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Elliptic Combinatorics of Lattice Paths, Domino Tilings and Rook Placements

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Abstract

In 2007, Michael Schlosser published an article in which lattice paths and nonintersecting nests of lattice paths where counted with respect to an elliptic weight function assigned to the paths. A convolution of paths leads to an identity equivalent to a fundamental summation formula for elliptic hypergeometric series discovered, in a different setting, by Frenkel and Turaev in 1997. These series form a natural generalization of ordinary hypergeometric series, which date back to Newton and Gauß and basic hypergeometric series, which date back to Heine. Elliptic hypergeometric series made their first explicit appearance in the late 1980's in the work of mathematical physicists and since Frenkel and Turaev's work, various researchers started to develop a yet expanding theory, which combines the theory of theta functions with the theory of basic hypergeometric series, even further. Basic hypergeometric series have many applications in number theory, statistical and mathematical physics and combinatorics and it appears natural to wonder if these applications also extend to elliptic hypergeometric series. In this context, several authors studied elliptic analogues of combinatorial models and combinatorial special numbers in the past few years.

In this thesis we contribute to the general task of finding elliptic analogues of combinatorial models by studying elliptic binomial coefficients with integer values, elliptic Fibonomial numbers and elliptic Stirling numbers, Lah numbers and rook numbers. We introduce appropriate combinatorial models and elliptic weight functions. Whenever possible, we generalize the elliptic weights even further and present general weight-dependent analogues of combinatorial models and special numbers.

Zusammenfassung

Michael Schlosser veröffentlichte 2007 einen Artikel, in dem gewichtete Gitterpunktwege abgezählt werden und die Gewichte elliptische Funktionen sind. Eine gewichtete Abzählung solcher Wege führt zu einer Identität, die äquivalent zu einer fundamentalen Formel aus dem Bereich der elliptischen hypergeometrischen Reihen ist und von Frenkel und Turaev 1997 in einem anderen Zusammenhang entdeckt wurde. Elliptische hypergeometrische Reihen verallgemeinern auf natürliche Weise gewöhnliche hypergeometrische Reihen, die auf Newton und Gauß zurückgehen, und q-hypergeometrische Reihen, die von Heine eingeführt wurden. Explizit kamen elliptische hypergeometrische Reihen zum ersten Mal in den späten 1980er Jahren in einer Arbeit aus dem Bereich der mathematischen Physik vor. Seit Frenkel und Turaevs Arbeit zu diesem Thema entwickeln viele Wissenschaftler*innen diese Theorie, welche die Theorie der Thetafunktionen mit der Theorie der q-hypergeometrischen Reihen verbindet, weiter. Da q-hypergeometrische Reihen viele Anwendungen in den Bereichen der Zahlentheorie, statistischen und mathematischen Physik und in der Kombinatorik haben, stellt sich die Frage, ob sich diese Anwendungen auch auf den elliptischen Fall übertragen. Beginnend mit Schlossers Arbeit von 2007 haben vor diesem Hintergrund in den vergangenen Jahren verschiedene Autor*innen elliptische Analoga von kombinatorischen Modellen und speziellen kombinatorischen Zahlen studiert.

In der vorliegenden Dissertation schließen wir uns dieser Aufgabe an, elliptische Verallgemeinerungen von kombinatorischen Modellen zu finden, und untersuchen elliptische Binomialkoeffizienten mit ganzzahligen Einträgen, elliptische Fibonomialkoeffizienten und elliptische Stirling-, Lah- und Rook-Zahlen. Wir führen passende kombinatorische Modelle und elliptische Gewichtsfunktionen ein. Wenn es möglich ist, verallgemeinern wir die elliptischen Gewichte sogar noch weiter und präsentieren allgemeine gewichtsabhängige Analoga von kombinatorischen Modellen und Zahlen.

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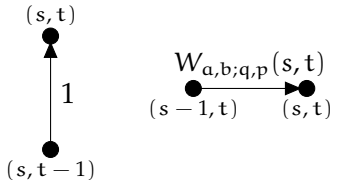
1 Introduction

In 2007, Michael Schlosser published an article called "Elliptic enumeration of nonintersecting lattice paths" [64]. In this article lattice paths and nonintersecting nests of lattice paths are counted with respect to an elliptic weight function assigned to the paths. The paths consist of north and east steps on the planar integer lattice. Each north step is assigned weight 1 and each east step is assigned an elliptic function $W_{a,b;q,p}(s,t)$, depending on the end point of the step, (s,t), and some additional parameters a, b, q and p, which will be discussed later.



The weight of a path is defined as the product of the weights of its steps. The sum of the weights of paths that start at a point $A \in \mathbb{Z}^2$ and end at a point $B \in \mathbb{Z}^2$ is denoted the elliptic enumeration of lattice path from A to B. By a clever choice of $W_{a,b;q,p}(s,t)$ this sum is factorizing nicely. We introduce the technical details in Chapter 2.

A convolution of paths leads to an identity equivalent to a fundamental summation formula for elliptic hypergeometric series discovered, in a different setting, by Frenkel and Turaev in 1997 [29]. These series form a natural generalization of ordinary hypergeometric series, which date back to Newton and Gauß and basic hypergeometric series, which date back to Heine. Elliptic hypergeometric series made their first explicit appearance in the late 1980's in the work of the mathematical physicists Date, Jimbo, Kunibo, Miwa and Okado [22] and since Frenkel and Turaev's work, various researchers started to develop a yet expanding theory, which combines the theory of theta functions with the theory of basic hypergeometric series, even further.

Basic hypergeometric series have many applications in number theory, statistical and mathematical physics and combinatorics. Identities in basic hypergeometric series are used for proving combinatorial arguments and on the other hand there exist various combinatorial proofs of summation and transformation formulae for basic hypergeometric series. It appears natural to wonder if these applications also extend to elliptic hypergeometric series. In this context, several authors studied elliptic analogues of combinatorial models and combinatorial special numbers (see Section 2.3.6 for a brief overview).

In this thesis we contribute to the general task of finding elliptic analogues of combinatorial models by studying elliptic binomial coefficients with integer values, elliptic Fibonomial numbers and elliptic Stirling and rook numbers. We introduce appropriate combinatorial models and elliptic weight functions. Whenever possible, we generalize the elliptic weights even further and present general weight-dependent analogues of combinatorial models and special numbers.

This thesis is structured as follows. Chapter 2 gives an introduction to elliptic enumerative combinatorics. We introduce generating functions, the concept of weighted counting and hypergeometric series and show how these concepts can be adapted to q- and elliptic analogues.

In this chapter we also introduce the most important notations and preliminary results used later in the thesis.

In Chapter 3 we introduce weight-dependent binomial coefficients with integer values. They generalize q-binomial coefficients with integer values from [28] as well as weight-dependent and elliptic binomial coefficients with nonnegative integer values from [64, 65]. We introduce a corresponding noncommutative algebra, a combinatorial model, reflection formulae, a binomial theorem and convolution formulae. We show some interesting special cases of this weight-dependent binomial coefficients including elementary and complete homogeneous symmetric functions, weighted Stirling numbers and elliptic binomial coefficients. We end by using the results of this chapter to find a weight-dependent matrix inversion. This chapter is based on joint work of the author with Michael Schlosser and Meesue Yoo [43].

In Chapter 4 we define q- and elliptic analogues of Fibonomial numbers. We generalize a combinatorial model of Sagan and Savage [60] to q- and elliptic weighted settings. We show several interesting examples and obtain a convolution formula for elliptic Fibonomial numbers. We also focus on q-Fibonacci analogues of Catalan numbers and a more recent combinatorial model for Fibonomials by Bennett, Carrillo, Machacek, and Sagan [7]. This chapter is based on joint work of the author with Nantel Bergeron and Cesar Ceballos [11].

Chapter 5 is devoted to solving the problem of finding an explicit formula for elliptic Stirling numbers of the second kind. This problem was posed by Schlosser and Yoo in [70]. We find the questioned explicit formula by revealing the connection between elliptic Stirling numbers and complete homogeneous symmetric functions. Furthermore, we generalize an operator proof for the explicit formula of ordinary and q-Stirling numbers of the second kind to complete homogeneous symmetric functions. This proof also generalizes to find formulas for more general polynomials and elliptic rook and Lah numbers. This chapter is based on unpublished work of the author.

2 Preliminaries

The focus of this chapter is to work out the concept of elliptic enumerative combinatorics. We will present the basic notions and definitions and explore the path that leads from enumerative combinatorics to elliptic enumerative combinatorics. We start by introducing generating functions and hypergeometric series in Section 2.1 and work out some basic results. We then discuss the concept of q-analogues and basic hypergeometric series in Section 2.2 to generalize results from the previous section. In Section 2.3 we first introduce basic notions, definitions and results concerning elliptic functions and theta functions and then focus on elliptic enumeration. We will discuss some results in the field of elliptic combinatorics and how they are connected to elliptic hypergeometric series. Finally, we introduce weight-dependent binomial coefficients in Section 2.4, which are the main focus of Chapter 3.

2.1 Background

Throughout this thesis we use the notations $\mathbb{N}$ for positive integers, $\mathbb{N}_0$ for nonnegative integers including 0 and $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{C}$ for integers, rational numbers and complex numbers, respectively.

2.1.1 Generating functions

Elliptic combinatorics is based on the combinatorial theory of generating functions and weighted enumeration. Generating functions are a very powerful method in enumerative combinatorics. In this section we give some basic definitions and refer to [30, 59, 77] for more details. Let x be an indeterminate and $\mathbb{C}[[x]]$ be the set of formal power series

$$\mathbb{C}[[x]] = \{\sum_{k\geqslant 0} a_k x^k : a_k \in \mathbb{C}\}.$$

Consider a sequence of complex numbers $(a_k)_{k\geqslant 0} = (a_0, a_1, a_2, \dots)$. We call the formal power series

$$\sum_{k\geqslant 0} a_k x^k$$

the *generating function* of $(a_k)_{k\geqslant 0}$. If the sequence consists of nonnegative integers, the numbers may count the number of elements in some sets A_k for a series of sets $(A_k)_{k\geqslant 0} = (A_0, A_1, A_2, \dots)$. In this case, the generating function of $(a_k)_{k\geqslant 0}$ is also called the generating function of $(A_k)_{k\geqslant 0}$. For example, let n and k be nonnegative integers and A_k be the set of all subsets of $[n] = \{1, 2, 3, \dots, n\}$ with k elements. Then $a_k = \binom{n}{k}$ and

$$\sum_{k\geqslant 0} \binom{n}{k} x^k = (1+x)^n \tag{2.1}$$

is the corresponding generating function of $(a_k)_{k\geqslant 0}$ and $(A_k)_{k\geqslant 0}$. The concept of formal power series and generating functions may also be generalized to any finite set of variables.

Now let $\mathcal{A}$ be a (possibly infinite) set. We assign every element A in $\mathcal{A}$ a specific weight $w(A)$. By summing up the weights of all $A \in \mathcal{A}$ we obtain the *weight generating function*

$$\sum_{A \in \mathcal{A}} w(A)$$

of the set $\mathcal{A}$. For example, let $\mathcal{A}_n$ be the set of all subsets of $[n]$. If we assign the weight $w(A) = x^{|A|}$ to each element $A \in \mathcal{A}_n$, we again obtain

$$\sum_{A \in \mathcal{A}_n} w(A) = \sum_{k \geqslant 0} \binom{n}{k} x^k = (1+x)^n.$$

By assigning the weight $w(A) = x^{|A|} y^{n-|A|}$ and obtain the weight generating function

$$\sum_{A \in \mathcal{A}_n} w(A) = \sum_{k \geqslant 0} \binom{n}{k} x^k y^{n-k} = (x+y)^n.$$

If $\mathcal{A}$ and $\mathcal{B}$ are disjoint sets, the weight generating function of $\mathcal{A} \cup \mathcal{B}$ is

$$\sum_{C \in \mathcal{A} \cup \mathcal{B}} w(C) = \sum_{A \in \mathcal{A}} w(A) + \sum_{B \in \mathcal{B}} w(B). \tag{2.2}$$

For arbitrary sets $\mathcal{A}$ and $\mathcal{B}$ and for a tuple $(A, B) \in \mathcal{A} \times \mathcal{B}$ we define the weight $w(A, B) = w(A)w(B)$. Then the weight generating function of $\mathcal{A} \times \mathcal{B}$ is

$$\sum_{(A,B) \in \mathcal{A} \times \mathcal{B}} w(A, B) = \Big(\sum_{A \in \mathcal{A}} w(A) \Big) \Big(\sum_{B \in \mathcal{B}} w(B) \Big). \tag{2.3}$$

The identities (2.2) and (2.3) can also be extended to any finite collection of sets and under suitable convergence conditions even to infinitely many sets. See [59, Proposition 4.1.6] for more details.

As in our previous example, let $m, n \geqslant 0$ and $\mathcal{A}_n$ be the set of subsets of $[n]$ with weights $w(A) = x^{|A|} y^{n-|A|}$ for $A \in \mathcal{A}_n$ and $\mathcal{A}_m$ be the set of subsets of $[m]$ with $w(B) = x^{|B|} y^{m-|B|}$ for $B \in \mathcal{A}_m$. Then the weight generating function of $\mathcal{A}_n \times \mathcal{A}_m$ is

$$\begin{aligned} \sum_{(A,B) \in \mathcal{A}_n \times \mathcal{A}_m} w(A, B) &= \Big(\sum_{A \in \mathcal{A}_n} w(A) \Big) \Big(\sum_{B \in \mathcal{A}_m} w(B) \Big) \\ &= (x+y)^n (x+y)^m = (x+y)^{n+m}, \end{aligned}$$

which is equal to the weight generating function of $\mathcal{A}_{n+m}$.

2.1.2 Hypergeometric series

Hypergeometric series play an important role in combinatorics, have a rich history and date back to Newton and Gauß. In fact, we have already seen an example of a hypergeometric identity in (2.1), as we will verify in a moment. In this section we introduce hypergeometric series and give some examples. We follow the notations and definitions of [2, 31].

Let $S = \sum_{k \geqslant 0} c_k$ be a series. We call S a *hypergeometric series* if c_{k+1}/c_k is a rational function of k. If we look at our example (2.1), we get for any $k \geqslant 0$

$$\frac{c_{k+1}}{c_k} = \frac{n-k}{k+1} x,$$

which is indeed a rational function of k. For general hypergeometric series we obtain by factorization that

$$\frac{c_{k+1}}{c_k} = \frac{(k+a_1)(k+a_2)\cdots(k+a_r)}{(k+b_1)(k+b_2)\cdots(k+b_s)(k+1)}z \tag{2.4}$$

for some a_i, b_i, r, s and z. We write the $(k+1)$ in the denominator by convention and it may be cancelled by a $(k+a_i)$ with $a_i = 1$ in the numerator. The z occurs because the polynomial may not be monic. By setting $r = 1$, $s = 0$, $a_1 = -n$ and $z = -x$, we obtain our previous example.

From (2.4) we obtain by iteration that in general we can write a summand of a hypergeometric series as

$$c_k = c_0 \frac{(a_1)_k (a_2)_k \cdots (a_r)_k}{(b_1)_k (b_2)_k \cdots (b_s)_k k!} z^k$$

where $(n)_k = n(n+1)(n+2)\cdots(n+k-1)$ and $(n)_0 = 1$ is the *Pochhammer symbol*. We define the general form of a hypergeometric series with $c_0 = 1$ as

$${}_rF_s\left[\begin{matrix} a_1, & \ldots, & a_r \\ b_1, & \ldots, & b_s \end{matrix}; z\right] := \sum_{k\geqslant 0} \frac{(a_1)_k (a_2)_k \cdots (a_r)_k}{(b_1)_k (b_2)_k \cdots (b_s)_k k!} z^k. \tag{2.5}$$

The left hand side of (2.1) may thus be written as

$${}_1F_0\left[\begin{matrix} -n \\ - \end{matrix}; -x\right] = \sum_{k\geqslant 0} \frac{(-n)_k}{k!}(-x)^k = \sum_{k\geqslant 0} \frac{(n)(n-1)\cdots(n-k+1)}{k!} x^k = \sum_{k\geqslant 0} \binom{n}{k} x^k$$

and (2.1) is our first example of a summation formula, namely the binomial theorem

$${}_1F_0\left[\begin{matrix} -c \\ - \end{matrix}; -x\right] = (1-x)^c, \tag{2.6}$$

which is true for $c \in \mathbb{C}$ and converges if $|x| < 1$. We prove this theorem bijectively for nonnegative integers n as follows. The left hand side of (2.6) is the weight generating function of all subsets of $[n]$, where each subset A is assigned the weight $x^{|A|}$. By (2.3), the right hand side is the weight generating function of n-tuples $(a_1, a_2, \ldots, a_n)$ with $a_i \in \{0, 1\}$, where each element a_i is assigned weight x^{a_i}, so the weight of the tuple is $x^{\#(\text{entries equal to } 1)}$. There is a simple weight-preserving bijection between subsets of $[n]$ and n-tuples with entries 0 and 1: We identify each tuple $(a_1, a_2, \ldots, a_n)$ with a subset A of $[n] = \{1, 2, \ldots, n\}$ by saying that $i \in [n]$ is an element of A if and only if $a_i = 1$.

Another important hypergeometric summation formula is the *Chu–Vandermonde* summation or convolution formula which can be stated in terms of hypergeometric series as

$${}_2F_1\left[\begin{matrix} -n, & a \\ b \end{matrix}; 1\right] = \frac{(b-a)_n}{(b)_n}. \tag{2.7}$$

By substituting $n \mapsto k$, $a \mapsto -n$, $b \mapsto m-k+1$ and multiplying both sides with $\binom{m}{k}$, we obtain the Chu–Vandermonde convolution in the well-known form

$$\binom{n+m}{k} = \sum_{j=0}^{k} \binom{n}{j}\binom{m}{k-j}. \tag{2.8}$$

There are several ways to prove this identity. We can again prove it by a combinatorial argument using subsets of sets. In Chapter 3 we will construct a proof using lattice paths even if n and m

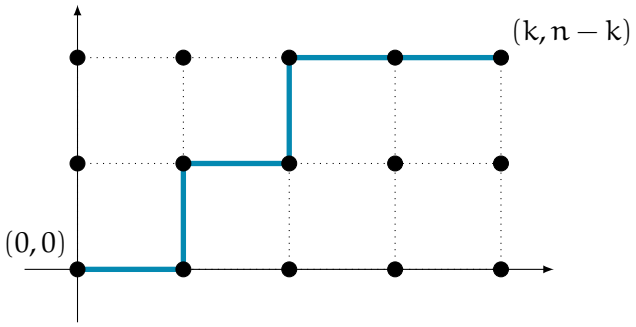
are negative integers. A common way of proving this identity algebraically is to compare the coefficients of x^k on both sides of the equation $(1+x)^{n+m} = (1+x)^n(1+x)^m$. Expanding by the binomial theorem and using the Cauchy product gives for $|x| < 1$

$$\sum_{k\geqslant 0}\binom{n+m}{k}x^k = \sum_{k\geqslant 0}\sum_{j=0}^{k}\binom{n}{j}\binom{m}{k-j}x^k.$$

We will construct a similar proof for a generalized version of this identity in Chapter 3.

2.1.3 Lattice paths

Another very interesting interpretation of binomial coefficients is the combinatorial family of lattice paths. For some $0 \leqslant k \leqslant n$ the binomial coefficient $\binom{n}{k}$ counts the number of lattice paths in the first quadrant starting from the origin and ending at $(k, n-k)$ using north steps $(0,1)$ and east steps $(1,0)$ only. We will denote this set of lattice paths by $\mathcal{P}_{\mathbb{N}}((0,0) \to (k, n-k))$ and call them *ordinary lattice paths* throughout this thesis. Each ordinary lattice path ending at $(k, n-k)$ has k east steps, $n-k$ north steps and n steps in total. We can identify each ordinary lattice path in $\mathcal{P}_{\mathbb{N}}((0,0) \to (k, n-k))$ with a subset A of $[n]$ by saying that i is an element of A if and only if the i-th step (starting from the origin) of the lattice path is an east step. For example, the subset $\{1,3,5,6\}$ of $\{1,2,3,4,5,6\}$ corresponds to the following path:



We assign each east step weight x and each north step weight y and the weight of a path is the product of the weights of its steps. Then it is not hard to see that $(x+y)^n$ is the generating function for all paths with n steps for a nonnegative integer n (by seeing a lattice path as an n-tuple of steps). On the other hand, there are $\binom{n}{k}$ paths with k east steps, $n-k$ north steps and weight $x^k y^{n-k}$ for $0 \leqslant k \leqslant n$, which yields the generating function $\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$. Comparing these two results, we obtain a combinatorial proof of the binomial theorem in the form

$$\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k} = (x+y)^n. \tag{2.9}$$

In the following sections we will see extensions of this interpretation to a q-, elliptic and more general weight-depending level. In Chapter 3 we extend these results to negative integers n and k.

2.2 q-Combinatorics

2.2.1 q-Enumeration

In combinatorics it is common to generalize enumeration formulae by assigning some weight involving a parameter q that gives back the original formula if we let $q \to 1$. This is probably best explained by an example. A classical example is the q-enumeration of subsets of $[n] = \{1, 2, \ldots, n\}$ with k elements. (See for example [38].) First, we define the q-analogue of a nonnegative integer n by $[0]_q = 0$ and

$$[n]_q := 1 + q + q^2 + \cdots + q^{n-1}.$$

We call $[n]_q$ the q-*number* of n. This definition also extends to complex numbers α by the more general definition

$$[\alpha]_q = \frac{1 - q^\alpha}{1 - q}.$$

and it is clear that $\lim_{q\to 1}[n]_q = n$. The q-analogue of $n!$, the q-*factorial* of n, is defined by

$$[n]_q! := [n]_q \cdot [n-1]_q \cdot [n-2]_q \cdots [1]_q$$

and $[0]_q! = 1$.

The binomial coefficient extends for $0 \leqslant k \leqslant n$ to the q-*binomial coefficient* by the definition

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

The q-binomial coefficient satisfies for $0 < k < n$ the two Pascal rules

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q, \tag{2.10a}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k\begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \tag{2.10b}$$

and the initial conditions

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_q = \begin{bmatrix} n \\ n \end{bmatrix}_q = 1 \qquad \text{for } n \geqslant 0\,, \tag{2.11}$$

from which it immediately follows that the q-binomial coefficient is a polynomial in q of degree $k(n-k)$ with leading coefficient 1 for $0 \leqslant k \leqslant n$ [38].

Consider again $\mathcal{A}_{n,k}$, the set of all subsets of $[n]$ with k elements. Every element $A = \{a_1, a_2, \cdots, a_k\}$ of $\mathcal{A}_{n,k}$ is assigned a weight $w(A)$ as follows: Each element a_i is assigned weight $q^{a_i - i}$ and the weight of A, $w(A)$, is the product of the weights of all its elements. We obtain $w(A) = q^{s(A)-k(k+1)/2}$, where $s(A) = a_1 + a_2 + \cdots + a_k$. The weight of an empty set is 1 by definition. Then we can prove for $0 \leqslant k \leqslant n$

$$\sum_{A\in\mathcal{A}_{n,k}} w(A) = \begin{bmatrix} n \\ k \end{bmatrix}_q. \tag{2.12}$$

This equality can be proved by showing that both sides of the equation fulfil the same recursion and initial conditions [38, Theorem 6.1.].

In the following we will make use of the q-*Pochhammer symbol* which is defined for nonnegative integers n by

$$(a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), \quad (a;q)_0 = 1. \tag{2.13}$$

and

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

By a small manipulation one can see that the q-binomial coefficient can be expressed by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}.$$

This definition can be extended for $|q| < 1$ to complex numbers α and β (see [31, (I.40)]) by

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}_q = \frac{(q^{\beta+1};q)_\infty (q^{\alpha-\beta+1};q)_\infty}{(q;q)_\infty (q^{\alpha+1};q)_\infty},$$

and the two Pascal rules (2.10) and the initial conditions (2.11) still hold for complex values.

2.2.2 Basic hypergeometric series

There is also a q-analogue of hypergeometric series, called the q-*hypergeometric series* or *basic hypergeometric series*. We call a series $S = \sum_{k\geqslant 0} c_k$ a basic hypergeometric series, if c_{k+1}/c_k is a rational function of q^k. Basic hypergeometric series play an important role in combinatorics and special functions and date back to Euler and Heine. In this section we introduce the notation and some simple examples and refer to [31] for further reading.

By taking similar steps as in Section 2.1.2 (see [31, 38] for more details) we obtain the conventional form of the summands of a basic hypergeometric series

$$c_k = c_0 \frac{(a_1;q)_k (a_2;q)_k \cdots (a_r;q)_k}{(b_1;q)_k (b_2;q);k \cdots (b_s;q)_k (q;q)_k} \left((-1)^k q^{\binom{k}{2}}\right)^{s-r+1} z^k$$

and we define the general form of basic hypergeometric series as

$${}_r\phi_s\left[\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix}; q,z\right] := \sum_{k\geqslant 0} \frac{(a_1;q)_k (a_2;q)_k \cdots (a_r;q)_k}{(b_1;q)_k (b_2;q)_k \cdots (b_s;q)_k (q;q)_k} \left((-1)^k q^{\binom{k}{2}}\right)^{s-r+1} z^k. \tag{2.14}$$

Many summation and transformation formulae for basic hypergeometric series are known and we again refer to [31] for an extensive treatment of this topic.

For ordinary hypergeometric series we studied the binomial theorem (2.6). The q-analogue of the binomial theorem, the q-binomial theorem, has for $|x| < 1$ and $|q| < 1$ the form

$${}_1\phi_0\left[\begin{matrix} c \\ - \end{matrix}; q,x\right] = \frac{(cx;q)_\infty}{(x;q)_\infty}. \tag{2.15}$$

Specializing $x \mapsto -q^n x$ and $c \mapsto q^{-n}$ we obtain after some calculation the terminating q-binomial theorem

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} x^k = (-x;q)_n = (1+x)(1+xq)(1+xq^2)\cdots(1+xq^{n-1}). \tag{2.16}$$

It is possible to prove this version of the q-binomial theorem combinatorially. Consider the set $\mathcal{A}_n$ of all subsets of $[n]$. Given a set $A = \{a_1, a_2, \cdots, a_j\}$ in $\mathcal{A}_n$ with cardinality j, we assign weight $w(a_i) = q^{a_i-1}x$ to each element a_i. The weight of a set is again the product of the weights of its elements, such that $w(A) = q^{s(a)-j}x^j$, where $s(a) = a_1+a_2+\cdots+a_j$. The weight generating function of every subset $\mathcal{A}_{n,k}$ of $\mathcal{A}_n$ is equal to $\begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}}x^k$, similarly to (2.12). By summing up the weights of all subsets of $[n]$ we obtain

$$\sum_{A\in\mathcal{A}_n} w(A) = \sum_{k=0}^{n} \sum_{A\in\mathcal{A}_{n,k}} w(A) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}}x^k, \tag{2.17}$$

which is the left side of (2.16). On the other hand, by modifying the weight-preserving bijection in the combinatorial interpretation of (2.6), we can identify a subset of $[n]$ with n-tuples $(a_1, a_2, \ldots, a_n)$ with $a_i \in \{0, 1\}$, where now each element a_i is assigned weight 1 if it is zero and $q^{i-1}x$ if it is 1. The weight generating function of these tuples gives the right side of (2.16). Since the bijection is weight-preserving, we proved (2.16).

It is not hard to see that $(-x;q)_{n+m} = (-x;q)_n(-xq^n;q)_m$. As an application of the q-binomial theorem we can derive the q-Chu–Vandermonde identity by comparing the coefficients of x^k on both sides of $(-x;q)_{n+m} = (-x;q)_n(-xq^n;q)_m$ (analogous to the ordinary case in Section 2.1.2). Applying the q-binomial theorem (2.16) to $(-x;q)_{n+m}$, $(-x;q)_n$ and $(-xq^n;q)_m$ yields for $|x| < 1$ and $|q| < 1$

$$\sum_{k\geqslant 0} \begin{bmatrix} n+m \\ k \end{bmatrix}_q q^{\binom{k}{2}}x^k = \sum_{k\geqslant 0} \sum_{j=0}^{k} \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} m \\ k-j \end{bmatrix}_q q^{(n-j)(k-j)+\binom{k}{2}}x^k.$$

By comparing the coefficients of x^k we obtain the q-analogue of the Chu–Vandermonde convolution

$$\begin{bmatrix} n+m \\ k \end{bmatrix}_q = \sum_{j=0}^{k} \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} m \\ k-j \end{bmatrix}_q q^{(n-j)(k-j)}, \tag{2.18}$$

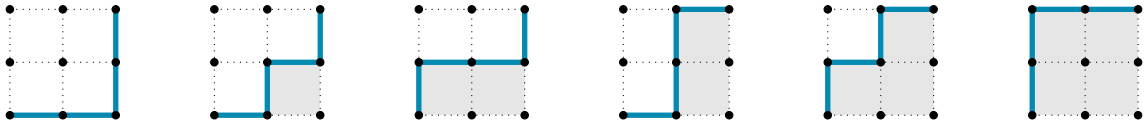
which can be reformulated in terms of q-hypergeometric series by (see [31, (1.5.2)])

$${}_2\phi_1\left[\begin{matrix} q^{-n}, a \\ b \end{matrix}; q, bq^n/a\right] = \frac{(b/a;q)_n}{(b;q)_n}, \tag{2.19}$$

where, by analytic continuation, the restriction of a and b being integer powers of q can be relaxed. One can derive (2.18) from (2.19) by simultaneously mapping $a \mapsto q^{-m}$, $b \mapsto q^{n-k+1}$ and $n \mapsto k$.

2.2.3 Lattice paths with q-weights

The q-binomial coefficient also has an elegant interpretation in terms of lattice paths analogous to Section 2.1.3 (see for example [51]). Consider the set of ordinary lattice paths from $(0,0)$ to $(k, n-k)$ with n steps. Each path P is assigned weight $q^{\text{area}(P)}$, where $\text{area}(P)$ is the number of unit squares that are below the path and above the x-axis. For example, the path in Figure 2.1 has weight q^5. For $n = 4$ and $k = 2$ there are the 6 paths



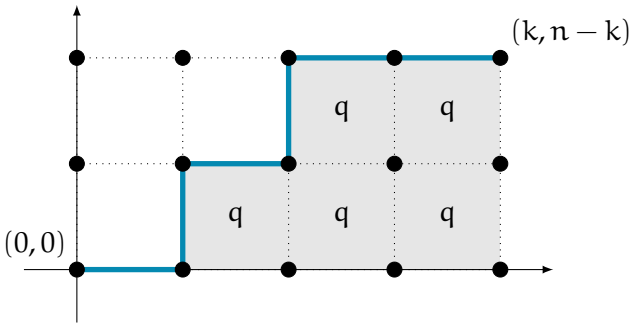


Figure 2.1: A lattice path with 5 unit squares between the path and the x-axis (area(P) = 5) and thus the q-weight according to area is q^5.

with weights 1, q, q^2, q^2, q^3 and q^4, respectively. By summing the weights of these paths we obtain the q-binomial coefficient $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + q^3 + q^4$.

Proposition 2.1. *For* $0 \leqslant k \leqslant n$ *we have*

$$\sum_{P \in \mathcal{P}_{\mathbb{N}}((0,0) \to (k, n-k))} q^{\text{area}(P)} = \begin{bmatrix} n \\ k \end{bmatrix}_q .$$

Proof. We prove the identity by comparing the recurrence relation and initial conditions of the left side of the equation with the recurrence relation and initial conditions of the q-binomial coefficient in (2.10) and (2.11). Let

$$G(n, k) = \sum_{P \in \mathcal{P}_{\mathbb{N}}((0,0) \to (k, n-k))} q^{\text{area}(P)}.$$

For $k = 0$ and $k = n$, there is only one path with weight 1 in each case, which yields $G(n, 0) = G(n, n) = 1$ and indeed $\begin{bmatrix} n \\ 0 \end{bmatrix}_q = \begin{bmatrix} n \\ n \end{bmatrix}_q = 1$ for $n \geqslant 0$. For $0 < k < n$ the last step of each path is either a north step or an east step. If the path ends with a north step, it consists of a weighted path from $(0, 0)$ to $(k, n - k - 1)$ (counted by $G(n - 1, k)$) and an additional north step which does not change the weight of the path. If the last step is an east step, it consists of a weighted path from $(0, 0)$ to $(k - 1, n - k)$ (counted by $G(n - 1, k - 1)$) and an additional east step which adds $n - k$ unit squares between the path and the x-axis, so we multiply the weight with q^{n-k}. Hence, the weighted enumeration fulfils $G(n, k) = G(n - 1, k) + q^{n-k} G(n - 1, k - 1)$ which is the Pascal rule (2.10a) and we proved the proposition. □

Another way of interpreting the q-binomial coefficient is by weighting lattice paths corresponding to their major index (see for example [51, Section 6.3]). A *valley* of a lattice path is an east step followed by a north step (if we say that the path starts from the origin). We say that a valley v lies at position (s, t), if the east step ends and the north step starts at the point (s, t). For example, the path in Figure 2.1 has two valleys at the positions $(1, 0)$ and $(2, 1)$. Now we define the *major index* of a path P, maj(P), by summing up $s + t$ for all valleys v at (s, t) in P. The path in Figure 2.1 thus has major index $\text{maj}(P) = 1 + 3 = 4$. If we assign the weight $q^{\text{maj}(P)}$ to each lattice path it is well-known that we again obtain the q-binomial coefficient as the weight generating function.

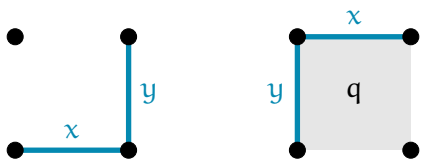
Proposition 2.2. *For $0 \leqslant k \leqslant n$ we have*

$$\sum_{P \in \mathcal{P}_{\mathbb{N}}((0,0)\to(k,n-k))} q^{\mathrm{maj}(P)} = \begin{bmatrix} n \\ k \end{bmatrix}_q .$$

One can prove this by a bijection on the set of lattice paths that takes area to major index (see [51, Problem 6.4]) or by induction using a particular recursion for the q-binomial coefficient. We will see a more general version of this proof in Section 2.4.2.

2.2.4 Noncommutative binomial theorem

Now recall the interpretation of the binomial theorem (2.9), where we assigned each east step in an ordinary lattice path weight x and each north step weight y. Following a path from the origin to the end point $(k, n-k)$ and reading off the labels x for each east step and y for each north step we obtain a word in x and y of length n. For example, the path in Figure 2.1 corresponds to the word $xyxyxx$. If we interchange an east step x with a north step y in a path with weight $q^{\mathrm{area}(P)}$,



the weight of the path changes by a factor q. Formally, this yields the noncommutative rule (see for example [2, Chapter 10])

$$yx = qxy, \tag{2.20}$$

where yx is according to a north step followed by an east step and xy is according to an east step followed by a north step. Changing the order of two consecutive x steps will not change the weight of a path and the same is true for two y steps. In order to collect q-weights we assume that

$$xq = qx, \tag{2.21}$$

$$yq = qy. \tag{2.22}$$

We will bring words in a canonical form $q^j x^k y^l$. For instance, transforming the word $xyxyxx$ associated to the example from Figure 2.1 to the canonical form yields

$$xyxyxx = xqxyqxyx = q^2xxqxyqxy = q^5x^4y^2$$

and the weight according to the area of the path is indeed q^5. Transforming the word in canonical form reveals the q-weight according to area of a path in general. We denote by $\mathbb{C}_q[x,y]$ the complex associative unital algebra generated by x and y with relation (2.20). Now we want to evaluate $(x+y)^n$ in $\mathbb{C}_q[x,y]$ and collect the variables in canonical form $c_k x^k y^{n-k}$. For example,

$$(x+y)^2 = x^2 + xy + yx + y^2 = x^2 + xy + qxy + y^2 = x^2 + (1+q)xy + y^2$$

It turns out that if we are considering $(x+y)^n$ in $\mathbb{C}_q[x,y]$, the coefficient of $x^k y^{n-k}$ is always $\begin{bmatrix} n \\ k \end{bmatrix}_q$. This is the content of the general noncommutative q-binomial theorem:

Theorem 2.3. *In* $\mathbb{C}_q[x,y]$ *there holds*

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k}. \tag{2.23}$$

This theorem, which dates back to Schützenberger [73] and Potter [53], plays an important role in q-special functions and quantum groups [2, 19, 38, 41]. It can be proved by induction on n and multiplying both sides of the equation with $(x+y)$ in the induction step. This corresponds to adding a north or an east step in the combinatorial model. Cigler [19] showed that the (commutative) q-binomial theorem (2.16) follows from the noncommutative q-binomial theorem and Koornwinder [41] showed that these theorems are even equivalent.

2.3 Elliptic combinatorics

In the previous sections we have called a series $\sum_{k\geqslant 0} c_k$ hypergeometric, if c_{k+1}/c_k is a rational function of k and basic hypergeometric, if c_{k+1}/c_k is a rational function of q^k. It is natural to ask how we can generalize these definitions even further. An addition theorem of Weierstraß indicates a possible direction to add a third level of hypergeometry (see [31, Section 11.1]). Consider meromorphic functions $f(z)$ and nonzero polynomials P that satisfy an algebraic addition theorem of the form

$$P(f(x), f(y), f(x+y)) = 0,$$

where the coefficients of P are independent of x and y. The theorem of Weierstraß states that the function $f(z)$ is then either a rational function of z, a rational function of q^z or an elliptic function. In this manner we will call a series $\sum_{k\geqslant 0} c_k$ elliptic hypergeometric, if c_{k+1}/c_k is an elliptic function of k. Elliptic hypergeometric series first appeared in the late 1980's in the work of Date, Jimbo, Kuniba, Miwa and Okado [22] while studying elliptic solutions of the Yang–Baxter equation and a decade later in the work of Frenkel and Turaev [29]. We refer to [31] and [57] for a more detailed discussion. In this section we will focus on elliptic functions, elliptic hypergeometric series and discuss how the combinatorics of q-enumeration and basic hypergeometric series can be generalized to an elliptic level.

2.3.1 Elliptic functions

Before we define elliptic functions and discover how elliptic functions can be factorized using theta functions, we recall some basic definitions. We also refer to classical textbooks on elliptic functions and complex analysis [44, 45, 79].

Let $f(z)$ be a function in the complex variable z defined on a domain $D \subset \mathbb{C}$. The function $f(z)$ is called *holomorphic* on D if $f(z)$ is complex differentiable at every point of D. If $f(z)$ is holomorphic on D except for a discrete set of isolated poles P, we call it *meromorphic* on D.

For instance, $f(z) = \cos(z)$ is differentiable on $\mathbb{C}$ and thus holomorphic on $\mathbb{C}$. The function $f(z) = \frac{1}{\cos(z)}$ is a meromorphic function on $\mathbb{C}$ since it is differentiable on $\mathbb{C}$ except for the discrete set of points $\{n\pi + \pi/2 : n \in \mathbb{Z}\}$.

For the rest of this subsection we will follow the lecture notes of Rosengren [57]. For elliptic functions there are two equivalent definitions. The classical additive definition goes back to unpublished work of Gauß. A function $f(z)$ has an additive period η, if

$$f(z) = f(z+\eta)$$

and is called *doubly periodic* if it has two periods η and τ, such that

$$f(z) = f(z+\eta) = f(z+\tau)$$

for all $z \in \mathbb{C}$, where we assume that η and τ are nonzero, $\tau/\eta \notin \mathbb{R}$ and $\mathrm{Im}(\eta/\tau) > 0$ or $\mathrm{Im}(\tau/\eta) > 0$ to avoid trivialities. A function $f(z)$ is called *additive elliptic function* if it is a doubly periodic meromorphic function on $\mathbb{C}$.

The parallelogram on the complex plane

$$\{c_1\eta + c_2\tau : 0 \leqslant c_1, c_2 \leqslant 1\}$$

is called the fundamental domain of an additive elliptic function with periods η and τ. If we tile the complex plane with parallelograms of this form, the function repeats its values in each parallelogram.

Throughout this thesis we will use a second definition of elliptic functions. First, we take $\eta = 1$ without loss of generality (this can be obtained from a general η by the change of variables $z \mapsto \eta z$). In this case an additive elliptic function $f(z)$ fulfils

$$f(z) = f(z+1) = f(z+\tau),$$

where $\mathrm{Im}(\tau) > 0$. We define a new function $g(x)$ by $f(z) = g(e^{2\pi i z})$. This function $g(x)$ is meromorphic on $\mathbb{C} \setminus \{0\}$. The relation $f(z) = f(z+1)$ corresponds to the relation $g(e^{2\pi i z}) = g(e^{2\pi i(z+1)})$ or $g(x) = g(e^{2\pi i}x)$. In the same manner, the relation $f(z) = f(z+\tau)$ is equivalent to the relation $g(x) = g(px)$ for $p = e^{2\pi i\tau}$. From $\mathrm{Im}(\tau) > 0$ it follows that $0 < |p| < 1$.

A function $g(x)$ is called *(multiplicative) elliptic* if it is meromorphic on $\mathbb{C} \setminus \{0\}$ with period p, satisfying $0 < |p| < 1$, such that

$$g(x) = g(px)$$

for all $x \in \mathbb{C} \setminus \{0\}$. In this thesis we will use the term elliptic function for multiplicative elliptic functions.

We call the period annulus

$$\{x : |p| < |x| < 1\}$$

the fundamental domain of an elliptic function g with period p. If we tile the complex plane with period annuli of the form

$$A = \{x : |p| \cdot r < |x| < r\},$$

the function repeats its values in each period annulus. We also have an interesting property: In each period annulus A every elliptic function with period p has as many poles as zeroes, counted with multiplicity. A proof of this property is given in [57].

2.3.2 Theta functions

Theta functions can be seen as the building blocks of elliptic functions and we will see that each elliptic function can be factorized as a fraction of products of theta functions. We define the *modified Jacobi theta function* as

$$\theta(x;p) := \prod_{j \geqslant 0}\left((1-p^j x)(1-\frac{p^{j+1}}{x})\right), \quad \theta(x_1,\ldots,x_\ell;p) = \prod_{k=1}^{\ell}\theta(x_k;p),$$

where $x, x_1, \ldots, x_\ell \neq 0$ and $|p| < 1$. Modified Jacobi theta functions are the only theta functions that appear in this thesis, therefore we will call them *theta functions* for short. In terms of the q-Pochhammer symbol we can write the theta function alternatively as $\theta(x;p) = (x;p)_\infty (p/x;p)_\infty$.

The modified Jacobi theta function satisfies some fundamental relations [83, cf. p. 451, Example 5]. One can see directly from the definition that

$$\theta(x;0) = 1 - x, \tag{2.24a}$$
$$\theta(1;p) = 0. \tag{2.24b}$$

Furthermore, we have the inversion formula

$$\theta(x;p) = -x\theta(1/x;p). \tag{2.24c}$$

The theta function is itself not an elliptic function but fulfils the quasi-periodicity relation

$$\theta(px;p) = -\frac{1}{x}\theta(x;p). \tag{2.24d}$$

Another important property of the theta function is the following (see [57] for a proof).

Proposition 2.4. $\theta(x;p)$ *is holomorphic for* $x \neq 0$ *and has zeroes precisely at* $x \in p^{\mathbb{Z}} = \{p^z : z \in \mathbb{Z}\}$.

The probably most fundamental result for elliptic combinatorics is the three-term identity due to Weierstraß. This identity allows us to have simple recursions on elliptic weight generating functions. Since we use this identity several times in this thesis we will give a proof of it for completeness.

Proposition 2.5 (Three-term identity)**.** *For the modified Jacobi theta function there holds*

$$\theta(xy, x/y, uz, u/z; p) = \theta(uy, u/y, xz, x/z; p) + \frac{x}{z}\theta(zy, z/y, ux, u/x; p). \tag{2.25}$$

Proof. We follow the proof of [57] using Liouville's theorem from complex analysis. Liouville's theorem states that any bounded function, which is differentiable on $\mathbb{C}$, is constant. An elliptic function, which is periodic by definition, either has poles and thus is not differentiable on $\mathbb{C}$ or the function is bounded and thus constant. Let

$$f(y) \coloneqq \theta(xy, x/y, uz, u/z; p) - \theta(uy, u/y, xz, x/z; p) - \frac{x}{z}\theta(zy, z/y, ux, u/x; p).$$

Using (2.24b) it is not hard to see that

$$f(z) = f(z^{-1}) = \theta(xz, x/z, uz, u/z; p) - \theta(uz, u/z, xz, x/z; p) - \frac{x}{z}\theta(z^2, z/z, ux, u/x; p) = 0.$$

So $f(y)$ vanishes at z and z^{-1}. Moreover, using (2.24c) and (2.24d) we obtain

$$f(py) = \frac{f(y)}{py^2},$$

so $f(y)$ vanishes at $zp^{\mathbb{Z}} \cup z^{-1}p^{\mathbb{Z}}$. The function

$$g(y) = \frac{f(y)}{\theta(zy, zy^{-1}; p)}$$

is thus differentiable for $y \neq 0$. Furthermore, we have $g(py) = g(y)$, so $g(y)$ is an elliptic function with period p. By Liouville's theorem, $g(y)$ is constant and since $f(u) = 0$ we get $g(y) = 0$ for all $y \neq 0$, which completes the proof. □

Now we come to the factorization theorem for elliptic functions.

Theorem 2.6. *Any elliptic function* f *with period* p *can be factorized as*

$$f(x) = C\frac{\theta(x/a_1, x/a_2 \ldots, x/a_n; p)}{\theta(x/b_1, x/b_2 \ldots, x/b_n; p)} \tag{2.26a}$$

where $C \in \mathbb{C}$ *is a constant and* $a_i, b_i \in \mathbb{C} \setminus \{0\}$ *fulfil the condition*

$$a_1 a_2 \cdots a_n = b_1 b_2 \cdots b_n. \tag{2.26b}$$

Rosengren gives a constructive proof in [57] where $a_1, a_2, \ldots, a_n$ are the zeroes and $b_1, b_2, \ldots, b_n$ are the poles in one arbitrary period annulus that has no zeroes or poles at the boundary. We omit the proof here.

By using (2.24d) one can see that if a function $f(x)$ is of the form (2.26), it clearly has period p:

$$f(px) \overset{(2.26a)}{=} C\frac{\theta(px/a_1, px/a_2 \ldots, px/a_n; p)}{\theta(px/b_1, px/b_2 \ldots, px/b_n; p)} \tag{2.27}$$

$$\overset{(2.24d)}{=} C\frac{x^n a_1 a_2 \cdots a_n}{x^n b_1 b_2 \cdots b_n}\frac{\theta(x/a_1, x/a_2 \ldots, x/a_n; p)}{\theta(x/b_1, x/b_2 \ldots, x/b_n; p)} \tag{2.28}$$

$$\overset{(2.26b)}{=} C\frac{\theta(x/a_1, x/a_2 \ldots, x/a_n; p)}{\theta(x/b_1, x/b_2 \ldots, x/b_n; p)} \tag{2.29}$$

$$\overset{(2.26a)}{=} f(x) \tag{2.30}$$

Furthermore, from the fact that $\theta(x; p)$ is holomorphic for $x \neq 0$ with zeroes at $a_1 p^{\mathbb{Z}}, a_2 p^{\mathbb{Z}}, \ldots, a_n p^{\mathbb{Z}}$ and poles at $b_1 p^{\mathbb{Z}}, b_2 p^{\mathbb{Z}}, \ldots, b_n p^{\mathbb{Z}}$ it follows that $f(x)$ is meromorphic on $\mathbb{C} \setminus \{0\}$. So $f(x)$ is an elliptic function in x with period p for $|p| < 1$, if it is of the form (2.26).

One can check that elliptic functions with period p form a field.

There is an analogue of the Pochhammer symbol for theta functions. We define the *theta shifted factorial* as

$$(x; q, p)_n := \begin{cases} \prod_{k=0}^{n-1} \theta(xq^k; p) & n > 0 \\ 1 & n = 0 \\ \prod_{k=0}^{-n-1} \theta(xq^{n+k}; p)^{-1} & n < 0. \end{cases} \tag{2.31}$$

For brevity, we write

$$(x_1, x_2, \ldots, x_\ell; q, p)_k = (x_1; q, p)_k (x_2; q, p)_k \cdots (x_\ell; q, p)_k.$$

2.3.3 Elliptic enumeration

In this subsection we look at some examples of elliptic functions and see how they can be interpreted as weights and weight generating functions.

One way to define an elliptic analogue of natural numbers is the following. The *elliptic analogue* of a nonnegative integer n (or simply *elliptic number*) is defined as:

$$[n]_{a,b;q,p} := \frac{\theta(q^n, aq^n, bq, aq/b; p)}{\theta(q, aq, bq^n, aq^n/b; p)}, \tag{2.32}$$

which also extends to complex numbers. Here, a and b are two additional parameters. This definition follows the definition in [66] and corresponds to the definition in [70] subject to the

substitution $b \mapsto bq^{-1}$. It turned out that this definition has several nice applications (see for example [11, 70]). The elliptic number is a generalization of the q-number as follows. Taking the limit $p \to 0$, then $a \to 0$ and then $b \to 0$, one recovers the q-analogue $[n]_q$:

$$\begin{aligned}
&\lim_{b\to 0}\Big(\lim_{a\to 0}\Big(\lim_{p\to 0}\frac{\theta(q^n, aq^n, bq, aq/b; p)}{\theta(q, aq, bq^n, aq^n/b; p)}\Big)\Big)\\
&= \lim_{b\to 0}\Big(\lim_{a\to 0}\frac{(1-q^n)(1-aq^n)(1-bq)(1-aq/b)}{(1-q)(1-aq)(1-bq^n)(1-aq^n/b)}\Big)\\
&= \lim_{b\to 0}\frac{(1-q^n)(1-bq)}{(1-q)(1-bq^n)} = \frac{1-q^n}{1-q} = [n]_q.
\end{aligned}$$

The elliptic number is indeed an elliptic function in its parameters (see also [70, Remark 4]). Functions that are elliptic in all its free parameters with equal periods are called *totally elliptic* [74]. The function $[n]_{a,b;q,p}$ is elliptic in a, b, q and q^n each with period p which follows from the fact that it is of the form (2.26) for each parameter. Therefore, $[n]_{a,b;q,p}$ is totally elliptic.

We also define the elliptic weight

$$W_{a,b;q,p}(k) := \frac{\theta(aq^{2k+1}, b, bq, a/b, aq/b; p)}{\theta(aq, bq^k, bq^{k+1}, aq^k/b, aq^{k+1}/b; p)}q^k, \tag{2.33}$$

which is again a totally elliptic function and reduces to q^k by taking the limit $p \to 0$, $a \to 0$ and $b \to 0$ (in this order).

We are now ready to elaborate our first small result in elliptic enumerative combinatorics: Consider the subsets $\mathcal{A}_{n,1}$ of the set $[n] = \{1, 2, \dots, n\}$ with only one element. If the single element in a subset $A \in \mathcal{A}_{n,1}$ is k, we assign weight $w(A) = W_{a,b;q,p}(k-1)$.

Proposition 2.7. *For $n \geqslant 1$ we have*

$$\sum_{A\in\mathcal{A}_{n,1}} w(A) = [n]_{a,b;q,p}. \tag{2.34}$$

Proof. We have to show that

$$\sum_{k=1}^{n} W_{a,b;q,p}(k-1) = [n]_{a,b;q,p}, \tag{2.35}$$

which we will do by induction on n using the properties of theta functions. For $n = 1$ the statement is clear and both sides are equal to 1. For $n > 1$ the set $\mathcal{A}_{n,1}$ consists of the subset $\{n\}$ and all subsets in $\mathcal{A}_{n-1,1}$, so it suffices to show

$$[n]_{a,b;q,p} = [n-1]_{a,b;q,p} + W_{a,b;q,p}(n-1), \tag{2.36}$$

or, expressed by theta functions,

$$\begin{aligned}
\frac{\theta(q^n, aq^n, bq, aq/b; p)}{\theta(q, aq, bq^n, aq^n/b; p)} &= \frac{\theta(q^{n-1}, aq^{n-1}, bq, aq/b; p)}{\theta(q, aq, bq^{n-1}, aq^{n-1}/b); p)}\\
&\quad + \frac{\theta(aq^{2n-1}, b, bq, a/b, aq/b; p)}{\theta(aq, bq^{n-1}, bq^n, aq^{n-1}/b, aq^n/b; p)}q^{n-1}.
\end{aligned}$$

Multiplying both sides by $\theta(q, aq, bq^n, aq^n/b, bq^{n-1}, aq^{n-1}/b; p)$ and dividing by $\theta(bq, aq/b; p)$ gives

$$\theta(q^n, aq^n, bq^{n-1}, aq^{n-1}/b; p) = \theta(q^{n-1}, aq^{n-1}, bq^n, aq^n/b; p) + \theta(a/b, b, aq^{2n-1}, q; p)q^{n-1}.$$

Now we use (2.24c) on the rightmost summand to obtain

$$\theta(q^n, aq^n, bq^{n-1}, aq^{n-1}/b; p) = \theta(q^{n-1}, aq^{n-1}, bq^n, aq^n/b; p) + \frac{aq^n}{b}\theta(b/a, b, aq^{2n-1}, 1/q; p),$$

which is precisely the three-term identity (2.25) with $x = a^{1/2}q^n$, $y = a^{-1/2}$, $u = a^{1/2}q^{n-1}$ and $z = ba^{-1/2}$. Therefore, we have shown (2.36) and (2.35) and completed the proof of the proposition. □

We can generalize this result to k-elementary subsets. Recall the definition of the theta shifted factorial from (2.31). We introduce the *elliptic binomial coefficient* for nonnegative integers n, k as

$$\begin{bmatrix} n \\ k \end{bmatrix}_{a,b;q,p} := \frac{(q^{n-k+1}, aq^{n-k+1}, bq^{1+k}, aq^{1-k}/b; q, p)_k}{(q, aq, bq^{n+1}, aq^{n-2k+1}/b; q, p)_k} \tag{2.37}$$

together with the elliptic weights

$$W_{a,b;q,p}(s, t) := \frac{\theta(aq^{s+2t}, bq^{2s-1}, bq^{2s}, aq^{1-s}/b, aq^{-s}/b; p)}{\theta(aq^s, bq^{2s+t-1}, bq^{2s+t}, aq^{t-s}/b, aq^{1+t-s}/b; p)}q^t. \tag{2.38}$$

The elliptic binomial coefficient was introduced by Schlosser [64, 65] and is an elliptic function in a, b, q, q^k and q^n and thus a totally elliptic function. We obtain the elliptic number $[n]_{a,b;q,p}$ if we set $k = 1$ and substitute $b \mapsto bq^{-1}$. Furthermore,

$$W_{a,bq^{-1};q,p}(1, k) = W_{a,b;q,p}(k).$$

To see the connection to q-binomial coefficients, consider the limits

$$\lim_{b\to 0}\Big(\lim_{a\to 0}\Big(\lim_{p\to 0}\begin{bmatrix} n \\ k \end{bmatrix}_{a,b;q,p}\Big)\Big) = \begin{bmatrix} n \\ k \end{bmatrix}_q, \tag{2.39}$$

$$\lim_{b\to 0}\Big(\lim_{a\to 0}\Big(\lim_{p\to 0} W_{a,b;q,p}(s, t)\Big)\Big) = q^t. \tag{2.40}$$

For some purposes it is also interesting to consider other limits of the elliptic binomial coefficients (for example only taking the limit $p \mapsto 0$). Some work in this direction was done for example in [66] or in the appendix of [65].

For nonnegative integers n and k the elliptic binomial coefficient satisfies

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}_{a,b;q,p} = 1, \quad \begin{bmatrix} n \\ k \end{bmatrix}_{a,b;q,p} = 0 \qquad \text{for } k < 0 \text{ or } k > n\,, \tag{2.41a}$$

and for $n, k > 0$

$$\begin{bmatrix} n \\ k \end{bmatrix}_{a,b;q,p} = \begin{bmatrix} n-1 \\ k \end{bmatrix}_{a,b;q,p} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{a,b;q,p} W_{a,b;q,p}(k, n-k), \tag{2.41b}$$

which can be checked very similar to the proof of (2.36) by applying the three-term identity (2.25).

Consider again the subsets $\mathcal{A}_{n,k}$ of $[n]$ with k elements. We write them in canonical form $A = \{a_1, a_2, \ldots, a_k\}$, such that $a_1 < a_2 < \cdots < a_k$. We assign the weight $W_{a,b;q,p}(i, a_i - i)$ to each element a_i and the weight of a subset A, $w(A)$, is the product of the weights of its entries. The weight of an empty set is defined to be 1.

Proposition 2.8. *For* $0 \leqslant k \leqslant n$ *we have*

$$\sum_{A \in \mathcal{A}_{n,k}} w(A) = \begin{bmatrix} n \\ k \end{bmatrix}_{a,b;q,p}. \tag{2.42}$$

Proof. We prove this proposition by checking that the initial conditions and recurrence relations of the weighted counting coincide with (2.41). By considering whether the element n is an element of a subset or not we obtain the following recurrence relation for the weighted counting

$$\sum_{A \in \mathcal{A}_{n,k}} w(A) = \sum_{A \in \mathcal{A}_{n-1,k}} w(A) + W_{a,b;q,p}(k, n-k) \sum_{A \in \mathcal{A}_{n-1,k-1}} w(A) \tag{2.43}$$

for $n, k > 0$ which is equivalent to (2.41b). Since also the initial conditions of the weighted counting and the elliptic binomial coefficient coincide the proposition is true. □

Proposition 2.8 is a generalization of (2.12) and reduces to that case by taking again the limits $p \to 0$, $a \to 0$ and $b \to 0$ in this order.

2.3.4 Elliptic hypergeometric series

As explained in the beginning of Section 2.3, we can consider a third level of hypergeometric series, elliptic hypergeometric series. We call a series $S = \sum_{k \geqslant 0} c_k$ an *elliptic hypergeometric series* if c_{k+1}/c_k is an additive elliptic function. Introductions to elliptic hypergeometric series can for example be found in [31, 57, 75]. In the following we stick to the notations and definitions of [31] and [57].

The additive elliptic function $f(k) = c_{k+1}/c_k$ can be expressed by a multiplicative elliptic function $g(x)$ (as explained in Section 2.3.1). Let τ/ν and $1/\nu$ be the periods of f and assume that 1, ν and τ are linearly independent over $\mathbb{Q}$. Define $p = e^{2\pi i \tau}$ and $q = e^{2\pi i \nu}$ and let g be a function, such that $f(k) = g(q^k)$. Then g is a multiplicative elliptic function and by Theorem 2.6 we can write

$$\frac{c_{k+1}}{c_k} = f(k) = g(q^k) = \frac{\theta(a_1 q^k, a_2 q^k, \ldots, a_{m+1} q^k; p)}{\theta(b_1 q^k, b_2 q^k, \ldots, b_{m+1} q^k; p)} z,$$

where

$$a_1 a_2 \cdots a_{m+1} = b_1 b_2 \cdots b_{m+1}.$$

By again taking similar steps as in Section 2.1.2 and recalling the definition of the theta shifted factorial (2.31), we obtain

$$c_k = c_0 \frac{(a_1, a_2, \ldots, a_{m+1}; q, p)}{(b_1, b_2, \ldots, b_{m+1}; q, p)} z^k. \tag{2.44}$$

By convention we assume $c_0 = 1$ and $b_{m+1} = q$ and define the general form of elliptic hypergeometric series as

$${}_{m+1}E_m \left[\begin{matrix} a_1, \ldots, a_{m+1} \\ b_1, \ldots, b_m \end{matrix} ; q, p; z \right] := \sum_{k \geqslant 0} \frac{(a_1, a_2, \ldots, a_{m+1}; q, p)_k}{(b_1, b_2, \ldots, b_m, q; q, p)_k} z^k, \tag{2.45a}$$

where the parameters satisfy the balancing condition

$$a_1 a_2 \cdots a_{m+1} = q b_1 b_2 \cdots b_m. \tag{2.45b}$$

By specializing $p \mapsto 0$ one recovers basic hypergeometric series of the form

$${}_{m+1}\phi_m\left[\begin{matrix} a_1,\ldots,a_{m+1} \\ b_1,\ldots,b_m \end{matrix}; q,z\right],$$

which satisfy the balancing condition (2.45b). Also for elliptic hypergeometric series there are various summation and transformation formulae and they tend to be more involved than in the q-case (see [57, Section 2.3] for a discussion).

Usually, we will focus on finite sums and therefore take $a_{m+1} = q^{-n}$ for some nonnegative integer n, such that the series is finite. Because of convergence reasons the theory of infinite elliptic hypergeometric series is not very well developed. To overcome these obstacles with infinite series one usually switches to a definition using integrals [57] but in this thesis we are only dealing with finite series and, in fact, only with very-well poised series, which we define next.

We call a series ${}_{m+1}E_m$ *well-poised* if

$$a_1 q = a_2 b_1 = a_3 b_2 = \cdots = a_{m+1} b_m, \tag{2.46}$$

and we call it *very-well-poised* if it is well-poised, $m \geqslant 4$, and

$$a_2 = q a_1^{1/2}, \quad a_3 = -q a_1^{1/2}, \quad a_4 = q(a_1/p)^{1/2}, \quad a_5 = -q(a_1 p)^{1/2}. \tag{2.47}$$

A basic hypergeometric series ${}_{m+1}\phi_m$ is called well-poised if (2.46) is true and very-well poised if in addition

$$a_2 = q a_1^{1/2}, \quad a_3 = -q a_1^{1/2}. \tag{2.48}$$

If an elliptic hypergeometric series is very-well poised, and therefore (2.46) and (2.47) are true, we introduce the notation

$${}_{m+1}V_m(a_1; a_6, a_7, \ldots, a_{m+1}; q, p; z) := {}_{m+1}E_m\left[\begin{matrix} a_1,\ldots,a_{m+1} \\ b_1,\ldots,b_m \end{matrix}; q,p; -z\right]. \tag{2.49}$$

We can use the identity

$$\frac{\theta(aq^{2n};p)}{\theta(a;p)} = \frac{(qa^{1/2}, -qa^{1/2}, q(a/p)^{1/2}, -q(ap)^{1/2}; q, p)_n}{(a^{1/2}, -a^{1/2}, (ap)^{1/2}, -(a/p)^{1/2}; q, p)_n}(-q)^{-n}$$

(see [31, (11.2.17)]) to find that

$${}_{m+1}V_m(a_1; a_6, a_7, \ldots, a_{m+1}; q, p; z) = \sum_{k\geqslant 0} \frac{\theta(a_1 q^{2n};p)}{\theta(a_1;p)} \frac{(a_1, a_6, a_7, \ldots, a_{m+1}; q, p)_n}{(q, a_1 q/a_6, a_1 q/a_7, \ldots, a_1 q/a_{m+1}; q, p)_n}(qz)^n. \tag{2.50}$$

We are now equipped to state one of the most fundamental identities in the theory of elliptic hypergeometric series, the Frenkel–Turaev summation [29]:

Theorem 2.9 (Frenkel and Turaev's ${}_{10}V_9$ summation)**.** *Let n be a nonnegative integer. When $bcde = a^2q^{n+1}$, then*

$$ {}_{10}V_9(a; b, c, d, e, q^{-n}; q, p; 1) = \frac{(aq, aq/bc, aq/bd, aq/cd; q, p)_n}{(aq/b, aq/c, aq/d, aq/bcd; q, p)_n} \tag{2.51} $$

holds.

One should read expressions like aq/bc as $\frac{aq}{bc}$. By setting $p = 0$, this particular identity extends Jackson's ${}_8\phi_7$ summation which itself extends Dougall's ${}_7F_6$ summation [31, (2.6.2),(2.1.6)]. In the next section we show a combinatorial proof of Frenkel and Turaev's sum due to Schlosser [64, 65], where some parameters are restricted to be nonnegative integers. This proof also simplifies to combinatorial proofs of Jackson's and Dougall's summation formulae as special cases. In Chapter 3 we will extend this proof to a more general setting.

The Frenkel–Turaev summation is a direct consequence of the following transformation formula also due to Frenkel and Turaev [29] which can be found under some restrictions on the parameters in [23].

Theorem 2.10. *Let n be a nonnegative integer. When $bcdefg = a^3q^{n+2}$ and $\lambda = a^2q/bcd$, then*

$$ \begin{aligned} &{}_{12}V_{11}(a; b, c, d, e, f, g, q^{-n}; q, p; 1) \\ &\quad = \frac{(aq, aq/ef, \lambda q/e, \lambda q/f; q, p)_n}{(aq/e, aq/f, \lambda q, \lambda q/ef; q, p)_n} {}_{12}V_{11}(\lambda; \lambda b/a, \lambda c/a, \lambda d/a, e, f, g, q^{-n}; q, p; 1) \end{aligned} \tag{2.52} $$

holds.

This identity is an elliptic analogue of Bailey's ${}_{10}\phi_9$ transformation formula [31, (2.9.1)]. A combinatorial proof of this formula remains an open problem in elliptic combinatorics.

2.3.5 Lattice paths with elliptic weights

Like the q-binomial coefficient, also the elliptic binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_{a,b;q,p}$ has a combinatorial interpretation in terms of lattice paths by area. Recall the definition of the elliptic weights from (2.38)

$$ W_{a,b;q,p}(s,t) = \frac{\theta(aq^{s+2t}, bq^{2s-1}, bq^{2s}, aq^{1-s}/b, aq^{-s}/b; p)}{\theta(aq^s, bq^{2s+t-1}, bq^{2s+t}, aq^{t-s}/b, aq^{1+t-s}/b; p)} q^t, $$

which we will call *big elliptic weights* and we define *small elliptic weights* as

$$ w_{a,b;q,p}(s,t) := \frac{W_{a,b;q,p}(s,t)}{W_{a,b;q,p}(s,t-1)} = \frac{\theta(aq^{s+2t}, bq^{2s+t-2}, aq^{t-s-1}/b; p)}{\theta(aq^{s+2t-2}, bq^{2s+t}, aq^{t-s+1}/b; p)} q, \tag{2.53} $$

for $s, t \in \mathbb{N}$ and, due to $W_{a,b;q,p}(s,0) = 1$, we have

$$ W_{a,b;q,p}(s,t) = \prod_{j=1}^{t} w_{a,b;q,p}(s,j). $$

Consider the set of lattice paths $\mathcal{P}_{\mathbb{N}}((0,0) \to (k, n-k))$ from $(0,0)$ to $(k, n-k)$. Each unit square in the first quadrant, which has its north-east corner at position (s,t), is assigned weight $w_{a,b;q,p}(s,t)$. The elliptic weight of a lattice path P, $w(P)$, is defined as the product over all weights of unit squares between the path and the x-axis. See Figure 2.2 for an example. The empty product is defined to be 1. If there are no unit squares between the path and the x-axis or the path is the empty path, the weight is defined to be 1.

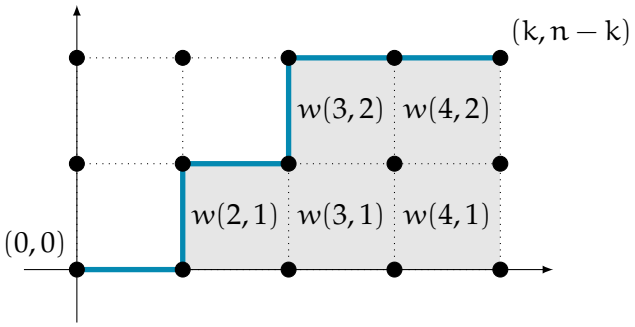


Figure 2.2: A lattice path with weight $w(P) = w(2,1)w(3,1)w(3,2)w(4,1)w(4,2)$, where we use the short notation $w(s,t) = w_{a,b;q,p}(s,t)$.

Proposition 2.11 (Schlosser [64]). *For* $0 \leqslant k \leqslant n$ *the equality*

$$\sum_{P \in \mathcal{P}_{\mathbb{N}}((0,0)\to(k,n-k))} w(P) = \begin{bmatrix} n \\ k \end{bmatrix}_{a,b;q,p} \tag{2.54}$$

holds.

Proof. As in the q-case and in the elliptic enumeration of subsets of $[n]$, one can see that this is true by checking that both sides fulfil the same recurrence relation and initial conditions. Let

$$G(n,k) = \sum_{P \in \mathcal{P}_{\mathbb{N}}((0,0)\to(k,n-k))} w(P).$$

For $k = 0$ and $k = n$ there is only one path with weight 1 in each case, which yields $G(n,0) = G(n,n) = 1$ and indeed $\left[\begin{smallmatrix} n \\ 0 \end{smallmatrix}\right]_{a,b;q,p} = \left[\begin{smallmatrix} n \\ n \end{smallmatrix}\right]_{a,b;q,p} = 1$ for $n \geqslant 0$. For $0 < k < n$ the last step of each path is either a north step or an east step. If the path ends with a north step, it consists of a weighted path from $(0,0)$ to $(k, n-k-1)$ (counted by $G(n-1,k)$) and an additional north step, which does not change the weight of the path. If the last step is an east step, it consists of a weighted path from $(0,0)$ to $(k-1, n-k)$ (counted by $G(n-1,k-1)$) and an additional east step, which adds $n-k$ unit squares between the path and the x-axis, so we multiply the weight with

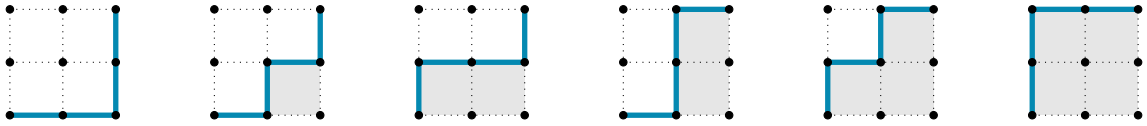
$$\prod_{j=1}^{n-k} w_{a,b;q,p}(k,j) = W_{a,b;q,p}(k, n-k).$$

Hence, the weighted enumeration fulfils

$$G(n,k) = G(n-1,k) + W_{a,b;q,p}(k,n-k)G(n-1,k-1),$$

which is the elliptic Pascal rule (2.41b) and we proved the proposition. □

For example, if we take $n = 4$ and $k = 2$ there are the 6 paths



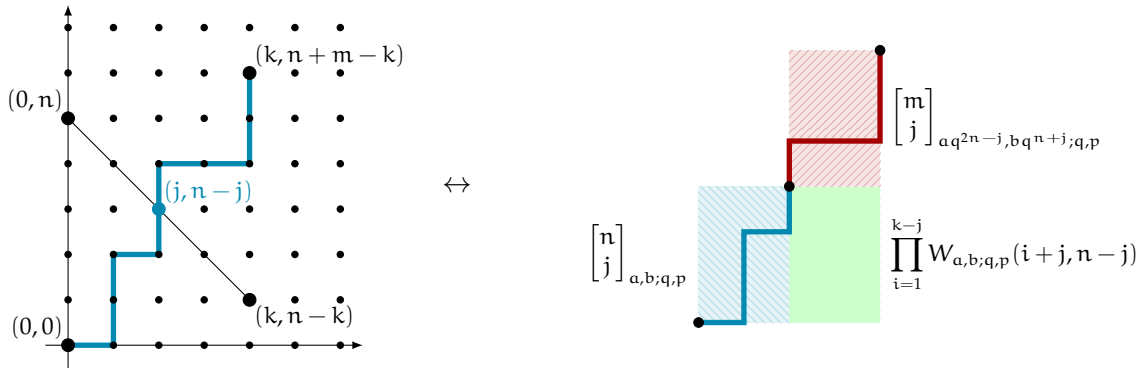


Figure 2.3: The decomposition of a path from $(0,0)$ to $(k, n+m-k)$ into three parts as in the proof of Theorem 2.9.

with weights 1, $W_{a,b;q,p}(2,1)$, $W_{a,b;q,p}(1,1)W_{a,b;q,p}(2,1)$, $W_{a,b;q,p}(2,2)$, $W_{a,b;q,p}(1,1)W_{a,b;q,p}(2,2)$ and $W_{a,b;q,p}(1,2)W_{a,b;q,p}(2,2)$, respectively. By summing up the weights of these paths one obtains the elliptic binomial coefficient $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_{a,b;q,p}$.

With this model we are able to give a very illustrative combinatorial proof of the identity (2.51). The proof was first described by Schlosser in [64]. First, observe that we can shift weights with the identity

$$w_{aq^{i+2j},bq^{2i+j};q,p}(s,t) = w_{a,b;q,p}(s+i,t+j).$$

Therefore, $\begin{bmatrix} n \\ k \end{bmatrix}_{aq^{i+2j},bq^{2i+j};q,p}$ is the weighted counting of lattice paths starting at (i,j) and ending at $(k+i, n-k+j)$, where the weight is the product of all weights of unit squares between the path and the line $y = j$. This corresponds to shifting an ordinary path and its weights i positions to the east and j positions to the north.

With this considerations we are ready to derive Theorem 2.9. For nonnegative integers n, m, k consider all elliptic weighted paths from $(0,0)$ to $(k, n+m-k)$, counted by $\begin{bmatrix} n+m \\ k \end{bmatrix}_{a,b;q,p}$. Observe that each path has to cross the line $y = n - x$. The weighted counting of all paths that touch this line at $(j, n-j)$ is

$$\begin{bmatrix} n \\ j \end{bmatrix}_{a,b;q,p} \begin{bmatrix} m \\ k-j \end{bmatrix}_{aq^{2n-j},bq^{n+j};q,p} \prod_{i=1}^{k-j} W_{a,b;q,p}(i+j, n-j),$$

where $\begin{bmatrix} n \\ j \end{bmatrix}_{a,b;q,p}$ corresponds to the weight of the first part of the paths between $(0,0)$ and $(j, n-j)$, the factor $\begin{bmatrix} m \\ k-j \end{bmatrix}_{aq^{2n-j},bq^{n+j};q,p}$ corresponds to the weight of the second part of the paths between $(j, n-j)$ and $(k, n+m-k)$ and the product of big elliptic weights takes care of the weights in the rectangle between the second part of the path and the x-axis (this will get clear in Figure 2.3). If we sum over all possible $j \in \{0, 1, 2, \ldots, k\}$, we obtain the convolution formula

$$\begin{bmatrix} n+m \\ k \end{bmatrix}_{a,b;q,p} = \sum_{j=0}^{k} \begin{bmatrix} n \\ j \end{bmatrix}_{a,b;q,p} \begin{bmatrix} m \\ k-j \end{bmatrix}_{aq^{2n-j},bq^{n+j};q,p} \prod_{i=1}^{k-j} W_{a,b;q,p}(i+j, n-j) \qquad (2.55)$$

for nonnegative integers n, m and k. As described in [65], we can extend this formula to complex n and m by repeated analytic continuation. If we take the substitution $(a,b,c,d,e,n) \mapsto (bq^{-n}/a, q^{-n}/a, bq^{1+m+n}, bq^{-n-m+k}/a, q^{-n}, k)$ in Frenkel and Turaev's summation (2.51), we obtain (2.55) by doing a careful computation. If we treat q^n and q^m as

complex variables, this substitution is reversible and therefore, these equations are equivalent. Thus we proved Frenkel and Turaev's summation (2.51).

2.3.6 Elliptic combinatorics in the literature

Elliptic hypergeometric series first appeared in the 1980's and 1990's in the work of Date et al. [22] and Frenkel and Turaev [29] while studying elliptic solutions of the Yang–Baxter equation.

In an article published in 2007 [64], Schlosser studied an elliptic enumeration of nonintersecting lattice paths using the Lindström–Gessel–Viennot theorem (see for example [59]), elliptic weight functions and an elliptic determinant evaluation due to Warnaar [82]. This work by Schlosser already contained the elliptic binomial coefficient (2.37) and the combinatorial proof for Frenkel and Turaev's summation (2.51) and can be seen as the starting point for elliptic combinatorics.

A few years later, Borodin, Gorin and Rains [14] took a similar approach by enumerating lozenge tilings of a hexagon using elliptic weight functions. This approach was further studied by Betea [12, 13].

In a series of articles [67, 68, 69, 70], Schlosser and Yoo generalized the combinatorial model of rook theory to an elliptic level. These articles contained many interesting results and studied elliptic analogues of combinatorial special numbers involving elliptic rook numbers, elliptic (generalized) Stirling numbers of the first and second kind and elliptic Lah numbers. In [72], Schlosser and Yoo developed elliptic Lucas and Fibonacci numbers and polynomials. The approach of generalizing combinatorial special numbers was also pursued by Bergeron, Ceballos and the author in [11], where q- and elliptic analogues of Fibonacci and Fibonomial numbers were studied and a combinatorial model using elliptic weight functions was introduced (see Chapter 4).

In 2018, Baba and Katori [5] modified the original approach of Schlosser [64] and studied excursions on $\mathbb{Z}$ starting and ending at the origin and studied asymptotic probability laws. In a paper that was published in 2021, Schlosser, Senapati and Uncu studied the log-concavity of the elliptic binomial coefficient (2.37) and limit cases of the same.

Schlosser generalized the elliptic binomial coefficient (2.37) in [65] using more general weights that are not necessarily elliptic functions leading to weight-dependent binomial coefficients that are described in the following section. This article was published in 2020 but an earlier version was already uploaded to the arXiv in 2011. These weight-dependent binomial coefficients were further studied by Schlosser and Yoo in [71] and generalized to a setting were the arguments might be negative integers by Schlosser, Yoo and the author in [43] (see Chapter 3).

2.3.7 Useful identities for elliptic combinatorics

In this section we want to collect some useful identities in elliptic combinatorics we use throughout this thesis. Most of them appear in [65] and [70]. For the elliptic number

$$[n]_{a,b;q,p} = \frac{\theta(q^n, aq^n, bq, aq/b; p)}{\theta(q, aq, bq^n, aq^n/b; p)}, \tag{2.56}$$

the big elliptic weight

$$W_{a,b;q,p}(k) = \frac{\theta(aq^{2k+1}, b, bq, a/b, aq/b; p)}{\theta(aq, bq^k, bq^{k+1}, aq^k/b, aq^{k+1}/b; p)} q^k, \tag{2.57}$$

and the small elliptic weight

$$w_{a,b;q,p}(k) = \frac{\theta(aq^{2k+1}, bq^{k-1}, aq^{k-1}/b; p)}{\theta(aq^{2k-1}, bq^{k+1}, aq^{k+1}/b; p)} q, \tag{2.58}$$

we have

$$[n+m]_{a,b;q,p} = [n]_{a,b;q,p} + W_{a,b;q,p}(n)[m]_{aq^{2n},bq^{n};q,p}, \tag{2.59}$$
$$[n]_{a,b;q,p} = [n-1]_{a,b;q,p} + W_{a,b;q,p}(n-1), \tag{2.60}$$
$$W_{a,b;q,p}(k) = \prod_{j=1}^{k} w_{a,b;q,p}(j), \tag{2.61}$$
$$W_{a,b;q,p}(k+j) = W_{a,b;q,p}(j) W_{aq^{2j},bq^{j};q,p}(k), \tag{2.62}$$
$$w_{a,b;q,p}(k+j) = w_{aq^{2j},bq^{j};q,p}(k). \tag{2.63}$$
$$\tag{2.64}$$

For the elliptic binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_{a,b;q,p} = \frac{(q^{n-k+1}, aq^{n-k+1}, bq^{1+k}, aq^{1-k}/b; q, p)_k}{(q, aq, bq^{n+1}, aq^{n-2k+1}/b; q, p)_k} \tag{2.65}$$
$$= \frac{(q^{1+k}, aq^{1+k}, bq^{1+k}, aq^{1-k}/b; q, p)_{n-k}}{(q, aq, bq^{1+2k}, aq/b; q, p)_{n-k}}, \tag{2.66}$$

the big elliptic weight

$$W_{a,b;q,p}(s,t) = \frac{\theta(aq^{s+2t}, bq^{2s-1}, bq^{2s}, aq^{1-s}/b, aq^{-s}/b; p)}{\theta(aq^{s}, bq^{2s+t-1}, bq^{2s+t}, aq^{t-s}/b, aq^{1+t-s}/b; p)} q^t, \tag{2.67}$$

and the small elliptic weight

$$w_{a,b;q,p}(s,t) = \frac{\theta(aq^{s+2t}, bq^{2s+t-2}, aq^{t-s-1}/b; p)}{\theta(aq^{s+2t-2}, bq^{2s+t}, aq^{t-s+1}/b; p)} q, \tag{2.68}$$

we have

$$\begin{bmatrix} n \\ k \end{bmatrix}_{a,b;q,p} = \begin{bmatrix} n-1 \\ k \end{bmatrix}_{a,b;q,p} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{a,b;q,p} W_{a,b;q,p}(k, n-k), \tag{2.69}$$
$$W_{a,b;q,p}(s,t) = \prod_{j=1}^{t} w_{a,b;q,p}(s,j), \tag{2.70}$$
$$W_{a,b;q,p}(s,t+j) = W_{a,b;q,p}(s,t) W_{aq^{2t},bq^{t};q,p}(s,j), \tag{2.71}$$
$$w_{a,b;q,p}(s+i,t+j) = w_{aq^{i+2j},bq^{2i+j};q,p}(s,t), \tag{2.72}$$
$$\begin{bmatrix} n+m \\ k \end{bmatrix}_{a,b;q,p} = \sum_{j=0}^{k} \begin{bmatrix} n \\ j \end{bmatrix}_{a,b;q,p} \begin{bmatrix} m \\ k-j \end{bmatrix}_{aq^{2n-j},bq^{n+j};q,p} \prod_{i=1}^{k-j} W_{a,b;q,p}(i+j, n-j). \tag{2.73}$$

2.4 Weight-dependent binomial coefficient

It turns out that the elliptic binomial coefficient and the elliptic enumeration of lattice paths can be generalized to an even more general setting. We will consider weight-dependent binomial coefficients that depend on a series of indeterminate weights $(w(s,t))_{s,t\in\mathbb{N}}$, which can be specialized to 1, q or, more generally, to the elliptic weights $w_{a,b;q,p}(s,t)$. These very general binomial coefficients were introduced by Schlosser [65] and further developed by Schlosser, Yoo and the author in [43]. In this section we discuss a weight-dependent generalization of the noncommutative q-binomial theorem (2.23) and a generalization of the q-Chu–Vandermonde Identity (2.18) and Frenkel and Turaev's ${}_{10}V_9$ summation (2.51).

Let $(w(s,t))_{s,t\in\mathbb{N}}$ be a series of indeterminate weights, which we denote *small weights*, and

$$W(s,t) := \prod_{j=1}^{t} w(s,j) \tag{2.74}$$

are denoted *big weights*. We define the *weight-dependent binomial coefficients* or *w-binomial coefficients* recursively by

$$ {}_w\begin{bmatrix}0\\0\end{bmatrix} = 1, \quad {}_w\begin{bmatrix}n\\k\end{bmatrix} = 0 \qquad \text{for } k<0 \text{ or } k>n, \tag{2.75a}$$

and for $n,k>0$

$$ {}_w\begin{bmatrix}n\\k\end{bmatrix} = {}_w\begin{bmatrix}n-1\\k\end{bmatrix} + {}_w\begin{bmatrix}n-1\\k-1\end{bmatrix} W(k,n-k). \tag{2.75b}$$

By following the proof of Proposition 2.11, it follows from this recursion that for $n,k\geqslant 0$ the w-binomial coefficient counts weighted lattice paths in the first quadrant, where the weight of a path is defined equally as in the elliptic setting (2.54), with $w_{a,b;q,p}(s,t)$ replaced by the more general $w(s,t)$. The path in Figure 2.2 now has the correct labels and the weight of this path is $W(2,1)W(3,2)W(4,2)$.

By specializing $w(s,t)=1$ we obtain the ordinary binomial coefficients. Taking $w(s,t)=q$ yields the q-binomial coefficients. Furthermore, by specializing $w(s,t)=w_{a,b;q,p}(s,t)$ we obtain the elliptic binomial coefficients. We will see specializations to complete homogeneous and elementary symmetric functions in Chapter 3.

2.4.1 Weight-dependent binomial theorem

Before we state an extension of the noncommutative q-binomial theorem, we extend the algebra $\mathbb{C}_q[x,y]$:

Definition 2.12. For a doubly-indexed sequence of indeterminates $(w(s,t))_{s,t\in\mathbb{N}}$ let $\mathbb{C}_w[x,y]$ be the associative unital algebra over $\mathbb{C}$ generated by x,y and $w(s,t)$ for $s,t\in\mathbb{N}$ satisfying the following relations:

$$yx = w(1,1)\,xy, \tag{2.76a}$$
$$x\,w(s,t) = w(s+1,t)\,x, \tag{2.76b}$$
$$y\,w(s,t) = w(s,t+1)\,y, \tag{2.76c}$$

for all $(s,t)\in\mathbb{N}^2$.

Schlosser [65] showed that in this algebra the following theorem holds:

Theorem 2.13 ([65]). *Let* $n \in \mathbb{N}_0$. *Then, as an identity in* $\mathbb{C}_w[x,y]$,

$$(x+y)^n = \sum_{k=0}^{n} {}_w\begin{bmatrix} n \\ k \end{bmatrix} x^k y^{n-k}. \tag{2.77}$$

This theorem can be shown by using induction and has a nice interpretation in terms of lattice paths. In the induction step one multiplies both sides of the identity with $(x+y)$, this corresponds to adding an east step x or a north step y to a lattice path.

Recall again the interpretation of the binomial theorem (2.9) where we assigned each east step in an ordinary lattice path weight x and each north step weight y. As observed in Section 2.2.4 we obtained a word in x and y of length n by following a path from the origin to the end point $(k, n-k)$ and reading off the labels x for each east step and y for each north step. If we interchange an east step x with a north step y at the beginning of a path,

the weight of the path changes by a factor $w(1,1)$. Formally, this yields the noncommutative rule (2.76a), where yx accords to a north step followed by an east step and xy accords to an east step followed by a north step. Interchanging weights with north and east steps yields the relations (2.76b) and (2.76c) because the positions of the weights is changing. We want to bring words in a canonical form $c_{k,l}x^k y^l$ and collect the weights in the coefficient $c_{k,l}$. We use the running example and transform the word $xyxyxx$ associated to the example from Figure 2.1 to the canonical form

$$\begin{aligned} xyxyxx &= xw(1,1)xyw(1,1)xyx = w(2,1)w(3,2)xxw(1,1)xyw(1,1)xy \\ &= w(2,1)w(3,2)w(3,1)w(4,2)xxxw(1,1)xyy = w(2,1)w(3,2)w(3,1)w(4,2)w(4,1)x^4y^2 \\ &= W(2,1)W(3,2)W(4,2)x^4y^2 \end{aligned}$$

and the weight of the path is indeed $W(2,1)W(3,2)W(4,2)$.

Theorem 2.13 then states that if we are considering $(x+y)^n$ in $\mathbb{C}_w[x,y]$ and we collect the variables in canonical form $c_k x^k y^{n-k}$, for $n \geqslant k \geqslant 0$, the coefficient of $x^k y^{n-k}$ is the w-binomial coefficient ${}_w\begin{bmatrix} n \\ k \end{bmatrix}$.

For example,

$$\begin{aligned} (x+y)^2 &= x^2 + xy + yx + y^2 = x^2 + xy + w(1,1)xy + y^2 = x^2 + (1+w(1,1))xy + y^2 \\ &= {}_w\begin{bmatrix} 2 \\ 2 \end{bmatrix} x^2 + {}_w\begin{bmatrix} 2 \\ 1 \end{bmatrix} xy + {}_w\begin{bmatrix} 2 \\ 0 \end{bmatrix} y^2. \end{aligned}$$

In Section 2.3.5 we gave a combinatorial proof for the convolution formula (3.71) by splitting up a lattice path into two parts. In this proof we can replace all weights $w_{a,b;q,p}(s,t)$ by $w(s,t)$ and $W_{a,b;q,p}(s,t)$ by $W(s,t)$. The elliptic binomial coefficients then correspond to the w-binomial coefficients and the shift of the weights by

$$(a,b) \mapsto (aq^{i+2j}, bq^{2i+j})$$

corresponds to the formal expression

$$x^i y^j w(s,t) y^{-j} x^{-i} = w(s+i, t+j).$$

The lattice path proof of (3.71) then yields the following weight-dependent extension of the Chu–Vandermonde convolution (2.8), obtained by Schlosser in [65].

Corollary 2.14. *Let $n, m \in \mathbb{N}_0$ and $k \geqslant 0$. For the w-binomial coefficients we have the following formal identity in $\mathbb{C}[(w(s,t))_{s,t\in\mathbb{N}}]$:*

$$\begin{bmatrix} n+m \\ k \end{bmatrix}_w = \sum_{j=0}^{k} \begin{bmatrix} n \\ j \end{bmatrix}_w \left(x^j y^{n-j} \begin{bmatrix} m \\ k-j \end{bmatrix}_w y^{j-n} x^{-j} \right) \prod_{i=1}^{k-j} W(i+j, n-j). \tag{2.78}$$

Compared to the corresponding identity in [65, Corollary 1], the sum in the above identity is bounded by k instead of $\min(k, n)$. But since $\begin{bmatrix} n \\ j \end{bmatrix}_w$ is zero if $j > n \geqslant 0$, this makes no difference in the case $n \geqslant 0$. The above identity is formal because it contains noncommuting variables x and y, which are understood to be shift operators shifting the w-binomial coefficient in the expression

$$x^j y^{n-j} \begin{bmatrix} m \\ k-j \end{bmatrix}_w y^{j-n} x^{-j}.$$

Evaluating this expression will shift all weights in $\begin{bmatrix} m \\ k-j \end{bmatrix}_w$ by $w(s,t) \mapsto w(s+j, t+n-j)$ and afterwards $x^j y^{n-j}$ will cancel with $y^{j-n} x^{-j}$ so we obtain an identity in $\mathbb{C}[(w(s,t))_{s,t\in\mathbb{N}}]$ since all x and y vanish.

Corollary 2.14 can also be derived from Theorem 2.13 by expanding $(x+y)^{n+m}$ on the one hand and $(x+y)^n (x+y)^m$ on the other hand. Then, comparing the coefficients of $x^k y^{n+m-k}$ yields Corollary 2.14.

2.4.2 Major index

In Section 2.2.3 we have seen that the q-binomial coefficient counts lattice paths according to area but also according to the major index. For the elliptic binomial coefficient we have worked out an analogue of the area generating function in Proposition 2.11 and we extended this proposition to w-binomial coefficients by replacing all $w_{a,b;q,p}(s,t)$ by $w(s,t)$. In this section we will see that we can also generalize the major index approach.

First, we define the *hook weights*

$$H(s,t) := \Big(\prod_{j=1}^{s-1} w(j,1) \Big) \Big(\prod_{j=1}^{t} w(s,j) \Big) \tag{2.79}$$

to derive the following lemma.

Lemma 2.15. *For $0 < k < n$ we have*

$$\begin{bmatrix} n \\ k \end{bmatrix}_w = x \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_w x^{-1} + \sum_{j=1}^{n-k} \Big(y \begin{bmatrix} n-j-1 \\ k-1 \end{bmatrix}_w y^{-1} \Big) H(k, n-k-j+1)$$

Proof. The left hand side of the equation counts weighted lattice paths from $(0,0)$ to $(k, n-k)$.

To get the right hand side of the equation, we distinguish whether the first step of a lattice path is an east step or a north step. The first part $x \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_w x^{-1}$ counts the subset of all lattice

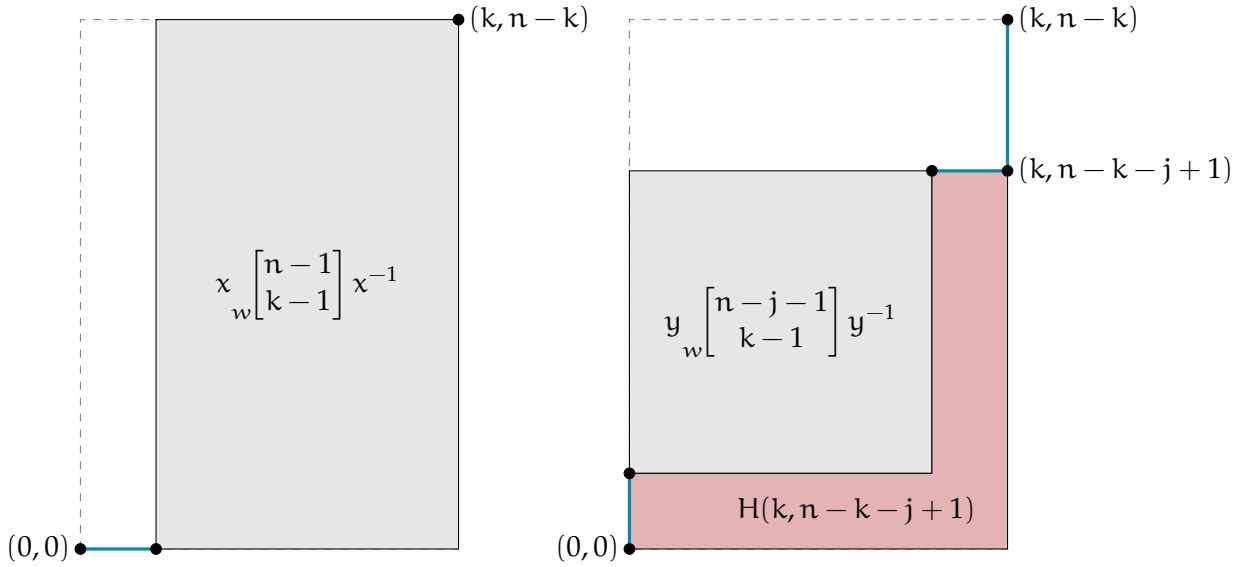


Figure 2.4: Illustration of the proof of Lemma 2.15, if the first step is an east step (left) and if the first step is a north step and the last east step ends at position $(k, n-k-j+1)$ (right).

paths beginning with an east step, since the first east step contributes weight 1 and we are left with a lattice path from $(1,0)$ to $(k, n-k)$. The shift by x takes care of the fact that the lattice path is shifted one step east. For an illustration see Figure 2.4 (left).

Each summand

$$\Big(y \begin{bmatrix} n-j-1 \\ k-1 \end{bmatrix}_w y^{-1}\Big) H(k, n-k-j+1)$$

counts the subset of lattice paths beginning with a north step, where the last east step ends at $(k, n-k-j+1)$ for $1 \leqslant j \leqslant n-k$: The lattice path from $(0,1)$ to $(k-1, n-k-j+1)$ contributes $y \begin{bmatrix} n-j-1 \\ k-1 \end{bmatrix}_w y^{-1}$. The shift by y takes care of the fact that the lattice path is shifted one step north. The unit squares below and to the right of the path contribute $H(k, n-k-j+1)$. (See Figure 2.4 (right).) By summing over all possible j, we obtain the identity. □

From Lemma 2.15 we derive an elliptic extension of the major counting of lattice paths. Recall that a valley v is an east step followed by a north step and the touching point of these two steps is the position of the valley. Let $\mathrm{EV}(P, v)$ be the number of valleys in the lattice path P that are east of v and $\mathrm{EN}(P, v)$ be the number of east steps east of v, that are not part of a valley.

Now we assign weights to the valleys of a lattice path. Each valley at position (s, t) of a lattice path $P \in \mathcal{P}_{\mathbb{N}}((0,0) \to (k, n-k))$ is assigned weight

$$x^{\mathrm{EN}(P,v)} y^{\mathrm{EV}(P,v)} H(s, t+1) y^{-\mathrm{EV}(P,v)} x^{-\mathrm{EN}(P,v)}.$$

The weight of the path by valleys, $w_{\mathrm{maj}}(P)$, is defined to be the product of the weights of its valleys. The construction of this weight function may seem obscure but will get clearer in the proof of Theorem 2.16.

For an example consider again the path from Figure 2.2. The path has a valley v_1 at position $(2,1)$ with $\mathrm{EN}(P,v_1)=2$ and $\mathrm{EV}(P,v_1)=0$ and a valley v_2 at position $(1,0)$ with $\mathrm{EN}(P,v_2)=2$ and $\mathrm{EV}(P,v_2)=1$. The weight of v_1 is $x^2H(2,2)x^{-2}$ and the weight of v_2 is $x^2y^1H(1,1)y^{-1}x^{-2}$. Therefore, the weight of the path is

$$w_{\mathrm{maj}}(P) = \left(x^2w(1,1)w(2,1)w(2,2)x^{-2}\right)\left(x^2y^1w(1,1)y^{-1}x^{-2}\right) \\ = w(3,1)w(3,2)w(4,1)w(4,2) = W(3,2)W(4,2).$$

If $w(s,t)=q$ for all $s,t\in\mathbb{N}$, the weight of a valley reduces to q^{s+t} and therefore $w_{\mathrm{maj}}(P)$ reduces to $q^{\mathrm{maj}(p)}$.

Theorem 2.16. *For $n\geqslant k\geqslant 0$ we have*

$$ {}_w\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{P\in\mathcal{P}_{\mathbb{N}}((0,0)\to(k,n-k))} w_{\mathrm{maj}}(P). $$

Proof. The proof is by induction on n and k, using Lemma 2.15 and initial conditions of the w-binomial coefficient. An illustration of the induction step is given in Figure 2.5.

For $k=0$ or $n=k$ the only possible lattice path does not have valleys and therefore has weight 1 and it follows by induction using (2.75) that ${}_w\left[\begin{smallmatrix} n \\ 0 \end{smallmatrix}\right] = {}_w\left[\begin{smallmatrix} n \\ n \end{smallmatrix}\right] = 1$ for $n\geqslant 0$.

For $0<k<n$ we look at the last east step of a given lattice path. If the east step ends at $(k,n-k)$, it is not part of a valley. (See Figure 2.5 (left).) Therefore, the step increases the value of $\mathrm{EN}(P,v)$ of all valleys that come before this step by 1. So we can shift the weight of these valleys by x and delete the last step of the lattice path without changing the weight of the path. By induction, lattice paths from $(0,0)$ to $(k,n-k)$ weighted by valleys that end with an east step are counted by

$$ x\,{}_w\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} x^{-1}. $$

If the last east step ends at $(k,n-k-j)$ for $1\leqslant j\leqslant n-k$, it is part of a valley and contributes weight $H(k,n-k-j+1)$ (See Figure 2.5 (right).). It increases $\mathrm{EV}(P,v)$ of all valleys that come before this step by 1. So we can shift the weight of these valleys by y and delete the last east step and all of the following north steps of the lattice path without changing the weight of the path. By induction, lattice paths from $(0,0)$ to $(k,n-k)$ weighted by valleys, where the last east step ends at $(k,n-k-j)$, for $1\leqslant j\leqslant n-k$, are counted by

$$ \left(y\,{}_w\begin{bmatrix} n-j-1 \\ k-1 \end{bmatrix} y^{-1}\right) H(k,n-k-j+1). $$

By summing over all possible j, we complete the proof. □

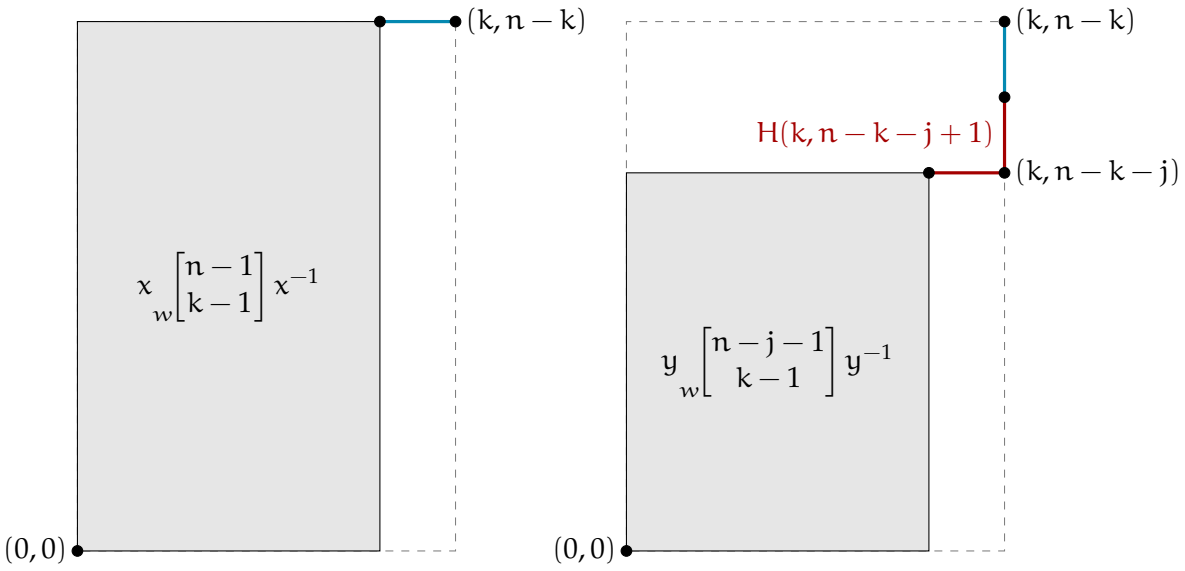


Figure 2.5: Illustration of the proof of Theorem 2.16, if the last east step ends at $(k, n-k)$ and is not part of a valley (left) and if the last east step is part of a valley v at $(k, n-k-j)$ (right).

3 Lattice paths and binomial coefficients with integer values

3.1 Introduction

Loeb [47] studied a generalization of the binomial coefficients $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, where n and k are allowed to be negative integers. He defined them for integers n and k in terms of

$$\binom{n}{k} := \lim_{\epsilon \mapsto 0} \frac{\Gamma(n+1+\epsilon)}{\Gamma(k+1+\epsilon)\Gamma(n-k+1+\epsilon)}.$$

From this definition it follows that the binomial coefficients with integer values satisfy the recursion

$$\binom{n}{0} = \binom{n}{n} = 1 \qquad \text{for } n \in \mathbb{Z}, \tag{3.1a}$$

and for $n, k \in \mathbb{Z}$, provided that $(n+1, k) \neq (0, 0)$,

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}, \tag{3.1b}$$

and they can be fully characterized by this recursion. The binomial coefficients with integer values appear in the power series expansion of $(x+y)^n$ (n being an arbitrary integer) in two ways. Suppose $f_n(x, y)$ is a function with power series expansions

$$f_n(x, y) = \sum_{k \geqslant 0} a_k x^k y^{n-k} \quad \text{in } \mathbb{C}[[x, y, y^{-1}]] \tag{3.2a}$$

or

$$f_n(x, y) = \sum_{k \leqslant n} b_k x^k y^{n-k} \quad \text{in } \mathbb{C}[[x, x^{-1}, y]], \tag{3.2b}$$

then we extract coefficients of the expansions by writing $[x^k y^{n-k}] f_n(x, y) = a_k$ for $k \geqslant 0$ and $[x^k y^{n-k}] f_n(x, y) = b_k$ for $k < 0$ (see [28]). We have the following extension of the binomial theorem.

Theorem 3.1 ([28, 47]). *For* $n, k \in \mathbb{Z}$,

$$[x^k y^{n-k}](x+y)^n = \binom{n}{k}.$$

This theorem essentially means that we have the two expansions

$$(x+y)^n = \sum_{k \geqslant 0} \binom{n}{k} x^k y^{n-k}$$

and

$$(x+y)^n = \sum_{k \leqslant n} \binom{n}{k} x^k y^{n-k}.$$

If $n \geqslant 0$, the expansions coincide, but if $n < 0$, they are different. For $n < 0$ the first expansion is an expansion in x and y^{-1}, while the second is an expansion in x^{-1} and y.

Recently, Formichella and Straub [28] extended this theorem to a q-binomial expansion. They considered an extension of the q-binomial coefficients to arbitrary integer values which can be characterized by the recursive definition

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_q = \begin{bmatrix} n \\ n \end{bmatrix}_q = 1 \qquad \text{for } n \in \mathbb{Z}, \tag{3.3a}$$

and for $n, k \in \mathbb{Z}$, provided that $(n+1, k) \neq (0, 0)$,

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q q^{n+1-k}. \tag{3.3b}$$

They proved the following generalization of Theorem 3.1.

Theorem 3.2. *Suppose we have* $yx = qxy$ *for invertible variables* x *and* y*, and an invertible indeterminate* q*. Then, for* $n, k \in \mathbb{Z}$*,*

$$[x^k y^{n-k}](x+y)^n = \begin{bmatrix} n \\ k \end{bmatrix}_q. \tag{3.4}$$

In their paper they gave a combinatorial interpretation of the q-binomial coefficients (3.3) in terms of hybrid sets. This interpretation is a q-analogue of a model by Loeb for (ordinary) binomial coefficients.

Theorem 3.2 also extends the noncommutative q-binomial theorem for nonnegative integers n and k: Let $\mathbb{C}_q[x, y]$ be the associative unital algebra over $\mathbb{C}$ generated by noncommutative variables x and y, satisfying the relation

$$yx = qxy, \tag{3.5}$$

for an indeterminate q. Recall from Theorem 2.3 that we have, as an identity in $\mathbb{C}_q[x, y]$,

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k}, \tag{3.6}$$

which is the $n, k \geqslant 0$ case of Theorem 3.2. In Section 2.4 we worked out a generalization of the noncommutative q-binomial theorem (3.6) to a weight-dependent binomial theorem for weight-dependent binomial coefficients and gave a combinatorial interpretation of these coefficients in terms of lattice paths due to Schlosser [65]. Specializing the general weights of the weight-dependent binomial coefficients, one obtains some interesting special cases, such as elliptic binomial coefficients and symmetric functions as well as the q- and ordinary binomial coefficients. In the conclusion of their paper [28], Formichella and Straub ask whether these elliptic binomial coefficients also permit a natural extension to negative numbers. We will show that such an extension is possible even for the more general weight-dependent binomial coefficients.

In this chapter, we extend the weight-dependent binomial coefficients to negative integer values, analogous to the work of Loeb [47] and Formichella and Straub [28]. Remarkably, many of their results continue to hold in the general weighted setting. We study reflection formulae inspired by an involution which recently appeared in [72]. We give a combinatorial interpretation of the weight-dependent binomial coefficients in terms of lattice paths and prove a weight-dependent generalization of Theorem 3.2. As a corollary of the binomial theorem, we obtain convolution formulae analogous to the Chu–Vandermonde convolution formula, which we also are able to interpret combinatorially. Our combinatorial interpretation of the weight-dependent binomial coefficients in terms of lattice paths translates, via a one-to-one correspondence, to a combinatorial interpretation in terms of hybrid sets (as studied by Loeb [47] in the classical case and by Formichella and Straub [28] in the q-case) which we make explicit. Finally, we study some important special cases of the weight-dependent binomial coefficients, such as elementary and complete homogeneous symmetric functions (with application of these cases to Stirling numbers), and elliptic binomial coefficients. This chapter is based on work of the author in collaboration with Michael Schlosser and Meesue Yoo [43].

3.2 Weight-dependent commutation relations

3.2.1 A noncommutative algebra

Let $(w(s,t))_{s,t\in\mathbb{Z}}$ be a sequence of invertible variables. Note that, unlike in the previous chapter, s and t are now allowed to be any integers, including 0 and negative integers. We start by extending the algebra $\mathbb{C}_w[x,y]$ from Definition 2.12 to invertible weights with integer indices and invertible variables x and y.

Definition 3.3. For a doubly-indexed sequence of invertible variables $(w(s,t))_{s,t\in\mathbb{Z}}$, let $\mathbb{C}_w[x,x^{-1},y,y^{-1}]$ be the associative unital algebra over $\mathbb{C}$ generated by x, x^{-1}, y and y^{-1} and the sequence of invertible variables $(w(s,t)^{\pm1})_{s,t\in\mathbb{Z}}$ satisfying the following relations:

$$x^{-1}x = xx^{-1} = 1 \tag{3.7a}$$
$$y^{-1}y = yy^{-1} = 1 \tag{3.7b}$$
$$yx = w(1,1)xy, \tag{3.7c}$$
$$xw(s,t) = w(s+1,t)x, \tag{3.7d}$$
$$yw(s,t) = w(s,t+1)y, \tag{3.7e}$$

for all $s,t\in\mathbb{Z}$.

Note that from the above relations also the following relations can be obtained:

$$x^{-1}y = w(0,1)yx^{-1}, \tag{3.7f}$$
$$xy^{-1} = w(1,0)y^{-1}x, \tag{3.7g}$$
$$y^{-1}x^{-1} = w(0,0)x^{-1}y^{-1}, \tag{3.7h}$$
$$x^{-1}w(s,t) = w(s-1,t)x^{-1}, \tag{3.7i}$$
$$y^{-1}w(s,t) = w(s,t-1)y^{-1}. \tag{3.7j}$$

In this chapter, we will frequently restrict the just defined algebra $\mathbb{C}_w[x,x^{-1},y,y^{-1}]$ to $\mathbb{C}_w[x,y,y^{-1}]$ (where we allow no negative powers of x and omit the relation (3.7a)) or to $\mathbb{C}_w[x,x^{-1},y]$ (where

we allow no negative powers of y and omit the relation (3.7b)), respectively. In addition we find it convenient to work in the extensions of these algebras to the algebras of formal power series $\mathbb{C}_w[[x, x^{-1}, y, y^{-1}]]$, $\mathbb{C}_w[[x, y, y^{-1}]]$ and $\mathbb{C}_w[[x, x^{-1}, y]]$ (keeping the same relations). Note that expressions of the form $(x + y)^n$ with $n < 0$ do not have a unique power series expansion in $\mathbb{C}_w[[x, x^{-1}, y, y^{-1}]]$. For $n < 0$ we have to decide how to expand $(x + y)^n$, say as $(x+y)^n = ((1+xy^{-1})y)^n$ as an element in $\mathbb{C}_w[[x, y, y^{-1}]]$, or rather as $(x+y)^n = (x(1+x^{-1}y))^n$ as an element in $\mathbb{C}_w[[x, x^{-1}, y]]$, respectively. By default, we shall always choose the algebra of formal power series $\mathbb{C}_w[[x, y, y^{-1}]]$ (allowing no negative powers of x) and only resort to $\mathbb{C}_w[[x, x^{-1}, y]]$ (allowing no negative powers of y) if we explicitly mention that.

For $l, m \in \mathbb{Z} \cup \{\pm\infty\}$ we define products of (possibly noncommutative) invertible variables A_j as follows:

$$\prod_{j=l}^{m} A_j = \begin{cases} A_l A_{l+1} \dots A_m & m > l-1 \\ 1 & m = l-1 \\ A_{l-1}^{-1} A_{l-2}^{-1} \dots A_{m+1}^{-1} & m < l-1 \end{cases} . \tag{3.9}$$

Note that

$$\prod_{j=l}^{m} A_j = \prod_{j=m+1}^{l-1} A_{m+l-j}^{-1}, \tag{3.10}$$

for all $l, m \in \mathbb{Z} \cup \{\pm\infty\}$.

Especially for the reflection formulae it will be necessary to define $\operatorname{sgn}(n)$ for $n \in \mathbb{Z}$, following [28, 76], as

$$\operatorname{sgn}(n) = \begin{cases} 1 & n \geqslant 0 \\ -1 & n < 0. \end{cases} \tag{3.11}$$

The definition of the big weights in (2.74),

$$W(s, t) = \prod_{j=1}^{t} w(s, j), \tag{3.12}$$

extends to $s, t \in \mathbb{Z}$ and the sequence of invertible weights $(w(s, t))_{s,t\in\mathbb{Z}}$.

Lemma 3.4. *Let $(w(s, t))_{s,t\in\mathbb{Z}}$ be a doubly-indexed sequence of invertible variables, and x and y two additional invertible variables, together forming the associative algebra $A_{x,y} = \mathbb{C}_{w_{x,y}}[x, x^{-1}, y, y^{-1}]$ where $w_{x,y}(s, t) = w(s, t)$. Then the following six homomorphisms are involutive algebra isomorphisms.*

$$\phi_{y,x} : \quad A_{x,y} \to A_{y,x} \quad \textit{with} \quad w_{y,x}(s, t) = w(t, s)^{-1}, \tag{3.13a}$$

$$\phi_{x^{-1},y} : \quad A_{x,y} \to A_{x^{-1},y} \quad \textit{with} \quad w_{x^{-1},y}(s, t) = w(1 - s, t)^{-1}, \tag{3.13b}$$

$$\phi_{x^{-1},x^{-1}y} : \quad A_{x,y} \to A_{x^{-1},x^{-1}y} \quad \textit{with} \quad w_{x^{-1},x^{-1}y}(s, t) = w(1 - s - t, t)^{-1}, \tag{3.13c}$$

$$\phi_{x,y^{-1}} : \quad A_{x,y} \to A_{x,y^{-1}} \quad \textit{with} \quad w_{x,y^{-1}}(s, t) = w(s, 1 - t)^{-1}, \tag{3.13d}$$

$$\phi_{x^{-1},y^{-1}} : \quad A_{x,y} \to A_{x^{-1},y^{-1}} \quad \textit{with} \quad w_{x^{-1},y^{-1}}(s, t) = w(1 - s, 1 - t), \tag{3.13e}$$

$$\phi_{y^{-1}x,y^{-1}} : \quad A_{x,y} \to A_{y^{-1}x,y^{-1}} \quad \textit{with} \quad w_{y^{-1}x,y^{-1}}(s, t) = w(s, 1 - s - t)^{-1}. \tag{3.13f}$$

It is straightforward to check that the simultaneous replacement of $w_{x,y}(s, t)$ ($s, t \in \mathbb{Z}$), x and y in (3.13a)–(3.13f) by $w_{x',y'}(s, t)$, x' and y', respectively, again satisfies the conditions in (3.7). Note that the involutions (3.13d)–(3.13f) can be realized as compositions of (3.13a)–(3.13c). For example, $\phi_{y^{-1}x,y^{-1}} = \phi_{y,x} \circ \phi_{x^{-1},x^{-1}y} \circ \phi_{y,x}$. Further note that Lemma 3.4 also holds for the

algebras $\mathbb{C}_w[x, x^{-1}, y]$ and $\mathbb{C}_w[x, y, y^{-1}]$ (when the respective homomorphisms can be defined) as well as for their respective formal power series extensions.

Lemma 3.4 is an extension of [72, Lemma 2]. There, the involution $\phi_{x^{-1},x^{-1}y}$ with corresponding weight function $\widetilde{w}(s,t) \coloneqq w_{x^{-1},x^{-1}y}(s,t) = w(1-s-t,t)^{-1}$ was used in the algebra $\mathbb{C}_w[x, x^{-1}, y]$ to construct weight-dependent Fibonacci polynomials satisfying a noncommutative weight-dependent Euler–Cassini identity.

It is clear that, given an identity in the variables $w(s,t)$ ($s,t \in \mathbb{Z}$), x and y, subject to the commutation relations (3.7), a new valid identity can be obtained by applying the isomorphism ϕ to each side of the identity where the variables still satisfy the commutation relations (3.7). We will apply such isomorphisms in the proofs of Lemma 3.6 and Theorem 3.16.

Remark 3.5. While the six isomorphisms in Lemma 3.4 are involutions, i.e., are of order 2, it is also possible to specify isomorphisms of order 3. In particular, for the homomorphisms

$$\phi_{y^{-1},xy^{-1}}: \quad A_{x,y} \to A_{y^{-1},xy^{-1}} \quad \text{with} \quad w_{y^{-1},xy^{-1}}(s,t) = w(t, 2-s-t), \tag{3.14a}$$

$$\phi_{yx^{-1},x^{-1}}: \quad A_{x,y} \to A_{yx^{-1},x^{-1}} \quad \text{with} \quad w_{yx^{-1},x^{-1}}(s,t) = w(2-s-t, s), \tag{3.14b}$$

$$\phi_{y,x^{-1}y^{-1}}: \quad A_{x,y} \to A_{y,x^{-1}y^{-1}} \quad \text{with} \quad w_{y,x^{-1}y^{-1}}(s,t) = w(1-t, 1+s-t), \tag{3.14c}$$

$$\phi_{x^{-1}y^{-1},x}: \quad A_{x,y} \to A_{x^{-1}y^{-1},x} \quad \text{with} \quad w_{x^{-1}y^{-1},x}(s,t) = w(t-s, 1-s), \tag{3.14d}$$

we have

$$\phi^3_{y^{-1},xy^{-1}} = \phi^3_{yx^{-1},x^{-1}} = \phi^3_{y,x^{-1}y^{-1}} = \phi^3_{x^{-1}y^{-1},x} = \mathrm{id}_{A_{x,y}},$$

moreover,

$$\phi^{-1}_{y^{-1},xy^{-1}} = \phi_{yx^{-1},x^{-1}}$$

and

$$\phi^{-1}_{y,x^{-1}y^{-1}} = \phi_{x^{-1}y^{-1},x}.$$

We will not make use of these isomorphisms of order 3 in this thesis (nor do we provide combinatorial interpretations in terms of lattice paths or hybrid sets here).

The following rule for interchanging powers of x and y is an extension to integer values of a corresponding lemma in [65, Lemma 1]:

Lemma 3.6. *For all $k, \ell \in \mathbb{Z}$ we have*

$$y^k x^\ell = \left(\prod_{i=1}^{\ell} \prod_{j=1}^{k} w(i,j) \right) x^\ell y^k = \left(\prod_{i=1}^{\ell} W(i,k) \right) x^\ell y^k.$$

Proof. The case $k, \ell \geqslant 0$ is already given in [65] and is easy to prove by induction; we therefore omit the proof. For $k \geqslant 0$ and $\ell < 0$ we combine the involution (3.13b) with the $k, \ell \geqslant 0$ case and use identity (3.10) to obtain:

$$y^k x^\ell = y^k (x^{-1})^{-\ell} = \left(\prod_{i=1}^{-\ell} \prod_{j=1}^{k} w(1-i,j)^{-1} \right) (x^{-1})^{-\ell} y^k = \left(\prod_{i=1}^{\ell} \prod_{j=1}^{k} w(i,j) \right) x^\ell y^k.$$

The remaining cases can be proved in the same manner by applying the involutions (3.13d) and (3.13e) to the $k, \ell \geqslant 0$ case. □

3.2.2 Weight-dependent binomial coefficients with integer values

Now we are ready to define weight-dependent binomial coefficients for all integer values. We extend the definition of the weight-dependent binomial coefficients or w-binomial coefficients in (2.75) by

$$ {}_w\begin{bmatrix} n \\ 0 \end{bmatrix} = {}_w\begin{bmatrix} n \\ n \end{bmatrix} = 1 \qquad \text{for } n \in \mathbb{Z}\,, \tag{3.15a} $$

and for $n, k \in \mathbb{Z}$, provided that $(n+1, k) \neq (0, 0)$,

$$ {}_w\begin{bmatrix} n+1 \\ k \end{bmatrix} = {}_w\begin{bmatrix} n \\ k \end{bmatrix} + {}_w\begin{bmatrix} n \\ k-1 \end{bmatrix} W(k, n+1-k). \tag{3.15b} $$

Example 3.7 ($n = -1$). Using induction separately for $k \geqslant 0$ and for $k < 0$ we obtain that the weight-dependent binomial coefficient for $n = -1$ evaluates to

$$ {}_w\begin{bmatrix} -1 \\ k \end{bmatrix} = (-1)^k \operatorname{sgn}(k) \prod_{j=1}^{k} W(j, -j), $$

where $\operatorname{sgn}(k)$ is defined in (3.11).

For $n, k \geqslant 0$ in (3.15), the w-binomial coefficients coincide with the weight-dependent binomial coefficients defined in Equation 2.75, which have a combinatorial interpretation in terms of lattice paths (see Section 2.4) due to Schlosser [65].

Interpreting the x-variable as an east step and the y-variable as a north step, we derived the weight-dependent binomial theorem (Theorem 2.13). Our goal is to extend this weight-dependent binomial theorem to arbitrary integer exponents and therefore generalize Theorems 3.1 and 3.2.

Before stating the general binomial theorem, let us extend the lattice path model to all $n, k \in \mathbb{Z}$. Let a *weighted hybrid lattice path* be a sequence of steps in the xy-plane starting at the origin $(0, 0)$ and ending at (n, m) with $n, m \in \mathbb{Z}$ using the following steps:

1. If $n, m \geqslant 0$, we use north and east steps $(\uparrow, \rightarrow)$,
2. if $n \geqslant 0$ and $m < 0$, we use south steps and east-south step combinations $(\downarrow, \urcorner\!\downarrow)$ and every path starts with a south step,
3. if $n < 0$ and $m \geqslant 0$, we use north-west step combinations and west steps $(\urcorner\!\leftarrow, \leftarrow)$ and every path starts with a west step,
4. if $n, m < 0$, there are no allowed steps.

Figure 3.1 shows the possible steps of a hybrid lattice path. The arrows indicate the direction of the steps. Note that in the region $m < 0 \leqslant n$ every east step has to be followed by a south step and in the region $n < 0 \leqslant m$ every north step has to be followed by a west step (which is not evident in the figure but follows from the step combinations in the definition). The figure also shows that some points are not reachable using these steps. This corresponds to the fact that the w-binomial coefficient with integer values is 0 in some regions which we will specify in Equation (3.17).

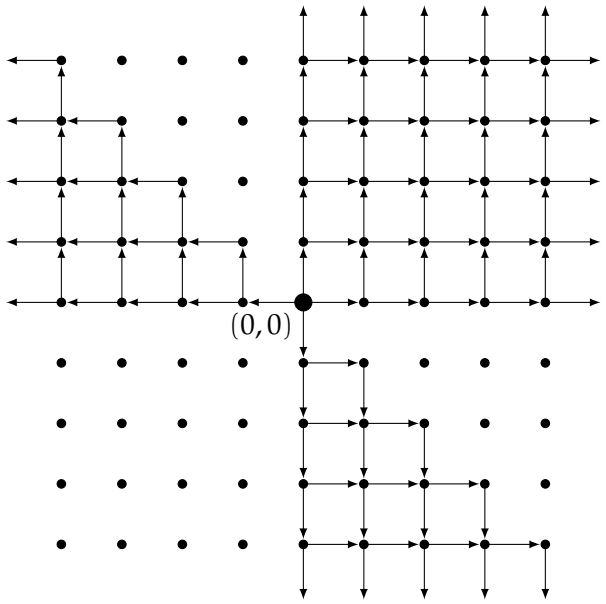
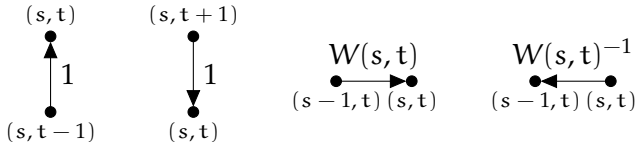


Figure 3.1: The possible steps of a hybrid lattice path.

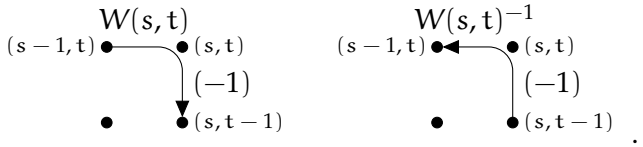
We give weights to such paths by assigning the weights

- 1 to each regular north step and south step,
- (-1) to each north and south step in a north-west or an east-south step combination,
- $W(s,t)$ to each east step $(s-1,t) \to (s,t)$, and
- $W(s,t)^{-1}$ to each west step $(s-1,t) \leftarrow (s,t)$.

To illustrate it graphically, we assign the weights



and, if we highlight the step combinations by rounded corners,



The weight of a path P, $w(P)$, is defined as the product of the weights of all its steps.

Example 3.8. Figure 3.2 shows a hybrid lattice path in the area $n, m \geqslant 0$ with weight $W(1,0) \cdot 1 \cdot W(2,1) \cdot 1 \cdot W(3,2) \cdot W(4,2) = w(2,1)w(3,1)w(3,2)w(4,1)w(4,2)$. Paths in this area correspond to (ordinary) weighted lattice paths. The left side of Figure 3.3 shows a hybrid lattice path in the area $m < 0 \leqslant n$ with weight $1 \cdot W(1,-1) \cdot (-1) \cdot 1 \cdot W(2,-3) \cdot (-1) = (-1)^2 w(1,0)^{-1} w(2,0)^{-1} w(2,-1)^{-1} w(2,-2)^{-1}$. The right side of Figure 3.3 shows a hybrid lattice path in the area $n < 0 \leqslant m$ with weight $W(0,0)^{-1} \cdot (-1) \cdot W(-1,1)^{-1} \cdot W(-2,1)^{-1} \cdot (-1) \cdot W(-3,2)^{-1} = (-1)^2 w(-1,1)^{-1} w(-2,1)^{-1} w(-3,1)^{-1} w(-3,2)^{-1}$.

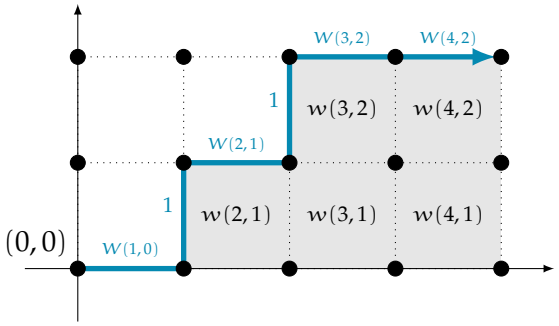


Figure 3.2: A hybrid lattice path in the area $n, m \geqslant 0$.

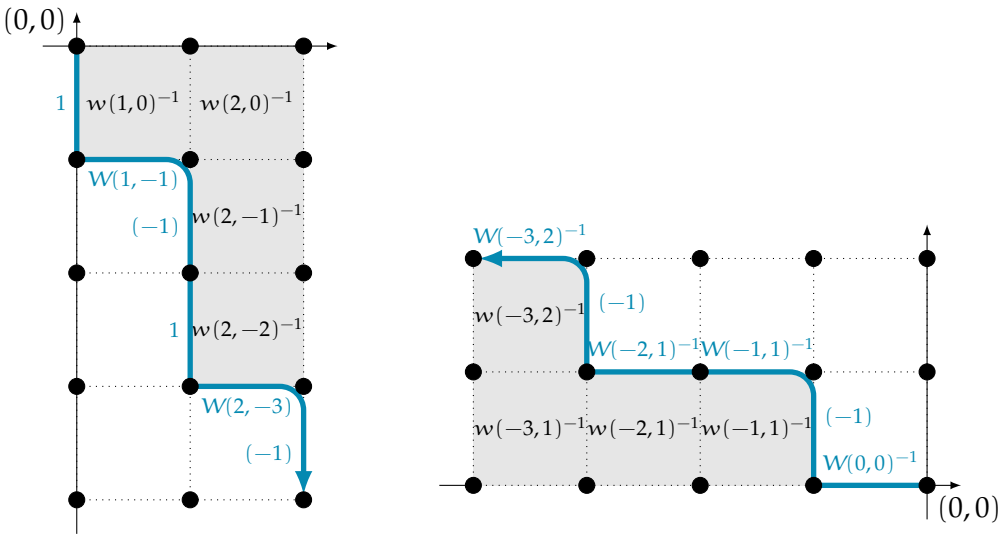


Figure 3.3: A hybrid lattice path in the area $m < 0 \leqslant n$ (left) and a path in the area $n < 0 \leqslant m$ (right). The (diagonal) step combinations are indicated by rounded corners. The negative inner corners (see Remark 3.10) are marked with red diagonal lines.

Given two points $A, B \in \mathbb{Z}^2$, let $\mathcal{P}_{\mathbb{Z}}(A \to B)$ be the set of all hybrid lattice paths from A to B, and define

$$w(\mathcal{P}_{\mathbb{Z}}(A \to B)) := \sum_{P \in \mathcal{P}_{\mathbb{Z}}(A \to B)} w(P).$$

Theorem 3.9. *Let $n, k \in \mathbb{Z}$. Then,*

$$w(\mathcal{P}_{\mathbb{Z}}((0,0) \to (k, n-k))) = {}_w\begin{bmatrix} n \\ k \end{bmatrix}. \tag{3.16}$$

Proof. For $0 \leqslant k \leqslant n$ this reduces to the ordinary case we described in Section 2.4. Let $0 \leqslant n < k$, $k < 0 \leqslant n$ or $n < k < 0$, then there are no possible hybrid lattice paths and indeed, it is not hard to use induction to prove that

$${}_w\begin{bmatrix} n \\ k \end{bmatrix} = 0 \qquad \text{for } 0 \leqslant n < k,\ k < 0 \leqslant n \text{ or } n < k < 0. \tag{3.17}$$

The remaining cases $n < 0 \leqslant k$ and $k \leqslant n < 0$ can be proved using induction.

For $n < 0 \leqslant k$ we are dealing with the set of lattice paths from $(0,0)$ to $(k, n-k)$ using south steps and east-south step combinations. We begin with the initial conditions

$${}_w\begin{bmatrix} n \\ 0 \end{bmatrix} = 1 \text{ and } {}_w\begin{bmatrix} -1 \\ k \end{bmatrix} = (-1)^k \prod_{j=1}^{k} W(j, -j)$$

for $n < 0$ and $k \geqslant 0$ and indeed, there is only one path from $(0,0)$ to $(0,n)$ consisting only of south steps with weight 1 and one path from $(0,0)$ to $(k, -1-k)$ consisting only of east-south step combinations with weight $(-1)^k \prod_{j=1}^{k} W(j,-j)$.

Now assume the result holds when $n+1 \leqslant -1$ or $k-1 \geqslant 0$. The last step of the hybrid lattice path ending at $(k, n-k)$ is either a south step starting at $(k, n-k+1)$ or an east-south step combination starting at $(k-1, n-k+1)$. We obtain from (3.15b) that

$${}_w\begin{bmatrix} n \\ k \end{bmatrix} = {}_w\begin{bmatrix} n+1 \\ k \end{bmatrix} - {}_w\begin{bmatrix} n \\ k-1 \end{bmatrix} W(k, n+1-k). \tag{3.18}$$

By induction, the first term of the right hand side is the weighted counting of hybrid lattice paths ending at $(k, n-k+1)$ combined with a south step with weight 1. The second term is the weighted counting of paths ending at $(k-1, n-k+1)$ combined with an east-south step combination with weight $(-1) \cdot W(k, n-k+1)$. This completes the proof for the case $n < 0 \leqslant k$.

The case $k \leqslant n < 0$ can be proved analogously using the following third form of the recurrence relation (3.15b):

$${}_w\begin{bmatrix} n \\ k \end{bmatrix} = {}_w\begin{bmatrix} n+1 \\ k+1 \end{bmatrix} W(k+1, n-k)^{-1} - {}_w\begin{bmatrix} n \\ k+1 \end{bmatrix} W(k+1, n-k)^{-1}, \tag{3.19}$$

where the first term of the sum corresponds to the weighted counting of hybrid lattice paths ending at $(k+1, n-k)$ combined with a west step with weight $W(k+1, n-k)^{-1}$ and the second term to the weighted counting of paths ending at $(k+1, n-k-1)$ combined with a north-west step combination with weight $(-1) \cdot W(k+1, n-k)^{-1}$. □

Remark 3.10. As the Figures 3.2 and 3.3 show, we can interpret the weight of the path as a weight function corresponding to the area between the path and the x-axis. For $s, t \in \mathbb{Z}$, let the cell (s, t) be the unit square with north-east corner at $x = s$ and $y = t$. For a hybrid lattice path P we define the set $\mathcal{C}(P)$ as the collection of cells (s, t) between the path and the x-axis. Further, we say that (s, t) is an *inner corner* of $\mathcal{C}(P)$, if the path touches the cell from two sides (the path in Figure 3.2 has the two inner corners $(2, 1)$ and $(3, 2)$) and it is a *negative inner corner* if $s \leqslant 0$ or $t \leqslant 0$ (in Figure 3.3 there are the negative inner corners $(1, 0)$ and $(2, -2)$ in the left path and $(-1, 1)$ and $(-3, 2)$ in the right path). For a cell (s, t) we define $N(s, t)$ to be 1 if it is a negative inner corner and 0 otherwise. Then, the definition of the weight of a path is equivalent to

$$w(P) = \prod_{(i,j)\in\mathcal{C}(P)} (-1)^{N(i,j)} w(i, j)^{\operatorname{sgn}((i,j))} \tag{3.20}$$

where $\operatorname{sgn}((i, j)) := \operatorname{sgn}(i-1)\operatorname{sgn}(j-1)$. Therefore, $w(\mathcal{P}_{\mathbb{Z}}((0, 0) \to (k, n-k)))$ can be seen as an area generating function. If a hybrid lattice path ends at $(k, n-k)$, the exponents of the small weights in (3.20) are always 1 if $0 \leqslant k \leqslant n$ and -1 otherwise. Moreover, the product over all $(-1)^{N(i,j)}$ is $(-1)^k$ if $n < 0 \leqslant k$, it is $(-1)^{n-k}$ if $k \leqslant n < 0$, and 1 otherwise.

Remark 3.11. A recent preprint by O'Sullivan [50, Figure 1] contains a figure similar to our Figure 3.1, showing the values of the *harmonic multiset numbers* $\left\| {n \atop k} \right\|$ (that extend the Stirling numbers) for integers n and k.

Example 3.12. By setting $w(s, t) = 1$ for all $s, t \in \mathbb{Z}$, the generating function of hybrid lattice paths is given by the binomial coefficients with integer values defined by (3.1). By setting $w(s, t) = q$ for all $s, t \in \mathbb{Z}$, we obtain a q-weighting of hybrid lattice paths with the generating function given by the q-binomial coefficients with integer values defined by (3.3).

3.2.3 Reflection formulae

As pointed out in [65], the weight-dependent binomial coefficients do not satisfy the symmetry ${}_w[{n \atop k}] = {}_w[{n \atop n-k}]$. Nevertheless, we can prove the following reflection formula using the weight

$$\widehat{w}(s, t) := w_{y,x}(s, t) = w(t, s)^{-1} \tag{3.21}$$

from the involution (3.13a) and

$$\widehat{W}(s, t) := \prod_{j=1}^{t} \widehat{w}(s, j). \tag{3.22}$$

Theorem 3.13. *Let $n, k \in \mathbb{Z}$ and $\widehat{w}(s, t) = w(t, s)^{-1}$. Then,*

$${}_w\begin{bmatrix} n \\ k \end{bmatrix} = {}_{\widehat{w}}\begin{bmatrix} n \\ n-k \end{bmatrix} \prod_{j=1}^{k} W(j, n-k).$$

Proof. We proceed by showing that both sides of the equation satisfy the same recursion and initial conditions.

For $k = 0$ and $n = k$ we obtain from (3.15a) and the definition for products (3.9) that

$${}_{\widehat{w}}\begin{bmatrix} n \\ n \end{bmatrix} \prod_{j=1}^{0} W(j, n) = 1 = {}_w\begin{bmatrix} n \\ 0 \end{bmatrix},$$

$${}_{\widehat{w}}\begin{bmatrix} n \\ 0 \end{bmatrix} \prod_{j=1}^{n} W(j, 0) = 1 = {}_w\begin{bmatrix} n \\ n \end{bmatrix}.$$

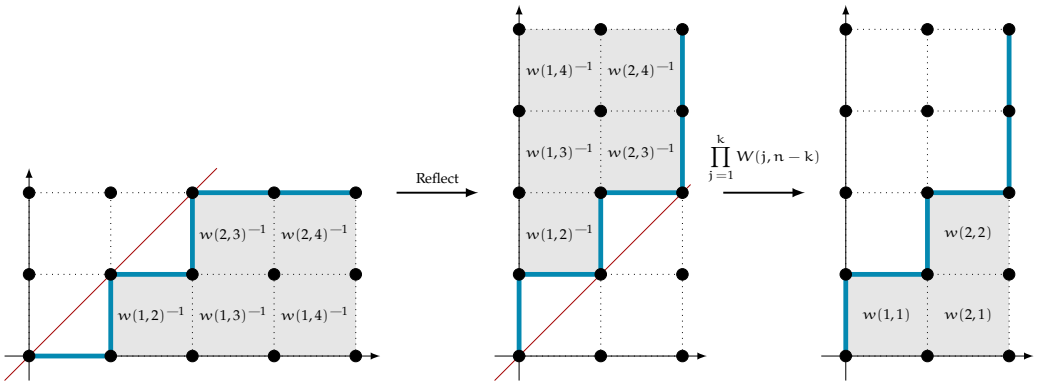


Figure 3.4: An illustration of the combinatorial interpretation of Theorem 3.13 for $0 \leqslant k \leqslant n$.

For $(n+1,k) \neq (0,0)$, note that

$$\widehat{W}(n+1-k,k) = \prod_{j=1}^{k} \widehat{w}(n+1-k,j) = \prod_{j=1}^{k} w(j,n+1-k)^{-1}$$

to deduce the recurrence relation

$$\begin{aligned}
A_{n+1,k} &:= \begin{bmatrix} n+1 \\ n+1-k \end{bmatrix}_{\widehat{w}} \prod_{j=1}^{k} W(j,n+1-k) \\
&= \left(\begin{bmatrix} n \\ n+1-k \end{bmatrix}_{\widehat{w}} + \begin{bmatrix} n \\ n-k \end{bmatrix}_{\widehat{w}} \widehat{W}(n+1-k,k) \right) \prod_{j=1}^{k} W(j,n+1-k) \\
&= \begin{bmatrix} n \\ n-(k-1) \end{bmatrix}_{\widehat{w}} \left(\prod_{j=1}^{k-1} W(j,n-(k-1)) \right) W(k,n+1-k) + \begin{bmatrix} n \\ n-k \end{bmatrix}_{\widehat{w}} \prod_{j=1}^{k} W(j,n-k) \\
&= A_{n,k-1} W(k,n+1-k) + A_{n,k},
\end{aligned}$$

which is equal to the recurrence relation (3.15b) for $\begin{bmatrix} n \\ k \end{bmatrix}_{w}$. □

Theorem 3.13 also has a combinatorial interpretation in terms of hybrid lattice paths weighted by area (3.20). The w-binomial coefficient on the right side of the equation corresponds to lattice paths from $(0,0)$ to $(n-k,k)$, where the weights are inverted and reflected by the line $y = x$, see the leftmost path in Figure 3.4 and 3.5. If we reflect such a path with its weights by the line $y = x$, we obtain a lattice path to $(k,n-k)$, where the weight of the path corresponds to the unit squares between the path and the y-axis and all weights are still inverted compared to the usual weighting, see the middle path in Figure 3.4 and 3.5. By multiplying with the weights $\prod_{j=1}^{k} W(j,n-k)$, which are the corresponding weights to the rectangle with vertices $(0,0)$, $(k,0)$, $(k,n-k)$ and $(0,n-k)$, the weights between the y-axis and the path cancel and we are left with the (non-inverted) weights between the path and the x-axis, see the rightmost path in Figure 3.4 and 3.5. So we are left with a path from $(0,0)$ to $(k,n-k)$ which is weighted as in (3.20), since the signs do not change.

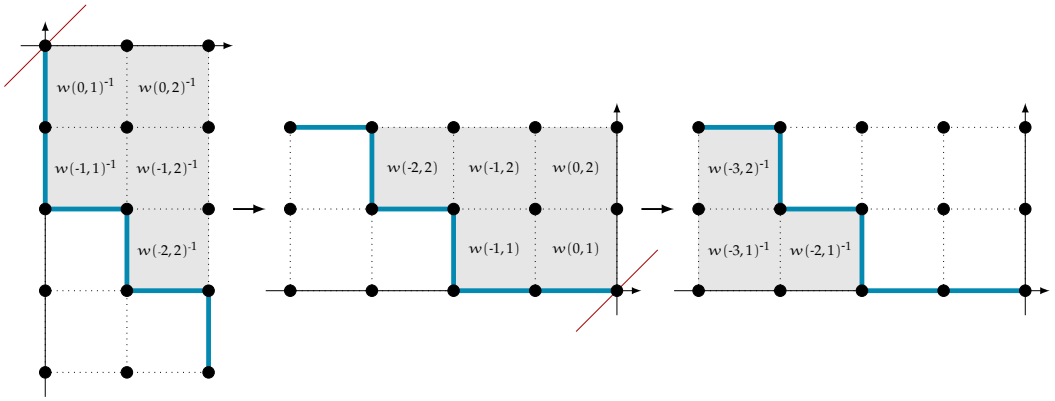


Figure 3.5: An illustration of the combinatorial interpretation of Theorem 3.13 for $k \leqslant n < 0$.

In the case of $w(s,t) = q$, Theorem 3.13 reduces to the well-known formula

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_{q^{-1}} q^{k(n-k)}.$$

In the following theorem we give two reflection formulae from which we can deduce the behaviour of w-binomial coefficients with negative values (see Proposition 3.15). Recall the involutions (3.13c) and (3.13f) to define the dual weight functions

$$\widetilde{w}(s,t) := w_{x^{-1},x^{-1}y}(s,t) = w(1-s-t,t)^{-1}, \tag{3.23}$$

$$\check{w}(s,t) := w_{y^{-1}x,y^{-1}}(s,t) = w(s,1-s-t)^{-1}. \tag{3.24}$$

The proof of both equations can be done analogously to the proof of Theorem 3.13; we therefore omit the proof.

Theorem 3.14. *Let $n, k \in \mathbb{Z}$. Then,*

$$\begin{aligned} {}_w\begin{bmatrix} n \\ k \end{bmatrix} &= (-1)^{n-k}\operatorname{sgn}(n-k) \begin{bmatrix} -k-1 \\ -n-1 \end{bmatrix}_{\widetilde{w}} \prod_{j=1}^{n-k} W(n+1-j,j)^{-1} && (3.25) \\ &= (-1)^k \operatorname{sgn}(k) \begin{bmatrix} k-n-1 \\ k \end{bmatrix}_{\check{w}} \prod_{j=1}^{k} W(j,-j) && (3.26) \end{aligned}$$

where $\widetilde{w}(s,t) = w(1-s-t,t)^{-1}$, $\check{w}(s,t) = w(s,1-s-t)^{-1}$ and $\operatorname{sgn}(k)$ is defined by (3.11).

By a careful case-by-case analysis one can also find combinatorial interpretations of the formulas in Theorem 3.14 similar to the interpretation of Theorem 3.13 above. We leave the details to the reader.

Given these reflection formulae, we can give a weight-dependent generalization of a Proposition by Loeb [47, Proposition 4.1].

Proposition 3.15 (The six regions). *Let $n, k \in \mathbb{Z}$, $\widehat{w}(s,t) = w(t,s)^{-1}$, $\widetilde{w}(s,t) = w(1-s-t,t)^{-1}$ and $\check{w}(s,t) = w(s,1-s-t)^{-1}$. Depending on the signs of n and k, the following formulae apply:*

1. *If $0 \leqslant k \leqslant n$, then ${}_w\begin{bmatrix} n \\ k \end{bmatrix} = {}_w\begin{bmatrix} n \\ k \end{bmatrix}$.*

2. *If* $n < 0 \leqslant k$, *then* ${}_w\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] = (-1)^k {}_{\check{w}}\left[\begin{smallmatrix} k-n-1 \\ k \end{smallmatrix}\right] \prod_{j=1}^{k} W(j,-j)$.

3. *If* $k \leqslant n < 0$, *then* ${}_w\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] = (-1)^{n-k} {}_{\widetilde{w}}\left[\begin{smallmatrix} -k-1 \\ -n-1 \end{smallmatrix}\right] \prod_{j=1}^{n-k} W(n+1-j,j)^{-1}$.

4. *If* $0 \leqslant n < k$, *then* ${}_w\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] = 0$.

5. *If* $n < k < 0$, *then* ${}_w\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] = 0$.

6. *If* $k < 0 \leqslant n$, *then* ${}_w\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] = 0$.

Case (1) corresponds to ordinary weighted lattice paths. The cases (2) and (3) show that weighted hybrid lattice paths from $(0,0)$ to (n,m), where n or m is negative, can be realized as ordinary weighted lattice paths where the weights $w(s,t)$ are replaced by $w(s,1-s-t)^{-1}$ or $w(1-s-t,t)^{-1}$ and additionally, the weight of the path is multiplied by some weight given in the above equations. Cases $(4)-(6)$ are already discussed in the proof of Theorem 3.9.

3.2.4 An extension of the noncommutative binomial theorem

For the purpose of the following theorem, we consider the algebras of formal power series $\mathbb{C}_w[[x,x^{-1},y]]$ and $\mathbb{C}_w[[x,y,y^{-1}]]$ of formal power series. Suppose $f_n(x,y)$ is a function with power series expansions

$$f_n(x,y) = \sum_{k\geqslant 0} a_k x^k y^{n-k} \quad \text{in } \mathbb{C}_w[[x,y,y^{-1}]] \tag{3.27a}$$

or

$$f_n(x,y) = \sum_{k\leqslant n} b_k x^k y^{n-k} \quad \text{in } \mathbb{C}_w[[x,x^{-1},y]]. \tag{3.27b}$$

Following [28], we extract coefficients of the expansions by writing $[x^k y^{n-k}]f_n(x,y) = a_k$ for $k \geqslant 0$ and $[x^k y^{n-k}]f_n(x,y) = b_k$ for $k < 0$. Now we can state the weight-dependent noncommutative binomial theorem for integer values which generalizes the results of [28], [47] and [65].

Theorem 3.16. *Let* $n,k \in \mathbb{Z}$ *and* x,y *be invertible variables satisfying the commutation relations* (3.3), *then we have*

$$[x^k y^{n-k}](x+y)^n = {}_w\begin{bmatrix} n \\ k \end{bmatrix}. \tag{3.28}$$

Proof. We will prove the theorem by showing that for $n \in \mathbb{Z}$,

$$(x+y)^n = \sum_{k\geqslant 0} {}_w\begin{bmatrix} n \\ k \end{bmatrix} x^k y^{n-k} \quad \text{in } \mathbb{C}_w[[x,y,y^{-1}]] \tag{3.29}$$

or

$$(x+y)^n = \sum_{k\leqslant n} {}_w\begin{bmatrix} n \\ k \end{bmatrix} x^k y^{n-k} \quad \text{in } \mathbb{C}_w[[x,x^{-1},y]]. \tag{3.30}$$

We begin with (3.29). The case $n \geqslant 0$ is just Theorem 2.13 combined with the fact that ${}_w\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] = 0$ if $0 \leqslant n < k$. We prove the case $n < 0$ by induction.

For $n = -1$, we use the geometric series and apply the relations (3.7) to obtain

$$(x+y)^{-1} = y^{-1}(xy^{-1}+1)^{-1} = y^{-1}\sum_{k\geqslant 0}(-1)^k(xy^{-1})^k$$
$$= \sum_{k\geqslant 0}(-1)^k\Big(\prod_{j=1}^{k} W(j,-j)\Big)x^k y^{-k-1}.$$

From Example 3.7 we know that for $k \geqslant 0$

$$(-1)^k \prod_{j=1}^{k} W(j,-j) = {}_w\begin{bmatrix} -1 \\ k \end{bmatrix}.$$

Suppose (3.29) holds for some $n+1 \leqslant -1$. Then

$$\begin{aligned}(x+y)^{n+1} &= \sum_{k\geqslant 0} {}_w\begin{bmatrix} n+1 \\ k \end{bmatrix} x^k y^{n+1-k} \\ &= \sum_{k\geqslant 0} {}_w\begin{bmatrix} n \\ k \end{bmatrix} x^k y^{n+1-k} + \sum_{k\geqslant 0} {}_w\begin{bmatrix} n \\ k-1 \end{bmatrix} W(k, n+1-k) x^k y^{n+1-k} \\ &= \sum_{k\geqslant 0} {}_w\begin{bmatrix} n \\ k \end{bmatrix} x^k y^{n+1-k} + \sum_{k\geqslant 0} {}_w\begin{bmatrix} n \\ k \end{bmatrix} W(k+1, n-k) x^{k+1} y^{n-k} \\ &= \Big(\sum_{k\geqslant 0} {}_w\begin{bmatrix} n \\ k \end{bmatrix} x^k y^{n-k}\Big)(x+y)\end{aligned}$$

Hence, (3.29) is true for n as well and therefore, it is true for all $n \in \mathbb{Z}$. Since we pulled out y^{-1} in the initial step, the expansion is true in $\mathbb{C}_w[[x, y, y^{-1}]]$.

In order to prove the second expansion (3.30) we apply the involution (3.13a) to (3.29) to obtain

$$(y+x)^n = \sum_{k\geqslant 0} {}_{\widehat{w}}\begin{bmatrix} n \\ k \end{bmatrix} y^k x^{n-k},$$

where $\widehat{w}(s,t) = w(t,s)^{-1}$. We use the reflection formula of Theorem 3.13 to obtain

$$\begin{aligned}(x+y)^n = (y+x)^n &= \sum_{k\geqslant 0} {}_{\widehat{w}}\begin{bmatrix} n \\ k \end{bmatrix} y^k x^{n-k} = \sum_{k\geqslant 0} {}_{\widehat{w}}\begin{bmatrix} n \\ k \end{bmatrix} \Big(\prod_{i=1}^{n-k} W(i,k)\Big) x^{n-k} y^k \\ &= \sum_{k\geqslant 0} {}_w\begin{bmatrix} n \\ n-k \end{bmatrix} x^{n-k} y^k = \sum_{k\leqslant n} {}_w\begin{bmatrix} n \\ k \end{bmatrix} x^k y^{n-k}\end{aligned}$$

in $\mathbb{C}_w[[x, x^{-1}, y]]$, as claimed. □

Recall the interpretation of expansion (3.29) in the case $n \geqslant 0$ in terms of weighted lattice paths in Section 2.4.1. It turns out that this interpretation is also valid for the case $n < 0$ and for expansion (3.30). In the following we will extend the details of this interpretation.

In Theorem 3.9 we interpreted the w-binomial coefficients as the weighted sum over hybrid lattice paths. Identify expressions (or words) in $\mathbb{C}_w[x, x^{-1}, y, y^{-1}]$ with hybrid lattice paths

locally (variable by variable, or step by step) as follows:

$$\begin{aligned} x &\leftrightarrow \text{east step} \\ x^{-1} &\leftrightarrow \text{west step} \\ y &\leftrightarrow \text{north step} \\ y^{-1} &\leftrightarrow \text{south step} \end{aligned}$$

This means, reading from left to right, that xy^{-1} corresponds to a path where an east step is followed by a south step and $y^{-1}x$ corresponds to a path where the south step is followed by an east step. The relations of the algebra $\mathbb{C}_w[x, x^{-1}, y, y^{-1}]$ take into account the changes of the respective weights when two consecutive steps are being interchanged. For instance, the lattice path P_0 from the left side of Figure 3.3 corresponds to the following expression:

$$\begin{aligned} &y^{-1}xy^{-1}y^{-1}xy^{-1} = w(1,0)^{-1}xy^{-2}w(1,0)^{-1}xy^{-2} \\ &= w(1,0)^{-1}w(2,-2)^{-1}w(2,-1)^{-1}w(2,0)^{-1}x^2y^{-4} = (-1)^2w(P_0)x^2y^{-4} \end{aligned}$$

where $w(P_0)$ is the weight of the path P_0 assigned to hybrid lattice paths in Theorem 3.9.

In summary, the expression $(x+y)^n$ translates into the sum over all expressions in $\mathbb{C}_w[x, x^{-1}, y, y^{-1}]$ corresponding to a hybrid lattice path from $(0,0)$ to $(k, n-k)$, where k can be any nonnegative integer (in (3.29)) or any integer $\leqslant n$ (in (3.30)), and the expression is multiplied by $\epsilon = \operatorname{sgn}(n)^k \operatorname{sgn}(k)^n$.

3.2.5 Convolution formulae

In [65], Schlosser derived a weight-dependent generalization of the Chu–Vandermonde convolution formula

$$\binom{n+m}{k} = \sum_{j=0}^{k} \binom{n}{j}\binom{m}{k-j}$$

by expanding $(x+y)^{n+m}$ on the one hand and $(x+y)^n(x+y)^m$ on the other hand using Theorem 2.13 and comparing the coefficients of $x^k y^{n+m-k}$ for $n, m, k \geqslant 0$ (see Corollary 2.14). Given Theorem 3.16, we can expand $(x+y)^{n+m}$ and $(x+y)^n(x+y)^m$ using (3.29) for all $n, m \in \mathbb{Z}$ to obtain the following weight-dependent convolution formula by again comparing the coefficients of $x^k y^{n+m-k}$ for $k \geqslant 0$. The proof is similar to the proof of [65, Corollary 1] and therefore we omit the details.

Corollary 3.17. *Let $n, m \in \mathbb{Z}$ and $k \geqslant 0$. For the w-binomial coefficients, defined by the doubly-indexed sequence of invertible variables $(w(s,t))_{s,t\in\mathbb{Z}}$, we have the following formal identity in $\mathbb{C}[(w(s,t))_{s,t\in\mathbb{Z}}]$:*

$$\begin{bmatrix} n+m \\ k \end{bmatrix}_w = \sum_{j=0}^{k} \begin{bmatrix} n \\ j \end{bmatrix}_w \left(x^j y^{n-j} \begin{bmatrix} m \\ k-j \end{bmatrix}_w y^{j-n} x^{-j} \right) \prod_{i=1}^{k-j} W(i+j, n-j). \tag{3.31}$$

As described in Section 2.4.1, the above identity is formal because it contains noncommuting variables x and y defined by (3.7), which can be understood to be shift operators shifting the weight-dependent binomial coefficient and cancel afterwards, so we obtain an identity in $\mathbb{C}[(w(s,t))_{s,t\in\mathbb{Z}}]$ since all x and y vanish.

By expanding $(x+y)^{n+m}$ and $(x+y)^n(x+y)^m$ using (3.30) and comparing coefficients as before, we obtain a second w-Chu–Vandermonde convolution formula.

Corollary 3.18. *Let $n, m \in \mathbb{Z}$ and $k \leqslant n+m$. For the w-binomial coefficients, defined by the doubly-indexed sequence of invertible variables $(w(s,t))_{s,t\in\mathbb{Z}}$, we have the following formal identity in $\mathbb{C}[(w(s,t))_{s,t\in\mathbb{Z}}]$:*

$$\begin{bmatrix} n+m \\ k \end{bmatrix}_w = \sum_{j=k-m}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_w \left(x^j y^{n-j} \begin{bmatrix} m \\ k-j \end{bmatrix}_w y^{j-n} x^{-j} \right) \prod_{i=1}^{k-j} W(i+j, n-j). \tag{3.32}$$

If $n+m<k<0$, both sides of the equation vanish, therefore (3.32) is true for all $k<0$. This equation generalizes an identity in [28, Lemma 4.9]. There the corresponding identity is limited to the case $n, m, k<0$ whereas Corollary 3.18 is even true if n, m are positive or have mixed signs.

In fact, Corollary 3.17 and 3.18 are equivalent. To see the correspondence, let $k \leqslant n+m$ and apply Corollary 3.17 to ${}_w\left[\begin{smallmatrix} n+m \\ n+m-k \end{smallmatrix}\right]$. Additionally, we apply the involution from (3.13a) with $\widehat{w}(s,t) = w(t,s)^{-1}$ to obtain

$$\begin{aligned}
&\begin{bmatrix} n+m \\ n+m-k \end{bmatrix}_{\widehat{w}} \\
&= \sum_{j=0}^{n+m-k} \begin{bmatrix} n \\ j \end{bmatrix}_{\widehat{w}} \left(y^j x^{n-j} \begin{bmatrix} m \\ n+m-k-j \end{bmatrix}_{\widehat{w}} x^{j-n} y^{-j} \right) \prod_{i=1}^{n+m-k-j} \widehat{W}(i+j, n-j) \\
&= \sum_{j=0}^{n+m-k} \begin{bmatrix} n \\ n+m-k-j \end{bmatrix}_{\widehat{w}} \left(x^{-m+k+j} y^{n+m-k-j} \begin{bmatrix} m \\ j \end{bmatrix}_{\widehat{w}} y^{-n-m+k+j} x^{m-k-j} \right) \\
&\qquad \times \prod_{i=1}^{j} \widehat{W}(i+n+m-k-j, -m+k+j) \\
&= \sum_{j=k-m}^{n} \begin{bmatrix} n \\ n-j \end{bmatrix}_{\widehat{w}} \left(x^j y^{n-j} \begin{bmatrix} m \\ m-k+j \end{bmatrix}_{\widehat{w}} y^{j-n} x^{-j} \right) \prod_{i=1}^{m-k+j} \widehat{W}(i+n-j, j).
\end{aligned}$$

Then we apply the reflection formula from Theorem 3.13 on both sides to obtain

$$\begin{aligned}
&\begin{bmatrix} n+m \\ k \end{bmatrix}_w \left(\prod_{i=1}^{k} W(i, n+m-k)^{-1} \right) = \sum_{j=k-m}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_w \left(\prod_{i=1}^{j} W(i, n-j)^{-1} \right) \\
&\times \left(x^j y^{n-j} \begin{bmatrix} m \\ k-j \end{bmatrix}_w \left(\prod_{i=1}^{m-k+j} W(i, m-k+j)^{-1} \right) y^{j-n} x^{-j} \right) \prod_{i=1}^{m-k+j} \widehat{W}(i+n-j, j).
\end{aligned}$$

It is a straightforward calculation to see that this expression is equivalent to Corollary 3.18.

We can apply the same method to the second and third weight-dependent binomial convolution formula in [65] to see that these two identities (extended to integers) are also equivalent to Corollary 3.17. There we can use the reflection formulae from Theorem 3.14 and the involutions from (3.13c) and (3.13f). The equivalence of the three convolution formulae in [65] also explains why the elliptic case of all three formulae are variants of the same elliptic hypergeometric summation formula, namely Frankel and Turaev's ${}_{10}V_9$ summation.

3.2.6 Combinatorial interpretation

In Theorem 3.9 we interpret w-binomial coefficients in terms of lattice paths. If the signs of n and m are identical, Corollaries 3.17 and 3.18 translate into convolutions of paths. If $n, m \geqslant 0$,

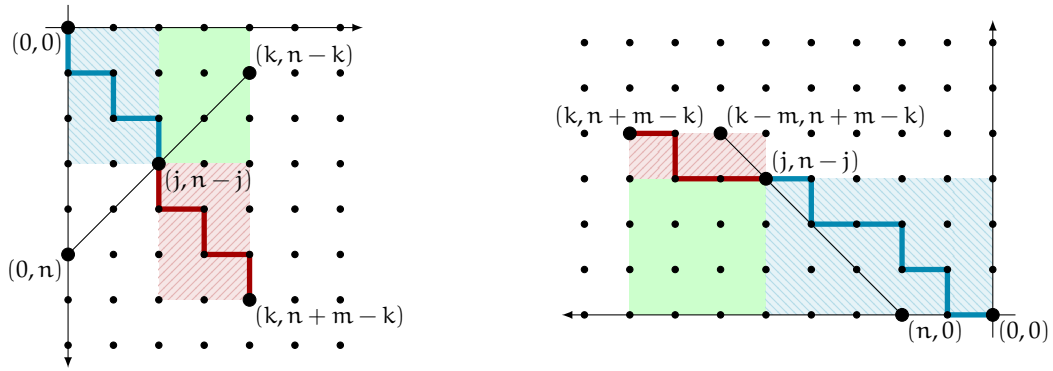


Figure 3.6: For $n, m < 0$, Corollary 3.17 (left) and Corollary 3.18 (right) translate into convolutions of paths.

both identities translate to convolutions corresponding to a diagonal in the first quadrant (see Figure 2.3 in Section 2.3.5). As we described in Section 2.3.5, a weighted path counted by ${}_w\left[\begin{smallmatrix} n+m \\ k \end{smallmatrix}\right]$, which crosses the point $(j, n-j)$ for some $0 \leqslant j \leqslant k$, can be decomposed into three parts: A weighted path from $(0,0)$ to $(j, n-j)$ (counted by ${}_w\left[\begin{smallmatrix} n \\ j \end{smallmatrix}\right]$), a weighted path from $(j, n-j)$ to $(k, n+m-k)$ (counted by $x^j y^{n-j} {}_w\left[\begin{smallmatrix} m \\ k-j \end{smallmatrix}\right] y^{j-n} x^{-j}$) and the weights corresponding to the rectangle between the x-axis and the second path (which contributes $\prod_{i=1}^{k-j} W(i+j, n-j)$). By summing over all $0 \leqslant j \leqslant k$ we obtain the convolution.

If $n, m < 0$, Corollary 3.17 translates to a convolution corresponding to a diagonal in the fourth quadrant and Corollary 3.18 to a diagonal in the second quadrant (see Figure 3.6). The decomposition is similar to the $n, m \geqslant 0$ case.

If n and m have mixed signs, the combinatorial interpretation is less obvious, but we can give a combinatorial proof of the Corollaries by defining a sign-reversing involution. Since both Corollaries are equivalent, we will only give a combinatorial proof of Corollary 3.17. Note that constructing an involution to prove Corollary 3.18 works very similar.

Let $k \geqslant 0$ and $n, m \in \mathbb{Z}$, where $\operatorname{sgn}(n) \neq \operatorname{sgn}(m)$. Then we define $P(n, m, k)$ to be the set of tuples of weighted hybrid lattice paths (P_1, P_2), where the first path, P_1, goes from $(0,0)$ to $(j, n-j)$ for some $0 \leqslant j \leqslant k$ and the second path, P_2, goes from $(j, n-j)$ to $(k, n+m-k)$. (For the second path it is assumed that the origin is shifted to $(j, n-j)$, such that the path is not restricted to the same step set as P_1.) Recall the weight-assignment to hybrid lattice paths (3.20) from Remark 3.10. Let $\mathcal{C}(P)$ be the collection of cells between the x-axis and the two paths. We define the weight of an element $(P_1, P_2) \in P(n, m, k)$ as

$$w((P_1, P_2)) = (-1)^{S(P_1, P_2)} \prod_{(i,j) \in \mathcal{C}(P)} w(i,j)^{\operatorname{sgn}((i,j))}$$

where $\operatorname{sgn}((i,j)) = \operatorname{sgn}(i-1)\operatorname{sgn}(j-1)$ and $S(P_1, P_2)$ is the number of east-south step combinations in either P_1 or P_2. See Figures 3.7 and 3.8 for examples, where the cells in $\mathcal{C}(P)$ are shaded. By construction, the sum over all weights of tuples in $P(n, m, k)$ is given by the right hand side of Corollary 3.17. Let $Q(n, m, k)$ be the subset of $P(n, m, k)$ where, after deleting all intersecting north and south steps of P_1 and P_2, one of the paths is completely deleted and we are left with a valid hybrid lattice path from $(0,0)$ to $(k, n+m-k)$ and an empty path. See

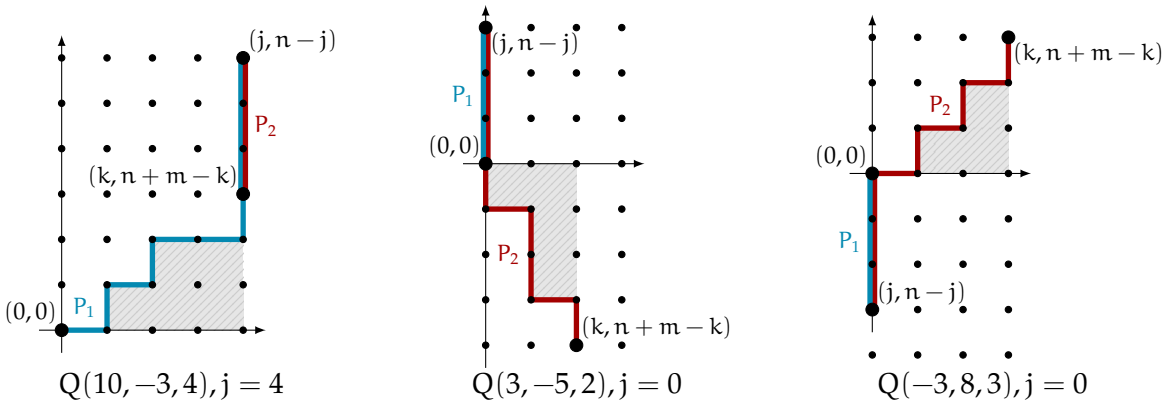


Figure 3.7: Three pairs of paths (P_1, P_2) in the fixed-point set.

Figure 3.7 for three examples. The sum over all weights of tuples in $Q(n, m, k)$ is given by the left hand side of Corollary 3.17, since in this case the weighting corresponds to the weighting of hybrid lattice paths (3.20). In order to give a combinatorial interpretation of the Corollary, we will define a sign-reversing involution ι on $P(n, m, k)$ whose fixed-point set is $Q(n, m, k)$.

Consider the case $m < 0 \leqslant n$. Given an element (P_1, P_2) of $P(n, m, k)$, P_1 ending at $(j, n-j)$, we look at the last east step, e_1, of P_1 and the first east step, e_2, of P_2. If both paths have an east step, we construct a new pair of paths (P_1', P_2') as follows. If e_1 is at the same height or below e_2 (see the right side of Figure 3.8 for an example), then P_1' starts by following P_1 until e_1. P_1' is following e_1 and afterwards it's going north until it reaches e_2. (As in the example in the figure, both east steps might as well be at the same height, then no north step is taken.) P_1' ends by following e_2 and by adding north steps until the path reaches $(j+1, n-j-1)$. The path P_2' begins at $(j+1, n-j-1)$ with south steps until it reaches P_2 and then follows all remaining steps of P_2. If e_1 lies above e_2 (see the left side of Figure 3.8 for an example), then P_1' is the path that follows P_1 until the starting point of e_1. Instead of following e_1, P_1' is going north until it reaches $(j-1, n-j+1)$. P_2' then starts at $(j-1, n-j+1)$ with south steps until it reaches e_1. P_2' follows e_1 and afterwards it follows all remaining steps of P_2 until it reaches $(k, n+m-k)$. If P_1 does not have an east step and e_2 is at height 0 or higher, we construct P_1' and P_2' as in the case when e_1 is weakly below e_2 (so e_2 will be part of P_1'). If P_2 does not have an east step and e_1 is at height $n+m-k+1$ or higher, we construct P_1' and P_2' as in the case when e_1 is above e_2 (so e_1 will be part of P_2). In all other cases, $(P_1, P_2) \in Q(n, m, k)$ and we set $(P_1', P_2') = (P_1, P_2)$. Let ι be the map that maps (P_1, P_2) to (P_1', P_2'). By construction, (P_1', P_2') is an element of $P(n, m, k)$. Also by construction, $\iota^2 = \mathrm{id}$, so ι is an involution. It is not hard to check that ι is weight-preserving, while P_2' has one more or one less east-south step combination than P_2 (see Figure 3.8 for an example), except if (P_1, P_2) is in $Q(n, m, k)$. This makes ι sign-reversing outside of the fixed-point set $Q(n, m, k)$. We therefore are finished with the case $m < 0 \leqslant n$.

The case $n < 0 \leqslant m$ works very similar and is therefore left to the reader.

3.2.7 Hybrid Sets

While it is classical that binomial coefficients count the number of lattice paths, it is also classical that binomial coefficients $\binom{n}{k}$ with nonnegative integer values count the number of

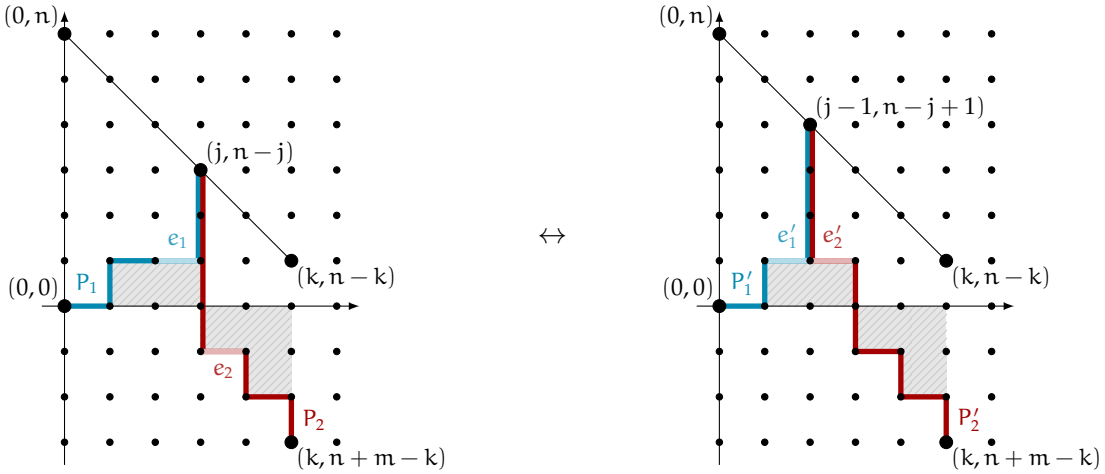


Figure 3.8: Visualization of the sign-reversing weight-preserving involution on a tuple of paths $(P_1, P_2) \in P(6, -4, 5)$ and $j = 3$. Both tuples of paths have the same weight (marked with grey diagonal lines), while P_2' has one more east south step combination than P_2, so the sign is reversed.

subsets with k elements of a set with n elements. In [47], Loeb showed that this interpretation can be extended to all integers n, k by a generalization of the definition of sets. Formichella and Straub [28] introduced a q-weighting of these sets. After briefly introducing the notation of Loeb (see [28, 47] for more details and examples), we will show that there is a simple bijection between hybrid lattice paths and subsets of hybrid new sets that preserves the q-weighting.

Let U be a collection of elements, then any map $X : U \mapsto \mathbb{Z}$ is called a *hybrid set*. The value of $X(a)$ is called the *multiplicity* of a in X. Hybrid sets are usually denoted in the form $\{\cdots|\cdots\}$, where elements $a \in U$ with positive multiplicity are listed $X(a)$ times before the bar and elements with negative multiplicity are listed $-X(a)$ times after the bar. The *cardinality* of a hybrid set, $|\{\cdots|\cdots\}|$, is the sum of all multiplicities of the set.

For example, given $U = \{1, 2, 3, 4, 5, 6\}$, the hybrid set $\{1, 1, 1, 3, 3|2, 5, 5\}$ contains the elements $1, 2, 3, 5$ with multiplicity $3, -1, 2, -2$, respectively, while the elements $4, 6$ have multiplicity 0. The cardinality of the set is $|\{1, 1, 1, 3, 3|2, 5, 5\}| = 3 - 1 + 2 - 2 = 2$.

A hybrid *new set* is a hybrid set where all multiplicities are either in $\{0, 1\}$ or in $\{0, -1\}$. We define the new set $\{a_l \cdots a_m\}$ by

$$\{a_l \cdots a_m\} = \begin{cases} \{a_l, a_{l+1}, \ldots, a_m|\} & m > l - 1 \\ \emptyset & m = l - 1 \\ \{|a_{l-1}, a_{l-2}, \ldots, a_{m+1}\} & m < l - 1 \end{cases} . \tag{3.33}$$

A hybrid set Y is a *subset* of a hybrid set X, if one can repeatedly decrease the multiplicity of elements in X with nonzero multiplicity to obtain Y or if one has removed Y.

Example 3.19. From the hybrid new set $\{a_1 \cdots a_2\} = \{a_1, a_2|\}$ one can remove the subsets $\emptyset$, $\{a_1|\}$, $\{a_2|\}$ and $\{a_1, a_2|\}$ to obtain the subsets $\{a_1, a_2|\}$, $\{a_2|\}$, $\{a_1|\}$, $\emptyset$. So there are 4 different subsets of $\{a_1 \cdots a_2\}$.

From the hybrid new set $\{a_1 \cdots a_{-2}\} = \{|a_0, a_{-1}\}$ one can remove three subsets with cardinality 2, $\{a_0, a_0|\}$, $\{a_0, a_{-1}|\}$ and $\{a_{-1}, a_{-1}|\}$, to obtain three subsets with cardinality -4,

$\{|a_0, a_0, a_0, a_{-1}\}$, $\{|a_0, a_0, a_{-1}, a_{-1}\}$, $\{|a_0, a_{-1}, a_{-1}, a_{-1}\}$. So there are 3 subsets of $\{a_1 \cdots a_{-2}\}$ with cardinality 2 and 3 subsets with cardinality -4, while $\{a_1 \cdots a_{-2}\}$ has infinitely many subsets in total.

Loeb [47] proved that, given a hybrid new set with cardinality n, the number of subsets with cardinality k is equal to the absolute value of $\binom{n}{k}$ for all $n, k \in \mathbb{Z}$. Formichella and Straub [28] extended this result to a q-weighted version. In the following, we will establish a general weighted extension by showing that subsets of hybrid new sets are in one-to-one correspondence with hybrid lattice paths.

For all $n \in \mathbb{Z}$ let $[n]$ be the hybrid new set

$$[n] = \begin{cases} \{1, 2, 3, \ldots, n|\}, & n > 0 \\ \emptyset, & n = 0 \\ \{|0, -1, -2, \ldots, n+1\}, & n < 0. \end{cases}$$

Let Y be a subset of $[n]$ with k elements, then we write it in a canonical form with ordered elements as follows. If $n \geqslant 0$, we write $Y = \{y_1, y_2, \ldots, y_k|\}$, where $y_i \leqslant y_{i+1}$ and, if $n < 0$, we write $Y = \{y_1, y_2, \ldots, y_k|\}$ (for $k \geqslant 0$) or $Y = \{|y_1, y_2, \ldots, y_{-k}\}$ (for $k < 0$) with $y_i \geqslant y_{i+1}$.

We identify k-element subsets Y of $[n]$ with hybrid lattice paths ending at $(k, n-k)$ as follows: If $k \geqslant 0$, each element y_i of Y corresponds to an east step ending at $(i, y_i - i)$. If $k < 0$, each element y_i of Y corresponds to a west step starting at $(i, y_i - i)$. By connecting the horizontal steps, the origin and the end-point $(k, n-k)$ with north or south steps wherever possible, one can check, region by region, that we obtain a unique hybrid lattice path from $(0, 0)$ to $(k, n-k)$, since the lattice path is determined by its horizontal steps. For example, the path in Figure 3.2 corresponds to the subset $\{1, 3, 5, 6|\}$ of $\{1, 2, 3, 4, 5, 6|\}$. The left path in Figure 3.3 corresponds to the subset $\{0, -1|\}$ of $\{|0, -1\}$ whereas the right path corresponds to the subset $\{|0, 0, -1, -1\}$ of $\{|0, -1\}$.

This correspondence immediately yields

$$w\begin{bmatrix} n \\ k \end{bmatrix} = \epsilon \sum_{Y} \prod_{i=1}^{k} W(i, y_i - i), \tag{3.34}$$

where the sum ranges over all k-element subsets $Y = \{y_1 \cdots y_k\}$ (in canonical form) of $[n]$ and $\epsilon = \operatorname{sgn}(k)^n \operatorname{sgn}(n)^k$. For $w(s, t) = q$ (and $W(s, t) = q^t$) this reduces to the weighted model of Formichella and Straub [28, Theorem 4.5], which can be stated as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \epsilon \sum_{Y} q^{s(Y) - k(k+1)/2},$$

where the sum ranges over all k-element subsets $Y = \{y_1 \cdots y_k\}$ of $[n]$ and $s(Y) = \sum_{y_i \in Y} y_i$. This is a generalization of the q-enumeration of subsets for nonnegative integers (2.12).

3.3 Symmetric Functions

In [65] two important specializations of the weights $w(s, t)$ involving symmetric functions were worked out in the case $n, k \geqslant 0$. Here we discuss how this approach generalizes naturally to the case $n, k \in \mathbb{Z}$.

In [21] Damiani, D'Antona and Loeb define generalized complete homogeneous symmetric functions and elementary symmetric functions over hybrid sets. For our purposes it is sufficient to look at the following simplified definitions. See [21] for a more extensive discussion.

Recall the definition of the hybrid set $\{a_1 \cdots a_n\}$ from (3.33) and define the *elementary symmetric function* $e_k(\{a_1 \cdots a_n\})$ and the *complete homogeneous symmetric function* $h_k(\{a_1 \cdots a_n\})$ for $n \in \mathbb{Z}$ and $k \geqslant 0$ by

$$\prod_{i=1}^{n}(1 + a_i t) = \sum_{k \geqslant 0} e_k(\{a_1 \cdots a_n\})t^k \tag{3.35a}$$

and

$$\prod_{i=1}^{n}(1 - a_i t)^{-1} = \sum_{k \geqslant 0} h_k(\{a_1 \cdots a_n\})t^k. \tag{3.35b}$$

In the spirit of Theorem 3.16 it makes sense to extend these definitions for all integers n and $k \leqslant n$ or $k \leqslant -n$ by

$$\prod_{i=1}^{n}(1 + a_i t) = \sum_{k \leqslant n} e_k(\{a_1 \cdots a_n\})t^k \tag{3.36a}$$

and

$$\prod_{i=1}^{n}(1 - a_i t)^{-1} = \sum_{k \leqslant -n} h_k(\{a_1 \cdots a_n\})t^k, \tag{3.36b}$$

where we expand the left side in the variable t^{-1}. For example, for $n = -1$ we get:

$$\begin{aligned}\prod_{i=1}^{-1}(1 + a_i t) = \frac{1}{1 + a_0 t} &= a_0^{-1}t^{-1}\frac{1}{1 + a_0^{-1}t^{-1}} \\ &= a_0^{-1}t^{-1}\sum_{k \geqslant 0}(-1)^k a_0^{-k}t^{-k} = \sum_{k \leqslant -1}(-1)^k a_0^k t^k.\end{aligned}$$

For convenience, we will also use the abbreviated notations $e_k(n)$ and $h_k(n)$, for $e_k(\{a_1 \cdots a_n\})$ and $h_k(\{a_1 \cdots a_n\})$, respectively. The elementary and complete homogeneous symmetric functions satisfy for all $n, k \in \mathbb{Z}$, provided that $(n+1, k) \neq (0, 0)$, the recurrence relations

$$\begin{aligned}e_k(n+1) &= e_k(n) + a_{n+1}e_{k-1}(n) \\ h_k(n) &= h_k(n-1) + a_n h_{k-1}(n)\end{aligned}$$

with initial conditions

$$\begin{aligned}e_0(n) = 1, \quad h_0(n) &= 1 \\ e_n(n) = \prod_{i=1}^{n} a_i, \quad h_n(1) &= \operatorname{sgn}(n)a_1^n.\end{aligned}$$

We can deduce directly from the definitions that for all $n, k \in \mathbb{Z}$ there holds (see also [21, Proposition 1])

$$e_k(\{a_1 \cdots a_n\}) = h_k(\{(-a_{n+1}) \cdots (-a_0)\}) \tag{3.37a}$$

$$e_k(\{a_1 \cdots a_n\}) = e_{n-k}(\{a_1^{-1} \cdots a_n^{-1}\}) \prod_{i=1}^{n} a_i. \tag{3.37b}$$

$$h_k(\{a_1 \cdots a_n\}) = h_{-n-k}(\{a_1^{-1} \cdots a_n^{-1}\})(-1)^n \prod_{i=1}^{n} a_i^{-1} \tag{3.37c}$$

Remark 3.20. Loeb [47] defined ordinary binomial coefficients for $n \in \mathbb{Z}$ and $k \geqslant 0$ as $\binom{n}{k} = e_k(\{a_1 \cdots a_n\})$ with $a_i = 1$ for all $i \in \mathbb{Z}$. He extended this definition for negative k by an explicit formula containing gamma functions. With the extension (3.36) of the definition of $e_k(n)$ and $a_i = 1$ for all $i \in \mathbb{Z}$ we now have the equality $\binom{n}{k} = e_k(\{a_1 \cdots a_n\})$ which holds for all $n, k \in \mathbb{Z}$.

For nonnegative integers n and k the definitions (3.35) correspond to the sum formulae

$$e_k(\{a_1, a_2, \ldots, a_n\}) = \sum_{1 \leqslant j_1 < j_2 < \cdots < j_k \leqslant n} a_{j_1} a_{j_2} \cdots a_{j_k} \tag{3.38}$$

$$h_k(\{a_1, a_2, \ldots, a_n\}) = \sum_{1 \leqslant j_1 \leqslant j_2 \leqslant \cdots \leqslant j_k \leqslant n} a_{j_1} a_{j_2} \cdots a_{j_k} \tag{3.39}$$

and $e_0(n) = h_0(n) = 1$ (see for example [59, Proposition 4.3.5]).

This definition extends even to the case where n might be negative as follows. Analogously to the extended definition for products in (3.9), we define sums generally by

$$\sum_{j=l}^{m} A_j = \begin{cases} A_l + A_{l+1} + \cdots + A_m & m > l-1 \\ 0 & m = l-1 \\ -A_{l-1} - A_{l-2} - \cdots - A_{m+1} & m < l-1 \end{cases}. \tag{3.40}$$

Then, for $n < 0$ and $k > 0$, the sum in (3.38) becomes

$$\begin{aligned} \sum_{1 \leqslant j_1 < j_2 < \cdots < j_k \leqslant n} a_{j_1} a_{j_2} \cdots a_{j_k} &= \sum_{j_1=1}^{n-k+1} \sum_{j_2=j_1+1}^{n-k+2} \cdots \sum_{j_k=j_{k-1}+1}^{n} a_{j_1} a_{j_2} \cdots a_{j_k} \\ &= \sum_{j_1=n-k+2}^{0} \sum_{j_2=n-k+3}^{j_1} \cdots \sum_{j_k=n+1}^{j_{k-1}} (-1)^k a_{j_1} a_{j_2} \cdots a_{j_k} \\ &= \sum_{n+1 \leqslant j_k \leqslant j_{k-1} \leqslant \cdots \leqslant j_1 \leqslant 0} (-a_{j_1})(-a_{j_2}) \cdots (-a_{j_k}) \end{aligned}$$

which is by (3.39) equal to $h_k(\{-a_{n+1}, -a_{n+2}, \ldots, -a_0\})$. Indeed, also by (3.37) we have $e_k(\{|a_0, a_{-1}, \ldots, a_{n+1}\} = h_k(\{-a_{n+1}, -a_{n+2}, \ldots, -a_0|\})$.

3.3.1 Elementary symmetric functions

The first choice of the weights is $w(s,t) = \frac{a_{s+t}}{a_{s+t-1}}$. In this case we have ${}_w\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] = e_k(n) \prod_{i=1}^{k} a_i^{-1}$. The noncommutative binomial theorem in Theorem 3.16 now gives the expansions

$$(x+y)^n = \sum_{k \geqslant 0} e_k(n) \Big(\prod_{i=1}^{k} a_i \Big) x^k y^{n-k} \tag{3.41a}$$

and

$$(x+y)^n = \sum_{k\leqslant n} e_k(n)\Big(\prod_{i=1}^{k} a_i\Big)x^k y^{n-k} \tag{3.41b}$$

for x and y noncommutative invertible variables satisfying

$$yx = \frac{a_2}{a_1}xy, \tag{3.42a}$$

$$x\frac{a_{s+t}}{a_{s+t-1}} = \frac{a_{s+t+1}}{a_{s+t}}x, \tag{3.42b}$$

$$y\frac{a_{s+t}}{a_{s+t-1}} = \frac{a_{s+t+1}}{a_{s+t}}y. \tag{3.42c}$$

Example 3.21. Define ϵ_a to be an operator that shifts all a_i to a_{i+1}, $i \in \mathbb{Z}$. For example, $\epsilon_a(a_1 + \frac{a_{-2}a_3}{a_5}) = a_2 + \frac{a_{-1}a_4}{a_6}$. Now let $x = a_1 t\epsilon_a$ and $y = \epsilon_a$, such that all relations in (3.42) are satisfied. Evaluating (3.41) by applying all operators on both sides recovers the generating functions

$$\prod_{i=1}^{n}(1+a_i t) = \sum_{k\geqslant 0} e_k(n)t^k \quad \text{and} \quad \prod_{i=1}^{n}(1+a_i t) = \sum_{k\leqslant n} e_k(n)t^k$$

and applying the duality (3.37) and $h_k(\{a_1\cdots a_n\})(-1)^k = h_k(\{(-a_1)\cdots(-a_n)\})$ yields

$$\prod_{i=1}^{n}(1-a_i t)^{-1} = \sum_{k\geqslant 0} h_k(n)t^k \quad \text{and} \quad \prod_{i=1}^{n}(1-a_i t)^{-1} = \sum_{k\leqslant -n} h_k(n)t^k,$$

A second choice of x and y would be $x = -a_1\epsilon_a$ and $y = t\epsilon_a$. Again, all relations in (3.42) are satisfied and we obtain another well-known generating function (see [21, p. 208])

$$(t-a_1)(t-a_2)\cdots(t-a_n) = \sum_{k\geqslant 0} e_k(\{(-a_1)\cdots(-a_n)\})t^{n-k} \tag{3.43}$$

for $n \geqslant 0$ as well as

$$(t-a_1)^{-1}(t-a_2)^{-1}\cdots(t-a_n)^{-1} = \sum_{k\geqslant 0} e_k(\{(-a_1)\cdots(-a_n)\})t^{n-k}$$

and

$$(t-a_1)^{-1}(t-a_2)^{-1}\cdots(t-a_n)^{-1} = \sum_{k\leqslant n} e_k(\{(-a_1)\cdots(-a_n)\})t^{n-k}$$

for $n < 0$.

The convolutions in Corollary 3.17 and 3.18, respectively, give

$$e_k(n+m) = \sum_{j=0}^{k} e_j(\{a_1\cdots a_n\})e_{k-j}(\{a_{n+1}\cdots a_{n+m}\}) \tag{3.44}$$

and

$$e_k(n+m) = \sum_{j=k-m}^{n} e_j(\{a_1\cdots a_n\})e_{k-j}(\{a_{n+1}\cdots a_{n+m}\}). \tag{3.45}$$

Note that these convolutions are extensions of the corresponding convolution in [65, (3.6a)], since n, m and k can be negative. One consequence of this observation is the following example.

Example 3.22. Choose $m = -n$ in (3.44). Then, by using the duality of e_k and h_k (3.37), we recover the well-known identity (cf. [59])

$$\sum_{j=0}^{k} (-1)^j e_j(\{a_1 \cdots a_n\}) h_{k-j}(\{a_1 \cdots a_n\}) = \delta_{k,0}.$$

More generally, we obtain

$$\sum_{j=0}^{k} e_j(\{a_1 \cdots a_n\}) h_{k-j}(\{(-a_{n+m+1}) \cdots (-a_n)\}) = e_k(n+m).$$

3.3.2 Complete homogeneous symmetric functions

The second choice of the weights is $w(s,t) = \frac{a_{t+1}}{a_t}$. In this case we have ${}_w\begin{bmatrix} n \\ k \end{bmatrix} = \operatorname{sgn}(k) h_k(n-k+1) a_1^{-k}$. The noncommutative binomial theorem in Theorem 3.16 now gives the expansions

$$(x+y)^n = \sum_{k \geqslant 0} h_k(n-k+1) a_1^{-k} x^k y^{n-k} \tag{3.46a}$$

and

$$(x+y)^n = \sum_{k \leqslant n} \operatorname{sgn}(k) h_k(n-k+1) a_1^{-k} x^k y^{n-k} \tag{3.46b}$$

for x and y noncommutative invertible variables satisfying

$$yx = \frac{a_2}{a_1} xy, \tag{3.47a}$$

$$x \frac{a_{t+1}}{a_t} = \frac{a_{t+1}}{a_t} x, \tag{3.47b}$$

$$y \frac{a_{t+1}}{a_t} = \frac{a_{t+2}}{a_{t+1}} y. \tag{3.47c}$$

Example 3.23. Let $x = a_1$ and $y = (t - a_1)\epsilon_a$, such that all relations in (3.47) are satisfied. Evaluating (3.46) by applying all operators on both sides recovers the generating functions

$$t^n = \sum_{k \geqslant 0} h_k(\{a_1 \cdots a_{n-k+1}\})(t-a_1)(t-a_2)\cdots(t-a_{n-k}) \tag{3.48}$$

for $n \geqslant 0$ (see also [21, p. 208]) as well as

$$t^n = \sum_{k \geqslant 0} h_k(\{a_1 \cdots a_{n-k+1}\})(t-a_1)^{-1}(t-a_2)^{-1}\cdots(t-a_{n-k})^{-1}$$

and

$$t^n = \sum_{k \leqslant n} \operatorname{sgn}(k) h_k(\{a_1 \cdots a_{n-k+1}\})(t-a_1)(t-a_2)\cdots(t-a_{n-k})$$

for $n < 0$.

The convolutions in Corollary 3.17 and 3.18, respectively, give

$$\begin{aligned}&h_k(n+m-k+1)\\&=\sum_{j=0}^{k} h_j(\{a_1\cdots a_{n-j+1}\})h_{k-j}(\{a_{n-j+1}\cdots a_{n+m-k+1}\})\end{aligned}\tag{3.49}$$

and

$$\begin{aligned}&h_k(n+m-k+1)\mathrm{sgn}(k)\\&=\sum_{j=k-m}^{n} \mathrm{sgn}(j)\mathrm{sgn}(k-j)h_j(\{a_1\cdots a_{n-j+1}\})h_{k-j}(\{a_{n-j+1}\cdots a_{n+m-k+1}\}).\end{aligned}\tag{3.50}$$

Remark 3.24. By applying the involution (3.13f) to the weight $w(s,t)=\frac{a_{s+t}}{a_{s+t-1}}$, we obtain

$$\check{w}(s,t)=w(s,1-s-t)^{-1}=\frac{a_{-t}}{a_{1-t}},$$

which is the weight for the complete homogeneous symmetric functions subject to the substitution $a_n\mapsto a_{-n+1}$. The reflection formula (3.26) reduces to (3.37a) in this case.

3.3.3 Application to Stirling numbers

Here we define weighted integers

$${}_w[n]:={}_w\begin{bmatrix}n\\1\end{bmatrix}.\tag{3.51}$$

Note that by Proposition 3.15, ${}_w[n]$ is well-defined for all $n\in\mathbb{Z}$. ${}_w[n]$ is equal to the elliptic number $[n]_{a,b;q,p}$ from (2.32) for the choice $w(s,t)=w_{a,b;q,p}(s,t)$ and equal to the q-number $[n]_q$ for $w(s,t)=q$.

It is well known that the Stirling numbers of the first kind and the second kind, denoted by $s(n,k)$ and $S(n,k)$, respectively, can be defined as the connecting coefficients of the following identities

$$t(t-1)\cdots(t-n+1)=\sum_{k=0}^{n} s(n,k)t^k\tag{3.52a}$$

and

$$t^n=\sum_{k=0}^{n} S(n,k)t(t-1)\cdots(t-k+1).\tag{3.52b}$$

We also have the recursions

$$s(n,k)=s(n-1,k-1)-(n-1)s(n-1,k)$$

and

$$S(n,k)=kS(n-1,k)+S(n-1,k-1)$$

which uniquely determine the Stirling numbers, given the initial conditions $s(n,0)=\delta_{n,0}$ and $s(n,k)=0$ for $k>n$, for the Stirling numbers of the first kind, and $S(n,0)=\delta_{n,0}$ and $S(n,k)=0$

for $k > n$, for the Stirling numbers of the second kind. Comparing (3.52a) to the generating function of the elementary symmetric function given in (3.43) gives the well known fact that $s(n,k)$ is the $(n-k)^{\text{th}}$ elementary symmetric function of $-1,-2,\ldots,-n+1$ (cf. [48, I.2, Ex. 11 (a)]). Similarly, comparison of (3.52b) and (3.48) gives the fact that $S(n,k)$ is the $(n-k)^{\text{th}}$ complete homogeneous symmetric function of $1,2,\ldots,k$ (cf. [48, I.2, Ex. 11 (b)]).

We replace each integer showing up in (3.52) by the weighted integer defined in (3.51) to define weighted analogues of the Stirling numbers

$$t(t-{}_w[1])\cdots(t-{}_w[n-1]) = \sum_{k=0}^{n} s_w(n,k)t^k \tag{3.53a}$$

and

$$t^n = \sum_{k=0}^{n} S_w(n,k)t(t-{}_w[1])\cdots(t-{}_w[k-1]). \tag{3.53b}$$

From (3.53), we can obtain the recurrence relations

$$s_w(n,k) = s_w(n-1,k-1) - {}_w[n-1]s_w(n-1,k) \tag{3.54a}$$

and

$$S_w(n,k) = {}_w[k]S_w(n-1,k) + S_w(n-1,k-1). \tag{3.54b}$$

By comparing (3.53a) to the generating function of the elementary symmetric function given in (3.43), for any nonnegative integers n and k, we get

$$s_w(n,k) = e_{n-k}(\{-{}_w[0],-{}_w[1],-{}_w[2],\ldots,-{}_w[n-1]\}). \tag{3.55}$$

Furthermore, the recurrence relation (3.54a) is valid for negative n and k values with $k \leqslant n$. As a result of using the recurrence relation for $k \leqslant n < 0$, we can prove that (3.55) extends to hold for $n,k \in \mathbb{Z}$ with $k \leqslant n$.

Similarly, for the weighted analogue of the Stirling numbers of the second kind, we compare (3.53b) to (3.48) and obtain, for any nonnegative integers n and k,

$$S_w(n,k) = h_{n-k}(\{{}_w[0],{}_w[1],\ldots,{}_w[k]\}). \tag{3.56}$$

Also, the recurrence relation (3.54b) allows us to extend the definition of $S_w(n,k)$ for negative n and k values, with $k \leqslant n$. As a result, we can prove that (3.56) holds for $k \leqslant n < 0$ as well.

Identities (3.44), (3.45), (3.49) and (3.50) satisfied by elementary symmetric functions and complete homogeneous symmetric functions coming from the convolution formulas give relations satisfied by weighted analogues of the Stirling numbers. To do that, however, we need a slightly more generalized definition of the weighted Stirling numbers. Namely, we define *α-shifted weighted Stirling numbers* by

$$(t-{}_w[\alpha])(t-{}_w[\alpha+1])\cdots(t-{}_w[\alpha+n-1]) = \sum_{k=0}^{n} s_w^{\alpha}(n,k)t^k \tag{3.57a}$$

and

$$t^n = \sum_{k=0}^{n} S_w^{\alpha}(n,k)(t-{}_w[\alpha])(t-{}_w[\alpha+1])\cdots(t-{}_w[\alpha+k-1]), \tag{3.57b}$$

for some parameter α. Note that we recover the weighted Stirling numbers when $\alpha = 0$. Then again by comparing to the generating functions of symmetric functions, we get

$$s_w^\alpha(n,k) = e_{n-k}(\{-{}_w[\alpha], -{}_w[\alpha+1], \ldots, -{}_w[\alpha+n-1]\}), \tag{3.58a}$$

and

$$S_w^\alpha(n,k) = h_{n-k}(\{{}_w[\alpha], {}_w[\alpha+1], \ldots, {}_w[\alpha+k]\}), \tag{3.58b}$$

for any $n, k \in \mathbb{Z}$ with $k \leqslant n$. These two identities, combined with (3.37a), immediately imply the relation

$$s_w^\alpha(n,k) = S_w^{\alpha+n}(-k-1,-n-1), \tag{3.59}$$

valid for $n, k \in \mathbb{Z}$ with $k \leqslant n$, that connects the first and second kinds of α-shifted weighted Stirling numbers. This is actually an extension of a well-known duality for the Stirling numbers which takes the form

$$s(n,k) = (-1)^{n-k} S(-k,-n), \tag{3.60}$$

that can be recovered from (3.59) by letting $\alpha = 0$ and the letting the weights $w(s,t) \to 1$. The duality relation in (3.60) is very classical and appears, for instance, in the (almost) two centuries old treatise [81, p. 305] by von Ettingshausen, in different notation. See also the discussion by Knuth [40].

The recurrence relations for the α-shifted weighted Stirling numbers that extend those in (3.54) are

$$s_w^\alpha(n,k) = s_w^\alpha(n-1,k-1) - {}_w[\alpha+n-1]\, s_w^\alpha(n-1,k) \tag{3.61a}$$

and

$$S_w^\alpha(n,k) = {}_w[\alpha+k]\, S_w^\alpha(n-1,k) + S_w^\alpha(n-1,k-1). \tag{3.61b}$$

The identities (3.44), (3.45), (3.49) and (3.50) can be written in terms of weighted (and shifted weighted) Stirling numbers as follows:

$$S_w(n+m, n+m-k) = \sum_{j=0}^{k} S_w(n, n-j)\, s_w^{n+m-k+1}(k-j-m-1, -m-1),$$

$$S_w(n+m, n+m-k) = \sum_{j=k-m}^{n} S_w(n, n-j)\, s_w^{n+m-k+1}(k-j-m-1, -m-1),$$

and

$$s_w(n+m, k-n-m) = \sum_{j=0}^{k} s_w(n, n-j)\, S_w^{n+m}(k-j-m-1, -m-1),$$

$$s_w(n+m, k-n-m) = \sum_{j=k-m}^{n} s_w(n, n-j)\, S_w^{n+m}(k-j-m-1, -m-1).$$

Remark 3.25. The α-shifted Stirling numbers have been defined by Remmel and Wachs in [56] when the weighted integer ${}_w[n]$ is given by $[n]_{p,q}$ where

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}.$$

3.4 Elliptic Hypergeometric Series

A very important specialization of the weights in [65], which also served as a major motivation for the present work, is the "elliptic" specialization. In this section we study the results from the previous sections in the elliptic case.

We refer to the introduction on elliptic functions and elliptic hypergeometric series in Section 2.3. Recall that a totally elliptic function is elliptic in all its parameters $u_1, u_2, \ldots, u_n$. The *field of totally elliptic multivariate functions* with period p is denoted by $\mathbb{E}_{u_1,\ldots,u_n;q,p}$.

3.4.1 Elliptic weights

Recall the definition of theta functions $\theta(x;p)$ and theta shifted factorials $(x;q,p)_n$ from Subsection 2.3.2. We consider the elliptic specialization of the weights $w(s,t)$ from [65] and introduced in Section 2.3, here defined for all $s,t \in \mathbb{Z}$ by

$$w_{a,b;q,p}(s,t) = \frac{\theta(aq^{s+2t}, bq^{2s+t-2}, aq^{t-s-1}/b;p)}{\theta(aq^{s+2t-2}, bq^{2s+t}, aq^{t-s+1}/b;p)}q, \tag{3.62}$$

to obtain the big weights

$$W_{a,b;q,p}(s,t) = \frac{\theta(aq^{s+2t}, bq^{2s}, bq^{2s-1}, aq^{1-s}/b, aq^{-s}/b;p)}{\theta(aq^{s}, bq^{2s+t}, bq^{2s+t-1}, aq^{1+t-s}/b, aq^{t-s}/b;p)}q^t \tag{3.63}$$

for all $s,t \in \mathbb{Z}$ and the *elliptic binomial coefficients*

$$\begin{bmatrix} n \\ k \end{bmatrix}_{a,b;q,p} = \frac{(q^{1+k}, aq^{1+k}, bq^{1+k}, aq^{1-k}/b;q,p)_{n-k}}{(q, aq, bq^{1+2k}, aq/b;q,p)_{n-k}}. \tag{3.64}$$

It is not hard to check that the elliptic binomial coefficient satisfies

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_{a,b;q,p} = \begin{bmatrix} n \\ n \end{bmatrix}_{a,b;q,p} = 1 \qquad \text{for } n \in \mathbb{Z},$$

and for $n,k \in \mathbb{Z}$, provided that $(n+1,k) \neq (0,0)$,

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_{a,b;q,p} = \begin{bmatrix} n \\ k \end{bmatrix}_{a,b;q,p} + \begin{bmatrix} n \\ k-1 \end{bmatrix}_{a,b;q,p} W_{a,b;q,p}(k, n+1-k),$$

where the recurrence relation is a consequence of the three-term relation (2.25). The elliptic binomial coefficient (3.64) is totally elliptic, i.e., periodic in each of a, b, q^k and q^n with period p. We denote the field of totally elliptic functions over $\mathbb{C}\setminus\{0\}$, in the complex variables a and b with equal period p, by $\mathbb{E}_{a,b;q,p}$.

We now focus on an elliptic version of the noncommutative algebra from Definition 3.3. The following definition is not a direct specialization of the weight-dependent algebra but a convenient extension of the specialization.

Definition 3.26. Let $x, x^{-1}, y, y^{-1}, a, b$ be noncommuting variables such that a and b commute with each other. Further let q and p be two complex numbers with $|p| < 1$. Let $\mathbb{C}_{a,b;q,p}[x, x^{-1}, y, y^{-1}]$ denote the associative unital algebra over $\mathbb{C}$, generated by the invertible

variables x, x^{-1}, y, y^{-1} and the set of all totally elliptic functions $\mathbb{E}_{a,b;q,p}$, satisfying the relations

$$x^{-1}x = xx^{-1} = 1, \tag{3.65a}$$
$$y^{-1}y = yy^{-1} = 1, \tag{3.65b}$$
$$yx = w_{a,b;q,p}(1,1)xy, \tag{3.65c}$$
$$xf(a,b) = f(aq, bq^2)x, \tag{3.65d}$$
$$yf(a,b) = f(aq^2, bq)y, \tag{3.65e}$$

for all $f \in \mathbb{E}_{a,b;q,p}$.

We refer to the variables $x, x^{-1}, y, y^{-1}, a, b$ forming $\mathbb{C}_{a,b;q,p}[x, x^{-1}, y, y^{-1}]$ as *elliptic-commuting* variables. The actions of x and y on a weight $w(s,t)$ in (3.7d) and (3.7e) correspond to the shifts of the parameters a and b as described in (3.65d) and (3.65e).

In this section we extend the binomial theorem to negative values of n and k and derive reflection and convolution formulae for the elliptic binomial coefficients. Since we proved the weight-dependent convolution formulae combinatorially in Section 3.2.6, we extend the combinatorial proof of Schlosser in [65] of Theorem 2.9 (see Section 2.3.5) to negative values.

3.4.2 Reflection formulae for elliptic binomial coefficients

In (3.13) we defined several involutions in the algebra $\mathbb{C}_w[x, x^{-1}, y, y^{-1}]$. Some of these involutions turn out to be particularly nice in the elliptic case. Recall that

$$\widehat{w}(s,t) = w(t,s)^{-1},$$
$$\widetilde{w}(s,t) = w(1-s-t,t)^{-1},$$
$$\check{w}(s,t) = w(s,1-s-t)^{-1}.$$

For the elliptic small weights we obtain

$$\widehat{w}_{a,b;q,p}(s,t) = w_{b,a;q,p}(s,t) \tag{3.66a}$$
$$\widetilde{w}_{a,b;q,p}(s,t) = w_{a/b,1/b;q,p}(s,t) \tag{3.66b}$$
$$\check{w}_{a,b;q,p}(s,t) = w_{1/a,b/a;q,p}(s,t), \tag{3.66c}$$

which follows from (2.24c). Equation (3.66b) was recently introduced by Schlosser and Yoo in [72]. We are now ready to derive the following reflection formulae from Theorem 3.13 and 3.14

Corollary 3.27. *Let $n, k \in \mathbb{Z}$. Then,*

$$\begin{aligned}
\begin{bmatrix} n \\ k \end{bmatrix}_{a,b;q,p} &= \begin{bmatrix} n \\ n-k \end{bmatrix}_{b,a;q,p} \prod_{j=1}^{k} W_{a,b;q,p}(j, n-k) \\
&= (-1)^{n-k}\operatorname{sgn}(n-k) \begin{bmatrix} -k-1 \\ -n-1 \end{bmatrix}_{a/b,1/b;q,p} \prod_{j=1}^{n-k} W_{a,b;q,p}(n+1-j, j)^{-1} \\
&= (-1)^{k}\operatorname{sgn}(k) \begin{bmatrix} k-n-1 \\ k \end{bmatrix}_{1/a,b/a;q,p} \prod_{j=1}^{k} W_{a,b;q,p}(j, -j),
\end{aligned}$$

where $\operatorname{sgn}(k)$ *is defined by* (3.11).

3.4.3 Elliptic binomial theorem

Recall the definition of $\mathbb{C}_{a,b;q,p}[x,y]$ from Definition 3.26. We will also consider the algebra of formal power series $\mathbb{C}_{a,b;q,p}[[x,y,y^{-1}]]$ and $\mathbb{C}_{a,b;q,p}[[x,x^{-1},y]]$. Then, the weight-dependent noncommutative binomial theorem (Theorem 3.16) reduces to the following theorem.

Theorem 3.28. *Let $n,k\in\mathbb{Z}$. Then we have*

$$[x^k y^{n-k}](x+y)^n=\begin{bmatrix}n\\k\end{bmatrix}_{a,b;q,p}. \tag{3.67}$$

In particular, we have the expansions

$$(x+y)^n=\sum_{k\geqslant 0}\begin{bmatrix}n\\k\end{bmatrix}_{a,b;q,p}x^k y^{n-k}\quad \textit{in } \mathbb{C}_{a,b;q,p}[[x,y,y^{-1}]] \tag{3.68}$$

or

$$(x+y)^n=\sum_{k\leqslant n}\begin{bmatrix}n\\k\end{bmatrix}_{a,b;q,p}x^k y^{n-k}\quad \textit{in } \mathbb{C}_{a,b;q,p}[[x,x^{-1},y]]. \tag{3.69}$$

The theorem reduces to the noncommutative q-binomial theorem by Formichella and Straub [28] if we formally let $p\to 0$, $a\to 0$, then $b\to 0$ (in this order).

In [65] the elliptic case of the convolution formula in Corollary 3.17 was already derived for nonnegative n and m, but the restriction to nonnegative values was removed by analytic continuation. In the following Corollary, n and m can also be negative integers.

Corollary 3.29. *Let $n,m\in\mathbb{Z}$, $k\geqslant 0$ and $a,b,q,p\in\mathbb{C}$ with $|p|<1$. Then we have*

$$\begin{bmatrix}n+m\\k\end{bmatrix}_{a,b;q,p}=\sum_{j=0}^{k}\begin{bmatrix}n\\j\end{bmatrix}_{a,b;q,p}\begin{bmatrix}m\\k-j\end{bmatrix}_{aq^{2n-j},bq^{n+j};q,p}\prod_{i=1}^{k-j}W_{a,b;q,p}(i+j,n-j), \tag{3.70}$$

where the elliptic binomial coefficients and $W_{a,b;q,p}(s,t)$ are defined by (3.64) *and* (3.63).

By changing the summation index k to j, substituting the parameters (a,b,c,d,e,n) appearing in Equation (2.51) by $(bq^{-n}/a,q^{-n}/a,bq^{1+n+m},bq^{-n-m+k}/a,q^{-n},k)$ and doing some basic manipulations one obtains Corollary 3.29. As described in [65] and Section 2.3.5, this substitution is reversible if q^{-n} and q^{-m} are treated as complex variables. Therefore, Corollary 3.29 is equivalent to Frenkel and Turaev's ${}_{10}V_9$ summation (2.51) and we provided a combinatorial proof for $n,m\in\mathbb{Z}$ and $k\geqslant 0$ in Section 3.2.6.

Corollary 3.30. *Let $n,m\in\mathbb{Z}$, $k\leqslant n+m$ and $a,b,q,p\in\mathbb{C}$ with $|p|<1$. Then we have*

$$\begin{bmatrix}n+m\\k\end{bmatrix}_{a,b;q,p}=\sum_{j=k-m}^{n}\begin{bmatrix}n\\j\end{bmatrix}_{a,b;q,p}\begin{bmatrix}m\\k-j\end{bmatrix}_{aq^{2n-j},bq^{n+j};q,p}\prod_{i=1}^{k-j}W_{a,b;q,p}(i+j,n-j), \tag{3.71}$$

where the elliptic binomial coefficients and $W_{a,b;q,p}(s,t)$ are defined by (3.64) *and* (3.63).

As worked out in Section 3.2.5, these two Convolution formulae are equivalent. Therefore, also Corollary 3.30 is equivalent to Frenkel and Turaev's ${}_{10}V_9$ summation (2.51). By repeated analytic continuation we can extend both corollaries to complex n and m.

3.5 A matrix inversion

In this section we will use the fact that Corollary 3.17 is true for all integers n, m to obtain a weight-dependent matrix inversion. Matrix inversions are an important tool in combinatorics and special functions. They are important, for instance, in the theory of ordinary, basic and elliptic hypergeometric functions [42, 63, 82]. Given an infinite-dimensional lower-triangular matrix $F = (f_{n,k})_{n,k\in\mathbb{Z}}$ with $f_{n,k} = 0$ unless $n \geqslant k$, the matrix $G = (g_{k,l})_{k,l\in\mathbb{Z}}$ is the inverse matrix of F if and only if

$$\sum_{k=l}^{n} f_{n,k} g_{k,l} = \delta_{n,l} \quad \text{for all } n, l \in \mathbb{Z}.$$

For convenience, in this section sums of the form $\sum_{j=u}^{v}$ are defined to be 0 if $v \leqslant u-1$.

Theorem 3.31. *Let $(w(s,t))_{s,t\in\mathbb{Z}}, x, y$ be noncommuting variables as in* (3.7) *and m be some integer. If $F = (f_{n,k})_{n,k\in\mathbb{Z}}$ with*

$$f_{n,k} = x^k y^{-m-k-1} \begin{bmatrix} m+n \\ n-k \end{bmatrix}_w y^{m+k+1} x^{-k} \prod_{i=1}^{n-k} W(i+k, -m-k-1), \tag{3.72a}$$

and $f_{n,k} = 0$ if $k > n$, then $G = (g_{k,l})_{k,l\in\mathbb{Z}}$ with

$$g_{k,l} = x^l \begin{bmatrix} -m-l-1 \\ k-l \end{bmatrix}_w x^{-l} \tag{3.72b}$$

and $g_{k,l} = 0$ if $l > k$ is the inverse matrix of F.

Both $f_{n,k}$ and $g_{k,l}$ are formal in $\mathbb{C}[(w(s,t))_{s,t\in\mathbb{Z}}]$, since they contain noncommuting variables x and y, which can be understood to be shift operators that cancel after shifting the weights $w(s,t)$.

Proof. We begin with Corollary 3.17 and substitute $(n, m, k) \mapsto (-m-l-1, m+n, n-l)$ for some $m, l, n \in \mathbb{Z}$ and $l \leqslant n$ to obtain that $\begin{bmatrix} n-l-1 \\ n-l \end{bmatrix}_w$ is equal to

$$\begin{aligned}
&\sum_{k=0}^{n-l} \Bigg(\begin{bmatrix} -m-l-1 \\ k \end{bmatrix}_w \left(x^k y^{-m-l-k-1} \begin{bmatrix} m+n \\ n-l-k \end{bmatrix}_w y^{m+l+k+1} x^{-k} \right) \\
&\qquad \times \prod_{i=1}^{n-l-k} W(i+k, -m-l-k-1) \Bigg) \\
&= \sum_{k=l}^{n} \Bigg(\left(x^{k-l} y^{-m-k-1} \begin{bmatrix} m+n \\ n-k \end{bmatrix}_w y^{m+k+1} x^{-k+l} \right) \\
&\qquad \times \prod_{i=1}^{n-k} W(i+k-l, -m-k-1) \begin{bmatrix} -m-l-1 \\ k-l \end{bmatrix}_w \Bigg).
\end{aligned}$$

Multiplying x^l from the left and x^{-l} from the right yields

$$\sum_{k=l}^{n} f_{n,k} g_{k,l} = x^l \begin{bmatrix} n-l-1 \\ n-l \end{bmatrix}_w x^{-l} = \delta_{n,l}$$

for all $l \leqslant n \in \mathbb{Z}$. For $l > n$, the sum is defined to be 0, which completes the proof. □

4 Domino tilings and Fibonomial numbers

4.1 Introduction

The Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$ is one of the most important and beautiful sequences in mathematics. It starts with the numbers $F_0 = 0$ and $F_1 = 1$, and is recursively defined by the formula $F_n = F_{n-1} + F_{n-2}$.

Fibonacci analogues of famous numbers, such as the binomial coefficients and Catalan numbers:

$$\binom{m+n}{n} = \frac{(m+n)!}{m! \cdot n!} \qquad \text{and} \qquad \frac{1}{n+1}\binom{2n}{n},$$

have intrigued some mathematicians over the last few years [1, 6, 7, 17, 60, 62]. The *Fibonomial* and *Fibo-Catalan* numbers are defined, respectively, as:

$$\binom{m+n}{n}_{\mathcal{F}} := \frac{F^!_{m+n}}{F^!_m \cdot F^!_n} \qquad \text{and} \qquad \frac{1}{F_{n+1}}\binom{2n}{n}_{\mathcal{F}},$$

where $F^!_n := \prod_{k=1}^{n} F_k$ is the Fibonacci analogue of $n!$. These rational expressions turn out to be positive integers. In [60], Sagan and Savage introduced a combinatorial model to interpret the Fibonomial numbers in terms of certain tilings of an $m \times n$ rectangle. A *path-domino tiling* of an $m \times n$ rectangle is a tiling with monominos and dominos and a lattice path from $(0, 0)$ to (m, n) is specified, and such that:

- all tiles above the path are either monominos or horizontal dominos;
- all tiles below the path are either monominos or vertical dominos; and
- all tiles that touch the path from below are vertical dominos.

We call these last tiles touching the path from below *special vertical dominos*, and denote by $\mathcal{T}_{m,n}$ the collection of all path-domino tilings of an $m \times n$ rectangle. An example is illustrated on the left of Figure 4.1. The following result is a special case of [60, Theorem 3].[1]

Theorem 4.1 ([60])**.** *The Fibonomial number* $\binom{m+n}{n}_{\mathcal{F}}$ *counts the number of path-domino tilings of an* $m \times n$ *rectangle.*

The main objective of this chapter is to present a q-analogue and an elliptic analogue generalization of this result. The resulting q-Fibonomial and elliptic Fibonomial numbers count the number of path-domino tilings of an $m \times n$ rectangle according to their q-weights and elliptic weights, respectively. This chapter is based on work of the author in collaboration with Nantel Bergeron and Cesar Ceballos [11].

[1] The path-domino tilings here are a slight modification of the tilings used in [60]. The only difference is that in [60] the tiles below the path touching the bottom of the rectangle are required to be vertical dominos, while here this condition is required for the tiles below the path touching the path itself. This modification is essential to make our combinatorial model work.

4.2 q-Analogue of the Fibonomial numbers

Recall the definition of the q-analogue of a natural number n:

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1}.$$

Before studying the q-analogue of the Fibonomial numbers, let us recall some useful and known straightforward lemmas.

Lemma 4.2. *For $m, n \in \mathbb{N}$, the following identities hold:*

$$[m+n]_q = [m]_q + q^m [n]_q, \tag{4.1}$$
$$[m \cdot n]_q = [m]_q [n]_{q^m}. \tag{4.2}$$

It is well known that the Fibonacci number F_n counts the number of tilings of an $(n-1)$-*strip* (a rectangle with diagonal endpoints $(0,0)$ and $(n-1,1)$) using dominos and monominos. Given such a tiling T, we define the *weight* $\omega(T)$ of T as the product of the weights of its tiles, where a monomino has weight 1 and a domino whose top-right coordinate is $(i,1)$ has weight q^{F_i}. The weight of a 0-strip is by definition equal to 1.

Lemma 4.3 (*cf.* [60])**.** *For $n \in \mathbb{N}$, the q-analogue of the Fibonacci numbers*[2] *can be computed as*

$$[F_n]_q = \sum_T \omega(T),$$

where $[F_n]_q = 1 + q + q^2 + \cdots + q^{F_n - 1}$ and the sum ranges over all tilings of an $(n-1)$-strip using dominos and monominos.

Proof. The result is clearly true for $n = 1, 2$. Let $n > 2$, applying Equation (4.1) from Lemma 4.2 we get:

$$[F_n]_q = [F_{n-1} + F_{n-2}]_q = [F_{n-1}]_q + q^{F_{n-1}} [F_{n-2}]_q.$$

By induction, the first term of this sum corresponds to the tilings of an $(n-1)$-strip that finish with a monomino, while the second term to the tilings of an $(n-1)$-strip that finish with a domino. □

Lemma 4.4. *For $m, n \in \mathbb{N}$, the following identities hold:*

$$F_{m+n} = F_n F_{m+1} + F_m F_{n-1}, \tag{4.3}$$
$$[F_{m+n}]_q = [F_n]_q [F_{m+1}]_{q^{F_n}} + q^{F_n F_{m+1}} [F_m]_q [F_{n-1}]_{q^{F_m}}. \tag{4.4}$$

Proof. Equation (4.3) is a well known identity for Fibonacci numbers; see for instance [60, Lemma 1]. The Fibonacci number F_{m+n} counts the number of tilings of an $(m+n-1)$-strip with monominos and dominos. These tilings can be subdivided into two types: those containing a domino which is cut in two by the line $x = m$, and all other tilings. The first kind is counted by $F_m F_{n-1}$ (the number of tilings of an $(m-1)$-strip times the number of tilings of an $(n-2)$-strip), while the second kind is counted by $F_n F_{m+1}$ (the number of tilings of an m-strip times the number of tilings of an $(n-1)$-strip). Therefore, Equation (4.3) follows.

Applying Lemma 4.2 to (4.3) leads to Equation (4.4). □

[2] The elliptic and q-analogues of the Fibonacci numbers we use are different from the analogues considered *e.g.* in [71].

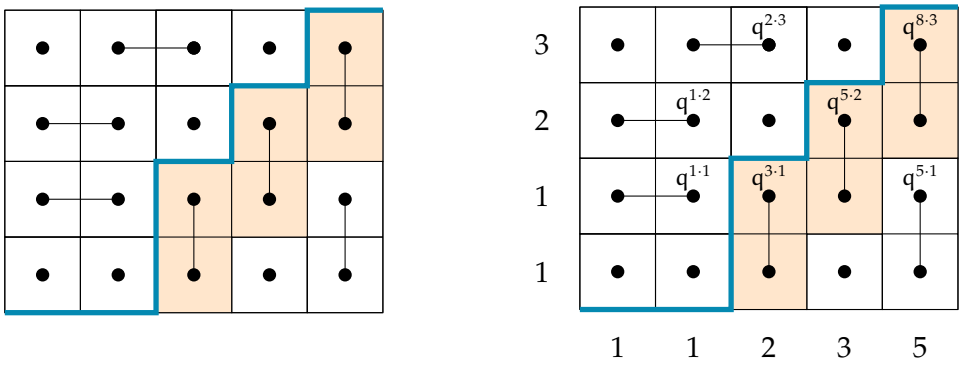


Figure 4.1: A path-domino tiling T of a 5×4 rectangle (left), and the q-Fibonacci weights of its tiles (right). The weight of the tiling is the product of the weights of its tiles, $\omega(T) = q^{1+2+6+3+10+24+5} = q^{51}$.

For $m, n \in \mathbb{N}$, the q-*analogue of the Fibonomial number* is defined as

$$\begin{bmatrix} m+n \\ n \end{bmatrix}_{\mathcal{F}} := \frac{[F_{m+n}]_q^!}{[F_m]_q^! \cdot [F_n]_q^!}, \tag{4.5}$$

where $[F_n]_q^! := \prod_{k=1}^{n} [F_k]_q$ is the q-Fibonacci analogue of $n!$. Surprisingly, this rational expression turns out to be a polynomial. Our objective is to present a combinatorial model to describe it. In order to achieve this, we will introduce some q-weights associated to path-domino tilings of an $m \times n$ rectangle (with m columns and n rows).

Let $T \in \mathcal{T}_{m,n}$ be a path-domino tiling of an $m \times n$ rectangle. The q-*weights* of the possible tiles in T are defined as follows:

$$\omega\left(\boxed{\bullet}\right) = 1, \qquad \omega\left(\boxed{\bullet\!-\!\bullet}\right) = q^{F_i F_j}, \qquad \omega\left(\begin{array}{|c|}\hline \bullet \\ \hline \\ \hline \end{array}\right) = q^{F_i F_j}, \qquad \omega\left(\begin{array}{|c|}\hline \bullet \\ \hline \\ \hline \end{array}\right) = q^{F_{i+1} F_j}$$

where (i, j) denotes the coordinate of the top-right corner of the tile, and the shaded vertical domino represents a special vertical domino touching the path from below. The q-*weight* of T is defined as the product of the weights of its tiles; see an example in Figure 4.1. The following theorem is one of our main results.

Theorem 4.5. *For $m, n \in \mathbb{N}$, the q-analogue of the Fibonomial number is a polynomial in q with non-negative integer coefficients. It can be computed as*

$$\begin{bmatrix} m+n \\ n \end{bmatrix}_{\mathcal{F}} = \sum_{T \in \mathcal{T}_{m,n}} \omega(T).$$

Proof. Let us start proving the result for the initial cases $m = 1$ or $n = 1$.

For $n = 1$, we have

$$\begin{bmatrix} m+1 \\ 1 \end{bmatrix}_{\mathcal{F}} = [F_{m+1}]_q.$$

The collection $\mathcal{T}_{m,1}$ coincides with the tilings of an m-strip with dominos and monominos, since only the last step of the specified lattice path can be a north step because of the special

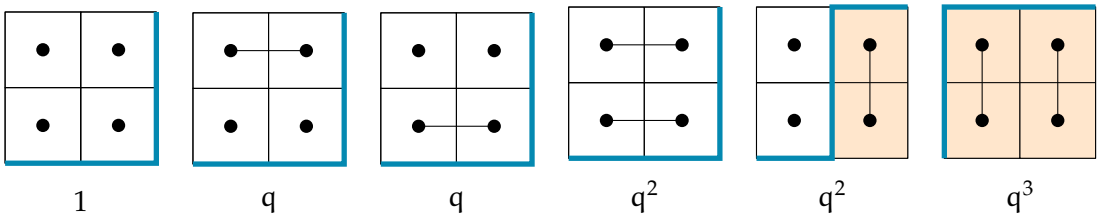


Figure 4.2: The six path-domino tilings of a 2×2 rectangle, and their contribution to the q-Fibonomial $\begin{bmatrix} 2+2 \\ 2 \end{bmatrix}_{\mathcal{F}} = \frac{[F_4][F_3]}{[F_2][F_1]} = \frac{[3][2]}{[1][1]} = 1 + 2q + 2q^2 + q^3$ when computed as a generating function $\sum_{T \in \mathcal{T}_{2,2}} \omega(T)$.

vertical domino condition. The weight of a domino in a tiling, whose top-right corner has coordinate $(i, 1)$, is $q^{F_i F_1} = q^{F_i}$. Therefore, the result follows from Lemma 4.3.

For $m = 1$, we have

$$\begin{bmatrix} 1+n \\ n \end{bmatrix}_{\mathcal{F}} = [F_{n+1}]_q.$$

The collection $\mathcal{T}_{1,n}$ can be identified with the collection of tilings of a vertical n-strip with dominos and monominos, where the topmost domino has a special weight. The weight of a usual vertical domino, whose top-right corner has coordinate $(1, j)$, is $q^{F_1 F_j} = q^{F_j}$, while the weight of a special vertical domino located at the same place is $q^{F_{1+1} F_j} = q^{F_j}$. Therefore, the result also follows from Lemma 4.3.

Now assume the result holds when $m = 1$ or $n = 1$. Letting $m, n > 1$ and using Equation (4.4) in the following equation we obtain:

$$\begin{aligned} \begin{bmatrix} m+n \\ n \end{bmatrix}_{\mathcal{F}} &= \frac{[F_{m+n}]_q [F_{m+n-1}]_q^!}{[F_m]_q^! \cdot [F_n]_q^!} \\ &= [F_{m+1}]_{q^{F_n}} \frac{[F_{m+n-1}]_q^!}{[F_m]_q^! \cdot [F_{n-1}]_q^!} + q^{F_n F_{m+1}} [F_{n-1}]_{q^{F_m}} \frac{[F_{m+n-1}]_q^!}{[F_{m-1}]_q^! \cdot [F_n]_q^!} \\ &= [F_{m+1}]_{q^{F_n}} \begin{bmatrix} m+n-1 \\ n-1 \end{bmatrix}_{\mathcal{F}} + q^{F_n F_{m+1}} [F_{n-1}]_{q^{F_m}} \begin{bmatrix} m-1+n \\ n \end{bmatrix}_{\mathcal{F}}. \end{aligned}$$

By induction (and using again Lemma 4.3), the first term of the sum is the weighted counting of the path-domino tilings of the $m \times n$ rectangle whose specified path ends with a north step, while the second term is the weighted counting of those finishing with an east step. Indeed, the path-domino tilings whose path ends with a north step have an extra contribution $[F_{m+1}]_{q^{F_n}}$ corresponding to the tilings of the last row with horizontal dominos and monominos. The path-domino tilings whose path ends with an east step have an extra contribution $q^{F_n F_{m+1}} [F_{n-1}]_{q^{F_m}}$; this corresponds to the weight of the forced special vertical domino ($q^{F_n F_{m+1}}$) and the tilings of the remaining $(n-2)$-strip in the last column ($[F_{n-1}]_{q^{F_m}}$). □

We have checked the following conjecture by computer for $m, n \leqslant 10$:

Conjecture 4.6. *The polynomials* $\begin{bmatrix} m+n \\ n \end{bmatrix}_{\mathcal{F}}$ *are unimodal.*

Example 4.7. Figure 4.2 illustrates an example of the q-Fibonomial for $m = n = 2$.

Example 4.8 ($n = 2$). Let $m \in \mathbb{N}$ and $n = 2$. Theorem 4.5 leads to the identity:

$$[F_{m+2}]_q [F_{m+1}]_q = \sum_{k=1}^{m+1} q^{c_k^m} [F_k]_q^2, \tag{4.6}$$

where $c_k^m = \sum_{i=k}^{m} F_{i+1}$.

The left hand side comes from the equality

$$\begin{bmatrix} m+2 \\ 2 \end{bmatrix}_{\mathcal{F}} = [F_{m+2}]_q [F_{m+1}]_q.$$

The right hand side is the sum of the weights of all path-domino tilings of an $m \times 2$ rectangle. In fact, the term $q^{c_k^m}[F_k]^2$ indicates the sum of the weights of the path-domino tilings whose specified path is $E^{k-1}N^2E^{m-(k-1)}$: $q^{c_k^m}$ is the product of the weights of the special vertical dominos, and $[F_k]^2$ is the weight of the two horizontal rows above the path. Since there are no more possibilities for the specified path due to the special vertical domino condition, the identity (4.6) follows. The evaluation at $q = 1$ recovers

$$F_{m+2}F_{m+1} = \sum_{k=1}^{m+1} F_k^2. \tag{4.7}$$

Remark 4.9. Equation (4.7) is a well known identity due to its relation with the golden ratio and golden spirals in nature, see for instance [24, 52]. The left hand side of the equation is the area of a $F_{m+2} \times F_{m+1}$ rectangle, which can be subdivided into a sequence of squares, with side lengths $F_1, F_2, \ldots, F_{m+1}$, forming a spiral as illustrated in Figure 4.3 (left). This Fibonacci spiral is an approximation of the golden spiral, a special case of logarithmic spirals which describe the shape of various natural phenomena such as galaxies, nautilus shells and hurricanes. On the other hand, Equation (4.6) also has a natural geometric interpretation. The left hand side represents the weighted area of a $F_{m+2} \times F_{m+1}$ rectangle, where a unit square whose bottom-left corner is located at (i, j) has weight q^{i+j}. This rectangle can be subdivided into a sequence of squares, with side lengths $F_1, F_2, \ldots, F_{m+1}$, in the north-east direction as illustrated in Figure 4.3 (right). The sum of their weighted areas is exactly the right hand side of Equation (4.6). This sum can also be interpreted as the "mass" of the rectangle, where the F_k-square has density $d(F_k) = q^{c_k^m}$. This density increases according to the ratio $\frac{d(F_k)}{d(F_{k+1})} = q^{F_{k+1}}$, satisfying the initial condition $d(F_{m+1}) = 1$. It would be interesting to assign these densities to the squares giving rise to the Fibonacci spiral on Figure 4.3 (left), and see if the resulting equation has some physical meaning. Or even more interesting, to have a continuous version of the equation representing the mass of the Fibonacci spiral (or golden spiral), in order to describe some physical phenomenon in nature. For instance, it is quite natural to think that the density of galaxies grows exponentially as it approaches the centre of the spiral. In Example 4.14, we additionally provide a generalization of Equation (4.6) using elliptic weight functions.

4.3 Elliptic analogues of the Fibonomial numbers

As in the previous section, the elliptic analogue of the Fibonomial number is obtained by replacing each term in the binomial coefficient by its corresponding Fibonacci elliptic number. In this section, we contribute to the field of elliptic combinatorics by studying an elliptic

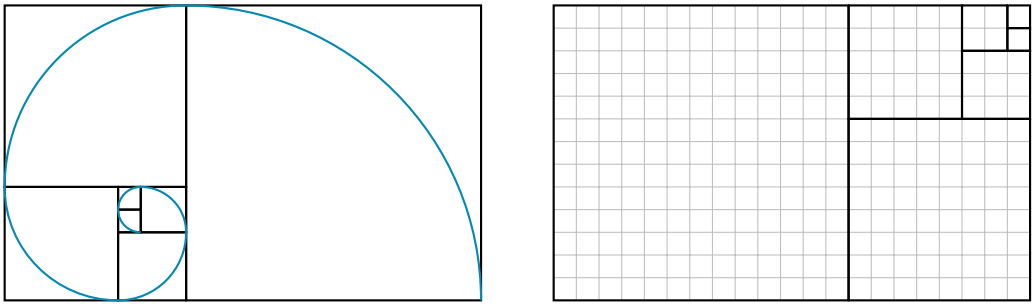


Figure 4.3: Geometric interpretation of Equations (4.6) and (4.7).

generalization of the Fibonomial number. Recall the elliptic analogue of a natural number $n \in \mathbb{N}$ from Equation (2.32), which we defined as:

$$[n]_{a,b;q,p} = \frac{\theta(q^n, aq^n, bq, aq/b; p)}{\theta(q, aq, bq^n, aq^n/b; p)}.$$

We define the elliptic weight function

$$\nu_{a,b;q,p}(m,n) := \frac{\theta(aq^{2m+n}, b, bq^n, aq^n/b, a/b; p)}{\theta(aq^n, bq^m, bq^{m+n}, aq^m/b, aq^{m+n}/b; p)} q^m \tag{4.8}$$

to obtain the following lemma.

Lemma 4.10. *For $m, n \in \mathbb{N}$, the following identities hold:*

$$[m+n]_{a,b;q,p} = [m]_{a,b;q,p} + \nu_{a,b;q,p}(m,n)[n]_{a,b;q,p}, \tag{4.9}$$

$$[m \cdot n]_{a,b;q,p} = [m]_{a,b;q,p}[n]_{a,b;q^m,p}. \tag{4.10}$$

Proof. Rearranging the left hand side of Equation (4.9) and applying the three-term identity from Proposition 2.5 (by taking $x = a^{\frac{1}{2}}q^m$, $y = a^{\frac{1}{2}}b^{-1}$, $u = a^{\frac{1}{2}}q^{m+n}$ and $z = a^{\frac{1}{2}}$) yields:

$$\begin{aligned}
[m+n]_{a,b;q,p} &= \frac{\theta(bq, aq/b; p)}{\theta(q, aq, bq^{m+n}, aq^{m+n}/b; p)} \cdot \frac{\theta(aq^m/b, bq^m, aq^{m+n}, q^{m+n}; p)}{\theta(aq^m/b, bq^m; p)} \\
&= \frac{\theta(bq, aq/b; p)}{\theta(q, aq, bq^{m+n}, aq^{m+n}/b; p)} \\
&\quad \times \frac{\theta(aq^{m+n}/b, bq^{m+n}, aq^m, q^m; p) + q^m\theta(a/b, b, aq^{2m+n}, q^n; p)}{\theta(aq^m/b, bq^m; p)} \\
&= [m]_{a,b;q,p} + \frac{\theta(aq^{2m+n}, b, bq^n, aq^n/b, a/b; p)}{\theta(aq^n, bq^m, bq^{m+n}, aq^m/b, aq^{m+n}/b; p)} q^m [n]_{a,b;q,p}.
\end{aligned}$$

Equation (4.10) follows from simple cancellations. □

Coming back to Fibonacci numbers, we simply define the *elliptic analogue of the Fibonacci number* F_n by $[F_n]_{a,b;q,p}$.

For $m, n \in \mathbb{N}$, the *elliptic analogue of the Fibonomial number* is defined as

$$\begin{bmatrix} m+n \\ n \end{bmatrix}_{\mathcal{F}_{a,b;q,p}} := \frac{[F_{m+n}]^!_{a,b;q,p}}{[F_m]^!_{a,b;q,p} \cdot [F_n]^!_{a,b;q,p}}, \tag{4.11}$$

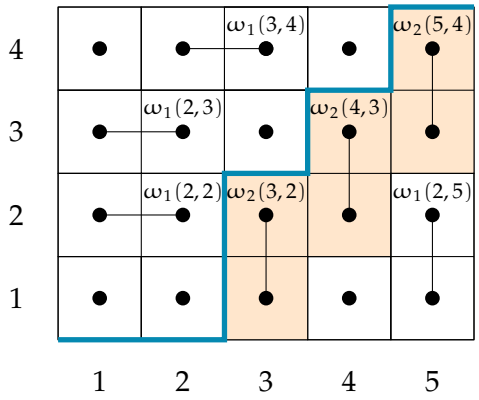


Figure 4.4: The path-domino tiling T from Figure 4.1 with the elliptic Fibonacci weights of its tiles. The weight of the tiling is the product of the weights of its tiles.

where $[F_n]^!_{a,b;q,p} := \prod_{k=1}^{n} [F_k]_{a,b;q,p}$ is the elliptic Fibonacci analogue of $n!$.

Similarly as before, the elliptic Fibonomial number counts path-domino tilings of an $m \times n$ rectangle according to certain elliptic weights. For $T \in \mathcal{T}_{m,n}$, the *elliptic weights* of the possible tiles in T are defined as follows:

$$\widetilde{\omega}\left(\boxed{\bullet}\right) = 1, \qquad \widetilde{\omega}\left(\boxed{\bullet\!-\!\bullet}\right) = \omega_1(i,j), \qquad \widetilde{\omega}\left(\begin{array}{|c|}\hline \bullet \\ \hline \bullet \\ \hline \end{array}\right) = \omega_1(j,i), \qquad \widetilde{\omega}\left(\begin{array}{|c|}\hline \bullet \\ \hline \bullet \\ \hline \end{array}\right) = \omega_2(i,j)$$

where (i,j) denotes the coordinate of the top-right corner of the tile, the shaded vertical domino represents a special vertical domino touching the path from below, and

$$\omega_1(i,j) := \nu_{a,b;q^{F_j},p}(F_i, F_{i-1}), \tag{4.12}$$

$$\omega_2(i,j) := \nu_{a,b;q,p}(F_{i+1}F_j, F_iF_{j-1}), \tag{4.13}$$

Note that the weight of a "regular" vertical domino is evaluated at (j,i) instead of (i,j). This transposition does not make any difference for the q-analogue of the Fibonomial numbers, but it does for the elliptic case. The *elliptic weight* $\widetilde{\omega}(T)$ of T is defined as the product of the weights of its tiles; see an example in Figure 4.4. The elliptic weight is a generalization of the q-weight, since we obtain the q-weight by taking the limit $p \to 0$, $a \to 0$ and $b \to 0$ in this order.

Theorem 4.11. *For $m, n \in \mathbb{N}$, the elliptic analogue of the Fibonomial number can be computed as*

$$\begin{bmatrix} m+n \\ n \end{bmatrix}_{\mathcal{F}_{a,b;q,p}} = \sum_{T \in \mathcal{T}_{m,n}} \widetilde{\omega}(T).$$

The proof of this theorem follows the same steps as the proof of Theorem 4.5. Before proving the theorem, we need to prove the following two lemmas, which are obtained by using the same arguments as in the q-case and replacing the weight of a domino whose top-right coordinate is $(i,1)$ by $\omega_1(i,1)$.

Lemma 4.12. *For $n \in \mathbb{N}$, the elliptic analogue of the Fibonacci numbers can be computed as*

$$[F_n]_{a,b;q,p} = \sum_T \widetilde{\omega}(T),$$

where the sum ranges over all tilings of an $(n-1)$-strip using dominos and monominos.

Proof. The result is clearly true for $n = 1, 2$. Let $n > 2$, applying Equation (4.9) from Lemma 4.10 yields:

$$\begin{aligned}[F_n]_{a,b;q,p} &= [F_{n-1} + F_{n-2}]_{a,b;q,p} \\ &= [F_{n-1}]_{a,b;q,p} + \nu_{a,b;q,p}(F_{n-1}, F_{n-2})[F_{n-2}]_{a,b;q,p} \\ &= [F_{n-1}]_{a,b;q,p} + \omega_1(n-1,1)[F_{n-2}]_{a,b;q,p}.\end{aligned}$$

By induction, the first term of this sum corresponds to the tilings of an $(n-1)$-strip that finish with a monomino, while the second term to the tilings of an $(n-1)$-strip that finish with a domino. □

Lemma 4.13. *For $m, n \in \mathbb{N}$, the following identity holds:*

$$[F_{m+n}]_{a,b;q,p} = [F_n]_{a,b;q,p}[F_{m+1}]_{a,b;q^{F_n},p} + \omega_2(m,n)[F_m]_{a,b;q,p}[F_{n-1}]_{a,b;q^{F_m},p}. \tag{4.14}$$

Proof. Applying Lemma 4.10 to Equation (4.3) leads to Equation (4.14). □

Now, we have all tools to prove the main theorem in this section.

Proof of Theorem 4.11. For $n = 1$, we have

$$\begin{bmatrix} m+1 \\ 1 \end{bmatrix}_{\mathcal{F}_{a,b;q,p}} = [F_{m+1}]_{a,b;q,p}.$$

The collection $\mathcal{T}_{m,1}$ coincides with the tilings of an m-strip with dominos and monominos, since only the last step of the specified lattice path can be a north step because of the special vertical domino condition. Therefore, the result follows from Lemma 4.12.

For $m = 1$, we have

$$\begin{bmatrix} 1+n \\ n \end{bmatrix}_{\mathcal{F}_{a,b;q,p}} = [F_{n+1}]_{a,b;q,p}.$$

The collection $\mathcal{T}_{1,n}$ can be identified with the collection of tilings of a vertical n-strip with dominos and monominos, where the topmost domino has a special weight. The weight of a usual vertical domino, whose top-right corner has coordinate $(1, j)$, is $\omega_1(j, 1)$ while the weight of a special vertical domino located at the same place is $\omega_2(1, j)$. Since $\omega_2(1, j) = \omega_1(j, 1)$, the result also follows from Lemma 4.12.

Now assume the result holds when $m = 1$ or $n = 1$. Letting $m, n > 1$ and using Equation (4.14) in the following equation we obtain:

$$\begin{aligned}\begin{bmatrix} m+n \\ n \end{bmatrix}_{\mathcal{F}_{a,b;q,p}} &= \frac{[F_{m+n}]_{a,b;q,p}[F_{m+n-1}]^!_{a,b;q,p}}{[F_m]^!_{a,b;q,p} \cdot [F_n]^!_{a,b;q,p}} \\ &= [F_{m+1}]_{a,b;q^{F_n},p} \begin{bmatrix} m+n-1 \\ n-1 \end{bmatrix}_{\mathcal{F}_{a,b;q,p}} \\ &\quad + \omega_2(m,n)[F_{n-1}]_{a,b;q^{F_m},p} \begin{bmatrix} m-1+n \\ n \end{bmatrix}_{\mathcal{F}_{a,b;q,p}}.\end{aligned}$$

By induction (and using again Lemma 4.12), the first term of the sum is the weighted counting of the path-domino tilings of the $m \times n$ rectangle whose specified path ends with a north step, while the second term is the weighted counting of those finishing with an east step.

Indeed, the path-domino tilings whose path ends with a north step have an extra contribution $[F_{m+1}]_{a,b;q^{F_n},p}$. This corresponds to the weighted enumeration of the tilings of the last row with horizontal dominos and monominos. This follows from the fact that both quantities satisfy the same initial conditions and recurrence relation, which is obtained by applying Equation (4.9) to $F_m + F_{m-1}$:

$$[F_{m+1}]_{a,b;q^{F_n},p} = [F_m]_{a,b;q^{F_n},p} + \omega_1(m,n)[F_{m-1}]_{a,b;q^{F_n},p}.$$

The path-domino tilings whose path ends with an east step have an extra contribution

$$\omega_2(m,n)[F_{n-1}]_{a,b;q^{F_m},p}.$$

This corresponds to the weight of the forced special vertical domino ($\omega_2(m,n)$) and the tilings of the remaining $(n-2)$-strip in the last column ($[F_{n-1}]_{a,b;q^{F_m},p}$). □

Example 4.14 ($n = 2$). The identity (4.6) in Example 4.8 for $m \in \mathbb{N}$ and $n = 2$ generalizes in the elliptic case to

$$[F_{m+2}]_{a,b;q,p}[F_{m+1}]_{a,b;q,p} = \sum_{k=1}^{m+1} \Omega_k^m [F_k]^2_{a,b;q,p}, \tag{4.15}$$

where $\Omega_k^m = \prod_{i=k}^m \omega_2(i,2)$ is the product of the weights of the special vertical dominos, and $[F_k]^2_{a,b;q,p} = [F_k]_{a,b;q,p} \cdot [F_k]_{a,b;q^{F_2},p}$ is the weight of the two horizontal rows above the path.

Example 4.15 ($a, b; p \to 0$). By computing the limits $p \to 0$, $a \to 0$ and $b \to 0$ (in this order) of $\widetilde{\omega}(T)$ and $\begin{bmatrix} m+n \\ n \end{bmatrix}_{\mathcal{F}_{a,b;q,p}}$ we obtain Theorem 4.5.

Remark 4.16. The elliptic analogue of the Fibonomial number is a Fibonacci analogue of the "regular" elliptic binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}'_{a,b;q,p} := \frac{[n]_{a,b;q,p}!}{[k]_{a,b;q,p}! \cdot [n-k]_{a,b;q,p}!}, \tag{4.16}$$

where $[n]_{a,b;q,p}! = \prod_{i=1}^n [i]_{a,b;q,p}$. This can be expressed as

$$\begin{bmatrix} n \\ k \end{bmatrix}'_{a,b;q,p} = \frac{(q^{n-k+1}, aq^{n-k+1}, bq, aq/b; q, p)_k}{(q, aq, bq^{n-k+1}, aq^{n-k+1}/b; q, p)_k}, \tag{4.17}$$

where the theta shifted factorial is defined in Equation (2.31).

Several elliptic versions of the binomial coefficient have been already considered in the literature, see for instance [54] or [64]. By substituting $b \mapsto bq^k$ in Equation (4.17), we obtain the elliptic binomial coefficient defined by Schlosser in [64]:

$$\begin{bmatrix} n \\ k \end{bmatrix}_{a,b;q,p} = \frac{(q^{n-k+1}, aq^{n-k+1}, bq^{1+k}, aq^{1-k}/b; q, p)_k}{(q, aq, bq^{n+1}, aq^{n-2k+1}/b; q, p)_k},$$

which we defined in Equation (2.37) and studied in Section 2.3. One can check that these elliptic binomial coefficients are different to one defined by Rains in [54, Definition 11], even in the simplest case where both partitions are single columns of respective lengths n and k, *cf.* [70, Equation 3.12].

An interesting task for future works on the elliptic Fibonomials could be to generalize Fibonomial identities to elliptic identities, see *e.g.* [6, 55] for potential candidates. For instance, we generalize a proof by Reiland [55] for the identity

$$\binom{m+n}{n}_{\mathcal{F}} = \sum_{j=0}^{n} F_{m+1}^{j} F_{n-j-1} \binom{m-1+n-j}{n-j}_{\mathcal{F}}.$$

in the following corollary.

Corollary 4.17. *For $m, n \geqslant 1$, the elliptic Fibonomial $\begin{bmatrix} m+n \\ n \end{bmatrix}_{\mathcal{F}_{a,b;q,p}}$ is equal to*

$$\sum_{j=0}^{n} \Big(\prod_{i=0}^{j-1} [F_{m+1}]_{a,b;q^{F_{n-i}},p} \Big) [F_{n-1-j}]_{a,b;q^{F_m},p} w_2(m, n-j) \begin{bmatrix} m-1+n-j \\ n-j \end{bmatrix}_{\mathcal{F}_{a,b;q,p}}.$$

Proof. For $0 \leqslant j \leqslant n$, consider all weighted path-domino tilings $\mathcal{T}^{j}_{m,n}$ of an $m \times n$ rectangle for $m, n \geqslant 1$ where the last horizontal step of the path ends at position $(m, n-j)$. To obtain the sum of the weights $\sum_{T \in \mathcal{T}^{j}_{m,n}} \widetilde{\omega}(T)$, we subdivide the tilings (as illustrated in Figure 4.5): The collection $\mathcal{T}^{j}_{m,n}$ can be identified with the collection of

- a path domino tiling of an $(m-1) \times (n-j)$ rectangle which adds weight $\begin{bmatrix} m-1+n-j \\ n-j \end{bmatrix}_{\mathcal{F}_{a,b;q,p}}$,
- j full rows on top which add weights $[F_{m+1}]_{a,b;q^{F_{n-i}},p}$ for $0 \leqslant i \leqslant j-1$ or 1 if $j = 0$,
- the tile below the last horizontal step, which is forced to be a special vertical domino with weight $\omega_2(m, n-j)$,
- and the column below the special vertical domino which adds weight $[F_{n-1-j}]_{a,b;q^{F_m},p}$

to the sum. Summing over all j, we obtain the required convolution formula. Note that in the case $j = n-1$ the number of tilings is 0 since there has to be a vertical domino below the last horizontal step and indeed, there appears the factor $[F_0]_{a,b;q^{F_m},p} = 0$ in the corresponding summand. For $j = n$ there is no special vertical domino and no column below the last east step of the path but $\omega_2(m, 0) = 1$ and $[F_{-1}]_{a,b;q^{F_m},p} = 1$ (if we define F_{-1} by $F_{-1} + F_0 = F_1$ to be 1). □

4.4 The q-Fibonacci analogue of the rational Catalan numbers

Given a pair of relatively prime numbers $m, n \in \mathbb{N}$ (that is, such that their greatest common divisor is $(m, n) := \gcd\{m, n\} = 1$), the m, n-*Catalan number* is defined as

$$\mathrm{Cat}_{m,n} := \frac{1}{m+n} \binom{m+n}{n}.$$

This number is equal to the number of lattice paths from $(0, 0)$ to (m, n) that stay weakly above the main diagonal of the $m \times n$ rectangle. The study of these numbers (also in the non-coprime case), and their q-analogue and q, t-analogue generalizations, is connected to numerous relevant topics including rectangular diagonal harmonics and Macdonald polynomials [4, 8, 9, 10, 49], Shi hyperplane arrangements and affine Weyl groups [3, 32, 80], affine

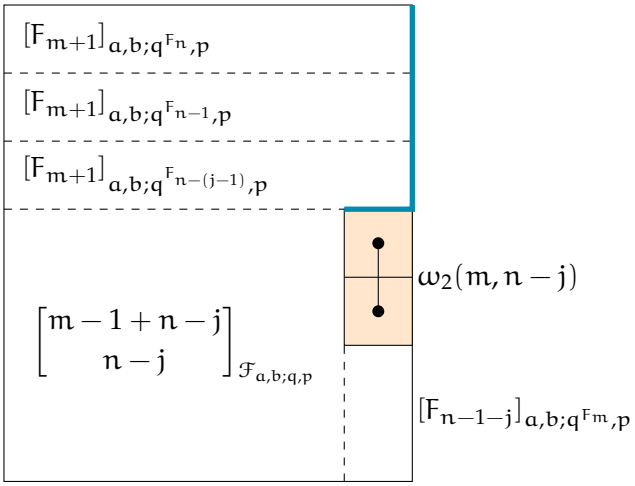


Figure 4.5: The parts of one summand of the right hand side of Corollary 4.17 for $m = 6$, $n = 7$ and $j = 3$.

Springer fibers [32, 36], affine Hecke algebras [18], knot theory [33, 34], and representation theory of Cherednik algebras [25, 26, 27].

The q-*Fibonacci analogue of the* m, n-*Catalan number* is defined as

$$[\mathrm{FCat}_{m,n}] := \frac{1}{[F_{m+n}]_q} \begin{bmatrix} m+n \\ n \end{bmatrix}_{\mathcal{F}} = \frac{[F_{m+n-1}]_q^!}{[F_m]_q^! [F_n]_q^!}.$$

Surprisingly, this rational expression also turns out to be a polynomial when the greatest common divisor $(m, n) \in \{1, 2\}$. Before proving this, we need the following lemma.

Lemma 4.18 ([37]). *For* $m, n \in \mathbb{N}$, *we have* $(F_m, F_n) = F_{(m,n)}$.

Proposition 4.19 (Shu Xiao Li [46]). *If the greatest common divisor* $(m, n) \in \{1, 2\}$, *then* $[\mathrm{FCat}_{m,n}]$ *is a polynomial in* q *with integer coefficients.*

Proof. First note that

$$[F_n]_q \begin{bmatrix} m+n \\ n \end{bmatrix}_{\mathcal{F}} = [F_{m+n}]_q \begin{bmatrix} m+n-1 \\ n-1 \end{bmatrix}_{\mathcal{F}}.$$

Since all the terms involved in this identity are polynomials, we know $[F_{m+n}]$ divides

$$[F_n]_q \begin{bmatrix} m+n \\ n \end{bmatrix}_{\mathcal{F}}.$$

But

$$(F_{m+n}, F_n) = F_{(m+n,n)} = F_{(m,n)} = 1$$

whenever (m, n) is equal to 1 or 2. Since the roots of $[n]_q$ are the n-th roots of unity that are different to 1, the polynomials $[F_{m+n}]_q$ and $[F_n]_q$ have no roots in common. Thus, $[F_{m+n}]_q$ divides $\begin{bmatrix} m+n \\ n \end{bmatrix}_{\mathcal{F}}$. □

Computational experimentation for $m, n \leqslant 15$ suggests that the coefficients of these polynomials are non-negative integers. However, we do not have a proof nor a combinatorial model to describe them.

Open Problem 4.20. Find a combinatorial interpretation of the q-Fibonacci analogue of the rational Catalan numbers.

Remark 4.21. The model of Sagan and Savage in [60] gives a combinatorial interpretation of the Lucas analogue of the binomial coefficients. The *Lucas polynomials* $\{n\}$ generalize the Fibonacci numbers, and are defined by the initial conditions $\{0\} = 0, \{1\} = 1$ and the recurrence $\{n\} = s\{n-1\}+t\{n-2\}$ for some variables s, t. It was proven in [7, Section 6.2], that the Lucas analogue of the rational m, n-Catalan numbers is also polynomial in s, t with integer coefficients.[3] However, to the best of our knowledge, no q-analogues of the Lucas-binomial coefficients have been studied in the literature. The proof of Proposition 4.19 was originally found by Shu Xiao Li, during discussions about this topic in the Algebraic Combinatorics Seminar at the Fields Institute in 2015. At this Seminar, Farid Aliniaeifard showed that Conjecture 4.6 implies the positivity of the coefficients.

Remark 4.22. Given a crystallographic Coxeter group W with Coxeter exponents $e_1 < e_2 < \ldots < e_n$, the rational W-Catalan number is defined as $C_W(a) = \prod_{i=1}^{n} \frac{a+e_i}{e_i+1}$, and this is an integer when a is relatively prime to $e_n + 1$. The Coxeter exponents for the crystallographic Coxeter groups are:

type of W	$e_1, e_2, \ldots, e_n$
A_n	$1, 2, 3, \ldots, n$
B_n	$1, 3, 5, \ldots, 2n-1$
D_n	$n-1, 1, 3, 5, \ldots, 2n-3$
E_6	$1, 4, 5, 7, 8, 11$
E_7	$1, 5, 7, 9, 11, 13, 17$
E_8	$1, 7, 11, 13, 17, 19, 23, 29$
F_4	$1, 5, 7, 11$
G_2	$1, 5$

The classical Catalan number corresponds to type A_n. We can now define a q-Fibonacci analogue as follows:

$$C_{W,\mathcal{F}}(a) = \prod_{i=1}^{n} \frac{[F_{a+e_i}]_q}{[F_{e_i+1}]_q}.$$

We have computationally checked that this is a polynomial with positive integer coefficients when a and e_n+1 are relatively prime, for each type and various values of a. It is interesting to note that although in type A_n we have shown that it is a polynomial as long as $(F_a, F_{e_n+1}) = 1$, for other types we must have the stronger condition $(a, e_n + 1) = 1$. For example $C_{F_4,\mathcal{F}}(2)$ is not a polynomial.

4.5 The model of Bennett, Carrillo, Machacek, and Sagan

Recently, Bennett, Carrillo, Machacek, and Sagan [7] gave another combinatorial interpretation of the Lucas analogue of the Binomial coefficient. A special case of their interpretation is that the Fibonomial number $\binom{n}{k}_{\mathcal{F}} = \frac{F_n^!}{F_k^! \cdot F_{n-k}^!}$ counts the number of certain partial tilings of a Young diagram of staircase shape of size n. Similarly as above, we can also assign q-weights and elliptic weights to this model, giving rise to other combinatorial interpretations for the q-analogue and elliptic analogue of the Fibonomial number, respectively.

[3] Their proof is reproduced from the proof of Proposition 4.19 with our permission.

We start by briefly recalling the Bennett–Carrillo–Machacek–Sagan model. For $n \in \mathbb{N}$, consider the Young diagram of staircase shape of size n (in French notation) associated to the partition $(n-1, n-2, \ldots, 2, 1)$: the position of the south-west corner is placed at the origin of the Cartesian coordinate system and the number of boxes in row i (counted from bottom to top) is $n - i$. We also include the lines from $(0, n-1)$ to $(0, n)$ and from $(n-1, 0)$ to $(n, 0)$ for convenience. An example of a Young diagram of staircase shape for $n = 9$ is illustrated on the left of Figure 4.6.

Given two natural numbers $n \geqslant k$, an (n, k)-*tiling* is a partial tiling of a staircase shape of size n with monominos and dominos where one lattice path is specified such that:

- the lattice path goes from $(k, 0)$ to $(0, n)$ using steps $(0, 1)$ and $(-1, 0)$. It stays inside the Young diagram and every west step $(-1, 0)$ must be followed by a north step $(0, 1)$.
- if a north step is not forced by a previous west step, the boxes that are on its left in the same row are tiled with monominos and horizontal dominos.
- if a north step is forced by a previous west step, the boxes that are on its right in the same row are tiled with monominos and horizontal dominos. The tile touching the north step is forced to be a domino (we call it a special domino).

We denote by $\mathcal{S}_{n,k}$ the collection of all (n, k)-tilings. An example for $n = 9$ and $k = 5$ is shown on the right of Figure 4.6.

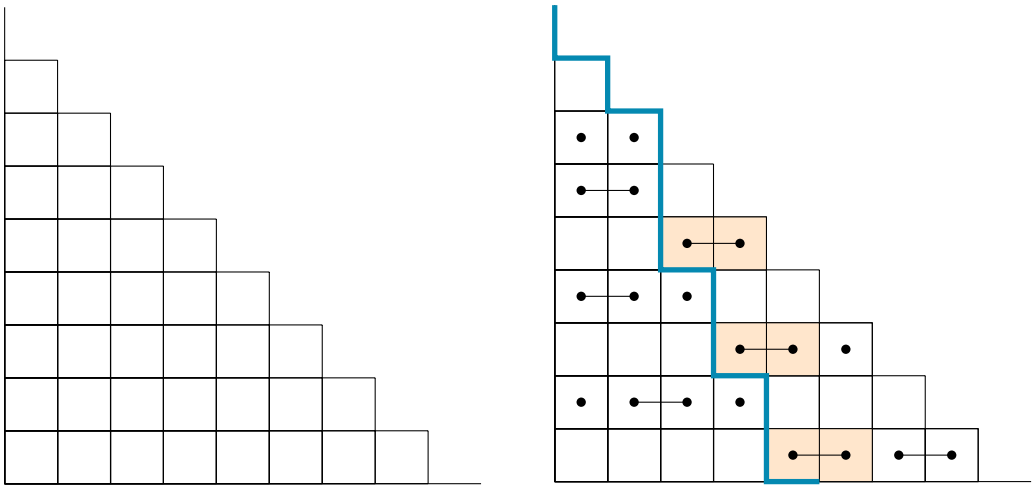


Figure 4.6: A Young diagram of staircase shape of size 9 (left) and a $(5, 9)$-tiling (right).

The tilings in the Bennett–Carrillo–Machacek–Sagan model are in natural bijection with the tilings of the Sagan–Savage model: the tilings on the right (resp. left) of the path in the staircase become the tilings of the columns below (resp. rows above) the path in the rectangle (see last paragraph before Section 3 in [7]). We have chosen our example to match that of Figure 4.1.

Using this bijection, one can easily reinterpret the q-weights and elliptic weights described in Section 4.2 and Section 4.3 for this new model. Let T be an (n, k)-tiling. For a tile t (monomino or domino) on the left of the path, we define the *floor* of t as the number of boxes in the same row from the western border of the Young diagram to the eastern border of t, and the *height* as 1 plus the number of boxes in the same row from the eastern border of the Young diagram to

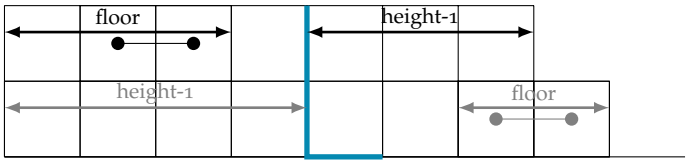


Figure 4.7: An illustration of the height and floor statistics of one domino on the left and one domino on the right of the path. The domino on the top row has floor 3 and height 4, while the domino on the bottom row has floor 2 and height 5.

the north step to the east of t. For a tile t on the right of the path the statistics are mirrored as illustrated in Figure 4.7.

The q-weights of the tiles are defined as follows:

$$\nu\left(\boxed{\bullet}\right) = 1, \qquad \nu\left(\boxed{\bullet\!-\!\bullet}\right) = q^{F_{\mathrm{floor}}F_{\mathrm{height}}}, \qquad \nu\left(\boxed{\bullet\!-\!\bullet}\right) = q^{F_{\mathrm{floor}}F_{\mathrm{height}+1}}$$

where the shaded domino represents a special domino forced by a west step of the path. In the elliptic case the weights are defined as:

$$\widetilde{\nu}\left(\boxed{\bullet}\right) = 1, \qquad \widetilde{\nu}\left(\boxed{\bullet\!-\!\bullet}\right) = \omega_1(\mathtt{floor}, \mathtt{height}), \qquad \widetilde{\nu}\left(\boxed{\bullet\!-\!\bullet}\right) = \omega_2(\mathtt{height}, \mathtt{floor}).$$

The q- and elliptic weights of a tiling T are defined as the product of the weights of its tiles.

As consequences of Theorem 4.5 and Theorem 4.11 we get the following two results.

Corollary 4.23. *For $n \geqslant k$, the q-analogue of the Fibonomial number can be computed as*

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\mathcal{F}} = \sum_{T \in \mathcal{S}_{n,k}} \nu(T).$$

Corollary 4.24. *For $n \geqslant k$, the elliptic analogue of the Fibonomial number can be computed as*

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\mathcal{F}_{a,b;q,p}} = \sum_{T \in \mathcal{S}_{n,k}} \widetilde{\nu}(T).$$

An example of Corollary 4.23, for $n = 4$ and $k = 2$, is given in Figure 4.8. We have chosen the order of the tilings to match the order of the tilings for the corresponding example in Figure 4.2.

Remark 4.25. The Bennett–Carrillo–Machacek–Sagan model also gives rise to combinatorial interpretations of the Lucas analogue of the (ordinary) Catalan numbers and Fuss–Catalan numbers described in [7]. They define a Catalan partial tiling C of a Young diagram of staircase shape of size $2n$ to be a $(2n, n-1)$-tiling where the first row (at the bottom) stays blank except for the forced special domino if the first step is a west step.

The Lucas analogue of the Catalan numbers

$$C_{\{n\}} := \frac{1}{\{n+1\}} \frac{\{2n\}!}{\{n\}!\{n\}!}$$

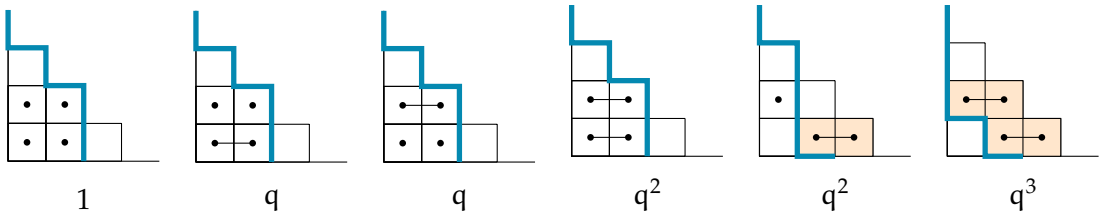


Figure 4.8: The six $(4,2)$-tilings of a staircase shape of size 4, and their contribution to the q-Fibonomial $\begin{bmatrix}4\\2\end{bmatrix}_{\mathcal{F}} = \frac{[F_4]_q[F_3]_q}{[F_2]_q[F_1]_q} = \frac{[3]_q[2]_q}{[1]_q[1]_q} = 1+2q+2q^2+q^3$ when computed as a generating function $\sum_{T\in\mathcal{S}_{4,2}} \nu(T)$.

(where $\{n\}! = \prod_{k=1}^{n}\{k\}$) is the sum over the weights of all Catalan partial tilings C of size $2n$ if all dominos have weight t, all monominos have weight s, and the weight of the tiling C is defined as the product of the weights of its tiles [7, Corollary 4.2.].

This model does not translate to the q-Fibonacci analogue of the Catalan numbers

$$[\mathrm{FCat}_n] := \frac{1}{[F_{n+1}]_q}\begin{bmatrix}2n\\n\end{bmatrix}_{\mathcal{F}} = \frac{[F_{2n}]_q^!}{[F_{n+1}]_q^![F_n]_q^!}.$$

For example, $[\mathrm{FCat}_3]$ is not equal to the sum of the weights of all Catalan partial tilings C of size 6 with the corresponding q-weights defined above.

Although the Bennett–Carrillo–Machacek–Sagan model does not seem to trivially solve the following problem, it still might be easier than Open Problem 4.20.

Open Problem 4.26. Find a combinatorial interpretation of the q-Fibonacci analogue of the Catalan numbers (and the Fuss–Catalan numbers).

5 An operator approach to elliptic rook and Stirling numbers

5.1 Introduction

The classical *Stirling numbers of the second kind*, $S(n,k)$, count set partitions of an n-elementary set into k non-empty sets. They fulfil the recurrence relation

$$S(n,k) = 0, \qquad \text{for } k < 0 \text{ or } k > n\,, \tag{5.1a}$$
$$S(0,0) = 1, \tag{5.1b}$$

and, for $k \geqslant 0$,

$$S(n+1,k) = S(n,k-1) + kS(n,k). \tag{5.1c}$$

Given a nonnegative integer n and a variable z, we define the corresponding *falling factorial* as

$$z{\downarrow_n} := z(z-1)\cdots(z-n+1)$$

and $z{\downarrow_0} = 1$. Then, $S(n,k)$ are the connection coefficients in

$$z^n = \sum_{k=0}^{n} S(n,k) z{\downarrow_k}\,. \tag{5.2}$$

Let E_z be the shift operator that shifts a polynomial $p(x)$ by $E_z p(z) = p(z+1)$ and Δ be the difference operator $\Delta = E_z - I$. Then it is a classical result that we can derive an explicit formula for $S(n,k)$ from (5.2) by applying the difference operator Δ on both sides of the equation to obtain

$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^j \binom{k}{j}(k-j)^n. \tag{5.3}$$

We recommend [19, 58] for further reading on the theory of difference operators.

Following Carlitz [15, 16], we can define q-*Stirling numbers of the second kind*, $S_q(n,k)$, by

$$[z]_q^n = \sum_{k=0}^{n} q^{\binom{k}{2}} S_q(n,k)[z]_q[z-1]_q\ldots[z-k+1]_q. \tag{5.4}$$

We define q-*falling factorials* by

$$z{\downarrow_n^q} := z(z-[1]_q)\cdots(z-[n-1]_q)$$

and $z{\downarrow_0^q} = 1$ and use the identity

$$[m]_q - [n]_q = [m-n]_q q^n$$

to reformulate equation (5.4) as

$$[z]_q^n = \sum_{k=0}^{n} S_q(n,k)[z]_q\downarrow_k^q . \tag{5.5}$$

One can derive from this relation that $S_q(n,k)$ satisfies

$$S_q(n,k) = 0, \qquad \text{for } k<0 \text{ or } k>n , \tag{5.6a}$$
$$S_q(0,0) = 1, \tag{5.6b}$$

and, for $k \geqslant 0$,

$$S_q(n+1,k) = S_q(n,k-1) + [k]_q S_q(n,k). \tag{5.6c}$$

In order to derive an explicit sum formula for these numbers, Carlitz generalizes the difference operator Δ by the recursive definition

$$\Delta_q p(z) = p(z+1) - p(z), \tag{5.7a}$$
$$\Delta_q^{n+1} p(z) = \Delta_q^n p(z+1) - q^n \Delta_q^n p(z), \tag{5.7b}$$

or explicitly by

$$\Delta_q^n = \prod_{i=0}^{n-1} (E_z - q^i). \tag{5.7c}$$

Applying Δ_q to both sides of (5.4), Carlitz derives [16, (3.3)]

$$S_q(n,k) = \frac{q^{-\binom{k}{2}}}{[k]_q!} \sum_{j=0}^{k} (-1)^j q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q [k-j]_q^n. \tag{5.8}$$

In [39], Kerekényiné Balogh and Schlosser study an elliptic analogue of Carlitz' q-Stirling numbers of the second kind. We introduce *elliptic falling factorials* by the definition

$$z\downarrow_n^{a,b;q,p} := z(z-[1]_{a,b;q,p})\cdots(z-[n-1]_{a,b;q,p}),$$

and $z\downarrow_0^{a,b;q,p} = 1$, where $[z]_{a,b;q,p}$ is the elliptic number of z (defined by (2.32)). Then, the *elliptic Stirling numbers of the second kind*, $S_{a,b;q,p}(n,k)$, can be defined by the relation

$$[z]_{a,b;q,p}^n = \sum_{k=0}^{n} S_{a,b;q,p}(n,k)[z]_{a,b;q,p}\downarrow_k^{a,b;q,p} . \tag{5.9}$$

Note that each factor $[z]_{a,b;q,p} - [j]_{a,b;q,p}$ in the elliptic falling factorial of $[z]_{a,b;q,p}$ factors by the relation

$$[z]_{a,b;q,p} - [j]_{a,b;q,p} = W_{a,b;q,p}(j)[z-j]_{aq^{2j},bq^j;q,p}. \tag{5.10}$$

This definition is actually slightly different to the definition in [39]. One obtains the original definition by multiplying $S_{a,b;q,p}(n,k)$ with $\prod_{j=0}^{k-1} W_{a,b;q,p}(j)$, but it turns out to be convenient to stick to the definition (5.9).

From (5.9) one can deduce the following recurrence relation for $S_{a,b;q,p}(n,k)$:

$$S_{a,b;q,p}(n,k) = 0, \qquad \text{for } k < 0 \text{ or } k > n, \tag{5.11a}$$
$$S_{a,b;q,p}(0,0) = 1, \tag{5.11b}$$

and, for $k \geqslant 0$,

$$S_{a,b;q,p}(n+1,k) = S_{a,b;q,p}(n,k-1) + [k]_{a,b;q,p} S_{a,b;q,p}(n,k). \tag{5.11c}$$

In [70], Schlosser and Yoo study elliptic Stirling numbers of the second kind in the context of rook theory. They ask for a generalization of the explicit sum formula (5.8) to $S_{a,b;q,p}(n,k)$ and work out special cases for $0 \leqslant k \leqslant 3$. Furthermore, the authors suggest to use an appropriate extension of the q-difference operator Δ_q to achieve this goal.

In this chapter, we deduce an explicit formula for $S_{a,b;q,p}(n,k)$ analogous to (5.8) by using the fact that elliptic Stirling numbers of the second kind are a specialization of the complete homogeneous symmetric functions h_k (similarly to the case of weighted Stirling numbers in Section 3.3.3). An explicit sum-formula for h_k is well-known and we obtain a formula for $S_{a,b;q,p}(n,k)$ as a special case. Furthermore, we will generalize Carlitz' q-difference operator appropriately to deduce a new operator proof of the formula for h_k.

We show that this operator proof also generalizes to prove formulas for more general polynomials with elliptic rook numbers and Lah numbers as special cases.

5.2 Complete homogeneous symmetric functions

In Section 3.3 we defined complete homogeneous symmetric functions $h_k(n)$ for $n, k \in \mathbb{Z}$. In this chapter we only consider the classical case, where $n \geqslant 0$. We will use the definition

$$h_k(a_0, a_1, a_2, \ldots, a_n) = \sum_{0 \leqslant j_1 \leqslant j_2 \leqslant \cdots \leqslant j_k \leqslant n} a_{j_1} a_{j_2} \ldots a_{j_k}.$$

We introduce a generalization of falling factorials and factorials for a series of variables $\mathbf{a} = a_0, a_1, a_2, \ldots$ (following [61]) as

$$z\downarrow_0^{\mathbf{a}} := 1, \quad z\downarrow_n^{\mathbf{a}} := \prod_{i=0}^{n-1} (z - a_i) \quad \text{and} \quad a_n! := a_n\downarrow_n^{\mathbf{a}}. \tag{5.12}$$

Then we can deduce from (3.48) that

$$z^n = \sum_{k=0}^{n} h_{n-k}(a_0, a_1, \ldots, a_k) z\downarrow_k^{\mathbf{a}}. \tag{5.13}$$

This equation is well-known and appears for example in [61, Theorem 2.2] and [21, p. 208]. By setting $a_i = [i]_q$ for all $i \geqslant 0$ and $z = [z]_q$ in this equation, we obtain the generating function (5.5) of Carlitz' q-Stirling numbers of the second kind. This leads to the specialization

$$S_q(n,k) = h_{n-k}([0]_q, [1]_q, \ldots, [k]_q),$$

that is also well-known and appeared for example recently in [61].

By setting $a_i = [i]_{a,b;q,p}$ for all i and $z = [z]_{a,b;q,p}$ in (5.13), we obtain the generating function for the elliptic Stirling numbers of the second kind (5.9), therefore

$$S_{a,b;q,p}(n,k) = h_{n-k}([0]_{a,b;q,p}, [1]_{a,b;q,p}, \dots, [k]_{a,b;q,p}). \tag{5.14}$$

By multiplying both sides of (5.13) with z it follows that

$$\begin{aligned} z^{n+1} &= \sum_{k=0}^{n} h_{n-k}(a_0, a_1, \dots, a_k) z\downarrow_k^{\mathbf{a}} \, (z - a_k + a_k) \\ &= \sum_{k=0}^{n} h_{n-k}(a_0, a_1, \dots, a_k) z\downarrow_{k+1}^{\mathbf{a}} + \sum_{k=0}^{n} a_k h_{n-k}(a_0, a_1, \dots, a_k) z\downarrow_k^{\mathbf{a}} \\ &= \sum_{k=0}^{n+1} \big(h_{n-k+1}(a_0, a_1, \dots, a_{k-1}) + a_k h_{n-k}(a_0, a_1, \dots, a_k) \big) z\downarrow_k^{\mathbf{a}} \end{aligned}$$

and we can derive the recursion

$$h_{n-k}(a_0, a_1, \dots, a_k) = 0 \quad \text{for } k < 0 \text{ or } k > n, \tag{5.15a}$$

$$h_0(a_0) = 1 \tag{5.15b}$$

and, for $k \geqslant 0$,

$$h_{n-k+1}(a_0, a_1, \dots, a_k) = h_{n-k+1}(a_0, a_1, \dots, a_{k-1}) + a_k h_{n-k}(a_0, a_1, \dots, a_k) \tag{5.15c}$$

analogous to (5.1), (5.6) and (5.11).

The complete homogeneous symmetric functions can be computed by the following explicit formula (*cf.* [35], where this formula was obtained as a special case of a Schur function identity, or [78, Exercise 7.4]).

Theorem 5.1. *Let n, k be nonnegative integers, $a_0, a_1, \dots, a_k$ be variables and $h_n(a_0, \dots, a_k)$ the n-th complete homogeneous symmetric function in $k+1$ variables. We then have*

$$h_n(a_0, \dots, a_k) = \sum_{j=0}^{k} \Big(\prod_{\substack{i=0 \\ i \neq j}}^{k} (a_j - a_i)^{-1} \Big) a_j^{n+k}. \tag{5.16}$$

By setting $a_i = i$ and $a_i = [i]_q$ for all i and doing some small manipulations, we recover the sum-formulas for $S(n,k)$ in (5.3) and $S_q(n,k)$ in (5.8), respectively.

To complete the task of finding an explicit formula for $S_{a,b;q,p}(n,k)$, described by Schlosser and Yoo [70], we set $a_i = [i]_{a,b;q,p}$, change the indices according to (5.14) and reverse the order of summation, to obtain the following corollary.

Corollary 5.2. *Let n, k be nonnegative integers, $[n]_{a,b;q,p}$ be the elliptic number of n and $S_{a,b;q,p}(n,k)$ be the elliptic Stirling numbers of the second kind defined by* (5.9)*. We then have*

$$S_{a,b;q,p}(n,k) = \sum_{j=0}^{k} \Big(\prod_{\substack{i=0 \\ i \neq k-j}}^{k} (W_{a,b;q,p}(i)[k-j-i]_{a q^{2i}, b q^i; q, p})^{-1} \Big) [k-j]_{a,b;q,p}^{n}. \tag{5.17}$$

This formula reduces to (5.8) by taking the limits $p \to 0$, then $a \to 0$ and then $b \to 0$.

Remark 5.3. In contrast to (5.3) and (5.8) it seems that there appears no binomial coefficient in the sum-formula for h_k (5.16). But we can define the **a**-binomial coefficient for $0 \leqslant k \leqslant n$ as

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\mathbf{x}} := (-1)^{n-k} \frac{a_n!}{\prod_{\substack{i=0 \\ i\neq k}}^{n} (a_k - a_i)} = (-1)^{n-k+1} \prod_{\substack{i=0 \\ i\neq k}}^{n-1} \frac{a_n - a_i}{a_k - a_i}$$

Then we obtain

$$h_n(a_0, \dots, a_k) = \frac{1}{a_k!} \sum_{j=0}^{k} (-1)^{k-j} \begin{bmatrix} k \\ j \end{bmatrix}_{\mathbf{a}} a_j^{n+k}. \tag{5.18}$$

This binomial coefficient fulfils the recurrence relation for $0 < k < n+1$

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_{\mathbf{a}} = \prod_{j=0}^{n-1} \frac{a_{n+1} - a_{j+1}}{a_n - a_j} \begin{bmatrix} n \\ k \end{bmatrix}_{\mathbf{a}} + E_{\mathbf{a}} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{\mathbf{a}}, \tag{5.19}$$

where $E_{\mathbf{a}}$ is an operator that maps all a_i to a_{i+1}. However, we chose not to use this notation in this work.

In the following section we describe an operator proof of Theorem 5.1.

5.3 A generalized difference operator

We consider functions $f(z; \mathbf{x})$ in the variables z and $\mathbf{x} = x_0, x_1, x_2 \dots$ and recall the generalized notation for the falling factorial and the factorial according to z and $\mathbf{x}$ as follows:

$$z{\downarrow_0^{\mathbf{x}}} = 1, \quad z{\downarrow_n^{\mathbf{x}}} = \prod_{i=0}^{n-1} (z - x_i) \quad \text{and} \quad x_n! = x_n{\downarrow_n^{\mathbf{x}}}. \tag{5.20}$$

We define the *shift operator* $E_{z,\mathbf{x}}$ as $E_{z,\mathbf{x}} f(z; x_0, x_1, x_2 \dots) = f(z+1; x_1, x_2, x_3 \dots)$. For example,

$$E_{z,\mathbf{x}} \frac{z - x_1 + a_2}{x_k - x_2} = \frac{z + 1 - x_2 + a_2}{x_{k+1} - x_3}.$$

The *difference operator* $\Delta_{\mathbf{x}}^n$ is recursively defined by

$$\Delta_{\mathbf{x}} f(z, \mathbf{x}) = E_{z,\mathbf{x}} f(z, \mathbf{x}) - f(z, \mathbf{x}), \tag{5.21a}$$

$$\Delta_{\mathbf{x}}^{n+1} f(z, \mathbf{x}) = E_{z,\mathbf{x}} \Delta_{\mathbf{x}}^n f(z, \mathbf{x}) - \Big(\prod_{j=0}^{n-1} \frac{x_{n+1} - x_{j+1}}{x_n - x_j} \Big) \Delta_{\mathbf{x}}^n f(z, \mathbf{x}), \tag{5.21b}$$

or explicitly by

$$\Delta_{\mathbf{x}}^n = \prod_{i=0}^{n-1} \Big(E_{z,\mathbf{x}} - \prod_{j=0}^{n-i-2} \frac{x_{n-i} - x_{j+1}}{x_{n-i-1} - x_j} \Big). \tag{5.21c}$$

It follows by induction on n (and (5.19)) that

$$\Delta_{\mathbf{x}}^n = \sum_{k=0}^{n} x_n! \Big(\prod_{\substack{i=0 \\ i\neq k}}^{n} (x_k - x_i)^{-1} \Big) E_{z,\mathbf{x}}^k. \tag{5.22}$$

We are now ready to prove the explicit formula for $h_k(a_0, a_1, \dots, a_k)$. Note that we introduced two different sequences of variables $\mathbf{x}$ and $\mathbf{a}$.

Proof of Theorem 5.1. Let C_k be the connection coefficients in

$$f(z) = \sum_{k=0}^{n} C_k a_z \downarrow_k^{\mathbf{a}}, \tag{5.23}$$

where $f(z)$ is a polynomial in a_z of degree at most n and independent of the variables $\mathbf{x}$. Then the C_k are independent of z and $\mathbf{x}$. We apply $\Delta_{\mathbf{x}}^j$ on both sides of (5.23) and set $z = 0$ and $x_i = a_i$ for $i \geqslant 0$ afterwards to obtain

$$\Delta_{\mathbf{x}}^j f(z)\big|_{z=0,x_i=a_i} = \sum_{k=0}^{n} C_k \Delta_{\mathbf{x}}^j a_z \downarrow_k^{\mathbf{a}} \big|_{z=0,x_i=a_i}$$

since C_k is independent of z and x_i. The above notation means that we set $z = 0$ and $x_i = a_i$ for $i \geqslant 0$ after applying the operators on both sides. Evaluating the operator on the right hand side gives

$$\Delta_{\mathbf{x}}^j a_z \downarrow_k^{\mathbf{a}} = \sum_{s=0}^{j} x_j! \prod_{\substack{i=0 \\ i\neq s}}^{j} (x_s - x_i)^{-1} \prod_{i=0}^{k-1} (a_{z+s} - a_i).$$

In this expression we can set $z = 0$ and $x_i = a_i$ for $i \geqslant 0$ to obtain

$$\Delta_{\mathbf{x}}^j a_z \downarrow_k^{\mathbf{a}} \big|_{z=0,x_i=a_i} = \sum_{s=0}^{j} a_j! \frac{\prod_{i=0}^{k-1}(a_s - a_i)}{\prod_{\substack{i=0 \\ i\neq s}}^{j}(a_s - a_i)}.$$

We want to show that this sum simplifies to $\delta_{j,k} a_j!$, where δ is the Kronecker delta.

Suppose $0 \leqslant j < k$. Then we have a factor $(a_s - a_s) = 0$ in the nominator of each summand and the sum vanishes.

For $k = j$ the sum reduces to $(a_k)!$ by the same argument.

Suppose $k < j \leqslant n$. Then

$$\sum_{s=0}^{j} a_j! \frac{\prod_{i=0}^{k-1}(a_s - a_i)}{\prod_{\substack{i=0 \\ i\neq s}}^{j}(a_s - a_i)} = \sum_{s=k}^{j} a_j! \frac{\prod_{i=0}^{k-1}(a_s - a_i)}{\prod_{\substack{i=0 \\ i\neq s}}^{j}(a_s - a_i)} = \sum_{s=k}^{j} a_j! \prod_{\substack{i=k \\ i\neq s}}^{j} (a_s - a_i)^{-1}.$$

In order to prove

$$\sum_{s=k}^{j} a_j! \prod_{\substack{i=k \\ i\neq s}}^{j} (a_s - a_i)^{-1} = 0$$

for $k < j$, we assume $a_j \downarrow_k^{\mathbf{a}} \neq 0$ (otherwise the equation would be true as well) and divide both sides by $a_j \downarrow_k^{\mathbf{a}}$. By a simple computation we arrive at the equivalent equation

$$\sum_{s=k}^{j-1} \prod_{\substack{i=k \\ i\neq s}}^{j-1} \frac{(a_j - a_i)}{(a_s - a_i)} = 1.$$

The sum on the left side can be seen as a polynomial in a_j of degree $j - k - 1$ which is equal to 1 in the $j - k$ values $a_k, a_{k+1}, \ldots, a_{j-1}$ inserted for a_j. In fact, it is the Lagrange interpolation

polynomial for the function $f(a_j) = 1$. Therefore, the equality holds and we have proven the case $k < j$.

In summary, we have shown that

$$\frac{\Delta_{\mathbf{x}}^{j} f(z)}{a_j!}\Big|_{z=0, x_i = a_i} = C_j.$$

To prove the theorem, we take $f(z) = a_z^n$ to obtain

$$C_k = \frac{\Delta_{\mathbf{x}}^{k} a_z^n}{a_k!}\Big|_{z=0, x_i = a_i} = \sum_{j=0}^{k} \Big(\prod_{\substack{i=0 \\ i \neq j}}^{k} (a_j - a_i)^{-1} \Big) a_j^n.$$

By comparing (5.23) with (5.13) (where z is replaced by a_z) we finally arrive at

$$h_n(a_0, \ldots, a_k) = \sum_{j=0}^{k} \Big(\prod_{\substack{i=0 \\ i \neq j}}^{k} (a_j - a_i)^{-1} \Big) a_j^{n+k}. \qquad \square$$

Remark 5.4 (Specializations)**.** Besides the elliptic and q-Stirling numbers, there are several interesting specializations of $h_n(a_0, a_1, \ldots, a_k)$. For example, we obtain (q, r)-Whitney numbers of the second kind $W_{m,r}[n, k]_q$ (see for example [20]) by

$$W_{m,r}[n, k]_q q^{-kr-m\binom{k}{2}} = W^*_{m,r}[n, k]_q = h_{n-k}([r]_q, [m + r]_q, [2m + r]_q, \ldots, [km + r]_q).$$

Then, Theorem 5.1 reduces to [20, (12)].

As mentioned in Remark 3.25, Remmel and Wachs defined a (p, q)-analogue of shifted Stirling numbers of the second kind. We obtain the (p, q)-shifted Stirling numbers of the second kind, $S^{r,m}_{n,k}(p, q)$ [56, (28)], by

$$S^{r,m}_{n,k}(p, q) = h_{n-k}([r]_{p,q}, [m + r]_{p,q}, [2m + r]_{p,q}, \ldots, [km + r]_{p,q}),$$

which are actually a (p, q)-analogue of r-Whitney numbers. Then, one can check by a careful computation that Theorem 5.1 reduces to [56, Theorem 13].

An elliptic analogue of $S^{r,m}_{n,k}(p, q)$ was defined by Schlosser and Yoo [70] and can be obtained by specializing

$$S^{r,m}_{a,b;q,p}(n, k) = h_{n-k}([r]_{a,b;q,p}, [m + r]_{a,b;q,p}, [2m + r]_{a,b;q,p}, \ldots, [km + r]_{a,b;q,p}),$$

up to a power of elliptic weights.

5.4 More general cases

The proof of Theorem 5.1 can be adapted to any polynomial f in a_z of degree at most n which are independent of the variables $\mathbf{x}$. Let C and $c_1, c_2, \ldots, c_n$ be complex numbers. If we take $f(z) = c_0(a_z - c_1)(a_z - c_2) \cdots (a_z - c_n)$, then (5.23) becomes

$$c_0 \prod_{i=1}^{n} (a_z - c_i) = \sum_{k=0}^{n} C_{n,k} a_z{\downarrow}_{\mathbf{a}}^{k}, \tag{5.24}$$

and, following the proof of Theorem 5.1, the coefficients have the explicit form

$$C_{n,k} = \frac{\Delta_x^k f(z)}{a_k!}\Big|_{z=0,x_i=a_i} = c_0 \sum_{j=0}^{k} \Big(\prod_{\substack{i=0 \\ i\neq j}}^{k} (a_j - a_i)^{-1} \Big) \prod_{i=1}^{n} (a_j - c_i). \tag{5.25}$$

If we see $a_0, a_1, \ldots, a_n$ as complex numbers, then C_k is the change of basis coefficient between the two series of polynomials over $\mathbb{C}$

$$\Big(c_0 \prod_{i=1}^{n} (a_z - c_i) \Big)_{n\geqslant 0} \qquad \text{and} \qquad \Big(\prod_{i=0}^{n-1} (a_z - a_i) \Big)_{n\geqslant 0}.$$

From (5.24) we can derive the recurrence relations

$$C_{n,k} = 0, \qquad \text{for } k < 0 \text{ or } k > n\,, \tag{5.26a}$$

$$C_{0,0} = c_0, \tag{5.26b}$$

and, for $k \geqslant 0$,

$$C_{n+1,k} = C_{n,k-1} + (a_k - c_{n+1}) C_{n,k}. \tag{5.26c}$$

Remark 5.5. For $c_0 = 1$, the coefficient C_k is equal to the complementary symmetric function $\mathrm{comp}_k(\{c_1, c_2, \ldots, c_n | a_0, a_1, \ldots, a_k\})$ defined over hybrid sets in [21]. This can be seen by comparing (5.24) with [21, Corollary 2]. The explicit formula (5.25) did not appear in [21] but is most likely already known, although the author has not been able to find it explicitly in the literature.

5.4.1 Rook numbers

In [70], Schlosser and Yoo define the k-th elliptic rook number, $r_k(a, b; q, p; B)$, for a Ferrers board

$$B = B(b_1, b_2, \ldots, b_n) = \{(i, j) : 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant b_i\}$$

as the sum of specific elliptic weights for rook placements of B. We neglect the details here and only focus on the following algebraic expression. The authors obtain the following generating function for the elliptic rook numbers [70, Theorem 12]:

$$\prod_{i=1}^{n} [z - i + 1 + b_i]_{a q^{2(i-1-b_i)}, b q^{i-1-b_i}; q, p} = \sum_{k=0}^{n} r_{n-k}(a, b; q, p; B) \prod_{j=1}^{k} [z - j + 1]_{a q^{2(j-1)}, b q^{j-1}; q, p}. \tag{5.27}$$

We can reformulate this equation by using (5.10) as

$$\prod_{i=1}^{n} W_{a,b;q,p}(i - 1 - b_i)^{-1} ([z]_{a,b;q,p} - [i - 1 - b_i]_{a,b;q,p}) = \sum_{k=0}^{n} r_{n-k}(a, b; q, p; B) \Big(\prod_{j=1}^{k} W_{a,b;q,p}(j-1)^{-1} \Big) [z]_{a,b;q,p} \downarrow_k^{a,b;q,p}. \tag{5.28}$$

Comparing this expression with (5.24) yields

$$r_{n-k}(a,b;q,p;B) = \frac{\prod_{j=1}^{k} W_{a,b;q,p}(j-1)}{\prod_{i=1}^{n} W_{a,b;q,p}(i-1-b_i)} \times \sum_{j=0}^{k} \Big(\prod_{\substack{i=0 \\ i\neq j}}^{k} ([j]_{a,b;q,p} - [i]_{a,b;q,p})^{-1} \Big) \prod_{i=1}^{n} ([j]_{a,b;q,p} - [i-1-b_i]_{a,b;q,p}). \quad (5.29)$$

as a special case of (5.25) with $c_0 = \prod_{i=1}^{n} W_{a,b;q,p}(i-1-b_i)$, $a_i = [i]_{a,b;q,p}$, $c_i = [i-1-b_i]_{a,b;q,p}$ and

$$C_{n,k} = r_{n-k}(a,b;q,p;B) \prod_{j=1}^{k} W_{a,b;q,p}(j-1)^{-1}.$$

These elliptic rook numbers generalize elliptic Stirling numbers of the second kind which can be recovered by considering the staircase Ferrers board $B(1,2,3,\dots,n-1)$ (with the original definition of elliptic Stirling numbers).

5.4.2 Lah numbers

Another interesting board considered in [70] is $B(b_1,b_2,\dots,b_n) = B(n-1,n-1,\dots,n-1)$. In this case, the elliptic rook numbers are equal to elliptic Lah numbers, defined in [39]. We will again use a slightly different definition (up to a power of elliptic weights). We introduce *elliptic rising factorials* by the definition

$$z{\uparrow}_n^{a,b;q,p} := z(z-[-1]_{a,b;q,p}) \cdots (z-[-n+1]_{a,b;q,p}),$$

and $z{\uparrow}_0^{a,b;q,p} = 1$, and define *elliptic Lah numbers*, $L_{a,b;q,p}(n,k)$, by

$$[z]_{a,b;q,p}{\uparrow}_n^{a,b;q,p} = \sum_{k=0}^{n} L_{a,b;q,p}(n,k)[z]_{a,b;q,p}{\downarrow}_k^{a,b;q,p} . \quad (5.30)$$

They fulfil the recurrence relation

$$L_{a,b;q,p}(n,k) = 0, \qquad \text{for } k<0 \text{ or } k>n\,, \quad (5.31a)$$

$$L_{a,b;q,p}(0,0) = 1, \quad (5.31b)$$

and, for $k \geqslant 0$,

$$L_{a,b;q,p}(n+1,k) = L_{a,b;q,p}(n,k-1) + W_{a,b;q,p}(-n)[n+k]_{aq^{-2n},bq^{-n};q,p} L_{a,b;q,p}(n,k). \quad (5.31c)$$

We derive the explicit expression

$$L_{a,b;q,p}(n,k) = \sum_{j=0}^{k} \Big(\prod_{\substack{i=0 \\ i\neq j}}^{k} ([j]_{a,b;q,p} - [i]_{a,b;q,p})^{-1} \Big) \prod_{i=1}^{n} ([j]_{a,b;q,p} - [-n+i]_{a,b;q,p}) \quad (5.32)$$

by comparing (5.30) with (5.27) or directly with (5.24).

Bibliography

[1] Tewodros Amdeberhan, Xi Chen, Victor H. Moll, and Bruce E. Sagan. Generalized Fibonacci polynomials and Fibonomial coefficients. *Ann. Comb.*, 18(4):541–562, 2014.

[2] George E. Andrews, Richard Askey, and Ranjan Roy. *Special functions*, volume 71 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1999.

[3] Christos A. Athanasiadis. On a refinement of the generalized Catalan numbers for Weyl groups. *Trans. Amer. Math. Soc.*, 357(1):179–196, 2005.

[4] Jean-Christophe Aval and François Bergeron. A note on: rectangular Schröder parking functions combinatorics. *Sém. Lothar. Combin.*, 79:Art. B79a, 13, 2018-2019.

[5] Hiroya Baba and Makoto Katori. Excursion processes associated with elliptic combinatorics. *J. Stat. Phys.*, 171(6):1035–1066, 2018.

[6] Arthur T. Benjamin and Elizabeth Reiland. Combinatorial proofs of Fibonomial identities. *Fibonacci Quart.*, 52(5):28–34, 2014.

[7] Curtis Bennett, Juan Carrillo, John Machacek, and Bruce E. Sagan. Combinatorial interpretations of Lucas analogues of binomial coefficients and Catalan numbers. *Ann. Comb.*, 24(3):503–530, 2020.

[8] François Bergeron. Open questions for operators related to rectangular Catalan combinatorics. *J. Comb.*, 8(4):673–703, 2017.

[9] Francois Bergeron, Adriano Garsia, Emily Sergel Leven, and Guoce Xin. Some remarkable new plethystic operators in the theory of Macdonald polynomials. *J. Comb.*, 7(4):671–714, 2016.

[10] Francois Bergeron, Adriano Garsia, Emily Sergel Leven, and Guoce Xin. Compositional (km, kn)-shuffle conjectures. *Int. Math. Res. Not. IMRN*, (14):4229–4270, 2016.

[11] Nantel Bergeron, Cesar Ceballos, and Josef Küstner. Elliptic and q-analogs of the Fibonomial numbers. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 16:Paper No. 076, 16, 2020.

[12] Dan Betea. *Elliptic Combinatorics and Markov Processes*. ProQuest LLC, Ann Arbor, MI, 2012. Thesis (Ph.D.)–California Institute of Technology.

[13] Dan Betea. Elliptically distributed lozenge tilings of a hexagon. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 14:Paper No. 032, 39, 2018.

[14] Alexei Borodin, Vadim Gorin, and Eric M. Rains. q-Distributions on boxed plane partitions. *Selecta Math. (N.S.)*, 16(4):731–789, 2010.

[15] Leonard Carlitz. On abelian fields. *Trans. Amer. Math. Soc.*, 35(1):122–136, 1933.

[16] Leonard Carlitz. q-Bernoulli numbers and polynomials. *Duke Math. J.*, 15:987–1000, 1948.

[17] Xi Chen and Bruce E. Sagan. The fractal nature of the Fibonomial triangle. *Integers*, 14:Paper No. A3, 12, 2014.

[18] Ivan Cherednik. *Double affine Hecke algebras*, volume 319 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2005.

[19] Johann Cigler. Operatormethoden für q-Identitäten. *Monatsh. Math.*, 88(2):87–105, 1979.

[20] Roberto B. Corcino, Jay M. Ontolan, and Maria Rowena S. Lobrigas. Explicit formulas for the first form (q, r)-Dowling numbers and (q, r)-Whitney-Lah numbers. *Eur. J. Pure Appl. Math.*, 14(1):65–81, 2021.

[21] Ernesto Damiani, Ottavio M. D'Antona, and Daniel E. Loeb. The complementary symmetric functions: connection constants using negative sets. *Adv. Math.*, 135(2):207–219, 1998.

[22] E. Date, M. Jimbo, A. Kuniba, T. Miwa, and M. Okado. Exactly solvable SOS models: local height probabilities and theta function identities. *Nuclear Phys. B*, 290(2):231–273, 1987.

[23] E. Date, M. Jimbo, A. Kuniba, T. Miwa, and M. Okado. Exactly solvable SOS models. II. Proof of the star-triangle relation and combinatorial identities. In *Conformal field theory and solvable lattice models (Kyoto, 1986)*, volume 16 of *Adv. Stud. Pure Math.*, pages 17–122. Academic Press, Boston, MA, 1988.

[24] Richard A. Dunlap. *The golden ratio and Fibonacci numbers*. World Scientific Publishing Co., Inc., River Edge, NJ, 1997.

[25] Pavel Etingof and Victor Ginzburg. Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism. *Invent. Math.*, 147(2):243–348, 2002.

[26] Pavel Etingof, Eugene Gorsky, and Ivan Losev. Representations of rational Cherednik algebras with minimal support and torus knots. *Adv. Math.*, 277:124–180, 2015.

[27] Pavel Etingof and Xiaoguang Ma. Lecture notes on Cherednik algebras. Preprint, arXiv: 1001.0432.

[28] Sam Formichella and Armin Straub. Gaussian binomial coefficients with negative arguments. *Ann. Comb.*, 23(3-4):725–748, 2019.

[29] Igor B. Frenkel and Vladimir G. Turaev. Elliptic solutions of the Yang-Baxter equation and modular hypergeometric functions. In *The Arnold-Gelfand mathematical seminars*, pages 171–204. Birkhäuser Boston, Boston, MA, 1997.

[30] Markus Fulmek. Kombinatorik. Vorlesungsskriptum, Universität Wien, 2015.

[31] George Gasper and Mizan Rahman. *Basic Hypergeometric Series*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2nd edition, 2004.

[32] Eugene Gorsky, Mikhail Mazin, and Monica Vazirani. Affine permutations and rational slope parking functions. *Trans. Amer. Math. Soc.*, 368(12):8403–8445, 2016.

[33] Eugene Gorsky and Andrei Neguţ. Refined knot invariants and Hilbert schemes. *J. Math. Pures Appl. (9)*, 104(3):403–435, 2015.

[34] Eugene Gorsky, Alexei Oblomkov, Jacob Rasmussen, and Vivek Shende. Torus knots and the rational DAHA. *Duke Math. J.*, 163(14):2709–2794, 2014.

[35] R. A. Gustafson and S. C. Milne. Schur functions, Good's identity, and hypergeometric series well poised in $\mathrm{SU}(n)$. *Adv. in Math.*, 48(2):177–188, 1983.

[36] Tatsuyuki Hikita. Affine Springer fibers of type A and combinatorics of diagonal coinvariants. *Adv. Math.*, 263:88–122, 2014.

[37] Verner E. Hoggatt, Jr. and Calvin T. Long. Divisibility properties of generalized Fibonacci polynomials. *Fibonacci Quart.*, 12:113–120, 1974.

[38] Victor Kac and Pokman Cheung. *Quantum calculus*. Universitext. Springer-Verlag, New York, 2002.

[39] Zsófia Kereskényiné Balogh and Michael J. Schlosser. Elliptic stirling numbers of the second and first kind. In preparation.

[40] Donald E. Knuth. Two notes on notation. *Amer. Math. Monthly*, 99(5):403–422, 1992.

[41] Tom H. Koornwinder. Special functions and q-commuting variables. In *Special functions, q-series and related topics (Toronto, ON, 1995)*, volume 14 of *Fields Inst. Commun.*, pages 131–166. Amer. Math. Soc., Providence, RI, 1997.

[42] Christian Krattenthaler. A new matrix inverse. *Proc. Amer. Math. Soc.*, 124(1):47–59, 1996.

[43] Josef Küstner, Michael Schlosser, and Meesue Yoo. Lattice paths and negatively indexed weight-dependent binomial coefficients. Preprint, arXiv:2204.05505, 2022.

[44] Serge Lang. *Elliptic functions*, volume 112 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1987. With an appendix by J. Tate.

[45] Serge Lang. *Complex analysis*, volume 103 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, fourth edition, 1999.

[46] Shu Xiao Li. Personal communication. Algebraic Combinatorics Seminar at the Fields Institute, 2015.

[47] Daniel E. Loeb. Sets with a negative number of elements. *Adv. Math.*, 91(1):64–74, 1992.

[48] Ian G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Classic Texts in the Physical Sciences. The Clarendon Press, Oxford University Press, New York, second edition, 2015.

[49] Anton Mellit. Toric braids and (m, n)-parking functions. *Duke Math. J.*, 170(18):4123–4169, 2021.

[50] Cormac O'Sullivan. Symmetric functions and a natural framework for combinatorial and number theoretic sequences. Preprint, arXiv:2203.03023, 2022.

[51] T. Kyle Petersen. *Eulerian numbers*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser/Springer, New York, 2015. With a foreword by Richard Stanley.

[52] Alfred S. Posamentier and Ingmar Lehmann. *The (fabulous) Fibonacci numbers*. Prometheus Books, Amherst, NY, 2007. With an afterword by Herbert A. Hauptman.

[53] H. S. A. Potter. On the latent roots of quasi-commutative matrices. *Amer. Math. Monthly*, 57:321–322, 1950.

[54] Eric M. Rains. BC_n-symmetric Abelian functions. *Duke Math. J.*, 135(1):99–180, 2006.

[55] Elizabeth Reiland. Combinatorial interpretations of Fibonomial identities. Master's thesis, Harvey Mudd College, Claremont, CA, 2011. Senior Thesis.

[56] Jeffrey B. Remmel and Michelle L. Wachs. Rook theory, generalized Stirling numbers and (p, q)-analogues. *Electron. J. Combin.*, 11(1):Research Paper 84, 48, 2004.

[57] Hjalmar Rosengren. Elliptic hypergeometric functions. In Howard S. Cohl and Mourad E. H. Ismail, editors, *Lectures on Orthogonal Polynomials and Special Functions*, pages 213–279. Cambridge University Press, Cambridge, 2020.

[58] Gian-Carlo Rota, editor. *Finite operator calculus*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1975. With the collaboration of P. Doubilet, C. Greene, D. Kahaner, A. Odlyzko and R. Stanley.

[59] Bruce E. Sagan. *The symmetric group*, volume 203 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 2001.

[60] Bruce E. Sagan and Carla D. Savage. Combinatorial interpretations of binomial coefficient analogues related to Lucas sequences. *Integers*, 10:A52, 697–703, 2010.

[61] Bruce E. Sagan and Joshua P. Swanson. q-Stirling numbers in type B. Preprint, arXiv:2205.14078, 2022.

[62] Bruce E. Sagan and Jordan Tirrell. Lucas atoms. *Adv. Math.*, 374:107387, 25, 2020.

[63] Michael Schlosser. Multidimensional matrix inversions and A_r and D_r basic hypergeometric series. *Ramanujan J.*, 1(3):243–274, 1997.

[64] Michael J. Schlosser. Elliptic enumeration of nonintersecting lattice paths. *J. Combin. Theory Ser. A*, 114(3):505–521, 2007.

[65] Michael J. Schlosser. A noncommutative weight-dependent generalization of the binomial theorem. *Sém. Lothar. Combin.*, 81:Art. B81j, 24, 2020.

[66] Michael J. Schlosser, Koushik Senapati, and Ali K. Uncu. Log-concavity results for a biparametric and an elliptic extension of the q-binomial coefficients. *Int. J. Number Theory*, 17(3):787–804, 2021.

[67] Michael J. Schlosser and Meesue Yoo. An elliptic extension of the general product formula for augmented rook boards. *European J. Combin.*, 58:247–266, 2016.

[68] Michael J. Schlosser and Meesue Yoo. Basic hypergeometric summations from rook theory. In *Analytic number theory, modular forms and* q*-hypergeometric series*, volume 221 of *Springer Proc. Math. Stat.*, pages 677–692. Springer, Cham, 2017.

[69] Michael J. Schlosser and Meesue Yoo. Elliptic extensions of the alpha-parameter model and the rook model for matchings. *Adv. in Appl. Math.*, 84:8–33, 2017.

[70] Michael J. Schlosser and Meesue Yoo. Elliptic rook and file numbers. *Electron. J. Combin.*, 24(1):Paper No. 1.31, 47, 2017.

[71] Michael J. Schlosser and Meesue Yoo. Weight-dependent commutation relations and combinatorial identities. *Discrete Math.*, 341(8):2308–2325, 2018.

[72] Michael J. Schlosser and Meesue Yoo. Elliptic solutions of dynamical Lucas sequences. *Entropy*, 23(2):Paper No. 183, 14, 2021.

[73] Marcel P. Schützenberger. Une interprétation de certaines solutions de l'équation fonctionnelle: $F(x+y) = F(x)F(y)$. *C. R. Acad. Sci. Paris*, 236:352–353, 1953.

[74] Vyacheslav P. Spiridonov. Theta hypergeometric series. In *Asymptotic combinatorics with application to mathematical physics (St. Petersburg, 2001)*, volume 77 of *NATO Sci. Ser. II Math. Phys. Chem.*, pages 307–327. Kluwer Acad. Publ., Dordrecht, 2002.

[75] Vyacheslav P. Spiridonov. Essays on the theory of elliptic hypergeometric functions. *Uspekhi Mat. Nauk*, 63(3(381)):3–72, 2008.

[76] Renzo Sprugnoli. Negation of binomial coefficients. *Discrete Math.*, 308(22):5070–5077, 2008.

[77] Richard P. Stanley. *Enumerative Combinatorics 1*. Cambridge Univ. Press, Cambridge, 1999.

[78] Richard P. Stanley. *Enumerative Combinatorics 2*. Cambridge Univ. Press, Cambridge, 2001.

[79] Ian Stewart and David Tall. *Complex Analysis*. Cambridge University Press, 2 edition, 2018.

[80] Marko Thiel. From Anderson to zeta. *Adv. in Appl. Math.*, 81:156–201, 2016.

[81] Andreas von Ettingshausen. *Die combinatorische Analysis*. Wallishausser Verlag, Wien, 1826.

[82] S. Ole Warnaar. Summation and transformation formulas for elliptic hypergeometric series. *Constr. Approx.*, 18(4):479–502, 2002.

[83] Heinrich Weber. *Elliptische Functionen und algebraische Zahlen*. Vieweg-Verlag, Braunschweig, 1891.