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## Condensation Monads in Low Dimension

verfasst von / submitted by

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# Abstract

Some systems of condensed matter show phase transitions which are not due to symmetry breaking; they are topologically ordered. Such topological orders can be described by higher categories. To account for condensation of these topological phases of matter, the notion of categorical $n$-condensation was defined model independently for weak $n$-categories in [GJ19]. We develop this in detail for $n = 2$.
We introduce 2-condensation monads in a weak 2-category $\mathcal{B}$, which may or may not condense, as categorification of idempotents in a category, which may or may not split. If in all hom-categories of $\mathcal{B}$ all idempotents split, we are able to construct its 2-Karoubi completion $\mathrm{Kar}(\mathcal{B})$ consisting of the 2-condensation monads of $\mathcal{B}$ and 2-condensation bimodules between them. It generalizes the Karoubi completion of an ordinary category: all 2-condensation monads of $\mathrm{Kar}(\mathcal{B})$ condense, and $\mathcal{B}$ and $\mathrm{Kar}(\mathcal{B})$ are 2-equivalent if in $\mathcal{B}$ already all 2-condensation monads condense.
We show that $\mathrm{Kar}(\mathcal{B})$ is 2-equivalent to the 2-category of nonunital, noncounital $\Delta$-separable Frobenius algebras, special Frobenius bimodules and bimodule maps.
We also discuss unital and counital 2-condensation monads which similarly form a 2-category which is 2-equivalent to the well known 2-category $\mathrm{Frob}_\Delta(\mathcal{B})$ of $\Delta$-separable Frobenius algebras, bimodules and bimodule maps. Finally, we show that if in $\mathcal{B}$ all 1-cells have adjoints, then $\mathrm{Kar}(\mathcal{B})$ and $\mathrm{Frob}_\Delta(\mathcal{B})$ are 2-equivalent.

# Abriss

Manche Systeme von kondensierter Materie zeigen Phasenübergänge, die nicht aufgrund von Symmetriebrechung passieren; sie sind topologisch geordnet. Solche topologische Ordnungen können durch höhere Kategorien beschrieben werden. Um die Kondensation dieser topologischen Materiephasen zu beschreiben, wurde in [GJ19] der Begriff von kategorieller $n$-Kondensation für schwache $n$-Kategorien modellunabhängig eingeführt. Wir untersuchen diesen im Detail für $n = 2$.
Wir führen 2-Kondensationsmonaden in einer schwachen 2-Kategorie $\mathcal{B}$ ein, welche kondensieren können aber nicht müssen. Diese sind eine Kategorifizierung von Idempotenten in einer Kategorie, welche spalten können aber nicht müssen. Wenn in allen hom-Kategorien von $\mathcal{B}$ alle Idempotenten spalten, können wir ihre 2-Karoubi-Vervollständigung $\mathrm{Kar}(\mathcal{B})$ konstruieren, bestehend aus den 2-Kondensationsmonaden von $\mathcal{B}$ und deren 2-Kondensationsbimoduln. Diese verallgemeinert die Karubi-Vervollständigung einer normalen Kategorie: alle 2-Kondensationsmonaden von $\mathrm{Kar}(\mathcal{B})$ kondensiern und $\mathcal{B}$ und $\mathrm{Kar}(\mathcal{B})$ sind 2-äquivalent, falls in $\mathcal{B}$ bereits alle 2-Kondensationsmonaden kondensieren.
Wir zeigen, dass $\mathrm{Kar}(\mathcal{B})$ 2-äquivalent zur 2-Kategorie nicht-unitaler, nicht-kounitaler spezieller Frobenius-Algebren, spezieller Frobenius-Bimoduln und Bimodulabbildungen ist.
Weiters besprechen wir unitale und kounitale 2-Kondensationsmonaden, die eine 2-Kategorie bilden, welche zur bekannten 2-Kategorie $\mathrm{Frob}_{\Delta}(\mathcal{B})$ spezieller Frobenius-Algebren, Bimoduln und Bimodulabbildungen 2-äquivalent ist. Letztlich zeigen wir, dass, falls in $\mathcal{B}$ alle 1-Morphismen Adjungierte haben, $\mathrm{Kar}(\mathcal{B})$ und $\mathrm{Frob}_{\Delta}(\mathcal{B})$ 2-äquivalent sind.

# Contents

# 1 Introduction

Ever since the emergence of physics as a scientific discipline, it has been in dynamic interaction with mathematics. On the one hand, mathematics provides the language and the tools used by theoretical physics, so that sometimes only new mathematical concepts enable new physical theories. On the other hand, it often is this need for improved techniques and the questions raised by physics, that drive the development of new mathematical fields. Amazingly, even the most abstract concepts of pure mathematics, seemingly mere inventions in the mathematicians mind and completely removed from material reality, end up useful to describe just that.

This work explores a concept from the mathematical field of higher category theory, which promises to be useful to understand topological phases of matter.

## 1.1 Category Theory

One of the most abstract fields of modern mathematics is category theory: an abstraction of mathematical structures themselves. Samuel Eilenberg and Saunders MacLane founded it in [EM45] to be able to formalize relations between topology and algebra.
Usually a mathematical theory is concerned with certain objects (e.g. vector spaces, groups, topological spaces, ...), and relates them via structure preserving maps (linear transformations, group homomorphisms, continuous functions, ...). There are some very basic properties all of these have in common. Given any two such structure preserving maps $f : A \to B$ and $g : B \to C$ (e.g. two linear transformations between vector spaces $A$, $B$ and $C$) they can be applied after each other to obtain $g \circ f : A \to C$. Usually this is defined element-wise by $(g \circ f)(x) := g(f(x))$, hence the notation in that order. This composition preserves the structure of our theory as well, because both its parts, $f$ and $g$, do. The resulting operation "$\circ$" of composing two maps naturally is associative: if we compose three structure preserving maps, it does not matter if we think about this as doing the first one followed by the composition of second and third, or as starting with the composition of the first two and then do the third:

$$(h \circ g) \circ f = h \circ (g \circ f).$$

Finally, for every object $X$ of our theory there is an identity map $\mathrm{id}_X$ which just leaves $X$ as it is. Clearly, composing this identity map with any structure preserving maps $f : X \to Y$ or $g : Z \to X$ does not change them:

$$f \circ \mathrm{id}_X = f \quad \text{and} \quad \mathrm{id}_X \circ g = g.$$

Category theory now takes these properties as axioms: a category is any collection of objects and so-called morphisms between them (both completely abstract, with no further structure or meaning) with an associative composition operation of morphisms and an identity morphism for each object which leaves any morphism as it is when composed with it. The examples from above give rise to the categories **Vec** of vector spaces and linear transformations (where usually one only considers $\mathbf{Vec}_{\Bbbk}$, the category of vector spaces over a fixed field $\Bbbk$, as there are no linear transformations between vector spaces over different fields), **Grp** of groups and group homomorphisms and **Top** of topological spaces and continuous maps.

This might seem rather trivial: the objects are just sets with additional structure satisfying some axioms and the morphisms are functions between these sets preserving the validity of the axioms. But category theory takes a fundamentally different approach: it suffices to know how all the objects relate to each other. What primarily matters is the category as a whole, not the details of its definition.
The objects of a category do not at all have to be sets with extra structure. There is a category whose objects are logical theorems and whose morphisms are proofs. For any group $G$ there is the category $\mathrm{B}G$ which has only a single object and the elements of $G$ as morphisms; composition is given by group multiplication (which by definition is associative) so that the unit of $G$ serves as the identity morphism for the single object. Another example are bordism categories: roughly, their objects are $n$-dimensional closed manifolds and morphisms are equivalence classes of $(n+1)$-dimensional manifolds (called bordisms) which have the manifolds they map between as boundary. They are fundamental to topological quantum field theory as described in Section 1.2.

The idea that the relations between things matter more than the things themselves, applies to category theory as well. Mathematical theories do not exist independently of each other. For example, in algebraic topology, one associates to each topological space (with chosen base point) its fundamental group. Thus for each object in the category of (pointed) topological spaces we get an object of the category of groups. A crucial property of this association is the following: any (base point preserving) continuous map between two (pointed) topological spaces, induces a group homomorphism between their fundamental groups. This furthermore happens in a way which respects the compositions and identities. We thus have a structure preserving map between categories and those are called functors. A functor $F : \mathcal{C} \to \mathcal{D}$ maps the objects of a category $\mathcal{C}$ to the ones of $\mathcal{D}$ and the morphisms of $\mathcal{C}$ to morphisms between the according objects in $\mathcal{D}$, respecting composition:

$$F(g \circ f) = F(g) \circ F(f)$$

and mapping identities to identities. Functors clearly can be composed element wise, so we can form the category **Cat** which has categories as objects and functors as morphisms.

Interestingly this is not the end of the story. One can reasonably define structure preserving maps between functors. So reasonably in fact, that they are called natural transformations. An example are intertwiners of group representations: a representation of a group $G$ on a $\Bbbk$-vector space $V$ is the same as a functor $\mathrm{B}G \to \mathbf{Vec}_{\Bbbk}$ mapping the single object of $\mathrm{B}G$ to $V$; an intertwiner between two representations is a natural transformation between the corresponding functors.
It was in order to understand such natural transformations in algebraic topology that category theory was formulated by Samuel Eilenberg and Saunders MacLane in 1945.

### 1.1.1 Higher Category Theory

We have seen that there are categories, functors between them and natural transformations between these. Thus we may wish to again do an abstraction of this situation and consider a category with not only objects and morphisms between them but also morphisms between morphisms. This leads to the concept of a strict 2-category. In order to keep track of what kind of morphism we are talking about, we refer to the morphisms between objects as 1-cells, and to the morphisms between those as 2-cells. In an ordinary category, two morphisms can only be equal or different, but in a 2-category 1-cells can be related via 2-cells, for example they may be different but isomorphic. This opens up the possibility to weaken the axioms for a strict 2-category to arrive at the definition of a weak 2-category (or bicategory), as given in Section 2.2. Associativity of the composition for 1-cells is demanded to hold only up to isomorphism. This isomorphism, called the associator, is then asked to satisfy the pentagon axiom which guarantees that the resulting isomorphisms between compositions of more than three 1-cells are all equal. Similarly, the composition of a 1-cell $f$ with the identity 1-cell, is only asked to be isomorphic to $f$ via a unitor 2-cell which has to satisfy the triangle axiom.

An example from algebraic topology is the fundamental groupoid of a topological space: objects are points in the space, 1-cells are paths connecting them and 2-cells are homotopies of paths. An example from algebra (which will play an important role in the following) is the bicategory which has algebras (vector spaces with a linear, associative multiplication) as objects, bimodules between them as 1-cells and their bimodule maps as 2-cells. An example from physics can be sketched as having 2-dimensional theories describing patches of spacetime as objects, defect lines separating these patches as 1-cells and junctions of defect lines as 2-cells. We say some more about this in Section 1.2.1.

Having started counting, there is no reason to stop: by introducing 3-cells between 2-cells and so on up to $n$-cells between $(n-1)$-cells one readily arrives at the notion of a strict $n$-category. It is however far from obvious how to weaken it.

For weak 3-categories one can replace the pentagon axiom of weak 2-categories by introducing a pentagonator 3-cell, but the coherence axioms this has to satisfy are already vastly more complicated. For $n > 4$ this becomes unfeasible. There exist various different approaches to define weak $n$-categories for arbitrary $n$ (some with a possible generalization even to $\infty$-categories), but it is an open problem to tell which is the right one in which situation and how they all relate to each other.

All three examples above can be generalized to $n$-categories: for the fundamental $n$-groupoid one can just keep considering homotopies between homotopies, algebras can be generalized to $E_n$-algebras (cf. [Hau17]), and for $n$-dimensional physical theories one can take hypersurfaces separating patches of spacetime as 1-cells and so on.

### 1.1.2 Categorification

When well-known concepts are generalized from set theory to category theory, or from categories to higher categories, one speaks of categorification. In this context sets are considered to be 0-categories.

The definitions of many algebraic notions (like group or algebra) can be phrased entirely in the language of category theory. By replacing the underlying set with an object in any category $\mathcal{C}$, these notions can be internalized to $\mathcal{C}$. Considering them internal to a higher category and weakening the defining axioms (similar to how strict 2-categories are weakened to bicategories) one arrives at the categorification. In this sense, considering groups internal to the 2-category of categories, functors and natural transformations gives rise to the notion of a 2-group.

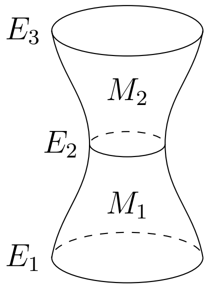
One of our main topics is to give a categorification of the notion of splitting idempotent. An idempotent is a morphism $e : X \to X$ which stays the same when applied multiple times, i.e. $e \circ e = e$. It splits, if there are $f : X \to Y$ and $g : Y \to X$ such that $g \circ f = e$ and $f \circ g = \mathrm{id}_Y$. A classic example are projections in vector spaces: a projection onto a $k$-dimensional subspace of an $n$-dimensional vector space, can be split by first doing the appropriate surjective map onto the $k$-dimensional vector space (viewed as a different object of **Vec**), and then embedding it back to the $n$-dimensional vector space. Clearly, first embedding and then surjecting gives the identity. It turns out that projections are the only idempotents in **Vec**, so that all its idempotents split. This is not the case for arbitrary categories. What is true however, is that every category $\mathcal{C}$ can be embedded into its so called Karoubi completion $\mathrm{Kar}(\mathcal{C})$ in which all idempotents do split.
We will categorify idempotents to 2-condensation monads in a bicategory $\mathcal{B}$ (defined in Section 3.1) and find a generalization of the Karoubi completion which is equivalent to the bicategory of (a certain kind of) algebras, bimodules and bimodule maps

internal to $\mathcal{B}$.

## 1.2 Topological Quantum Field Theory

Despite their outstanding success, the methods of quantum field theory still lack a solid mathematical foundation. In particular there is no sound definition of the path integral.
Classical mechanics is concerned with point particles moving through space which is described as some $n$-dimensional manifold. Classical field theory on the other hand works with fields which are simultaneously defined on all of space. To be in accordance with the theory of relativity, these fields are considered to be functions defined on some $(n+1)$-dimensional spacetime manifold. A field at a certain time is then given by the restriction of the function to a spacelike slice of the manifold. Quantum mechanics replaces the direct description of point particles by an analysis of their spaces of states by the means of linear algebra. Accordingly, in quantum field theory, we have for each spacelike slice of spacetime a Hilbert space describing the states of the field. In both cases, time evolution is given by a linear transformation between vector spaces of states and can be described in terms of a path integral. Actually, quantum mechanics can be interpreted as quantum field theory for a $(0+1)$-dimensional spacetime, and only for this case has it been possible to rigorously define the path integral.

There exist different approaches to axiomatize the ideas of quantum field theory in order to give them a precise mathematical footing. The one of topological quantum field theory (TQFT for short) can be outlined as a structure preserving map from spacetimes to algebra, mapping spacelike slices of a spacetime to spaces of states, and chunks between such slices to a time evolution operator. Using the language of category theory introduced above, we can say a TQFT is a functor $\mathcal{Z}$ from a category of spacetimes to some algebraic category (usually the one of complex vector spaces $\mathbf{Vec}_{\mathbb{C}}$). Given two consecutive chunks of spacetime $M_1$ and $M_2$ so that $M_1$ is between spacial slices $E_1$ and $E_2$ and $M_2$ is between $E_2$ and $E_3$, the time evolution operator corresponding to the whole chunk between $E_1$ and $E_3$ (i.e. $M_1$ and $M_2$ glued along $E_2$) should be the same as applying first $\mathcal{Z}(M_1)$ and then $\mathcal{Z}(M_2)$. Because of this locality property of the time evolution, the TQFT $\mathcal{Z}$ is a functor (a symmetric monoidal one to be precise).




$$\mapsto \mathcal{Z}(M_2 \circ M_1) = \mathcal{Z}(M_2) \circ \mathcal{Z}(M_1)$$

It remains to define what the categories of spacetimes are supposed to be. They are realized as bordism categories: their objects are closed $n$-dimensional manifolds (representing spacial slices of spacetime) and the morphisms between them are bordism classes. A bordism from $E_1$ to $E_2$ (both $n$-dimensional manifolds) is an $(n+1)$-dimensional manifold $M$ together with (sufficiently regular) embeddings of $E_1$ and $E_2$ into the boundary $\partial M$ of $M$, such that their images split $\partial M$ into a disjoint union of an ingoing and an outgoing part. Two bordisms belong to the same bordism class, if there is a diffeomorphism between them which (separately) keeps ingoing and outgoing boundaries fixed. Due to this definition up to diffeomorphism, possibly relevant geometric properties of the manifolds (e.g. a metric, its smooth structure, etc.) are discarded so that bordism categories only contain topological information. Hence the name *topological* quantum field theory. Restrictions on the kinds of manifolds to be considered give rise to different flavours of bordism categories, e.g. demanding orientability gives $\mathrm{Bord}^{\mathrm{or}}_{n+1,n}$ which consists of closed, oriented manifolds and oriented bordisms.

TQFT is by no means just a toy model for proper quantum field theory. On the one hand it has various applications in different fields of pure mathematics. Foremost it provides a powerful framework to study topological invariants of manifolds. It was in this vain that it was introduced, inspired by the work of Graeme Segal on conformal field theory [Seg88] and Edward Witten, who was able to use field theory to derive results about the topology of low dimensional manifolds [Wit88], by Michael Atiyah in [Ati88]. There he writes:

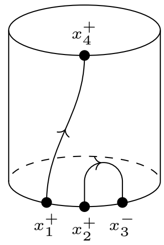
> "The new feature of present developments is that links are being established between *quantum* physics and *topology*. It is no longer the purely *local* aspects that are involved but their *global* counterparts. In a very general sense this should not be too surprising. Both quantum theory and topology are characterized by discrete phenomena emerging from a continuous background."[1]

[1][Ati88], p.175

On the other hand, there are actual, real world situations which can be described by TQFTs, like the quantum Hall effect, as briefly discussed in Section 1.3.

### 1.2.1 Defect TQFT and Generalized Orbifolds

One way to increase the information contained in a TQFT is to enlarge the bordism category it is defined on. The idea of defect TQFT, as developed in [DKR11], is to include labelled submanifolds which correspond to domain walls separating areas of spacetime which are described by different theories (e.g. matter in different phases). In two dimensions, given two sets of labels $D_1$ and $D_2$, we define the category of bordisms with defects, $\mathrm{Bord}_{2,1}^{\mathrm{def}}(D_2, D_1)$ as follows. Objects are closed, oriented, 1-dimensional manifolds (i.e. disjoint unions of circles) decorated with a finite number of marked points, each labelled by an element of $D_1$ and an orientation in $\{+, -\}$. The open intervals between the marked points are labelled by elements of $D_2$. Morphisms between these objects are (equivalence classes of) bordisms $M$, decorated with an oriented 1-dimensional submanifold consisting of non-intersecting defect lines, each labelled by an element of $D_1$, such that the boundaries of the defect lines are contained in the boundary of $M$ and meet marked points with the same label and matching orientation there (i.e. a defect line oriented away/towards the boundary has to meet a marked point with orientation $+/-$ on the ingoing boundary and vice versa on the outgoing boundary). The patches between the defect lines are labelled with elements of $D_2$ in a way that matches the $D_2$-labels of the boundary.

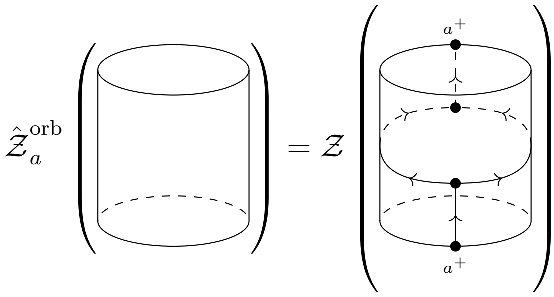



A (symmetric monoidal) functor $\mathrm{Bord}_{2,1}^{\mathrm{def}}(D_2, D_1) \to \mathbf{Vec}_{\Bbbk}$, into the category of vector spaces over a field $\Bbbk$, is called a 2-dimensional defect TQFT.

Given such a defect TQFT $\mathcal{Z}$, we can construct the corresponding bicategory of worldsheet phases $\mathcal{D}_{\mathcal{Z}}$. Its objects are the elements of $D_2$; as they are labelled patches of spacetime they are interpreted as different worldsheet phases. The 1-cells are given by all possible combinations of defect lines between them, which can be composed by horizontal concatenation. The space of 2-cells between them is given by applying $\mathcal{Z}$ to a single circle marked with points labelled in the same way as the defect lines corresponding to the 1-cells they map between.

This framework allows the generalization of the technique of orbifolding a symmetry, common in QFT and string theory. The starting point is a defect TQFT $\mathcal{Z} : \mathrm{Bord}^{\mathrm{def}}_{2,1}(D_2, D_1) \to \mathbf{Vec}_{\Bbbk}$ and a $\Delta$-separable, symmetric Frobenius algebra $(a, \mu, \Delta)$ internal to the bicategory of worldsheet phases $\mathcal{D}_{\mathcal{Z}}$; that is a line defect $a$ together with a trivalent junction $\mu$ with two ingoing and one outgoing $a$-line and a trivalent junction $\Delta$ with one ingoing and two outgoing $a$-lines satisfying some relations (i.e. the ones given in Section 2.3 and symmetry which is compatibility with adjoints). This orbifolding defect may (but does not have to) stem from the action of a symmetry group on a theory, in which case the generalized orbifolding procedure corresponds to the usual one. From $(a, \mu, \Delta)$ we can define a new TQFT $\mathcal{Z}^{\mathrm{orb}}_a : \mathrm{Bord}_{2,1} \to \mathbf{Vec}_{\Bbbk}$, the orbifolding of $\mathcal{Z}$ by $a$.

The details of this construction can be outlined as follows: First construct $\hat{\mathcal{Z}}^{\mathrm{orb}}_a$ which maps a disjoint union of circles to $\mathcal{Z}$ applied to the same number of circles each decorated with a single point labelled by $(a, +)$. A bordism is mapped to $\mathcal{Z}$ applied to the same bordism decorated with a network consisting of $a$-lines and the trivalent vertices $\mu$ and $\Delta$, matching the $a$-labelled points in the ingoing and outgoing boundary.




For any such network which is fine enough (i.e. is the Poincaré dual of a nontrivial triangulation), this gives the same result (because of the relations satisfied by $(a, \mu, \Delta)$) and therefore $\hat{\mathcal{Z}}^{\mathrm{orb}}_a(M)$ always is an idempotent. For an object $U \in \mathrm{Bord}_{2,1}$, we can thus define:

$$\mathcal{Z}^{\mathrm{orb}}_a(U) = \mathrm{im}(\hat{\mathcal{Z}}^{\mathrm{orb}}_a(C_U))$$

where $C_U$ is the cylinder over $U$ considered as a bordism $U \to U$. For any bordism $M : U \to V$ we define:

$$\mathcal{Z}^{\mathrm{orb}}_a(M) = f_V \circ \hat{\mathcal{Z}}^{\mathrm{orb}}_a(M) \circ g_U$$

for splittings of the idempotents $\hat{\mathcal{Z}}^{\mathrm{orb}}_a(C_i) = g_i \circ f_i$.

As this can be done for any $(a, \mu, \Delta)$, the bicategory of $\Delta$-separable, symmetric Frobenius algebras internal to $\mathcal{D}_{\mathcal{Z}}$ describes all possible TQFTs that can be obtained from $\mathcal{Z}$ by orbifolding and is, in this context, called the orbifold completion of $\mathcal{D}_{\mathcal{Z}}$.

The theory of generalized orbifolds was introduced in [Frö+10] and developed as outlined here in [CR16]. The latter work also discusses the situation where the algebra structure of the orbifolding defect is not required to be symmetric. This way one finds the equivariant completion which is the bicategory of internal $\Delta$-separable Frobenius algebras. It is precisely this bicategory of algebras that the bicategory of unital and counital 2-condensation monads constructed in Section 3.6 will turn out to be equivalent to.

## 1.3 Topological Phases of Matter

A third driving factor in the interplay between physics and mathematics, which has not been addressed so far, is given by experimental and technological applications which create the need for their theoretical description.

An ancient question of physics is to find out what different states of matter there are. Everybody knows that some materials, like water, can be in a solid, a liquid and a gaseous phase. But these are by no means all possible phases of matter: there are plasmas, ferromagnets can be in a magnetized and an unmagnetized phase, solids in different crystalline configurations are to be considered different phases and there are more exotic phases like superconductors and superfluids. If a material changes from one phase to another one speaks of a phase transition, everyday examples are melting and freezing.

A successful theoretical description of such phase transitions was given by Lew Landau in the 1930ies. The basic idea of his theory of spontaneous symmetry breaking can be seen in the case of freezing: a material in a liquid phase is unordered and therefore its description is rotationally symmetric. If it freezes to form a crystal described by a specific lattice configuration (of its atoms), there are distinguished directions breaking the rotational symmetry. Only the symmetries of the lattice remain after the phase transition. That happens suddenly at a critical temperature $T_c$, so one speaks of spontaneous symmetry breaking. The system is described by an order parameter which is non-zero only below the critical temperature. After the phase transition the ordered state breaks a symmetry of the system.

For a long time it was believed that Landau theory describes all possible phases of matter and their transitions. It turned out that this is not the case (but it might be possible to do so using generalized symmetries as discussed in [McG22]). An important example occurs in the so called $X$-$Y$-model which describes the magnetization of a 2-dimensional magnet by the alignment of spin vectors sitting on a 2-dimensional lattice. The configuration may contain a vortex, i.e. the spin vectors may rotate around a point, in the sense that if we follow a closed curve, the directions of the spins rotate (at least) a full circle. A configuration with a vortex (i.e. with non-zero vorticity) cannot be transformed into one without a vortex by locally changing the directions of the spins. Vorticity is a global, topological property of the configuration, it cannot be seen from looking only at a few lattice sites. John Kosterlitz and David Thouless describe in [KT73] a phase transition of a 2-dimensional magnet caused by the breaking up of vortex-antivortex pairs (which have zero vorticity for macroscopic contours and can be created by thermal fluctuations also at low temperatures) at a critical temperature. This transition does not break any symmetry, the corresponding order parameter depends on a topological property of the system so one speaks of topological order.

Of similar importance was the discovery of the quantum Hall effect: the Hall conductance occurring in a 2-dimensional conductor placed in a perpendicular magnetic field (at very low temperatures), can only take integer multiples of a constant with an accuracy far greater than the thermal fluctuations. Mahito Kohmoto explained in [Koh85], that occurs because the conductance depends only on the first Chern number (a classical topological invariant which always is an integer) of a principal $U(1)$-bundle related to the state.

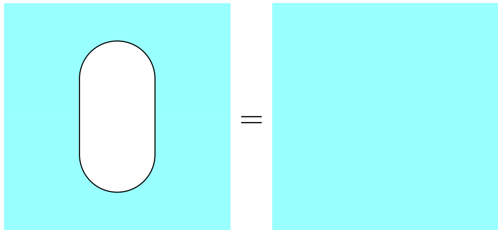
Since then many more such topological phases of matter have been found, including superfluids and (high $T_c$) superconductors. The description of their phases depends on global properties which were named topological order (originally in [Wen89]) in analogy to the order parameter of Landau theory. They are important to understand long-range entanglement (cf. [CGW10]) and promise to have applications in quantum information theory.

More concretely, topological orders are equivalence classes of (ground states of) gapped quantum systems. A system is gapped if it requires a non-zero amount of energy to create exited states from the ground states (there is an energy gap between states) and it usually has a low energy (infrared limit) effective description by a TQFT.

Gapped phases of matter in $n$ spacetime dimensions can be assembled into an $n$-category where, similar to the bicategory of worldsheet phases described above, objects are different phases, 1-cells are $(n-1)$-dimensional topological defects (or gapped interfaces) between them and higher cells are defects between those. In this setting, Davide Gaiotto and Theo Johnson-Freyd give in [GJ19] the following formalization of the concept of condensation of one phase to another. A condensation of a phase $X$ onto a phase $Y$ consists of an interface $f$ from $X$ to $Y$ and an interface $g$ from $Y$ to $X$ together with a condensation of the composite interface $f\,g$ between $Y$ and $Y$ to the trivial interface from $Y$ to $Y$. For this inductive definition to make sense one declares that the trivial interface between 1-dimensional phases is a condensation.

This formalization includes the possibility to arbitrarily create and destroy bubbles of the phase $X$ inside the phase $Y$, due to the topological nature of the defects.

In two dimensions this can be depicted as follows:



Here the phase $X$ is depicted as white area, the phase $Y$ as cyan area and the topological defect between them as black line. Later this equation will be interpreted as a string diagram (cf. Section 2.2.1) and will play a crucial role in this work, showing that the formalization of condensation of 2-dimensional topological phases of matter gives a categorification of splitting idempotents. Therefore a 2-category describing 2-dimensional topological phases of matter should be 2-Karoubi complete. What that means and how to construct the 2-Karoubi completion of a given 2-category is the aim of this thesis.

The study of topological orders actually has applications in the construction of fault tolerant quantum computers (cf. [Kit03]), closing the loop of fundamental physics, abstract mathematics and potentially useful technology.

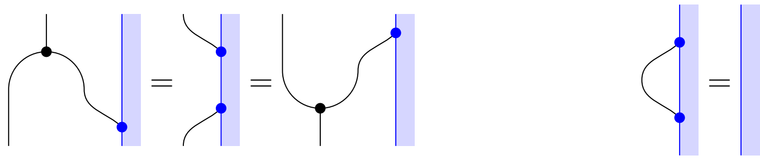
## 1.4 Outline and Summary of Results

In Section 2.1 the notions of category, functor and natural transformation sketched above are made precise. In Section 2.1.1 we review the concept of splitting idempotent and retraction for arbitrary categories $\mathcal{C}$ which are to be categorified in the following. We construct the well known Karoubi completion $\mathrm{Kar}(\mathcal{C})$ of $\mathcal{C}$ and show that in $\mathrm{Kar}(\mathcal{C})$ all idempotents split and that $\mathcal{C}$ embeds into $\mathrm{Kar}(\mathcal{C})$ (Proposition 2.12) and that this embedding is an equivalence if in $\mathcal{C}$ all idempotents split. Furthermore limits and colimits are defined in Section 2.1.2, in particular the coequalizer.

In Section 2.2 we review the notion of a (weak) 2-category, as well as 2-functors, adjunctions and internal equivalences.

The calculus of string diagrams, used extensively throughout the thesis, is introduced in Section 2.2.1.

Section 2.3 deals with algebras and their modules internal to a 2-category, including nonunital cases, coalgebras and comodules. We define the new notion of a *special Frobenius module*, as a module (and comodule) of an algebra (and coalgebra) which satisfies relations analogous to the ones defining a $\Delta$-separable Frobenius algebra. For a left module these read as follows (cf. Definition 2.38):



For modules between $\Delta$-separable Frobenius algebras (with unit) we establish the existence of a canonical coaction which makes it a special Frobenius module (Proposition 2.46).

Section 3 is the main part of the thesis: we make precise the notion of $n$-condensation, introduced model independently in [GJ19], in complete detail for $n = 2$. In Section 3.1 we define a 2-condensation in a 2-category $\mathcal{B}$, the generalization of a retract, as a functor from the walking 2-condensation $\spadesuit_2$ into $\mathcal{B}$ (Definition 3.1). An idempotent then corresponds to a 2-condensation monad, defined as a functor from a full subcategory $\clubsuit_2 \subset \spadesuit_2$. The generalization of an idempotent that splits, is a 2-condensation monad that condenses i.e. as a functor it can be lifted from $\clubsuit_2$ to all of $\spadesuit_2$.

Section 3.2 establishes the notion of 2-condensation bimodule between 2-condensation monads, which is similarly defined as a functor $\diamondsuit_2 \to \mathcal{B}$ (in Definition 3.6). Corollaries 3.5 and 3.8 establish the corresponding algebraic notion internal to $\mathcal{B}$: a 2-condensation monad gives rise to a nonunital, noncounital $\Delta$-separable Frobenius algebra and a 2-condensation bimodule to a special Frobenius bimodule between the corresponding algebras.

In Section 3.3 we introduce another small 2-category, the walking tensor product of 2-condensation bimodules $\heartsuit_2$, to define a natural composition of 2-condensation bimodules, which indeed corresponds to the relative tensor product of the associated bimodules (Proposition 3.16). The 2-categories $\spadesuit_2$, $\clubsuit_2$, $\diamondsuit_2$ and $\heartsuit_2$ are named after the French suits of playing cards[2] in continuation of [GJ19] and in reference to their shapes which are summarized in Remark 3.13.

In Section 3.4 we then go through all details of how to assemble all of this into the 2-category $\mathrm{Kar}(\mathcal{B})$ in order to prove the following theorem:

**Theorem 3.17.** *Given a 2-category $\mathcal{B}$ for which all hom-categories $\mathcal{B}(X, Y)$ are Karoubi complete (cf. Definition 2.10), there is a 2-category* $\mathrm{Kar}(\mathcal{B})$ *which has the 2-condensation monads of $\mathcal{B}$ (cf. Definition 3.3) as objects, their 2-condensation bimodules (cf. Definition 3.6) as 1-cells and the relative tensor product (cf. Definition 3.15) as horizontal composition.*

[2] If you are familiar with playing cards, you may understand the use of the letters $A$, $K$, $q$, $j$, $\tau$ and $\nu$ (in that order) for a walking 2-condensation.

We show that the units of $\mathrm{Kar}(\mathcal{B})$ are strict in Section 3.4.1, find a natural choice of 2-cells which corresponds to bimodule maps in Section 3.4.2 and construct its associator in Section 3.4.3, explicitly proving its naturality and the pentagon axiom.

The hence established 2-category $\mathrm{Kar}(\mathcal{B})$ can be seen to be 2-equivalent to the 2-category of nonunital, noncounital $\Delta$-separable Frobenius algebras and special Frobenius modules between them as discussed in Remark 3.32.

In Section 3.5 we then show that $\mathrm{Kar}(\mathcal{B})$ deserves the name 2-Karoubi completion by proving the following generalizations of what holds for the usual Karoubi completion of a 1-category:

**Proposition 3.34.** *For a 2-category $\mathcal{B}$ for which all hom-categories are idempotent complete, the canonical inclusion $J : \mathcal{B} \to \mathrm{Kar}(\mathcal{B})$ is a well-defined, strict, fully faithful 2-functor.*

**Proposition 3.35.** *If a 2-category $\mathcal{B}$, for which all hom-categories are idempotent complete, has all condensates, then the canonical inclusion $J : \mathcal{B} \to \mathrm{Kar}(\mathcal{B})$ is a 2-equivalence.*

**Theorem 3.37.** *The 2-Karoubi completion $\mathrm{Kar}(\mathcal{B})$ of a 2-category $\mathcal{B}$, all of whose hom-categories are idempotent complete, has all condensates.*

Section 3.6 introduces unital and counital 2-condensation monads by appropriately adding an adjunction in $\spadesuit_2$, leading to a 2-category $\mathrm{Kar}^{\mathbb{1},\mathbb{I}}(\mathcal{B})$ which is equivalent to the well known 2-category of $\Delta$-separable Frobenius algebras internal to $\mathcal{B}$, as explained in Remark 3.44. The last theorem we prove is the following, from which it follows that $\mathrm{Kar}^{\mathbb{1},\mathbb{I}}(\mathcal{B}) \cong \mathrm{Kar}(\mathcal{B})$ if in $\mathcal{B}$ all 1-cells have adjoints.

**Theorem 3.46.** *Given a 2-condensation monad $M : \clubsuit_2 \to \mathcal{B}$ in a 2-category $\mathcal{B}$ for which all hom-categories are idempotent complete, if the underlying 1-cell $a := M(e)$ admits a right adjoint, then $M$ is equivalent as an object of $\mathrm{Kar}(\mathcal{B})$ to a unital 2-condensation monad and to a counital 2-condensation monad.*

Finally Section 4 briefly touches upon the general $n$-dimensional case and states expectations on how to generalize what we did in Section 3 to $n = 3$, which happens to be vastly more cumbersome due to the far increased complexity of 3-categories compared to 2-categories.

# 2 Preliminaries

We introduce the concepts we want to work with, giving little more than the definitions which are actually needed. We begin with usual categories in Section 2.1 including (co)limits and the notions of retract and Karoubi completion which we later want to generalize to 2-categories (introduced in Section 2.2). Finally Section 2.3 defines different types of algebras internal to 2-categories and their modules.

## 2.1 Categories

A standard reference for category theory is [Lan78] by Saunders Mac Lane, one of the founders of the field.

**Definition 2.1** (Category)**.** A *category* $\mathcal{C}$ consists of:

- a class $\mathrm{Ob}(\mathcal{C})$ whose elements are called *objects*,
- for all $A, B \in \mathrm{Ob}(\mathcal{C})$ a class $\mathrm{Hom}_{\mathcal{C}}(A, B)$ whose elements are called *morphisms* (*from $A$ to $B$*),
- for all $A, B, C \in \mathrm{Ob}(\mathcal{C})$ a map

  $$\circ : \mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$$

  called *composition*,
- for all $A \in \mathrm{Ob}(\mathcal{C})$ there is a morphism $\mathrm{id}_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$ called *identity* (*on $A$*),

such that:

- for all $A, B, C, D \in \mathrm{Ob}(\mathcal{C})$ and all $f \in \mathrm{Hom}_{\mathcal{C}}(A, B), g \in \mathrm{Hom}_{\mathcal{C}}(B, C), h \in \mathrm{Hom}_{\mathcal{C}}(C, D)$ it holds that:

  $$h \circ (g \circ f) = (h \circ g) \circ f \tag{2.1}$$

- for all $A, B, B' \in \mathrm{Ob}(\mathcal{C})$ and all $f \in \mathrm{Hom}_{\mathcal{C}}(A, B), f' \in \mathrm{Hom}_{\mathcal{C}}(B', A)$ it holds that:

  $$f \circ \mathrm{id}_A = f \quad \text{and} \quad \mathrm{id}_A \circ f' = f' \tag{2.2}$$

A morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ is called *isomorphism*, if there exists $g \in \mathrm{Hom}_{\mathcal{C}}(B, A)$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$.

Note the abuse of notation using the same symbol for all the different composition maps.

We will also refer to this usual notion of category as 1-category in the following in order to distinguish it from the one of 2-category introduced in Section 2.2.

**Remark 2.2.** *Categories can be described using generators and relations. For a directed multigraph $G$, there is the category freely generated by $G$ which has the vertices of $G$ as objects, and for each edge between two vertices a morphism between the corresponding objects, as well as all their possible compositions (i.e. one for each path in $G$) and the necessary identities. If we want some of these morphisms to be equal, we can demand this by adding it as a relation, and construct the category generated by generators and relations as a quotient (the details can be found in [Lan78]).*

**Example 2.3.** *The trivial category, denoted* $\mathbf{1}$, *consists of a single object* $\mathrm{Ob}(\mathbf{1}) = \{\bullet\}$ *and a single morphism* $\mathrm{Hom}_{\mathbf{1}}(\bullet, \bullet) = \{\mathrm{id}_\bullet\}$ *with the only possible composition* $\mathrm{id}_\bullet \circ \mathrm{id}_\bullet = \mathrm{id}_\bullet$.

**Example 2.4.** *Given two categories $\mathcal{C}$ and $\mathcal{D}$, we can define their product category $\mathcal{C} \times \mathcal{D}$ whose object and morphism classes consist of the corresponding Cartesian products:* $\mathrm{Ob}(\mathcal{C} \times \mathcal{D}) = \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{D})$, $\mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((A, A'), (B, B')) = \mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{D}}(A', B')$ *and composition works component wise:* $(g, g') \circ (f, f') = (g \circ f, g' \circ f')$.

It turns out that categories themselves form the objects of a category. The appropriate notion of morphism between categories is given by the one of a functor.

**Definition 2.5** (Functor)**.** A *functor* $F : \mathcal{C} \to \mathcal{D}$ between two categories $\mathcal{C}$ and $\mathcal{D}$ consists of:

- a map $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$,
- for all $A, B \in \mathrm{Ob}(\mathcal{C})$ maps $F : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$

such that:

- for all $A, B, C \in \mathrm{Ob}(\mathcal{C})$ and all $f \in \mathrm{Hom}_{\mathcal{C}}(A, B), g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$ it holds that:
$$F(g \circ f) = F(g) \circ F(f) \tag{2.3}$$
- for all $A \in \mathrm{Ob}(\mathcal{C})$:
$$F(\mathrm{id}_A) = \mathrm{id}_{F(A)} \tag{2.4}$$

A functor is called *full*, if all maps on morphisms are surjective and *faithful* if they are injective. A functor is called an *embedding* if it is faithful and injective on objects.

Note the abuse of notation using the same symbol for the functor as well as all its components.

**Example 2.6.** *For each category $\mathcal{C}$ there is an identity functor, denoted always as $id : \mathcal{C} \to \mathcal{C}$, which is the identity map on objects as well as morphisms.*
*For each element $C \in \mathcal{C}$ there is the constant functor on $C$ $\Delta_C : \mathcal{D} \to \mathcal{C}$ defined on any category $\mathcal{D}$ sending all its objects to $C$ and all morphisms to $\mathrm{id}_C$.*

Fixing two categories $\mathcal{C}$ and $\mathcal{D}$, the functors between them are again objects of a category, the morphisms between them are called natural transformations and defined as follows. We will be able to recognize them as the 2-cells in the strict 2-category of categories, functors and natural transformations in the next section.

**Definition 2.7** (Natural transformation)**.** A *natural transformation* $\eta : F \to G$ between two functors $F, G : \mathcal{C} \to \mathcal{D}$ between the same two categories $\mathcal{C}$ and $\mathcal{D}$ consists of a morphism $\eta_A \in \mathrm{Hom}_{\mathcal{D}}(F(A), G(A))$ for all $A \in \mathrm{Ob}(\mathcal{C})$, such that for all $A, B \in \mathrm{Ob}(\mathcal{C})$ and all $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ the following diagram commutes:

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\eta_A"] & F(B) \arrow[d, "\eta_B"] \\ G(A) \arrow[r, "G(f)"] & G(B) \end{tikzcd} \tag{nat.}$$

$\eta$ is called a *natural isomorphism* if for all $A \in \mathrm{Ob}(\mathcal{C})$, $\eta_A$ is an isomorphism.

**Remark 2.8.** *Definition 2.7 includes the concept of a commutative diagram which is widely used in category theory. A diagram (a directed multigraph with objects at its vertices and morphisms between them at the edges) is said to commute, if composing the morphisms along any path between two vertices gives the same result.*
*For the square (nat.) above, that just means:*

$$\eta_B \circ F(f) = G(f) \circ \eta_A$$

### 2.1.1 Karoubi Completion

**Definition 2.9** (Idempotent)**.** A morphism $e \in \mathrm{Hom}_{\mathcal{C}}(A, A)$ is called *idempotent* if $e \circ e = e$.

**Definition 2.10** (Retract)**.** Given objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, a *retract of* $X$ *onto* $Y$ consists of morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$ such that $f \circ g = \mathrm{id}_Y$. In this case we also say $g$ is a *section* of $f$, $f$ is a *retraction* of $g$ and $(f, g)$ is a *splitting* of the idempotent $e = g \circ f$ with *image* $Y$.
A category in which there is a splitting of every idempotent is called *Karoubi complete* (or *idempotent complete*).

The objects and morphisms participating in a retract can be visualized in the following commutative diagram:

$$\begin{tikzcd}
X \arrow[loop above, "e"] \arrow[d, bend right, "f"'] \\
Y \arrow[loop below, "\mathrm{id}_Y"] \arrow[u, bend right, "g"']
\end{tikzcd}$$

The image of a splitting idempotent is unique up to isomorphism, as guaranteed by the following proposition.

**Proposition 2.11.** *If an idempotent* $e$ *in a category* $\mathcal{C}$ *has two different splittings* $(f, g)$ *and* $(f', g')$ *with images* $Y$ *and* $Y'$ *respectively, then there is an isomorphism in* $\mathrm{Hom}_{\mathcal{C}}(Y, Y')$.

*Proof.* By assumption we know that $g \circ f = g' \circ f' = e$ as well as $f \circ g = \mathrm{id}_Y$ and $f' \circ g' = \mathrm{id}'_Y$. From this we can show that the desired isomorphism and its inverse are given by $f' \circ g \in \mathrm{Hom}_{\mathcal{C}}(Y, Y')$ and $f \circ g' \in \mathrm{Hom}_{\mathcal{C}}(Y', Y)$:

$$f \circ g' \circ f' \circ g = f \circ e \circ g = f \circ g \circ f \circ g = \mathrm{id}_Y \circ \mathrm{id}_Y = \mathrm{id}_Y$$

$$f' \circ g \circ f \circ g' = f' \circ e \circ g' = f' \circ g' \circ f' \circ g' = \mathrm{id}'_Y \circ \mathrm{id}'_Y = \mathrm{id}'_Y$$

$\square$

**Proposition 2.12** (Karoubi Completion)**.** *For every category* $\mathcal{C}$*, there is a Karoubi complete category* $\mathrm{Kar}(\mathcal{C})$*, called its Karoubi completion, into which* $\mathcal{C}$ *embeds fully.*

*Proof.* $\mathrm{Kar}(\mathcal{C})$ can be constructed as follows:

$$\mathrm{Ob}(\mathrm{Kar}(\mathcal{C})) = \{(X, e), X \in \mathrm{Ob}(\mathcal{C}), e \in \mathrm{Hom}_{\mathcal{C}}(X, X) | e \circ e = e\}$$
$$\mathrm{Hom}_{\mathrm{Kar}(\mathcal{C})}((X, e), (X', e')) = \{(e', f, e), f \in \mathrm{Hom}_{\mathcal{C}}(X, X') | e' \circ f = f = f \circ e\}$$

identities are given by

$$\mathrm{id}_{(X,e)} = (e, e, e)$$

and composition is induced from $\mathcal{C}$

$$(e'', g, e') \circ (e', f, e) = (e'', g \circ f, e)$$

Then clearly $\mathcal{C} \to \mathrm{Kar}(\mathcal{C})$: $X \mapsto (X, \mathrm{id}_X)$, $f \mapsto (\mathrm{id}_Y, f, \mathrm{id}_X)$ is a full embedding (i.e. the functor is full and an embedding according to Definition 2.5).
Now given any idempotent $(e, m, e) \in \mathrm{Hom}_{\mathrm{Kar}(\mathcal{C})}((X, e), (X, e))$ we can construct the following splitting:

$$\begin{tikzcd} (X,e) \arrow[loop above, "{(e,m,e)}"] \arrow[d, bend right, "{(m,m,e)}"'] \\ (X,m) \arrow[u, bend right, "{(e,m,m)}"'] \arrow[loop below, "{(m,m,m)}"] \end{tikzcd} \tag{2.5}$$

which follows from the fact that in $\mathrm{Hom}_{\mathcal{C}}(X, X)$ we have $m \circ m = m$ and $m \circ e = m = e \circ m$. □

Using the notion of equivalence, which is the appropriate generalization of isomorphism between sets to categories, defined as internal equivalence in the 2-category of categories, functors and natural definitions as defined in Definition 2.23, we can proof the following.

**Proposition 2.13.** *If a category $\mathcal{C}$ is Karoubi complete, the embedding into* $\mathrm{Kar}(\mathcal{C})$ *is an equivalence.*

*Proof.* To prove this, we need to find for every object $(X, e) \in \mathrm{Kar}(\mathcal{C})$ an isomorphisms to an object in the image of the embedding. From the fact that $e$ is an idempotent in $\mathcal{C}$, which splits because $\mathcal{C}$ is Karoubi complete by assumption, we get the following commutative diagram:

$$\begin{tikzcd} X \arrow[loop above, "e"] \arrow[d, bend right, "f"'] \\ Y \arrow[u, bend right, "g"'] \arrow[loop below, "\mathrm{id}_Y"] \end{tikzcd} \tag{2.6}$$

From these morphisms we can now build a commutative diagram in $\mathrm{Kar}(\mathcal{C})$:

$$\begin{tikzcd}
(X,e) \arrow[loop above, "{(e,e,e)}"] \arrow[d, bend right, "{(\mathrm{id}_Y,f,e)}"'] \\
(Y,\mathrm{id}_Y) \arrow[u, bend right, "{(e,g,\mathrm{id}_Y)}"'] \arrow[loop below, "{(\mathrm{id}_Y,\mathrm{id}_Y,\mathrm{id}_Y)}"]
\end{tikzcd} \tag{2.7}$$

To see that $(\mathrm{id}_Y, f, e)$ and $(e, g, \mathrm{id}_Y)$ indeed are morphisms of $\mathrm{Kar}(\mathcal{C})$ we need to check that $\mathrm{id}_Y \circ f = f$ and $g \circ \mathrm{id}_Y = g$ which is trivial, as well as $f \circ e = f$ and $e \circ g = g$ which follows from (2.6):

$$f \circ e = f \circ g \circ f = \mathrm{id}_Y \circ f = f$$

$$e \circ g = g \circ f \circ g = g \circ \mathrm{id}_Y = g$$

Commutativity of (2.7) follows from commutativity of (2.6). As the automorphisms on the top and the bottom are both identities of $\mathrm{Kar}(\mathcal{C})$, this gives the desired isomorphisms between $(X, e)$ and $(Y, \mathrm{id}_Y)$ which is in the image of the embedding. □

From this it follows that $\mathrm{Kar}(\mathrm{Kar}(\mathcal{C}))$ is equivalent to $\mathrm{Kar}(\mathcal{C})$, justifying the name "completion".

### 2.1.2 Limits and Colimits

The dual notions of limit and colimit are of great importance in category theory. The general definition is as follows:

**Definition 2.14.** Given a functor $F : \mathcal{D} \to \mathcal{C}$, we call

- a pair $(L, \pi)$ consisting of an object $L \in \mathrm{Ob}(\mathcal{C})$ and a natural transformation $\pi : \Delta_L \to F$ a *limit* of $F$, if for any such pair $(Z, \xi)$ there is a unique morphism $h \in \mathrm{Hom}_{\mathcal{C}}(Z, L)$ such that
$$\xi_A = \pi_A \circ h$$
for all $A \in \mathrm{Ob}(\mathcal{D})$.

- a pair $(C, \iota)$ consisting of an object $C \in \mathrm{Ob}(\mathcal{C})$ and a natural transformation $\iota : F \to \Delta_C$ a *colimit* of $F$, if for any such pair $(Z, \zeta)$ there is a unique morphism $h \in \mathrm{Hom}_{\mathcal{C}}(C, Z)$ such that
$$\zeta_A = h \circ \iota_A$$
for all $A \in \mathrm{Ob}(\mathcal{D})$.

An important fact which follows directly from the definition, is that if a limit or colimit of a functor exits, it is unique up to (a unique) isomorphism.

**Example 2.15.** *Given a functor from the category consisting of two objects and no non-trivial morphisms ($\bullet \quad \bullet$) into the category of sets, its limit is the Cartesian product of the two sets and its colimit is their disjoint union. The natural transformations in these cases consist of the projections from the product and the inclusions into the disjoint union, respectively. Often, the limit is denoted by the object only, suppressing the natural transformation if the context is clear.*
*More generally, given such a functor into any category (i.e. just picking two objects) its limit is called a product and its colimit a coproduct, if they exist.*

We will make particular use of the following two colimits, defined on $\bullet \leftarrow \bullet \rightarrow \bullet$ and $\bullet \rightrightarrows \bullet$, respectively. The first is called a pushout:

**Definition 2.16** (Pushout)**.** The colimit of a functor $F$ defined on the category freely generated by $A \xleftarrow{f} C \xrightarrow{g} B$, is called the *pushout* of $F(A)$ and $F(B)$ along $F(f)$ and $F(g)$ and is denoted as $F(A) \sqcup_{F(C)} F(B)$.

The latter one, called coequalizer, will be of special importance so we give an explicit definition:

**Definition 2.17** (Coequalizer)**.** Given objects $A$ and $B$ of a category $\mathcal{C}$ and morphisms $f, g \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, an object $C$ together with a morphism $c \in \mathrm{Hom}_{\mathcal{C}}(B, C)$ is called *coequalizer* of $f$ and $g$, if $c \circ f = c \circ g$, and for all objects $Z$ and morphisms $z \in \mathrm{Hom}_{\mathcal{C}}(B, Z)$ for which $z \circ f = z \circ g$, there is a unique morphism $h \in \mathrm{Hom}_{\mathcal{C}}(C, Z)$ such that $z = h \circ c$.

A coequalizer $(C, c)$ of $f$ and $g$ can be visualized in the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "c"] \arrow[dr, "z"'] & C \arrow[d, dashed, "h"] \\
& & Z
\end{tikzcd}$$

## 2.2 2-Categories

In a category as in Definition 2.1, there are objects and morphisms between them. We can add a layer of complexity by introducing morphisms between morphisms (referred to 2-cells), giving what we will call a strict 2-category. Using these 2-cells

to weaken the associativity of the composition of 1-cells, demanding that (2.1) and (2.2) hold only up to a 2-cell isomorphism (which has to satisfy certain coherence axioms), one is lead to the notion of a weak 2-category, also commonly known as bicategory and simply called 2-category here and in the following.
A comprehensive reference for the theory of 2-categories is [JY20].

**Definition 2.18.** (2-Category) A *2-category* $\mathcal{B}$ consists of the following:

- a class $\mathrm{Ob}(\mathcal{B})$ whose elements are called *objects* (or *0-cells*),
- for all $A, B \in \mathrm{Ob}(\mathcal{B})$ a category $\mathcal{B}(A, B)$, whose objects are referred to as *1-cells* and whose morphisms are referred to as *2-cells*,
- for all $A, B, C \in \mathrm{Ob}(\mathcal{B})$ a functor

  $$c_{A,B,C} : \mathcal{B}(B, C) \times \mathcal{B}(A, B) \to \mathcal{B}(A, C)$$

  denoted for $a, a' \in \mathcal{B}(A, B)$, $b, b' \in \mathcal{B}(B, C)$, $\phi \in \mathrm{Hom}_{\mathcal{B}(A,B)}(a, a')$ and $\psi \in \mathrm{Hom}_{\mathcal{B}(B,C)}(b, b')$ as:

  $$\begin{aligned}(b, a) &\mapsto b \otimes a \in \mathcal{B}(A, C)\\ (\psi, \phi) &\mapsto \psi \otimes \phi \in \mathrm{Hom}_{\mathcal{B}(A,C)}(b \otimes a, b' \otimes a')\end{aligned}$$

- for all $A \in \mathrm{Ob}(\mathcal{B})$ a functor:

  $$\mathbb{1}_A : \mathbf{1} \to \mathcal{B}(A, A)$$

  its image is called the *unit* 1-cell corresponding to $A$ and is denoted by $\mathbb{1}_A$ as well,
- for all $A, B, C, D \in \mathrm{Ob}(\mathcal{B})$ a natural isomorphism, called the *associator*:

  $$\alpha^{A,B,C,D} : c_{A,B,D} \circ (c_{B,C,D} \times id) \to c_{A,C,D} \circ (id \times c_{A,B,C})$$

- for all $A, B \in \mathrm{Ob}(\mathcal{B})$ natural isomorphisms, called *left* and *right unitors*:

  $$\lambda^{A,B} : c_{B,A,A} \circ (\mathbb{1}_A \times id) \to id$$

  $$\rho^{A,B} : c_{A,A,B} \circ (id \times \mathbb{1}_A) \to id$$

such that for all $A, B, C, D, E \in \mathrm{Ob}(\mathcal{B})$ and all $d \in \mathcal{B}(A, B), c \in \mathcal{B}(B, C), b \in \mathcal{B}(C, D), a \in \mathcal{B}(D, E)$ the following diagrams commute:

- Pentagon identity:

$$\begin{tikzcd}
& (a \otimes b) \otimes (c \otimes d) \arrow[dr, "\alpha^{A,C,D,E}_{a,b,c\otimes d}"] & \\
((a \otimes b) \otimes c) \otimes d \arrow[ur, "\alpha^{A,B,C,E}_{a\otimes b,c,d}"] \arrow[d, "\alpha^{B,C,D,E}_{a,b,c} \otimes \mathrm{id}_d"'] & & a \otimes (b \otimes (c \otimes d)) \\
(a \otimes (b \otimes c)) \otimes d \arrow[rr, "\alpha^{A,B,D,E}_{a,b\otimes c,d}"] & & a \otimes ((b \otimes c) \otimes d) \arrow[u, "\mathrm{id}_a \otimes \alpha^{A,B,C,D}_{b,c,d}"']
\end{tikzcd} \tag{Pen.}$$

- Triangle identity:

$$\begin{tikzcd}
(a \otimes \mathbb{1}_D) \otimes b \arrow[rr, "\alpha^{C,D,D,E}_{a,\mathbb{1}_D,b}"] \arrow[dr, "\rho^{D,E}_{a,\bullet} \otimes \mathrm{id}_b"'] & & a \otimes (\mathbb{1}_D \otimes b) \arrow[dl, "\mathrm{id}_a \otimes \lambda^{D,C}_{\bullet,b}"] \\
& a \otimes b &
\end{tikzcd} \tag{Tri.}$$

**Remark 2.19.** *Given $A, B, C, D \in \mathrm{Ob}(\mathcal{B})$ and $c \in \mathcal{B}(A, B), b \in \mathcal{B}(B, C)$ and $a \in \mathcal{B}(C, D)$ the components of the corresponding associators and unitors are 2-cells:*

$$\alpha_{a,b,c} : (a \otimes b) \otimes c \to a \otimes (b \otimes c)$$

$$\rho_a : a \otimes \mathbb{1}_C \to a,\ \lambda_a : \mathbb{1}_D \otimes a \to a$$

*where we omitted the superscripts as we will do in the following.*

**Remark 2.20.** *Like 1-categories, also 2-categories can be described by generators and relations. Given some objects, 1-cells between them, 2-cells between those and relations for 1-cells and 2-cells, there is a 2-category consisting of these objects, 1-cells and 2-cells as well as all their possible compositions (where all the different bracketings of 1-cells are different if there is no relation stating that this is not the case) and all the necessary unit 1-cells and identity, associator and unitor 2-cells. This can be made precise using the notion of a computad (cf. [Str76]) which plays the role of the directed multigraph in Remark 2.2.*

**Example 2.21.** *The fact that the symbol chosen for the composition of 1-cells resembles the tensor product of vector spaces is no coincidence: There is a 2-category* $\mathrm{B}\,\boldsymbol{Vec}_{\mathbb{k}}$ *which has one object $\bullet_{\mathbb{k}}$, the category of $\mathbb{k}$-vector spaces as 1-cells (so that linear transformations are 2-cells) and their tensor product as composition.*

**Definition 2.22** (Adjunction)**.** A 1-cell $f \in \mathcal{B}(A,B)$ is called *left adjoint* to $g \in \mathcal{B}(B,A)$, denoted $f \dashv g$, if there exist 2-cells $\eta \in \mathrm{Hom}_{\mathcal{B}(A,A)}(\mathbb{1}_A, g \otimes f)$ and $\epsilon \in \mathrm{Hom}_{\mathcal{B}(B,B)}(f \otimes g, \mathbb{1}_B)$ such that:

$$\lambda^{B,A}_{\bullet,f} \circ (\epsilon \otimes \mathrm{id}_f) \circ (\alpha^{A,B,A,B}_{f,g,f})^{-1} \circ (\mathrm{id}_f \otimes \eta) \circ (\rho^{A,B}_{f,\bullet})^{-1} = \mathrm{id}_f \tag{2.8}$$

$$\rho^{B,A}_{g,\bullet} \circ (\mathrm{id}_g \otimes \epsilon) \circ (\alpha^{A,B,A,B}_{f,g,f}) \circ (\eta \otimes \mathrm{id}_g) \circ (\lambda^{A,B}_{\bullet,g})^{-1} = \mathrm{id}_g \tag{2.9}$$

in that case we also call $g$ *right adjoint* to $f$ and $(f, g, \eta, \epsilon)$ an *adjunction* with *unit* $\eta$ and *counit* $\epsilon$.

**Definition 2.23** (Internal equivalence)**.** Objects $A, B \in \mathrm{Ob}(\mathcal{B})$ of a 2-category $\mathcal{B}$ are called *equivalent*, if there are 1-cells $f \in \mathcal{B}(A,B)$ and $g \in \mathcal{B}(B,A)$ such that there are isomorphisms $\eta \in \mathrm{Hom}_{\mathcal{B}(A,A)}(\mathbb{1}_A, g \otimes f)$ and $\epsilon \in \mathrm{Hom}_{\mathcal{B}(B,B)}(f \otimes g, \mathbb{1}_B)$.

A classic result states that if $f$ and $g$ are part of an equivalence, one can construct an adjunction between them.

The notion of functor between categories is generalized to 2-functor between 2-categories.

**Definition 2.24** (2-functor)**.** A *2-functor* $F : \mathcal{A} \to \mathcal{B}$, between two 2-categories $\mathcal{A}$ and $\mathcal{B}$ consists of:

- a map $F : \mathrm{Ob}(\mathcal{A}) \to \mathrm{Ob}(\mathcal{B})$,
- for all $A, B \in \mathrm{Ob}(\mathcal{A})$ a functor $F_{A,B} : \mathcal{A}(A,B) \to \mathcal{B}(F(A), F(B))$, (where the subscript is usually omitted)
- for all $A, B, C \in \mathrm{Ob}(\mathcal{A})$ a natural isomorphism

$$\phi^{A,B,C} : c_{F(A),F(B),F(C)} \circ (F_{B,C} \times F_{A,B}) \to F_{A,C} \circ c_{A,B,C}$$

- for all $A \in \mathrm{Ob}(\mathcal{A})$ a natural isomorphism $\phi^A : \mathbb{1}_{F(A)} \to F \circ \mathbb{1}_A$,

such that for all $A, B, C, D \in \mathrm{Ob}(\mathcal{A})$ and all $c \in \mathcal{A}(A,B), b \in \mathcal{A}(B,C), a \in \mathcal{A}(C,D)$ the following diagrams commute:

$$
\begin{tikzcd}
& (F(a) \otimes F(b)) \otimes F(c) \arrow[dl, "\alpha^{F(A),F(B),F(C),F(D)}_{F(a),F(b),F(c)}"'] \arrow[dr, "\phi^{B,C,D}_{a,b} \otimes \mathrm{id}_{F(c)}"] & \\
F(a) \otimes (F(b) \otimes F(c)) \arrow[d, "\mathrm{id}_{F(a)} \otimes \phi^{A,B,C}_{b,c}"'] & & F(a \otimes b) \otimes F(c) \arrow[d, "\phi^{A,B,D}_{a \otimes b,c}"] \\
F(a) \otimes F(b \otimes c) \arrow[dr, "\phi^{A,C,D}_{a,b \otimes c}"'] & & F((a \otimes b) \otimes c) \arrow[dl, "F(\alpha^{A,B,C,D}_{a,b,c})"] \\
& F(a \otimes (b \otimes c)) &
\end{tikzcd}
\tag{Hex.}
$$

$$
\begin{tikzcd}
& \mathbb{1}_{F(D)} \otimes F(a) \arrow[dl, "\phi^{D}_{\bullet} \otimes \mathrm{id}_{F(a)}"] \arrow[dr, "\lambda^{F(D),F(C)}_{\bullet,F(a)}"] & & F(a) \otimes \mathbb{1}_{F(C)} \arrow[dl, "\rho^{F(C),F(D)}_{F(a),\bullet}"'] \arrow[dr, "\mathrm{id}_{F(a)} \otimes \phi^{C}_{\bullet}"] & \\
F(\mathbb{1}_D) \otimes F(a) \arrow[dr, "\phi^{C,D,D}_{\mathbb{1}_D,a}"] & & F(a) & & F(a) \otimes F(\mathbb{1}_C) \arrow[dl, "\phi^{C,C,D}_{a,\mathbb{1}_C}"] \\
& F(\mathbb{1}_D \otimes a) \arrow[ur, "F(\lambda^{D,C}_{\bullet,a})"'] & & F(a \otimes \mathbb{1}_D) \arrow[ul, "F(\rho^{C,D}_{a,\bullet})"] &
\end{tikzcd}
\tag{2.10}
$$

**Remark 2.25.** *We can form a 1-category which has 2-categories as objects and 2-functors as morphisms. There are the appropriate notions of "2-natural transformation" between 2-functors advancing this to a 2-category and "modification" between 2-natural transformations giving rise to the 3-category of 2-categories. As 2-natural transformations and modifications will not be used, their lengthy definitions are omitted.*

When applying the definitions of adjunction and internal equivalence in the 2-category of 1-categories, functors and natural transformations, one arrives at the common notions of adjoint functors and equivalence of categories, respectively. The Whitehead theorem for 1-categories states that two 1-categories are equivalent if and only if there is a fully faithful functor between them which induces a bijection on isomorphism classes of objects.
The natural generalization of the latter is the 2-equivalence of 2-categories: two 2-categories $\mathcal{A}$ and $\mathcal{B}$ are 2-equivalent if there are 2-functors $F : \mathcal{A} \to \mathcal{B}$ and

$G : \mathcal{B} \to \mathcal{A}$ such that both their compositions are equivalent to the appropriate identities in the respective 2-category of 2-functors, 2-natural transformations and modifications. Thanks to the Whitehead theorem for bicategories ([JY20, Theorem 7.4.1]), this can be equivalently defined as follows.

**Definition 2.26.** A 2-functor $F$ is called

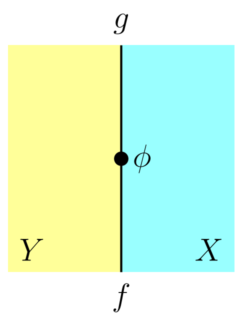
- *essentially surjective*, if the map on objects induces a surjection on equivalence classes,
- *essentially full on 1-cells*, if all component functors $F_{A,B}$ induce surjections on isomorphism classes,
- *fully faithful on 2-cells*, if all component functors $F_{A,B}$ are full and faithful,
- a *2-equivalence* if it is essentially surjective, essentially full on 1-cells and fully faithful on 2-cells. Accordingly two 2-categories are called 2-equivalent, or equivalent as 2-categories, if there is a functor between them which is a 2-equivalence.

**Definition 2.27.** A 2-category $\mathcal{A}$ is called a *full subcategory* of a 2-category $\mathcal{B}$, if there is a 2-functor $\mathcal{A} \to \mathcal{B}$ which is essentially full on 1-cells and fully faithful on 2-cells.

A main result in the theory of 2-categories is the coherence theorem: every 2-category is 2-equivalent to a strict 2-category ([JY20, Theorem 8.4.1]). An important consequence of this is that in every 2-category, any combination of identities, unitors and associators between the same 1-cells have to be equal. This will be exploited repeatedly in the following.

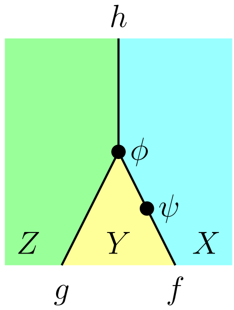
### 2.2.1 String Diagrams

To better visualize 2-cells in a 2-category we introduce string diagrams. Objects $X, Y \in \mathrm{Ob}(\mathcal{B})$ in a 2-category $\mathcal{B}$ are depicted as areas of different colours, 1-cells $f, g \in \mathcal{B}(X,Y)$ as strings separating these areas and 2-cells between these $\phi \in \mathrm{Hom}_{\mathcal{B}(X,Y)}(f,g)$ as junctions between the corresponding strings:



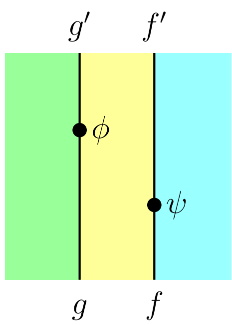
1-cells go from right to left and their composition is given by horizontal concatenation of diagrams, 2-cells go from bottom to top and their composition corresponds to vertical concatenation. In both cases the diagrams contain no information about the order of bracketing, i.e. associators are omitted. Unit 1-cells and identity 2-cells are not drawn (invisible) so that unitors are omitted as well. That this leads to no ambiguity is guaranteed by the coherence theorem.

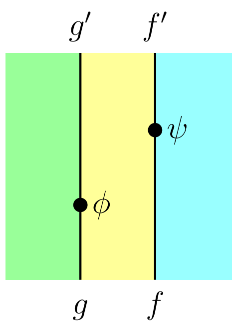
For example, given $f \in \mathcal{B}(X,Y)$, $g \in \mathcal{B}(Y,Z)$, $h \in \mathcal{B}(X,Z)$, $\psi \in \mathrm{Hom}_{\mathcal{B}(X,Y)}(f,f)$ and $\phi \in \mathrm{Hom}_{\mathcal{B}(X,Z)}(g \otimes f, h)$, the 2-cell $\phi \circ (\mathrm{id}_g \otimes \psi) \in \mathrm{Hom}_{\mathcal{B}(X,Z)}(g \otimes f, h)$ is depicted as:




Two string diagrams represent the same 2-cell, if they are related by a continuous and pointwise progressive deformation. This is a consequence of the axioms for a 2-category and greatly facilitates their application. In particular the following holds:

**Proposition 2.28** (Interchange Law). *For $f, f' \in \mathcal{B}(X,Y)$, $g, g' \in \mathcal{B}(Y,Z)$ and $\psi \in \mathrm{Hom}_{\mathcal{B}(X,Y)}(f,f')$ and $\phi \in \mathrm{Hom}_{\mathcal{B}(Y,Z)}(g,g')$:*

$(\phi \otimes \mathrm{id}_{f'}) \circ (\mathrm{id}_g \otimes \psi) =$


$=$


$= (\mathrm{id}_{g'} \otimes \psi) \circ (\phi \otimes \mathrm{id}_f)$

*Proof.* This is a consequence of the functoriality of the composition $c_{X,Y,Z} : \mathcal{B}(Y,Z) \times \mathcal{B}(X,Y) \to \mathcal{B}(X,Z)$

$$
\begin{aligned}
(\phi \otimes \mathrm{id}_{f'}) \circ (\mathrm{id}_g \otimes \psi) &= c_{X,Y,Z}((\phi, \mathrm{id}_{f'})) \circ c_{X,Y,Z}((\mathrm{id}_g, \psi)) = \\
&= c_{X,Y,Z}((\phi, \mathrm{id}_{f'}) \circ (\mathrm{id}_g, \psi)) = \\
&= c_{X,Y,Z}((\phi \circ \mathrm{id}_g, \mathrm{id}_{f'} \circ \psi)) = c_{X,Y,Z}((\phi, \psi)) = \\
&= c_{X,Y,Z}((\mathrm{id}_{g'} \circ \phi, \psi \circ \mathrm{id}_f)) = \\
&= c_{X,Y,Z}((\mathrm{id}_{g'}, \psi) \circ (\phi, \mathrm{id}_f)) = \\
&= c_{X,Y,Z}((\mathrm{id}_{g'}, \psi)) \circ c_{X,Y,Z}((\phi, \mathrm{id}_f)) = \\
&= (\mathrm{id}_{g'} \otimes \psi) \circ (\phi \otimes \mathrm{id}_f)
\end{aligned}
$$

□

Even though the idea of proving mathematical statements by manipulating drawings may seem heuristic, it can be made perfectly precise (originally done so in [JS91]) and even generalized to 3-categories (in [BMS18]).

**Remark 2.29.** *This string diagrammatic calculus can be used to express otherwise lengthy equations elegantly. For example Equations (2.8) & (2.9), defining an adjunction $f \dashv g$, are equivalent to the so called Zorro moves:*


## 2.3 Algebras

### 2.3.1 Algebras in 2-Categories

The well-known notion of an algebra, given as a $\Bbbk$-vector space with a $\Bbbk$-linear, associative multiplication, can be generalized to an algebra internal to any 2-category. A usual $\Bbbk$-algebra then is recognized as an algebra internal to the 2-category $\mathrm{B}\mathbf{Vec}_{\Bbbk}$.

**Definition 2.30** (Algebra)**.** Given a 2-category $\mathcal{B}$ and an object $A \in \mathrm{Ob}(\mathcal{B})$, a 1-cell $a \in \mathcal{B}(A, A)$ together with

- a 2-cell $\mu \in \mathrm{Hom}_{\mathcal{B}(A,A)}(a \otimes a, a)$ (called *multiplication*) such that:

$$\mu \circ (\mu \otimes \mathrm{id}_a) = \mu \circ (\mathrm{id}_a \otimes \mu) \circ \alpha_{a,a,a} \tag{A1}$$

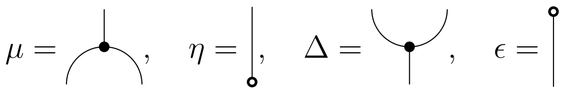
  is called an *nonunital algebra*.

- $\mu$ as above and in addition a 2-cell $\eta \in \mathrm{Hom}_{\mathcal{B}(A,A)}(\mathbb{1}_A, a)$ (called *unit*) such that:
$$\mu \circ (\eta \otimes \mathrm{id}_a) \circ \lambda_a^{-1} = \mathrm{id}_a = \mu \circ (\mathrm{id}_a \otimes \eta) \circ \rho_a^{-1} \tag{2.11}$$
is called a *(unital) algebra*.

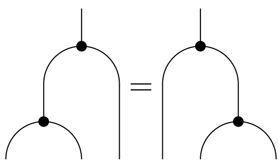
- instead a 2-cell $\Delta \in \mathrm{Hom}_{\mathcal{B}(A,A)}(a, a \otimes a)$ (called *comultiplication*) such that:
$$(\Delta \otimes \mathrm{id}_a) \circ \Delta = \alpha_{a,a,a}^{-1} \circ (\mathrm{id}_a \otimes \Delta) \circ \Delta \tag{A2}$$
is called a *noncounital coalgebra*.

- $\Delta$ as above and in addition a 2-cell $\epsilon \in \mathrm{Hom}_{\mathcal{B}(A,A)}(a, \mathbb{1}_A)$ (called *counit*) such that:
$$\lambda_a \circ (\epsilon \otimes \mathrm{id}_a) \circ \Delta = \mathrm{id}_a = \rho_a \circ (\mathrm{id}_a \otimes \epsilon) \circ \Delta \tag{2.12}$$
is called a *(counital) coalgebra*.

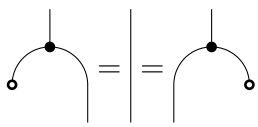
**Remark 2.31.** *The equations from Definition 2.30 can be conveniently stated using string diagrams as introduced in Section 2.2.1, if we depict the object A as the white background, the 1-cell a as black line and the 2-cells as:*



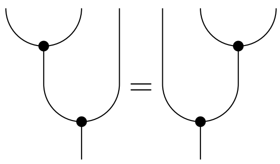

*The equations from above can then be beautifully stated as follows:*



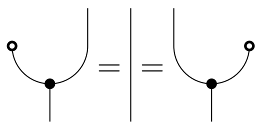
(A1)



(2.11)



(A2)



(2.12)

**Definition 2.32** (Frobenius Algebra)**.** Given $(a, \mu, \Delta)$ as above, both a nonunital algebra and a noncounital coalgebra, we say $a$ is:

- *Frobenius* if:

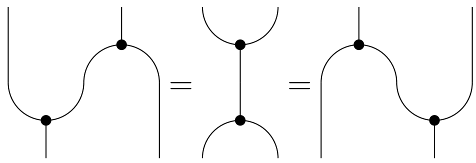
$$(\mathrm{id}_a \otimes \mu) \circ \alpha_{a,a,a} \circ (\Delta \otimes \mathrm{id}_a) = \Delta \circ \mu = (\mu \otimes \mathrm{id}_a) \circ \alpha^{-1}_{a,a,a} \circ (\mathrm{id}_a \otimes \Delta) \tag{F1}$$

- $\Delta$-*separable* if:

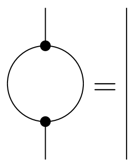
$$\mu \circ \Delta = \mathrm{id}_a \tag{S1}$$

Here and below we may refer to an algebra only by the 1-cell if the 2-cells and the kind of algebra we are talking about are clear from context.

**Remark 2.33.** *The equations defining a $\Delta$-separable Frobenius algebra can be stated as string diagrams (using the same notation as in Remark 2.31) as follows:*



(F1)



(S1)

### 2.3.2 Modules

The notion of a module on which an algebra acts can be generalized to arbitrary 2-categories as well.

**Definition 2.34** (Module)**.** Given a nonunital algebra $(a, \mu)$ in $\mathcal{B}(A, A)$,

- a 1-cell $m \in \mathcal{B}(B, A)$ together with a 2-cell $l_m \in \mathrm{Hom}_{\mathcal{B}(B,A)}(a \otimes m, m)$ (called the *left action* of $a$ on $m$), is a *left $a$-module* if:

$$l_m \circ (\mathrm{id}_a \otimes l_m) \circ \alpha_{a,a,m} = l_m \circ (\mu \otimes \mathrm{id}_m) \tag{2.13}$$

- a 1-cell $n \in \mathcal{B}(A, C)$ together with a 2-cell $r_n \in \mathrm{Hom}_{\mathcal{B}(A,C)}(n \otimes a, n)$ (called the *right action* of $a$ on $n$), is a *right $a$-module* if:

$$r_n \circ (r_n \otimes \mathrm{id}_a) \circ \alpha^{-1}_{n,a,a} = r_n \circ (\mathrm{id}_n \otimes \mu) \tag{2.14}$$

- if for a 1-cell $m \in \mathcal{B}(B, C)$, $(m, l_m)$ is a left $c$-module and $(m, r_m)$ is a right $b$-module for two nonunital algebras $c \in \mathcal{B}(C, C)$ and $b \in \mathcal{B}(B, B)$, $(m, l_m, r_m)$ is a *$c$-$b$-bimodule* if:

$$r_m \circ (l_m \otimes \mathrm{id}_b) = l_m \circ (\mathrm{id}_c \otimes r_m) \circ \alpha_{c,m,b} \tag{2.15}$$

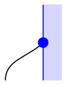
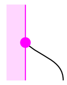
- If we instead consider a unital algebra $(a, \mu, \eta)$, we additionally require that the actions respect the units, i.e. for a left module:

$$l_m \circ (\eta \otimes \mathrm{id}_m) \circ \lambda_m^{-1} = \mathrm{id}_m \tag{2.16}$$

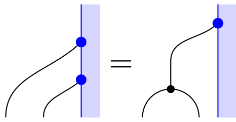
  and for a right module:

$$r_n \circ (\mathrm{id}_n \otimes \eta) \circ \rho_n^{-1} = \mathrm{id}_n \tag{2.17}$$

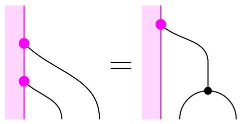
**Remark 2.35.** *As string diagrams, depicting the objects $A$ as white, $B$ as blue and $C$ as magenta backgrounds, the algebras $a$, $b$ and $c$ as black, $m$ as blue and $n$ as magenta lines and the 2-cells as:*

$l_m =$ , $\quad r_n =$ 

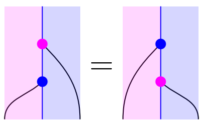
*the equations from Definition 2.34 take the following form:*



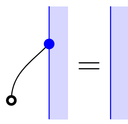
(2.13)



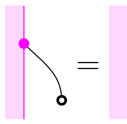
(2.14)



(2.15)



(2.16)



(2.17)

Just as the equations defining a coalgebra are the upside down versions of the ones defining an algebra, we get the notion of a comodule by turning all above string diagrams upside-down.

**Definition 2.36** (Comodule)**.** Given a noncounital coalgebra $(a, \Delta)$ in $\mathcal{B}(A, A)$,

- a 1-cell $m \in \mathcal{B}(B, A)$ together with a 2-cell $\check{l}_m \in \mathrm{Hom}_{\mathcal{B}(B,A)}(m, a \otimes m)$ (called the *left coaction* of $a$ on $m$), is a *left $a$-comodule* if:

$$\alpha^{-1}_{a,a,m} \circ (\mathrm{id}_a \otimes \check{l}_m) \circ \check{l}_m = (\Delta \otimes \mathrm{id}_m) \circ \check{l}_m \tag{2.18}$$

- a 1-cell $n \in \mathcal{B}(A, C)$ together with a 2-cell $\check{r}_n \in \mathrm{Hom}_{\mathcal{B}(A,C)}(n, n \otimes a)$ (called the *right coaction* of $a$ on $n$), is a *right $a$-comodule* if:

$$\alpha_{n,a,a} \circ (\check{r}_n \otimes \mathrm{id}_a) \circ \check{r}_n = (\mathrm{id}_n \otimes \Delta) \circ \check{r}_n \tag{2.19}$$

- if for a 1-cell $m \in \mathcal{B}(B, C)$, $(m, \check{l}_m)$ is a left $c$-comodule and $(m, \check{r}_m)$ is a right $b$-module for two noncounital coalgebras $b \in \mathcal{B}(B, B)$ and $c \in \mathcal{B}(C, C)$, $(m, \check{l}_m, \check{r}_m)$ is an *$c$-$b$-bicomodule* if:

$$(\check{l}_m \otimes \mathrm{id}_b) \circ \check{r}_m = \alpha^{-1}_{c,m,b} \circ (\mathrm{id}_c \otimes \check{r}_m) \circ \check{l}_m \tag{2.20}$$

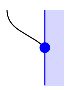
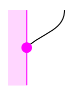
- If we instead consider a counital coalgebra $(a, \Delta, \epsilon)$, we require that the actions respect the units, i.e. for a left comodule:

$$\lambda_m \circ (\epsilon \otimes \mathrm{id}_m) \circ \check{l}_m = \mathrm{id}_m \tag{2.21}$$

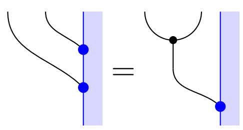
  and for a right module:

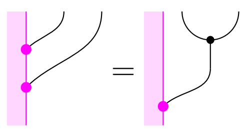
$$\rho_n \circ (\mathrm{id}_n \otimes \epsilon) \circ \check{r}_n = \mathrm{id}_n \tag{2.22}$$

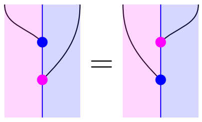
**Remark 2.37.** *As string diagrams, depicting the objects $A$ as white, $B$ as blue and $C$ as magenta backgrounds, the algebras $a$, $b$ and $c$ as black, $m$ as blue and $n$ as magenta lines and the 2-cells as:*

$\check{l}_m =$ , $\quad \check{r}_n =$ 

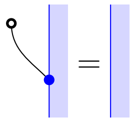
*the equations from Definition 2.36 take the following form:*

 (2.18)

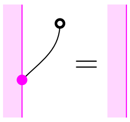
 (2.19)



$$(2.20)$$



$$(2.21)$$



$$(2.22)$$

**Definition 2.38** (Frobenius Module)**.** Given a nonunital algebra $(a, \mu)$ and a non-counital coalgebra $(a, \Delta)$ in $\mathcal{B}(A, A)$,

- we call $(m, l_m, \check{l}_m)$ such that $(m, l_m)$ is a left $a$-module and $(m, \check{l}_m)$ is a left $a$-comodule, a *left Frobenius module* if:

$$(\mu \otimes \mathrm{id}_m) \circ \alpha^{-1}_{a,a,m} \circ (\mathrm{id}_a \otimes \check{l}_m) = \check{l}_m \circ l_m = (\mathrm{id}_a \otimes l_m) \circ \alpha_{a,a,m} \circ (\Delta \otimes \mathrm{id}_m) \quad \text{(F2)}$$

- furthermore it is called *special* if:

$$l_m \circ \check{l}_m = \mathrm{id}_m \quad \text{(S2)}$$

- Analogously we call $(n, r_n, \check{r}_n)$ such that $(n, r_n)$ is a right $a$-module and $(n, \check{r}_n)$ is a right $a$-comodule, a *right Frobenius module* if:

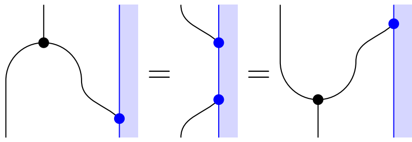
$$(\mathrm{id}_n \otimes \mu) \circ \alpha_{n,a,a} \circ (\check{r}_n \otimes \mathrm{id}_a) = \check{r}_n \circ r_n = (r_n \otimes \mathrm{id}_a) \circ \alpha^{-1}_{n,a,a} \circ (\mathrm{id}_n \otimes \Delta) \quad \text{(F3)}$$

- and *special* if:

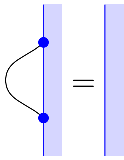
$$r_n \circ \check{r}_n = \mathrm{id}_n \quad \text{(S3)}$$

- $(m, l_m, r_m, \check{l}_m, \check{r}_m)$ is called a *special Frobenius bimodule*, if $(m, l_m, r_m)$ is a bimodule and $(m, \check{l}_m, \check{r}_m)$ is a bicomodule such that $(m, l_m, \check{l}_m)$ is a special left Frobenius module and $(m, r_m, \check{r}_m)$ is a special right Frobenius module.

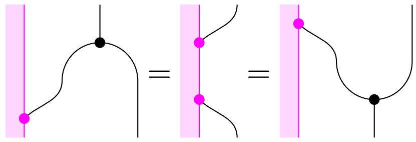
**Remark 2.39.** *Using the notation introduced above, the conditions of being a special Frobenius bimodule can be expressed elegantly as follows:*



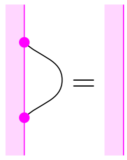
(F2)



(S2)



(F3)



(S3)

**Definition 2.40** (Module Map)**.** Given an algebra (or nonunital algebra) $a$ in $\mathcal{B}(A, A)$ and

- parallel left $a$-modules $(m, l_m)$ and $(n, l_n)$, a 2-cell $\phi \in \mathrm{Hom}_{\mathcal{B}(B,A)}(m, n)$ is a *module map* if:

$$\phi \circ l_m = l_n \circ (\mathrm{id}_a \otimes \phi) \tag{2.23}$$

- parallel right $a$-modules $(m, r_m)$ and $(n, r_n)$, a 2-cell $\psi \in \mathrm{Hom}_{\mathcal{B}(A,C)}(m, n)$ is a *module map* if:

$$\psi \circ r_m = r_n \circ (\psi \otimes \mathrm{id}_a) \tag{2.24}$$

- The subset of module maps inside $\mathrm{Hom}_{\mathcal{B}(B,A)}(m, n)$ is denoted $\mathrm{Hom}_a(m, n)$.

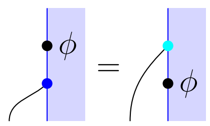
- For another algebra $b \in \mathcal{B}(B, B)$, given two parallel $a$-$b$-bimodules $(m, l_m, r_m)$ and $(n, l_n, r_n)$, a 2-cell $\phi \in \mathrm{Hom}_{\mathcal{B}(B,A)}(m, n)$ is a *bimodule map* if it is a module map for both the corresponding left and right module structures.

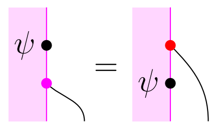
Again we can define the notion of comodule map analogously:

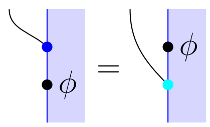
**Definition 2.41** (Comodule Map)**.** Given a coalgebra (or noncounital coalgebra) $a$ in $\mathcal{B}(A, A)$ and

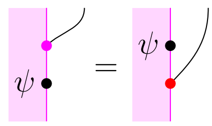
- parallel left $a$-comodules $(m, \check{l}_m)$ and $(n, \check{l}_n)$, a 2-cell $\phi \in \mathrm{Hom}_{\mathcal{B}(B,A)}(m,n)$ is a *comodule map* if:
$$\check{l}_n \circ \phi = (\mathrm{id}_a \otimes \phi) \circ \check{l}_m \tag{2.25}$$
- parallel right $a$-modules $(m, r_m)$ and $(n, r_n)$, a 2-cell $\psi \in \mathrm{Hom}_{\mathcal{B}(A,C)}(m,n)$ is a *comodule map* if:
$$\check{r}_n \circ \psi = (\psi \otimes \mathrm{id}_a) \circ \check{r}_m \tag{2.26}$$
- For another coalgebra $b \in \mathcal{B}(B,B)$, given two parallel $a$-$b$-bicomodules $(m, \check{l}_m, \check{r}_m)$ and $(n, \check{l}_n, \check{r}_n)$, a 2-cell $\phi \in \mathrm{Hom}_{\mathcal{B}(B,A)}(m,n)$ is a *bicomodule map* if it is a comodule map for both the corresponding left and right comodule structures.

**Remark 2.42.** *As before it is most convenient to express the equations of the previous two definitions in terms of string diagrams:*


(2.23)


(2.24)

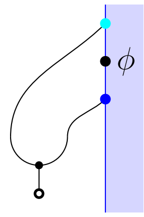

(2.25)


(2.26)

**Remark 2.43.** *From the string diagrammatic equations above, it is obvious that the property of being a module map is preserved by composition of 2-cells. Thus, given algebras $a \in \mathcal{B}(A,A)$ and $b \in \mathcal{B}(B,B)$, there is a 1-category which has a-b-bimodules as objects and their bimodule maps as morphism, with the same composition as in $\mathcal{B}(B,A)$.*
*Similarly there is a 1-category of a-b-bicomodules and their comodule maps.*

Modules between $\Delta$-separable Frobenius algebras have some useful properties. Firstly, there is the following description of module maps between them.

**Proposition 2.44.** *Given an algebra $(a, \mu, \eta)$ in $\mathcal{B}(A,A)$ and left a-modules $(m, l_m)$ and $(n, l_n)$, if a is $\Delta$-separable Frobenius then the map $\pi_a : \mathrm{Hom}_{\mathcal{B}(B,A)}(m,n) \to \mathrm{Hom}_{\mathcal{B}(B,A)}(m,n)$*

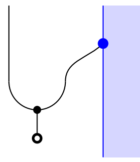
$$\pi_a : \phi \mapsto$$ 

*is an idempotent and has* $\mathrm{Hom}_a(X, Y)$ *as its image if it splits.*

Its proof is can be done by manipulating string diagrams and can be found like that for example in [CR16, Lemma 2.2].
Secondly, for modules of $\Delta$-separable Frobenius algebras, every module structure naturally also gives a comodule structure.

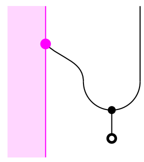
**Proposition 2.45.** *Given a* $\Delta$*-separable Frobenius algebra* $(a, \mu, \Delta, \eta, \epsilon)$*, and a left* $a$*-module* $(m, l_m)$*, the canonical coaction defined by:*

$$\check{l}_m^{can} := (\mathrm{id}_a \otimes l_m) \circ \alpha_{a,a,m} \circ ((\Delta \circ \eta) \otimes \mathrm{id}_m) \circ \lambda_m^{-1} = \tag{2.27}$$



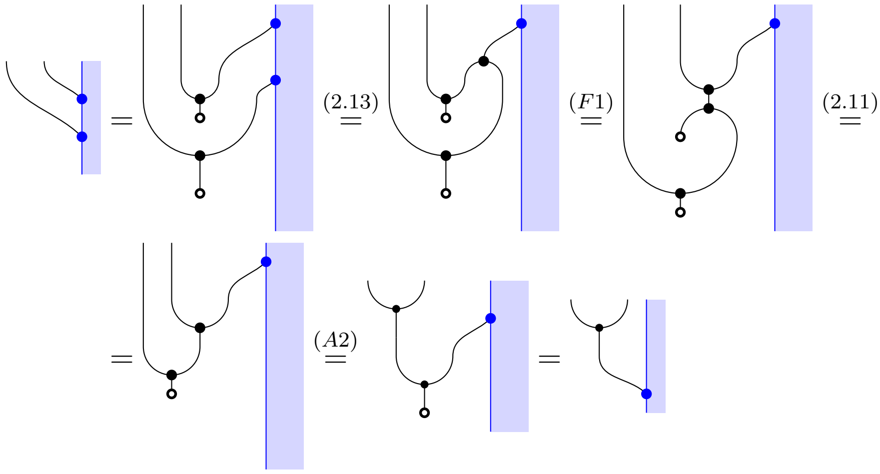
*makes* $(m, \check{l}_m^{can})$ *a left* $a$*-comodule.*
*Similarly for a right* $a$*-module* $(n, r_n)$*, the canonical coaction defined by:*

$$\check{r}_n^{can} := (r_n \otimes \mathrm{id}_a) \circ \alpha_{n,a,a}^{-1} \circ (\mathrm{id}_n \otimes (\Delta \circ \eta)) \circ \rho_n^{-1} = \tag{2.28}$$

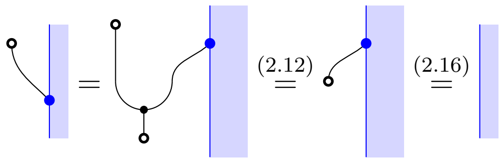


*makes* $(n, \check{r}_n^{can})$ *a right* $a$*-comodule.*

*Proof.* We prove the statement for the left module, the version for the right module follows analogously by mirroring all the sting diagrams. We first show compatibility with the comultiplication, using the compatibility of action and multiplication:



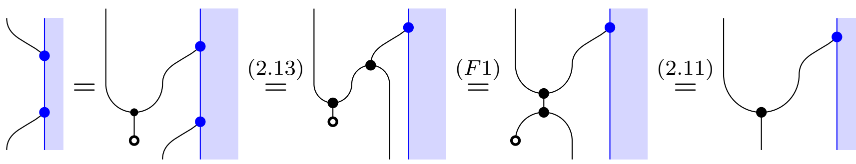

It remains to show the compatibility with the counit:



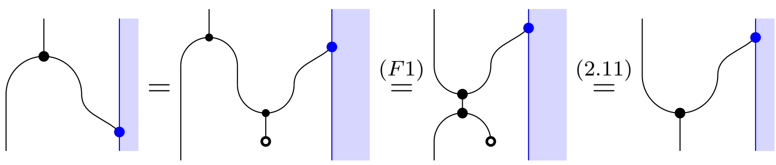

□

**Proposition 2.46.** *Let $(a, \mu, \Delta, \eta, \epsilon)$ be a $\Delta$-separable Frobenius algebra, $(m, l_m)$ a left $a$-module and $(n, r_n)$ a right $a$-module. Together with their canonical coactions, $(m, l_m, \check{l}_m^{can})$ forms a special left Frobenius module and $(n, r_n, \check{r}_n^{can})$ forms a special right Frobenius module.*

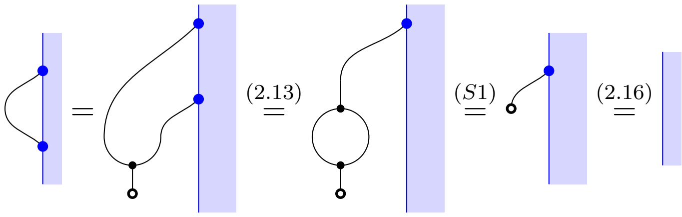
*Proof.* To prove this we need to check the equations from Definition 2.38. We do this for the left module, for the right module they follow analogously by mirroring all the sting diagrams. We start with the Frobenius property, first the right-hand equality in (F2):




and then show that also the left-hand side is equal to that:



The speciality of the module follows from the $\Delta$-separability of the algebra:




□

### 2.3.3 Tensor Products

In classical algebra, a left $a$-module $n$ and a right $a$-module $m$ can be combined to their relative tensor product over $a$ (or $a$-balanced product) $m \otimes_a n$. As this is defined via a universal property, it can be directly generalized to modules between algebras internal to arbitrary 2-categories, using the notion of coequalizer introduced in Definition 2.17.

**Definition 2.47** (Relative Tensor Product)**.** Given $a \in \mathcal{B}(A, A)$, $m \in \mathcal{B}(A, C)$ and $n \in \mathcal{B}(B, A)$, let $(a, \mu, \eta)$ be an algebra, $(m, r_m)$ a right $a$-module and $(n, l_n)$ a left $a$-module. The coequalizer of $r := r_m \otimes \mathrm{id}_n$ and $l := (\mathrm{id}_m \otimes l_n) \circ \alpha_{m,a,n}$ is called the *tensor product of $m$ and $n$ over $a$* and denoted $(m \otimes_a n, \theta)$, if it exits.

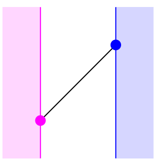
$$\begin{tikzcd}
(m \otimes a) \otimes n \arrow[r, shift left, "r"] \arrow[r, shift right, "l"'] & m \otimes n \arrow[r, "\theta"] \arrow[dr, "\zeta"'] & m \otimes_a n \arrow[d, dashed, "\zeta'"] \\
& & z
\end{tikzcd}$$

**Proposition 2.48.** *Given $a \in \mathcal{B}(A, A)$ which is a nonunital algebra as well as a noncounital coalgebra, the relative tensor product of a right special Frobenius module $(m, r_m, \check{r}_m)$ on $m \in \mathcal{B}(A, C)$ and a left special Frobenius module $(n, l_n, \check{l}_n)$ on $n \in \mathcal{B}(B, A)$, $m \otimes_a n$ exists if the idempotent*

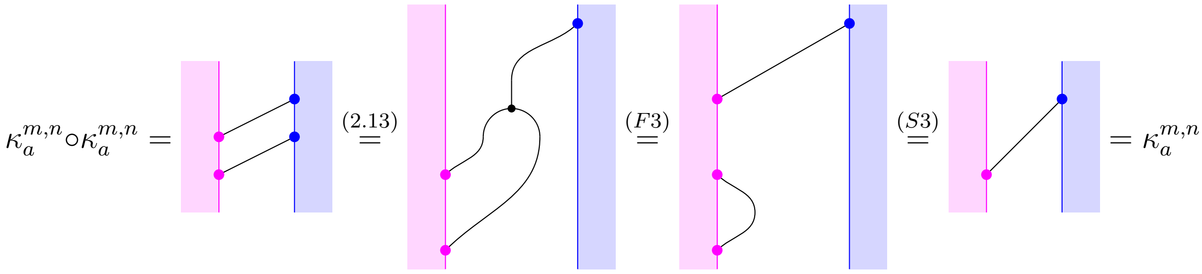
$$\kappa_a^{m,n} := (\mathrm{id}_m \otimes l_n) \circ \alpha_{m,a,n} \circ (\check{r}_m \otimes \mathrm{id}_n) \in \mathrm{Hom}_{\mathcal{B}(B,C)}(m \otimes n, m \otimes n)$$

*splits, and can be written as its image in that case.*

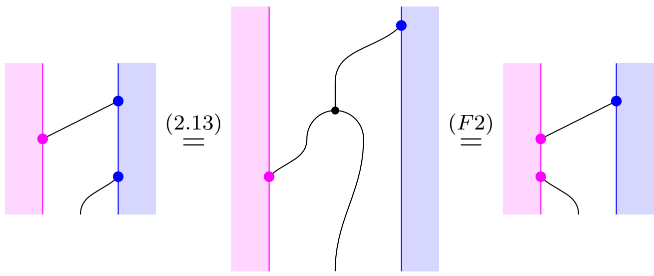
*Proof.* To see that $\kappa_a^{m,n}$ indeed is an idempotent, we write it as a string diagram

$\kappa_a^{m,n} =$ 

and calculate:




Assume it splits. That means there are 2-cells $\theta \in \mathrm{Hom}_{\mathcal{B}(C,B)}(m \otimes n, i)$ and $\psi \in \mathrm{Hom}_{\mathcal{B}(C,B)}(i, m \otimes n)$ such that $\psi \circ \theta = \kappa_a^{m,n}$ and $\theta \circ \psi = \mathrm{id}_i$, where we denoted the image of $\kappa_a^{m,n}$ by $i$. To see that $i = m \otimes_a n$, we first need to check that $\theta \circ l = \theta \circ r$ (for $r$ and $l$ as in Definition 2.47). This follows from $\kappa_a^{m,n} \circ l = \kappa_a^{m,n} \circ r$ (by post-composing with $\theta$ and applying both splitting properties) which can be shown using again the Frobenius module property:




Now we need to check the coequalizer property. For any $z \in \mathcal{B}(B, C)$ and $\zeta \in \mathrm{Hom}_{\mathcal{B}(B,C)}(m \otimes n, z)$ such that $\zeta \circ r = \zeta \circ l$, let $\zeta' := \zeta \circ \psi$. Then we have:

$$\zeta' \circ \theta = \zeta \circ \psi \circ \theta = \zeta \circ \kappa = \zeta \circ l \circ \check{r} = \zeta \circ r \circ \check{r} = \zeta$$

because $r \circ \check{r} = \mathrm{id}_{m \otimes b}$ by (S3).
Finally, to show uniqueness of $\zeta'$, let $\zeta'' \in \mathrm{Hom}_{\mathcal{B}(B,C)}(i, z)$ be another 2-cell such that $\zeta'' \circ \theta = \zeta$ to see:

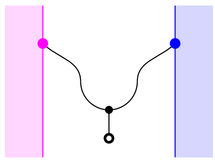
$$\zeta' = \zeta \circ \psi = \zeta'' \circ \theta \circ \psi = \zeta''$$

because $\theta \circ \psi = \mathrm{id}_i$ by the splitting property. $\square$

The more common version of this, for modules between $\Delta$-separable Frobenius algebras, is a direct consequence.

**Proposition 2.49.** *Given a $\Delta$-separable Frobenius algebra $(a, \mu, \Delta, \eta, \epsilon)$, the relative tensor product of a right $a$-module $(m, r_m)$ and a left $a$ module $(n, l_n)$, $m \otimes_a n$ exists if the idempotent*

$$\pi_a^{m,n} := (\mathrm{id}_m \otimes l_n) \circ \alpha_{m,a,n} \circ (((r_m \otimes \mathrm{id}_a) \circ \alpha_{m,a,a}^{-1} \circ (\mathrm{id}_m \otimes (\Delta \circ \epsilon)) \circ \rho_m^{-1}) \otimes \mathrm{id}_n) =$$

$=$  $\in \mathrm{Hom}_{\mathcal{B}(B,C)}(m \otimes n, m \otimes n)$

*splits, and can be written as its image.*

*Proof.* This follows directly from Proposition 2.48 by noting that

$$\pi_a^{m,n} = (\mathrm{id}_m \otimes l_n) \circ \alpha_{m,a,n} \circ (\check{r}_m^{can} \otimes \mathrm{id}_n)$$

is actually equal to $\kappa_a^{m,n}$ using the canonical coaction from Proposition 2.45, and the fact that these are indeed special Frobenius as proven in Proposition 2.46. □

### 2.3.4 2-Categories of Algebras

Given a 2-category $\mathcal{B}$ where in all hom-categories $\mathcal{B}(a, b)$ all idempotents split (i.e. are Karoubi complete), we can define the following 2-categories which have algebras internal to $\mathcal{B}$ as objects.

**Sketch 2.50.** *Given a 2-category $\mathcal{B}$ for which all hom-categories $\mathcal{B}(a, b)$ are Karoubi complete, we can define the following 2-category of nonunital, noncounital $\Delta$-separable Frobenius algebras internal to $\mathcal{B}$ and their special Frobenius bimodules,* $\mathrm{Frob}_{\Delta}^{\cancel{\mathbb{1}},\cancel{\mathbb{I}}}(\mathcal{B})$*:*

- *Objects are the nonunital, noncounital $\Delta$-separable Frobenius algebras $(a, \mu, \Delta)$ of $\mathcal{B}$,*

- $\mathrm{Frob}_{\Delta}^{\cancel{\mathbb{1}},\cancel{\mathbb{I}}}(\mathcal{B})((a, \mu, \Delta), (a', \mu', \Delta'))$ *is the category of special Frobenius bimodules between $a'$ and $a$ and the 2-cells of $\mathcal{B}$ which are both bimodule maps and bicomodule maps between those,*

- *composition of bimodules is given by their relative tensor product,*

- *the unit on $(a, \mu, \Delta)$ is $a$ as an $a$-$a$-bimodule (with $\mu$ as left and right action and $\Delta$ as left and right coaction).*

In order to have units which are indeed special Frobenius modules it is necessary to consider only nonunital, noncounital $\Delta$-separable Frobenius algebras. That the composition of 1-cells is well-defined is guaranteed by Proposition 2.48. Similarly, Proposition 2.49 enables the following definition.

**Sketch 2.51.** *Given a 2-category $\mathcal{B}$ for which all hom-categories $\mathcal{B}(a,b)$ are Karoubi complete, we can define the following 2-category of $\Delta$-separable Frobenius algebras internal to $\mathcal{B}$, $\mathrm{Frob}_\Delta(\mathcal{B})$:*

- *Objects are $\Delta$-separable Frobenius algebras $(a,\mu,\Delta,\eta,\epsilon)$ of $\mathcal{B}$,*
- *$\mathrm{Frob}_\Delta(\mathcal{B})((a,\mu,\Delta,\eta,\epsilon),(a',\mu',\Delta',\eta',\epsilon'))$ is the category of $a'$-$a$-bimodules and their bimodule maps,*
- *composition of bimodules is given by their relative tensor product,*
- *the unit on $(a,\mu,\Delta,\eta,\epsilon)$ is $a$ as an $a$-$a$-bimodule (with $\mu$ both as left and right action).*

The 2-category $\mathrm{Frob}_\Delta(\mathcal{B})$ is discussed in [CR16, Section 4] and referred to as the "equivariant completion" $\mathcal{B}_{\mathrm{eq}}$ there.
How the remaining details of the above 2-categories should look like will become clear from Section 3.4.

# 3 Condensation

The notions of condensation and condensation monad are an approach to generalize the concepts of retract and idempotent (cf. Definition 2.10) to higher categories and was introduced by Davide Gaiotto and Theo Johnson-Freyd in [GJ19]. It can be defined for $n$-categories inductively where one considers equality a 0-condensation and a retract a 1-condensation.
In this section we develop the theory of condensation in 2-categories. Starting with the definition of 2-condensation monads in Section 3.1 and their 2-condensation bimodules in Section 3.2, we establish in Section 3.4 the existence of a 2-category of 2-condensation monads and 2-condensation bimodules between them, where composition is given by their relative tensor product (introduced in Section 3.3). The justifications that it appropriately generalizes the Karoubi completion of 1-categories is given in Section 3.5. Finally Section 3.6 introduces unital condensations.

## 3.1 Condensation Monads

The notion of condensation in a higher category is a formalization of condensation of topological phases of matter and consists of a categorification of retraction in a 1-category.

**Definition 3.1** (2-Condensation)**.** Given objects $X, Y \in \mathrm{Ob}(\mathcal{B})$ in a 2-category $\mathcal{B}$, a *2-condensation* of $X$ onto $Y$ is given by 1-cells $f \in \mathcal{B}(X, Y)$ and $g \in \mathcal{B}(Y, X)$ and a retract of $f \otimes g$ onto $\mathbb{1}_Y$ in $\mathcal{B}(Y, Y)$, i.e. 2-cells $\gamma \in \mathrm{Hom}_{\mathcal{B}(Y,Y)}(f \otimes g, \mathbb{1}_Y)$ and $\delta \in \mathrm{Hom}_{\mathcal{B}(Y,Y)}(\mathbb{1}_Y, f \otimes g)$ such that $\gamma \circ \delta = \mathrm{id}_{\mathbb{1}_Y}$.

Following [GJ19], we introduce the walking 2-condensation $\spadesuit_2$ such that a 2-condensation in a 2-category $\mathcal{B}$ is equivalent to a 2-functor $\spadesuit_2 \to \mathcal{B}$.

**Definition 3.2** ($\spadesuit$ and $\clubsuit$)**.** The 2-category $\spadesuit_2$ is generated by two objects $A$ and $K$, two 1-cells $q \in \spadesuit_2(A, K)$ and $j \in \spadesuit_2(K, A)$ and two 2-cells $\tau \in \mathrm{Hom}_{\spadesuit_2(K,K)}(q \otimes j, \mathbb{1}_K)$ and $\nu \in \mathrm{Hom}_{\spadesuit_2(K,K)}(\mathbb{1}_K, q \otimes j)$ satisfying the relation $\tau \circ \nu = \mathrm{id}_{\mathbb{1}_K}$. The full subcategory on the object $A$ of $\spadesuit_2$ is denoted $\clubsuit_2$.

The 2-category $\spadesuit_2$ can be depicted as the following diagram which is not commutative as $q \otimes j$ is not equal to $\mathrm{id}_K$.

$$\begin{tikzcd}
A \arrow[d, bend right=60, "q"'] \arrow[d, phantom, shift right=8ex, "\tau"] \arrow[d, phantom, shift left=8ex, "\nu"] \\
K \arrow[u, bend right=60, "j"'] \arrow[loop below, "\mathbb{1}_K"]
\end{tikzcd}$$

Note that $(A, K, q, j, \tau, \nu)$ constitutes a 2-condensation in $\spadesuit_2$ according to Definition 3.1.

**Definition 3.3** (2-Condensation Monad)**.** A *2-condensation monad* in a 2-category $\mathcal{B}$ is a 2-functor $\clubsuit_2 \to \mathcal{B}$. We say a 2-condensation monad $(M, \phi) : \clubsuit_2 \to \mathcal{B}$ *condenses* onto $Y \in \mathrm{Ob}(\mathcal{B})$, if $M$ can be lifted to a 2-functor $\spadesuit_2 \to \mathcal{B}$ with $M(K) = Y$. If all 2-condensation monads of $\mathcal{B}$ do condensate, we say $\mathcal{B}$ has *all condensates.*

An idempotent which splits in a 1-category then corresponds to a 2-condensation monad which condenses in a 2-category.

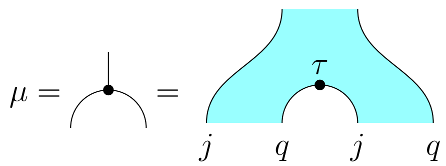
The word monad usually describes an algebra internal to a category of endofunctors of a given 1-category or more generally any endomorphism category. Its use here is therefore justified by the following two results (even though the common sense of the word would require a unital algebra structure).

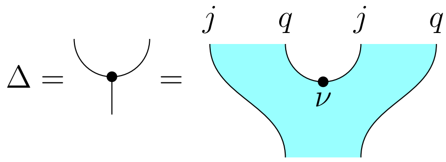
**Proposition 3.4.** *In the 2-category $\clubsuit_2$, $(e, \mu, \Delta)$ where*

$$\begin{aligned}
e &:= j \otimes q \\
\mu &:= (\mathrm{id}_j \otimes (\lambda_q \circ (\tau \otimes \mathrm{id}_q) \circ \alpha^{-1}_{q,j,q})) \circ \alpha_{j,q,j\otimes q} \\
\Delta &:= \alpha^{-1}_{j,q,j\otimes q} \circ (\mathrm{id}_j \otimes (\alpha_{q,j,q} \circ (\nu \otimes \mathrm{id}_q) \circ \lambda_q^{-1}))
\end{aligned}$$

*has the structure of a nonunital, noncounital $\Delta$-separable Frobenius algebra.*

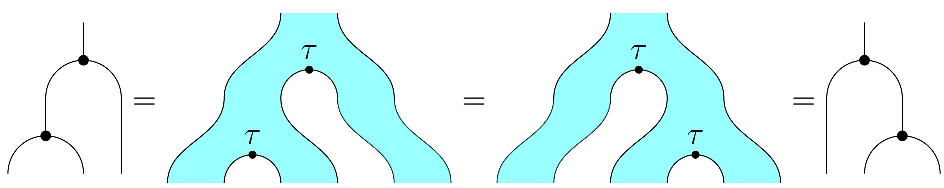
*Proof.* As $\clubsuit_2$ is a full subcategory of $\spadesuit_2$ we can equivalently prove the statement on $(e, \mu, \Delta)$ inside $\spadesuit_2$ using string diagrams where we depict $A$ as the white background, $K$ as cyan and:



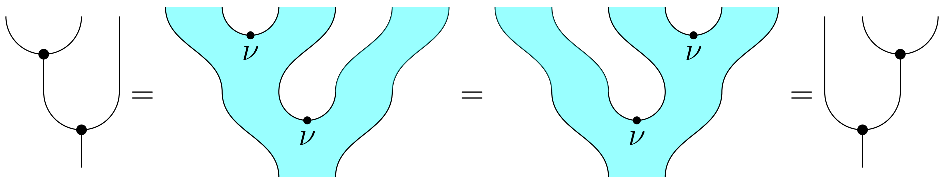




Making repeated use of the interchange law (Proposition 2.28) we can prove:

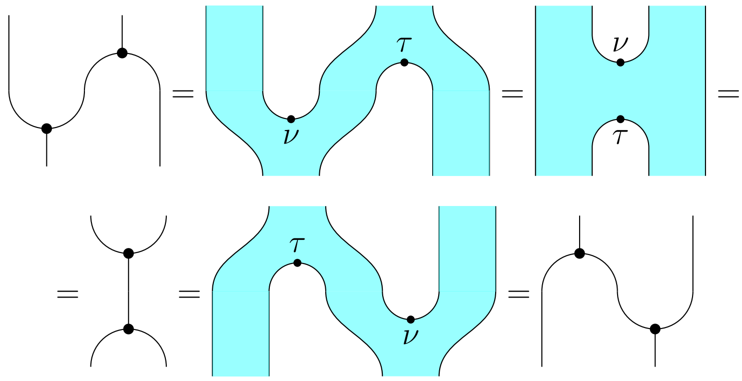
- Associativity:



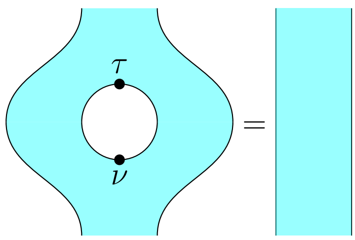
- Coassociativity:




- Frobenius property:




- $\Delta$-separability:

$\mu \circ \Delta =$  $= \mathrm{id}_e$


where we used $\tau \circ \nu = \mathrm{id}_{\mathbb{1}_K}$ from Definition 3.1.

□

**Corollary 3.5.** *For a 2-condensation monad $(M, \phi) : \clubsuit_2 \to \mathcal{B}$ in a 2-category $\mathcal{B}$, with:*

$$A \mapsto X, \quad e \mapsto a, \quad \mu \mapsto M(\mu), \quad \Delta \mapsto M(\Delta)$$

$(a, M(\mu) \circ \phi_{e,e}, \phi^{-1}_{e,e} \circ M(\Delta))$ *has the structure of a nonunital, noncounital $\Delta$-separable Frobenius algebra.*

*Proof.* This follows from Proposition 3.4 by a straightforward application of the naturality of the coherence isomorphism $\phi$ (nat.), the hexagon axiom (Hex.) and functoriality of $M$, writing $m := M(\mu) \circ \phi_{e,e}$ and $d := \phi^{-1}_{e,e} \circ M(\Delta)$.

- Associativity:

$$\begin{aligned}
m \circ (m \otimes \mathrm{id}_a) &= M(\mu) \circ \phi_{e,e} \circ (M(\mu) \otimes \mathrm{id}_a) \circ (\phi_{e,e} \otimes \mathrm{id}_a) \overset{\text{nat.}}{=} \\
&= M(\mu \circ (\mu \otimes \mathrm{id}_e)) \circ \phi_{e\otimes e,e} \circ (\phi_{e,e} \otimes \mathrm{id}_a) \overset{3.4}{=} \\
&= M(\mu \circ (\mathrm{id}_e \otimes \mu) \circ \alpha_{e,e,e}) \circ \phi_{e\otimes e,e} \circ (\phi_{e,e} \otimes \mathrm{id}_a) \overset{\text{Hex.}}{=} \\
&= M(\mu) \circ M(\mathrm{id}_e \otimes \mu) \circ \phi_{e,e\otimes e} \circ (\mathrm{id}_a \otimes \phi_{e,e}) \circ \alpha_{a,a,a} \overset{\text{nat.}}{=} \\
&= M(\mu) \circ \phi_{e,e} \circ (\mathrm{id}_a \otimes M(\mu)) \circ (\mathrm{id}_a \otimes \phi_{e,e}) \circ \alpha_{a,a,a} = \\
&= m \circ (\mathrm{id}_a \otimes m) \circ \alpha_{a,a,a}
\end{aligned}$$

The various applications of the naturality of the coherence isomorphism $\phi$ are summarized in the following diagram, where only the individual squares commute:

$$\begin{tikzcd}
M(e\otimes e)\otimes M(e) \arrow[r, "M(\mu)\otimes M(\mathrm{id}_e)"] \arrow[d, "\phi_{e\otimes e,e}"'] & M(e)\otimes M(e) \arrow[d, "\phi_{e,e}"] & M(e)\otimes M(e\otimes e) \arrow[l, "M(\mathrm{id}_e)\otimes M(\mu)"'] \arrow[d, "\phi_{e,e\otimes e}"] \\
M((e\otimes e)\otimes e) \arrow[r, shift left, "M(\mu\otimes\mathrm{id}_e)"] & M(e\otimes e) \arrow[l, shift left, "M(\Delta\otimes\mathrm{id}_e)"] \arrow[r, shift right, "M(\mathrm{id}_e\otimes\Delta)"'] & M(e\otimes(e\otimes e)) \arrow[l, shift right, "M(\mathrm{id}_e\otimes\mu)"'] \\
M(e\otimes e)\otimes M(e) \arrow[u, "\phi_{e\otimes e,e}"] & M(e)\otimes M(e) \arrow[l, "M(\Delta)\otimes M(\mathrm{id}_e)"] \arrow[u, "\phi_{e,e}"] \arrow[r, "M(\mathrm{id}_e)\otimes M(\Delta)"] & M(e)\otimes M(e\otimes e) \arrow[u, "\phi_{e,e\otimes e}"']
\end{tikzcd}$$

- Coassociativity:

$$\begin{aligned}
(d \otimes \mathrm{id}_a) \circ d =& (\phi_{e,e}^{-1} \otimes \mathrm{id}_a) \circ (M(\Delta) \otimes \mathrm{id}_a) \circ \phi_{e,e}^{-1} \circ M(\Delta) \overset{\text{nat.}}{=} \\
=& (\phi_{e,e}^{-1} \otimes \mathrm{id}_a) \circ \phi_{e\otimes e,e}^{-1} \circ M(\Delta \otimes \mathrm{id}_e) \circ M(\Delta) \overset{3.4}{=} \\
=& (\phi_{e,e}^{-1} \otimes \mathrm{id}_a) \circ \phi_{e\otimes e,e}^{-1} \circ M(\alpha_{e,e,e}^{-1} \circ (\mathrm{id}_e \otimes \Delta) \circ \Delta) \overset{\text{Hex.}}{=} \\
=& \alpha_{a,a,a}^{-1} \circ (\mathrm{id}_a \otimes \phi_{e,e}^{-1}) \circ \phi_{e,e\otimes e}^{-1} \circ M(\mathrm{id}_e \otimes \Delta) \circ M(\Delta) \overset{\text{nat.}}{=} \\
=& \alpha_{a,a,a}^{-1} \circ (\mathrm{id}_a \otimes \phi_{e,e}^{-1}) \circ (\mathrm{id}_a \otimes M(\Delta)) \circ \phi_{e,e}^{-1} \circ M(\Delta) = \\
=& \alpha_{a,a,a}^{-1} \circ (\mathrm{id}_a \otimes d) \circ d
\end{aligned}$$

- Frobenius:

$$
\begin{aligned}
&(\mathrm{id}_a \otimes m) \circ \alpha_{a,a,a} \circ (d \otimes \mathrm{id}_a) = \\
&= (\mathrm{id}_a \otimes M(\mu)) \circ (\mathrm{id}_a \otimes \phi_{e,e}) \circ \alpha_{a,a,a} \circ (\phi_{e,e}^{-1} \otimes \mathrm{id}_a) \circ (M(\Delta) \otimes \mathrm{id}_a) \overset{\text{Hex.}}{=} \\
&= (\mathrm{id}_a \otimes M(\mu)) \circ \phi_{e,e\otimes e}^{-1} \circ M(\alpha_{e,e,e}) \circ \phi_{e\otimes e,e} \circ (M(\Delta) \otimes \mathrm{id}_a) \overset{\text{nat.}}{=} \\
&= \phi_{e,e}^{-1} \circ M(\mathrm{id}_e \otimes \mu) \circ M(\alpha_{e,e,e}) \circ \phi_{e\otimes e,e} \circ (M(\Delta) \otimes \mathrm{id}_a) \overset{\text{nat.}}{=} \\
&= \phi_{e,e}^{-1} \circ M((\mathrm{id}_e \otimes \mu) \circ \alpha_{e,e,e} \circ (\Delta \otimes \mathrm{id}_e)) \circ \phi_{e,e} \overset{3.4}{=} \phi_{e,e}^{-1} \circ M(\Delta \circ \mu) \circ \phi_{e,e} = \\
&= d \circ m = \\
&= \phi_{e,e}^{-1} \circ M((\mu \otimes \mathrm{id}_e) \circ \alpha_{e,e,e}^{-1} \circ (\mathrm{id}_e \otimes \Delta)) \circ \phi_{e,e} \overset{\text{nat.}}{=} \\
&= (M(\mu) \otimes \mathrm{id}_a) \circ \phi_{e\otimes e,e}^{-1} \circ M(\alpha_{e,e,e}^{-1}) \circ M(\mathrm{id}_e \otimes \Delta) \circ \phi_{e,e} \overset{\text{nat.}}{=} \\
&= (M(\mu) \otimes \mathrm{id}_a) \circ \phi_{e\otimes e,e}^{-1} \circ M(\alpha_{e,e,e}^{-1}) \circ \phi_{e,e\otimes e} \circ (\mathrm{id}_a \otimes M(\Delta)) \overset{\text{Hex.}}{=} \\
&= (M(\mu) \otimes \mathrm{id}_a) \circ (\phi_{e,e} \otimes \mathrm{id}_a) \circ \alpha_{a,a,a}^{-1} \circ (\mathrm{id}_a \otimes \phi_{e,e}^{-1}) \circ (\mathrm{id}_a \otimes M(\Delta)) = \\
&= (m \otimes \mathrm{id}_a) \circ \alpha_{a,a,a}^{-1} \circ (\mathrm{id}_a \otimes d)
\end{aligned}
$$

- $\Delta$-separability:

$$
m \circ d = M(\mu) \circ \phi_{e,e} \circ \phi_{e,e}^{-1} \circ M(\Delta) = M(\mu \circ \Delta) \overset{3.4}{=} M(\mathrm{id}_e) = \mathrm{id}_a
$$

□

## 3.2 Condensation Bimodules

In the construction of the Karoubi completion $\mathrm{Kar}(\mathcal{C})$ of a 1-category $\mathcal{C}$ in Proposition 2.12, we had idempotents of $\mathcal{C}$ as objects and morphisms $f$ of $\mathcal{C}$ satisfying

$$
e' \circ f = f = f \circ e
$$

as morphisms between idempotents $e$ and $e'$.
As generalization of these, we introduce the notion of a 2-condensation bimodule between 2-condensation monads and see that they deserve their name, again following [GJ19].
This can be thought of as replacing the equality signs above with “1-condensation onto”, consisting of actions and coactions.

**Definition 3.6** (2-Condensation Bimodule)**.** Define the 2-category $\blacklozenge_2$ to be generated by two copies of $\spadesuit_2$, $(A_1, K_1, q_1, j_1, \tau_1, \nu_1)$ and $(A_2, K_2, q_2, j_2, \tau_2, \nu_2)$, a single 1-cell $n : K_2 \to K_1$ and their relations induced from $\spadesuit_2$. The full subcategory on the objects $A_1$ and $A_2$ is the *walking 2-condensation bimodule* $\diamondsuit_2$. A *2-condensation bimodule* in a 2-category $\mathcal{B}$ is a 2-functor $\diamondsuit_2 \to \mathcal{B}$.

The 2-category $\blacklozenge_2$ can be depicted as follows, introducing the 1-cell

$$m := (j_1 \otimes n) \otimes q_2$$

$$\begin{tikzcd}
A_1 \arrow[loop above, "e_1"] \arrow[d, bend right, "q_1"'] & A_2 \arrow[l, "m"'] \arrow[loop above, "e_2"] \arrow[d, bend right, "q_2"'] \\
K_1 \arrow[u, bend right, "j_1"'] & K_2 \arrow[l, "n"] \arrow[u, bend right, "j_2"']
\end{tikzcd}$$

**Proposition 3.7.** *In the walking 2-condensation bimodule $\diamondsuit_2$, the 1-cells:*

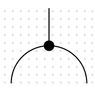
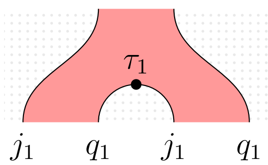
$$e_1 := j_1 \otimes q_1, \quad e_2 := j_2 \otimes q_2, \quad m := (j_1 \otimes n) \otimes q_2$$

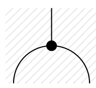
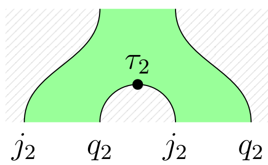
*together with the 2-cells $\mu_1$, $\Delta_1$, $\mu_2$, $\Delta_2$ from the two copies of $\clubsuit_2$ (cf. Proposition 3.4) as well as:*

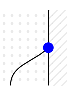
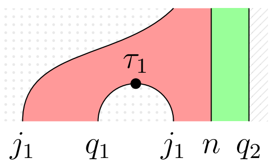
$$\begin{aligned}
l_m :=& \alpha^{-1}_{j_1,n,q_2} \circ ((\rho_{j_1} \circ (\mathrm{id}_{j_1} \otimes \tau_1) \circ \alpha_{j_1,q_1,j_1}) \otimes \mathrm{id}_{n\otimes q_2}) \circ \alpha^{-1}_{j_1\otimes q_1, j_1, n\otimes q_2} \circ (\mathrm{id}_{j_1\otimes q_1} \otimes \alpha_{j_1,n,q_2}) \\
r_m :=& (\mathrm{id}_{j_1\otimes n} \otimes (\lambda_{q_2} \circ (\tau_2 \otimes \mathrm{id}_{q_2}) \circ \alpha^{-1}_{q_2,j_2,q_2})) \circ \alpha_{j_1\otimes n, q_2, j_2\otimes q_2} \\
\check{l}_m :=& (\mathrm{id}_{j_1\otimes q_1} \otimes \alpha^{-1}_{j_1,n,q_2}) \circ \alpha_{j_1\otimes q_1, j_1, n\otimes q_2} \circ ((\alpha^{-1}_{j_1,q_1,j_1} \circ (\mathrm{id}_{j_1} \otimes \nu_1) \circ \rho^{-1}_{j_1}) \otimes \mathrm{id}_{n\otimes q_2}) \circ \alpha_{j_1,n,q_2} \\
\check{r}_m :=& \alpha^{-1}_{j_1\otimes n, q_2, j_2\otimes q_2} \circ (\mathrm{id}_{j_1\otimes n} \otimes (\alpha_{q_2,j_2,q_2} \circ (\nu_2 \otimes \mathrm{id}_{q_2}) \circ \lambda^{-1}_{q_2}))
\end{aligned}$$

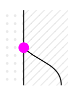
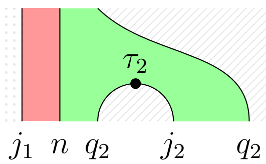
*satisfy the relations making $(e_1, \mu_1, \Delta_1)$ and $(e_2, \mu_2, \Delta_2)$ nonunital, noncounital $\Delta$-separable Frobenius algebras and $(m, l_m, r_m, \check{l}_m, \check{r}_m)$ a special Frobenius bimodule.*

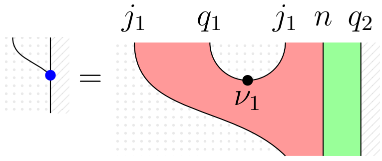
*Proof.* As the full subcategories of $\diamondsuit_2$ on the objects $A_1$ and $A_2$ are isomorphic to $\clubsuit_2$, the statements about $(e_1, \mu_1, \Delta_1)$ and $(e_2, \mu_2, \Delta_2)$ follow directly from Proposition 3.4. The other equalities are proven inside the bigger category $\blacklozenge_2$, using string diagrammatic notation where $A_1$ is represented by dotted, $K_1$ by red, $A_2$ by dashed and $K_2$ by green background:

$\mu_1 =$  $=$ 

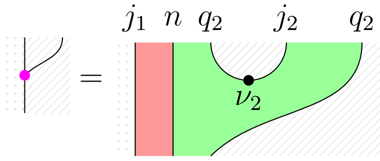

$\mu_2 =$  $=$ 


$l_m =$  $=$ 


$r_m =$  $=$ 

$\check{l}_m =$ 

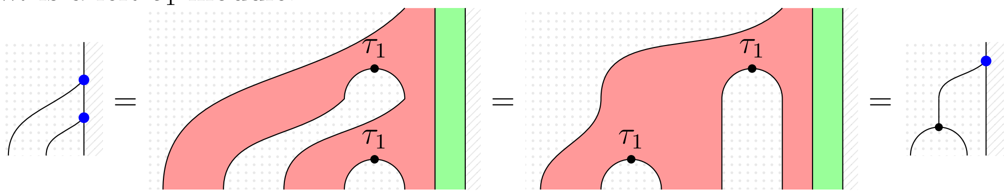


$\check{r}_m =$ 



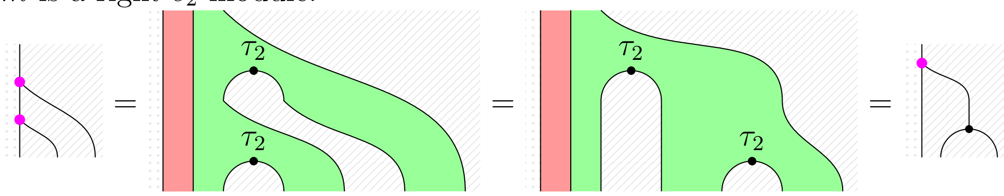
making again repeated use of Proposition 2.28.

- $m$ is a left $e_1$-module:



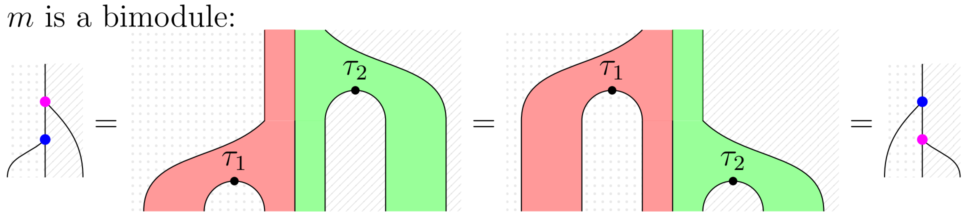


- $m$ is a right $e_2$-module:





- $m$ is a bimodule:





The comodule property follows by mirroring the string diagrams proving the module property upside down. Note that the sting diagrams for the module and comodule properties are obtained from the ones for associativity and coassociativity in Proposition 3.4 by just changing the backgrounds and adding a bar on one side. Doing the same to the diagrams showing $\Delta$-separability and the Frobenius property in Proposition 3.4, one proves that $m$ satisfies the relations making it a special Frobenius bimodule. $\square$

**Corollary 3.8.** *For a 2-condensation bimodule $(B,\phi) : \diamondsuit_2 \to \mathcal{B}$,*

$$(B(m), B(l_m) \circ \phi_{e_1,m}, B(r_m) \circ \phi_{m,e_2}, \phi^{-1}_{e_1,m} \circ B(\check{l}_m), \phi^{-1}_{m,e_2} \circ B(\check{r}_m))$$

*is a $(B(e_1), B(\mu_1) \circ \phi_{e_1,e_1})$-$(B(e_2), B(\mu_2) \circ \phi_{e_2,e_2})$ special Frobenius bimodule.*

*Proof.* This is analogous to the proof of Corollary 3.5, writing for $i \in \{1,2\}$ $b := B(m)$, $a_i := B(e_i)$, $m_i := B(\mu_i) \circ \phi_{e_i,e_i}$, $l_b := B(l_m) \circ \phi_{e_1,m}$ and $r_b := B(r_m) \circ \phi_{m,e_2}$.

- $b$ is a left $a_1$-module:

$$\begin{aligned}
l_b \circ (m_1 \otimes \mathrm{id}_b) &= B(l_m) \circ \phi_{e_1,m} \circ (B(\mu_1) \otimes \mathrm{id}_b) \circ (\phi_{e_1,e_1} \otimes \mathrm{id}_b) \overset{\text{nat.}}{=} \\
&= B(l_m) \circ B(\mu_1 \otimes \mathrm{id}_m) \circ \phi_{e_1 \otimes e_1,m} \circ (\phi_{e_1,e_1} \otimes \mathrm{id}_b) \overset{3.7}{=} \\
&= B(l_m \circ (\mathrm{id}_{e_1} \otimes l_m) \circ \alpha_{e_1,e_1,m}) \circ \phi_{e_1 \otimes e_1,m} \circ (\phi_{e_1,e_1} \otimes \mathrm{id}_b) \overset{\text{Hex.}}{=} \\
&= B(l_m) \circ B(\mathrm{id}_{e_1} \otimes l_m) \circ \phi_{e_1,e_1 \otimes m} \circ (\mathrm{id}_{a_1} \otimes \phi_{e_1,m}) \circ \alpha_{a_1,a_1,b} \overset{\text{nat.}}{=} \\
&= B(l_m) \circ \phi_{e_1,m} \circ (\mathrm{id}_{a_1} \otimes B(l_m)) \circ (\mathrm{id}_{a_1} \otimes \phi_{e_1,m}) \circ \alpha_{a_1,a_1,b} = \\
&= l_b \circ (\mathrm{id}_{a_1} \otimes l_b) \circ \alpha_{a_1,a_1,b}
\end{aligned}$$

- $b$ is a right $a_2$-module:

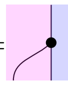
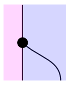
$$\begin{aligned}
r_b \circ (\mathrm{id}_b \otimes m_2) &= B(r_m) \circ \phi_{m,e_2} \circ (\mathrm{id}_b \otimes B(\mu_2)) \circ (\mathrm{id}_b \otimes \phi_{e_2,e_2}) \overset{\text{nat.}}{=} \\
&= B(r_m) \circ B(\mathrm{id}_m \otimes \mu_2) \circ \phi_{m,e_2 \otimes e_2} \circ (\mathrm{id}_b \otimes \phi_{e_2,e_2}) \overset{3.7}{=} \\
&= B(r_m \circ (r_m \otimes \mathrm{id}_{e_2}) \circ \alpha^{-1}_{m,e_2,e_2}) \circ \phi_{m,e_2 \otimes e_2} \circ (\mathrm{id}_b \otimes \phi_{e_2,e_2}) \overset{\text{Hex.}}{=} \\
&= B(r_m) \circ B(r_m \otimes \mathrm{id}_{e_2}) \circ \phi_{m \otimes e_2,e_2} \circ (\phi_{m,e_2} \otimes \mathrm{id}_{a_2}) \circ \alpha^{-1}_{b,a_2,a_2} \overset{\text{nat.}}{=} \\
&= B(r_m) \circ \phi_{m,e_2} \circ (B(r_m) \otimes \mathrm{id}_{a_2}) \circ (\phi_{m,e_2} \otimes \mathrm{id}_{a_2}) \circ \alpha^{-1}_{b,a_2,a_2} = \\
&= r_b \circ (r_b \otimes \mathrm{id}_{a_2}) \circ \alpha^{-1}_{b,a_2,a_2}
\end{aligned}$$

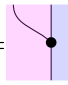
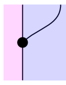
- $b$ is a bimodule:

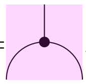
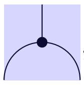
$$\begin{aligned}
r_b \circ (l_b \otimes \mathrm{id}_{a_2}) &= B(r_m) \circ \phi_{m,e_2} \circ (B(l_m) \otimes \mathrm{id}_{a_2}) \circ (\phi_{e_1,m} \otimes \mathrm{id}_{a_2}) \overset{\text{nat.}}{=} \\
&= B(r_m) \circ (B(l_m \mathrm{id}_{e_2}) \circ \phi_{e_\otimes m,e_2} \circ (\phi_{e_1,m} \otimes \mathrm{id}_{a_2}) \overset{3.7}{=} \\
&= B(l_m \circ (\mathrm{id}_{e_1} \otimes r_m) \circ \alpha_{e_1,m,e_2}) \circ \phi_{e_1 \otimes m,e_2} \circ (\phi_{e_1,m} \otimes \mathrm{id}_{a_2}) \overset{\text{Hex.}}{=} \\
&= B(l_m) \circ B(\mathrm{id}_{e_1} \otimes r_m) \circ \phi_{e_1,m \otimes e_2} \circ (\mathrm{id}_{a_1} \otimes \phi_{m,e_2}) \circ \alpha_{a_1,b,a_2} \overset{\text{nat.}}{=} \\
&= B(l_m) \circ \phi_{e_1,m} \circ (\mathrm{id}_{a_1} \otimes B(r_m)) \circ (\mathrm{id}_{a_1} \otimes \phi_{m,e_2}) \circ \alpha_{a_1,b,a_2} = \\
&= l_b \circ (\mathrm{id}_{a_1} \otimes r_b) \circ \alpha_{a_1,b,a_2}
\end{aligned}$$

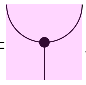
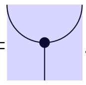
The proof of the module property has precisely the same structure as the proof of associativity in Corollary 3.5. Similarly, comodule properties and special Frobenius relations follow completely analogously to coassociativity and $\Delta$-separable Frobenius relations in Corollary 3.5. □

**Remark 3.9.** *From now on we will state equations exclusively in the form of string diagrams whenever convenient. To do so we will use the following notation:*

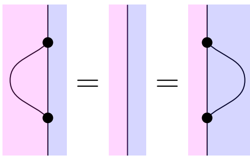
$B(l_m) \circ \phi_{e_1,m} =$ , $\quad B(r_m) \circ \phi_{m,e_2} =$ ,

$\phi^{-1}_{e_1,m} \circ B(\check{l}_m) =$ , $\quad \phi^{-1}_{m,e_2} \circ B(\check{r}_m) =$ ,

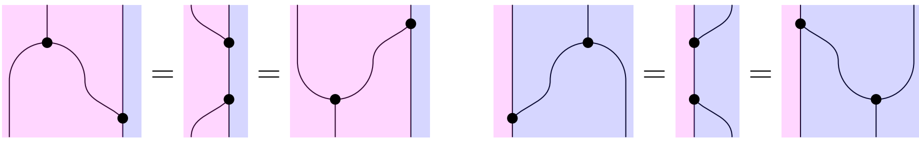
$B(\mu_1) \circ \phi_{e_1,e_1} =$ , $\quad B(\mu_2) \circ \phi_{e_2,e_2} =$ ,

$\phi^{-1}_{e_1,e_1} \circ B(\Delta_1) =$ , $\quad \phi^{-1}_{e_2,e_2} \circ B(\Delta_2) =$ .

*The statement from Corollary 3.8, that the bimodule b is special Frobenius then takes the following form:*



(S4)



(F4)

## 3.3 Relative Tensor Product of Condensation Bimodules

Having established that 2-condensation monads and 2-condensation bimodules have underlying algebras and bimodules between them, we look for a natural way to find a 2-condensation bimodule which corresponds to the relative tensor product of the underlying bimodules. Thus we want to find a way to combine two condensation bimodules, which have a condensation monad in common, into a new condensation bimodule. We start by gluing two copies of $\blacklozenge_2$ along $\spadesuit_2$, i.e. taking the pushout (cf. 2.16) in the 1-category of 2-categories and 2-functors.

**Definition 3.10** ($\blacktriangledown_2$ and $\heartsuit_2$)**.** Define $\blacktriangledown_2$ to be the pushout of two copies of $\blacklozenge_2$ along the two different canonical inclusions of $\spadesuit_2$, $\blacktriangledown_2 := \blacklozenge_2 \sqcup_{\spadesuit_2} \blacklozenge_2$ (schematically depicted below) and $\heartsuit_2$ to be the full subcategory on the objects $A$, $A_1$ and $A_2$ therein.
In $\heartsuit_2$ the *tensor product of condensation bimodules* is defined to be:

$$(j_1 \otimes n_1) \otimes (n_2 \otimes q_2) = N_1 \otimes N_2$$

where $N_1 := j_1 \otimes n_1$ and $N_2 := n_2 \otimes q_2$.

$$
\begin{tikzcd}
A_1 \arrow[loop above, "e_1"] \arrow[d, bend right, "q_1"'] & & A \arrow[ll, "m_1"'] \arrow[loop above, "e"] \arrow[dd, bend right, "q"'] & & A_2 \arrow[ll, "m_2"'] \arrow[loop above, "e_2"] \arrow[d, bend right, "q_2"'] \\
K_1 \arrow[u, bend right, "j_1"'] & & & & K_2 \arrow[u, bend right, "j_2"'] \arrow[dll, "n_2"] \\
& & K \arrow[ull, "n_1"] \arrow[uu, bend right, "j"'] & &
\end{tikzcd}
$$

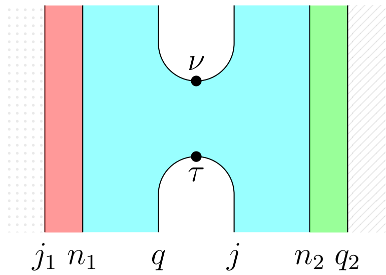
To summarize we have the following four different condensation bimodules $\diamondsuit_2 \to \heartsuit_2$: the two canonical inclusions mapping $m \mapsto m_1$ and $m \mapsto m_2$, their naive combination mapping $m \mapsto m_1 \otimes m_2$ and their relative tensor product defined above $m \mapsto N_1 \otimes N_2$. Note that the latter two are not equal, but rather that we have a 1-condensation from $m_1 \otimes m_2$ onto $N_1 \otimes N_2$, as the following lemma shows.

**Lemma 3.11.** *In $\heartsuit_2(A_2, A_1)$, $N_1 \otimes N_2$ is the image of the splitting idempotent*

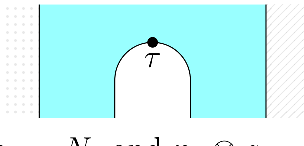
$$\kappa := \alpha^{-1} \circ ((\alpha^{-1}_{N_1,q,j} \circ (\mathrm{id}_{N_1} \otimes (\nu \circ \tau)) \circ \alpha_{N_1,q,j}) \otimes \mathrm{id}_{N_2}) \circ \alpha$$

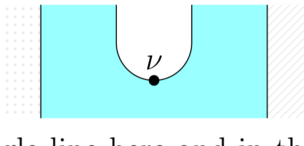
*where $\alpha := \alpha^{-1}_{m_1,j,N_2} \circ (\mathrm{id}_{m_1} \otimes \alpha_{j,n_2,q_2})$.*

*Proof.* Note that $m_1 \otimes m_2 = ((j_1 \otimes n_1) \otimes q) \otimes ((j \otimes n_2) \otimes q_2)$ by definition. As a string diagram we can write:

$\kappa =$ 

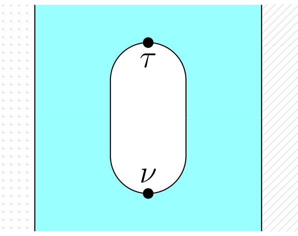

In the bigger category $\blacktriangledown_2$ we thus see that we can write $\kappa = \psi \circ \theta$ with:

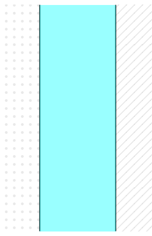
$\theta :=$ 

$\psi :=$ 


Denoting $j_1 \otimes n_1 = N_1$ and $n_2 \otimes q_2 = N_2$ by a single line here and in the following (note that this is an abuse of notation as $n_2$ and $q_2$ are not bracketed together in the domain of $\kappa$ which is $m_1 \otimes m_2$).

Then, because of $\tau \circ \nu = \mathrm{id}_{\mathbb{1}_K}$, we find:

$\theta \circ \psi =$ 

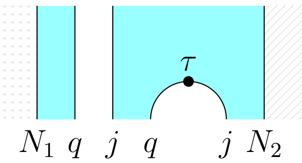
$=$ 
$= \mathrm{id}_{N_1 \otimes N_2}$

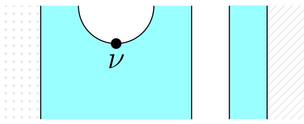
Note that while $N_1$ and $N_2$ on their own exist only in $\heartsuit_2$, $N_1 \otimes N_2$ as well as $\theta$ and $\psi$ do also exist in $\heartsuit_2$. $\square$

Now we are ready to see that the tensor product of condensation bimodules defined in Definition 3.10, actually deserves its name and is equal to the relative tensor product from Definition 2.47 of $m_1$ and $m_2$ with their bimodule structures as described in Proposition 3.7.

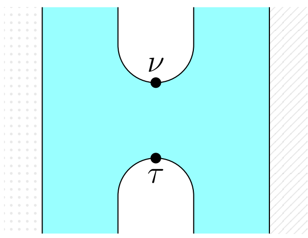
**Proposition 3.12.** *In $\heartsuit_2$, $(N_1 \otimes N_2, \theta)$ is the relative tensor product of $m_1$ and $m_2$ over $e$, $m_1 \otimes_e m_2$.*

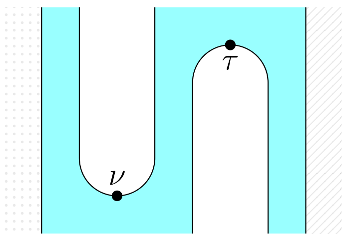
*Proof.* Writing the maps $l := (\mathrm{id}_{m_1} \otimes l_{m_2}) \circ \alpha_{m_1,e,m_2}$ and $\check{r} := \check{r}_{m_1} \otimes \mathrm{id}_{m_2} =$ as string diagrams:

$l =$ 

$\check{r} =$ 


we can see, using the interchange law (Proposition 2.28), that

$\kappa =$ 

$=$ 

$= l \circ \check{r}$

is equal to $\kappa_e^{m_1,m_2}$. The result now follows from Proposition 2.48 because $m_1$ and $m_2$ are special Frobenius modules as was proven in Proposition 3.7. $\square$

Next we want to use the tensor product of condensation bimodules in $\heartsuit_2$ from Definition 3.10 to construct a new 2-condensation bimodule from two given ones which are composable. Two 2-condensation bimodules $B_1, B_2 : \diamondsuit_2 \to \mathcal{B}$ are composable if $B_1(A_2) = B_2(A_1)$, $B_1(e_2) = B_2(e_1) = a$ and they give the same algebra structure on $a$. Put differently that means $B_1$ restricted to $A_2$ is the same

2-condensation monad as $B_2$ restricted to $A_1$. Thus $B_1$ and $B_2$ are composable if they can be combined to a single 2-functor $B : \diamondsuit_2 \sqcup_{\clubsuit_2} \diamondsuit_2 \to \mathcal{B}$ where $\diamondsuit_2 \sqcup_{\clubsuit_2} \diamondsuit_2$ is the pushout (in the 1-category of 2-categories and 2-functors) of two copies of $\diamondsuit_2$ along the two canonical inclusions of $\clubsuit_2$ mapping $A \mapsto A_1$ and $A \mapsto A_2$ respectively.

**Remark 3.13.** *So far we have considered the following small 2-categories:*

$$\clubsuit_2 = \begin{tikzcd} A \arrow[loop above, "e"] \arrow[loop left, "e\otimes e"] \arrow[loop right, "e\otimes e"] \arrow[loop below, "\mathbb{1}_A"] \arrow[phantom, "\mu", ul] \arrow[phantom, "\Delta", ur] \end{tikzcd} \subset \spadesuit_2 = \begin{tikzcd} A \arrow[loop above, "e"] \arrow[d, bend right, "q"'] \\ K \arrow[u, bend right, "j"'] \arrow[loop below, "\mathbb{1}_K"] \arrow[phantom, "\tau", l] \arrow[phantom, "\nu", r] \end{tikzcd}$$

$$\diamondsuit_2 = \begin{tikzcd} A_1 \arrow[loop above, "e_1"] & A_2 \arrow[loop above, "e_2"] \arrow[l, "m"'] \end{tikzcd} \subset \blacklozenge_2 = \begin{tikzcd} A_1 \arrow[loop above, "e_1"] \arrow[d, bend right, "q_1"'] & A_2 \arrow[loop above, "e_2"] \arrow[l, "m"'] \arrow[d, bend right, "q_2"'] \\ K_1 \arrow[u, bend right, "j_1"'] & K_2 \arrow[l, "n"] \arrow[u, bend right, "j_2"'] \end{tikzcd}$$

$$\diamondsuit_2 \sqcup_{\clubsuit_2} \diamondsuit_2 = \begin{tikzcd} A_1 \arrow[loop above, "e_1"] & A \arrow[loop above, "e"] \arrow[l, "m_1"'] & A_2 \arrow[loop above, "e_2"] \arrow[l, "m_2"'] \end{tikzcd} \subset \heartsuit_2 = \begin{tikzcd} A_1 \arrow[loop above, "e_1"] & A \arrow[loop above, "e"] \arrow[l, bend right, "m_1"'] & A_2 \arrow[loop above, "e_2"] \arrow[l, bend right, "m_2"'] \arrow[ll, bend left, "m_1\otimes_e m_2"] \end{tikzcd}$$

$$\subset \blacktriangledown_2 = \begin{tikzcd} A_1 \arrow[loop above, "e_1"] \arrow[d, bend right, "q_1"'] & A \arrow[loop above, "e"] \arrow[l, bend right, "m_1"'] \arrow[dd, bend right, "q"'] & A_2 \arrow[loop above, "e_2"] \arrow[l, bend right, "m_2"'] \arrow[d, bend right, "q_2"'] \\ K_1 \arrow[u, bend right, "j_1"'] & & K_2 \arrow[u, bend right, "j_2"'] \\ & K \arrow[ul, "n_1"] \arrow[uu, bend right, "j"'] & \arrow[from=ul, to=l] \end{tikzcd}$$

**Remark 3.14.** *While $\spadesuit_2$ and $\blacklozenge_2$ are given in terms of generators and relations, this is not the case for $\clubsuit_2$ and $\diamondsuit_2$.*
*One may think that $\clubsuit_2$ is generated by $A$, $e$, $\mu$ and $\Delta$ defined in Proposition 3.4 and the relations making it a nonunital, noncounital $\Delta$-separable Frobenius algebra,*

*but that is only almost the case. While in the latter 2-category, the only 1-cells are powers of $e = (j \otimes q)$, $\clubsuit_2$ contains also other bracketings of combinations of $j \otimes q$, for example $j \otimes ((q \otimes j) \otimes q)$ which is not the same as $e \otimes e = (j \otimes q) \otimes (j \otimes q)$. But as those are just rebracketings of 1-cells which are powers of $e$, they are all isomorphic via suitable combinations of associators. Thus it is clear that $\clubsuit_2$ (while not equal too) is equivalent as a 2-category to the one generated by $A$, $e$, $\mu$ and $\Delta$ and the relations making it a nonunital, noncounital $\Delta$-separable Frobenius algebra.*
*Completely analogously one sees that $\diamondsuit_2$ is equivalent as a 2-category to the one generated by $A_1$, $A_2$, $e_1$, $e_2$, $m$, $\mu_1$, $\Delta_1$, $\mu_2$, $\Delta_2$, $r_m$, $\check{r}_m$, $l_m$ and $\check{l}_m$ and the relations making $(e_1, \mu_1, \Delta_1)$, $(e_2, \mu_2, \Delta_2)$ nonunital, noncounital $\Delta$-separable Frobenius algebras and $(m, r_m, \check{r}_m, l_m, \check{l}_m)$ a special Frobenius bimodule.*
*Thus, if we want to define a 2-condensation monad or a 2-condensation bimodule, it suffices to suitably (i.e. so that they satisfy the relations) assign images to the generators named above. This is possible as follows: map all 1-cells which are isomorphic via a combination of associators, to the same fixed bracketing (e.g. all brackets to the right) of the images of the appropriate generators. The images of said associators then all are the identity, and the hexagon axiom can be satisfied by defining the coherence isomorphism of the 2-functor to be the appropriate combination of associators of the codomain category.*
*For example, after choosing a nonunital, noncounital $\Delta$-separable Frobenius algebra $a \in \mathcal{B}(A, A)$ we can define a 2-condensation monad $M : \clubsuit_2 \to \mathcal{B}$ which maps $(j \otimes ((q \otimes j) \otimes q)) \otimes (j \otimes q)$ as well as $e \otimes (e \otimes e)$ and $(e \otimes e) \otimes e$ to $a \otimes (a \otimes a)$.*
*This will be used repeatedly when 2-condensation monads and 2-condensation bimodules are constructed explicitly.*
*Similar arguments apply to $\heartsuit_2$ if we add to the three algebras and the two bimodules from the overlapping copies of $\diamondsuit_2$ the 1-cell $m_1 \otimes_e m_2$, the 2-cells $\theta$ and $\psi$ and the relations so that this indeed constitutes the relative tensor product of $m_1$ and $m_2$ over $e$.*
*The generators discussed here are precisely the ones depicted in the diagrams, e.g. in Remark 3.13 above (where 2-cells are omitted in the bigger diagrams).*

**Definition 3.15** (Tensor Product of Condensation Bimodules)**.** Let $B_1$ and $B_2$ be two composable 2-condensation bimodules, combined to a single 2-functor $(B, \phi)$ : $\diamondsuit_2 \sqcup_{\clubsuit_2} \diamondsuit_2 \to \mathcal{B}$. Consider

$$\mathfrak{l} := (\mathrm{id}_{b_1} \otimes (B(l_{m_2}) \circ \phi_{e,m_2})) \circ \alpha_{b_1,a,b_2} \quad \text{and} \quad \check{\mathfrak{r}} := (\phi^{-1}_{m_1,e} \circ B(\check{r}_{m_1})) \otimes \mathrm{id}_{b_2}$$

to form the idempotent $\mathfrak{k} := \mathfrak{l} \circ \check{\mathfrak{r}}$.
If $\mathfrak{k}$ splits, its image is defined to be the *relative tensor product* $b_1 \otimes_a b_2$ of $b_1 := B(m_1)$ and $b_2 := B(m_2)$ over $a := B(e)$.
In this case $B$ can be extended to a 2-functor on $\heartsuit_2$ via $B(N_1 \otimes N_2) := b_1 \otimes_a b_2$, the restriction of which along the appropriate inclusion $\diamondsuit_2 \hookrightarrow \heartsuit_2$ (which maps

$m \mapsto N_1 \otimes N_2$) gives the corresponding 2-condensation bimodule $B_1 \otimes_M B_2 : \diamondsuit_2 \to \mathcal{B}$, where the 2-condensation monad $B_1|_{\clubsuit_2} = B_2|_{\clubsuit_2}$ is denoted by $M$.

**Proposition 3.16.** *The tensor product of Definition 3.15 is well-defined and equal to the relative tensor product of the corresponding modules as in Definition 2.47 (with module structure as in Corollary 3.8).*

*Proof.* That the image of $\mathfrak{k}$, if it exists, indeed is the relative tensor product is given by Proposition 2.48. To see that we get a well-defined 2-condensation monad, we need to check that $\mathfrak{k}$ and $B(\kappa)$ are suitably compatible. This is a calculation using the naturality of the coherence isomorphism $\phi$ (nat.) and the hexagon axiom (Hex.):

$$\begin{aligned}
\mathfrak{k} = \mathfrak{l} \circ \check{\mathfrak{r}} &= (\mathrm{id}_{b_1} \otimes (B(l_{m_2}) \circ \phi_{e,m_2})) \circ \alpha_{b_1,a,b_2} \circ ((\phi^{-1}_{m_1,e} \circ B(\check{r}_{m_1})) \otimes \mathrm{id}_{b_2}) = \\
&= (\mathrm{id}_{b_1} \otimes B(l_{m_2})) \circ (\mathrm{id}_{b_1} \otimes \phi_{e,m_2}) \circ \alpha_{b_1,a,b_2} \circ (\phi^{-1}_{m_1,e} \otimes \mathrm{id}_{b_2}) \circ (B(\check{r}_{m_1}) \otimes \mathrm{id}_{b_2}) \overset{\text{nat.}}{=} \\
&= \phi^{-1}_{m_1,m_2} \circ B(\mathrm{id}_{m_1} \otimes l_{m_2}) \circ \phi_{m_1,e\otimes m_2} \circ (\mathrm{id}_{b_1} \otimes \phi_{e,m_2}) \circ \\
&\quad \circ \alpha_{b_1,a,b_2} \circ (\phi^{-1}_{m_1,e} \otimes \mathrm{id}_{b_2}) \circ \phi^{-1}_{m_1 \otimes e,m_2} \circ B(\check{r}_{m_1} \otimes \mathrm{id}_{m_2}) \circ \phi_{m_1,m_2} \overset{\text{Hex.}}{=} \\
&= \phi^{-1}_{m_1,m_2} \circ B(\mathrm{id}_{m_1} \otimes l_{m_2}) \circ B(\alpha_{m_1,e,m_2}) \circ B(\check{r}_{m_1} \otimes \mathrm{id}_{m_2}) \circ \phi_{m_1,m_2} = \\
&= \phi^{-1}_{m_1,m_2} \circ B(\kappa) \circ \phi_{m_1,m_2}
\end{aligned}$$

Now assume $\mathfrak{k}$ splits along some $\Theta$ and $\Psi$ so that $\mathfrak{k} = \Psi \circ \Theta$ and call its image $b_1 \otimes_a b_2$. Then we can extend $B$ to a functor on $\heartsuit_2$ by defining $B(N_1 \otimes N_2) := b_1 \otimes_a b_2$, $B(\theta) := \Theta \circ \phi^{-1}_{m_1,m_2}$ and $B(\psi) := \phi_{m_1,m_2} \circ \Psi$. This is sufficient as explained in Remark 3.14. $\square$

## 3.4 2-Karoubi Completion

The goal of this section is to establish the existence of a 2-category $\mathrm{Kar}(\mathcal{B})$, which will turn out to be the appropriate generalization of the ordinary Karoubi completion described in Proposition 2.12.

**Theorem 3.17.** *Given a 2-category $\mathcal{B}$ for which all hom-categories $\mathcal{B}(X, Y)$ are Karoubi complete (cf. Definition 2.10), there is a 2-category* $\mathrm{Kar}(\mathcal{B})$ *which has the 2-condensation monads of $\mathcal{B}$ (cf. Definition 3.3) as objects, their 2-condensation bimodules (cf. Definition 3.6) as 1-cells and the relative tensor product (cf. Definition 3.15) as horizontal composition.*

Because of this definition, assume from now on that $\mathcal{B}$ is a 2-category for which all hom-categories $\mathcal{B}(X, Y)$ are Karoubi complete.

### 3.4.1 Units

The relative tensor product of 2-condensation bimodules is defined only up to unique isomorphism and it turns out to be useful and possible to make the choice so that $\mathrm{Kar}(\mathcal{B})$ possesses strict units.

**Lemma 3.18.** *Given a 2-condensation monad $M : \clubsuit_2 \to \mathcal{B}$ with $M(A) = X$, $M(e) = a$, there is a 2-condensation bimodule $\mathbb{1}_M : \diamondsuit_2 \to \mathcal{B}$ with $\mathbb{1}_M(A_1) = \mathbb{1}_M(A_2) = X$ and $\mathbb{1}_M(e_1) = \mathbb{1}_M(e_2) = \mathbb{1}_M(m) = a$ which acts as a strict unit for the relative tensor product.*

*Proof.* By Remark 3.14 it suffices to specify the multiplication and comultiplication on $\mathbb{1}_M(e_1)$ and $\mathbb{1}_M(e_2)$ and the left and right actions and coactions on $\mathbb{1}_M(m)$. But all of those can trivially be chosen to be the multiplication and comultiplication on $a$ given by $M$.

Given now any 2-condensation bimodule $B : \diamondsuit_2 \to \mathcal{B}$ such that its restriction to $A_1$ agrees with $M$ as well (i.e. $\mathbb{1}_M$ and $B$ are composable) and denoting $B(m) = b$, we want to compute $a \otimes_a b$. As $\mathbb{1}_M(\check{r}_m) = M(\Delta) = B(\Delta_1)$ we can write the crucial idempotent $\mathfrak{k}$ (cf. Definition 3.15) as follows and compute:

$$\begin{aligned}
\mathfrak{k} =&(\mathrm{id}_a \otimes (B(l_m) \circ \phi_{e_1,m})) \circ \alpha_{a,a,b} \circ ((\phi^{-1}_{e_1,e_1} \circ B(\Delta_1)) \otimes \mathrm{id}_b) = \\
=&(\mathrm{id}_a \otimes B(l_m)) \circ (\mathrm{id}_a \otimes \phi_{e_1,m}) \circ \alpha_{a,a,b} \circ (\phi^{-1}_{e_1,e_1} \otimes \mathrm{id}_b) \circ (B(\Delta_1) \otimes \mathrm{id}_b) \overset{\text{nat.}}{=} \\
=&\phi^{-1}_{e_1,m} \circ B(\mathrm{id}_{e_1} \otimes l_m) \circ \phi_{e_1,e_1 \otimes m} \circ (\mathrm{id}_a \otimes \phi_{e_1,m}) \circ \alpha_{a,a,b} \circ \\
&(\phi^{-1}_{e_1,e_1} \otimes \mathrm{id}_b) \circ \phi^{-1}_{e_1 \otimes e_1,m} \circ B(\Delta_1 \otimes \mathrm{id}_m) \circ \phi_{e_1,m} \overset{\text{Hex.}}{=} \\
=&\phi^{-1}_{e_1,m} \circ B(\mathrm{id}_{e_1} \otimes l_m) \circ B(\alpha_{e_1,e_1,m}) \circ B(\Delta_1 \otimes \mathrm{id}_m) \circ \phi_{e_1,m} \overset{3.7}{=} \\
=&\phi^{-1}_{e_1,m} \circ B(\check{l}_m \circ l_m) \circ \phi_{e_1,m} = \phi^{-1}_{e_1,m} \circ B(\check{l}_m) \circ B(l_m) \circ \phi_{e_1,m} \overset{3.7}{=} \check{l}_b \circ l_b
\end{aligned}$$

It is easy to see that this indeed gives a splitting $\mathfrak{k}$, by Proposition 3.7:

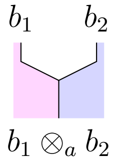
$$l_b \circ \check{l}_b = B(l_m) \circ \phi_{e_1,m} \circ \phi^{-1}_{e_1,m} \circ B(\check{l}_m) = B(l_m \circ \check{l}_m) = \mathrm{id}_b$$

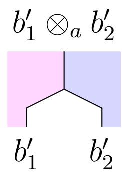
We see it has image $b$ and we thus get that $a \otimes_a b = b$. Completely analogously one can show that $b \otimes_a a = b$. $\square$

### 3.4.2 2-Cells

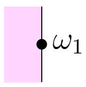
To find the appropriate notion of 2-cell between two parallel 2-condensation bimodules $B$ and $B'$, we assume by analogy to the 1-dimensional case, that they are 2-cells $B(m) \to B'(m)$ in $\mathcal{B}$ fulfilling a certain property.

A natural way to define horizontal composition of 2-cells then is the following: given composable 2-condensation bimodules $B_1$ and $B_2$ as well as $B'_1$ and $B'_2$ (combining $B_1$ and $B_2$ as well as $B'_1$ and $B'_2$ to functors $B$ and $B'$ defined on $\heartsuit_2$) we denote

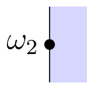
$$\Psi = \phi^{-1}_{m_1,m_2} \circ B(\psi) =$$ 

, $$\Theta' = B'(\theta) \circ \phi'_{m_1,m_2} =$$ 


where in these string diagrams the black lines denote the 1-cells $b_1 := B(m_1)$, $b_2 := B(m_2)$, $b'_1 := B'(m_1)$, $b'_2 := B'(m_2)$ and $a := B(e) = B'(e)$ and the backgrounds denote the objects: $B(A_1) = B'(A_1)$ in magenta, $B(A) = B'(A)$ in white and $B(A_2) = B'(A_2)$ in blue.
Then we can define for

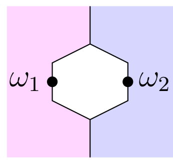


$$\in \mathrm{Hom}_{\mathcal{B}(B_1(A_2),B_1(A_1))}(B_1(m), B'_1(m))$$



$$\in \mathrm{Hom}_{\mathcal{B}(B_2(A_2),B_2(A_1))}(B_2(m), B'_2(m))$$

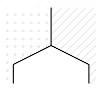
the following horizontal composition:

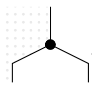
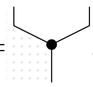
$$\omega_1 \otimes_a \omega_2 := \Theta' \circ (\omega_1 \otimes \omega_2) \circ \Psi = \tag{3.1}$$ 

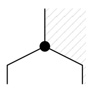
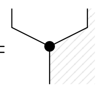

As the property which the 2-cells need to fulfil we now demand that this is well-defined.

Inside the category $\heartsuit_2$ it can be proven that this condition is equivalent to the usual property of bimodule maps by creating, destroying and moving around bubbles. The consequences of these moves are captured by the equations resembling a nonunital, noncounital $\Delta$-separable Frobenius algebra (Lemma 3.19) and those get carried over to the target category due to functoriality (Corollary 3.20).

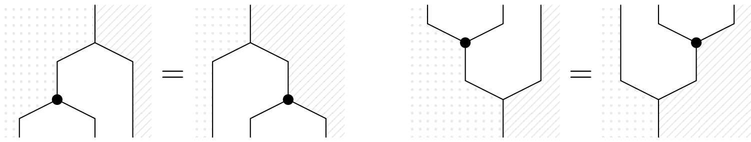
**Lemma 3.19.** *In $\heartsuit_2$, denoting $\theta$ and $\psi$ defined in the proof of Lemma 3.11 and $l_{m_1}$, $\check{l}_{m_1}$, $r_{m_2}$ and $\check{r}_{m_2}$ inherited from $\diamondsuit_2$ (see Proposition 3.7) as:*

$\theta =$ , $\psi =$ ,

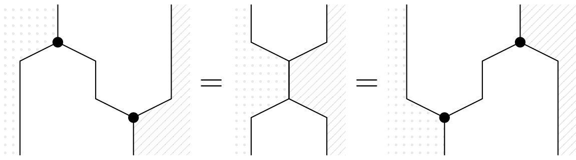
$r_{m_1} =$ , $\check{r}_{m_1} =$ ,

$l_{m_2} =$ , $\check{l}_{m_2} =$ .

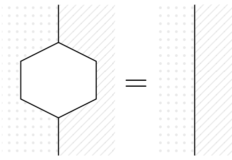
*they satisfy the following properties resembling those of a nonunital, noncounital* $\Delta$*-separable Frobenius algebra:*



(A3)

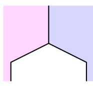
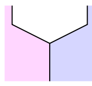


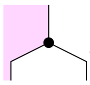
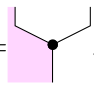
(F5)

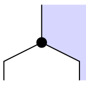
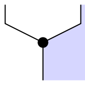
 $= \mathrm{id}_{m_1 \otimes_e m_2}$ (S5)

*Proof.* Using the graphical calculus inside $\blacktriangledown_2$, completely analogous to the proof of Proposition 3.4 by just adding the appropriate backgrounds on the left and right of all diagrams. $\square$

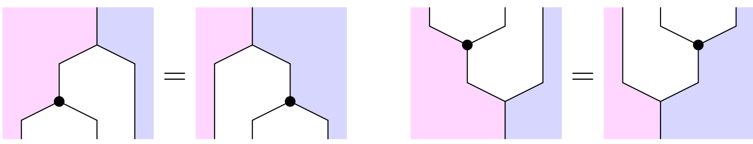
**Corollary 3.20.** *Let* $B_1$ *and* $B_2$ *be two composable 2-condensation bimodules so that their relative tensor product exists and they thus lift to a functor* $B : \heartsuit_2 \to \mathcal{B}$ *with coherence isomorphism* $\phi$*. Denoting:*

$\Theta = B(\theta) \circ \phi_{m_1,m_2} =$ , $\quad \Psi = \phi^{-1}_{m_1,m_2} \circ B(\psi) =$ ,

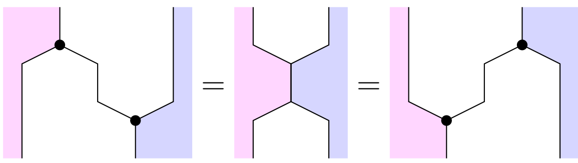
$r_{b_1} = B(r_{m_1}) \circ \phi_{m_1,e} =$ , $\quad \check{r}_{b_1} = \phi^{-1}_{m_1,e} \circ B(\check{r}_{m_1}) =$ ,

$l_{b_2} = B(l_{m_2}) \circ \phi_{e,m_2} =$ , $\quad \check{l}_{b_2} = \phi^{-1}_{e,m_2} \circ B(\check{l}_{m_2}) =$ .

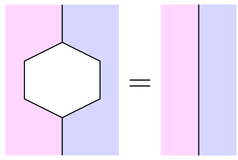
*they satisfy the following properties resembling those of a nonunital, noncounital* $\Delta$*-separable Frobenius algebra:*



(A4)



(F6)



$$= \mathrm{id}_{b_1 \otimes_a b_2} \tag{S6}$$

*Proof.* As the structure of the equations is the same as in Corollary 3.5 this can be proven completely analogous by replacing $M$ by $B$, $\mu$ by $\Theta$, $r_{b_1}$ or $l_{b_2}$ and $\Delta$ by $\Psi$, $\check{r}_{b_1}$ or $\check{l}_{b_2}$ appropriately. The occurring associator $\alpha_{e,e,e}$ has to be replaced with $\alpha_{m_1,e,m_2}$. □

The essential part about the well-definedness of (3.1) concerns $\mathrm{id}_a \otimes_a \omega$. By naturality of the unitors (ignoring for a second that they are identities) the following square needs to commute:

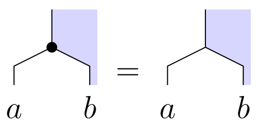
$$\begin{tikzcd}
a \otimes_a b \arrow[r, "\mathrm{id}_a \otimes_a \omega"] \arrow[d, "\lambda_b"] & a \otimes_a b' \arrow[d, "\lambda_{b'}"] \\
b \arrow[r, "\omega"] & b'
\end{tikzcd}$$

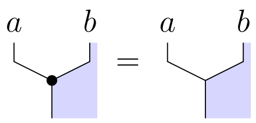
As we have chosen the units to be strict (Lemma 3.18), the unitors are in fact identities and this reads $\mathrm{id}_a \otimes_a \omega = \omega$, and similar using the right unitor $\omega \otimes_a \mathrm{id}_a = \omega$. As string diagrams we thus have:


(3.2)

Here and in the following note the change of colour when referring to the left and right module structure respectively.

As we are dealing with the unit bimodule, we can use the following facts which we saw in the proof of Lemma 3.18:

$l =$  $= \Theta$ (3.3)

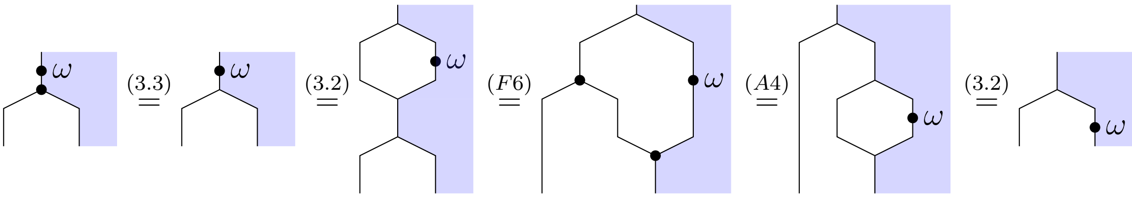

$\check{l} =$  $= \Psi$ (3.4)


and similar for a right module structure.

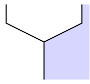
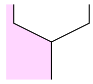
**Proposition 3.21.** *Given two 2-condensation bimodules between the same 2-condensation monads, a 2-cell $\omega$ between the corresponding bimodules satisfies property (3.2) if and only if it is a bimodule map, i.e.:*

(Mod.)

*Proof.* Assuming property (3.2) holds, by Corollary 3.20 we can compute:




Analogously (by mirroring) one proves the property for the right module structure. Assuming (Mod.) holds, we obtain (3.2) simply by pre-composing with

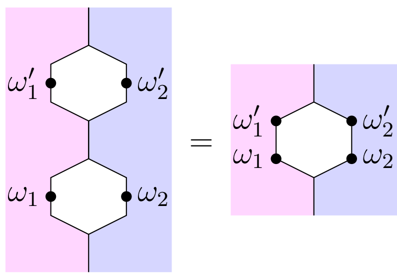
 and 

respectively and popping the resulting bubble on the left-hand side according to Corollary 3.20. ☐

**Remark 3.22.** *By turning all diagrams in the above proof upside down, one proofs that property (3.2) and therefore also property (Mod.) is equivalent to the statement that $\omega$ is a bicomodule map, meaning that:*

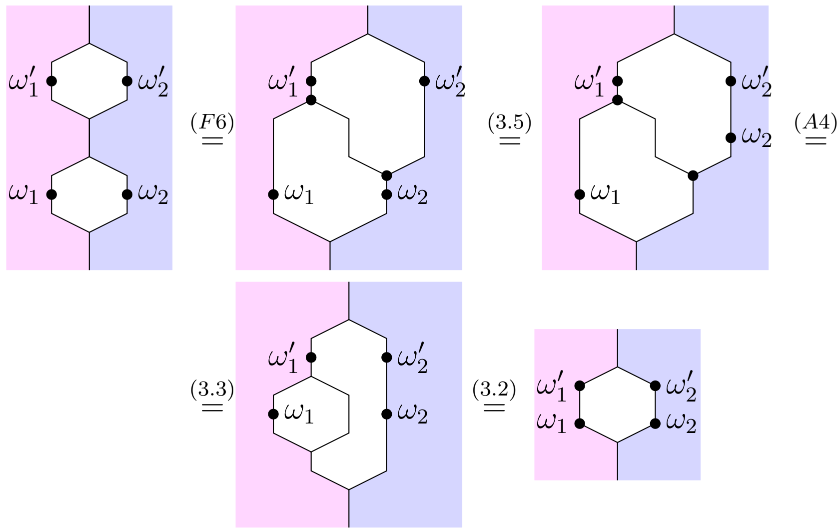
(3.5)

**Proposition 3.23.** *The property (3.2) implies that the above defined horizontal composition is functorial in the sense that:*



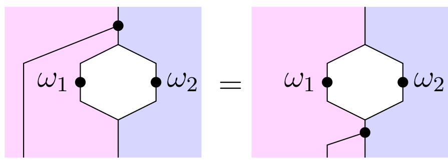
(3.6)

*Proof.* This also is an application of Corollary 3.20 and property (3.2):




□

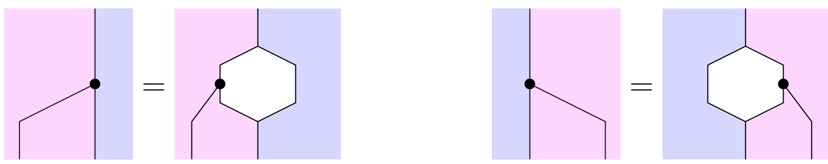
Finally, that the horizontal composition is well-defined in the sense that for bimodule maps $\omega_1$ and $\omega_2$, $\omega_1 \otimes_a \omega_2$ is a bimodule map as well, as string diagram:




(3.7)

follows easily from Lemma 3.27, for the proof of which we need the following:

**Lemma 3.24.** *Given two composable 2-condensation bimodules $B_1$ and $B_2$ so that their relative tensor product exists, using the notation defined in Corollary 3.20 it holds that:*

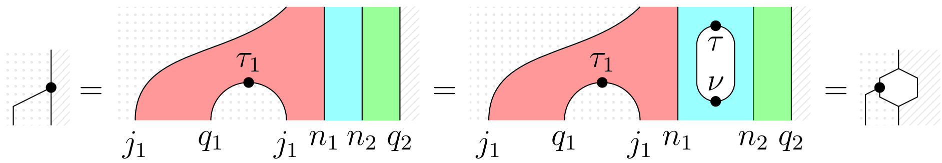


(3.8)

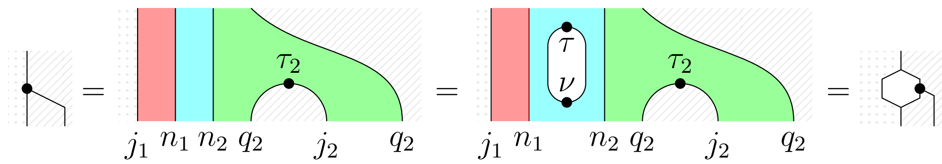
*as well as:*

(3.9)

*Proof.* As described in Definition 3.15, $B_1 \otimes_a B_2$ is given by first lifting the combination of $B_1$ and $B_2$ to a 2-functor $B$ on $\heartsuit_2$ and then restricting it along the inclusion $\iota : \diamondsuit_2 \hookrightarrow \heartsuit_2$.
In $\blacklozenge_2$ (cf. Remark 3.13) we can do the following computation, denoting $A_1$ dotted, $K_1$ red, $A$ white, $K$ cyan, $A_2$ dashed and $K_2$ green:

$\iota(l_m) =$ 


and similarly:

$\iota(r_m) =$ 

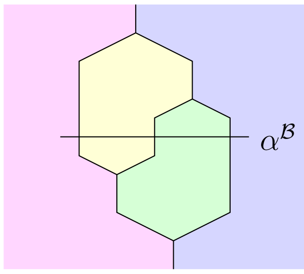

That these relations are preserved by $B$ follows analogously to the proofs of Corollary 3.8 and Proposition 3.16.
The second pair of equalities follows in the same way by flipping everything upside down and switching the roles of taus and nus accordingly. $\square$

### 3.4.3 Associator

The final ingredient to our desired 2-category is its associator. To find it consider three composable 2-condensation bimodules $b_i$ between 2-condensation monads $a_i$ in the following diagram:

$$\begin{tikzcd}
(b_1 \otimes b_2) \otimes b_3 \arrow[r, "\Theta_{1,2}\otimes \mathrm{id}_{b_3}"] \arrow[d, "\alpha^{\mathcal{B}}"'] & (b_1 \otimes_{a_1} b_2) \otimes b_3 \arrow[r, "\Theta_{12,3}"] & (b_1 \otimes_{a_1} b_2) \otimes_{a_2} b_3 \arrow[d, dashed, "\alpha^{\mathrm{Kar}(\mathcal{B})}"] \\
b_1 \otimes (b_2 \otimes b_3) \arrow[r, "\mathrm{id}_{b_1}\otimes\Theta_{2,3}"'] & b_1 \otimes (b_2 \otimes_{a_2} b_3) \arrow[r, "\Theta_{1,23}"'] & b_1 \otimes_{a_1} (b_2 \otimes_{a_2} b_3)
\end{tikzcd}$$

As all the occurring horizontal arrows have right inverses we can use these to move through the diagram counter-clockwise from the top right to the bottom right and define $\alpha^{\mathrm{Kar}(\mathcal{B})}$ that way:

$\alpha^{\mathrm{Kar}(\mathcal{B})} :=$ 


**Remark 3.25.** *As we are dealing with associators explicitly, here and in the rest of this subsection we are extra careful and include in all string diagrams associators $\alpha^{\mathcal{B}}$ as horizontal lines, and inverse associators as horizontal lines with "$-1$" next to them. Note that these are usually omitted as described in Section 2.2.1. Once bracketings for the 1-cells corresponding to ingoing and outgoing strings are chosen, the appropriate associators can be added to the string diagram. This can be done unambiguously: any combination of associators matching the same bracketings need to be equal by the coherence theorem. This fact will be crucial to the proofs of this subsection.*

Another essential step is to note the following:

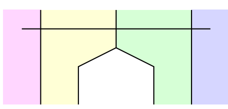
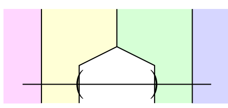
**Remark 3.26.** *The associator $\alpha^{\mathcal{B}}$ of the original 2-category $\mathcal{B}$ is a natural transformation. That means for all 1-cells $a, a', b, b', c, c'$ and all 2-cells $f : a \to a'$, $g : b \to b'$ and $h : c \to c'$ the following diagram commutes:*

$$\begin{tikzcd}
(a \otimes b) \otimes c \arrow[r, "(f\otimes g)\otimes h"] \arrow[d, "\alpha^{\mathcal{B}}_{a,b,c}"] & (a' \otimes b') \otimes c' \arrow[d, "\alpha^{\mathcal{B}}_{a',b',c'}"] \\
a \otimes (b \otimes c) \arrow[r, "f\otimes(g\otimes h)"] & a' \otimes (b' \otimes c')
\end{tikzcd}$$

*In particular for the projection on the relative tensor product $\Theta : b_2 \otimes b_3 \to b_2 \otimes_{a_2} b_3$ that means:*

$$\begin{tikzcd}
(b_1 \otimes (b_2 \otimes b_3)) \otimes b_4 \arrow[r, "(\mathrm{id}_{b_1}\otimes\Theta)\otimes\mathrm{id}_{b_4}"] \arrow[d, "\alpha^{\mathcal{B}}_{b_1,b_2\otimes b_3,b_4}"] & (b_1 \otimes (b_2 \otimes_{a_2} b_3)) \otimes b_4 \arrow[d, "\alpha^{\mathcal{B}}_{b_1,b_2\otimes_{a_2} b_3,b_4}"] \\
b_1 \otimes ((b_2 \otimes b_3) \otimes b_4) \arrow[r, "\mathrm{id}_{b_1}\otimes(\Theta\otimes\mathrm{id}_{b_4})"] & b_1 \otimes ((b_2 \otimes_{a_2} b_3) \otimes b_4)
\end{tikzcd}$$

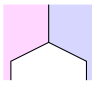
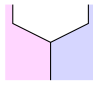
*which as a string diagram reads:*

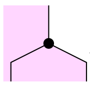
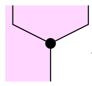
 $=$  (Nat.)

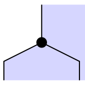
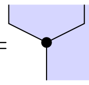
*where the brackets indicate that the associator on the right-hand side indeed is $\alpha^{\mathcal{B}}_{b_1,b_2\otimes b_3,b_4}$ which otherwise might be confused with $\alpha^{\mathcal{B}}_{b_1,b_2,b_3\otimes b_4}$ or $\alpha^{\mathcal{B}}_{b_1\otimes b_2,b_3,b_4}$, but are usually omitted in the following.*
*The analogous relations clearly also hold for the projections $b_1 \otimes b_2 \to b_1 \otimes_{a_1} b_2$ and $b_3 \otimes b_4 \to b_3 \otimes_{a_3} b_4$ and the appropriate bracketings. Furthermore, replacing $\Theta$ with $\Psi$ or any of the possible actions and coactions, one finds that in our notation all trivalent vertices can be "pulled through" the associator lines as long as the bracketings fit. The same holds for the inverse associator. In the proofs below steps applying this will be labeld by "Nat.".*

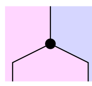
To prove the pentagon axiom in $\mathrm{Kar}(\mathcal{B})$, we first of all have to see that the above defined associator indeed is a 2-cell of $\mathrm{Kar}(\mathcal{B})$, i.e. a module map. To do so we need the following lemma about the action on a relative tensor product, which resemble associativity and Frobenius laws.

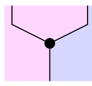
**Lemma 3.27.** *Let $(B_1, \phi^1)$ and $(B_2, \phi^2)$ be two composable 2-condensation bimodules such that their relative tensor product $(B_1 \otimes_e B_2, \phi^{\otimes})$ exists and they thus lift to a functor $(B, \phi) : \heartsuit_2 \to \mathcal{B}$. Denoting:*

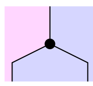
$\Theta = B(\theta) \circ \phi_{m_1,m_2} =$ , $\quad \Psi = \phi^{-1}_{m_1,m_2} \circ B(\psi) =$ ,

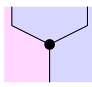
$l_{b_1} = B(l_{m_1}) \circ \phi_{e_1,m_1} =$ , $\quad \check{l}_{b_1} = \phi^{-1}_{e_1,m_1} \circ B(\check{l}_{m_1}) =$ ,

$r_{b_2} = B(r_{m_2}) \circ \phi_{m_2,e_2} =$ , $\quad \check{r}_{b_2} = \phi^{-1}_{m_2,e_2} \circ B(\check{r}_{m_2}) =$ ,

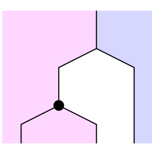
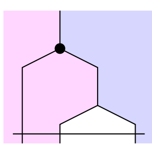
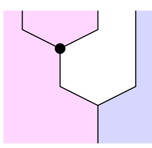
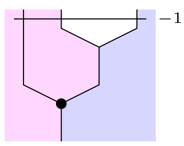
$l_{b_1 \otimes_a b_2} = (B_1 \otimes_e B_2)(l_m) \circ \phi^{\otimes}_{e_1,m} =$ ,

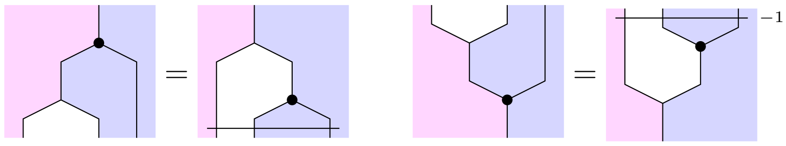
$\check{l}_{b_1 \otimes_a b_2} = (\phi^{\otimes}_{e_1,m})^{-1} \circ (B_1 \otimes_e B_2)(\check{l}_m) =$ ,

$r_{b_1 \otimes_a b_2} = (B_1 \otimes_e B_2)(r_m) \circ \phi^{\otimes}_{m,e_2} =$ ,

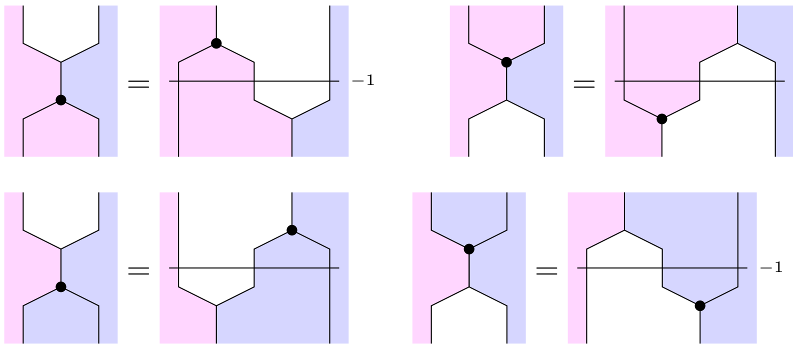
$\check{r}_{b_1 \otimes_a b_2} = (\phi^{\otimes}_{m,e_2})^{-1} \circ (B_1 \otimes_e B_2)(\check{r}_m) =$ 

*the actions $l$ and $r$ commute with the projection $\Theta$ onto the relative tensor product and the coactions $\check{l}$ and $\check{r}$ commute with the inclusion $\Psi$ of the relative tensor product, in the following sense:*

 $=$  $\qquad$  $=$  $\quad$ (A5)

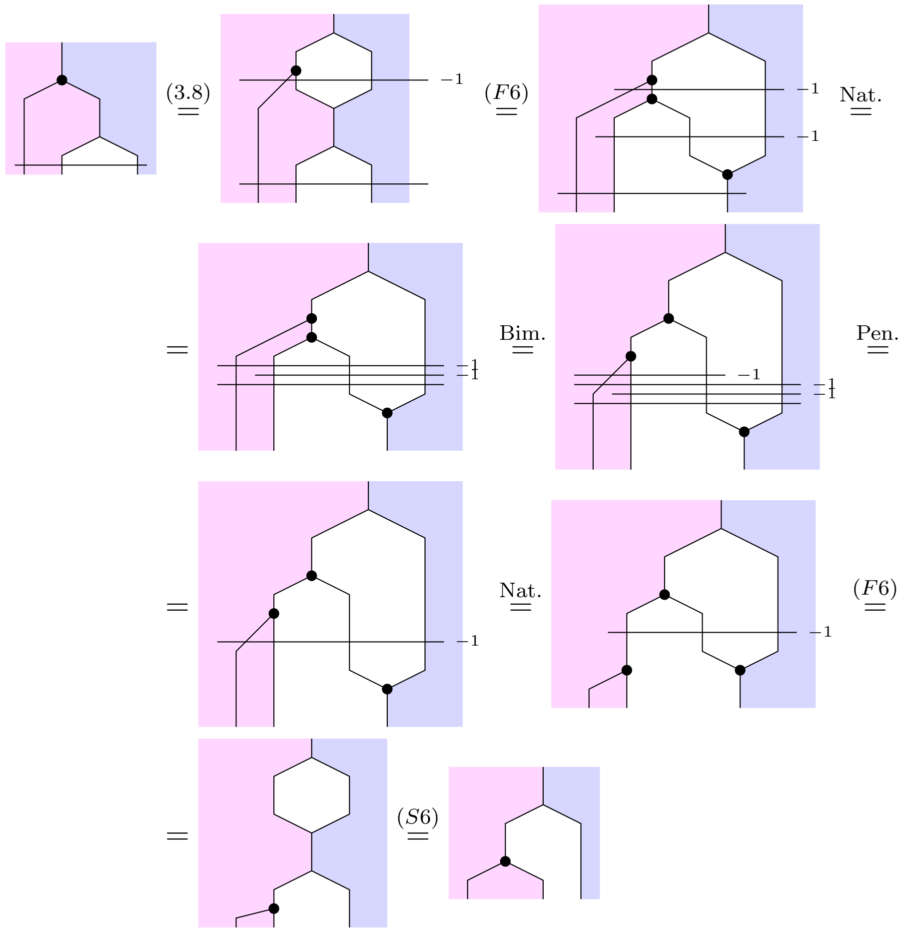



*Similarly also the coactions $\check{l}$ and $\check{r}$ commute with the projection $\Theta$ onto the relative tensor product and the actions $l$ and $r$ commute with the inclusion $\Psi$ of the relative tensor product, in the following sense:*




(F7)

*Proof.* We only prove the first equality each, as the other ones follow analogously (they are just mirrored horizontally, vertically or both, respectively).



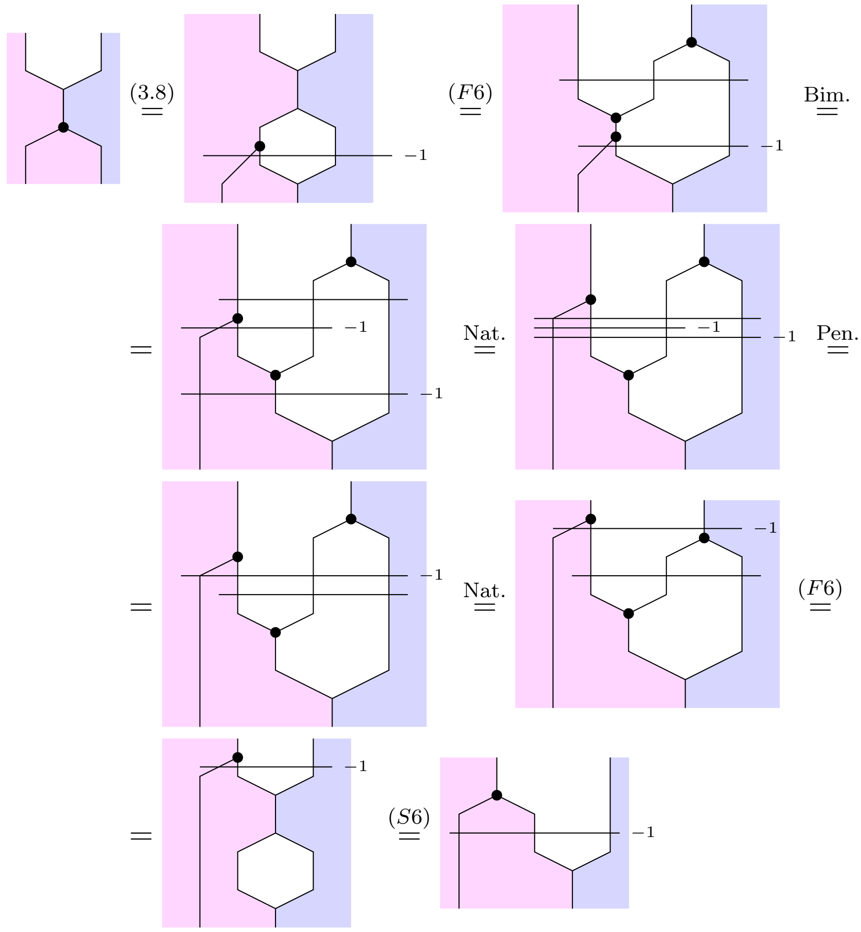

Note that after the first step there is an extra line for an inverse associator which was omitted in equation (3.8) but is included now as described in Remark 3.25. The step labelled "Bim." is the bimodule property of $b_1$ proved in Corollary 3.8, the step labelled "Pen." is due the pentagon axiom of the original 2-category $\mathcal{B}$. One can see this as follows: the rebracketing on the left-hand side is the following long path through the pentagon:

$$\begin{aligned}(a_1 \otimes b_1) \otimes (a \otimes b_2) \to & a_1 \otimes (b_1 \otimes (a \otimes b_2)) \to a_1 \otimes ((b_1 \otimes a) \otimes b_2) \to \\ \to & (a_1 \otimes (b_1 \otimes a)) \otimes b_2 \to ((a_1 \otimes b_1) \otimes a) \otimes b_2\end{aligned}$$

which by the pentagon axiom is the same as following through the pentagon in the other direction:

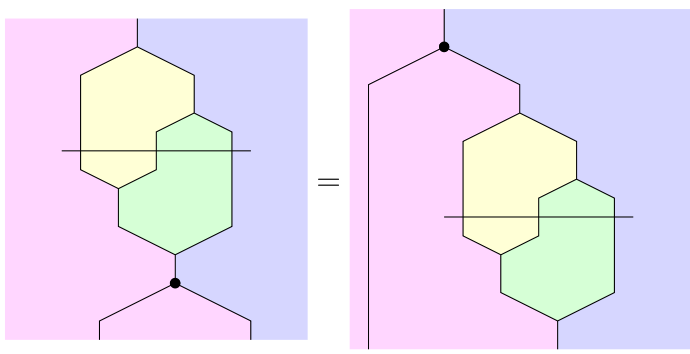
$$(a_1 \otimes b_1) \otimes (a \otimes b_2) \to ((a_1 \otimes b_1) \otimes a) \otimes b_2$$

The proof of the second statement is quite similar:



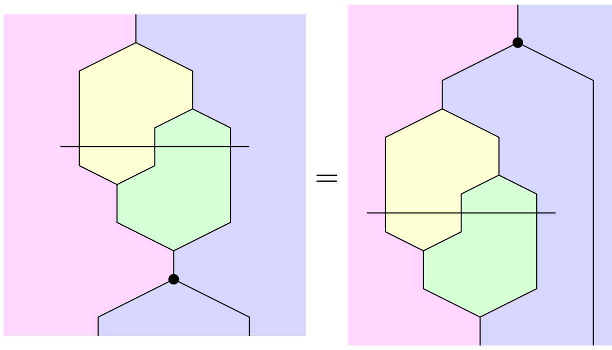

where in the step labelled "Pen." the equality of two different paths through the pentagon was used. □

**Lemma 3.28.** *The associator $\alpha^{\mathrm{Kar}(\mathcal{B})}$ is a bimodule map, i.e. it commutes with the left action:*

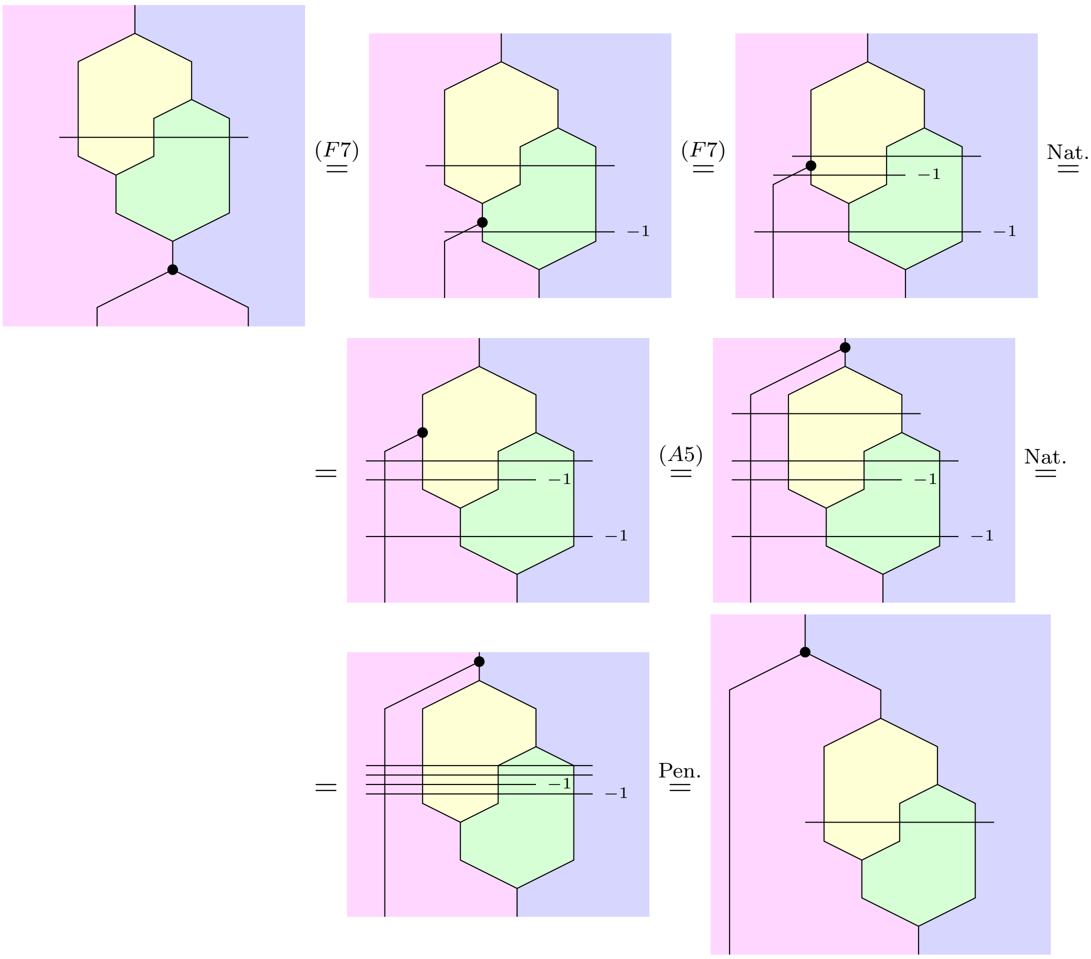


(3.10)

*as well as with the right action:*

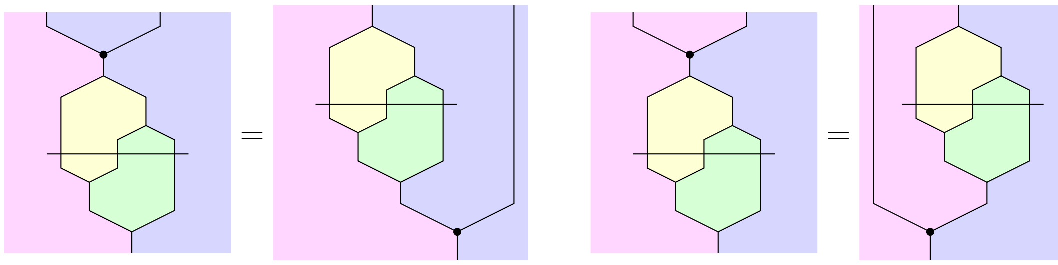


*Proof.* This is a calculation using string diagrams and what we have seen so far:



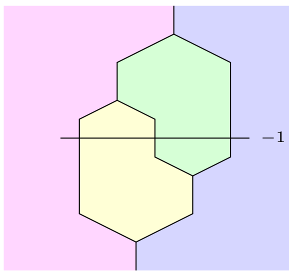

where the last step, labelled "Pen.", is an application of the pentagon axiom of the original 2-category $\mathcal{B}$.
That it commutes with the right action as well can be seen analogously by mirroring all the string diagrams above. □

Note that we have already shown in Remark 3.22 that this implies that $\alpha^{\mathrm{Kar}(\mathcal{B})}$ is also a bicomodule map:

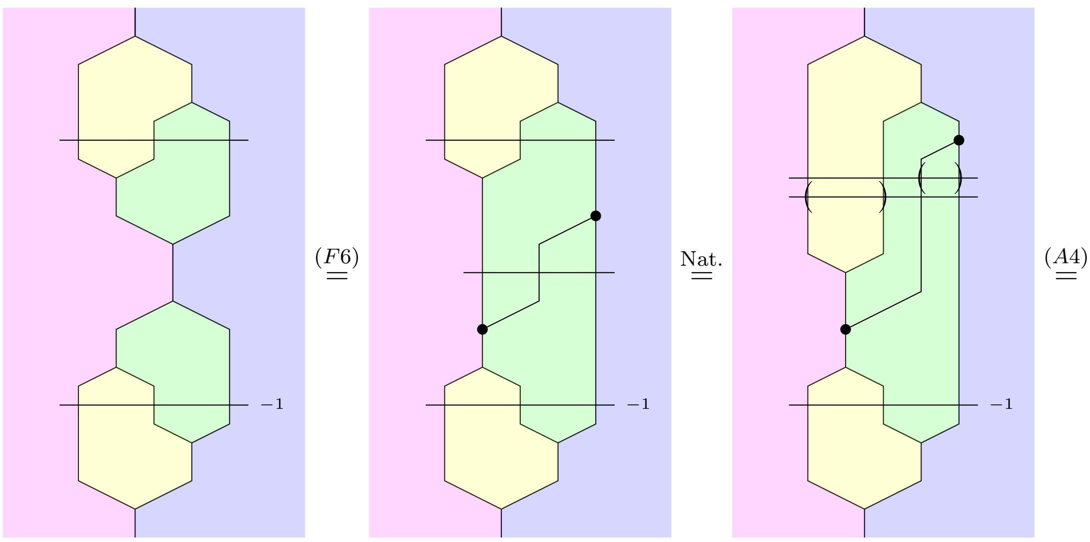



**Proposition 3.29.** *The associator $\alpha^{\mathrm{Kar}(\mathcal{B})}$ is an isomorphism with inverse:*

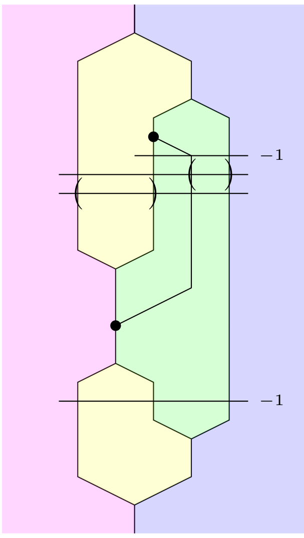
$$(\alpha^{\mathrm{Kar}(\mathcal{B})})^{-1} =$$



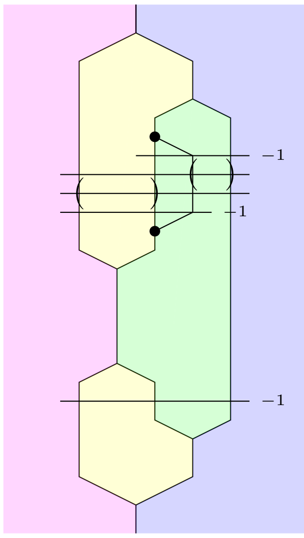

*Proof.* This is a direct calculation using what we found so far, $\alpha^{\mathrm{Kar}(\mathcal{B})} \circ (\alpha^{\mathrm{Kar}(\mathcal{B})})^{-1}$ is equal to:



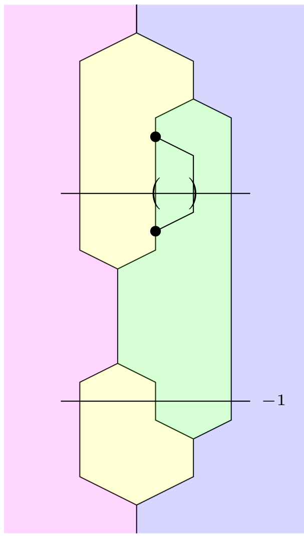
$=$




$\overset{(A5)}{=}$

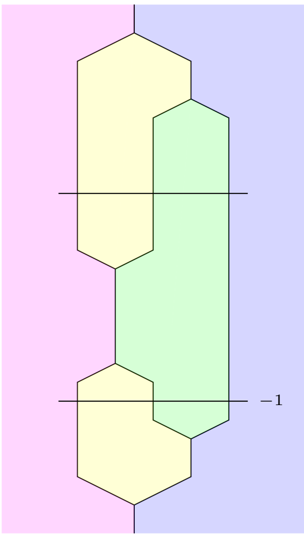



$\overset{\text{Pen.}}{=}$

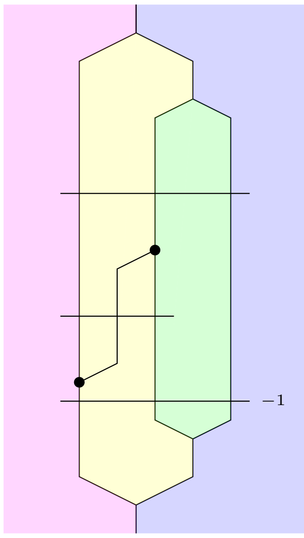



$\overset{(S4)}{=}$

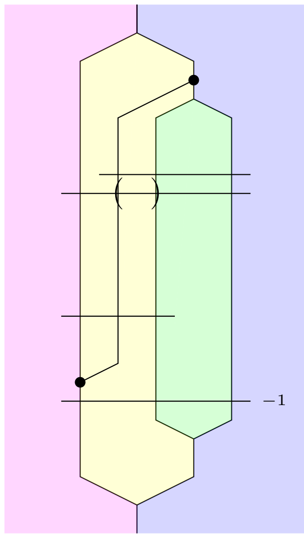
$=$




$\overset{(F6)}{=}$

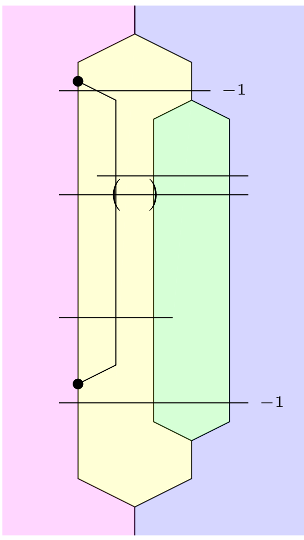



$\overset{(A5)}{=}$

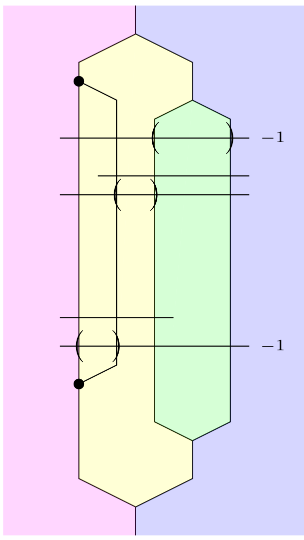



$\overset{(A4)}{=}$

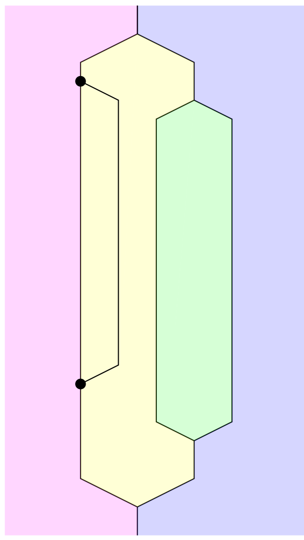
$=$



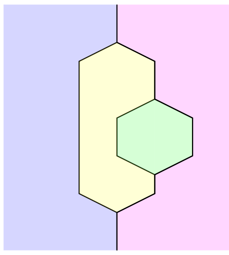
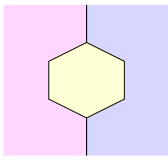

$\overset{\text{Nat.}}{=}$




$\overset{\text{Pen.}}{=}$



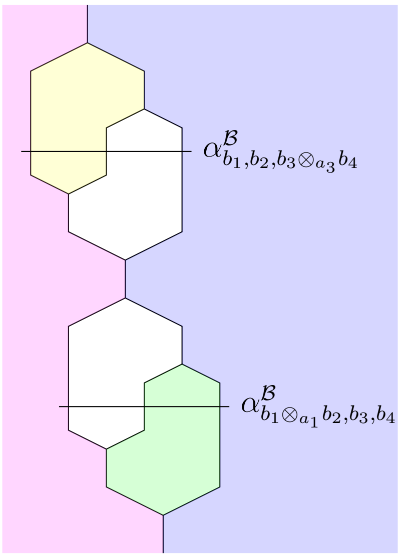
$\overset{(S4)}{=}$

$=$  $\overset{(S6)}{=}$  $\overset{(S6)}{=}$ 

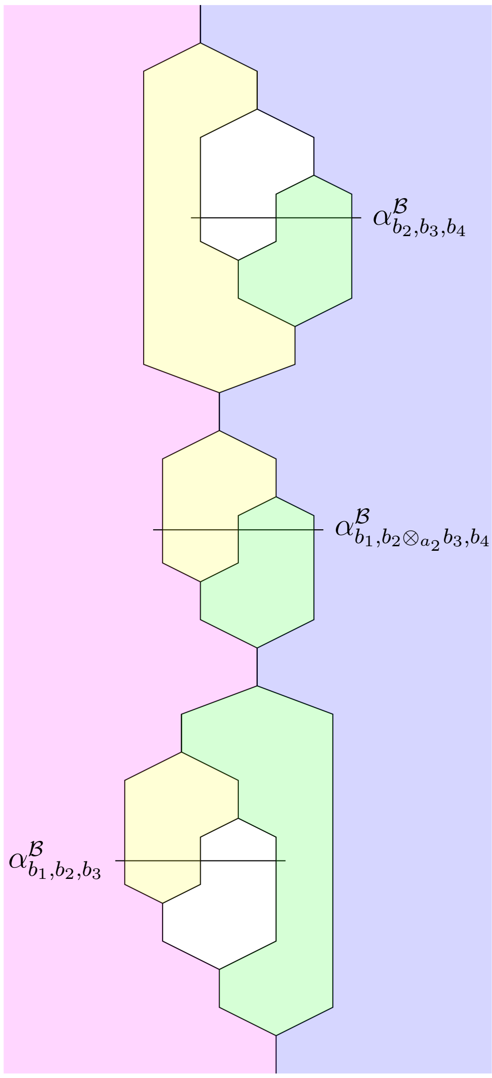
That $(\alpha^{\mathrm{Kar}(\mathcal{B})})^{-1} \circ \alpha^{\mathrm{Kar}(\mathcal{B})}$ is the identity as well can be seen by rotating the above proof upside-down. $\square$

**Proposition 3.30.** *The associator $\alpha^{\mathrm{Kar}(\mathcal{B})}$ satisfies the pentagon identity:*

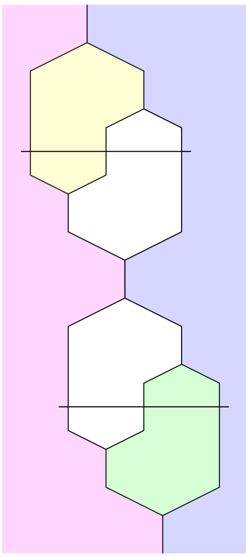
$$\alpha^{\mathrm{Kar}(\mathcal{B})}_{b_1,b_2,b_3\otimes_{a_3}b_4} \circ \alpha^{\mathrm{Kar}(\mathcal{B})}_{b_1\otimes_{a_1}b_2,b_3,b_4} = (\mathrm{id}_{b_1} \otimes_{a_1} \alpha^{\mathrm{Kar}(\mathcal{B})}_{b_2,b_3,b_4}) \circ \alpha^{\mathrm{Kar}(\mathcal{B})}_{b_1,b_2\otimes_{a_2}b_3,b_4} \circ (\alpha^{\mathrm{Kar}(\mathcal{B})}_{b_1,b_2,b_3} \otimes_{a_3} \mathrm{id}_{b_4})$$



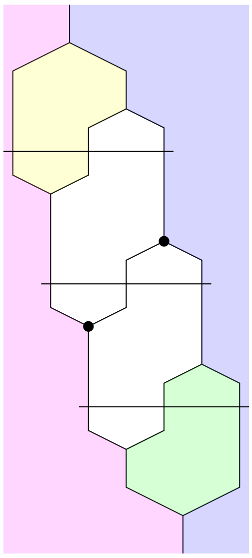

$=$



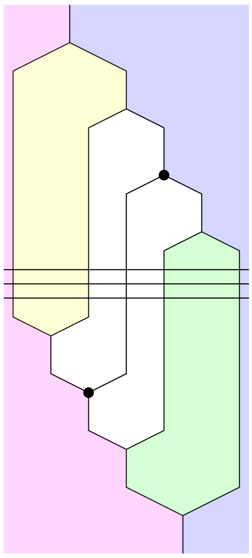
*Proof.* We bring the associators together in order to argue by the coherence theorem for the original 2-category $\mathcal{B}$. For the left-hand side this is easy:



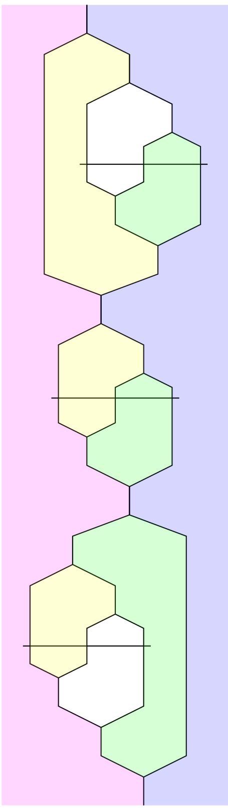
$\overset{(F6)}{=}$



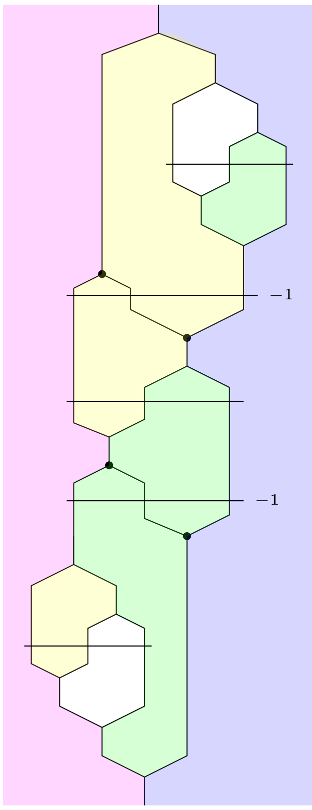
$\overset{\text{Nat.}}{=}$



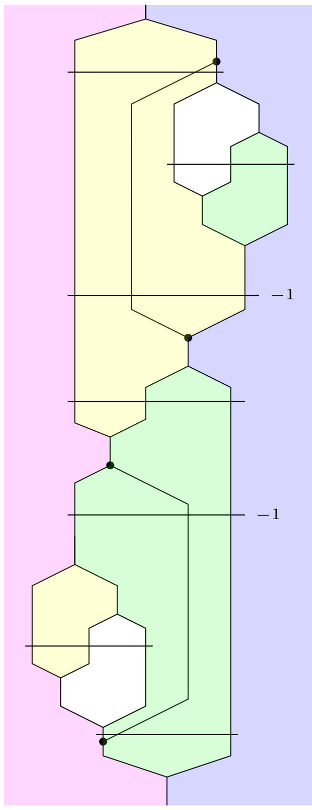
The right-hand side is more involved, mostly performing the same move in the upper and in the lower half at the same time:



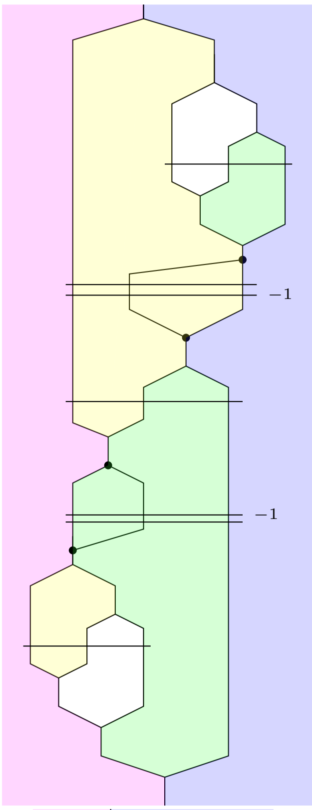
$\overset{(F6)}{=}$

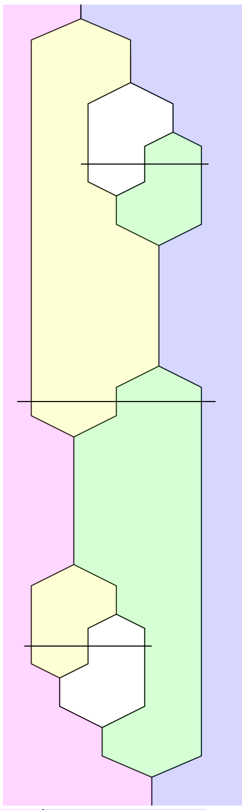


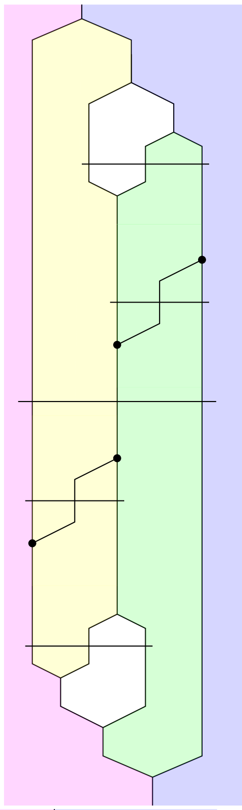

$\overset{(A5)}{=}$



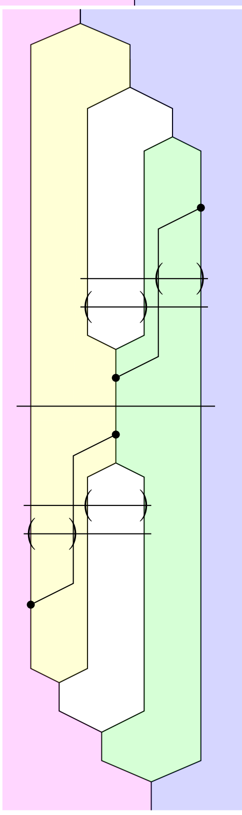

$\overset{(3.10)}{=}$

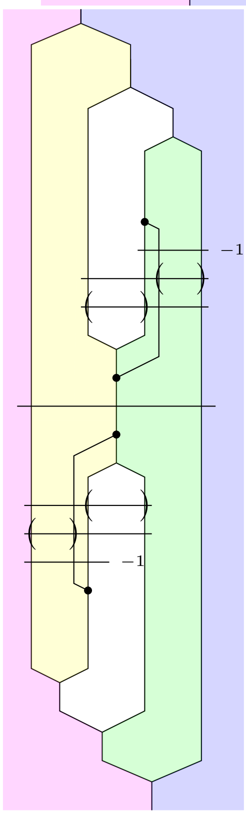
$=$ 

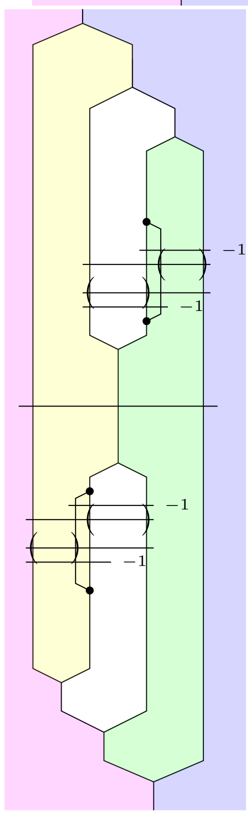

$\overset{(S4)}{=}$ 

$\overset{(F6)}{=}$ 

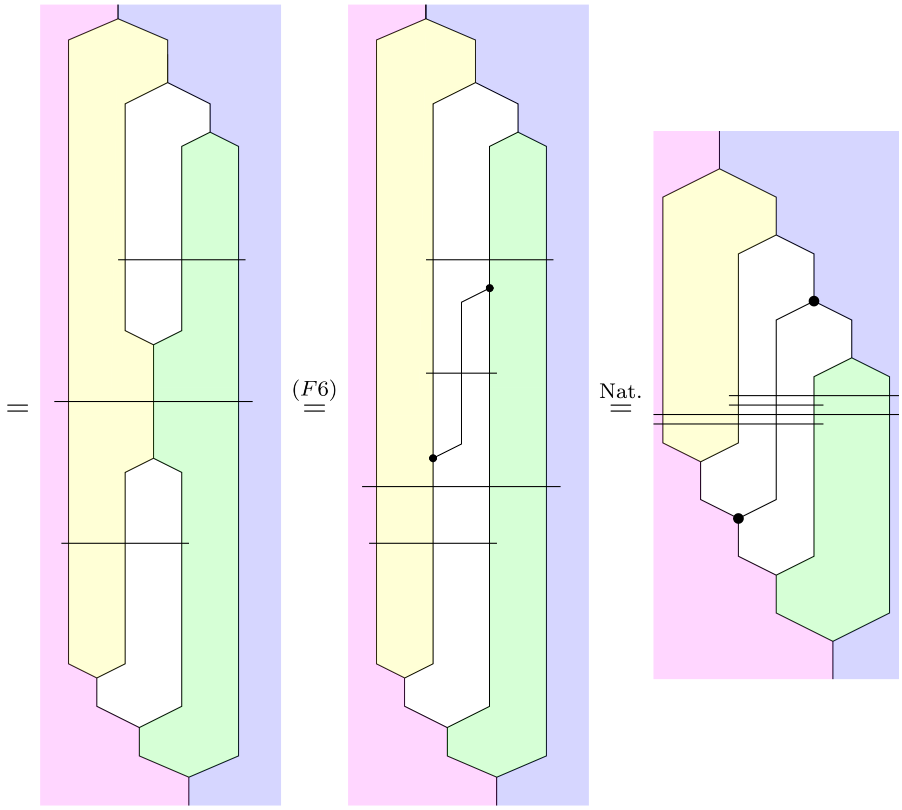
$\overset{\text{Nat.}}{=}$

$=$ 

$\overset{(A5)}{=}$ 

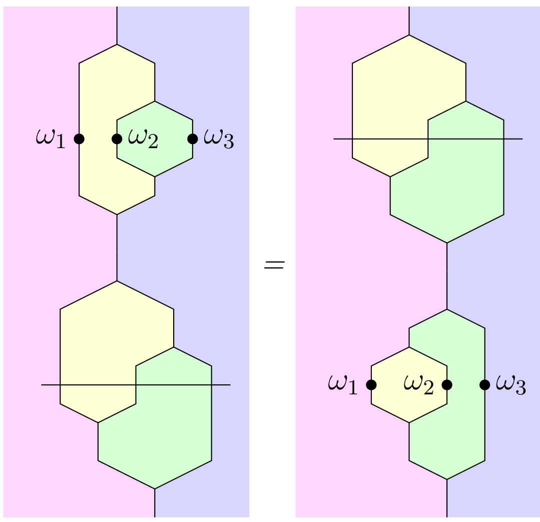

$\overset{(A5)}{=}$ 


$\overset{(*)}{=}$

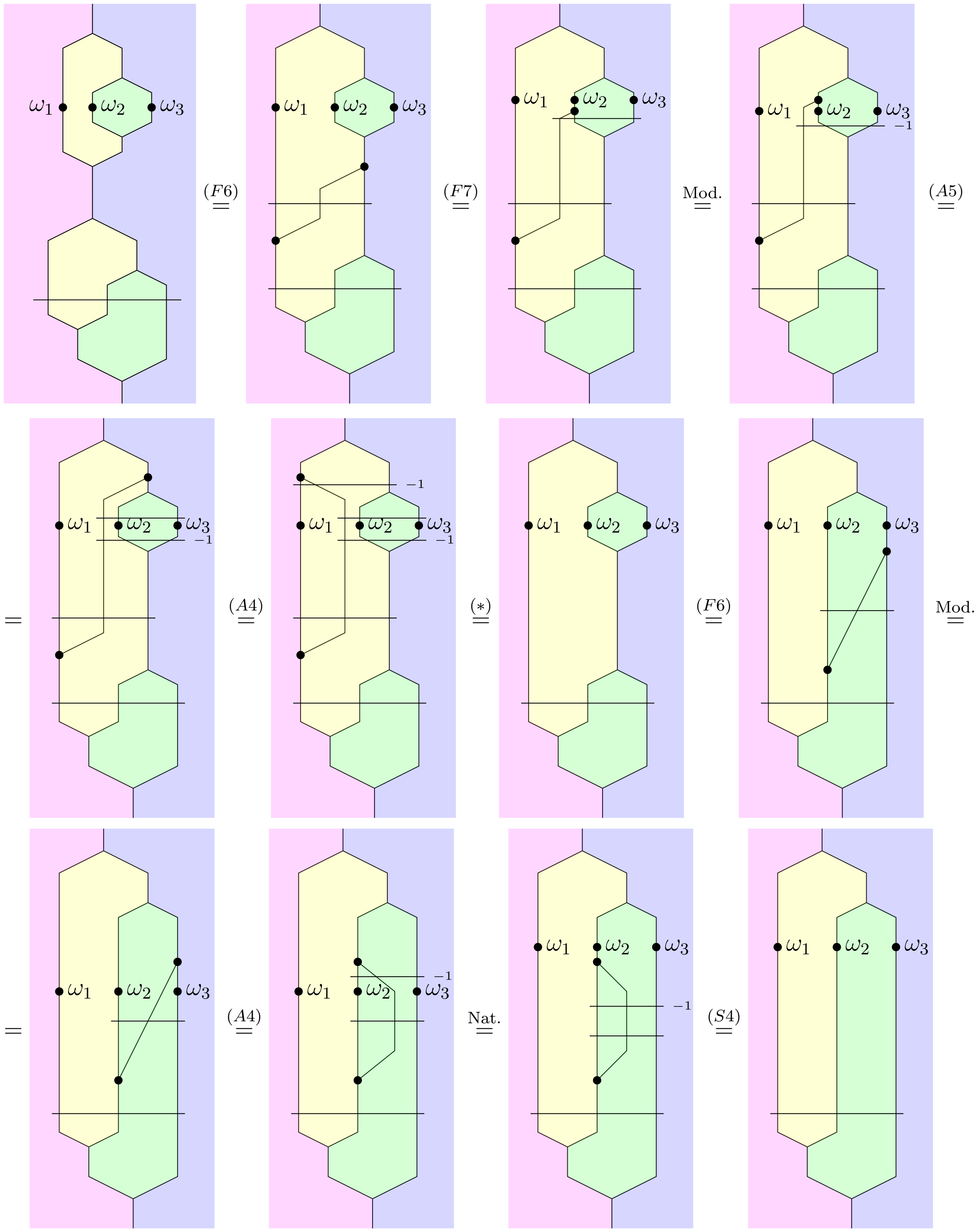



Which is equal to what we had before due to coherence in $\mathcal{B}$. In the step labelled $(*)$ we first apply the pentagon axiom of $\mathcal{B}$ to write the combinations of four associators $\alpha^{\mathcal{B}}$ as only one, which we then pull out of the yellow (respectively green) bubble by naturality (cf. Remark 3.26) and finally pop the now empty bubble according to (S4). $\square$

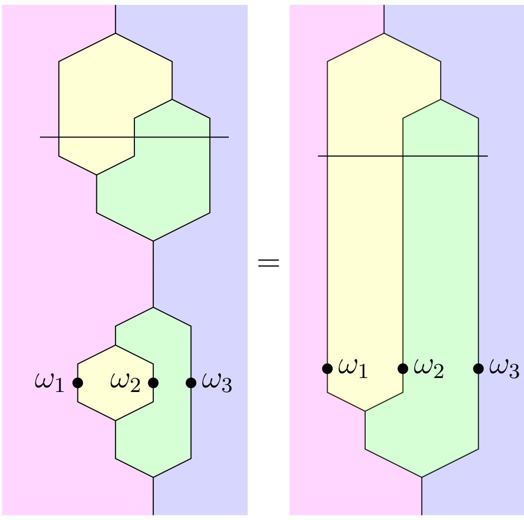
**Proposition 3.31.** *The associator $\alpha^{\mathrm{Kar}(\mathcal{B})}$ is a natural transformation, i.e. for bimodule maps $\omega_1$, $\omega_2$ and $\omega_3$ it holds that:*



*Proof.* We can calculate:



Where in $(*)$ we use the naturality of the associator of $\mathcal{B}$ to bring the two pairs of associator and its inverse together, and then applying (3.2). To complete the proof, note that up to colours and labels, the left- and right-hand side of the desired equality are 180 degree rotations of each other. As all the relations used above have rotated counterparts, by virtually the same computation we find:




Finally this is equal to what we had before precisely due to the naturality of the associator in the original 2-category $\mathcal{B}$. $\square$

This completes the proof of Theorem 3.17.

**Remark 3.32.** *Having established the existence of the 2-category* $\mathrm{Kar}(\mathcal{B})$ *of condensation monads and their bimodules, we might want to compare it to the 2-categories of algebras described in Section 2.3.4. It turns out that* $\mathrm{Kar}(\mathcal{B})$ *is actually equivalent to* $\mathrm{Frob}_{\Delta}^{\cancel{\perp},\cancel{\top}}(\mathcal{B})$*, the 2-category of nonunital, noncounital* $\Delta$*-separable Frobenius algebras internal to* $\mathcal{B}$*. We find a 2-functor* $\mathrm{Kar}(\mathcal{B}) \to \mathrm{Frob}_{\Delta}^{\cancel{\perp},\cancel{\top}}(\mathcal{B})$ *which is essentially surjective on objects, essentially full on 1-cells, and fully faithful on 2-cells, as follows:*
*on objects, map a given 2-condensation monad to the algebra described in Corollary 3.5. This is actually surjective as for a given object* $(a, m, d)$ *in* $\mathrm{Frob}_{\Delta}^{\cancel{\perp},\cancel{\top}}(\mathcal{B})$*, we find the 2-condensation monad given by the functor described in Remark 3.14, mapping* $e \mapsto a$*,* $\mu \mapsto m$ *and* $\Delta \mapsto d$*. Similarly by Corollary 3.8 we can map a 2-condensation bimodule between two 2-condensation monads to the underlying special Frobenius bimodule between the corresponding algebras. Fully faithfulness on 2-cells follows by Proposition 3.21.*

## 3.5 Completeness

Having established the existence of the 2-category $\mathrm{Kar}(\mathcal{B})$ of 2-condensation monads and their bimodules, we now want to see that it deserves to be called the 2-Karoubi completion of $\mathcal{B}$. That is the case if $\mathrm{Kar}(\mathcal{B})$ has all condensates and we can find a fully faithful 2-functor $J : \mathcal{B} \to \mathrm{Kar}(\mathcal{B})$ which is an equivalence if $\mathcal{B}$ already has all condensates. The 2-functor in question is easily constructed.

**Definition 3.33.** For a 2-category $\mathcal{B}$, there is a canonical 2-functor $J : \mathcal{B} \to \mathrm{Kar}(\mathcal{B})$ given by:

$$\begin{aligned}
\mathrm{Ob}(\mathcal{B}) \ni X \mapsto M^X &: \clubsuit_2 \to \mathcal{B} \\
& A \mapsto X \\
& e \mapsto \mathbb{1}_X \\
& \mu \mapsto \lambda_{\mathbb{1}_X},\, \Delta \mapsto \lambda^{-1}_{\mathbb{1}_X} \\
\mathcal{B}(X,Y) \ni f \mapsto B^f &: \diamondsuit_2 \to \mathcal{B} \\
& A_1 \mapsto Y,\, A_2 \mapsto X \\
& m \mapsto f,\, e_1 \mapsto \mathbb{1}_Y,\, e_2 \mapsto \mathbb{1}_X \\
& \mu_1 \mapsto \lambda_{\mathbb{1}_Y},\, \Delta_1 \mapsto \lambda^{-1}_{\mathbb{1}_Y},\, \mu_2 \mapsto \lambda_{\mathbb{1}_X},\, \Delta_2 \mapsto \lambda^{-1}_{\mathbb{1}_X} \\
& l_m \mapsto \lambda_f,\, \check{l}_m \mapsto \lambda_f^{-1},\, r_m \mapsto \rho_f,\, \check{r}_m \mapsto \rho_f^{-1} \\
\mathrm{Hom}_{\mathcal{B}(X,Y)}(f,g) \ni \phi \mapsto \phi &\in \mathrm{Hom}_{\mathrm{Kar}(\mathcal{B})(M^X,M^Y)}(B^f, B^g)
\end{aligned}$$

**Proposition 3.34.** *For a 2-category $\mathcal{B}$ for which all hom-categories are idempotent complete, the canonical inclusion $J : \mathcal{B} \to \mathrm{Kar}(\mathcal{B})$ is a well-defined, strict, fully faithful 2-functor.*

*Proof.* First of all note that the information given about $M^X$ and $B^f$ is sufficient to construct well-defined elements in $\mathrm{Ob}(\mathrm{Kar}(\mathcal{B}))$ and $\mathrm{Kar}(\mathcal{B})(M^X, M^Y)$, respectively, according to Remark 3.14 as long as the given 2-cells in $\mathcal{B}$ obey the relations of a nonunital, noncounital, $\Delta$-separable Frobenius algebra and a special Frobenius bimodule respectively. That these indeed are satisfied follows directly from the triangle axiom (Tri.) and the fact that for all $X \in \mathrm{Ob}(\mathcal{B})$ it holds that $\lambda_{\mathbb{1}_X} = \rho_{\mathbb{1}_X}$. This fact follows from the coherence theorem, a direct proof is given in [Koc05, Lemma 1.5].

To see that any $\phi \in \mathrm{Hom}_{\mathcal{B}(X,Y)}(f,g)$ is indeed a bimodule map between $B^f(m)$ and $B^g(m)$, note that by Definition 2.40 that is the case if:

$$\phi \circ \lambda_f = \lambda_g \circ (\mathrm{id}_{\mathbb{1}_Y} \otimes \phi)$$

and

$$\phi \circ \rho_f = \rho_g \circ (\phi \otimes \mathrm{id}_{\mathbb{1}_X})$$

This follows directly from the naturality of left and right unitors in $\mathcal{B}$:

$$\begin{tikzcd} \mathbb{1}_Y \otimes f \arrow[r, "\mathrm{id}_{\mathbb{1}_Y} \otimes \phi"] \arrow[d, "\lambda_f"] & \mathbb{1}_Y \otimes g \arrow[d, "\lambda_g"] \\ f \arrow[r, "\phi"] & g \end{tikzcd} \qquad \begin{tikzcd} f \otimes \mathbb{1}_X \arrow[r, "\phi \otimes \mathrm{id}_{\mathbb{1}_X}"] \arrow[d, "\rho_f"] & g \otimes \mathbb{1}_X \arrow[d, "\rho_g"] \\ f \arrow[r, "\phi"] & g \end{tikzcd}$$

Next let $f \in \mathcal{B}(X, Y)$ and $h \in \mathcal{B}(Y, Z)$ in order to compare $J(h \otimes f) = B^{h \otimes f}$ with $J(h) \otimes_{J(Y)} J(f) = B^h \otimes_{M^Y} B^f = h \otimes_{\mathbb{1}_Y} f$. The latter is by Definition 3.15 the image of the splitting idempotent

$$\mathfrak{k} = (\mathrm{id}_h \otimes \lambda_f) \circ \alpha_{h,\mathbb{1}_Y,f} \circ (\rho_h^{-1} \otimes \mathrm{id}_f)$$

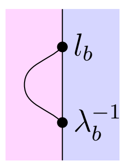
but due to the triangle axiom (Tri.) of $\mathcal{B}$:

$$\begin{tikzcd} (h \otimes \mathbb{1}_Y) \otimes f \arrow[rr, "\alpha_{h,\mathbb{1}_Y,f}"] \arrow[dr, "\rho_h \otimes \mathrm{id}_f"'] & & h \otimes (\mathbb{1}_Y \otimes f) \arrow[dl, "\mathrm{id}_h \otimes \lambda_f"] \\ & h \otimes f & \end{tikzcd}$$

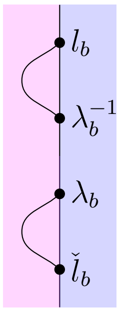
this is equal to the identity on $h \otimes f$, $\mathfrak{k} = \mathrm{id}_{h \otimes f} = \mathrm{id}_{h \otimes f} \circ \mathrm{id}_{h \otimes f}$. Thus $\mathfrak{k}$ splits with image $h \otimes f$ so that $B^h \otimes_{M^Y} B^f = B^{h \otimes f}$ and we see that $J$ strictly preserves composition.

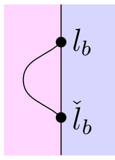
It also strictly preserves units: $J(\mathbb{1}_X) = B^{\mathbb{1}_X}$ is $\mathbb{1}_X$ as a $\mathbb{1}_X$-$\mathbb{1}_X$-bimodule and agrees with the unit $\mathbb{1}_{M^X}$ on $M^X = J(X)$ described in Lemma 3.18, again using that $\lambda_{\mathbb{1}_X} = \rho_{\mathbb{1}_X}$.

It remains to show that $J$ is fully faithful, i.e. it is essentially full on 1-cells: for all $X, Y \in \mathrm{Ob}(\mathcal{B})$ and all 2-condensation bimodules $B$ between $M^X$ and $M^Y$, there is an $f \in \mathcal{B}(X, Y)$ such that $B \cong B^f$; and it is fully faithful on 2-cells: for all $f, g \in \mathcal{B}(X, Y)$ there is a one-to-one correspondence between $\mathrm{Hom}_{\mathcal{B}(X,Y)}(f, g)$ and $\mathrm{Hom}_{\mathrm{Kar}(\mathcal{B})(M^X,M^Y)}(B^f, B^g)$.
That $J$ is fully faithful on 2-cells is clear as it is just the identity map. To show fullness on 1-cells let $(B, \phi) : \diamondsuit_2 \to \mathcal{B}$ be any 2-condensation bimodule between $M^X$ and $M^Y$ with underlying 1-cell $B(m) =: b \in \mathcal{B}(X, Y)$. According to Corollary 3.8, it comes with a left action $l_b := B(l_m) \circ \phi_{e1,m} : \mathbb{1}_Y \otimes b \to b$ and analogously a left coaction $\check{l}_b := \phi^{-1}_{e1,m} \circ B(\check{l}_m) : b \to \mathbb{1}_Y \otimes b$. A natural candidate for the desired isomorphism is $l_b \circ \lambda_b^{-1}$ from $B^b$ to $B$, as a string diagram:

$l_b \circ \lambda_b^{-1} =$ 

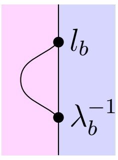
We first note that this has a right inverse given by $\lambda_b \circ \check{l}_b$:

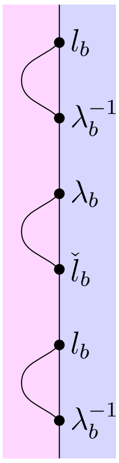
$(l_b \circ \lambda_b^{-1}) \circ (\lambda_b \circ \check{l}_b) =$ 

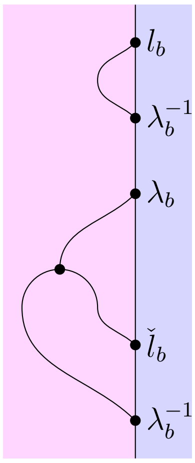
$=$ 

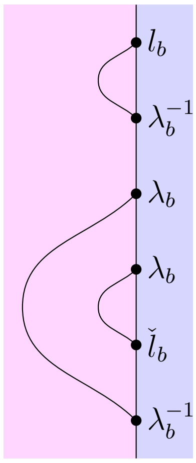
$\overset{(S4)}{=} \mathrm{id}_b$ (3.11)

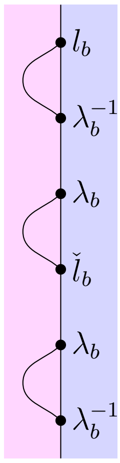
But instead of showing that these are proper inverses of each other, using the above relation we see that $l_b \circ \lambda_b^{-1}$ actually is the identity:



$\overset{(3.11)}{=}$ 

$\overset{(F4)}{=}$ 

$\overset{(2.13)}{=}$ 

$\overset{\text{Nat.}}{=}$ 

$= \mathrm{id}_b$

The third picture contains an unlabelled vertex which refers to the multiplication in $\mathbb{1}_Y$, and is followed by an application of the left module property of $B^b$. The second to last step is the naturality of $\lambda_b$.
As $\lambda_b$ is an isomorphism it follows that $l_b = \lambda_b$ and $\check{l}_b = \lambda_b^{-1}$. The analogous result for the right actions follows in precisely the same manner. Thus we see that the only possible $\mathbb{1}_X$-$\mathbb{1}_Y$-bimodule structure is given by the unitors and is therefore equal to the one of $B^b$, which completes the proof. □

**Proposition 3.35.** *If a 2-category $\mathcal{B}$, for which all hom-categories are idempotent complete, has all condensates, then the canonical inclusion $J : \mathcal{B} \to \mathrm{Kar}(\mathcal{B})$ is a 2-equivalence.*

*Proof.* As we already showed in Proposition 3.34 that $J$ is always fully faithful, it remains to show that it is essentially surjective on objects.
To this end let $(M, \phi) : ♣_2 \to \mathcal{B}$ be any object of $\mathrm{Kar}(\mathcal{B})$, a 2-condensation monad on $M(A) =: X$ with underlying 1-cell $M(e) =: a$. As by assumption all 2-condensation monads condensate, so does $M$. That means that $M$ lifts to a 2-functor

defined on $\spadesuit_2$, or explicitly that there is $Y \in \mathrm{Ob}(\mathcal{B})$, $f \in \mathcal{B}(X,Y)$, $g \in \mathcal{B}(Y,X)$, $\gamma \in \mathrm{Hom}_{\mathcal{B}(Y,Y)}(f \otimes g, \mathbb{1}_Y)$ and $\delta \in \mathrm{Hom}_{\mathcal{B}(Y,Y)}(\mathbb{1}_Y, f \otimes g)$ forming a 2-condensation according to Definition 3.1 such that $\phi_{j,q} : g \otimes f \cong a$. We now claim $J(Y) = M^Y$ is the desired object in the image of $J$ equivalent to $M$. To witness the equivalence define a $\mathbb{1}_Y$-$a$-bimodule with underlying 1-cell $f$ and the following actions:

$$l_f := \lambda_f$$
$$r_f := \lambda_f \circ (\gamma \otimes \mathrm{id}_f) \circ \alpha^{-1}_{f,g,f} \circ (\mathrm{id}_f \otimes \phi^{-1}_{j,q})$$
$$\check{l}_f := \lambda^{-1}_f$$
$$\check{r}_f := (\mathrm{id}_f \otimes \phi_{j,q}) \circ \alpha_{f,g,f} \circ (\delta \otimes \mathrm{id}_f) \circ \lambda^{-1}_f$$

as well as an $a$-$\mathbb{1}_Y$-bimodule with $g$ as underlying 1-cell and analogous actions:

$$l_g := \rho_g \circ (\mathrm{id}_g \otimes \gamma) \circ \alpha_{g,f,g} \circ (\phi^{-1}_{j,q} \otimes \mathrm{id}_g)$$
$$r_g := \rho_g$$
$$\check{l}_f := (\phi_{j,q} \otimes \mathrm{id}_g) \circ \alpha^{-1}_{g,f,g} \circ (\mathrm{id}_g \otimes \delta) \circ \rho^{-1}_g$$
$$\check{r}_f := \rho^{-1}_g$$

That these indeed satisfy the relations of a special Frobenius bimodule can be seen by expressing them in terms of string diagrams once the following fact has been established: the multiplication on $a$ given by $M$, which by definition is given by:

$$M((\mathrm{id}_j \otimes (\lambda_q \circ (\tau \otimes \mathrm{id}_q) \circ \alpha^{-1}_{q,j,q})) \circ \alpha_{j,q,j\otimes q}) \circ \phi_{e,e}$$

can be written as:

$$\phi_{j,q} \circ (\mathrm{id}_g \otimes (\lambda_f \circ (\gamma \otimes \mathrm{id}_f) \circ \alpha^{-1}_{f,g,f})) \circ \alpha_{g,f,g\otimes f} \circ (\phi^{-1}_{j,q} \otimes \phi^{-1}_{j,q})$$

(and the analogue for the comultiplication) which follows from a lengthy but straightforward calculation applying naturality of the occurring isomorphisms and twice the hexagon axiom (similar to the calculations occurring in the proof of Corollary 3.5).

Next note that also $M^Y$ condenses onto some $Z \in \mathrm{Ob}(\mathcal{B})$ along some $h \in \mathcal{B}(Y,Z)$ and $i \in \mathcal{B}(Z,Y)$. Everything then fits into a $\heartsuit_2$-shaped diagram:

$$\begin{tikzcd}
Y \arrow[loop above, "\mathbb{1}_Y"] \arrow[d, bend right, "h"'] & X \arrow[l, "f"'] \arrow[loop above, "a"] \arrow[dd, bend right, "f"'] & Y \arrow[loop above, "\mathbb{1}_Y"] \arrow[l, "g"'] \arrow[d, bend right, "h"'] \\
Z \arrow[u, bend right, "i"'] & & Z \arrow[u, bend right, "i"'] \arrow[dl, "i"] \\
& Y \arrow[ul, "h"] \arrow[uu, bend right, "g"'] &
\end{tikzcd}$$

from which we can read off, according to Definition 3.15, the relative tensor product

$$f \otimes_a g = (i \otimes h) \otimes (i \otimes h) \cong \mathbb{1}_Y \otimes \mathbb{1}_Y \cong \mathbb{1}_Y \tag{3.12}$$

which is thus isomorphic the unit on $M^Y$. That the relevant 2-cell isomorphisms are indeed 2-cells of $\mathrm{Kar}(\mathcal{B})$ is trivial as is the condition to be a $\mathbb{1}_Y$-$\mathbb{1}_Y$-bimodule map according to the proof of Proposition 3.34.

In that proof we furthermore saw that the relative tensor product over $\mathbb{1}_Y$ is trivial so that we have:

$$g \otimes_{\mathbb{1}_Y} f = g \otimes f \cong a \tag{3.13}$$

which is the unit on $M$ according to Lemma 3.18. That the isomorphism $\phi_{j,q}$ is a 2-cell of $\mathrm{Kar}(\mathcal{B})$ can also be seen using string diagrams exploiting the fact stated above. □

Having the associator of $\mathrm{Kar}(\mathcal{B})$ and the relative tensor product of condensation bimodules at hand, we can now prove the following analogue of Proposition 2.11.

**Proposition 3.36.** *Assume a given 2-condensation monad condenses in two different ways, $M, M' : \spadesuit_2 \to \mathcal{B}$ onto $Y := M(K)$ and onto $Y' := M'(K)$. Then $Y$ and $Y'$ are equivalent in $\mathcal{B}$.*

*Proof.* This works along the lines of the proof of Proposition 2.11. We denote $M(q) =: f$, $M(j) =: g$, $M'(q) =: f'$, $M'(j) =: g'$ and $M(e) = M'(e) = a$. As we are dealing with condensations and not just condensation monads, we can define four different condensation bimodules which have $f$ as underlying $\mathbb{1}_Y$-$a$-bimodule, $f'$ as underlying $\mathbb{1}_{Y'}$-$a$-bimodule, $g$ as underlying $a$-$\mathbb{1}_Y$-bimodule and $g'$ as underlying $a$-$\mathbb{1}_{Y'}$-bimodule, in the same way as in the proof of Proposition 3.35. Using these we can construct the desired 1-cells as $(f \otimes_a g')$ and $(f' \otimes_a g)$. That they constitute an equivalence can then be checked as follows:

$$(f \otimes_a g') \otimes (f' \otimes_a g) \xrightarrow{\alpha^{\mathrm{Kar}(\mathcal{B})}} (f \otimes_a (g' \otimes f')) \otimes_a g \xrightarrow{\phi} (f \otimes_a a) \otimes_a g = f \otimes_a g \cong \mathbb{1}_Y$$

where $\alpha^{\mathrm{Kar}(\mathcal{B})}$ stands for the appropriate rebracketing, and can be applied because the composition of 1-cells in $\mathcal{B}$ is the same as calculating the relative tensor product over $\mathbb{1}_{Y'}$ as we have seen in the proof of Proposition 3.34. $\phi$ stands for the application of the coherence isomorphism of $M'$. The second to last equality holds because $a$ acts as a strict unit for the relative tensor product over a, which can be seen along the lines of Lemma 3.18. Finally, the last isomorphism holds according to Equation (3.12) in the proof of Proposition 3.35.

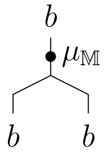
That $(f' \otimes_a g) \otimes (f \otimes_a g') \cong \mathrm{id}_{\mathbb{1}_{Y'}}$ follows in exactly the same manner. □

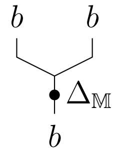
**Theorem 3.37.** *The 2-Karoubi completion* $\mathrm{Kar}(\mathcal{B})$ *of a 2-category $\mathcal{B}$, all of whose hom-categories are idempotent complete, has all condensates.*

*Proof.* The idea of the proof is the same as for the 1-dimensional case in Proposition 2.12.

Let $(\mathbb{M}, \phi) : \clubsuit_2 \to \mathrm{Kar}(\mathcal{B})$ be any condensation monad of $\mathrm{Kar}(\mathcal{B})$ so that $\mathbb{M}(A) = M : \clubsuit_2 \to \mathcal{B}$ is a condensation monad on $M(A) = X \in \mathrm{Ob}(\mathcal{B})$ with underlying 1-cell $M(e) = a$, and $\mathbb{M}(e) = E : \diamondsuit_2 \to \mathcal{B}$ is an $a$-$a$-condensation bimodule. The key to the proof is to construct a condensation monad with underlying 1-cell $b := E(m)$, which will be the desired object of $\mathrm{Kar}(\mathcal{B})$ onto which $\mathbb{M}$ condenses.

To this end consider $\mu_{\mathbb{M}} := \mathbb{M}(\mu) \circ \phi_{e,e}$ and $\Delta_{\mathbb{M}} := \phi_{e,e}^{-1} \circ \mathbb{M}(\Delta)$ which are 2-cells in $\mathrm{Hom}_{\mathcal{B}(X,X)}(b \otimes_a b, b)$ and $\mathrm{Hom}_{\mathcal{B}(X,X)}(b, b \otimes_a b)$, respectively. Here, $b \otimes_a b$ is defined (in Definition 3.15) as the image of an idempotent, which splits (by the assumption of Karoubi completeness of all hom-categories of $\mathcal{B}$) along some $\Theta \in \mathrm{Hom}_{\mathcal{B}(X,X)}(b \otimes b, b \otimes_a b)$ and $\Psi \in \mathrm{Hom}_{\mathcal{B}(X,X)}(b \otimes_a b, b \otimes b)$. We can thus use all of the above to construct:

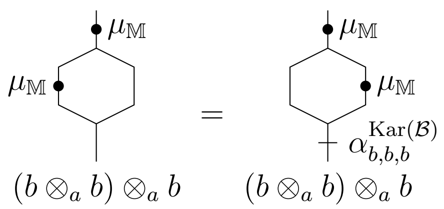
$\mu_b := \mu_{\mathbb{M}} \circ \Theta =$ 


$\Delta_b := \Psi \circ \Delta_{\mathbb{M}} =$ 


We now want to see that these turn $b$ into a nonunital, noncounital $\Delta$-separable Frobenius algebra. To show associativity of $\mu_b$ we are going to use the associativity of $\mu_{\mathbb{M}}$:

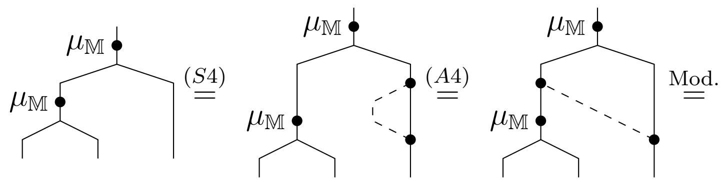
$$\mu_{\mathbb{M}} \circ (\mu_{\mathbb{M}} \otimes_a \mathrm{id}_b) = \mu_{\mathbb{M}} \circ (\mathrm{id}_b \otimes_a \mu_{\mathbb{M}}) \circ \alpha^{\mathrm{Kar}(\mathcal{B})}_{b,b,b}$$

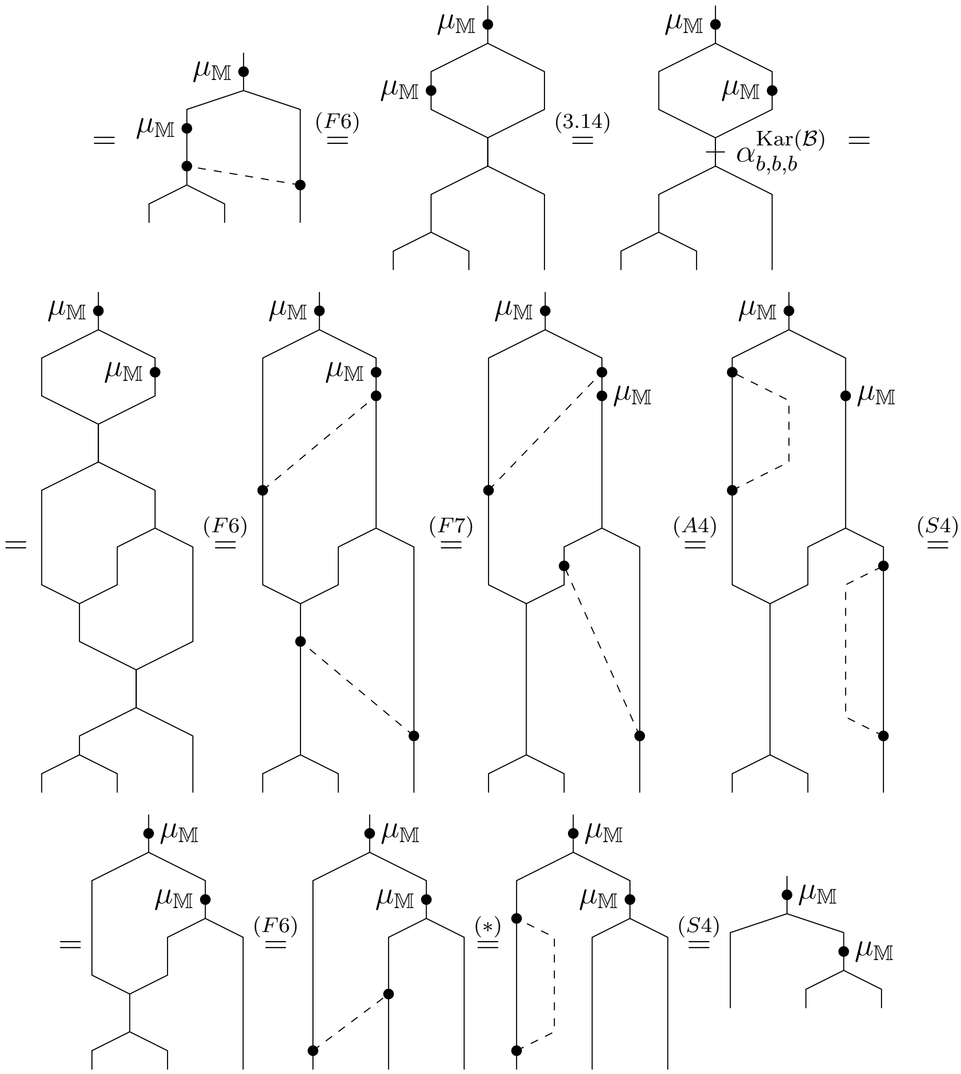
in the following form:



(3.14)

where in the string diagrams (here as well as in the rest of the proof) are in $\mathcal{B}$ and the white background corresponds to $X$, dashed lines to $a$ and solid lines to $b$ or relative tensor products over $a$ of $b$ with itself. Then we can compute:

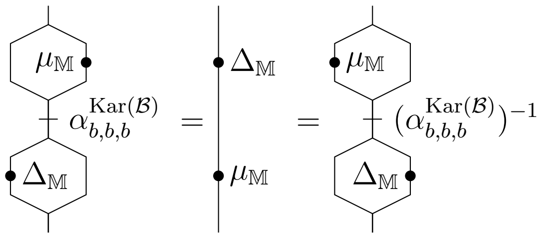
$\mu_b \circ (\mu_b \otimes \mathrm{id}_b) =$ 




Where the step labelled "$Mod.$" is due to the fact that $\mu_{\mathbb{M}}$ is not only a 2-ell of $\mathcal{B}$ but a 2-cell of $\mathrm{Kar}(\mathcal{B})$ and thus a module map. The step labelled $(*)$ is a combination of Equation $(A5)$, the module map property of $\mu_{\mathbb{M}}$ and Equation $(A4)$.
Coassociativity of $\Delta_{\mathbb{M}}$ is shown in exactly the same way just by mirroring the above proof upside-down.
For the Frobenius property of $\mu_b$ and $\Delta_b$ we similarly use the Frobenius property of $\mu_{\mathbb{M}}$ and $\Delta_{\mathbb{M}}$:

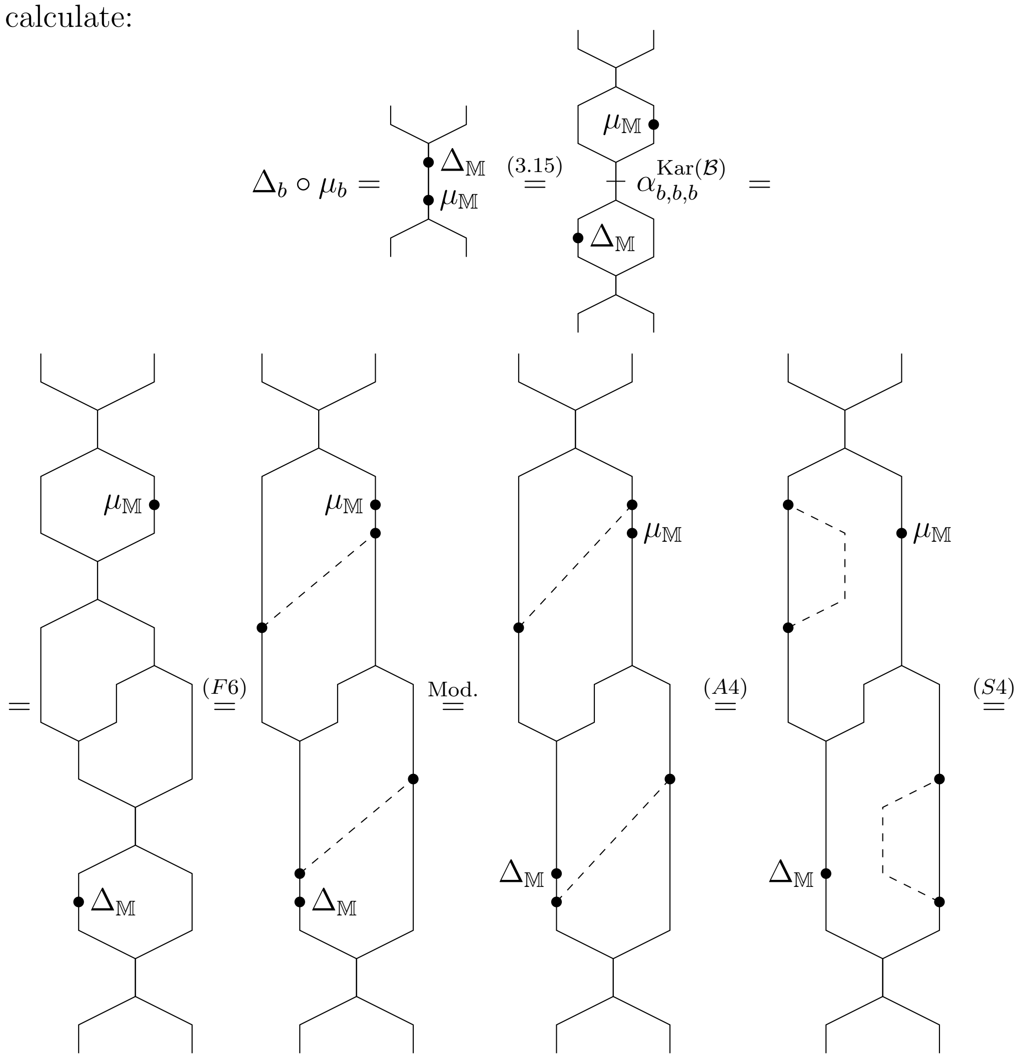
$$(\mathrm{id}_b \otimes_a \mu_{\mathbb{M}}) \circ \alpha_{b,b,b}^{\mathrm{Kar}(\mathcal{B})} \circ (\Delta_{\mathbb{M}} \otimes_a \mathrm{id}_b) = \Delta_{\mathbb{M}} \circ \mu_{\mathbb{M}} = (\mu_{\mathbb{M}} \otimes_a \mathrm{id}_b) \circ (\alpha_{b,b,b}^{\mathrm{Kar}(\mathcal{B})})^{-1} \circ (\mathrm{id}_b \otimes_a \Delta_{\mathbb{M}})$$

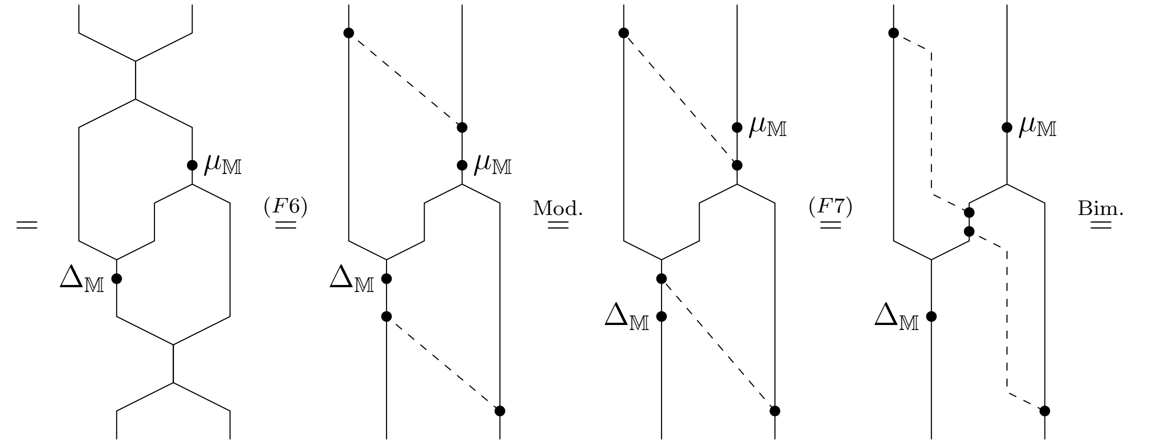
expressed as string diagrams in $\mathcal{B}$:

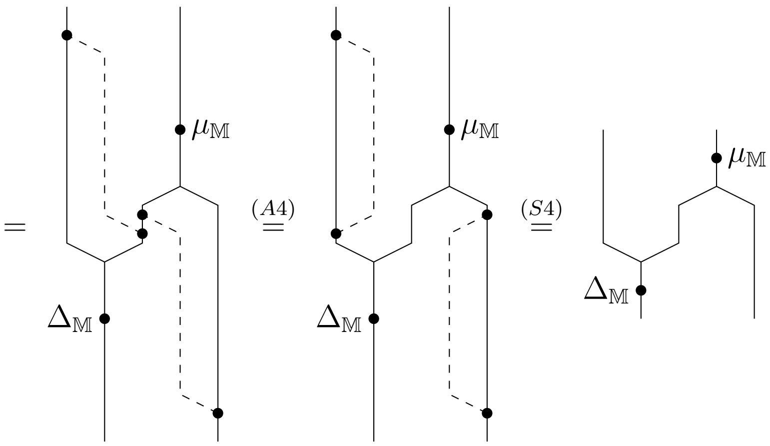



(3.15)

and calculate:



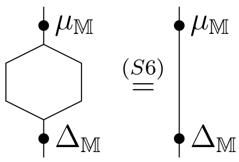



$$= (\mathrm{id}_b \otimes \mu_b) \circ \alpha_{b,b,b} \circ (\Delta_b \otimes \mathrm{id}_b)$$

The other equality to check for the Frobenius property is proven in the same way, by mirroring the above from left to right, using the other equality in (3.15) and noting that as a string diagram $(\alpha_{b,b,b}^{\mathrm{Kar}(\mathcal{B})})^{-1}$ indeed is the appropriate mirror image of $\alpha_{b,b,b}^{\mathrm{Kar}(\mathcal{B})}$ according to Proposition 3.29.
$\Delta$-separability follows easily from the $\Delta$-separability of $\mu_{\mathbb{M}}$ and $\Delta_{\mathbb{M}}$:

$$\mu_b \circ \Delta_b =$$




$$= \mathrm{id}_b$$

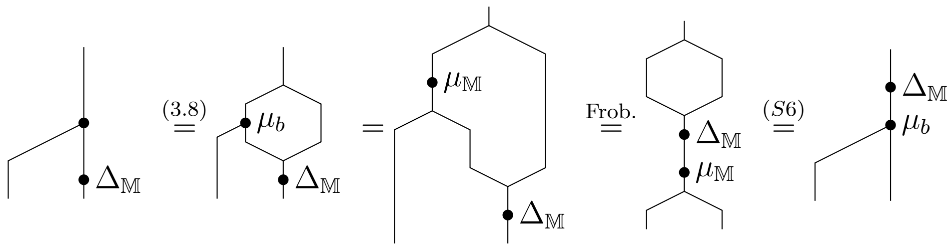
Thus (according to Remark 3.14) we can construct a condensation monad $N : ♣_2 \to \mathcal{B}$ with $N(A) = X$ and $N(e) = b$. To see that $M$ condenses onto $N$, define $F : \diamondsuit_2 \to \mathcal{B}$ to be $b$ as a $b$-$a$-bimodule where the left action and coaction are just $\mu_b$ and $\Delta_b$ and the right action and coaction are the ones given by $E$, and vice versa define $G : \diamondsuit_2 \to \mathcal{B}$ to be $b$ as an $a$-$b$-bimodule. All of this then fits into the following diagram, displayed on the left-hand side in $\mathrm{Kar}(\mathcal{B})$ with their corresponding cells of $\mathcal{B}$ on the right-hand side:

$$
\begin{tikzcd}
M \arrow[loop above, "E"] \arrow[d, bend right, "F"'] \\
N \arrow[u, bend right, "G"'] \arrow[loop below, "\mathbb{1}_N"]
\end{tikzcd}
\qquad \widehat{=} \qquad
\begin{tikzcd}
(X,a) \arrow[loop above, "b"] \arrow[d, bend right, "b"'] \\
(X,b) \arrow[u, bend right, "b"'] \arrow[loop below, "b"]
\end{tikzcd}
$$

To prove that $M$ actually condenses onto $N$, it remains to show that the above actually fits into a $\spadesuit_2$-shaped diagram, that is on the one hand we need to check that $G \otimes_N F = E$ and on the other hand we need to find $\gamma_{\mathbb{M}} \in \mathrm{Hom}_{\mathrm{Kar}(\mathcal{B})(N,N)}(F \otimes_E G, \mathbb{1}_N)$ and $\delta_{\mathbb{M}} \in \mathrm{Hom}_{\mathrm{Kar}(\mathcal{B})(N,N)}(\mathbb{1}_N, F \otimes_E G)$ such that $\gamma_{\mathbb{M}} \circ \delta_{\mathbb{M}} = \mathrm{id}_{\mathbb{1}_N}$.
$G \otimes_N F$ corresponds to $b \otimes_b b$ as an $a$-$a$-bimodule and is given according to Definition 3.15 as the image of the splitting idempotent:

$$\mathfrak{k} = (\mathrm{id}_b \otimes \mu_b) \circ \alpha_{b,b,b} \circ (\Delta_b \otimes \mathrm{id}_b)$$

but precisely due to the Frobenius property of $\mu_b$ and $\Delta_b$ which we showed above, we see that the desired splitting is given by $\mathfrak{k} = \Delta_b \circ \mu_b$ which indeed has $b$ as an $a$-$a$-bimodule (i.e. $E$) as its image.
As the underlying 1-cells of $\mathbb{1}_N$ and $F \otimes_E G$ are $b$ and $b \otimes_a b$ respectively, we might define $\delta_{\mathbb{M}} := \Delta_{\mathbb{M}}$ and $\gamma_{\mathbb{M}} := \mu_{\mathbb{M}}$ so that the desired equality $\gamma_{\mathbb{M}} \circ \delta_{\mathbb{M}} = \mu_{\mathbb{M}} \circ \Delta_{\mathbb{M}} = \mathrm{id}_b$ follows from $\Delta$-separability. But $\mu_{\mathbb{M}}$ and $\Delta_{\mathbb{M}}$ are by definition only maps of $a$-$a$-bimodules; it remains to show that they are also maps of $b$-$b$-bimodules.
We start with $\Delta_{\mathbb{M}}$:




where the step labelled "Frob." is an application of the Frobenius property of $\mu_b$ and $\Delta_b$ which we showed above. Analogously for the right action using the other Frobenius equality. Turning the above proof upside down and switching the roles of $\mu$ and $\Delta$, one shows that $\mu_{\mathbb{M}}$ commutes with the coaction, which suffices as explained in Remark 3.22. ☐

Together with Proposition 3.35, Theorem 3.37 implies the following:

**Corollary 3.38.** *For a 2-category $\mathcal{B}$ where all hom-categories are idempotent complete, its 2-Karoubi completion* $\mathrm{Kar}(\mathcal{B})$ *and the 2-Karoubi completion of its 2-Karoubi completion* $\mathrm{Kar}(\mathrm{Kar}(\mathcal{B}))$ *are 2-equivalent.*

## 3.6 Unital Condensations

The notion of condensation introduced in Definition 3.1, has led us to a 2-category of condensation monads which is equivalent to the 2-category of nonunital, noncounital, $\Delta$-separable Frobenius algebras. In order to drop the "nonunital" and "noncounital", we need to ask an extra property of the condensation and will end up with a 2-category of unital and counital condensation monads which is equivalent to the usual 2-category of $\Delta$-separable Frobenius algebras.

**Definition 3.39** (Unital and Counital 2-Condensation)**.** Given a 2-condensation $(X, Y, f, g, \gamma, \delta)$ in a 2-category $\mathcal{B}$, we say that

- $(X, Y, f, g, \gamma, \delta)$ is a *unital 2-condensation*, if $\gamma \in \mathrm{Hom}_{\mathcal{B}(Y,Y)}(f \otimes g, \mathbb{1}_Y)$ is the counit of an adjunction $f \dashv g$, i.e. if there is a 2-cell $\eta \in \mathrm{Hom}_{\mathcal{B}(X,X)}(\mathbb{1}_X, g \otimes f)$ such that:

$$\lambda_f \circ (\gamma \otimes \mathrm{id}_f) \circ \alpha^{-1}_{f,g,f} \circ (\mathrm{id}_f \otimes \eta) \circ \rho_f^{-1} = \mathrm{id}_f \tag{3.16}$$
$$\rho_g \circ (\mathrm{id}_g \otimes \gamma) \circ \alpha_{g,f,g} \circ (\eta \otimes \mathrm{id}_g) \circ \lambda_g^{-1} = \mathrm{id}_g \tag{3.17}$$

- $(X, Y, f, g, \gamma, \delta)$ is a *counital 2-condensation* if $\delta \in \mathrm{Hom}_{\mathcal{B}(Y,Y)}(\mathbb{1}_Y, f \otimes g)$ is the unit of an adjunction $g \dashv f$, i.e. if there is a 2-cell $\epsilon \in \mathrm{Hom}_{\mathcal{B}(X,X)}(g \otimes f, \mathbb{1}_X)$ such that:

$$\lambda_g \circ (\epsilon \otimes \mathrm{id}_g) \circ \alpha^{-1}_{g,f,g} \circ (\mathrm{id}_g \otimes \delta) \circ \rho_g^{-1} = \mathrm{id}_g \tag{3.18}$$
$$\rho_f \circ (\mathrm{id}_f \otimes \epsilon) \circ \alpha_{f,g,f} \circ (\delta \otimes \mathrm{id}_f) \circ \lambda_f^{-1} = \mathrm{id}_f \tag{3.19}$$

We can then define the notions of unital and counital condensation monads along the lines of Section 3.1.

**Definition 3.40.** (Unital and Counital 2-Condensation Monads).

- Define the 2-category $\spadesuit_2^{\mathbb{1}}$ by adding a 2-cell $\eta : \mathbb{1}_A \to j \otimes q$ to the generators of $\spadesuit_2$ and analogues of (3.16) & (3.17) to its relations so that $(A, K, q, j, \tau, \nu)$ becomes a unital 2-condensation. Denoting the full subcategory on the object $A$ of $\spadesuit_2^{\mathbb{1}}$ as $\clubsuit_2^{\mathbb{1}}$, a 2-functor from $\clubsuit_2^{\mathbb{1}}$ to a 2-category $\mathcal{B}$ is called a *unital condensation monad* in $\mathcal{B}$.

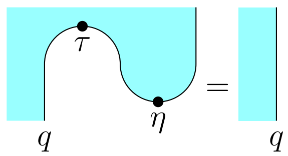
- Define the 2-category $\spadesuit_2^{\mathbb{I}}$ by adding a 2-cell $\epsilon : j \otimes q \to \mathbb{1}_A$ to the generators of $\spadesuit_2$ and analogies of (3.18) & (3.19) to its relations so that $(A, K, q, j, \tau, \nu)$ becomes a counital 2-condensation. Denoting the full subcategory on the object $A$ of $\spadesuit_2^{\mathbb{I}}$ as $\clubsuit_2^{\mathbb{I}}$, a 2-functor from $\clubsuit_2^{\mathbb{I}}$ to a 2-category $\mathcal{B}$ is called a *counital condensation monad* in $\mathcal{B}$.

It turns out that these actually deserve their names so that unital condensation monads give rise to unital algebras and counital condensation monads to counital algebras.

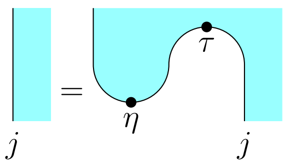
**Proposition 3.41.** *Consider a 2-condensation monad* $(M, \phi) : \clubsuit_2 \to \mathcal{B}$ *and the corresponding nonunital, noncounital* $\Delta$*-separable Frobenius algebra* $(a, m, d)$*.*

- *If* $(M, \phi)$ *lifts to a 2-functor* $\clubsuit_2^{\mathbb{1}} \to \mathcal{B}$*, the 2-cell* $\eta_M := M(\eta) \circ \phi^A$ *makes* $(a, m, \eta_M)$ *a unital algebra.*
- *If* $(M, \phi)$ *lifts to a 2-functor* $\clubsuit_2^{\mathbb{I}} \to \mathcal{B}$*, the 2-cell* $\epsilon_M := (\phi^A)^{-1} \circ M(\epsilon)$ *makes* $(a, d, \epsilon_M)$ *a counital coalgebra.*

*Proof.* We first show that $\eta$ is a unit for the algebra structure $(e, \mu)$ given in Proposition 3.4 (for the subcategory $\clubsuit_2 \subset \clubsuit_2^{\mathbb{1}}$). This can be done using string diagrams in the bigger category $\spadesuit_2^{\mathbb{1}}$, using the relations (3.16) and (3.17) which can be expressed there as follows:

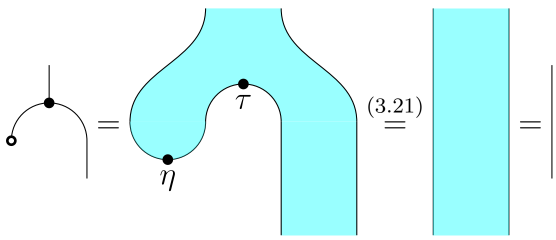


(3.20)



(3.21)

The left-hand side of the desired unitality condition (2.11) then is immediate from the so called Zorro move (3.21) above:



(3.22)

Similarly the right-hand side follows from (3.20).
For the first case it remains to show that this is persevered by the 2-functor $(M, \phi)$, so that $M(\eta) \circ \phi^A$ indeed is a unit for $(M(e), M(\mu) \circ \phi_{e,e})$. This is a calculation similar to the proof of Corollary 3.5, using the compatibility with the unitors instead

of the hexagon axiom:

$$\begin{aligned}\mu_M \circ (\eta_M \otimes \mathrm{id}_a) \circ \lambda_a^{-1} &= M(\mu) \circ \phi_{e,e} \circ (M(\eta) \otimes \mathrm{id}_a) \circ (\phi^A \otimes \mathrm{id}_a) \circ \lambda_a^{-1} \overset{(2.10)}{=} \\ &= M(\mu) \circ \phi_{e,e} \circ (M(\eta) \otimes \mathrm{id}_a) \circ \phi^{-1}_{\mathbb{1}_A,e} \circ M(\lambda_e^{-1}) \overset{\text{nat.}}{=} \\ &= M(\mu) \circ M(\eta \otimes \mathrm{id}_e) \circ M(\lambda_e^{-1}) = \\ &= M(\mu \circ (\eta \otimes \mathrm{id}_e) \circ \lambda_e^{-1}) \overset{(3.22)}{=} M(\mathrm{id}_e) = \mathrm{id}_a\end{aligned}$$

And similar for the right-hand side of (2.11).
The statement about the counit is proven completely analogously, using the upside-down versions of the string diagrams. □

We can describe categories of unital and counital condensation monads as full subcategories of the category of all condensation monads $\mathrm{Kar}(\mathcal{B})$ established in Section 3.4.

**Definition 3.42.** Given a 2-category $\mathcal{B}$ where all hom-categories are idempotent complete, we define the following subcategories of $\mathrm{Kar}(\mathcal{B})$:

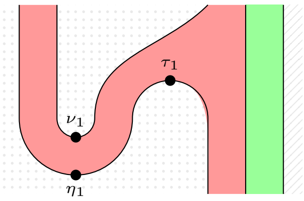
- the full subcategory of $\mathrm{Kar}(\mathcal{B})$ that contains only those objects which are unital 2-condensation monads is denoted $\mathrm{Kar}^{\mathbb{1}}(\mathcal{B})$,
- the full subcategory of $\mathrm{Kar}(\mathcal{B})$ that contains only those objects which are counital 2-condensation monads is denoted $\mathrm{Kar}^{\mathbb{I}}(\mathcal{B})$,
- the full subcategory of $\mathrm{Kar}(\mathcal{B})$ that contains only those objects which are both unital and counital 2-condensation monads is denoted $\mathrm{Kar}^{\mathbb{1},\mathbb{I}}(\mathcal{B})$.

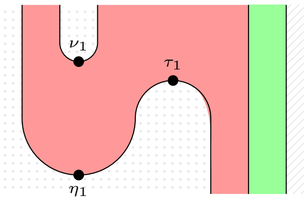
What remains to show in order to see that $\mathrm{Kar}^{\mathbb{1},\mathbb{I}}(\mathcal{B})$ and $\mathrm{Frob}_\Delta(\mathcal{B})$ are equivalent, is the following.

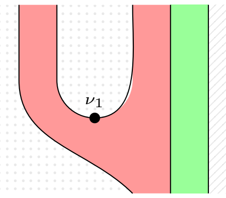
**Proposition 3.43.** *Given a 2-condensation bimodule $(B, \phi) : \diamondsuit_2 \to \mathcal{B}$ between two objects, $M$ and $N$, of $\mathrm{Kar}^{\mathbb{1},\mathbb{I}}(\mathcal{B})$, which has $(b, l_b, r_b, \check{l}_b, \check{r}_b)$ as underlying special Frobenius bimodule between the $\Delta$-separable Frobenius algebras $(a_M, \mu_M, \Delta_M, \eta_M, \epsilon_M)$ and $(a_N, \mu_N, \Delta_N, \eta_N, \epsilon_N)$, the bimodule $(b, l_b, r_b, \check{l}_b, \check{r}_b)$ is compatible with the unital and counital structures and its coactions equal the canonical coactions: $\check{l}_b = \check{l}_b^{\mathrm{can}}$ and $\check{r}_b = \check{r}_b^{\mathrm{can}}$.*

*Proof.* The fact that left- and right-action are compatible with the units, i.e. Equations (2.16) & (2.17) hold, follows completely analogous to the proof of Proposition 3.41 by changing the the background colour on the right-hand side in the string diagrams of (3.22) and noting that the equations have precisely the same structure so the fact that they are persevered by a 2-functor can be shown in the same way. The statement that the coactions are compatible with the counits is just the upside down version of that and thus follows similarly.

As the 2-condensation bimodule under consideration maps between two 2-condensation monads which both are unital and counital, it lifts to a 2-functor defined on the category $\blacklozenge_2^{\mathbb{1},\mathbb{I}}$ which is obtained from $\blacklozenge_2$ by adding 2-cells $\eta_1$, $\eta_2$, $\epsilon_1$ and $\epsilon_2$ so that the two canonical inclusions of $\clubsuit_2$ become unital and counital 2-condensation monads. Inside $\blacklozenge_2^{\mathbb{1},\mathbb{I}}$ we can calculate, using the same notation as in the proof of Proposition 3.7:

$\check{l}_m^{\text{can}} =$ 

$=$ 

$\overset{(3.21)}{=}$ 


(3.23)

Which is equal to $\check{l}_m$ as given in the proof of Proposition 3.7. It remains to see that this equality is preserved by the 2-functor $B$. We need to show that

$$\check{l}_b^{\text{can}} = \check{l}_b \overset{3.8}{:=} \phi_{e_1,m}^{-1} \circ B(\check{l}_m)$$

We bring the isomorphism to the other side and calculate:

$$\begin{aligned}
\phi_{e_1,m} \circ \check{l}_b^{\text{can}} =& \phi_{e_1,m} \circ (\mathrm{id}_{a_M} \otimes l_b) \circ \alpha_{a_M,a_M,b} \circ ((\Delta_M \circ \eta_M) \otimes \mathrm{id}_b) \circ \lambda_b^{-1} = \\
=& \phi_{e_1,m} \circ (B(\mathrm{id}_{e_1}) \otimes B(l_m)) \circ (B(\mathrm{id}_{e_1}) \otimes \phi_{e_1,m}) \circ \alpha_{a_M,a_M,b} \circ \\
& ((\phi_{e_1,e_1}^{-1} \circ B(\Delta_1 \circ \eta_1)) \otimes B(\mathrm{id}_m)) \circ (\phi^{A_1} \otimes \mathrm{id}_b) \circ \lambda_b^{-1} \overset{\text{nat.}}{=} \\
=& B(\mathrm{id}_{e_1} \otimes l_m) \circ \phi_{e_1,e_1 \otimes m} \circ (\mathrm{id}_{a_M} \otimes \phi_{e_1,m}) \circ \alpha_{a_M,a_M,b} \circ \\
& ((\phi_{e_1,e_1}^{-1} \circ B(\Delta_1 \circ \eta_1)) \otimes B(\mathrm{id}_m)) \circ (\phi^{A_1} \otimes \mathrm{id}_b) \circ \lambda_b^{-1} \overset{(2.10)}{=} \\
=& B(\mathrm{id}_{e_1} \otimes l_m) \circ \phi_{e_1,e_1 \otimes m} \circ (\mathrm{id}_{a_M} \otimes \phi_{e_1,m}) \circ \alpha_{a_M,a_M,b} \circ \\
& ((\phi_{e_1,e_1}^{-1} \circ B(\Delta_1 \circ \eta_1)) \otimes B(\mathrm{id}_m)) \circ \phi_{\mathbb{1}_{A_1},m}^{-1} \circ B(\lambda_m^{-1}) \overset{\text{nat.}}{=} \\
=& B(\mathrm{id}_{e_1} \otimes l_m) \circ \phi_{e_1,e_1 \otimes m} \circ (\mathrm{id}_{a_M} \otimes \phi_{e_1,m}) \circ \alpha_{a_M,a_M,b} \circ \\
& (\phi_{e_1,e_1}^{-1} \otimes B(\mathrm{id}_m)) \circ \phi_{e_1 \otimes e_1,m}^{-1} \circ B((\Delta_1 \circ \eta_1) \otimes \mathrm{id}_m) \circ B(\lambda_m^{-1}) \overset{\text{Hex.}}{=} \\
=& B(\mathrm{id}_{e_1} \otimes l_m) \circ B(\alpha_{e_1,1,m}) \circ B((\Delta_1 \circ \eta_1) \otimes \mathrm{id}_m) \circ B(\lambda_m^{-1}) = \\
=& B((\mathrm{id}_{e_1} \otimes l_m) \circ \alpha_{e_1,1,m} \circ ((\Delta_1 \circ \eta_1) \otimes \mathrm{id}_m) \circ \lambda_m^{-1}) = B(\check{l}_m^{\text{can}}) \overset{(3.23)}{=} B(\check{l}_m)
\end{aligned}$$

The statement about $\check{r}_b^{\text{can}}$ follows analogously. □

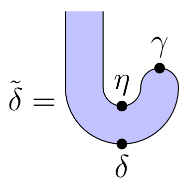
**Remark 3.44.** *Using Proposition 3.41 and Proposition 3.43 we can find a 2-equivalence between* $\mathrm{Kar}^{\mathbb{1},\mathbb{I}}(\mathcal{B})$ *and* $\mathrm{Frob}_\Delta(\mathcal{B})$ *analogously to the one constructed in Remark 3.32. We can send a unital and counital 2-condensation monad to its underlying* *algebra (which is in* $\mathrm{Frob}_\Delta(\mathcal{B})$ *by Proposition 3.41) and a 2-condensation bimodule to its underlying bimodule, forgetting the comodule structure. This forgetting causes no problem as for every bimodule between objects of* $\mathrm{Frob}_\Delta(\mathcal{B})$ *there is a canonical comodule structure making it a special Frobenius module (and thus it is in the image of the proposed equivalence) by Proposition 2.45 and there are no other comodule structures occurring in the image by Proposition 2.46.*

**Lemma 3.45.** *Let* $(X, Y, f, g, \gamma, \delta)$ *be a 2-condensation in* $\mathcal{B}$*. If* $f$ *admits a right adjoint* $f^R$*, there is a unital 2-condensation* $(X, Y, f, f^R, \tilde{\gamma}, \tilde{\delta})$*. Similarly, if* $g$ *admits a right adjoint* $g^R$*,* $g$ *and* $g^R$ *participate in a counital 2-condensation, if* $f$ *admits a left adjoint* $f^L$*,* $f$ *and* $f^L$ *participate in a counital 2-condensation and if* $g$ *admits a left adjoint* $g^L$*,* $g$ *and* $g^L$ *participate in a unital 2-condensation.*

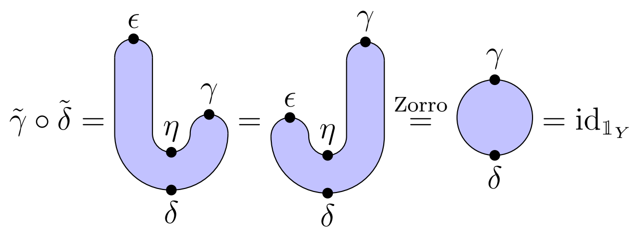
*Proof.* For the first statement, let $\eta \in \mathrm{Hom}_{X,X}(\mathbb{1}_X, f^R \otimes f)$ and $\epsilon \in \mathrm{Hom}_{X,X}(f \otimes f^R, \mathbb{1}_X)$ be the unit and counit of the adjunction $f \dashv f^R$. Defining $\tilde{\gamma} := \epsilon$ we need to find $\tilde{\delta}$ so that $\tilde{\gamma} \circ \tilde{\delta} = \mathrm{id}_{\mathbb{1}_Y}$. We can define it al follows:

$$\tilde{\delta} := (\mathrm{id}_f \otimes (\rho_{f^R} \circ (\mathrm{id}_{f^R} \otimes \gamma) \circ \alpha_{f^R,f,g} \circ (\eta \otimes \mathrm{id}_g) \circ \lambda_g^{-1})) \circ \delta$$

Which can be drawn as a string diagram as follows, denoting $Y$ by the white background and $X$ by blue:




in order to calculate:



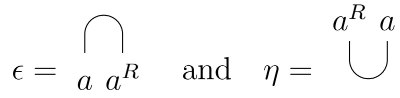

The other statements can be shown similarly by using the upside-down and/or left-right mirror images of the string diagrams. $\square$

**Theorem 3.46.** *Given a 2-condensation monad* $M : \clubsuit_2 \to \mathcal{B}$ *in a 2-category* $\mathcal{B}$ *for which all hom-categories are idempotent complete, if the underlying 1-cell* $a := M(e)$ *admits a right adjoint, then* $M$ *is equivalent as an object of* $\mathrm{Kar}(\mathcal{B})$ *to a unital 2-condensation monad and to a counital 2-condensation monad.*

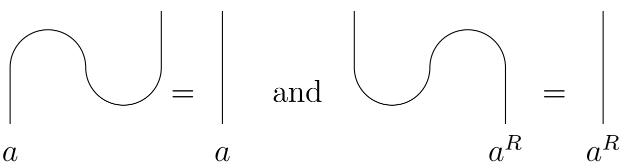
*Proof.* Let $X := M(A)$ be the underlying object of $M$ and consider $M^X$, the 2-condensation monad with underlying 1-cell $\mathbb{1}_X$ (as in Definition 3.33). Define the following 2-condensation bimodules between them: $F$ with $a$ as an $a$-$\mathbb{1}_X$-bimodule as underlying 1-cell and $G$ with $a$ as a $\mathbb{1}_X$-$a$-bimodule as underlying 1-cell. Then $(M^X, M, F, G, \mu_a, \Delta_a)$ is a condensation of $M^X$ onto $M$ in $\mathrm{Kar}(\mathcal{B})$, where we can use the multiplication and comultiplication given by $M$, $\mu_a$ and $\Delta_a$, because $F \otimes_{\mathbb{1}_X} G$ has $a \otimes a$ as underlying 1-cell. That $\mu_a$ and $\Delta_a$ are 2-cells of $\mathrm{Kar}(\mathcal{B})$ follows from (co)associativity, that they constitute a condensation from $\Delta$-separability.

We claim to be done once we can construct right adjoints for $F$ and $G$. This can be seen as follows: from a right adjoint $F^R$ for $F$, by Lemma 3.45 we can construct a unital 2-condensation of $M^X$ onto $M$. This gives rise to a unital 2-condensation monad $\mathbb{M}$ with underlying 1-cell (in $\mathrm{Kar}(\mathcal{B})$) $F^R \otimes_M F$ on $M^X$ which by construction condenses onto $M$. The underlying 1-cell (in $\mathcal{B}$) $\tilde{a}$ of $F^R \otimes_M F$ in turn, as the inclusion $J : \mathcal{B} \to \mathrm{Kar}(\mathcal{B})$ is fully faithful, gives rise to a unital 2-condensation monad $\tilde{M}$ in $\mathcal{B}$. Taking 2-condensation bimodules $\tilde{F}$ with $\tilde{a}$ as a $\mathbb{1}_X$-$\tilde{a}$-bimodule and $\tilde{G}$ with $\tilde{a}$ as an $\tilde{a}$-$\mathbb{1}_X$-bimodule as underlying 1-cell, we get a condensation $(M^X, \tilde{M}, \tilde{F}, \tilde{G}, \mu_{\tilde{a}}, \Delta_{\tilde{a}})$ for the same reasons as above. So we see that $\mathbb{M}$ condenses both onto $M$ and onto $\tilde{M}$, so these must be equivalent objects of $\mathrm{Kar}(\mathcal{B})$ by Proposition 3.36. Similarly from a right adjoint for $G$ we can show that $M$ is equivalent to a counital condensation monad (again by Lemma 3.45), proving the claim.

It remains to construct right adjoints for $F$ and $G$. Denote the unit and counit of the adjunction $a \dashv a^R$ by:

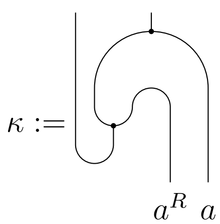



so that their Zorro relations read:

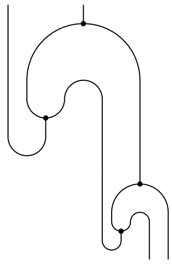
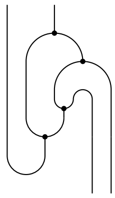
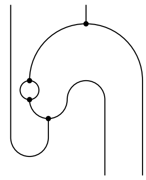


(3.24)

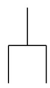
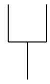
Define the underlying 1-cell, $f^R$, of $F^R$ as the image of the following idempotent



where here and in the following the vertices marked with a small dot denote the multiplication and comultiplication on $a$ given by $M$. First of all we need to see that this indeed is an idempotent:

$\kappa \circ \kappa =$  $\overset{(3.24)}{=}$  $\overset{(A1),(A2)}{=}$  $\overset{(S1)}{=} \kappa$ (3.25)

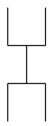
By the assumption on $\mathcal{B}$ we know that it splits; so there are 2-cells $\Theta \in \mathrm{Hom}_{\mathcal{B}(a,a)}(a^R \otimes a, f^R)$ and $\Psi \in \mathrm{Hom}_{\mathcal{B}(a,a)}(f^R, a^R \otimes a)$ which we denote by:

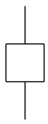
$\Theta =$  and $\Psi =$ 



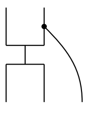
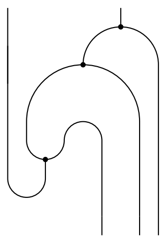
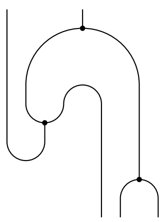
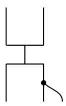


such that:

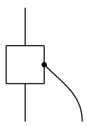
$\Psi \circ \Theta =$  $= \kappa$ (3.26)

$\Theta \circ \Psi =$  $= \mathrm{id}_{f^R}$ (3.27)

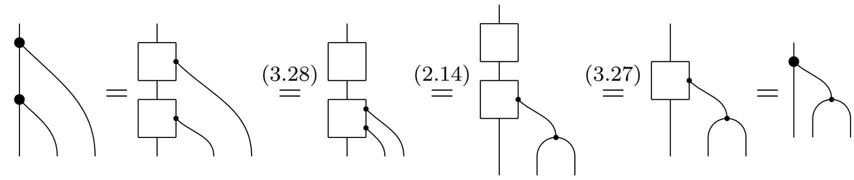
Using them we can express the fact that $\kappa$ commutes with the multiplication $\mu_M$ on $a$ given by $M$ as follows:

 $=$  $\overset{(A1)}{=}$  $=$  (3.28)

Now we can define a right action on $f^R$:

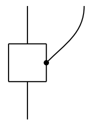
$r_{f^R} :=$ 

where the trivalent vertex is just the multiplication $\mu_M$ (which also is the right action of $a$ on $a$ as a $\mathbb{1}_X$-$a$-bimodule) and see that this indeed makes $f^R$ a right $a$-module:

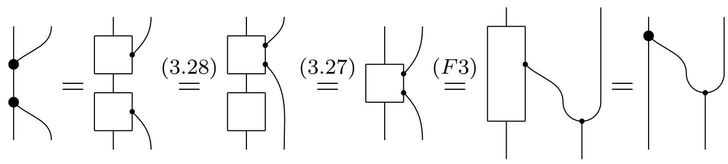



(3.29)

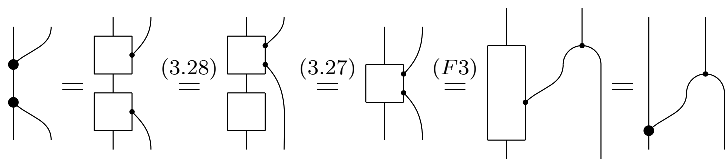
Here $r_{f^R}$ is denoted by the trivalent vertex marked with a bigger dot (analogous notation is used in the following). As according to Remark 3.22, it follows from (3.28) that $\kappa$ also commutes with the comultiplication $\Delta_M$ (which is the coaction of $a$ on $a$ as a $\mathbb{1}_X$-$a$-bimodule), we can similarly make $f^R$ a right $a$-comodule by defining:

$\check{r}_{f^R} :=$ 

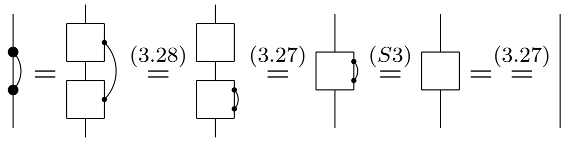
Then, by applying (3.28), we get that these make $f^R$ a special Frobenius bimodule (the left actions of $\mathbb{1}_X$ are trivially given by unitors):




(3.30)

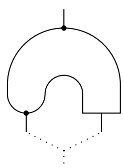
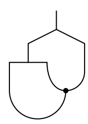



(3.31)

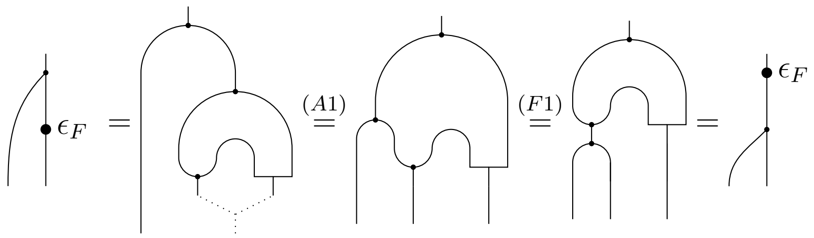



(3.32)

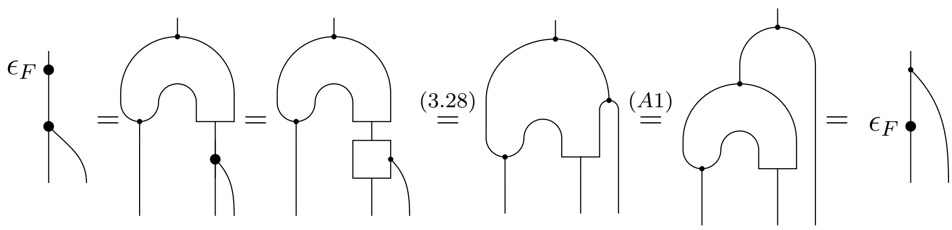
Therefore we get a 2-condensation bimodule $F^R$ between $M$ and $M^X$ with underlying 1-cell $f^R$. To see that it actually is a right adjoint of $F$, we need to find a unit $\epsilon_F \in \mathrm{Hom}_{\mathrm{Kar}(\mathcal{B})(M,M)}(F \otimes_{M^X} F^R, \mathbb{1}_M)$ and a counit $\eta_F \in \mathrm{Hom}_{\mathrm{Kar}(\mathcal{B})(M^X,M^X)}(\mathbb{1}_{M^X}, F^R \otimes_M F)$. We define their corresponding 2-cells in $\mathrm{Hom}_{\mathcal{B}(X,X)}(a \otimes f^R, a)$ and $\mathrm{Hom}_{\mathcal{B}(X,X)}(\mathbb{1}_X, f^R)$ respectively, as follows:

$\epsilon_F :=$  and $\eta_F :=$  (3.33)

As the relative tensor product over $\mathbb{1}_X$ is just the composition of 1-cells in $\mathcal{B}$, the projection and inclusion $\Theta_{\mathbb{1}_X}$ and $\Psi_{\mathbb{1}_X}$ are both the identity. In order to keep track of this, here and in the following we denote them with the usual trivalent vertex drawn dotted. Before checking the Zorro moves, we need to make sure that this unit and counit indeed are 2-cells of Kar($\mathcal{B}$). For $\eta_F$ this is clear, because being a map of $\mathbb{1}_X$-$\mathbb{1}_X$-bimodules is trivial, as we have seen in the proof of Proposition 3.35. For $\epsilon_F$ we need to check that it commutes white the left action:

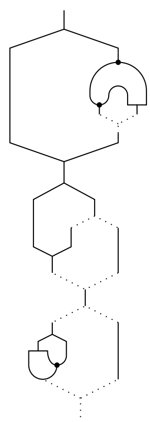



omitting the dotted vertex which is just the identity. And that it commutes with the right action:

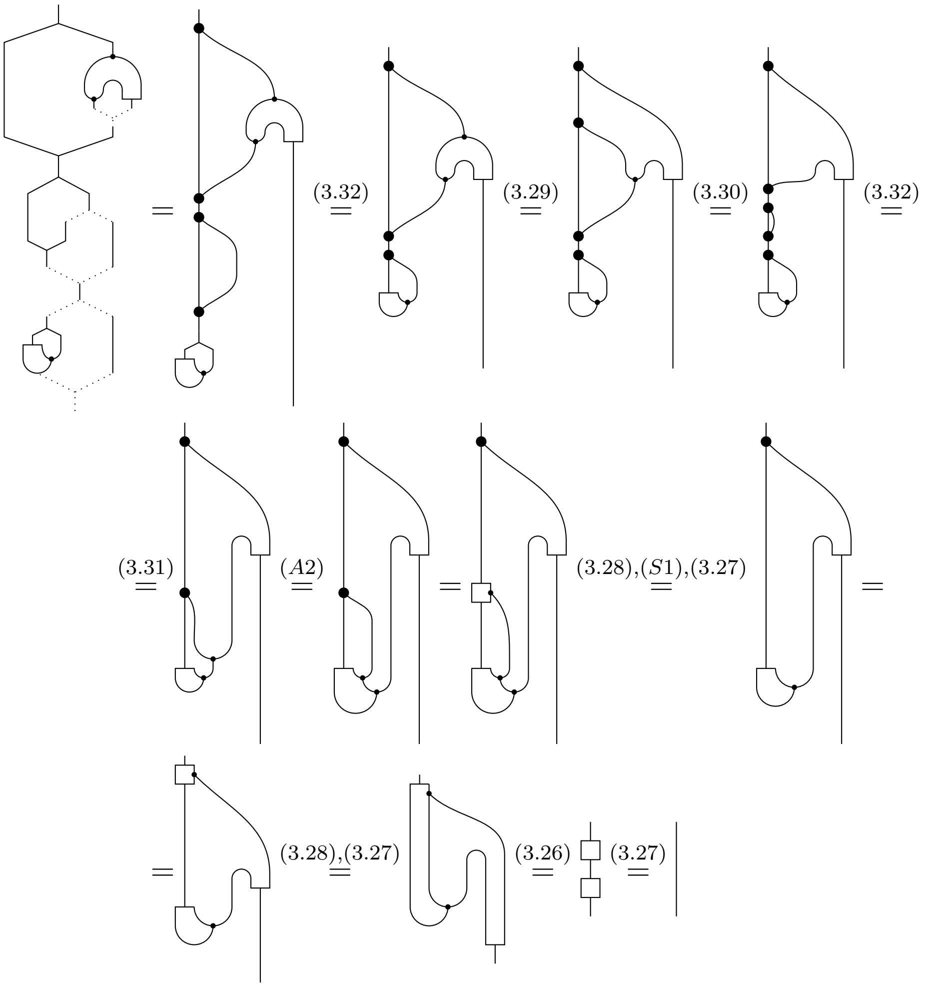



Now we can check that $\eta_F$ and $\epsilon_F$ are unit and counit of the desired adjunction $F \dashv F^R$. To do so, we express the Zorro moves in Kar($\mathcal{B}$) as string diagrams in $\mathcal{B}$, using the associator as introduced in Section 3.4.3 and the horizontal composition of 2-cells of Kar($\mathcal{B}$) as given in Section 3.4.2. To check equation 2.8, we need to see that the following is equal to the identity 2-cell on $f^R$ (omitting both unitors as they are identities in Kar($\mathcal{B}$)):

$$(\mathrm{id}_{f^R} \otimes_a \epsilon_F) \circ \alpha^{\mathrm{Kar}(\mathcal{B})}_{f^R,a,f^R} \circ (\eta_F \otimes_{\mathbb{1}_X} \mathrm{id}_{f^R}) = \tag{3.34}$$



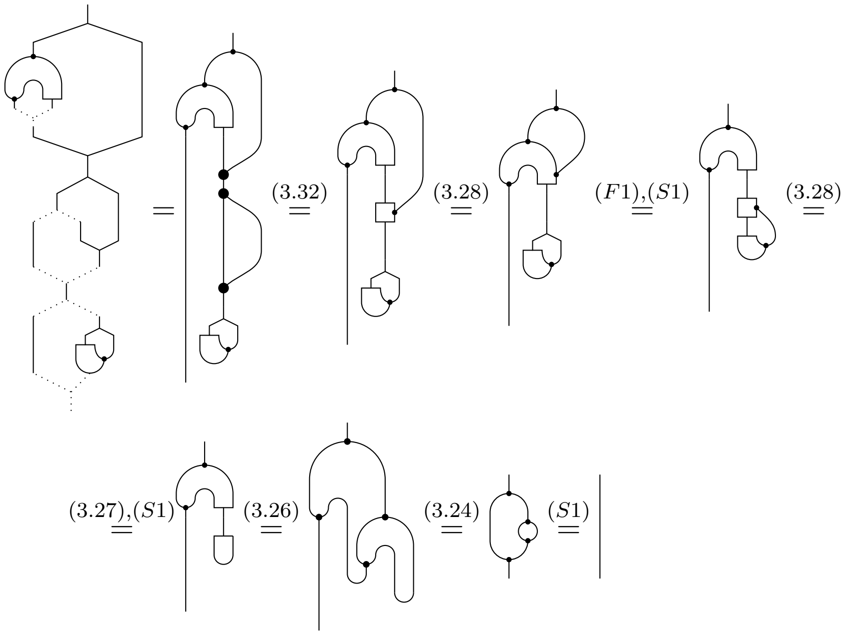
This string diagram can be simplified by noting that the dotted vertices are just identities, and that the other occurring projections and inclusions of relative tensor products, concern relative tensor products with $a$ over $a$ so that these are just the appropriate actions and coactions, which can be seen analogously to Lemma 3.18. Furthermore we use Lemma 3.24 when denoting the action on a tensor product. Then we can calculate:



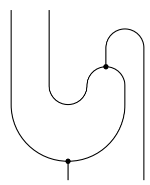

Similarly, equality 2.9 in this case reads:

$$(\epsilon_F \otimes_a \mathrm{id}_a) \circ (\alpha_{a,f^R,a})^{-1} \circ (\mathrm{id}_a \otimes \eta_F) = \mathrm{id}_a \tag{3.35}$$

and is checked as follows:




Completely analogously we find a right adjoint for $G$ which has the image of the following idempotent, given by rotating the string diagram defining $\kappa$ by 180 degrees:



as underlying 1-cell. This can be verified by accordingly rotating all the string diagrams above and interchanging the roles of $\epsilon$ and $\eta$. This completes the proof. $\square$

From this we see that if in $\mathcal{B}$ all 1-cells have adjoints, all objects of $\mathrm{Kar}(\mathcal{B})$ are equivalent to objects of $\mathrm{Kar}^{\mathbb{1},\mathbb{I}}(\mathcal{B})$, so as $\mathrm{Kar}^{\mathbb{1},\mathbb{I}}(\mathcal{B})$ is a full subcategory of $\mathrm{Kar}(\mathcal{B})$ we immediately get the following.

**Corollary 3.47.** *If in a 2-category $\mathcal{B}$ all hom-categories are idempotent complete and all 1-cells admit right adjoints, then* $\mathrm{Kar}(\mathcal{B})$ *and* $\mathrm{Kar}^{\mathbb{1},\mathbb{I}}(\mathcal{B})$ *are 2-equivalent.*

# 4 Outlook towards Higher Condensation

Having established 2-condensation as a categorification of retraction, it is natural to ask how to further categorify this to higher categories. As this is beyond the scope of this work we can only give the basic idea for general $n$-dimensional condensation and state possible approaches and expectations for the case $n = 3$.

## 4.1 $n$-Condensation

As mentioned in Section 1.3 of the introduction, the notion of condensations in higher categories was introduced to describe actual condensation between topological phases of matter for arbitrary dimension. The definition of $n$-condensation in a weak $n$-category can be given inductively, where sets are considered to be 0-categories. The concepts of condensation monad, condensation bimodule and their relative tensor product can then be generalized accordingly.

**Definition 4.1.** Given a weak $n$-category $\mathcal{D}$, an $n$-condensation of an object $X$ onto an object $Y$ consists of 1-cells $f \in \mathcal{D}(X,Y)$ and $g \in \mathcal{D}(Y,X)$ and an $(n-1)$-condensation of $f \otimes g$ onto $\mathbb{1}_Y$. 0-condensation is defined to be equality of elements of a set.

In order to use this definition without referring to a specific realization of weak $n$-categories, the authors of [GJ19] take the following, model independent, inductive point of view: in a weak $n$-category, two $k$-cells do not have a unique composition but a weak $(n-k)$-category of compositions which is required to be contractible (i.e. equivalent to the trivial $(n-k)$-category which has just one object and only identity cells). In this way they are able to inductively prove the following (omitting the model dependent details), for a weak $n$-category $\mathcal{D}$ in which all hom-$(n-1)$-categories have all condensates:

- the $n$-category of possible $n$-condensations of an $n$-condensation monad in $\mathcal{D}$ is either empty or contractible (generalizing Proposition 3.36),
- there is an $n$-category $\mathrm{Kar}(\mathcal{D})$ of $n$-condensation monads and $n$-condensation bimodules with their relative tensor product (generalizing Theorem 3.17),
- there is a fully faithful $n$-functor $\mathcal{D} \to \mathrm{Kar}(\mathcal{D})$ which is an $n$-equivalence if $\mathcal{D}$ has all condensates (generalizing Proposition 3.35),
- $\mathrm{Kar}(\mathcal{D})$ has all condensates (generalizing Theorem 3.37),
- if the underlying 1-cell of an $n$-condensation monad $M : \clubsuit_n \to \mathcal{D}$ admits a right adjoint, then $M$ is equivalent as an object of $\mathrm{Kar}(\mathcal{D})$ to a unital $n$-condensation monad as well as to a counital $n$-condensation monad (generalizing Theorem 3.46).

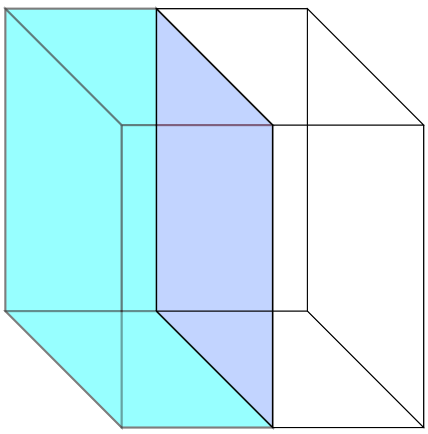
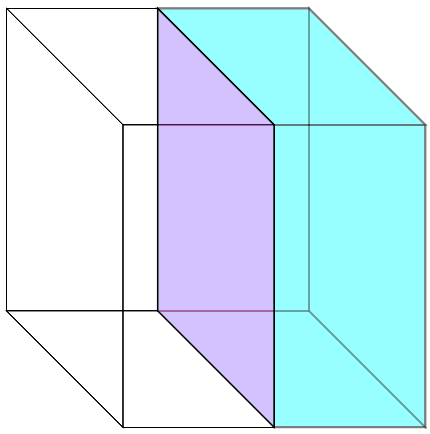
## 4.2 3-Condensation

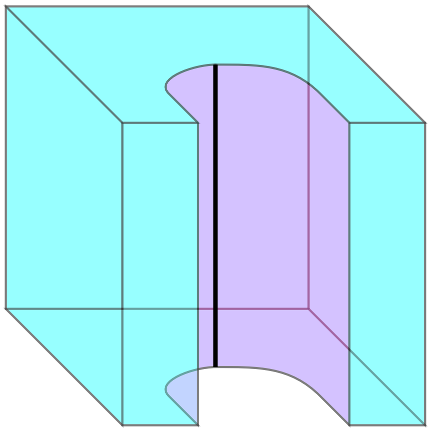
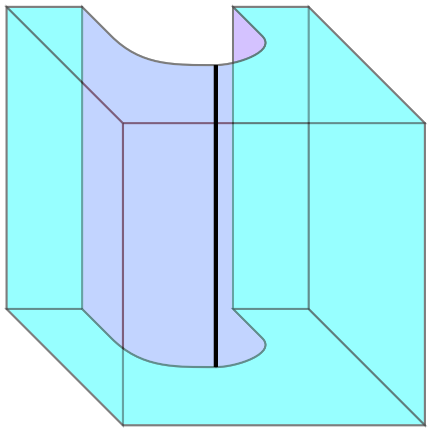
For $n = 3$, Definition 4.1 can be unravelled as follows: a 3-condensation of an object $X$ onto an object $Y$ is given by 1-cells $f : X \to Y$ and $g : Y \to X$, 2-cells $\gamma : f \Box g \to \mathbb{1}_Y$ and $\delta : \mathbb{1}_Y \to f \Box g$ and 3-cells $\Gamma : \gamma \otimes \delta \to \mathbb{1}_{\mathbb{1}_Y}$ and $\Lambda : \mathbb{1}_{\mathbb{1}_Y} \to \gamma \otimes \delta$ such that $\Gamma \circ \Lambda = \mathrm{id}_{\mathbb{1}_{\mathbb{1}_Y}}$.

For weak 3-categories there is the well established model of tricategories (introduced in [GPS95]), which can be stated explicitly, even though it is notably more complicated than the one of bicategories (given in Section 2.2). Thus the above can be made precise in a way where the composition of 1-cells, denoted by "$\Box$", is a 2-functor between 2-categories of 1-cells, 2-cells and 3-cells.

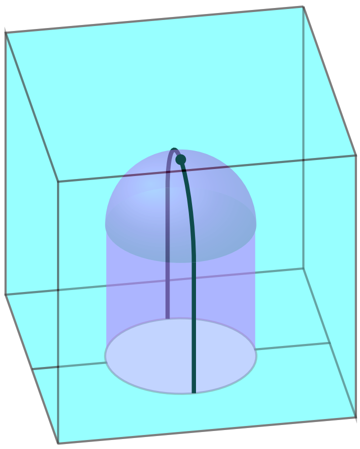
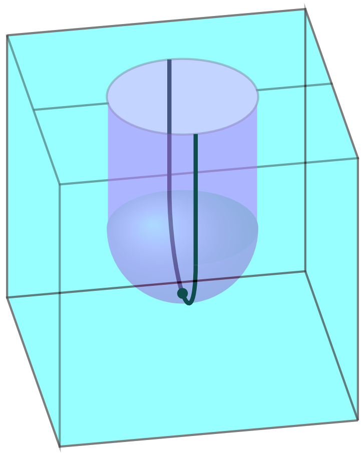
In the derivation of our results on 2-condensation, the use of string diagrams was hugely beneficial. Therefore we expect that a similar diagrammatic calculus for tricategories is necessary to enable a generalization. Such a calculus, using 3-dimensional diagrams where 3-dimensional regions correspond to objects, surfaces between them to 1-cells, lines between those to 2-cells and junctions of lines to 3-cells, can be made precise for Gray categories (as introduced in [BMS18]). A Gray category is a tricategory where all compositions are strictly associative, but the interchange law (cf. Proposition 2.28) for the $\Box$-composition of 1-cells only holds up to a 3-cell isomorphism called the tensorator. Fortunately it turns out that every tricategory is triequivalent to a Gray category (this was shown in [GPS95], generalizing the coherence theorem which states that every bicategory is biequivalent to a strict 2-category).

Thus, in a Gray category, the data making up a 3-condensation can be visualized as follows, where $\Box$-composition goes horizontally from right to left, $\otimes$-composition in depth from front to back and $\circ$-composition vertically from bottom to top:

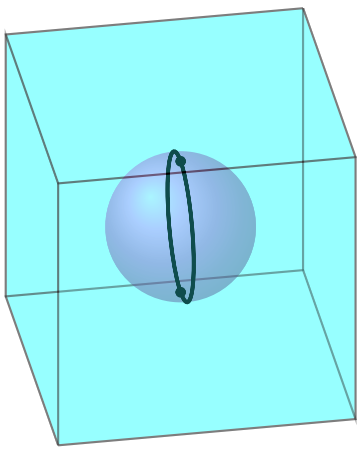
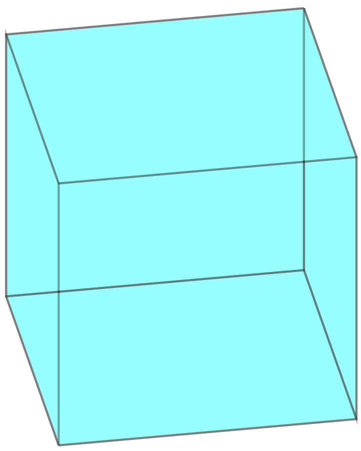
$\mathrm{id}_{\mathbb{1}_f} =$  $\mathrm{id}_{\mathbb{1}_g} =$ 

$\mathrm{id}_\gamma =$  $\mathrm{id}_\delta =$ 

$\mathrm{id}_{\gamma\otimes\delta}$ then is expressed by putting the latter two diagrams together in a way that creates a cube with a tunnel from top to bottom. $\Gamma$ and $\Lambda$ can then be visualized as a cup and a cap, closing this tunnel from above and from below respectively:

$\Gamma =$  $\Lambda =$ 

From this we see that the relation $\Gamma \circ \Lambda = \mathrm{id}_{\mathbb{1}_{\mathbb{1}_Y}}$ corresponds to the fact that bubbles may be created and destroyed:

$\Gamma \circ \Lambda =$  $=$  $= \mathrm{id}_{\mathbb{1}_{\mathbb{1}_Y}}$

quite like in the 2-dimensional case (cf. Proposition 3.4).

From the diagrams describing the 2-cells, it can be seen that $g\Box f$, the underlying 1-cell of the corresponding 3-condensation monad, carries a structure, generalizing the one of an algebra, where associativity of the multiplication holds only up to an isomorphism which in this case is given by the tensorator of the Gray category.

Further investigation is needed to describe the relative tensor product of 3-condensation bimodules, but it is expected to be closely related to the balanced tensor product of module categories described in [DSS19].

Due to the more complicated nature of tricategories, establishing the existence of a 3-Karoubi completion is notably harder than what we did for bicategories in Section 3.4. Nevertheless it might be worth to do so considering possible applications in condensed matter physics and topological quantum computation.

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