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Proper forcing and countable support iterations

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Abstract

The main goal of this thesis is to present countable support iterations of proper forcing notions in a comprehensible manner. Necessary prerequisites are explained and proofs are given with enough details, such that anyone with a basic understanding of forcing should be able to understand the content of this thesis without relying on additional resources.
At the beginning of the thesis a short overview of the basics of the forcing theory is given. Then, separative quotients, forcing equivalence, projections and two step iterations are each discussed briefly in the second section. The third sections contains the basic results about proper forcing notions, a comprehensive discussion of three equivalent definitions of properness, and the central theorems on countable support iterations of proper forcing notions. In the last section Miller forcing is introduced and a proof for the properness of Miller forcing is given. A major part of the last section is dedicated to showing that "$\omega_1 = \mathfrak{a} < \mathfrak{d} = \mathfrak{c} = \omega_2$" holds in the forcing extensions of a ω_2-length countable support iteration of Miller forcing, if we assume that $\mathtt{CH}$ holds in the ground model.

Das Ziel dieser Arbeit ist es, Abzählbarer-Träger-Iterationen aus proper Forcings auf leicht verständliche Weise zu päsentieren. Notwendige Grundkonzepte werden erklärt und Beweise werden detailiert ausgeführt, sodass jeder mit einem grundliegenden Verständis von Forcing den Inhalt dieser Arbeit verstehen kann ohne auf weitere Ressourcen angewiesen zu sein.
Am Anfang der Arbeit wird ein kurzer Überblick der Grundkonzepte der Forcingtheorie gegeben. Im zweiten Abschnitt werden separative Quotienten, Forcing-Äquivalenz, Projektionen, und Zwei-Schritt-Iterationen jeweils kurz besprochen. Der dritte Abschnitt beinhält grundliegende Resultate über proper Forcings, eine ausführliche Abhandlung zu drei äquivalenten Definitionen von Properness und die zentralen Theoreme zu Abzählbarer-Träger-Iterationen aus proper Foricngs. Im letzten Abschnitt wird das Miller-Forcing vorgestellt und bewiesen, dass das Miller-Forcing proper ist. Ein Großteil des letzten Abschnitts bafasst sich, damit zu zeigen, dass in Forcingerweiterungen einer Äbzählbarer-Träger-Iteration der Länge ω_2 aus Miller-Forcings die Aussage "$\omega_1 = \mathfrak{a} < \mathfrak{d} = \mathfrak{c} = \omega_2$" gilt, wenn man $\mathtt{CH}$ *für das Grundmodell annimmt.*

Contents

1 Introduction

In 1874 Cantor [1] published his first proof of the uncountability of the real numbers. The revelation that $\mathfrak{c}$ is strictly greater than $\aleph_0$ lead to a natural question: Is there an uncountable set whose cardinality is strictly less than $\mathfrak{c}$? Four years later Cantor [2] advanced the hypothesis that such a set could not exist. This hypothesis became known as the *Continuum Hypothesis* ($\mathtt{CH}$) and its resolutions turned into one of the most pondered puzzles of 20th century mathematics. Nonetheless, the solution to this puzzle stayed out of reach for many decades. A first ray of hope in the pursuit of a definite answer appeared in 1940, when Gödel [3] published his proof of the consistency of the Continuum Hypothesis. However, a complete solution to the problem still eluded mathematics for another 20 years. Then, in 1963 Cohen [4] published his paper titled "The independence of the Continuum Hypothesis", in which he proved the consistency of $\mathtt{ZFC} + \neg\mathtt{CH}$, which by Gödel's previous result established what the name of Cohen's paper promised i.e. $\mathtt{ZFC}$ can neither prove nor disprove $\mathtt{CH}$. At the heart of Cohen's proof was the technique called *forcing*, through which Cohen was able to show the existence of a models of set theory in which $\mathfrak{c}$ is strictly larger than $\aleph_1$.
Today, the independence of the Continuum Hypothesis is one of the most well known results of set theory. However, some might argue that the introduction of forcing, the technique Cohen used to establish this result, was even more impactful. Ever since it was conceived forcing has time after time been shown to be a very effective tool for establishing consistency results. Hence, the analysis of models produced by forcing is one of the major tasks of contemporary set theory. In this thesis we will examine some tools for forcing constructions. As forcing is central to our discussion we start off by taking a look at the intuition behind it, while going over some of the necessary prerequisites.

1.1 The basic idea behind forcing

The standard purpose of forcing is the establishment of the consistency of a statement $\mathtt{A}$ with $\mathtt{ZFC}$ by establishing, for each finite $\mathtt{S}' \subseteq \mathtt{ZFC}$, the existence of a model V' which models $\mathtt{S}' + \mathtt{A}$, i.e. a set V' such that in the structure $(V', \in)$ all sentences of $\mathtt{S}' + \mathtt{A}$ hold.[1] This is done by starting with a model V for a large enough finite fragment $\mathtt{S} \subseteq \mathtt{ZFC}$, which depends only on $\mathtt{S}'$, and then extending the model V to a superset of V which includes a new object

[1] For an explanation of how this establishes the consistency of $\mathtt{ZFC} + \mathtt{A}$ see Remark 1.20.

c. V is referred to as the *ground model*, while the extension is called a *forcing extension* and is sometimes denoted by $V[c]$. This notation is reminiscent of the notation $R[a]$ for the ring obtained by adjoining an element a to R, and the similarity in the notation already hints at the nature of $V[c]$. As $R[a]$ is the smallest ring which includes $R \cup \{a\}$, the forcing extension $V[c]$ is a model of $\mathtt{S}'$ and it is the smallest such model which includes $V \cup \{c\}$. Unfortunately, when comparing $V[c]$ to $R[a]$ we lack an analogue for the way in which c relates to V. Essentially, c can be thought of as a limit which has approximations in V.[2] The type of c is fixed and a set of potential approximations is chosen in advance. For example, in Cohen forcing we approximate a function $c : \omega \to \omega$ by finite partial functions from ω to ω. In general, the potential approximations are referred to as *conditions* and the set of all conditions is some $P \in V$. Each condition can be thought of as consisting of a small, often finite or countable, segment as well as a restriction on the possible continuations of this segment. As we want to extend approximations to better approximations, we need a relation $\leq$ on P. Clearly, $\leq$ should be transitive, and it is common convention to also assume that $\leq$ is reflexive i.e. $\leq$ is a preorder. For practical reasons we always assume that P contains an "empty approximation", which is denoted by 1 and which is the $\leq$-greatest element of P i.e. $1 \geq p$ holds for all $p \in P$. Then, $\mathbb{P} = (P, \leq, 1)$ is called a *forcing notion*. For $p, q \in P$, if $p \leq q$ we say "p is below q", "p extends q", "p is an extension of q", "p is stronger than q", "q is above p" or "q is weaker than p". Intuitively speaking, p is a better approximation than q. If two conditions are approximations for the same object c, we want to have an approximation combining the two conditions, and two conditions $p, q \in P$ are said to be *compatible* if they have such a common extension i.e. there is some $r \in P$ such that $r \leq p, q$. If two conditions do not have a common extension, we call them *incompatible*. The object c will be completely determined by the collection $G \subseteq P$ of all conditions that are approximations for c, and vice versa, and it turns out that it is easier to work with this G. Therefore, we often denote the forcing extension by $V[G]$ instead of $V[c]$. As $G \subseteq P$ is supposed to contain all conditions that approximate c, G is *upward closed* and *directed* i.e. $\forall p \in G \forall q \in P\ (p \leq q \Rightarrow q \in G)$ and $\forall p, q \in G \exists r \in G\ (r \leq p, q)$. An upward closed and directed subset of a preorder is called a *filter*. By envisioning G as the collection which contains all conditions that approximate c, one can already suspect that G will be a maximal filter, in the sense that there is no proper superset $H \subseteq P$ which is a filter. Indeed, G will always be a maximal filter. However, we require an even stronger property if we want the below fundamental theorems to hold. A set $D \subseteq P$ such that $\forall p \in P \exists q \in D\ (q \leq p)$, is called *dense*,

[2]Here, neither "limit" nor "approximation" need to be thought of in any mathematically precise way, but usually the corresponding construction is that of a directed limit.

and we need that $G \cap D \neq \emptyset$ holds for every dense $D \subseteq P$ in V. If G is as such, we say it is $\mathbb{P}$-*generic* over V. In case that the choice of forcing notion or the choice of ground model is clear from context we may say "G is generic over V", "G is $\mathbb{P}$-generic" or simply "G is generic". Intuitively speaking, this property guarantees that G approximates c as close as $\mathbb{P}$ allows.

Definition 1.1. So far we have the following:

1. a *forcing notion* $\mathbb{P} = (P, \leq, 1)$ consists of an underlying set P, a preorder $\leq$ on P and a greatest element $1 \in P$;
2. two conditions $p, q \in P$ are said to be *compatible* if they have a common extension $r \leq p, q$, otherwise they are said to be *incompatible*;
3. a subset $X \subseteq P$ is directed if for every two conditions $p, q \in X$, there is $r \in X$ such that $r \leq p, q$;
4. a subset $X \subseteq P$ is upward closed if $q \geq p \in X$ implies $q \in X$;
5. a *filter* is a directed and upward closed set;
6. a subset $D \subseteq P$ is *dense* if for every $p \in P$ there is $q \in D$ such that $q \leq D$;
7. a filter $G \subseteq P$ is *generic* over V if $G \cap D$ is non-empty for all dense D subsets of $\mathbb{P}$ in V.

Additionally, we say that $D \subseteq P$ is *dense below* $p \in P$ if for every $q \leq p$ there is $r \in D$ such that $r \leq q$.

Remark 1.2. Note that if G is a generic filter, p is a condition in G and D is dense below p, the set $D' = \{r \in \mathbb{P} \mid r \in D \lor r \text{ is incompatible with } p\}$ is dense. Hence, any generic filter that contains p, meets every set which is dense below p.

From now on we will follow the slightly misleading but common notational convention of writing $p \in \mathbb{P}$ instead of $p \in P$ and $A \subseteq \mathbb{P}$ instead of $A \subseteq P$, as well as using the symbols $\leq$ and 1 for all forcing notions. If we want to be more more careful we write $\leq_{\mathbb{P}}$ and $1_{\mathbb{P}}$ to specify to which forcing notion the symbols refer.

1.2 Forcing fundamentals

In this subsection, we give a quick overview of the fundamental concepts of forcing. We loosely follow Chapter 14 of the book "Combinatorial Set Theory - With a Gentle Introduction to Forcing" by Halbeisen [5]. We skip most proofs in this subsection, and refer the interested reader to [5, Chapter 14].

First, we examine the construction of the forcing extension $V[G]$. To better comprehend this construction, let us again compare $V[G]$ to $R[a]$. One way to construct $R[a]$ is to start with $R[X]$, the ring of all polynomials (in one variable) over R, and then consider $R[a]$ as the image of the evaluation homomorphism, which sends a polynomial $p(X) \in R[X]$ to $p(a)$, the value of the polynomial $p(X)$ at a. In the case of forcing extensions we start with

$$V^{\mathbb{P}} = \{\tau \in V \mid \tau \text{ is a } \mathbb{P}\text{-name}\},$$

where τ is a $\mathbb{P}$-*name* if all its elements are of the form (p, σ) with $p \in \mathbb{P}$ and σ a $\mathbb{P}$-name. Note that the definition of $\mathbb{P}$-names works recursively on rank, which is to say the definition makes sense because $\in$ is well-founded. Then we define

$$V[G] = \{\tau^G \mid \tau \in V^{\mathbb{P}}\},$$

where $\tau^G = \{\sigma^G \mid (p, \sigma) \in \tau,\ p \in G\}$. The definition of τ^G also works recursively on rank. We say that τ^G is the evaluation of τ by G. Hence, $V[G]$ is the collection of all evaluations (by G) of members of $V^{\mathbb{P}}$, just as $R[a]$ is the collection of all evaluations (at a) of members of $R[X]$.

Remark 1.3. In the standard approach to forcing we only consider particularly well-behaved models. In particular, we assume that the ground model V is a *transitive* set i.e. V is such that $\forall x \in V\ (x \subseteq V)$. Transitive models have the advantage that most basic concepts behave the same "internally", from the point of the transitive model, and "externally", in the surrounding universe of all sets. For example, assuming that V is a transitive model, f is a function if and only if $V \models$ "f is a function". More generally, for any transitive model V, $x \in V$ and any Δ_1 formula φ, the statement $\varphi(x)$ holds if and only if $V \models \varphi(x)$. The symbol $\models$ relativises the subsequent formula to the preceding model, meaning that "$V \models \varphi(x)$" is the formula obtained from $\varphi(x)$ by restricting all its quantifiers to V. The property that $V \models \varphi(x)$ is equivalent to $\varphi(x)$ for all $x \in V$ is often phrased as "φ is *absolute* for V". Hence, the above mentioned property of transitive models can be stated more concisely as "Δ_1 formulas are absolute for all transitive models".[3]

The first fundamental theorem lists a few basic properties of forcing extensions:

Theorem 1.4. *For each finite fragment $S' \subseteq$ ZFC there exists a corresponding finite fragment $S \subseteq$ ZFC, such that, if V is a transitive model of S and $\mathbb{P} \in V$ is a forcing notion, then the following hold, for any filter $G \subseteq \mathbb{P}$ which is generic over V:*

[3] A detailed discussion of Δ_1 formulas and their absoluteness for transitive models can be found under "Transitive Models and Δ_0 Formulas" and "The Lévy Hierarchy" in [6, Part I, 12 and Part II, 13].

1. the forcing extension $V[G]$ is a transitive model of $\mathcal{S}'$;
2. $V \subseteq V[G]$ and $G \in V[G]$;
3. $Ord^{V[G]} = Ord^V$;
4. if V' is a transitive model of $\mathcal{S}'$ which includes $V \cup \{G\}$, then $V[G]$ is a subset of V'.

Remark 1.5. In 1.4.3, Ord refers to the class of all ordinals, Ord^V is defined by the same formula but relativised to V, and $Ord^{V[G]}$ is defined analogously. As V and $V[G]$ are transitive and "being an ordinal" is a Δ_0-property, we have $Ord^V = Ord \cap V$ and $Ord^{V[G]} = Ord \cap V[G]$. Even though 1.4.3 might not seem as significant as the other three statements, it is very useful. In particular, we will often switch from "an arbitrary ordinal in $V[G]$" to "an arbitrary ordinal in V" (or vice versa) without further explanation.

Even though one can understand most forcing arguments without remembering the full proof of Theorem 1.4, it is important keep in mind how to prove 1.4.2. First, we define the *check name* for $x \in V$ by

$$\check{x} = \{(1, \check{y}) \mid y \in x\}$$

by recursion on rank. Note that $\check{x} \in V^{\mathbb{P}}$ and $\check{x}^G = x$, for each $x \in V$ and any generic filter $G \subseteq \mathbb{P}$. Hence, we have $V \subseteq V[G]$. Next, we define

$$\dot{G} = \{(p, \check{p}) \mid p \in \mathbb{P}\},$$

which serves as a canonical name for the generic filter. The notation $\dot{G}$ is according to common convention, however it is somewhat unfortunately chosen, as $\check{x}$ depends on x, while $\dot{G}$ does not depend on G. We have $\dot{G} \in V^{\mathbb{P}}$ and $\dot{G}^G = G$, but also $\dot{G}^H = H$ for any other generic filter $H \subseteq \mathbb{P}$.

As established by 1.4, forcing serves as a method to generate models for finite fragments of ZFC. Of course, this approach would be a hopeless endeavor if there were no generic filters. Fortunately, the Löwenheim-Skolem theorem[4] allows us to assume that V is countable, and we have the following:

Proposition 1.6. *Assume that V is a countable transitive model for a large enough finite fragment of ZFC and let $\mathbb{P} \in V$ be a forcing notion. Then, for any condition $p \in \mathbb{P}$, there is a generic filter $G \subseteq \mathbb{P}$ such that $p \in G$.*

Proof. Fix $p \in \mathbb{P}$, and let $\{D_n \mid n \in \omega\}$ enumerate all dense subsets of $\mathbb{P}$ which are in V. Choose $p_0 \in D_0$ such that $p_0 \leq p$, and inductively for $n \in \omega$ take

[4] The Löwenheim-Skolem can be found in [6, Part I, 12.1]

$p_{n+1} \in D_{n+1}$ such that $p_{n+1} \leq p_n$. Note that $G = \{q \mid \exists n \in \omega \ (p_n \leq q)\}$ is a filter and $p \in G$. Furthermore, G is generic as $p_n \in G \cap D_n$ for each n. □

Note that the above proof uses the axiom of choice. Consequently, we only know that G exists, but we do not get any more information about G. In general, the generic filter G is very obscure, and this carries over to $V[G]$. Hence, we usually do not want to analyse $V[G]$ for a fixed G. Instead, we fix some condition $p \in \mathbb{P}$ and consider the collection of all $V[G]$ such that G contains p. This motivates the following definition:

Definition 1.7. Suppose V is a transitive model for a large enough finite fragment of ZFC and $\mathbb{P} \in V$ is a forcing notion. Let $n \in \omega$, $p \in \mathbb{P}$ be a condition, $\tau_0, ..., \tau_{n-1} \in V^{\mathbb{P}}$ $\mathbb{P}$-names, and φ a formula with n free variables. If $V[G] \models \varphi(\tau_0^G, ..., \tau_{n-1}^G)$ holds for every generic filter $G \subseteq \mathbb{P}$ such that $p \in G$, then we say p *forces* $\varphi(\tau_0, ..., \tau_{n-1})$ and we write $p \Vdash \varphi(\tau_0, ..., \tau_{n-1})$. If we want to emphasise the forcing notion and the ground model, we write $p \Vdash_{\mathbb{P}}^{V} \varphi(\tau_0, ..., \tau_{n-1})$.

Remark 1.8. To avoid repeating the same assumptions, from now on V is a transitive model for a large enough finite fragment of ZFC and $\mathbb{P} \in V$ is a forcing notion.

The relation $\Vdash$ is fittingly called *forcing relation*, or sometimes *external forcing relation*. It is "external" in the sense that the definition does not make sense in the structure $(V, \in)$, as the quantifier for G is not restricted to V and $V[G] \models \varphi(\tau_0, ..., \tau_{n-1})$ cannot be evaluated in V. Hence, it would seem that we cannot use the forcing relation to define sets in V. We need the following:

Definition 1.9. The relation $\Vdash^*$ is defined for each $p \in \mathbb{P}$ and every formula, by recursion on the ranks of the names and on the length of the formula, as follows:

(a) for $\sigma, \tau \in V^{\mathbb{P}}$,

$$p \Vdash^* \sigma \in \tau$$

if and only if the set

$$\{q \in \mathbb{P} \mid \exists (r, \sigma') \in \tau \ (q \leq r \ \wedge \ q \Vdash^* \sigma = \sigma')\}$$

is dense below p;

(b) for $\sigma, \tau \in V^{\mathbb{P}}$,

$$p \Vdash^* \sigma = \tau$$

if and only if for all $(r, \rho) \in \sigma$ the set

$$\{q \in \mathbb{P} \mid q \leq r \Rightarrow \exists (r', \rho') \in \tau \ (q \leq r' \ \wedge \ q \Vdash^* \rho = \rho')\}$$

is dense below p, and for all $(r, \rho) \in \tau$ the set

$$\{q \in \mathbb{P} \mid q \leq r \Rightarrow \exists (r', \rho') \in \sigma \ (q \leq r' \ \wedge \ q \Vdash^* \rho = \rho')\}$$

is dense below p;

(c) for formulas φ, ψ and $\mathbb{P}$-names $\sigma_0, ..., \sigma_{m-1}, \tau_0, ..., \tau_{n-1}$,

$$p \Vdash^* \varphi(\sigma_0, ..., \sigma_{m-1}) \wedge \psi(\tau_0, ..., \tau_{n-1})$$

if and only if $p \Vdash^* \varphi(\sigma_0, ..., \sigma_{m-1})$ and $p \Vdash^* \psi(\tau_0, ..., \tau_{n-1})$;

(d) for any formula φ and $\mathbb{P}$-names $\tau_0, ..., \tau_{n-1}$,

$$p \Vdash^* \neg\varphi(\tau_0, ..., \tau_{n-1})$$

if and only if $q \not\Vdash^* \varphi(\tau_0, ..., \tau_{n-1})$ holds for all $q \leq p$, i.e.

$$\forall q \leq p \ \neg(q \Vdash^* \varphi(\tau_0, ..., \tau_{n-1}));$$

(e) for any formula φ and $\mathbb{P}$-names $\tau_0, ..., \tau_{n-1}$,

$$p \Vdash^* \exists x \ \varphi(x, \tau_0, ..., \tau_{n-1})$$

if and only if the set

$$\{q \in \mathbb{P} \mid \exists \sigma \in V^{\mathbb{P}} \ (q \Vdash^* \varphi(\sigma, \tau_0, ..., \tau_{n-1}))\}$$

is dense below p.

Remark 1.10. Note that the set $\{(p, \tau_0, ..., \tau_n) \mid p \Vdash^* \tau(0, ..., \tau_{n-1})\}$ can be defined in V, for each formula φ.

The relation $\Vdash^*$ is called *internal forcing relation*, and the following theorem, which is often referred to as the *Definability Lemma*, justifies this:

Theorem 1.11. *Suppose φ is a formula, $\tau_0, ..., \tau_{n-1}$ are $\mathbb{P}$-names and $p \in \mathbb{P}$. If for each $p \in \mathbb{P}$ there exists a generic filter G that contains p, we have*

$$p \Vdash \varphi(\tau_0, ..., \tau_{n-1})) \text{ if and only if } V \models p \Vdash^* \varphi(\tau_0, ..., \tau_{n-1}).$$

As a consequence of Theorem 1.11 we can expand the language we use to define sets in V by a symbol for the forcing relation. Furthermore, we can infer absoluteness for formulas that contain this symbol. For example, if φ is equivalent to a Δ_0-formula then so is the formula "$p \Vdash \varphi$", and hence it is absolute for every transitive model. In other cases, the absoluteness might be less obvious. Then, revisiting the exact definition of $\Vdash^*$ can be helpful.

The next theorem, called the *Truth Lemma* is the heart of forcing. This theorem is so crucial to proofs that involve forcing that it is often simply referred to as the "forcing theorem", and even more often than that it is applied without further mention.

Theorem 1.12. *Suppose φ is a formula, $\tau_0, ..., \tau_{n-1}$ are $\mathbb{P}$-names, and $G \subseteq \mathbb{P}$ is a generic filter. If for each $p \in \mathbb{P}$ there exists a generic filter G that contains p, we have*

$$V[G] \models \varphi(\tau_0^G, ..., \tau_{n-1}^G) \text{ if and only if } p \Vdash \varphi(\tau_0, ..., \tau_{n-1}) \text{ for some } p \in G.$$

Remark 1.13. Note that the implication from right to left follows straight from the definition of $\Vdash$. Hence, the main point of Theorem 1.12 is the other implication.

For an intuitive understanding of Theorem 1.12, think again of approximating some object c. We fix an approximation p of c, thereby partly fixing G which in turn partly fixes $V[G]$. The further we approximate c the more the structure of $V[G]$ precipitates. Then, the theorem says that for each φ there is some stage of the approximation process, where enough structure of $V[G]$ has been determined such that the truth value of φ can be deduced.

At this point we want to mention another perspective that also helps with the intuition about the the forcing relation. The forcing relation behaves similar to the proof theoretic entailment $\vdash$. In particular, we want to call attention to two specific properties. Firstly, if p is stronger than q, and $q \Vdash \varphi$ holds, then also $p \Vdash \varphi$ holds. Secondly, Modus pones applies i.e. if $p \Vdash \varphi$ and $p \Vdash \varphi \Rightarrow \psi$, then $p \Vdash \psi$. This is because in "$p \Vdash \varphi$", p can be understood as an assumption under which we deduce the statement φ. Indeed,"$p \Vdash \varphi$" is defined as "under the assumption $p \in G$ we have $V[G] \models \varphi$". Here, φ and ψ are a sentence in the language of set theory with $\mathbb{P}$-names as additional constant symbols. In "$p \Vdash \varphi$", these constant symbols are simply interpreted as $\mathbb{P}$-names, and in "$V[G] \models \varphi$" a $\mathbb{P}$-name τ is interpreted as τ^G. This language is fittingly referred to as the *forcing language*.

There is more to the perspective on forcing from a pure logic point of view. The details are beyond the scope of this thesis. However, it is true that conditions are truth values of a multi-valued logic for formulas in the forcing language.[5] In this context, the proposition below tells us that for each formula of the form $\exists x\ \varphi(x)$, there is a name τ such that $\varphi(\tau)$ and $\exists x\ \varphi(x)$ are equivalent in the multi-valued logic. Regardless of this perspective the following proposition is of great importance and it is usually referred to as *existential completeness of names*.

Proposition 1.14. *If $\varphi(x)$ is a formula in the forcing language $p \in \mathbb{P}$ is such that $p \Vdash \exists x\ \varphi(x)$, then there is a name $\tau \in V^{\mathbb{P}}$ such that $p \Vdash \varphi(\tau)$.*

The method by which we construct τ in the proof of existential completeness, is a versatile tool. Hence, it is very useful to remember, how to prove this proposition. In the proof we need the following definition and the subsequent lemma:

Definition 1.15. Let A be a subset of $\mathbb{P}$. We call A an *antichain* if, for all $p, q \in A$ such that $p \neq q$, we have that p and q are incompatible. If A is an antichain and for each $p \in \mathbb{P}$ there is some $q \in A$ such that p and q are compatible, we call A a *maximal antichain*.

Lemma 1.16. *A filter G is generic over V if and only if $G \cap A \neq \emptyset$ holds for every maximal antichains $A \subseteq \mathbb{P}$ in V.*

Proof of Proposition 1.14. Fix a formula φ and $p \in \mathbb{P}$ such that $p \Vdash \exists x\ \varphi(x)$. We claim that $D = \{q \in \mathbb{P} \mid \exists \sigma \in V^{\mathbb{P}}\ (q \Vdash \varphi(\sigma))\}$ is dense below p. To see this, fix $r \leq p$ and a generic filter $G \subseteq \mathbb{P}$ such that $r \in G$. As $r \leq p$, we have $p \in G$ and hence $V[G] \models \exists x\ \varphi(x)$. By the definition of $V[G]$, there is a name σ such that $V[G] \models \varphi(\sigma^G)$. Hence, by the Truth Lemma, there is $r' \in G$ such that $r \Vdash \varphi(\sigma)$. Now, r' and r have a common extension q in G. Then, $q \Vdash \varphi(\sigma)$ and $q \leq r$. As r was arbitrary, D is dense below p, and thus $D' = \{q \in \mathbb{P} \mid q \in D\ \vee\ q \text{ is incompatible with } p\}$ is dense. By Zorn's lemma, we find a maximal antichain A such that $A \subseteq D'$. Furthermore, for each $q \in A$ we find a name τ_q such that $q \Vdash \varphi(\tau_q)$ holds whenever q is in D. Define

$$\tau = \{(r, \rho) \mid \exists q \in D' \exists r' \in \mathbb{P}\ (r \leq q\ \wedge\ (r', \rho) \in \tau_q\ \wedge\ r \Vdash \rho \in \tau_q)\}$$

[5]The trick is to construct a so called "complete Boolean algebra" which is equivalent to the original forcing notion in the sense of Definition 2.5. Then, each sentence in the forcing language has a unique maximal condition in the Boolean algebra which forces the sentence. This condition can be regarded as the truth value of the sentence. For details see [6, Part II, 14].

and note that τ is a $\mathbb{P}$-name. We leave it to the reader to check that $q \Vdash \tau = \tau_q$. Now, assume that G is a generic filter such that $p \in G$. Then, there is some $q \in G \cap A$. Such a q must be in D, since it is compatible with p. Thus, we have $q \Vdash \tau = \tau_q \ \wedge \ \varphi(\tau_q)$. This shows that, for any generic filter G that contains p, we have $V[G] \models \varphi(\tau^G)$. Therefore, $p \Vdash \varphi(\tau)$ holds. □

Remark 1.17. Usually, when we apply existential completeness, the formula φ is not given explicitly. Instead, we construct $s \in V[G]$ for an arbitrary generic filter G which contains p, such that s satisfies a finite list of properties. Then, these properties make up the formula φ. To indicate that the name produced by existential completeness inherits these properties from s we often call it $\dot{s}$. Similarly, we write $\dot{q}$ for a name corresponding to $q \in V[G]$, $\dot{t}$ for a name corresponding to $t \in V[G]$, etc.

An important application of existential completeness uses the following slightly stronger version (which has essentially the same proof): If $G \subseteq \mathbb{P}$ is a generic filter, $t \in V[G]$ such that $\varphi(t)$ holds in $V[G]$, and $p \Vdash \exists x \ \varphi(x)$, then there is a name $\dot{t} \in V^{\mathbb{P}}$ such that $\dot{t}^G = t$ and $p \Vdash \varphi(\dot{t})$. This version of existential completeness, legitimises the common practise of supposing that a name is of a particular kind. For example, if $f \in V[G]$ is a function, we can say "let $\dot{f}$ be a name for a function such that $\dot{f}^G = f$", since for any function $f \in V[G]$, by the stronger version of existential completeness, there is a name $\dot{f}$ which is forced to be a function.
To illustrate another application, we draw a comparison between existential completeness and constructions in $R[X]$. As explained above, $R[X]$ can be viewed as analogous to $V^{\mathbb{P}}$. A typical construction in $R[X]$ is the definition of a new polynomial using ring operations, and with existential completeness we have a method to define new names in $V^{\mathbb{P}}$. For example, from $p(X), q(X) \in R[X]$ we can construct the polynomial $(p+q)(X)$, which has the property that $(p+q)(a) = p(a)+q(a)$ holds for all a. Similarly, by using existential completeness with $p = 1$, from $\sigma, \tau \in V^{\mathbb{P}}$ we get a name $\langle \sigma, \tau \rangle$ such that $\langle \sigma, \tau \rangle^G = (\sigma^G, \tau^G)$ holds for all G i.e. $\langle \sigma, \tau \rangle$ is a name for the corresponding ordered pair.

Lastly, we want to mention a lemma which is less profound, but which makes many proofs more concise. The lemma gives a few equivalences, in the style of 1.16, and it involves the following additional definitions:

Definition 1.18. Let D be a subset of $\mathbb{P}$.

1. D is called *open* if it is downward closed i.e. $\forall p \in D \forall q \leq p \ (q \in D)$.

2. D is called *predense* if for each $p \in \mathbb{P}$ there is $q \in D$ such that p and q are compatible. (Note that this is equivalent to D having dense downward closure i.e. D is predense if and only if $\{r \in \mathbb{P} \mid \exists d \in D\ (r \leq d)\}$ is dense.)

Lemma 1.19. *For any filter $G \subseteq \mathbb{P}$ the following are equivalent:*

(i). $G \cap A \neq \emptyset$ holds for all maximal antichains A in V;
(ii). $G \cap D \neq \emptyset$ holds for all predense D in V;
(iii). $G \cap D \neq \emptyset$ holds for all dense D in V i.e. G is generic over V;
(iv). $G \cap D \neq \emptyset$ holds for all dense open D in V.

We sketch a proof of Lemma 1.19: Note that dense sets and maximal antichains are predense, and dense open sets are dense. Hence, "$(ii) \Rightarrow (i)$", "$(ii) \Rightarrow (iii)$", $(ii) \Rightarrow (iv)$ and "$(iii) \Rightarrow (iv)$" hold. Furthermore, the downward closure of any predense set is dense and open. Thus, "$(iv) \Rightarrow (ii)$" follows from the fact that ever filter is upward closed. Using Zorn's lemma one can construct a maximal antichain inside any dense set to prove "$(i) \Rightarrow (iii)$", which completes the proof of Lemma 1.19.
Almost the same proof applies to a generalisation of Lemma 1.19, which uses the following definitions: We say that $D \subseteq \mathbb{P}$ is predense below p if every $q \leq p$ is compatible with some element of D; and we call $A \subseteq \{q \in \mathbb{P} \mid q \leq p\}$ a *maximal antichain below* p if it is an antichain which is predense below p. Then, we have the following generalisation: A filter G meets all maximal antichains below p iff it meets all subsets which are predense below p iff it meets all subsets which are dense below p iff it meets all subsets which are open and dense below p.

Remark 1.20. As a consequence of Gödel's second incompleteness theorem, the existence of set models for `ZFC` is not provable from `ZFC`. To get around this problem, the ground model is only assumed to satisfy a finite fragment of `ZFC`. However, we can still obtain consistency results for `ZFC`. To do this, we use the compactness theorem, which states that any inconsistent theory has a finite fragment that is inconsistent. Hence, a consistency proof for arbitrary large finite fragments of `ZFC` suffices. By the reflection theorem, for each finite fragment of `ZFC`, there exist models that satisfy all the axioms of the finite fragment.[6] The minimal list of axioms we need in a given argument, is almost never stated explicitly. However, by making sure that the arguments can be carried out internally in the ground model V, we guarantee that each proof only relies on a finite list of axioms that need to hold in V. We only have to keep in mind

[6]Further details on models of finite fragments of `ZFC` and the reflection principle can be found in [5, Chapter 15]

that each statement we make about $V[G]$ must have an analogue that can be stated internally in V using the forcing relation. Then, since all proofs are finite in nature, there must be a corresponding finite list of axioms which V needs to satisfy.
As the forcing arguments need to work internally in the ground model V, we often treat V as our universe. In particular, when we assume the existence of an object, we usually implicitly assume that the object is an element of V. Obvious exceptions to this are generic filters or sets which are explicitly assumed to be in $V[G]$.

2 Prerequisites & finite iterations

This section contains a short introduction on finite iterations as well as a discussion of preliminary concepts. The section is based on [7, 1 Forcing basics] which in turn loosely follows [6, Part II, 14].

2.1 Separativity

For two conditions p, q of a forcing notion $\mathbb{P}$ it might happen that p forces everything that is forced by q i.e. for all φ, $q \Vdash \varphi$ implies $p \Vdash \varphi$. In this case, we have $p \Vdash q \in \dot{G}$, since $q \Vdash q \in \dot{G}$ always holds. In fact, the converse also holds, i.e. $p \Vdash q \in \dot{G}$ implies that p forces everything that is forced by q. There is yet another equivalent way to state that this relation between p and q holds. On one hand, if there is some $r \leq p$ which is incompatible with q, any generic filter G that contains r is such that $p \in G$ but $q \notin G$. On the other hand, if every $r \leq p$ is compatible with q, the set $\{s \in \mathbb{P} \mid s \leq q\}$ is dense below p, and thus $p \Vdash q \in \dot{G}$. Therefore, $p \Vdash q \in \dot{G}$ holds if and only if every $r \leq p$ is compatible with q. This motivates the following:

Definition 2.1. Assume $\mathbb{P}$ is a forcing notion.

1. $\mathbb{P}$ is called *separative* if it is antisymmetric ($p \leq q \ \wedge \ q \leq p \Rightarrow p = q$) and for all $p, q \in \mathbb{P}$ with $p \not\leq q$ there exists $r \leq p$ such that r is incompatible with q.
2. Define
   a. the sep-order $\leq_{\text{sep}}$ by $p \leq_{\text{sep}} q \Leftrightarrow \forall r \leq p$ (r is compatible with q), or equivalently $p \Vdash q \in \dot{G}$;
   b. the sep-equivalence relation $\sim_{\text{sep}}$ by $p \sim_{\text{sep}} q \Leftrightarrow (p \leq_{\text{sep}} q) \wedge (q \leq_{\text{sep}} p)$;
   c. the *separative quotient* $\mathbb{P}_{\text{sep}}$ of $\mathbb{P}$ by $\mathbb{P}_{\text{sep}} = (\{[p]_{\text{sep}} \mid p \in \mathbb{P}\}, \leq_{\mathbb{P}_{\text{sep}}}, [1]_{\text{sep}})$, where $[p]_{\text{sep}}$ denotes the $\sim_{\text{sep}}$-equivalence class of $p \in \mathbb{P}$ and the order

of equivalence classes is induced by the sep-order of representatives i.e. $[p]_{\text{sep}} \leq [q]_{\text{sep}} \Leftrightarrow p \leq_{\text{sep}} q$.

Clearly, $p \leq_{\text{sep}} p$ holds for all p, as each $r \leq p$ is compatible with p. The proof that $\leq_{\text{sep}}$ is transitive is straightforward and similar to the proof of 2.2.2. This establishes that $\leq_{\text{sep}}$ is a preorder. Then, it immediately follows that $\sim_{\text{sep}}$ is an equivalence relation, and that $\leq_{\text{sep}}$ defines a partial order (i.e. an antisymmetric preorder) on the set of $\sim_{\text{sep}}$-equivalence classes. As 1 is compatible with every condition, we have $[p]_{\text{sep}} \leq [1]_{\text{sep}}$ for all p. Hence, $\mathbb{P}_{\text{sep}}$ is a forcing notion.

Next, we prove some basic properties of the separative quotient.

Proposition 2.2. *For the notions defined in 2.1.2 the following hold:*

1. *the* sep*-relation $\leq_{\text{sep}}$ extends the underlying forcing relation $\leq$, i.e. $p \leq q$ implies $p \leq_{\text{sep}} q$;*
2. *two conditions $p, q \in \mathbb{P}$ are $\leq$-compatible if and only if $[p]_{\text{sep}}, [q]_{\text{sep}}$ are compatible in $\mathbb{P}_{\text{sep}}$;*
3. *the separative quotient $\mathbb{P}_{\text{sep}}$ is separative;*
4. *if $\mathbb{P}$ is separative, $\mathbb{P}_{\text{sep}}$ is isomorphic to $\mathbb{P}$.*

Proof. 1: Assume $p, q \in \mathbb{P}$ are such that $p \leq q$. Then, for any $r \leq p$, we have $r \leq q$ and $r \leq r$, that is r witnesses that q and r are compatible. As this works for all $r \leq p$, it follows that $p \leq_{\text{sep}} q$ holds.

2: First, consider two compatible conditions $p, q \in \mathbb{P}$. Then, there is $r \in \mathbb{P}$ such that $r \leq p, q$. By the first part, $r \leq_{\text{sep}} p, q$, and thus $[r]_{\text{sep}} \leq [p]_{\text{sep}}, [q]_{\text{sep}}$, that is $[p]_{\text{sep}}$ and $[q]_{\text{sep}}$ are compatible. Now, consider $p, q \in \mathbb{P}$ such that $[p]_{\text{sep}}$ is compatible with $[q]_{\text{sep}}$. Then, there exists $r \in \mathbb{P}$ such that $[r]_{\text{sep}} \leq [p]_{\text{sep}}, [q]_{\text{sep}}$, and hence $r \leq_{\text{sep}} p, q$. By definition of $\leq_{\text{sep}}$, for every $s \in \mathbb{P}$ such that $s \leq r$ holds, the condition s is compatible with p and with q. In particular, r is compatible with p, and thus there exists some $p' \in \mathbb{P}$ such that $p' \leq p, r$. As $p' \leq r$ holds, p' is compatible with q. Therefore, there exists $p'' \in \mathbb{P}$ such that $p'' \leq p', q$. Hence, $p'' \leq p, q$ holds, that is p'' witnesses that p and q are compatible.

3: To see that $\mathbb{P}_{\text{sep}}$ is separative, consider $p, q \in \mathbb{P}$ such that $[p]_{\text{sep}} \nleq [q]_{\text{sep}}$, or equivalently $p \nleq_{\text{sep}} q$. By definition of $\leq_{\text{sep}}$, there exists $r \leq p$ such that r is not $\leq$-compatible with q. For such an r, by part 1 of this proposition, $r \leq_{\text{sep}} p$ holds, and thus $[r]_{\text{sep}} \leq [p]_{\text{sep}}$. By part 2, $[r]_{\text{sep}}$ is not compatible with $[q]_{\text{sep}}$. As p and q were arbitrary, $\mathbb{P}_{\text{sep}}$ is separative.

4: Assume that $\mathbb{P}$ is separative and consider the function $[-]_{\mathrm{sep}} : \mathbb{P} \to \mathbb{P}_{\mathrm{sep}}$, $p \mapsto [p]_{\mathrm{sep}}$, which maps an element $p \in \mathbb{P}$ to its equivalence class in the separative quotient. By definition of $\mathbb{P}_{\mathrm{sep}}$, the function $[-]_{\mathrm{sep}}$ is surjective. To see that $[-]_{\mathrm{sep}}$ is also injective, consider $p, q \in \mathbb{P}$ such that $[p]_{\mathrm{sep}} = [q]_{\mathrm{sep}}$. Then, we have $p \leq_{\mathrm{sep}} q$ and $q \leq_{\mathrm{sep}} p$. As $p \leq_{\mathrm{sep}} q$ holds, every $r \leq p$ is compatible with q, and since $\mathbb{P}$ is separative, this implies $p \leq q$. Similarly, $q \leq_{\mathrm{sep}} p$ implies $q \leq p$. By assumption, $\leq$ is antisymmetric, and thus p must be equal to q. This shows that $[-]_{\mathrm{sep}}$ is a bijection. The only thing left to show is that, $p \leq q$ holds if and only if $[p]_{\mathrm{sep}} \leq [q]_{\mathrm{sep}}$ holds. By part 1, we know that $p \leq q$ implies $[p]_{\mathrm{sep}} \leq [q]_{\mathrm{sep}}$. Now, let us assume $[p]_{\mathrm{sep}} \leq [q]_{\mathrm{sep}}$ holds. Then, $p \leq_{\mathrm{sep}} q$ hods, and as in the argument for the injectivity of $[-]_{\mathrm{sep}}$, we get $p \leq q$. □

As $p \leq_{\mathrm{sep}} q$ is defined as $p \Vdash q \in \dot{G}$, we have $p \in G$ if and only if $[p]_{\mathrm{sep}} \subseteq G$, for any generic filter $G \subseteq \mathbb{P}$. In particular, each generic filter $G \subseteq \mathbb{P}$ is the preimage of some $H \subseteq \mathbb{P}_{\mathrm{sep}}$ under $[-]_{\mathrm{sep}}$. Now, one might suspect that H is again a generic filter. This is the case, and in fact all $\mathbb{P}_{\mathrm{sep}}$-generic filters are of this form. To prove this we need the following:

Lemma 2.3. *A subset $D \subseteq \mathbb{P}$ is predense in $\mathbb{P}$ if and only if $\{[p]_{\mathrm{sep}} \mid p \in D\}$ is predense in $\mathbb{P}_{\mathrm{sep}}$.*

Proof. First, let us assume $D \subseteq \mathbb{P}$ is predense. We need to show that each condition in $\mathbb{P}_{\mathrm{sep}}$ is compatible with some $[q]_{\mathrm{sep}}$ such that $q \in D$. Hence, fix $p \in \mathbb{P}$ and let $q \in D$ such that p and q are $\leq$-compatible. Such a q always exists, as D is predense. By 2.2.2, the conditions $[p]_{\mathrm{sep}}$ and $[q]_{\mathrm{sep}}$ are compatible, and as p was arbitrary, the set $\{[s]_{\mathrm{sep}} \mid s \in D\}$ is predense in $\mathbb{P}_{\mathrm{sep}}$.

Now, assume that $\{[s]_{\mathrm{sep}} \mid s \in D\}$ is predense, and fix $p \in \mathbb{P}$. Then, there is some $q \in D$ such that $[p]_{\mathrm{sep}}$ and $[q]_{\mathrm{sep}}$ are compatible. Hence, p and q are $\leq$-compatible, by 2.2.2. As p was arbitrary, D is predense. □

Theorem 2.4. *The following two statements hold for every forcing notion $\mathbb{P}$:*

1. *If G is a $\mathbb{P}$-generic filter, then $\{[s]_{\mathrm{sep}} \mid s \in G\}$ is a $\mathbb{P}_{\mathrm{sep}}$-generic filter.*
2. *If H is a $\mathbb{P}_{\mathrm{sep}}$-generic filter, then $\{s \mid [s]_{\mathrm{sep}} \in H\}$ is a $\mathbb{P}$-generic filter.*

Proof. 1: Suppose $G \subseteq \mathbb{P}$ is a generic filter and let $H = \{[s]_{\mathrm{sep}} \mid s \in G\}$. First, note that $\{[s]_{\mathrm{sep}} \mid s \in G\}$ is directed. Indeed, this follows straight from G being directed together with the fact that $\leq_{\mathrm{sep}}$ extends $\leq$. In particular, for $p, q \in G$ we find $r \in G$ such that $r \leq p, q$, and thus $[r]_{\mathrm{sep}} \leq [p]_{\mathrm{sep}}, [q]_{\mathrm{sep}}$. Next, we show that H is upward closed. Fix $p, q \in \mathbb{P}$ such that $[p]_{\mathrm{sep}} \in H$ and $[p]_{\mathrm{sep}} \leq [q]_{\mathrm{sep}}$.

Then, there is some $r \in [p]_{\text{sep}} \cap G$, and we have $r \sim_{\text{sep}} p \leq_{\text{sep}} q$. Hence, r is in G and $r \Vdash q \in \dot{G}$. Therefore, q is in G, and $[q]_{\text{sep}}$ is in H. As p and q were arbitrary, H is upward closed. To see that H is generic, fix some predense $D \subseteq \mathbb{P}_{\text{sep}}$, and let D' be its preimage under $[-]_{\text{sep}}$, i.e. $D' = \{p \in \mathbb{P} \mid [p]_{\text{sep}} \in D\}$. Then, $D = \{[p]_{\text{sep}} \mid p \in D'\}$, and thus D' is predense, by Lemma 2.3. As G is generic there is some $p \in G \cap D'$. For this p, we have $[p]_{\text{sep}} \in H \cap D$. Hence, H meets every predense subset of $\mathbb{P}_{\text{sep}}$.

2: Suppose $H \subseteq \mathbb{P}_{\text{sep}}$ is a generic filter and let $G = \{s \mid [s]_{\text{sep}} \in H\}$. As $\leq_{\text{sep}}$ extends $\leq$, the fact that H is upward closed in $\mathbb{P}_{\text{sep}}$ implies that G is upward closed in $\mathbb{P}$. To see that G is also directed, fix $p, q \in G$ and consider $D = \{r \in \mathbb{P} \mid r \leq p, q \;\vee\; r \text{ is incompatible with } p \text{ or with } q\}$. Note that D is predense. In fact, D is dense, as for any $s \in \mathbb{P}$, for which all extensions are compatible with p and with q, we find $s' \leq p, s$ and $s'' \leq q, s'$. By Lemma 2.3, the set $D' = \{[r]_{\text{sep}} \mid r \in D\}$ is predense in $\mathbb{P}_{\text{sep}}$. As H is generic, $H \cap D'$ is non-empty. Thus, there is $r \in D$ such that $[r]_{\text{sep}} \in H$. By definition of G, the condition r is in G. Note that the three conditions $[r]_{\text{sep}}$, $[p]_{\text{sep}}$, $[q]_{\text{sep}}$ are all in H, and thus they are pairwise compatible. By 2.2.2, it follows that r is $\leq$-compatible with p and with q. As r is in D, we must have $r \leq p, q$. Hence, r is a common extension of p and q in G. We conclude that G is directed, since p and q were arbitrary. To see that G is generic, consider a predense set $D \subseteq \mathbb{P}$ and set $D' = \{[p]_{\text{sep}} \mid p \in d\}$. By Lemma 2.3, D' is predense. As H is generic, $H \cap D'$ is non-empty. Hence, there is some $p \in D$ such that $[p]_{\text{sep}} \in H$. Then, p is in $G \cap D$, and thus $G \cap D$ is non-empty. As D was arbitrary, G is generic. $\square$

Note that, for any generic filter $H \subseteq \mathbb{P}_{\text{sep}}$ and $G = \{p \mid [p]_{\text{sep}} \in H\}$ we always have $H = \{[p]_{\text{sep}} \mid p \in G\}$. Further note that, if $G' \subseteq \mathbb{P}$ is a generic filter and $H' = \{[p]_{\text{sep}} \mid p \in G'\}$, we have $G' = \{p \mid [p]_{\text{sep}} \in H'\}$, as $[p]_{\text{sep}} \subseteq G'$ holds for every $p \in G'$. Hence, it follows from 1.4.4, that every $\mathbb{P}$-forcing extension is a $\mathbb{P}_{\text{sep}}$-forcing extension, and vice versa.

2.2 Forcing equivalence

In the previous subsection, we have seen that forcing with $\mathbb{P}$ is essentially the same as forcing with $\mathbb{P}_{\text{sep}}$. In particular, the two forcing notions produce the same forcing extensions. Moreover, we can translate the canonical names for the generic filters in the following sense: $\dot{h}_{\text{sep}} = \{(p, \check{[p]_{\text{sep}}}) \mid p \in \mathbb{P}\}$ is a $\mathbb{P}$-name for a $\mathbb{P}_{\text{sep}}$-generic filter; and $\dot{g}_{\text{sep}} = \{([p]_{\text{sep}}, \check{p}) \mid p \in \mathbb{P}\}$ is a $\mathbb{P}_{\text{sep}}$-name for a $\mathbb{P}$-generic filter. Hence, $\mathbb{P}$ and $\mathbb{P}_{\text{sep}}$ are in some sense equivalent, and this can be expressed internally in the ground model. More precisely, the equivalence is of the following form:

Definition 2.5. Assume that $\mathbb{P}, \mathbb{Q}$ are two forcing notions. We say $\mathbb{P}$ and $\mathbb{Q}$ are *forcing equivalent*, if there are a $\mathbb{Q}$-name $\dot{g}$ and a $\mathbb{P}$-name $\dot{h}$ such that the following hold:

(a) $\dot{g}$ is a name for a filter that is $\mathbb{P}$-generic over V, i.e. for every dense $D \subseteq \mathbb{P}$ in V, we have $1_{\mathbb{Q}} \Vdash_{\mathbb{Q}} \dot{g} \cap \check{D}$;
(b) $\dot{h}$ is a name for a filter that is $\mathbb{Q}$-generic over V, i.e. for every dense $D \subseteq \mathbb{Q}$ in V, we have $1_{\mathbb{P}} \Vdash_{\mathbb{P}} \dot{h} \cap \check{D} \neq \emptyset$;
(c) $1_{\mathbb{P}} \Vdash_{\mathbb{P}} (\check{\dot{g}})^{\dot{h}} = \dot{G}$ and $1_{\mathbb{Q}} \Vdash_{\mathbb{Q}} (\check{\dot{h}})^{\dot{g}} = \dot{H}$;

where $\dot{G}$ denotes the canonical name for the $\mathbb{P}$-generic filter, and $\dot{H}$ denotes the canonical name for the $\mathbb{Q}$-generic filter, that is $\dot{G} = \{(p, \check{p}) \mid p \in \mathbb{P}\}$ and $\dot{H} = \{(q, \check{q}) \mid q \in \mathbb{Q}\}$.

Remark 2.6. It follows from 1.4.4 that forcing equivalent forcing notions produce the same forcing extensions.

For the two forcing notions $\mathbb{P}$, $\mathbb{P}_{\text{sep}}$, we have the names $\dot{h}_{\text{sep}}$,$\dot{g}_{\text{sep}}$, and they are such that $\dot{h}_{\text{sep}}^{G} = \{[p]_{\text{sep}} \mid p \in G\}$ for each generic filter $G \subseteq \mathbb{P}$, and $\dot{g}_{\text{sep}}^{H} = \{p \mid [p]_{\text{sep}} \in H\}$ for each generic filter $H \subseteq \mathbb{P}_{\text{sep}}$. Hence, $\mathbb{P}$ and $\mathbb{P}_{\text{sep}}$ are forcing equivalent by Theorem 2.4.

In many cases the forcing equivalence of $\mathbb{P}$ and $\mathbb{Q}$ is due to some function between the two forcing notions and we can ask whether the function also preserves the forcing relation. To this end, we need to define how a function between forcing notions acts on the names. If $f : \mathbb{P} \to \mathbb{Q}$ is a function between forcing notion, we define a corresponding translation from $\mathbb{P}$-names to $\mathbb{Q}$-names by

$$\tau_f = \{(f(p), \sigma_f) \mid (p, \sigma) \in \tau)\}.$$

The dual translation from $\mathbb{Q}$-names to $\mathbb{P}$-names is defined by

$$\tau^f = \{(p, \sigma^f) \mid \sigma \in \text{names}(\tau),\ (f(p) \Vdash \sigma \in \tau)\};$$

where $\text{names}(\tau) = \{\sigma \mid \exists p\ (p, \sigma) \in \tau\}$. Now, we can ask whether

$$\forall \tau \in V^{\mathbb{P}} \forall p \in \mathbb{P}\ ((f(p) \Vdash_{\mathbb{Q}} \varphi(\tau_f)) \Leftrightarrow (p \Vdash_{\mathbb{P}} \varphi(\tau))) \tag{2.1}$$

and

$$\forall \tau \in V^{\mathbb{Q}} \forall p \in \mathbb{P}\ ((p \Vdash_{\mathbb{P}} \varphi(\tau^f)) \Leftrightarrow (f(p) \Vdash_{\mathbb{Q}} \varphi(\tau))) \tag{2.2}$$

hold for every formula φ.

For $f = [-]_{\mathrm{sep}} : \mathbb{P} \to \mathbb{P}_{\mathrm{sep}}$, we can compute that $(\tau_f)^H = \tau^G$ holds for $\tau \in V^{\mathbb{P}}$ and $(\tau^f)^G = \tau^H$ holds for $\tau \in V^{\mathbb{P}_{\mathrm{sep}}}$, whenever G, H are generic filters for $\mathbb{P}, \mathbb{P}_{\mathrm{sep}}$ respectively, and such that $H = f[G]$, or equivalently $G = f^{-1}[H]$. Since in this case also $p \in G \Leftrightarrow f(p) \in H$ and $V[G] = V[H]$ hold, it follows that 2.1 and 2.2 hold for all φ.

Another example is the case where $f : \mathbb{P} \to \mathbb{Q}$ is an isomorphism of forcing notions. More generally, we have the following theorem, where $f : \mathbb{P} \to \mathbb{Q}$ is a dense embedding of preorders:

Theorem 2.7. *Assume that $\mathbb{P}, \mathbb{Q}$ are forcing notions and $f : \mathbb{P} \to \mathbb{Q}$ is such that $f(p) \leq f(q) \Leftrightarrow p \leq q$ for all $p, q \in \mathbb{P}$, and $f[\mathbb{P}]$ is dense in $\mathbb{Q}$. Then the following hold:*

1. *if $G \subseteq \mathbb{P}$ is a generic filter, the upward closure H of $f[G]$ is a generic filter for $\mathbb{Q}$;*
2. *if $H \subseteq \mathbb{Q}$ is a generic filter, $G = f^{-1}[H]$ is a generic filter for $\mathbb{P}$;*
3. *$\mathbb{P}$ and $\mathbb{Q}$ are forcing equivalent;*
4. *$(\tau_f)^H = \tau^G$ holds for all $\tau \in V^{\mathbb{P}}$ and $(\tau^f)^G = \tau^H$ holds for all $\tau \in V^{\mathbb{Q}}$, whenever G, H are generic filters for $\mathbb{P}, \mathbb{Q}$ respectively, and such that H is the upward closure of $f[G]$, or equivalently $G = f^{-1}[H]$.*

Proof. 1: Let $G \subseteq \mathbb{P}$ be a generic filter and define H to be the upward closure of $f[G]$ in $\mathbb{Q}$. Note that $f[G]$ is directed, since G is directed and $r \leq p, q$ implies $f(r) \leq f(p), f(q)$ for all r, p, q. Then, H is a filter, as it is the upward closure of a directed set. To see that H is generic, fix a dense open set $D \subseteq \mathbb{Q}$. Let $D' = f^{-1}[D]$ and note that D' is dense in $\mathbb{P}$. Indeed, if $p \in \mathbb{P}$, there is $q \in D$ such that $q \leq f(p)$, and since $f[\mathbb{P}]$ is dense in $\mathbb{Q}$, there is $p' \in \mathbb{P}$ such that $f(p') \leq q$. Now, since D is open, $f(p')$ is in D, and thus p' is in D'. Furthermore, $f(p') \leq f(p)$ implies $p' \leq p$. As p was arbitrary, this shows that D' is dense in $\mathbb{P}$. Therefore, $G \cap D'$ is non-empty, and since $f[G \cap D']$ is a subset of $H \cap D$, the intersection $H \cap D$ is non-empty. As D was arbitrary, H is generic.

2: Let $H \subseteq \mathbb{Q}$ be a generic filter and define $G = f^{-1}[H]$. As H is upward closed in $\mathbb{Q}$, we see that G is upward closed in $\mathbb{P}$. To see that G is directed, fix $p, q \in G$ and take $q' \in H$ such that $q' \leq f(p), f(q)$. Now, since $f[\mathbb{P}]$ is dense below q' and $q' \in H$, the generic filter H meets $f[\mathbb{P}]$ below q', and thus there exists $r \in G$ such that $f(r) \leq q'$. Then, $f(r) \leq f(p), f(q)$ implies $r \leq p, q$. And since p, q were arbitrary, G is directed. Next, note that for any set $D \subseteq \mathbb{P}$ that is dense in $\mathbb{P}$, the set $f[D]$ is dense in $\mathbb{Q}$, since $f[\mathbb{P}]$ is dense in $\mathbb{Q}$. Thus, there is some $q \in H \cap f[D]$ and there is $p \in D$ such that $f(p) = q$. This p is in $G \cap D$. Hence, $G \cap D$ is non empty. As D was arbitrary, G is generic.

3: Note that for $\dot{g} = \{(f(p), \check{p}) \mid f(p) \in \mathbb{Q}\}$, we have $\dot{g}^H = f^{-1}[H]$ for any generic filter $H \subseteq \mathbb{Q}$. Further note that, for $\dot{h} = \{(p, \check{q}) \mid q \geq f(p),\ p \in \mathbb{P}\}$, we have that $\dot{h}^G$ is the upward closure of $f[G]$, for any generic filter $G \subseteq \mathbb{P}$. Now, assume that G, H are generic filters for $\mathbb{P}, \mathbb{Q}$ respectively. If H is the upward closure of $f[G]$, we have $f^{-1}[H] = G$, since G is upward closed. Conversely, if $G = f^{-1}[H]$, the upward closure of $f[G]$ is included in H, as H is upward closed. To see that H is included in the upward closure of $f[G]$, fix $q \in H$. Since $f[\mathbb{P}]$ is dense below q, the generic filter H meets $f[\mathbb{P}]$ below q, and thus there is $p \in \mathbb{P}$ such that $f(p) \in H$ and $f(p) \leq q$. As $f(p)$ is in $f[G]$, q is in the upward closure of $f[G]$. Hence, we have $G = f^{-1}[H]$ if and only if H is the upward closure of $f[G]$. It follows that the names $\dot{g}, \dot{h}$ are as in Definition 2.5.

4: Assume that $G \subseteq \mathbb{P}$, $H \subseteq \mathbb{Q}$ are generic filters such that $G = f^{-1}[H]$. We prove the claim $(\tau_f)^H = \tau^G$ by induction on the rank of $\tau \in V^{\mathbb{P}}$. Assume that the claim holds for all names of rank strictly less than the rank of τ. If x is in τ^G, there is $(p, \sigma) \in \tau$ such that $p \in G$ and $x = \sigma^G$. Then $x = (\sigma_f)^G$ by induction hypothesis, and $f(p)$ is in H. Hence, x is in $(\tau_f)^H$. Conversely, if y is in $(\tau_f)^H$, there is $(f(q), \rho_f) \in \tau_f$ such that $f(q) \in H$ and $y = (\rho_f)^H$. Then $y = \rho^G$ by the induction hypothesis, and $p \in G$. Hence, y is in τ. As x and y were arbitrary, we have $(\tau_f)^H = \tau^G$. Similarly, we prove the claim $(\tau^f)^H = \tau^G$ by induction on the rank of $\tau \in V^{\mathbb{Q}}$. If x is in $(\tau^f)^G$, there is $(p, (\sigma^f)) \in \tau^f$ such that $p \in G$ and $x = (\sigma^f)^G$. By definition of τ^f, we have $f(p) \Vdash \sigma \in \tau$. As $f(p)$ is in H, σ^H is in τ^H. Conversely, if y is in τ^H, there is $(q, \rho) \in \tau$ such that $q \in H$ and $y = \rho^H$. Since $f[\mathbb{P}]$ is dense below q, there is $q' \in H \cap f[\mathbb{P}]$ below q, and there is $p \in \mathbb{P}$ such that $f(p) = q'$. We have $p \in G$, and $f(p) \leq q \Vdash \rho \in \tau$. Hence, (p, ρ^f) is in τ^f. Then $y = (\rho^f)^G$ by induction hypothesis, and $p \in G$, since $f(p) \in H$. Hence, y is in $(\tau^f)^G$. As x and y were arbitrary, we have $(\tau_f)^H = \tau^G$. $\square$

2.3 Projections

If we have two forcing notions $\mathbb{P}, \mathbb{P}'$ we can form the product forcing $\mathbb{Q} = \mathbb{P} \times \mathbb{P}'$ which consists of all pairs (p, p') such that $p \in \mathbb{P}$ and $p' \in \mathbb{P}'$. The corresponding order is defined by $(p, p') \leq (q, q') \Leftrightarrow p \leq q \ \wedge \ p' \leq q'$. We can project $\mathbb{Q}$ onto $\mathbb{P}$ by the associated order preserving map $\pi : \mathbb{Q} \to \mathbb{P}$, $(p, p') \mapsto p$. Analogously, we get a map $\pi' : \mathbb{Q} \to \mathbb{P}'$, $(p, p') \mapsto p'$. Then, for any generic filter $H \subseteq \mathbb{Q}$, we get two corresponding generic filters, $G = \pi[H] = \{p \mid (p, p') \in H\}$ for $\mathbb{P}$ and a $G' = \pi'[H] = \{p' \mid (p, p') \in H\}$ for $\mathbb{P}'$. Conversely, from two generic filters $G \subseteq \mathbb{P}$ and $G' \subseteq \mathbb{P}'$ we can form a generic filter H for $\mathbb{Q}$ by $H = G \times G' = \{(p, p') \mid p \in G,\ p' \in G'\}$. In the latter situation, one might ask whether G' is generic over $V[G]$, or analogously whether G is generic over $V[G']$. Indeed, this is the case, and it generalises to other situations, where we have a map $\pi : \mathbb{Q} \to \mathbb{P}$ between two forcing notions $\mathbb{P}, \mathbb{Q}$. In particular, we can consider the following:

Definition 2.8. Assume that $\mathbb{P}$ and $\mathbb{Q}$ are forcing notions. A map $\pi : \mathbb{Q} \to \mathbb{P}$ is called a *projection* if the following three conditions hold:

(a) π is order preserving i.e. $q \leq q'$ implies $\pi(q) \leq \pi(q')$;
(b) $\pi(1_{\mathbb{Q}}) = 1_{\mathbb{P}}$;
(c) whenever $p \leq \pi(q)$ holds, there is $q' \leq q$ such that $\pi(q') \leq p$ holds.

Even though, a projection π might not be surjective, it is not hard to see that $\pi[\mathbb{Q}]$ must be dense in $\mathbb{P}$. In fact, $\pi[D]$ is dense for every dense $D \subseteq \mathbb{Q}$. We do not necessarily get that $\pi^{-1}[D]$ is dense in $\mathbb{Q}$ for every dense $D \subseteq \mathbb{P}$. However, we have the following lemma:

Lemma 2.9. *Assume that $\pi : \mathbb{Q} \to \mathbb{P}$ is a projection. Then the following hold:*

1. if D is a dense subset of $\mathbb{Q}$, then $\pi[D]$ is dense in $\mathbb{P}$;
2. if U is a dense and open subset of $\mathbb{P}$, then $\pi^{-1}[U]$ is dense in $\mathbb{Q}$.

Proof. 1: Let $D \subseteq \mathbb{Q}$ be dense. Fix $p \in \mathbb{P}$ and note that $p \leq 1_{\mathbb{P}} = \pi(1_{\mathbb{Q}})$. Then, since π is a projection, there is $q \leq 1_{\mathbb{Q}}$ such that $\pi(q) \leq p$. As D is dense, there is $q' \in D$ such that $q' \leq q$. It follows that $\pi(q') \leq p$, by the fact that π is order preserving. As p was arbitrary, this shows that $\pi[D]$ is dense in $\mathbb{P}$.

2: Let $U \subseteq \mathbb{P}$ be dense and open. Fix $q \in \mathbb{Q}$. As U is dense, there is $p \in U$ such that $p \leq \pi(q)$. Then, there exists a $q' \leq q$ such that $\pi(q') \leq p$. Since U is open, we get $\pi(q') \in U$, and thus $q' \in \pi^{-1}[U]$. Now, $q' \leq q$ and $q' \in \pi^{-1}[U]$ hold. And as q was arbitrary, $\pi^{-1}[U]$ is dense in $\mathbb{Q}$. □

Remark 2.10. Note that we can use similar arguments to prove that $\pi[D]$ is dense below $\pi(q)$ in $\mathbb{P}$ if D is dense below q in $\mathbb{Q}$, and that $\pi^{-1}[U]$ is dense below p in $\mathbb{P}$ if U is dense open below $\pi(p)$ in $\mathbb{Q}$.

One can easily check that π and π' from the example at the beginning of this subsection are indeed projections. In this example, if we "factor out" $\mathbb{P}$ from $\mathbb{Q} = \mathbb{P} \times \mathbb{P}'$ we get $\mathbb{P}'$. Intuitively, it makes sense that we need to combine a generic filter $G \subseteq \mathbb{P}$ with a generic filter $G' \subseteq \mathbb{P}'$ to obtain a generic filter for $\mathbb{Q}$.

However, for a projection $\pi : \mathbb{Q} \to \mathbb{P}$ between two arbitrary forcing notions, we cannot factor out $\mathbb{P}$ from $\mathbb{Q}$ in any sensible way. Hence, the following question arises: How do we obtain a generic filter $H \subseteq \mathbb{Q}$ from a generic filter $G \subseteq \mathbb{P}$ such that H projects to G? To answer this question we define in $V[G]$, the quotient $\mathbb{Q}/G = \pi^{-1}[G]$. The quotient $\mathbb{Q}/G$ inherits the order from $\mathbb{Q}$ and is a forcing notion in $V[G]$. We have the following correspondence between generic filters of $\mathbb{P}$, $\mathbb{Q}$ and $\mathbb{Q}/G$.

Theorem 2.11. *Assume that $\pi : \mathbb{Q} \to \mathbb{P}$ is a projection. Then the following hold:*

1. *If G is a $\mathbb{P}$-generic filter over V and H is a $\mathbb{Q}/G$-generic filter over $V[G]$, then H is $\mathbb{Q}$-generic over V, and G is the upward closure of $\pi[H]$.*
2. *If H is a $\mathbb{Q}$-generic filter over V and G is the upward closure of $\pi[H]$, then G is $\mathbb{P}$-generic over V, and H is $\mathbb{Q}/G$-generic over $V[G]$.*

Proof. 1: Assume G is a $\mathbb{P}$-generic filter over V and H is a $\mathbb{Q}/G$-generic filter over $V[G]$. First, note that $\mathbb{Q}/G = \pi^{-1}[G]$ is upward closed in $\mathbb{Q}$, as π is order preserving and G is upward closed in $\mathbb{P}$. Therefore, and since H is upward closed in $\mathbb{Q}/G$, H is upward closed in $\mathbb{Q}$. Since H is also directed, H is a filter for $\mathbb{Q}$. To see that H is $\mathbb{Q}$-generic over V, consider a dense set $D \subseteq \mathbb{Q}$ in V. In $V[G]$ define $D' = D \cap \pi^{-1}[G]$. We claim that D' is dense in $\mathbb{Q}/G$. To prove this claim, fix $q \in \mathbb{Q}/G$ and consider $D_q = \{r \in D \mid r \leq q\}$. As D_q is dense below q in $\mathbb{Q}$, $\pi[D_q]$ is dense below $\pi(q)$ in $\mathbb{P}$. As $\pi(q)$ is in the generic filter G, there is some $p \in \pi[D_q] \cap G$. Let $q' \in D$ be such that $q' \leq q$ and $\pi(q) = p$. Then, we have $\pi(q') \in G$, and thus q' is in D'. As q was arbitrary, this proves the claim that D' is dense in $\mathbb{Q}/G$. Hence, $H \cap D'$ is non-empty, and since $D' \subseteq D$, also $H \cap D$ is non-empty. As D was arbitrary, H is $\mathbb{Q}$-generic over V.
To see that G is the upward closure of $\pi[H]$, first note that H is a subset of $\pi^{-1}[G]$ and hence $\pi[H]$ is a subset of G. Hence, it suffices to show that for each $p \in G$, there is $q \in H$ such that $\pi(q) \leq p$. Fix $p \in G$ and consider the set $D = \{r \in \mathbb{Q} \mid \pi(r) \leq p \ \vee \ \pi(r) \text{ is incompatible with } p\}$. If $r \in \mathbb{Q}$ is such that $\pi(r)$ and p are compatible, there is $s \leq \pi(r), p$ and we find $r' \leq r$ such that $\pi(r') \leq s$. Hence, D is dense in $\mathbb{Q}$ and there is some $q \in H \cap D$. We cannot have that $\pi(q)$ is incompatible with p, since $\pi(q)$ and p are both in G. As q is in D, we must have $\pi(q) \leq p$.

2: Assume H is a $\mathbb{Q}$-generic filter over V and let G be the upward closure of $\pi[H]$ in $\mathbb{P}$. Since π is order preserving and H is directed, also $\pi[H]$ is directed. Then, G is a filter, as it is the upward closure of a directed set. To see that G is generic, consider a dense open set $U \subseteq \mathbb{P}$ in V. By Lemma 2.3, $\pi^{-1}[U]$ is dense

in $\mathbb{Q}$. Hence, there is some $p \in H \cap \pi^{-1}[U]$. We have $\pi(p) \in G \cap U$. As U was arbitrary, G is generic.
By definition of G, we have $H \subseteq \mathbb{Q}/G$, and thus H is a filter in $\mathbb{Q}/G$. To see that H is generic over $V[G]$, fix a dense open set $D \subseteq \mathbb{Q}/G$ in $V[G]$. Let $\dot{D} \in V^{\mathbb{P}}$ be a $\mathbb{P}$-name for a dense open subset of $\mathbb{Q}/G$ such that $\dot{D}^G = D$. Consider $E = \{r \in \mathbb{Q} \mid \pi(r) \Vdash \check{r} \in \dot{D}\}$. We claim that E is dense. To prove this claim, fix $q \in \mathbb{Q}$ and let G' be a $\mathbb{P}$-generic filter that contains $\pi(q)$. Note that $\dot{D}^{G'}$ is dense below q, and hence $\{\pi(d) \mid d \in \dot{D}^{G'},\ d \leq q\}$ is dense below $\pi(q)$. Thus, there is $d \in \dot{D}^{G'}$ such that $d \leq q$ and $\pi(d) \in G'$. By the Truth Lemma, there is some $s \in G'$ such that $s \Vdash \check{d} \in \dot{D}$. As $\pi(d)$ is in G', we can find such an s below $\pi(d)$. Then, there is some $r \leq d$ with $\pi(r) \leq s$, since π is a projection. As $\pi(r) \leq s$ holds and $\dot{D}$ is forced to be open, we get $\pi(r) \Vdash \check{r} \in \dot{D}$. We have $r \leq q$ and q was arbitrary. Thus, E is dense and there is some q in $H \cap E$. Then, $\pi(q)$ is in G, we have $\pi(q) \Vdash \check{q} \in \dot{D}$, and therefore $q \in H \cap D$. As D was arbitrary, H is $\mathbb{Q}/G$-generic over $V[G]$. $\square$

Note that in case of $\mathbb{Q} = \mathbb{P} \times \mathbb{P}'$, $\pi : \mathbb{Q} \to \mathbb{P}$, it is not hard to show that $\mathbb{Q}/G$ and $\mathbb{P}'$ have isomorphic separative quotients. Hence, 2.11 also applies to the example described at the beginning of this subsection.

2.4 Finite iterations

The basic idea in this subsection is the iteration of forcings in the following sense: If $\mathbb{Q}$ is a forcing notion in $V[G]$ for some $\mathbb{P}$-generic filter G, and if H is a $\mathbb{Q}$-generic filter over $V[G]$, we obtain a forcing extension $V[G][H]$. However, $\mathbb{Q}$ is not necessarily in V. As we want to work with the iterations in V, we need the following definition:

Definition 2.12. Assume that $\mathbb{P}$ is a forcing notion and $\dot{\mathbb{Q}}$ is $\mathbb{P}$-name for a forcing notion. The *two step iteration* $\mathbb{P} * \dot{\mathbb{Q}}$ is given by the set

$$\{(p, \dot{q}) \mid p \in \mathbb{P},\ \dot{q} \textit{ is a canonical name such that } \mid p \Vdash \dot{q} \in \dot{\mathbb{Q}}\}$$

and the preorder defined by $(p, \dot{q}) \leq (p', \dot{q}') \Leftrightarrow (p \leq p' \wedge p \Vdash \dot{q} \leq \dot{q}')$. We also define $1_{\mathbb{P}*\dot{\mathbb{Q}}} = (1_{\mathbb{P}}, \dot{1})$, where $\dot{1}$ is some fixed canonical $\mathbb{P}$-name for a greatest element of $\dot{\mathbb{Q}}$.

The reason why we restrict $\dot{q}$ to "canonical names", is that we need to make sure that the conditions form a set and not a proper class.[7] We are not going to give a definition of canonical names, and instead we refer the interested reader to [8, I §5]. We just state the following as fact:

Fact 2.13. *Assume λ is an uncountable regular cardinal and $\mathbb{P}$ is a forcing notion. Then, the following hold:*

1. *for every $\mathbb{P}$-name τ there exists a canonical $\mathbb{P}$-name τ' such that $1 \Vdash \tau = \tau'$;*
2. *if the forcing notion $\mathbb{P}$ is in $H(\lambda)$ and the the cardinality of $\mathbb{P}$ is strictly less than the cofinality of λ, then for every canonical $\mathbb{P}$-name τ we have*

$$\tau \in H(\lambda) \text{ if and only if, for every generic filter } G,\ \tau^G \in H(\lambda)^{V[G]},$$

and hence $H(\lambda)[G] = H(\lambda)^{V[G]}$ for every generic filter $G \subseteq \mathbb{P}$.

Remark 2.14. In 2.13.2 we use the following standard definitions:

$$H(\lambda) = \{x \mid \ |\operatorname{trcl}(x)| < \lambda\} \ \text{ and } \ H(\lambda)[G] = \{\tau^G \mid \tau \in H(\lambda) \cap V^{\mathbb{P}}\},$$

where $\operatorname{trcl}(x) = \{x\} \cup \bigcup\{\operatorname{trcl}(y) \mid y \in x\}$ is called the *transitive closure* of x. Recall that we say a set t is transitive if $x \subseteq t$ holds for every $x \in t$. Hence, $\operatorname{trcl}(x)$ is the minimal transitive set which contains x. It is not hard to see that $H(\lambda)$ is itself a transitive set, and one can prove that, if λ is an uncountable regular cardinal, $H(\lambda)$ models `ZFC` except for some instances of the power set axiom [9, Chapter IV, Theorem 6.5].
Note that, by 2.13.1, whenever we work with a name $\dot{q}$ such that $1 \Vdash \dot{q} \in \dot{\mathbb{Q}}$ we may switch to a canonical name $\dot{q}'$ which evaluates to the same condition as $\dot{q}$ in any generic extension. Usually we neglect to mention that we make this switch. Instead, we silently assume that the name $\dot{q}$ is already canonical.

We want to prove that the iteration $\mathbb{P} * \dot{\mathbb{Q}}$ indeed captures the idea which we presented at the beginning of this subsection. To this end we need the following:

Lemma 2.15. *Assume that $\mathbb{P}$ is a forcing notion and $\dot{\mathbb{Q}}$ a $\mathbb{P}$-name for a forcing notion. Then the function $\pi : \mathbb{P} * \dot{\mathbb{Q}} \to \mathbb{P}$, given by $\pi(p, \dot{q}) = p$, is a projection.*

Proof. The greatest element of $\mathbb{P} * \dot{\mathbb{Q}}$ is $(1_{\mathbb{P}}, \dot{1})$, and $\pi(1_{\mathbb{P}}, \dot{1}) = 1_{\mathbb{P}}$. That π is order preserving, follows straight from the definition of the order on $\mathbb{P} * \dot{\mathbb{Q}}$. To see that π satisfies also the third defining property of a projection, consider $(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}}$ and $p' \in \mathbb{P}$ such that $p' \leq \pi(p, \dot{q})$, i.e. $p' \leq p$. Since $\dot{q} \leq \dot{q}$ is forced,

[7] For a detailed description of how one can end up with a proper class, as well as an alternative approach to fixing this problem see Eskew [7, 1.5 Iterations].

it follows that $(p', \dot{q}) \leq (p, \dot{q})$. And we have $\pi(p', \dot{q}) = p' \leq p = \pi(p, \dot{q})$. This works for arbitrary p, p' and q. Thus, π is a projection. $\square$

The following theorem is the main point of this subsection:

Theorem 2.16. *Assume $\mathbb{P} * \dot{\mathbb{Q}}$ is a two step iteration and $G \subseteq \mathbb{P}$ is a generic filter. Then, $(\mathbb{P} * \dot{\mathbb{Q}})/G$ and $\dot{\mathbb{Q}}^G$ have isomorphic separative quotients.*

Proof. First, note that $(\mathbb{P} * \dot{\mathbb{Q}})/G = \pi^{-1}[G] = G \times \{\dot{q} \mid (1, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}}\}$ and $\dot{\mathbb{Q}}^G = \{\dot{q}^G \mid (1, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}}\}$. We claim that $\Phi : (\dot{\mathbb{Q}}^G)_{\text{sep}} \to ((\mathbb{P} * \dot{\mathbb{Q}})/G)_{\text{sep}}$ given by $\Phi([\dot{q}^G]_{\text{sep}}) = [(1, \dot{q})]_{\text{sep}}$ is an isomorphism.
Note that we use $\leq_{\text{sep}}$ to denote the sep-order of $(\mathbb{P} * \dot{\mathbb{Q}})/G$ and of $\dot{\mathbb{Q}}^G$. To see that Φ is well-defined and order-preserving, we will show that $\dot{q}_1^G \leq_{\text{sep}} \dot{q}_0^G$ implies $(1, \dot{q}_1) \leq_{\text{sep}} (1, \dot{q}_0)$, for all $\dot{q}_0, \dot{q}_1$. Suppose $\dot{q}_1^G \leq_{\text{sep}} \dot{q}_0^G$ and take $p \in G$ such that $p \Vdash \dot{q}_1 \leq_{\text{sep}} \dot{q}_0$. Consider any $(p_2, \dot{q}_2) \in (\mathbb{P} * \dot{\mathbb{Q}})/G$ such that $(p_2, \dot{q}_2) \leq (1, \dot{q}_1)$. Then, by definition of the order of the iteration $\mathbb{P} * \dot{\mathbb{Q}}$, we have $p_2 \Vdash \dot{q}_2 \leq \dot{q}_1$, and thus $\dot{q}_2^G \leq \dot{q}_1^G$. Since $\dot{q}_0^G \leq_{\text{sep}} \dot{q}_1^G$, this implies that $\dot{q}_2^G$ is compatible with $\dot{q}_0^G$. Take a $\mathbb{P}$-name $\dot{q}_3$ such that $\dot{q}_3^G \leq \dot{q}_0^G, \dot{q}_2^G$ holds. Then, there is some $p_3 \in G$ such that $p_3 \leq p, p_2$ and $p_3 \Vdash \dot{q}_3 \leq \dot{q}_0, \dot{q}_2$. Note that we have $(p_3, \dot{q}_3) \leq (p_2, \dot{q}_2), (1, \dot{q}_0)$. As $(p_2, \dot{q}_2)$ was arbitrary, it follows that $(1, \dot{q}_1) \leq_{\text{sep}} (1, \dot{q}_0)$. This works for all $\dot{q}_0, \dot{q}_1$, and hence $[(1, \dot{1})]_{\text{sep}}$ does not depend on the choice of the representative $\dot{q}$, which makes Φ well-defined. The argument in this paragraph also shows that Φ is order-preserving.
Next, we show that the converse of the above implication also holds, i.e. $(1, \dot{q}_1) \leq_{\text{sep}} (1, \dot{q}_0)$ implies $\dot{q}_1^G \leq_{\text{sep}} \dot{q}_0^G$, for all $\dot{q}_0, \dot{q}_1$. To see this, suppose $\dot{q}_1^G \not\leq_{\text{sep}} \dot{q}_0^G$ holds. Then, there is a name $\dot{q}_2$ such that $\dot{q}_2^G \leq \dot{q}_1^G$ and $\dot{q}_2^G$ is incompatible with $\dot{q}_0^G$. Take $p \in G$ such that $p \Vdash$ "$\dot{q}_2 \leq \dot{q}_1$ and $\dot{q}_2$ is incompatible with $\dot{q}_0$". Then $(p, \dot{q}_2) \leq (p_1, \dot{q}_1)$ and $(p, \dot{q}_2)$ is incompatible with $(p_0, \dot{q}_0)$. Thus, $(1, \dot{q}_1) \not\leq_{\text{sep}} (1, \dot{q}_0)$ holds, whenever $\dot{q}_1^G \not\leq_{\text{sep}} \dot{q}_0^G$ holds. Consequently, we have that $[(1, \dot{q})]_{\text{sep}} = [(1, \dot{q}')]_{\text{sep}}$ implies $[\dot{q}^G]_{\text{sep}} = [\dot{q}'^G]_{\text{sep}}$. Therefore, Φ is injective.
Lastly, we show that $G \times \{\dot{q}\} \subseteq [(1, \dot{q})]_{\text{sep}}$ holds for every $\dot{q}$. Consider any $p \in G$. On one hand, for any $(p', \dot{q}') \in (\mathbb{P} * \dot{\mathbb{Q}})/G$, if $(p', \dot{q}') \leq (p, \dot{q})$ holds, also $(p', \dot{q}') \leq (1, \dot{q})$ holds. This implies $(p, \dot{q}) \leq_{\text{sep}} (1, \dot{q})$. On the other hand, if $(p', \dot{q}') \leq (1, \dot{q})$ holds, take $p'' \in G$ such that $p'' \leq p, p'$. Then $(p'', \dot{q}') \leq (p, \dot{q}), (p', \dot{q}')$. Thus, also $(1, \dot{q}) \leq_{\text{sep}} (p, \dot{q})$ holds. Now, any element of $(\mathbb{P} * \dot{\mathbb{Q}})/G$ is contained in an equivalence class $[(1, \dot{q})]_{\text{sep}}$, for some $\dot{q}$. Therefore, Φ is surjective onto the separative quotient of $(\mathbb{P} * \dot{\mathbb{Q}})/G$. Then Φ has an inverse, which by the argument of the previous paragraph is order-preserving. Thus, Φ is an isomorphism. $\square$

As a consequence of Theorem 2.4 and Theorem 2.11, if $G \subseteq \mathbb{P}$ is generic over V and $H \subseteq \dot{\mathbb{Q}}^G$ is generic over $V[G]$, we obtain a generic filter $G * H$ for $\mathbb{P} * \dot{\mathbb{Q}}$, and we have $V[G][H] = V[G * H]$. The generic filter $G * H$ is simply the closure with respect to sep-equivalence in $\mathbb{P} * \dot{\mathbb{Q}}/G$ of $\{(1, \dot{q}) \mid \dot{q}^G \in H\}$. Conversely, if $K \subseteq \mathbb{P} * \dot{\mathbb{Q}}$ is generic over V, we obtain a generic filter $G \subseteq \mathbb{P}$ over V and a generic filter $H \subseteq \dot{\mathbb{Q}}^G$ over $V[G]$, and such that $G * H = K$.

Corollary 2.17. *For any two step iteration $\mathbb{P} * \dot{\mathbb{Q}}$ the following hold:*

1. *If $G \subseteq \mathbb{P}$ is a generic filter over V and $H \subseteq \dot{\mathbb{Q}}^G$ is a generic filter over $V[G]$, then $G * H = \{(p, \dot{q}) \mid p \in G,\ \dot{q}^G \in H\}$ is a $\mathbb{P} * \dot{\mathbb{Q}}$-generic filter over V.*
2. *If $K \subseteq \mathbb{P} * \dot{\mathbb{Q}}$ is a generic filter over V, then $G = \{p \mid \exists \dot{q}\ ((p, \dot{q}) \in K)\}$ is a $\mathbb{P}$-generic filter over V and $H = \{\dot{q}^G \mid (1, \dot{q}) \in K\}$ is a $\dot{\mathbb{Q}}^G$-generic filter over $V[G]$.*

Proof. 1: Fix two generic filters G and H for $\mathbb{P}$ and $\dot{\mathbb{Q}}^G$, respectively. By the previous theorem together with Theorem 2.4, we get the $\mathbb{P} * \dot{\mathbb{Q}}/G$-generic filter $K = \{(p, \dot{q}) \mid \exists \dot{q}'\ (\dot{q}'^G \in H \wedge (p, \dot{q}) \sim_{\text{sep}} (1, \dot{q}'))\}$. Hence, it suffices to show, for all $(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}}/G$, that there exists some $\dot{q}'$ such that $\dot{q}'^G \in H$ and $(p, \dot{q}) \sim_{\text{sep}} (1, \dot{q}')$ if and only if $\dot{q}^G$ is in H. To this end, fix some $(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}}$. First, note that we have already seen in the proof of of Theorem 2.16 that $(p, \dot{q}) \sim_{\text{sep}} (1, \dot{q})$ holds. Thus, we only need to prove the other implication. Assume $\dot{q}'$ is such that $\dot{q}'^G$ is in H and $(p, \dot{q}) \sim_{\text{sep}} (1, \dot{q}')$ holds. We claim that $\dot{q}'^G \leq_{\text{sep}} \dot{q}^G$, where $\leq_{\text{sep}}$ is the sep-order of $\dot{\mathbb{Q}}^G$. Consider any condition $s \leq \dot{q}'^G$ and let $\dot{s}$ be a name such that $\dot{r}^G = r$. Take $r \in G$ such that $r \Vdash \dot{s} \leq \dot{q}'$. Now, $(r, \dot{s})$ is in $\mathbb{P} * \dot{\mathbb{Q}}/G$ and we have $(r, \dot{s}) \leq (1, \dot{q}')$. Since $(p, \dot{q}) \leq_{\text{sep}} (1, \dot{q}')$ holds, we find $(r', \dot{s}')$ in $\mathbb{P} * \dot{\mathbb{Q}}/G$ such that $(r', \dot{s}') \leq (r, \dot{s}), (p, \dot{q})$. As r' is in G, we have $\dot{s}'^G \leq \dot{s}^G, \dot{q}^G$, and since s was arbitrary, this concludes the proof of the claim. Then, $\dot{q}^G$ must be in H, since we have $\dot{q}'^G \leq_{\text{sep}} \dot{q}^G$. As $\dot{q}$ was arbitrary, we see that K is equal to $\{(p, \dot{q}) \mid p \in G,\ \dot{q}^G \in H\}$.

2: Fix a generic filter $K \subseteq \mathbb{P} * \dot{\mathbb{Q}}$. Then, $G = \{\{p \mid \exists \dot{q}\ ((p, \dot{q}) \in K)\}$ is simply the image of K under the natural projection $\mathbb{P} * \dot{\mathbb{Q}} \to \mathbb{P}$. Clearly, G is upward closed in $\mathbb{P}$ and thus G is a generic filter for $\mathbb{P}$, by Theorem 2.11. Now, K is $\mathbb{P} * \dot{\mathbb{Q}}/G$-generic over $V[G]$, and thus by the previous theorem we know that $\{q \mid \exists (1, \dot{q}) \in K\ (q \sim_{\text{sep}} \dot{q}^G)\}$ is a generic filter for $\dot{\mathbb{Q}}^G$. Hence, we need to show that $\exists (1, \dot{q}) \in K\ (q \sim_{\text{sep}} \dot{q}^G)$ implies that there is a $\mathbb{P}$-name τ such that $\tau^G = q$ and $(1, \tau) \in K$. Fix a name τ such that $\tau^G = q$ and such that $1 \Vdash \tau \sim_{\text{sep}} \dot{q}$. Consider any condition $(p, \dot{q}') \in \mathbb{P} * \dot{\mathbb{Q}}/G$ such that $(p, \dot{q}') \leq (1, \tau)$. Note that $p \Vdash \dot{q}' \leq \dot{q} \sim_{\text{sep}} \tau$, and thus $p \Vdash \exists r \in \dot{\mathbb{Q}}\ (r \leq \tau, \dot{q}')$. Let $\dot{r}$ be a corresponding name defined by existential completeness, and note that $(p, \dot{r}) \leq (p, \dot{q}'), (1, \tau)$.

This shows that every condition below $(1, \dot{q})$ is compatible with $(1, \tau)$, that is $(1, \tau) \leq_{\text{sep}} (1, \dot{q})$. Hence, if $(1, \dot{q})$ is in K, so is $(1, \tau)$. As this works for arbitrary $\dot{q}$, we see that $\{q \mid \exists (1, \dot{q}) \in K \ (q \sim_{\text{sep}} \dot{q}^G)\}$ is equal to $\{\dot{q}^G \mid (1, \dot{q}) \in K\}$. $\square$

To round off this chapter on finite iterations, we briefly discuss the translation from $\mathbb{P} * \dot{\mathbb{Q}}$-names to $\mathbb{P}$-names for $\dot{\mathbb{Q}}$-names, and the associativity of $*$. If τ is a $\mathbb{P} * \dot{\mathbb{Q}}$-name, we can define the $\mathbb{P}$-name $\tau_* = \{(p, \langle \dot{q}, \sigma_* \rangle) \mid ((p, \dot{q}), \sigma) \in \tau\}$ by recursion on rank, where $\langle \dot{q}, \sigma_* \rangle$ denotes the $\mathbb{P}$-name for the corresponding ordered pair. A straightforward computation shows that

$$(\tau_*^G)^H = \tau^{G*H}$$

holds whenever $G \subseteq \mathbb{P}$ is generic over V and $H \subseteq \dot{\mathbb{Q}}^G$ is generic over $V[G]$. Now, if $\dot{\mathbb{R}}$ is a $\mathbb{P} * \dot{\mathbb{Q}}$-name for a forcing notion, we can use existential completeness to define a $\mathbb{P}$-name $(\dot{\mathbb{Q}} * \dot{\mathbb{R}}_*)$ for the tow-step iteration of $\dot{\mathbb{Q}}$ and $\dot{\mathbb{R}}_*$. Then, $\iota : (\mathbb{P} * \dot{\mathbb{Q}}) * \dot{\mathbb{R}} \to \mathbb{P} * (\dot{\mathbb{Q}} * \dot{\mathbb{R}}_*)$, $((p, \dot{q}), \dot{r}) \mapsto (p, \langle \dot{q}, \dot{r}_* \rangle)$ is an order preserving map, and such that $\iota[(\mathbb{P} * \dot{\mathbb{Q}}) * \dot{\mathbb{R}}]$ is dense in $\mathbb{P} * (\dot{\mathbb{Q}} * \dot{\mathbb{R}}_*)$. In particular, we get that $(\mathbb{P} * \dot{\mathbb{Q}}) * \dot{\mathbb{R}}$ and $\mathbb{P} * (\dot{\mathbb{Q}} * \dot{\mathbb{R}}_*)$ are forcing equivalent.

3 Countable support iterations and proper forcing

In this chapter we discuss iterations of transfinite length. In particular, we examine so called *countable support iterations*. To make these iterations well-behaved, one usually imposes additional assumptions on the iterants. In our case we will use *proper* forcing notions as iterants. Our discussion follows Chapter III of Shelah [8].

Before we give a precise definition of countable support iterations we explain the motivation behind this definition. Given a forcing notion $\mathbb{P}_1$ and a $\mathbb{P}_1$-name $\dot{\mathbb{Q}}_1$ for a forcing notion, we can set $\mathbb{P}_2 = \mathbb{P}_1 * \dot{\mathbb{Q}}_1$ and continue iterating recursively, by picking for each natural number $n \geq 2$, a $\mathbb{P}_n$-name $\dot{\mathbb{Q}}_n$ and setting $\mathbb{P}_{n+1} = \mathbb{P}_n * \dot{\mathbb{Q}}_n$. By identifying each $p \in \mathbb{P}_n$ with $(p, \dot{1}) \in \mathbb{P}_{n+1}$ we can tread $\mathbb{P}_n$ as a subset of $\mathbb{P}_{n+1}$, and and we obtain a forcing notion $\mathbb{P}_\omega$ as the union over the increasing chain $\mathbb{P}_1 \subseteq \mathbb{P}_2 \subseteq \mathbb{P}_2 \cdots$. This construction illustrates the basic idea behind a so called *finite support iteration* of length ω. Naturally, we can continue the same process to obtain an iteration of length $\alpha > \omega$. In the corresponding forcing notion $\mathbb{P}_\alpha$, any condition p can be encoded as a sequence of length α, where for each $i < \alpha$, either $p(i)$ is the trivial name $\dot{1}$ for the greatest element of $\dot{\mathbb{Q}}_i$, or $p(i)$ is some non-trivial $\mathbb{P}_i$-name for an element of $\dot{\mathbb{Q}}_i$. Then, $\mathbb{P}_\alpha$ has finite support in the sense that, for each $p \in \mathbb{P}_\alpha$, $p(i)$ is non-trivial for only finitely many i. Finite support iterations have the drawback that we cannot construct "limit conditions", in the following way: Given a sequence $\{p_n \mid n \in \omega\}$ of conditions in $\mathbb{P}_\alpha$ such that $p_{n+1}(i) = p_n(i)$ whenever $p_n(i)$ is non-trivial, let p_ω be such that $\forall i < \alpha\ (p_\omega(i) = p_n(i)$ for all but finitely many $n)$. The problem is that such a p_ω might need infinitely many non-trivial components, and hence we cannot guarantee that p_ω is an element of $\mathbb{P}_\alpha$. To ensure the existence of such "limit conditions", it is natural to consider the following:

Definition 3.1. Assume α is an ordinal. A *countable support iteration* of length α is a forcing notion $\mathbb{P}_\alpha$, sometimes called the *limit* of the iteration, together with a sequence $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_i \mid i < \alpha \rangle$ such that the following hold:

(a) For $i < \alpha$, $\mathbb{P}_i$ is a forcing notion, $\dot{\mathbb{Q}}_i$ is a $\mathbb{P}_i$-name for a forcing notion, and we have a fixed canonical name $\dot{1}_i$ for a greatest element of $\dot{\mathbb{Q}}_i$;
(b) For $i \leq \alpha$, $\mathbb{P}_i$ contains precisely those functions f with domain i for which $\{j \mid f(j) \neq \dot{1}_j\}$ is countable and $f(j)$ is a canonical $\mathbb{P}_j$-name such that $1 \Vdash_{\mathbb{P}_j} f(j) \in \dot{\mathbb{Q}}_j$, for each $j < i$;[8]
(c) For $i \leq \alpha$, the preorder $\leq_i$ on $\mathbb{P}_i$ is such that

$$f \leq_i g \Leftrightarrow \forall j < i\ (f \restriction j \Vdash_{\mathbb{P}_j} f(j) \leq_{\dot{\mathbb{Q}}_j} g(j)).$$

The set $\mathrm{supp}(f) = \{j \mid f(j) \neq \dot{1}_j\}$ is called the *support* of f.

Remark 3.2.

(1) The forcing notion $\mathbb{P}_0 = \{\emptyset\}$ is always (isomorphic to) the trivial forcing notion.
(2) For $i \leq j$, we have a canonical embedding $e_{ij} : \mathbb{P}_i \hookrightarrow \mathbb{P}_j$ which is given by $e_{ij}(f)(k) = f(k)$ whenever $k < i$ and by $e_{ij}(f)(k) = \dot{1}_k$ otherwise, and hence we can treat $\mathbb{P}_i$ as a subset of $\mathbb{P}_j$. In particular, we will often say "f is in $\mathbb{P}_j$" for some $f \in \mathbb{P}_i$, instead of explicitly invoking e_{ij}.
(3) As e_{ij} is an embedding, it does not matter whether we compare conditions of $\mathbb{P}_i$ in $\mathbb{P}_i$ or in $\mathbb{P}_j$. Hence, we usually drop the subscript and simply use the symbol $\leq$.
(4) The forcing notion $\mathbb{P}_\alpha$ can be defined from $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_i \mid i < \alpha \rangle$, and more generally $\mathbb{P}_j$ can be defined from $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_i \mid i < j \rangle$ for any $j \leq \alpha$. Hence, the

[8] To make sure that we do not end up with a proper class, we have restricted ourselves to canonical names, as we did for two step iterations. Again, we usually disregard mentioning that the names for conditions need to be canonical in the upcoming proofs. Instead, we silently invoke 2.13.1.

sequence of names $\langle \dot{\mathbb{Q}}_i \mid i \in \alpha \rangle$ suffices to reconstruct $\mathbb{P}_j$ recursively for any $j \leq \alpha$.

We can rigorously define finite support iterations in the same way we defined countable support iterations, by requiring $\operatorname{supp} f$ to be finite instead of countable. Similarly, we can define I-support iterations for an arbitrary ideal $I \subseteq \mathcal{P}(\alpha)$, by requiring $\operatorname{supp} f \in I$. Then, we get the following general associativity property:

Proposition 3.3. *Assume $I \subseteq \mathcal{P}(\alpha)$ is an ideal that contains all singletons, and $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j \mid i \leq \alpha,\ j < \alpha \rangle$. Let $\xi < \alpha$ be an ordinal, and let $\dot{J}$ be a $\mathbb{P}_\xi$-name for the ideal generated by closing the set $I \restriction (\alpha \setminus \xi) = \{x \cap (\alpha \setminus \xi) \mid x \in I\}$ with respect to subsets. Then, there is a $\mathbb{P}_\xi$-name $\dot{\mathbb{R}}_\beta$ for a limit of a $\dot{J}$-support iteration of length β and a dense embedding $\iota : \mathbb{P}_\alpha \to \mathbb{P}_\xi * \dot{\mathbb{R}}_\beta$, where β is the unique ordinal which is order isomorphic to $\alpha \setminus \xi$ (i.e. β is the order type of $\alpha \setminus \xi$).*

We do not need the above proposition in any of the upcoming proofs. Hence we do not give a proof of the proposition, and instead refer the interested reader to Eskew [7], Lemma 3.31.

Remark 3.4. Proposition 3.3 hints at one difficulty with countable support iterations. In the above, if I is the ideal of countable subsets of α, $\dot{J}$ is not necessarily a name for the ideal of countable subsets of $\alpha \setminus \xi$ in the extension, since forcing with $\mathbb{P}_\xi$ might add new countable subsets of $\alpha \setminus \xi$. Therefore, $\dot{\mathbb{R}}_\beta$ does not necessarily correspond to the limit of a countable support iteration. In the case of finite support iterations, we get the associativity property, without running into the same problem, as forcing does not add finite subsets of $\alpha \setminus \xi$.
A second complication with countable support iterations is the preservation of ω_1. Usually, we want to guarantee that ω_1^V is still a cardinal in the forcing extension, thereby making sure that $\omega_1^V = \omega_1^{V[G]}$. For finite support iterations, everything works out nicely if we iterate forcing notions that have the *countable chain condition* (ccc), i.e. forcing notions that have no uncountable antichains. In fact, the limit of any finite support iteration of ccc forcing notions has the ccc. Every ccc forcing notion preserves all cardinals, and hence forcing with the limit of a finite support iteration of ccc forcing notions preserves all cardinals.[9] In the case of countable support, this approach already fails for infinite products of forcing notions, let alone infinite iterations. In particular, any countable support product with infinitely many factors, which do not have trivial separative quotient, does not have the ccc. Hence, for countable support, we need a different approach to

[9]A proof of the fact that ccc forcing notions preserve all cardinals can be found in Jech [6], Theorem 14.34. Shelah [8] discusses finite support iterations in §2 of Chapter 2. In particular, Shelah's discussion contains a proof of the fact that a limit of a finite support iteration of ccc forcing notions has the ccc.

make the iterations well-behaved. A common solution is *proper forcing*, which we discuss in the rest of this section.

3.1 Clubs, stationary sets and properness

To define properness, we first need to discuss *closed unbounded* (club) sets and stationary sets. Originally, a closed unbounded subset of some limit ordinal α is a set $C \subseteq \alpha$, such that C is unbounded in the sense that there is no $i < \alpha$ such $j \leq i$ holds for all $j \in C$, and such that C is closed in the sense that $\bigcup B$ is in C for all $B \subseteq C$ with $\bigcup B < \alpha$. Note that instead of requiring that C is unbounded, we can equivalently require that C is cofinal in the sense that for each $i < \alpha$ there is $j \in C$ such that $i \leq j$. Furthermore, in the case that α is ω_1, instead of requiring that C is closed in the above sense, we can equivalently require that for each $\leq$-increasing ω-sequence $\{i_n \mid n \in \omega\} \subseteq C$ we get $\bigcup\{i_n \mid n \in \omega\} \in C$. This leads us to the following generalisation which is due to Jech:

Definition 3.5. Assume X is a set. A set $C \subseteq [X]^{\leq\omega}$ is said to be *club* in $[X]^{\leq\omega}$ if the following hold:

(a) C is closed under $\subseteq$-increasing ω-sequences, i.e.

$$\forall \{c_n \mid n \in \omega\} \subseteq C \ (\forall n \ (c_n \subseteq c_{n+1}) \Rightarrow \bigcup\{c_n \mid n \in \omega\} \in C);$$

(b) C is cofinal in $[X]^{\leq\omega}$, i.e.

$$\forall s \in [X]^{\leq\omega} \exists c \in C \ (s \subseteq c).$$

Remark 3.6. Note that clubs in $[X]^{\leq\omega}$ are always non-empty, even if X is empty. Furthermore, if X is any countable set, then $\{X\}$ is club in $[X]^{\leq\omega}$ and every set which is club in $[X]^{\leq\omega}$ contains X.

The following proposition shows how clubs in $[X]^{\leq\omega}$ can be understood as a generalisation of clubs in the original sense.

Proposition 3.7. *A subset of ω_1 is club in $[\omega_1]^{\leq\omega}$ if and only if it is club in ω_1.*

Proof. Fix a set $C \subseteq \omega_1$. First, assume that C is club in ω_1. Then, C is closed under $\subseteq$-increasing ω-sequences, as the order $\leq$ on ordinals is the subset relation. Thus, it suffices to show that C is cofinal in $[\omega_1]^{\leq\omega}$. Consider $s \in [\omega_1]^{\leq\omega}$ and define $\alpha = \bigcup\{\beta + 1 \mid \beta \in s\}$. As α is a countable union of countable ordinals, it is a countable ordinal, and since $\beta \in \beta + 1$ holds for every ordinal β, s is a subset of α. Now, $\alpha \in \omega_1$, and since C is unbounded in ω_1, there is some $\gamma \in C$, such that $\alpha \subseteq \gamma$ and thus $s \subseteq \gamma$. Since s was arbitrary, C is cofinal in $[\omega_1]^{\leq\omega}$. Now, assume $C \subseteq \omega_1$ is club in $[\omega_1]^{\leq\omega}$. Analogously to the first paragraph, it suffices to show that C is $\leq$-unbounded in ω_1. However, this follows directly from the fact that C is $\subseteq$-cofinal in a superset of ω_1, namely in $[\omega_1]^{\leq\omega}$. $\square$

Sets that have a club subset are often compared to the subsets of the unit interval $[0, 1]$ that have Lebesgue measure 1. This is to say, that clubs (and their supersets) can be understood as "large" sets. More precisely, we have the following:

Proposition 3.8. *The intersection of ω many clubs in $[X]^{\leq\omega}$ is club in $[X]^{\leq\omega}$.*

Proof. Let $\{C_n \mid n \in \omega\}$ be a set of clubs in $[X]^{\leq\omega}$. To see that the intersection $C = \bigcap\{C_n \mid n \in \omega\}$ is cofinal, consider some $s \in [X]^{\leq\omega}$ and recursively define a function $f : \omega \times \omega \to [X]^{\leq\omega}$ such that:

$$\begin{aligned}
&f(i, j) \in C_j \text{ for all } i, j,\\
&f(0, 0) \supseteq s,\\
&f(i, j + 1) \supseteq f(i, j) \text{ for all } i, j,\\
&f(i + 1, 0) \supseteq \bigcup\{f(i, k) \mid k \in \omega\} \text{ for all } i.
\end{aligned}$$

Note that, for every n, $\{f(k, n) \mid k \in \omega\}$ is an increasing sequence in C_n. Hence, $c_n = \bigcup\{f(k, n) \mid k \in \omega\}$ is an element of C_n, since C_n is closed under increasing ω-sequences. Furthermore, $c_m \subseteq c_n$ holds for all m, n, since $f(k, m) \subseteq f(k+1, n)$ holds for all k, m, n. Thus, $c_m = c_n$ must hold for all m, n. Hence, we see that c_0 is an element of C_n, for every n, and thus c_0 is in C. Since $s \subseteq c_0$ and s was arbitrary, it follows that C is cofinal. To see that C is closed under increasing ω-sequences, note that $S \subseteq C$, implies $S \subseteq C_n$ for all n. Then, if $\bigcup S \in C_n$ holds for all n, then $\bigcup S \in C$. In particular, this holds whenever S is an increasing ω-sequence. $\square$

Remark 3.9. It is an immediate consequence of the above lemma, that the set $\{C \mid C \text{ is club in } [X]^{\leq\omega}\}$ is $\subseteq$-directed, i.e. for every two clubs A, B there is a club $C \subseteq A, B$. In fact, whenever A and B are clubs, also $C = A \cap B$ is club. Hence, $\mathcal{D}(X) = \{A \subseteq [X]^{\leq\omega} \mid \exists C \subseteq A\ (C \text{ is club in } [X]^{\leq\omega})\}$ is a filter for the powerset of $[X]^{\leq\omega}$ with respect to the partial order $\subseteq$. The filter $\mathcal{D}(X)$ is sometimes referred to as the *club filter* (of $[X]^{\leq\omega}$), and the above lemma is often phrased as "the club filter is countably closed". This justifies the aforementioned comparison to sets of measure 1, as these sets also form a countably closed filter.

Next, we introduce a very versatile technique of constructing clubs.

Definition 3.10. Let X be a set.

1. A *partial algebra on* X is a set $\mathcal{A}$ of functions, such that for every $f \in \mathcal{A}$, we have $\operatorname{range}(f) \subseteq X$ and there exists a natural number n, such that $\operatorname{dom}(f) \subseteq X^n$ and.
2. An *algebra on* X is a partial algebra $\mathcal{A}$, such that for every $f \in \mathcal{A}$, there is a natural number n, such that $\operatorname{dom}(f) = X^n$.
3. For a partial algebra $\mathcal{A}$ on X, we call a subset $Y \subseteq X$ closed under $\mathcal{A}$, if for each natural number n and every n-place function $f \in \mathcal{A}$, the set $f[Y] = \{f(y_0, ..., y_{n-1}) \mid (y_0, ..., y_{n-1}) \in Y^n \cap \operatorname{dom}(f)\}$ is a subset of Y.
4. For a partial algebra $\mathcal{A}$ on X, denote by $\operatorname{club}(\mathcal{A})$ the set of all countable subsets of X that are closed under $\mathcal{A}$.

Proposition 3.11. *Assume X is an uncountable set and $\mathcal{A}$ is a partial algebra on X, such that $|\mathcal{A}| \leq \omega$. Then* $\operatorname{club}(\mathcal{A})$ *is club in* $[X]^{\leq\omega}$.

Proof. First, note that the union over an increasing ω-sequence of sets that are closed under $\mathcal{A}$ is closed under $\mathcal{A}$. Thus, for an increasing ω-sequence of countable sets that are closed under $\mathcal{A}$ the union is countable and closed under $\mathcal{A}$. Therefore, $\operatorname{club}(\mathcal{A})$ is closed under increasing ω-sequences.
To see that $\operatorname{club}(\mathcal{A})$ is cofinal, fix $Y \in [X]^{\leq\omega}$ and recursively define a sequence $(Y_n)_{n\in\omega}$ by:

$$
\begin{aligned}
&Y_0 = Y \\
&Y_{n+1} = Y_n \cup \bigcup\{f[Y_n] \mid f \in \mathcal{A}\}, \text{ for } n \in \omega.
\end{aligned}
$$

Then, since $\mathcal{A}$ is countable, a simple induction argument shows that Y_n is countable, for every $n \in \omega$. Thus, also $Y_\omega = \bigcup_{n\in\omega} Y_n$ is countable. Now, since $f[Y_n] \subseteq Y_{n+1}$ for each $n \in \omega$ and every $f \in \mathcal{A}$, it follows that $Y_\omega \in \operatorname{club}(\mathcal{A})$. Clearly, also $Y \subseteq Y_\omega$ holds. And since Y was arbitrary, this shows that $\operatorname{club}(\mathcal{A})$ is cofinal in $[X]^{\leq\omega}$. □

Remark 3.12. The set Y_ω in the above proof is called the *closure of* Y *under* $\mathcal{A}$.

By definition the filter $\mathcal{D}(X)$ is generated from the set of all clubs by closing under supersets. The next lemma shows that the sets of the form $\operatorname{club}(\mathcal{A})$, with $\mathcal{A}$ a countable algebra on X, suffice to generate $\mathcal{D}(X)$.

Lemma 3.13. *Let X be an uncountable set. For every C which is club in* $[X]^{\leq\omega}$ *there is a countable algebra $\mathcal{A}$ on X such that* $\operatorname{club}(\mathcal{A}) \subseteq C$.

Proof. Fix a club $C \subseteq [X]^{\leq\omega}$ and, by induction on length of the argument, define a function $f : X^{<\omega} \to C \setminus \{\emptyset\}$, such that $f(x_0, ..., x_n) \supseteq \{x_0, ..., x_n\} \cup f(x_0, ..., x_{n-1})$, where $(x_0, ..., x_{n-1})$ is the empty sequence when $n = 0$. Clearly, this is possible, since C is cofinal in $[X]^{\leq\omega}$. Now, for every pair of natural numbers $(l, n) \in \omega^2$ choose a function $F^n_l : X^n \to X$ such that $f(x_0, ..., x_{n-1}) = \{F^n_l(x_0, ..., x_{n-1}) \mid l \in \omega\}$ holds for every $n \in \omega$ and all $x_0, ..., x_{n-1} \in X$. Then, $\mathcal{A} = \{F^n_l \mid l, n \in \omega\}$ is an algebra on X, and hence club$(\mathcal{A})$ is club. We claim that club$(\mathcal{A})$ is a subset of C. To see this, consider a countable set $Y = \{y_0, y_1, ...\} \subseteq X$ that is closed under $\mathcal{A}$. Denote $s_n = f(y_0, ..., y_{n-1}) = \{F^n_l(y_0, ..., y_{n-1}) \mid l \in \omega\}$. Note that $s_n \subseteq Y$ holds for all n, since Y is closed under $\mathcal{A}$. By definition of f, also $\{y_0, ..., y_{n-1}\} \subseteq s_n$ holds for every n. Thus, $Y = \bigcup_{n\in\omega} s_n$. Furthermore, it follows from the definition of f that s_n is an element of C and $s_n \subseteq s_{n+1}$ hold for all n. Therefore, Y is the union of an increasing ω-sequence of elements of C, and thus $Y \in C$. This works for any countable set $Y \subseteq X$ which is closed under $\mathcal{A}$. Therefore, club$(\mathcal{A})$ is a subset of C. □

The main goal of this section is the introduction of a suitable replacement for the countable chain condition, which will involve the club filter. Hence, it is of interest to us, how the club filter behaves under ccc forcing. If $\mathbb{P}$ is a ccc forcing notion, $G \subseteq \mathbb{P}$ is a generic filter and λ is an uncountable cardinal, then $\mathcal{D}(\lambda)^{V[G]}$ is generated by $\mathcal{D}(\lambda)^V$. This follows directly from the following:

Proposition 3.14. *Assume $\mathbb{P}$ is a ccc forcing notion, $G \subseteq \mathbb{P}$ is a generic filter and λ is an uncountable cardinal. Then, for every $A \in \mathcal{D}(\lambda)^{V[G]}$ there is $B \in \mathcal{D}(\lambda)^V$, such that B is a subset of A.*

Proof. Let A be in $\mathcal{D}^{V[G]}$. Then, by the previous lemma, there is an algebra $\mathcal{A}$ in $V[G]$, such that $V[G] \models \text{club}(\mathcal{A}) \subseteq A \ \wedge\ |\mathcal{A}| \leq \omega$. Note that if we have $\mathcal{A} = \emptyset$, then A must be equal to $[\lambda]^{\leq\omega}$, and we have $A \in V$. Hence, it suffices to consider the case where $\mathcal{A}$ is non-empty. Then, in $V[G]$ there is a function such that $\text{dom}(f) = \omega$ and $\{f(n) \mid n < \omega\} = \mathcal{A}$. Take a $\mathbb{P}$-name $\dot{f}$ for a function from ω to $\bigcup_{k\in\omega} \lambda^{\lambda^k}$ such that $\dot{f}^G = f$. By existential completeness, for each n there is a $\mathbb{P}$-name $\dot{k}_n$ for a natural number and a $\mathbb{P}$-name τ_n for a function from $\lambda^{\dot{k}_n}$ to λ such that $1 \Vdash \dot{f}(n) = \tau_n$. As $\mathbb{P}$ is ccc, for any $k < \omega$ and $\alpha_0, ..., \alpha_{k-1} < \lambda$ the set $\{\beta \in \lambda \mid \exists p \in \mathbb{P}\ (p \Vdash \dot{k}_n = k \ \wedge\ \tau_n(\alpha_0, ..., \alpha_{k-1}) = \beta)\}$ is countable. To see this fix $k < \omega$ as well as $\alpha_0, ..., \alpha_{k-1} < \lambda$ and, for each $\beta < \lambda$ for which there exists $p \in \mathbb{P}$ with $p \Vdash \dot{k}_n = k \ \wedge\ \tau_n(\alpha_0, ..., \alpha_{k-1}) = \beta$ pick p_β such that $p_\beta \Vdash \dot{k}_n = k \ \wedge\ \tau_n(\alpha_0, ..., \alpha_{k-1}) = \beta$. Note that the p_β is different from $p_{\beta'}$ whenever β is different from β'. Furthermore, the set of all the p_β is an antichain and hence countable. Therefore, for all $k, n < \omega$ and all $\alpha_0, ..., \alpha_{k-1} < \lambda$, the set $\{\beta \in \lambda \mid \exists p \in \mathbb{P}\ (p \Vdash \dot{k}_n = k\ \wedge\ \tau_n(\alpha_0, ..., \alpha_{k-1}))\}$ is countable. Then, for every pair (n, k), there is in V a set $\{g^l_{(n,k)} \mid l < \omega\}$ of partial functions from λ^k to λ, such that $\{\beta \in \lambda \mid \exists p \in \mathbb{P}\ (p \Vdash \dot{k}_n = k\ \wedge\ \tau_n(\alpha_0, ..., \alpha_{k-1}))\} = \{g^l_{(n,k)}(\alpha_0, ..., \alpha_{k-1}) \mid l \in \omega\}$ holds for all $\alpha_0, ..., \alpha_{k-1} \in \lambda$. Set $\mathcal{B} = \{g^l_{(n,k)} \mid k, l, n \in \omega\}$ and note that $\mathcal{B}$ is a partial algebra on λ in V. Clearly, $\mathrm{club}(\mathcal{B}) \subseteq \mathrm{club}(\mathcal{A})$ holds in $V[G]$, as any set that is closed under $\mathcal{B}$ is closed under $\mathcal{A}$. Since the property of being closed under $\mathcal{B}$ is absolute between V and $V[G]$, also $\mathrm{club}(\mathcal{B})^V \subseteq \mathrm{club}(\mathcal{B})^{V[G]}$ holds. Setting $B = \mathrm{club}(\mathcal{B})^V$, we have $B \subseteq A$ and $B \in \mathcal{D}(\lambda)^V$. $\square$

Before we can define properness we need one more auxiliary definition:

Definition 3.15. A set $S \subseteq [X]^{\leq\omega}$ is called *stationary* if $S \cap C$ is non-empty for every club $C \subseteq [X]^{\leq\omega}$.

Remark 3.16. In the aforementioned analogy between the club filer and the filter of measure-1 subsets of $[0, 1]$, the analogue of the stationary sets are the subsets of $[0, 1]$ that do not have measure 0, since a set has measure 0 if and only if it is included in the complement of a measure-1 set.

An important consequence of the above proposition, is the fact that any stationary set S in V, also meets all clubs in any ccc forcing extension, i.e. S will still be stationary in $V[G]$. It turns out that we can deduce the fact that ccc forcing notions preserve ω_1 already from this preservation of stationary sets. This is at least one of the motivations for the following definition:

Definition 3.17. A forcing notion $\mathbb{P}$ is said to be *proper* if for every uncountable cardinal λ forcing with $\mathbb{P}$ preserves stationarity of subsets of $[\lambda]^{\leq\omega}$ i.e. for every stationary $S \subseteq [\lambda]^{\leq\omega}$ in V, $1 \Vdash$ "S is stationary" holds.

Corollary 3.18. *Every ccc forcing notion is proper.*

Proof. As mentioned above, this is a direct consequence of proposition 3.14. $\square$

The fact that proper forcing notions preserve ω_1 follows from the next proposition:

Proposition 3.19. *Assume that $\mathbb{P}$ is a proper forcing notion and G is a $\mathbb{P}$-generic filter. Then, for every countable set $A \subseteq Ord$ in $V[G]$, there is a countable set $B \subseteq Ord$ in V such that $A \subseteq B$.*

Proof. Assume that A is a countable set of ordinals in $V[G]$ and let λ be the least ordinal such that A is a subset of λ. If λ is countable, simply set $B = \lambda$. Otherwise, λ is uncountable and $C = \{s \in [\lambda]^{\leq\omega} \mid A \subseteq s\}$ is club in $([\lambda]^{\leq\omega})^{V[G]}$, as C is non-empty and upward closed in $([\lambda]^{\leq\omega})^{V[G]}$. Since $([\lambda]^{\leq\omega})^{V}$ is stationary in V and $\mathbb{P}$ is proper, $([\lambda]^{\leq\omega})^{V}$ is stationary in $V[G]$ and we find $B \in ([\lambda]^{\leq\omega})^{V} \cap C$. Then, B is a countable set in V and A is a subset of B. □

Corollary 3.20. *Assume that $\mathbb{P}$ is a proper forcing notion. Then, $\omega_1^V = \omega_1^{V[G]}$ holds for any generic filter $G \subseteq \mathbb{P}$.*

Proof. Assume for the sake of contradiction that there is some generic filter $G \subseteq \mathbb{P}$ such that ω_1^V is countable in $V[G]$. By Proposition 3.19, ω_1^V is included in a countable set in V, and conclude that ω_1^V is countable in V, which is a contradiction. For any generic filter $G \subseteq \mathbb{P}$, the extension $V[G]$ has the same ordinals as the ground model V, and any ordinal strictly below ω_1^V is already countable in V. Hence, ω_1^V is still the least uncountable ordinal in $V[G]$. □

3.2 Characterising properness through $\lambda \times \lambda$-matrices

Before we discuss countable support iterations of proper forcing notions, we present a few equivalent ways to define properness. To this end, we fix a forcing notion $\mathbb{P}$. In this subsection, the main task is to characterise when $\mathbb{P}$ is proper, using $\lambda \times \lambda$-matrices with components in $\mathbb{P}$ and with each row a predense set, where λ ranges over all uncountable cardinals. More precisely, we consider functions $F : \lambda \to \mathbb{P}^\lambda$, and we say that F has *predense rows* if $F(i)$ is predense for all i. For $F : \lambda \to \mathbb{P}^\lambda$ with predense rows, and $p \in \mathbb{P}$ we consider the following set:

$$W_p(F) = \{s \in [\lambda]^{\leq\omega} \mid \exists q \leq p \forall i \in s \ (F(i)[s] \text{ is predense below } q)\},$$

where $F(i)[s]$ denotes the image of s under $F(i)$. Then, we want to compare the following two conditions for each uncountable cardinal λ:

$$Con_1(\lambda): \quad \forall \text{ stationary } S \subseteq [\lambda]^{\leq\omega} \ (1 \Vdash \text{“}S \text{ is stationary”});$$

$$Con_2(\lambda): \quad \forall F : \lambda \to \mathbb{P}^\lambda \text{ with predense rows } \forall p \in \mathbb{P} \ (W_p(F) \in \mathcal{D}(\lambda)).$$

If λ is countable, we take both $Con_1(\lambda)$ and $Con_2(\lambda)$ to be the trivially true statement. Note that $\mathbb{P}$ is proper if and only if $Con_1(\lambda)$ holds for all λ. Furthermore, we have the following two propositions:

Proposition 3.21. *For every uncountable cardinal λ, $Con_1(\lambda)$ implies $Con_2(\lambda)$.*

Proof. Assume that $Con_2(\lambda)$ does not hold, for some uncountable cardinal λ. Hence, there is a condition $p \in \mathbb{P}$ and a function $F : \lambda \to \mathbb{P}^\lambda$ with predense rows such that $W_p(F)$ is not in $\mathcal{D}(\lambda)$. Then, $S = [\lambda]^{\leq\omega} \setminus W_p(F)$ is stationary. Let G be a filter containing p. Note that, since G meets every predense set in V, there exists a function $g : \lambda \to \lambda$ in $V[G]$ such that $F(i)(g(i)) \in G$ holds for all $i \in \lambda$. The set containing only this g is an algebra on λ in $V[G]$. Let $C = \mathrm{club}(\{g\})$ be the associated club in $V[G]$. Consider any $s \in C \cap V$. Note that s is closed under g. Thus, there exists a condition $r \in G$ such that $r \Vdash \forall i \in s \exists j \in s\ (F(i)(j) \in \dot{G})$. Take $q \in G$, such that $q \leq r, p$. Then, also $q \Vdash \forall i \in s \exists j \in s\ (F(i)(j) \in \dot{G})$ must hold. Therefore, $F(i)[s]$ is predense below q, for all $i \in s$. To see this, assume toward contradiction that, for some $i \in s$, $F(i)[s]$ is not predense below q. Then, there is $q' \leq q$, such that q' is incompatible with $F(i)(j)$, for all $j \in s$. Consider a generic filter G' containing q'. Now, there cannot exist a $j \in s$, such that $F(i)(j) \in G'$. However, the existence of such a j is forced by q'. Since this is a contradiction, $F(i)[s]$ must be predense below q, for all $i \in s$. Then, s is in $W_p(F)$ and not in S. Since $s \in C \cap V$ was arbitrary and S is in V, it follows that S does not meet C. Thus, S is not stationary in $V[G]$, and we see that $Con_1(\lambda)$ does not hold. Since this works for any uncountable λ, the claim of the theorem follows. $\square$

Proposition 3.22. *For every uncountable cardinal $\lambda \geq |\mathbb{P}|$, $Con_2(\lambda)$ implies $Con_1(\lambda)$.*

Proof. Assume $Con_2(\lambda)$ holds, let S be a stationary subset of $[\lambda]^{\leq\omega}$ in V, and let $G \subseteq \mathbb{P}$ be a generic filter. We will show that S is stationary in $V[G]$. For each $n \in \omega$, let $\dot{k}_n$ be a name for a natural number and let τ_n be a name for a function from $\lambda^{\dot{k}_n}$ to λ. Note that $\{\tau_n \mid n < \omega\}$ is forced to be a countable algebra on λ. To show that $\mathrm{club}(\{\tau_n \mid n < \omega\})$ is forced to meet S, we define a countable algebra $\mathcal{A}$ on λ and a function $F : \lambda \to \mathbb{P}^\lambda$ with predense rows.
Fix a bijection $h : \lambda^{<\omega} \to \lambda$. For each $i < \lambda$, if $h^{-1}(i) = (n, \alpha_1, ..., \alpha_k)$ with $n, k \in \omega$ and $\alpha_1, ..., \alpha_k \in \lambda$, then let $A_i \subseteq \mathbb{P}$ be a maximal antichain, such that each member of A_i forces a value on $\tau_n(\alpha_0, ..., \alpha_{k-1})$. More precisely, for each $r \in A_i$, either $r \Vdash \dot{k}_n \neq k$, or there exists $\beta_{i,r} < \lambda$ such that $r \Vdash \tau_n(\alpha_1, ..., \alpha_k) = \beta_{i,r}$. If $h^{-1}(i)$ is the empty sequence or if the 0-th entry of $h^{-1}(i)$ is not a natural number, take A_i to be any predense subset of $\mathbb{P}$. For each $i < \lambda$, take $F(i) : \lambda \to \mathbb{P}$ enumerating A_i, that is $\mathrm{range}(F(i)) = A_i$. Note that F is a function from λ to $\mathbb{P}^\lambda$, and F has predense rows. Define the partial function g from λ^2 to λ by $g(i, j) = \beta_{i,F(i)(j)}$. Then, set $\mathcal{A} = \{h^n, \overline{n} \mid n \in \omega\} \cup \{g\}$, where

$h^n = h \restriction \lambda^n$ and $\overline{n}$ is the function that sends the empty sequence to n. Note that $\mathcal{A}$ is a countable partial algebra on λ.
Fix some $p \in \mathbb{P}$ and note that, by $Con_2(\lambda)$, the set $W_p(F)$ is in $\mathcal{D}(\lambda)$. Hence, we find some $s \in S \cap W_p(F) \cap \text{club}(\mathcal{A})$, as $W_p(F) \cap \text{club}(\mathcal{A})$ is in $\mathcal{D}(\lambda)$ and S is stationary. Since s is in $W_p(F)$, there is $q \leq p$ such that $F(i)[s]$ is predense below q, for every $i \in s$.

Claim: The condition q forces that s is in $\text{club}(\{\tau_n \mid n < \omega\})$.

Proof of Claim: Assume k, n are two natural numbers and $\alpha_1, ..., \alpha_k$ are in s. As s is in $\text{club}(\mathcal{A})$, we see that s contains all natural numbers, and thus n is in s. Furthermore, since s is closed under h^{k+1}, also $i = h(n, \alpha_1, ..., \alpha_k)$ is in s. For every $j < \lambda$, we have $F(i)(j) \Vdash \tau_n(\alpha_1, ..., \alpha_k) = g(i,j)$ or $F(i)(j) \Vdash \dot{k}_n \neq k$. Since s is closed under g, if j is in s and (i,j) is in $\text{dom}(g)$, then $g(i,j)$ is in s. Note that any generic filter that contains q has to contain $F(i)(j)$ for some $j \in s$, since $F(i)[s]$ is predense below q. As k, n and $\alpha_1, ..., \alpha_k$ were arbitrary, it follows that, for any generic filter $G \subseteq \mathbb{P}$ that contains q, in $V[G]$ the set s is closed under the algebra $\{\tau_n^G \mid n < \omega\}$. This concludes the proof of the claim.

As q was such that $q \leq p$ and p was arbitrary, we have that the set of $q \in \mathbb{P}$ such that $q \Vdash S \cap \text{club}(\{\tau_n \mid n < \omega\}) \neq \emptyset$ is dense. Therefore, it is forced that S meets $\text{club}(\{\tau_n \mid n < \omega\})$. Finally, note that for any club $C \subseteq [\lambda]^{\leq\omega}$ in any generic extension $V[G]$, by Lemma 3.13 there is a countable algebra $\mathcal{B}$ in $V[G]$ such that the corresponding club is included in C. If $\mathcal{B}$ is non-empty, we find appropriate $\{\tau_n \mid n < \omega\}$ such that $\{\tau_n^G \mid n < \omega\} = \mathcal{B}$, and the above argument shows that $\text{club}(\mathcal{B})$ meets S. Otherwise, $\mathcal{B}$ is empty and C must be equal to $([\lambda]^{\leq\omega})^{V[G]}$. In either case $C \cap S$ is non-empty. As C and S were arbitrary, $Con_1(\lambda)$ holds. □

We will soon see that $\lambda > |\mathbb{P}|$, does not really pose a restriction. First, we have two lemmas using the following notation: For two sets A, B, we denote the set $\{a \cup b \mid a \in A,\ b \in B\}$ by $A \overline{\cup} B$.

Lemma 3.23. *Assume that X, Y are two uncountable sets, such that $X \cap Y = \emptyset$. Then, a subset $W \subseteq A$ is club (resp. stationary) in $[X]^{\leq\omega}$ if and only if $W \overline{\cup} [Y]^{\leq\omega}$ is club (resp. stationary) in $[X \cup Y]^{\leq\omega}$.*

Proof. First, assume that W is club in $[X]^{\leq\omega}$ and fix some $s \in [X \cup Y]^{\leq\omega}$. Since $s \cap X$ is a countable subset of X, there is $w \in W$ such that $s \cap X \subseteq w$. Then $s \subseteq w \cup (s \cap Y)$ holds, and since $s \cap Y$ is a countable subset of Y, also $w \cup (s \cap Y) \in W \overline{\cup} [Y]^{\leq\omega}$ holds. As this works for every $s \in [X \cup Y]^{\leq\omega}$, it follows that $W\overline{\cup}[Y]^{\leq\omega}$ is cofinal in $[X \cup Y]^{\leq\omega}$. Now, consider an increasing sequence $\{s_n \mid n \in \omega\} \subseteq W\overline{\cup}[Y]^{\leq\omega}$. Note that $\bigcup_{n\in\omega} s_n = (\bigcup_{n\in\omega} s_n \cap X) \cup (\bigcup_{n\in\omega} s_n \cap Y)$. For any n, since X and Y are disjoint, $s_n \cap X \in W$ holds. Then, since W is closed under increasing ω-sequences, it follows that $\bigcup_{n\in\omega} s_n \cap X$ is in W. As the set $\bigcup_{n\in\omega} s_n \cap Y$ is a countable union of countable sets, it is countable, and thus $\bigcup_{n\in\omega} s_n \cap Y$ is an element of $[Y]^{\leq\omega}$. Therefore, $\bigcup_{n\in\omega} s_n \in W\overline{\cup}[Y]^{\leq\omega}$ holds. As $\{s_n \mid n \in \omega\}$ was arbitrary, $W\overline{\cup}[Y]^{\leq\omega}$ is closed under increasing ω-sequences.

Now, assume that there is an increasing ω-sequence $\{w_n \mid n \in \omega\} \subseteq W$ such that $\bigcup_{n\in\omega} w_n \notin W$. Then, $\{w_n \mid n \in \omega\}$ is an increasing ω-sequence in $W\overline{\cup}[Y]^{\leq\omega}$, and since X and Y are disjoint, $\bigcup_{n\in\omega} w_n$ cannot be an element of $W\overline{\cup}[Y]^{\leq\omega}$. Hence, we see that W is closed under increasing ω-sequences, whenever $W\overline{\cup}[Y]^{\leq\omega}$ is. Similarly, if $W \subseteq [X]^{\leq\omega}$ is not cofinal, there is some $x \in [X]^{\leq\omega}$, such that $x \not\subseteq w$, for all $w \in W$. Then, since X and Y are disjoint, $x \not\subseteq s$ holds for all $s \in W\overline{\cup}[Y]^{\leq\omega}$. Hence, if $W\overline{\cup}[Y]^{\leq\omega}$ is cofinal in $[X \cup Y]^{\leq\omega}$, then W is cofinal in $[X]^{\leq\omega}$.

Next, we consider the case where $W \subseteq [X]^{\leq\omega}$ is stationary. If X is empty, we have $W\overline{\cup}[Y]^{\leq\omega} = [Y]^{\leq\omega} = [X \cup Y]^{\leq\omega}$. Hence, only the case were $X \neq \emptyset$ remains. Fix a countable set of functions $\mathcal{A}$, such that $\mathcal{A}$ is an algebra on $X \cup Y$. Let $\mathcal{B}$ be the closure under finite composition of functions in $\mathcal{A}$, and note that $\mathcal{B}$ is countable. Further note that $\mathrm{club}(\mathcal{A}) = \mathrm{club}(\mathcal{B})$, since a subset of $X \cup Y$ is closed under $\mathcal{A}$ if and only if it is closed under $\mathcal{B}$. To see this, first note that every subset of $X \cup Y$ that is closed under $\mathcal{B}$ is closed under $\mathcal{A}$, as $\mathcal{A}$ is a subset of $\mathcal{B}$. Since every function in $\mathcal{B}$ is a finite composition of functions in $\mathcal{A}$, a straight forward proof by induction on the number of composed functions shows that closedness under $\mathcal{A}$ implies closedness under $\mathcal{B}$. Now, fix $x \in X$, and for each $n \in \omega$ and every n-place function $f \in \mathcal{B}$ define $\tilde{f} : X^n \to X$ as follows:

$$\tilde{f}(x_0, ..., x_{n-1}) = \begin{cases} f(x_0, ..., x_{n-1}) & \text{if } f(x_0, ..., x_{n-1}) \in X, \\ x & \text{otherwise} \end{cases}.$$

Then, let $\tilde{\mathcal{B}} = \{\tilde{f} \mid f \in \mathcal{B}\}$. As W is stationary there is some $s \in W \cap \mathrm{club}(\tilde{\mathcal{B}})$. Let t be the closure of s under $\mathcal{B}$. Then, t is countable and hence $t \in [X \cup Y]^{\leq\omega}$. For $z \in t \setminus s$, since $\mathcal{B}$ is closed under finite compositions, there is some natural number n, some $f \in \mathcal{B}$ and $x_0, ..., x_{n-1} \in s$, such that $z = f(x_0, ..., x_{n-1})$. If z is in X, then $\tilde{f}(x_0, ..., x_{n-1}) = z$, by definition of $\tilde{f}$. However, since $x_0, ..., x_{n-1} \in s$ and $z \notin s$, this contradicts the closedness of s under $\tilde{\mathcal{B}}$. Hence, z must be an element of Y. As z was arbitrary, $t \setminus s$ is a subset of Y. Hence, $t = s \cup (t \setminus s) \in W\overline{\cup}[Y]^{\leq\omega}$. In particular, $W\overline{\cup}[Y]^{\leq\omega}$ meets $\mathrm{club}(\mathcal{B}) = \mathrm{club}(\mathcal{A})$. As $\mathcal{A}$ was arbitrary, $W\overline{\cup}[Y]^{\leq\omega}$

meets club($\mathcal{A}$) for every countable algebra $\mathcal{A}$ on $X \cup Y$. By 3.13, it follows that $W \overline{\cup} [Y]^{\leq\omega}$ is stationary.

Lastly, assume that $W \subseteq [X]^{\leq\omega}$ is not stationary. Then, there is a club $C \subseteq [X]^{\leq\omega} \setminus W$. By the argument in the first paragraph of this proof, $C \overline{\cup} [Y]^{\leq\omega}$ is club in $[X \cup Y]^{\leq\omega}$. As X and Y are disjoint, for any $s \in (C \overline{\cup} [Y]^{\leq\omega}) \cap (W \overline{\cup} [Y]^{\leq\omega})$, the set $s \cap X$ must be an element of $C \cap W$. However, as C is included in the complement of W, $C \cap W$ is empty, and thus $(C \overline{\cup} [Y]^{\leq\omega}) \cap (W \overline{\cup} [Y]^{\leq\omega})$ is empty. Hence, $W \overline{\cup} [Y]^{\leq\omega}$ is not stationary. $\square$

Lemma 3.24. *Assume X, Y are two uncountable sets and $f : X \to Y$ is injective. Then, the following hold:*

1. *$W \subseteq [X]^{\leq\omega}$ is stationary in $[X]^{\leq\omega}$ if and only if $\{s \in [Y]^{\leq\omega} \mid f^{-1}[s] \in W\}$ is stationary in $[Y]^{\leq\omega}$;*
2. *if f is a bijection, f induces an isomorphism $f^* : \mathcal{P}([X]^{\leq\omega}) \to \mathcal{P}([Y]^{\leq\omega})$ of Boolean algebras, defined by $f^*(A) = \{f[s] \mid s \in A\}$, and f^* preserves stationarity;*
3. *if $V' \supseteq V$ is a larger transitive model and $V \models |X| = |Y|$, then there is some $A \subseteq [X]^{\leq\omega}$ which is stationary in V but not stationary in V' if and only if there is some $B \subseteq [Y]^{\leq\omega}$ that is stationary in V but not stationary in V'.*

Proof. 1: Without loss of generality assume that X and Y are disjoint.[10] Note that $\{f\}$ is a partial algebra on $X \cup Y$, and thus we obtain an associated club subset of $[X \cup Y]^{\leq\omega}$.
Assume $W \subseteq [X]^{\leq\omega}$ is stationary and $C \subseteq [Y]^{\leq\omega}$ is club. By the previous lemma, $W \overline{\cup} [Y]^{\leq\omega}$ is stationary and $[X]^{\leq\omega} \overline{\cup} C$ is club. Then, $([X]^{\leq\omega} \overline{\cup} C) \cap \mathrm{club}(\{f\})$ is also club. Thus, there is some t in $(W \overline{\cup} [Y]^{\leq\omega}) \cap ([X]^{\leq\omega} \overline{\cup} C) \cap \mathrm{club}(\{f\})$. As X and Y are disjoint, there is $w \in W$ and $c \in C$ such that t is the disjoint union of w and c. Since t is closed under f, the image $f[w]$ is a subset of $t \cap Y = c$. Hence, we have $f^{-1}[c] = w$, and c is an element of $\{s \in [Y]^{\leq\omega} \mid f^{-1}[s] \in W\}$. This shows that any club $C \subseteq [Y]^{\leq\omega}$ meets $\{s \in [Y]^{\leq\omega} \mid f^{-1}[s] \in W\}$. Hence, we conclude that $\{s \in [Y]^{\leq\omega} \mid f^{-1}[s] \in W\}$ is stationary, whenever $W \subseteq [X]^{\leq\omega}$ is stationary.
For the converse, assume $W \subseteq [X]^{\leq\omega}$ is such that $\{s \in [Y]^{\leq\omega} \mid f^{-1}[s] \in W\}$ is stationary. Let C be a club subset of $[X]^{\leq\omega}$ and let t be a member of the intersection $(C \overline{\cup} [Y]^{\leq\omega}) \cap ([X]^{\leq\omega} \overline{\cup} \{s \in [Y]^{\leq\omega} \mid f^{-1}[s] \in W\}) \cap \mathrm{club}(\{f\})$. Then, t is the disjoint union of some $c \in C$ and some $w \in \{s \in [Y]^{\leq\omega} \mid f^{-1}[s] \in W\}$.

[10] To ensure that we are working with disjoint sets, we can replace X by $X' = X \times \{0\}$, Y by $Y' = Y \times \{1\}$, and f by $f' : X' \to Y'$ given by $f'(x, 0) = (f(x), 1)$. One can easily check that $A \subseteq [X]^{\leq\omega}$ is stationary in $[X]^{\leq\omega}$ if and only if $A \times \{0\}$ is stationary in $[X']^{\leq\omega}$, and that $B \subseteq [Y]^{\leq\omega}$ is stationary in $[Y]^{\leq\omega}$ if and only if $B \times \{1\}$ is stationary in $[Y']^{\leq\omega}$.

Note that $f^{-1}[w]$ is in W. As t is closed under f, the image $f[c]$ is included in w, and thus $c = f^{-1}[w]$. Hence, c is in $C \cap W$. As C was arbitrary, W is stationary.

2: Assume that f is bijective. Then, we have $\forall s, t \subseteq X$ $(s = t \Leftrightarrow f[s] = f[t])$ and $\forall s, t \subseteq Y$ $(s = t \Leftrightarrow f^{-1}[s] = f^{-1}[t])$. Furthermore, we have $|f[s]| = |s|$ for every $s \subseteq X$, and $|f^{-1}[t]| = |t|$ for every $t \subseteq Y$. Hence, we see that f^* is indeed a function from $\mathcal{P}([X]^{\leq\omega})$ to $\mathcal{P}([Y]^{\leq\omega})$. Furthermore, we get a bijective function $(s \mapsto f[s])$ between $[X]^{\leq\omega}$ and $[Y]^{\leq\omega}$, with inverse $(t \mapsto f^{-1}[t])$. It follows that $(f^{-1})^*$ is the inverse function of f^*. As $f^*, (f^{-1})^*$ preserve Boolean operations, f^* is an isomorphism of Boolean algebras. Now, consider a stationary subset $W \subseteq [X]^{\leq\omega}$. As f is bijective, $\{f[s] \mid s \in W\}$ is the same as $\{t \in [Y]^{\leq\omega} \mid f^{-1}[t] \in W\}$. Then, by part 1, $f^*(W)$ is stationary.

3: If $V \models |X| = |Y|$, there is a bijection $g : X \to Y$ in V. Then, we can define $(g^*)^V$ and its inverse in V, as well as $(g^*)^{V'}$ and its inverse in V'. By part 2, V and V' model the statement that g^* and its inverse preserve stationarity, and hence the claim follows. $\square$

Proposition 3.25. *Assume that $\mu < \lambda$ are two uncountable cardinals. Then $Con_1(\lambda)$ implies $Con_1(\mu)$.*

Proof. Assume $Con_1(\lambda)$ holds, and let W be a stationary subset of $[\mu]^{\leq\omega}$. By 3.23, we know that $W\overline{\cup}[\lambda \setminus \mu]^{\leq\omega}$ is stationary in $[\lambda]^{\leq\omega}$. By $Con_1(\lambda)$, $(W\overline{\cup}[\lambda \setminus \mu]^{\leq\omega})^V$ is stationary in $([\lambda]^{\leq\omega})^{V[G]}$, for any generic filter $G \subseteq \mathbb{P}$. Now, fix a generic filter $G \subseteq \mathbb{P}$, and in $V[G]$ take an arbitrary club $C \subseteq [\mu]^{\leq\omega}$. Then, in $V[G]$ the set $C\overline{\cup}[\lambda \setminus \mu]^{\leq\omega}$ is a club subset of $[\lambda]^{\leq\omega}$, by 3.23. Thus, there is some $s \in (C\overline{\cup}[\lambda \setminus \mu]^{\leq\omega})^{V[G]} \cap (W\overline{\cup}[\lambda \setminus \mu]^{\leq\omega})^V$. Then, $s \cap \mu$ is in $C \cap W$. As C was arbitrary, W is stationary in $V[G]$. And as W and G were arbitrary, $Con_1(\mu)$ holds. $\square$

So far we have "$Con_2(\lambda) \Rightarrow Con_1(\lambda)$" for all λ, "$Con_1(\lambda) \Rightarrow Con_2(\lambda)$" for all sufficiently large λ (i.e. $\lambda \geq |\mathbb{P}|$), and "$Con_1(\lambda) \Rightarrow Con_1(\mu)$" whenever $\mu < \lambda$. Hence, we know that $\mathbb{P}$ is proper if and only if $Con_2(\lambda)$ holds for all λ. Next, we will show that it suffices to exhibit $Con_2(\lambda)$ for a single sufficiently large λ.

From now on we assume that $\mathbb{P}$ has infinite cardinality. This is hardly a restriction, as $V[G]$ is equal to V whenever G is finite.

Proposition 3.26. *$Con_2(2^{|\mathbb{P}|})$ implies $Con_2(\lambda)$ for every cardinal $\lambda > 2^{|\mathbb{P}|}$.*

Proof. Set $\mu = 2^{|\mathbb{P}|}$ and fix $\lambda > \mu$. Assume $Con_2(\mu)$ holds and let $E : \mu \to \mathbb{P}^\mu$ be such that $E(i)$ is predense for every $i < \mu$, and such that, for each predense

$A \subseteq \mathbb{P}$, there is $i \in \mu$ with $\mathrm{range}(E(i)) = A$. To see that $Con_2(\lambda)$ holds, fix some $F : \lambda \to \mathbb{P}^\lambda$ with predense rows and let $p \in \mathbb{P}$ be an arbitrary condition. Define $f : \lambda \to \mu$ by $f(i) = \min\{j \in \mu \mid \mathrm{range}(F(i)) = \mathrm{range}(E(j))\}$ and define $g : \lambda \times \mu \to \lambda$ by $g(i,j) = \min\{k \in \lambda \mid F(i)(k) = E(f(i))(j)\}$. Note that $W_p(E)$ is in $\mathcal{D}(\mu)$, by $Con_2(\mu)$. Let $\mathcal{A}$ be an algebra on μ, such that $\mathrm{club}(\mathcal{A}) \subseteq W_p(E)$. Set $\mathcal{B} = \mathcal{A} \cup \{f, g\}$, and note that $\mathcal{B}$ is a partial algebra on λ. We claim that $\mathrm{club}(\mathcal{B})$ is a subset of $W_p(F)$. To see this, consider $s \in \mathrm{club}(\mathcal{B})$, and note that s is closed under $\mathcal{A}$. Furthermore, $s \cap \mu$ is closed under $\mathcal{A}$, since $\mathrm{range}(h)$ is a subset of μ, for every $h \in \mathcal{A}$. Thus, $s \cap \mu$ is an element of $\mathrm{club}(\mathcal{A})$. Since $\mathrm{club}(\mathcal{A})$ is a subset of $W_p(E)$, there is some $q \leq p$ such that $E(i)[s \cap \mu]$ is predense below q for all $i \in s \cap \mu$. To see that, for this q, we have $\forall i \in s\ (F(i)[s]$ is predense below $q)$, first note that $f(i)$ is in $s \cap \mu$ for every $i \in s$, since s is closed under f and $\mathrm{range}(f)$ is a subset of μ. It follows that $E(f(i))[s \cap \mu]$ is predense below q for every $i \in s$. Note that $E(f(i))[s \cap \mu]$ is a subset of $F(i)[s]$, since i is in s and s is closed under g. Hence, $F(i)[s]$ is predense below q, for every $i \in s$. This works for every $s \in \mathrm{club}(\mathcal{B})$, and thus $\mathrm{club}(\mathcal{B})$ is a subset of $W_p(F)$. As F and p were arbitrary, $Con_2(\lambda)$ holds. $\square$

As a consequence of the last three propositions, to check for properness of $\mathbb{P}$, it suffices to check for $Con_1(\lambda)$ or $Con_2(\lambda)$ at a single $\lambda \geq 2^{|\mathbb{P}|}$.

3.3 Characterising properness through elementary submodels of $H(\lambda)$

In this subsection, we again fix a forcing notion $\mathbb{P}$. We want to characterise properness of $\mathbb{P}$ using countable elementary submodels of $H(\lambda)$, for some sufficiently large regular cardinal λ. For a model N we say that a subset $M \subseteq N$ is an *elementary submodel* of N, and we write $M \prec N$, if for every formulas φ and all $x_0, ..., x_{n-1} \in M$, we have $(M \models \varphi(x_0, ..., x_{n-1})) \Leftrightarrow (N \models \varphi(x_0, ..., x_{n-1}))$. Recall that $H(\lambda) = \{x \mid |\mathrm{trcl}(x)| < \lambda\}$ is transitive and models `ZFC` except for some instances of the power set axiom.

To characterise properness, we use the following definition:

Definition 3.27. Assume that M is an elementary submodel of $H(\lambda)$ for some regular uncountable cardinal λ, and such that $\mathbb{P}$ is in M. A condition q is said to be *$(M, \mathbb{P})$-generic*, if for every $A \in M$ which is predense in $\mathbb{P}$, the intersection $A \cap M$ is predense below q.

Remark 3.28. Note that we do not assume that M is transitive. However, many of the properties, that make transitive models convenient to work with, are also exhibited by elementary submodels of transitive models. One scenario that comes up repeatedly is the following: For $M \prec N$ where N is transitive, we use a formula φ (possibly with parameters in M) for which there is a unique $x \in N$ with $N \models \varphi(x)$, to establish that x is in M. Indeed, by elementarity we always find some $x' \in M$ such that $M \models \varphi(x')$, and thus also $N \models \varphi(x')$. Then, by uniqueness, we must have $x = x'$. To avoid making this argument explicit, we just say that "x can be defined in M".

Generic conditions are characterised by the following:

Proposition 3.29. *Assume λ is a sufficiently large uncountable cardinal and M is an elementary submodel of $H(\lambda)$ such that $\mathbb{P} \in M$. For a condition $q \in \mathbb{P}$ the following are equivalent:*

(i) *q is $(M, \mathbb{P})$-generic;*
(ii) *for every $\tau \in M$, such that τ is a $\mathbb{P}$-name for an ordinal, $q \Vdash \tau \in M$ holds;*
(iii) *for every $\mathbb{P}$-name $\tau \in M$, $q \Vdash \tau \in V \Rightarrow \tau \in M$ holds, i.e. if there is $r \leq q$ and $x \in V$ such that $r \Vdash \tau = \check{x}$ then $x \in M$ holds.*

Proof. "*(i)* $\Rightarrow$ *(iii)*": Assume q is $(M, \mathbb{P})$-generic and let $\tau \in M$ be a $\mathbb{P}$-name. Define $A = \{r \in \mathbb{P} \mid \exists x\ (r \Vdash \tau = \check{x})\}$ and let A' be the union of A with the set $\{r \in \mathbb{P} \mid \forall s \in A(s \text{ is incompatible with } r)\}$. Note that A' is predense. Further note that the formula "$\exists x\ (r \Vdash \tau = \check{x})$" is absolute for $H(\lambda)$. Hence, we have

$$H(\lambda) \models A = \{r \in \mathbb{P} \mid \exists x\ (r \Vdash \tau = \check{x})\}$$
$$H(\lambda) \models A' = A \cup \{r \in \mathbb{P} \mid \forall s \in A(s \text{ is incompatible with } r)\}.$$

By elementarity, A and A' can be defined in M. Next, define $f : A \to V$, such that $q \Vdash \tau = f(q)$ holds. Then, the definition of f is absolute for $H(\lambda)$, and thus f can be defined in M. Now, assume $r \leq q$ is such that $r \Vdash \tau = \check{x}$ for some $x \in V$, and let $G \subseteq \mathbb{P}$ be a generic filter that contains r. Then G contains q, and hence G meets $A' \cap M$. As G is directed and contains r, it follows that $G \cap (A' \setminus A)$ is empty. Thus, G meets $A \cap M$. Let s be an element of $A \cap M \cap G$, and note that $s \Vdash \tau = f(s)$. As s and f are in M, so is $f(s)$. Hence, τ^G is in M. As G contains r and $r \vdash \tau = \check{x}$, we have $x \in M$.

"*(iii)* $\Rightarrow$ *(ii)*": Assume that q is as in *(iii)*, and let $\tau \in M$ be a name for an ordinal. Then, for any generic filter $G \subseteq \mathbb{P}$ that contains q, there is some ordinal

α and some $r \in G$ with $r \leq q$ such that $r \Vdash \tau = \check{\alpha}$. Hence, τ^G is in M. As G was arbitrary, we have $q \Vdash \tau \in M$.

"$(ii) \Rightarrow (i)$": Assume that $q \Vdash \tau \in M$ holds for every $\mathbb{P}$-name $\tau \in M$ that is a name for an ordinal. Let $A \subseteq \mathbb{P}$ be predense and such that $A \in M$. Then, in $H(\lambda)$ there is a bijection from $|A|$ to A. Hence, by elementarity, there is a bijection $f : |A| \to A$ in M. With this f define

$$\tau = \bigcup_{r \in \mathbb{P}} (\{r\} \times \min\{i \in |A| \mid r \text{ is compatible with } f(i)\}),$$

and note that τ is a $\mathbb{P}$-name for an ordinal. Further note that the definition of τ is absolute for $H(\lambda)$, and thus τ can be defined in M. By assumption, we have $q \Vdash \tau \in M$. Now, consider any generic filter $G \subseteq \mathbb{P}$. Then, G meets A, and thus $\{i \mid f(i) \in G\}$ is non-empty. Let α be the minimum of $\{i \mid f(i) \in G\}$ and note that $f(\alpha)$ is in G. Furthermore, we must have $r \Vdash \check{\alpha} = \min\{i \mid f(i) \in \dot{G}\}$, for some $r \in G$. Then, r is compatible with $f(\alpha)$, but incompatible with $f(i)$ for all $i < \alpha$. Hence, $r \times \alpha$ is a subset of τ, and α is a subset of τ^G. To see that also $\tau^G \subseteq \alpha$ holds, note that every $s \in G$ is compatible with $f(\alpha)$, and hence $\min\{i \mid s \text{ is compatible with } f(i)\} \leq \alpha$ holds for every $s \in G$. Therefore, we have $\tau^G = \alpha$. Since G was arbitrary, it follows that $1 \Vdash \tau = \min\{i \mid f(i) \in \dot{G}\}$. Hence, also $1 \Vdash f(\tau) \in \dot{G}$ holds. Then, $q \Vdash f(\tau) \in M \cap \dot{G}$ holds, since f is in M and $q \Vdash \tau \in M$. Therefore, $q \Vdash A \cap M \cap \dot{G} \neq \emptyset$ holds. It follows that $A \cap M$ is predense below q. As A was arbitrary, q is $(M, \mathbb{P})$-generic. □

Before we prove the main theorem of this subsection, we have one more lemma.

Lemma 3.30. *Assume that N is an uncountable model and let $A \subseteq N$ be a countable set. Then, $\{M \in [N]^{\leq\omega} \mid M \prec N,\ A \subseteq M\}$ is in $\mathcal{D}(N)$.*

Proof. By the Tarski-Vaught test, for any $M \subseteq N$, the condition $M \prec N$ is equivalent to the following statement: For each natural number n, every $(n+1)$-place formula φ, and all $\overline{x} \in M^n$, if $N \models \exists y\ \varphi(\overline{x}, y)$ then there is $z \in M$ such that $N \models \varphi(\overline{x}, z)$.[11] For each formula φ choose a function $f_\varphi : N^{<\omega} \to N$ such that $\varphi(\overline{x}, f_\varphi(\overline{x}))$ holds whenever $N \models \exists y\ \varphi(\overline{x}, y)$, and set $\mathcal{A} = \{f_\varphi \mid \varphi \text{ is a formula}\}$. Note that $\mathcal{A}$ is a countable algebra on N, and if $M \subseteq N$ is closed under $\mathcal{A}$, then $M \prec N$ holds. Further note that $C = \{M \in [N]^{\leq\omega} \mid A \subseteq M\}$ is non-empty and upward closed in $[N]^{\leq\omega}$, and thus C is a club. Now, simply note that $\mathrm{club}(\mathcal{A}) \cap C$ is a subset of $\{M \in [N]^{\leq\omega} \mid M \prec N,\ A \subseteq M\}$. □

[11] The proof of this is a straight forward argument by induction on the length of the formula φ, and it can be found for example in [10, Part 4, 19.16].

The following theorem gives us an equivalent definition of properness, which is often more convenient than working with Con_1 or Con_2.

Theorem 3.31. *Assume $\mathbb{P} \in H(\lambda)$ for some sufficiently large regular cardinal λ with $\lambda > 2^{|\mathbb{P}|}$. Then the following are equivalent:*

(i) $\mathbb{P}$ is proper;

(ii) For each countable elementary submodel $M \prec H(\lambda)$ and every $p \in \mathbb{P} \cap M$, if $\mathbb{P}$ is in M then there exists $q \leq p$ such that q is $(M, \mathbb{P})$-generic.

Proof. "$(i) \Rightarrow (ii)$": Assume that $\mathbb{P}$ is proper, fix $p \in \mathbb{P}$ and consider a countable elementary submodel $M \prec H(\lambda)$ such that $p, \mathbb{P} \in M$. Set $\mu = |2^{\mathbb{P}}|$ and note that, since $\mathbb{P}$ is proper, in particular $Con_2(\mu)$ holds. Take a function $F : \mu \to \mathbb{P}^{\mu}$ with predense rows, and such that, for each predense $A \subseteq \mathbb{P}$, there is $i \in 2^{|\mathbb{P}|}$ for which $A = \text{range}(F(i))$. Note that F is in $H(\lambda)$, since λ is regular and $\lambda > \mu$ holds. By $Con_2(\mu)$, the set $W_p(F)$ is in $\mathcal{D}(\lambda)$. Let $\mathcal{A}$ be a countable set of functions, such that $\mathcal{A}$ is an algebra on μ and $\text{club}(\mathcal{A}) \subseteq W_p(F)$. Take a surjection $f : \omega \to \mathcal{A}$. Note that also f is in $H(\lambda)$. Now, by elementarity, there are such functions $F, f \in M$. Since any natural number is definable in M, we have $\omega \subseteq M$. In M we can evaluate f at every natural number, and thus $\text{range}(f)$ is a subset of M. Similarly, evaluating $h \in \text{range}(f)$ on $\mu \cap M$ shows that $\text{range}(h \restriction M)$ is a subset of $\mu \cap M$ i.e. $\mu \cap M$ is closed under the algebra $\text{range}(f)$. Hence, $\mu \cap M \in \text{club}(\text{range}(f)) \subseteq W_p(F)$ holds, and thus there is $q \leq p$ such that $\forall i \in \mu \cap M(F(i)[\mu \cap M]$ is predense below $q)$. Fix such a q and consider a predense set $A \subseteq \mathbb{P}$ such that $A \in M$. By the definition of F, there is $i \in \mu$ such that $A = \text{range}(F(i))$, and in particular $H(\lambda)$ models that there exists such i. Thus, by elementarity, $M \models \exists i \in \mu(A = \text{range}(F(i)))$, i.e. $A = \text{range}(F(i))$ for some $i \in \mu \cap M$. Note that we have $F(i)[\mu \cap M] = \text{range}(F(i)) \cap M = A \cap M$, and thus $A \cap M$ is predense below q. As A was arbitrary, q is $(M, \mathbb{P})$-generic, and as M and p were arbitrary, the implication "$(i) \Rightarrow (ii)$" holds.

"$(ii) \Rightarrow (i)$": Assume that for each countable elementary submodel $M \prec H(\lambda)$ and every condition $p \in \mathbb{P} \cap M$ there exists a condition $q \leq p$ such that q is $(M, \mathbb{P})$-generic. Set $\mu = |2^{\mathbb{P}}|$ and consider any $F : \mu \to \mathbb{P}^{\mu}$ with predense rows. Fix $p \in \mathbb{P}$ and take $M \prec H(\lambda)$, such that $p, \mathbb{P}, F \in M$. Note that, if $q \leq p$ is an $(M, \mathbb{P})$-generic condition, then for every $i \in \mu \cap M$, $\text{range}(F(i)) \cap M$ is predense below q, since $\text{range}(F(i))$ is predense and in M. Now, consider the sets $W = \{M \in [H(\lambda)]^{\leq\omega} \mid p, \mathbb{P}, F \in M,\ M \prec H(\lambda)\}$ and $W' = \{\mu \cap M \mid M \in W\}$. The set W' is a subset of $W_p(F)$, since, for each $M \in A$ and every $i \in \mu \cap M$, we have $F(i)[\mu \cap M] = \text{range}(F(i)) \cap M$ and $\text{range}(F(i)) \cap M$ is predense below any $(M, \mathbb{P})$-generic condition $q \leq p$. Further note that the condition p and the function F were arbitrary. Hence, to show that $Con_2(\mu)$ holds, it suffices to show that W' is in $\mathcal{D}(\mu)$. To this end, first note that μ is a subset of $H(\lambda)$, since $\mu < \lambda$. Hence, μ injects into $H(\lambda)$ by the inclusion map. Furthermore, W is in $\mathcal{D}(H(\lambda))$, by Lemma 3.30, and thus $[H(\lambda)]^{\leq\omega} \setminus W$ is a non-stationary subset of $[H(\lambda)]^{\leq\omega}$. Then, it follows from Lemma 3.24 that $[\mu]^{\leq\omega} \setminus W'$ is non-stationary subset of $[\mu]^{\leq\omega}$. Indeed, if we let f be the inclusion map from μ to $H(\lambda)$, we get $\{M \in [H(\lambda)]^{\leq\omega} \mid \mu \cap M \notin W'\} = \{M \in [H(\lambda)]^{\leq\omega} \mid f^{-1}[M] \in [\mu]^{\leq\omega} \setminus W'\}$, and $\{M \in [H(\lambda)]^{\leq\omega} \mid \mu \cap M \notin W'\}$ is a subset of $[H(\lambda)]^{\leq\omega} \setminus W$. Now, since $[\mu]^{\leq\omega} \setminus W'$ is non-stationary, W' is in $\mathcal{D}(\mu)$. This proves that $Con_2(2^{|\mathbb{P}|})$ holds and so $\mathbb{P}$ is proper. $\square$

3.4 Iterating proper forcing notions

In this subsection, our main goal is to show that a countable support iteration of proper forcing notions is proper. We are going to use the characterisation of properness by elementary submodels of $H(\lambda)$. Hence, we will have to consider $M[G] = \{\tau^G \mid \tau \in M \cap V^{\mathbb{P}}\}$, where M is some elementary submodel of $H(\lambda)$ and $G \subseteq \mathbb{P}$ is a generic filter. The next two propositions are essential in this approach to analysing iterations of proper forcing notions.

Proposition 3.32. *Assume that λ is a sufficiently large regular cardinal, M is an elementary submodel model of $H(\lambda)$ with $\mathbb{P} \in M$, and $G \subseteq \mathbb{P}$ is a generic filter. Then, $M[G]$ is an elementary submodel of $H(\lambda)[G]$.*

Proof. By the Tarski-Vaught test, it suffices to show that, for each natural number n and every $(n+1)$-place formula φ, if there exists $\overline{x} \in H(\lambda)[G]^n$ such that $H(\lambda)[G] \models \exists y\ \varphi(\overline{x}, y)$, then there exists $z \in M[G]$ such that $H(\lambda)[G] \models \varphi(\overline{x}, z)$. Hence, assume that $\overline{x} = (x_0, ..., x_{n-1})$ is such that, for some formula φ, we have $H(\lambda)[G] \models \exists y\ \varphi(\overline{x}, y)$. Let $\tau_0, ..., \tau_{n-1} \in M$ be $\mathbb{P}$-names and such that $\tau_i^G = x_i$ for $i < n$. For sufficiently large regular λ, the Truth Lemma can be proved for $H(\lambda)$, we can use Zorn's lemma to construct a maximal antichain A in $H(\lambda)$, such that for each $p \in A$ either $H(\lambda) \models (p \Vdash \exists y\ \varphi(\tau_0, ..., \tau_{n-1}, y))$ or $H(\lambda) \models (p \Vdash \neg\exists y\ \varphi(\tau_0, ..., \tau_{n-1}, y))$, and we can apply existential completeness in $H(\lambda)$ to get, for each $p \in A$ with $H(\lambda) \models (p \Vdash \exists y\ \varphi(\tau_0, ..., \tau_{n-1}, y))$, some $\mathbb{P}$-name σ_p in $H(\lambda)$ such that $H(\lambda) \models (p \Vdash \varphi(\tau_0, ..., \tau_{n-1}, \sigma_p))$. If $p \in A$ is such that $H(\lambda) \models (p \Vdash \neg\exists y\ \varphi(\tau_0, ..., \tau_{n-1}, y))$, then let $\sigma_p = \emptyset$. Now, define

$$\sigma = \{(\rho, q) \mid \exists p \in A\ (q \leq p\ \wedge\ \rho \in \text{names}(\sigma_p)\ \wedge\ q \Vdash \rho \in \sigma_p)\}.$$

Note that σ is in $H(\lambda)$. Furthermore, $H(\lambda)$ models $\forall p \in A\ (p \Vdash \sigma = \sigma_p)$. Hence,

$$H(\lambda) \models \forall p \in A\ (p \Vdash \exists y\ \varphi(\tau_0, ..., \tau_{n-1}, y) \Rightarrow \varphi(\tau_0, ..., \tau_{n-1}, \sigma))$$

holds. By elementarity, there exist such a maximal antichain A and a name σ in M. Then, σ^G is in $M[G]$, and as $H(\lambda)[G] \models \exists y\ \varphi(\tau_0^G, ..., \tau_{n-1}^G, y)$, we have $H(\lambda)[G] \models \varphi(\tau_0^G, ..., \tau_{n-1}^G, \sigma^G)$. As this works for arbitrary φ, we conclude $M[G] \prec H(\lambda)[G]$. $\square$

Remark 3.33. Recall that, for a sufficiently large regular cardinal λ, any generic filter $G \subseteq \mathbb{P}$, $H(\lambda)[G]$ is equal to $H(\lambda)^{V[G]}$. Hence, it is a consequence of the above proposition, that for every proper forcing notion $\mathbb{Q}$ in $V[G]$ and any sufficiently large regular cardinal λ, if M is a countable elementary submodel of $H(\lambda)$ such that $\mathbb{Q} \in M[G]$, then for every $p \in \mathbb{Q} \cap M$, there is a $(M[G], \mathbb{Q})$-generic condition $q \leq p$.

At successor steps of an iteration we essentially perform a two step iteration. Hence, we also need the following:

Proposition 3.34. *Assume $\mathbb{P} * \dot{\mathbb{Q}}$ is a two step iteration, and M is an elementary submodel of $H(\lambda)$, for some sufficiently large regular cardinal λ. If M contains $\mathbb{P} * \dot{\mathbb{Q}}$, then we have $M[G][H] = M[G * H]$, for any two generic filters G, H for $\mathbb{P}, \dot{\mathbb{Q}}^G$, respectively.*

Proof. Let M, G and H be as assumed. Then, we can define $\mathbb{P}$, $\dot{\mathbb{Q}}$ and the canonical name $\dot{K}$ for the generic filter of $\mathbb{P} * \dot{\mathbb{Q}}$, from $\mathbb{P} * \dot{\mathbb{Q}}$ in M. In $M[G * H]$, we can define G from $G * H$, and thus we get $M[G] \subseteq M[G * H]$, by evaluating every $\mathbb{P}$-name $\tau \in M$ by G in $M[G * H]$. As one can also define H from $G * H$ in $M[G * H]$, we get $M[G][H] \subseteq M[G * H]$, by evaluating every $\mathbb{Q}^G$-name $\tau \in M[G]$ by H in $M[G][H]$. Similarly, in $M[G][H]$ we can define $G * H$ from G and H, and we get $M[G * H] \subseteq M[G][H]$, by evaluating each $\mathbb{P} * \dot{\mathbb{Q}}$-name $\tau \in M$ by $G * H$ in $M[G][H]$. $\square$

In general, $M[G]$ cannot be understood as a forcing extension of M. In particular, if M is not transitive, $G \cap A \cap M$ might be empty for some predense set $A \in M$ i.e. $G \cap M$ might not be generic over M. In fact, $G \cap M$ is generic over M if and only if G contains an $(M, \mathbb{P})$-generic condition.

Proposition 3.35. *Assume that λ is a sufficiently large regular cardinal, M is an elementary submodel of $H(\lambda)$ with $\mathbb{P} \in M$, and $G \subseteq \mathbb{P}$ is a generic filter. Then, G contains an $(M, \mathbb{P})$-generic condition if and only if one of the following equivalent conditions holds:*

(i) $G \cap M$ *meets every* $A \in M$ *which is predense in* $\mathbb{P}$*;*

(ii) $M[G] \cap Ord = M \cap Ord$*;*

(iii) $M[G] \cap V = M$*;*

(iv) $G \cap M$ *meets every* $A \in M$ *which is dense in* $\mathbb{P}$*;*

(v) $G \cap M$ *meets every* $A \in M$ *which is dense open in* $\mathbb{P}$*;*

(vi) $G \cap M$ *meets every* $A \in M$ *which is a maximal antichain in* $\mathbb{P}$*;*

Proof. The fact that (i) holds if and only if G contains an $(M, \mathbb{P})$-generic condition follows directly from the Truth Lemma. Then, (ii) and (iii) are equivalent to (i) by Proposition 3.29. To show "$(iv) \Rightarrow (v) \Rightarrow (i) \Rightarrow (vi)$" we can argue as in 1.19: Any maximal antichain is predense; the downward closure of any predense set is dense open and, if the predense set is in M, then its downward closure can be defined in M; any dense open set is dense. Hence, it suffices to prove "$(vi) \Rightarrow (iv)$".

"$(vi) \Rightarrow (iv)$": Assume that $D \in M$ is a dense subset of $\mathbb{P}$. Then, D is in $H(\lambda)$, and we can apply Zorn's lemma in $H(\lambda)$ to obtain a maximal antichain $A \subseteq D$ in $H(\lambda)$. By elementarity, there is such a maximal antichain A in M. By assumption, $G \cap M$ meets A. $\square$

Remark 3.36. Note that, if $q \in \mathbb{P}$ is such that $q \Vdash$ "$\dot{G}$ contains an $(M, \mathbb{P})$-generic condition", then q is already $(M, \mathbb{P})$-generic, since such q forces that $\dot{G}$ meets $A \cap M$ (i.e. $A \cap M$ is predense below q) for each predense $A \in M$. Hence, to show that q is $(M, \mathbb{P})$-generic, it suffices to prove that q forces one of the above equivalent conditions (i)-(vi).

From the next lemma one can deduce that, if $\mathbb{P} * \dot{\mathbb{Q}}$ is a two step iteration such that $\mathbb{P}$ is proper and $1 \Vdash$ "$\dot{\mathbb{Q}}$ is proper", then $\mathbb{P} * \dot{\mathbb{Q}}$ is proper.[12]

Lemma 3.37. *Assume* $\mathbb{P} * \dot{\mathbb{Q}}$ *is a two step iteration. Let* λ *be a sufficiently large regular cardinal and let* M *be a countable elementary submodel of* $H(\lambda)$ *with* $\mathbb{P}, \dot{\mathbb{Q}} \in M$*. If* $(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}}$ *is such that* p *is* $(M, \mathbb{P})$*-generic and* p *forces that* $\dot{q}$ *is* $(M[\dot{G}], \dot{Q})$*-generic, then* $(p, \dot{q})$ *is* $(M, \mathbb{P} * \dot{\mathbb{Q}})$*-generic.*

Proof. By 3.35, it suffices to show that, for every generic filter $K \subseteq \mathbb{P} * \dot{\mathbb{Q}}$ which contains $(p, \dot{q})$, we have $M[G] \cap Ord = M \cap Ord$ holds. Fix a generic filter $K \subseteq \mathbb{P} * \dot{\mathbb{Q}}$ which contains $(p, \dot{q})$. First, note that we have $K = G * H$, where $G = \{r \mid (r, \dot{s}) \in K\}$ and $H = \{\dot{s}^G \mid (1, \dot{s}) \in K\}$. By Proposition 3.34, we get $M[K] = M[G][H]$. As G contains the $(M, \mathbb{P})$-generic condition p, we

[12]In fact, we essentially present this argument in the proof of 3.38.

see that $M[G] \cap Ord = M \cap Ord$ holds, and as H contains the $(M[G], \dot{\mathbb{Q}}^G)$-generic condition $\dot{q}^G$, also $M[G][H] \cap Ord = M[G] \cap Ord$ holds. This shows that $M[K] \cap Ord = M \cap Ord$, and as K was arbitrary, we conclude that $(p, \dot{q})$ is $(M, \mathbb{P} * \dot{\mathbb{Q}})$-generic. □

The main point of this section is the following theorem:

Theorem 3.38. *Assume α is an ordinal and $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j \mid i \leq \alpha,\ j < \alpha \rangle$ is a countable support iteration. If $1_{\mathbb{P}_i} \Vdash$ "$\dot{\mathbb{Q}}_i$ is proper" holds for all $i < \alpha$, then $\mathbb{P}_\alpha$ is proper.*

Proof. Let λ be a sufficiently large regular cardinal and consider any countable $M \prec H(\lambda)$ which contains α and $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j \mid i \leq \alpha,\ j < \alpha \rangle$. Note that this implies $\mathbb{P}_\alpha \in M$ and $\mathbb{P}_i, \dot{\mathbb{Q}}_i \in M$ for all $i \in \alpha \cap M$. We need that for all $p \in \mathbb{P}_\alpha \cap M$ there is a condition $q \leq p$ in $\mathbb{P}_\alpha$ such that q is $(M, \mathbb{P}_\alpha)$-generic. To show this, we will prove the statement $\forall i < j\ Stat(i, j)$ by induction for every $j \leq \alpha$, where $Stat(i, j)$ is the following statement:

> If i, j are in M, then for every $p \in M \cap \mathbb{P}_j$ and each $(M, \mathbb{P}_i)$-generic $q_0 \in \mathbb{P}_i$ such that $q_0 \leq p \restriction i$, there is some condition $q \in \mathbb{P}_j$ such that $q \restriction i = q_0$, $q \leq p$ and q is $(M, \mathbb{P}_j)$-generic.

For $j = 0$, there is no $i < j$, and hence there is nothing to show.

Successor step: Fix $j + 1 \in M$ and assume that $\forall i < j'\ Stat(i, j')$ holds for every $j' \leq j$. Note that j is in M and for any $i \in j \cap M$, if $q_0 \in \mathbb{P}_i$ is $(M, \mathbb{P}_i)$-generic, and $p \in M \cap \mathbb{P}_{j+1}$ is such that $q_0 \leq p \restriction i$, then there is $q \in \mathbb{P}_j$ such that $q \restriction i = q_0$, $q \leq p \restriction j$ and q is $(M, \mathbb{P}_j)$-generic. To get such an q, simply apply $Stat(i, j)$ to $p \restriction j$ and q_0. Hence, to complete the inductive step from j to $j + 1$ it suffices to prove $Stat(j, j + 1)$.
Fix an $(M, \mathbb{P}_j)$-generic condition q_0 and a condition $p \in \mathbb{P}_{j+1} \cap M$ such that $q_0 \leq p \restriction j$. Let $G \subseteq \mathbb{P}_j$ be a generic filter such that $q_0 \in G$. Then, $\dot{\mathbb{Q}}_j^G$ is in $M[G]$, since $\dot{\mathbb{Q}}_j$ is in M. By assumption, $1 \Vdash$ "$\dot{\mathbb{Q}}_j$ is proper", and thus $\dot{\mathbb{Q}}_j^G$ is proper in $V[G]$. As p and j are in M, so is the name $p(j)$. Then, $p(j)^G$ is in $M[G]$, and since $p(j)$ is forced to be in $\dot{\mathbb{Q}}_j$, $p(j)^G$ is in $M[G] \cap \dot{\mathbb{Q}}_j^G$. Note that $M[G]$ is a countable elementary submodel of $H(\lambda)[G] = H(\lambda)^{V[G]}$, by Proposition 3.32. Hence, by properness of $\dot{\mathbb{Q}}_j^G$, there is $r \in \dot{\mathbb{Q}}_j^G$ such that $r \leq p(j)^G$ and r is $(M[G], \dot{\mathbb{Q}}_j^G)$-generic. As this works for any generic filter that contains q, we have

$$q \Vdash \exists r \in \dot{\mathbb{Q}}_j\ (r \leq p(j)\ \wedge\ r \text{ is } (M[\dot{G}], \dot{\mathbb{Q}}_j)\text{-generic}).$$

By existential completeness, there is a $\mathbb{P}_j$-name $\dot{r}$ for a condition in $\dot{\mathbb{Q}}_j$ such that q forces $\dot{r} \le p(j)$ and $\dot{r}$ is $(M[\dot{G}], \dot{\mathbb{Q}}_j)$-generic. Then, $q = q_0 \cup \{(j, \dot{r})\}$ is in $\mathbb{P}_{j+1}$ and $q \le p$ holds. By Lemma 3.37, we see that q is $(M, \mathbb{P}_{j+1})$-generic. This concludes the induction step for successor ordinals.

Notation: For the limit case, we first introduce some additional notation. Assume i, j are two ordinals such that $i < j \le \alpha$. If $r \in \mathbb{P}_i$ and $s \in \mathbb{P}_j$, let $r \cup^* s$ denote the condition $r \cup (s \restriction (j \setminus i))$. Note that $r \cup^* s$ is in $\mathbb{P}_j$ and $r \cup^* s \le s$ holds whenever $r \le s \restriction i$ holds. Furthermore, for $s, s' \in \mathbb{P}_j$ and $G \subseteq \mathbb{P}_i$ a generic filter such that $s \restriction i, s' \restriction i \in G$, we write $s \sim_G s'$ if s and s' belong to the same equivalence class in the separative quotient of $\mathbb{P}_j/G$, or equivalently, if $s \in H \Leftrightarrow s' \in H$ holds for all generic filters $H \subseteq \mathbb{P}_j/G$. Note that $s \sim_G s'$ implies $s \restriction k \sim_G s' \restriction k$ for all $k \in [i, j)$. Lastly, we write $s \sim s'$ for $s \le s' \ \wedge \ s' \le s$. Note that $s \sim s'$ implies $s \sim_G s'$.

Limit step: Assume that $j \in M$ is a limit ordinal and that $\forall i < j' \ Stat(i, j')$ holds for all $j' < j$. Fix an $i \in j \cap M$ and let $\{D_n \mid n \in \omega\}$ enumerate all dense open subsets of $\mathbb{P}_j$ that are in M. Note that $j \cap M$ has no maximal element, as M models that j is a limit ordinal. Take a strictly increasing and unbounded sequence $(i_n)_{n \in \omega}$ in $M \cap j$ such that $i_0 = i$. Fix $p \in M \cap \mathbb{P}_j$ and an $(M, \mathbb{P}_i)$-generic condition $q \le p \restriction i$. We will recursively define two sequences $(\dot{p}_n)_{n \in \omega}$ and $(q_n)_{n \in \omega}$ with $\dot{p}_0 = \check{p}$, such that following properties hold for all $n \in \omega$:

(a) $q_n \in \mathbb{P}_{i_n}$ and $q_{n+1} \restriction i_n = q_n$;
(b) $\dot{p}_{n+1}$ is a $\mathbb{P}_{i_n}$-name for a condition in $\mathbb{P}_j \cap M$ such that $\dot{p}_{n+1} \le \dot{p}_n$ is forced;
(c) $1 \Vdash \dot{p}_{n+1} \in D_n$;
(d) $q_n \Vdash \dot{p}_n \restriction i_n \in \dot{G}$;
(e) q_n is $(M, \mathbb{P}_{i_n})$-generic.

Note that, for $i' < j'$, if $\dot{p}$ is a $\mathbb{P}_{i'}$-name, we can also interpret it as a $\mathbb{P}_{j'}$-name, using the canonical injection $\mathbb{P}_{i'} \hookrightarrow \mathbb{P}_{j'}$, and we have $\dot{p}^G = \dot{p}^{G \restriction i'}$, for any generic filter $G \subseteq \mathbb{P}_j$.

Assume that $\dot{p}_n, q_n$ are defined and satisfy the above properties. To define $\dot{p}_{n+1}$ and q_{n+1}, we first construct the $\mathbb{P}_{i_n}$-name $\dot{p}_{n+1}$, using existential completeness, then we construct a $\mathbb{P}_{i_n}$-name $\dot{q}_{n+1}$, which encodes how to extend q_n to q_{n+1}.

Construction of $\dot{p}_{n+1}$: Fix a generic filter $G \subseteq \mathbb{P}_{i_n}$. There is $p_{n+1} \in D_n$ such that $p_{n+1} \le \dot{p}_n^G$. As M is an elementary submodel of $H(\lambda)$ and since $D_n, \dot{p}_n^G$ are in M, we can take p_{n+1} in $D_n \cap M$. If q_n is in G, we may pick p_{n+1} such that $p_{n+1} \restriction i_n$ is in G. To see this, first note that $\dot{p}_n^G \restriction i_n$ is in G, whenever $q_n \in G$. Next, note that $D'_n = \{r \restriction i_n \mid r \in D_n,\ r \leq \dot{p}_n^G\}$ is dense below $\dot{p}_n^G \restriction i_n$ in $\mathbb{P}_{i_n}$, and $D'_n \in M$. Then, there is some $p' \in D'_n \cap M \cap G$, since G contains the $(M, \mathbb{P}_{i_n})$-generic condition q_n. Now, we take p_{n+1} such that $p_{n+1} \restriction i_n = p'$. By elementarity, we find such p_{n+1} in M. As G was arbitrary, we have

$$1 \Vdash \exists p_{n+1} \in D_n \cap M \ (\ p_{n+1} \leq \dot{p}_n \ \wedge \ (q_n \in \dot{G} \Rightarrow p_{n+1} \restriction i_n \in \dot{G})\ ).$$

This defines a $\mathbb{P}_{i_n}$-name $\dot{p}_{n+1}$, by existential completeness.

Construction of $\dot{q}_{n+1}$*:* By $Stat(i_n, i_{n+1})$, for each $s \in \mathbb{P}_j \cap M$, there is an $(M, \mathbb{P}_{i_{n+1}})$-generic condition q^s such that $q^s \restriction i_n = q_n$ and $q^s \leq 1_{\mathbb{P}_{i_n}} \cup^* s \restriction i_{n+1}$. Let $\dot{q}_{n+1}$ be the $\mathbb{P}_{i_n}$-name for $q^{\dot{p}_{n+1}}$, defined by existential completeness.

Definition of q_{n+1}*:* Set $q_{n+1} \restriction i_n = q_n \restriction i_n$ and $q_{n+1}(\gamma) = \dot{1}_\gamma$ whenever γ is not in $\bigcup\{\mathrm{supp}(q^s) \mid s \in \mathbb{P}_j \cap M\}$. For $\gamma \in [i_n, j) \cap \bigcup\{\mathrm{supp}(q^s) \mid s \in \mathbb{P}_j \cap M\}$, let $q_{n+1}(\gamma)$ be the $\mathbb{P}_\gamma$-name for a condition in $\dot{\mathbb{Q}}_\gamma$ determined by the $\mathbb{P}_{i_n}$-name $\dot{q}_{n+1}$. More precisely, let $q_{n+1}(\gamma)$ be the following $\mathbb{P}_\gamma$-name:

$$\{(t, \tau) \mid t \in \mathbb{P}_\gamma, \exists s\ (\ (t \restriction i_n \Vdash \dot{q}_{n+1} = s) \wedge \tau \in \mathrm{names}(s(\gamma)) \wedge t \restriction \gamma \Vdash \tau \in s(\gamma)\ )\}.$$

Note that the support of q_{n+1} is countable, since it is included in the countable set $\bigcup\{\mathrm{supp}(q^s) \mid s \in \mathbb{P}_j \cap M\}$. Hence, q_{n+1} is in $\mathbb{P}_j$.

Claim 1: The condition q_{n+1} has the following properties:

(1) $q_{n+1}(\gamma)^G = (\dot{q}_{n+1}^{G \restriction i_n}(\gamma))^G$, for $\gamma \in [i_n, i_{n+1})$ and every generic filter $G \subseteq \mathbb{P}_\gamma$;
(2) $q_{n+1} \sim_G \dot{q}_{n+1}^G$, for every generic filter $G \subseteq \mathbb{P}_{i_n}$ which contains q_n;
(3) $q_{n+1} \restriction \gamma \Vdash \dot{q}_{n+1} \restriction \gamma \in \dot{G}$, for $\gamma \in [i_n, i_{n+1}]$.

Proof of (1): We have $1 \Vdash \dot{q}_{n+1} \restriction i_n = q_{n+1} \restriction i_n$, and the support of q_{n+1} is a subset of $\bigcup\{\mathrm{supp}(q^s) \mid s \in \mathbb{P}_j \cap M\}$, and thus it suffices to consider $\gamma \in [i_n, i_{n+1}) \cap \bigcup\{\mathrm{supp}(q^s) \mid s \in \mathbb{P}_j \cap M\}$. Fix such a γ, let $G \subseteq \mathbb{P}_\gamma$ be a generic filter. If x is an element of $q_{n+1}(\gamma)^G$, then there are $t \in G$, $s \in \mathbb{P}_{i_{n+1}}$ and $\tau \in \mathrm{names}(s)$ such that $t \restriction i_n \Vdash \dot{q}_{n+1} = s$, $t \Vdash \tau \in s(\gamma)$ and $\tau^G = x$. Then, $G \restriction i_n$ contains $t \restriction i_n$, and thus $\dot{q}_{n+1}^{G \restriction i_n}$ is equal to s. As G contains t, we have $\tau^G \in s(\gamma)^G$, and therefore $x \in (\dot{q}_{n+1}^{G \restriction i_n}(\gamma))^G$. Conversely, if x is in $(\dot{q}_{n+1}^{G \restriction i_n}(\gamma))^G$, take $r \in G \restriction i_n$ and $s \in \mathbb{P}_j$ such that $r \Vdash \dot{q}_{n+1} = s$. Then, there is $(r', \tau) \in s(\gamma)$ such that $r' \in G$ and $\tau^G = x$. Now, r and r' are in G, and thus there is a common extension $t \leq r, r'$. Note that $t \restriction i_n \Vdash \dot{q}_{n+1} = s$ and $t \Vdash \tau \in s(\gamma)$. By definition, $q_{n+1}(\gamma)$ contains (t, τ), and hence τ^G is in $q_{n+1}(\gamma)^G$.

Proof of (2): Fix a generic filter $G \subseteq \mathbb{P}_{i_n}$ which contains q_n, and set $s = \dot{q}_{n+1}^G$. It suffices to show that, for each $r \in \mathbb{P}_{i_{n+1}}/G$, the following two hold:

1. $r \leq q_{n+1}$ implies that r and s are compatible in $\mathbb{P}_{i_{n+1}}/G$;
2. $r \leq s$ implies that r and q_{n+1} are compatible in $\mathbb{P}_{i_{n+1}}/G$.

First, note that for any $r \in \mathbb{P}_{i_{n+1}}/G$, the restriction $r \restriction i_n$ is in G, and hence there is $r' \in G$ such that $r' \leq q_n, r \restriction i_n$ and $r' \Vdash \dot{q}_{n+1} = s$. Further note that $r' \leq r \restriction i_n$ implies $r' \cup r \leq r$. We will first show that $r' \cup^* s \sim r' \cup^* q_{n+1}$ holds. Then, to obtain a common extension in both cases, we will prove the following two implications:

$$r \leq q_{n+1} \Rightarrow r' \cup^* r \leq s, \tag{3.1}$$

$$r \leq s \Rightarrow r' \cup^* r \leq q_{n+1}. \tag{3.2}$$

Fix $\gamma \in [i_n, i_{n+1})$, and consider any generic filter $H \subseteq \mathbb{P}_\gamma$. Note that if r' is in $H \restriction i_n$, we have $s = \dot{q}_{n+1}^{H \restriction i_n}$, and thus $s(\gamma)^H = (\dot{q}_{n+1}^{H \restriction i_n}(\gamma))^H = (q_{n+1}(\gamma))^H$, by part (1). Since H was arbitrary, we have

$$r' \cup^* s \restriction \gamma \Vdash s(\gamma) \leq q_{n+1}(\gamma) \text{ and } r' \cup^* q_{n+1} \restriction \gamma \Vdash q_{n+1}(\gamma) \leq s(\gamma),$$

As this works for all $\gamma \in [i_n, i_{n+1})$, it follows that $r' \cup^* s \sim r' \cup^* q_{n+1}$ holds. To check that we have (3.1), assume that $r \leq q_{n+1}$ holds, and note that we have

$$r' \cup^* r \;\leq\; r' \cup^* q_{n+1} \;\leq\; r' \cup^* s \;\leq\; s.$$

Similarly, for (3.2), $r \leq s$ yields

$$r' \cup^* r \;\leq\; r' \cup^* s \;\leq\; r' \cup^* q_{n+1} \;\leq\; q_{n+1}.$$

Proof of (3): Fix $\gamma \in [i_n, i_{n+1}]$ and a generic filter $G \subseteq \mathbb{P}_\gamma$ which contains q_{n+1}. Note that part (2) implies $q_{n+1} \restriction \gamma \sim_{G \restriction i_n} \dot{q}_{n+1}^G \restriction \gamma$. Hence, $\dot{q}_{n+1}^G$ is in G.
This concludes the proof of Claim 1.

Checking that $\dot{p}_{n+1}, q_{n+1}$ satisfy (a)-(e): That the conditions (a)-(c) hold, follows straight from the construction of $\dot{p}_{n+1}$ and q_{n+1}. Hence, we only need to check that (d) and (e) hold.
(d): Let $G \subseteq \mathbb{P}_{i_{n+1}}$ be a generic filter that contains q_{n+1}. Then, q_n is in G. Hence, $\dot{p}_{n+1}^G \restriction i_n$ is in G, by definition of $\dot{p}_{n+1}$. Furthermore, we have $\dot{q}_{n+1}^G \in G$, by Claim 1. By the definition of $\dot{q}_{n+1}$, we have $\dot{q}_{n+1}^G \leq 1_{\mathbb{P}_{i_n}} \cup^* \dot{p}_{n+1}^G \restriction i_{n+1}$. Therefore, $1_{\mathbb{P}_{i_n}} \cup^* \dot{p}_{n+1}^G \restriction i_{n+1}$ is also in G. As $\dot{p}_{n+1}^G \restriction i_n$ and $1_{\mathbb{P}_{i_n}} \cup^* \dot{p}_{n+1}^G \restriction i_{n+1}$ are in G, we have $\dot{p}_{n+1}^G \restriction i_{n+1} \in G$.

(e): By Claim 1, q_{n+1} forces that $\dot{G}$ contains the $(M, \mathbb{P}_{i_{n+1}})$-generic condition $\dot{q}_{n+1}$. Hence, q_{n+1} is itself $(M, \mathbb{P}_{i_{n+1}})$-generic.

The generic condition q: We set $q = \bigcup_{n\in\omega} q_n$. Note that q is well-defined, has countable domain, and is such that $q \restriction i_n$ is q_n for all n. To show that q is $(M, \mathbb{P}_j)$-generic we will prove the following claim:

Claim 2: For ever $n < \omega$, we have $q \Vdash \dot{p}_{n+1} \in \dot{G}$.

Proof of Claim 2: To see this, fix $n < \omega$ and consider $D = \{r \in \mathbb{P}_j \mid r \Vdash \dot{p}_{n+1} \in G\}$. We will show that D is dense below q in $\mathbb{P}_j$. Fix $r \leq q$ in $\mathbb{P}_j$. Let $p_{n+1} \in \mathbb{P}_j \cap M$ and $q' \in \mathbb{P}_{i_n}$ with $q' \leq r \restriction i_n$ be such that $q' \Vdash \dot{p}_{n+1} = p_{n+1}$. Note that $q' \cup^* r \Vdash \dot{p}_{n+1} \restriction i_{n+1} \in \dot{G}$ holds, by (d). Further note that $q' \cup^* r \leq r$ and $q' \cup^* r \Vdash \dot{p}_{n+1} = p_{n+1}$ hold. Hence, it suffices to show that $q' \cup^* r \leq 1_{\mathbb{P}_{i_{n+1}}} \cup^* p_{n+1}$ holds. Fix $\gamma \in [i_{n+1}, j) \cap M$ and a generic filter $G \subseteq \mathbb{P}_\gamma$ that contains $q' \cup^* r \restriction \gamma$. Let $k \in \omega$ be such that $\gamma \in [i_k, i_{k+1})$. Note that we have $\dot{q}^G_{k+1} \restriction \gamma \in G$, since $q_{k+1} \restriction \gamma$ is in G. Then, also $p_{k+1} \restriction \gamma$ is in G, since $q_k \Vdash \dot{p}_{k+1} \restriction i_k \in \dot{G}$ and $\dot{q}^G_{k+1} \restriction \gamma \leq 1_{\mathbb{P}_{i_k}} \cup^* p_{k+1} \restriction \gamma$ hold. Hence, we have

$$q_{k+1}(\gamma)^G \;=\; (\dot{q}^G_{k+1}(\gamma))^G \;\leq\; (\dot{p}^G_{k+1}(\gamma))^G \;\leq\; (\dot{p}^G_{n+1}(\gamma))^G. \tag{3.3}$$

Since $q' \cup^* r \leq q_{k+1}$ holds and since G was arbitrary, we see that $q' \cup^* r \restriction \gamma$ forces $r(\gamma) \leq p_{n+1}(\gamma)$ for every $\gamma \in [i_{n+1}, j) \cap M$. As the support of p_{n+1} is a subset of M, we conclude $q' \cup^* r \leq 1_{\mathbb{P}_{i_{n+1}}} \cup^* p_{n+1}$, and thus $q' \cup^* r \Vdash \dot{p}_{n+1} \in \dot{G}$. As r was arbitrary, D is dense below q, and hence $q \Vdash \dot{p}_{n+1} \in \dot{G}$.

Claim 2 together with (c) shows that $q \Vdash D \cap M \cap \dot{G} \neq \emptyset$ holds for every dense $D \in M$ which is dense in $\mathbb{P}_j$. Thus, q is $(M, \mathbb{P}_j)$-generic. Since $\dot{p}_0$ is $\check{p}$, from the proof of Claim 2, we also see that $q \leq p$ holds. In particular, we can apply (3.3) for each $\gamma \in [i_0, j) \cap M$ and every generic filter $G \subseteq \mathbb{P}_\gamma$ which contains $q \restriction \gamma$, and with k such that $\gamma \in [i_k, i_{k+1})$. This concludes the limit step of the induction.

Final argument: To finish the proof of the theorem, we still need to show that $\mathbb{P}_\alpha$ is proper. Fix some $p \in \mathbb{P}_\alpha \cap M$, and note that $1_{\mathbb{P}_0}$ is $(M, \mathbb{P}_0)$-generic, as it is the only element of $\mathbb{P}_0$. Recall that $j = \alpha$ is in M. Clearly, also $i = 0$ is in M. Let $q_0 = 1_{\mathbb{P}_0}$. By $\mathit{Stat}(0, j)$, there is $q \in \mathbb{P}_\alpha$ such that $q \leq p$ and q is $(M, \mathbb{P}_\alpha)$-generic. This proves that, for any countable elementary submodel M of $H(\lambda)$ such that $\alpha, \langle \mathbb{P}_i, \dot{\mathbb{Q}}_j \mid i \leq \alpha,\ j < \alpha \rangle \in M$, and for each $p \in \mathbb{P}_\alpha \cap M$ there is an $(M, \mathbb{P}_\alpha)$-generic condition q such that $q \leq p$. Here, the assumptions on M are slightly stronger than in Theorem 3.31. In particular, we have additionally assumed that M always contains $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j \mid i \leq \alpha,\ j < \alpha \rangle$ and α. Nonetheless, the same argument as in the proof of Theorem 3.31 applies, and hence $\mathbb{P}_\alpha$ is proper. $\square$

Remark 3.39. Recall that, by Proposition 3.19, proper forcing notions preserve ω_1, and hence, by the above theorem, we conclude that any countable support iteration of proper forcing notions also preserves ω_1. Furthermore, Proposition 3.19 and Theorem 3.38 present a solution to the problem with the "general associativity law" for countable support iterations: In Proposition 3.3, if I is the ideal of countable subsets of α, and if $\mathbb{P}_\xi$ is proper, then $\dot{J}$ is a $\mathbb{P}_\xi$-name for the ideal of countable subsets. Hence, when iterating proper forcing notions, for $\xi < \alpha$, we can "split" any countable support iteration of length α into a countable support iteration of length ξ followed by a countable support iteration of length $\alpha \setminus \xi$.

Countable support iterations of proper forcing notions do not not collapse ω_1. For cardinals strictly greater than ω_1 one can use the following:

Definition 3.40. Let κ be a cardinal, We say that a forcing notion $\mathbb{P}$ has the *κ-chain condition* (κ-cc) if every antichain in $\mathbb{P}$ is of size $< \kappa$.

Proposition 3.41. *Assume* CH*. Let $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j \mid i \leq \omega_2,\ j < \omega_2 \rangle$ be a countable support iteration such "$\dot{\mathbb{Q}}_j$ is proper and $|\dot{\mathbb{Q}}_j| \leq \omega_1$" is forced for each $j < \omega_2$. Then, for every $i < \omega_2$ there is a dense set $D_i \subseteq \mathbb{P}_i$ with $|D_i| \leq \omega_1$, and $\mathbb{P}_{\omega_2}$ has the ω_2-cc.*

Proof. For each $i < \omega_2$, let $\dot{f}_i$ be a $\mathbb{P}_i$-name such that $1 \Vdash$"$\dot{f}_i$ is a surjection from ω_1 to $\dot{\mathbb{Q}}_i$ with $\dot{f}_i(0) = \dot{1}_i$", and for each $\mathbb{P}_i$-name τ for an ordinal $< \omega_1$, let $f_i(\tau)$ be a $\mathbb{P}_i$-name such that $1 \Vdash f_i(\tau) = \dot{f}_i(\tau)$. Recursively on $i < \omega_2$ define D_i and $\mathcal{N}_i$ as follows:

1. set $D_i = \{p \in \mathcal{P}_i \mid \forall k \in \operatorname{supp}(p) \exists \tau \in \mathcal{N}_k\ (p(k) = f_k(\tau))\}$, and
2. define $\mathcal{N}_i$ to be the $\subseteq$-minimal superset of $\{\check{\alpha} \mid \alpha < \omega_1\}$ with the following two closure properties:
   (A) for every countable sequence $\{p_n \mid n \in \omega\} \subseteq D_i$ and each countable sequence $\{\tau_n \mid n \in \omega\} \subseteq \mathcal{N}_i$, there is a $\mathbb{P}_i$-name $\sigma \in \mathcal{N}_i$ such that $\sigma^G = \tau_n$ for the minimal n with $p_n \in G$, if there is such n, and $\sigma^G = 0$ otherwise.
   (B) for each $\tau \in \mathcal{N}_i$, each $q \in D_i$ and every $\{\{\tau_n^m \mid n \in \omega\} \mid m \in \omega\} \subseteq [\mathcal{N}_i]^{\leq\omega}$ there is $\sigma \in \mathcal{N}_i$ with

$$q \Vdash \dot{f}_i(\sigma) \leq \dot{f}_i(\tau)\ \wedge\ \forall m\ (\{\dot{f}_i(\tau_n^m) \mid n \in \omega\} \text{ is predense below } \dot{f}_i(\sigma)),$$

if such a name σ exists in $\mathbb{P}_i$.

By applying `CH`, a straightforward induction on i shows that $|D_i| \leq \omega_1$ holds for all $i < \omega_2$.

We show that D_i is dense in $\mathbb{P}_i$ for all $i \leq \omega_2$, by induction on i. Fix $\alpha \leq \omega_2$, and assume D_i is dense for all $i < \alpha$. Let M be any countable elementary submodel of $H(\lambda)$, for some sufficiently large regular cardinal λ, such that α, $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j \mid i \leq \alpha,\ j < \alpha\rangle$, $\langle \dot{f}_i \mid i \leq \alpha\rangle$ and $\langle D_i \mid i \leq \alpha\rangle$ are all in M. To see that D_α is dense, we prove the statement $\forall i < j\ Stat(i,j)$ by induction for $j \leq \alpha$, where $Stat(i,j)$ is the following statement:

> If i, j are in M, then for each $p \in \mathbb{P}_j \cap M$, and every $(M, \mathbb{P}_i)$-generic $q_0 \in D_i$ such that $q_0 \leq p \restriction i$, there is $q \in D_j$ such that q is $(M, \mathbb{P}_j)$-generic, $q \leq p$ and $q \restriction i = q_0$.

Successor step: Note that this step reduces to $Stat(j, j+1)$. Fix $p \in \mathbb{P}_j \cap M$ and a $(M, \mathbb{P}_j)$-generic condition $q_0 \in D_j$ such that $q_0 \leq p \restriction j$. Let $\tau \in M$ be a $\mathbb{P}_j$-name for an ordinal $< \omega_1$ such that $1 \Vdash \dot{f}_j(\tau) = p(j)$. Fix a maximal antichain $A \subseteq D_j$ in M that decides the value of τ, i.e. for each $r \in A$ there is a check name τ_r for an ordinal $< \omega_1$ such that $r \Vdash \tau = \tau_r$. Note that $\tau' = \{(r, \tau_r) \mid r \in A \cap M\}$ is in $\mathcal{N}_j$ and τ' is either 0 or τ. Further note that $q_0 \Vdash \tau' = \tau$, since q_0 is $(M, \mathbb{P}_j)$-generic.
As $\dot{\mathbb{Q}}_j$ is forced to be proper, there is a $\mathbb{P}_j$-name $\dot{q}$ for an $(M[\dot{G}], \dot{\mathbb{Q}}_j)$-generic condition below $p(j)$. Let $\{\dot{D}^m \mid m \in \omega\}$ enumerate all $\mathbb{P}_j$-names in M that are names for dense subsets of $\dot{\mathbb{Q}}_j$, and for each m pick an enumeration $\{\tau^m_k \mid k < \omega_1\}$ in M such that $1 \Vdash \dot{D}^m = \{\dot{f}_j(\tau^m_k) \mid k < \omega_1\}$. For each $k \in \omega_1 \cap M$, define $(\tau^m_k)' \in \mathcal{N}_j$ from τ^m_k, as we defined τ' from τ. Since $\dot{q}$ is forced to be generic, we see that $\{\dot{f}_j(\tau^m_k) \mid k \in \omega_1 \cap M\}$ is forced to be predense below $\dot{q}$, for each $m < \omega$. Let σ be a $\mathbb{P}_j$-name for an ordinal $< \omega_1$ such that $1 \Vdash \dot{f}_j(\sigma) = \dot{q}$. Then, we have

$$q_0 \Vdash \dot{f}_j(\sigma) \leq \dot{f}_j(\tau') \ \wedge\ \forall m\ (\{\dot{f}_j((\tau^m_k)') \mid k \in \omega_1 \cap M\} \text{ is predense below } \dot{f}_j(\sigma)).$$

Hence, there is such a σ in $\mathcal{N}_j$. Set $q = q_0 \cup \{(j, f_j(\sigma))\}$. Then, we have $q \leq p$, since $q_0 \Vdash f_j(\sigma) = \dot{f}_j(\sigma) \leq \dot{f}_j(\tau') = p(j)$ holds. Furthermore, q_0 forces that $f_j(\sigma)$ is $(M[\dot{G}], \dot{\mathbb{Q}})$-generic, and thus q is $(M, \mathbb{P}_{j+1})$-generic, by 3.37. This concludes the successor step.

Limit step: Let $j \leq \gamma$ be a limit ordinal in M. Fix $i \in j \cap M$, and let $\{E_n \mid n \in \omega\}$ enumerate all dense open subsets of $\mathbb{P}_j$ that are in M. Take a strictly increasing sequence $\{i_n \mid n < \omega\}$ that is unbounded in $j \cap M$ and such that $i_0 = i$. Fix $p \in \mathbb{P}_j \cap M$ and an $(M, \mathbb{P}_i)$-generic condition $q_0 \leq p \restriction i$. We construct sequences $(\dot{p}_n)_{n\in\omega}$ and $(q_n)_{n\in\omega}$ with $\dot{p}_0 = \check{p}$ such that the following hold for all $n \in \omega$:

(a) $q_n \in D_{i_n}$ and $q_{n+1} \restriction i_n = q_n$;
(b) $\dot{p}_{n+1}$ is a $\mathbb{P}_{i_n}$-name for a condition in $\mathbb{P}_j \cap M$ such that $\dot{p}_{n+1} \leq \dot{p}_n$ is forced;
(c) $q_n \Vdash \dot{p}_{n+1} \in E_n$;
(d) $q_n \Vdash \dot{p}_n \restriction i_n \in \dot{G}$;
(e) q_n is $(M, \mathbb{P}_{i_n})$-generic;
(f) there are a maximal antichain $A_{n+1} \subseteq D_{i_n}$ and a set $\{p^r_{n+1} \mid r \in A_{n+1}\} \subseteq \mathbb{P}_j$ in M such that for any generic filter $G \subseteq \mathbb{P}_{i_n}$, $\dot{p}^G_{n+1} = p^r_{n+1}$ if $r, q_n \in G$, and $\dot{p}^G_{n+1} = \dot{p}^G_n$ otherwise.

Assume $q_n, \dot{p}_n$ have been defined according to the conditions (a)-(f), and define $\dot{p}_{n+1}$ and q_{n+1} as described below.

Construction of $\dot{p}_{n+1}$*:* Fix $r \in A_n$. Pick a $\mathbb{P}_{i_n}$-name $\dot{p}^r$ for a condition in $\mathbb{P}_j$ such that $\dot{p}^r \leq p^r_n$ and $\dot{p}^r \in E_n$ are forced and such that also $\dot{p}^r \restriction i_n \in \dot{G}$ if possible. Take a maximal antichain $A^r_n \subseteq D_n$ below r in $\mathbb{P}_{i_n}$ such that A^r_n forces a value on $\dot{p}^r$, i.e. for all $s \in A^r_n$ exists $p^s_{n+1} \in \mathbb{P}_j$ such that $s \Vdash \dot{p}^r = p^s_{n+1}$. By elementarity, we can assume that $\{\dot{p}^r \mid r \in A_n\}$ and $\{A^r_n \mid r \in A_n\}$ are in M, and such that $r \in M$ implies $\dot{p}^r, A^r_n \in M$. Then, use existential completeness to define $\dot{p}_{n+1}$ such that $\dot{p}^G_{n+1} = (\dot{p}^r)^G$ for the unique $r \in G \cap A_n \cap M$, whenever $q_n \in G$, and $\dot{p}^G_{n+1} = \dot{p}^G_n$ otherwise. Set $A_{n+1} = \{s \mid \exists r \in A_n\ (s \in A^r_n)\}$, and note that A_{n+1} is a maximal antichain, and A_{n+1} can be defined in M. Further note that $\dot{p}_{n+1}$ is a name for a condition in $\mathbb{P}_j \cap M$, since $q_n \Vdash M[\dot{G}] \cap V = M$ and thus $q_n \Vdash \dot{p}^r \in M$ for every $r \in A_n \cap M$. Furthermore, if $G \subseteq \mathbb{P}_{i_n}$ is a generic filter that contains q_n, then also $\dot{p}^G_n \restriction i_n$ is in G, and as $\dot{p}^G_n = p^r_n$ for the unique $r \in G \cap A_n$, we find $p' \leq p^r_n$ in E_n such that $p' \restriction i_n$ is in G (cf. *Construction of* $\dot{p}_{n+1}$ in the proof of Theorem 3.38). Hence, we have $q_n \Vdash \dot{p}_{n+1} \restriction i_n \in \dot{G}$.

Definition of q_{n+1}: For each s in $A_{n+1} \cap M$, we can find an $(M, \mathbb{P}_{i_{n+1}})$-generic condition $q^s \in D_{i_{n+1}}$ such that $q^s \leq 1_{\mathbb{P}_{i_n}} \cup^* p^s_{n+1} \restriction i_{n+1}$ and $q^s \restriction i_n = q_n$, by $Stat(i_n, i_{n+1})$. For for each $k \in \operatorname{supp}(q^s)$ there is $\tau^s_k \in \mathcal{N}_k$ such that $q^s(k) = f_k(\tau^s_k)$. Then, for each $k \in \bigcup\{\operatorname{supp}(q^s) \mid s \in A_{n+1} \cap M\}$ we find $\sigma_k \in \mathcal{N}_k$ such that $\sigma^G_k = (\tau^s_k)^G$ if $k \in \operatorname{supp}(q^s)$ for some $s \in G \cap A_{n+1} \cap M$, and $\sigma^G_k = 0$ otherwise. Set $q_{n+1} \restriction i_n = q_n$. Let $q_{n+1}(k) = f_k(\sigma_k)$ if $k \in [i_n, i_{n+1}) \cap \bigcup_{s\in A_{n+1}\cap M} \operatorname{supp}(q^s)$, and $q_{n+1}(k) = \dot{1}_k$ for the remaining k.

Let $\dot{q}_{n+1}$ be a $\mathbb{P}_{i_n}$-name such that $\dot{q}^G_{n+1} = q^s$ whenever there is some $s \in A_{n+1} \cap M \cap G$ and $\dot{q}^G_{n+1} = q_n \cup^* 1_{\mathbb{P}_{n+1}}$ otherwise. Then, q_n forces that $\dot{q}_{n+1}$ is $(M, \mathbb{P}_{i_{n+1}})$-generic, and we have:

Claim 1: The condition q_{n+1} has the following properties:

(1) $q_{n+1}(\gamma)^G = (\dot{q}_{n+1}(\gamma))^G$, for $\gamma \in [i_n, i_{n+1})$ and every generic filter $G \subseteq \mathbb{P}_\gamma$;
(2) $q_{n+1} \sim_G \dot{q}_{n+1}^G$, for every generic filter $G \subseteq \mathbb{P}_{i_n}$ which contains q_n;
(3) $q_{n+1} \restriction \gamma \Vdash \dot{q}_{n+1} \restriction \gamma \in \dot{G}$, for $\gamma \in [i_n, i_{n+1}]$.

Proof of Claim 1: We have $1 \Vdash \dot{q}_{n+1} \restriction i_n = q_{n+1} \restriction i_n$. Fix such a $\gamma \in [i_n, i_{n+1})$ and let $G \subseteq \mathbb{P}_\gamma$ be a generic filter. If there is is no $s \in G \cap A_{n+1} \cap M$ or if there is such s but $\gamma \notin \operatorname{supp}(q^s)$, then $q_{n+1}(\gamma)^G = \dot{f}_\gamma^G(0) = 1_{\dot{\mathbb{Q}}_\gamma^G} = (\dot{q}_{n+1}^G(\gamma))^G$. Otherwise, there is a unique $s \in G \cap A_{n+1} \cap M$ and we have $q_{n+1}(\gamma)^G = f_\gamma(\tau_k^s)^G = q^s(\gamma)^G$. This concludes the proof of (1). Then, (2) and (3) follow as in the proof of 3.38.

That the conditions $(a) - (c)$ and (f) hold is clear from the construction of $\dot{p}_{n+1}$ and q_{n+1}. For (d) and (e) we can argue as in the proof of 3.38.

As in the proof of 3.38 set $q = \bigcup_{n \in \omega} q_n$. Then, q is the desired condition in D_j, and we are finished with the induction argument for $\forall j \leq \alpha \forall i < j \; Stat(i, j)$. To see that D_α is dense in $\mathbb{P}_\alpha$, first fix $p \in \mathbb{P}_\alpha$, then pick an appropriate countable model M with $p \in M$ and use $Stat(0, \alpha)$ to obtain $q \in D_\alpha$ such that $q \leq p$.

$\mathbb{P}_{\omega_2}$ **has the ω_2-cc:** Fix a subset $\{p_i \mid i < \omega_2\} \subseteq \mathbb{P}_{\omega_2}$, define $F : \omega_2 \to \omega_2$ by $F(i) = \sup(i \cap \operatorname{supp}(p_i))$, and note that F is regressive on $S = \{i < \omega_2 \mid i \text{ has uncountable cofinality}\}$, i.e. $F(i) < i$ for all $i \in S$. Next, note that S is a stationary subset of ω_2, since for every closed unbounded $C \subseteq \omega_2$ and any strictly increasing sequence of length ω_1 in C, the supremum of the sequence is in $C \cap S$ and has cofinality ω_1. By Fodor's lemma, there is a stationary subset S' of ω_2 such that $S' \subseteq S$, and such that $F(i) = \alpha$ for all $i \in S'$ and some fixed $\alpha < \omega_2$.[13] Next, we choose a strictly increasing sequence $\{c_i \mid i < \omega_2\} \subseteq \omega_2$ such that $c_0 = 0$, $c_{j+1} > c_j$ is such that $\bigcup_{i<j} \operatorname{supp}(p_{c_i}) \subseteq c_{j+1}$, and if j is a limit ordinal then $c_j = \sup\{c_i \mid i < j\}$. Then, $C = \{c_i \mid i < \omega_2\}$ is a closed unbounded subset of ω_2, and hence $S' \cap C$ is still stationary. Now, consider $i, j \in S' \cap C$ such that $i < j$, and note that $\operatorname{supp}(p_i) \subseteq j$ while $\operatorname{supp}(p_j) \setminus \alpha \subseteq \omega_2 \setminus j$. Hence, we have $\operatorname{supp}(p_i) \cap \operatorname{supp}(p_j) \subseteq \alpha$. It follows that p_i is compatible with p_j if and only if $p_i \restriction \alpha$ is compatible with $p_j \restriction \alpha$. As $\mathbb{P}_\alpha$ has a dense subset of cardinality $\leq \omega_1$, we see that $\mathbb{P}_\alpha$ has the ω_2-cc, and since $S' \cap C$ has cardinality ω_2, there must be two distinct ordinals i, j in $S' \cap C$ such that $p_i \setminus \alpha$ and $p_j \setminus \alpha$ are compatible. Hence, there is no antichain of size ω_2 in $\mathbb{P}_{\omega_2}$. $\square$

[13] A proof of Fodor's lemma can be found in [6, Part I, Theorem 8.7].

Remark 3.42. Forcing notions that have the κ-cc preserve all cardinals $\geq \kappa$.[14] Hence, under $\mathtt{CH}$, any countable support iteration as in the premise of 3.41, does not collapse any cardinals.

In remaining part of this section we show that a proper countable support iterations can only add reals at successor steps and limit steps of countable cofinality, and we prove that $\mathtt{CH}$ is preserved when we force with a countable support iterations of length $< \omega_2$, if the iterants are forced to be proper and of size $\leq \mathfrak{c}$.

Proposition 3.43. *Let $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j \mid i \leq \alpha,\ j < \alpha \rangle$ be a countable support iteration with $\mathrm{cf}(\alpha) \geq \omega_1$ and such that $\mathbb{P}_\alpha$ is proper. For any generic filter $G \subseteq \mathbb{P}_\alpha$ and any real $f \in 2^\omega$ in $V[G]$, there is $\beta < \alpha$ such that f is in $V[G \restriction \beta]$.*

Proof. Let $\dot{f}$ be a name for a function from ω to 2. Fix a generic filter $G \subseteq \mathbb{P}_\alpha$ and a sufficiently large regular cardinal λ. Consider the set

$$D = \{q \in \mathbb{P}_\alpha \mid \exists M\ (|M| = \omega\ \wedge\ M \prec H(\lambda)\ \wedge\ \mathbb{P}, \dot{f} \in M\ \wedge\ q \text{ is } (M, \mathbb{P}_\alpha)\text{-generic}\},$$

and note that D is dense, since $\mathbb{P}_\alpha$ is proper. Hence, there is some $q \in D \cap G$ and a countable elementary submodel M of $H(\lambda)$ such that q is $(M, \mathbb{P}_\alpha)$-generic and $\dot{f} \in M$. For each $n < \omega$, let $A_n \in M$ be a maximal antichain such that A_n decides $\dot{f}(n)$, i.e. for each $r \in A_n$ there is $i^r_n < 2$ such that $r \Vdash \dot{f}(n) = i^r_n$. Set $\tau = \{(r, \langle n, i^r_n \rangle) \mid n < \omega,\ r \in A_n \cap M\}$, where $\langle n, i^r_n \rangle$ is a name for the pair (n, i^r_n). Note that $q \Vdash \dot{f} = \tau$, since q is $(M, \mathbb{P}_\alpha)$-generic. Let β be the supremum of $\bigcup\{\mathrm{supp}(r) \mid n < \omega,\ r \in A_n \cap M\}$, and note that $\beta < \alpha$ holds, as $\bigcup\{\mathrm{supp}(r) \mid n < \omega,\ r \in A_n \cap M\}$ is countable. Now, τ can be regarded as a $\mathbb{P}_\beta$-name and we have $\tau^{G \restriction \beta} = \tau^G = \dot{f}^G$. Hence, $\dot{f}^G$ is in $V[G \restriction \beta]$. □

Lemma 3.44. *Assume $\mathtt{CH}$. Let $\mathbb{P}$ be a proper forcing notion such that $|\mathbb{P}| \leq \omega_1$. Then, forcing with $\mathbb{P}$ preserves $\mathtt{CH}$.*

Proof. Let $\mathcal{N}$ be the set of all names of the form $\{(r, \langle n, i^r_n \rangle) \mid n < \omega, r \in A_n\}$, where each A_n is some countable antichain and $\langle n, i^r_n \rangle$ is a name for the pair $(n, i^r_n) \in \omega \times 2$. Note that $|\mathcal{N}| \leq |(\omega_1^\omega \times 2^\omega)^\omega| = \omega_1$, by $\mathtt{CH}$.
Fix a name $\dot{f}$ for a function from ω to 2. Let M be a countable elementary submodel of $H(\lambda)$ for some sufficiently large regular cardinal λ, such that $\mathbb{P}$ and $\dot{f}$ are in M. Define τ from $\dot{f}$ as in the proof of Proposition 3.43, and note that τ is in $\mathcal{N}$. Further note that, for any $(M, \mathbb{P})$-generic condition q, we have $q \Vdash \tau = \dot{f}$ and $q \Vdash \exists \tau \in \mathcal{N}\ \dot{f} = \tau$. As $\mathbb{P}$ is proper, the set $\{q \in \mathbb{P} \mid q \Vdash \exists \tau \in \mathcal{N}\ \dot{f} = \tau\}$ is

[14] A proof of this can be found in [9, Chapter VII]

dense, and as $\dot{f}$ was arbitrary, we have $1 \Vdash |2^\omega| \leq |\mathcal{N}|$. Hence, $1 \Vdash |2^\omega| = \omega_1$ must hold. $\square$

Proposition 3.45. *Assume* CH. *Let* $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j \mid i \leq \alpha,\ j < \alpha \rangle$ *be a countable support iteration with* $\alpha < \omega_2$ *and such that* $1 \Vdash$ *"*$\dot{\mathbb{Q}}_j$ *is proper and of size* $\leq \mathfrak{c}$*" holds for every* $j < \alpha$*. Then,* $\mathbb{P}_\alpha$ *has a dense subset of size* $\leq \omega_1$*, and hence* $\mathbb{P}_\alpha$ *preserves* CH*.*

Proof. We prove, by induction on $j \leq \alpha$, that $1 \Vdash |\dot{\mathbb{Q}}_j| \leq \omega_1 = \mathfrak{c}$ holds for all j. Note that $\mathbb{P}_0$ is trivial, and thus $|\dot{\mathbb{Q}}_0^G| \leq \mathfrak{c}^{V[G]} = \mathfrak{c}^V = \omega_1$ holds for any generic filter $G \subseteq \mathbb{P}_0$. Now, assume $1 \Vdash |\dot{\mathbb{Q}}_i| \leq \omega_1$ holds for all $i < j$. By 3.41, there is a dense subset $D_j \subseteq \mathbb{P}_j$ such that $|D_j| \leq \omega_1$. By 2.7, forcing with D_j is equivalent to forcing with $\mathbb{P}_j$. Hence, $\mathbb{P}_j$ preserves CH, by Lemma 3.44. Thus, we get $1 \Vdash |\dot{\mathbb{Q}}_j| \leq \mathfrak{c} = \omega_1$. $\square$

4 A countable support iteration of Miller forcing

In this last section, we will study the forcing extension formed by a ω_2-length countable support iteration of so called *Miller forcing*. In particular, we will determine the values of two *cardinal characteristics* for this extension. The discussion of Miller forcing and the proof for the value of the two cardinal characteristics in the forcing extension, are based on Dow [11, 8. Iterations - case studies].

4.1 Three cardinal characteristics

All cardinal characteristics that we will consider are so called "cardinal characteristics of the continuum". These are properties of the real numbers that are expressed as cardinals. Two simple examples are the *uniformity of the Lebesgue-measure-0 ideal* and the *uniformity of the meager ideal*, which are usually denoted by $\text{non}(\mathcal{L})$ and $\text{non}(\mathcal{B})$, respectively: $\text{non}(\mathcal{L})$ is the minimal cardinality of a non-measure-0 set of reals and $\text{non}(\mathcal{B})$ is the minimal cardinality of a non-meager set of reals. Clearly, both of these cardinals are at most $\mathfrak{c}$, and since the two ideals are closed under countable unions and they contain all singletons, we can also deduce that $\text{non}(\mathcal{L})$ and $\text{non}(\mathcal{B})$ are uncountable cardinals. However, their exact values are not determined by ZFC. In particular, it is consistent with ZFC that $\text{non}(\mathcal{L})$ and $\text{non}(\mathcal{B})$ are strictly below $\mathfrak{c}$ (see Chapter 6 by Blass in "Handbook of Set Theory" [12]).

Usually cardinal characteristics of the continuum can be defined in the following way: First, we define a nom-empty family $\mathcal{X}$ of subsets of the reals; then, this family defines a cardinal characteristic $\mathfrak{x} = \min\{|X| \mid X \in \mathcal{X}\}$, where "the reals" usually refers to 2^ω, ω^ω or $[\omega]^\omega$. In particular, all three cardinal characteristics that will become relevant to the ongoing discussion can be defined in this way.

Definition 4.1.

1. a. A family $\mathcal{F}$ has the *strong finite intersection property* (SFIP) if for every finite $\mathcal{F}' \subseteq \mathcal{F}$ the intersection $\bigcap \mathcal{F}'$ is infinite.
   b. A family $\mathcal{F}$ has a *pseudo-intersection* if there is an infinite set S such that $S \setminus X$ is finite for every $X \in \mathcal{F}$.
   c. The *pseudo-intersection number*, $\mathfrak{p}$, is the smallest cardinality of a family of infinite subsets of ω that has the SFIP but has no pseudo-intersection.
2. a. A family $\mathcal{A} \subseteq [\omega]^\omega$ is *almost disjoint* if $X \cap Y$ is finite for all $X, Y \in \mathcal{A}$.
   b. A *maximal almost disjoint* (mad) family is an infinite almost disjoint family $\mathcal{A}$ such that there exists no $X \in [\omega]^\omega$ which is almost disjoint from every $Y \in \mathcal{A}$, i.e. $\forall X \in [\omega]^\omega \exists Y \in \mathcal{A}\ (X \cap Y$ is infinite).
   c. The *almost disjoint number*, $\mathfrak{a}$, is the smallest cardinality of a mad family.
3. a. For $f, g \in \omega^\omega$ we write $f \leq^* g$ if f is eventually below g, i.e. there exists a natural number m such that $\forall n \geq m\ f(n) \leq g(n)$. Note that $\leq^*$ defines a preorder on ω^ω.
   b. A family $\mathcal{D} \subseteq \omega^\omega$ is *dominating* if for every $f \in \omega^\omega$ there is some $g \in \mathcal{D}$ such that $f \leq^* g$.
   c. The *dominating number*, $\mathfrak{d}$, is the smallest cardinality of a dominating family.

Remark 4.2. Clearly, ω^ω is dominating, and hence $\mathfrak{d}$ exists. Any infinite partition of ω into infinite subsets, can be extended to a mad family using Zorn's lemma, and thus also $\mathfrak{a}$ exists. The construction of a family of infinite subsets of ω that has the SFIP but has no pseudo-intersection can also be done using Zorn's lemma. To see this, note that we can partition a pseudo-intersection S of a family $\mathcal{F}$ into two infinite sets S_0 and S_1. Now, S is not a pseudo-intersection of $\mathcal{F} \cup \{S_0\}$, and if $\mathcal{F}$ has the SFIP, then so does $\mathcal{F} \cup \{S_0\}$. This shows that any $\mathcal{F} \subseteq [\omega]^\omega$, which is $\subseteq$-maximal among all families of infinite subsets of ω that have the SFIP, has no pseudo-intersection. By Zorn's lemma there is such a maximal family, and thus $\mathfrak{p}$ exists.

We can immediately tell that all three cardinal characteristics in question are at most $\mathfrak{c}$, as each of them refers to the size of a set of reals. Furthermore, the following proposition shows that $\mathfrak{a}$, $\mathfrak{d}$ and $\mathfrak{p}$ are uncountable:

Proposition 4.3.

1. *If a countable family of infinite subsets of ω has the SFIP, then it has a pseudo-intersection.*
2. *If $\mathcal{A}$ is a countably infinite, almost disjoint family, then there is some $X \in [\omega]^\omega$ which is almost disjoint from every $Y \in \mathcal{A}$.*
3. *If $\mathcal{D} \subseteq \omega^\omega$ is countable then there is some $f \in \omega^\omega$ which is not eventually below any $g \in \mathcal{D}$.*

Proof. 1: Let $\mathcal{F} = \{X_n \mid n \in \omega\} \subseteq [\omega]^\omega$ be a countable family which has the SFIP. Note that, for all $n \in \omega$, $Y_n = \bigcap_{k \leq n} X_k$ is infinite. Therefore, we can choose $s_n \in Y_n \setminus \{s_k \mid k < n\}$ for every $n \in \omega$. Then, $S = \{s_n \mid n \in \omega\}$ is a pseudo-intersection for $\mathcal{F}$.

2: Let $\mathcal{A} = \{A_n \mid n \in \omega\} \subseteq [\omega]^{\leq\omega}$ be an almost disjoint family, such that $A_m \neq A_n$ whenever $m \neq n$. Then, for all $n \in \omega$, $Y_n = \omega \setminus \bigcup_{k \leq n} A_k$ is infinite. To see this, simply note that Y_n is a superset of $A_{n+1} \setminus \bigcup_{k \leq n}(A_k \cap A_{n+1})$. Now, we can choose $x_n \in Y_n \setminus \{x_k \mid k < n\}$ for every $n \in \omega$, and $X = \{x_n \mid n \in \omega\}$ is almost disjoint from every member of $\mathcal{A}$.

3: Let $\mathcal{D} = \{g_n \mid n \in \omega\} \subseteq \omega^\omega$ be a countable family. Define $f(n) = 1 + \sum_{k \leq n} g_k(n)$. Then, $\exists m \in \omega \forall k > m \; g_n(k) < f(k)$ holds for every $n \in \omega$, and thus $f \not\leq^* g_n$ holds for all n. □

Note that, under $\mathtt{CH}$, the three cardinal characteristics, $\mathfrak{a}$, $\mathfrak{d}$, $\mathfrak{p}$, are all equal to ω_1. In particular, we can consistently add $\mathfrak{p} = \mathfrak{c}$ to the assumptions of a theorem. Under this assumption we can construct an infinite almost disjoint family $\mathcal{A}$ with a particular combinatorial property which implies the maximality of $\mathcal{A}$. As we will see later, this property also guarantees that $\mathcal{A}$ is still maximal in iterated Miller forcing extensions.

Proposition 4.4. *Assume $\mathfrak{p} = \mathfrak{c}$ and $\{\{B_i^n \mid n < \omega\} \mid i < \mathfrak{c}\}$ is an enumeration of all infinite subsets of $[\omega]^{<\omega}$ such that $B_i^m \neq B_i^n$ whenever $m \neq n$. For $j < \mathfrak{c}$, denote by $\mathcal{I}_j$ the closure of $\{A_k \mid k < j\} \cup [\omega]^{<\omega}$ w.r.t. finite unions. Then, there is a mad family $\mathcal{A} = \{A_i \mid i < \mathfrak{c}\}$ such that the following holds for all ordinals i, j with $i < j < \mathfrak{c}$:*

$$\forall I \in \mathcal{I}_j \;\; (\{n \mid B_i^n \cap I = \emptyset\} \text{ is infinite}) \Rightarrow \{n \mid B_i^n \subseteq A_j\} \text{ is infinite.}$$

Proof. Take any infinite partition $\{A_n \mid n \in \omega\}$ of ω into infinite sets. We will recursively construct A_j for $j \in \mathfrak{c} \setminus \omega$.

Assume $\{A_i \mid i < j\}$ have been chosen and let J be the set of all $i < j$ such that $\{n \mid B_i^n \cap I = \emptyset\}$ is infinite for all $I \in \mathcal{I}_j$. For $i < j$ and $n < \omega$, define

$$X_i = \{x \in [\omega]^{<\omega} \mid x \cap A_i = \emptyset\} \quad \text{and} \quad Y_i^n = \{x \in X_i \mid \exists m > n\ (B_i^m \subseteq x)\}.$$

Now, set $\mathcal{F} = \{X_i \mid i < j\} \cup \{Y_i^n \mid i \in J,\ n \in \omega\}$. We claim that $\mathcal{F}$ has the the SFIP. First, consider $X_i \cap X_{i'}$ for $i, i' < j$, and note that $X_i \cap X_{i'}$ contains every finite subset of $A_k \setminus (A_i \cup A_{i'})$, for any $k < j$, and if k is neither i nor i', then $A_k \setminus (A_i \cup A_{i'})$ is infinite. Clearly, this generalises to any finite intersection of the form $\bigcap_{k<N} X_{i_k}$. Next, consider $Y_i^n \cap Y_{i'}^{n'}$ for $i, i' \in J$ and $n, n' \in \omega$. Note that $I = A_i \cup A_{i'}$ is a member of $\mathcal{I}_j$, and thus the two sets $\{m \mid B_i^m \cap I = \emptyset\}$ and $\{m \mid B_{i'}^m \cap I = \emptyset\}$ are infinite. As $\{B_i^m \cup B_{i'}^{m'} \mid m > n,\ m' > n',\ (B_i^m \cup B_{i'}^{m'}) \cap I = \emptyset\}$ is a subset of $Y_i^n \cap Y_{i'}^{n'}$, we see that $Y_i^n \cap Y_{i'}^{n'}$ is infinite. Finally, consider $X_i \cap Y_{i'}^n$ for $i < j$, $i' \in J$ and $n < \omega$. Simply note that $\{B_{i'}^m \mid m > n,\ B_{i'}^m \cap I = \emptyset\}$ is an infinite subset of $X_i \cap Y_{i'}^n$, where $I = A_i \cup A_{i'}$. In the general case, where we consider a finite intersection of the form $\bigcap_{k<N} X_{i_k} \cap \bigcap_{k'<N'} Y_{i'_{k'}}^{n_{k'}}$ with $N' > 0$, we note that the set

$$\{\bigcup_{k'<N'} B_{i'_{k'}}^{m'_{k'}} \mid \forall k' < N'\ (m'_{k'} > n_{k'}) \ \wedge \bigcup_{k'<N'} B_{i'_{k'}}^{m'_{k'}} \cap I = \emptyset\}$$

is infinite, where $I = \bigcup_{k<N} A_{i_k} \cup \bigcup_{k'<N'} A_{i'_{k'}}$.This proves that $\mathcal{F}$ has the SFIP. Since we have $|\mathcal{F}| < \mathfrak{c} = \mathfrak{p}$ (and $|[\omega]^{<\omega}| = \omega$), there is a pseudo-intersection $S \subseteq [\omega]^{<\omega}$ for $\mathcal{F}$. Take $A_j = \bigcup S$. Note that A_j is almost disjoint from A_i for every $i < j$, since we have $A_i \cap A_j = (\bigcup S) \setminus (\bigcup X_i) \subseteq \bigcup (S \setminus X_i)$. Furthermore, if i is in J, then for each $n < \omega$, there is some $x \in S \cap Y_i^n$ which must be such that $B_i^m \subseteq x$ for some $m > n$, and hence $\{m \mid B_i^m \subseteq A_j\}$ is infinite. This shows that A_j is as desired.

The only thing left to prove, is the maximality of $\mathcal{A} = \{A_i \mid i < \mathfrak{c}\}$. Assume for the sake of contradiction that there is some $A \in [\omega]^\omega$ which is almost disjoint from ever member of $\mathcal{A}$. Set $B = \{\{a\} \mid a \in A\}$. Then, there is $i < \mathfrak{c}$ such that $B = \{B_i^n \mid n < \omega\}$. Fix j such tat $i < j < \mathfrak{c}$, and note that $A \cap I$ is finite for each $I \in \mathcal{I}_j$, since A is almost disjoint from every member of $\mathcal{A}$. Then, $\{n \mid B_i^n \cap I = \emptyset\}$ is infinite for every $I \in \mathcal{I}_j$, and thus $\{n \mid B_i^n \subseteq A_j\}$ is infinite, which contradicts the assumption that A is almost disjoint from A_j. $\square$

4.2 Miller forcing

In this subsection, we discuss the so called *Miller forcing*, which is a proper forcing notion and adds a function $d : \omega \to \omega$, such that $d \not\leq^* f$ holds for every $f : \omega \to \omega$ in the ground model. First, we have the following definitions:

Definition 4.5. Let X be a set.

1. A *tree* on X is a set $T \subseteq X^{<\omega}$, such that T is closed with respect to initial segments, i.e. $\forall s, t \in X^{<\omega}$ $(t \in T \wedge s \subseteq t \Rightarrow s \in T)$. Elements of T are called *nodes*. If $S \subseteq T$ is closed w.r.t initial segments, then we call S a *subtree* of T.
2. If s is a finite sequence of length n, then $s^\frown x$ denotes the finite sequence $s \cup \{(n, x)\}$ of length $n + 1$. For a tree T, if $s^\frown x$ is in T, we call $s^\frown x$ an *immediate successor* of s in T. The set of all immediate successors of s in T is denoted by $\mathrm{Succ}_T(s)$.
3. For a tree T, a node $s \in T$ is called *splitting* if $\mathrm{Succ}_T(s)$ is infinite, and T is called *splitting*, if every node of T can be extended to a splitting node, i.e. $\forall s \in T \exists t \in T$ $(s \subseteq t$ and t is splitting$)$.

Now, we define the Miller forcing notion as follows:

Definition 4.6. *Miller forcing* consists of the set of all non-empty, splitting trees on ω together with the partial order given by inclusion, i.e. $S \leq T \Leftrightarrow S \subseteq T$. We denote this forcing notion by $\mathbb{M}$. The maximal element $1_{\mathbb{M}}$ is $\omega^{<\omega}$.

We also use the following definitions:

Definition 4.7. Assume T is a tree.

1. We say two nodes $s, t \in T$ are *comparable* if $s \subseteq t$ or $t \subseteq s$ holds. If s and t are not comparable, we say that they are *incomparable*.
2. If T has at least two incomparable nodes, then the $\subseteq$-greatest element of the set $\{s \in T \mid s \text{ is comparable with every } t \in T\}$ is called the *root* of T.
3. A *branch* of T is a subtree $B \subseteq T$ such that all members of B are pairwise comparable.
4. The n-th *splitting level* of T is the set

$$\mathrm{Lev}_T(n) = \{t \in T \mid t \text{ is splitting and } |\{s \subseteq t \mid t \text{ is splitting}\}| = n + 1\}.$$

   A node $s \in \mathrm{Lev}_T(n)$ is called an *n-th splitting node* of T.
5. For $s \in T$, the *restriction* of T to s is the set

$$T_s = \{t \in T \mid t \text{ is comparable with } s\}.$$

Note that, T_s is a subtree of T, and if T is splitting, then so is T_s.

As mentioned at the beginning of the chapter, Miller forcing adds a particular real $d \in \omega^\omega$ which is not eventually bounded by any ground model real.

Proposition 4.8. *For any generic filter $G \subseteq \mathbb{M}$, we can define the so called Miller real $d : \omega \to \omega$ by $d = \bigcup \bigcap G$, and for every $f \in (\omega^\omega)^V$ the set $\{n \in \omega \mid f(n) < d(n)\}$ is infinite.*

Proof. Fix a generic filter $G \subseteq \mathbb{M}$. Note that $\bigcap G$ is downward closed, i.e. $\bigcap G$ is a tree. Next, note that if $S, T \in \mathbb{M}$ are two trees such that the root r of S is not a node in T, then $S \cap T$ is a finite set of initial segments of r. Hence, any two compatible conditions $S, T \in \mathbb{M}$, are such that the root of S is a member of T, and the root of T is a member of S. Therefore, the set $\{r \mid \exists T \in G \ (r \text{ is the root of } T)\}$ is a subset of $\bigcap G$. Next, note that, for each $n \in \omega$, the set $A_n = \{(\omega^{<\omega})_s \mid s \in \omega^n\}$ is a maximal antichain in $\mathbb{M}$, and thus $G \cap A_n$ has exactly one element. For $n \in \omega$, let r_n be the unique finite sequence such that $G \cap A_n = \{(\omega^{<\omega})_{r_n}\}$. Now, if s is in $\omega^n \cap \bigcap G$ for some $n \in \omega$, then s is a member of $(\omega^{<\omega})_{r_n}$, and thus s must be r_n. We conclude that $\bigcap G$ is the branch $\{r_n \mid n \in \omega\}$, and hence $d = \bigcup \bigcap G$ is indeed a function from ω to ω. We have $d(k) = r_n(k)$ whenever $k < n$.
For the second part, fix a ground model real $f \in (\omega^{<\omega})^V$ and some natural number $n \in \omega$. Let $\dot{d}$ be a name for the Miller real. We claim that the set $D_n = \{T \mid \exists k \geq n \ (T \Vdash \dot{d}(k) > f(k))\}$ is dense in $\mathbb{M}$, for every $n \in \omega$. Fix $T \in \mathbb{M}$ and let $r \in T$ be a splitting node of T with length $k \geq n$. Now, define $T' = \{s \in T_r \mid k \in \mathrm{dom}(s) \Rightarrow s(k) > f(k)\}$. Note that T' a non-empty, splitting Tree with root r and $T' \leq T$. Now, if G is a generic filter that contains T', then $d(k) = s(k)$ for some $s \in T'$, and hence $T' \Vdash \dot{d}(k) > f(k)$. This shows that D_n is dense for every n, and thus $\{n \mid d(n) > f(n)\}$ is infinite in any forcing extension. □

In some proofs it easier to consider only those conditions of $\mathbb{M}$ for which each node is either splitting or has a unique immediate successor. The below proposition justifies this approach.

Proposition 4.9. *The set of all non-empty trees $T \in \mathbb{M}$, such that each node of T is either splitting or has a unique immediate successor, is dense in $\mathbb{M}$.*

Proof. Let $T \in \mathbb{M}$ and fix a well-order R on T. Let s_0 be the R-minimal splitting node of T, and recursively define a decreasing sequence of subtrees $(T_n)_{n\in\omega}$ by $T_0 = T_{s_0}$ and for each $n \in \omega$:

$$T_{n+1} = \bigcup\{(T_n)_{s(n,t)} \mid s \in \mathrm{Lev}_{T_n}(n),\ t \in \mathrm{Succ}_{T_n}(s)\},$$

where $s(n,t)$ is the R-minimal splitting node in T_n that extends t. As T_{n+1} is the union over a non-empty set of non-empty splitting trees, T_{n+1} is itself non-empty and splitting. Furthermore, we have $(T_{n+1})_t = (T_n)_{s(n,t)}$ holds whenever t is an immediate successor of some $s \in \mathrm{Lev}_n(T_n)$. Set $T_\omega = \bigcap_{n\in\omega} T_n$. By induction on n, we get $\mathrm{Lev}_{T_\omega}(n) = \mathrm{Lev}_{T_n}(n)$ for every $n \in \omega$. In particular, T_ω is a non-empty, splitting tree.
To see that every node of T_ω is either splitting or has a unique immediate successor, fix any node $s \in T_\omega$ that is not splitting. If s is an initial segment of s_0, then s has a unique immediate successor in T_{s_0}, and thus also in T_ω. Otherwise, there is some $n \in \omega$, such that s is strictly between the n-th splitting level of T_ω and the $n+1$-th splitting level of T_ω, i.e. there is $t_n \in \mathrm{Lev}_{T_\omega}(n)$ and $t_{n+1} \in \mathrm{Lev}_{T_\omega}(n+1)$ such that $t_n \subsetneq s \subsetneq t_{n+1}$. Then, t_{n+1} is $s(n,t')$ for the unique $t' \in \mathrm{Succ}_{T_n}(t)$ with $t' \subseteq s$. We have $(T_{n+1})_{t'} = (T_n)_{s(n,t')} = (T_n)_{t_{n+1}}$, and hence s has a unique immediate successor in T_{n+1}. As T_ω is a subtree of T_{n+1}, it follows that s has a unique immediate successor in T_ω. □

Remark 4.10. By the above proposition, the set of non-empty, splitting trees T, such that each each node of T is splitting or has a unique immediate successor, forms a forcing notion $\mathbb{M}'$ which densely embeds into $\mathbb{M}$. Recall that this implies that they are forcing equivalent, and in fact essentially interchangeable in forcing arguments, by Theorem 2.7.

The sequence $(T_n)_{n\in\omega}$ in the proof of Proposition 4.9 is decreasing and such that for each $k \in \omega$ the sequence $(\mathrm{Lev}_{T_n}(k))_{n\in\omega}$ is eventually constant, which guarantees that $\bigcap_{n\in\omega} T_n$ is in $\mathbb{M}$. We call such a sequence a *fusion sequence*. We can also generalise the construction of T_{n+1} from T_n, which gives us a handy method for building fusion sequences.

Lemma 4.11. *Fix $T \in \mathbb{M}$ and $I \subseteq T$. For every $s \in I$ take a subtree $T^s \in \mathbb{M}$ of T_s and define:*

$$T' = (T \setminus \bigcup_{s\in I} T_s) \cup \bigcup_{s\in I} T^s.$$

Then T' is a non-empty, splitting subtree of T.

Proof. First note that T' is non-empty, since either $T' = T$, or I is non-empty and then T^s is a subset of T', for any $s \in I$. To see that T' is a subset of T, simply note that, for each $s \in I$, T^s is a subset of T_s, and thus $\bigcup_{s\in I} T^s$ is a subset of $\bigcup_{s\in I} T_s$.
To prove that T' is closed w.r.t. initial segment, take any node $t \in T'$. Note that $\bigcup_{s\in I} T^s$ is closed w.r.t. initial segments, and thus it suffices to consider $t \in T \setminus \bigcup_{s\in I} T_s$ such that t has an initial segment $r \in T_s$ for some $s \in I$. As t does not extend s, neither does r. Hence, $r \subsetneq s$ must hold. Note that T^s contains the root of T_s, and since the root of T_s extends s, we see that r is in T^s.
Lastly, we need to show that T' is splitting. Fix some node $t \in T$. Note that if there is some $s \in I$ such that t is a member of T^s, then T^s contains a splitting node t' which extends t, and since T^s is a subset of T', the node t' is splitting in T'. Otherwise, if $t \in T \setminus \bigcup_{s\in I} T_s$, then there is a splitting node t' of T, such that t' extends t. As $T \setminus \bigcup_{s\in I} T_s$ is closed w.r.t. extensions in T, the set $T \setminus \bigcup_{s\in I} T_s$ contains t' and all of its immediate successors. Hence, t' is a splitting node in T'. $\square$

Remark 4.12. If we want to use Lemma 4.11 to recursively build a fusion sequence $(T_n)_{n\in\omega}$, we simply pick for each $n \in \omega$ a set $I_n \subseteq T_n$ such that the following holds for every $k \in \omega$: For sufficiently large $n \in \omega$, each member of I_n extends some k-th splitting node of T_n.
In Lemma 4.11, if we additionally require that distinct members of I are incomparable, we can replace $\mathbb{M}$ by $\mathbb{M}'$ and the lemma still holds. Furthermore, for any fusion sequence $(T_n)_{n\in\omega}$ with $T_n \in \mathbb{M}'$ for all n, we get $\bigcap_{n\in\omega} T_n$ in $\mathbb{M}'$.

We can use fusion sequences in $\mathbb{M}'$ to show that $\mathbb{M}'$ is proper. As properness is preserved between forcing equivalent forcing notions, this also shows that $\mathbb{M}$ is proper.

Proposition 4.13. *Miller forcing is proper.*

Proof. Let λ be a sufficiently large regular cardinal. Fix a countable model $M \prec H(\lambda)$ such that $\mathbb{M}' \in M$. Let T be in $\mathbb{M}' \cap M$, and let $\{D_n \mid n \in \omega\}$ enumerate all dense subsets of $\mathbb{M}'$ in M. Recursively define a fusion sequence $(T_n)_{n\in\omega}$ by $T_0 = T$ and for every $n \in \omega$:

$$T_{n+1} = (T_n \setminus \bigcup\{(T_n)_s \mid s \in \mathrm{Lev}_{T_n}(n)\}) \cup \bigcup\{(T_n)^s \mid s \in \mathrm{Lev}_{T_n}(n)\},$$

where we take $(T_n)^s \subseteq (T_n)_s$ such that $(T_n)^s \in D_n \cap M$. For $n = 0$, we have $\mathrm{Lev}_{T_0}(0) = \{r\}$, where r is the root of T_0. Then, we find $(T_0)^r \in D_0 \cap M$,

since $(T_0)_r$ can be defined from T_0 and s in M, and M models that D_0 is dense. Now, assume we have chosen $(T_n)^s \in D_n \cap M$ for every $s \in \mathrm{Lev}_{T_n}(n)$. Then, any $s' \in \mathrm{Lev}_{T_{n+1}}(n+1)$ extends some $s \in \mathrm{Lev}_{T_n}(n)$ and we have $(T_{n+1})_{s'} = ((T_n)^s)_{s'}$. As $((T_n)^s)_{s'}$ can be defined in M, there exists an appropriate condition $(T_{n+1})^{s'} \in D_{n+1} \cap M$ such that $(T_{n+1})^{s'} \leq (T_{n+1})_{s'}$.
Note that $(T_n)_{n\in\omega}$ is a fusion sequence. Thus, we get $T_\omega = \bigcap_{n\in\omega} T_n$ in $\mathbb{M}'$. We claim that $T_{n+1} \Vdash D_n \cap M \cap \dot{G} \neq \emptyset$ holds for every $n \in \omega$. Fix $n \in \omega$, and note that it suffices to show that every $S \leq T_{n+1}$ is compatible with some $S' \in D_n \cap M$, since then the conditions, that force $D_n \cap M \cap \dot{G}$ to be non-empty, are dense below T_{n+1}. Fix $S \leq T_{n+1}$ and let $r \in \mathrm{Lev}_{T_{n+1}}(n)$ be a node that is comparable with the root of S. Then S_r is a subtree of both $(T_{n+1})_r$ and S. Let $s \in \mathrm{Lev}_{T_n}(n)$ be such that $s \subseteq r$, and note that $(T_{n+1})_r = (T_n)^s$. Hence, S is compatible with $(T_n)^s$ which is in $D_n \cap M$. As n and S were arbitrary, this proves $T_{n+1} \Vdash D_n \cap M \cap \dot{G} \neq \emptyset$ for all n. Since $T_\omega \leq T_n$ holds for all n, it follows that T_ω is a generic condition below T. As T and M were arbitrary, we conclude that Miller forcing is proper. □

Remark 4.14. The above proof can be generalised to a large family of forcings, which includes many tree forcings (i.e. forcing notions whose conditions are trees) and was first defined by Baumgartner [13]. The defining property of this family of forcings is called *Axiom A*. A forcing notion $\mathbb{P}$ is said to satisfy Axiom A, if there are preorders $\{\leq_n \mid n < \omega\}$ with $\leq_0 = \leq$ and such that the following hold[15]:

(a) For $m < n$, we have $\leq_n \subseteq \leq_m$.
(b) For each $p \in \mathbb{P}$, every $n < \omega$, and any dense $D \subseteq \mathbb{P}$, there are $p' \leq_n p$ and a countable $D' \subseteq D$ such that D' is predense below p' in $\mathbb{P}$.
(c) For any sequence $(p_n)_{n\in\omega}$ in $\mathbb{P}$ with $p_{n+1} \leq_n p_n$, there is $p_\omega \in \mathbb{P}$ such that $p_\omega \leq_n p_n$ holds for all n.

In the case of Miller forcing, we can define $\leq_n$ by

$$S \leq_n T \Leftrightarrow (\ S \leq T \ \wedge \ \forall k < n \ \ \mathrm{Lev}_S(k) = \mathrm{Lev}_T(k)\ ).$$

We have discussed that (c) holds in this instance. Clearly, also (a) holds. To see that (b) holds, we fix a countable elementary submodel M of $H(\lambda)$ with λ a sufficiently large regular cardinal and $p, D \in M$. Then, we can construct p' from p, as we constructed T_{n+1} from T_n in the proof of 4.13, and we get that $D \cap M$ is predense below p'.
It is not hard to adjust the proof of 4.13 to show that all forcing notions that

[15] Baumgartner originally gave a sliglty different but equivalent definition of Axiom A. The definition given here is due to Abraham [14]

satisfy Axiom A are proper. Alternatively, one can find a proof of this fact in [13].

4.3 $\mathfrak{a}$ and $\mathfrak{d}$ in the iterated Miller model

In this final part of the thesis, we will work with the countable support iteration $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j \mid i \leq \omega_2,\ j < \omega_2 \rangle$, where for each $j < \omega_2$, the name $\dot{\mathbb{Q}}_j$ is a $\mathbb{P}_j$-name for the Miller forcing notion of the extension. We will show that if we force with $\mathbb{P}_{\omega_2}$ over a ground model of $\mathtt{CH}$, then in the forcing extension the almost disjoint number is ω_1, while the dominating number is $\mathfrak{c} = \omega_2$.

To show that the maximality of the almost disjoint family from Proposition 4.4 is preserved when forcing with $\mathbb{P}_{\omega_2}$, we use the following three lemmas and one auxiliary definition:

Lemma 4.15. *If M is an elementary submodel of $H(\lambda)$ for some sufficiently large regular cardinal λ, then $\omega_1 \cap M$ is an ordinal.*

Proof. We need to show that $\omega_1 \cap M$ is downward closed w.r.t. the natural order on ordinals. Take $\alpha \in M \cap \omega_1$ and note that $H(\lambda)$ models that α is countable. By elementarity, there is an enumeration $f : \omega \to \alpha$ in M. As every $n \in \omega$ is definable in M, we get $f(n) \in M$ for all n. Hence, $M \cap \omega_1$ is a downward closed set of ordinals, i.e. $M \cap \omega_1$ is an ordinal. $\square$

Definition 4.16. Assume $\mathcal{A} \subseteq [\omega]^\omega$ is enumerated by $\langle A_i \mid i < \omega_1 \rangle$. Let λ be a sufficiently large regular cardinal and let M be a countable elementary submodel of $H(\lambda)$. Set $\delta = \omega_1 \cap M$ and let $\mathcal{I}_\delta$ be the closure of $\{A_i \mid i < \delta\} \cup [\omega]^{<\omega}$ w.r.t. finite unions. We say that M is *good* for $\mathcal{A}$, if the following two conditions hold:

(a) $\mathcal{A}$ and its enumeration are in M;
(b) for each infinite $X \subseteq [\omega]^{<\omega}$ in M, either there is $I \in \mathcal{I}_\delta$ such that $I \cap x$ is non-empty for all but finitely many $x \in X$, or there are infinitely many $x \in X$ such that x is a subset of A_δ.

Lemma 4.17. *Assume $\mathtt{CH}$. Let $\mathcal{A}$, with the enumeration $\langle A_i \mid i < \omega_1 \rangle$, be the mad family constructed in Proposition 4.4. Let λ be a sufficiently large regular cardinal and let M be a countable elementary submodel of $H(\lambda)$, such that $\mathcal{A}$ and its enumeration are in M. Then M is good for $\mathcal{A}$.*

Proof. First, note that in M we find an enumeration $\{\{B_i^n \mid n < \omega\} \mid i < \omega_1\}$ of all infinite subsets of $[\omega]^{<\omega}$, such that $\{\{B_i^n \mid n < \omega\} \mid i < \omega_1\}$ and $\{A_i \mid i < \omega_1\}$

are as in Proposition 4.4. This is because there is such an enumeration in $H(\lambda)$, and M is an elementary submodel. Now, take an infinite subset $X \subseteq [\omega]^{<\omega}$ in M. Then, X is equal to $\{B_i^n \mid n \in \omega\}$ for some $i < \omega_1$. By elementarity, there is such an i in M. Then, we have $i < \delta = \omega_1 \cap M$. Recall that $\mathcal{A}$ is such that $\forall I \in \mathcal{I}_\delta$ ($\{n \mid B_i^n \cap I = \emptyset\}$ is infinite) implies $\{n \mid B_i^n \subseteq A_\delta\}$ is infinite. Hence, either there is some $I \in \mathcal{I}_\delta$ such that $I \cap x$ is non-empty for all but finitely many $x \in X$, or there are infinitely many $x \in X$ such that $x \subseteq A_\delta$. $\square$

Lemma 4.18. *Let $\mathcal{A}$ be a subset of $[\omega]^\omega$ with an enumeration $\langle A_i \mid i < \omega_1\rangle$. Let λ be a sufficiently large regular cardinal and let M be a countable elementary submodel of $H(\lambda)$, such that $\mathbb{M}' \in M$ and M is good for $\mathcal{A}$. Then, for any $S \in \mathbb{M}' \cap M$, there is a $(M, \mathbb{M}')$-generic condition $T \in \mathbb{M}'$ such that $T \leq S$ and T forces that $M[\dot{G}]$ is good for $\mathcal{A}$.*

Proof. Note that, for any name $\dot{f} \in M$ for a function from ω to $[\omega]^{<\omega}$, we can find a sequence of names $\{\tau_n \mid n \in \omega\} \in M$ such that $1 \Vdash \dot{f}(n) = \tau_n$ for each n. Hence, we find names $\{\tau_n^m \mid m, n \in \omega\} \subseteq M$, such that, for each m, the sequence $\{\tau_n^m \mid n \in \omega\}$ is in M with $1 \Vdash \tau_n^m \neq \tau_{n'}^m$ whenever $n \neq n'$, and such that, for every name $\tau \in M$ that is name for an infinite subset of $[\omega]^{<\omega}$, there is $m \in \omega$ such that $1 \Vdash \tau = \{\tau_n^m \mid n \in \omega\}$. Fix an enumeration $\{D_n \mid n \in \omega\}$ of all dense open subsets of $\mathbb{M}'$ that are in M. Denote by $\mathcal{I}$ the closure of $\mathcal{A} \cup [\omega]^{<\omega}$ w.r.t. finite unions, and note that $\mathcal{I}$ is in M. We define a fusion sequence $(T_n)_{n \in \omega}$ with $T_0 = S$. Note that while constructing T_{n+1} from T_n we ensure that $(T_{n+1})_s$ is in M for each $s \in \mathrm{Lev}_{T_{n+1}}(n+1)$. Assume T_n has been constructed, and define T_{n+1} as follows:

> For every splitting node $s \in \mathrm{Lev}_{T_n}(n)$, set $f_s(0) = (T_n)_s$; for each natural number $k \leq n$, recursively define $f_s(k+1)$ to be a root preserving extension of $f_s(k)$ in $D_k \cap M$, if such an extension exists, and $f_s(k+1) = f_s(k)$ otherwise; next, set $g_s(0) = f_s(n+1)$, and for each natural number $k \leq n$, recursively define $g_s(k+1)$ to be a root preserving extension of $g_s(k)$ in M, which forces that there exists $I \in \mathcal{I}$ with $\tau_n^k \cap I \neq \emptyset$ for all but finitely many $n \in \omega$, if such an extension exists, and otherwise let $g_s(k+1)$ be the condition $\bigcup\{T'_j \mid j \in J\}$, where $J \in M$ is an infinite subset of ω such that $s^\frown j \in g_s(k)$ for each $j \in J$, and $T'_j \in \mathbb{M}' \cap M$ is such that $T'_j \leq (g_s(k))_{s^\frown j}$ and such that $T'_j \Vdash \tau_l^k \subseteq A_\delta$ for some $l \geq n$; then set $T_{n+1} = \bigcup\{g_s(n+1) \mid s \in \mathrm{Lev}_{T_n}(n)\}$.

To see that we can always find J and T'_j for each $j \in J$, as in the definition of $g_s(k+1)$, assume that there is no root preserving extension of $(g_s(k))_s$ which forces that there exists some $I \in \mathcal{I}$ with $\tau_n^k \cap I \neq \emptyset$ for all but finitely many

n. Then, there are at most finitely many j with $s^\frown j \in g_s(k)$ such that there is $R \leq (g_s(k))_{s^\frown j}$ with $R \Vdash \exists I \in \mathcal{I}\ (\tau_n^k \cap I \neq \emptyset$ for all but finitely many $n)$. Take

$$J = \{j \mid s^\frown j \in g_s(k),\ (g_s(k))_{s^\frown j} \Vdash \forall I \in \mathcal{I}\ (\tau_n^k \cap I = \emptyset \text{ for infinitely many } n)\},$$

and note that J is infinite and in M. Fix any $j \in J$ and consider the set $X = \{x \mid \exists R \leq (g_s(k))_{s^\frown j} \exists l \geq n\ (R \Vdash \tau_l^k = x)\}$. For each $I \in \mathcal{I}$, there must be infinitely many $x \in X$ such that $x \cap I = \emptyset$, and $H(\lambda)$ models this. Since X can be defined in M and M is good for $\mathcal{A}$, there are infinitely many $x \in X$ such that $x \subseteq A_\delta$, where $\delta = \omega_1 \cap M$. Fix such an $x \in X$. By definition of X, we find $T'_j \in \mathbb{M}'$ such that $T'_j \leq (g_s(k))_{s^\frown j}$ and $T'_j \Vdash \tau_l^k = x$ for some $l \geq n$, and by elementarity, we find such a T'_j in M.
Set $T = \bigcap_{n\in\omega} T_n$ and note that T is in $\mathbb{M}'$, as $(T_n)_{n\in\omega}$ is a fusion sequence. To finish the proof, we check that T has the desired properties.

Claim 1: T is $(M, \mathbb{M}')$-generic.

Proof of Claim 1: Fix $k \in \omega$ and a generic filter $G \subseteq \mathbb{M}'$ such that $T \in G$. Note that the set

$$D'_k = \{R \in D_k \mid \exists n \geq k \text{ (the root of } R \text{ is in } \mathrm{Lev}_T(n)\}$$

is dense below T, since D_k is dense open. Hence, there is some condition $R \leq T$ in $D'_k \cap G$. Let r be the root of R, and let n be such that $r \in \mathrm{Lev}_T(n)$. Note that in the construction of T_{n+1}, we had $r \in \mathrm{Lev}_n(T_n)$, and hence we defined $f_r(k+1)$ accordingly. In particular, $f_r(k+1) \in D_k \cap M$ holds, since $H(\lambda)$ models that R is a root preserving extension of $f_r(k)$, and thus, by elementarity, there is a root preserving extension of $f_r(k)$ in $D_k \cap M$. As $R \leq T_r \leq f_r(k+1)$ and R is in G, we see that $G \cap D_k \cap M$ contains $f_r(k+1)$. As G and k were arbitrary, T is $(M, \mathbb{M}')$-generic.

Claim 2: $M[G]$ is good for every generic filter $G \subseteq \mathbb{M}'$ that contains T.

Proof of Claim 2 Note that $(\omega_1)^{V[G]} \cap M[G] = \omega_1 \cap M$, since $\mathbb{M}'$ preserves ω_1 and G contains the generic condition T. Set $\delta = \omega_1 \cap M$ and let $\mathcal{I}_\delta$ be the closure of the set $\{A_i \mid i < \delta\} \cup [\omega]^{<\omega}$ w.r.t. finite unions. Fix an infinite set $X \subseteq [\omega]^{<\omega}$ in $M[G]$ and let $m \in \omega$ be such that $\{(\tau_k^m)^G \mid k \in \omega\} = X$. Assume that for each $I \in \mathcal{I}_\delta$, there are infinitely many k so that $I \cap (\tau_k^m)^G$ is empty. For the sake of contradiction, additionally assume that there is $l < \omega$ such that $(\tau_k^m)^G$ is not a subset of A_δ for any $k \geq l$. Let $R \in G$ with $R \leq T$ force both of these assumptions, and take $R' \leq R$ such that the root of R' is in $\mathrm{Lev}_T(n)$ for some $n > l, m$. The set of such R' is dense below R, and thus we can find such an R' in G. Let r be the root of R'. In the construction of T_{n+1}, we did not find a root preserving extension of $g_r(m)$ that forces $\forall I \in \mathcal{I}$ ($\tau_k^m \cap I \neq \emptyset$ for all but finitely many k). To see this, first note that $R' \leq T_r \leq g_r(m+1)$ holds, and thus $g_r(m+1)$ is in G. By assumption, for each I in $\mathcal{I}_\delta$ there are infinitely many k such that $(\tau_k^m)^G \cap I$ is empty. Since G contains a generic condition, we have $\mathcal{I} \cap M[G] = \mathcal{I} \cap M = \mathcal{I}_\delta$, and hence

$$M[G] \models \forall I \in \mathcal{I}\ (I \cap (\tau_n^m)^G = \emptyset \text{ for infinitely many } n).$$

By elementarity, $H(\lambda)[G]$ models the same sentence, and thus $g_r(m+1)$ cannot force the opposite. Therefore, in the construction of $g_r(k+1)$ we chose $J \subseteq \omega$ and $\{T'_j \mid j \in J\}$ accordingly. Since R' extends T, there is some $j \in J$ such that $r^\frown j$ is in R'. Note that we have $R'_{r^\frown j} \leq (T_{n+1})_{r^\frown j} = T'_j$, and thus T'_j and R' are compatible. Furthermore, T'_j forces that τ_k^m is a subset of A_δ for some $k \geq n$, contradicting the fact that R' forces the opposite. This concludes the proof of Claim 2, and thus also the proof of the lemma. □

Remark 4.19. By Theorem 2.7, we see that the above lemma still holds if we replace $\mathbb{M}'$ by $\mathbb{M}$.

We use the mad family $\mathcal{A}$ constructed in Proposition 4.4 to show that forcing with $\mathbb{P}_{\omega_2}$, over a ground model of CH, preserves $\mathfrak{a} = \omega_1$.

Theorem 4.20. *Assume* CH *holds (in V). Then, the mad family $\mathcal{A}$, from Proposition* 4.4*, is mad in every $\mathbb{P}_{\omega_2}$-forcing extension (of V).*

Proof. The main part of this proof is dedicated to showing that, there are enough good models in the extension. To this end, fix a sufficiently large regular cardinal λ and an elementary submodel M of $H(\lambda)$ such that M contains ω_2, $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j \mid i \leq \omega_2,\ j < \omega_2 \rangle$, $\mathcal{A}$ and its enumeration. We will show by induction on j, that $\forall i < j\ Stat(i,j)$ holds for every $j \leq \omega_2$, where $Stat(i,j)$ is the following statement :

> If i and j are in M, then for each $(M, \mathbb{P}_i)$-generic condition $q_0 \in \mathbb{P}_i$ such that $q_0 \Vdash$"$M[\dot{G}]$ is good", and every $p \in \mathbb{P}_j \cap M$ such that $q_0 \leq p \restriction i$, there is $q \in \mathbb{P}_j$ with the following properties: $q \restriction i = q_0$, $q \leq p$, q is $(M, \mathbb{P}_j)$-generic, and $q \Vdash$"$M[\dot{G}]$ is good for $\mathcal{A}$".

Note that the induction statement is vacuously true for $j = 0$.

Successor step: Assume $j+1$ is in M and $\forall i < j'\ Stat(i,j')$ holds for every $j' \leq j$. First, note that for any $i < j$, and q_0 and p as in the premise of $Stat(i, j+1)$. By $Stat(i,j)$, we find $q' \in \mathbb{P}_j$ such that $q' \restriction i = q_0$, $q' \leq p \restriction j$, q'

is $(M, \mathbb{P}_j)$-generic, and such that $q' \Vdash$"$M[\dot{G}$ is good for $\mathcal{A}$". Hence, it suffices to prove $Stat(j, j+1)$.
Fix $i < j$, q_0 and p as in the premise of $Stat(j, j+1)$. We will use existential completeness to define an appropriate name $\dot{T}$ for a condition in $\dot{\mathbb{Q}}_j$, such that $q = q_0 \cup \{(j, \dot{T})\}$ is as in $Stat(j, j+1)$. Hence, let $G_j \subseteq \mathbb{P}_j$ be a generic filter that contains q_0. Set $\mathbb{Q} = \dot{\mathbb{Q}}_j^{G_j}$ and recall that $\mathbb{Q}$ is Miller forcing in $V[G_j]$. Note that $S = p(j)^{G_j}$ is in $\mathbb{Q} \cap M[G_j]$ and we have $M[G_j] \prec H(\lambda)[G_j] = H(\lambda)^{V[G_j]}$. By Lemma 4.18, there is $T \in \mathbb{Q}$ such that $T \leq S$, T is $(M[G_j], \mathbb{Q})$-generic, and T forces that $M[G_j][\dot{G}]$ is good for $\mathcal{A}$. If q_0 is not in G take T to be any element of $\mathbb{Q}$. We obtain a $\mathbb{P}_j$-name $\dot{T}$, by existential completeness. Define $q = q_0 \cup \{(j, \dot{T})\}$. We immediately see that $q \leq p$ holds, since $q_0 \leq p \restriction j$ holds and q_0 forces that $\dot{T}$ is below $p(j)$. By 3.37, q is $(M, \mathbb{P}_{j+1})$-generic. To see that $M[G_{j+1}]$ is good for $\mathcal{A}$, for every generic filter $G_{j+1} \subseteq \mathbb{P}_{j+1}$ that contains q, fix any generic filter $G_{j+1} \subseteq \mathbb{P}_{j+1}$ that contains q. Set $G_j = G_{j+1} \restriction j$ and $H = \{r(j)^{G_j} \mid r \in G_{j+1}\}$. Note that we have $G_{j+1} = G_j * H$, and thus $M[G_{j+1}] = M[G_j][H]$. As q_0 is in G_j and T is in H, we see that $M[G_j][H]$ is good for $\mathcal{A}$. Hence, q is as desired. This concludes the proof of the successor step.

Limit step: Let $j \leq \omega_2$ be a limit ordinal such that $j \in M$. Assume that $\forall i < j'$ $Stat(i, j')$ holds for every $j' < j$. Fix $i < j$, q_0 and p as in the premise of $Stat(i, j)$. Note that M models that j is a limit ordinal and take a strictly increasing unbounded sequence $\{i_n \mid n \in \omega\} \subseteq j \cap M$ with $i_0 = i$. Let $\{D_n \mid n \in \omega\}$ enumerate all dense open subsets of $\mathbb{P}_j$ that are in M, and let $\{\{\tau_n^m \mid n \in \omega\} \mid m \in \omega\}$ enumerate all $\mathbb{P}_j$-names in M that are names for infinite subsets of $[\omega]^{<\omega}$, and such that $1 \Vdash \tau_n^m \neq \tau_{n'}^m$ whenever $n \neq n'$. Set $\delta = \omega_1 \cap M$, denote by $\mathcal{I}$ the closure of $\mathcal{A} \cup [\omega]^{<\omega}$ w.r.t. finite unions, and denote by $\mathcal{I}_\delta$ the closure of $\{A_k \mid k < \delta\} \cup [\omega]^{<\omega}$ w.r.t. finite unions. We construct sequences $(\dot{p}_n)_{n \in \omega}$ and $(q_n)_{n \in \omega}$. Set $\dot{p}_0 = \check{p}$ and recursively define $\dot{p}_n$ and q_n for $n \in \omega$, such that that the following hold for all n:

(a) $q_n \in \mathbb{P}_{i_n}$ and $q_{n+1} \restriction i_n = q_n$;
(b) $\dot{p}_{n+1}$ is a $\mathbb{P}_{i_n}$-name for a condition in $\mathbb{P}_j \cap M$ such that $\dot{p}_{n+1} \leq \dot{p}_n$ is forced;
(c) $1 \Vdash \dot{p}_{n+1} \in D_n$;
(d) $q_n \Vdash \dot{p}_n^{\dot{G}} \restriction i_n \in \dot{G}$;
(e) q_n is $(M, \mathbb{P}_{i_n})$-generic and $q_n \Vdash$"$M[\dot{G}]$ is good";
(f) for every $m \leq n$, if $G \subseteq \mathbb{P}_j$ is a generic filter containing $\dot{p}_{n+1}^G$ then at least one of the following holds:
    (1) $\exists k \geq n$ $(\tau_k^m)^G \subseteq A_\delta$
    (2) $\exists I \in \mathcal{I}$ $((\tau_k^m)^G \cap I \neq \emptyset$ for all but finitely many $k)$.

Assume $q_n, \dot{p}_n$ have been defined according to the conditions (a)-(h), and define $\dot{p}_{n+1}$ and q_{n+1} as described below.

Definition of $\dot{p}_{n+1}$*:* Fix a generic filter $G \subseteq \mathbb{P}_{i_n}$ and work in $V[G]$ to define a condition p^{n+1} as follows. Let $p^0 \in D_n \cap M$ such that $p^0 \leq \dot{p}_n^G$. We recursively define $p^{m+1} \in \mathbb{P}_j \cap M$ for $m \leq n$. Assume p^m has been defined. Consider $X(m) = \{x \mid \exists r \leq p^m \exists k \geq n\ (r \Vdash \tau_k^m = x)\}$ and note that $X(m)$ is in M. If there is some $x \in X(m)$ such that $x \subseteq A_\delta$, take $p^{m+1} \leq p^m$ such that $p^{m+1} \Vdash \tau_k^m = x$ for some $k \geq n$. Otherwise, let $p^{m+1} \leq p^m$ be such that p^{m+1} forces that there is some $I \in \mathcal{I}$ such that $I \cap \tau_k^m \neq \emptyset$ holds for all but finitely many k. This works, because M is good. To see this, note that, if no $r \leq p^m$ forces that there is $I \in \mathcal{I}$ such that $\tau_k^m \cap I \neq \emptyset$ for all but finitely many k, then p^m forces $\forall I \in \mathcal{I}\ (\tau_k^m \cap I = \emptyset$ for infinitely many $k)$, and hence for each $I \in \mathcal{I}$ there are infinitely many $x \in X(m)$ such that $x \cap I = \emptyset$.
If we additionally have $q_n \in G$, we may pick p^{n+1} such that $p^{n+1} \restriction i_n$ is in G. To see this, assume $p^m \restriction i_n$ is in G. Note that the appropriate conditions for p^{m+1} form a set D^m which is dense open below p_m, and D^m can be defined in M. Then, $D' = \{r \restriction i_n \mid r \in D^m\}$ is dense below $p^m \restriction i_n$ in $\mathbb{P}_{i_n}$, and D' is in M. As G contains the $(M, \mathbb{P}_{i_n})$-generic condition q_n, there is some $p' \in D'_m \cap M \cap G$. By elementarity of M, there is $p^{m+1} \in D_m \cap M$ such that $p^{m+1} \restriction i_n = p'$. The same argument applies to p^0, since D_n is dense open, D_n is in M and $q_n \Vdash \dot{p}_n \restriction i_n \in \dot{G}$.
We define $\dot{p}_{n+1}$ from p^{n+1} by existential completeness.

Definition of q_{n+1}: For each $s \in \mathbb{P}_j \cap M$, use $Stat(i_n, i_{n+1})$ to obtain an $(M, \mathbb{P}_{i_{n+1}})$-generic condition $q^s \in \mathbb{P}_{i_n}$ such that $q^s \restriction i_n = q_n$, $q^s \leq 1_{\mathbb{P}_{i_n}} \cup^* s \restriction i_{n+1}$ and $q^s \Vdash$"$M[\dot{G}]$ is good". By existential completeness, there exists a $\mathbb{P}_{i_n}$-name $\dot{q}$ such that $1 \Vdash$"$\dot{q} = q^{\dot{p}_{n+1}}$. Define q_{n+1} from q_n and $\dot{q}$, as in the proof of Theorem 3.38. Then, q_{n+1} has countable domain, since there are only countably many q^s. Hence, q_{n+1} is in $\mathbb{P}_{i_{n+1}}$.

That the conditions (a)-(c) and (f) hold, follows straight from the definitions of $\dot{p}_{n+1}$ and q_{n+1}. That also (d) and (e) hold, follow by the same arguments as in the proof of Theorem 3.38.

Set $q = \bigcup_{n \in \omega} q_n$ and note that we get $q \leq p$, q is $(M, \mathbb{P}_j)$-generic and $q \Vdash \dot{p}_{n+1} \in G$ for all n, as we did in the proof of Theorem 3.38. To see that also $q \Vdash$ "$M[G]$ is good for $\mathcal{A}$" holds, fix a generic filter $G \subseteq \mathbb{P}_j$ that contains q. Note that $\omega_1 \cap M[G] = \delta$, since G contains the generic condition q. Let X be an infinite subset of $[\omega]^{<\omega}$ in $M[G]$ such that $x \subseteq A_\delta$ only holds for finitely many $x \in X$, and let m be such that $X = \{(\tau_n^m)^G \mid n < \omega\}$. Take $n < \omega$

such that $(\tau_n^m)^G \not\subseteq A_\delta$ for all $k \geq n$. Since $\dot{p}_{n+1}^G$ is in G, there must be some $I \in \mathcal{I}$ such that $x \cap I$ is non-empty for all but finitely many $x \in X$, and since $M[G] \prec H(\lambda)[G]$, there is such an I in $\mathcal{I} \cap M[G] = \mathcal{I}_\delta$.
This concludes the limit step, and hence the proof of $\forall j \leq \omega_2 \forall i < j\ Stat(i,j)$.

Final argument: We show that $\mathcal{A}$ is forced to be maximal. Assume for the sake of contradiction that there is some $\mathbb{P}_{\omega_2}$-name $\dot{A}$ for an infinite subset of $[\omega]^{<\omega}$, such that $\dot{A}$ is not forced to have infinite intersection with each member of $\mathcal{A}$. Let $M \prec H(\lambda)$ be a countable model such that $\mathcal{A}$, $\dot{A}$ and $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j \mid i \leq \omega_2, i < \omega_2 \rangle$ are in M. There is a condition $p \in \mathbb{P}_{\omega_2}$ such that $p \Vdash \forall B \in \mathcal{A}$ ($\dot{A} \cap B$ is finite), and as M is an elementary submodel of $H(\lambda)$, we find such a p in M. By $Stat(0, \omega_2)$, there is $q \leq p$ such that $q \Vdash$"$M[\dot{G}]$ is good". Take a generic filter $G \subseteq \mathbb{P}_{\omega_2}$ that contains q, and define $X = \{\{n\} \mid n \in \dot{A}^G\}$. Then, there are finitely many $x \in X$ such that $x \subseteq A_\delta$, where $\delta = \omega_1 \cap M[G]$. As $M[G]$ is good, we find $I \in \mathcal{I}_\delta$ such that $I \cap x$ is non-empty for all but finitely many $x \in X$. Then, there is a finite subset $\mathcal{F} \subseteq \mathcal{A}$ such that $\dot{A}^G \setminus \bigcup \mathcal{F}$ is finite, and hence $\dot{A}^G \cap B$ is infinite for some $B \in \mathcal{F}$. This is a contradiction, and hence $\mathcal{A}$ is forced to be maximal. $\square$

Proposition 4.21. *Assume* CH *(in V). Then, $\mathfrak{d} = \mathfrak{c} = \omega_2$ holds in any $\mathbb{P}_{\omega_2}$ forcing extension (of V).*

Proof. First, note that for each $j < \omega_2$, there is a dense $D_j \subseteq \mathbb{P}_j$, and $\mathbb{P}_{\omega_2}$ has the ω_2-cc, by 3.41 and 3.45. Then, $\mathbb{P}_j$ preserves all cardinals, for every $j \leq \omega_2$.

Let $G \subseteq \mathbb{P}_{\omega_2}$ be a generic filter. By Proposition 3.43, each new real in $V[G]$ already appears in $V[G \restriction j]$ for some $j < \omega_2$. Furthermore, $V[G \restriction j]$ has only ω_1 many reals, by 3.44. Hence, we calculate in $V[G]$:

$$|2^\omega| = \sum_{j<\omega_2} |2^\omega \cap V[G \restriction j]| \leq \omega_2.$$

Now, consider set $\{f_i \mid i < \omega_1\} \subseteq 2^\omega$ in $V[G]$. For each $i < \omega_1$, let $j_i < \omega_2$ be such that f_i is in $V[G \restriction j_i]$. Then, set $j = \sup\{j_i \mid i < \omega_1\}$ and note that $j < \omega_2$ holds. Forcing with $\dot{\mathbb{Q}}_j^{G \restriction j}$ adds a real that is not dominated by any real in $V[G \restriction j]$. Hence, $\{f_i \mid i < \omega_1\}$ is not dominating in $V[G]$. This proves that in $V[G]$ we have $\mathfrak{d} > \omega_1$ and thus $\mathfrak{d} = \mathfrak{c} = \omega_2$. $\square$

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