldsymbol{a}_i + H^{\leq}_{(\boldsymbol{a}_i)^c}) \cap H^{\geq}_{(\boldsymbol{a}_i)^c} \right) \text{ and} \tag{3.81}$$

$$\Pi_Z = \bigcap_{\boldsymbol{a}_i \in Z} \left((\boldsymbol{a}_i + H^{<}_{(\boldsymbol{a}_i)^c}) \cap H^{\geq}_{(\boldsymbol{a}_i)^c} \right). \tag{3.82}$$

By analogy to the construction in the proof of Theorem 1.47, we can now translate the zonotope $B(Z)$ by a vector $\boldsymbol{v} \in \mathbb{R}^s$, with the property that $-1 < \langle \boldsymbol{h}_{(\boldsymbol{a}_i)^c}, \boldsymbol{v} \rangle < 0$, for all $1 \leq i \leq s$, such that

$$(\boldsymbol{v} + B(Z)) \cap \mathbb{Z}^s = \Pi_Z \cap \mathbb{Z}^s. \tag{3.83}$$

The vector $\boldsymbol{u} := \boldsymbol{v} + \sum_{i=1}^{s} \boldsymbol{a}_i$ is then contained in a unique Z–region $\mathfrak{c}_* \subseteq \mathbb{R}^s$ with the property that $\delta(\mathfrak{c}_*|Z) = \Pi_Z \cap \mathbb{Z}^s$. □

Lemma 3.61. *Suppose that the sublist $Z := (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_s) \subseteq X$ forms a basis of $\mathbb{R}^s$. Then, for any point $\boldsymbol{p} \in B(X) \backslash B(Z)$ there exists an element $\boldsymbol{a}_i \in Z$ such that either the point $\boldsymbol{p} - \boldsymbol{a}_i$ or $\boldsymbol{p} + \boldsymbol{a}_i$ lies within the zonotope $B(X)$.*

Proof. We review some properties of the polytope $B(Z)$. Due to the fact that the elements of the list Z form a basis of $\mathbb{R}^s$, the cocircuits of this list are the singleton sublists, and the essential hyperplane arrangement associated with the list Z is given by $\mathcal{C}(Z) = \{H_{(\boldsymbol{a}_i)^c} : \boldsymbol{a}_i \in Z\}$. Again, we choose the vector $\boldsymbol{h}_{(\boldsymbol{a}_i)^c} \in \mathbb{R}^s$ such that $\langle \boldsymbol{h}_{(\boldsymbol{a}_i)}, \boldsymbol{a}_i \rangle = 1$. Accordingly,

$$B(Z) = \bigcap_{\boldsymbol{a}_i \in Z} \left((\boldsymbol{a}_i + H^{\leq}_{(\boldsymbol{a}_i)^c}) \cap H^{\geq}_{(\boldsymbol{a}_i)^c} \right). \tag{3.84}$$

The facets of the polytope $B(Z)$ are given by the translated zonotopes $B(Z \backslash (\boldsymbol{a}_i))$ and $\boldsymbol{a}_i + B(Z \backslash (\boldsymbol{a}_i))$ for $1 \leq i \leq s$, cf. Lemma 3.31. Moreover, according to the definition of the zonotope associated with the list X, a point $\boldsymbol{p}$ lies within $B(X)$ if and only if

$$\boldsymbol{p} = \underbrace{\mu_1 \boldsymbol{a}_1 + \ldots + \mu_s \boldsymbol{a}_s}_{=:\boldsymbol{p}_Z} + \underbrace{\mu_{s+1} \boldsymbol{a}_{s+1} + \ldots + \mu_d \boldsymbol{a}_d}_{=:\boldsymbol{p}_{X \backslash Z}}, \tag{3.85}$$

for some reals $0 \leq \mu_1, \ldots, \mu_d \leq 1$. Note that the vector $\boldsymbol{p}_Z$ lies within the polytope $B(Z)$, while $\boldsymbol{p}_{X \backslash Z} \in B(X \backslash Z)$. If $\boldsymbol{p} \in B(X) \backslash B(Z)$, then we have that $\boldsymbol{p}_{X \backslash Z} \neq \boldsymbol{0}$, for all such representations of $\boldsymbol{p}$.

In order to prove our claim, we need a few more considerations. Since the elements of the list Z form a basis of $\mathbb{R}^s$, there are uniquely determined reals $\lambda_1^k, \ldots, \lambda_s^k$ such that $\boldsymbol{a}_k = \lambda_1^k \boldsymbol{a}_1 + \ldots + \lambda_s^k \boldsymbol{a}_s$, for all $s+1 \leq k \leq d$. Thus, we have that

$$\begin{aligned} \boldsymbol{p}_{X\backslash Z} &= \sum_{k=s+1}^{d} \mu_k \left(\lambda_1^k \boldsymbol{a}_1 + \ldots + \lambda_s^k \boldsymbol{a}_s \right) \\ &= \left(\sum_{k=s+1}^{d} \mu_k \lambda_1^k \right) \boldsymbol{a}_1 + \ldots + \left(\sum_{k=s+1}^{d} \mu_k \lambda_s^k \right) \boldsymbol{a}_s. \end{aligned} \tag{3.86}$$

Hence, there exists a maximal non–negative real M such that the point $\boldsymbol{p}'_Z := \boldsymbol{p}_Z + M\boldsymbol{p}_{X\backslash Z} \in B(Z)$. Note that $0 < M < 1$, for all $\boldsymbol{p} \in B(X)\backslash B(Z)$. That is, there is at least one $1 \leq i \leq s$ such that either

$$\begin{aligned} &\mu_i + K \left(\sum_{k=s+1}^{d} \mu_k \lambda_i^k \right) > 1 \text{ in case that } \left\langle \boldsymbol{h}_{(\boldsymbol{a}_i)^c}, \boldsymbol{p} \right\rangle > 1 \text{ or} \\ &\mu_i + K \left(\sum_{k=s+1}^{d} \mu_k \lambda_i^k \right) < 0 \text{ in case that } \left\langle \boldsymbol{h}_{(\boldsymbol{a}_i)^c}, \boldsymbol{p} \right\rangle < 0, \end{aligned} \tag{3.87}$$

for all $K > M$. Therefore, there are non–negative reals $0 \leq \mu'_1, \ldots, \mu'_s \leq 1$ such that $\boldsymbol{p}'_Z = \mu'_1 \boldsymbol{a}_1 + \ldots + \mu'_s \boldsymbol{a}_s$, where μ'_i equals either 0 or 1. Takeing (3.87) into account, we obtain that

$$\begin{aligned} \boldsymbol{p} + \boldsymbol{a}_i &= \boldsymbol{p}'_Z + \boldsymbol{a}_i + (1-M)\boldsymbol{p}_{X\backslash Z} \in B(X) \quad \text{if } \left\langle \boldsymbol{h}_{(\boldsymbol{a}_i)^c}, \boldsymbol{p} \right\rangle < 0, \text{ while} \\ \boldsymbol{p} - \boldsymbol{a}_i &= \boldsymbol{p}'_Z - \boldsymbol{a}_i + (1-M)\boldsymbol{p}_{X\backslash Z} \in B(X) \quad \text{for } \left\langle \boldsymbol{h}_{(\boldsymbol{a}_i)^c}, \boldsymbol{p} \right\rangle > 1. \end{aligned} \tag{3.88}$$

□

Theorem 3.62. *Let $\mathfrak{c} \subseteq \mathbb{R}^s$ be an X–region and suppose that*

$$g_{\delta(\mathfrak{c}|X)} : \delta(\mathfrak{c}|X) \to \mathbb{C} \tag{3.89}$$

is an arbitrary complex valued function. Then there exists a unique determined function $f \in \mathrm{DM}(X)$ such that the restriction of f to the set $\delta(\mathfrak{c}|X)$ agrees with $g_{\delta(\mathfrak{c}|X)}$.

Proof. We already proved that a function $f \in \mathrm{DM}(X)$ is uniquely determined by the values on the set $\delta(\mathfrak{c}|X)$, cf. Theorem 3.45. What is left to prove is the existence of a continuation of the function $g_{\delta(\mathfrak{c}|X)}$ into $\mathbb{Z}^s$ which lies in the space $\mathrm{DM}(X)$.

We use the same notation as in the proof of Theorem 3.57, and proceed once more by induction on the number of elements in the list X. Without loss of generality, we assume again that the sublist $Z := (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_s) \subseteq X$ forms a basis of $\mathbb{R}^s$. We carry out the proof for the X–region $\mathfrak{c}_* \subseteq \mathbb{R}^s$ which meets the condition that $\Pi_Z \cap \mathbb{Z}^s = \delta(\mathfrak{c}_*|Z)$, cf. Lemma 3.60. Then, the general statement follows by the wall crossing formula, Theorem 3.44.

First suppose the list X contains s vectors. Then, we have $Z = X$ and the cocircuits are given by singleton sublists. Consequently, we obtain that a function $f : \mathbb{Z}^s \to \mathbb{C}$ lies in the space $\mathrm{DM}(X)$ if and only if $\nabla_{\boldsymbol{a}_i} f \equiv 0$ for all $\boldsymbol{a}_i \in X$. In other words, a function $f \in \mathrm{DM}(X)$ if and only if it takes constant values on each coset of $\mathbb{Z}^s/\Lambda_Z$. Since $\delta(\mathfrak{c}_*|X) = \Pi_Z \cap \mathbb{Z}^s$, this set contains exactly one representative of each coset of $\mathbb{Z}^s/\Lambda_Z$. Therefore, we may conclude that there exists indeed a unique function $f \in \mathrm{DM}(X)$ which agrees with the function $g_{\delta(\mathfrak{c}_*|X)}$ on the set $\delta(\mathfrak{c}_*|X)$.

In order to perform our induction step, we need the following observations. The sets $\delta(\mathfrak{c}_*|X \backslash \{\boldsymbol{a}_j\})$ and $\delta(\mathfrak{c}_*|X \backslash \{\boldsymbol{a}_j\}) - \boldsymbol{a}_j$ are subsets of $\delta(\mathfrak{c}_*|X)$. Moreover, for any point $\boldsymbol{z} \in \delta(\mathfrak{c}_*|X)$ the vector $\boldsymbol{z} - \boldsymbol{a}_j$ is an element of the set $\delta(\mathfrak{c}_*|X)$ if and only if $\boldsymbol{z} \in \delta(\mathfrak{c}_*|X \backslash \{\boldsymbol{a}_j\})$. Therefore, the function

$$\nabla_{\boldsymbol{a}_j} g_{\delta(\mathfrak{c}_*|X)}(\boldsymbol{b}) = g_{\delta(\mathfrak{c}_*|X)}(\boldsymbol{b}) - g_{\delta(\mathfrak{c}_*|X)}(\boldsymbol{b} - \boldsymbol{a}_j) \tag{3.90}$$

is only determined for all points in the set $\delta(\mathfrak{c}_*|X \backslash \{\boldsymbol{a}_j\})$. Hence, by our induction hypothesis, there exists a uniquely determined continuation of $\nabla_{\boldsymbol{a}_j} g_{\delta(\mathfrak{c}_*|X)}(\boldsymbol{b})$ to a function in the space $\mathrm{DM}(X \backslash \{\boldsymbol{a}_j\})$.

We assumed that the sublist $Z \subseteq X$ forms a basis of $\mathbb{R}^s$. Thus, any integral point $\boldsymbol{b} \in \mathbb{Z}^s$ has a unique representation as

$$\boldsymbol{b} = \{\lambda_{1,Z}(\boldsymbol{b})\}\, \boldsymbol{a}_1 + \ldots + \{\lambda_{s,Z}(\boldsymbol{b})\}\, \boldsymbol{a}_s + \lfloor \lambda_{1,Z}(\boldsymbol{b}) \rfloor\, \boldsymbol{a}_1 + \ldots + \lfloor \lambda_{s,Z}(\boldsymbol{b}) \rfloor\, \boldsymbol{a}_s, \tag{3.91}$$

c.f. Proposition 3.49. Note that the vector $\{\lambda_{1,Z}(\boldsymbol{b})\}\, \boldsymbol{a}_1 + \ldots + \{\lambda_{s,Z}(\boldsymbol{b})\}\, \boldsymbol{a}_s$ is located in the set $\delta(\mathfrak{c}_*|Z)$, for all $\boldsymbol{b} \in \mathbb{Z}^s$. We claim that the function $f \in \mathrm{DM}(X)$, defined by

$$\begin{aligned} f(\boldsymbol{b}) := {} & g_{\delta(\mathfrak{c}_*|X)}\left(\sum_{i=1}^{s} \{\lambda_{i,Z}(\boldsymbol{b})\}\, \boldsymbol{a}_i \right) \\ & + \sum_{j=1}^{s} \sum_{k=1}^{\lfloor \lambda_{j,Z}(\boldsymbol{b}) \rfloor} \nabla_{\boldsymbol{a}_j} g_{\delta(\mathfrak{c}_*|X)} \left(\boldsymbol{b} - \sum_{i=1}^{j} \boldsymbol{a}_i \lfloor \lambda_{i,Z}(\boldsymbol{b}) \rfloor + k\, \boldsymbol{a}_j \right), \end{aligned} \tag{3.92}$$

agrees with the function $g_{\delta(\mathfrak{c}_*|X)}$ at all points in the set $\delta(\mathfrak{c}_*|X)$. The fact that this function is contained in the space $\mathrm{DM}(X)$ follows since, on the one hand, any cocitcuit of the list X contains at least one element $\boldsymbol{a}_j \in Z$, since $Z \in \mathcal{B}(X)$, and the function $\nabla_{\boldsymbol{a}_j} g_{\delta(\mathfrak{c}_*|X)}\left(\sum_{i=1}^s \{\lambda_i(\boldsymbol{b})\}\, \boldsymbol{a}_i\right) = 0$ for all $\boldsymbol{a}_j \in Z$, on the other hand, we have the inclusion $\mathrm{DM}(X\backslash\{\boldsymbol{a}_j\}) \subseteq \mathrm{DM}(X)$ for $1 \leq j \leq s$. Now note that $\nabla_Y(g+h) = \nabla_Y g + \nabla_Y h$, for all functions $g, h : \mathbb{Z}^s \to \mathbb{C}$ and all lists Y.

To prove our assertion (3.92) we proceed by induction on $S(\boldsymbol{b}) := \sum_{i=1}^s |\lfloor \lambda_i(\boldsymbol{b}) \rfloor|$. Now $S(\boldsymbol{b}) = 0$ if and only if $\boldsymbol{b} \in \delta(\mathfrak{c}_*|Z) \subseteq \delta(\mathfrak{c}_*|X)$, and in this case we have that

$$f(\boldsymbol{b}) = g_{\delta(\mathfrak{c}_*|X)}(\boldsymbol{b})\,. \tag{3.93}$$

Now suppose that $\boldsymbol{b} \in \delta(\mathfrak{c}_*|X)\backslash\delta(\mathfrak{c}_*|Z)$, i.e. $\boldsymbol{u} - \boldsymbol{b} \in B(X)\backslash B(Z)$ for $\boldsymbol{u} \in \mathfrak{c}_*$. Then, by Lemma 3.61, there exists an index $1 \leq i \leq s$ such that either the point $\boldsymbol{u} - \boldsymbol{b} - \boldsymbol{a}_i$ or $\boldsymbol{u} - \boldsymbol{b} + \boldsymbol{a}_i$ lies within the zonotope $B(X)$. In order to apply our induction hypothesis, we have to show that $\boldsymbol{u} - \boldsymbol{b} - \boldsymbol{a}_i \in B(X)$ if $\lambda_{i,Z}(\boldsymbol{b}) > 0$ and that $\boldsymbol{u} - \boldsymbol{b} + \boldsymbol{a}_i \in B(X)$ in case that $\lambda_{i,Z}(\boldsymbol{b}) < 0$. Using the notation of the proof of Lemma 3.61, we have that

$$\left\langle \boldsymbol{h}_{(\boldsymbol{a}_i)^c}, \boldsymbol{u} - \boldsymbol{b} \right\rangle = \left\langle \boldsymbol{h}_{(\boldsymbol{a}_i)^c}, \boldsymbol{u} \right\rangle - \left\langle \boldsymbol{h}_{(\boldsymbol{a}_i)^c}, \boldsymbol{b} \right\rangle = \left\langle \boldsymbol{h}_{(\boldsymbol{a}_i)^c}, \boldsymbol{u} \right\rangle - \lambda_{i,Z}(\boldsymbol{b}). \tag{3.94}$$

As pointed out in the proof of Lemma 3.60, we have that $\boldsymbol{u} = \boldsymbol{v} + \sum_{j=1}^s \boldsymbol{a}_j$ where $-1 < \langle \boldsymbol{h}_{(\boldsymbol{a}_i)^c}, \boldsymbol{v} \rangle < 0$. Recall that $\langle \boldsymbol{h}_{(\boldsymbol{a}_i)^c}, \sum_{i=j}^s \boldsymbol{a}_j \rangle = \langle \boldsymbol{h}_{(\boldsymbol{a}_i)^c}, \boldsymbol{a}_i \rangle = 1$, by our assumptions on the vector $\boldsymbol{h}_{(\boldsymbol{a}_i)^c}$. Thus, we have that

$$\left\langle \boldsymbol{h}_{(\boldsymbol{a}_i)^c}, \boldsymbol{u} \right\rangle - \lambda_{i,Z}(\boldsymbol{b}) < 0 \Leftrightarrow 0 < \left\langle \boldsymbol{h}_{(\boldsymbol{a}_i)^c}, \boldsymbol{v} \right\rangle + 1 < \lambda_i(\boldsymbol{b}), \text{ and} \tag{3.95}$$

$$\left\langle \boldsymbol{h}_{(\boldsymbol{a}_i)^c}, \boldsymbol{u} \right\rangle - \lambda_{i,Z}(\boldsymbol{b}) > 1 \Leftrightarrow 0 > \left\langle \boldsymbol{h}_{(\boldsymbol{a}_i)^c}, \boldsymbol{v} \right\rangle > \lambda_{i,Z}(\boldsymbol{b}). \tag{3.96}$$

Hence, if $\lfloor \lambda_{i,Z}(\boldsymbol{b}) \rfloor < 0$ Lemma 3.61 implies that the point $\boldsymbol{b} + \boldsymbol{a}_i$ lies within $\delta(\mathfrak{c}_*|X)$ while the point $\boldsymbol{b} - \boldsymbol{a}_i$ is located in the set $\delta(\mathfrak{c}_*|X)$ in case that $\lfloor \lambda_{i,Z}(\boldsymbol{b}) \rfloor \geq 1$. This shows that for each point $\boldsymbol{b} \in \delta(\mathfrak{c}_*|X)$ there exists an index $1 \leq i \leq s$ such that the point $\boldsymbol{b} - \mathrm{sgn}(\lambda_i(\boldsymbol{b}))\boldsymbol{a}_i$ is also an element of $\delta(\mathfrak{c}_*|X)$. Note that $S(\boldsymbol{b}) := S(\boldsymbol{b} - \mathrm{sgn}(\lambda_i(\boldsymbol{b}))\boldsymbol{a}_i) + 1$. Therefore, our induction hypothesis implies that

$$f(\boldsymbol{b} - \mathrm{sgn}(\lambda_i(\boldsymbol{b}))\boldsymbol{a}_i) = g_{\delta(\mathfrak{c}_*|X)}(\boldsymbol{b} - \mathrm{sgn}(\lambda_i(\boldsymbol{b}))\boldsymbol{a}_i). \tag{3.97}$$

After all, we conclude that

$$f(\boldsymbol{b}) \quad = \quad f(\boldsymbol{b} - \boldsymbol{a}_i) + \nabla_{\boldsymbol{a}_i} f(\boldsymbol{b}) = g_{\delta(\mathfrak{c}_*|X)}(\boldsymbol{b} - \boldsymbol{a}_i) + \nabla_{\boldsymbol{a}_i} g_{\delta(\mathfrak{c}_*|X)}(\boldsymbol{b}) = g_{\delta(\mathfrak{c}_*|X)}(\boldsymbol{b}),$$

in case that $\mathrm{sgn}(\lambda_i(\boldsymbol{b})) > 0$, or

$$\begin{aligned} f(\boldsymbol{b}) &= f(\boldsymbol{b} + \boldsymbol{a}_i) + \nabla_{\boldsymbol{a}_i} f(\boldsymbol{b} + \boldsymbol{a}_i) = g_{\delta(\mathfrak{c}_*|X)}(\boldsymbol{b} + \boldsymbol{a}_i) + \nabla_{\boldsymbol{a}_i} g_{\delta(\mathfrak{c}_*|X)}(\boldsymbol{b} + \boldsymbol{a}_i) \\ &= g_{\delta(\mathfrak{c}_*|X)}(\boldsymbol{b}), \end{aligned}$$

if $\mathrm{sgn}(\lambda_i(\boldsymbol{b})) < 0$, for all integral points $\boldsymbol{b} \in \delta(\mathfrak{c}_*|X)$.

$\square$

3.4 Vector Partition Functions and the Space DM(X)

Applying the wall crossing formula, we may now conclude that the restriction of the vector partition function Φ_X to chamber of X agrees with a unique function from the space $\mathrm{DM}(X)$. Furthermore, in case of adjacent chambers we obtain that the according functions agree on a strip containing their common boundary. We deduce our proof from an analog result for a certain refinement of the chamber subdivision of $\mathbb{R}^s$, defined as follows.

Definition 3.63. We call a maximal connected subset

$$\mathfrak{C} \subseteq \{\boldsymbol{x} \in \mathbb{R}^s : \boldsymbol{x} \notin H_{Y^c}, \text{ for all } Y \in \mathcal{Y}(X)\}, \tag{3.98}$$

i.e. a pointed, polyhedral cone bounded by hyperplanes spanned by sublists of X, an *X–cone.*

Remark 3.64. The elements of an X–cone are regular points of $\mathbb{R}^s$ with respect to the list X but the chambers of X are in general bigger sets.

Example 3.65. We consider once more the list of the column vectors of the 3×6 matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}, \tag{3.99}$$

c.f. Example 3.16. The chamber subdivision of $\mathrm{cone}(X)$ is displayed in Figure 3.6. Figure 3.7 shows a 2–dimensional slice of $\mathrm{cone}(X)$, where the different chambers are shaded in distinct colors. The gray lines picture the hyperplane–arrangement $\{\mathrm{span}(X \backslash Y) : Y \in \mathcal{Y}(X)\}$, the points located in 2–dimensional cones generated by sublists of X are highlighted in black. Note that that the bottommost chamber is decomposed into two X–regions.

Comparing the upper definition with the notion of X–regions, we observe that any X–region is located within a unique X–cone. Vice versa, we gain the following result.

Lemma 3.66. *Every X–cone $\mathfrak{C} \subseteq \mathbb{R}^s$ contains a unique initial X–region $\mathfrak{c}_0 \subseteq \mathbb{R}^s$.*

Proof. The closure of an X–cone equals the union of the closures of the X–regions contained within this cone. Therefore, and since the origin lies in the closure of every

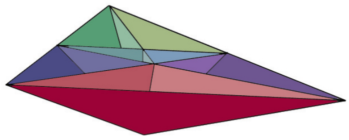


Figure 3.6: The cone(X)

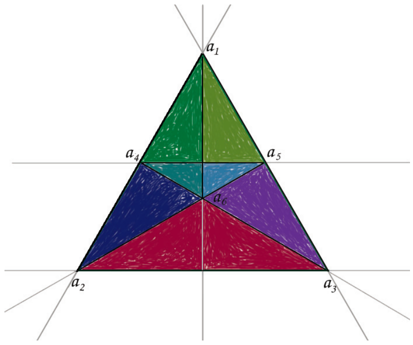


Figure 3.7: The X–regions

X–cone, the cone $\mathfrak{C}$ contains at least one initial X–region. However, because X–cones are regions of $\mathbb{R}^s$ which do not include any element of the hyperplane arrangement $\mathcal{C}(X)$, the X–cone $\mathfrak{C} \subseteq \mathbb{R}^s$ can not contain another initial X–region. $\square$

To show that the vector partition function Φ_X agrees on each X–cone $\mathfrak{C} \subseteq \text{cone}(X)$ with a given function $f_{\mathfrak{C}} \in \text{DM}(X)$, we need another lemma. To achieve the needed result, we introduce the following notation.

Definition 3.67. We set

$$\Lambda_X^+ := \{n_1 \boldsymbol{a}_1 + \ldots + n_d \boldsymbol{a}_d : n_1, \ldots, n_d \in \mathbb{Z}_{\geq 0}\}. \tag{3.100}$$

Note that $\text{supp}(\Phi_X) = \Lambda_X^+ \subseteq \text{cone}(X)$

Definition 3.68. To shorten notation, let

$$\Xi := \text{cone}(X) \backslash \bigcup_{\boldsymbol{z} \in \Lambda_X^+ \backslash \{\boldsymbol{0}\}} (\boldsymbol{z} + \text{cone}(X)). \tag{3.101}$$

For any integral vector $\boldsymbol{z} \in \Lambda_X^+ \backslash \{\boldsymbol{0}\}$ there exists a list–element $\boldsymbol{a}_i \in X$ such that $\boldsymbol{z} = \boldsymbol{a}_i + \boldsymbol{y}$ for some point $\boldsymbol{y} \in \Lambda_X^+$. Thus, we may alternatively say that the set $\Xi = \text{cone}(X) \backslash \bigcup_{\boldsymbol{a}_i \in X} (\boldsymbol{a}_i + \text{cone}(X))$. It emerges from this description and the definition of the zonotope $B(X)$ that the set Ξ lies within the zonotope. Furthermore, note that Ξ is a non–empty set, since we assumed cone(X) to be pointed. The importance of this notation is emphasized by the following lemma.

Lemma 3.69. *Given an element* $\boldsymbol{u} \in \operatorname{cone}(X)$, *we have that*

$$\delta(\boldsymbol{u}|X) \cap \Lambda_X^+ = \{\mathbf{0}\} \Leftrightarrow \boldsymbol{u} \in \Xi. \tag{3.102}$$

Proof. Since $\Xi \subseteq B(X)$, the set $\delta(\boldsymbol{u}|X)$ includes the origin if $\boldsymbol{u} \in \Xi$, by Lemma 3.40. On the other hand, if $\boldsymbol{u} \in \operatorname{cone}(X)\backslash\Xi$, there are non–negative reals $\alpha_1, \ldots, \alpha_d$ such that $\boldsymbol{u} = \alpha_1 \boldsymbol{a}_1, \ldots, \alpha_d \boldsymbol{a}_d$ and $\alpha_i \geq 1$ for at least one $1 \leq i \leq d$. Thus, the point $\boldsymbol{z} := \boldsymbol{u} - (\{\alpha_1\}\, \boldsymbol{a}_1 + \ldots + \{\alpha_d\}\, \boldsymbol{a}_d) \in \delta(\boldsymbol{u}|X) \cap \Lambda_X^+\backslash\{\mathbf{0}\}$.

Now, recall that an integral vector $\boldsymbol{z} \in \delta(\boldsymbol{u}|X)$ if and only if $\boldsymbol{u} \in \boldsymbol{z} + B(X) \subseteq \boldsymbol{z} + \operatorname{cone}(X)$. Therefore, we conclude that a vector $\boldsymbol{u} \in \operatorname{cone}(X)$ can not be an element of the set Ξ in case that the set $\delta(\boldsymbol{u}|X)$ contains an integral point $\boldsymbol{z} \in \Lambda_X^+\backslash\{\mathbf{0}\}$. □

Corollary 3.70. *Let* $\mathfrak{c}_0 \subseteq \operatorname{cone}(X)$ *be an initial* X*–region. Then the intersection*

$$\delta(\mathfrak{c}_0|X) \cap \Lambda_X^+ = \{\mathbf{0}\}. \tag{3.103}$$

Proof. Observe that the set Ξ includes all initial X–regions within $\operatorname{cone}(X)$. □

Proposition 3.71. *Suppose that* $\mathfrak{C} \subseteq \operatorname{cone}(X)$ *is an* X*–cone, and denote by* $\mathfrak{c}_0$ *the unique initial* X*–region in* $\mathfrak{C}$*. Moreover, let* $f_{\mathfrak{C}} \in \operatorname{DM}(X)$ *be the function which is uniquely determined by the condition that*

$$f_{\mathfrak{C}}(\boldsymbol{b}) = \begin{cases} 1 & \text{if } \boldsymbol{b} = \mathbf{0} \\ 0 & \text{if } \boldsymbol{b} \in \delta(\mathfrak{c}_0|X)\backslash\{\mathbf{0}\}. \end{cases} \tag{3.104}$$

Then the vector partition function Φ_X *agrees with the function* $f_{\mathfrak{C}}$ *on* $(\mathfrak{C} - B(X)) \cap \mathbb{Z}^s$.

Proof. At first recall that the vector partition function $\Phi_X(\boldsymbol{b})$ vanishes at all integral points $\boldsymbol{b} \notin \Lambda_X^+$ and that $\Phi_X(\mathbf{0}) = 1$. Therefore, and since $\delta(\mathfrak{c}_0|X) \cap \Lambda_X^+ = \{\mathbf{0}\}$, we have that the vector partition function Φ_X agrees with the function $f_{\mathfrak{C}}$ on the set $\delta(\mathfrak{c}_0|X)$.

Now, the vector partition function solves the difference equation

$$\nabla_Y \Phi_X(\boldsymbol{b}) = 0 \quad \text{at all points } \boldsymbol{b} \notin \operatorname{cone}(X\backslash Y) \cap \mathbb{Z}^s \subsetneq H_{Y^c}. \tag{3.105}$$

Hence, if $\mathfrak{c}_j$ and $\mathfrak{c}_{j+1}$ are adjacent X–regions, separated by a hyperplane $\boldsymbol{a}_j + H_{Y_j^c}$, with $\boldsymbol{a}_j \in \mathbb{Z}^s\backslash\{\mathbf{0}\}$ and $Y_j \subseteq X$ a cocircuit, we can apply the wall crossing formula, Theorem 3.44, to calculate the values that the vector partition function Φ_X takes on the points in the set $\delta(\mathfrak{c}_{j+1}|X)$ from the function values on the set $\delta(\mathfrak{c}_j|X)$.

Since the function $f_{\mathfrak{C}}$ lies in the space $\mathrm{DM}(X)$, we may calculate the values of $f_{\mathfrak{C}}$ on $\delta(\mathfrak{c}_{j+1}|X)$ in the same way from the values the function $f_{\mathfrak{C}}$ takes on $\delta(\mathfrak{c}_j|X)$. Thus, both functions take the same values on $\delta(\mathfrak{c}_{j+1}|X)$, provided they agree on the set $\delta(\mathfrak{c}_j|X)$.

Now, suppose that $\boldsymbol{z}$ is an integral point in $(\mathfrak{C}-B(X))\cap\mathbb{Z}^s$. Then, there exists at least one X–region $\mathfrak{c}\subseteq\mathfrak{C}$ with $\boldsymbol{z}\in\delta(\mathfrak{c}|X)$. Thus, we may conclude our proof by showing that there exists a finite sequence of X–regions, starting with $\mathfrak{c}_0$ and ending with $\mathfrak{c}$, where consecutive sequence elements are adjacent X–regions which are separated by affine hyperplanes $\boldsymbol{a}_i+H_{Y_i^c}$, where the integral vector $\boldsymbol{a}_j\in\mathbb{Z}^s\backslash\{\mathbf{0}\}$.

To this end, observe that X–cones are convex sets. Therefore, given arbitrary points $\boldsymbol{u}\in\mathfrak{c}$ and $\boldsymbol{u}_0\in\mathfrak{c}_0$, the line segment $[\boldsymbol{u},\boldsymbol{u}_0]$ lies entirely within the X–cone $\mathfrak{C}$. Traveling from $\boldsymbol{u}_0$ toward $\boldsymbol{u}$ along this line segment, we meet finitely many distinct X–regions $\mathfrak{c}_0,\mathfrak{c}_1,\ldots,\mathfrak{c}_n=\mathfrak{c}$, all of them located within $\mathfrak{C}$. In between the X–regions $\mathfrak{c}_{i-1}$ and $\mathfrak{c}_i$, the line segment $[\boldsymbol{u},\boldsymbol{u}_0]$ has to cross at least one hyperplane $\boldsymbol{a}_{i,1}+H_{Y_{i,1}^c}$, where the vector $\boldsymbol{a}_{i,1}\in\mathbb{Z}^s\backslash\{\mathbf{0}\}$, since both X– regions, $\mathfrak{c}_{i-1}$ and $\mathfrak{c}_i$, lie in the interior of the X–cone $\mathfrak{C}$, for all $1\le i\le n$.

Assume that the set $[\boldsymbol{u},\boldsymbol{u}_0]\cap(\overline{\mathfrak{c}_i}\cap\overline{\mathfrak{c}_{i-1}})$ intersects the affine hyperplanes $\boldsymbol{a}_{i,1}+H_{Y_{i,1}^c},\ldots,$ $\boldsymbol{a}_{i,m}+H_{Y_{i,m}^c}$. That is, the X–regions $\mathfrak{c}_{i-1}$ and $\mathfrak{c}_i$ lie on opposite sides of each of the hyperplanes $\boldsymbol{a}_{i,j}+H_{Y_{i,j}^c}$, for $1\le j\le m$. So starting in $\mathfrak{c}_i$ and crossing one of these hyperplanes at a time, we obtain again a sequence of adjacent X–regions $(\mathfrak{c}_i,\mathfrak{c}_{i,1},\ldots,\mathfrak{c}_{i,m})$, ending with $\mathfrak{c}_{i,m}=\mathfrak{c}_{i+1}$. Furthermore, all of these X–regions lie within the X–cone $\mathfrak{C}$, since we did not cross any of the linear hyperplanes spanned by a sublist of X.

Assembling these sequences, in ascending order starting with $i=1$, we obtain a finite sequence of X–regions starting with $\mathfrak{c}_0$ and ending with $\mathfrak{c}=\mathfrak{c}_0+\boldsymbol{z}$. Moreover, all X–regions in this sequence are contained within the X–cone $\mathfrak{C}$ and consecutive sequence elements are given by adjacent X–regions, separated by affine hyperplanes $\boldsymbol{a}_i+H_{Y_i^c}$, where the integral vector $\boldsymbol{a}_i\in\mathbb{Z}^s\backslash\{\mathbf{0}\}$. □

Theorem 3.72. *Let $\mathcal{C}\subseteq$ cone(X) be a chamber of X and $\mathfrak{c}_0$ an initial X–region in $\mathcal{C}$. Then the vector partition function $\Phi_X(\boldsymbol{b})$ agrees with the function $f_{\mathcal{C}}\in\mathrm{DM}(X)$ which is uniquely determined by the condition that*

$$f_{\mathcal{C}}(\boldsymbol{b})=\begin{cases}1 & \text{if } \boldsymbol{b}=\mathbf{0}\\ 0 & \text{if } \boldsymbol{b}\in\delta(\mathfrak{c}_0|X)\setminus\{\mathbf{0}\}\end{cases}\tag{3.106}$$

at all integral points $\boldsymbol{b}\in(\mathcal{C}-B(X))\cap\mathbb{Z}^s$.

Proof. At first, we briefly review some facts about the chambers of a list X. A chamber is defined as a maximal connected subset of $\mathrm{cone}(X)$, that consists solely of points which are regular with respect to the list X. Any regular point contained in $\mathrm{cone}(X)$ is either located in an X–cone or they lie in an set given as $(H_{Y_i^c} \cap \mathrm{cone}(X)) \setminus \mathrm{cone}(Y_i^c)$, where Y_i denotes a cocircuit of the list X. We have that $\overline{\mathcal{C}} = \bigcup_{\mathfrak{C}_i \subseteq \mathcal{C}} \overline{\mathfrak{C}_i}$, where the $\mathfrak{C}_i$'s denote the X–cones located within the chamber $\mathcal{C}$.

Now, let $\mathfrak{C}_0 \subseteq \mathcal{C}$ be the X–cone containing the initial X–region $\mathfrak{c}_0$, and $f_{\mathfrak{C}_0} \in \mathrm{DM}(X)$ the uniquely determined function which agrees with the vector partition function Φ_X on the set $(\mathfrak{C}_0 - B(X)) \cap \mathbb{Z}^s$, cf. Proposition 3.71. Thus, in order to finish our proof, we have to show that $\Phi_X(\boldsymbol{b}) = f_{\mathfrak{C}_0}(\boldsymbol{b})$, for all integral points

$$\boldsymbol{b} \in (\mathcal{C} - B(X)) \cap \mathbb{Z}^s = \bigcup_{\mathfrak{C}_i \subseteq \mathcal{C}} (\mathfrak{C}_i - B(X)) \cap \mathbb{Z}^s. \tag{3.107}$$

Now, if the chamber $\mathcal{C}$ does not contain any X–region besides $\mathfrak{C}_0$, there is nothing left to prove. Else, let $\mathfrak{C} \subseteq \mathcal{C}$ denote an X–cone different from $\mathfrak{C}_0$. In this case there exists a finite sequence of pairwise distinct X–cones $\mathfrak{C}_0, \mathfrak{C}_1, \ldots, \mathfrak{C}_{n-1}, \mathfrak{C}_n = \mathfrak{C}$, each of them contained in the chamber $\mathcal{C}$, with the property that consecutive elements of this sequence meet the condition that

$$\overline{\mathfrak{C}_{i-1}} \cap \overline{\mathfrak{C}_i} \subsetneq (H_{Y_i^c} \cap \mathrm{cone}(X)) \setminus \mathrm{cone}(Y_i^c), \tag{3.108}$$

where Y_i denotes an appropriate cocircuit of X. The existence of such a sequence follows from the fact that chambers of X are convex sets and that $\overline{\mathcal{C}} = \bigcup_{\mathfrak{C}_i \subseteq \mathcal{C}} \overline{\mathfrak{C}_i}$.

We may now conclude our proof by induction with respect to the length n of the sequence, once more by applying the wall crossing formula, Theorem 3.44. The initial step of our induction follows by Proposition 3.71, as already mentioned above. For the induction step, we assume that the vector partition function $\Phi_X(\boldsymbol{b})$ agrees with the function $f_{\mathfrak{C}_0}(\boldsymbol{b})$ on all points $\boldsymbol{b} \in (\mathfrak{C}_i - B(X)) \cap \mathbb{Z}^s$, for $0 \leq i \leq n-1$, and demonstrate that this implies that $\Phi_X(\boldsymbol{b}) = f_{\mathfrak{C}_0}(\boldsymbol{b})$ for all integral points $\boldsymbol{b} \in (\mathfrak{C}_n - B(X)) \cap \mathbb{Z}^s$. To this end, we need the following observation.

The set $\overline{\mathfrak{C}}_{n-1} \cap \overline{\mathfrak{C}}_n \subseteq H_{Y_n^c}$ is an $s-1$ dimensional cone. Thus, there are $s-1$ linearly independent points $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_{s-1}$ contained in $\overline{\mathfrak{C}}_{n-1} \cap \overline{\mathfrak{C}}_n$. Now, let $\mathcal{P} \subseteq H_{Y_n^c}$ be an $(s-1)$–polytope with vertices $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$. Then, each of these vertices has a unique representation as $\boldsymbol{v}_i = \lambda_{i,1}\boldsymbol{c}_1 + \ldots + \lambda_{i,s-1}\boldsymbol{c}_{s-1}$, for some reals $\lambda_{i,1}, \ldots, \lambda_{i,s-1}$. Setting $\mu_j := \max_{1 \leq i \leq m}\{0, |\lambda_{i,j}| : \lambda_{i,j} \leq 0\}$ for all $1 \leq j \leq s-1$, we obtain that

$\boldsymbol{p}+\mathcal{P} \subseteq \overline{\mathfrak{C}}_{n-1} \cap \overline{\mathfrak{C}}_n$ for all points $\boldsymbol{p} = \nu_1 \boldsymbol{c}_1 + \ldots + \nu_{i,s-1}\boldsymbol{c}_{s-1}$ with $\nu_j \geq \mu_j$ for all $1 \leq j \leq s-1$. Hence, any polytope $\mathcal{P} \subseteq H_{Y_n^c}$ can be translated by some vector $\boldsymbol{p} \in \overline{\mathfrak{C}}_{n-1} \cap \overline{\mathfrak{C}}_n$, such that the resulting set $\boldsymbol{z}+\mathcal{P}$ is located in the cone $\overline{\mathfrak{C}}_{n-1} \cap \overline{\mathfrak{C}}_n$.

Now let $\mathfrak{c}_{0,n-1}$ and $\mathfrak{c}_{0,n}$ be the unique initial X–regions in the X–cones $\mathfrak{C}_{n-1}$ and $\mathfrak{C}_n$, respectively. Note that (3.108) implies that $\mathfrak{c}_{0,n-1}$ and $\mathfrak{c}_{0,n}$ are adjacent and Y_n separated. Hence, we have that the set $\mathfrak{f}_{0,n} := (\bar{\mathfrak{c}}_{0,n-1} \cap \bar{\mathfrak{c}}_{0,n})^\circ$ is located on the hyperplane $H_{Y_n^c}$. According to our previous considerations there exists an integral vector $\boldsymbol{p} \in \overline{\mathfrak{C}}_{n-1} \cap \overline{\mathfrak{C}}_n$ such that the translated polytope $\boldsymbol{p} + \mathfrak{f}_{0,n} - B(Y_n^c)$ is located in the set $\overline{\mathfrak{C}}_{n-1} \cap \overline{\mathfrak{C}}_n$. Setting $\mathfrak{c}_{n-1} := \mathfrak{c}_{0,n-1} + \boldsymbol{p}$ and $\mathfrak{c}_n := \mathfrak{c}_{0,n} + \boldsymbol{p}$ we obtain two adjacent X–regions $\mathfrak{c}_{n-1} \subseteq \mathfrak{C}_{n-1}$ and $\mathfrak{c}_n \subseteq \mathfrak{C}_n$ with the property that

$$\delta(\mathfrak{f}_n|Y_n^c) = (\mathfrak{f}_n - B(Y_n^c)) \cap \mathbb{Z}^s \subseteq (H_{Y_n^c} \cap \operatorname{cone}(X)) \setminus \operatorname{cone}(Y_n^c), \tag{3.109}$$

where we set $\mathfrak{f}_n = \boldsymbol{p}+\mathfrak{f}_{0,n}$. Since the latter set contains solely regular points of $\operatorname{cone}(X)$, the vector partition function solves the difference equation

$$\nabla_{Y_n^c}\Phi_X(\boldsymbol{b}) = 0 \text{ for all } \boldsymbol{b} \in \delta(\mathfrak{f}_n|Y_n^c). \tag{3.110}$$

Therefore, we may apply the wall crossing formula to calculate the values of the function Φ_X on $\delta(\mathfrak{c}_n|X)$ from the according function values on the set $\delta(\mathfrak{c}_{n-1}|X)$. Now, by our induction hypothesis, we have that the vector partition function Φ_X agrees with the function $f_{\mathfrak{C}_0} \in \mathrm{DM}(X)$ at all points in $\delta(\mathfrak{c}_{n-1}|X)$. Therefore, by the wall crossing formula, we may calculate the function values of $f_{\mathfrak{C}_0}$ on the set $\delta(\mathfrak{c}_n|X)$ from the same values. Thus we get that $\Phi_X(\boldsymbol{b}) = f_{\mathfrak{C}_0}(\boldsymbol{b})$ for all points $\boldsymbol{b} \in \delta(\mathfrak{c}_n|X)$. On the other hand, by Proposition 3.71, there exists a unique function $f_{\mathfrak{C}_n} \in \mathrm{DM}(X)$ such that

$$\Phi_X(\boldsymbol{b}) = f_{\mathfrak{C}_n}(\boldsymbol{b}) \text{ for all } \boldsymbol{b} \in (\mathfrak{C}_n - B(X)) \cap \mathbb{Z}^s. \tag{3.111}$$

Since the functions in the space $\mathrm{DM}(X)$ are uniquely determined by the function values on a set $\delta(\mathfrak{c}|X)$, we conclude that $f_{\mathfrak{C}_n}(\boldsymbol{b}) = f_{\mathfrak{C}_0}(\boldsymbol{b})$ for all $\boldsymbol{b} \in \mathbb{Z}^s$. Therefore, and by Proposition 3.71, we conclude that the vector partition function Φ_X agrees also with the function $f_{\mathfrak{C}_0}$ on the set $\boldsymbol{b} \in (\mathfrak{C}_n - B(X)) \cap \mathbb{Z}^s$. □

Summarizing, we may formulate the following result as a corollary of the previous theorem and the properties of the functions in $\mathrm{DM}(X)$.

Theorem 3.73. *There exists a finite decomposition of $\mathbb{R}^s$ into chambers such that the vector partition function Φ_A agrees with a multivariate quasi–polynomial in $\boldsymbol{b} \in \mathbb{Z}^s$ on each of these chambers. The degree of these quasi–polynomials equals $d - s$ which gives at the same time the dimension of the polytope $P_A(\boldsymbol{b}) = \{\boldsymbol{x} \in \mathbb{R}^d : A\boldsymbol{x} = \boldsymbol{b}\}$. Moreover, for neighboring chambers the corresponding quasi–polynomials coincide along a strip near their common boundary.*

Remark 3.74. The specific functions in the space DM(X) which agree, by Theorem 3.72, with the vector partition function Φ_X on particular chambers of cone(X), satisfy a remarkable recurrence relation.

In fact, suppose that $\mathfrak{c}_0 \subseteq \text{cone}(X)$ is an initial X–region. The function $f_{\mathfrak{c}_0} \in \text{DM}(X)$, determined by the condition that

$$f_{\mathfrak{c}_0}(\boldsymbol{b}) = \begin{cases} 1 & \text{if } \boldsymbol{b} = \boldsymbol{0} \\ 0 & \text{if } \boldsymbol{b} \in \delta(\mathfrak{c}_0|X)\backslash\{0\}, \end{cases} \tag{3.112}$$

satisfies the relation

$$(-1)^{d-s} f_{\mathfrak{c}_0}(-\boldsymbol{b}) = f_{\mathfrak{c}_0}(\boldsymbol{b} - \boldsymbol{a}_X) \text{ for all } \boldsymbol{b} \in \mathbb{Z}^s. \tag{3.113}$$

A simple proof of this fact may be found in [6]. To enlighten the meaning of this relation with regard to the number integral points in rational polytopes, we make the following observation.

Let A be the $s \times d$ matrix whose column vectors are given by the elements of the list X and $\boldsymbol{b} \in \mathbb{Z}^s$ an integral vector. Then, the integral points in the interior of the polytope $P_A(\boldsymbol{b})$ are given by

$$\begin{aligned} P_A(\boldsymbol{b})^\circ \cap \mathbb{Z}^s &= \{\boldsymbol{x} \in \mathbb{Z}^d_{>0} : x_1\boldsymbol{a}_1 + \ldots + x_d\boldsymbol{a}_d = \boldsymbol{b}\} \\ &= \{\boldsymbol{x} \in \mathbb{Z}^d_{\geq 0} : x_1\boldsymbol{a}_1 + \ldots + x_d\boldsymbol{a}_d = \boldsymbol{b} - \boldsymbol{a}_x\}. \end{aligned} \tag{3.114}$$

Now, if the vector $\boldsymbol{b} \in \mathbb{Z}^s$ lies in the chamber $\mathcal{C}$, then $\boldsymbol{b} - \boldsymbol{a}_X \in (\mathcal{C} - B(X)) \cap \mathbb{Z}^s$. Thus, the vector partition function Φ_X agrees with the function $f_{\mathcal{C}} \in \text{DM}(X)$ at the point $\boldsymbol{b}$ as well as at the point $\boldsymbol{b} - \boldsymbol{a}_X$, cf. Theorem 3.72. Therefore, we conclude from (3.113) and (3.114), that the evaluation of the function $f_{\mathcal{C}} \in \text{DM}(X)$ at the point $-\boldsymbol{b}$ gives $-(1)^{\dim(P_A(\boldsymbol{b}))}$ times the number of integral points in the interior of the polytope $P_A(\boldsymbol{b})$. Note that this result generalizes the Ehrhart–Macdonald Reciprocity of Ehrhart quasi–polynomials, cf. Theorem 2.28.

List of Symbols

$\mathrm{proj}_{\mathrm{k}}(P)$	the projection in direction $\boldsymbol{e}_k$ of a set $P \subseteq \mathbb{R}^n$, page 19
$\mathrm{reg}(X)$	the set of regular points with respect to the list X, page 100
$\mathcal{P}^\circ$	the relative interior of a polytope $\mathcal{P}$, page 38
$\partial\mathcal{P}$	the boundary of a polytope $\mathcal{P}$, page 38
$\Phi_A(\boldsymbol{b})$	the vector partition function of a matrix A, page 97
$\Pi_{\mathcal{K}}$	the fundamental parallelepiped of the cone $\mathcal{K}$, page 67
Π_Λ	the fundamental parallelepiped of the lattice Λ, page 59
$\mathbb{R}$	the set of real numbers, page 12
$\mathbb{R}^n$	the n–dimensional real vector space, page 12
$\mathbb{R}^n_\geq$	the non–negative orthant of $\mathbb{R}^n$, page 12
$\sigma_S(\boldsymbol{z})$	the integer–point transform of a set $S \subseteq \mathbb{R}^n$, page 66
$\mathrm{span}(M)$	the linear span of a set $M \subseteq \mathbb{R}^n$, page 13
$\mathrm{vert}(\mathcal{P})$	the vertex set of a polytope $\mathcal{P}$, page 26
$\lvert\lvert\boldsymbol{x}\rvert\rvert$	the Euclidian norm of the vector $\boldsymbol{x} \in \mathbb{R}^n$, page 15
$\lvert\lvert\boldsymbol{x}\rvert\rvert_1$	the 1–norm of a vector $\boldsymbol{x} \in \mathbb{R}^s$, $\lvert\lvert\boldsymbol{x}\rvert\rvert_1 := \sum_{i=1}^s \lvert x_i\rvert$, page 30
$\mathrm{vol}(S)$	the volume of the set S, page 77
$\mathrm{vol}_{\mathcal{R}}(\mathcal{P})$	the relative volume of a polytope $\mathcal{P}$, page 80
$\mathbb{Z}^n$	the set of integral points in $\mathbb{R}^n$, page 12
$\mathbb{Z}_{\geq 0}$	the set of non–negative integers, page 12
$B(X)$	the zonotope associated with the list X, page 106
d–polytope	a d–dimensional polytope, page 18
H_{Y^c}	the hyperplane related to a cocircuit $Y \in \mathcal{Y}(X)$, page 104
$L \oplus M$	direct sum of two disjoint sets L and M, page 44
$L^\perp$	the orthogonal complement of a linear sub–space $L \subseteq \mathbb{R}^n$, page 16
$P = P(A, \boldsymbol{b})$	the $\mathcal{H}$–polyhedron specified by the matrix A and the vector $\boldsymbol{b}$, page 17
$P \cong Q$	affinely isomorphic sets $P \subseteq \mathbb{R}^n$ and $Q \subseteq \mathbb{R}^m$, page 15
$P_A(\boldsymbol{b})$	the variable polytope related to the matrix A and the vector $\boldsymbol{b}$, page 97
tS	the t-th dilation of a subset $S \subseteq \mathbb{R}^n$, page 63

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