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Approximation of CR Functions

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Abstract

In this present work we are going to discuss several approximation results in Complex Analysis. Starting with the classical approximation theorems by Runge and Mergelyan we mainly investigate approximation of CR functions on CR submanifolds in $\mathbb{C}^n$ by entire functions. These generalized versions of holomorphic functions on CR submanifolds can always be approximated locally on CR submanifolds by entire functions. This result is covered by the Baouendi-Treves-Approximation Theorem. More recent works treat global approximation of CR functions. We will present the hypersurface case and discuss the situation in higher codimensions.

Zusammenfassung

In dieser Arbeit werden wir einige Approximationsresultate in der Komplexen Analysis diskutieren. Beginnend mit den klassischen Sätzen von Runge und Mergelyan wird hauptsächlich Approximation von CR Funktionen auf CR Teilmannigfaltigkeiten des $\mathbb{C}^n$ durch ganze Funktionen behandelt. Diese verallgemeinerten holomorphen Funktionen auf CR Teilmannigfaltigkeiten können lokal immer durch ganze Funktionen approximiert werden, was mit dem Satz von Baouendi-Treves bewiesen wird. Neuere Arbeiten untersuchen globale Approximation von CR Funktionen. Hier werden wir den Fall einer Hyperfläche und die Situation in höheren Kodimensionen betrachten.

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0 Introduction and Preparations

0.1 Introduction

An interesting question in function theory is under which conditions holomorphic functions can be approximated uniformly by entire functions.

The classical Runge's Theorem gives a complete solution in the complex plane: The uniform approximation of holomorphic functions in a neighbourhood of a compactum K, denoted by $\mathcal{H}(K)$, by entire functions is equivalent for K being "holomorphic convex". Now one could think of different generalizations of this way of posing a problem. On the one hand we can replace the set $\mathcal{H}(K)$ by a different function algebra $A(K)$, which consists of functions being holomorphic in the interior of K and continuous up to the boundary of K. The question is under which assumptions on K the entire functions are dense in $A(K)$. Again the notion of "holomorphic convexity" of K turns out to be useful and is at least sufficient solving this problem. This considerations are treated in Mergelyan's Theorem and both classical Theorems are presented in Chapter 1.

Another way of generalizing the problem is, if we want to approximate functions on real submanifolds of $\mathbb{C}^n$. Here CR submanifolds proved to be useful. These objects are characterized by the dimension of the space of antiholomorphic tangent vectors, which has to be independent of the point p on the submanifold. The functions which we want to approximated by polynomials in this setting are so called CR functions, that means functions, which vanish locally along the anitholomorphic vector fields. But that means that CR functions are generalizations of holomorphic functions on CR submanifolds. Since the corresponding one dimensional approximation results are of global character a local result as treated in Chapter 2 in the Baouendi-Treves-Approximation Theorem is interesting, but unsatisfactory in this approach.

More recent results investigate global approximation on CR submanifolds and seek for sufficient conditions for which compacta such approximation processes are possible. In Chapter 3 we present such a work from [1] in the case, where the CR submanifold is a hypersurface graph.

In higher codimensions the situation is more involved. In the case of a special global graph, a so called "Bloom-Graham model graph", global approximation of CR functions is possible, as can be found in [2]. We will show, following [1], that in general in higher codimension CR functions on compact subsets of global graphs cannot be approximated uniformly by entire functions and the analogous statement as in the hypersurface case fails to hold.

0.2 Strict Convexity and Pseudoconvexity

Introduction 0.1. This section is dedicated to investigate the connection between strict convexity and strict pseudoconvexity we need in Chapter 3. We will see that strict pseudoconvexity is a generalization of strict convexity. The notion of pseudoconvexity is of great importance in Complex Analysis, since one can solve the $\bar{\partial}-$equation on a set which is pseudoconvex and has $\mathcal{C}^2$-boundary. We follow the presentation in [3].

Definition 0.2. The open set $D \subset \mathbb{R}^n$ has a **differentiable boundary bD of class $\mathcal{C}^k$ for $1 \leq k \leq \infty$ at the point $p \in bD$** if there is an open neighbourhood U of p and a real-valued function $r \in \mathcal{C}^k(U)$, such that

(i) $U \cap D = \{x \in U : \ r(x) < 0\}$

(ii) $dr_x := \sum_{j=1}^{n} \frac{\partial r}{\partial x_j}(x) dx_j \neq 0$ for $x \in U$

hold. bD is of class $\mathcal{C}^k$ if it is of class $\mathcal{C}^k$ at every $p \in bD$.
A function $r \in \mathcal{C}^k(U)$, which satisfies (i) and (ii) from above is called a **local defining function for D at p**. If U is a neighbourhood of bD a function $r \in \mathcal{C}^k(U)$, which satisfies (i) and (ii) from above is called a **global defining function for D**.

Remark 0.3. As in Remark 2.3 one can show that for two local defining functions $r_1, r_2 \in \mathcal{C}^k(U)$ for D at $p \in bD$, where U is a neighbourhood of p, one can find a positive function $h \in \mathcal{C}^{k-1}(U)$, such that $r_1 = h \cdot r_2$ on U. Moreover the differentials fulfill $dr_1(x) = h(x) \cdot dr_2(x)$ for $x \in U \cap bD$.

Lemma 0.4. *For $D \subset\subset \mathbb{R}^n$ with $\mathcal{C}^k$ boundary there exists a global defining function r for D.*

Proof. We will glue local defining functions together with a partition of unity to obtain a global defining function.
For every $p \in bD$ we obtain an open set U_p and a local defining function $r_p \in \mathcal{C}^k(U_p)$ with all the properties of Definition 0.2. $(U_p)_{p \in bD}$ is an open cover of bD and by compactness of bD we obtain finitely many sets U_j, $1 \leq j \leq N$, such that $bD \subset U \subset \bigcup_{j=1}^{N} U_j$. Now we take a partition of unity subordinate to this cover φ_j and define $r := \sum_{j=1}^{N} \varphi_j r_j$.
Then $r \in \mathcal{C}^k(\mathbb{R}^n)$. Next we want to see, that $dr_x \neq 0$ for all $x \in U'$, where U' is an appropriate neighbourhood of bD. We begin with some $x \in bD$, then $x \in U_{j_1} \cap U_{j_2}$ for $1 \leq j_1 < j_2 \leq N$, where the local defining functions r_{j_1} and r_{j_2} have the properties from Remark 0.3. Any local defining function r_j, where $j \neq j_1, j_2$ is zero at x. Then we translate D, such that $x = 0$ and rotate $\mathbb{R}^n$, such that grad $r_{j_i}(x) = \left(0, \ldots, 0, \frac{\partial r_{j_i}}{\partial x_n}(x) > 0\right)$ for $i = 1, 2$. Thus $dr_x = dr_{j_1 x} + dr_{j_2 x} = \sum_{i=1}^{2} \frac{\partial r_{j_i}}{\partial x_n}(x) dx_n \neq 0$. Since dr_x is continuous, $dr_x \neq 0$ holds in a small neighbourhood of bD in $U_{j_1} \cap U_{j_2}$. Hence we obtain $dr_x \neq 0$ for all $x \in U$, where U is a neighbourhood of bD.
If $x \in U \cap D$, then $x \in U_j \cap D$ and $\varphi_j(x) > 0$ for some j, thus $r(x) \leq \varphi_j(x) r_j(x) < 0$. Conversely, if $r(x) < 0$ for $x \in U$, then there exist some j with $r_j(x) < 0$, hence $x \in U_j \cap D \subset U \cap D$. □

Remark 0.5. (i) From now on we do not distinguish between "local" and "global" defining functions.

(ii) Definition 0.2 says that $bD \cap U$ is a closed $\mathcal{C}^k$ real submanifold of U of dimension $n-1$ for some neighbourhood U of $p \in bD$. More precisely $U \cap bD = \{x \in U : \ r(x) = 0\}$, thus we can define the tangent space as in Definition 2.4 to be $T_p(bD) := \left\{\xi \in \mathbb{R}^n :\ dr_p(\xi) := \sum_{j=1}^n \frac{\partial r}{\partial x_j}(p)\xi_j = 0\right\}$, which is called the **tangent space $T_p(bD)$ to bD at p**.

Definition 0.6. Let $p \in bD$ and r a defining function for D.

$$T_p^c(bD) := T_p(bD) \cap iT_p(bD)$$

is called the **complex tangent space $T_p^c(bD)$ to bD at p**.

Remark 0.7. Compare this definition with Remark 2.10 (iii). The complex tangent space has complex dimension $n-1$ as can be seen in the following

Lemma 0.8. *Let $p \in bD$ and r be a defining function for $D \subset \mathbb{C}^n$, then*

$$T_p^c(bD) = \left\{ t \in \mathbb{C}^n :\ \partial r_p(t) := \sum_{j=1}^n \frac{\partial r}{\partial z_j}(p) t_j = 0 \right\}.$$

Proof. Since r is real-valued it is $dr_p(t) = \partial r_p(t) + \bar{\partial} r_p(t) = 2\mathrm{Re}(\partial r_p(t))$ together with $\mathrm{Re}(\partial r_p(it)) = \mathrm{Re}(i\partial r_p(t)) = -\mathrm{Im}(\partial r_p(t))$ we obtain

$$T_p^c(bD) = \{t \in \mathbb{C}^n :\ dr_p(t) = dr_p(it) = 0\} = \{t \in \mathbb{C}^n :\ \mathrm{Re}(\partial r_p(t)) = \mathrm{Im}(\partial r_p(t)) = 0\}.$$

□

Definition 0.9 (strict convexity)**.** Let $p \in bD$ and r be a defining function for $D \subset \mathbb{R}^n$.
Then D is called **strictly convex at $p \in bD$**, if

$$d_2 r_p(\xi) := \sum_{j,k=1}^n \frac{\partial^2 r}{\partial x_j \partial x_k}(p)\xi_j \xi_k > 0 \tag{0.2.1}$$

for all $\xi \in T_p(bD)$ with $\xi \neq 0$.

Definition 0.10 (strict pseudoconvexity)**.** Let r be a real-valued $\mathcal{C}^2$ function defined in an open set $D \subset \mathbb{C}^n$.
Then r is called **strictly plurisubharmonic at $p \in bD$**, if

$$L_p(r,t) := \sum_{j,k=1}^n \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k}(p) t_j \bar{t}_k > 0$$

for all $t \in \mathbb{C}^n$ with $t \neq 0$.
If r is strictly plurisubharmonic at every $p \in bD$, then r is called **strictly plurisubharmonic on D**.
A bounded domain D in $\mathbb{C}^n$ is called **strictly pseudoconvex**, if there is a neighbourhood U of bD and a strictly plurisubharmonic function $r \in \mathcal{C}^2(U)$, such that

$$D \cap U = \{z \in U :\ r(z) < 0\}.$$

Remark 0.11. The function r in the definition above is in general not a defining function for D, since we do not assume that $d_z r \neq 0$, hence a strictly pseudoconvex domain need not have a $\mathcal{C}^2$ boundary. Nevertheless we will show that a bounded, strictly convex domain with smooth boundary is strictly pseudoconvex.

Theorem 0.12. *A bounded, strictly convex domain $D \subset\subset \mathbb{C}^n$ with smooth boundary (of class at least $\mathcal{C}^2$) is strictly pseudoconvex.*

Proof. Let $p \in bD$ and let r be a defining function for D as in Lemma 0.4. First we will show that the complex analogous of (0.2.1) holds, i. e.,

$$L_p(r,t) = \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k}(p) t_j \bar{t}_k > 0 \tag{0.2.2}$$

for $t \in T_p^c(bD)$ with $t \neq 0$. This condition is refered as "strict Levi pseudoconvexity". We assume to have coordinates $(z_1, \dots, z_n)$ for $\mathbb{C}^n$ resp. $\mathbb{R}^{2n}$ with coordinates $(x_1, x_2, \dots, x_{2n-1}, x_{2n})$, where $x_{2j-1} = \frac{1}{2}(z_j + \bar{z}_j)$ and $x_{2j} = \frac{1}{2i}(z_j - \bar{z}_j)$ for $1 \leq j \leq n$. Now we Taylor expand r at p in the complex form. For $t \in \mathbb{C}^n$ we have

$$\begin{aligned} r(p+t) &= r(p) + \sum_{j=1}^{n} \left(\frac{\partial r}{\partial z_j}(p) t_j + \frac{\partial r}{\partial \bar{z}_j}(p) \bar{t}_j \right) \\ &\quad + \sum_{j,k=1}^{n} \left(\frac{1}{2} \frac{\partial^2 r}{\partial z_j \partial z_k}(p) t_j t_k + \frac{1}{2} \frac{\partial^2 r}{\partial \bar{z}_j \partial \bar{z}_k}(p) \bar{t}_j \bar{t}_k + \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k}(p) t_j \bar{t}_k \right) + o(|t|^2) \\ &= r(p) + 2\mathrm{Re}(\partial_p r(t) + Q_p(r,t)) + L_p(r,t) + o(|t|^2), \end{aligned}$$

where $Q_p(r,t) := \frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial z_k}(p) t_j t_k$. This shows that the real Hessian of r at $p \in \mathbb{R}^{2n}$ from (0.2.1) in terms of $Q_p(r,t)$ and $L_p(r,t)$ is given by

$$\frac{1}{2} \sum_{j,k=1}^{2n} \frac{\partial^2 r}{\partial x_j \partial x_k}(p) \xi_j \xi_k = 2\mathrm{Re} Q_p(r,t) + L_p(r,t),$$

where $t = (t_1, \dots, t_n)$ with $t_j = \xi_{2j-1} + i\xi_{2j}$ for $1 \leq j \leq n$. This implies, by the strict convexity of D, $2\mathrm{Re}Q_p(r,t) + L_p(r,t) > 0$. We have $Q_p(r, it) = -Q_p(r,t)$ and $L_p(r, it) = L_p(r,t)$. Thus $L_p(r,t) > 0$ for any $t \in T_p^c(bD)$ with $t \neq 0$.

Next we need to find a defining function $\tilde{r}$ for D on a neighbourhood U of bD, which fulfills $L_p(\tilde{r}, t) > 0$ for every $t \in \mathbb{C}^n$ with $t \neq 0$ and $p \in U$. We will show that $\tilde{r} := \exp(Ar) - 1$ for some appropriate $A > 0$ is a correct choice.

We first show that $\tilde{r}$ is a defining function for D. Let U denote the neighbourhood of bD for r, where all the properties of Definition 0.2 hold. Then $\tilde{r}(x) < 0$ only for $x \in U \cap D$ and $d\tilde{r}(x) = dr(x) A \exp(Ar(x)) \neq 0$ for all $x \in U$. Now we compute

$$\begin{aligned} L_z(\tilde{r}, t) &= A \exp(Ar(z)) \left(\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k}(z) t_j \bar{t}_k + A \sum_{j,k=1}^{n} \frac{\partial r}{\partial z_j}(z) t_j \frac{\partial r}{\partial \bar{z}_k}(z) \bar{t}_k \right) \\ &= A \exp(Ar(z)) \left(L_z(r,t) + A |\partial_z r(t)|^2 \right) \end{aligned}$$

We claim that for an appropriate choice of A, this implies $L_z(\tilde{r}, t) > 0$ for all $0 \neq t \in \mathbb{C}^n$ and $z \in bD$. Define the compact set $S := bD \times \{t \in \mathbb{C}^n : \ |t| = 1\}$. Since $L_z(r, t)$ is continuous on S, the closed subset $K := \{(z, t) \in S : \ L_z(r, t) \leq 0\}$ is compact. For $(z, t) \in K$ we obtain, since $L_{z'}(r, t') > 0$ for every $0 \neq t' \in T^c_{z'}(bD)$ by the first part of the proof, $\partial_z r(t) \neq 0$, i. e., $|\partial_z r(t)|^2 > 0$ using Lemma 0.8. By compactness $C := \min\{|\partial_z r(t)|^2 : \ (z, t) \in K\} > 0$ and $M := \min\{L_z(r, t) : \ (z, t) \in S\}$ exist. For $(z, t) \in K$ we choose A large enough to satisfy $A \cdot C + M > 0$, then we obtain

$$L_z(\tilde{r}, t) = A^2 \exp(Ar)|\partial_z r(t)|^2 + A \exp(Ar) L_z(r, t) \geq A \exp(Ar)(A \cdot C + M) > 0.$$

For $(z, t) \in S \setminus K$, where $\partial_z r(t) = 0$, we have by Lemma 0.8 that

$$L_z(\tilde{r}, t) = A \exp(Ar) L_z(r, t) > 0.$$

In total we obtain that $L_z(\tilde{r}, t) > 0$ for $0 \neq t \in \mathbb{C}^n$ and $z \in bD$. We also showed that $M > 0$ from above. Let $t \in \mathbb{C}^n$ with $|t| = r > 0$, then we have

$$L_z(\tilde{r}, t) = r^2 L_z\left(\tilde{r}, \frac{t}{r}\right) \geq M r^2 = M|t|^2.$$

By the continuity of the second order derivative of $\tilde{r}$ the above statement holds for $0 < c < M$ and $z \in U$, a neighbourhood of bD. □

0.3 A Sufficient Condition for Approximation by Polynomials

Introduction 0.13. In this section we want to show that holomorphic functions defined in a polynomially convex set can be approximated uniformly by polynomials. The part about the polynomial convexity is taken from [4]. The proof for the existence of a polynomial polyhedron is taken from [5] and the rest of this chapter we follow [6].

Analogous to the definition of holomorphic convexity in Definition 1.4 we define the polynomially convex hull. Let $\mathcal{P}$ denote the algebra of holomorphic polynomials on $\mathbb{C}^n$.

Definition 0.14. Let $K \subset \mathbb{C}^n$ be compact.

(i) The **polynomially convex hull $\widehat{K}_{\mathcal{P}}$ of K** is defined as

$$\widehat{K}_{\mathcal{P}} := \{z \in \mathbb{C}^n : \ |p(z)| \leq \|p\|_K, \ \forall p \in \mathcal{P}\}.$$

(ii) If $K = \widehat{K}_{\mathcal{P}}$, then K is called **polynomially convex**.

Remark 0.15. (i) Since polynomials are dense in the entire functions $\mathcal{H}(\mathbb{C}^n)$ we could consider entire functions instead of polynomials in Definition 0.14 and see, that polnomial convexity is a special case of holomorphic convexity.

(ii) If K is polynomially convex, then it is compact. Also similarly as in the case of holomorphic convexity we have the following

Lemma 0.16. *Every compact and convex set in $\mathbb{C}^n$ is polynomially convex.*

Proof. Let $K \subset \mathbb{C}^n$ be compact and convex and $z_0 \in \complement K$. There exist a ball B around z_0 with $B \cap K = \varnothing$. By the Hahn-Banach separation theorem there exists a continuous, $\mathbb{C}$-linear functional l, such that, $\mathrm{Re}(l(z)) > 0$ for $z \in U$ and $\mathrm{Re}(l(z)) < 0$ for $z \in K$. The function $f(z) := \exp(l(z))$ is an entire function and has the property that

$$|\exp(l(z_0))| = \exp(\mathrm{Re}(l(z_0))) > 1 > \exp(\mathrm{Re}(l(z))) = |\exp(l(z))|, \ \forall z \in K.$$

Hence $|f(z_0)| > \|f\|_K$, i. e., K is polynomially convex. □

Lemma 0.17. *Let K be polynomially convex and U a neighbourhood of K.*
Then there exist polynomials $p_1, \dots, p_m$, such that

$$K \subset L := \{z \in \mathbb{C}^n : \ |p_j(z)| \le 1, \ 1 \le j \le m\} \subset U.$$

L is called a ***polynomial polyhedron****, which is polynomially convex.*

Proof. Let U be a neighbourhood of K. W. l. o. g. we assume, that $\bar{U}$ is compact, otherwise take a bounded neighbourhood U' of K with $U' \subset U$. Then bU is compact and $bU \cap K = \varnothing$. Since K is polynomially convex, we have for every $z \in bU$, that there exists a polynomial p_z, such that $|p_z(z)| > \|p_z\|_K$. This inequality holds in a neighbourhood U_z of z by continuity of the polynomials. Next we cover bU with finitely many neighbourhoods $U_{z_1}, \dots, U_{z_m}$ and obtain $p_1, \dots, p_m$ polynomials, such that $|p_j(z)| > \|p_j\|_K$ for $z \in U_{z_j}$ and $1 \le j \le m$. Since K is compact we can multiply p_j with $1/\|p_j\|_K$ such that $\|p_j\|_K \le 1$ and that for any $z \in U_j$ we have $|p_j(z)| > 1$. Now we define $L := \{z \in U : \ |p_j(z)| \le 1, \ 1 \le j \le m\}$. Then we have $K \subset L$ and L is a compact subset of U.
Moreover L is polynomially convex. L is compact and the inclusion $L \subseteq \widehat{L}_{\mathcal{P}}$ is always true. If $z \in \widehat{L}_{\mathcal{P}}$, we have that $|p_l(z)| \le \|p_l\|_L$ holds for $1 \le l \le m$. Define $\tilde{p}_l(z) := p_l(z)/\|p_l\|_L$, which fulfills $|\tilde{p}_l(z)| \le 1$, hence $z \in L$. □

Theorem 0.18. *Let $D := \{z \in \mathbb{C}^n : \ 0 < |z_j| < r_j, 1 \le j \le n\}$ be an open polydisc and $f \in \mathcal{C}^\infty_{(p,q+1)}(D)$ for $p, q \ge 0$ with $\bar{\partial} f = 0$.*
If $D' \subset\subset D$, then there exists $u \in \mathcal{C}^\infty_{(p,q)}(D)$ such that $\bar{\partial} u = f$.

Proof. We prove the statement by induction over k and assume that f doesn't involve $d\bar{z}_{k+1}, \dots, d\bar{z}_n$. For $k = 0$, we have since $f \in \mathcal{C}^\infty_{(p,q+1)}(D)$ and $q + 1 > 0$ that f must be the $0-$form and $u = 0$ solves this $\bar{\partial}-$equation.
For proving the induction step from $k - 1$ to k we reorder f and write the $(p, q + 1)-$form as $f = d\bar{z}_k \wedge g + h$, where $g \in \mathcal{C}^\infty_{(p,q)}(D)$, $h \in \mathcal{C}^\infty_{(p,q+1)}(D)$ and both are independent of $d\bar{z}_k$ and $d\bar{z}_{k+1}, \dots, d\bar{z}_n$ by assumption. We write

$$g = \sum_{|\alpha|=p} \sum_{|\beta|=q} g_{\alpha\beta} dz^\alpha \wedge d\bar{z}^\beta,$$

where we sum over increasing multi-indices $1 \le \alpha_1 < \ldots < \alpha_p \le n$ and $1 \le \beta_1 < \ldots < \beta_q \le k - 1$. $0 = \bar{\partial} f = \bar{\partial}(d\bar{z}_k \wedge g) + \bar{\partial} h$ implies that

$$0 = \bar{\partial}(d\bar{z}_k \wedge g) = \sum_{|\alpha|=p} \sum_{|\beta|=q} \sum_{j=1}^{n} \frac{\partial g_{\alpha\beta}}{\partial \bar{z}_j} d\bar{z}_j \wedge d\bar{z}_k \wedge dz^\alpha \wedge d\bar{z}^\beta.$$

Thus $\frac{\partial g_{\alpha\beta}}{\partial \bar{z}_j} = 0$ for $j \geq k+1$, i. e., $g_{\alpha\beta}$ is analytic in $z_{k+1}, \ldots, z_n$.
Now we want to solve $\frac{\partial G_{\alpha\beta}}{\partial \bar{z}_k} = g_{\alpha\beta}$. Let $\psi \in \mathcal{D}(D_k)$ be a cut-off, which is one in the set $V_k := pr_k(D'')$, which consists of the projection of $D'' \subseteq D$, a neighbourhood of $\overline{D'}$, to the z_k-coordinate. Let

$$\begin{aligned}G_{\alpha\beta}(z) :=& \frac{1}{2\pi i}\int_{D_k} \frac{\psi(\zeta)}{\zeta - z_k} g_{\alpha\beta}(z_1, \ldots, z_{k-1}, \zeta, z_{k+1}, \ldots, z_n) d\zeta \wedge d\bar{\zeta} \\ =& -\frac{1}{2\pi i}\int_{D_k} \frac{\psi(z_k - \zeta)}{\zeta} g_{\alpha\beta}(z_1, \ldots, z_{k-1}, z_k - \zeta, z_{k+1}, \ldots, z_n) d\zeta \wedge d\bar{\zeta},\end{aligned}$$

and since $z \mapsto \psi(z_k) g_{\alpha\beta}(z)$ is smooth and $\zeta \mapsto g_{\alpha\beta}/\zeta$ is integrable in D_k, we can differentiate under the integral sign, hence $G_{\alpha\beta}$ is $\mathcal{C}^\infty(D)$. Moreover, because $\zeta \mapsto \zeta^{-1}$ is integrable on any compact set we, have

$$\begin{aligned}\frac{\partial G_{\alpha\beta}}{\partial \bar{z}_k}(z) =& -\frac{1}{2\pi i}\int_{D_k} \frac{1}{\zeta}\frac{\partial}{\partial \bar{z}_k}(\psi(z_k - \zeta) g_{\alpha\beta}(z_1, \ldots, z_{k-1}, z_k - \zeta, z_{k+1}, \ldots, z_n)) d\zeta \wedge d\bar{\zeta} \\ =& \frac{1}{2\pi i}\int_{D_k} \frac{1}{z_k - \zeta}\frac{\partial}{\partial \bar{\zeta}}(\psi(\zeta) g_{\alpha\beta}(z_1, \ldots, z_{k-1}, \zeta, z_{k+1}, \ldots, z_n)) d\zeta \wedge d\bar{\zeta} \\ =& \psi(z_k) g_{\alpha\beta}(z_1, \ldots, z_k, \ldots, z_n).\end{aligned}$$

The last equality follows from Pompeiu's Formula and that ψ is compactly supported in D_k. The above equation means that $\frac{\partial G_{\alpha\beta}}{\partial \bar{z}_k} = g_{\alpha\beta}$ in D''.
If we differentiate under the integral sign w. r. t. $\bar{z}_j$ for $j \geq k+1$ we obtain that $G_{\alpha\beta}$ is analytic in $z_{k+1}, \ldots, z_n$ by the analyticity of $g_{\alpha\beta}$. Now we define

$$G := \sum_{|\alpha|=p}\sum_{|\beta|=q} G_{\alpha\beta} dz^\alpha \wedge d\bar{z}^\beta,$$

where the sum again runs over increasing multi-indices as above. Then we have in D''

$$\begin{aligned}\bar{\partial} G =& \sum_{\alpha,\beta}\sum_{j=1}^{n} \frac{\partial G_{\alpha\beta}}{\partial \bar{z}_j} d\bar{z}_j \wedge dz^\alpha \wedge d\bar{z}^\beta \\ =& \sum_{\alpha,\beta} g_{\alpha\beta} d\bar{z}_k \wedge dz^\alpha \wedge d\bar{z}^\beta + \sum_{\alpha,\beta}\sum_{j=1}^{k-1} \frac{\partial G_{\alpha\beta}}{\partial \bar{z}_j} d\bar{z}_j \wedge dz^\alpha \wedge d\bar{z}^\beta \\ =& d\bar{z}_k \wedge g + h_1,\end{aligned}$$

where h_1 is independent of $d\bar{z}_k, \ldots, d\bar{z}_n$. Thus $\bar{\partial} G - f = h_1 - h$ and since the right-hand side is independent of $d\bar{z}_k, \ldots, d\bar{z}_n$ and $\bar{\partial}(\bar{\partial} G - f) = 0 - \bar{\partial} f = 0$ we use the induction hypothesis to obtain $v \in \mathcal{C}^\infty_{(p,q)}(D')$, such that $\bar{\partial} v = \bar{\partial} G - f$. Define $u := G - v$, which solves $\bar{\partial} u = f$ in D''.
For $k = n$ we obtain the case formulated as in the theorem. □

Definition 0.19. Let K be a compact set.
K is said to have the **Cousin property**, if for every open neighbourhood U of K and every $f \in \mathcal{C}^\infty_{(p,q+1)}(U)$ for $p, q \geq 0$ with $\bar{\partial} f = 0$ in U the equation $\bar{\partial} u = f$ has a solution $u \in \mathcal{C}^\infty_{(p,q)}(U)$.

Remark 0.20. Theorem 0.18 says that any closed polydisc has the Cousin property.

Lemma 0.21. *Let D be an open polydisc in $\mathbb{C}^n$, $\mathbb{D}$ the closed unit disc in $\mathbb{C}$ and $p_1, \dots, p_m$ polynomials. Let the* ***Oka map*** $\boldsymbol{\mu}$ *be defined as*

$$\mu: \ \mathbb{C}^n \ni z \mapsto (z, p_1(z), \dots, p_m(z)) \in \mathbb{C}^{n+m}$$

and define $L_{\overline{D}} := \{z \in \overline{D}: \ |p_j(z)| \leq 1, \ 1 \leq j \leq m\}$. Let U be an open neighbourhood of $L_{\overline{D}}$. Then for every $f \in \mathcal{C}^\infty_{(p,q)}(U)$ with $\bar{\partial} f = 0$ there exists $F \in \mathcal{C}^\infty_{(p,q)}(V)$ with $\bar{\partial} F = 0$ in a neighbourhood V of $\overline{D} \times \mathbb{D}^m$, such that $f = F \circ \mu$ in U.

Proof. We prove the statement by induction over m.
For $m = 1$ let $\pi: \ \mathbb{C}^{n+1} \ni (z, w) \mapsto z \in \mathbb{C}^n$ be the projection onto $\mathbb{C}^n$, then $\pi \circ \mu = id_{\mathbb{C}^n}$. Note that $L_{\overline{D}} = \mu^{-1}(\overline{D} \times \mathbb{D})$.
Let U be a neighbourhood of $L_{\overline{D}}$, then $f \circ \pi \in \mathcal{C}^\infty_{(p,q)}(\pi^{-1}U)$ and for any $(z_0, w_0) \in \pi^{-1}U$ it is $\bar{\partial}(f \circ \pi)(z_0, w_0) = \bar{\partial} f(z_0) = 0$. Moreover $\pi^{-1}U$ is an open neighbourhood of $\mu(L_{\overline{D}}) = \{(z, w) \in \overline{D} \times \mathbb{D}: \ w = p_1(z)\}$. Let $\varphi \in \mathcal{D}(\pi^{-1}U)$ with $\varphi = 1$ in a smaller neighbourhood $\pi^{-1}U'$ of $\mu(L_{\overline{D}})$ and define $F := \varphi(f \circ \pi) - QG$, where $Q(z, w) := w - p_1(z)$ is an analytic function and G is a $(p, q)-$form defined in V.
In U' we have $F \circ \mu = (\varphi \circ \mu)(f \circ (\pi \circ \mu)) - (QG) \circ \mu = 1 \cdot f - 0 = f$, which proves the second property of F.
G is a correction term to obtain that $\bar{\partial} F = 0$ and will be determined as follows:
In V we have $\bar{\partial} F = 0$ iff $0 = \bar{\partial}\varphi \wedge (f \circ \pi) + \varphi \wedge \bar{\partial}(f \circ \pi) - \bar{\partial}(QG) = \bar{\partial}\varphi \wedge (f \circ \pi) - Q\bar{\partial} G$ iff $Q\bar{\partial} G = \bar{\partial}\varphi \wedge (f \circ \pi)$.
Then we have $\bar{\partial} G = \frac{1}{Q}\bar{\partial}\varphi \wedge (f \circ \pi) =: H$. The poles of H, which appears only in $\mu(L_{\overline{D}})$ get cancelled in $\pi^{-1}U'$, because $\bar{\partial}\varphi = 0$ there. It follows $H \in \mathcal{C}^\infty_{(p,q+1)}(V')$, where V' is a neighbourhood of $\overline{D} \times \mathbb{D}$. Moreover it is $\bar{\partial} H = \frac{1}{Q}\bar{\partial}^2\varphi \wedge (f \circ \pi) = 0$ in V'. By Theorem 0.18 the set $\overline{D} \times \mathbb{D}$ has the Cousin property, hence there exists $G \in \mathcal{C}^\infty_{(p,q)}(V')$, such that $\bar{\partial} G = H$ in V'.
For proving the induction step from $m - 1$ to m apply the first step to p_m, define the Oka map $\mathbb{C}^{n+m-1} \ni z \mapsto (z, p_m(z)) \in \mathbb{C}^m$ and take $L_{\overline{D}}$, U and V as in the theorem. □

Theorem 0.22. *Let $K \subset \mathbb{C}^n$ be a polynomially convex set. Then any $f \in \mathcal{H}(K)$ can be approximated by polynomials uniformly on K.*

Proof. Let U denote the neighbourhood of K, where f is analytic. By Lemma 0.17 there exists a polynomial polyhedron $\tilde{L}$ such that $K \subset \tilde{L} \subset U$. Now we take an open polydisc D centered at 0 with $K \subset \tilde{L} \subseteq \overline{D}$ and define $L := \{z \in \overline{D}: \ |p_j(z)| \leq 1, \ 1 \leq j \leq m\}$. Applying Lemma 0.21 to L we obtain a function F, which is analytic in a neighbourhood of $\overline{D} \times \mathbb{D}^m$ and has the property that

$$f(z) = F(z, p_1(z), \dots, p_m(z))$$

in a neighbourhood U of L. Denote as $F_k(z, w)$ the partial sums of the power series of F, then we have $F_k \to F$ uniformly in $\overline{D} \times \mathbb{D}^m$. Thus

$$\mathcal{P} \ni F_k(z, p_1(z), \dots, p_m(z)) \to F(z, p_1(z), \dots, p_m(z)) = f(z)$$

uniformly in $L \supset K$. □

1 Classical Approximation Theorems

Introduction 1.1. In the first part of this chapter we want to discuss a classical global approximation result in $\mathbb{C}$, called Runge's Theorem, which answers the question which pairs (Ω, K) fulfill that the restrictions to K of holomorphic functions in Ω are dense in the algebra of functions holomorphic in a neighbourhood of K. Then an immediate consequence is that, if $\Omega = \mathbb{C}$, holomorphic functions in K can be approximated uniformly by polynomials.
In the second part we want to describe the set of restrictions to K of holomorphic functions in Ω for $\Omega = \mathbb{C}$. This result is well-known as Mergelyan's Theorem. We follow the presentation of [7] and [8].

1.1 Runge's Theorem

Definition 1.2. Let $\Omega \subseteq \mathbb{C}$ open, $K \subset \Omega$ compact. We define

$$\begin{aligned}
&\mathcal{H}(\Omega)|_K := \{f|_K : f \in \mathcal{H}(\Omega)\} \text{ and} \\
&f \in \mathcal{H}(K) :\Longleftrightarrow \exists\, U \text{ an open set, with } K \subseteq U,\ f \text{ is defined in } U \text{ and } f \in \mathcal{H}(U).
\end{aligned}$$

Remark 1.3. We have that $\mathcal{H}(\Omega)|_K \subseteq \mathcal{H}(K)$. The question whether $\overline{\mathcal{H}(\Omega)|_K}^{\mathcal{C}(K)} = \mathcal{H}(K)$ holds true is answered by Runge's theorem.

Definition 1.4. Let $\Omega \subseteq \mathbb{C}$ open, $K \subset \Omega$ compact.

(i) The **holomorphic convex hull $\widehat{K}_\Omega$ of K in Ω** is defined as

$$\widehat{K}_\Omega := \{z \in \Omega : |f(z)| \leq \|f\|_K,\ \forall f \in \mathcal{H}(\Omega)\}$$

(ii) If $K = \widehat{K}_\Omega$, then K is called **holomorphically convex in Ω**.

Example 1.5. To motivate the notion of holomorphic convexity, we prove that the holomorphic convex hull of K is contained in the convex hull of K, which implies that if K is compact and convex, then it is holomorphically convex.

Proof. The claim follows if we prove that

$$\operatorname{conv}(K) = \{z \in \mathbb{C} : |e^{az}| \leq \sup_{\zeta \in K} |e^{a\zeta}|,\ \forall a \in \mathbb{C}\} \tag{1.1.1}$$

and for $K \subset \Omega_1 \subseteq \Omega_2$, Ω_j open sets, we have $\widehat{K}_{\Omega_1} \subseteq \widehat{K}_{\Omega_2} \cap \Omega_1$.
By denoting the right-hand side of (1.1.1) with $\widetilde{K}$ and taking $z, w \in \widetilde{K}$, $\lambda, \mu > 0$ with $\lambda + \mu = 1$ we

have

$$|e^{a(\lambda z+\mu w)}| = |e^{az}|^{\lambda}|e^{aw}|^{\mu} \leq \sup_{\zeta\in K}|e^{a\zeta}|,$$

hence $\widetilde{K}$ is convex and since $K \subseteq \widetilde{K}$ we obtain $\operatorname{conv}(K) \subseteq \widetilde{K}$.
For the converse direction we let $z_0 \in \complement(\operatorname{conv}(K))$. By the Hahn-Banach separation theorem we obtain a straight line $g = \{z \in \mathbb{C}|\ l(z) := \operatorname{Re}((a-ib)z + c) = 0,$ for some $a, b, c \in \mathbb{R}\}$ such that $\operatorname{conv}(K) \subset \{w \in \mathbb{C}|\ l(w) < 0\}$ and $l(z_0) > 0$. Now we choose $u = a - ib$ and add c from the definition of g to the inequality at the right-hand side and we obtain since $K \subseteq \operatorname{conv}(K)$

$$l(z_0) > 0 \geq \sup_{\zeta\in K} l(\zeta) = \sup_{\zeta\in K} \operatorname{Re}((a-ib)\zeta + c).$$

Using the equivalence $|e^{uz_0}| > \sup_{\zeta\in K}|e^{u\zeta}| \iff \operatorname{Re}(uz_0) > \sup_{\zeta\in K}\operatorname{Re}(u\zeta)$ for $u \in \mathbb{C}$, we get $z_0 \in \complement\widetilde{K}$, i. e., $\widetilde{K} \subseteq \operatorname{conv}(K)$. Since $z \mapsto e^{az}$ is an entire function for all $a \in \mathbb{C}$, by monotonicity of the exponential function and by (1.1.1) we have that

$$\widehat{K}_{\mathbb{C}} \subseteq \operatorname{conv}(K) = \bigcap_{\varphi\in L_{\mathbb{R}}(\mathbb{C})} \{z \in \mathbb{C}|\ |\varphi(z)| \leq \sup_K \|\varphi\|\},$$

where $L_{\mathbb{R}}(\mathbb{C})$ is the set of all real-linear maps over $\mathbb{C}$.
For $K \subset \Omega_1 \subseteq \Omega_2$ we have that $\mathcal{H}(\Omega_1) \supseteq \mathcal{H}(\Omega_2)$ and $\bigcap_{f\in\mathcal{H}(\Omega_1)}\{z \in \Omega_1|\ |f(z)| \leq \sup_K \|f\|\} \subseteq \bigcap_{f\in\mathcal{H}(\Omega_2)}\{z \in \Omega_2|\ |f(z)| \leq \sup_K \|f\|\} \cap \Omega_1$, hence $\widehat{K}_{\Omega_1} \subseteq \widehat{K}_{\Omega_2} \cap \Omega_1$.
In total we obtain for $\Omega_1 = \Omega$ and $\Omega_2 = \mathbb{C}$

$$\widehat{K}_{\Omega} \subseteq \widehat{K}_{\mathbb{C}} \cap \Omega \subseteq \operatorname{conv}(K) \cap \Omega \subseteq \operatorname{conv}(K).$$

□

Remark 1.6. One can show that if K is compact in Ω, then $\widehat{K}_{\Omega}$ is compact in Ω and that $\widehat{K}_{\Omega}$ is the union of K and the connected components of $\Omega\backslash K$ which are relatively compact in Ω.

Theorem 1.7 (Runge's Theorem)**.** *Let $\Omega \subseteq \mathbb{C}$ open, $K \subset \Omega$ compact. TFAE:*

(a) Every $f \in \mathcal{H}(K)$ is the uniform limit of functions in $\mathcal{H}(\Omega)$.

(b) The open set $\Omega\backslash K$ has no connected component which is relatively compact in Ω.

(c) K is holomorphically convex in Ω.

Proof. We will show $(a) \Leftrightarrow (b) \Leftrightarrow (c)$.

$(c) \Rightarrow (b)$:

We assume that $\Omega\backslash K$ has a connected component $\mathcal{O}$, which is relatively compact in Ω. Note that $\mathcal{O}$ is the biggest set with these properties. We show that $b\mathcal{O} \subseteq K$. Assume that there exist $z_0 \in b\mathcal{O}$, but $z_0 \notin K$. Since $z_0 \in \overline{\mathcal{O}} \subset \Omega$, $z_0 \notin b\Omega$. Hence $z_0 \in \Omega\backslash K$, which is open, and we can find $r > 0$, such that $B_r(z_0) \subseteq \Omega\backslash K$. Again since $z_0 \in \overline{\mathcal{O}}$, we have that $B_r(z_0) \cap \mathcal{O} \neq \varnothing$ and thus $B_r(z_0) \cup \mathcal{O} \subseteq \Omega\backslash K$

is a bigger connected component of $\Omega\backslash K$ than $\mathcal{O}$, a contradiction. Now for every $f \in \mathcal{H}(\Omega)$ the maximum principle implies $\|f\|_{\overline{\mathcal{O}}} = \|f\|_{b\mathcal{O}} \leq \|f\|_K$, contradicting (c).

$(a) \Rightarrow (b)$:

As in the proof of $(c) \Rightarrow (b)$ we assume that $\Omega\backslash K$ has a connected component $\mathcal{O}$, which is relatively compact in Ω with $b\mathcal{O} \subseteq K$. For any fixed $\zeta \in \mathcal{O}$ the function $f : z \mapsto 1/(z-\zeta)$ is holomorphic in a neighbourhood of K. By statement (a) there exist a sequence $(f_n)_{n\geq 1}$ each f_n in $\mathcal{H}(\Omega)$, such that $\|f - f_n\|_K \to 0$. As in the proof of $(c) \Rightarrow (b)$ we have that

$$\|f_n - f_m\|_{\overline{\mathcal{O}}} \leq \|f_n - f_m\|_K,$$

hence $(f_n)_{n\geq 1}$ converges uniformly on $\overline{\mathcal{O}}$ to a function $F \in \mathcal{H}(\overline{\mathcal{O}})$. We have

$$\begin{aligned} 0 = \lim_{n\to\infty} \sup_{z\in K} |f_n(z)(z-\zeta) - 1| &= \sup_{z\in K} |F(z)(z-\zeta) - 1| \\ \geq \sup_{z\in b\mathcal{O}} |F(z)(z-\zeta) - 1| &\geq \sup_{z\in \mathcal{O}} |F(z)(z-\zeta) - 1|, \end{aligned}$$

by the maximum principle, hence $F(z)(z-\zeta) = 1$ for every $z \in \mathcal{O}$. This gives a contradiction for $z = \zeta$.

$(b) \Rightarrow (a)$:

The idea of proving this implication is to show that every continuous functional in $\mathcal{C}'(K)$, which vanishes on $\mathcal{H}(\Omega)|_K$, already vanishes on $\mathcal{H}(K)$. The Hahn-Banach theorem, then implies that $\overline{\mathcal{H}(\Omega)|_K} = \mathcal{H}(K)$, which is precisely statement (a). By the Riesz representation theorem we have $\mathcal{C}'(K) \cong M(K)$, where $M(K)$ is the space of Radon measures on K. To prove this implication we show that every $\mu \in M(K)$ vanishing on $\mathcal{H}(\Omega)|_K$, vanishes on $\mathcal{H}(K)$.
Let $\mu \in \mathcal{C}'(K)$ orthogonal to $\mathcal{H}(\Omega)|_K$, i. e., $\int_K f d\mu = 0$ for all $f \in \mathcal{H}(\Omega)|_K$ and define

$$\hat{\mu}(\zeta) := -\frac{1}{\pi}\int_K \frac{d\mu(z)}{z-\zeta},$$

which is called the Cauchy transform of μ. We show that $\hat{\mu} = 0$ on $\complement K = \mathbb{C}\backslash K$. We deal with the unbounded and bounded connected components of $\complement K$ separately.
In any bounded connected component $\mathcal{O}$ of $\complement K$ we have that $\mathcal{O} \cap \complement\Omega \neq \varnothing$, because otherwise since $b\mathcal{O} \subseteq K \subset \Omega$, it follows that $\overline{\mathcal{O}} \subset \Omega$ and hence $\mathcal{O}$ would be a relatively compact component of $\Omega\backslash K$, which contradicts the assumption of (b).
Let $\zeta \in \mathcal{O} \cap \complement\Omega$, then the function $z \mapsto 1/(z-\zeta)$ is in $\mathcal{H}(\Omega)|_K$, hence

$$\hat{\mu}(\zeta) = -\frac{1}{\pi}\int_K \frac{d\mu(z)}{z-\zeta} = 0.$$

Let $\zeta > \sup_{z\in K}|z|$ be contained in the unbounded connected component of $\complement K$. Therefore we have for $z \in K$ that

$$\frac{1}{z-\zeta} = -\sum_{n\geq 0} \frac{z^n}{\zeta^{n+1}}$$

converges uniformly. Thus we can interchange summation and integration in the following way to obtain

$$\hat{\mu}(\zeta) = \frac{1}{\pi}\sum_{n\geq 0}\left(\int_K z^n d\mu(z)\right)/\zeta^{n+1}.$$

But since $z \mapsto z^n$ is in $\mathcal{H}(\Omega)|_K$ for every $n \geq 0$, $\int_K z^n d\mu(z) = 0$, hence $\hat{\mu} = 0$ in the unbounded connected component of $\complement K$ by the principle of analytic continuation.
Summarizing we found that $\hat{\mu} = 0$ in $\complement K$ and now we show that this implies μ vanishing on $\mathcal{H}(K)$.
Let $f \in \mathcal{H}(K)$, i. e., $f \in \mathcal{H}(U)$ for some open neighbourhood U of K and $\varphi \in \mathcal{D}(U)$, such that $\varphi = 1$ in a compact neighbourhood of K contained in U. Define $K_1 := \operatorname{supp}\left(\frac{\partial\varphi}{\partial\bar{z}}\right)$. For $z \in K$ we obtain by Theorem 1.10 that

$$f(z) = \varphi(z)f(z) = \frac{1}{2\pi i}\int_{K_1} f(\zeta)\frac{\partial\varphi(\zeta)}{\partial\bar{\zeta}}\frac{1}{\zeta - z}d\zeta\wedge d\bar{\zeta}.$$

Since $K \cap K_1 = \varnothing$ we obtain that the function $(z,\zeta) \mapsto f(\zeta)\frac{\partial\varphi(\zeta)}{\partial\bar{\zeta}}\frac{1}{\zeta-z}$ is continuous on $K \times K_1$ and we can apply Fubini's theorem. Thus we have

$$\begin{aligned}\int_K f(z)d\mu(z) &= \int_K \left(\frac{1}{2\pi i}\int_{K_1} f(\zeta)\frac{\partial\varphi(\zeta)}{\partial\bar{\zeta}}\frac{1}{\zeta - z}d\zeta\wedge d\bar{\zeta}\right)d\mu(z)\\ &= -\frac{1}{2i}\int_{K_1} f(\zeta)\frac{\partial\varphi(\zeta)}{\partial\bar{\zeta}}\hat{\mu}(\zeta)d\zeta\wedge d\bar{\zeta} = 0,\end{aligned}$$

since $K_1 \subseteq \complement K$.

***(b)* $\Rightarrow$ *(c)* :**

Let $z \in \Omega\backslash K$ and $L := \overline{B_r(z)} \subset \Omega\backslash K$. The connected components of $\Omega\backslash(K\cup L)$ are the same as those of $\Omega\backslash K$, except the one, where we take away L. Our assumption (b) is equivalent that every connected component of $\Omega\backslash K$ gets arbitrarily near $\partial\Omega$, hence the statement in (b) stays the same if we replace K by $K \cup L$. Now we use the implication $(b) \Rightarrow (a)$ and apply it to $(\Omega, K\cup L)$. Since $K \cap L = \varnothing$ we choose U_1 an open neighbourhood of K and U_2 an open neighbourhood of L, such that $U_1 \cap U_2 = \varnothing$ and define the function f with $f = 0$ on U_1 and $f = 1$ on U_2. Then f belongs to $\mathcal{H}(K\cup L)$, hence there exist a sequence $(f_n)_{n\geq 1}$ in $\mathcal{H}(\Omega)$, which approximates f uniformly in $K\cup L$. If n is chosen to be big enough we obtain $\|f_n\|_K \leq \frac{1}{3}$ and $\|f_n\|_L \geq \frac{2}{3}$. Hence

$$|f_n(z)| \geq \frac{2}{3} > \|f_n\|_K.$$

$\square$

Corollary 1.8. *Every function holomorphic in a neighbourhood of a compact set K can be approximated uniformly on K by polynomials if and only if $\complement K$ is connected.*

Proof. Apply Runge's Theorem with $\Omega = \mathbb{C}$ and use that polynomials are dense in $\mathcal{H}(\mathbb{C})$. $\square$

1.2 Mergelyan's Theorem

Definition 1.9. Let $\Omega \subseteq \mathbb{C}$ open, $K \subset \Omega$ compact, and $K = \widehat{K}_\Omega$. We define

$$A(K) := \{f \in \mathcal{C}(K) : \ f \in \mathcal{H}(\mathring{K})\}.$$

In the proof of Mergelyan's Theorem we need the following generalization of Cauchy's formula:

Theorem 1.10 (Pompeiu's Formula, Inhomogeneous Cauchy Integralformula)**.** *Let G be a domain in $\mathbb{C}$, bounded with piecewise smooth and regular boundary and $f \in \mathcal{C}^1(U)$, where U is an open neighbourhood of $\overline{G}$.*
Then we have for every $z \in G$:

$$f(z) = \frac{1}{2\pi i}\int_{\partial G}\frac{f(\zeta)}{\zeta - z}d\zeta + \frac{1}{2\pi i}\int_G \frac{\partial f}{\partial \bar{\zeta}}(\zeta)\frac{1}{\zeta - z}d\zeta \wedge d\bar{\zeta}$$

Proof. Let $G_\varepsilon := \{\zeta \in G : \ |z-\zeta| > \varepsilon\}$, where $0 < \varepsilon < d(z, \complement G)$. We apply Stoke's formula to the 1-form $\frac{f(\zeta)}{\zeta - z}d\zeta$ and the open set G_ε, whose boundary is piecewise smooth and regular. bG_ε consists of the positively orientated boundary of G and the negatively orientated boundary of $B_\varepsilon(z)$. $bB_\varepsilon(z)$ is parametrized by $(0, 2\pi] \ni \theta \mapsto z + \varepsilon e^{i\theta} \in bB_\varepsilon(z)$. Since

$$d\left(\frac{f(\zeta)}{\zeta - z}d\zeta\right) = -\frac{\partial f}{\partial \bar{\zeta}}(\zeta)\frac{d\zeta \wedge d\bar{\zeta}}{\zeta - z}$$

we obtain

$$-\int_{G_\varepsilon}\frac{\partial f}{\partial \bar{\zeta}}(\zeta)\frac{d\zeta \wedge d\bar{\zeta}}{\zeta - z} = \int_{bG}\frac{f(\zeta)}{\zeta - z}d\zeta - i\int_0^{2\pi} f(z + \varepsilon e^{i\theta})d\theta. \tag{1.2.1}$$

Since for $\delta > 0$

$$\int_{B_{\delta(0)}}\frac{1}{|z|}dz \wedge d\bar{z} = -2i\int_{B_{\delta(0)}}\frac{1}{|z|}dx \wedge dy = -2i\int_0^\delta\int_0^{2\pi}\frac{1}{r}rd\theta dr = -4\pi i\delta$$

the function $\zeta \mapsto \frac{\partial f}{\partial \bar{\zeta}}(\zeta)\frac{1}{\zeta - z}$ is integrable on G. Therefore the left-handside of (1.2.1) converges to the integral over G as $\varepsilon \to 0$.
The second term at the right-handside of (1.2.1) converges to $-2\pi i f(z)$ as $\varepsilon \to 0$. $\square$

Theorem 1.11 (Mergelyan's Theorem 1)**.** *Let K be holomorphically convex in $\mathbb{C}$. Then*

$$A(K) = \overline{\mathcal{H}(\mathbb{C})|_K}.$$

Observation 1.12. (i) The closure in Mergelyan's Theorem is taken with respect to $\mathcal{C}(K)$. $A(K)$ is a subalgebra of $\mathcal{C}(K)$.

(ii) By Runge's Theorem we have that $\overline{\mathcal{H}(\Omega)|_K} = \mathcal{H}(K) \subseteq A(K)$ for every $\Omega \subseteq \mathbb{C}$ open.

(iii) We can now use Corollary 1.8 to give an equivalent formulation of Mergelyan's Theorem considering the previous remark:

Theorem 1.13 (Mergelyan's Theorem 2)**.** *Let K be a compact set.*
If $\complement K$ is connected, then every $f \in A(K)$ can be approximated uniformly by polynomials.

(iv) The idea of the proof is, that we first construct a function φ_δ, which is holomorphic and in fact coincides with $f \in A(K)$ in a big subset of $\mathring{K}$, denoted by Ω_δ. Then we have to take care of the terms outside of Ω_δ. For this purpose we rewrite φ_δ by the inhomogeneous Cauchy integral formula. With a Lemma which uses Koebe's One-Quarter Theorem we obtain functions in $\mathcal{H}(K)$ which approximate φ_δ. Then Runge's Theorem gives the approximation by polynomials on K.

Proof of Theorem 1.13. We start with some observations to simplify the setting:

(a) A given function $f \in \mathcal{C}(K)$ can always be extended to a continuous function on $\mathbb{C}$ by the Tietze extension theorem. Then we multiply this $f \in \mathcal{C}(\mathbb{C})$ by a test function $\varphi \in \mathcal{D}(\mathbb{C})$ satisfying $\varphi \equiv 1$ in a neighbourhood of K to obtain $f \in \mathcal{C}_c(\mathbb{C})$.

(b) If we assume that $f \in A(\overline{B_\delta(z_0)})$ and $k(z) = k(|z|) \in \mathcal{C}_c(\overline{B_\delta(0)})$ with $\int_{\mathbb{C}} k(z) dz \wedge d\bar{z} = 1$, then

$$f(z_0) = 2\pi \int_0^\delta f(z_0)k(r)rdr = \int_0^\delta \int_0^{2\pi} f(z_0 - re^{it})dt\ k(r)rdr = \int_{B_\delta(0)} f(z_0 - z)k(z)dz \wedge d\bar{z}. \tag{1.2.2}$$

(c) We will use the notion of the **modulus of continuity** $\omega(\delta)$, which is defined as

$$\omega(\delta) := \sup_{0<|z-w|<\delta} |f(z) - f(w)|.$$

Note that $f \in \mathcal{C}_c(\mathbb{C})$ is uniformly continuous, hence $\omega(\delta) \to 0$ for $\delta \to 0$.

Let $f \in A(K)$. By observation (a) we can assume $f \in \mathcal{C}_c(\mathbb{C})$ and let ω be the modulus of continuity of f. We define $k \in \mathcal{C}_c^1(\mathbb{C})$ with $k(z) = k(|z|)$ as in observation (b):

$$k(z) := \begin{cases} \frac{3}{\pi}(1 - |z|^2)^2 & \text{for } 0 \le |z| \le 1 \\ 0 & \text{for } |z| > 1. \end{cases}$$

Then $\int_{\mathbb{C}} k(z)dz \wedge d\bar{z} = 1$. Now fix any $0 < \delta < 1$ and let $k_\delta(z) := \delta^{-2}k(z/\delta)$. Hence $\operatorname{supp} k_\delta = \overline{B_\delta(0)}$ and $\int_{\mathbb{C}} k_\delta(z)dz \wedge d\bar{z} = 1$. Now we consider

$$\varphi_\delta(z) := \int_{\mathbb{C}} f(z - \zeta)k_\delta(\zeta)d\zeta \wedge d\bar{\zeta}.$$

Step 1: Approximation of $f \in A(K)$ by φ_δ

By the uniform continuity we obtain

$$|\varphi_\delta(z) - f(z)| \leq \int_{\mathbb{C}} |f(z-\zeta) - f(z)| k_\delta(\zeta) d\zeta \wedge d\bar{\zeta} \leq \omega(\delta) \int_{\mathbb{C}} k_\delta(\zeta) d\zeta \wedge d\bar{\zeta} \to 0 \quad (\delta \to 0).$$

Thus φ_δ approximates f uniformly.

Step 2: Properties of φ_δ and $\frac{\partial}{\partial \bar{z}}\varphi_\delta$

Since φ_δ is a convolution of a continuous and a continuously differentiable, both compactly supported, function, we get $\varphi_\delta \in \mathcal{C}_c^1(\mathbb{C})$. Hence we obtain $\frac{\partial}{\partial \bar{z}}\varphi_\delta(z) = -\int_{\mathbb{C}} f(z-\zeta) \frac{\partial}{\partial \bar{\zeta}} k_\delta(\zeta) d\zeta \wedge d\bar{\zeta}$.
Let $\Omega_\delta := \{z \in K : \; d(z, \complement K) > \delta\}$ and $S := \operatorname{supp}\left(\frac{\partial}{\partial \bar{z}}\varphi_\delta\right)$. Then $\Omega_\delta \subseteq \mathring{K}$, thus $\varphi_\delta = f$ in Ω_δ by (1.2.2) in observation (b), hence $S \cap \Omega_\delta = \varnothing$.

Step 3: Existence of $\Phi_\delta \in \mathcal{H}(K)$

Lemma 1.14. *Let E be a compact, connected set with $\operatorname{diam} E \geq r > 0$, $\complement E$ connected and B an open disk with radius r such that $E \subseteq B$.*
Then there exists a function $Q \in \mathcal{H}(S^2 \backslash E)$, $\beta \in \mathbb{C}$ and $c_1,\; c_2 \in \mathbb{R}^+$ such that

$$R(\zeta, z) = Q(z) + (\zeta - \beta)(Q(z))^2$$

satisfies

$$|R(\zeta, z)| \leq \frac{c_1}{r} \quad \text{and} \quad \left| R(\zeta, z) - \frac{1}{z-\zeta} \right| \leq \frac{c_2 \; r^2}{|\zeta - z|^3}$$

for all $z \in \complement E$ and $\zeta \in B$.

Proof. W. l. o. g. we assume B to be $B_r(0)$, if we translate E and B by the center of B denoted by ζ_0. The right-hand side of the inequalities are independent of ζ_0 and only β is affected.
By a Corollary of Koebe's One-Quarter Theorem we have a biholomorphic mapping

$$\begin{aligned} F &: B_1(0) \to S^2 \backslash E \\ z = F(w) &= \frac{a}{w} + \sum_{j \geq 0} b_j w^j, \quad \text{where } a \geq \frac{r}{4}. \end{aligned}$$

Now we define

$$\begin{aligned} Q &: S^2 \backslash E \to B_{\frac{1}{a}}(0) \\ w = Q(z) &:= \frac{1}{a} F^{-1}(z), \end{aligned}$$

which is again a biholomorphic mapping. For $z = F(w)$ we have $aQ(z) = w$. Hence $zQ(z) = \frac{wF(w)}{a}$ and since $Q(\infty) = 0$ we have $\lim_{z \to \infty} zQ(z) = 1$. Moreover for any fixed ζ it still holds that $\lim_{z\to\infty}(z-\zeta)Q(z) = 1$. Since $\complement E$ is connected and $E \subseteq B_r(0)$, we have that $\complement B_r(0) \subseteq S^2 \backslash E$, where

$S^2 \backslash E$ is simply connected, and therefore we can expand Q in the following way

$$Q(z) = \frac{1}{z} + \frac{\beta}{z^2} + \sum_{j \geq 3} a_{-j} z^{-j}$$

for all $z \in \complement B_r(0)$. The coefficient β is given by

$$\beta = \frac{1}{2\pi i} \int_{|z|=\rho} zQ(z)dz, \quad \text{where } r \leq \rho < \infty.$$

We have $|Q(z)| \leq \frac{1}{a} \leq \frac{4}{r}$ for $z \in \complement E$, thus $|\beta| \leq 4r$ if we take $\rho = r$ in the above integral.
The first claimed estimate

$$|R(\zeta, z)| \leq |Q(z)| + (|\zeta| + |\beta|)|Q(z)|^2 < \frac{84}{r}$$

then holds for $|\zeta| < r$ and $z \in \complement E$.
For the second estimate we fix any $\zeta \in B_r(0)$ and expand Q into its Laurent series for $|z - \zeta| > 2r$

$$Q(z) = \frac{1}{z - \zeta} + \sum_{j \geq 2} c_{-j}(z - \zeta)^{-j},$$

where the coefficient c_{-2} is given by

$$c_{-2} = \frac{1}{2\pi i} \int_{|z|=3r} (z - \zeta)Q(z)dz = \frac{1}{2\pi i} \int_{|z|=3r} zQ(z)dz - \frac{\zeta}{2\pi i} \int_{|z|=3r} Q(z)dz = \beta - \zeta.$$

Hence the following representations of

$$Q(z) = \frac{1}{z - \zeta} + \frac{\beta - \zeta}{(z - \zeta)^2} + \sum_{j \geq 3} c_{-j}(z - \zeta)^{-j}$$

and

$$(Q(z))^2 = \frac{1}{(z - \zeta)^2} + \sum_{j \geq 3} d_{-j}(z - \zeta)^{-j}$$

hold. Then the function

$$z \mapsto (z - \zeta)^3 \left(R(\zeta, z) - \frac{1}{z - \zeta} \right)$$

is holomorphic for $|z - \zeta| > 2r$. For $z \in \complement E$ the possible pole for $z = \zeta$ gets cancelled by the factor $(z - \zeta)^3$, hence the above mapping is holomorphic in $S^2 \backslash E$.
By the maximum principle we only need to find an upper bound for $(S^2 \backslash E) \cap B_r(0)$ to obtain an upper bound for $S^2 \backslash E$. Indeed for $z \in B_r(0)$ we have $|z - \zeta| \leq 2r$ and

$$\left| (z - \zeta)^3 \left(R(\zeta, z) - \frac{1}{z - \zeta} \right) \right| \leq |(z - \zeta)|^3 |R(\zeta, r)| + |z - \zeta|^2 \leq 676r^2$$

by the first estimate. $\square$

We want to construct such sets as E and B in Lemma 1.14 for $S = \operatorname{supp}\left(\frac{\partial}{\partial \bar{z}}\varphi_\delta\right)$. Since φ_δ is compactly supported we cover S with finitely many $B_1, \ldots, B_m$, each with radius 2δ and such that the centers M_j lie in $\complement K$. Since K is compact, $S^2 \backslash K$ is connected. Thus we can define a path Γ_j starting at the point $\{\infty\} \in S^2$ and passing M_j, such that $S^2 \backslash \Gamma_j$ is connected. Now we perturb Γ_j a little bit near M_j to gain a new path, denoted by $\widetilde{\Gamma}_j$. $\widetilde{\Gamma}_j$ should stay inside B_j, starting near the point in bB_j, where Γ_j intersects with bB_j. Then $S^2 \backslash \widetilde{\Gamma}_j$ is still connected. Within Γ_j and $\widetilde{\Gamma}_j$ we define a compact, connected set $E_j \subseteq B_j$, diam $E_j \geq 2\delta$, $S^2 \backslash E_j$ connected and $E_j \cap K = \varnothing$.

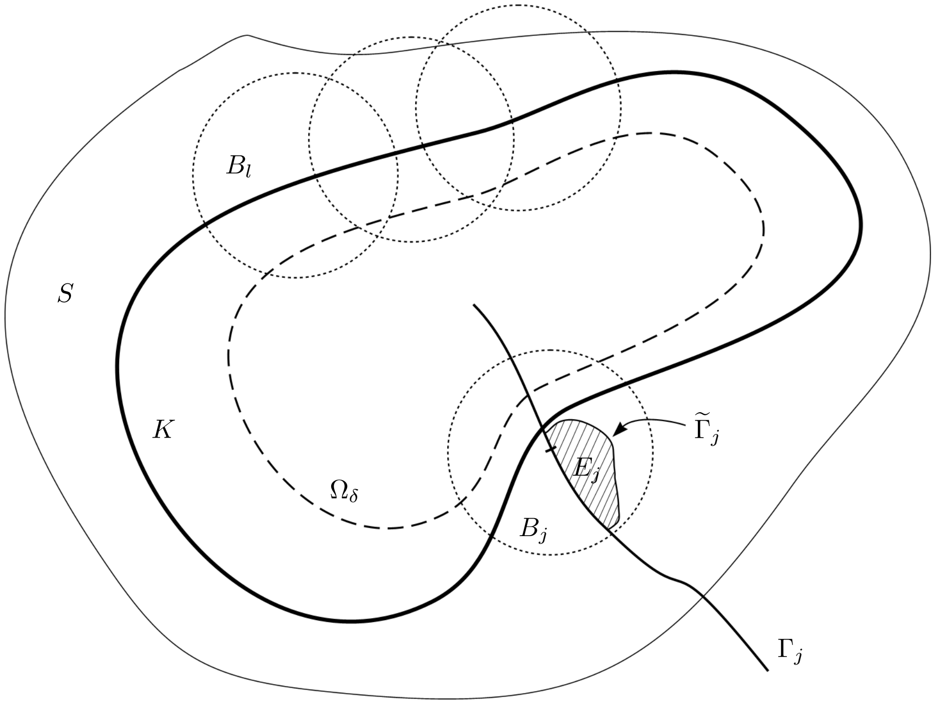

Figure 1.1: Construction of E_j

Every E_j together with B_j and $r = 2\delta$ fulfill the assumptions of Lemma 1.14, which gives us functions $R_j(\zeta, z) = Q_j(z) + (\zeta - \beta_j)(Q_j(z))^2$ satisfying the estimates stated in Lemma 1.14.
Using **Step 2** we obtain by Pompeiu's Formula

$$\varphi_\delta(z) = \frac{1}{2\pi i} \int_{\complement\Omega_\delta} \frac{\partial \varphi_\delta}{\partial \bar{\zeta}}(\zeta) \frac{d\zeta \wedge d\bar{\zeta}}{\zeta - z} = \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{S_j} \frac{\partial \varphi_\delta}{\partial \bar{\zeta}}(\zeta) \frac{d\zeta \wedge d\bar{\zeta}}{\zeta - z}.$$

Where $S_j := B_j \backslash \bigcup_{k=1}^{j-1} S_k$ are disjoint Borel-measurable sets, $S = \bigcup_j S_j$ and $S_j \subseteq B_j$ for every $1 \leq j \leq m$. Now we define

$$\Phi_\delta(z) := \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{S_j} \frac{\partial \varphi_\delta}{\partial \bar{\zeta}}(\zeta) R_j(\zeta, z) d\zeta \wedge d\bar{\zeta}.$$

This is a linear combination of Q_j and Q_j^2, which are holomorphic in $S^2 \backslash E_j$ and hence in $\Omega := S^2 \backslash \bigcup_j E_j$, which is an open neighbourhood of K by the construction of the E_j's. That means $\Phi_\delta \in \mathcal{H}(K)$ and since by assumption $\complement K$ is connected we apply the Corollary of Runge's Theorem, thus Φ_δ can be approximated uniformly by polynomials on K.

Step 4: Approximation of φ_δ by Φ_δ

We want to approximate φ_δ uniformly by $\Phi_\delta \in \mathcal{H}(K)$. Therefore we need an uniform estimate for $\|\Phi_\delta - \varphi_\delta\|_K$. For this purpose we use the estimates from Lemma 1.14 and require an upper bound for $\frac{\partial}{\partial \bar{z}}\varphi_\delta$.
From the very beginning of the proof we recall k_δ and obtain by Stoke's Theorem and due to the compact support of k_δ that $\int_{\mathbb{C}} \frac{\partial}{\partial \bar{\zeta}} k_\delta(\zeta) d\zeta \wedge d\bar{\zeta} = 0$. Using the formula for $\frac{\partial}{\partial \bar{z}}\varphi_\delta(z)$ of **Step 2** we rewrite

$$\frac{\partial}{\partial \bar{z}}\varphi_\delta(z) = -\int_{\mathbb{C}} f(z-\zeta)\frac{\partial}{\partial \bar{\zeta}} k_\delta(\zeta) d\zeta \wedge d\bar{\zeta} = -\int_{\mathbb{C}} (f(z-\zeta) - f(z))\frac{\partial}{\partial \bar{\zeta}} k_\delta(\zeta) d\zeta \wedge d\bar{\zeta}$$

Since

$$\left|\frac{\partial}{\partial \bar{\zeta}} k_\delta(\zeta)\right| = \frac{1}{2\delta^3}\left|\frac{dk}{d|\zeta|}\left(\frac{|\zeta|}{\delta}\right)\right| = \frac{6|\zeta|}{\pi\delta^4}\left(1 - \frac{|\zeta|^2}{\delta^2}\right)$$

we have

$$\left|\frac{\partial}{\partial \bar{z}}\varphi_\delta(z)\right| \le \frac{6\omega(\delta)}{\delta}$$

for $0 \le |\zeta| \le \delta$ and zero otherwise.
Let $z \in \Omega$ be a fixed point. We split each S_j into the disjoint subsets $S_j' := \{\zeta : |z-\zeta| \le 3\delta\} \cap S_j$ and $S_j'' := S_j \backslash S_j'$. We write $\zeta = z + \rho e^{i\theta}$ for the points in S.
For $\zeta \in S_j'$, we have

$$\sum_{j=1}^{m} \int_{S_j'} \left|R_j(\zeta, z) - \frac{1}{z-\zeta}\right| d\zeta \le \frac{c_1(3\delta)^2\pi}{2\delta} + \int_{B_{3\delta}(0)} \frac{1}{|z-\zeta|} d\zeta = \delta\left(\frac{9\pi c_1}{2} + 6\pi\right) = c'\delta.$$

Where we used the first inequality of Lemma 1.14 for $|R_j(\zeta, z)|$.
For $\zeta \in S_j''$ we have

$$\sum_{j=1}^{m} \int_{S_j''} \left|R_j(\zeta, z) - \frac{1}{z-\zeta}\right| d\zeta \le 2\pi \int_{3\delta}^{\infty} \frac{4\delta^2 c_2}{\rho^2} d\rho = c''\delta.$$

Where we used the second inequality of Lemma 1.14 for $|R_j(\zeta, z) - \frac{1}{z-\zeta}|$.
Finally we consider for any $z \in \Omega$

$$|\Phi_\delta(z) - \varphi_\delta(z)| \le \sum_{j=1}^{m} \frac{1}{2\pi} \int_{S_j} \left|\frac{\partial \varphi_\delta}{\partial \bar{\zeta}}(\zeta)\right| \left|R_j(\zeta, z) - \frac{1}{z-\zeta}\right| d\zeta \le \frac{3\omega(\delta)}{\pi\delta}(c' + c'')\delta = c\,\omega(\delta).$$

Step 5: Approximation of $f \in A(K)$ by polynomials

Using **Step 1** we obtain $\|f - \Phi_\delta\|_K \le c\omega(\delta)$, where c is an independent constant. For given $\varepsilon > 0$ we choose $\delta > 0$ such that $c\omega(\delta) \le \varepsilon/2$. Then applying the Corollary of Runge's Theorem and considering **Step 3** we approximate Φ_δ in K by a polynomial p such that $\|\Phi_\delta - p\|_K \le \varepsilon/2$. Finally

$\|f-p\|_K < \varepsilon$. $\square$

2 Local Approximation of Continuous Functions on CR Submanifolds

Introduction 2.1. Starting with this chapter we leave the one dimensional setting and deal with real submanifolds in $\mathbb{C}^n$. A natural question again is if continuous functions can be approximated uniformly by polynomials on a compact subset of a real submanifold. In this chapter we deal with local approximation and give an answer for so called "CR functions", which vanish locally along certain vector fields. The main result will be that every continuous "CR functions" can be approximated locally by polynomials. We follow the presentation in [9] resp. [10] for the original proof of the Baouendi-Treves-Approximation Theorem.

2.1 Real Submanifolds and their Tangent Spaces

Definition 2.2. A **(smooth) real submanifold M of $\mathbb{C}^n$ of codimension d** is a subset of $\mathbb{C}^n$, such that for every $p_0 \in M$ there exists a neighbourhood U of p_0 and a (smooth) real-valued function $\rho = (\rho_1, \dots, \rho_d)$ defined in U such that

$$M \cap U = \{z \in U : \ \rho(z, \bar{z}) = 0\}.$$

Additionally the differentials $d\rho_1, \dots, d\rho_d$ have to be linearly independent in U.
A real submanifold of codimension 1 is called a **real hypersurface**.
ρ is called a **local defining function for M near p_0**.

Remark 2.3. (i) After an affine transformation we can assume that $p_0 = 0$. By defining new coordinates near 0, again denoted by $(x_1, \dots, x_{2n}) \in \mathbb{R}^{2n}$, where $x_i := \rho_i$, for $1 \leq i \leq d$, M is given by $x_1 = \dots = x_d = 0$ in a neighbourhood of 0.

(ii) If we have two defining functions ρ', ρ for M, there is a smooth, invertible real-valued $d \times d$-Matrix a, such that $\rho' = a\rho$. Indeed, if we assume the coordinate system given by ρ in (i), then $\rho'(0, x') = 0$, where $x' \in \mathbb{R}^{2n-d}$. Let $1 \leq i \leq d$, then we have

$$\begin{aligned} \rho_i'(x_1, \dots, x_d, x') =& \rho_i'(x_1, \dots, x_d, x') - \rho_i'(0, x') = \rho_i'(x_1, \dots, x_d, x') - \rho_i'(0, x_2, \dots, x_d, x') + \\ & \rho_i'(0, x_2, \dots, x_d, x') - \dots - \rho_i'(0, x_d, x') + \rho_i'(0, x_d, x') - \rho_i'(0, x') \\ =& \int_0^1 D_1\rho_i'(tx_1, x_2, \dots, x_d, x')x_1 dt + \dots + \int_0^1 D_d\rho_i'(0, tx_d, x')x_d dt. \end{aligned}$$

We write $\sum_k a_{ik}x_k$ for the last sum, thus $\rho' = a\rho$, where a is smooth and real-valued. To see that a is invertible, we differentiate the above equation and obtain $D\rho' = aD\rho$. Since the differentials of ρ' and ρ are linearly independent, the matrix a must be of full rank.

Definition 2.4. Let $p \in \mathbb{C}^n$.

$$T_p\mathbb{C}^n := \left\{ X = \sum_{j=1}^{n} a_j \left.\frac{\partial}{\partial x_j}\right|_p + b_j \left.\frac{\partial}{\partial y_j}\right|_p , \ a_j, b_j \in \mathbb{R} \right\} \tag{2.1.1}$$

is called the **real tangent space $T_p\mathbb{C}^n$ of $\mathbb{C}^n$ at p**. The **complexified tangent space $\mathbb{C}T_p\mathbb{C}^n$ of $\mathbb{C}^n$ at p** is defined by allowing the coefficients $a_j,\ b_j \in \mathbb{C}$ in (2.1.1).
For $p \in M$ and $X \in T_p\mathbb{C}^n$ resp. $\mathbb{C}T_p\mathbb{C}^n$, then **$X$ is tangent to M at p** if $X\rho_i = 0$ for every ρ_i of a defining function $\rho = (\rho_1, \dots, \rho_d)$ of M near p.
The **real tangent space T_pM of M at p** is the set of all real tangent vectors $X \in T_p\mathbb{C}^n$, which are tangent to M at p. The **complexified tangent space $\mathbb{C}T_pM$ to M** is the set of all $X \in \mathbb{C}T_p\mathbb{C}^n$, which are tangent to M at p.
The **real tangent bundle TM of M** is defined by $TM := \coprod_{p\in M} T_pM$. The **complexified tangent bundle $\mathbb{C}TM$ of M** is defined by $\mathbb{C}TM := \coprod_{p\in M} \mathbb{C}T_pM$.

Remark 2.5. (i) The definition of $T_p\mathbb{C}^n$ is independent of the choice of the defining funtion ρ. If ρ' is another local defining function of M, then by 2.3 (ii) we have that $X\rho' = (Xa)\rho + aX\rho = 0$ near p.

(ii) Every $X \in \mathbb{C}T_p\mathbb{C}^n$ can be uniquely written as

$$X = \sum_{j=1}^{n} a_j \left.\frac{\partial}{\partial z_j}\right|_p + b_j \left.\frac{\partial}{\partial \bar{z}_j}\right|_p$$

for $a_j, b_j \in \mathbb{C}$. Hence the following definition is obvious.

Definition 2.6. Let $p \in \mathbb{C}^n$.
The **space of holomorphic tangent vectors $T_p^{1,0}\mathbb{C}^n$ at p** resp. the **space of antiholomorphic tangent vectors $T_p^{0,1}\mathbb{C}^n$ at p** is defined by

$$T_p^{1,0}\mathbb{C}^n := \left\{ X = \sum_{j=1}^{n} a_j \left.\frac{\partial}{\partial z_j}\right|_p , \ a_j \in \mathbb{C} \right\} \text{ resp. } T_p^{0,1}\mathbb{C}^n := \left\{ X = \sum_{j=1}^{n} b_j \left.\frac{\partial}{\partial \bar{z}_j}\right|_p , \ b_j \in \mathbb{C} \right\}.$$

A tangent vector $X \in \mathbb{C}T_p\mathbb{C}^n$ is called **holomorphic or of type (1,0)** resp. **antiholomorphic or of type (0,1)** if $X \in T_p^{1,0}\mathbb{C}^n$ resp. $X \in T_p^{0,1}\mathbb{C}^n$.
$\mathcal{V}_p := T_p^{0,1}\mathbb{C}^n \cap \mathbb{C}T_pM$ is called the **space of antiholomorphic vectors tangent $\mathcal{V}_p$ to M at p**.

Remark 2.7. (i) The above definition is independent of the choice of holomorphic coordinates in $\mathbb{C}^n$. If we take $Z'_j := f_j(Z_1, \dots, Z_n)$, where f_j are holomorphic functions, then $X = \sum_k a_k \frac{\partial}{\partial Z_k} = \sum_j \left(\sum_k a_k \frac{\partial Z'_j}{\partial Z_k}\right) \frac{\partial}{\partial Z'_j} =: \sum_j a'_j \frac{\partial}{\partial Z'_j}$ is again a holomorphic tangent vector near p. Analogous for antiholomorphic tangent vectors.

(ii) Applying the Rank-Nullity Theorem to the matrix given by the entries $\left(\frac{\partial \rho_k}{\partial \bar{z}_j}(p, \bar{p})\right)$ for $1 \le k \le d$ and $1 \le j \le n$ we obtain

$$\dim_{\mathbb{C}} \mathcal{V}_p = \dim_{\mathbb{C}} \ker \left(\frac{\partial \rho_k}{\partial \bar{z}_j}(p, \bar{p})\right)_{\substack{1\le k\le d \\ 1\le j\le n}} = n - \operatorname{rank}_{\mathbb{C}} \left(\frac{\partial \rho_k}{\partial \bar{z}_j}(p, \bar{p})\right)_{\substack{1\le k\le d \\ 1\le j\le n}}.$$

As above $\dim_{\mathbb{C}} \mathcal{V}_p$ is independent of the choice of the defining function ρ and holomorphic coordinates for $\mathbb{C}^n$.

$\dim_{\mathbb{C}} \mathcal{V}_p$ is in general not constant with respect to $p \in M$ as can be seen in the following

Example 2.8. (i) Consider $\rho_1(z_1, z_2) := \mathrm{Im} z_2$ and $\rho_2(z_1, z_2) := \mathrm{Re} z_2 + |z_1|^2$ and M as the subset of $\mathbb{C}^2$ where ρ_1 and ρ_2 vanish. Then M is a real submanifold of codimension 2 and $\dim_{\mathbb{C}} \mathcal{V}_p = 0$ for $z_1 = 0$ and 1 otherwise.

(ii) Nevertheless there are real submanifolds where $\dim_{\mathbb{C}} \mathcal{V}_p$ doesn't vary with $p \in M$, e. g., the real submanifold $M \subset \mathbb{C}^2$ of codimension 2 given by $\rho_1(z_1, z_2) := \mathrm{Re} z_2$ and $\rho_2(z_1, z_2) := \mathrm{Im} z_2$ describes $\mathbb{C}$ as a subset of $\mathbb{C}^2$ and has $\dim_{\mathbb{C}} \mathcal{V}_p = 1$ for all $p \in M$. Real submanifolds where $\dim_{\mathbb{C}} \mathcal{V}_p$ is constant with respect to $p \in M$ will be our main objects of study in the following chapters. Thus we introduce the following

Definition 2.9. A real submanifold M of $\mathbb{C}^n$ is called **CR submanifold M of $\mathbb{C}^n$** if $\dim_{\mathbb{C}} \mathcal{V}_p$ is constant for $p \in M$.

For M a CR submanifold of $\mathbb{C}^n$, $\dim_{\mathbb{C}} \mathcal{V}_p$ is called the **CR dimension of M**.

CR submanifolds with CR dimension 0 are called **totally real**.

The **CR bundle $\mathcal{V}$ of M** is defined as the complex subbundle $\mathcal{V} \subset \mathbb{C}TM$, whose fiber at $p \in M$ is $\mathcal{V}_p$.

Remark 2.10. (i) Examples for totally real submanifolds are submanifolds given as a graph over the imaginary axes. Let $h = (h_1, \ldots, h_n) : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth function and $\mathbb{C}^n$ given by the coordinates $z = (z_1 = x_1 + iy_1, \ldots, z_n = x_n + iy_n)$. Let $M := \{z \in \mathbb{C}^n : \; x_k = h_k(y_1, \ldots, y_n), \; 1 \le k \le n\}$. The codimension is n, since the defining functions are given by $\rho_k(z) = x_k - h_k(y_1, \ldots, y_n)$ for $1 \le k \le n$.

For $n = 1$ we have the defining function $\rho(z) = x - h(y)$ and $\frac{\partial \rho}{\partial \bar{z}} = \frac{1}{2}(1 - ih'(y))$, which can never be 0, hence $\dim_{\mathbb{C}} \mathcal{V}_p = 0$ for all $p \in M$.

For $n = 2$ let $p \in M \subseteq \mathbb{C}^2$. We denote by Dh the Jacobian of h at p with real entries $h_{ij} := \frac{\partial h_i}{\partial y_j}(p)$ for $i, j \in \{1, 2\}$. Then we have that $\mathrm{rk}_{\mathbb{C}} \left(\frac{\partial \rho_l}{\partial \bar{z}_k}(p) \right)_{\substack{1 \le l \le 2 \\ 1 \le k \le 2}}$ is strictly less than 2, if $\det(I_2 - iDh) = 0$. The last condition says that $\det Dh = 1$ and $h_{22} = -h_{11}$. Since both h_{12} and h_{21} cannot be 0 by the first equation, we obtain $h_{21} = -\frac{1+h_{11}^2}{h_{12}}$. Transforming $\mathbb{R}^2$ to new coordinates $(\tilde{y}_1, \tilde{y}_2) := (y_1, y_2/\varepsilon)$ for any $\varepsilon > 0$ we get $\tilde{h}_{21} = -\varepsilon \left(\frac{1+\tilde{h}_{11}^2}{\tilde{h}_{12}} \right)$. Thus $h_{21} = 0$, which implies, that $1 = \det Dh = -h_{11}^2 \le 0$ by the reality of h_{11}, a contradiction. Hence we have $\dim_{\mathbb{C}} \mathcal{V}_p = 2 - 2 = 0$.

For the case $n \ge 3$ the procedure is analogous as for $n = 2$.

(ii) Let $h : \mathbb{R} \times \mathbb{C}^{n-1} \to \mathbb{R}$ be a smooth function and $\mathbb{C}^n$ given by the coordinates $(z = x + iy, w = u + iv) \in \mathbb{C} \times \mathbb{C}^{n-1}$. Then $M = \{(z, w) \in \mathbb{C} \times \mathbb{C}^{n-1} : \; x = h(y, w), \; y \in \mathbb{R}, \; w \in \mathbb{C}^{n-1}\}$ is called **hypersurface graph M of h**. The codimension is 1, since the defining function is given by $\rho(z, w) = x - h(y, w)$.

For $n = 1$ we are in the business of (i) above, which says, that M is a totally real submanifold of $\mathbb{C}$.

For $n = 2$ the CR-dimension of M is 1, since $\mathrm{rk}_{\mathbb{C}} \left(\frac{\partial \rho}{\partial \bar{z}}, \frac{\partial \rho}{\partial \bar{w}} \right) = \mathrm{rk}_{\mathbb{C}} \left(-2 + 2ih_y, h_u + ih_v \right) = 1$ at every $p \in M \subseteq \mathbb{C}^2$.

For $n \ge 3$ we have that the CR-dimension of M is $n - 1$ analogously to the above case.

(iii) A different approach to $\mathcal{V}_p$ is the following. Via introducing a complex structure J on $T_p\mathbb{C}^n$, which has the property that $J^2 = -id$, one can show that $\mathcal{V}_p = \{X \in \mathbb{C}T_pM |\ J(X) \in T_pM\}$. Motivated by this characterization of $\mathcal{V}_p$ one could think of totally real submanifolds as submanifolds with "no complex directions" in their tangent spaces.

2.2 CR Functions and CR Vector Fields

Definition 2.11. Let M be a real submanifold of $\mathbb{C}^n$.
A **complex vector field** $\boldsymbol{X}$ **on** $\boldsymbol{M}$ is a smooth mapping X, which has the property that $X(p) \in \mathbb{C}T_pM$ for every $p \in M$.

Remark 2.12. Hence a vector field X on M can be written as

$$X(z, \bar{z}) = \sum_{k=1^n} a_k(z, \bar{z}) \frac{\partial}{\partial z} + b_k(z, \bar{z}) \frac{\partial}{\partial \bar{z}},$$

where $a_k(z, \bar{z}),\ b_k(z, \bar{z})$ are smooth and complex valued functions defined on M such that

$$\sum_{k=1}^{n} a_k(p) \frac{\partial \rho}{\partial z}(p) + b_k(p) \frac{\partial \rho}{\partial \bar{z}}(p) = 0$$

for every defining function ρ of M.

Definition 2.13. Let M be a CR submanifold of $\mathbb{C}^n$ and $\mathcal{V}$ the CR bundle of M.
A vector field L on M is called a **CR vector field** $\boldsymbol{L}$ **on** $\boldsymbol{M}$, if $L(p) \in \mathcal{V}_p$ for every $p \in M$.

This definition leads us to the main object of our studies.

Definition 2.14. Let M be a CR submanifold of $\mathbb{C}^n$.
$f \in \mathcal{C}^k(M)$ for $k \geq 1$ is called a **CR function** $\boldsymbol{f}$ **on** $\boldsymbol{M}$ if $Lf \equiv 0$ for every CR vector field on M.
$g \in \mathcal{D}'(M)$ is called a **CR distribution** $\boldsymbol{g}$ **on** $\boldsymbol{M}$ if $Lg \equiv 0$ in the sense of distributions for every CR vector field on M.

Example 2.15. (i) Any holomorphic function defined in an open neighbourhood U of a CR submanifold M restricted to M is a CR function.

(ii) If M is a totally real submanifold, then any function defined on M is a CR function.

(iii) A product of CR functions is again a CR function by the product rule.

2.3 Baouendi-Treves-Approximation Theorem

Theorem 2.16. *Let M be a CR submanifold of $\mathbb{C}^n$ of codimension d and $p_0 \in M$. Then there exist smooth, complex-valued CR functions $Z = (Z_1, \ldots, Z_m)$ defined in an open neighbourhood of p_0 and a compact neighbourhood K of p_0 in M, such that for any continuous CR function f on M, there is a sequence of polynomials p_ν with*

$$\sup_{u \in K} |f(u) - p_\nu(Z(u))| \to 0 \textit{ for } \nu \to \infty.$$

Moreover the functions $Z_1, \dots, Z_m$ vanish at p_0 and have the property that their differentials are $\mathbb{C}$-linearly independent.

Remark 2.17. (i) Theorem 2.16 can be shown for "integrable structures", which are certain subbundles $\mathcal{V}$ of $\mathbb{C}TM$, instead of CR submanifolds. The set $Z_1, \dots, Z_m$ which occur in Theorem 2.16 are then called "basic solutions" and have to vanish on every section of the subbundle $\mathcal{V}$. f from above is called "solution". With this generalizations the statement stays the same, but the beginning of the proof has to be modified slightly.

(ii) The idea of the proof is to first choose appropriate coordinates near p_0, which basically are the functions Z_k and to find a sequence of entire functions, which are given by an integral representation. Then the compact set K can be defined to assure that the integral kernels can be estimated properly and approximate f uniformly on K.

Proof of Theorem 2.16. Let M be a CR submanifold of $\mathbb{C}^n$ of codimension d with local defining functions $\rho_1, \dots, \rho_d$, $p_0 \in M$ and let r be the rank of $\partial\rho_1, \dots, \partial\rho_d$. After reordering we may assume that the first r are $\mathbb{C}$-linearly independent. Let f be a CR function on M, for which we assume that $f \in \mathcal{C}^1(M)$ and afterwards we will discuss the continuous case.

Step 1: Regular coordinates of M near p_0

After a holomorphic linear translation we may assume that $p_0 = 0$ and that we have given coordinates $\mathbb{C}^n \ni (z_1, \dots, z_n) \cong (x_1', y_1', \dots, x_n', y_n') \in \mathbb{R}^{2n}$.
Now we expand every ρ_j into its Taylor series. Since $\rho_j(0) = 0$ and by defining $a_{jl} := \frac{\partial \rho_j}{\partial x_l'}$ and $b_{jl} := \frac{\partial \rho_j}{\partial y_l'}$ we obtain

$$\rho_j(x', y') = \sum_{l=1}^{n} a_{jl} x_l' + b_{jl} y_l' \ + \text{higher order terms vanishing at } 0.$$

Since ρ_j is real valued, so is every a_{jl} and b_{jl}. Now we define $w_j := \sum_{l=1}^{n} c_{jl} z_l$, where $c_{jl} := b_{jl} + i a_{jl}$ and $z_l = x_l' + i y_l'$ for $1 \le j \le r$.
By definition $(z, w) = (z_1, \dots, z_{n-r}, w_1, \dots, w_r) \in \mathbb{C}^n$ vanish at 0. To see that (z, w) are coordinates of M near 0 we will show that (z, w) is obtained by applying an invertible matrix A' to the coordinates $(z_1, \dots, z_n)$ from the very first line of **Step 1**.
For $1 \le j \le r$ the rows of the $r \times n$ matrix C with entries $c_{jl} = 2i \frac{\partial \rho_j}{\partial z_l}$ for $1 \le l \le n$ are assumed to be $\mathbb{C}$-linearly independent. Hence the matrix C has rank r, so we choose a matrix $A'' \in \mathbb{C}^{r \times n}$ with rank r, which is of the form $(0, C') \in \mathbb{C}^{r \times n-r} \times \mathbb{C}^{r \times r}$, where C' is invertible. Then there exist an invertible $r \times r$ matrix B and an invertible $n \times n$ matrix D such that $BCD = A''$. So we assume that the coordinates w are given by $A''z$ and depend only on $(z_{n-r+1}, \dots, z_n)$. Now define

$$A' := \begin{pmatrix} I_{n-r} & 0 \\ 0 & C' \end{pmatrix}$$

which is an invertible $n \times n$ matrix and transforms $(z_1, \dots, z_n)$ into $(z_1, \dots, z_{n-r}, w_1, \dots, w_r)$. Hence $(z, w) \in \mathbb{C}^n$ are coordinates of M in a neighbourhood U of 0.

Step 2: Defining $Z_1, \ldots, Z_m$

Define $m := n - d + r,\ k := n - r$ and $\tilde{Z}_j := z_j$ for $1 \leq j \leq n - r$ and $\tilde{Z}_{j+n-r} := w_j$ for $1 \leq j \leq 2r - d$. Then every $\tilde{Z}_j$ vanishes on all CR vector fields on M by definition and $d\tilde{Z}_1, \ldots, d\tilde{Z}_m$ are $\mathbb{C}$-linearly independent by **Step 1**. If we restrict every $\tilde{Z}_j$ to $M \cap U$, then $\tilde{Z}_j$ is a function of $(x'_1, \ldots, x'_m, y'_1, \ldots, y'_k) \in \mathbb{R}^{m+k}$ and still has the properties from above.
So we have found coordinate functions $\tilde{Z} = (\tilde{Z}_1, \ldots, \tilde{Z}_m)$ with the same properties as claimed for the Z_j in the theorem, now we want to transform $\tilde{Z}$ into a more convenient shape for the remaining proof. To see that the matrix

$$A := \left(\frac{\partial \tilde{Z}_j}{\partial x'_l}(0) \right)_{1 \leq j, l \leq m}$$

is invertible, one has to proceed analogously to **Step 1**. Then we set $x := \mathrm{Re}(A^{-1}\tilde{Z}(x', y')) \in \mathbb{R}^m$ and $y_l := y'_l$ for $1 \leq l \leq k$. Then define $\Phi(x, y) := \mathrm{Im}(A^{-1}\tilde{Z}(x', y')) \in \mathbb{R}^m$ and $Z := x + i\Phi(x, y)$, which is a linear transformation, so the properties of $\tilde{Z}$ are passed on to Z. Moreover Φ has the properties that $\Phi(0) = 0$, $\Phi_x(0) = 0$, it's real valued and smooth in a neighbourhood of 0. From now on we assume the coordinates $(x, y) \in \mathbb{R}^{m+k}$ for M near 0, which vanish at 0.

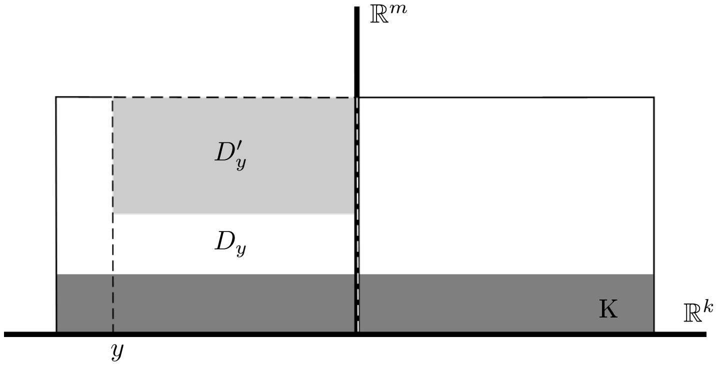

Figure 2.1: Sets involved in the proof

Step 3: Introducing the approximating sequence

For $j = 1, 2$ let $r_j,\ d_j \in \mathbb{R}^+ \backslash \{0\}$, which we will determine in the course of the proof, and $\varphi \in \mathcal{D}(\mathbb{R}^m)$, such that $\varphi(x) \equiv 1$ for $|x| \leq \frac{r_j}{2}$ and $|x| > r_j$. Then we define the set

$$K := \left\{ (x, y) \in M : \ |x| \leq \frac{r_j}{4},\ |y| \leq d_j \right\},$$

a compact neighbourhood of 0. Now set $dZ(x, y) := dZ_1(x, y) \wedge \ldots \wedge dZ_m(x, y)$ and define the m-form

$$\alpha_\nu(x, y, z) := \left(\frac{\nu}{\pi}\right)^{\frac{m}{2}} \exp\left(-\nu(z - Z(x, y))^2\right) \varphi(x) f(x, y) dZ(x, y)$$

for $z \in \mathbb{C}^m$, $\nu \in \mathbb{N}$ and for a vector $v = (v_1, \ldots, v_m) \in \mathbb{C}^m$ we denote $v^2 := \sum_{j=1}^m v_j^2$.
Now we fix a $y \in \mathbb{R}^k$ with $0 < |y| < d_j$ and define

$$D_y := \left\{ (x', y') \in \mathbb{R}^{m+k} : \ |x'| < r_j,\ y' = ty,\ t \in (0, 1) \right\}$$

a $(m+1)$-real submanifold of $\mathbb{R}^{m+k}$ with boundary given by $\partial D_y = \{(x',0) \in \mathbb{R}^{m+k} : |x'| < r_j\} \cup \{(x',y) \in \mathbb{R}^{m+k} : |x'| < r_j\} \cup \{(x',y') \in \mathbb{R}^{m+k} : |x'| = r_j,\ y' = ty,\ t \in [0,1]\}$. Hence we use Stoke's Theorem to obtain

$$\begin{aligned}\int_{D_y} d\alpha_\nu(x',y',z) &= \int_{\partial D_y} \alpha_\nu(x',y',z) \\ &= \left(\frac{\nu}{\pi}\right)^{\frac{m}{2}} \int_{\mathbb{R}^m} \exp\left(-\nu(z - Z(x',y))^2\right) \varphi(x') f(x',y) d_{x'} Z(x',y) \\ &- \left(\frac{\nu}{\pi}\right)^{\frac{m}{2}} \int_{\mathbb{R}^m} \exp\left(-\nu(z - Z(x',0))^2\right) \varphi(x') f(x',0) d_{x'} Z(x',0) \\ &=: \beta_\nu(f,z) - H_\nu(z)\end{aligned}$$

since $\varphi(x') \equiv 0$ for $|x'| \geq r_j$. We have that $d_{x'}Z(x',y) = \det(Z_{x'}(x',y))dx'$.
For every $\nu \in \mathbb{N}$ the functions $H_\nu(z)$ are entire in $\mathbb{C}^m$. Now we replace z in the above equation by $Z(x,y)$ to obtain

$$\int_{D_y} d\alpha_\nu(x',y',Z(x,y)) = \beta_\nu(f,Z(x,y)) - H_\nu(Z(x,y)). \tag{2.3.1}$$

We will show that the left-hand side of (2.3.1) converges to 0 uniformly in K as $\nu \to \infty$ and that $\beta_\nu(f,Z(x,y)) \to f(x,y)$ uniformly in K as $\nu \to \infty$.

Step 4: Left-hand side of (2.3.1)

To compute $d\alpha_\nu(x',y',Z(x,y))$ we characterize CR functions by their behaviour under the action of the total derivative.

Lemma 2.18. *Let $\Omega \subseteq \mathbb{R}^{m+k}$ be open and $Z_1,\dots,Z_m$ defined as above. Then we have*

$$f \text{ is a CR function in } \Omega \iff d(fdZ_j) = 0 \text{ in } \Omega \text{ for every } 1 \leq j \leq m.$$

Proof. Since $2n - d = \dim_{\mathbb{C}} \mathbb{C}T_pM = \dim_{\mathbb{C}} \mathcal{V}_p + \dim_{\mathbb{C}} \mathcal{V}_p^\perp = n - r + m$, we have that $m = n - d + r$, so we use $Z_1,\dots,Z_m$ as a basis for $\mathcal{V}_p^\perp$. Now the following equivalence holds

$$\begin{aligned} f \text{ is a CR function in } \Omega &\iff \forall p \in \Omega:\ df(p) \in \mathcal{V}_p^\perp \\ &\iff \forall p \in \Omega:\ df(p) \in \operatorname{span}\{dZ_1(p),\dots,dZ_m(p)\}.\end{aligned}$$

If necessary we scale down Ω to use **Step 2** and obtain that the set $\{dZ_1,\dots,dZ_m,dy_1,\dots,dy_k\}$ is a basis of $\mathbb{C}T^*_{(x,y)}\Omega$ for every $(x,y) \in \Omega$, i. e., there exist R_i for $1 \leq i \leq m$ and S_l for $1 \leq l \leq k$ vector fields, such that

$$df = \sum_{i=1}^m R_i f dZ_i + \sum_{l=1}^k S_l f dy_l. \tag{2.3.2}$$

Then $0 = d(fdZ_j) = df \wedge dZ_j = \sum_{l=1}^k S_l f dy_l \wedge dZ_j$ if and only if $S_l f = 0$ for every $1 \leq l \leq k$. Hence we have the following last equivalence

$$\begin{aligned} f \text{ is a CR function in } \Omega \quad &\iff \quad \forall p \in \Omega: \ S_l f(p) = 0 \text{ for every } 1 \leq l \leq k \\ &\iff \quad d(f dZ_j) = 0 \text{ in } \Omega, \end{aligned}$$

which completes the proof of this Lemma. □

Since $\exp(-\nu(Z(x,y)-Z(x',y'))^2)$ is a holomorphic function of the $Z_j(x',y')$, hence a CR function, and by Example 2.15 and applying Lemma 2.18 we obtain that

$$d\alpha_\nu(x',y',Z(x,y)) = \left(\frac{\nu}{\pi}\right)^{\frac{m}{2}} \exp(-\nu(Z(x,y) - Z(x',y'))^2) f(x',y') d\varphi(x') \wedge dZ(x',y')$$

Now we will estimate the above expression to apply the Dominated Convergence Theorem, which is done in the following

Lemma 2.19. *For any CR function f in Ω there exist r_2 and d_2 positive, real numbers, such that*

$$\left(\frac{\nu}{\pi}\right)^{\frac{m}{2}} \int_{D_y} \exp(-\nu(Z(x,y) - Z(x',y'))^2) f(x',y') d\varphi(x') dZ(x',y') \to 0$$

uniformly for $(x,y) \in K$ as $\nu \to \infty$.

Proof. Since $d\varphi(x') = 0$ for $|x'| \leq \frac{r_j}{2}$ and $|x'| > r_j$ the integral in Lemma 2.19 is evaluated over the set $D_y' := \{(x',y') \in \mathbb{R}^{m+k} : \ \frac{r_j}{2} < |x'| \leq r_j, \ y' = ty, \ t \in (0,1)\}$. From **Step 2** we use the representation of $Z(.,.)$ to obtain

$$\begin{aligned} |\exp(-\nu(Z(x,y) - Z(x',y'))^2)| &= \exp(-\nu \mathrm{Re}\,(Z(x,y) - Z(x',y'))^2) \\ &= \exp(-\nu\left((x-x')^2 - (\Phi(x,y) - \Phi(x',y'))^2\right)). \end{aligned}$$

Since the $\Phi_j(.,.)$ are smooth we apply the Mean Value Theorem for

$$|\Phi(x,y) - \Phi(x',y')| \leq |\Phi(x,y) - \Phi(x',y)| + |\Phi(x',y) - \Phi(x',y')| \leq C_{1j}|x-x'| + C_{2j}|y-y'|,$$

where

$$C_{1j} := \sup_{\substack{|x'| \leq r_j \\ |y'| \leq d_j}} |\Phi_{x'}(x',y')| \text{ and } C_{2j} := \sup_{\substack{|x'| \leq r_j \\ |y'| \leq d_j}} |\Phi_{y'}(x',y')|.$$

In **Step 2** we had that $\Phi_{x'}(0) = 0$, hence we can find r_1 and d_1, such that $C_{11} \leq \frac{1}{8}$ and obtain C_{21}. Then we choose $d_2 := \min\left(\frac{r_1}{32C_{21}}, d_1\right)$. Setting $r_2 := r_1$ we obtain that $C_{12} \leq \frac{1}{8}$ and $d_2 \leq \frac{r_2}{32C_{22}}$. Hence if we take $j = 2$ in K in the beginning of **Step 3** we have that

$$|\Phi(x,y) - \Phi(x',y')| \leq \frac{1}{8}|x-x'| + C_{22}|y-y'| \leq \frac{1}{8}\left(\frac{r_2}{4} + r_2\right) + C_{22}2d_2 \leq \frac{7r_2}{32} < \frac{r_2}{4}.$$

The term $|x-x'|$ is bounded from below by $\frac{r_2}{4}$. Thus

$$\exp(\nu\left(-(x-x')^2+(\Phi(x,y)-\Phi(x',y'))^2\right)) \le \exp\left(\nu\left(-\frac{r_2^2}{16}+\frac{49r_2^2}{1024}\right)\right) = \exp\left(-\frac{15\nu r_2^2}{1024}\right),$$

which is a uniform upper bound for the exponential in D'_y and $(x,y)\in K$. We estimate the integrand in Lemma 2.19 and have that

$$\left(\frac{\nu}{\pi}\right)^{\frac{m}{2}} \exp\left(-\frac{15\nu r_2^2}{1024}\right) \sup_{D'_y} \|f\| \to 0 \text{ as } \nu\to\infty$$

by L'Hôpital's rule. Thus we can find an $L\in\mathbb{R}^+$, such that for every $\nu\in\mathbb{N}$ the integrand in Lemma 2.19 is bounded by L, and since D'_y is bounded, Dominated Convergence gives the claim. □

Step 5: β_ν approximates the identity

We will show the remaining part of (2.3.1), which is formulated in the following

Lemma 2.20. *For any CR function f in Ω and r_2, d_2 from Lemma 2.19 we have that*

$$\left(\frac{\nu}{\pi}\right)^{\frac{m}{2}} \int_{\mathbb{R}^m} \exp(-\nu(Z(x,y)-Z(x',y))^2) f(x',y)\varphi(x')\det(Z_{x'}(x',y))dx' \to f(x,y)$$

uniformly for $(x,y)\in K$ as $\nu\to\infty$.

Proof. The integral in the claim is evaluated over the bounded set $D_{x'} := \{x'\in\mathbb{R}^m : \frac{r_2}{2} < |x'| \le r_2\}$. Then we make a change of variables and define $\xi := \sqrt{\nu}(x'-x)$ to obtain

$$\begin{aligned}\beta_\nu(f,Z(x,y)) = \frac{1}{\pi^{\frac{m}{2}}}\int_{\mathbb{R}^m} &\exp\left(-\left(\xi+i\sqrt{\nu}\left(\Phi\left(x+\frac{\xi}{\sqrt{\nu}},y\right)-\Phi(x,y)\right)\right)^2\right)\\ &f\left(x+\frac{\xi}{\sqrt{\nu}},y\right)\varphi\left(x+\frac{\xi}{\sqrt{\nu}}\right)\det\left(D_1Z\left(x+\frac{\xi}{\sqrt{\nu}},y\right)\right)d\xi.\end{aligned}$$

Considering $D_{x'}$ in the new coordinates implies that $|\xi|\le r_2$. From the proof of Lemma 2.19 in **Step 4** we use the constant $C_{12}\le\frac{1}{4}$ to obtain that

$$\left|\Phi\left(x+\frac{\xi}{\sqrt{\nu}},y\right)-\Phi(x,y)\right| \le \frac{|\xi|}{4\sqrt{\nu}}.$$

Then we have

$$\begin{aligned}&\left|\exp\left(-\left(\xi+i\sqrt{\nu}\left(\Phi\left(x+\frac{\xi}{\sqrt{\nu}},y\right)-\Phi(x,y)\right)\right)^2\right)\right|\\ &=\exp\left(-\xi_1^2+\xi_2^2+2\xi_2\sqrt{\nu}\left(\Phi\left(x+\frac{\xi}{\sqrt{\nu}},y\right)-\Phi(x,y)\right)+\nu\left(\Phi\left(x+\frac{\xi}{\sqrt{\nu}},y\right)-\Phi(x,y)\right)^2\right)\\ &\le \exp\left(2|\xi|^2+2|\xi|\sqrt{\nu}\frac{|\xi|}{4\sqrt{\nu}}+\nu\frac{|\xi|^2}{16\nu}\right)=\exp\left(\frac{41r_2^2}{16}\right),\end{aligned}$$

which is an uniform upper bound for every $\nu\in\mathbb{N}$ and $(x,y)\in K$. Since $D_{x'}$ is bounded dominated convergence gives

$$\lim_{\nu\to\infty}\beta_\nu(f,Z(x,y))=\frac{1}{\pi^{\frac{m}{2}}}\int_{\mathbb{R}^m}\exp\left(-\left(\xi+i\Phi_x(x,y)\xi\right)^2\right)f(x,y)\det(Z_x(x,y))d\xi,$$

where we used that

$$\lim_{\nu\to\infty}\left(\Phi\left(x+\frac{\xi}{\sqrt{\nu}},y\right)-\Phi(x,y)\right)=\lim_{h\to 0}\frac{\Phi\left(x+\xi h,y\right)-\Phi(x,y)}{h}=\Phi_x(x,y)\xi.$$

Now we note that $\xi+i\Phi_x(x,y)\xi=:A(x,y)\xi=:A\xi$ with the properties from **Step 2**, hence we may apply the following

Lemma 2.21. *For every invertible matrix $A=I_m+iB$, where B is a real $m\times m$ matrix with entries arbitrarily near 0, we have that*

$$\frac{\det A}{\pi^{\frac{m}{2}}}\int_{\mathbb{R}^m}\exp\left(-(A\xi)^2\right)d\xi=1.$$

Proof. We are going to show that for any $\varepsilon>0$ and the matrices A and B as in the Lemma we can find an appropriate $\delta>0$ with

$$\frac{1}{\pi^{\frac{m}{2}}}\int_{\mathbb{R}^m}\exp\left(-(A\xi)^2\right)d\xi\in B_\delta(1)\text{ and }\det A\in B_\delta(1)$$

to obtain $\frac{\det A}{\pi^{\frac{m}{2}}}\int_{\mathbb{R}^m}\exp\left(-(A\xi)^2\right)d\xi\in B_\varepsilon(1)$, which proves the Lemma.
We think of A as a matrix, which is nearly the idendity matrix up to a perturbation matrix B, which has entries arbitrarily near 0 by **Step 2**. The compact set K from **Step 3** is a neighbourhood of 0, so we may shrink K to obtain the claimed δ. Denoting $\tilde{b}_l:=\sum_{k=1}^m b_{lk}\xi_k$, where the b_{lk} are entries of B we have

$$(A\xi)^2=\sum_{j=1}^m(\xi_j+i\tilde{b}_j)^2=\sum_{j=1}^m\left(\xi_j^2-\sum_{k=1}^m b_{jk}^2\xi_k^2-2\sum_{1\le k<n\le m}b_{jk}b_{jn}\xi_k\xi_n+2i\xi_j\sum_{k=1}^m b_{jk}\xi_k\right).$$

Now we split up the above sum into terms which involve ξ_m and the rest of the sum denoted by B_{m-1}, which only depends on $\xi_1,\dots,\xi_{m-1}$. Hence we have

$$\begin{aligned}(A\xi)^2&=B_{m-1}+\xi_m^2(1-b_{mm}^2+2ib_{mm})-2\xi_m\left(\sum_{k=1}^{m-1}(b_{mm}b_{mk}-ib_{mk})\xi_k\right)\\&=:B_{m-1}+\xi_m^2\widetilde{a}_m-2\xi_m a_{m-1},\end{aligned}$$

where $\widetilde{a}_m$ is a non-zero complex number and a_{m-1} only depends on $\xi_1,\dots,\xi_{m-1}$. We write

$$\int_{\mathbb{R}^m}\exp\left(-(A\xi)^2\right)d\xi=\int_{\mathbb{R}^{m-1}}\exp(-B_{m-1})\int_{\mathbb{R}}\exp(-\xi_m^2\widetilde{a}_m+2\xi_m a_{m-1})d\xi_m d\xi_{m-1}\dots d\xi_1.$$

We calculate the inner integral w. r. t. ξ_m as follows

$$\begin{aligned}\int_{\mathbb{R}} \exp(-\xi_m^2 \widetilde{a}_m + 2\xi_m a_{m-1}) d\xi_m &= \int_{\mathbb{R}} \exp\left(-\widetilde{a}_m \left(\xi_m - \frac{a_{m-1}}{\widetilde{a}_m}\right)^2 + \frac{a_{m-1}^2}{\widetilde{a}_m}\right) d\xi_m \\ &= \exp\left(\frac{a_{m-1}^2}{\widetilde{a}_m}\right) \int_{\mathbb{R}} \exp\left(-\widetilde{a}_m \left(\xi_m - \frac{a_{m-1}}{\widetilde{a}_m}\right)^2\right) d\xi_m \\ &= \exp\left(\frac{a_{m-1}^2}{\widetilde{a}_m}\right) \frac{\sqrt{\pi}}{\sqrt{\widetilde{a}_m}}.\end{aligned}$$

The exponential function involving a_{m-1} has to be taken into account when the integral w. r. t. ξ_{m-1} is evaluated. Repeating this computations till ξ_1 we obtain

$$\frac{1}{\pi^{\frac{m}{2}}} \int_{\mathbb{R}^m} \exp\left(-(A\xi)^2\right) d\xi = \prod_{j=1}^{m} \frac{1}{\sqrt{\widetilde{a}_j}} \exp\left(\frac{b_{11}^2}{\widetilde{a}_1^2}\right),$$

where $\widetilde{a}_1 = 1 - b_{11}^2 + 2ib_{11} - \frac{\widetilde{b}_1}{\widetilde{a}_2}$ and $\widetilde{b}_1$ consists of sums of products of components of B. Now we shrink K such that the right-hand side of the above equation is contained in the δ-ball around 1, where $\delta = \sqrt{1+\varepsilon} - 1 > 0$.

Since $\det A = \det I_m + \det(iB) = 1 +$ sums of products of entries of B, which again is contained in $B_\delta(1)$ by shrinking K if necessary. In total the product of $\det A$ and $\frac{1}{\pi^{\frac{m}{2}}} \int_{\mathbb{R}^m} \exp\left(-(A\xi)^2\right) d\xi$ is contained in the ε-ball around 1. $\square$

This completes **Step 5**. $\square$

Summing up, we obtained a sequence of entire functions $H_\nu(Z(x,y))$, which approximates $f(x,y)$ uniformly in K. Using the fact that polynomials are dense in $\mathcal{H}(\mathbb{C}^m)$ we conclude the proof of the Theorem. $\square$

Remark 2.22. (i) For a function $f \in \mathcal{C}(M)$ one has to consider Lemma 2.18 resp. formula (2.3.2) in the sense of distributions. The integrands on both sides of (2.3.1) are continuous, hence integrable and the Baouendi-Treves-Theorem holds for this class of functions.

(ii) In the case where M is totally real, the Baouendi-Treves-Theorem says, that any continuous function defined on M can be approximated locally by polynomials.

3 Global Approximation of Continuous Functions on Hypersurface Graphs

Introduction 3.1. As we have seen in the previous chapter local approximation of CR functions on CR manifolds by polynomials is always possible. One could ask, whether a similar global statement holds true, i. e., if a CR function, which is defined on a given compacta, can be approximated uniformly by polynomials. Here we will deal with the special case of graphed hypersurfaces and give a positive answer, if we assume certain convexity properties. We will also see, that in higher codimension global approximation is in general not possible. We follow the work by [1]. The part about the strictly convex exhaustion of convex sets is taken from [11].

3.1 Mergelyan's Theorem Revisited

Using the methods of the proof in $\mathbb{C}^n$ first we show a special case of Mergelyan's Theorem in $\mathbb{C}$.

Theorem 3.2 (Mergelyan's Theorem 3 - A special case)**.** *Let $h : \mathbb{R} \to \mathbb{R}$ be a smooth function and M be given as a graph over the imaginary axis, i. e., $M := \{z = x + iy \in \mathbb{C} : \ x = h(y)\}$.*
Then every continuous function f on M can be approximated uniformly on every compact subset of M by entire functions.

Remark 3.3. (i) Since M in Theorem 3.2 is totally real by Remark 2.10, we omit the CR property and show the Theorem for continuous functions.
For K a compact subset of M we have $\mathring{K} = \varnothing$, hence $A(K)$ from Definition 1.9 consists of the continuous functions on K. That means Theorem 3.2 is a special case of Mergelyan's Theorem.

(ii) The idea of the proof is to define a certain Gauß/Laplacian kernel, which fulfills the approximation identity. To obtain a negative exponent of the Gauß/Laplacian kernel we integrate the kernel w. r. t. a certain angle. Then with a partition of unity we glue all those "local" approximations together and show the approximation on the given compacta.

Proof. Let $f \in \mathcal{C}(M)$ and K be a compact subset of M.
W. l. o. g. we assume that f has compact support in a neighbourhood of K in M, otherwise we multiply it with a cut-off function. Let K' be a compact subset of M, which contains the support of f. Since we want to show the Theorem for K we multiply the graphing function h of M with a cut-off function and assume that $h(y) = 0$ for $y \in \complement K''$, where K'' is a compact subset of M containing K'.

Step 1: Introducing the approximating sequence

We denote for $s, y \in \mathbb{R}$, $z = h(y) + iy$ and $\zeta = h(s) + is$ as points on M, $\alpha \in \mathbb{C}$ and $p > 2$, $\varepsilon > 0$ real numbers. Then we define the integral kernel as

$$E^p_\varepsilon(\zeta, z, \alpha) := \exp\left(\left(\frac{\alpha(\zeta - z)}{\varepsilon}\right)^2 - \alpha^p\right).$$

The approximating sequence is defined as

$$F^p_\varepsilon(f)(z) := \frac{-iC_p}{\varepsilon}\int_M \int_0^\infty E^p_\varepsilon(\zeta, z, \alpha) f(\zeta)\alpha d\alpha d\zeta,$$

where C_p is a normalizing constant, such that $C_p\sqrt{\pi}\int_0^\infty \exp(-\alpha^p)d\alpha = 1$.

Step 2: Kernel estimates

The main idea to get a negative exponent in the integral kernel can be seen in the following

Observation 3.4. For $z = re^{i\theta} \in \mathbb{C}$ with $0 < |\theta| < \pi$ one can always find $\alpha = r'e^{i\theta'} \in \mathbb{C}$ with $|\theta'| < \frac{\pi}{4}$, such that $\mathrm{Re}((\alpha z)^2) < 0$.

Proof. For $0 < \theta < \pi$ take e. g. $\alpha := \exp\left(i\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right)$, then $\frac{\pi}{4} < \mathrm{Arg}(\alpha z) < \frac{3\pi}{4}$. Hence $\mathrm{Re}((\alpha z)^2) < 0$. If $-\pi < \theta < 0$ take e. g. $\alpha := \exp\left(-i\left(\frac{\pi}{4} + \frac{\theta}{2}\right)\right)$. □

Since M is graphed over the imaginary axis, we have that $0 < |\mathrm{Arg}(\zeta - z)| < \pi$ for $\zeta \neq z$ two fixed points in M and we may apply Observation 3.4. The chosen α_ζ will depend on ζ and z, but the same α_ζ will work even for points $\widetilde{\zeta} - z$ with $\widetilde{\zeta} \in U_\zeta$, where U_ζ is a small neighbourhood of ζ in M. It follows that $\mathrm{Re}((\alpha_\zeta(\widetilde{\zeta} - z))^2) < 0$ for $\widetilde{\zeta} \in U_\zeta$. Moreover we can find some $\delta > 0$, such that $\mathrm{Re}((\alpha_\zeta(\widetilde{\zeta} - z))^2) \leq -\delta|\alpha_\zeta|^2|\widetilde{\zeta} - z|^2$ for $\widetilde{\zeta} \in U_\zeta$, since $\frac{\pi}{2} < |\mathrm{Arg}(\alpha_\zeta(\widetilde{\zeta} - z))^2)| < \frac{3\pi}{2}$ and hence $\cos(\alpha_\zeta(\widetilde{\zeta} - z))^2) < -\delta < 0$.
$|\mathrm{Arg}(\alpha_\zeta)| < \frac{\pi}{4}$ implies that for $p > 2$ we have $\mathrm{Re}((\alpha_\zeta)^p) > 0$. And again we can find some $\delta > 0$, such that $\mathrm{Re}((\alpha_\zeta)^p) > \delta|\alpha|^p$.
Since the statement in Observation 3.4 doesn't depend on the absolute value of α, the estimate holds for any complex number in the set $\Gamma_\zeta := \{\alpha = r\exp(i\mathrm{Arg}(\alpha_\zeta)) : \ r \geq 0\}$.
Summing up we obtained for $z \in M$ that

$$|E^p_\varepsilon(\widetilde{\zeta}, z, \alpha_\zeta)| \leq \exp\left(-\delta\frac{|\alpha_\zeta|^2|\widetilde{\zeta} - z|^2}{\varepsilon^2} - \delta\alpha^p\right) \tag{3.1.1}$$

for $\widetilde{\zeta} \in U_\zeta$ a small neighbourhood of $\zeta \in M$ and $\alpha_\zeta \in \Gamma_\zeta$ chosen according to Observation 3.4.

Step 3: $F^p_\varepsilon(1)(z') = 1$

Let $z' = x' + iy' \in \mathbb{C}$. We deform the domain of integration in $F^p_\varepsilon(1)(z')$ from M to $M_{z'} := \{z \in \mathbb{C} : \ \mathrm{Re}z = \mathrm{Re}z'\}$ by Cauchy's Theorem. The horizontal parts of the integration path, which connect M and $M_{z'}$ are going to vanish. For example assume we want to handle the path $\gamma(t) := t\mathrm{Re}z' + iR$

for $t \in [0,1]$ and $R > 0$ big enough, such that $\overline{B_R(0)} \supset K''$. Then we have

$$\begin{aligned}
&\left|\frac{-iC_p}{\varepsilon}\int_0^\infty \alpha\exp(-\alpha^p)\int_0^1 \exp\left(\frac{\alpha^2((t-1)\mathrm{Re}z' + i(R-\mathrm{Im}z'))^2}{\varepsilon^2}\right)|\mathrm{Re}z'|dtd\alpha\right| \\
\leq&\frac{C_p|\mathrm{Re}z'|}{\varepsilon}\int_0^\infty \alpha\exp\left(-\left(\frac{\alpha(R-\mathrm{Im}z')}{\varepsilon}\right)^2 - \alpha^p\right)\int_0^1 \exp\left(\left(\frac{\alpha(t-1)\mathrm{Re}z'}{\varepsilon}\right)^2\right)dtd\alpha \\
\leq&\frac{C_p|\mathrm{Re}z'|}{\varepsilon}\int_0^\infty \alpha\exp\left(-\left(\frac{\alpha(R-\mathrm{Im}z')}{\varepsilon}\right)^2 + \left(\frac{\alpha\mathrm{Re}z'}{\varepsilon}\right)^2 - \alpha^p\right)d\alpha.
\end{aligned}$$

Now we set $t := \frac{\alpha(R-\mathrm{Im}z')}{\varepsilon}$, which gives

$$=\frac{C_p|\mathrm{Re}z'|\varepsilon}{(R-\mathrm{Im}z')^2}\int_0^\infty t\exp\left(t^2\left(-1+\left(\frac{\mathrm{Re}z'}{R-\mathrm{Im}z'}\right)^2\right) - t^p\left(\frac{\varepsilon}{R-\mathrm{Im}z'}\right)^p\right)dt$$

For R big enough, the first expression in the exponential is negative and bounded by 1. Then we set $s := \frac{t\varepsilon}{R-\mathrm{Im}z'}$ and obtain

$$\leq\frac{C_p|\mathrm{Re}z'|\varepsilon}{(R-\mathrm{Im}z')^2}\int_0^\infty s\exp(-s^p)ds,$$

where the last integral is bounded, hence for $R \to \infty$ the integral over γ vanishes.
Now if we calculate $F^p_\varepsilon(1)(z')$ after deforming the domain of integration to $M_{z'}$ we obtain

$$\begin{aligned}
F^p_\varepsilon(1)(z') =&\frac{-iC_p}{\varepsilon}\int_0^\infty \alpha\exp(-\alpha^p)\int_{M'_z}\exp\left(-\left(\frac{\alpha\mathrm{Im}(\zeta-z')}{\varepsilon}\right)^2\right)d\zeta d\alpha \\
=&C_p\int_0^\infty \exp(-\alpha^p)d\alpha\int_{\mathbb{R}}\exp(-t^2)dt = 1,
\end{aligned}$$

where we transformed the integral w. r. t. s to an integral over $\mathbb{R}$ setting $t := \frac{\alpha(s-y')}{\varepsilon}$.

Step 4: F^p_ε approximates the identity on M

Let $z = h(y) + iy \in K$. By **Step 3** we write

$$F^p_\varepsilon(f)(z) - f(z) = \frac{-iC_p}{\varepsilon}\int_M\int_0^\infty E^p_\varepsilon(\zeta,z,\alpha)(f(\zeta)-f(z))\alpha d\alpha d\zeta. \tag{3.1.2}$$

Now we fix $R > 0$, such that $\overline{B_R(0)} \supset K''$ and $h(y) \leq \delta'|s-y|$ for some $0 < \delta' < 1$ and $|s| \geq R$. Then we divide up the integral over M into the set $K_R := \{\zeta \in M : |\mathrm{Im}\zeta| \leq R\}$ and its complement. On $\complement K_R$ we have that $\mathrm{Re}\zeta = 0$, hence the right-hand side of (3.1.2), integrated over $\complement K_R$, can be estimated by

$$\begin{aligned}
\leq & \frac{2C_p\|f\|_\infty}{\varepsilon}\int_0^\infty \alpha\exp(-\alpha^p)\int_R^\infty \exp\left(\frac{\alpha^2}{\varepsilon^2}(-(s-y)^2+h(y)^2)\right)dsd\alpha \\
\leq & \frac{2C_p\|f\|_\infty}{\varepsilon}\int_0^\infty \alpha\exp(-\alpha^p)\int_R^\infty \exp\left(-\frac{\alpha^2\delta(s-y)^2}{\varepsilon^2}\right)dsd\alpha \\
= & 2C_p\|f\|_\infty\sqrt{\delta}\int_0^\infty \exp(-\alpha^p)\int_{R'}^\infty \exp(-t^2)dtd\alpha,
\end{aligned}$$

where $R' := \frac{\sqrt{\delta}\alpha(R-y)}{\varepsilon}$ and $\delta = 1-\delta' > 0$. The function $g_\varepsilon(\alpha) := \int_{R'}^\infty \exp(-t^2)dt$ is bounded by $\frac{\sqrt{\pi}}{2}$ and converges to 0 as $\varepsilon \to 0$. Since $\exp(-\alpha^p)g_\varepsilon(\alpha)$ is integrable on $\mathbb{R}^+$ for every $\varepsilon > 0$, the Dominated Convergence Theorem shows that

$$\int_0^\infty \exp(-\alpha^p)\int_{R'}^\infty \exp(-t^2)dtd\alpha \to 0$$

for $\varepsilon \to 0$. And the analogous computation for $(-\infty,-R) \subset \complement K_R$ gives that the right-hand side of (3.1.2) vanishes over $\complement K_R$.

Now we have to deal with the integral over the compact set K_R. To this end we cover K_R with finitely many of the sets U_ζ, say $U_1,\ldots,U_m$, given in **Step 2**. Then we take a partition of unity ψ_j subordinate to the cover U_j and we deform the integral over α from $\mathbb{R}^+$ to an integral over the ray $\Gamma_j := \{\alpha = r\exp(i\theta_j):\ r \geq 0\}$ by Cauchy's Theorem to obtain

$$\begin{aligned}
& \frac{-iC_p}{\varepsilon}\int_{K_R}\int_0^\infty E_\varepsilon^p(\zeta,z,\alpha)(f(\zeta)-f(z))\alpha d\alpha d\zeta \\
= & \frac{-iC_p}{\varepsilon}\sum_{j=1}^m\int_{U_j}\int_{\Gamma_j}\psi_j(\zeta)E_\varepsilon^p(\zeta,z,\alpha)(f(\zeta)-f(z))\alpha d\alpha d\zeta.
\end{aligned}$$

The integral over the arc $\gamma_j(t) := R\exp(it)$ for $t \in [0,\theta_j]$ connecting the real line and the ray Γ_j with angle θ_j vanishes as can be seen as follows

$$\begin{aligned}
& \left|\int_0^{\theta_j}\exp\left(\frac{R^2e^{2it}(\zeta-z)^2}{\varepsilon^2} - R^pe^{ipt}\right)Re^{it}Rdt\right| \\
\leq & R^2\int_0^{\theta_j}\exp(CR^2-\delta R^p)dt \to 0
\end{aligned}$$

as $R \to \infty$, where we used $p > 2$ together with $|\theta_j| < \frac{\pi}{4}$ and hence $\mathrm{Re}\alpha^p \geq \delta|\alpha|^p$ for some $\delta > 0$.

Now we split up every integral over U_j into the part, where $|s-y| < \nu$ and its complement in U_j for $\nu > 0$. Parametrizing $M \ni \zeta = \zeta(s) = h(s)+is$, we estimate the integral over U_j and $|s-y| < \nu$ as follows

$$
\begin{aligned}
&\frac{C_p}{\varepsilon}\int_{\Gamma_j}\int_{|s-y|<\nu}\exp\left(-\delta\left(\frac{|\alpha|^2|s-y|^2}{\varepsilon^2}+|\alpha|^p\right)\right)\psi_j(s)|f(s)-f(y)||\alpha||\zeta'(s)|dsd|\alpha|\\
\leq & C_p\|\zeta'\|_\infty\int_0^\infty\int_{|t|<\frac{\nu}{\varepsilon}}\exp\left(-\delta\left(r^2t^2+r^p\right)\right)|f(\varepsilon t+y)-f(y)|rdtdr\\
< & C_p\varepsilon\|\zeta'\|_\infty\int_0^\infty r\exp(-\delta r^p)\int_{\mathbb{R}}\exp(-\delta r^2t^2)dtdr\\
= & \varepsilon C_p\|\zeta'\|_\infty\sqrt{\frac{\pi}{\delta}}\int_0^\infty\exp(-\delta r^p)dr,
\end{aligned}
$$

where we used the uniform continuity of f on K_R, the estimate in (3.1.1) and the last integral is bounded. Hence this term gets arbitrarily small, if we choose $\nu > 0$ small enough.
The terms in the integral over U_j, where $|s-y| \geq \nu$ are treated as follows

$$
\begin{aligned}
&\frac{C_p}{\varepsilon}\int_{\Gamma_j}\int_{|s-y|\geq\nu}\exp\left(-\delta\left(\frac{|\alpha|^2|s-y|^2}{\varepsilon^2}+|\alpha|^p\right)\right)\psi_j(s)|f(s)-f(y)||\alpha||\zeta'(s)|dsd|\alpha|\\
\leq & \frac{C_p\|f\|_\infty\mathrm{vol}(U_j)\|\zeta'\|_\infty}{\varepsilon}\int_0^\infty\exp\left(-\delta\left(\frac{r^2\nu^2}{\varepsilon^2}-r^p\right)\right)rdr\\
= & \varepsilon C\int_0^\infty\exp(-\delta(s^2\nu^2+\varepsilon^ps^p))sds\\
\leq & \varepsilon C\int_0^\infty\exp(-\delta\nu^2s^2)sds,
\end{aligned}
$$

where the last integral is bounded, hence the integral over the set $\{\zeta \in U_j : \ |s-y| \geq \nu\}$, gets arbitrarily small.
The preceding computations can be done for every U_j, hence in total we obtained, for any $\varepsilon > 0$, that $|F_\varepsilon(f)(z) - f(z)| < \varepsilon$ for every $z \in K$. $\square$

Remark 3.5. (i) The nice thing about the one dimensional case is that the approximating sequence F_ε^p is an entire function. This stands in contrast to the higher dimensional case, where the initial approximating sequence is not holomorphic at all.

(ii) The term α^p in the integral kernel plays an essential role in the convergence of F_ε^p, especially when we integrated over the set $\{\zeta \in U_j : \ |s-y| < \nu\}$.

(iii) Another important property of the integral kernel is its **approximation identity**. This means, that if the kernel E_ε^p is integrated e. g. over M then it sums up to 1, c. f. **Step 3**, and that the volume outside of a small neighbourhood of 0 gets arbitrarily small as $\varepsilon \to 0$.

3.2 The General Case

Recall from Remark 2.10 the notion of hypersurface graphs.

Definition 3.6. Let M be a hypersurface graph of h.
An open set $\omega \subseteq M$ is called **CR-Runge** in $\mathbb{C}^n$, if every continuous CR function on ω can be approximated uniformly by entire functions on any compact subset of ω.

Theorem 3.7. *Let M be a hypersurface graph of h and $\omega \subseteq M$ an open set, which is given as the graph of h over a convex set ω', i. e., $\omega = \{x = h(y, w) : (y, w) \in \omega' \subseteq \mathbb{R} \times \mathbb{C}^{n-1}\}$.*
Then ω is CR-Runge in $\mathbb{C}^n$.

Remark 3.8. (i) From now on we will deal with the case of $n \geq 2$.

(ii) The notion of convexity in Theorem 3.7 is the real euclidean convexity in $\mathbb{R}^{2n-1}$, i. e., $\omega' \subseteq \mathbb{R}^{2n-1}$ is said to be convex, if for any two points $x, y \in \omega'$ the straight line connecting x and y is contained in ω'.

(iii) The idea of the proof is to first show "local approximation" of f of the non-holomorphic initial approximation sequence on "slices" of M, then by the CR property of f we can show that the "local approximation" is independent of the slice up to a small error term. With a partition of unity we glue the "local approximations" together to a global approximaton sequence on the given compact set and solve a $\bar{\partial}-$equation to obtain a resulting holomorphic global approximation sequence.

Proof. We write $z = x + iy \in \mathbb{C}$ and $w = u + iv \in \mathbb{C}^{n-1}$.
Let ω be given as in the Theorem and K be any compact subset of ω. Then we assume w. l. o. g. that f has compact support in $\omega \subseteq M$, otherwise we multiply it with a cut-off function, which is one on K. And as in the one dimensional case we assume that $h(y, u + iv) = 0$, for large $|y|$ or $|v|$, outside of the compact set K, where we want to approximate. Moreover we assume that $f \in \mathcal{C}^1(M)$ and discuss the continuous case afterwards.

Step 1: Introducing the approximation sequence

We denote (z, w), $(\zeta, \eta) \in \mathbb{C} \times \mathbb{C}^{n-1}$, $\alpha \in \mathbb{C}$, $\varepsilon > 0$ and $p > 2$ real numbers and $\Lambda > 0$ a large constant.
Then we define the integral kernel to be

$$E_\varepsilon^{\Lambda,p}(\zeta, \eta, z, w, \alpha) := \frac{C_\Lambda}{\varepsilon^n} \exp\left(\alpha^2 \left(\frac{\zeta - z}{\varepsilon}\right)^2 + \Lambda\alpha^2 \left(\frac{\eta - w}{\varepsilon}\right)^2 - \alpha^p\right),$$

where C_Λ is normalizing constant, such that

$$C_\Lambda \int_0^\infty \int_{\mathbb{R}^n} \exp(-r^2 x^2 - \Lambda r^2 y^2 - r^p) r^n d(x, y) dr = 1, \tag{3.2.1}$$

for $\Lambda \geq 1$ to be chosen in **Step 2**. Now we define the initial approximation sequence

$$F_\varepsilon^u(f)(z, w) := \int_{M_u} \int_0^\infty f(\zeta, \eta) E_\varepsilon^{\Lambda,p}(\zeta, \eta, z, w, \alpha) \alpha^n d\alpha d(\zeta, \eta),$$

where $u = \mathrm{Re} w$ and $M_u := \{(\zeta, \eta) \in M : \mathrm{Re}\eta = u\}$. This means M_u is a totally real, n real dimensional slice of M, where we fix the real part of η at u. The term $d(\zeta, \eta)$ denotes $d\zeta \wedge d\eta_1 \wedge \ldots \wedge d\eta_{n-1}$, a $(n, 0)-$form. The term F_ε^u is finite for any $\Lambda, \varepsilon > 0$, since f has compact support and $p > 2$.
Note that F_ε^u is not holomorphic, since the domain of integration depends on $u = \mathrm{Re} w$.

Step 2: Kernel estimates

As above let $\mathbb{C}^{n-1}\times\mathbb{C}\ni(\zeta,\eta)=(h(s,u',t)+is,u'+it)\in M$ and $(z,w)=(h(y,u,v)+q+iy,u+iv)\in\mathbb{C}\times\mathbb{C}^{n-1}$, where $q\in\mathbb{R}$. Then we have $(\zeta,\eta)\in M_u$ iff $u'=u$ and $(z,w)\in M$ iff $q=0$. We have the following

Lemma 3.9. *Let L be a compact subset of $\mathbb{C}^n$ and $d>0$.*
Then there exist constants $p>2$, $\delta, C, \Lambda_0>0$ depending only on K and d, such that the following hold for $\Lambda>\Lambda_0$:
For any $(z_0,w_0)\in L$ and $(\zeta_0,\eta_0)\in L\cap M$ there exist neighbourhoods Q and Q' of (z_0,w_0) and (ζ_0,η_0) in $\mathbb{C}^n$, both with diameter less than $\frac{d}{4}$, and an angle θ_0 with $|p\theta_0|\le\frac{\pi}{2}-\delta$, such that for all $(z,w)=(h(y,u,v)+q+iy,u+iv)\in Q$, $(\zeta,\eta)=(h(s,u',t)+is,u'+it)\in Q'\cap M$ and $\alpha\in\Gamma_{\theta_0}:=\{r\exp(i\theta_0):\ r\ge 0\}$ we have

1. *if $|s_0-y_0|\le d$, then*

$$\left|E_\varepsilon^{\Lambda,p}(\zeta,\eta,z,w,\alpha)\right|\le\exp\left(-\delta\left(|\alpha|^2\left|\frac{s-y}{\varepsilon}\right|^2+\Lambda|\alpha|^2\left|\frac{t-v}{\varepsilon}\right|^2+|\alpha|^p\right)\right.$$
$$\left.+\Lambda C|\alpha|^2\left|\frac{u-u'}{\varepsilon}\right|^2+C|\alpha|^2\left(\frac{q}{\varepsilon}\right)^2\right),$$

and

2. *if $|s_0-y_0|\ge d$, then*

$$\left|E_\varepsilon^{\Lambda,p}(\zeta,\eta,z,w,\alpha)\right|\le\exp\left(-\delta\left(|\alpha|^2\left|\frac{s-y}{\varepsilon}\right|^2+\Lambda|\alpha|^2\left|\frac{t-v}{\varepsilon}\right|^2+|\alpha|^p\right)\right.$$
$$\left.+\Lambda C|\alpha|^2\left|\frac{u-u'}{\varepsilon}\right|^2\right).$$

Proof. First we fix $(z_0,w_0)\in L$ and $(\zeta_0,\eta_0)\in L\cap M$ and choose a smooth function $\varphi:\mathbb{R}\to[0,1]$ with $\varphi(s-y)=0$ for $|s-y|\le\frac{d}{4}$ and $\varphi(s-y)=1$ for $|s-y|\ge\frac{d}{2}$. Then we write

$$\begin{aligned}\zeta-z=&h(s,u',t)-h(y,u,v)-q+i(s-y)\\=&\underbrace{\left(\frac{h(s,u,v)-h(y,u,v)}{s-y}-\frac{\varphi(s-y)q}{s-y}+i\right)}_{=:W}(s-y)\\&+(h(s,u',v)-h(s,u,v))+(h(s,u',t)-h(s,u',v))-(1-\varphi(s-y))q\end{aligned}\tag{3.2.2}$$

Since h is smooth and $\varphi(s-y)=0$ for $|s-y|\le\frac{d}{4}$ the real part of W is bounded on $K\times K$ and the imaginary part is positive, hence $|\mathrm{Arg}W|>0$. We will apply the ideas of Observation 3.4 in this setting. This gives us some $\delta>0$, depending only on L, an angle θ_0 with $|\theta_0|<\frac{\pi}{4}-\delta$ and neighbourhoods Q' and Q of (ζ_0,η_0) and (z_0,w_0), both with diameter less than $\frac{d}{4}$, such that if $(\zeta,\eta)\in Q'\cap M$ and $(z,w)\in Q$ we have

$$\mathrm{Re}(e^{2i\theta_0}W^2)\le-\delta|W|^2.\tag{3.2.3}$$

Using (3.2.2) we obtain

$$\begin{aligned}\mathrm{Re}(e^{2i\theta_0}(\zeta-z)^2) =&\mathrm{Re}\left(e^{2i\theta_0}(W^2(s-y)^2+(h(s,u',v)-h(s,u,v))^2\right.\\ &\left.+(h(s,u',t)-h(s,u',v))^2+(1-\varphi)^2q^2)+\text{ cross terms}\right).\end{aligned} \tag{3.2.4}$$

The terms $|h(s,u',v)-h(s,u,v)|$ resp. $|h(s,u',t)-h(s,u',v)|$ are bounded by $C|u'-u|$ resp. $C|t-v|$ for some $C>0$ depending only on L.
The cross terms are estimated as follows. For example the term $|(h(s,u',t)-h(s,u',v))(1-\varphi)q|\leq \frac{1}{2}(C^2|t-v|^2+|q|^2)$ or

$$|W(s-y)(1-\varphi)q|=\sqrt{\frac{\delta}{4}}|W||s-y|\sqrt{\frac{4}{\delta}}|1-\varphi||q|\leq\frac{\delta}{8}|W|^2|s-y|^2+\frac{2}{\delta}q^2$$

and the term $\frac{\delta}{8}|W|^2|s-y|^2$ gets absorbed by $-\delta|W|^2|s-y|^2$ from (3.2.4) resp. (3.2.3). We are going to write δ again for this slightly smaller δ. The other term $\frac{2}{\delta}q^2$ becomes $C'q^2$ for some constant $C'>0$ depending on L and d. In total we obtain

$$\mathrm{Re}(e^{2i\theta_0}(\zeta-z)^2)\leq-\delta|W|^2|s-y|^2+\widetilde{C}(|u-u'|^2+|t-v|^2)+C'q^2. \tag{3.2.5}$$

Now we want to estimate the $(\eta-w)-$terms.
Since $|2\theta_0|<\frac{\pi}{2}-2\delta$ it is

$$\begin{aligned}\mathrm{Re}(e^{2i\theta_0}(\eta-w)^2)&=\mathrm{Re}\left(e^{2i\theta_0}(u'-u+i(t-v))^2\right)\\ &\leq(u'-u)^2-\cos(2\theta_0)(t-v)^2+2\sin(2\theta_0)|u'-u||t-v|\\ &\leq(u'-u)^2-\delta(t-v)^2+2\sqrt{\frac{2}{\delta}}|u'-u|\sqrt{\frac{\delta}{2}}|t-v|\\ &\leq-\frac{\delta}{2}|t-v|^2+\left(1+\frac{2}{\delta}\right)|u'-u|^2\end{aligned} \tag{3.2.6}$$

for any $\eta,\ w\in\mathbb{C}^{n-1}$.
Take some p with $2<p<\frac{2}{1-\delta}$ and since $|\theta_0|<\frac{\pi}{4}-\delta$ we have $|\theta_0|<\frac{\pi}{2}$. Hence there exists some $\delta'>0$, such that

$$\text{for } \alpha=re^{i\theta_0},\ \text{with } r\geq0,\ \text{then } \mathrm{Re}(\alpha^p)\geq\delta' r^p. \tag{3.2.7}$$

If we are in the case, where $|s_0-d_0|\leq d$, we use (3.2.5), (3.2.6) and (3.2.7) to obtain, since $|W|^2\geq1$, with $\alpha=re^{i\theta_0}$ for $r\geq0$

$$\begin{aligned}&\mathrm{Re}\left(\alpha^2\left(\frac{\zeta-z}{\varepsilon}\right)^2+\Lambda\alpha^2\left(\frac{\eta-w}{\varepsilon}\right)^2-\alpha^p\right)\\ &\leq-\delta r^2\left|\frac{s-y}{\varepsilon}\right|^2+\widetilde{C}r^2\left(\left|\frac{u-u'}{\varepsilon}\right|^2+\left|\frac{t-v}{\varepsilon}\right|^2\right)+C'r^2\left(\frac{q}{\varepsilon}\right)^2\\ &-\Lambda\frac{\delta}{2}r^2\left|\frac{t-v}{\varepsilon}\right|^2+\Lambda r^2\left(1+\frac{2}{\delta}\right)\left|\frac{u-u'}{\varepsilon}\right|^2-\delta' r^p.\end{aligned} \tag{3.2.8}$$

Take $\Lambda \geq 1$ big enough, such that $C - \Lambda\frac{\delta}{2} \leq -\delta\Lambda$ and set $C := \max(C', \widetilde{C} + \Lambda\left(1 + \frac{2}{\delta}\right))$ and we obtain the first part of the Lemma.
In the second case, where $|s_0 - y_0| \geq d$, then we have $|s - y| > \frac{d}{2}$ for all $(\zeta, \eta) \in Q'$ and $(z, w) \in Q$ and $\varphi(s - y) = 1$. Hence from (3.2.5) we obtain since $(1 - \varphi)q = 0$

$$\mathrm{Re}(e^{2i\theta_0}(\zeta - z)^2) \leq -\delta|W|^2|s - y|^2 + \widetilde{C}(|u - u'|^2 + |t - v|^2). \tag{3.2.9}$$

Then we proceed as in the first case using the estimates from (3.2.9), (3.2.6) and (3.2.7) to complete the proof. □

We will need an application of the previous Lemma, where (ζ, η) lies near the boundary of ω and (z, w) in a compact subset of ω'.
Define $H(y, u, v) := h(y, u, v) + iy, u + iv$. To prove Theorem 3.7 we have to show that for every $\omega_1' \subset\subset \omega'$ any continuous CR function on $\omega = H(\omega')$ can be approximated by entire functions on $H(\omega_1')$. Now we are going to apply the following

Theorem 3.10. *Let $\Omega \subset \mathbb{R}^n$ be an open and convex set.*
Then Ω can be exhausted by a sequence of strictly convex sets with smooth boundary, i. e., there exist a sequence Ω_j of strictly convex sets with smooth boundary, such that $\Omega_j \subset\subset \Omega_{j+1} \subset\subset \Omega$ and $\bigcup_{j=1}^{\infty} \Omega_j = \Omega$.

Proof. e. g. [11], Corollary 2. □

Remark 3.11. Let $\Omega \subset \mathbb{R}^n$ be a convex set with smooth boundary and local defining function μ, then for $\delta > 0$ the function $\tilde{\mu}(x) := \mu(x) + \delta|x|^2$ is strictly convex. To see this we check (0.2.1) for $\tilde{\mu}$, so let $p \in b\Omega$ and $0 \neq \xi \in T_p(b\Omega)$. We obtain

$$d_2\tilde{\mu}_p(\xi) = d_2\mu_p(\xi) + 2\delta\sum_{k=1}^{n} \xi_k^2 \geq 2\delta|\xi|^2 > 0,$$

thus $\tilde{\mu}$ is strictly convex. A natural choice for μ would be the signed distance function, which is defined as

$$\mu(x) := \begin{cases} -d(x, b\Omega) & \text{for } x \in \overline{\Omega} \\ d(x, b\Omega) & \text{for } x \notin \overline{\Omega}, \end{cases}$$

where $d(.,.)$ is the euclidean distance of $\mathbb{R}^n$. Then we have to see, that the signed distance function is convex and one has to consider a smoothed version of μ. Then we could take the level sets of $\tilde{\mu}$ with a sufficiently small $\delta > 0$ as exhaustion sequence for the convex set Ω.

Since ω' is open and convex, it can be exhausted by strictly convex sets with smooth boundary, hence we assume ω_1' to be strictly convex and has smooth boundary. Then we choose another strictly convex set ω_2', such that $\omega_1' \subset\subset \omega_2' \subset\subset \omega'$ and denote the distance from ω_1' to $b\omega_2'$ by $3d$. Let $I \subset \mathbb{R}$ be an interval, such that $\omega = H(\omega')$ is contained in $I \times \omega'$. Define $L := I \times \overline{\omega'}$, which is a compact and convex set.

Lemma 3.12. *Let $\omega_1' \subset\subset \omega_2' \subset\subset \omega'$ and L as above.*
Then there exist constants $p > 2$, $\delta, C, \Lambda, d > 0$, such that if $(y_0, u_0, v_0) \in \omega_1'$ and $(s_0, u_0', t_0) \in \omega' \backslash \omega_2'$ with $|u_0 - u_0'| \leq d$, then there exist neighbourhoods Q_1 of $H(y_0, u_0, v_0) + (q, 0) \in L$ and Q_1' of $H(s_0, u_0', t_0) \in L \cap M$ and an angle θ with $|p\theta| \leq \frac{\pi}{2} - \delta$, such that

$$\left|E_\varepsilon^{\Lambda,p}(\zeta, \eta, z, w, \alpha)\right| \leq \exp\left(-|\alpha|^2 \frac{\delta}{8}\left(\frac{d}{\varepsilon}\right)^2 + \Lambda C|\alpha|^2 \left|\frac{u-u'}{\varepsilon}\right|^2 - \delta|\alpha|^p\right),$$

for all $H(y, u, v) + (q, 0) \in Q_1$ and $H(s, u', t) \in Q_1' \cap M$ and $\alpha \in \Gamma_\theta$.

Proof. The preparations preceding the Lemma imply that (y_0, u_0, v_0) and (s_0, u_0', t_0) are separated by a distance of $3d$. Since $|u_0 - u_0'| \leq d$ we obtain the case, where $|s_0 - y_0| \geq d$ or the case, where $|t_0 - v_0| \geq d$ and $|s_0 - y_0| \leq d$.
By Lemma 3.9 we obtain neighbourhoods Q' of $(z_0, w_0) = H(y_0, u_0, v_0) + (q, 0) \in L$ and Q of $(\zeta_0, \eta_0) = H(s_0, u_0', t_0) \in L \cap M$, both with diameter less than $\frac{d}{4}$ and two possible estimates for the integral kernel.
If $|s_0 - y_0| \geq d$ we obtain by the second case of Lemma 3.9 and because $|s - y| \geq \frac{d}{2}$ on Q' and Q, that

$$\left|E_\varepsilon^{\Lambda,p}(\zeta, \eta, z, w, \alpha)\right| \leq \exp\left(-\delta\left(|\alpha|^2\left(\frac{d}{2\varepsilon}\right)^2 + |\alpha|^p\right) + \Lambda C|\alpha|^2\left|\frac{u-u'}{\varepsilon}\right|^2\right),$$

for all $(\zeta, \eta) \in Q' \cap M$, $(z, w) \in Q$ and $\alpha \in \Gamma_{\theta_0}$.
In the second case, where $|t_0 - v_0| \geq d$ and $|s_0 - y_0| \leq d$, we apply the first case of Lemma 3.9 and since $|t - v| \geq \frac{d}{2}$ on Q' and Q, we have

$$\left|E_\varepsilon^{\Lambda,p}(\zeta, \eta, z, w, \alpha)\right| \leq \exp\left(-\delta\left(\Lambda|\alpha|^2\left(\frac{d}{2\varepsilon}\right)^2 + |\alpha|^p\right) + \Lambda C|\alpha|^2\left|\frac{u-u'}{\varepsilon}\right|^2 + C|\alpha|^2\left(\frac{q}{\varepsilon}\right)^2\right),$$

for all $(\zeta, \eta) \in Q' \cap M$, $(z, w) \in Q$ and $\alpha \in \Gamma_{\theta_0}$.
Now we choose $\Lambda \geq 1$ big enough and depending only on K, C and d, to obtain

$$-\delta\Lambda|\alpha|^2\left(\frac{d}{2\varepsilon}\right)^2 + C|\alpha|^2\left(\frac{q}{\varepsilon}\right)^2 \leq -\frac{\delta}{2}\Lambda|\alpha|^2\left(\frac{d}{2\varepsilon}\right)^2 \leq -\frac{\delta}{2}|\alpha|^2\left(\frac{d}{2\varepsilon}\right)^2,$$

for any $q \in \mathbb{R}$, such that $(z, w) = H(y, u, v) + (q, 0) \in L$.
Finally we obtain $p > 2$ and θ from Lemma 3.9 and set $Q_1 := Q$ and $Q_1' := Q'$. □

Step 3: $E_\varepsilon^{\Lambda,\delta,p}$ fulfills the approximation identity

For $s \in \mathbb{R}$, $t \in \mathbb{R}^{n-1}$ and $r \geq 0$ we define

$$E_\varepsilon^{\Lambda,\delta,p}(s, t, r) := \frac{C_\Lambda}{\varepsilon^n} r^n \exp\left(-\delta\left(r^2\left(\frac{s}{\varepsilon}\right)^2 + \Lambda r^2\left(\frac{t}{\varepsilon}\right)^2 + r^p\right)\right).$$

Lemma 3.13. *1. Let $\delta > 0$ be fixed, then for any $\varepsilon > 0$ we have*

$$\int_0^\infty \int_{\mathbb{R}^n} E_\varepsilon^{\Lambda,\delta,p}(s,t,r)d(s,t)dr = C_\delta,$$

where C_δ only depends on δ.

2. Let $\nu > 0$ be fixed, then

$$\lim_{\varepsilon\to 0}\left(\int_0^\infty \int_{\mathbb{R}^n\setminus B_\nu(0)} E_\varepsilon^{\Lambda,\delta,p}(s,t,r)d(s,t)dr\right) = 0$$

Proof. 1. We transform the integral by setting $x := \frac{s}{\varepsilon}$ and $y := \frac{t}{\varepsilon}$. Then using the property of C_Λ from (3.2.1) we obtain by another variable transformation $C_\delta = \frac{1}{\sqrt{\delta^n}\sqrt[p]{\delta}}$.

2. We transform the integral as above to obtain

$$C_\Lambda \int_0^\infty \exp(-\delta r^p) \underbrace{\int_{\mathbb{R}^n\setminus B_{R_{r,\varepsilon}}(0)} \exp(-\delta(x^2+\Lambda y^2)d(x,y)\,dr}_{=:g_\varepsilon(r)}$$

where $R_{r,\varepsilon} := \frac{r\nu}{\varepsilon}$. For every $r \geq 0$ $g_\varepsilon(r)$ is uniformly bounded, hence $\exp(-\delta r^p)g_\varepsilon(r)$ is an integrable function on $\mathbb{R}^+$. Since for any $r > 0$ $g_\varepsilon(r)$ goes to 0 as $\varepsilon \to 0$, we apply the Dominated Convergence Theorem and the Lemma is proved.

□

Step 4: F_ε^u approximates the identity on M_u

Similar to the one dimensional case, c.f. Theorem 3.2, we have the following

Lemma 3.14. *Let $(z,w) \in M_u$, then $F_\varepsilon^u(f)(z,w) \to f(z,w)$ as $\varepsilon \to 0$.*

Proof. First we will show

$$\int_{M_u}\int_0^\infty E_\varepsilon^{\Lambda,p}(\zeta,\eta,z,w,\alpha)\alpha^n d\alpha d(\zeta,\eta) = 1 \tag{3.2.10}$$

for every $(z,w) \in \mathbb{C}^n$. Here we deform the $\zeta-$integral from $M_u := \{(h(s,u+it)+is, u+it) : s \in \mathbb{R},\ t \in \mathbb{R}^{n-1}\}$ to $M_u^z := \{(\mathrm{Re} z + is, u+it) : s \in \mathbb{R},\ t \in \mathbb{R}^{n-1}\}$ by Cauchy's Theorem and since $h(y,u+it) = 0$ for $|y|$ large enough, hence we can show as in **Step 4** of the one-dimensional case, that the horizontal terms vanishes. We obtain

$$\begin{aligned}&\frac{C_\Lambda}{\varepsilon^n}\int_0^\infty\int_{\mathbb{R}^n}\exp\left(-r^2\left(\frac{s-y}{\varepsilon}\right)^2 - \Lambda r^2\left(\frac{t-v}{\varepsilon}\right)^2 - r^p\right)r^n d(s,t)dr\\ =&C_\Lambda\int_0^\infty\int_{\mathbb{R}^n}\exp(-r^2x^2-\Lambda r^2y^2-r^p)r^n d(x,y)dr = 1,\end{aligned}$$

due to the presence of the constant C_Λ from (3.2.1). Hence using (3.2.10) we obtain

$$F_\varepsilon^u(f)(z,w) - f(z,w) = \int_0^\infty \int_{M_u} (f(\zeta,\eta) - f(z,w)) E_\varepsilon^{\Lambda,p}(\zeta,\eta,z,w,\alpha)\alpha^n d(\zeta,\eta) d\alpha,$$

where $(\zeta,\eta) = H(s,u,t) = (h(s,u,t)+is, u+it)$ and $(z,w) = H(y,u,v)$ with $s,y \in \mathbb{R}$ and $t,v \in \mathbb{R}^{n-1}$. Then we split up the integral over M_u into the part, where $\complement M_u^R := \{(\zeta,\eta) \in M_u : \ |s-y|^2+|t-v|^2 \geq R^2\}$ and its complement.

The integral over $\complement M_u^R$ is treated as follows. If $R > 0$ is large enough, such that $\mathrm{Re}\zeta = h(s,u,t) = 0$ and $|h(y,u,v)| \leq \delta'|s-y|$ for some $0 < \delta' < 1$ and $\zeta \in \complement M_u^R$. Then we have by setting $\widetilde{s} := s-y$ and $\widetilde{t} := t-v$

$$\begin{aligned}
&\frac{C_\Lambda}{\varepsilon^n}\int_0^\infty r^n \exp(-r^p) \int_{\complement M_u^R} |f(\zeta,\eta) - f(z,w)| \exp\left(r^2\left(\frac{h(y,u,v)^2 - \widetilde{s}^2}{\varepsilon^2}\right) - \Lambda r^2 \left(\frac{\widetilde{t}}{\varepsilon}\right)^2\right) d(\widetilde{s},\widetilde{t})dr \\
&\leq 2C_\Lambda \|f\|_\infty \int_0^\infty r^n \exp(-r^p) \int_{\mathbb{R}^{n-1}\setminus B_{R'}(0)} \exp(-\delta r^2 \widetilde{x}^2 - \Lambda r^2 \widetilde{y}^2) d(\widetilde{x},\widetilde{y}) dr,
\end{aligned}$$

where we transformed the integral from $\widetilde{s}$ to $\widetilde{x} := \frac{\widetilde{s}}{\varepsilon}$ resp. from $\widetilde{t}$ to $\widetilde{y} := \frac{\widetilde{t}}{\varepsilon}$ and we set $\delta := 1-\delta'$ and $R' := \frac{R}{\varepsilon}$. By Lemma 3.13, part 2 the above integral converges to 0 as $\varepsilon \to 0$.

Now we want to show that the integral over M_u^R also tends to 0 as $\varepsilon \to 0$. Here we apply Lemma 3.9 to the compact set M_u^R and some $d > 0$ to obtain finitely many neighbourhoods Q_j' covering M_u^R. Moreover we get for every Q_j' a ray Γ_j to which we transform the integral over the positive real axis by Cauchy's Theorem,where the boundary terms vanishes as in the one-dimensional case in **Step 4**. Now we take a partition of unity ψ_j subordinate to the cover Q_j' and write $\sum_j \int_{M_u^R}\int_{\Gamma_j} \psi_j(\zeta,\eta)(f(\zeta,\eta) - f(z,w))E_\varepsilon^{\Lambda,p}(\zeta,\eta,z,w,\alpha)\alpha^n d\alpha d(\zeta,\eta)$ for the integral over M_u^R.

For any $(\zeta,\eta) \in M_u^R$ we apply Lemma 3.9 with $q = 0$ and $u' = u$. Then we split up every integral over M_u^R into $M_u^\nu := \{(\zeta,\eta) \in M_u^R : \ |s-y|^2 + |t-v|^2 < \nu^2\}$ and its complement in M_u^R, denoted by $M_u^{\nu,R} := \{(\zeta,\eta) \in M_u^R : \ \nu^2 \leq |s-y|^2 + |t-v|^2 < R^2\}$, where $\nu > 0$. In the first case we obtain

$$\begin{aligned}
&\int_{M_u^\nu}\int_{\Gamma_j} \psi_j(\zeta,\eta)(f(\zeta,\eta) - f(z,w))E_\varepsilon^{\Lambda,p}(\zeta,\eta,z,w,\alpha)\alpha^n d\alpha d(\zeta,\eta) \\
\leq &\varepsilon \int_{\Gamma_j}\int_{M_u^\nu} \frac{C_\Lambda}{\varepsilon^n} \exp\left(-\delta\left(|\alpha|^2\left|\frac{s-y}{\varepsilon}\right|^2 + |\alpha|^2\Lambda\left|\frac{t-v}{\varepsilon}\right|^2 + |\alpha|^p\right)\right)|\alpha|^n d\alpha d(\zeta,\eta) \\
\leq &\varepsilon \int_0^\infty \int_{\mathbb{R}^n} E_\varepsilon^{\Lambda,\delta,p} d(s,t) dr = \varepsilon C_\delta,
\end{aligned}$$

where $\nu > 0$ is taken small enough, using the uniform continuity of f and the first part of Lemma 3.13.

In the second case we have

$$\begin{aligned}
&\int_{M_u^{\nu,R}}\int_{\Gamma_j} \psi_j(\zeta,\eta)(f(\zeta,\eta) - f(z,w))E_\varepsilon^{\Lambda,p}(\zeta,\eta,z,w,\alpha)\alpha^n d\alpha d(\zeta,\eta) \\
\leq &2\|f\|_\infty \int_{\Gamma_j}\int_{\mathbb{R}^n\setminus B_\nu(0)} \frac{C_\Lambda}{\varepsilon^n}\exp\left(-\delta\left(|\alpha|^2\left|\frac{s-y}{\varepsilon}\right|^2 + |\alpha|^2\Lambda\left|\frac{t-v}{\varepsilon}\right|^2 + |\alpha|^p\right)\right)|\alpha|^n d\alpha d(\zeta,\eta),
\end{aligned}$$

which tends to 0 as $\varepsilon \to 0$ by the second part of Lemma 3.13.

This completes the proof of the Lemma. □

Remark 3.15. The proof shows, that the convergence of Lemma 3.14 is uniform in u, if u is contained in a compact subset of $\mathbb{R}^{n-1}$.

Step 5: F_ε^u is locally independent of u up to an arbitrarily small error term

In this section we will use the CR-property of f to obtain a lower bound of the point (z,w), where we want to approximate, and the domain of integration, denoted by integration coordinates (ζ,η).
We recall the sets $\omega_1' \subset\subset \omega_2' \subset\subset \omega'$ from Lemma 3.12. Then we multiply f with a cut-off function, which is one in a neighbourhood of $\overline{H(\omega_2')}$ and denote the resulting function again by f.
The function f is CR on $H(\omega_2') \subseteq M$, hence it can be extended to a function, again denoted by f, in a neighbourhood V of $H(\omega_2')$ in $\mathbb{C}^n$, which fulfills $\bar\partial f = 0$ in V. This technique is called "almost holomorphic extension".
Thus we can assume, that $\bar\partial f$ has compact support in $H(\omega')\backslash H(\omega_2')$ and define the compact, convex set $L_1 := I \times \overline{\omega_1'}$, where I is the real interval from **Step 2**, defined above Lemma 3.12.

Lemma 3.16. *Let f be a continuous, compactly supported function in ω with supp$(\bar\partial f) \subseteq \omega\backslash H(\omega_2')$. Then there exists some $d_1 > 0$ and a constant $C > 0$, depending on L_1, such that if $|u-u_0| \le d_1$, then $|F_\varepsilon^u(f)(z,w) - F_\varepsilon^{u_0}(f)(z,w)| \le C\varepsilon$ for all $(z,w) \in L_1$.*

Remark 3.17. Lemma 3.16 says, that $F_\varepsilon^{u_0}$ approximates the identity on M_u if $|u_0 - u| \le d_1$.

Proof. We need an object $\widetilde{M}_{uu_0}$ to be a real submanifold, where the boundary of $\widetilde{M}_{uu_0}$ is given by M_u and M_{u_0}. We connect u and u_0 by a real line segment γ_{uu_0} and define $\mathbb{R}^{2n-1}_{u_0} := \{(\mathrm{Im}\zeta,\eta) \in \mathbb{R}\times\mathbb{R}^{2n-2} :\ \mathrm{Re}\eta = u_0\}$, which is a $n-$dimensional subspace of $\mathbb{R}^{2n-1}$. Then we define $\widetilde{M}_{uu_0} \subseteq M$ as the graph of h over $\mathrm{span}_{\mathbb{R}}\{\gamma_{uu_0}, \mathbb{R}^{2n-1}_{u_0}\}$, which is a real $(n+1)-$dimensional real submanifold of $\mathbb{R}^{2n}$.
With Stoke's Theorem we write

$$F_\varepsilon^u(f)(z,w) - F_\varepsilon^{u_0}(f)(z,w) = \int_{\widetilde{M}_{uu_0}} \int_0^\infty \bar\partial f(\zeta,\eta) E_\varepsilon^{\Lambda,p}(\zeta,\eta,z,w,\alpha)\alpha^n d\alpha d(\zeta,\eta), \tag{3.2.11}$$

because applying the exterior derivative $d_{\zeta,\eta}$ to $f(\zeta,\eta)E_\varepsilon^{\Lambda,p}(\zeta,\eta,z,w,\alpha)\alpha^n$ the $\partial-$terms vanish, since $d(\zeta,\eta)$ is a $(n,0)-$form, and the kernel $E_\varepsilon^{\Lambda,p}(\zeta,\eta,z,w,\alpha)$ is holomorphic in (ζ,η).
Since $(z,w) = H(y,u,v)+(q,0) \in L_1$, i. e., $(y,u,v) \in \omega_1'$, and $(\zeta,\eta) = H(s,u',t) \in (\omega\backslash H(\omega_2'))\cap\widetilde{M}_{uu_0}$, i. e. $(s,u',t) \in \omega'\backslash\omega_2'$, hence $|u-u'| \le d_1$, if $|u-u_0| \le d_1$, and if we choose $d_1 \le d$, we can apply Lemma 3.12. Since the domain of integration, i. e. supp$(\bar\partial f)$, is compact, we obtain a finite, open cover Q_j' of supp$(\bar\partial f)$. Choosing a partition of unity ψ_j subordinate to the cover Q_j', we can write

$$F_\varepsilon^u(f)(z,w) - F_\varepsilon^{u_0}(f)(z,w) = \sum_j \int_{\widetilde{M}_{uu_0}} \int_{\Gamma_j} \psi_j(\zeta,\eta)\bar\partial f(\zeta,\eta) E_\varepsilon^{\Lambda,p}(\zeta,\eta,z,w,\alpha)\alpha^n d\alpha d(\zeta,\eta),$$

where Γ_j is the ray given in Lemma 3.12. The integral is deformed from $\mathbb{R}^+$ to Γ_j by Cauchy's Theorem, again as in the one-dimensional case in **Step 4** the horizontal terms vanish. Every integrand of the above integrals is bounded by

$$\left|E_\varepsilon^{\Lambda,p}(\zeta,\eta,z,w,\alpha)\right| \le \exp\left(-|\alpha|^2\frac{\delta}{8}\left(\frac{d}{\varepsilon}\right)^2 + \Lambda C|\alpha|^2\left|\frac{u-u'}{\varepsilon}\right|^2 - \delta|\alpha|^p\right)$$

for the given $(z,w)\in L_1$, $(\zeta,\eta)\in Q'_j\cap\widetilde{M}_{uu_0}$ and $\alpha\in\Gamma_j$. Taking d_1 even smaller, to fulfill $\Lambda C d_1^2 \le \delta\frac{d^2}{16}$ we obtain,

$$\left|E_\varepsilon^{\Lambda,p}(\zeta,\eta,z,w,\alpha)\right| \le \exp\left(-|\alpha|^2\frac{\delta}{16}\left(\frac{d}{\varepsilon}\right)^2 - \delta|\alpha|^p\right),$$

hence

$$|F_\varepsilon^{u}(f)(z,w) - F_\varepsilon^{u_0}(f)(z,w)| \le \frac{\widetilde{C}}{\varepsilon^n}\int_0^\infty \exp\left(-r^2\frac{\delta}{16}\left(\frac{d}{\varepsilon}\right)^2\right) r^n dr = C\varepsilon,$$

where C depends on d,δ and L_1. □

Step 6: Globalization and making the approximation sequence holomorphic

In the last step we define a approximation sequence, which approximates the CR function f on $H(\omega_1')$ and is holomorphic in a strictly convex, compact neighbourhood of $H(\omega_1')$.
We cover the compact, convex set L_1 with finitely many open sets of the form $I_j\times N_j$, where I_j is an open set in $\mathbb{R}^{n-1}$ with diameter less than $\frac{d_1}{2}$, where d_1 is chosen according to Lemma 3.16, and N_j is an open set in $\mathbb{R}^{n+1}$. Then we choose for each j a point $u_j\in I_j$ and a partition of unity ψ_j subordinate to the open cover $I_j\times N_j$ of L_1. We define

$$F_\varepsilon(f)(z,w) := \sum_j \psi_j(z,w)F_\varepsilon^{u_j}(f)(z,w) - v_\varepsilon(z,w),$$

where v_ε will be an error term with arbitrarily small sup-norm, which makes F_ε holomorphic.
In the following we want to find a strictly convex set inside of $L_1 = I\times\overline{\omega_1'}$ to be able to solve a $\bar\partial-$equation on L_1. First we choose an exhaustion sequence of L_1, where every member of the sequence is of the form $L_1' := I'\times\overline{\omega_1''}+a$, where $I'\subset I$ is a real interval, $\omega_1''\subset\subset\omega_1'$ a strictly convex set and $a\in\mathbb{R}^n$. Moreover we assume that $L_1'\cap M\neq\varnothing$. Then we want to apply the following

Theorem 3.18. *Let $\Omega\subset\mathbb{R}^n$ be a bounded, open and convex set and $K\subset\Omega$ a compact and convex subset.*
Then there exists a strictly convex set D with smooth boundary, such that $K\subset D\subset\Omega$.

Proof. e. g. [11], Theorem 2. □

Setting $\Omega := \mathring{L}_1$ and $K := L_1'$ we obtain a strictly convex set $D\equiv D_1$ with $L_1'\subset D_1\subset\subset L_1$ and $D_1\cap M\neq\varnothing$.

Lemma 3.19. *Let $(z,w)\in D_1$.*
Then there exists $v_\varepsilon\in\mathcal{C}^\infty(D_1)$, with $\sup|v_\varepsilon|_{D_1} = O(\varepsilon)$, such that $F_\varepsilon(f)\in\mathcal{H}(D_1)$ and $F_\varepsilon(f)(z,w)\to f(z,w)$ as $\varepsilon\to 0$.

Proof. If $F_\varepsilon(f)$ is holomorphic in D_1 we have

$$\begin{aligned}\bar\partial v_\varepsilon(z,w) =& \bar\partial\left(\sum_j \psi_j(z,w)F_\varepsilon^{u_j}(f)(z,w)\right) = \sum_j \bar\partial(\psi_j(z,w))F_\varepsilon^{u_j}(f)(z,w)\\ =& \sum_{j,l}\psi_l(z,w)\bar\partial(\psi_j(z,w))F_\varepsilon^{u_j}(f)(z,w),\end{aligned}$$

since $F_\varepsilon^{u_j}$ is locally independent of u_j by Lemma 3.16, hence holomorphic in $I_j \times N_j$. The terms $\psi_l\bar\partial\psi_j$ are non-zero iff $I_l \cap I_j \neq \varnothing$, which is the case iff $|u_l - u_j| < d_1$. Lemma 3.16 implies

$$\begin{aligned}\bar\partial v_\varepsilon(z,w) &= \sum_{j,l}\psi_l(z,w)\bar\partial(\psi_j(z,w))F_\varepsilon^{u_l}(f)(z,w) + O(\varepsilon)\\ &= \sum_l \psi_l(z,w)F_\varepsilon^{u_l}(f)(z,w)\sum_j \bar\partial(\psi_j(z,w)) + O(\varepsilon) = O(\varepsilon),\end{aligned}$$

since $\sum_j \bar\partial\psi_j = \bar\partial\left(\sum_j \psi_j\right) = \bar\partial 1 = 0$. Now we want to solve the equation

$$\bar\partial v_\varepsilon(z,w) = \bar\partial\left(\sum_j \psi_j(z,w)F_\varepsilon^{u_j}(f)(z,w)\right). \tag{3.2.12}$$

Excursus: d-bar-Equation with Estimates

For $D \subseteq \mathbb{C}^n$ let $L^s_{p,q}(D)$ denote the space of $(p,q)-$forms with coefficients in $L^s(D)$ for $1 \le s \le \infty$. Analogously we define $\mathcal{C}^s_{p,q}$ the space of $(p,q)-$forms with coefficients in $\mathcal{C}^s(D)$ for $1 \le s \le \infty$.

Theorem 3.20. *Let $D \subset\subset \mathbb{C}^n$ be a strictly pseudoconvex set with boundary of class $\mathcal{C}^3$.*
For $1 \le q \le n$ there exist linear integral operators $S_q : L^1_{0,q}(D) \to L^1_{0,q-1}(D)$ and a constant $C > 0$, such that the following properties hold:

(i) If $f \in \mathcal{C}^1_{0,q}(D) \cap L^1_{0,q}(D)$ and $\bar\partial f = 0$, then $\bar\partial(S_q f) = f$, i. e. S_q solves the $\bar\partial-$equation $\bar\partial u = f$.

(ii) For $1 \le s \le \infty$ we have that if $f \in \mathcal{C}^s_{0,q}(D) \cap L^1_{0,q}(D)$, then $S_q f \in \mathcal{C}^s_{0,q-1}(D)$.

(iii) We have the sup-norm estimate for the solution $\|S_q f\|_{L^\infty(D)} \le C\|f\|_{L^\infty(D)}$.

Proof. e. g. [3], Chapter VII., Theorem 5.6. □

We have that D_1 is bounded and strictly convex with smooth boundary, which is strictly pseudoconvex, by Theorem 0.12. The term on the right-hand side of the $\bar\partial-$equation in (3.2.12) vanishes, if we apply $\bar\partial$ once again and the sup-norm is $O(\varepsilon)$. Hence we may apply Theorem 3.20 with $s = \infty$ and $q = 1$ and obtain a smooth function v_ε on D_1, which fulfills $\sup|v_\varepsilon|_{D_1} = O(\varepsilon)$. This computation shows that $F_\varepsilon \in \mathcal{H}(D_1)$.
Remark 3.17 and the sup-norm estimate of v_ε gives

$$F_\varepsilon(f)(z,w) = \sum_j \psi_j(z,w)F_\varepsilon - v_\varepsilon(z,w) \to \sum_j \psi_j(z,w)f(z,w) = f(z,w)$$

for $(z,w) \in D_1$. □

To obtain an entire approximation sequence, we apply Runge's Theorem 0.22 in $\mathbb{C}^n$, since the compact, strictly convex set D_1 is polynomially convex by Lemma 0.16. And since L_1' was arbitrary member of an exhaustion sequence of L_1 this completes the proof of Theorem 3.7. □

Remark 3.21. In the case where $f \in \mathcal{C}(M)$ one has to read the support of $\bar{\partial} f$ at the beginning of **Step 5** in the sense of distributions and since both sides of (3.2.11) in the proof of Lemma 3.16 are continuous the proof of Theorem 3.7 is true for continuous CR functions as well.

3.3 Counterexamples in Higher Codimensions

Motivation 3.22. In the previous section we have seen, that uniform global approximation of CR functions on hypersurfaces is possible, if we assume certain convexity properties of the set, where we want to approximate. In higher codimensions it is possible to construct a totally real submanifold M and to find a compact set of M, denoted by K, and a continuous function f, where f cannot be approximated uniformly by entire functions. The tool for constructing such a K is to attach an analytic discs A to M.

Definition 3.23. An **analytic disc** $\boldsymbol{A}$ in $\mathbb{C}^n$ is a continous mapping $A : \mathbb{C} \supset \overline{B_1(0)} \to \mathbb{C}^n$, which is holomorphic in $B_1(0)$.
An analytic disc is **attached to** a subset $M \subseteq \mathbb{C}^n$, if $A(S^1) \subseteq M$.

Recall the definition of a totally real submanifold M, which is given as a graph over the imaginary axes from Remark 2.10 (i). Here only the case for $n = 2$ is discussed, the higher dimensional case is similar.

Theorem 3.24. *There exists a totally real $\mathcal{C}^1$-submanifold M in $\mathbb{C}^2$ of codimension 2, which is the graph over the imaginary axes, i. e. $M = \{(z_1, z_2) = (x_1 + iy_1, x_2 + iy_2) \in \mathbb{C}^2 : \ x_1 = h_1(y_1, y_2), \ x_2 = h_2(y_1, y_2)\}$, where $h = (h_1, h_2)$ is a $\mathcal{C}^1-$function, a compact set K and a continuous CR-function f, defined in a neighbourhood of K, such that f cannot be approximated uniformly on K by entire functions.*

Proof. First we assume, that we already attached the following analytic disc A to M, and show, that this implies the claim in the Theorem.

Let A be defined as $A : \overline{B_1(0)} \to \mathbb{C}^2$, $A(\zeta) = (i\zeta, \zeta)$, which is analytic in its components, hence an analytic disc. The image of A is given by $\mathrm{im} A = \{(-y_2 + iy_1, y_1 + iy_2) : \ y_1^2 + y_2^2 \leq 1\}$.

Define $K := \{z = x + iy = (x_1, x_2) + i(y_1, y_2) \in M : \ y_1^2 + y_2^2 \leq 1\}$. Since A is attached to M, we have that the boundary of A, denoted by bA, agrees with the boundary of K in M, denoted by bK, and both A and K are graphed over the set $U = \{(y_1, y_2) \in \mathbb{R}^2 : \ y_1^2 + y_2^2 \leq 1\}$.

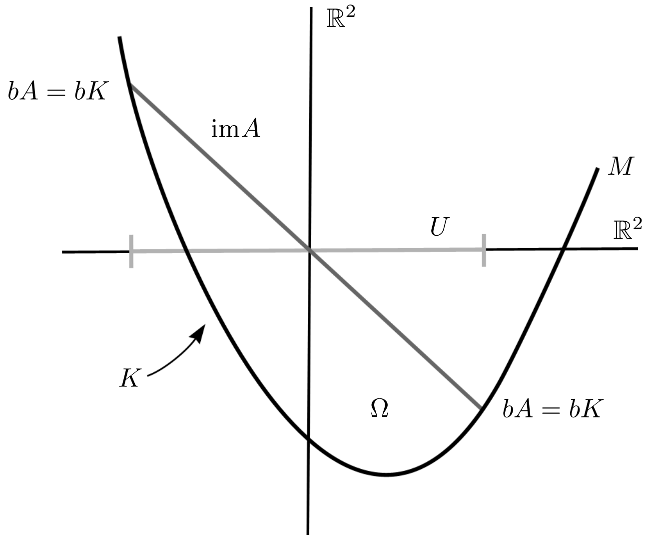

Figure 3.2: Attaching the analytic disc A to M

As in Remark 2.10 (i) we have $\det(Dh(y)+iI_2) \neq 0$ for $y \in U$, where Dh denotes the Jacobimatrix of h, hence $f(z) := (\det(Dh(y)+iI_2)^{-1}$ for $z = x+iy \in M$ is a continuous function in a neighbourhood of K.

Now we suppose, that f can be approximated by a sequence F_n of entire functions on K.

Then we have

$$\int_K f(z)dz_1 \wedge dz_2 = \lim_{n\to\infty} \int_K F_n(z)dz_1 \wedge dz_2. \tag{3.3.1}$$

Define Ω to be set in $\mathbb{C}^2$, such that K and im$A$ form the boundary of Ω. For every $n \in \mathbb{N}$ we write by Stoke's Theorem

$$0 = \int_\Omega dF_n(z)dz_1 \wedge dz_2 = \int_\Omega F_n(z)dz_1 \wedge dz_2 = \int_K F_n(z)dz_1 \wedge dz_2 - \int_{\mathrm{im}A} F_n(z)dz_1 \wedge dz_2,$$

since $d(F_n(z)dz_1 \wedge dz_2) = 0$, because F_n is holomorphic and $F_n(z)dz_1 \wedge dz_2$ is a $(2,0)-$form. The integral over im$A$ vanishes, since the form $dz_1 \wedge dz_2$ pulled back to a form $A^*(dz_1 \wedge dz_2)$ over $B_1(0)$ gives 0. Hence we obtain $\int_K f(z)dz_1 \wedge dz_2 = 0$ from (3.3.1).

If we compute the same integral by pulling back K to U via the parametrization $\varphi : U \to K$, $\varphi(y) = z = h(y) + iy$, then the form $dz_1 \wedge dz_2$ becomes $\det(Dh(y) + iI_2)dy_1 \wedge dy_2$. Hence we have that

$$0 = \int_K f(z)dz_1 \wedge dz_2 = \int_U dy_1 \wedge dy_2 \neq 0,$$

since U is the unit ball in $\mathbb{R}^2$, which has positive measure, a contradiction. Hence f cannot be approximated by entire functions on K.

Now we are going to construct M, such that bA is contained in M.

Since we used in the above proof, that K and A are graphed over U and $bK = bA$, we want $h_1(y) + iy_1 = i\zeta$ and $h_2(y) + iy_2 = \zeta$ on bU. This implies $h_1(y_1, y_2) = -y_2$ and $h_2(y_1, y_2) = y_1$ on

bU. If we transform U into polar coordinates, writing $y_1 = r\cos t$ and $y_2 = r\sin t$ for $0 \le r \le 1$ and $0 \le t < 2\pi$, the equation means that

$$h_1(r=1,t) = -\sin t \text{ and } h_2(r=1,t) = \cos t. \tag{3.3.2}$$

Now we make the following ansatz and write

$$\begin{aligned} h_1(r,t) &= \varphi(r)\cos t - \psi(r)\sin t \\ h_2(r,t) &= \varphi(r)\sin t + \psi(r)\cos t \end{aligned} \tag{3.3.3}$$

for φ and ψ real valued $\mathcal{C}^1$-functions and denote $H(r,t) := (h_1(r,t), h_2(r,t))$. Hence $\varphi(1) = 0$ and $\psi(1) = 1$ to fulfill the condition on h in (3.3.2).
Transforming the ansatz of $H(r,t)$ into cartesian coordinates, we see that the first derivative of h is continuous at the origin, only if φ and ψ together with their first derivatives vanish at 0.
Another condition h has to satisfy is that $\det(Dh(y) + iI_2) \neq 0$ for $y \in U$. First we compute $\det(Dh(y) + iI_2)$ in polar coordinates.

Lemma 3.25. $\det(Dh(y) + iI_2) = \frac{1}{r}\underbrace{[\varphi(r)\varphi'(r) + \psi(r)\psi'(r) - r + i(r\varphi'(r) + \varphi(r))]}_{=:B(r)}$.

Proof. We define $H : [0,1] \times [0,2\pi) \to \mathbb{C}^2$ by

$$H(r,t) := A(t)\begin{pmatrix} \varphi(r) + ir \\ \psi(r) \end{pmatrix}, \text{ where } A(t) := \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

If we transform $h(y) + iy$ from cartesian into polar coordinates, we obtain $H(r,t)$ from (3.3.3) and since the determinant of the Jacobimatrix of the transformation from cartesian to polar coordinates is $\frac{1}{r}$, we only need to show, $\det D_{r,t}H(r,t) = B(r)$.
Using the property of $A(t)$, that $A'(t)(x,y) = A(t)(-y,x)$ we obtain

$$D_{r,t}H(r,t) = A(t)\begin{pmatrix} \varphi'(r) + i & -\psi(r) \\ \psi'(r) & \varphi(r) + ir \end{pmatrix}.$$

Thus $\frac{1}{r}\det(D_{r,t}H(r,t))$ is established as claimed in the Lemma, since $\det A(t) = 1$. □

To sum up $h = (h_1, h_2)$ given by the ansatz in (3.3.3) has to fulfill

$$\begin{aligned} &\varphi(0) = \varphi(1) = \psi(0) = 0 \text{ and } \psi(1) = 1 \\ &\varphi'(0) = \psi'(0) = 0 \\ &\det(Dh(y) + iI_2) = \frac{1}{r}\det(DH(r,t)) \neq 0, \end{aligned} \tag{3.3.4}$$

for $0 \le y_1^2 + y_2^2 \le 1$ resp. $0 \le r \le 1$.
Define e. g. $\psi(r) := r^2$ and $\varphi(r) := 4r^2$ for $r \in [0, \frac{1}{2}]$ and $\varphi(r) := -4(r-1)^2$ for $r \in [\frac{1}{2}, 1]$. Both are $\mathcal{C}^1$ in $[0,1]$ and satisfy the conditions given in (3.3.4), since the imaginary part and the real part of $\frac{1}{r}\det(D_{r,t}H(r,t))$ do not vanish simultaneously. □

4 Appendix

4.1 Koebe's One-Quarter Theorem

In order to fill the gap in the proof of Mergelyan's Theorem we will prove Koebe's One-Quarter Theorem, which gives us the minimal size of the image of certain holomorphic functions on the unit disc in $\mathbb{C}$. Then we will prove the precise statement which was used in the proof of Mergelyan's Theorem. We follow the presentation in [7].

Lemma 4.1 (Area Theorem). *Let $g(z) = z + \sum_{n\geq 0} b_n z^{-n}$ be holomorphic and injective in $\{z \in \mathbb{C} : 1 < |z| < \infty\}$, then $\sum_{n\geq 1} n|b_n|^2 \leq 1$.*

Proof. Since g is holomorphic in the open set $T := \{z \in \mathbb{C} : 1 < |z| < \infty\}$ the set $\mathrm{im}(g(T))$ is open. Let $E := \complement(\mathrm{im}(g(T)))$. Since g is injective, the set $\mathrm{im}(g(\{z \in \mathbb{C} : |z| = r > 1\}))$ is a simple, closed curve and the interior E_r contains E, hence E is compact. The area of E_r can be computed as follows

$$\begin{aligned}
\mathrm{area}(E_r) &= \int_{E_r} dx \wedge dy = \frac{1}{2i}\int_{E_r} d\bar{w} \wedge dw = \frac{1}{2i}\int_{bE_r} \bar{w}dw = \frac{1}{2i}\int_{|z|=r} g^*(\bar{w}dw) \\
&= \frac{1}{2i}\int_{|z|=r} \overline{g(z)}g'(z)dz = \frac{1}{2}\int_0^{2\pi} \left(re^{-i\theta} + \sum_{n\geq 0} \bar{b}_n \frac{e^{in\theta}}{r^n}\right)\left(1 - \sum_{n\geq 1} nb_n \frac{e^{-i(n+1)\theta}}{r^{n+1}}\right) re^{i\theta} d\theta \\
&= \frac{1}{2}\int_0^{2\pi} \left(r^2 - \sum_{n\geq 0}\sum_{k\geq 0} \frac{(n-k)e^{i(2k-n)\theta}}{r^n}\bar{b}_k b_{n-k}\right) d\theta = \frac{1}{2}\int_0^{2\pi}\left(r^2 - \sum_{k\geq 0}\frac{k}{r^{2k}}\bar{b}_k b_k\right) d\theta \\
&= \pi\left(r^2 - \sum_{n\geq 1}\frac{n}{r^{2n}}|b_n|^2\right).
\end{aligned}$$

Now we let $r \to 1$ to obtain

$$0 \leq \mathrm{area}(E) \leq \lim_{r\to 1} \mathrm{area}(E_r) = \pi\left(1 - \sum_{n\geq 1} n|b_n|^2\right).$$

□

Definition 4.2. $\mathcal{S} := \{f \in \mathcal{H}(B_1(0)) :\ f \text{ injective},\ f(0) = 0,\ f'(0) = 1\}$

Remark 4.3. Any $f \in \mathcal{S}$ is given by a power series developement $f(z) = z + \sum_{n\geq 2} a_n z^n$.

Proposition 4.4. *For every $f \in \mathcal{S}$ we have $|a_2| \leq 2$.*

Proof. We want to see that we can extract a square root of f, denoted by the symbol $h(w) := \sqrt{f(w^2)}$.

It is

$$f(w^2) = w^2 + \sum_{n\geq 2} a_n w^{2n} = w^2 \left(1 + \sum_{n\geq 2} a_n w^{2(n-1)}\right).$$

The function $\tilde{f}(w) := 1 + \sum_{n\geq 2} a_n w^{2(n-1)}$ is holomorphic in $B_1(0)$ and since f is injective, $f(w^2)$ only vanishes at $w = 0$, hence $\tilde{f}$ can never vanish. So $\sqrt{\tilde{f}(w)} := \exp(\frac{1}{2}\log(\tilde{f}(w)))$ makes sense and is a holomorphic square root, which takes the value 1 at $w = 0$. The function $h(w) = w\sqrt{\tilde{f}(w)}$ is holomorphic in $B_1(0)$ and vanishes only for $w = 0$.
In fact $h \in \mathcal{S}$, since $h(0) = 0$, $h'(0) = 1$ and we have to check the injectivity. If $h(w_1) = h(w_2)$ for $w_1,\ w_2 \in B_1(0)$, then $f(w_1^2) = f(w_2^2)$. Therefore $w_1^2 = w_2^2$ and in the unrequested case, where $w_1 \neq w_2$ it is $w_2 = -w_1$, but $h(w_2) = h(-w_1) = -h(w_1)$. This can only happen if $h(w_1) = 0$, which only occurs for $w_1 = 0$. The same for w_2 implies $w_1 = 0 = w_2$.
Let $g(z) := 1/h(1/z)$, which is holomorphic and injective in the set $\{z \in \mathbb{C} :\ 1 < |z| < \infty\}$ and we have the power series expansion

$$g(z) = \frac{z}{\sqrt{\tilde{f}(1/z)}} = \frac{z}{1 + \frac{a_2}{2}\frac{1}{z^2} + O(z^4)} = z - \frac{a_2}{2}\frac{1}{z} + O(z^3).$$

The assumptions of the Area Theorem are fulfilled and we obtain for $b_1 = -\frac{a_2}{2}$, that $|b_1| \leq 1$. $\square$

Theorem 4.5 (Koebe's One-Quarter Theorem). *Let $f \in \mathcal{S}$, then $B_{\frac{1}{4}}(0) \subseteq f(B_1(0))$.*

Proof. Let $\zeta \notin f(B_1(0))$ and define

$$F(z) := \frac{\zeta f(z)}{\zeta - f(z)} = z + \left(a_2 + \frac{1}{\zeta}\right) z^2 + O(z^3).$$

Since $F \in \mathcal{S}$ we obtain by Proposition 4.4 $|a_2 + 1/\zeta| \leq 2$. Together with $|a_2| \leq 2$ we get $|1/\zeta| \leq 4$. $\square$

Corollary 4.6. *Let K be a compact, connected set in $\mathbb{C}$ with diam$(K) \geq r > 0$ and $\complement K$ is connected. Then there exist a biholomorphic mapping $F :\ B_1(0) \to S^2 \setminus K$ such that*

$$F(z) = \frac{a}{z} + \sum_{n\geq 0} b_n z^n$$

with $a \geq \frac{r}{4}$.

Proof. The assumptions imply that $S^2 \setminus K$ is simply connected, hence there exist a biholomorphic mapping $F :\ B_1(0) \to S^2 \setminus K$ by the Riemann mapping theorem. Since $\{\infty\} \in S^2 \setminus K$, we assume $F(0) = \infty$, hence the coefficient of z^{-1} is $a > 0$. For $w_0 \in K$ we define

$$\varphi(z) = \frac{a}{F(z) - w_0} = z - \frac{b_0 - w_0}{a} z^2 + O(z^3).$$

φ is holomorphic in $B_1(0)$ and a biholomorphism, because it is a composition of a Moebiustransformation $\psi :\ S^2 \to S^2$, $\psi(w) := \frac{a}{w - w_0}$ and the biholomorphism F. It follows that $\varphi \in \mathcal{S}$. For any $w_1 \in K$ it is $\psi(w_1) \notin \varphi(B_1(0))$, since $\mathrm{im}(F) \subseteq \complement K$. Applying Koebe's One-Quarter-Theorem to φ, we obtain that

$$\frac{1}{4} \leq |\psi(w_1)| = \left| \frac{a}{w_1 - w_0} \right|$$

and hence $a \geq \frac{1}{4}|w_1 - w_0|$. The diameter of K was at least r, which implies that $a \geq \frac{r}{4}$. $\square$

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