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„Existence results and dimension reduction problems in large-strain magnetoelasticity“

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Abstract

The thesis concerns the analysis of the variational model of Brown for magnetoelasticity at large-strains. A characteristic feature of this model consists in its mixed Eulerian-Lagrangian fomulation: while deformations are defined on the reference configuration (Lagrangian), magnetizations are defined on the deformed configuration in the actual space (Eulerian).

The thesis is subdivided into two parts. In the first part, we investigate the existence of solutions, both in the static and in the quasistatic setting. In the static setting, solutions correspond to minimizers of the magnetoelastic energy while, in the quasistatic setting, these are understood in the energetic sense according to the theory of rate-independent processes. We establish a compactness result which, in particular, yields the convergence of the compositions of magnetizations with deformations. This enables us to prove the existence of minimizers by means of classical lower semicontinuity methods. Our compactness result also allows us to address the analysis in the quasistatic setting, where we examine rate-independent evolutions driven by applied loads and boundary conditions. In this case, we prove the existence of energetic solutions. At first, we restrict ourselves to the case of continuous deformations; subsequently, we assume that admissible deformations belong to the class of possibly discontinuous deformations for which cavitation, i.e., the formation of voids inside the material, is excluded introduced by Barchiesi, Henao and Mora-Corral (2017).

In the second part of the thesis, we investigate the problem of dimension reduction for magnetoelastic plates. The aim is to identify a reduced two-dimensional model, as the thickness of the plate goes to zero, describing the asymptotic behaviour of minimizers of the magnetoleastic energy. We focus on low-energy configurations by rescaling the elastic energy according to the linearized von Kármán regime. First, we identify a reduced model by computing the Γ-limit of the magnetoleastic energy, as the thickness of the plate goes to zero. Then, we introduce applied loads given by mechanical forces and external magnetic fields, and we show that, under clamped boundary conditions, sequences of almost minimizers of the total energy converge to minimizers of the corresponding energy in the reduced model. Finally, we consider time-dependent applied loads and dissipation functionals as in the first part of the thesis, and we prove that energetic solutions of the three-dimensional model converge, in the sense of dimension reduction, to energetic solutions of the reduced two-dimensional model. This last result provides a further justification of the latter in the framework of evolutionary Γ-convergence.

Zusammenfassung

Die Dissertation befasst sich mit der Analyse des Variationsmodells von Brown für die Magnetoelastizität bei großer Dehnung. Ein charakteristisches Merkmal dieses Modells besteht in seiner gemischten Euler-Lagrange-Formulierung: während Deformationen in der Referenzkonfiguration definiert sind (Lagrange-Bild), werden Magnetisierungen in der deformierten Konfiguration im realen Raum beschrieben (Euler-Bild). Die Doktorarbeit ist in zwei Teile gegliedert. Im ersten Teil untersuchen wir die Existenz von Lösungen, sowohl im statischen als auch im quasistatischen Setting. Im statischen Fall entsprechen die Lösungen den Minimierern der magnetoelastischen Energie. Im quasistatischen Umfeld werden Lösungen im energetischen Sinne mit der Theorie von Ratenunabhängigen Prozessen definiert. Wir zeigen ein Kompaktheitsresultat, welches insbesondere die Konvergenz der Zusammensetzungen von Magnetisierungen mit Deformationen garantiert. Dies ermöglicht uns, die Existenz von Minimierern mittels klassischer Methoden der Unterhalbstetigkeit zu beweisen. Das Kompaktheitsresultat erlaubt es auch, den quasistatischen Fall zu analysieren, wobei wir Ratenunabhängige Prozesse betrachten, die durch aufgebrachte Lasten und Randbedingungen angetrieben werden. Wir beweisen die Existenz energetischer Lösungen, wobei wir uns zunächst auf kontinuierliche Verformungen beschränken. Danach betrachten wir zulässige Verformungen in der von Barchiesi, Henao und Mora-Corral (2017) eingeführten Klasse der möglicherweise unstetigen Deformationen, bei denen Kavitation, also die Bildung von Hohlräumen im Material, ausgeschlossen ist.

Im zweiten Teil der Arbeit untersuchen wir das Problem der Dimensionsreduktion für magnetoeelastische Platten. Ziel ist es, ein reduziertes zweidimensionales Modell zu identifizieren, welches das asymptotische Verhalten von Minimierern der magnetoleastischen Energie beschreibt, wenn die Dicke der Platte gegen Null strebt. Wir konzentrieren uns auf Niedrigenergiekonfigurationen durch Umskalierung der elastischen Energie gemäß dem linearisierten von Kármán-Regime. Zuerst identifizieren wir ein reduziertes Modell, indem wir den Γ-Limes der magnetoelastischen Energie bei verschwindender Plattendicke berechnen. Dann führen wir aufgebrachte Lasten ein, die durch mechanische Kräfte und externe Magnetfelder gegeben sind, und zeigen, dass unter geklemmten Randbedingungen Folgen von "Fast-Minimierern" der Gesamtenergie zu Minimierern der entsprechenden Energie im reduzierten Modell konvergieren. Schließlich betrachten wir zeitabhängige aufgebrachte Lasten und Dissipationsfunktionale wie im ersten Teil der Arbeit, und beweisen, dass energetische Lösungen des dreidimensionalen Modells unter Dimensionsreduktion zu energetischen Lösungen des reduzierten zweidimensionalen Modells konvergieren. Somit ist unser reduziertes zweidimensionales Modell auch im Rahmen der evolutionären Γ-Konvergenz bestätigt.

Contents

Introduction

Magnetoelastic materials are characterized by their tendency to experience mechanical deformations in response to external magnetic fields. This peculiar behaviour is termed magnetostriction and constitutes the foundation of the technology behind many devices such as sensors and actuators.
A first phenomenological theory of magnetoelasticity at large-strains has been proposed by Brown in the form of a variational principle [31]. The theory takes as independent variables the deformation and the magnetization. The first variable is set on the reference configuration (Lagrangian), while the second one is defined on the deformed configuration (Eulerian). This setting constitutes one of the characteristic features of the theory.
The rigorous analysis of mixed Eulerian-Lagrangian models provides several substantial difficulties that require the development of novel strategies to be overcome. For this reason, in recent years, such models have attracted also the attention of the mathematical community. Apart from magnetoelasticity [18, 95, 129], problems with mixed formulations have been studied in various other contexts such as the theory of liquid crystals [17, 83], phase transitions [74, 134] and plasticity [91, 135], but also in electroelasticity [47, 133] and thermoelasticity [9, 111].
Specifically, the analytical investigation of magnetoelasticity poses very challenging problems that can be qualified as: nonlinear, as magnetostrictive materials can experience large deformations; nonconvex, as such problems are subjected to nonconvex constraints due to magnetic saturation; and nonlocal, as the magnetic response of the material depends on the shape assumed by the deformed body. Therefore, despite magnetoelastic materials are nowadays largely employed in applications [58], their theoretical understanding from the mathematical point of view is still in its infancy.
In this thesis, we advance the variational theory of magnetoelasticity at large-strains by focusing on two topics: the problem of existence of equilibrium configurations and the identification of reduced models for thin plates. We investigate both issues first in the static setting and then in the quasistatic setting. In the latter, we adopt the framework of the theory of rate-independent processes [110], which appears to be adequate in view of the hysteretic nature of magnetostrictive phenomena [45].
In addition to being object of pure mathematical interest, these two topics are of fundamental importance for applications. On the one hand, proving the existence of solutions ensures the legitimacy of the model and, at the same time, might suggest possible approximation techniques for such solutions. On the other hand, reduced models are extremely useful in the applied sciences as they significantly diminish the computational complexity of the original model while preserving its main properties.
The content of the present work is based on a recasting of the results published

by the Ph. D. candidate, either as a single author [27, 28] or in collaboration with Elisa Davoli and Martin Kružík [29, 30]. The results in these papers have been reorganized and, in part, slightly improved in order to achieve an overall consistency of the assumptions. A detailed comparison between the results contained in the thesis and the ones published in the aforementioned papers is provided in the last section of this introduction together with the outline of the thesis.
Before moving to the scientific contributions of the present work, we start by giving some basic phenomenological background on magnetoelasticity and by describing the variational model of Brown. Then, we provide an overview of the mathematical literature concerning this model with particular emphasis on the two problems addressed here. Subsequently, we detail the contributions contained in the thesis. Eventually, we delineate the structure of this work and we compare its content with the publications mentioned above.

Magnetoelasticity

Magnetoelasticity concerns the study of deformable solids in presence of magnetic fields. The principal aim is to understand how magnetic and mechanical properties of materials influence each other. Apart from [32], classical references in the physical literature are [96, 107, 139].
In absence of external fields, ferromagnetic materials spontaneously arrange themselves in regions of uniform magnetization, known as Weiss domains, which are separated by thin transition layers, the domain walls [84]. The formation of these structures is the result of the competition of two effects: the magnetocrystalline anisotropy, that is, the existence of preferred magnetization directions determined by the underlying crystal lattice, and the long-range interactions between magnetic dipoles which disfavour configurations with constant magnetization throughout all the specimen. As the orientation of magnetic dipoles causes the local distortion of the crystal lattice, each domain exhibits a uniform strain, thus determining the stress-free configuration of the sample.
Upon the application of a magnetic field, a reorganization of the domain structure is observed. This is mainly due to the more favourable orientation of certain easy axes with respect to the direction of the external field. Thus, the domain walls shift and the domains themselves rotate. These movements produce a macroscopic strain and, in turn, lead to the change of the overall shape of the specimen, see Figure 1. This peculiar effect for which the application of a magnetic field might induce a reversible deformation of the body is termed magnetostriction.
By means of a similar mechanism, mechanical stresses can give rise to the reorientation of the magnetic dipoles. In this way, it is possible to modify the magnetic response of the sample by exerting mechanical forces on it. This phenomenon is usually referred to as inverse magnetostriction.
Magnetostrictive effects characterize magnetomechanical couplings. In the case of ferromagnetic shape-memory alloys, magnetostriction can also yield the phase transformation from one martensitic variant to another [86, 145].
From the engineering point of view, magnetostrictive effects lie at the basis of the operating principle of many technological devices. Indeed, magnetoelastic materials constitute a prominent example of the so-called “smart” or “active” materials, which

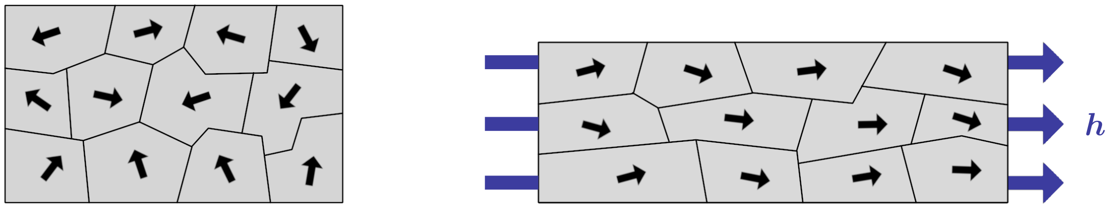

Figure 1: Schematic representation of magnetostriction. The application of the magnetic field $\boldsymbol{h}$ induces a deformation of the body.

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have the peculiar property of being able to activate one of their features in response to external stimuli of different type. On the one hand, magnetostriction allows for the remote activation of actuators by means of the imposition of magnetic fields [6]. In this regard, there has been a great interest in specific rare-earth alloys such as TerFeNOL and GalFeNOL, as well as in ferromagnetic shape-memory alloys like Ni_2MnGa and NiMnInCo, because they are capable of experience remarkably large strains, of order 10^{-3}, upon the application of relatively moderate magnetic fields [39, 40]. As the average magnetostrictive strains of ferromagnets are generally of order 10^{-5}, in this case, one speaks about "giant magnetostriction" [58].
On the other hand, inverse magnetostriction is exploited in the design of sensors that measure force and torque by estimating the corresponding variation of magnetic field [75]. Further applications concern transducers and energy harvesters [48].

The variational model of Brown

The mathematical model of magnetoelasticity that is investigated in this thesis is usually attributed to Brown [31, 33]. He systematically developed this continuum theory, which he refers to as "the conventional theory of magnetostriction", starting from the principles of micromagnetics [32] and elaborating some ideas by Toupin [143]. An equivalent theory has been developed, basically contemporarily, also by Tiersten starting from a different modeling approach [140, 141]. We refer to [53] for a comparison between the two approaches.
Regarding the theory of Brown, we are interested in its formulation as a variational principle [31]. Brown calls this principle "the energy method" in contraposition with "the stress method". The latter refers to the formulation based on the concept of forces that he also investigated. This variational model takes as independent variables the deformation and the magnetization, where the latter should be interpreted as the local density of magnetic dipoles per unit volume. We mention that, in recent years, alternative variational formulations, having the magnetic field in place of the magnetization vector as independent variable [136] or taking the magnetic vector potential as an additional variable [88], have been proposed. We refer to [56] for a general overview on alternative formulations and to [132] for a comparison between the corresponding variational principles.

The model under consideration contemplates finite strains and features a strong interplay between Lagrangian end Eulerian coordinates. Indeed, while deformations are classically defined as maps on the refence configuration, magnetic quantities are naturally set in the actual space.
Let $\Omega \subset \mathbb{R}^N$ represent the reference configuration of a magnetoelastic body. This is subjected to elastic deformations $\boldsymbol{y}\colon \Omega \to \mathbb{R}^N$ and magnetizations $\boldsymbol{m}\colon \boldsymbol{y}(\Omega) \to \mathbb{R}^N$. As customary in nonlinear elasticity, deformations are assumed to be orientation-preserving, namely with $\det D\boldsymbol{y} > 0$, and invertible in order to exclude the interpenetration of matter [37]. According to the theory of micromagnetics [32], at sufficiently low constant temperature, magnetizations are subjected to a magnetic saturation constraint. In the case of rigid bodies, up to normalization, these are required to be sphere-valued. Indeed, the modulus of magnetizations corresponds to the spontaneous magnetic moment per unit mass that, for simplicity, is assumed to depend only on the temperature. In the case of deformable bodies, especially at large strains, this observation calls for a reformulation of the magnetic saturation constraint. In this regard, employing our notation, Brown writes [32, p. 73]: *"since perfect alignment of spins produces a definite magnetic moment per unit mass, not per unit volume, it is* $|\boldsymbol{m} \circ \boldsymbol{y}| \det D\boldsymbol{y}$ *and not* $|\boldsymbol{m}|$ *that must be supposed constant".* Therefore, the magnetic saturation constraint should take the following form:

$$|\boldsymbol{m} \circ \boldsymbol{y}| \det D\boldsymbol{y} = 1 \text{ in } \Omega. \tag{1}$$

In particular, for incompressible materials, the constraint (1) is equivalent to the requirement of magnetizations being sphere-valued, namely

$$|\boldsymbol{m}| = 1 \text{ in } \boldsymbol{y}(\Omega). \tag{2}$$

The theory of Brown is based on the assumption that equilibrium configurations correspond to minimizers of the following magnetoelastic energy functional:

$$(\boldsymbol{y}, \boldsymbol{m}) \mapsto \int_\Omega W(D\boldsymbol{y}, \boldsymbol{m} \circ \boldsymbol{y})\, \mathrm{d}\boldsymbol{x} + \int_{\boldsymbol{y}(\Omega)} |D\boldsymbol{m}|^2\, \mathrm{d}\boldsymbol{\xi} + \frac{1}{2}\int_{\mathbb{R}^N} |\boldsymbol{\Psi}_{\boldsymbol{m}}|^2\, \mathrm{d}\boldsymbol{\xi}. \tag{3}$$

The first term in (3) stands for the elastic energy of the system. The expression of the elastic energy density W should exhibit a strong coupling between its two variables. In particular, it is desirable to consider constitutive relations that are able to describe magnetostrictive effects. Taking into account (1), a prototypical example inspired by the modeling of nematic elastomers [54] is given by

$$W(\boldsymbol{F}, \boldsymbol{\lambda}) := \Phi\left((\alpha \boldsymbol{z} \otimes \boldsymbol{z} + \beta(\boldsymbol{I} - \boldsymbol{z} \otimes \boldsymbol{z}))^{-1} \boldsymbol{F} \right), \tag{4}$$

where $\boldsymbol{z} = \boldsymbol{\lambda}(\det \boldsymbol{F})$. Here, $\alpha, \beta > 0$ are material parameters, while Φ is a frame-indifferent density that is minimized at the identity and that blows up as the determinant of its argument approaches zero. Also, in accordance with the principles of continuum mechanics [85], the function W must fulfill the two requirements of frame-indifference

$$\forall\, \boldsymbol{R} \in SO(N),\ \forall\, \boldsymbol{F} \in \mathbb{R}^{N \times N},\ \forall\, \boldsymbol{\lambda} \in \mathbb{R}^N, \quad W(\boldsymbol{R}\boldsymbol{F}, \boldsymbol{R}\boldsymbol{\lambda}) = W(\boldsymbol{F}, \boldsymbol{\lambda})$$

and magnetic parity

$$\forall\, \boldsymbol{F} \in \mathbb{R}^{N \times N},\ \forall\, \boldsymbol{\lambda} \in \mathbb{R}^N, \quad W(\boldsymbol{F}, -\boldsymbol{\lambda}) = W(\boldsymbol{F}, \boldsymbol{\lambda}).$$

The second term in (3) represent the exchange energy which is derived by microscopic considerations. This energy contribution arises from the pairwise interaction of magnetic dipoles and favours their alignment.
Eventually, the third term in (3) accounts for the magnetostatic energy; the stray field $\boldsymbol{\Psi}_{\boldsymbol{m}} \colon \mathbb{R}^N \to \mathbb{R}$ is given by a solution to the static Maxwell equations:

$$\begin{cases} \mathrm{curl} \boldsymbol{\Psi}_{\boldsymbol{m}} = 0, \\ \mathrm{div} \left(\boldsymbol{\Psi}_{\boldsymbol{m}} - \chi_{\boldsymbol{y}(\Omega)} \boldsymbol{m} \right) = 0, \end{cases} \quad \text{in } \mathbb{R}^N. \tag{5}$$

Here, $\chi_{\boldsymbol{y}(\Omega)} \boldsymbol{m}$ simply denotes the extension of $\boldsymbol{m}$ by zero outside of $\boldsymbol{y}(\Omega)$. The previous system is equivalent to the existence of a stray field potential $\psi_{\boldsymbol{m}} \colon \mathbb{R}^N \to \mathbb{R}$, so that $\boldsymbol{\Psi}_{\boldsymbol{m}} = -D\psi_{\boldsymbol{m}}$, which is a solution of the following equation:

$$\mathrm{div} \left(D\psi_{\boldsymbol{m}} - \chi_{\boldsymbol{y}(\Omega)} \boldsymbol{m} \right) = 0 \text{ in } \mathbb{R}^N. \tag{6}$$

In (3), the energy contribution coming from the magnetocrystalline anisotropy has not been accounted. This is described by the functional

$$(\boldsymbol{y}, \boldsymbol{m}) \mapsto \int_{\boldsymbol{y}(\Omega)} \phi \left(\frac{\boldsymbol{m}}{|\boldsymbol{m}|} \right) \mathrm{d}\boldsymbol{\xi}, \tag{7}$$

where $\phi \colon \mathbb{S}^{N-1} \to [0, +\infty)$ is a continuous function vanishing on a finite set of directions corresponding to the easy axes. In view of what has been said in the previous section, this contribution is extremely important from the modeling point of view and should be included into the magnetoelastic energy. However, since the analytical treatment of this contribution in relation to the analysis presented here does not introduce any additional difficulty, we neglect it. We will come back to this point later on.
Possibly, the work of applied loads is determined by the functional

$$(\boldsymbol{y}, \boldsymbol{m}) \mapsto \int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{y} \, \mathrm{d}\boldsymbol{x} + \int_{\partial\Omega} \boldsymbol{g} \cdot \boldsymbol{y} \, \mathrm{d}\boldsymbol{a} + \int_{\boldsymbol{y}(\Omega)} \boldsymbol{h} \cdot \boldsymbol{m} \, \mathrm{d}\boldsymbol{\xi}. \tag{8}$$

The first two term in (8) stem for the work of mechanical body and surface forces $\boldsymbol{f} \colon \Omega \to \mathbb{R}^N$ and $\boldsymbol{g} \colon \partial\Omega \to \mathbb{R}^N$, respectively, which are described according to the assumption of dead loads [37]. The last term in (8), which is of Eulerian type, measures the effect of the external magnetic field $\boldsymbol{h} \colon \mathbb{R}^N \to \mathbb{R}^N$ and it is known as Zeeman energy [84]. The resulting total energy is expressed by taking the difference between the two functionals in (3) and (8).
Having described the variational model, we immediately realize that its rigorous analysis encounters several difficulties. These are mainly due to its mixed Eulerian-Lagrangian formulation that requires the knowledge of the deformed configuration which constitutes by itself one of the unknowns of the problem. Also, as we can expect, this formulation makes the question of the invertibility of deformations of central importance.
Some issues are already evident at the modeling stage. Indeed, if deformations are assumed to be Sobolev maps, thus defined almost everywhere, as customary in nonlinear elasticity theory, then the notion of deformed configuration is ambiguous. Also, the second term on the energy functional (3) suggests that magnetizations

should be given by Sobolev maps on the deformed configuration. However, if deformations are not homeomorphisms, these sets are not necessarily open, making such a regularity requirement meaningless. Both these issues are solved by appealing to the notion of topological image [18, 137].
In general, the mixed Eulerian-Lagrangian setting makes standard compactness and lower semincontinuity methods of the calculus of variations not applicable. Some specific aspects of our model that are particularly delicate from the analytical point of view are the following. First, apart from having the typical nonlinear and nonconvex character postulated in finite elasticity, the elastic energy density W depends on the composition of magnetization with deformations, which is usually hard to handle. Second, altough the solution operator $\boldsymbol{m} \mapsto \boldsymbol{\Psi}_{\boldsymbol{m}}$ corresponding to (5) enjoys good continuity properties [127], the qualitative description of the stray field starting from the knowledge of the deformed configuration and the magnetization field is bewildering. Indeed, even in the case of rigid bodies with simple geometries, the explicit computation of the solution of (6) constitutes an extremely difficult task. Third, the magnetic saturation constraint (1) inolves the composition of magnetization with deformations and the Jacobian determinant of deformation; given its nonconvex nature, the fact that this constraint is preserved along converging sequences is anything but trivial.
Depending on the problem under consideration, the anisotropy energy in (7) may or may not be of interest. For example, this term certainly has to be taken into account when one aims to determine qualitative properties of ground states. Differently, the anisotropy term is not relevant for the topics addressed here. Indeed, the functional in (7) turns out to be continuous both in the setting of our existence theory and in our dimension reduction procedure, so that this term is easily handled. For this reason, this term will be systematically neglected in what follows.
In the small-strain regime, the variational model simplifies substantially. In this case, when deformations are given by small perturbations of the identity, it is acceptable to assume that magnetizations are defined on the reference configuration. This semplification rules out various analytical obstacles. Also, in the linearized setting, the elastic energy density W is supposed to be quadratic and convex. Nevertheless, the typical difficulties of the variational theory of micromagnetics [55] persist and the magnetomechanical coupling, modeled by the introduction of a magnetic prestrain term in the expression of the elastic energy [52, 89], makes the analysis nontrivial.

Overview of the literature

We now describe the state of the art concerning the mathematical analysis of magnetoelasticity. Without any claim of completeness, we only mention works from the literature investigating variational models close to the one described in the previous section. We mainly focus on the problems analyzed in this thesis, namely the existence of solutions and the rigorous derivation of plate models, in order to frame our results into the literature.
In the following overview, we will include some works on variational models for liquid crystals. By replacing the nematic variable with the magnetization, this models are formally analogous to the one of Brown, except that the magnetostatic term is not included into the expression of the energy. However, since these works tackle some

of the mathematical difficulties that are encountered in the analysis of the model of Brown, they are significant for our purposes. For an overview on the mathematical theory of liquid crystals, we recommend [12].

Existence of solutions

The program of extending the existence theory of elastic materials [10, 14] to variational models for magnetostrictive solids has been initiated in [5] and then pursued in [126, 127]. In these works, magnetic quantities are pulled back to the reference configuration in order to rewrite the magnetoelastic energy as a Lagrangian functional. This approach has the disadvantage that imposes specific constitutive assumptions on the elastic energy density, so that nonstandard growth conditions must be postulated.

In [129], this issues are partially solved. In this case, no restrictive structure of the elastic energy density is assumed. The compactness of magnetization is achieved by working on the deformed configuration. However, the analysis requires the control of the Jacobian determinant of deformations, which is recovered by adding some higher-order term to the energy functional. This sets the model into the framework of nonsimple materials [14, 142, 144]. See [24] for a related study focusing on nearly incompressible materials.

For incompressible materials, the existence of minimizers has been proved, almost concurrently, in [17, 95] without introducing any regularization. The first work actually concerns liquid crystals, while the second one studies precisely the model described in the previous section. Clearly, the incompressibility assumption becomes particularly convenient while applying the Change-of-variable Formula.

In [95], admissible deformations belong to $W^{1,p}(\Omega;\mathbb{R}^N)$ with $p > N$, where the spatial dimension is denoted by $N \in \mathbb{N}$. This ensures the continuity of deformations and simplifies localization arguments in view of the Morrey embedding. In the same paper, the authors observe that, having the convergence of sequences of admissible states with respect to a suitable topology, the incompressibility yields the strong convergence of the composition of magnetizations with deformations; this allows them to apply classical lower semicontinuity results [14, 57].

In [17], deformations are given by maps in the space $W^{1,N}(\Omega;\mathbb{R}^N)$. Under the orientation-preserving constraint, such maps admit a continuous representative which does not necessarily admit a continuous extension up to the boundary [72]. The strategy of the authors relies on the local invertibility resul for Sobolev maps by Fonseca and Gangbo [60, 61]. After localization, the elastic energy is rewritten as a functional defined on the actual space. In this way, its dependence on the composition of the nematic variable with deformations is removed and the gradient of the inverse deformations appears in the expression. Then, the proof exploits the stability of the inverse with respect to the weak convergence of deformations [61].

The results in [17] have been improved in [18] by allowing for possibly discontinuous deformations and by removing the incompressibility constraint. In order to do this, the authors have generalized the local invertibility result in [60] to the case of Sobolev maps with subcritical integrability. Starting from the investigations in [78, 79, 80, 81], they have identified a class of locally invertible deformations in $W^{1,p}(\Omega;\mathbb{R}^N)$ with $p > N-1$ which can be characterized as the class of orientation-preserving Sobolev maps for which cavitation is excluded. Recall that this term refers to the phenomena of creation of voids inside materials in response to mechanical stresses [11] and that, if the creation of new surface through deformation is not penalized, cavitation may preclude the existence of minimizers in nonlinear elasticity [15, 120]. Additionally, they have extended standard degree-theoretic arguments of nonlinear elasticity [13, 90, 121, 122, 138], see also [82], to this new setting and they have further developed the notion of topological image studied in [82, 122, 138, 137]. We will comment more about this class of deformations and the concept of topological image in the next section. By means of the techniques developed in [18], the authors have been able to establish the existence of minimizers of the magnetoelastic energy functional (3) by following the strategy in [17]. However, even though their model contemplates compressible materials, in this work the magnetic saturation constraint takes the form in (2) instead of the one in (1).

The existence result in [18] has been recently extended in [83] by enlarging the class of admissible deformations to the scale of Sobolev-Orlicz spaces.

The literature investigating the variational model of Brown under quasistatic evolution is scarce. In this context, we seek for solutions in the energetic sense according the theory of rate-independent processes. Therefore, dissipation effects have to be taken into account. The mechanical response of the material is supposed to be purely elastic, so that energy dissipation results from magnetic reorientation only.

In the small-strain regime, the existence of energetic solutions has been proved in [94]. The same kind of result for a similar model of piezoelectricity has been established in [116]. Previous related studies on magnetic shape memory alloys are given in [8, 23]. In all these cases, the analysis takes advantage of the linearity of the model.

Eventually, the existence of energetic solutions for the nonlinear model has been firstly achieved in [95], still in the incompressible case. Also in the quasistatic setting, the key point is the convergence of the composition of magnetization with deformations which entails the continuity of the dissipation distance. We are not aware about other contributions regarding the large-strain model under quasistatic evolution.

Dimension reduction

In the calculus of variation, the dimension reduction is usually understood in the sense of Γ-convergence [26, 44]. This notion has nowadays reached a prominent role in the analytical validation of reduced models.

Up to now, few works have been published on the rigorous derivation of effective models for thin plates starting from the variational model of Brown. For the case of small-strains, the dimension reduction for plates in the Kirchoff-Love regime has been investigated in [94].

For a nonlinear but purely Lagrangian model in the context of liquid crystals, with the nematic variable defined on the reference configuration, the membrane regime has been studied in [2], while the bending regime is the subject of [19]. In the first case, the nematic variable is allowed to change only in the vertical direction. In the second case, this restriction is not present and some numerical simulations are also demonstrated.

In the framework of nonsimple materials, the problem of dimension reduction for

magnetoelastic plates in the membrane regime undergoing large deformations have been addressed in [101], under some a priori constraint on the Jacobian determinant of deformations, and then subsequently numerically investigated in [102, 103]. In [46], the same kind of analysis is carried out without any constraint on the Jacobian determinant of deformation and a partial upper bound is established.
A recent contribution, still in the framework of nonsimple materials, focusing on the derivation of a reduced model for magnetoelastic rods in the membrane regime, has been given in [36].
We remark that the introduction of regularizing terms involving the Hessian of deformations into the energy entails the Sobolev regularity of the composition of magnetizations with deformations. This regularity is exploited to identify limiting magnetizations by rescaling. In general, without regularizing terms, similar compactness arguments are not available, thus it is not clear how to identify limiting magnetizations; this point makes the study of dimension reduction problems for Eulerian-Lagrangian energies particularly challenging.
In the quasistatic setting, a suitable notion of converge of variational evolutions is the one of evolutionary Γ-convergence in the context of rate-independent processes [112]. Given two sequences of energy an dissipation functionals, this notion refers to the convergence of the corresponding sequence of energetic solutions to energetic solutions driven by the Γ-limits of the given energy and dissipation functionals. In dimension reduction problems, evolutionary Γ-convergence can provide a further justification of the reduced models under time evolution.
To our knowledge, the only work in this spirit about the dimension reduction of magnetoelastic plates available in the literature is the aforementioned paper [94]. There, apart from the static Γ-convergence, the corresponding evolutionary result is also established.

Other contributions

We briefly mention few works from the literature focusing on other problems related to the variational model which are not going to be addressed in the present work.
The problem of relaxation, in the sense of lower semicontinuous envelope, for a variational model of liquid crystals has been investigated in [117]. The exchange term involves a more general density which experiences relaxation together with the elastic energy density. Their analysis has been recently extended to the case of deformations with Orlicz integrability in [131].
Other works concern the analysis of the large-body limit [50], which corresponds to neglecting the exchange term from the expression of the magnetoelastic energy. This clearly leads to a lack of coercivity and lower semicontinuity. In [51], the existence of minimizers in two dimensions is proved by means of convex integration methods. In classical works, the asymptotic behaviour of minimizing sequences is studied by employing Young measures. In [85], the anisotropy is described starting from lattice considerations and the resulting domain structures are predicted precisely; in [52], materials with high anisotropy are studied, within the linearized framework, by computing the corresponding relaxed energy for simple geometries. In a two-dimensional setting, the problem of relaxation in the sense of Young measures has been also investigated in [125].

Finally, we list some related contribution on the dynamical problem. In the literature on micromagnetics, some works on the Landau-Lifschitz-Gilbert equation [32, 96] accounting for magnetostrictive effects are available [16, 35, 146]. All these works confine themselves to the small-strain setting. Starting from the model of Brown, systems of partial differential equations coupling elastodynamics and the Landau-Lifschitz-Gilbert equation have been derived and studied in [21, 20, 63, 87, 128]. Given the complexity of the problem, several simplifications are unavoidable.

Original contributions of the thesis

We now proceed by describing the original contributions contained in this thesis. As we said, our analysis addresses two problems, the existence of solutions and the dimension reduction for plates, both in the static and in the quasistatic setting.
The thesis is based on the two publications [27, 29] and on the two submitted preprints [28, 30]. The papers [27, 28] are single-author works of the Ph. D. candidate, [29] is joint work with E. Davoli and M. Kružík, whereas [30] has been obtained in collaboration with M. Kružík.

Existence of solutions

Concerning the problem of the existence of solutions, in this thesis we extend the analysis in [95] by removing the incompressibility constraint and by requiring less restrictive growth conditions on the elastic energy density. It is worth to notice that, apart for the sake of mathematical generality, these two improvements are also interesting from the modeling point of view. On the one hand, the incompressibility assumption is not very realistic for typical magnetostrictive alloys; on the other hand, relaxing the coercivity assumptions on the elastic energy density makes the analysis compatible with a broader range of mechanical models acknowledged in the literature in which the elastic energy density has subcritical, even quadratic, growth [37].
As we mentioned before, remarkable progresses in this direction have been achieved in [18] for the static setting. In this thesis, we push the analysis even further by tackling two open questions. First, we provide a mathematical treatment of the magnetic saturation constraint in its more realistic form (1). Note that, in [18], the sphere-valued constraint (2) was enforced. Second, the compactness result provided in [18] was not strong enough to extend the analysis from the static setting to the quasistatic one. The results in this thesis, instead, building upon [18], allow also the study of rate-independet evolutions.
Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. For some fixed $p > N-1$, admissible deformations belong to the same class considered in [18], namely:

$$\mathcal{Y}_p(\Omega) := \Big\{ \boldsymbol{y} \in W^{1,p}(\Omega;\mathbb{R}^N) : \ \det D\boldsymbol{y} \in L^1(\Omega),\ \det D\boldsymbol{y} > 0 \text{ a.e.}, \\ \boldsymbol{y} \text{ satisfies (DIV)} \Big\}.$$

The divergence identities [122] read as follows:

$$\forall\, \boldsymbol{\psi} \in C^1_c(\mathbb{R}^N;\mathbb{R}^N), \quad \mathrm{Div}((\mathrm{adj}\, D\boldsymbol{y})\, \boldsymbol{\psi} \circ \boldsymbol{y}) = \mathrm{div}\boldsymbol{\psi} \circ \boldsymbol{y}\, \det D\boldsymbol{y} \quad \text{in } \Omega. \tag{DIV}$$

These identities are a generalization of the well known identity $\mathrm{Det} D\boldsymbol{y} = \det D\boldsymbol{y}$ [118], asserting the coincidence of the distributional Jacobian with the pointwise Jacobian. As a result of the investigations in [18, 78, 80], this class can be characterized as the class of orientation-preserving Sobolev maps for which cavitation is excluded.
The restriction $p > N-1$ on the integrability exponent comes from technical reasons. Loosely speaking, this assumption entails the continuity of deformations on almost every surface and allows us to employ the topological degree.
The notion of topological image has been introduced in [18, 81], but the basic concept is due to Šverák [137]. Given $\boldsymbol{y} \in \mathcal{Y}_p(\Omega)$, the topological image of Ω under $\boldsymbol{y}$, denoted by $\mathrm{im}_{\mathrm{T}}(\boldsymbol{y}, \Omega)$, represent a suitable substitute of the deformed set $\boldsymbol{y}(\Omega)$ in the case of Sobolev maps. In particular, as a consequence of the continuity of the topological degree, the set $\mathrm{im}_{\mathrm{T}}(\boldsymbol{y}, \Omega)$ turns out to be open. Given a deformation $\boldsymbol{y} \in \mathcal{Y}_p(\Omega)$, this enables us to define admissible magnetizations as maps $\boldsymbol{m} \in W^{1,2}(\mathrm{im}_{\mathrm{T}}(\boldsymbol{y}, \Omega); \mathbb{R}^N)$ subjected to the constraint

$$|\boldsymbol{m} \circ \boldsymbol{y}| \det D\boldsymbol{y} = 1 \text{ a.e. in } \Omega. \tag{9}$$

Hence, the resulting class of admissible states is given by

$$\begin{aligned} \mathcal{Q} \coloneqq \big\{ (\boldsymbol{y}, \boldsymbol{m}) :\ \boldsymbol{y} \in \mathcal{Y}_p(\Omega), \quad & \boldsymbol{m} \in W^{1,2}(\mathrm{im}_{\mathrm{T}}(\boldsymbol{y}, \Omega); \mathbb{R}^N), \\ & |\boldsymbol{m} \circ \boldsymbol{y}| \det D\boldsymbol{y} = 1 \text{ a.e. in } \Omega \big\}. \end{aligned} \tag{10}$$

Recalling (3), we define the magnetoelastic energy functional by setting

$$E(\boldsymbol{q}) \coloneqq \int_\Omega W(D\boldsymbol{y}, \boldsymbol{m} \circ \boldsymbol{y})\,\mathrm{d}\boldsymbol{x} + \int_{\mathrm{im}_{\mathrm{T}}(\boldsymbol{y},\Omega)} |D\boldsymbol{m}|^2\,\mathrm{d}\boldsymbol{\xi} + \frac{1}{2}\int_{\mathbb{R}^N} |D\psi_{\boldsymbol{m}}|^2\,\boldsymbol{\xi}, \tag{11}$$

where $\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m}) \in \mathcal{Q}$ and $\psi_{\boldsymbol{m}}$ denotes a weak solution to the Maxwell equation:

$$\mathrm{div}\left(D\psi_{\boldsymbol{m}} - \chi_{\mathrm{im}_{\mathrm{T}}(\boldsymbol{y},\Omega)}\boldsymbol{m} \right) = 0 \text{ in } \mathbb{R}^N.$$

Our first main contribution consists in establishing the existence of minimizers of the energy in (11) within the class in (10) under Dirichlet boundary conditions on the deformations. The precise statement is given in Theorem 2.5. This refines the existence result in [18] by taking into account the more realistic constraint (9). Also, our proof strategy seems to be more direct.
The main ingredient to prove our existence theorem is the compactness result for sequences of admissible states with equibounded energy provided in Theorem 2.6 which, by itself, constitutes one of the main contributions in the first part of the thesis. The limiting states are identified by arguing substantially as in [18]. The hard part is to show that these satisfy the constraint (9). Moreover, our compactness result yields the convergence of the compositions of magnetizations with deformations to the corresponding limiting quantity. This convergence constitutes a very delicate issue as both deformations and magnetizations are generally discontinuous. We note that, in [18], this issue was circumvented by following the strategy in [17].
We provide two proofs of our compactness result: a first one for $p > N$ assuming the global invertibility of deformations and a second one under no additional assumptions.
In the first case, we exploit the continuity of deformation and the uniform convergence inferred by the Morrey embedding, together with the precise description of

the geometry of the topological image available in this setting. The proof of the convergence of composition relies on a peculiar use of the Area Formula to estimate the preimage of sets in the deformed configuration. A similar strategy has been employed in [74], where admissible deformations are given by homeomorphisms.
Clearly, also in the second case, the most delicate points are to show that the constraint (9) is preserved in the limit and to prove the convergence of the compositions of magnetizations with deformations. The strategy to tackle both these issues is based on a localization argument involving the topological image of nested balls. Given a converging sequence of deformations, this allows us to consider the sequence of the local inverses on the topological image of a fixed ball under the limiting deformation. This argument is presented in Lemma 2.7. This last result relies on the fine properties of admissible deformations, precisely on their regular approximate differentiability [71]. We point out that the local invertibility of deformations as well as the stability of the local inverse with respect to the weak convergence are essential ingredients in our proofs.
In the second cases, key difficulties are originated by the fact that admissible deformations may violate the Lusin property (N), i.e. they may map sets of measure zero to sets of positive measure. This fact represents an essential obstacle for the validiy of the Change-of-variable Formula [59]. This issue is tackled by means of the techniques developed in [18]. The concepts of topological image and geometric image play a central role in relation to this techniques. The former allows us to employ the Change-of-variable Formula from the deformed configuration to the reference one and backwards, whereas the latter enjoys good continuity properties with respect to the convergence of deformations. In this regard, the topological degree constitutes an indispensable tool. For almost everz subdomain, the corresponding topological image and geometric image coincide up to sets of measure zero. Indeed, this coincidence is typically violated when deformations experience cavitation.
With the aid of our compactness result, the proof of the existence of minimizers becomes not very demanding. The main issues, like the weak continuity of the Jacobian determinant and the lower semicontinuity of the elastic energy, are resolved by appealing to results available in the literature [14, 57, 119]. Overall, our existence theory requires physical growth conditions and standard polyconvexity assumptions on the elastic energy density.
Subsequently, we move to the quasistatic setting by studying evolutions driven by time-dependent boundary conditions and applied loads. Let $T > 0$ be the time horizon and let $\Sigma \subset \partial\Omega$ denote the portion of the boundary subjected to traction. Recalling (8), for applied body and surface forces

$$\boldsymbol{f} \in W^{1,1}(0,T;L^p(\Omega;\mathbb{R}^N)), \qquad \boldsymbol{g} \in W^{1,1}(0,T;L^p(\Sigma;\mathbb{R}^N))$$

and an external magnetic field

$$\boldsymbol{h} \in W^{1,1}(0,T;L^2(\mathbb{R}^N;\mathbb{R}^N)),$$

the total energy functional reads

$$\mathcal{F}(t,\boldsymbol{q}) := E(\boldsymbol{q}) - \int_\Omega \boldsymbol{f}(t)\cdot\boldsymbol{y}\,\mathrm{d}\boldsymbol{x} - \int_\Sigma \boldsymbol{g}(t)\cdot\boldsymbol{y}\,\mathrm{d}\boldsymbol{a} - \int_{\mathrm{im}_{\mathrm{T}}(\boldsymbol{y},\Omega)} \boldsymbol{h}(t)\cdot\boldsymbol{m}\,\mathrm{d}\boldsymbol{\xi}, \tag{12}$$

where $t \in [0,T]$ and $\boldsymbol{q} = (\boldsymbol{y},\boldsymbol{m}) \in \mathcal{Q}$. Our analysis is set within the framework of rate-independent processes [110]. In view of our mixed Eulerian-Lagrangian structure and the expression of the elastic energy in (11), the natural candidate as dissipative variable is given by the composition of magnetization and deformation. This modeling choice has been taken in [95]. Taking into account the compressibility of the material and the constraint (9), we define the dissipation distance as

$$\mathcal{D}(\boldsymbol{q},\widehat{\boldsymbol{q}}) := \int_\Omega |\boldsymbol{m}\circ\boldsymbol{y}\det D\boldsymbol{y} - \widehat{\boldsymbol{m}}\circ\widehat{\boldsymbol{y}}\det D\widehat{\boldsymbol{y}}|\,\mathrm{d}\boldsymbol{x}, \tag{13}$$

where $\boldsymbol{q}=(\boldsymbol{y},\boldsymbol{m}),\widehat{\boldsymbol{q}}=(\widehat{\boldsymbol{m}},\widehat{\boldsymbol{y}})\in\mathcal{Q}$. Looking at this expression, we immediately realize that the convergence of compositions provided by our compactness result is essential in order to address the analysis in the quasistatic setting.
Our second main contribution, namely Theorem 3.2, asserts the existence of energetic solutions corresponding to the energy functional in (12) and the dissipation distance in (13). Recall that such solutions are formulated on the two principles of global miniminality and energy-dissipation balance [110].
Thanks to our compactness result, the dissipation distance turns out to be continuous on the sublevel sets of the total energy. Hence, the construction of mutual recovery sequences [110, 112] represents a simple task. The existence of energetic solutions is established by following the standard time-discretization scheme [64, 105, 110, 115] based on a incremental minimization problems. Minor adaptations, similar to the ones in [95], are needed to establish the compactness of time-discrete solutions.
We remark that our model contemplates deformations that are locally invertible but not necessarily globally invertible. However, given the significance of this last requirement from the modeling point of view, it is desirable to incorporate the global invertibiliy constraint into our analysis. We show how this can be done in a fairly simple way. Our approach is based on the Ciarlet-Nečas condition [38] and is based on the observations made in [68, 138]. However, other equivalent possibilities, such as the (INV) condition [120], can be considered.

Dimension reduction

For the problem of dimension reduction, the main focus has been the rigorous derivation, via Γ-convergence techniques, of a reduced model for plates without resorting to any higher-order regularization of the energy. To our knowledge, these represent the first results on the dimension reduction for mixed Eulerian-Lagrangian energies that do not require the introduction of regularizing terms.
The bulk model is the same considered for our existence theory. For simplicity, we fix the dimension $N=3$. We restrict ourselves to low-energy configurations by rescaling the elastic energy according to the linearized von Kármán regime, where we adopt the terminology in [65].
Let $\Omega_h := S\times hI$ represent a thin magnetoelatic plate in its reference configuration. The section $S\subset\mathbb{R}^2$ is a bounded Lipschitz domain, the parameter $h>0$ specifies the thickness of the plate and $I:=(-1/2,1/2)$. After a customary change of variable, the problem is formulated on the domain $\Omega:=S\times I$. For some fixed $p>3$, deformations are given by maps $\boldsymbol{y}\in W^{1,p}(\Omega;\mathbb{R}^3)$ with $\det D\boldsymbol{y}>0$ that are almost everywhere injective, i.e. that are injective out of a set of measure zero. The restriction on the integrability exponent is merely technical but necessary. In this setting, the topological image of Ω under $\boldsymbol{y}$ coincides with the set $\Omega^{\boldsymbol{y}}:=\boldsymbol{y}(\Omega)\setminus\boldsymbol{y}(\partial\Omega)$, so that magnetizations are given by maps $\boldsymbol{m}\in W^{1,2}(\Omega^{\boldsymbol{y}};\mathbb{R}^3)$. Because of rescaling, the

magnetic saturation constraint takes the form

$$|\boldsymbol{m} \circ \boldsymbol{y}| \det D\boldsymbol{y} = h \text{ a.e. in } \Omega. \tag{14}$$

Summarizing, admissible states belong to the class

$$\begin{aligned} \mathcal{Q}_h \coloneqq \Big\{ (\boldsymbol{y}, \boldsymbol{m}) :\ & \boldsymbol{y} \in W^{1,p}(\Omega; \mathbb{R}^3),\ \det D\boldsymbol{y} > 0 \text{ a.e.},\ \boldsymbol{y} \text{ is a.e. injective}, \\ & \boldsymbol{m} \in W^{1,2}(\Omega^{\boldsymbol{y}}; \mathbb{R}^3),\ |\boldsymbol{m} \circ \boldsymbol{y}| \det D\boldsymbol{y} = h \text{ a.e.} \Big\}. \end{aligned} \tag{15}$$

The magnetoelastic energy functional is defined by setting

$$\begin{aligned} E_h(\boldsymbol{q}) \coloneqq & \frac{1}{h^\beta} \int_\Omega W_h(D_h \boldsymbol{y}, \boldsymbol{m} \circ \boldsymbol{y}) \, \mathrm{d}\boldsymbol{x} + \frac{1}{h} \int_{\Omega^{\boldsymbol{y}}} |D\boldsymbol{m}|^2 \, \mathrm{d}\boldsymbol{\xi} \\ & + \frac{1}{2h} \int_{\mathbb{R}^3} |D\psi_{\boldsymbol{m}}|^2 \, \mathrm{d}\boldsymbol{\xi}, \end{aligned} \tag{16}$$

where $\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m}) \in \mathcal{Q}_h$. Here, $\psi_{\boldsymbol{m}}$ is a weak solution to the Maxwell equation:

$$\operatorname{div}(D\psi_{\boldsymbol{m}} - \chi_{\Omega^{\boldsymbol{y}}} \boldsymbol{m}) = 0 \quad \text{in } \mathbb{R}^3. \tag{17}$$

In (16), the scaled gradient is defined as $D_h \coloneqq (D', h^{-1}\partial_3)$, where D' denotes the gradient with respect to the variable $\boldsymbol{x}' \in S$. The scaling $\beta > 4$ determines the linearized von Kármán regime but, for technical reasons, we restrict ourselves to the case $\beta > 6 \vee p$.

We observe that the elastic energy density depends on the thickness. Precisely, this is supposed to have the following structure:

$$W_h(\boldsymbol{F}, \boldsymbol{\lambda}) \coloneqq \Phi\left((\boldsymbol{I} + c_h \boldsymbol{\lambda} \det \boldsymbol{F} \otimes \boldsymbol{\lambda} \det \boldsymbol{F})^{-1} \boldsymbol{F} \right), \tag{18}$$

where $\boldsymbol{F} \in \mathbb{R}^{3\times 3}$ and $\boldsymbol{\lambda} \in \mathbb{R}^3$ are placeholders for the deformation gradient and for the composition of magnetizations with deformations, respectively. In (18), the function Φ is a nonlinear density and c_h is a material parameter of order $h^{\beta/2}$. This structure of the elastic energy density is similar to the one of our prototypical example (4) and resembles the constitutive assumptions usually assumed in the dimension reduction of prestrained materials [98, 99] and heterogeneous multilayers [49, 130]. See [2] for a specific example in a dimension reduction problem for liquid crystals. The function Φ is assumed to attain its minimum at the identity and is supposed to be frame-indifferent and of class C^2 in a neighborhood of the set of rotations. Additionally, Φ satisfies growth conditions involving the distance from the set of rotations, with global p-growth and quadratic growth close to rotations, and controls the inverse of the determinant of its argument. We remark that this set of assumptions is compatible with our existence theory.

In our third main contribution, we identify a reduced plate model by computing the Γ-limit of the energies (E_h) in (16), as $h \to 0^+$. The statement is given in Theorem 4.3. The Γ-limit is given by the functional E_0, see (16) below, defined on triplets

$$(\boldsymbol{u}, v, \boldsymbol{\zeta}) \in W^{1,2}(S; \mathbb{R}^2) \times W^{2,2}(S) \times W^{1,2}(S; \mathbb{S}^2).$$

These triplets are determined as limits of sequences $(\boldsymbol{q}_h)$ of admissible states $\boldsymbol{q}_h = (\boldsymbol{y}_h, \boldsymbol{m}_h) \in \mathcal{Q}_h$ with respect to the following notion of convergence:

$$\boldsymbol{u}_h \rightharpoonup \boldsymbol{u} \text{ in } W^{1,2}(S; \mathbb{R}^3), \tag{19}$$

$$v_h \to v \text{ in } W^{1,2}(S), \tag{20}$$

$$\boldsymbol{z}_h \to \boldsymbol{\zeta} \text{ in } L^q(\Omega; \mathbb{R}^3) \text{ for every } 1 \leq q < \infty, \tag{21}$$

where we set

$$\boldsymbol{u}_h(\boldsymbol{x}') := \frac{1}{h^{\beta/2}} \int_I (\boldsymbol{y}'(\boldsymbol{x}', x_3) - \boldsymbol{x}')\, \mathrm{d}x_3, \tag{22}$$

$$v_h(\boldsymbol{x}') := \frac{1}{h^{\beta/2-1}} \int_I y^3(\boldsymbol{x}', x_3)\, \mathrm{d}x_3, \tag{23}$$

$$\boldsymbol{z}_h(\boldsymbol{x}) := \boldsymbol{m}_h(\boldsymbol{y}_h(\boldsymbol{x})) \det D_h \boldsymbol{y}(\boldsymbol{x}), \tag{24}$$

for every $\boldsymbol{x}' \in S$ and $\boldsymbol{x} \in \Omega$. The quantities in (22)–(23) are known as averaged displacements [65], while the one in (24) is exactly the function involved in the magnetic saturation constraint (14). The expression of the limiting functional is the following:

$$\begin{aligned} E_0(\boldsymbol{q}_0) := \frac{1}{2} \int_S Q_{\mathrm{red}}(\mathrm{sym} D'\boldsymbol{u} - c_0 \boldsymbol{\zeta}' \otimes \boldsymbol{\zeta}')\, \mathrm{d}\boldsymbol{x}' + \frac{1}{24} \int_S Q_{\mathrm{red}}((D')^2 v)\, \mathrm{d}\boldsymbol{x}' \\ + \int_S |D'\boldsymbol{\zeta}|^2\, \mathrm{d}\boldsymbol{x}' + \frac{1}{2} \int_S |\zeta^3|^2\, \mathrm{d}\boldsymbol{x}', \end{aligned} \tag{25}$$

where $\boldsymbol{q}_0 = (\boldsymbol{u}, v, \boldsymbol{\zeta})$, and $\boldsymbol{\zeta}'$ and ζ^3 denote the first two and the third component of $\boldsymbol{\zeta}$, respectively. The reduced quadratic form Q_{red} is constructed from the quadratic from $Q := D^2\Phi(\boldsymbol{I})$ by minimizing among all the possible ways to complete a matrix in $\mathbb{R}^{2\times 2}$ to a matrix in $\mathbb{R}^{3\times 3}$ as in [65]. Also, we set $c_0 := \lim_{h\to 0^+} h^{-\beta/2} c_h$.
In (25), the first two terms are the result of the linearization of the elastic energy density typical of von Kármán regimes. The coupling between mechanical and magnetic quantities inside Q_{red} results from the prestrain term in (18). In particular, we observe that the expression of the magnetostatic energy simplifies substantially. This is analogous to what happens in the dimension reduction of micromagnetic energies [34, 70].
Concerning the proof of Theorem 4.3, the main difficulties lie in the proof of the equi-coercivity of the sequence (E_h). At first sight, the problem seems unapproachable, since it is not clear how to identify limiting magnetizations. These have to be Sobolev maps defined on the section S. In particular, the techniques based on the uniform convergence developed in [95] are useless for this problem where the deformed set is crushing on a set of measure zero.
Our approach is based on careful considerations on the geometry of the deformed sets and exploits the peculiar feature of von Kármán regimes to entail the convergence of deformation to the projection onto the section [65]. The rigidity estimate of Friesecke, James and Müller [66] occupies a central role in establishing the compactness of deformations, which is proved as in [65]. Combining the uniform convergence estimate of the deformations towards the projection with some elementary properties of the topological degree, we prove that for every h sufficiently small, up to rigid motions, the deformed sets $\Omega^{\boldsymbol{y}}$ contain a cylinder of height of order h whose section is obtained by shrinking S. On this cylinder, the compactness of magnetizations can be deduced by standard methods. The limiting object, locally identified by this procedure, turns out to be globally well defined. The same approach is used to deduce the compactness of the compositions of magnetizations with deformations. This is another very subtle point, since the magnetizations are not necessarily continuous. The issue is overcome by considering the restrictions of magnetizations to the previously determined cylinder, which is a regular domain, and extending them

to the whole space. Then, we obtain the desired convergence by using Lipschitz truncation arguments in the actual space. These techniques require an extensive use of the Area Formula and, in turn, require admissible deformations to be almost everywhere injective. In analogy with [65], compactness is obtained up to composition with rigid motions.
The proof of the lower bound employs the techniques in [65] for the elastic term, while the magnetic terms are treated by adapting the strategies in [34, 70]. In particular, the magnetostatic term actually converges to the corresponding limiting term.
The proof of the upper bound gives rise to some difficulties. Indeed, while constructing recovery sequences, we have to take into account that magnetizations have to been defined on the corresponding deformed set and have to satisfy the saturation constraint (14). The construction involves the inverse deformations. What helps here is the specific structure of deformations of the recovery sequence which are defined according the classical ansatz of the von Kármán regime [65].
Subsequently, we investigate the asymptotic behaviour of sequences of minimizers of the magnetoelastic energy, as $h \to 0^+$, in the presence of applied loads.
For fixed $h > 0$, these include horizontal forces , vertical forces and external magnetic fields respectively described by

$$\boldsymbol{f}_h \in L^2(S;\mathbb{R}^2), \qquad g_h \in L^2(S), \qquad \boldsymbol{h}_h \in L^2(\mathbb{R}^3;\mathbb{R}^3).$$

The resulting total energy is given by

$$F_h(\boldsymbol{q}) := E_h(\boldsymbol{q}) - \frac{1}{h^\beta}\int_\Omega \boldsymbol{f}_h \cdot (\boldsymbol{y}' - \boldsymbol{x}')\,\mathrm{d}\boldsymbol{x} - \frac{1}{h^\beta}\int_\Omega g_h\, y^3\,\mathrm{d}\boldsymbol{x} - \frac{1}{h}\int_{\Omega^{\boldsymbol{y}}} \boldsymbol{h}_h \cdot \boldsymbol{m}\,\mathrm{d}\boldsymbol{\xi}, \quad (26)$$

where $\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m}) \in \mathcal{Q}_h$. As in [97], we impose clamped boundary conditions by requiring admissible deformations to satisfy

$$\boldsymbol{y} = \boldsymbol{\pi}_h + h^{\beta/2}\begin{pmatrix}\overline{\boldsymbol{u}}\\ 0\end{pmatrix} + h^{\beta/2-1}\begin{pmatrix}\boldsymbol{0}'\\ \overline{v}\end{pmatrix} - h^{\beta/2}x_3\begin{pmatrix}D'\overline{v}\\ 0\end{pmatrix} \quad \text{on } \Lambda \times I \qquad (27)$$

for some fixed boundary data $\overline{\boldsymbol{u}} \in W^{2,\infty}(S;\mathbb{R}^2)$ and $\overline{v} \in W^{3,\infty}(S)$. In (27), $\boldsymbol{\pi}_h$ denotes the rescaling map $\boldsymbol{x} \mapsto (\boldsymbol{x}', hx_3)$ and $\Lambda \subset \partial S$ is a regular subset of the boundary of the section with $\mathscr{H}^1(\Lambda) > 0$.
Concerning the asymptotic behaviour of the applied loads, we suppose the existence of maps $\boldsymbol{f}_0 \in L^2(S;\mathbb{R}^2)$, $g_0 \in L^2(S)$ and $\boldsymbol{h}_0 \in L^2(\mathbb{R}^2;\mathbb{R}^3)$ such that the following convergences hold, as $h \to 0^+$:

$$h^{-\beta/2}\boldsymbol{f}_h \rightharpoonup \boldsymbol{f}_0 \text{ in } L^2(S;\mathbb{R}^2), \qquad (28)$$

$$h^{-\beta/2-1}g_h \rightharpoonup g_0 \text{ in } L^2(S), \qquad (29)$$

$$\boldsymbol{h}_h \circ \boldsymbol{\pi}_h \rightharpoonup \chi_I \boldsymbol{h}_0 \text{ in } L^2(\mathbb{R}^3;\mathbb{R}^3). \qquad (30)$$

Hence, the reduced total energy is described by the functional

$$F_0(\boldsymbol{q}_0) := E_0(\boldsymbol{q}_0) - \int_S \boldsymbol{f}_0 \cdot \boldsymbol{u}\,\mathrm{d}\boldsymbol{x}' - \int_S g_0\, v\,\mathrm{d}\boldsymbol{x}' - \int_S \boldsymbol{h}_0 \cdot \boldsymbol{\zeta}\,\mathrm{d}\boldsymbol{x}', \qquad (31)$$

where $\boldsymbol{q}_0 = (\boldsymbol{u}, v, \boldsymbol{\zeta}) \in W^{1,2}(S;\mathbb{R}^2) \times W^{2,2}(S) \times W^{1,2}(S;\mathbb{S}^2)$. Moreover, in view of (19)–(20), (22)–(23) and (27), the limiting averaged displacements are subjected to the following boundary conditions:

$$\boldsymbol{u} = \overline{\boldsymbol{u}} \text{ on } \Lambda, \qquad v = \overline{v} \text{ on } \Gamma, \qquad D'v = D'\overline{v} \text{ on } \Gamma. \tag{32}$$

Our fourth main result, namely Theorem 4.14, states that every sequence of almost minimizers of the energies (F_h) in (26) under the clamped boundary conditions in (27) convergence with respect to the notion of convergence in (19)–(21), as $h \to 0^+$, to a minimizers of the reduced total energy in (31) under the boundary conditions in (32).

The clamped boundary conditions in (27) allow us to obtain the compactness of sequences of admissible states with equi-bounded energies without composing with rigid motions.

The principal difficulty in proving Theorem 4.14 is to deduce the equi-boundedness of the elastic energy in (16) from the equi-boundedness of the total energy in (26). This is accomplished by arguing by contradiction in a similar way as it is done in [97]. The argument, which is given in Lemma 4.20, is quite involved and exploits the rigidity of the linearized regime. Notably, we are able to obtain estiamates for the elastic energy that are independent on the sequence considered but depend only on the uniform bound on the total energy.

Finally, we investigate our dimension reduction problem in the quasistatic setting. The bulk model has the same structure of the quasistatic model considered in the first part of the thesis which is scaled as in the dimension reduction in the static case.

For every $h > 0$, we consider time-dependent applied loads

$$\boldsymbol{f}_h \in W^{1,1}(0,T;L^2(S;\mathbb{R}^2)), \qquad g_h \in W^{1,1}(0,T;L^2(S)),$$
$$\boldsymbol{h}_h \in W^{1,1}(0,T;L^2(\mathbb{R}^3;\mathbb{R}^3)),$$

and we define the total energy functional as

$$\begin{aligned}\mathcal{F}_h(t,\boldsymbol{q}) := E_h(\boldsymbol{q}) &- \frac{1}{h^\beta}\int_\Omega \boldsymbol{f}_h(t)\cdot(\boldsymbol{y}'-\boldsymbol{x}')\,\mathrm{d}\boldsymbol{x} \\ &- \frac{1}{h^\beta}\int_\Omega g_h(t)\,y^3\,\mathrm{d}\boldsymbol{x} - \frac{1}{h}\int_{\Omega^{\boldsymbol{y}}} \boldsymbol{h}_h(t)\cdot\boldsymbol{m}\,\mathrm{d}\boldsymbol{\xi},\end{aligned} \tag{33}$$

where $t \in [0,T]$ and $\boldsymbol{q} = (\boldsymbol{y},\boldsymbol{m}) \in \mathcal{Q}_h$. Recalling (13)–(14), the dissipation distance at thickness h is defined as

$$\mathcal{D}_h(\boldsymbol{q},\widehat{\boldsymbol{q}}) := \int_\Omega |\boldsymbol{m}\circ\boldsymbol{y}\det D_h\boldsymbol{y} - \widehat{\boldsymbol{m}}\circ\widehat{\boldsymbol{y}}\det D_h\widehat{\boldsymbol{y}}|\,\mathrm{d}\boldsymbol{x}, \tag{34}$$

where $\boldsymbol{q} = (\boldsymbol{y},\boldsymbol{m}), \widehat{\boldsymbol{q}} = (\widehat{\boldsymbol{m}},\widehat{\boldsymbol{y}}) \in \mathcal{Q}_h$. Similarly to (28)–(30), we assume that

$$h^{-\beta/2}\boldsymbol{f}_h \to \boldsymbol{f}_0 \text{ in } W^{1,1}(0,T;L^2(S;\mathbb{R}^2)), \tag{35}$$

$$h^{-\beta/2-1}g_h \to g_0 \text{ in } W^{1,1}(0,T;L^2(S)), \tag{36}$$

$$\boldsymbol{h}_h\circ\boldsymbol{\pi}_h \to \chi_I\boldsymbol{h}_0 \text{ in } W^{1,1}(0,T;L^2(\mathbb{R}^3;\mathbb{R}^3)), \tag{37}$$

for some limiting loads

$$\boldsymbol{f}_0 \in W^{1,1}(0,T;L^2(S;\mathbb{R}^2)), \quad g_0 \in W^{1,1}(0,T;L^2(S)), \quad \boldsymbol{h}_0 \in W^{1,1}(0,T;L^2(S;\mathbb{R}^3)).$$

Therefore, the reduced total energy reads

$$\mathcal{F}_0(t,\boldsymbol{q}_0) := E_0(\boldsymbol{q}_0) - \int_S \boldsymbol{f}_0(t)\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}' - \int_S g_0(t)\,v\,\mathrm{d}\boldsymbol{x}' - \int_S \boldsymbol{h}_0(t)\cdot\boldsymbol{\zeta}\,\mathrm{d}\boldsymbol{x}', \tag{38}$$

where $t\in[0,T]$ and $\boldsymbol{q}_0=(\boldsymbol{u},v,\boldsymbol{\zeta})\in W^{1,2}(S;\mathbb{R}^2)\times W^{2,2}(S)\times W^{1,2}(S;\mathbb{S}^2)$. In view of (21), (24) and (34), the dissipation distance for the reduced model is defined as

$$\mathcal{D}_0(\boldsymbol{q}_0,\widehat{\boldsymbol{q}}_0) := \int_S |\boldsymbol{\zeta}-\widehat{\boldsymbol{\zeta}}|\,\mathrm{d}\boldsymbol{x}', \tag{39}$$

where $\boldsymbol{q}_0=(\boldsymbol{u},v,\boldsymbol{\zeta}),\widehat{\boldsymbol{q}}_0=(\widehat{\boldsymbol{u}},\widehat{v},\widehat{\boldsymbol{\zeta}})\in W^{1,2}(S;\mathbb{R}^2)\times W^{2,2}(S)\times W^{1,2}(S;\mathbb{S}^2)$.
Also in the quasistatic setting, we impose clamped boundary conditions as in (27) for the bulk model and the boundary conditions in (32) for the reduced model.
The main results in the quasistatic setting are Theorem 5.4 and Theorem 5.11 which constitute the fifth and the sixth main contribution of the thesis. These results provide a further justification of the reduced model in the spirit of evolutionary Γ-convergence.
The first result states that every sequence $(\boldsymbol{q}_h)$, where $\boldsymbol{q}_h\colon[0,T]\to\mathcal{Q}_h$ is an energetic solution of the bulk model corresponding to the energy functional $\mathcal{F}_h$ in (33) and to the dissipation distance $\mathcal{D}_h$ in (34) subjected to the clamped boundary conditions in (27), converge to evolutions $\boldsymbol{q}_0\colon[0,T]\to W^{1,2}(S;\mathbb{R}^2)\times W^{2,2}(S)\times W^{1,2}(S;\mathbb{S}^2)$ which are energetic solution of the reduced model corresponding to the reduced total energy $\mathcal{F}_0$ in (38) and to the reduced dissipation distance $\mathcal{D}_0$ in (39) and satisfy the boundary conditions in (32).
The second result represents a variant of the first one for the case in which the existence of energetic solutions cannot be ensured and relies of the concept of approximate incremental minimization problem [113]. Theorem 5.11 states that, given a sequence of partitions of the time interval whose sizes converges to zero, as $h\to0^+$, every sequence of solutions of the approximate incremental minimization problem determined by $\mathcal{F}_h$ and $\mathcal{D}_h$, or better, of their piecewise-constant interpolants, converge to energetic solutions of the reduced quasistatic model corresponding to $\mathcal{F}_0$ and $\mathcal{D}_0$.
The biggest challenges for the analysis of our dimension reduction problem in the quasistatic setting are encountered while proving that the total energy functional satisfies suitable controls with respect to time. These are similar to the difficulties tackled in the static setting to establish uniform bounds on the elastic energy starting from bounds on the total energy. In particular, in this case, we need to obtain bounds that are also uniform with respect to time and this makes the proofs quite technical. Also here, the desired estimates are achieved by arguing by contradiction in the spirit of [97].
Once we show that the energy functionals satisfy the basic assumptions of evolutionary Γ-convergence, the proof follows the classical scheme [112]. Another technical point is the proof of the measurability with respect to time of the limiting evolution, which is established by means of tools of set-valued analysis [7].
We repeat that the setting of our dimension reduction problem is compatible with the existence theory presented in the first part of the thesis. Precisely, if the function Φ in (18) satisfies standard polyconvexity assumptions, then the existence of both minimizers and energetic solutions for the bulk model is ensured.

The major limitation of our analysis is the restriction $\beta > 6 \vee p$ that is assumed on the scaling. Unfortunately, so far, it does not seem possible to remove it. Indeed, our compactness arguments heavily rely on the fact that the rate of convergence of the deformations to the projection onto the section is of order bigger than one. This allows us to approximate the deformed sets with cylindrical sets. An alternative approach to extend the analysis to the regime $\beta > 2$ is currently being investigated by the Ph. D. candidate in collaboration with Maria Giovanna Mora.

Outline

The outline of the thesis is the following. In Chapter 1, we collect various result that are going to be employed in the rest of the thesis; these are mostly results available in the literature. The remainder of the thesis is dived into two parts. Part I deals with the problem of existence of solutions, while Part II focuses on the dimension reduction. Assumptions and notation are maintained through each part.
Part I includes Chapter 2 and Chapter 3 concerning the existence of minimizers and the existence of energetic solutions, respectively. The content of this first part is based on [28, 29]. Part II consists of Chapter 4 and Chapter 5 which address the problem of dimension reduction in the static setting and in the quasistatic setting, respectively. This second part is based on [27, 30].
Eventually, we recall the definition of Sobolev maps on the boundary of smooth domains in the Appendix.
We conclude the Introduction with a detailed comparison between the results published by the author and the ones presented here.
The static existence result in Chapter 2 has been published first in [29] for the case $p > N$ under global invertibility of deformations, and then extended in [28] without extra assumptions. In [29], the simpler constraint (2) was imposed, while here we always require (9) to be satisfied. The proof of the compactness result provided in Subsection 2.2.1 has been adapted accordingly.
The quasistatic existence result in Chapter 3 has been published first in [29] for the case $p > N$ assuming the global invertibility of deformations and imposing the constraint (2). In that paper, a different notion of dissipation is considered and this requires the introduction of some regularizing terms to achieve the existence of energetic solutions; we briefly describe this setting in Section 3.4. Here, the setting ensures the existence of energetic solutions with no need of regularization. This existence result is taken from [28].
The static Γ-convergence result in Chapter 4 has been published first in [27] for the case of incompressible materials under less realistic assumption on the elastic energy density and imposing the constraint (2). Then, the result has been extended to compressible materials under more physical assumptions on the density in [30]. In the same paper, the convergence of almost minimizers has been investigated as well as the evolutionary Γ-convergence in the quasistatic setting. The notion of dissipation considered there resembles the one in [29]. Here, the more realistic constraint in (9) is enforced and a notion of dissipation analogous to that of Chapter 3 is considered. This requires to impose some growth conditions tono Φ with respect to the determinant of its argument and to adapt the construction of magnetizations in the recovery sequence.

Notation and conventions

General notation

- Given $a, b \in \mathbb{R}$, we set $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$.
- Vectors and matrices are denoted in boldface and we use superscripts for their components. The only exception are space variables whose components are indicated with subscripts.
- Given $M, N \in \mathbb{N}$, the set of $M \times N$ matrices is denoted by $\mathbb{R}^{M\times N}$. The set of square matrices of order N with positive determinant is denoted by $\mathbb{R}^{N\times N}_+$. The set of rotation on $\mathbb{R}^N$ is denoted by $SO(N)$.
- The null vector is denoted by $\mathbf{0} \in \mathbb{R}^N$. The vectors of the canonical basis of $\mathbb{R}^N$ are denoted by $\boldsymbol{e}_1, \dots \boldsymbol{e}_N \in \mathbb{R}^N$. The null matrix and the identity matrix are denoted by $\boldsymbol{O} \in \mathbb{R}^{N\times N}$ and $\boldsymbol{I} \in \mathbb{R}^{N\times N}$, respectively. The identity map on $\mathbb{R}^N$ is denoted by $\boldsymbol{id}$.
- Given $\boldsymbol{A} \in \mathbb{R}^{N\times N}$, we denote its transpose by $\boldsymbol{A}^\top$. If $\boldsymbol{A}$ is invertible, we denote its inverse by $\boldsymbol{A}^{-1}$. The determinant and the trace of $\boldsymbol{A}$ are denoted by $\det \boldsymbol{A}$ and $\mathrm{tr}\boldsymbol{A}$, respectively. The adjugate matrix of $\boldsymbol{A}$ is the matrix $\mathrm{adj}\boldsymbol{A} \in \mathbb{R}^{N\times N}$ such that $(\det \boldsymbol{A})\boldsymbol{I} = \boldsymbol{A}(\mathrm{adj}\boldsymbol{A})$ and the the cofactor matrix of $\boldsymbol{A}$ is given by $\mathrm{cof}\boldsymbol{A} = (\mathrm{adj}\boldsymbol{A})^\top$.
- Given $\boldsymbol{A} \in \mathbb{R}^{N\times N}$, its symmetric and skew-symmetric part are denoted by $\mathrm{sym}\,\boldsymbol{A} = (\boldsymbol{A} + \boldsymbol{A}^\top)/2$ and $\mathrm{skew}\,\boldsymbol{A} = (\boldsymbol{A} - \boldsymbol{A}^\top)/2$.
- Given $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^N$, their scalar product is given by $\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{i=1}^N a^i b^i$. The norm of $\boldsymbol{a}$ is given by $|\boldsymbol{a}| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}}$.
- Given $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{N\times N}$, their scalar product is given by $\boldsymbol{A} : \boldsymbol{B} = \sum_{i,j=1}^N A^{ij} : B^{ij}$. The norm of $\boldsymbol{A}$ is given by $|\boldsymbol{A}| = \sqrt{\boldsymbol{A} : \boldsymbol{A}}$.
- The tensor product of $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^N$, is given the matrix $\boldsymbol{a} \otimes \boldsymbol{b} \in \mathbb{R}^{N\times N}$ whose elements are $a^i b^j$ for $i, j \in \{1, \dots, N\}$.
- The symmetric difference of two sets $A, B \subset \mathbb{R}^N$ is given by
$$A \triangle B = (A \setminus B) \cup (B \setminus A).$$
- Given $A \subset \mathbb{R}^N$, we denote by $\overline{A}$ and ∂A its closure and its boundary. If $B \subset \mathbb{R}^N$, we write $A \subset\subset B$ if A is bounded and $\overline{A} \subset B$.
- The open ball centered at $\boldsymbol{x}_0 \in \mathbb{R}^N$ with radius $r > 0$ is denoted by $B(\boldsymbol{x}_0, r)$, while its closure is denoted by $\overline{B}(\boldsymbol{x}_0, r)$.

Measure theory

- The m-dimensional Lebesgue measure is denoted by $\mathscr{L}^m$, where $m \in \mathbb{N}$. The s-dimenstional Hausdorff measure is denoted by $\mathscr{H}^s$, where $s > 0$. We will mainly consider $m = N$ and $s = N - 1$.

- Expressions like "almost everywhere" or "almost every" are referred to $\mathscr{L}^N$ or $\mathscr{H}^{N-1}$ depending on the context without any explicit specification.
- Generic points in the reference space and in the actual space are denoted by $\boldsymbol{x}$ and $\boldsymbol{\xi}$, respectively. Accordingly, integration with respect to $\mathscr{L}^N$ is indicated by $\mathrm{d}\boldsymbol{x}$ or $\mathrm{d}\boldsymbol{\xi}$. Integration with respect to $\mathscr{H}^{N-1}$ and $\mathscr{H}^1$ in the reference space is indicated by $\mathrm{d}\boldsymbol{a}$ and $\mathrm{d}\boldsymbol{l}$, respectively.
- The characteristic function of a set $A \subset \mathbb{R}^N$ is denoted by χ_A.
- We do not identify functions that coincide almost everywhere.
- For every exponent $1 \leq q < \infty$, we denote by q' its Hölder conjugate, i.e. $q' := q/(1-q)$ if $1 < q < \infty$ and $q' = \infty$ if $q = 1$.

Function spaces

- The distrbutional gradient is denoted by D. The gradient of scalar functions is a column vector.
- We use standard notation for Lebesgue and Sobolev spaces, i.e. L^p and $W^{m,p}$, where $1 \leq p \leq \infty$ and $m \in \mathbb{N}$, and for their local counterparts. Standard notation is used also for spaces of continuously differentiable functions and for their counterparts with compact support, i.e. C^k, and C^k_{c}, where $k \in \mathbb{N}$, as well as for spaces of functions of bounded variation, i.e. BV. In the notation, domain and codomain are separated by a semicolon.
- Given $S \subset \mathbb{R}^N$ open and $1 \leq p \leq \infty$, we define

  $$\begin{aligned} W^{1,p}(S; SO(N)) &:= \left\{ \boldsymbol{U} \in W^{1,p}(S; \mathbb{R}^{N\times N}) : \, \boldsymbol{U}(\boldsymbol{x}) \in SO(N) \text{ for a.e. } \boldsymbol{x} \in S \right\}, \\ W^{1,p}(S; \mathbb{S}^{N-1}) &:= \left\{ \boldsymbol{u} \in W^{1,p}(S; \mathbb{R}^N) : \, \boldsymbol{u}(\boldsymbol{x}) \in \mathbb{S}^{N-1} \text{ for a.e. } \boldsymbol{x} \in S \right\}. \end{aligned}$$

  Here, we denote by $\mathbb{S}^{N-1}$ the unit sphere in $\mathbb{R}^N$, i.e. $\mathbb{S}^{N-1} = \partial B(\boldsymbol{0}, 1)$.
- We denote by $V^{1,2}(\mathbb{R}^N)$ the homogeneous Sobolev space

  $$V^{1,2}(\mathbb{R}^N) := \left\{ \varphi \in L^2_{\mathrm{loc}}(\mathbb{R}^N) : \, D\varphi \in L^2(\mathbb{R}^N; \mathbb{R}^N) \right\}.$$
- We use standard notation for Bochner and Bochner-Sobolev spaces, i.e. $L^p(0,T;X)$ and $W^{1,p}(0,T;X)$, where $T > 0$, $1 \leq p \leq \infty$ and X is a Banach space, as well as for absolutely continuous functions, i.e. $AC([0,T];X)$. Derivatives with respect to time are indicated using a dot.
- Strong convergence is denoted by $\to$, while weak convergence and weak-* convergence are denoted by $\rightharpoonup$ and $\overset{*}{\rightharpoonup}$, respectively.

Other conventions

- We use the Landau symbols "o" and "O". When referred to vectors and matrices, these are understood with respect to the maximum of their components.
- We adopt the common convention of denoting by $C, C_0, C_1, C_2, \ldots$ positive constants that may change from line to line. Whenever necessary, we specify their dependence on given parameters by means of parentheses.

Chapter 1

Preliminaries

In this section, we introduce some preliminary notions that are going to be instrumental in the following. Henceforth, $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain, where the spatial dimension $N \in \mathbb{N}$ satisfies $N \geq 2$. The exponent $p > N-1$ is fixed.

1.1 Lebesgue points

Let $\boldsymbol{y} \in W^{1,p}(\Omega;\mathbb{R}^N)$. We denote by $L_{\boldsymbol{y}}$ the set of Lebesgue points of $\boldsymbol{y}$. Namely, $\boldsymbol{x}_0 \in L_{\boldsymbol{y}}$ if there exists $\boldsymbol{y}^*(\boldsymbol{x}_0) \in \mathbb{R}^N$ such that

$$\lim_{r\to 0^+} \fint_{B(\boldsymbol{x}_0,r)} |\boldsymbol{y}(\boldsymbol{x}) - \boldsymbol{y}^*(\boldsymbol{x}_0)| \,\mathrm{d}\boldsymbol{x} = 0.$$

In this case, $\boldsymbol{y}^*(\boldsymbol{x}_0)$ is termed the Lebesgue value of $\boldsymbol{y}$ at $\boldsymbol{x}_0$ and is characterized as

$$\boldsymbol{y}^*(\boldsymbol{x}_0) = \lim_{r\to 0^+} \fint_{B(\boldsymbol{x}_0,r)} \boldsymbol{y}(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}.$$

It is proved that $L_{\boldsymbol{y}} \subset \Omega$ is a Borel set and $\mathscr{L}^N(\Omega \setminus L_{\boldsymbol{y}}) = 0$. Actually, we have a precise estimate for the Hausdorff dimension of this set.

Proposition 1.1 ([69, Theorem 2, Section 1.1, Chapter 3]). *Let $\boldsymbol{y}$ belong to $W^{1,p}(\Omega;\mathbb{R}^N)$. Then, for every $s > N-p$, there holds $\mathscr{H}^s(\Omega \setminus L_{\boldsymbol{y}}) = 0$. In particular, we have $\mathscr{H}^1(\Omega \setminus L_{\boldsymbol{y}}) = 0$.*

We will use the following consequence of the previous result.

Corollary 1.2. *Let $\boldsymbol{y} \in W^{1,p}(\Omega)$. Then, for every $\boldsymbol{x}_0 \in \Omega$ and for almost every $0 < r < \mathrm{dist}(\boldsymbol{x}_0;\partial\Omega)$, there holds $\partial B(\boldsymbol{x}_0,r) \subset L_{\boldsymbol{y}}$.*

Proof. For simplicity, set $\overline{r} := \mathrm{dist}(\boldsymbol{x}_0;\partial\Omega)$. By [59, Corollary 2.10.11], we have

$$\int_0^{\overline{r}} \#(\partial B(\boldsymbol{x}_0,r) \setminus L_{\boldsymbol{y}}) \,\mathrm{d}r \leq \mathscr{H}^1(B(\boldsymbol{x}_0,\overline{r}) \setminus L_{\boldsymbol{y}}),$$

where the right-hand side equals zero by Proposition 1.1. Therefore, we have $\#(\partial B(\boldsymbol{x}_0,r) \setminus L_{\boldsymbol{y}}) = 0$ for almost every $0 < r < \overline{r}$, which proves the claim. $\square$

1.2 Lusin properties

We recall the definitions of the Lusin properties.

Definition 1.3 (Lusin properties). Let $X \subset \mathbb{R}^N$ and $\boldsymbol{y}\colon X \to \mathbb{R}^N$ be measurable.

(i) The map $\boldsymbol{y}$ is termed to have the Lusin property (N) if, for every $A \subset X$ measurable with $\mathscr{L}^N(A) = 0$, there holds $\mathscr{L}^N(\boldsymbol{y}(A)) = 0$;

(ii) The map $\boldsymbol{y}$ is termed to have the Lusin property (N^{-1}) if, for every $B \subset \mathbb{R}^N$ measurable with $\mathscr{L}^N(B) = 0$, there holds $\mathscr{L}^N(\boldsymbol{y}^{-1}(B)) = 0$.

Some simple consequences of the Lusin properties are given in the next result.

Lemma 1.4. *Let $A \subset \mathbb{R}^N$ and $\boldsymbol{y}\colon A \to \mathbb{R}^N$ be measurable.*

(i) Let $\boldsymbol{y}$ have the Lusin property (N). Then, for every $X \subset A$ measurable, the set $\boldsymbol{y}(X)$ is measurable.

(ii) Let $\boldsymbol{y}$ have the Lusin property (N^{-1}) and let $\boldsymbol{\psi}\colon \mathbb{R}^N \to \mathbb{R}^N$ be measurable. Then, $\boldsymbol{\psi} \circ \boldsymbol{y}$ is measurable. Moreover, given $\widetilde{\boldsymbol{\psi}}\colon \mathbb{R}^N \to \mathbb{R}^N$ measurable such that $\widetilde{\boldsymbol{\psi}} = \boldsymbol{\psi}$ almost everywhere in $\mathbb{R}^N$, there holds $\widetilde{\boldsymbol{\psi}} \circ \boldsymbol{y} = \boldsymbol{\psi} \circ \boldsymbol{y}$ almost everywhere in A.

Proof. For the proof of (i), we refer to [25, Theorem 3.6.9]. For (ii), let $B \subset \mathbb{R}^N$ be a Borel set. Given the measurability of $\boldsymbol{\psi}$, the set $\boldsymbol{\psi}^{-1}(B)$ is measurable. Hence, there exists $C \subset \mathbb{R}^N$ Borel and $Y \subset \mathbb{R}^N$ measurable with $\mathscr{L}^N(Y) = 0$ such that $\boldsymbol{\psi}^{-1}(B) = C \cup Y$. The measurability of $\boldsymbol{y}$ entails the one of $\boldsymbol{y}^{-1}(C) \subset A$ while, in view of the Lusin property (N^{-1}), we have $\mathscr{L}^N(\boldsymbol{y}^{-1}(Y)) = 0$ and, in particular, the set $\boldsymbol{y}^{-1}(Y) \subset A$ is measurable. Therefore, $(\boldsymbol{\psi} \circ \boldsymbol{y})^{-1}(B) = \boldsymbol{y}^{-1}(C) \cup \boldsymbol{y}^{-1}(Y)$ is measurable and this proves the measurability of $\boldsymbol{\psi} \circ \boldsymbol{y}$. For the last claim, set $Z \coloneqq \{\widetilde{\boldsymbol{\psi}} \neq \boldsymbol{\psi}\}$. By assumption $\mathscr{L}^N(Z) = 0$ so that, thanks to the Lusin property (N^{-1}), there holds $\mathscr{L}^N(\boldsymbol{y}^{-1}(Z)) = 0$. Since $\widetilde{\boldsymbol{\psi}} \circ \boldsymbol{y} = \boldsymbol{\psi} \circ \boldsymbol{y}$ on $A \setminus \boldsymbol{y}^{-1}(Z)$, this proves the claim. ☐

1.3 Approximate differentiabiliy and Change-of-variable Formula

We briefly recall the notion of density and of approximate differentiability. For more information, we refer to [59, 69]. Given $A \subset \mathbb{R}^N$ measurable and $\boldsymbol{x}_0 \in \mathbb{R}^N$, the N-dimensional density of the set A at $\boldsymbol{x}_0$ is defined as the limit

$$\Theta^N(A, \boldsymbol{x}_0) \coloneqq \lim_{r \to 0^+} \frac{\mathscr{L}^N(A \cap B(\boldsymbol{x}_0, r))}{\omega_N r^N}$$

whenever it exists. Here, we set $\omega_N \coloneqq \mathscr{L}^N(B(\mathbf{0}, 1))$. Also, given $A \subset \mathbb{R}$ measurable and $x_0 \in \mathbb{R}$, we define the right 1-dimensional density of A at x_0 as

$$\Theta^1_+(A, x_0) \coloneqq \lim_{r \to 0^+} \frac{\mathscr{L}^1(A \cap (x_0, x_0 + r))}{r},$$

provided that this limit exists.

We recall the definition of approximate differentiability due to Federer [59].

Definition 1.5 (Approximate differentiability). Let $\boldsymbol{y}\colon \Omega \to \mathbb{R}^N$ be measurable and let $\boldsymbol{x}_0 \in \Omega$. The map $\boldsymbol{y}$ is termed approximately differentiable at $\boldsymbol{x}_0$ is there exists $\boldsymbol{F} \in \mathbb{R}^{N\times N}$ such that

$$\operatorname*{ap\,lim}_{\boldsymbol{x}\to\boldsymbol{x}_0} \frac{|\boldsymbol{y}(\boldsymbol{x}) - \boldsymbol{y}(\boldsymbol{x}_0) - \boldsymbol{F}(\boldsymbol{x}-\boldsymbol{x}_0)|}{|\boldsymbol{x}-\boldsymbol{x}_0|} = 0. \tag{1.1}$$

That is, for every $\eta > 0$, there holds

$$\Theta^N\left(\left\{\boldsymbol{x} \in \Omega \setminus \{\boldsymbol{x}_0\} : \frac{|\boldsymbol{y}(\boldsymbol{x}) - \boldsymbol{y}(\boldsymbol{x}_0) - \boldsymbol{F}(\boldsymbol{x}-\boldsymbol{x}_0)|}{|\boldsymbol{x}-\boldsymbol{x}_0|} < \eta\right\}, \boldsymbol{x}_0\right) = 1.$$

In this case, $\nabla\boldsymbol{y}(\boldsymbol{x}_0) \coloneqq \boldsymbol{F}$ is termed the approximate gradient of $\boldsymbol{y}$ at $\boldsymbol{x}_0$.

The approximate gradient is unique whenever it is defined . Also, $\boldsymbol{y}$ is approximately differentiable at $\boldsymbol{x}_0$ if there exists a measurable set $A \subset \Omega$ with $\Theta^N(A, \boldsymbol{x}_0) = 1$ such that $\boldsymbol{y}|_A$ is differentiable at $\boldsymbol{x}_0$ [69, Claim (iv), Definition 3, Section 1.4, Chapter 3]. In particular, if $\boldsymbol{y}$ is differentiable at $\boldsymbol{x}_0$, then $\boldsymbol{y}$ is approximately differentiable at the same point. It is proved that maps in $W^{1,1}(\Omega;\mathbb{R}^N)$ are almost everywhere approximately differentiable and that the approximate gradient agrees almost everywhere with the distributional gradient [69, Theorem 2, Section 1.4, Chapter 3].
As a consequence of the work of Federer, the Area Formula and the Change-of-variable Formula hold for almost everywhere approximately differentiable maps, see [69, Theorem 1, Section 1.5, Chapter 3] of [59, Corollary 3.2.20]. We will present this formula later in Proposition 1.9 in the exact form that we need. For the moment, we simply present the following consequence of the Area Formula.

Corollary 1.6 ([18, Lemma 2.8, Claim (c)]). *Let $X \subset \mathbb{R}^N$ and $\boldsymbol{y}\colon X \to \mathbb{R}^N$ be almost everywhere approximately differentiable with $\det\nabla\boldsymbol{y} \neq 0$ almost everywhere. Then, $\boldsymbol{y}$ has the Lusin property* (N^{-1}).

We introduce now the notion of geometric image. The following definition is taken from [78, Definition 3].

Definition 1.7 (Geometric domain and geometric image). Let $\boldsymbol{y}\colon \Omega \to \mathbb{R}^N$ be almost everywhere approximately differentiable with $\det\nabla\boldsymbol{y} \neq 0$ almost everywhere. We denote by $\mathrm{dom}_{\mathrm{G}}(\boldsymbol{y},\Omega)$ the set of points $\boldsymbol{x}_0 \in \Omega$ such that $\boldsymbol{y}$ is approximately differentiable in $\boldsymbol{x}_0$ with $\det\nabla\boldsymbol{y}(\boldsymbol{x}_0) \neq 0$ and there exist a compact set $K \subset \Omega$ with $\boldsymbol{x}_0 \in K$ and $\Theta^N(K,\boldsymbol{x}_0) = 1$, and a map $\boldsymbol{w} \in C^1(\mathbb{R}^N;\mathbb{R}^N)$ satisfying $\boldsymbol{w}|_K = \boldsymbol{y}|_K$ and $D\boldsymbol{w}|_K = \nabla\boldsymbol{y}|_K$. For every measurable set $A \subset \Omega$, we set

$$\mathrm{dom}_{\mathrm{G}}(\boldsymbol{y},A) \coloneqq A \cap \mathrm{dom}_{\mathrm{G}}(\boldsymbol{y},\Omega),$$

and we define the geometric image of A under $\boldsymbol{y}$ as

$$\mathrm{im}_{\mathrm{G}}(\boldsymbol{y},A) \coloneqq \boldsymbol{y}(\mathrm{dom}_{\mathrm{G}}(\boldsymbol{y},A)).$$

Note that the previous definition we also require that $\boldsymbol{x}_0 \in K$. This requirement is neither essential nor restrictive, but will be exploited in Lemma 1.36.
In the following result, we collect some basic properties of the geometric domain and the geometric image.

Lemma 1.8. *Let $\boldsymbol{y}\colon \Omega \to \mathbb{R}^N$ be almost everywhere approximately differentiable with $\det\nabla\boldsymbol{y} \neq 0$ almost everywhere and let $A \subset \Omega$ be measurable. Then, the following hold:*

(i) $\mathrm{dom}_{\mathrm{G}}(\boldsymbol{y},A)$ *is measurable and* $\mathscr{L}^N(A\setminus \mathrm{dom}_{\mathrm{G}}(\boldsymbol{y},A))=0$*;*

(ii) $\boldsymbol{y}|_{\mathrm{dom}_{\mathrm{G}}(\boldsymbol{y},A)}$ *has the Lusin property* (N) *and the set* $\mathrm{im}_{\mathrm{G}}(\boldsymbol{y},A)$ *is measurable;*

(iii) for every $\boldsymbol{x}_0\in\Omega$ *with* $\Theta^N(A,\boldsymbol{x}_0)=1$*, there holds* $\Theta^N(\mathrm{im}_{\mathrm{G}}(\boldsymbol{y},A),\boldsymbol{y}(\boldsymbol{x}_0))=1$*.*

Proof. Denote by $\Omega_{\mathrm{d}}(\boldsymbol{y})$ the set of points where $\boldsymbol{y}$ is approximately differentiable. In particular, $\mathscr{L}^N(\Omega\setminus\Omega_{\mathrm{d}}(\boldsymbol{y}))=0$. It is proved in [120, Lemma 2.5] that

$$\mathscr{L}^N(\Omega_{\mathrm{d}}(\boldsymbol{y})\setminus \mathrm{dom}_{\mathrm{G}}(\boldsymbol{y},\Omega))=0,$$

so that claim (i) follows. For (ii), the Lusin property (N) follows from the fact that the same property holds for $\boldsymbol{y}|_{\Omega_{\mathrm{d}}(\boldsymbol{y})}$, see [69, Proposition 1, Section 1.5, Chapter 3]. Also, this property entails the measurability of the geometric image by claim (i) of Lemma 1.4. Claim (iii) is proved as in [120, Lemma 2.5]. □

The notions of gometric domain and geometric image clearly depend on the representative of $\boldsymbol{y}$. However, if $\boldsymbol{y}=\widetilde{\boldsymbol{y}}$ almost everywhere, then

$$\mathscr{L}^N(\mathrm{dom}_{\mathrm{G}}(\boldsymbol{y},A)\,\triangle\,\mathrm{dom}_{\mathrm{G}}(\widetilde{\boldsymbol{y}},A))=0,\quad \mathscr{L}^N(\mathrm{im}_{\mathrm{G}}(\boldsymbol{y},A)\,\triangle\,\mathrm{im}_{\mathrm{G}}(\widetilde{\boldsymbol{y}},A))=0,$$

thanks to claims (i)–(ii) of Lemma 1.8.
We present the change-of-variable formula in the following form.

Proposition 1.9 (Change-of-variable Formula). *Let* $\boldsymbol{y}\colon\Omega\to\mathbb{R}^N$ *be almost everywhere approximately differentiable with* $\det\nabla\boldsymbol{y}\neq 0$ *almost everywhere. For every measurable set* $A\subset\Omega$*, the multiplicity function* $\mathrm{mult}(\boldsymbol{y},A,\cdot)\colon\mathbb{R}^N\to\mathbb{N}\cup\{0\}$ *defined by*

$$\mathrm{mult}(\boldsymbol{y},A,\boldsymbol{\xi}):=\#\{\boldsymbol{x}\in\mathrm{dom}_{\mathrm{G}}(\boldsymbol{y},A):\ \boldsymbol{y}(\boldsymbol{x})=\boldsymbol{\xi}\}$$

is measurable. Also, for every measurable function $\boldsymbol{\psi}\colon\mathrm{im}_{\mathrm{G}}(\boldsymbol{y},A)\to\mathbb{R}^M$*, there holds*

$$\int_A \boldsymbol{\psi}\circ\boldsymbol{y}(\boldsymbol{x})\,\det\nabla\boldsymbol{y}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}=\int_{\mathrm{im}_{\mathrm{G}}(\boldsymbol{y},A)}\boldsymbol{\psi}(\boldsymbol{\xi})\,\mathrm{mult}(\boldsymbol{y},A,\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}$$

whenever one of the two integrals exists.

Note that the map $\boldsymbol{y}$ has the Lusin property (N^{-1}) by Corollary 1.6 and, in turn, the composition $\boldsymbol{\psi}\circ\boldsymbol{y}$ is measurable by claim (ii) of Lemma 1.4.

Proof. Again, denote by $\Omega_{\mathrm{d}}(\boldsymbol{y})$ the set of points of approximate differentiability of $\boldsymbol{y}$. By the Federer Change-of-variable Formula [69, Theorem 1, Section 1.5, Chapter 3], we have

$$\int_A \boldsymbol{\psi}\circ\boldsymbol{y}(\boldsymbol{x})\,\det\nabla\boldsymbol{y}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}=\int_{\boldsymbol{y}(A\cap\Omega_{\mathrm{d}}(\boldsymbol{y}))}\boldsymbol{\psi}(\boldsymbol{\xi})\,\#\{\boldsymbol{x}\in A\cap\Omega_{\mathrm{d}}(\boldsymbol{y}):\ \boldsymbol{y}(\boldsymbol{x})=\boldsymbol{\xi}\}\,\mathrm{d}\boldsymbol{\xi}.$$

First, recall that $\boldsymbol{y}|_{\Omega_{\mathrm{d}}(\boldsymbol{y})}$ has the Lusin property (N). Thus

$$\mathscr{L}^N(\boldsymbol{y}(A\cap\Omega_{\mathrm{d}}(\boldsymbol{y})))=\mathscr{L}^N(\mathrm{im}_{\mathrm{G}}(\boldsymbol{y},A))$$

since $\mathscr{L}^N(\Omega_{\mathrm{d}}(\boldsymbol{y})\setminus\mathrm{dom}_{\mathrm{G}}(\boldsymbol{y},\Omega))=0$. Second, note that

$$\#\{\boldsymbol{x}\in A\cap\Omega_{\mathrm{d}}(\boldsymbol{y}):\ \boldsymbol{y}(\boldsymbol{x})=\boldsymbol{\xi}\}=\mathrm{mult}(\boldsymbol{y},A,\boldsymbol{\xi})$$

for every $\boldsymbol{\xi}\in\mathrm{im}_{\mathrm{G}}(\boldsymbol{y},A)\setminus\boldsymbol{y}((A\cap\Omega_{\mathrm{d}}(\boldsymbol{y}))\setminus\mathrm{dom}_{\mathrm{G}}(\boldsymbol{y},A))$. Hence, the desired formula follows in view of these two observations. □

We recall the notion of regular approximate differentiability [71].

Definition 1.10 (Regular approximate differentiability). Let $\boldsymbol{y}\colon \Omega \to \mathbb{R}^N$ be measurable and let $\boldsymbol{x}_0 \in \Omega$. The map $\boldsymbol{y}$ is termed regularly approximately differentiable at $\boldsymbol{x}_0$ if there exists $\boldsymbol{F} \in \mathbb{R}^{N\times N}$ such that

$$\operatorname*{ap\,lim}_{r\to 0^+} \operatorname*{ess\,sup}_{\boldsymbol{x}\in\partial B(\boldsymbol{x}_0,r)} \frac{|\boldsymbol{y}(\boldsymbol{x})-\boldsymbol{y}(\boldsymbol{x}_0)-\boldsymbol{F}(\boldsymbol{x}-\boldsymbol{x}_0)|}{|\boldsymbol{x}-\boldsymbol{x}_0|}=0.$$

That is, for every $\eta>0$, there holds

$$\Theta^1_+\left(\left\{r>0:\operatorname*{ess\,sup}_{\boldsymbol{x}\in\partial B(\boldsymbol{x}_0,r)} \frac{|\boldsymbol{y}(\boldsymbol{x})-\boldsymbol{y}(\boldsymbol{x}_0)-\boldsymbol{F}(\boldsymbol{x}-\boldsymbol{x}_0)|}{|\boldsymbol{x}-\boldsymbol{x}_0|}<\eta\right\},0\right)=1.$$

If this case, $\nabla\boldsymbol{y}(\boldsymbol{x}_0)\coloneqq\boldsymbol{F}$ is termed the regular approximate gradient of $\boldsymbol{y}$ at $\boldsymbol{x}_0$.

In the situation of the previous definition, there exists $R\subset(0,\operatorname{dist}(\boldsymbol{x}_0;\partial\Omega))$ measurable with $\Theta^1_+(R,0)=1$ such that

$$\lim_{\substack{r\to 0\\ r\in R}} \operatorname*{ess\,sup}_{\boldsymbol{x}\in\partial B(\boldsymbol{x}_0,r)} \frac{|\boldsymbol{y}(\boldsymbol{x})-\boldsymbol{y}(\boldsymbol{x}_0)-\nabla\boldsymbol{y}(\boldsymbol{x}_0)(\boldsymbol{x}-\boldsymbol{x}_0)|}{|\boldsymbol{x}-\boldsymbol{x}_0|}=0.$$

Note that, in these formulas, the essential supremum is meant with respect to the $(N-1)$-dimensional Hausdorff measure. The regular approximate gradient is uniquely defined whenever it exists and, in this case, it coincides with the approximate gradient of $\boldsymbol{y}$ at $\boldsymbol{x}_0$. This justifies the notation. To see this, for every $r\in R$, let $T_r\subset\partial B(\boldsymbol{x}_0,r)$ with $\mathscr{H}^{N-1}(T_r)=0$ be such that

$$\begin{aligned}&\operatorname*{ess\,sup}_{\boldsymbol{x}\in\partial B(\boldsymbol{x}_0,r)} \frac{|\boldsymbol{y}(\boldsymbol{x})-\boldsymbol{y}(\boldsymbol{x}_0)-\nabla\boldsymbol{y}(\boldsymbol{x}_0)(\boldsymbol{x}-\boldsymbol{x}_0)|}{|\boldsymbol{x}-\boldsymbol{x}_0|}\\ &=\sup_{\boldsymbol{x}\in\partial B(\boldsymbol{x}_0,r)\setminus T_r} \frac{|\boldsymbol{y}(\boldsymbol{x})-\boldsymbol{y}(\boldsymbol{x}_0)-\nabla\boldsymbol{y}(\boldsymbol{x}_0)(\boldsymbol{x}-\boldsymbol{x}_0)|}{|\boldsymbol{x}-\boldsymbol{x}_0|}\end{aligned}$$

and set $A\coloneqq\bigcup\{\partial B(\boldsymbol{x}_0,r)\setminus T_r:\ r\in R\}$. Then, $\Theta^N(A,\boldsymbol{x}_0)=1$ and $\boldsymbol{y}|_A$ is differentiable at $\boldsymbol{x}_0$, which proves the claim.
Note that Definition 1.10 is slightly different from the definition of regular approximate differentiability available in the literature, see [71, Section 3] or [120, Definition 8.2]. Specifically, in (1.10), we use the essential supremum instead of the supremum. With this choice, as stated in the following result, any map in $W^{1,p}(\Omega;\mathbb{R}^N)$ is almost everywhere regularly approximately differentiable independently on the representative chosen.

Proposition 1.11. *Let $\boldsymbol{y}\in W^{1,p}(\Omega;\mathbb{R}^N)$. Then, $\boldsymbol{y}$ is almost everywhere regularly approximately differentiable with $\nabla\boldsymbol{y}=D\boldsymbol{y}$ almost everywhere.*

Proof. Recall Corollary 1.2. By [71, Section 3], for almost every $\boldsymbol{x}_0\in L_{\boldsymbol{y}}$, there exists $R^*\subset(0,\operatorname{dist}(\boldsymbol{x}_0;\partial\Omega))$ with $\Theta^1_+(R^*,0)=1$ such that $\partial B(\boldsymbol{x}_0,r)\subset L_{\boldsymbol{y}}$ for every $r\in R^*$ and there holds

$$\lim_{\substack{r\to 0\\ r\in R^*}} \sup_{\boldsymbol{x}\in\partial B(\boldsymbol{x}_0,r)} \frac{|\boldsymbol{y}^*(\boldsymbol{x})-\boldsymbol{y}^*(\boldsymbol{x}_0)-\nabla\boldsymbol{y}^*(\boldsymbol{x}_0)(\boldsymbol{x}-\boldsymbol{x}_0)|}{|\boldsymbol{x}-\boldsymbol{x}_0|}=0.$$

Set $R \coloneqq \{r \in R^* : \boldsymbol{y} = \boldsymbol{y}^* \text{ a.e. on } \partial B(\boldsymbol{x}_0, r)\}$. In this case, $\mathscr{L}^1(R) = \mathscr{L}^1(R^*)$ and, in turn, $\Theta^1_+(R,0) = \Theta^1_+(R^*,0) = 1$. Also, if $\boldsymbol{y}(\boldsymbol{x}_0) = \boldsymbol{y}^*(\boldsymbol{x}_0)$, then we have

$$\begin{aligned}\lim_{\substack{r\to 0\\ r\in R}} \operatorname*{ess\,sup}_{\boldsymbol{x}\in\partial B(\boldsymbol{x}_0,r)} &\frac{|\boldsymbol{y}(\boldsymbol{x}) - \boldsymbol{y}(\boldsymbol{x}_0) - \nabla \boldsymbol{y}^*(\boldsymbol{x}_0)(\boldsymbol{x}-\boldsymbol{x}_0)|}{|\boldsymbol{x}-\boldsymbol{x}_0|} \\ &\leq \lim_{\substack{r\to 0\\ r\in R}} \sup_{\boldsymbol{x}\in\partial B(\boldsymbol{x}_0,r)} \frac{|\boldsymbol{y}^*(\boldsymbol{x}) - \boldsymbol{y}^*(\boldsymbol{x}_0) - \nabla \boldsymbol{y}^*(\boldsymbol{x}_0)(\boldsymbol{x}-\boldsymbol{x}_0)|}{|\boldsymbol{x}-\boldsymbol{x}_0|} = 0.\end{aligned}$$

Thus, $\boldsymbol{y}$ is regularly approximately differentiable at $\boldsymbol{x}_0$ with $\nabla\boldsymbol{y}(\boldsymbol{x}_0) = \nabla \boldsymbol{y}^*(\boldsymbol{x}_0)$. As

$$\mathscr{L}^N(\Omega \setminus \{\boldsymbol{x}_0 \in L_{\boldsymbol{y}} :\ \boldsymbol{y}(\boldsymbol{x}_0) = \boldsymbol{y}^*(\boldsymbol{x}_0),\ D\boldsymbol{y}(\boldsymbol{x}_0) = \nabla\boldsymbol{y}^*(\boldsymbol{x}_0)\}) = 0,$$

this concludes the proof. □

1.4 Convergence of Sobolev maps on boundaries

In this section, we deal with Sobolev maps on the boundary of smooth domains. We start with some geometric preliminaries. Let $U \subset\subset \Omega$ be a domain of class C^2. In this case, ∂U is a compact hypersurface of class C^2 without boundary. The signed distance function $d_U \colon \mathbb{R}^N \to \mathbb{R}$ is defined by setting

$$d_U(\boldsymbol{x}) \coloneqq \begin{cases} \operatorname{dist}(\boldsymbol{x};\partial U), & \boldsymbol{x} \in U, \\ 0, & \boldsymbol{x}\in\partial U, \\ -\operatorname{dist}(\boldsymbol{x};\partial U), & \boldsymbol{x}\in\mathbb{R}^N\setminus\overline{U}. \end{cases}$$

For every $\ell \in \mathbb{R}$, we set $U_\ell \coloneqq \{\boldsymbol{x}\in\mathbb{R}^N :\ d_U(\boldsymbol{x}) > \ell\}$. As d_U is continuous, the set U_ℓ is open and $\partial U_\ell = \{\boldsymbol{x}\in\mathbb{R}^N :\ d_U(\boldsymbol{x}) = \ell\}$. In particular, we have $U \subset\subset U_\ell$ for $\ell<0$, $U_\ell \subset\subset U$ for $\ell > 0$ and $U_0 = U$. Given $\delta>0$, we consider the tubular neighborhood

$$T(\partial U,\delta) \coloneqq \{\boldsymbol{x}\in\mathbb{R}^N :\ -\delta < d_U(\boldsymbol{x}) < \delta\}.$$

For $\delta \ll 1$ depending on U, the following properties hold [3, Section 4]:

(i) the function d_U is of class C^2 on $T(\partial U,\delta)$;

(ii) the projection onto the boundary $\boldsymbol{\Pi}_U \colon T(\partial U,\delta)\to\partial U$ given by

$$\boldsymbol{\Pi}_U(\boldsymbol{x}) \coloneqq \operatorname*{argmin}_{\boldsymbol{z}\in\partial U} |\boldsymbol{x}-\boldsymbol{z}|$$

is well defined;

(iii) for every $\boldsymbol{x}\in T(\partial U,\delta)$ there hold

$$\boldsymbol{x} = \boldsymbol{\Pi}_U(\boldsymbol{x}) + d_U(\boldsymbol{x})\boldsymbol{n}_U(\boldsymbol{\Pi}_U(\boldsymbol{x})), \qquad Dd_U(\boldsymbol{x}) = -\boldsymbol{n}_U(\boldsymbol{\Pi}_U(\boldsymbol{x})),$$

and $\boldsymbol{n}_{U_\ell}(\boldsymbol{x}) = \boldsymbol{n}_U(\boldsymbol{\Pi}_U(\boldsymbol{x}))$ whenever $\boldsymbol{x}\in\partial U_\ell$.

The usual definition of Sobolev maps on the boundary of domains with Lipschitz regularity is given in terms of local charts. In our situation, where we are dealing with smooth domains, it is possible to give an equivalent definition which is analogous to the one of Sobolev maps on domains. This definition precisely requires regularity

of class C^2 since it is based on the Integration-by-parts Formula on hypersurfaces which involves the concept of mean curvature. For convenience of the reader, we recall this definition in the Appendix.
Let $\boldsymbol{y} \in W^{1,p}(\Omega;\mathbb{R}^N)$. Let $U \subset\subset \Omega$ be a domain of class C^2 and suppose that $\boldsymbol{y}|_{\partial U} \in W^{1,p}(\partial U;\mathbb{R}^N)$. In this case, as $p > N-1$, the map $\boldsymbol{y}|_{\partial U}$ admits a representative in $C^0(\partial U;\mathbb{R}^N)$ by the Morrey embedding.

Convention 1.12. Let $\boldsymbol{y} \in W^{1,p}(\Omega;\mathbb{R}^N)$ with $p > N-1$ and let $U \subset\subset \Omega$ be a domain of class C^2 such that $\boldsymbol{y}|_{\partial U} \in W^{1,p}(\partial U;\mathbb{R}^N)$. In this case, we identify $\boldsymbol{y}|_{\partial U}$ with its representative in $C^0(\partial U;\mathbb{R}^N)$. Also, we will often omit the restriction to ∂U in the notation.

The next result is taken from [120] and will be instrumental in our analysis.

Lemma 1.13 ([120, Lemma 2.9]). *Let $(\boldsymbol{y}_n) \subset W^{1,p}(\Omega;\mathbb{R}^N)$ and $\boldsymbol{y} \in W^{1,p}(\Omega;\mathbb{R}^N)$. Suppose that $\boldsymbol{y}_n \rightharpoonup \boldsymbol{y}$ in $W^{1,p}(\Omega;\mathbb{R}^N)$. Let $U \subset\subset \Omega$ be a domain of class C^2. Then, there exists $\delta > 0$ such that for almost every $\ell \in (-\delta,\delta)$ there holds*

$$\forall\, n \in \mathbb{N}, \quad \boldsymbol{y}_n|_{\partial U_\ell} \in W^{1,p}(\partial U_\ell;\mathbb{R}^N), \quad \boldsymbol{y}|_{\partial U_\ell} \in W^{1,p}(\partial U_\ell;\mathbb{R}^N). \tag{1.2}$$

Moreover, for any such $\ell \in (-\delta,\delta)$, there exists a subsequence $(\boldsymbol{y}_{n_k})$, possibly depending on ℓ, such that

$$\boldsymbol{y}_{n_k} \rightharpoonup \boldsymbol{y} \text{ in } W^{1,p}(\partial U_\ell;\mathbb{R}^N), \quad \boldsymbol{y}_{n_k} \to \boldsymbol{y} \text{ uniformly on } \partial U_\ell. \tag{1.3}$$

We stress that in (1.3) we adopt Convention 1.12.

Proof. Let $\delta > 0$ be such that the tubular neighborhood $T(\partial U,\delta)$ is defined. Claim (1.2) is established by mollification and by applying the Coarea Formula [4, Equation (2.74)] with the signed distance function. By the Rellich embedding, up to subsequences, we have $\boldsymbol{y}_n \to \boldsymbol{y}$ in $L^p(T(\partial U,\delta);\mathbb{R}^N)$. Hence, by the Coarea Formula [4, Equation (2.74)], we have

$$\boldsymbol{y}_n \to \boldsymbol{y} \text{ in } L^p(\partial U_\ell;\mathbb{R}^N) \text{ for every } \ell \in (-\delta,\delta) \setminus Z_1, \tag{1.4}$$

for some $Z_1 \subset (-\delta,\delta)$ with $\mathscr{L}^1(Z_1) = 0$. For every $n \in \mathbb{N}$, define $f_n\colon (-\delta,\delta) \to \mathbb{R}$ and $f\colon (-\delta,\delta) \to \mathbb{R}$ by setting

$$f_n(\ell) := \int_{\partial U_\ell} |D^{\partial U_\ell} \boldsymbol{y}_n|^p \, \mathrm{d}\boldsymbol{a}, \qquad f(\ell) := \liminf_{n\to\infty} f_n(\ell).$$

By the Fatou Lemma and by the Coarea Formula [4, Equation (2.74)], $f \in L^1(-\delta,\delta)$ and, in turn, $f(\ell) < +\infty$ for every $\ell \in (-\delta,\delta) \setminus Z_2$ for some $Z_2 \subset (-\delta,\delta)$ with $\mathscr{L}^1(Z_2) = 0$. Fix $\ell \in (-\delta,\delta) \setminus (Z_1 \cup Z_2)$ and select a subsequence indexed by (n_k), possibly depending on ℓ, such that

$$\lim_{k\to\infty} f_{n_k}(\ell) = f(\ell) < +\infty.$$

Thus, for $k \gg 1$, the sequence $(D^{\partial U_\ell} \boldsymbol{y}_{n_k}) \subset L^p(\partial U_\ell;\mathbb{R}^{N\times N})$ is bounded so that, up to subsequences, we have

$$D^{\partial U_\ell} \boldsymbol{y}_{n_k} \rightharpoonup \boldsymbol{F} \text{ in } L^p(\partial U_\ell;\mathbb{R}^{N\times N}) \tag{1.5}$$

for some $\boldsymbol{F} \in L^p(\partial U_\ell; \mathbb{R}^{N\times N})$. Now, given (1.2), for every $\boldsymbol{\Phi} \in C^1(\partial U_\ell; \mathbb{R}^{N\times N})$ and $k \in \mathbb{N}$ the Integration-by-parts Formula yields

$$-\int_{\partial U_\ell} \boldsymbol{y}_{n_k} \cdot \mathrm{div}^{\partial U_\ell} \boldsymbol{\Phi} \, \mathrm{d}\boldsymbol{a} = \int_{\partial U_\ell} D^{\partial U_\ell} \boldsymbol{y}_{n_k} : \boldsymbol{\Phi} \, \mathrm{d}\boldsymbol{a} - \int_{\partial U_\ell} H_{U_\ell} \, \boldsymbol{y}_{n_k} \cdot \boldsymbol{\Phi} \boldsymbol{n}_{U_\ell} \, \mathrm{d}\boldsymbol{a},$$

where H_{U_ℓ} denotes the scalar mean curvature of ∂U_ℓ. Passing to the limit, as $k \to \infty$, in the previous equation with the aid of (1.4)–(1.5), we deduce that $\boldsymbol{F} = D^{\partial U_\ell} \boldsymbol{y}$. Therefore, $\boldsymbol{y}_{n_k} \rightharpoonup \boldsymbol{y}$ in $W^{1,p}(\partial U_\ell; \mathbb{R}^N)$. Finally, since $p > N-1$, we have $\boldsymbol{y}_{n_k} \to \boldsymbol{y}$ uniformly on ∂U_ℓ by the Morrey embedding. □

Remark 1.14. Let $\boldsymbol{y} \in W^{1,p}(\Omega; \mathbb{R}^N)$ and let $\boldsymbol{x}_0 \in \Omega$. As in Lemma 1.13, for almost every $r \in (0, \mathrm{dist}(\boldsymbol{x}_0; \partial\Omega))$, we have $\boldsymbol{y}|_{\partial B(\boldsymbol{x}_0, r)} \in W^{1,p}(\partial B(\boldsymbol{x}_0, r); \mathbb{R}^N)$. In view of Corollary 1.2, without loss of generality, we can assume that $\partial B(\boldsymbol{x}_0, r) \subset L_{\boldsymbol{y}}$. In this case, by exploiting [120, Claim (iii), Property 2.8] it is possible to show that the continuous representative of $\boldsymbol{y}|_{\partial B(\boldsymbol{x}_0, r)}$ is given by $\boldsymbol{y}^*|_{\partial B(\boldsymbol{x}_0, r)}$.

1.5 Topological degree

In this section, we recall the notion of the topological degree which represents an essential tool for our analysis. We begin with the case of continuous maps and we collect some of its properties. Then, we present the case of Sobolev maps.

1.5.1 Topological degree of continuous maps

Let $U \subset\subset \mathbb{R}^N$ be open and let $\boldsymbol{y} \in C^0(\overline{U}; \mathbb{R}^N)$. The topological degree of $\boldsymbol{y}$ on U is a map $\deg(\boldsymbol{y}, U, \cdot) \colon \mathbb{R}^N \setminus \boldsymbol{y}(\partial U) \to \mathbb{Z}$ which can be defined axiomatically by means of the following properties [124, page 39]:

(i) Normalization: for every $\boldsymbol{\xi} \in U$, there holds $\deg(\boldsymbol{id}|_U, U, \boldsymbol{\xi}) = 1$;

(ii) Additivity: given $U_1, U_2 \subset U$ open with $U_1 \cap U_2 = \emptyset$, for every $\boldsymbol{\xi} \in \mathbb{R}^N \setminus \boldsymbol{y}(\partial U_1 \cup \partial U_2)$, there holds

$$\deg(\boldsymbol{y}, U_1 \cup U_2, \boldsymbol{\xi}) = \deg(\boldsymbol{y}, U_1, \boldsymbol{\xi}) + \deg(\boldsymbol{y}, U_2, \boldsymbol{\xi});$$

(iii) Homotopy invariance: given $\boldsymbol{H} \in C^0([0,1] \times \overline{U}; \mathbb{R}^N)$ and $\boldsymbol{\gamma} \in [0,1] \to \mathbb{R}^N$, for every $t \in [0,1]$ satisfying $\boldsymbol{\gamma}(t) \notin \boldsymbol{H}(\{t\} \times \partial U)$, there holds

$$\deg(\boldsymbol{H}(0, \cdot), U, \boldsymbol{\gamma}(0)) = \deg(\boldsymbol{H}(t, \cdot), U, \boldsymbol{\gamma}(t));$$

(iv) Solvability: for every $\boldsymbol{\xi} \in \mathbb{R}^N \setminus \boldsymbol{y}(\partial U)$ with $\deg(\boldsymbol{y}, U, \boldsymbol{\xi}) \neq 0$, there exists $\boldsymbol{x} \in U$ such that $\boldsymbol{\xi} = \boldsymbol{y}(\boldsymbol{x})$.

Starting from these properties, several others can be deduced. For example, we have the following:

(v) Continuity: the map $\deg(\boldsymbol{y}, U, \cdot)$ is continuous and hence constant on each connected component of $\mathbb{R}^N \setminus \boldsymbol{y}(\partial U)$;

(vi) Stability: given $\widetilde{\boldsymbol{y}} \in C^0(\overline{U};\mathbb{R}^N)$, for every $\boldsymbol{\xi} \in \mathbb{R}^N \setminus \boldsymbol{y}(\partial U)$ satisfying

$$||\boldsymbol{y} - \widetilde{\boldsymbol{y}}||_{C^0(\overline{U};\mathbb{R}^N)} \leq \mathrm{dist}(\boldsymbol{\xi}; \boldsymbol{y}(\partial U)),$$

there hold $\boldsymbol{\xi} \notin \widetilde{\boldsymbol{y}}(\partial U)$ and $\deg(\widetilde{\boldsymbol{y}}, U, \boldsymbol{\xi}) = \deg(\boldsymbol{y}, U, \boldsymbol{\xi})$;

(vii) Dependence on boundary values: given $\widetilde{\boldsymbol{y}} \in C^0(\overline{U};\mathbb{R}^N)$ such that $\widetilde{\boldsymbol{y}}|_{\partial U} = \boldsymbol{y}|_{\partial U}$, there holds $\deg(\widetilde{\boldsymbol{y}}, U, \cdot) = \deg(\boldsymbol{y}, U, \cdot)$;

(viii) Compact support: for every $\boldsymbol{\xi} \in \mathbb{R}^N \setminus \boldsymbol{y}(\overline{U})$, there holds $\deg(\boldsymbol{y}, U, \boldsymbol{\xi}) = 0$.

For a more constructive definition of the topological degree of a continuous map, we refer to [60]. The topological degree of regular maps can be computed explicitly. Precisely, given $\boldsymbol{y} \in C^1(\overline{U};\mathbb{R}^N)$, for every $\boldsymbol{\xi} \in \mathbb{R}^N \setminus \boldsymbol{y}(\partial U)$ with $\boldsymbol{\xi} \notin \boldsymbol{y}(\{\det D\boldsymbol{y} = 0\})$, the following formula holds:

$$\deg(\boldsymbol{y}, U, \boldsymbol{\xi}) = \sum_{\boldsymbol{x} \in U:\, \boldsymbol{y}(\boldsymbol{x}) = \boldsymbol{\xi}} \mathrm{sgn} \det D\boldsymbol{y}(\boldsymbol{x}).$$

Also, for every $\boldsymbol{\xi} \in \mathbb{R}^N \setminus \boldsymbol{y}(\partial U)$, there holds

$$\deg(\boldsymbol{y}, U, \boldsymbol{\xi}) = \int_U \psi \circ \boldsymbol{y} \, \det D\boldsymbol{y} \, \mathrm{d}\boldsymbol{x}, \tag{1.6}$$

where $\psi \in C^0_{\mathrm{c}}(\mathbb{R}^N)$ is any positive function with integral equal to one which is supported in the connected component of $\mathbb{R}^N \setminus \boldsymbol{y}(\partial U)$ containting $\boldsymbol{\xi}$.
The property of dependence on boundary values allows us to define the topological degree for maps $\boldsymbol{y} \in C^0(\partial U;\mathbb{R}^N)$. Indeed, by the Tiezte Theorem, any such map admists an extension $\boldsymbol{Y} \in C^0(\overline{U};\mathbb{R}^N)$ with $\|\boldsymbol{Y}\|_{C^0(\overline{U};\mathbb{R}^N)} = \|\boldsymbol{y}\|_{C^0(\partial U;\mathbb{R}^N)}$. In this case, we define the topological degree of $\boldsymbol{y}$ on U by setting $\deg(\boldsymbol{y}, U, \cdot) := \deg(\boldsymbol{Y}, U, \cdot)$. The well posedness of this definition comes from the abovementioned property. Also, the norm-preserving feature of the extension provided by the Tiezte Theorem ensures that the stability property still holds true for the topological degree of maps in $C^0(\partial U;\mathbb{R}^N)$, see [18, Lemma 3.1].

Definition 1.15 (Topological image). Let $\boldsymbol{y} \in C^0(\partial U;\mathbb{R}^N)$. We define the topological image of U via $\boldsymbol{y}$ by setting

$$\mathrm{im}_{\mathrm{T}}(\boldsymbol{y}, U) := \left\{ \boldsymbol{\xi} \in \mathbb{R}^N \setminus \boldsymbol{y}(\partial U) : \ \deg(\boldsymbol{y}, U, \boldsymbol{\xi}) \neq 0 \right\}.$$

Since $\deg(\boldsymbol{y}, U, \cdot)$ has compact support, we deduce that the topological image is a bounded set. Also, by the continuity of the degree, this set is open and there holds $\partial\, \mathrm{im}_{\mathrm{T}}(\boldsymbol{y}, U) \subset \boldsymbol{y}(\partial U)$. In the next two lemmas, we collect some elementary observations.

Lemma 1.16. *Let $U \subset\subset \Omega$ be open and let $\boldsymbol{y} \in C^0(\partial U;\mathbb{R}^N)$. Let $O \subset\subset \mathbb{R}^N$ be open and such that $\mathbb{R}^N \setminus O$ is connected. Suppose that $\boldsymbol{y}(\partial U) \subset O$. Then, $\mathrm{im}_{\mathrm{T}}(\boldsymbol{y}, U) \cup \boldsymbol{y}(\partial U) \subset O$.*

Proof. Since $\mathbb{R}^N \setminus O \subset \mathbb{R}^N \setminus \boldsymbol{y}(\partial U)$ is connected and unbounded, by maximality, there holds $\mathbb{R}^N \setminus O \subset V$, where V is the unique unbounded component of $\mathbb{R}^N \setminus \boldsymbol{y}(\partial U)$. Thus, $\mathrm{im}_{\mathrm{T}}(\boldsymbol{y}, U) \subset \mathbb{R}^N \setminus V \subset O$. □

Lemma 1.17. *Let $U \subset\subset \Omega$ be open and let $\boldsymbol{y} \in C^0(\partial U;\mathbb{R}^N)$. Also, let $\{V_i : i \in \mathbb{N}\}$ be the collection of the connected components of $\mathbb{R}^N \setminus \boldsymbol{y}(\partial U)$. Then:*

(i) for every $i \in \mathbb{N}$, there holds $\partial V_i \subset \boldsymbol{y}(\partial U)$;

(ii) if ∂U is connected, then, for every $I \subset \mathbb{N}$, the set $\bigcup_{i\in I} V_i \cup \boldsymbol{y}(\partial U)$ is connected and, in particular, $\mathrm{im_T}(\boldsymbol{y},U) \cup \boldsymbol{y}(\partial U)$ and $\mathbb{R}^N \setminus \mathrm{im_T}(\boldsymbol{y},U)$ are connected.

Proof. To prove (i), suppose that there exists $\boldsymbol{\xi} \in \partial V_i \setminus \boldsymbol{y}(\partial U)$. Then, $\boldsymbol{\xi} \in V_j$ for some $j \in \mathbb{N}$ with $j \neq i$. As V_j is open, $B(\boldsymbol{\xi},r) \subset V_j$ for some $r > 0$. However, as $\boldsymbol{\xi} \in \partial V_i$, there holds $B(\boldsymbol{\xi},r) \cap V_i \neq \emptyset$ and, in turn, $V_i \cap V_j \neq \emptyset$. This provides a contradiction. To prove (ii), note that, if ∂U is connected, then $\boldsymbol{y}(\partial U)$ is connected. For every $i \in \mathbb{N}$, the set $\overline{V}_i$ is connected being the closure of a connected set. By (i), we have $\overline{V}_i \cap \boldsymbol{y}(\partial U) = \partial V_i$, so that $V_i \cup \boldsymbol{y}(\partial U) = \overline{V}_i \cup \boldsymbol{y}(\partial U)$ is connected being the union of two connected set with nonempty intersection. For the same reason, given $I \subset \mathbb{N}$, the set $\bigcup_{i\in I} V_i \cup \boldsymbol{y}(\partial U)$ is connected since $\bigcap_{i\in I}(V_i \cap \boldsymbol{y}(\partial U)) = \boldsymbol{y}(\partial U)$. □

The continuity of the topological degree allows to describe the asymptotic behaviour of topological images of uniformly converging sequences of maps.

Lemma 1.18. *Let $U \subset\subset \mathbb{R}^N$ be open and let $(\boldsymbol{y}_n) \subset C^0(\partial U;\mathbb{R}^N)$ and $\boldsymbol{y} \in C^0(\partial U;\mathbb{R}^N)$. Suppose that $\boldsymbol{y}_n \to \boldsymbol{y}$ uniformly on ∂U and that $\mathscr{L}^N(\boldsymbol{y}(\partial U)) = 0$. Then:*

(i) for every $K \subset \mathrm{im_T}(\boldsymbol{y},U)$ compact, there holds $K \subset \mathrm{im_T}(\boldsymbol{y}_n,U)$ for $n \gg 1$ depending on K;

(ii) for every $K \subset \mathbb{R}^N \setminus (\mathrm{im_T}(\boldsymbol{y},U) \cup \boldsymbol{y}(\partial U))$ compact, there holds $K \subset \mathbb{R}^N \setminus (\mathrm{im_T}(\boldsymbol{y}_n,U) \cup \boldsymbol{y}_n(\partial U))$ for $n \gg 1$ depending on K;

(iii) for every $O \subset\subset \mathbb{R}^N$ open with $\mathbb{R}^N \setminus O$ connected such that $\boldsymbol{y}(\partial U) \subset O$, there hold $\mathrm{im_T}(\boldsymbol{y},U) \subset O$ and $\mathrm{im_T}(\boldsymbol{y}_n,U) \cup \boldsymbol{y}(\partial U) \subset O$ for $n \gg 1$ depending on O;

(iv) $\chi_{\mathrm{im_T}(\boldsymbol{y}_n,U)} \to \chi_{\mathrm{im_T}(\boldsymbol{y},U)}$ in $L^1(\mathbb{R}^N)$ and a.e. in $\mathbb{R}^N$.

Proof. Claims (i) and (ii) are shown in [18, Lemma 3.6]. To prove (iii), observe that, by uniform convergence, $\boldsymbol{y}_n(\partial U) \subset O$ for $n \gg 1$. Then, the claim follows by Lemma 1.16. To prove (iv), note first that

$$\chi_{\mathrm{im_T}(\boldsymbol{y}_n,U)} \to \chi_{\mathrm{im_T}(\boldsymbol{y},U)} \text{ pointwise in } \mathbb{R}^N \setminus \boldsymbol{y}(\partial U)$$

by (i)–(ii). Let B be a ball such that $\mathrm{im_T}(\boldsymbol{y},U) \subset B$. Then $\mathrm{im_T}(\boldsymbol{y}_n,U) \subset\subset B$ for $n \gg 1$ by (iii), so that (iv) follows by applying the Lebesgue Dominated Convergence Theorem thanks to the assumption $\mathscr{L}^N(\boldsymbol{y}(\partial U)) = 0$. □

1.5.2 Topological degree of Sobolev maps

Let $\boldsymbol{y} \in W^{1,p}(\Omega;\mathbb{R}^N)$ and let $U \subset\subset \Omega$ be a domain of class C^2. Suppose that $\boldsymbol{y} \in W^{1,p}(\partial U;\mathbb{R}^N)$. In this case, adopting Convention 1.12, we have $\boldsymbol{y} \in C^0(\partial U;\mathbb{R}^N)$. In particular, the map $\deg(\boldsymbol{y},U,\cdot)\colon \mathbb{R}^N \setminus \boldsymbol{y}(\partial U) \to \mathbb{Z}$ is defined and, in turn, also the set $\mathrm{im_T}(\boldsymbol{y},U)$ according to Definition 1.15.
For technical reasons, following [78, Definition 6], we introduce the class of regular subdomains. We employ the notation introduced in Section 1.4.

Definition 1.19 (Regular subdomains). Let $\boldsymbol{y} \in W^{1,p}(\Omega;\mathbb{R}^N)$. We define $\mathcal{U}_{\boldsymbol{y}}$ as the class of domains $U \subset\subset \Omega$ of class C^2 satisfying the following properties:

(i) $\boldsymbol{y}|_{\partial U} \in W^{1,p}(\partial U;\mathbb{R}^N)$ and $(\operatorname{cof}\nabla\boldsymbol{y})|_{\partial U} \in L^1(\partial U;\mathbb{R}^{N\times N})$;

(ii) $\mathscr{H}^{N-1}(\partial U \setminus \mathrm{dom}_{\mathrm{G}}(\boldsymbol{y},\Omega)) = 0$ and $\nabla^{\partial U}\boldsymbol{y}|_{\partial U} = \nabla\boldsymbol{y}(\boldsymbol{I} - \boldsymbol{n}_U \otimes \boldsymbol{n}_U)$ a.e. on ∂U;

(iii) there holds

$$\lim_{\delta\to 0^+} \fint_{-\delta}^{\delta} \left| \int_{\partial U_\ell} |\operatorname{cof}\nabla\boldsymbol{y}|\,\mathrm{d}\boldsymbol{a} - \int_{\partial U} |\operatorname{cof}\nabla\boldsymbol{y}|\,\mathrm{d}\boldsymbol{a} \right| \mathrm{d}\ell = 0;$$

(iv) for every $\boldsymbol{\psi} \in C^1_{\mathrm{c}}(\mathbb{R}^N;\mathbb{R}^N)$, there holds

$$\lim_{\delta\to 0^+} \fint_{-\delta}^{\delta} \left| \int_{\partial U_\ell} \boldsymbol{\psi}\circ\boldsymbol{y}\cdot(\operatorname{cof}\nabla\boldsymbol{y})\boldsymbol{n}_{U_\ell}\,\mathrm{d}\boldsymbol{a} - \int_{\partial U} \boldsymbol{\psi}\circ\boldsymbol{y}\cdot(\operatorname{cof}\nabla\boldsymbol{y})\boldsymbol{n}_U\,\mathrm{d}\boldsymbol{a} \right| \mathrm{d}\ell = 0.$$

Property (i) in the previous definition ensures the regularity of the map $\boldsymbol{y}|_{\partial U}$. In (ii), we denote by $\nabla^{\partial U}\boldsymbol{y}|_{\partial U}$ the approximate tangential gradient of $\boldsymbol{y}|_{\partial U}$, see [78, Definition 4] or [59, Paragraph 3.2.16]. Since we are not going to use it explicitly, we do not introduce this notion. For our purposes, the only important fact is that, in view of (ii), there holds $\mathscr{L}^N(\boldsymbol{y}(\partial U)) = 0$ [18, Remark 2.16, Claim (b)]. This is a consequence of the Area Formula for surface integrals, see [78, Proposition 2] or [59, Corollary 3.2.20], that, for the same reason as before, we do not present. Eventually, properties (iii) and (iv) are crucial for the validity of Theorem 1.26 below.
The following result ensures the abundance of regular subdomains as in Definition 1.19.

Lemma 1.20 ([78, Lemma 2]). *Let $\boldsymbol{y} \in W^{1,p}(\Omega;\mathbb{R}^N)$ be such that $\det D\boldsymbol{y} > 0$ almost everywhere and let $U \subset\subset \Omega$ be a domain of class C^2. Then, there exists $\delta > 0$ such that , for almost every $\ell \in (-\delta,\delta)$, there holds $U_\ell \in \mathcal{U}_{\boldsymbol{y}}$.*

The next result allows us to compute the topological degree of Sobolev maps at almost every point of the domain Ω with respect to balls of specific radii. The result is taken from [18] and it is complemented by an elementary observation.

Lemma 1.21 ([18, Lemma 3.2]). *Let $\boldsymbol{y} \in W^{1,p}(\Omega;\mathbb{R}^N)$ and let $\boldsymbol{x}_0 \in \Omega$ be such that $\boldsymbol{y}$ is regularly approximately differentiable at $\boldsymbol{x}_0$ with $\det\nabla\boldsymbol{y}(\boldsymbol{x}_0) \neq 0$. Then, there exists $P \subset (0,\mathrm{dist}(\boldsymbol{x}_0;\partial\Omega))$ with $\Theta^1_+(P,0) = 1$ such that the following hold:*

(i) for every $r \in P$, we have $B(\boldsymbol{x}_0,r) \in \mathcal{U}_{\boldsymbol{y}}$, $\boldsymbol{y}(\boldsymbol{x}_0) \notin \boldsymbol{y}(\partial B(\boldsymbol{x}_0,r))$ and also

$$\deg(\boldsymbol{y}, B(\boldsymbol{x}_0,r), \boldsymbol{y}(\boldsymbol{x}_0)) = \operatorname{sgn}\det\nabla\boldsymbol{y}(\boldsymbol{x}_0);$$

(ii) for every $r, r' \in P$ with $r' < r/2$, the set $\boldsymbol{y}(\partial B(\boldsymbol{x}_0,r'))$ is included in the connected component of $\mathbb{R}^N \setminus \boldsymbol{y}(\partial B(\boldsymbol{x}_0,r))$ containing $\boldsymbol{y}(\boldsymbol{x}_0)$ and, in turn, we have

$$\mathrm{im}_{\mathrm{T}}(\boldsymbol{y}, B(\boldsymbol{x}_0,r')) \cup \boldsymbol{y}(\partial B(\boldsymbol{x}_0,r')) \subset \mathrm{im}_{\mathrm{T}}(\boldsymbol{y}, B(\boldsymbol{x}_0,r)).$$

Proof. Claim (i) is proved in [18, Lemma 3.2]. Here, we exploit the fact that, for every $r > 0$ such that $B(\boldsymbol{x}_0,r) \in \mathcal{U}_{\boldsymbol{y}}$, the map $\boldsymbol{y}|_{\partial B(\boldsymbol{x}_0,r)}$ admits a continuous representative which, according to Convention 1.12, is identified with $\boldsymbol{y}$. In this case, the essential supremum of $\boldsymbol{y}$ on $\partial B(\boldsymbol{x}_0, r)$ conincide with the supremum on the same set. For claim (ii), let V denote the connected component of $\mathbb{R}^N \setminus \boldsymbol{y}(\partial B(\boldsymbol{x}_0, r))$ such that $\boldsymbol{y}(\boldsymbol{x}_0) \in V$. The inclusion $\boldsymbol{y}(\partial B(\boldsymbol{x}_0, r')) \subset V$ is shown as in [18, Lemma 3.2]. As $\deg(\boldsymbol{y}, B(\boldsymbol{x}_0, r), \boldsymbol{y}(\boldsymbol{x}_0)) \neq 0$, we have $V \subset \mathrm{im}_{\mathrm{T}}(\boldsymbol{y}, B(\boldsymbol{x}_0, r))$. Therefore, thanks to Lemma 1.16, we deduce that $\mathrm{im}_{\mathrm{T}}(\boldsymbol{y}, B(\boldsymbol{x}_0, r')) \subset \mathrm{im}_{\mathrm{T}}(\boldsymbol{y}, B(\boldsymbol{x}_0, r))$ since $\mathbb{R}^N \setminus \mathrm{im}_{\mathrm{T}}(\boldsymbol{y}, B(\boldsymbol{x}_0, r))$ is connected by Lemma 1.17. □

1.6 Admissible deformations

In the next section, we introduce the class of admissible deformations that will be considered in the following. Most of the results of this section are taken from [18]. Here, we limit ourselves to recall the results that will be employed in our analysis.

1.6.1 Divergence Identities

Let $\boldsymbol{y} \in W^{1,p}(\Omega; \mathbb{R}^N)$. Following the terminology in [122], the Divergence Identities read as follows:

$$\forall\, \boldsymbol{\psi} \in C^1_{\mathrm{c}}(\mathbb{R}^N; \mathbb{R}^N), \quad \mathrm{Div}((\mathrm{adj}\, D\boldsymbol{y})\, \boldsymbol{\psi} \circ \boldsymbol{y}) = \mathrm{div}\boldsymbol{\psi} \circ \boldsymbol{y}\, \det D\boldsymbol{y} \quad \text{in } \Omega. \tag{DIV}$$

The divergence on the left-hand side of the previous equation is intended in the sense of distributions. Namely, $\boldsymbol{y}$ satisfies (DIV) whenever, for every $\boldsymbol{\psi} \in C^1_{\mathrm{c}}(\mathbb{R}^N; \mathbb{R}^N)$ and $\varphi \in C^1_{\mathrm{c}}(\Omega)$, there holds

$$-\int_\Omega ((\mathrm{adj}\, D\boldsymbol{y})\, \boldsymbol{\psi} \circ \boldsymbol{y}) \cdot D\varphi \,\mathrm{d}\boldsymbol{x} = \int_\Omega \mathrm{div}\boldsymbol{\psi} \circ \boldsymbol{y}\, \det D\boldsymbol{y}\, \varphi \,\mathrm{d}\boldsymbol{x}.$$

We will consider the following class of admissible deformations:

$$\mathcal{Y}_p(\Omega) := \Big\{ \boldsymbol{y} \in W^{1,p}(\Omega; \mathbb{R}^N) : \det D\boldsymbol{y} \in L^1(\Omega), \\ \det D\boldsymbol{y} > 0 \text{ a.e.}, \boldsymbol{y} \text{ satisfies (DIV)} \Big\}. \tag{1.7}$$

The class in (1.7) has been introduced in [18], where the authors carried out a thorough study of fine and local invertibility properties of such maps. As a result of their analysis, deformations in this class turn out to enjoy a surprising degree of regularity.
The next two examples give us an impression on how large the class in (1.7) is.

Example 1.22 ([119]). *The following inclusion holds:*

$$\Big\{ \boldsymbol{y} \in W^{1,p}(\Omega; \mathbb{R}^N) : \mathrm{cof} D\boldsymbol{y} \in L^{p'}(\Omega; \mathbb{R}^{N\times N}),\, \det D\boldsymbol{y} > 0 \text{ a.e.} \Big\} \subset \mathcal{Y}_p(\Omega).$$

To see this, let $\boldsymbol{y}$ belong to the class on the left hand-side. Using the identity

$$(\det \boldsymbol{F})\boldsymbol{I} = \boldsymbol{F}(\mathrm{adj}\boldsymbol{F}), \tag{1.8}$$

which holds for every $\boldsymbol{F} \in \mathbb{R}^{N\times N}$, together with the Hölder inequality, we deduce that $\det D\boldsymbol{y} \in L^1(\Omega)$. To check that $\boldsymbol{y}$ satisfies (DIV), we recall the Piola identity:

$$\forall\, \boldsymbol{\eta} \in C^1_{\mathrm{c}}(\Omega; \mathbb{R}^N), \quad \int_\Omega \mathrm{cof} D\boldsymbol{y} : D\boldsymbol{\eta} \,\mathrm{d}\boldsymbol{x} = 0. \tag{1.9}$$

As $\mathrm{cof}D\boldsymbol{y} \in L^{p'}(\Omega;\mathbb{R}^{N\times N})$, *by density, the previous identity actually holds for every* $\boldsymbol{\eta} \in W^{1,p}_0(\Omega;\mathbb{R}^N)$. *Let* $\varphi \in C^1_{\mathrm{c}}(\Omega)$ *and* $\boldsymbol{\psi} \in C^1_{\mathrm{c}}(\mathbb{R}^N;\mathbb{R}^N)$. *By the chain rule for Sobolev maps, there holds* $\varphi(\boldsymbol{\psi}\circ\boldsymbol{y}) \in W^{1,p}_0(\Omega;\mathbb{R}^N)$. *Thus, testing* (1.9) *with* $\boldsymbol{\eta} = \varphi(\boldsymbol{\psi}\circ\boldsymbol{y})$ *and observing that*

$$\begin{aligned}\mathrm{cof}D\boldsymbol{y} : D\boldsymbol{\eta} &= \mathrm{cof}D\boldsymbol{y} : ((\boldsymbol{\psi}\circ\boldsymbol{y})\otimes D\varphi + \varphi(D\boldsymbol{\psi}\circ\boldsymbol{y})D\boldsymbol{y})\\ &= (\mathrm{adj}D\boldsymbol{y})(\boldsymbol{\psi}\circ\boldsymbol{y})\cdot D\varphi + \mathrm{div}\boldsymbol{\psi}\circ\boldsymbol{y}\,\det D\boldsymbol{y}\,\varphi\end{aligned}$$

thanks to (1.8), *we prove the claim.*

Example 1.23 ([122]). *The following inclusion holds:*

$$\left\{\boldsymbol{y} \in W^{1,p}(\Omega;\mathbb{R}^N):\ \mathrm{cof}D\boldsymbol{y} \in L^{\frac{N}{N-1}}(\Omega;\mathbb{R}^{N\times N}),\ \det D\boldsymbol{y} > 0 \ a.e.\right\} \subset \mathcal{Y}_p(\Omega). \quad (1.10)$$

First, observe that, taking the determinant at both sides of (1.8), *yields the identity* $\det(\mathrm{adj}\boldsymbol{F}) = (\det\boldsymbol{F})^{N-1}$ *for every* $\boldsymbol{F} \in \mathbb{R}^{N\times N}$. *From this, by the Hadamard inequality, we obtain* $|\det\boldsymbol{F}| \leq C|\mathrm{adj}\boldsymbol{F}|^{N/N-1}$. *This last inequality ensures that every map* $\boldsymbol{y}$ *belonging to the class on the left-hand side of* (1.10) *satisfies* $\det D\boldsymbol{y} \in L^1(\Omega)$. *The proof that* $\boldsymbol{y}$ *satisfies* (DIV) *is not immediate and we refer to [122, Theorem 3.2].*

The Divergence Identities have been firstly examined in [119] in connection with the problem of the weak continuity of the Jacobian determinant. In this regard, we have the following.

Theorem 1.24 ([119, Theorem 4]). *Let* $(\boldsymbol{y}_n) \subset \mathcal{Y}_p(\Omega)$. *Suppose that there exist* $\boldsymbol{y} \in W^{1,p}(\Omega;\mathbb{R}^N)$ *and* $h \in L^1(\Omega)$ *such that*

$$\boldsymbol{y}_n \rightharpoonup \boldsymbol{y} \ \text{in } W^{1,p}(\Omega;\mathbb{R}^N), \quad \det D\boldsymbol{y}_n \rightharpoonup h \ \text{in } L^1(\Omega).$$

Then, $\boldsymbol{y}$ *satisfies* (DIV) *and* $h = \det D\boldsymbol{y}$ *almost everywhere. In particular, if* $\det D\boldsymbol{y} > 0$ *almost everywhere, then* $\boldsymbol{y} \in \mathcal{Y}_p(\Omega)$.

As explained in the next remark, the Divergence Identities characterize the class of possibly discontinuous Sobolev maps for which cavitation is excluded.

Remark 1.25 (Excluding cavitation). The condition (DIV) is a generalization of the more popular identity $\mathrm{Det}D\boldsymbol{y} = \det D\boldsymbol{y}$ [118], asserting the coincidence of the distributional Jacobian with the pointwise Jacobian. Before the pathological example exhibited in [120], it was thought that the validity of the identity this identity excluded the phenomena of cavitation. More generally, the condition (DIV) turns out to be intimately connected to the phenomena of creation of new surface by means of deformation. Indeed, it has been proved in [78, 80] that, for $\boldsymbol{y} \in \mathcal{Y}_p(\Omega)$, the validity of (DIV) is equivalent to the property that $\boldsymbol{y}$ does not create new surface. Of course, the concept of created surface itself needs to be suitably defined [78]. As fracture is ruled out by the Sobolev regularity, this characterize the class in (1.7) as the class of maps in $W^{1,p}(\Omega;\mathbb{R}^N)$ for which cavitation is excluded. Moreover, one of the contributions in [18] is to have shown that the (DIV) holds whenever the identity $\mathrm{Det}D\boldsymbol{y} = \det D\boldsymbol{y}$ is statisfied and $\boldsymbol{y}$ is orientation-preserving in the topological sense. Loosely speaking, this means that the topological degree of $\boldsymbol{y}$ is positive on almost every surface.

1.6.2 Topological image

The divergence identities entail the coincidence of the topological image and the geometric image of regular subdomains up to sets of measure zero. Recall Definition 1.7 and the definition of multiplicity function in Proposition 1.9.

Theorem 1.26 ([18, Theorem 4.1]). *Let $\boldsymbol{y} \in \mathcal{Y}_p(\Omega)$. Then, for every $U \in \mathcal{U}_{\boldsymbol{y}}$, there holds*

$$\deg(\boldsymbol{y}, U, \cdot) = \mathrm{mult}(\boldsymbol{y}, U, \cdot) \ \text{ a.e. in } \mathbb{R}^N \setminus \overline{\boldsymbol{y}}(\partial U).$$

Therefore, $\mathscr{L}^N(\mathrm{im}_{\mathrm{T}}(\boldsymbol{y}, U) \triangle \mathrm{im}_{\mathrm{G}}(\boldsymbol{y}, U)) = 0$. In particular, $\boldsymbol{y} \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$.

The following result collects some consequences of the previous theorem.

Corollary 1.27. *Let $\boldsymbol{y} \in \mathcal{Y}_p(\Omega)$. Then:*

(i) for every $U \in \mathcal{U}_{\boldsymbol{y}}$, there holds $\deg(\boldsymbol{y}, U, \cdot) \geq 0$ almost everywhere in $\mathbb{R}^N \setminus \overline{\boldsymbol{y}}(\partial U)$;

(ii) for every $U_1, U_2 \in \mathcal{U}_{\boldsymbol{y}}$ such that $U_1 \subset\subset U_2$, there holds $\overline{\mathrm{im}_{\mathrm{T}}(\boldsymbol{y}, U_1)} \subset \overline{\mathrm{im}_{\mathrm{T}}(\boldsymbol{y}, U_2)}$;

(iii) for every $U \in \mathcal{U}_{\boldsymbol{y}}$, there holds $\mathrm{im}_{\mathrm{G}}(\boldsymbol{y}, U) \subset \overline{\mathrm{im}_{\mathrm{T}}(\boldsymbol{y}, U)}$.

Proof. Claim (i) is immediate. Claims (ii) and (iii) are proved in [18, Proposition 4.3, Claim (d)] and [18, Proposition 4.3, Claim (e)], respectively, by exploiting (i). □

For technical reasons, we give the following definition.

Definition 1.28 (Singularity set). Let $\boldsymbol{y} \in W^{1,p}(\Omega; \mathbb{R}^N)$. We define the singularity set of $\boldsymbol{y}$ as

$$S_{\boldsymbol{y}} := \left\{ \boldsymbol{x} \in \Omega : \limsup_{\substack{r \to 0 \\ B(\boldsymbol{x}, r) \in \mathcal{U}_{\boldsymbol{y}}}} \mathrm{diam}(\mathrm{im}_{\mathrm{T}}(\boldsymbol{y}, B(\boldsymbol{x}, r))) > 0 \right\}.$$

The set $S_{\boldsymbol{y}}$ does not depend on the specific representative of $\boldsymbol{y}$ but only on its equivalence class [18, Remark 5.7, Claim (c)]. Moreover, one can prove the following [18, Proposition 5.9]:

$$\mathscr{H}^{\max\{N-p,0\}}(S_{\boldsymbol{y}}) = 0, \qquad \Omega \setminus S_{\boldsymbol{y}} \subset L_{\boldsymbol{y}}, \qquad \boldsymbol{y}^*|_{\Omega \setminus S_{\boldsymbol{y}}} \text{ is continuous.} \tag{1.11}$$

We denote by $\mathcal{U}^*_{\boldsymbol{y}}$ the class of subdomains $U \in \mathcal{U}_{\boldsymbol{y}}$ such that $\partial U \cap S_{\boldsymbol{y}} = \emptyset$. Combining Lemma 1.20 with (1.11), we deduce the following result.

Lemma 1.29 ([18, Lemma 5.12]). *Let $\boldsymbol{y} \in \mathcal{Y}_p(\Omega)$ and let $U \subset\subset \Omega$ be a domain of class C^2. Then, there exists $\delta > 0$ such that, for almost every $\ell \in (-\delta, \delta)$, there holds $U_\ell \in \mathcal{U}^*_{\boldsymbol{y}}$.*

We define the topological image of Sobolev maps as in [18, Definition 5.17].

Definition 1.30 (Topological image). Let $\boldsymbol{y} \in \mathcal{Y}_p(\Omega)$. The topological image of $\boldsymbol{y}$ of Ω under $\boldsymbol{y}$ is defined as

$$\mathrm{im}_{\mathrm{T}}(\boldsymbol{y}, \Omega) := \bigcup_{U \in \mathcal{U}^*_{\boldsymbol{y}}} \mathrm{im}_{\mathrm{T}}(\boldsymbol{y}, U).$$

Being the union of open sets, the set $\mathrm{im}_{\mathrm{T}}(\boldsymbol{y},\Omega)$ is open. Notably, as shown in [18, Lemma 5.18, Claim (b)], this set depends only on the equivalence class of $\boldsymbol{y}$. This fact relies on the restriction taken in Definition 1.30 to the regular subdomains whose boundary does not intersect the singularity set.
Exploiting Theorem 1.26 and Lemma 1.29, we prove the following.

Corollary 1.31 ([18, Lemma 5.18, Claim (c)]). *Let $\boldsymbol{y} \in \mathcal{Y}_p(\Omega)$. Then, there holds*
$$\mathscr{L}^N\left(\mathrm{im}_{\mathrm{T}}(\boldsymbol{y},\Omega) \triangle \mathrm{im}_{\mathrm{G}}(\boldsymbol{y},\Omega)\right) = 0.$$

The next result will be instrumental in the proof of our compactness theorem. For the validity of the first claim, the fact that in Definition 1.30 we only considered regular subdomains whose boundaries do not intersect the singularity set is essential. This is the reason why this restriction will be crucial also in this paper.

Proposition 1.32. *Let $(\boldsymbol{y}_n) \subset \mathcal{Y}_p(\Omega)$ and let $\boldsymbol{y} \in \mathcal{Y}_p(\Omega)$. Suppose that $\boldsymbol{y}_n \rightharpoonup \boldsymbol{y}$ in $W^{1,p}(\Omega;\mathbb{R}^N)$.*

(i) for every $U \in \mathcal{U}_{\boldsymbol{y}}^*$ and for every $V \subset\subset \mathrm{im}_{\mathrm{T}}(\boldsymbol{y},U)$ open, there holds $V \subset\subset \mathrm{im}_{\mathrm{T}}(\boldsymbol{y}_n,\Omega)$ for $n \gg 1$;

(ii) if $(\det D\boldsymbol{y}_n)$ is equi-integrable, then, up to subsequences, there holds
$$\chi_{\mathrm{im}_{\mathrm{T}}(\boldsymbol{y}_n,\Omega)} \to \chi_{\mathrm{im}_{\mathrm{T}}(\boldsymbol{y},\Omega)} \text{ in } L^1(\mathbb{R}^N),$$
and also
$$\chi_{\mathrm{im}_{\mathrm{G}}(\boldsymbol{y}_n,\Omega)} \to \chi_{\mathrm{im}_{\mathrm{G}}(\boldsymbol{y},\Omega)} \text{ in } L^1(\mathbb{R}^N).$$

Proof. Claim (i) has been proved in [18, Theorem 6.3, Claim (a)]. For (ii), the first convergence has been proved in [18, Theorem 6.3, Claim (d)], and the second one is easily deduced from the first one thanks to Corollary 1.31. □

Remark 1.33 (Fine properties). The maps in the class (1.7) enjoy remarkable continuity and differentiability properties. Precisely, up to the choice of a suitable representative, any such map is continuous out of a set of zero $(N-p)$-dimensional Hausdorff measure and almost everywhere differentiable [18, Proposition 5.9]. Additionally, the same representative has the Lusin property (N).

1.7 Local invertibility

Let $\boldsymbol{y}\colon \Omega \to \mathbb{R}^N$ be measurable and let $A \subset \Omega$ be measurable. The map $\boldsymbol{y}$ is termed almost everywhere injective in A if there exists a subset $X \subset A$ with $\mathscr{L}^N(X) = 0$ such that $\boldsymbol{y}|_{A\setminus X}$ is injective. We observe that almost everywhere injectivity depends only on the equivalence class of $\boldsymbol{y}$.
The following result constitutes one of the main motivations for the definition of the geometric domain.

Lemma 1.34 ([78, Lemma 3]). *Let $\boldsymbol{y}\colon \Omega \to \mathbb{R}^N$ be almost everywhere approximately differentiable with $\det \nabla\boldsymbol{y} \neq 0$ almost everywhere. Suppose that $\boldsymbol{y}$ is almost everywhere injective on A for some measurable set $A \subset \Omega$. Then, $\boldsymbol{y}|_{\mathrm{dom}_{\mathrm{G}}(\boldsymbol{y},A)}$ is injective.*

Let $\boldsymbol{y} \in W^{1,p}(\Omega;\mathbb{R}^N)$ and recall Definition 1.19. We denote by $\mathcal{U}_{\boldsymbol{y}}^{\mathrm{inj}}$ the class of regular subdomains $U \in \mathcal{U}_{\boldsymbol{y}}$ such that $\boldsymbol{y}$ is almost everywhere injective in U. Also, we set $\mathcal{U}_{\boldsymbol{y}}^{*,\mathrm{inj}} := \mathcal{U}_{\boldsymbol{y}}^{*} \cap \mathcal{U}_{\boldsymbol{y}}^{\mathrm{inj}}$.
The following local invertibility result has been proven in [18] and is going to be instrumental in our proofs.

Theorem 1.35 ([18, Proposition 4.5, Claim (d)]). *Let $\boldsymbol{y} \in \mathcal{Y}_p(\Omega)$ and let $\boldsymbol{x}_0 \in \Omega$ be such that $\boldsymbol{y}$ is regularly approximately differentiable at $\boldsymbol{x}_0$ with $\det \nabla \boldsymbol{y}(\boldsymbol{x}_0) > 0$. Let $P \subset (0, \mathrm{dist}(\boldsymbol{x}_0;\partial\Omega))$ be the set given by Lemma 1.21. Then, for every $r \in P$ such that there exists $\widetilde{r} \in P$ with $r < \widetilde{r}/2$, we have $B(\boldsymbol{x}_0, r) \in \mathcal{U}_{\boldsymbol{y}}^{\mathrm{inj}}$.*

Let $\boldsymbol{y}\colon \Omega \to \mathbb{R}^N$ be almost everywhere approximately differentiable with $\det \nabla \boldsymbol{y} \neq 0$ almost everywhere. Let $A \subset \Omega$ be measurable and suppose that $\boldsymbol{y}$ is almost everywhere injective in A. By Lemma 1.34, the map $\boldsymbol{y}|_{\mathrm{dom}_{\mathrm{G}}(\boldsymbol{y},A)}$ is injective and, in turn, its inverse $(\boldsymbol{y}|_{\mathrm{dom}_{\mathrm{G}}(\boldsymbol{y},A)})^{-1}\colon \mathrm{im}_{\mathrm{G}}(\boldsymbol{y},A) \to \mathbb{R}^N$ is defined. Concerning the approximate differentiablity of this map, we make the following observation.

Lemma 1.36. *Let $\boldsymbol{y}\colon \Omega \to \mathbb{R}^N$ be almost everywhere approximately differentiable with $\det \nabla \boldsymbol{y} \neq 0$ almost everywhere and let $A \subset \Omega$ be measurable. Suppose that $\boldsymbol{y}$ is almost everywhere injective in A. Then, for every $\boldsymbol{x}_0 \in \mathrm{dom}_{\mathrm{G}}(\boldsymbol{y},A)$ with $\Theta^N(A,\boldsymbol{x}_0) = 1$, the map $(\boldsymbol{y}|_{\mathrm{dom}_{\mathrm{G}}(\boldsymbol{y},A)})^{-1}$ is approximately differentiable at $\boldsymbol{y}(\boldsymbol{x}_0)$ with $\nabla(\boldsymbol{y}|_{\mathrm{dom}_{\mathrm{G}}(\boldsymbol{y},A)})^{-1}(\boldsymbol{y}(\boldsymbol{x}_0)) = (\nabla \boldsymbol{y}(\boldsymbol{x}_0))^{-1}$.*

Proof. Consider $\boldsymbol{x}_0 \in \mathrm{dom}_{\mathrm{G}}(\boldsymbol{y},A)$ with $\Theta^N(A,\boldsymbol{x}_0) = 1$. By Definition 1.7, there exists a compact set $K \subset \Omega$ with $\boldsymbol{x}_0 \in K$ and $\Theta^N(K,\boldsymbol{x}_0) = 1$, and a map $\boldsymbol{w} \in C^1(\mathbb{R}^N;\mathbb{R}^N)$ such that $\boldsymbol{y}|_K = \boldsymbol{w}|_K$ and $\nabla \boldsymbol{y}|_K = D\boldsymbol{w}|_K$. Since $\det D\boldsymbol{w}(\boldsymbol{x}_0) = \det \nabla \boldsymbol{y}(\boldsymbol{x}_0) \neq 0$, by the Local Diffeomorphism Theorem, there exists $r > 0$ such that, setting $B := B(\boldsymbol{x}_0, r)$, the map $\boldsymbol{w}|_B$ is a diffeomorphism of class C^1 onto its image. In particular, $D(\boldsymbol{w}|_B)^{-1} = (D\boldsymbol{w})^{-1} \circ (\boldsymbol{w}|_B)^{-1}$ on the open set $\boldsymbol{w}(B)$. Thus, setting $\boldsymbol{\xi}_0 := \boldsymbol{y}(\boldsymbol{x}_0)$, there holds

$$\lim_{\boldsymbol{\xi}\to\boldsymbol{\xi}_0} \frac{|(\boldsymbol{w}|_B)^{-1}(\boldsymbol{\xi}) - (\boldsymbol{w}|_B)^{-1}(\boldsymbol{\xi}_0) - (D\boldsymbol{w}(\boldsymbol{x}_0))^{-1}(\boldsymbol{\xi} - \boldsymbol{\xi}_0)|}{|\boldsymbol{\xi} - \boldsymbol{\xi}_0|} = 0. \tag{1.12}$$

We have $\Theta^N(K \cap B \cap \mathrm{dom}_{\mathrm{G}}(\boldsymbol{y},A), \boldsymbol{x}_0) = 1$ so that, by Lemma 1.8, there holds $\Theta^N(\boldsymbol{y}(K \cap B) \cap \mathrm{im}_{\mathrm{G}}(\boldsymbol{y},A), \boldsymbol{\xi}_0) = 1$. Observing that $(\boldsymbol{w}|_B)^{-1} = (\boldsymbol{y}|_{\mathrm{dom}_{\mathrm{G}}(\boldsymbol{y},A)})^{-1}$ and $D(\boldsymbol{w}|_B)^{-1} = (\nabla \boldsymbol{y})^{-1} \circ (\boldsymbol{y}|_{\mathrm{dom}_{\mathrm{G}}(\boldsymbol{y},A)})^{-1}$ on $\boldsymbol{y}(K \cap B) \cap \mathrm{im}_{\mathrm{G}}(\boldsymbol{y},A)$, from (1.12), we obtain

$$\lim_{\substack{\boldsymbol{\xi}\to\boldsymbol{\xi}_0 \\ \boldsymbol{\xi} \in \boldsymbol{y}(K\cap B)\cap \mathrm{im}_{\mathrm{G}}(\boldsymbol{y},A)}} \frac{|(\boldsymbol{y}|_{\mathrm{dom}_{\mathrm{G}}(\boldsymbol{y},A)})^{-1}(\boldsymbol{\xi}) - (\boldsymbol{y}|_{\mathrm{dom}_{\mathrm{G}}(\boldsymbol{y},A)})^{-1}(\boldsymbol{\xi}_0) - (\nabla \boldsymbol{y}(\boldsymbol{x}_0))^{-1}(\boldsymbol{\xi} - \boldsymbol{\xi}_0)|}{|\boldsymbol{\xi} - \boldsymbol{\xi}_0|} = 0.$$

Therefore, the claim is proved. □

Let $\boldsymbol{y} \in W^{1,p}(\Omega;\mathbb{R}^N)$ be such that $\det D\boldsymbol{y} \neq 0$ almost everywhere and let $U \in \mathcal{U}_{\boldsymbol{y}}^{\mathrm{inj}}$. As before, we can consider the inverse $(\boldsymbol{y}|_{\mathrm{dom}_{\mathrm{G}}(\boldsymbol{y},U)})^{-1} : \mathrm{im}_{\mathrm{G}}(\boldsymbol{y},U) \to \mathbb{R}^N$. Recalling Theorem 1.26, we give the following definition.

Definition 1.37 (Local inverse). Let $\boldsymbol{y} \in \mathcal{Y}_p(\Omega)$ and let $U \in \mathcal{U}_{\boldsymbol{y}}^{\mathrm{inj}}$. We define $\boldsymbol{y}_U^{-1}\colon \mathrm{im}_{\mathrm{T}}(\boldsymbol{y},U) \to \mathbb{R}^N$ to be an arbitrary extension of the map

$$(\boldsymbol{y}|_{\mathrm{dom}_{\mathrm{G}}(\boldsymbol{y},U)})^{-1}|_{\mathrm{im}_{\mathrm{T}}(\boldsymbol{y},U)\cap \mathrm{im}_{\mathrm{G}}(\boldsymbol{y},U)}.$$

Note that, for every possible choice of the extension, the resulting map $\boldsymbol{y}_U^{-1}$ is measurable thanks to the completeness of the Lebesgue measure [41, Proposition 2.2.2]. As a consequence of (DIV), the map $\boldsymbol{y}_U^{-1}$ turns out to have Sobolev regularity. This is meaningful since the set $\mathrm{im_T}(\boldsymbol{y}, U)$ is open.

Proposition 1.38 ([18, Proposition 5.3]). *Let $\boldsymbol{y} \in \mathcal{Y}_p(\Omega)$ and let $U \in \mathcal{U}_{\boldsymbol{y}}^{\mathrm{inj}}$. Then, $\boldsymbol{y}_U^{-1} \in W^{1,1}(\mathrm{im_T}(\boldsymbol{y}, U); \mathbb{R}^N)$ and $D\boldsymbol{y}_U^{-1} = (D\boldsymbol{y})^{-1} \circ \boldsymbol{y}_U^{-1}$ almost everywhere in $\mathrm{im_T}(\boldsymbol{y}, U)$. Moreover, each Jacobian minor of $\boldsymbol{y}_U^{-1}$ belongs to $L^1(\mathrm{im_T}(\boldsymbol{y}, U))$.*

From Proposition 1.38, we deduce that $\boldsymbol{y}_U^{-1}$ is almost everywhere approximately differentiable [69, Theorem 2, Section 1.4, Chapter 3], so that the change-of-variable formula holds. More precisely, we have the following.

Proposition 1.39 (Change-of-variable Formula for the inverse). *Let $\boldsymbol{y} \in \mathcal{Y}_p(\Omega)$ and let $U \in \mathcal{U}_{\boldsymbol{y}}^{\mathrm{inj}}$. Then, for every $\boldsymbol{\varphi} \colon U \to \mathbb{R}^M$ measurable, there holds*

$$\int_U \boldsymbol{\varphi}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_{\mathrm{im_T}(\boldsymbol{y},U)} \boldsymbol{\varphi} \circ \boldsymbol{y}_U^{-1}(\boldsymbol{\xi})\, \det D\boldsymbol{y}_U^{-1}(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}.$$

In the previous formula, we observe that $\boldsymbol{\varphi} \circ \boldsymbol{y}_U^{-1}$ is measurable by claim (ii) of Lemma 1.4. Indeed, by Proposition 1.38, we have $\det D\boldsymbol{y}_U^{-1} = (\det D\boldsymbol{y}) \circ \boldsymbol{y}_U^{-1} > 0$ almost everywhere in $\mathrm{im_T}(\boldsymbol{y}, U)$.

Proof. By Lemma 1.36, the map $(\boldsymbol{y}|_{\mathrm{dom_G}(\boldsymbol{y},U)})^{-1}$ is approximately differentiable on $\mathrm{im_G}(\boldsymbol{y}, U)$. The result follows by applying the Federer Change-of-variable Formula [69, Theorem 1, Section 1.5, Chapter 3] to this map taking into account Theorem 1.26 and Definition 1.37. □

The next result show that the radius of local invertibility given by Theorem 1.35 can be chosen uniformly for sequences of deformations converging weakly in $W^{1,p}(\Omega; \mathbb{R}^N)$. The result is taken from [18] and it is reformulated according to our needs.

Proposition 1.40 ([18, Theorem 6.3, Claim (b)]). *Let $\boldsymbol{y} \in \mathcal{Y}_p(\Omega)$ and let $\boldsymbol{x}_0 \in \Omega$ be such that $\boldsymbol{y}$ is regularly approximately differentiable at $\boldsymbol{x}_0$ with $\det \nabla\boldsymbol{y}(\boldsymbol{x}_0) > 0$. Let $P \subset (0, \mathrm{dist}(\boldsymbol{x}_0; \partial\Omega))$ be the set given by Lemma 1.21. Also, let $(\boldsymbol{y}_n) \subset \mathcal{Y}_p(\Omega)$ and suppose that $\boldsymbol{y}_n \rightharpoonup \boldsymbol{y}$ in $W^{1,p}(\Omega; \mathbb{R}^N)$. Then, for every $r \in P$ such that there exist $\widetilde{r} \in P$ with $r < \widetilde{r}/2$ and a subsequence $(\boldsymbol{y}_{n_k})$ for which we have*

$$\boldsymbol{y}_{n_k} \to \boldsymbol{y} \text{ uniformly on } \partial B(\boldsymbol{x}_0, r) \cup \partial B(\boldsymbol{x}_0, \widetilde{r}),$$

there holds

$$B(\boldsymbol{x}_0, r) \in \bigcap_{k=1}^{\infty} \mathcal{U}_{\boldsymbol{y}_{n_k}}^{\mathrm{inj}} \cap \mathcal{U}_{\boldsymbol{y}}^{\mathrm{inj}}$$

The following result yields the stability of the local inverse with respect to the weak convergence of deformations in $W^{1,p}(\Omega; \mathbb{R}^N)$.

Theorem 1.41 ([18, Theorem 6.3, Claim (c)]). *Let $(\boldsymbol{y}_n) \subset \mathcal{Y}_p(\Omega)$ and $\boldsymbol{y} \in \mathcal{Y}_p(\Omega)$ be such that $\boldsymbol{y}_n \rightharpoonup \boldsymbol{y}$ in $W^{1,p}(\Omega; \mathbb{R}^N)$. Let $U \in \mathcal{U}_{\boldsymbol{y}}^{\mathrm{inj}} \cap \bigcap_{n=1}^{\infty} \mathcal{U}_{\boldsymbol{y}_n}^{\mathrm{inj}}$ and let $V \subset\subset \mathrm{im_T}(\boldsymbol{y}, U)$ be open, so that $V \subset\subset \mathrm{im_T}(\boldsymbol{y}_n, U)$ for $n \gg 1$. Then, there holds*

$$\boldsymbol{y}_{n,U}^{-1} \to \boldsymbol{y}_U^{-1} \text{ a.e. in } V \text{ and strongly in } L^1(V; \mathbb{R}^N).$$

Moreover, each Jacobian minor of $\boldsymbol{y}_{n,U}^{-1}$ *converges weakly-$*$ in* $\mathcal{M}_{\mathrm{b}}(V)$ *to the corresponding Jacobian minor of* $\boldsymbol{y}_U^{-1}$*. In particular, if* $(\det D\boldsymbol{y}_{n,U}^{-1}) \subset L^1(V)$ *is equi-integrable, then the Jacobian minors converge also weakly in* $L^1(V)$*. In this case, we have*

$$\boldsymbol{y}_{n,U}^{-1} \rightharpoonup \boldsymbol{y}_U^{-1} \ \text{in } W^{1,1}(V;\mathbb{R}^N).$$

1.8 Global invertibility

The requirement of global injectivity of admissible deformations can be easily incorporated in our analysis. Our approach relies on the Ciarlet-Nečas condition [38].
Let $\boldsymbol{y}\colon \Omega \to \mathbb{R}^N$ be almost everywhere approximately differentiable and recall Definition 1.7. The map $\boldsymbol{y}$ satisfies the Ciarlet-Nečas condition when the following holds:

$$\int_\Omega |\det \nabla \boldsymbol{y}|\,\mathrm{d}\boldsymbol{x} \le \mathscr{L}^N(\mathrm{im}_{\mathrm{G}}(\boldsymbol{y},\Omega)). \tag{CN}$$

Note that, by Proposition 1.9, the opposite inequality is automatically satisfied, so that, in the previous equation, we could equivalently replace the inequality with an equality. Also, we observe that the validity of (CN) depends only on the equivalence class of $\boldsymbol{y}$.
The following result clarifies the relationship between almost everywhere injectivity and the Ciarlet-Nečas condition and it is extremely useful in applications.

Proposition 1.42 (Ciarlet-Nečas condition). *Let* $\boldsymbol{y}\colon \Omega \to \mathbb{R}^N$ *be almost everywhere approximately differentiable. Then the following hold:*

(i) if $\boldsymbol{y}$ *is almost everywhere injective in* Ω*, then it satisfies* (CN)*;*

(ii) if $\boldsymbol{y}$ *satisfies* (CN) *and also* $\det \nabla \boldsymbol{y} \neq 0$ *almost everywhere, then* $\boldsymbol{y}$ *is almost everywhere injective.*

Proof. First, recall that $\mathscr{L}^N(\boldsymbol{y}(\Omega_{\mathrm{d}}(\boldsymbol{y})) \setminus \mathrm{im}_{\mathrm{G}}(\boldsymbol{y},\Omega)) = 0$. Indeed, this follows since $\mathscr{L}^N(\Omega_{\mathrm{d}}(\boldsymbol{y}) \setminus \mathrm{im}_{\mathrm{G}}(\boldsymbol{y},\Omega)) = 0$ by claim (i) of Lemma 1.8 and $\boldsymbol{y}|_{\Omega_{\mathrm{d}}(\boldsymbol{y})}$ has the Lusin property (N), see [69, Proposition 1, Section 1.5, Chapter 3]. Thus, the result follows by [68, Proposition 1.5]. □

Recalling (1.7), the class of globally injective admissible deformations is defined as

$$\mathcal{Y}_p^{\mathrm{inj}}(\Omega) := \{\boldsymbol{y} \in \mathcal{Y}_p(\Omega) :\ \boldsymbol{y} \text{ a.e. injective in } \Omega\}. \tag{1.13}$$

The following simple result provides the stability of almost everywhere injectivity with respect to the weak convergence in $W^{1,p}(\Omega;\mathbb{R}^N)$ and the weak convergence in $L^1(\Omega)$ of the Jacobian determinant.

Corollary 1.43. *Let* $(\boldsymbol{y}_n) \subset \mathcal{Y}_p^{\mathrm{inj}}(\Omega)$*. Suppose that there exists* $\boldsymbol{y} \in \mathcal{Y}_p(\Omega)$ *such that*

$$\boldsymbol{y}_n \rightharpoonup \boldsymbol{y} \ \text{in } W^{1,p}(\Omega;\mathbb{R}^N), \qquad \det D\boldsymbol{y}_n \rightharpoonup \det D\boldsymbol{y} \ \text{in } L^1(\Omega).$$

Then, $\boldsymbol{y}$ *is almost everywhere injective, so that* $\boldsymbol{y} \in \mathcal{Y}_p^{\mathrm{inj}}(\Omega)$*.*

Proof. By Proposition 1.42, for every $n \in \mathbb{N}$, there holds

$$\int_\Omega \det D\boldsymbol{y}_n \,\mathrm{d}\boldsymbol{x} \leq \mathscr{L}^N(\mathrm{im}_\mathrm{G}(\boldsymbol{y}_n, \Omega)). \tag{1.14}$$

Thanks to claim (ii) of Proposition 1.32, passing to the limit, as $n \to \infty$, in (1.14), we obtain

$$\int_\Omega \det D\boldsymbol{y} \,\mathrm{d}\boldsymbol{x} \leq \mathscr{L}^N(\mathrm{im}_\mathrm{G}(\boldsymbol{y}, \Omega)). \tag{1.15}$$

Since $\det D\boldsymbol{y} > 0$ almost everywhere by assumption, this implies that $\boldsymbol{y}$ is almost everywhere injective by Proposition 1.42. □

In the case of global injectivity, it is natural to wonder if the global inverse enjoys the same kind of regularity of the local inverse established in Theorem 1.38. The answer is actually affermative. Given $\boldsymbol{y} \in \mathcal{Y}_p^{\mathrm{inj}}(\Omega)$, it follows from [81, Theorem 3.3] that the global inverse $\boldsymbol{y}_\Omega^{-1}$, defined as an arbitrary extension of $(\boldsymbol{y}|_{\mathrm{dom}_\mathrm{G}(\boldsymbol{y},\Omega)})^{-1}|_{\mathrm{im}_\mathrm{T}(\boldsymbol{y},\Omega)\cap\mathrm{im}_\mathrm{G}(\boldsymbol{y},\Omega)}$ to $\mathrm{im}_\mathrm{T}(\boldsymbol{y},\Omega)$, belongs to $W^{1,1}(\mathrm{im}_\mathrm{T}(\boldsymbol{y},\Omega);\mathbb{R}^N)$. The question of the regularity of the global inverse for the case $p > N$ is addressed in the next section.

1.9 Sobolev maps with supercritical integrability

In this subsection, we specialize the results presented in the previous subsections in the case $p > N$. This setting will be considered in Subsection 2.2.1, but also in Chapter 4 and Chapter 5.
Since we are assuming that Ω is a bounded Lipschitz domain, by the Morrey embedding, any map in $W^{1,p}(\Omega;\mathbb{R}^N)$ with $p > N$ admits a representative which is continuous up to the boundary.

Convention 1.44. Let $\boldsymbol{y} \in W^{1,p}(\Omega;\mathbb{R}^N)$ with $p > N$. In this case, $\boldsymbol{y}$ is identified with its representative in $C^0(\overline{\Omega};\mathbb{R}^N)$.

Standing the previous convention, we have the following result.

Proposition 1.45 ([106, Corollary 1]). *Let $\boldsymbol{y} \in W^{1,p}(\Omega;\mathbb{R}^N)$. Then, $\boldsymbol{y}$ has the Lusin property* (N)*.*

In view of Proposition 1.45, the Change-of-variable Formula can be rewritten in the following simplified form.

Proposition 1.46 ([106, Theorem 2]). *Let $\boldsymbol{y} \in W^{1,p}(\Omega;\mathbb{R}^N)$ with $p > N$ with $\det D\boldsymbol{y} \neq 0$ almost everywhere. Then, for every $A \subset \Omega$ measurable, the function $\boldsymbol{\xi} \mapsto \#(\boldsymbol{y}^{-1}(\boldsymbol{\xi}) \cap A)$ is measurable. Also, for every $\boldsymbol{\psi}\colon \boldsymbol{y}(A) \to \mathbb{R}^M$ measurable, there holds*

$$\int_A \boldsymbol{\psi} \circ \boldsymbol{y}(\boldsymbol{x})\,|\det D\boldsymbol{y}(\boldsymbol{x})|\,\mathrm{d}\boldsymbol{x} = \int_{\boldsymbol{y}(A)} \boldsymbol{\psi}(\boldsymbol{\xi})\,\#(\boldsymbol{y}^{-1}(\boldsymbol{\xi}) \cap A)\,\mathrm{d}\boldsymbol{\xi},$$

whenever one of the two integrals exists.

Let $\boldsymbol{y} \in W^{1,p}(\Omega;\mathbb{R}^N)$. Recalling Convention 1.44, the map $\deg(\boldsymbol{y},\Omega,\cdot)$ is well defined. In the present setting, as observed in [13, p. 317], the degree can be computed by using the integral formula (1.6). This is easily checked by approximating $\boldsymbol{y}$ with maps in $C^1(\overline{\Omega};\mathbb{R}^N)$. Additionally, in this setting, the topological image of Ω via

$\boldsymbol{y}$ can be defined as in Definition 1.15. For orientation-preserving maps, the next result provides an explicit characterization of the topological image. We set

$$\Omega^{\boldsymbol{y}} := \boldsymbol{y}(\Omega) \setminus \boldsymbol{y}(\partial\Omega), \qquad \Omega_{\boldsymbol{y}} := \Omega \setminus \boldsymbol{y}^{-1}(\boldsymbol{y}(\partial\Omega)).$$

Lemma 1.47. *Let $\boldsymbol{y} \in W^{1,p}(\Omega;\mathbb{R}^N)$ with $p > N$ be such that $\det D\boldsymbol{y} > 0$ almost everywhere. Then, $\mathrm{im_T}(\boldsymbol{y},\Omega) = \Omega^{\boldsymbol{y}}$. Thus, $\Omega^{\boldsymbol{y}}$ and $\Omega_{\boldsymbol{y}}$ are open sets that differ from $\boldsymbol{y}(\Omega)$ and Ω, respectively, by at most a set of measure zero. Moreover, there hold:*

$$\overline{\Omega^{\boldsymbol{y}}} = \boldsymbol{y}(\overline{\Omega}), \quad \overline{\Omega_{\boldsymbol{y}}} = \overline{\Omega}, \quad \partial\Omega^{\boldsymbol{y}} = \boldsymbol{y}(\partial\Omega), \quad \partial\Omega_{\boldsymbol{y}} = \partial\Omega \cup \boldsymbol{y}^{-1}(\boldsymbol{y}(\partial\Omega)).$$

Proof. Let $\boldsymbol{\xi}_0 \in \mathrm{im_T}(\boldsymbol{y},\Omega)$. Then, by the solvability property, $\boldsymbol{\xi}_0 \in \boldsymbol{y}(\Omega)$ and, in turn, $\boldsymbol{\xi}_0 \in \Omega^{\boldsymbol{y}}$. Conversely, let $\boldsymbol{\xi}_0 \in \Omega^{\boldsymbol{y}}$. Denote by V the connected component of $\mathbb{R}^N \setminus \boldsymbol{y}(\partial\Omega)$ containing $\boldsymbol{\xi}_0$. Let $\psi \in C^\infty_c(\mathbb{R}^N)$ be positive, with integral equal to one and such that $K := \operatorname{supp}\psi \subset V$. Then, using formula (1.6), we compute

$$\deg(\boldsymbol{y},\Omega,\boldsymbol{\xi}_0) = \int_\Omega \psi\circ\boldsymbol{y}\,\det D\boldsymbol{y}\,\mathrm{d}\boldsymbol{x} = \int_{\boldsymbol{y}^{-1}(K)} \psi\circ\boldsymbol{y}\,\det D\boldsymbol{y}\,\mathrm{d}\boldsymbol{x} > 0,$$

which yields $\boldsymbol{\xi}_0 \in \mathrm{im_T}(\boldsymbol{y},\Omega)$. Thus, the equaility $\mathrm{im_T}(\boldsymbol{y},\Omega) = \Omega^{\boldsymbol{y}}$ is proved.
Recalling that the topological image is an open set, we deduce that $\Omega^{\boldsymbol{y}}$ is an open set. Also, since $\Omega_{\boldsymbol{y}} = \boldsymbol{y}^{-1}(\Omega^{\boldsymbol{y}})$, so is $\Omega_{\boldsymbol{y}}$. By the Lusin property (N), we have $\mathscr{L}^N(\boldsymbol{y}(\Omega)\setminus\Omega^{\boldsymbol{y}}) \leq \mathscr{L}^N(\boldsymbol{y}(\partial\Omega)) = 0$. Similarly, thanks to both Lusin properties (N) and (N^{-1}), we have $\mathscr{L}^N(\Omega\setminus\Omega_{\boldsymbol{y}}) = \mathscr{L}^N(\boldsymbol{y}^{-1}(\boldsymbol{y}(\partial\Omega))) = 0$. Hence, $\overline{\Omega_{\boldsymbol{y}}} = \overline{\Omega}$.
We prove that $\overline{\Omega^{\boldsymbol{y}}} = \boldsymbol{y}(\overline{\Omega})$. As $\Omega^{\boldsymbol{y}} \subset \boldsymbol{y}(\Omega)$, we immediately have $\overline{\Omega^{\boldsymbol{y}}} \subset \overline{\boldsymbol{y}(\Omega)} = \boldsymbol{y}(\overline{\Omega})$. Let $\boldsymbol{\xi} \in \boldsymbol{y}(\overline{\Omega})$ and consider $\boldsymbol{x} \in \overline{\Omega}$ such that $\boldsymbol{y}(\boldsymbol{x}) = \boldsymbol{\xi}$. By density, there exists $(\boldsymbol{x}_n) \subset \Omega_{\boldsymbol{y}}$ such that $\boldsymbol{x}_n \to \boldsymbol{x}$ and, in turn, $\boldsymbol{\xi}_n := \boldsymbol{y}(\boldsymbol{x}_n) \to \boldsymbol{\xi}$. As $(\boldsymbol{\xi}_n) \subset \Omega^{\boldsymbol{y}}$ by definition, this yields $\boldsymbol{\xi} \in \overline{\Omega^{\boldsymbol{y}}}$.
Finally, we prove that $\partial\Omega^{\boldsymbol{y}} = \boldsymbol{y}(\partial\Omega)$ and $\partial\Omega_{\boldsymbol{y}} = \partial\Omega\cup\boldsymbol{y}^{-1}(\boldsymbol{y}(\partial\Omega))$. The first identity is checked as follows

$$\partial\Omega^{\boldsymbol{y}} = \overline{\Omega^{\boldsymbol{y}}}\setminus\Omega^{\boldsymbol{y}} = \boldsymbol{y}(\overline{\Omega})\setminus(\boldsymbol{y}(\Omega)\setminus\boldsymbol{y}(\partial\Omega)) = \big(\boldsymbol{y}(\overline{\Omega})\setminus\boldsymbol{y}(\Omega)\big)\cup\boldsymbol{y}(\partial\Omega) = \boldsymbol{y}(\partial\Omega),$$

while, for the second one, we have

$$\partial\Omega_{\boldsymbol{y}} = \overline{\Omega_{\boldsymbol{y}}}\setminus\Omega_{\boldsymbol{y}} = \overline{\Omega}\setminus\big(\Omega\setminus\boldsymbol{y}^{-1}(\boldsymbol{y}(\partial\Omega))\big) = \partial\Omega\cup\boldsymbol{y}^{-1}(\boldsymbol{y}(\partial\Omega).$$

$\square$

Remark 1.48 (Positive degree). In the proof of Lemma 1.47, we have implicitly proved that $\deg(\boldsymbol{y},\Omega,\cdot)$ equals zero only on $\mathbb{R}^N\setminus\boldsymbol{y}(\overline{\Omega})$, while it is strictly positive everywhere else in $\mathbb{R}^N\setminus\boldsymbol{y}(\partial\Omega)$. If we additionally assume that $\boldsymbol{y}$ is almost everywhere injective, then employing Proposition 1.9, one shows that $\deg(\boldsymbol{y},\Omega,\cdot) = 1$ on $\Omega^{\boldsymbol{y}}$.

The next two examples clarifies the significance of the set $\Omega^{\boldsymbol{y}}$. In the first one, we consider a locally invertible deformation while, in the second one, the deformation is globally invertible.

Example 1.49. *Let $N = 2$ and set $\Omega := (1/2,1)\times(-\pi,\pi)$. Define $\boldsymbol{y}\colon\Omega\to\mathbb{R}^2$ by setting*

$$\boldsymbol{y}(\boldsymbol{x}) := \begin{cases} \boldsymbol{x} & \text{if } x_2 \leq 0, \\ (x_1\cos(4x_2), x_1\sin(4x_2))^\top & \text{if } x_2 \geq 0. \end{cases}$$

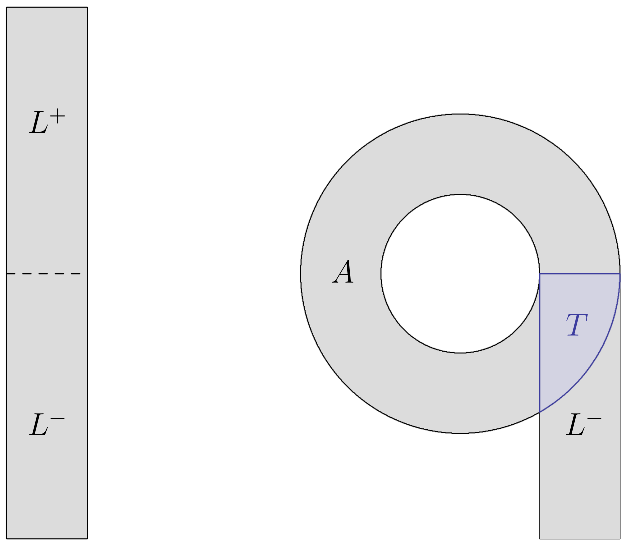

Figure 1.1: The deformation in Example 1.49.

The corresponding deformed set is depicted in Figure 1.1. In this case, $\boldsymbol{y} \in W^{1,\infty}(\Omega;\mathbb{R}^2)$*. Setting*

$$L^- := (1/2, 1) \times (-\pi, 0], \qquad L^+ := (1/2, 1) \times [0, \pi),$$

there holds $\boldsymbol{y}(L^-) = L^-$ *and* $\boldsymbol{y}(L^+) = A$*, where we set* $A := B(\mathbf{0}, 1) \setminus \overline{B}(\mathbf{0}, 1/2)$*. The deformation* $\boldsymbol{y}$ *is orientation-preserving. Indeed, for every* $\boldsymbol{x} \in L^+$*, we have* $\det D\boldsymbol{y}(\boldsymbol{x}) = x_1 > 0$*. Observe that the injectivity of* $\boldsymbol{y}$ *is violated on the preimage of the set*

$$T := \left\{ \boldsymbol{x} \in \overline{A} : \ x_1 \geq 1/2, \ x_2 \leq 0 \right\},$$

which has positive measure. Also, $\boldsymbol{y}(\Omega) = L^- \cup A$*, while* $\Omega^{\boldsymbol{y}} = (L^- \cup A) \setminus \partial T$*. In particular, the* $\boldsymbol{y}(\Omega)$ *is necessarily connected, while* $\Omega^{\boldsymbol{y}}$ *is not.*

Example 1.50 (Ball's example). *The following is inspired by [13, Example 1]. Let* $N = 3$ *and set* $\Omega := (-1, 1)^3$*. We write* $\Omega = \Omega^+ \cup P \cup \Omega^-$*, where*

$$\Omega^+ := (0, 1) \times (-1, 1)^2, \quad P := \{0\} \times (-1, 1)^2, \quad \Omega^- := (-1, 0) \times (-1, 1)^2.$$

Define $\boldsymbol{y} \colon \Omega \to \mathbb{R}^3$ *by* $\boldsymbol{y}(\boldsymbol{x}) := (x_1, x_2, |x_1|\, x_3)$*. The corresponding deformed set is depicted in Figure 1.2. In this case,* $\boldsymbol{y} \in W^{1,\infty}(\Omega;\mathbb{R}^3)$ *and, for every* $\boldsymbol{x} \in \Omega \setminus P$*, we have*

$$D\boldsymbol{y}(\boldsymbol{x}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_1\, x_3/|x_1| & 0 & |x_1| \end{pmatrix}.$$

In particular, $\det D\boldsymbol{y} > 0$ *on* $\Omega \setminus P$*. We see that* $\boldsymbol{y}(\Omega^+) = V^+$*,* $\boldsymbol{y}(P) = S$ *and*

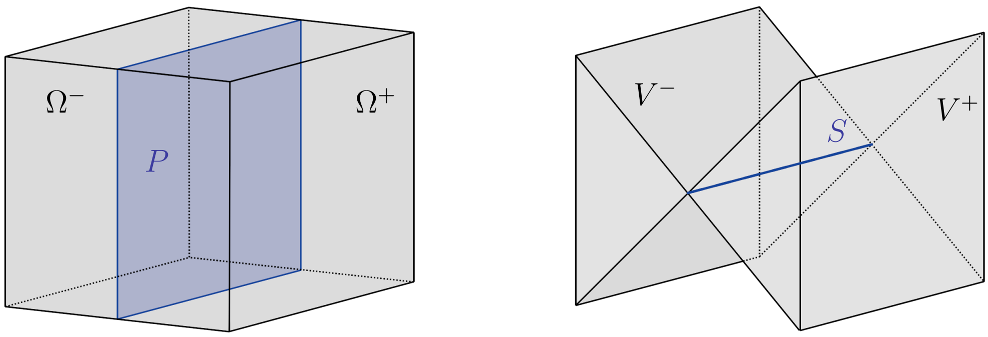

Figure 1.2: The deformation in Example 1.50.

$\boldsymbol{y}(\Omega^-) = V^-$, *where we set*

$$\begin{aligned} V^+ &:= \{\boldsymbol{\xi} \in \mathbb{R}^3 : 0 < \xi_1 < 1,\ -1 < \xi_2 < 1,\ |\xi_3| < \xi_1\}, \\ S &:= \{0\} \times (-1,1) \times \{0\}, \\ V^- &:= \{\boldsymbol{\xi} \in \mathbb{R}^3 : -1 < \xi_1 < 0,\ -1 < \xi_2 < 1,\ |\xi_3| < -\xi_1\}. \end{aligned}$$

Note that $\boldsymbol{y}|_{\Omega \setminus P}$ is injective, but $\boldsymbol{y}$ is not a homeomorphism. The image is given by $\boldsymbol{y}(\Omega) = V^+ \cup S \cup V^-$, which is not open. Instead, we have $\Omega^{\boldsymbol{y}} = V^+ \cup V^-$ since $S \subset \boldsymbol{y}\left(\overline{P} \cap \partial\Omega\right)$. Again, the set $\Omega^{\boldsymbol{y}}$ is not connected.

At this point, the reader must be warned: *given a map $\boldsymbol{y} \in \mathcal{Y}_p(\Omega)$ with $p > N$, Definition 1.15 for $U = \Omega$ is not consistent with Definition 1.30.* A simple counterexample is provided in Example 1.51 below. For this reason, when $p > N$, we prefer the notation $\Omega^{\boldsymbol{y}}$ for the topological image. However, under the additional assumption of $\boldsymbol{y}$ being almost everywhere injective, the two definitions turn out to be consistent. Indeed, in that case, there holds:

$$\Omega^{\boldsymbol{y}} = \bigcup_{U \subset\subset \Omega \text{ open}} \mathrm{im}_{\mathrm{T}}(\boldsymbol{y}, U) = \bigcup_{U \in \mathcal{U}_{\boldsymbol{y}}^*} \mathrm{im}_{\mathrm{T}}(\boldsymbol{y}, U).$$

The first equality is established in [90, Theorem 5.10]. We refer to the same paper for more information about the topological properties of orientation-preserving Sobolev maps with supercritical integrability. The second equality is checked thanks to Lemma 1.29.

Example 1.51. *Let $N = 2$ and let Ω, T and $\boldsymbol{y}$ be as in Example 1.49. The map $\boldsymbol{y}$ is locally but not globally injective. Also, we have*

$$\bigcup_{U \subset\subset \Omega \text{ open}} \mathrm{im}_{\mathrm{T}}(\boldsymbol{y}, U) = L^- \cup A, \qquad \Omega^{\boldsymbol{y}} = (L^- \cup A) \setminus \partial T.$$

We observe that, when $p > N$, the divergence identities are automatically satisfied. Indeed, any map $\boldsymbol{y} \in W^{1,p}(\Omega; \mathbb{R}^N)$ satisfies

$$\operatorname{cof} D\boldsymbol{y} \in L^{p/(N-1)}(\Omega; \mathbb{R}^{N \times N}) \subset L^{p'}(\Omega; \mathbb{R}^{N \times N}), \quad \det D\boldsymbol{y} \in L^{p/N}(\Omega) \subset L^1(\Omega).$$

Hence, in view of Example 1.22, $\boldsymbol{y}$ satisfies (DIV). Recalling (1.7), for $p > N$, we have the following equality

$$\mathcal{Y}_p(\Omega) = \left\{ \boldsymbol{y} \in W^{1,p}(\Omega;\mathbb{R}^N) : \, \det D\boldsymbol{y} > 0 \text{ a.e.} \right\}.$$

In particular, in the case of globally invertible deformations, we have

$$\mathcal{Y}_p^{\mathrm{inj}}(\Omega) = \left\{ \boldsymbol{y} \in W^{1,p}(\Omega;\mathbb{R}^N) : \, \det D\boldsymbol{y} > 0 \text{ a.e., } \boldsymbol{y} \text{ a.e. injective} \right\}. \tag{1.16}$$

Let $\boldsymbol{y} \in \mathcal{Y}_p^{\mathrm{inj}}(\Omega)$. By assumption, there exists $X \subset \Omega$ with $\mathscr{L}^N(X) = 0$ such that $\boldsymbol{y}|_{\Omega\setminus X}$ is injective. In this case, recalling Lemma 1.47, we define the global inverse $\boldsymbol{y}_\Omega^{-1} \colon \Omega^{\boldsymbol{y}} \to \mathbb{R}^N$ to be any extension of the map $(\boldsymbol{y}|_{\Omega\setminus X})^{-1}|_{\Omega^{\boldsymbol{y}}\cap \boldsymbol{y}(\Omega\setminus X)}$. Of course, this definition depends on the choice of the set X which is not unique. Recalling Lemma 1.34, a possible choice is given by $X = \Omega \setminus \mathrm{dom}_{\mathrm{G}}(\boldsymbol{y},\Omega)$. However, in view of Proposition 1.45, the choice of the set X is not relevant since $\mathscr{L}^N(\boldsymbol{y}(X)) = 0$ thanks to the Lusin property (N).
In any case, the map $\boldsymbol{y}_\Omega^{-1}$ is measurable [41, Proposition 2.2.2]. The next result show that this map has actually Sobolev regularity. This is meaningful as the set $\Omega^{\boldsymbol{y}}$ is open by Lemma 1.47. The regularity of the global inverse has been proved for more general classes of admissible deformations, see [137, Theorem 8] or [81, Theorem 3.3]. For convenience of the reader, we provide a self-contained proof in our setting. Note that, here, we do not assume any regularity of the deformation on the boundary.

Theorem 1.52 (Regularity of the global inverse). *Let $\boldsymbol{y} \in W^{1,p}(\Omega;\mathbb{R}^N)$ with $p > N$ be almost everywhere injective with $\det D\boldsymbol{y} > 0$ almost everywhere. Then, $\boldsymbol{y}_\Omega^{-1} \in W^{1,1}(\Omega^{\boldsymbol{y}};\mathbb{R}^N)$ with $D\boldsymbol{y}_\Omega^{-1} = (D\boldsymbol{y})^{-1} \circ \boldsymbol{y}_\Omega^{-1}$ almost everywhere in $\Omega^{\boldsymbol{y}}$. Moreover, each Jacobian minor of $\boldsymbol{y}_\Omega^{-1}$ belongs to $L^1(\Omega^{\boldsymbol{y}})$.*

Proof. Since Ω is bounded, it is clear that $\boldsymbol{y}_\Omega^{-1} \in L^\infty(\Omega^{\boldsymbol{y}};\mathbb{R}^N)$. Let $\varphi \in C^1(\overline{\Omega})$ and $\boldsymbol{\psi} \in C^1_{\mathrm{c}}(\Omega^{\boldsymbol{y}};\mathbb{R}^N)$, so that $\varphi(\boldsymbol{\psi}\circ\boldsymbol{y}) \in W^{1,p}_0(\Omega;\mathbb{R}^N)$. Testing the Piola identity (1.9) with $\boldsymbol{\eta} = \varphi(\boldsymbol{\psi}\circ\boldsymbol{y})$, after computations analogous to the ones in Example 1.22, we obtain

$$-\int_\Omega \mathrm{div}\boldsymbol{\psi}\circ\boldsymbol{y}\,\det D\boldsymbol{y}\,\varphi\,\mathrm{d}\boldsymbol{x} = \int_\Omega (\mathrm{adj}D\boldsymbol{y})(\boldsymbol{\psi}\circ\boldsymbol{y})\cdot D\varphi\,\mathrm{d}\boldsymbol{x} \tag{1.17}$$

Now, let $\boldsymbol{\Psi} \in C^1_{\mathrm{c}}(\Omega^{\boldsymbol{y}};\mathbb{R}^{N\times N})$ and denote its rows by $\boldsymbol{\Psi}^i \in C^1_{\mathrm{c}}(\Omega^{\boldsymbol{y}};\mathbb{R}^N)$ for every $i \in \{1,\dots,N\}$. Employing Proposition 1.46, we compute

$$\begin{aligned} -\int_{\Omega^{\boldsymbol{y}}} \boldsymbol{y}_\Omega^{-1}\cdot\mathrm{div}\boldsymbol{\Psi}\,\mathrm{d}\boldsymbol{\xi} &= -\int_\Omega \boldsymbol{x}\cdot\mathrm{div}\boldsymbol{\Psi}\circ\boldsymbol{y}\,\det D\boldsymbol{y}\,\mathrm{d}\boldsymbol{x} \\ &= -\sum_{i=1}^N \int_\Omega x_i\,\mathrm{div}\boldsymbol{\Psi}^i\circ\boldsymbol{y}\,\det D\boldsymbol{y}\,\mathrm{d}\boldsymbol{x}. \end{aligned}$$

Then, exploiting (1.17), we write

$$
\begin{aligned}
-\int_{\Omega^{\boldsymbol{y}}} \boldsymbol{y}_\Omega^{-1} \cdot \operatorname{div}\boldsymbol{\Psi} \, \mathrm{d}\boldsymbol{\xi} &= \sum_{i=1}^{N} \int_\Omega (\operatorname{adj} D\boldsymbol{y})(\boldsymbol{\Psi}^i \circ \boldsymbol{y}) \cdot \boldsymbol{e}_i \, \mathrm{d}\boldsymbol{x} \\
&= \int_\Omega (\operatorname{adj} D\boldsymbol{y}) : \boldsymbol{\Psi} \circ \boldsymbol{y} \, \mathrm{d}\boldsymbol{x} \\
&= \int_\Omega (D\boldsymbol{y})^{-1} : \boldsymbol{\Psi} \circ \boldsymbol{y} \, \det D\boldsymbol{y} \, \mathrm{d}\boldsymbol{x} \\
&= \int_{\Omega^{\boldsymbol{y}}} (D\boldsymbol{y})^{-1} \circ \boldsymbol{y}_\Omega^{-1} : \boldsymbol{\Psi} \, \mathrm{d}\boldsymbol{\xi},
\end{aligned}
$$

where in the last line we used Proposition 1.46. The previous identity shows that $\boldsymbol{y}_\Omega^{-1}$ admits a weak gradient which coincides with $(D\boldsymbol{y})^{-1} \circ \boldsymbol{y}_\Omega^{-1}$ almost everywhere in $\Omega^{\boldsymbol{y}}$. Thanks to Proposition 1.46, we have

$$
\begin{aligned}
\int_{\Omega^{\boldsymbol{y}}} |D\boldsymbol{y}_\Omega^{-1}| \, \mathrm{d}\boldsymbol{\xi} &= \int_{\Omega^{\boldsymbol{y}}} |(D\boldsymbol{y})^{-1} \circ \boldsymbol{y}_\Omega^{-1}| \, \mathrm{d}\boldsymbol{\xi} \\
&= \int_\Omega |(D\boldsymbol{y})^{-1}| \, \det D\boldsymbol{y} \, \mathrm{d}\boldsymbol{x} = \int_\Omega |\operatorname{adj} D\boldsymbol{y}| \, \mathrm{d}\boldsymbol{x},
\end{aligned}
$$

so that $\boldsymbol{y}_\Omega^{-1} \in W^{1,1}(\Omega^{\boldsymbol{y}};\mathbb{R}^N)$. The identity $D\boldsymbol{y}_\Omega^{-1} = (D\boldsymbol{y})^{-1} \circ \boldsymbol{y}_\Omega^{-1}$ almost everywhere in $\Omega^{\boldsymbol{y}}$ entails that $\det D\boldsymbol{y}_\Omega^{-1} = (\det D\boldsymbol{y})^{-1} \circ \boldsymbol{y}_\Omega^{-1}$ almost everywhere in $\Omega^{\boldsymbol{y}}$. In particular, $\det D\boldsymbol{y}_\Omega^{-1} > 0$ almost everywhere.
To prove the integrability of the Jacobian minors of $\boldsymbol{y}_\Omega^{-1}$, we recall the following formula for the minors of inverse matrices [67, Section 1.4, Chapter 1]: for every $\boldsymbol{F} \in \mathbb{R}^{N\times N}_+$ and $r \in \{1,\dots,N\}$, we have

$$
\mathbf{M}_r(\boldsymbol{F}^{-1}) = \frac{\widetilde{\mathbf{M}}_{N-r}(\boldsymbol{F})}{\det \boldsymbol{F}},
$$

where the elements of $\widetilde{\mathbf{M}}_{N-r}(\boldsymbol{F}) \in \mathbb{R}^{\binom{N}{N-r}\times\binom{N}{N-r}}$ are given, up to a possible change of sign, by elements of $\mathbf{M}_{N-r}(\boldsymbol{F})$. In particular, for $r = N$, we set $\widetilde{\mathrm{M}}_{N-r}(\boldsymbol{F}) = 1$. Therefore, employing Proposition 1.46, we compute

$$
\begin{aligned}
\int_{\Omega^{\boldsymbol{y}}} |\mathbf{M}_r(D\boldsymbol{y}_\Omega^{-1})| \, \mathrm{d}\boldsymbol{\xi} &= \int_{\Omega^{\boldsymbol{y}}} |\mathbf{M}_r((D\boldsymbol{y})^{-1} \circ \boldsymbol{y}_\Omega^{-1})| \, \mathrm{d}\boldsymbol{\xi} \\
&= \int_{\Omega^{\boldsymbol{y}}} \frac{|\widetilde{\mathbf{M}}_{N-r}(D\boldsymbol{y} \circ \boldsymbol{y}_\Omega^{-1})|}{\det D\boldsymbol{y} \circ \boldsymbol{y}_\Omega^{-1}} \, \mathrm{d}\boldsymbol{\xi} = \int_\Omega |\widetilde{\mathbf{M}}_{N-r}(D\boldsymbol{y})| \, \mathrm{d}\boldsymbol{x},
\end{aligned}
$$

where the right-hand side is bounded. □

Remark 1.53 (Sobolev homeomorphisms). Let $\boldsymbol{y} \in W^{1,p}(\Omega;\mathbb{R}^N)$ be injective with $\det D\boldsymbol{y} > 0$. In this case, it follows that $\boldsymbol{y}(\Omega) = \Omega^{\boldsymbol{y}}$ and $\boldsymbol{y}^{-1} \in W^{1,1}(\Omega^{\boldsymbol{y}};\mathbb{R}^N)$.

As a consequence of Theorem 1.52, the inverse $\boldsymbol{y}_\Omega^{-1}$ is almost everywhere approximately differentiable. In particular, the Area Formula and the Change-of-variable Formula hold for this map. The former can be used to estimate the measure of preimages of set via $\boldsymbol{y}$ as shown in the next result.

Corollary 1.54 (Measure of preimages). *Let $\boldsymbol{y} \in W^{1,p}(\Omega;\mathbb{R}^N)$ with $p > N$ be almost everywhere injective with $\det D\boldsymbol{y} > 0$ almost everywhere. Then, for every $B \subset \mathbb{R}^N$ measurable, there holds*

$$\mathscr{L}^N(\boldsymbol{y}^{-1}(B)) = \mathscr{L}^N(\boldsymbol{y}_\Omega^{-1}(B \cap \Omega^{\boldsymbol{y}})) = \int_{B\cap\Omega^{\boldsymbol{y}}} \det D\boldsymbol{y}_\Omega^{-1}\,\mathrm{d}\boldsymbol{\xi}.$$

Proof. Let $X \subset \Omega$ with $\mathscr{L}^N(X) = 0$ be such that $\boldsymbol{y}|_{\Omega\setminus X}$ is injective, so that $\boldsymbol{y}_\Omega^{-1}$ is an extension of the map $(\boldsymbol{y}|_{\Omega\setminus X})^{-1}|_{\Omega^{\boldsymbol{y}}\setminus \boldsymbol{y}(X)}$ to $\Omega^{\boldsymbol{y}}$. Recalling Proposition 1.45, we have

$$\begin{aligned}\mathscr{L}^N(\boldsymbol{y}^{-1}(B)) &= \mathscr{L}^N(\boldsymbol{y}^{-1}(B\cap\Omega^{\boldsymbol{y}})) = \mathscr{L}^N\left(y^{-1}(B\cap\Omega^{\boldsymbol{y}})\setminus \boldsymbol{y}(X)\right)\\ &= \mathscr{L}^N\left(\boldsymbol{y}^{-1}(B\cap(\Omega^{\boldsymbol{y}}\setminus\boldsymbol{y}(X))\right) = \mathscr{L}^N\left(\boldsymbol{y}_\Omega^{-1}(B\cap(\Omega^{\boldsymbol{y}}\setminus\boldsymbol{y}(X))\right)\\ &= \mathscr{L}^N(\boldsymbol{y}_\Omega^{-1}(B\cap\Omega^{\boldsymbol{y}})).\end{aligned}$$

Thus, the claim follows by applying Proposition 1.46. □

1.10 Helly Compactness Theorem

We recall below the the Helly Compactness Theorem. We state the result in a form which is sufficient for our purpouses, however more general versions are avalable, see [105, Theorem 3.2].

Let $T > 0$ and let Z be a Banach space. Given a function $\boldsymbol{z}\colon [0,T] \to Z$ and $[s,t] \subset [0,T]$, the variation of $\boldsymbol{z}$ on the interval $[s,t]$ is defined as

$$\mathrm{Var}_Z(\boldsymbol{z};[s,t]) := \sup\left\{\sum_{i=1}^{l} \|\boldsymbol{z}(t^{i-1}) - \boldsymbol{z}(t^i)\|_Z :\ \Pi = (t^0,t^1,\dots,t^l)\right\},$$

where the supremum is taken over all the partitions Π of $[s,t]$. We define the space $BV([0,T];Z)$ as the set of functions $\boldsymbol{z}\colon [0,T] \to Z$ such that $\mathrm{Var}_Z(\boldsymbol{z};[0,T]) < +\infty$.

Theorem 1.55 (Helly Compactness Theorem). *Let Z be a Banach space and let $\mathcal{K} \subset Z$ be compact. Let $(\boldsymbol{z}_n) \subset BV([0,T];Z)$ be such that the following hold:*

$$\forall\, n \in \mathbb{N},\ \forall\, t \in [0,T], \quad \boldsymbol{z}_n(t) \in \mathcal{K},$$

$$\sup_{n\in\mathbb{N}} \mathrm{Var}_Z(\boldsymbol{z}_n;[0,T]) < +\infty.$$

Then, there exists $\boldsymbol{z} \in BV([0,T];Z)$ such that, up to subsequences, there holds:

$$\forall\, t \in [0,T], \quad \boldsymbol{z}_n(t) \to \boldsymbol{z}(t) \text{ in } Z.$$

Proof. This is a special case of [105, Theorem 3.2]. □

1.11 Measurable Selection Theorem

In this section, we briefly recall some notions of set-valued analysis. Without seeking generality, these notions are presented according to our specific needs. For more information, we refer to [7].

Let $\mathcal{X}$ be a complete separable metric space with distance d. Given $A \subset \mathcal{X}$, for every $\boldsymbol{x}_0 \in \mathcal{X}$ we set $\mathrm{dist}(\boldsymbol{p}_0;A) := \inf\{d(\boldsymbol{p}_0,\boldsymbol{p}) :\ \boldsymbol{p} \in A\}$.

Definition 1.56 (Lower and upper limits of sets). Let $(A_n) \subset \mathcal{P}(\mathcal{X})$. The lower and upper limits of (A_n) in the sense of Kuratowski are defined as

$$\operatorname*{Liminf}_{n\to\infty} A_n \coloneqq \{\boldsymbol{p}_0 \in \mathcal{X} :\ \lim_{n\to\infty} \operatorname{dist}(\boldsymbol{p}_0; A_n) = 0\},$$
$$\operatorname*{Limsup}_{n\to\infty} A_n \coloneqq \{\boldsymbol{p}_0 \in \mathcal{X} :\ \liminf_{n\to\infty} \operatorname{dist}(\boldsymbol{p}_0; A_n) = 0\}.$$

Equivalently, $\operatorname{Liminf}_{n\to\infty} A_n$ is the set of limits of sequences $(\boldsymbol{p}_n) \subset \mathcal{X}$ with $\boldsymbol{p}_n \in A_n$ for every $n \in \mathbb{N}$, while $\operatorname{Limsup}_{n\to\infty} A_n$ consists of all limit points, i.e. all limits of subsequences, of such sequences [7, Proposition 1.1.2]. Note that both sets $\operatorname{Liminf}_{n\to\infty} A_n$ and $\operatorname{Limsup}_{n\to\infty} A_n$ are necessarily closed.
Let $T > 0$. A set-valued map from $[0,T]$ to $\mathcal{X}$ is a function $\mathcal{S}\colon [0,T] \to \mathcal{P}(\mathcal{X})$. In this context, the concept of measurability is defined as follows.

Definition 1.57 (Measurability of set-valued maps). A map $\mathcal{S}\colon [0,T] \to \mathcal{P}(\mathcal{X})$ is termed measurable if, for every $A \subset \mathcal{X}$ open, the set

$$\mathcal{S}^{-1}(A) \coloneqq \{t \in [0,T] :\ \mathcal{S}(t) \cap A \neq \emptyset\}$$

is measurable.

Concerning the measurability of lower and upper limits, we have the following result.

Proposition 1.58 ([7, Theorem 8.2.5]). *Let $(\mathcal{S}_n)$ be a sequence of measurable set-valued maps from $\mathcal{S}_n\colon [0,T] \to \mathcal{P}(\mathcal{X})$. Then, the set-valued maps*

$$t \mapsto \operatorname*{Liminf}_{n\to\infty} \mathcal{S}_n(t), \qquad t \mapsto \operatorname*{Limsup}_{n\to\infty} \mathcal{S}_n(t),$$

are both measurable.

We now give the definition of measurable selection.

Definition 1.59 (Measurable selection). Let $\mathcal{S}\colon [0,T] \to \mathcal{P}(\mathcal{X})$. A measurable map $\boldsymbol{s}\colon [0,T] \to \mathcal{X}$ such that $\boldsymbol{s}(t) \in \mathcal{S}(t)$ for every $t \in [0,T]$ is termed a measurable selection of $\mathcal{S}$.

The existence of measurable selections is ensured by the following result which will be employed to prove the existence of measurable energetic solutions.

Theorem 1.60 (Measurable Selection Theorem). *Let $\mathcal{S}\colon [0,T] \to \mathcal{P}(\mathcal{X})$ be measurable and suppose that, for every $t \in [0,T]$, the set $\mathcal{S}(t)$ is closed and nonempty. Then, the set-valued map $\mathcal{S}$ admits a measurable selection.*

Proof. We refer to [7, Theorem 8.1.3]. □

Part I

Existence results

Chapter 2

Existence of minimizers

In this chapter, we investigate the problem of the existence of solutions in the static setting. These correspond to minimizers of the magnetoelastic energy. The main results of the chapter are two: Theorem 2.6, which is a compactness result, and Theorem 2.5, stating the existence of minimizers under Dirichlet boundary conditions on the deformations. The results of this chapter are going to be instrumental also for the analysis in the quasistatic setting.
In Section 2.1, we describe the variational model and we list all the assumptions. Then, in Section 2.2, we prove a compactness result which will be employed in Section 2.3 to establish the existence of minimizers.

2.1 Setting of the problem

Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain and let $p > N-1$ be fixed. Recall (1.7). For each $\boldsymbol{y} \in \mathcal{Y}_p(\Omega)$, magnetizations are given by maps $\boldsymbol{m} \in W^{1,2}(\mathrm{im}_\mathrm{T}(\boldsymbol{y},\Omega);\mathbb{R}^N)$ satisfying the constraint

$$|\boldsymbol{m} \circ \boldsymbol{y}| \det D\boldsymbol{y} = 1 \text{ a.e. in } \Omega. \tag{2.1}$$

As $\det D\boldsymbol{y} > 0$ almost everwhere in Ω, condition (2.1) entails that $\boldsymbol{m} \neq \boldsymbol{0}$ almost everywhere in $\mathrm{im}_\mathrm{T}(\boldsymbol{y},\Omega)$. Admissible states belong to the class

$$\begin{aligned} \mathcal{Q} \coloneqq \big\{ (\boldsymbol{y},\boldsymbol{m}) :\ \boldsymbol{y} \in \mathcal{Y}_p(\Omega), \quad & \boldsymbol{m} \in W^{1,2}(\mathrm{im}_\mathrm{T}(\boldsymbol{y},\Omega);\mathbb{R}^N), \\ & |\boldsymbol{m} \circ \boldsymbol{y}| \det D\boldsymbol{y} = 1 \text{ a.e. in } \Omega \big\}. \end{aligned} \tag{2.2}$$

This class is endowed with the topology that makes the map

$$(\boldsymbol{y},\boldsymbol{m}) \mapsto (\boldsymbol{y},\ \chi_{\mathrm{im}_\mathrm{T}(\boldsymbol{y},\Omega)}\boldsymbol{m},\ \chi_{\mathrm{im}_\mathrm{T}(\boldsymbol{y},\Omega)}D\boldsymbol{m})$$

from $\mathcal{Q}$ to $W^{1,p}(\Omega;\mathbb{R}^N) \times L^2(\mathbb{R}^N;\mathbb{R}^N) \times L^2(\mathbb{R}^N;\mathbb{R}^{N\times N})$ a homeomorphism onto its image, where the latter space is equipped with the product weak topology. When considering only globally injective deformations, we restrict ourselves to the class

$$\mathcal{Q}^{\mathrm{inj}} \coloneqq \big\{ (\boldsymbol{y},\boldsymbol{m}) \in \mathcal{Q} :\ \boldsymbol{y} \in \mathcal{Y}_p^{\mathrm{inj}}(\Omega) \big\}, \tag{2.3}$$

where we recall (1.13).
The magnetoelastic energy functional $E\colon \mathcal{Q} \to [0,+\infty]$ is defined by setting

$$E(\boldsymbol{q}) \coloneqq \int_\Omega W(D\boldsymbol{y},\boldsymbol{m}\circ\boldsymbol{y})\,\mathrm{d}\boldsymbol{x} + \int_{\mathrm{im}_\mathrm{T}(\boldsymbol{y},\Omega)} |D\boldsymbol{m}|^2\,\mathrm{d}\boldsymbol{\xi} + \frac{1}{2}\int_{\mathbb{R}^N} |D\psi_{\boldsymbol{m}}|^2\,\mathrm{d}\boldsymbol{\xi}, \tag{2.4}$$

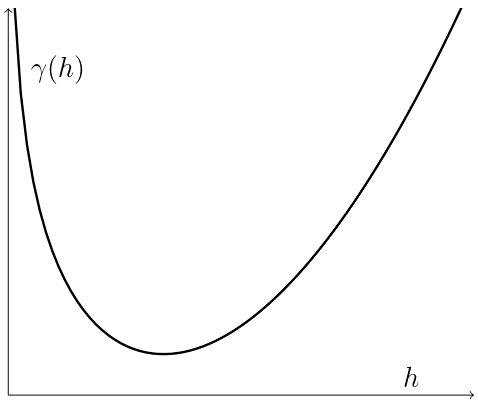

Figure 2.1: Qualitative behaviour of the functions γ in (2.8).

where $\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m})$. We denote the functionals determined by the three terms in the previous formula by E^{el}, E^{exc} and E^{mag}, respectively.
The first term in (2.4) accounts for the elastic energy and is determined by the elastic energy density $W\colon \mathbb{R}^{N\times N}_+ \times \mathbb{R}^N \setminus \{\boldsymbol{0}\} \to [0,+\infty]$. On the function W, we take the following two assumptions:

(i) Coercivity: there exist $C>0$ and $\gamma\colon (0,+\infty) \to (0,+\infty)$ Borel satisfying

$$\lim_{h\to 0^+} h\,\gamma(h) = +\infty, \qquad \lim_{h\to +\infty} \frac{\gamma(h)}{h} = +\infty, \tag{2.5}$$

such that

$$\forall\, \boldsymbol{F} \in \mathbb{R}^{N\times N}_+,\ \forall\, \boldsymbol{\lambda} \in \mathbb{R}^N, \quad W(\boldsymbol{F},\boldsymbol{\lambda}) \geq C|\boldsymbol{F}|^p + \gamma(\det \boldsymbol{F}); \tag{2.6}$$

(ii) Partial polyconvexity: there exists a continuous function

$$\widehat{W}\colon \mathbb{R}^{N\times N}_+ \times \cdots \times \mathbb{R}^{N\times N}_+ \times (0,+\infty) \times \mathbb{R}^N \setminus \{\boldsymbol{0}\} \to [0,+\infty),$$

such that the map $\boldsymbol{F} \mapsto \widehat{W}(\boldsymbol{F}, \mathbf{M}_2(\boldsymbol{F}), \dots, \mathbf{M}_{N-2}(\boldsymbol{F}), \mathrm{cof}\boldsymbol{F}, h, \boldsymbol{\lambda})$ is convex for every $h>0$ and $\boldsymbol{\lambda} \in \mathbb{R}^N \setminus \{\boldsymbol{0}\}$, and, for every $\boldsymbol{F} \in \mathbb{R}^{N\times N}_+$ and $\boldsymbol{\lambda} \in \mathbb{R}^N \setminus \{\boldsymbol{0}\}$ with $|\boldsymbol{\lambda}| \det \boldsymbol{F} = 1$, there holds

$$W(\boldsymbol{F},\boldsymbol{\lambda}) = \widehat{W}(\boldsymbol{F}, \mathbf{M}_2(\boldsymbol{F}), \dots, \mathbf{M}_{N-2}(\boldsymbol{F}), \mathrm{cof}\boldsymbol{F}, \det \boldsymbol{F}, \boldsymbol{\lambda}). \tag{2.7}$$

Concerning (i), prototypical examples of functions γ satifying (2.5) are given by

$$\gamma(h) \coloneqq h^{-a} + h^b, \qquad \gamma(h) \coloneqq ch^2 - d\log h, \tag{2.8}$$

for some $a,b>1$ and $c,d>0$ (see Figure 2.1). In particular, it is not restrictive, but also not necessary, to assume that γ is convex.
Observe that the first limit condition in (2.5) entails the more natural one

$$\lim_{h\to 0^+} \gamma(h) = +\infty. \tag{2.9}$$

This condition is fundamental from the modeling point of view, as it ensures that the elastic energy blows up in case of extreme compressions. Additionally, in view of (2.9), we can extend the map W by continuity as $W\colon \mathbb{R}^{N\times N}\times\mathbb{R}^N\to[0,+\infty]$ by setting $W(\boldsymbol{F},\boldsymbol{\lambda})=+\infty$ for every $\boldsymbol{F}\in\mathbb{R}^{N\times N}$ with $\det\boldsymbol{F}\leq 0$ and $\boldsymbol{\lambda}\in\mathbb{R}^N$. Moreover, thanks to (2.5), each $\boldsymbol{q}=(\boldsymbol{y},\boldsymbol{m})\in\mathcal{Q}$ with $E^{\mathrm{el}}(\boldsymbol{q})<+\infty$ satisfies $\det D\boldsymbol{y}\in L^1(\Omega)$ and $1/\det D\boldsymbol{y}\in L^1(\Omega)$.

Condition (ii) is close to the assumption of $W(\cdot,\boldsymbol{\lambda})$ being polyconvex for every $\boldsymbol{\lambda}\in\mathbb{R}^N$. On the one hand, the function $\widehat{W}$ does not depend on $\boldsymbol{\lambda}\in\mathbb{R}^N$. On the other hand, in (ii), the convexity requirement does not involve the determinant of $\boldsymbol{F}$. An example of function W satisfying (ii) is provided in Example 2.1. In (ii), for every $r\in\{1,\dots,N\}$, we denote by $\mathbf{M}_r(\boldsymbol{F})$ the square matrix of order $\binom{N}{r}$ whose terms are all the minors of order r of $\boldsymbol{F}$ with a given sign and order. A possible choice in this sense is described in [43, Section 5.4]. In particular, there hold $\mathbf{M}_1(\boldsymbol{F})=\boldsymbol{F}$, $\mathbf{M}_{N-1}(\boldsymbol{F})=\mathrm{cof}\boldsymbol{F}$ and $\mathrm{M}_N(\boldsymbol{F})=\det\boldsymbol{F}$. To clarify, in the case $N=2$, the identity (2.7) reads

$$W(\boldsymbol{F},\boldsymbol{\lambda})=\widehat{W}(\boldsymbol{F},\det\boldsymbol{F},\boldsymbol{\lambda}),$$

while, for $N=3$, we have

$$W(\boldsymbol{F},\boldsymbol{\lambda})=\widehat{W}(\boldsymbol{F},\mathrm{cof}\boldsymbol{F},\det\boldsymbol{F},\boldsymbol{\lambda}).$$

Note that the continuity of $\widehat{W}$ entails the one of W. Since $\boldsymbol{y}$ has the Lusin property (N^{-1}), the composition $\boldsymbol{m}\circ\boldsymbol{y}$ is measurable and well defined, in the sense that the equivalence class of the composition does not depend on the representative of $\boldsymbol{m}$ nor of the one of $\boldsymbol{y}$, by claim (ii) of Lemma 1.4. This ensures the measurability of the integrand in $E^{\mathrm{el}}(\boldsymbol{q})$.

Example 2.1. *Let $N=3$. Define $\boldsymbol{L}\colon\mathbb{S}^2\to\mathbb{R}^{3\times3}_+$ by setting $\boldsymbol{L}(\boldsymbol{z}):=\boldsymbol{I}+c\boldsymbol{z}\otimes\boldsymbol{z}$, where $c>0$ is given. Let $\Phi\colon\mathbb{R}^{N\times N}_+\to[0,+\infty]$ be continuous and define the function $W\colon\mathbb{R}^{N\times N}_+\times\mathbb{R}^N\setminus\{\boldsymbol{0}\}\to[0,+\infty]$ by setting*

$$W(\boldsymbol{F},\boldsymbol{\lambda}):=\Phi(\boldsymbol{L}(\boldsymbol{\lambda}\det\boldsymbol{F})^{-1}\boldsymbol{F}).$$

Observe that, for every $\boldsymbol{z}\in\mathbb{S}^2$, the matrix $\boldsymbol{L}(\boldsymbol{z})$ is actually invertible since

$$\det\boldsymbol{L}(\boldsymbol{z})=1+c.$$

Suppose that the function Φ satisfies the following property of partial polyconvexity: there exists $\widehat{\Phi}\colon\mathbb{R}^{N\times N}_+\times\mathbb{R}^{N\times N}_+\times(0,+\infty)\to[0,+\infty)$ continuous such that the map $\Xi\mapsto\widehat{\Phi}(\Xi,\mathrm{cof}\Xi,d)$ is convex for every $d>0$ and, for every $\Xi\in\mathbb{R}^{3\times3}_+$, there holds

$$\Phi(\Xi)=\widehat{\Phi}(\Xi,\mathrm{cof}\,\Xi,\det\Xi).$$

In this case, W satisfies condition (ii) above. This fact is very easy to check. Let $\boldsymbol{F}\in\mathbb{R}^{N\times N}_+$ and $\boldsymbol{\lambda}\in\mathbb{R}^N\setminus\{\boldsymbol{0}\}$ with $|\boldsymbol{\lambda}|\det\boldsymbol{F}=1$. Using the Cauchy-Binet formula together with the identities

$$\mathrm{cof}\,(\boldsymbol{A}^{-1})=\frac{1}{\det\boldsymbol{A}}\boldsymbol{A}^\top,\quad \det(\boldsymbol{A}^{-1})=\frac{1}{\det\boldsymbol{A}},$$

for every $\boldsymbol{A}\in\mathbb{R}^{3\times3}_+$, we have

$$\begin{aligned}W(\boldsymbol{F},\boldsymbol{\lambda})&=\widehat{\Phi}\left(\boldsymbol{L}(\boldsymbol{\lambda}\det\boldsymbol{F})^{-1}\boldsymbol{F},\mathrm{cof}(\boldsymbol{L}(\boldsymbol{\lambda}\det\boldsymbol{F})^{-1})\mathrm{cof}\boldsymbol{F},\det(\boldsymbol{L}(\boldsymbol{\lambda}\det\boldsymbol{F})^{-1})\det\boldsymbol{F}\right)\\&=\widehat{\Phi}\left(\boldsymbol{L}(\boldsymbol{\lambda}\det\boldsymbol{F})^{-1}\boldsymbol{F},\frac{1}{\det\boldsymbol{L}(\boldsymbol{\lambda}\det\boldsymbol{F})}\boldsymbol{L}(\boldsymbol{\lambda}\det\boldsymbol{F})^\top\mathrm{cof}\boldsymbol{F},\frac{\det\boldsymbol{F}}{\det\boldsymbol{L}(\boldsymbol{\lambda}\det\boldsymbol{F})}\right).\end{aligned}$$

Therefore, we define $\widehat{W}\colon \mathbb{R}^{N\times N}_+ \times \mathbb{R}^{N\times N}_+ \times (0,+\infty)\times \mathbb{R}^N \setminus \{\mathbf{0}\} \to [0,+\infty)$ *by setting*

$$\widehat{W}(\boldsymbol{F},\boldsymbol{G},h,\boldsymbol{\lambda}) := \widehat{\Phi}\left(\boldsymbol{L}(\boldsymbol{\lambda}h)^{-1}\boldsymbol{F}, \frac{1}{\det \boldsymbol{L}(\boldsymbol{\lambda}h)}\boldsymbol{L}(\boldsymbol{\lambda}h)^{\top}\boldsymbol{G}, \frac{h}{\det \boldsymbol{L}(\boldsymbol{\lambda}h)}\right).$$

From the convexity of $\Xi \mapsto \widehat{\Phi}(\Xi,\operatorname{cof}\Xi,d)$ *for every* $d>0$*, we deduce the one of* $(\boldsymbol{F},\boldsymbol{G}) \mapsto \widehat{W}(\boldsymbol{F},\boldsymbol{G},h,\boldsymbol{\lambda})$ *for every* $h>0$ *and* $\boldsymbol{\lambda}\in\mathbb{R}^N$*. Moreover, the previous computations give the identity*

$$W(\boldsymbol{F},\boldsymbol{\lambda}) = \widehat{W}(\boldsymbol{F},\operatorname{cof}\boldsymbol{F},\det\boldsymbol{F},\boldsymbol{\lambda})$$

for every $\boldsymbol{F}\in\mathbb{R}^{N\times N}_+$ *and* $\boldsymbol{\lambda}\in\mathbb{R}^N\setminus\{\mathbf{0}\}$ *with* $|\boldsymbol{\lambda}|\det\boldsymbol{F}=1$*. This proves the claim. An analogous example is valid for arbitrary dimension* $N\in\mathbb{N}$*.*

The second term in (2.4) constitutes the exchange energy and is given by an integral on $\mathrm{im}_{\mathrm{T}}(\boldsymbol{y},\Omega)$, where we refer to Definition 1.30. Since this set depends only on the equivalence class of $\boldsymbol{y}$, we see that the exchange energy is well defined.
The third term in (2.4) represents the magnetostatic energy and involves the stray field potential $\psi_{\boldsymbol{m}}\colon \mathbb{R}^N \to \mathbb{R}^N$. This is given by a weak solution of the Maxwell equation:

$$\operatorname{div}\left(D\psi_{\boldsymbol{m}} - \chi_{\mathrm{im}_{\mathrm{T}}(\boldsymbol{y},\Omega)}\boldsymbol{m}\right) = 0 \quad \text{in } \mathbb{R}^N. \tag{2.10}$$

Equivalently, $\psi_{\boldsymbol{m}}\in V^{1,2}(\mathbb{R}^N)$ satisfies the following:

$$\forall\,\varphi\in V^{1,2}(\mathbb{R}^N), \qquad \int_{\mathbb{R}^N}\left(D\psi_{\boldsymbol{m}} - \chi_{\mathrm{im}_{\mathrm{T}}(\boldsymbol{y},\Omega)}\boldsymbol{m}\right)\cdot D\varphi\,\mathrm{d}\boldsymbol{\xi} = 0. \tag{2.11}$$

The next result provides the existence of weak solutions to the Maxwell equation, which are shown to be unique up to additive constants. As a consequence, the magnetostatic energy is actually well defined.

Lemma 2.2 (Weak solutions of the Maxwell equation). *Let* $\boldsymbol{q}=(\boldsymbol{y},\boldsymbol{m})\in\mathcal{Q}$*. Then, the Maxwell equation* (2.10) *admits a weak solution* $\psi_{\boldsymbol{m}}\in V^{1,2}(\mathbb{R}^N)$ *which is unique up to additive constants and satisfies*

$$\|D\psi_{\boldsymbol{m}}\|_{L^2(\mathbb{R}^N;\mathbb{R}^N)} \le \|\chi_{\mathrm{im}_{\mathrm{T}}(\boldsymbol{y},\Omega)}\boldsymbol{m}\|_{L^2(\mathbb{R}^N;\mathbb{R}^N)}. \tag{2.12}$$

Proof. The proof is taken from [18, Proposition 8.8]. First, observe that the quotient space $V^{1,2}(\mathbb{R}^N)/\mathbb{R}$ is an Hilbert space with respect to the inner product

$$([\varphi],[\psi])_{V^{1,2}(\mathbb{R}^N)/\mathbb{R}} = \int_{\mathbb{R}^N} D\varphi\cdot D\psi\,\mathrm{d}\boldsymbol{\xi}.$$

Also, the linear functional $T_{\boldsymbol{m}}\colon V^{1,2}(\mathbb{R}^N)/\mathbb{R}\to\mathbb{R}$ defined by

$$T_{\boldsymbol{m}}([\varphi]) := \int_{\mathbb{R}^N}\chi_{\mathrm{im}_{\mathrm{T}}(\boldsymbol{y},\Omega)}\boldsymbol{m}\cdot D\varphi\,\mathrm{d}\boldsymbol{\xi}$$

is well defined and continuous on this space. Indeed, by the Hölder inequality, we have

$$\begin{aligned}|T_{\boldsymbol{m}}([\varphi])| &\le \|\chi_{\mathrm{im}_{\mathrm{T}}(\boldsymbol{y},\Omega)}\boldsymbol{m}\|_{L^2(\mathbb{R}^N;\mathbb{R}^N)}\,\|D\varphi\|_{L^2(\mathbb{R}^N;\mathbb{R}^N)}\\ &= \|\chi_{\mathrm{im}_{\mathrm{T}}(\boldsymbol{y},\Omega)}\boldsymbol{m}\|_{L^2(\mathbb{R}^N;\mathbb{R}^N)}\,\|[\varphi]\|_{V^{1,2}(\mathbb{R}^N)/\mathbb{R}}\,.\end{aligned} \tag{2.13}$$

Therefore, by the Riesz Representation Theorem, there exists $\psi_{\boldsymbol{m}} \in V^{1,2}(\mathbb{R}^N)$ such that $T_{\boldsymbol{m}}([\varphi]) = ([\psi_{\boldsymbol{m}}],[\varphi])_{V^{1,2}(\mathbb{R}^N)/\mathbb{R}}$ for every $\varphi \in V^{1,2}(\mathbb{R}^N)$, so that (2.11) is satisfied. As its class in $V^{1,2}(\mathbb{R}^N)/\mathbb{R}$ is uniquely defined, the function $\psi_{\boldsymbol{m}}$ is unique up to additive constants. Estimate (2.12) follows since, in view of (2.13), we have

$$\|[\psi_{\boldsymbol{m}}]\|_{V^{1,2}(\mathbb{R}^N)/\mathbb{R}} = \|T_{\boldsymbol{m}}\|_{(V^{1,2}(\mathbb{R}^N)/\mathbb{R})'} \leq \|\chi_{\mathrm{im}_{\mathrm{T}}(\boldsymbol{y},\Omega)}\boldsymbol{m}\|_{L^2(\mathbb{R}^N;\mathbb{R}^N)}.$$

□

Remark 2.3 (Variational characterization). The weak solution $\psi_{\boldsymbol{m}}$ of (2.10) can be equivalently characterized as a minimizer of the convex functional

$$\varphi \mapsto \int_{\mathbb{R}^N} |D\varphi - \chi_{\mathrm{im}_{\mathrm{T}}(\boldsymbol{y},\Omega)}\boldsymbol{m}|^2 \,\mathrm{d}\boldsymbol{\xi}$$

defined on $V^{1,2}(\mathbb{R}^N)$. Indeed, (2.11) represents the weak form of the corresponding Euler-Lagrange equation. Thus, the Direct Method provides an alternative way to prove the existence of weak solutions of (2.10).

Remark 2.4 (Integrability of the stray field potential). Since the datum of the Maxwell equation, i.e. the map $\chi_{\mathrm{im}_{\mathrm{T}}(\boldsymbol{y},\Omega)}\boldsymbol{m}$, has trivially compact support, it can be shown that (2.10) admits a solution belonging to $W^{1,2}(\mathbb{R}^N)$. For a proof, we refer to [55, Theorem 2 and Remark 3.1]

The aim of the chapter is to establish the existence of minimizers of the magnetoelastic energy functional under Dirichlet boundary conditions on the deformations. The result reads as follows.

Theorem 2.5 (Existence of mimimizers). *Assume* (2.6)–(2.7). *Let* $\Gamma \subset \partial\Omega$ *be measurable with* $\mathscr{H}^{N-1}(\Gamma) > 0$ *and let* $\overline{\boldsymbol{y}} \in \mathcal{Y}_p(\Omega)$. *Set*

$$\overline{\mathcal{Q}} \coloneqq \{\boldsymbol{q} = (\boldsymbol{y},\boldsymbol{m}) \in \mathcal{Q} : \ \boldsymbol{y} = \overline{\boldsymbol{y}} \text{ on } \Gamma\} \tag{2.14}$$

Then, the functional E *admits a minimizer in* $\overline{\mathcal{Q}}$. *Analogously, if* $\overline{\boldsymbol{y}} \in \mathcal{Y}_p^{\mathrm{inj}}(\Omega)$, *then the functional* E *admits a minimizer in* $\overline{\mathcal{Q}}^{\mathrm{inj}} \coloneqq \overline{\mathcal{Q}} \cap \mathcal{Q}^{\mathrm{inj}}$.

Some variants of the previous theorem will be discussed at the end of Section 2.3.

2.2 Compactness

In this section, we establish a compactness result that is going to be instrumental for the proof of the existence of minimizers as well as for the analysis in the quasistatic setting.

Theorem 2.6 (Compactness). *Let* $(\boldsymbol{q}_n) \subset \mathcal{Q}$ *with* $\boldsymbol{q}_n = (\boldsymbol{y}_n,\boldsymbol{m}_n)$ *for every* $n \in \mathbb{N}$. *Suppose that*

$$\boldsymbol{y}_n \rightharpoonup \boldsymbol{y} \text{ in } W^{1,p}(\Omega;\mathbb{R}^N), \qquad \det D\boldsymbol{y}_n \rightharpoonup \det D\boldsymbol{y} \text{ in } L^1(\Omega). \tag{2.15}$$

for some $\boldsymbol{y} \in \mathcal{Y}_p(\Omega)$. *Additionally, suppose that*

$$(1/\det D\boldsymbol{y}_n) \text{ is equi-integrable,} \tag{2.16}$$

$$(\chi_{\mathrm{im}_{\mathrm{T}}(\boldsymbol{y}_n,\Omega)}D\boldsymbol{m}_n) \text{ is bounded in } L^2(\mathbb{R}^N;\mathbb{R}^{N\times N}). \tag{2.17}$$

Then, there exists $\boldsymbol{m} \in W^{1,2}(\mathrm{im}_{\mathrm{T}}(\boldsymbol{y},\Omega);\mathbb{R}^N)$ *such that* $\boldsymbol{q} = (\boldsymbol{y},\boldsymbol{m}) \in \mathcal{Q}$ *and, up to subsequences, the following convergences hold:*

$$\chi_{\mathrm{im}_{\mathrm{T}}(\boldsymbol{y}_n,\Omega)}\boldsymbol{m}_n \rightharpoonup \chi_{\mathrm{im}_{\mathrm{T}}(\boldsymbol{y},\Omega)}\boldsymbol{m} \text{ in } L^2(\mathbb{R}^N;\mathbb{R}^N); \tag{2.18}$$

$$\chi_{\mathrm{im}_{\mathrm{T}}(\boldsymbol{y}_n,\Omega)}D\boldsymbol{m}_n \rightharpoonup \chi_{\mathrm{im}_{\mathrm{T}}(\boldsymbol{y},\Omega)}D\boldsymbol{m} \text{ in } L^2(\mathbb{R}^N;\mathbb{R}^{N\times N}); \tag{2.19}$$

$$\boldsymbol{m}_n \circ \boldsymbol{y}_n \to \boldsymbol{m} \circ \boldsymbol{y} \text{ a.e. in } \Omega \text{ and in } L^1(\Omega;\mathbb{R}^N). \tag{2.20}$$

Moreover, up to subsequences, there hold:

$$\det D\boldsymbol{y}_n \to \det D\boldsymbol{y} \text{ in } L^1(\Omega), \qquad \frac{1}{\det D\boldsymbol{y}_n} \to \frac{1}{\det D\boldsymbol{y}} \text{ in } L^1(\Omega); \tag{2.21}$$

$$\boldsymbol{m}_n \circ \boldsymbol{y}_n \det D\boldsymbol{y}_n \to \boldsymbol{m} \circ \boldsymbol{y} \det D\boldsymbol{y} \text{ in } L^q(\Omega;\mathbb{R}^N) \text{ for } 1 \leq q < \infty; \tag{2.22}$$

$$\chi_{\mathrm{im}_{\mathrm{T}}(\boldsymbol{y}_n,\Omega)}\boldsymbol{m}_n \to \chi_{\mathrm{im}_{\mathrm{T}}(\boldsymbol{y},\Omega)}\boldsymbol{m} \text{ a.e. in } \mathbb{R}^N \text{ and in } L^q(\mathbb{R}^N;\mathbb{R}^N) \text{ for } 1 \leq q < 2. \tag{2.23}$$

In particular, if $(\boldsymbol{q}_n) \subset \mathcal{Q}^{\mathrm{inj}}$*, then* $\boldsymbol{q} \in \mathcal{Q}^{\mathrm{inj}}$ *and, up to subsequences, there holds:*

$$\chi_{\mathrm{im}_{\mathrm{T}}(\boldsymbol{y}_n,\Omega)}\boldsymbol{m}_n \to \chi_{\mathrm{im}_{\mathrm{T}}(\boldsymbol{y},\Omega)}\boldsymbol{m} \text{ in } L^2(\mathbb{R}^N;\mathbb{R}^N). \tag{2.24}$$

For convenience of the reader, we will provide two different proofs of this result: a first one for the case $p > N$ assuming the global invertibility of deformations, and a second one without any additional assumption. The two proof are presented in the next two subsections.

2.2.1 Proof of Theorem 2.6 for $p > N$ assuming global invertibility

We provide a first proof of Theorem 2.6 under the two additional assumptions:

$$p > N, \qquad (\boldsymbol{q}_n) \subset \mathcal{Q}^{\mathrm{inj}}. \tag{2.25}$$

Note that, given (2.25), the second assumption in (2.15) is redundant as it follows from the first one by the weak continuity of minors.
In this first proof of Theorem 2.6, the identification of the limiting magnetization as well as the verification of the magnetic saturation constraint for the limiting state simplify a bit in view of the precise description of the topological image provided by Lemma 1.47. However, the general strategy does not depend on the additional assumptions (2.25). Otherwise, the proof of the convergence of the compositions provided here exploits substantially the continuity of admissible deformations and their uniform convergence inferred by the Morrey embedding.

Proof of Theorem 2.6 for $p > N$ *assuming global invertibility.* For convenience of the reader, the proof is subdivided into five steps.

Step 1 (Compactness of magnetizations). By the Morrey embedding, up to subsequences. there holds $\boldsymbol{y}_n \to \boldsymbol{y}$ uniformly in Ω. As $(\boldsymbol{y}_n) \subset \mathcal{Y}_p^{\mathrm{inj}}(\Omega)$ by (2.25), from (2.15) and Corollary 1.43, we deduce that $\boldsymbol{y} \in \mathcal{Y}_p^{\mathrm{inj}}(\Omega)$. For every $n \in \mathbb{N}$, set

$$\boldsymbol{v}_n := \chi_{\Omega^{\boldsymbol{y}_n}}\boldsymbol{m}_n, \qquad \boldsymbol{V}_n := \chi_{\Omega^{\boldsymbol{y}_n}}D\boldsymbol{m}_n.$$

Applying Proposition 1.46 and exploiting (2.1) and (2.25), we compute

$$\int_{\mathrm{im}_{\mathrm{T}}(\boldsymbol{y}_n,\Omega)} |\boldsymbol{m}_n|^2 \,\mathrm{d}\boldsymbol{\xi} = \int_\Omega |\boldsymbol{m}_n \circ \boldsymbol{y}_n|^2 \det D\boldsymbol{y}_n \,\mathrm{d}\boldsymbol{x} = \int_\Omega \frac{1}{\det D\boldsymbol{y}_n} \,\mathrm{d}\boldsymbol{x}. \tag{2.26}$$

By (2.16), the right-hand side of the previous equation is uniformly bounded, so that $(\boldsymbol{v}_n)$ is bounded in $L^2(\mathbb{R}^N;\mathbb{R}^N)$. Also $(\boldsymbol{V}_n) \subset L^2(\mathbb{R}^N;\mathbb{R}^{N\times N})$ is bounded thanks to (2.17). Therefore, there exist $\boldsymbol{v} \in L^2(\mathbb{R}^N;\mathbb{R}^N)$ and $\boldsymbol{V} \in L^2(\mathbb{R}^N;\mathbb{R}^{N\times N})$ such that, up to subsequences, we have

$$\boldsymbol{v}_n \rightharpoonup \boldsymbol{v} \text{ in } L^2(\mathbb{R}^N;\mathbb{R}^N), \qquad \boldsymbol{V}_n \rightharpoonup \boldsymbol{V} \text{ in } L^2(\mathbb{R}^N;\mathbb{R}^{N\times N}). \tag{2.27}$$

Let $V \subset\subset \Omega^{\boldsymbol{y}}$ be a smooth domain. By claim (i) of Lemma 1.18, we have $V \subset\subset \Omega^{\boldsymbol{y}_n}$ for $n \gg 1$ depending on V. From this and from the boundedness of $(\boldsymbol{v}_n)$ and $(\boldsymbol{V}_n)$ in $L^2(\mathbb{R}^N;\mathbb{R}^N)$ and $L^2(\mathbb{R}^N;\mathbb{R}^{N\times N})$, respectively, we see that

$$\sup_{n\gg 1} ||\boldsymbol{m}_n||_{W^{1,2}(V;\mathbb{R}^N)} = \sup_{n\gg 1} \left\{ ||\boldsymbol{v}_n||^2_{L^2(V;\mathbb{R}^N)} + ||\boldsymbol{V}_n||^2_{L^2(V;\mathbb{R}^{N\times N})} \right\}^{1/2} \leq C, \tag{2.28}$$

for some constant $C > 0$ which does not depend on V. Thus, there exist a subsequence $(\boldsymbol{m}_{n_k})$ and a map $\boldsymbol{m} \in W^{1,2}(V;\mathbb{R}^N)$, both possibly depending on V, such that

$$\boldsymbol{m}_{n_k} \rightharpoonup \boldsymbol{m} \text{ in } W^{1,2}(V;\mathbb{R}^N), \quad \boldsymbol{m}_{n_k} \to \boldsymbol{m} \text{ a.e. in } V. \tag{2.29}$$

Here, we employed the Sobolev embedding which is available in view of the smoothness of V. Given (2.27), we deduce $\boldsymbol{m} = \boldsymbol{v}|_V$ and $D\boldsymbol{m} = \boldsymbol{V}|_V$ almost everywhere, so that the limiting map does not actually depend on V. Hence, can assume $\boldsymbol{m} \in W^{1,2}_{\mathrm{loc}}(\Omega^{\boldsymbol{y}};\mathbb{R}^N)$. In fact, since the constant on the right-hand side of (2.28) is also independent on V, there holds $\boldsymbol{m} \in W^{1,2}(\Omega^{\boldsymbol{y}};\mathbb{R}^N)$.
Now, we write $\Omega^{\boldsymbol{y}} = \bigcup_{i=1}^{\infty} V_i$ for a sequence of smooth domains (V_i) with $V_i \subset\subset \Omega^{\boldsymbol{y}}$. Proceeding as in (2.29) and then by means of a diagonal argument, we select a not relabeled subsequence of $(\boldsymbol{m}_n)$ such that

$$\forall\, i \in \mathbb{N}, \quad \boldsymbol{m}_n \rightharpoonup \boldsymbol{m} \text{ in } W^{1,2}(V_i;\mathbb{R}^N), \quad \boldsymbol{m}_n \to \boldsymbol{m} \text{ a.e. in } V_i.$$

This clearly yields the following:

$$\forall\, V \subset\subset \Omega^{\boldsymbol{y}} \text{ open}, \quad \boldsymbol{m}_n \rightharpoonup \boldsymbol{m} \text{ in } W^{1,2}(V;\mathbb{R}^N), \quad \boldsymbol{m}_n \to \boldsymbol{m} \text{ a.e. in } V. \tag{2.30}$$

We stress that no smoothness of V is required in the previous equation.

Step 2 (Magnetic saturation constraint). In order to state that $\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m}) \in \mathcal{Q}$, we are left to show that (2.1) holds true. For every $n \in \mathbb{N}$, we have

$$|\boldsymbol{m}_n \circ \boldsymbol{y}_n| \det D\boldsymbol{y}_n = 1 \text{ a.e. in } \Omega. \tag{2.31}$$

Thanks to claim (i) of Lemma 1.18 and Theorem 1.41, there holds

$$\forall\, V \subset\subset \Omega^{\boldsymbol{y}} \text{ open}, \quad \boldsymbol{y}^{-1}_{n,\Omega} \to \boldsymbol{y}^{-1}_{\Omega} \text{ a.e. in } V. \tag{2.32}$$

Let $\varphi \in C^\infty_{\mathrm{c}}(\Omega_{\boldsymbol{y}})$ and set $K := \operatorname{supp}\varphi$. Let $V \subset\subset \Omega^{\boldsymbol{y}}$ be open with $\boldsymbol{y}(K) \subset V$. By continuity, $\boldsymbol{y}(K)$ is compact and hence closed. Thus, by uniform convergence, we have $\boldsymbol{y}_n(K) \subset V \subset \Omega^{\boldsymbol{y}}$ for $n \gg 1$. Recalling (2.25) and (2.31), with the aid of Proposition 1.46, we compute

$$\begin{aligned}\int_{\Omega_{\boldsymbol{y}}} \varphi \,\mathrm{d}\boldsymbol{x} &= \int_K \varphi\, |\boldsymbol{m}_n \circ \boldsymbol{y}_n| \det D\boldsymbol{y}_n \,\mathrm{d}\boldsymbol{x} \\ &= \int_{\boldsymbol{y}_n(K)} \varphi \circ \boldsymbol{y}^{-1}_{n,\Omega}\, |\boldsymbol{m}_n| \,\mathrm{d}\boldsymbol{\xi} = \int_V \varphi \circ \boldsymbol{y}^{-1}_{n,\Omega}\, |\boldsymbol{m}_n| \,\mathrm{d}\boldsymbol{\xi}.\end{aligned}$$

Taking into account (2.30) and (2.32), we pass to the limit, as $n \to \infty$, in the previous equation by applying the Lebesgue Dominated Convergence Theorem. Then, using once more Proposition 1.46, we obtain

$$\begin{aligned}\int_{\Omega_{\boldsymbol{y}}} \varphi \, \mathrm{d}\boldsymbol{x} = \int_V \varphi \circ \boldsymbol{y}_\Omega^{-1} \, |\boldsymbol{m}| \, \mathrm{d}\boldsymbol{\xi} &= \int_{\boldsymbol{y}(K)} \varphi \circ \boldsymbol{y}_\Omega^{-1} \, |\boldsymbol{m}| \, \mathrm{d}\boldsymbol{\xi} \\ &= \int_K \varphi \, |\boldsymbol{m} \circ \boldsymbol{y}| \det D\boldsymbol{y} \, \mathrm{d}\boldsymbol{x} = \int_{\Omega_{\boldsymbol{y}}} \varphi \, |\boldsymbol{m} \circ \boldsymbol{y}| \det D\boldsymbol{y} \, \mathrm{d}\boldsymbol{x}.\end{aligned}$$

As the function φ is arbitrary and $\mathscr{L}^N(\Omega \setminus \Omega_{\boldsymbol{y}}) = 0$ by Lemma 1.47, the previous equality proves (2.1).

Step 3 (Convergence of the extensions by zero). From Step 1, we know that $\boldsymbol{v}|_{\Omega^{\boldsymbol{y}}} = \boldsymbol{m}$ and $\boldsymbol{V}|_{\Omega^{\boldsymbol{y}}} = D\boldsymbol{m}$. If we show that $\boldsymbol{v}|_{\mathbb{R}^N \setminus \overline{\Omega^{\boldsymbol{y}}}} = \boldsymbol{0}$ and $\boldsymbol{V}|_{\mathbb{R}^N \setminus \overline{\Omega^{\boldsymbol{y}}}} = \boldsymbol{O}$, then (2.18)–(2.19) follow from (2.27). Note that this argument implicitly exploits that $\mathscr{L}^N(\partial\Omega^{\boldsymbol{y}}) = \mathscr{L}^N(\boldsymbol{y}(\partial\Omega)) = 0$ by Proposition 1.45 and Lemma 1.47.
Let $W \subset\subset \mathbb{R}^N \setminus \boldsymbol{y}(\overline{\Omega})$ be open. Then, by claim (ii) of Lemma 1.18, there holds $W \subset \mathbb{R}^N \setminus \boldsymbol{y}_n(\overline{\Omega})$ for $n \gg 1$ depending on W. This yields $\boldsymbol{v}_n|_W = \boldsymbol{0}$ and $\boldsymbol{V}_n|_W = \boldsymbol{O}$ for $n \gg 1$ and, given (2.27), also $\boldsymbol{v}|_W = \boldsymbol{0}$ and $\boldsymbol{V}|_W = \boldsymbol{O}$ for $n \gg 1$. Since W is arbitrary, this proves the claim.

Step 4 (Convergence of the compositions). First, from (2.16) and (2.31), we see that $(\boldsymbol{m}_n \circ \boldsymbol{y}_n)$ is equi-integrable. From this, we deduce the following:

$$\begin{gathered}\text{for every } \varepsilon > 0, \text{ there exists } \delta(\varepsilon) > 0 \text{ such that, for every } X \subset \Omega \text{ measurable} \\ \text{with } \mathscr{L}^N(X) < \delta(\varepsilon), \text{ there holds } \sup_{n \in \mathbb{N}} \int_X |\boldsymbol{m}_n \circ \boldsymbol{y}_n - \boldsymbol{m} \circ \boldsymbol{y}| \, \mathrm{d}\boldsymbol{x} < \varepsilon.\end{gathered} \tag{2.33}$$

Second, recalling Theorem 1.52 and employing Proposition 1.46, we compute

$$\int_{\Omega^{\boldsymbol{y}_n}} (\det D\boldsymbol{y}_{n,\Omega}^{-1})^2 \, \mathrm{d}\boldsymbol{\xi} = \int_{\Omega^{\boldsymbol{y}_n}} \frac{1}{(\det D\boldsymbol{y}_n \circ \boldsymbol{y}_{n,\Omega}^{-1})^2} \, \mathrm{d}\boldsymbol{\xi} = \int_\Omega \frac{1}{\det D\boldsymbol{y}_n} \, \mathrm{d}\boldsymbol{x}.$$

The right-hand side of the previous inequality is uniformly bounded by (2.16). Hence, by the De la Vallée-Poussin Criterion [62, Theorem 2.29] we deduce that, for every $V \subset\subset \Omega^{\boldsymbol{y}}$ open, the sequence $(\det D\boldsymbol{y}_{n,\Omega}^{-1})$, where we consider only $n \gg 1$ depending on V given by claim (i) of Lemma 1.18, is equi-integrable on V. Taking into account Corollay 1.54, the following holds:

$$\begin{gathered}\text{for every } \varepsilon > 0, \text{ there exists } \overline{\delta}(V, \varepsilon) > 0 \text{ such that,} \\ \text{for every } Y \subset V \text{ measurable with } \mathscr{L}^N(Y) < \overline{\delta}(\varepsilon, V), \\ \text{there holds } \mathscr{L}^N(\boldsymbol{y}^{-1}(Y)) + \sup_{n \gg 1} \mathscr{L}^N(\boldsymbol{y}_n^{-1}(Y)) < \varepsilon,\end{gathered} \tag{2.34}$$

where the supremum is taken among $n \gg 1$ as above.
We now prove (2.20). Let $\eta > 0$ be arbitrary. We take $V \subset\subset \Omega^{\boldsymbol{y}}$ open such that $\mathscr{L}^N(\Omega \setminus \boldsymbol{y}^{-1}(V)) < \delta(\eta)$ with $\delta(\eta) > 0$ chosen according to (2.33). Recalling claim (i) of Lemma 1.18, for $n \gg 1$ such that $V \subset\subset \Omega^{\boldsymbol{y}_n}$, we write

$$\begin{aligned}\int_\Omega |\boldsymbol{m}_n \circ \boldsymbol{y}_n - \boldsymbol{m} \circ \boldsymbol{y}| \, \mathrm{d}\boldsymbol{x} &= \int_{\Omega \setminus \boldsymbol{y}^{-1}(V)} |\boldsymbol{m}_n \circ \boldsymbol{y}_n - \boldsymbol{m} \circ \boldsymbol{y}| \, \mathrm{d}\boldsymbol{x} \\ &\quad + \int_{\boldsymbol{y}^{-1}(V)} |\boldsymbol{m}_n \circ \boldsymbol{y}_n - \boldsymbol{m} \circ \boldsymbol{y}| \, \mathrm{d}\boldsymbol{x}.\end{aligned} \tag{2.35}$$

By (2.33), the first integral on the right-hand side of the previous equations is less that η. For the second one, we split it as

$$\begin{aligned}\int_{\boldsymbol{y}^{-1}(V)} |\boldsymbol{m}_n \circ \boldsymbol{y}_n - \boldsymbol{m} \circ \boldsymbol{y}| \, \mathrm{d}\boldsymbol{x} = &\int_{\boldsymbol{y}^{-1}(V) \setminus \boldsymbol{y}_n^{-1}(V)} |\boldsymbol{m}_n \circ \boldsymbol{y}_n - \boldsymbol{m} \circ \boldsymbol{y}| \, \mathrm{d}\boldsymbol{x} \\ &+ \int_{\boldsymbol{y}^{-1}(V) \cap \boldsymbol{y}_n^{-1}(V)} |\boldsymbol{m}_n \circ \boldsymbol{y}_n - \boldsymbol{m} \circ \boldsymbol{y}| \, \mathrm{d}\boldsymbol{x}.\end{aligned} \tag{2.36}$$

We claim that $\mathscr{L}^N(\boldsymbol{y}^{-1}(V) \setminus \boldsymbol{y}_n^{-1}(V)) < \delta(\eta)$ for $n \gg 1$ depending only on η. In that case, we see from (2.33) that the second integral on the right-hand side of (2.36) is less that η. To check the claim, let $W \subset \mathbb{R}^N$ be open and such that $V \subset\subset W \subset\subset \Omega^{\boldsymbol{y}}$. Then, $\boldsymbol{y}(\boldsymbol{y}^{-1}(V)) = V \subset\subset W$ so that, by uniform convergence, $\boldsymbol{y}_n(\boldsymbol{y}^{-1}(V)) \subset W$ and, in turn, $\boldsymbol{y}^{-1}(V) \subset \boldsymbol{y}_n^{-1}(W)$ for $n \gg 1$. Thus, we have

$$\boldsymbol{y}^{-1}(V) \setminus \boldsymbol{y}_n^{-1}(V) \subset \boldsymbol{y}_n^{-1}(W) \setminus \boldsymbol{y}_n^{-1}(V) = \boldsymbol{y}_n^{-1}(W \setminus V), \tag{2.37}$$

for $n \gg 1$. In particular, if W is chosen such that $\mathscr{L}^N(W \setminus V) < \overline{\delta}(W, \delta(\eta))$ with $\overline{\delta}(W, \delta(\eta)) > 0$ chosen according to (2.34), then the desiderd claim follows from (2.37).

To estimate the second integral on the right-hand side of (2.36) we proceed as follows. Let $\overline{\delta}(V, \delta(\eta)) > 0$ be chosen according to (2.34). By the Lusin Theorem, there exists $K_1 \subset V$ compact with $\mathscr{L}^N(V \setminus K_1) < \overline{\delta}(V, \delta(\eta))/2$ such that $\boldsymbol{m}|_{K_1}$ is continuous. Also, by the Egorov Theorem, there exists $K_2 \subset V$ compact with $\mathscr{L}^N(V \setminus K_2) < \overline{\delta}(V, \delta(\eta))/2$ such that $\boldsymbol{m}_n \to \boldsymbol{m}$ uniformly on K_2. Set $K \coloneqq K_1 \cap K_2$, so that $K \subset V$ is compact with $\mathscr{L}^N(V \setminus K) < \overline{\delta}(V, \delta(\eta))$. Observe that

$$\boldsymbol{y}^{-1}(V) \cap \boldsymbol{y}_n^{-1}(V) \subset \big(\boldsymbol{y}^{-1}(K) \cap \boldsymbol{y}_n^{-1}(K)\big) \cup \boldsymbol{y}^{-1}(V \setminus K) \cup \boldsymbol{y}_n^{-1}(V \setminus K),$$

so that for the second integral on the right-hand side of (2.36) we have

$$\begin{aligned}\int_{\boldsymbol{y}^{-1}(V) \cap \boldsymbol{y}_n^{-1}(V)} &|\boldsymbol{m}_n \circ \boldsymbol{y}_n - \boldsymbol{m} \circ \boldsymbol{y}| \, \mathrm{d}\boldsymbol{x} \\ &\leq \int_{\boldsymbol{y}^{-1}(K) \cap \boldsymbol{y}_n^{-1}(K)} |\boldsymbol{m}_n \circ \boldsymbol{y}_n - \boldsymbol{m} \circ \boldsymbol{y}| \, \mathrm{d}\boldsymbol{x} \\ &+ \int_{\boldsymbol{y}^{-1}(V \setminus K) \cup \boldsymbol{y}_n^{-1}(V \setminus K)} |\boldsymbol{m}_n \circ \boldsymbol{y}_n - \boldsymbol{m} \circ \boldsymbol{y}| \, \mathrm{d}\boldsymbol{x}.\end{aligned} \tag{2.38}$$

By (2.34), given our choice of the set K, the second integral on the right-hand side of (2.38) is less than η. For the first one, using the triangle inequality, we write

$$\begin{aligned}\int_{\boldsymbol{y}^{-1}(K) \cap \boldsymbol{y}_n^{-1}(K)} &|\boldsymbol{m}_n \circ \boldsymbol{y}_n - \boldsymbol{m} \circ \boldsymbol{y}| \, \mathrm{d}\boldsymbol{x} \\ &\leq \int_{\boldsymbol{y}^{-1}(K) \cap \boldsymbol{y}_n^{-1}(K)} |\boldsymbol{m}_n \circ \boldsymbol{y}_n - \boldsymbol{m} \circ \boldsymbol{y}_n| \, \mathrm{d}\boldsymbol{x} \\ &+ \int_{\boldsymbol{y}^{-1}(K) \cap \boldsymbol{y}_n^{-1}(K)} |\boldsymbol{m} \circ \boldsymbol{y}_n - \boldsymbol{m} \circ \boldsymbol{y}| \, \mathrm{d}\boldsymbol{x}.\end{aligned} \tag{2.39}$$

Note that, in the previous equation, the composition $\boldsymbol{m} \circ \boldsymbol{y}_n$ is meaningful, at least for $n \gg 1$, since the domain of integration is a subset of $\boldsymbol{y}^{-1}(V) \cap \boldsymbol{y}_n^{-1}(V)$ and

$V \subset \Omega^{\boldsymbol{y}_n} \cap \Omega^{\boldsymbol{y}}$. Thanks to the uniform convergence, for $n \gg 1$ depending on η, we have

$$\sup_{\boldsymbol{\xi} \in K} |\boldsymbol{m}_n(\boldsymbol{\xi}) - \boldsymbol{m}(\boldsymbol{\xi})| < \eta,$$

which yields

$$\int_{\boldsymbol{y}^{-1}(K)\cap\boldsymbol{y}_n^{-1}(K)} |\boldsymbol{m}_n \circ \boldsymbol{y}_n - \boldsymbol{m} \circ \boldsymbol{y}_n| \,\mathrm{d}\boldsymbol{x} < \eta \mathscr{L}^N(\boldsymbol{y}^{-1}(K) \cap \boldsymbol{y}_n^{-1}(K)) \leq \eta \, \mathscr{L}^N(\Omega).$$

On the other hand, $\boldsymbol{m}$ is uniformly continuous on K. Namely

for every $\varepsilon > 0$, there exsits $\widetilde{\delta}(\varepsilon) > 0$ such that, for every $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in K$ with $|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2| < \widetilde{\delta}(\varepsilon)$, there holds $|\boldsymbol{m}(\boldsymbol{\xi}_1) - \boldsymbol{m}(\boldsymbol{\xi}_2)| < \varepsilon$.

As a consequence, for $n \gg 1$ such that $\|\boldsymbol{y}_n - \boldsymbol{y}\|_{C^0(\overline{\Omega};\mathbb{R}^N)} < \widetilde{\delta}(\eta)$, there holds

$$\int_{\boldsymbol{y}^{-1}(K)\cap\boldsymbol{y}_n^{-1}(K)} |\boldsymbol{m} \circ \boldsymbol{y}_n - \boldsymbol{m} \circ \boldsymbol{y}| \,\mathrm{d}\boldsymbol{x} < \eta \, \mathscr{L}^N(\Omega).$$

Therefore, combining (2.35)–(2.36) and (2.38)–(2.39), we deduce that

$$\limsup_{n\to\infty} \int_{\Omega} |\boldsymbol{m}_n \circ \boldsymbol{y}_n - \boldsymbol{m} \circ \boldsymbol{y}| \,\mathrm{d}\boldsymbol{x} \leq (3 + 2\mathscr{L}^N(\Omega))\eta.$$

As η is arbitrary, this establishes (2.20).

Step 5 (Improved convergences). By (2.1) and (2.31), we have

$$\begin{aligned} \int_{\Omega} \left| \frac{1}{\det D\boldsymbol{y}_n} - \frac{1}{\det D\boldsymbol{y}} \right| \mathrm{d}\boldsymbol{x} &= \int_{\Omega} \big| |\boldsymbol{m}_n \circ \boldsymbol{y}_n| - |\boldsymbol{m} \circ \boldsymbol{y}| \big| \,\mathrm{d}\boldsymbol{x} \\ &\leq \int_{\Omega} |\boldsymbol{m}_n \circ \boldsymbol{y}_n - \boldsymbol{m} \circ \boldsymbol{y}| \,\mathrm{d}\boldsymbol{x}. \end{aligned} \tag{2.40}$$

Given (2.20), the right-hand side goes to zero, as $n \to \infty$, and this yields the second convergence in (2.21). In particular, up to subsequences, we have $\det D\boldsymbol{y}_n \to \det D\boldsymbol{y}_n$ almost everywhere. On the other hand, $(\det D\boldsymbol{y}_n)$ is is bounded in $L^{p/N}(\Omega)$ and hence equi-integrable by the De la Vallée-Poussin Criterion [62, Theorem 2.29]. Therefore, the first convergence in (2.21) follows by the Vitali Convergence Theorem. Given (2.1) and (2.31), claim (2.22) is proved by applying the Dominated Convergence Theorem to a subsequence converging almost everywhere whose existence is ensured by (2.20)–(2.21).
Eventually, employing Proposition 1.46 as in (2.26), from (2.1), (2.31) and (2.20), we deduce that

$$\lim_{n\to\infty} \int_{\Omega^{\boldsymbol{y}_n}} |\boldsymbol{m}_n|^2 \,\mathrm{d}\boldsymbol{\xi} = \lim_{n\to\infty} \int_{\Omega} |\boldsymbol{m}_n \circ \boldsymbol{y}_n| \,\mathrm{d}\boldsymbol{x} = \int_{\Omega} |\boldsymbol{m} \circ \boldsymbol{y}| \,\mathrm{d}\boldsymbol{x} = \int_{\Omega^{\boldsymbol{y}}} |\boldsymbol{m}|^2 \,\mathrm{d}\boldsymbol{\xi}.$$

This, together with (2.18), yields (2.24). In particular, thanks to claim (iii) of Lemma 1.18, we see that all the maps of the sequence $(\boldsymbol{v}_n)$ are supported in a common bounded set for $n \gg 1$, so that (2.23) follows from (2.24). $\square$

2.2.2 Proof of Theorem 2.6

We now go back to the general setting with $p > N - 1$ without assuming the global invertibility of admissible deformations. Here, the identification of a limiting magnetization and the verification of the magnetic saturation constraint for the limiting state become more delicate in view of the local character of Definition 1.30. In this setting, the convergence of compositions is established with a different argument with respect to the one employed in Subsection 2.2.1.
The proof of Theorem 2.6 makes use of the following construction involving the topological image of nested balls.

Lemma 2.7 (Topological image of nested balls). *Let $\boldsymbol{y} \in \mathcal{Y}_p(\Omega)$ and let $\boldsymbol{x}_0 \in \Omega$ be such that $\boldsymbol{y}$ is regularly approximately differentiable at $\boldsymbol{x}_0$ with $\det \nabla \boldsymbol{y}(\boldsymbol{x}_0) > 0$. Then, there exists $r_{\boldsymbol{x}_0} > 0$ with the following property: for every $(\boldsymbol{y}_n) \subset \mathcal{Y}_p(\Omega)$ such that $\boldsymbol{y}_n \rightharpoonup \boldsymbol{y}$ in $W^{1,p}(\Omega;\mathbb{R}^N)$, there exist $r, r', r'' > 0$ with*

$$r_{\boldsymbol{x}_0} < r'' < r' < r < \operatorname{dist}(\boldsymbol{x}_0; \partial\Omega)$$

and a subsequence $(\boldsymbol{y}_{n_k})$, possibly depending on r, r' and r'', such that there hold:

$$B(\boldsymbol{x}_0, r), B(\boldsymbol{x}_0, r'), B(\boldsymbol{x}_0, r'') \in \bigcap_{k=1}^{\infty} \mathcal{U}^{*,\,\mathrm{inj}}_{\boldsymbol{y}_{n_k}} \cap \mathcal{U}^{*,\,\mathrm{inj}}_{\boldsymbol{y}}, \tag{2.41}$$

$$\boldsymbol{y}_{n_k} \to \boldsymbol{y} \text{ uniformly on } \partial B(\boldsymbol{x}_0, r) \cup \partial B(\boldsymbol{x}_0, r') \cup \partial B(\boldsymbol{x}_0, r''), \tag{2.42}$$

$$\operatorname{im}_{\mathrm{T}}(\boldsymbol{y}, B(\boldsymbol{x}_0, r'')) \subset\subset \operatorname{im}_{\mathrm{T}}(\boldsymbol{y}, B(\boldsymbol{x}_0, r')) \subset\subset \operatorname{im}_{\mathrm{T}}(\boldsymbol{y}, B(\boldsymbol{x}_0, r)), \tag{2.43}$$

$$\forall\, k \in \mathbb{N}, \quad \operatorname{im}_{\mathrm{T}}(\boldsymbol{y}_{n_k}, B(\boldsymbol{x}_0, r'')) \subset\subset \operatorname{im}_{\mathrm{T}}(\boldsymbol{y}, B(\boldsymbol{x}_0, r')) \subset\subset \operatorname{im}_{\mathrm{T}}(\boldsymbol{y}_{n_k}, B(\boldsymbol{x}_0, r)). \tag{2.44}$$

In this case, up to subsequences, we have

$$\boldsymbol{y}^{-1}_{n_k, B(\boldsymbol{x}_0, r)} \to \boldsymbol{y}^{-1}_{B(\boldsymbol{x}_0, r)} \ \text{a.e. in } \operatorname{im}_{\mathrm{T}}(\boldsymbol{y}, B(\boldsymbol{x}_0, r')) \text{ and strongly in } L^1(\operatorname{im}_{\mathrm{T}}(\boldsymbol{y}, B(\boldsymbol{x}_0, r')); \mathbb{R}^N).$$

We stress that, in the previous statement, the radius $r_{\boldsymbol{x}_0}$ does not depend on the sequence $(\boldsymbol{y}_n)$, but only on $\boldsymbol{y}$ and $\boldsymbol{x}_0$. A graphical representation of (2.43)–(2.44) is provided in Figure 2.2.

Proof of Lemma 2.7. We begin by checking the following claim:

$$\forall\, \alpha > 0,\ \exists\, r, r' > 0:\ 0 < r' < \frac{r}{2} < r < \alpha, \quad \mathscr{L}^1(P \cap (r', r)) > 0. \tag{2.45}$$

Recall that $\Theta^1_+(P, 0) = 1$. Thus, by definition, there holds

$$\forall\, \varepsilon > 0,\ \exists\, \delta(\varepsilon) > 0:\ \forall\, 0 < \rho < \delta(\varepsilon), \quad (1-\varepsilon)\rho < \mathscr{L}^1(P \cap (0, \rho)) < (1+\varepsilon)\rho. \tag{2.46}$$

Let $\varepsilon > 0$. Given $0 < r < \alpha \wedge \delta(\varepsilon)$, by (2.46), we have

$$(1-\varepsilon) r < \mathscr{L}^1(P \cap (0, r)) < (1+\varepsilon) r. \tag{2.47}$$

Similarly, for $0 < r' < r/2 \wedge \delta(\varepsilon)$, we obtain

$$(1-\varepsilon) r' < \mathscr{L}^1(P \cap (0, r')) < (1+\varepsilon) r'. \tag{2.48}$$

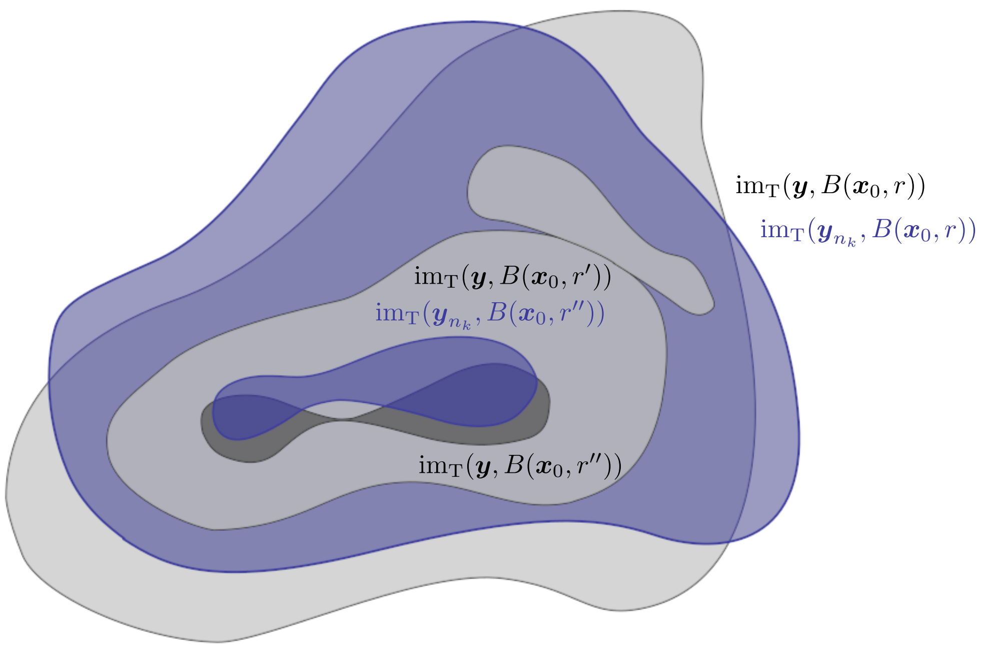

Figure 2.2: The sets $\mathrm{im}_{\mathrm{T}}(\boldsymbol{y}, B(\boldsymbol{x}_0, r''))$ and $\mathrm{im}_{\mathrm{T}}(\boldsymbol{y}_{n_k}, B(\boldsymbol{x}_0, r''))$ are both contained in $\mathrm{im}_{\mathrm{T}}(\boldsymbol{y}, B(\boldsymbol{x}_0, r'))$ which, in turn, is contained in the intersection of $\mathrm{im}_{\mathrm{T}}(\boldsymbol{y}, B(\boldsymbol{x}_0, r))$ and $\mathrm{im}_{\mathrm{T}}(\boldsymbol{y}_{n_k}, B(\boldsymbol{x}_0, r))$.

Combining (2.47)–(2.48), we estimate

$$\begin{aligned}\mathscr{L}^1(P \cap (r', r)) = \mathscr{L}^1(P \cap [r', r)) &= \mathscr{L}^1(P \cap (0, r)) - \mathscr{L}^1(P \cap (0, r')) \\ &> (1-\varepsilon)r - (1+\varepsilon)r' > \left(\frac{1}{2} - \frac{3}{2}\varepsilon\right) r.\end{aligned}$$

As we may assume $\varepsilon < 1/3$, this proves (2.45).
First, by applying (2.45) with $\alpha = \mathrm{dist}(\boldsymbol{x}_0; \partial\Omega)$, we find $\rho_1, \rho_2 > 0$ such that

$$\rho_2 < \frac{\rho_1}{2} < \rho_1 < \mathrm{dist}(\boldsymbol{x}_0; \partial\Omega), \quad \mathscr{L}^1(P \cap (\rho_2, \rho_1)) > 0. \tag{2.49}$$

Analogously, by a repeated application of (2.45), we find $\rho_3, \rho_4, \rho_5, \rho_6, \rho_7, \rho_8, > 0$ satisfying

$$\rho_8 < \frac{\rho_7}{2} < \rho_7 < \frac{\rho_6}{2} < \rho_6 < \frac{\rho_5}{2} < \rho_5 < \frac{\rho_4}{2} < \rho_4 < \frac{\rho_3}{2} < \rho_3 < \frac{\rho_2}{2} \tag{2.50}$$

and

$$\mathscr{L}^1(P \cap (\rho_4, \rho_3)) > 0, \quad \mathscr{L}^1(P \cap (\rho_6, \rho_5)) > 0, \quad \mathscr{L}^1(P \cap (\rho_8, \rho_7)) > 0 \tag{2.51}$$

by taking $\alpha = \rho_2/2$, $\alpha = \rho_4/2$ and $\alpha = \rho_6/2$. Then, we set $r_{\boldsymbol{x}_0} := \rho_8 > 0$.
Now, let $(\boldsymbol{y}_n) \subset \mathcal{Y}_p(\Omega)$ be such that $\boldsymbol{y}_n \rightharpoonup \boldsymbol{y}$ in $W^{1,p}(\Omega; \mathbb{R}^N)$. Thanks to (2.49) and (2.51), by Lemma 1.13 and Lemma 1.29, there exist $\widetilde{r} \in P \cap (\rho_2, \rho_1)$, $r \in P \cap (\rho_4, \rho_3)$, and a subsequence $(\boldsymbol{y}_{n_k})$, possibly depending on $\widetilde{r}$ and r, such that

$$B(\boldsymbol{x}_0, \widetilde{r}), B(\boldsymbol{x}_0, r) \in \bigcap_{k=1}^{\infty} \mathcal{U}^*_{\boldsymbol{y}_{n_k}} \cap \mathcal{U}^*_{\boldsymbol{y}}$$

and

$$\boldsymbol{y}_{n_k} \to \boldsymbol{y} \text{ uniformly on } \partial B(\boldsymbol{x}_0, \widetilde{r}) \cup \partial B(\boldsymbol{x}_0, r).$$

Note that, by (2.50), there holds $r < \widetilde{r}/2$. Therefore, thanks to Proposition 1.40, we conclude that each map $\boldsymbol{y}_{n_k}$ and also $\boldsymbol{y}$, are almost everywhere injective on $B(\boldsymbol{x}_0, r)$. At this point, taking into account (2.51), we apply once more Lemma 1.13 and Lemma 1.29 to select $r' \in P \cap (\rho_6, \rho_5)$ and $r'' \in P \cap (\rho_8, \rho_7)$, as well as a further subsequence that we do not relabel, so that (2.41)–(2.42) are satisfied.
From (2.50), we see that $r' < r/2$ and $r'' < r'/2$. Thus, by claim (ii) of Lemma 1.21, there hold

$$\mathrm{im}_{\mathrm{T}}(\boldsymbol{y}, B(\boldsymbol{x}_0, r')) \cup \boldsymbol{y}(\partial B(\boldsymbol{x}_0, r')) \subset \mathrm{im}_{\mathrm{T}}(\boldsymbol{y}, B(\boldsymbol{x}_0, r)) \tag{2.52}$$

and

$$\mathrm{im}_{\mathrm{T}}(\boldsymbol{y}, B(\boldsymbol{x}_0, r'')) \cup \boldsymbol{y}(\partial B(\boldsymbol{x}_0, r'')) \subset \mathrm{im}_{\mathrm{T}}(\boldsymbol{y}, B(\boldsymbol{x}_0, r')). \tag{2.53}$$

These yield (2.43). The first inclusion in (2.44) follows from (2.52) by claim (iii) of Lemma 1.18, whereas the second inclusion in (2.44) is deduced from (2.53) by claim (i) of Lemma 1.18. Finally, the last statement follows by Theorem 1.41. □

We are now ready to present the proof of our compactness result.

Proof of Theorem 2.6. This proof has the same structure of the one provided in Subsection 2.2.1 and is subdivided into six steps.

Step 1 (Compactness of magnetizations). In this step, we argue as in [18, Proposition 7.1]. For every $n \in \mathbb{N}$, set

$$\boldsymbol{v}_n := \chi_{\mathrm{im}_{\mathrm{T}}(\boldsymbol{y}_n, \Omega)} \boldsymbol{m}_n, \qquad \boldsymbol{V}_n := \chi_{\mathrm{im}_{\mathrm{T}}(\boldsymbol{y}_n, \Omega)} D\boldsymbol{m}_n.$$

By (2.17), $(\boldsymbol{V}_n)$ is bounded in $L^2(\mathbb{R}^N; \mathbb{R}^{N\times N})$. Also, $(1/\det D\boldsymbol{y}_n)$ is bounded in $L^1(\Omega)$ thanks to (2.16). Applying Corollary 1.31 and Proposition 1.9, we compute

$$\int_{\mathrm{im}_{\mathrm{T}}(\boldsymbol{y}_n, \Omega)} |\boldsymbol{m}_n|^2 \,\mathrm{d}\boldsymbol{\xi} \leq \int_\Omega |\boldsymbol{m}_n \circ \boldsymbol{y}_n|^2 \det D\boldsymbol{y}_n \,\mathrm{d}\boldsymbol{x} = \int_\Omega \frac{1}{\det D\boldsymbol{y}_n} \,\mathrm{d}\boldsymbol{x},$$

where the last equality is justified by (2.31), which holds for every $n \in \mathbb{N}$. Thus, $(\boldsymbol{v}_n)$ is bounded in $L^2(\mathbb{R}^N; \mathbb{R}^N)$. Therefore, there exist $\boldsymbol{v} \in L^2(\mathbb{R}^N; \mathbb{R}^N)$ and $\boldsymbol{V} \in L^2(\mathbb{R}^N; \mathbb{R}^{N\times N})$ such that, up to subsequences, we have

$$\boldsymbol{v}_n \rightharpoonup \boldsymbol{v} \text{ in } L^2(\mathbb{R}^N; \mathbb{R}^N), \qquad \boldsymbol{V}_n \rightharpoonup \boldsymbol{V} \text{ in } L^2(\mathbb{R}^N; \mathbb{R}^{N\times N}). \tag{2.54}$$

Let $U \in \mathcal{U}^*_{\boldsymbol{y}}$ and let $V \subset\subset \mathrm{im}_{\mathrm{T}}(\boldsymbol{y}, U)$ be a smooth domain. By claim (i) of Proposition 1.32, there holds $V \subset\subset \mathrm{im}_{\mathrm{T}}(\boldsymbol{y}_n, \Omega)$ for $n \gg 1$. In this case, we have

$$\sup_{n\gg 1} ||\boldsymbol{m}_n||_{W^{1,2}(V;\mathbb{R}^N)} = \sup_{n\gg 1} \left\{ ||\boldsymbol{v}_n||^2_{L^2(V;\mathbb{R}^N)} + ||\boldsymbol{V}_n||^2_{L^2(V;\mathbb{R}^{N\times N})} \right\}^{1/2} \leq C, \tag{2.55}$$

for some constant $C > 0$ which is independent on V. Thus, there exist a subsequence $(\boldsymbol{m}_{n_k})$ and a map $\boldsymbol{m} \in W^{1,2}(V; \mathbb{R}^N)$, both possibly depending on V, such that

$$\boldsymbol{m}_{n_k} \rightharpoonup \boldsymbol{m} \text{ in } W^{1,2}(V; \mathbb{R}^N) \tag{2.56}$$

$$\boldsymbol{m}_{n_k} \to \boldsymbol{m} \text{ in } L^q(V; \mathbb{R}^N) \text{ for every } 1 \leq q < 2^* \text{ and a.e. in } V. \tag{2.57}$$

Here, we applied the Sobolev Embedding Theorem. In view of (2.54), there hold $\boldsymbol{m} = \boldsymbol{v}|_V$ and $D\boldsymbol{m} = \boldsymbol{V}|_V$ almost everywhere. In particular, we deduce that the map $\boldsymbol{m}$ does not depend on V. As U and V are arbitrary and the constant on the right-hand side of (2.55) does not depend on these sets, we infer that $\boldsymbol{m} \in W^{1,2}(\mathrm{im}_{\mathrm{T}}(\boldsymbol{y},\Omega);\mathbb{R}^N)$. In turn, there hold $\boldsymbol{v} = \boldsymbol{m}$ and $\boldsymbol{V} = D\boldsymbol{m}$ almost everywhere in $\mathrm{im}_{\mathrm{T}}(\boldsymbol{y},\Omega)$.
Now, we write $\mathrm{im}_{\mathrm{T}}(\boldsymbol{y},\Omega) = \bigcup_{i=1}^{\infty} V_i$, where (V_i) denotes a sequence of smooth domains with the property that for every $i \in \mathbb{N}$ there exists $U_i \in \mathcal{U}^*_{\boldsymbol{y}}$ such that $V_i \subset\subset \mathrm{im}_{\mathrm{T}}(\boldsymbol{y},U_i)$. This is possible thanks to the Lindelöf property. Arguing as in (2.56) and applying a diagonal argument, we select a subsequence of $(\boldsymbol{m}_n)$, that we do not relabel, for which there holds

$$\begin{aligned} \forall\, i \in \mathbb{N}, \quad & \boldsymbol{m}_n \rightharpoonup \boldsymbol{m} \text{ in } W^{1,2}(V_i;\mathbb{R}^N), \\ & \boldsymbol{m}_n \to \boldsymbol{m} \text{ in } L^q(V_i;\mathbb{R}^N) \text{ for every } 1 \leq q < 2^* \text{ and a.e. in } V_i. \end{aligned}$$

This clearly entails the following:

$$\begin{gathered} \forall\, V \subset\subset \mathrm{im}_{\mathrm{T}}(\boldsymbol{y},\Omega) \text{ open}, \qquad \boldsymbol{m}_n \rightharpoonup \boldsymbol{m} \text{ in } W^{1,2}(V;\mathbb{R}^N), \\ \boldsymbol{m}_n \to \boldsymbol{m} \text{ in } L^q(V;\mathbb{R}^N) \text{ for every } 1 \leq q < 2^* \text{ and a.e. in } V. \end{gathered} \tag{2.58}$$

Step 2 (Magnetic saturation constraint). We check that (2.1) holds. In this case, $\boldsymbol{q} = (\boldsymbol{y},\boldsymbol{m}) \in \mathcal{Q}$. Let $\boldsymbol{x}_0 \in \Omega$ be such that $\boldsymbol{y}$ is regularly approximately differentiable at $\boldsymbol{x}_0$ with $\det \nabla \boldsymbol{y}(\boldsymbol{x}_0) > 0$. Let $r, r', r'' > 0$ and $(\boldsymbol{y}_{n_k})$ be given by Lemma 2.7. For convenience, we set

$$B \coloneqq B(\boldsymbol{x}_0, r), \quad B' \coloneqq B(\boldsymbol{x}_0, r'), \quad B'' \coloneqq B(\boldsymbol{x}_0, r'').$$

Thus

$$\mathrm{im}_{\mathrm{T}}(\boldsymbol{y},B'') \subset\subset \mathrm{im}_{\mathrm{T}}(\boldsymbol{y},B') \subset\subset \mathrm{im}_{\mathrm{T}}(\boldsymbol{y},B) \tag{2.59}$$

and, for every $k \in \mathbb{N}$, there holds

$$\mathrm{im}_{\mathrm{T}}(\boldsymbol{y}_{n_k},B'') \subset\subset \mathrm{im}_{\mathrm{T}}(\boldsymbol{y},B') \subset\subset \mathrm{im}_{\mathrm{T}}(\boldsymbol{y}_{n_k},B). \tag{2.60}$$

Also, we have

$$\boldsymbol{y}^{-1}_{n_k,B} \to \boldsymbol{y}^{-1}_B \text{ a.e. in } \mathrm{im}_{\mathrm{T}}(\boldsymbol{y},B'). \tag{2.61}$$

Let $\varphi \in C^0_{\mathrm{c}}(B'')$ and set $K \coloneqq \operatorname{supp}\varphi$. With the aid of Proposition 1.9, we compute

$$\begin{aligned} \int_{B''} \varphi \, \mathrm{d}\boldsymbol{x} &= \int_{B''} \varphi\, |\boldsymbol{m}_{n_k} \circ \boldsymbol{y}_{n_k}| \det D\boldsymbol{y}_{n_k} \, \mathrm{d}\boldsymbol{x} \\ &= \int_{\mathrm{im}_{\mathrm{G}}(\boldsymbol{y}_{n_k},B'')} \varphi \circ \boldsymbol{y}^{-1}_{n_k,B}\, |\boldsymbol{m}_{n_k}| \, \mathrm{d}\boldsymbol{\xi} \\ &= \int_{\mathrm{im}_{\mathrm{T}}(\boldsymbol{y},B')} \varphi \circ \boldsymbol{y}^{-1}_{n_k,B}\, |\boldsymbol{m}_{n_k}| \, \mathrm{d}\boldsymbol{\xi}. \end{aligned}$$

Here, we exploited (2.31) together with the identity

$$\varphi \circ \boldsymbol{y}^{-1}_{n_k,B} = 0 \text{ on } \mathrm{im}_{\mathrm{G}}(\boldsymbol{y}_{n_k},B'') \setminus \mathrm{im}_{\mathrm{G}}(\boldsymbol{y}_{n_k},K)$$

and the inclusion $\mathrm{im}_{\mathrm{G}}(\boldsymbol{y}_{n_k}, B'') \subset \mathrm{im}_{\mathrm{T}}(\boldsymbol{y}, B')$. The latter is justified by (2.60) and by claim (iii) of Corollary 1.31. Given (2.58) and (2.61), taking the limit, as $k \to \infty$, in the previous equation, we obtain

$$\begin{aligned}\int_{B''} \varphi \, \mathrm{d}\boldsymbol{x} &= \int_{\mathrm{im}_{\mathrm{T}}(\boldsymbol{y}, B')} \varphi \circ \boldsymbol{y}_B^{-1} \, |\boldsymbol{m}| \, \mathrm{d}\boldsymbol{\xi} \\ &= \int_{\mathrm{im}_{\mathrm{G}}(\boldsymbol{y}, B'')} \varphi \circ \boldsymbol{y}_B^{-1} \, |\boldsymbol{m}| \, \mathrm{d}\boldsymbol{\xi} \\ &= \int_{B''} \varphi \, |\boldsymbol{m} \circ \boldsymbol{y}| \, \det D\boldsymbol{y} \, \mathrm{d}\boldsymbol{x}.\end{aligned}$$

Here, we applied Proposition 1.9. Similarly to before, we exploited the identity

$$\boldsymbol{\varphi} \circ \boldsymbol{y}_B^{-1} = 0 \text{ on } \mathrm{im}_{\mathrm{G}}(\boldsymbol{y}, B'') \setminus \mathrm{im}_{\mathrm{G}}(\boldsymbol{y}, K)$$

and the inclusion $\mathrm{im}_{\mathrm{G}}(\boldsymbol{y}, B'') \subset \mathrm{im}_{\mathrm{T}}(\boldsymbol{y}, B')$ which comes from (2.59) and claim (iii) of Corollary 1.31. As φ is arbitrary, this entails

$$|\boldsymbol{m} \circ \boldsymbol{y}| \det D\boldsymbol{y} = 1 \text{ a.e. in } B''.$$

As almost every point in Ω has the same property of $\boldsymbol{x}_0$ by Proposition 1.11, this proves (2.1).

Step 3 (Convergence of the extensions by zero). In order to establish (2.18)–(2.19), it is sufficient to show that $\boldsymbol{v} = \boldsymbol{0}$ and $\boldsymbol{V} = \boldsymbol{O}$ almost everywhere in $\mathbb{R}^N \setminus \mathrm{im}_{\mathrm{T}}(\boldsymbol{y}, \Omega)$. To do this, we argue as in [18, Proposition 7.1]. Recall (2.54) and let $Y \subset \mathbb{R}^N \setminus \mathrm{im}_{\mathrm{T}}(\boldsymbol{y}, \Omega)$ be measurable with $\mathscr{L}^N(Y) < +\infty$. Then,

$$\left| \int_Y \boldsymbol{v} \, \mathrm{d}\boldsymbol{\xi} \right| = \lim_{n \to \infty} \left| \int_Y \boldsymbol{v}_n \, \mathrm{d}\boldsymbol{\xi} \right| \leq \lim_{n \to \infty} \int_{\mathrm{im}_{\mathrm{T}}(\boldsymbol{y}_n, \Omega) \setminus \mathrm{im}_{\mathrm{T}}(\boldsymbol{y}, \Omega)} |\boldsymbol{v}_n| \, \mathrm{d}\boldsymbol{\xi} = 0,$$

where, in the last equality, we exploited claim (ii) of Corollary 1.31 and the equi-integrability of $(\boldsymbol{v}_n)$. This last property follows from the boundedness of $(\boldsymbol{v}_n)$ in $L^2(\mathbb{R}^N; \mathbb{R}^N)$. The identity $\boldsymbol{V} = \boldsymbol{O}$ almost everywhere in $\mathbb{R}^N \setminus \mathrm{im}_{\mathrm{T}}(\boldsymbol{y}, \Omega)$ is proved in the same way by exploiting (2.17).

Step 4 (Convergence of compositions). From (2.31) and (2.16), we see that $(\boldsymbol{m}_n \circ \boldsymbol{y}_n)$ is equi-integrable. If we show that

$$\boldsymbol{m}_n \circ \boldsymbol{y}_n \to \boldsymbol{m} \circ \boldsymbol{y} \text{ a.e. in } \Omega, \tag{2.62}$$

then (2.20) follows by the Vitali Convergence Theorem.

First, we argue locally. Let $\boldsymbol{x}_0 \in \Omega$ be such that $\boldsymbol{y}$ is regularly approximately differentiable at $\boldsymbol{x}_0$ with $\det \nabla \boldsymbol{y}(\boldsymbol{x}_0) > 0$. Let $r, r', r'', r_{\boldsymbol{x}_0} > 0$ and $(\boldsymbol{y}_{n_k})$ be given by Lemma 2.7. Let B, B' and B'' be defined as in Step 2, so that (2.59)–(2.61) hold true. Fix $1 < q < 2^* - 1$. Exploiting (2.31) and applying Proposition 1.9, we compute

$$\begin{aligned}\int_{B''} |\boldsymbol{m}_{n_k} \circ \boldsymbol{y}_{n_k}|^q \, \mathrm{d}\boldsymbol{x} &= \int_{B''} |\boldsymbol{m}_{n_k} \circ \boldsymbol{y}_{n_k}|^{q+1} \det D\boldsymbol{y}_{n_k} \, \mathrm{d}\boldsymbol{x} \\ &= \int_{\mathrm{im}_{\mathrm{G}}(\boldsymbol{y}_{n_k}, B'')} |\boldsymbol{m}_{n_k}|^{q+1} \, \mathrm{d}\boldsymbol{\xi} \\ &= \int_{\mathrm{im}_{\mathrm{T}}(\boldsymbol{y}, B')} \chi_{\mathrm{im}_{\mathrm{G}}(\boldsymbol{y}_{n_k}, B'')} \, |\boldsymbol{m}_{n_k}|^{q+1} \, \mathrm{d}\boldsymbol{\xi}.\end{aligned}$$

By Lemma 2.7, there holds $\boldsymbol{y}_{n_k} \to \boldsymbol{y}$ uniformly on $\partial B''$. Thus, by claim (iv) of Lemma 1.18, we obtain

$$\chi_{\mathrm{im_G}(\boldsymbol{y}_{n_k}, B'')} \to \chi_{\mathrm{im_G}(\boldsymbol{y}, B'')} \text{ a.e. in } \mathbb{R}^N.$$

Recalling (2.58), we pass to the limit, as $k \to \infty$, in the previous equation with the aid of the Dominated Convergence Theorem. Then, employing once again Proposition 1.9 together with (2.1), we obtain

$$\begin{aligned}\lim_{k\to\infty} \int_{B''} |\boldsymbol{m}_{n_k} \circ \boldsymbol{y}_{n_k}|^q \,\mathrm{d}\boldsymbol{x} &= \int_{\mathrm{im_T}(\boldsymbol{y}, B')} \chi_{\mathrm{im_G}(\boldsymbol{y}, B'')} \, |\boldsymbol{m}|^{q+1} \,\mathrm{d}\boldsymbol{\xi} \\ &= \int_{\mathrm{im_G}(\boldsymbol{y}, B'')} |\boldsymbol{m}|^{q+1} \,\mathrm{d}\boldsymbol{\xi} \\ &= \int_{B''} |\boldsymbol{m} \circ \boldsymbol{y}|^{q+1} \det D\boldsymbol{y} \,\mathrm{d}\boldsymbol{x} \\ &= \int_{B''} |\boldsymbol{m} \circ \boldsymbol{y}|^q \,\mathrm{d}\boldsymbol{x}.\end{aligned} \tag{2.63}$$

In particular, we see that $(\boldsymbol{m}_{n_k} \circ \boldsymbol{y}_{n_k})$ is bounded in $L^q(B''; \mathbb{R}^N)$. Hence, there exist a subsequence of $(\boldsymbol{m}_{n_k} \circ \boldsymbol{y}_{n_k})$, that we do not relabel, and a map $\boldsymbol{\lambda} \in L^q(B''; \mathbb{R}^N)$, both possibly depending on B'', such that $\boldsymbol{m}_{n_k} \circ \boldsymbol{y}_{n_k} \rightharpoonup \boldsymbol{\lambda}$ in $L^q(B''; \mathbb{R}^N)$. We claim that $\boldsymbol{\lambda} = \boldsymbol{m} \circ \boldsymbol{y}$ almost everywhere in B''. To see this, let $\boldsymbol{\varphi} \in C^0_\mathrm{c}(B''; \mathbb{R}^N)$. With the aid of Proposition 1.9 and exploiting (2.31), we compute

$$\begin{aligned}\int_{B''} \boldsymbol{m}_{n_k} \circ \boldsymbol{y}_{n_k} \cdot \boldsymbol{\varphi} \,\mathrm{d}\boldsymbol{x} &= \int_{B''} |\boldsymbol{m}_{n_k} \circ \boldsymbol{y}_{n_k}| \, \boldsymbol{m}_{n_k} \circ \boldsymbol{y}_{n_k} \cdot \boldsymbol{\varphi} \det D\boldsymbol{y}_{n_k} \,\mathrm{d}\boldsymbol{x} \\ &= \int_{\mathrm{im_G}(\boldsymbol{y}_{n_k}, B'')} |\boldsymbol{m}_{n_k}| \, \boldsymbol{m}_{n_k} \cdot \boldsymbol{\varphi} \circ \boldsymbol{y}^{-1}_{n_k, B} \,\mathrm{d}\boldsymbol{\xi} \\ &= \int_{\mathrm{im_T}(\boldsymbol{y}, B')} |\boldsymbol{m}_{n_k}| \, \boldsymbol{m}_{n_k} \cdot \boldsymbol{\varphi} \circ \boldsymbol{y}^{-1}_{n_k, B} \,\mathrm{d}\boldsymbol{\xi}\end{aligned}$$

Recalling (2.58) and (2.61), applying the Lebesgue Dominated Convergence Theorem, we pass to the limit, as $k \to \infty$, on the right-hand side of the previous equation. Note that this is justified as $(|\boldsymbol{m}_{n_k}|^2)$ admits a integrable majorant by (2.58) and $(\boldsymbol{\varphi} \circ \boldsymbol{y}^{-1}_{n_k, B})$ is uniformly bounded. We obtain

$$\begin{aligned}\int_{B''} \boldsymbol{\lambda} \cdot \boldsymbol{\varphi} \,\mathrm{d}\boldsymbol{x} &= \int_{\mathrm{im_T}(\boldsymbol{y}, B')} |\boldsymbol{m}| \, \boldsymbol{m} \cdot \boldsymbol{\varphi} \circ \boldsymbol{y}^{-1}_B \,\mathrm{d}\boldsymbol{\xi} \\ &= \int_{\mathrm{im_G}(\boldsymbol{y}, B'')} |\boldsymbol{m}| \, \boldsymbol{m} \cdot \boldsymbol{\varphi} \circ \boldsymbol{y}^{-1}_B \,\mathrm{d}\boldsymbol{\xi} \\ &= \int_{B''} |\boldsymbol{m} \circ \boldsymbol{y}| \, \boldsymbol{m} \circ \boldsymbol{y} \cdot \boldsymbol{\varphi} \det D\boldsymbol{y} \,\mathrm{d}\boldsymbol{x} \\ &= \int_{B''} \boldsymbol{m} \circ \boldsymbol{y} \cdot \boldsymbol{\varphi} \,\mathrm{d}\boldsymbol{x}\end{aligned}$$

This entails $\boldsymbol{\lambda} = \boldsymbol{m} \circ \boldsymbol{y}$ almost everywhere in B''. Therefore, thanks to the Urysohn property, the whole sequence $(\boldsymbol{m}_{n_k} \circ \boldsymbol{y}_{n_k})$ converges weakly to $\boldsymbol{m} \circ \boldsymbol{y}$ in $L^q(B''; \mathbb{R}^N)$. Hence, as $q > 1$, the convergence in (2.63) yields

$$\boldsymbol{m}_{n_k} \circ \boldsymbol{y}_{n_k} \to \boldsymbol{m} \circ \boldsymbol{y} \text{ in } L^q(B''; \mathbb{R}^N)$$

so that, up to extraction of a further subsequence, we have

$$\boldsymbol{m}_{n_k} \circ \boldsymbol{y}_{n_k} \to \boldsymbol{m} \circ \boldsymbol{y} \text{ a.e. in } B''.$$

At this point, we stress that both the radius $r'' > 0$ of B'' as well as the sequence of indices (n_k) depend on the point $\boldsymbol{x}_0$. However, $r'' > r_{\boldsymbol{x}_0} > 0$, where the latter depends only on $\boldsymbol{y}$ and $\boldsymbol{x}_0$.
To achieve (2.62), we employ a diagonal argument. Let Ω_0 be the set of points $\boldsymbol{x}_0 \in \Omega$ such that $\boldsymbol{y}$ is regularly approximately differentiable at $\boldsymbol{x}_0$ with $\det \nabla \boldsymbol{y}(\boldsymbol{x}_0) > 0$. In this case, $\mathscr{L}^N(\Omega \setminus \Omega_0) = 0$ by Proposition 1.11. For every $\boldsymbol{x}_0 \in \Omega_0$, let $r_{\boldsymbol{x}_0} > 0$ be given by Lemma 2.7. Thanks to the Lindelöf property, we find a sequence $(\boldsymbol{x}_i) \subset \Omega_0$ such that

$$\Omega_0 \subset \bigcup_{i=1}^{\infty} B(\boldsymbol{x}_i, r_{\boldsymbol{x}_i}). \tag{2.64}$$

Now, by means of the previous local argument, for every $i \in \mathbb{N}$, we select a subsequence, indexed by $\left(n_k^{(i)}\right)$, such that

$$\boldsymbol{m}_{n_k^{(i)}} \circ \boldsymbol{y}_{n_k^{(i)}} \to \boldsymbol{m} \circ \boldsymbol{y} \text{ a.e. in } B(\boldsymbol{x}_i, r_{\boldsymbol{x}_i}).$$

For each $i \in \mathbb{N}$, we choose $\left(n_k^{(i)}\right)$ to be a subsequence of $\left(n_k^{(i-1)}\right)$. In this way, we have

$$\forall\, j \in \mathbb{N} : j \leq i, \quad \boldsymbol{m}_{n_k^{(i)}} \circ \boldsymbol{y}_{n_k^{(i)}} \to \boldsymbol{m} \circ \boldsymbol{y} \text{ a.e. in } B(\boldsymbol{x}_j, r_{\boldsymbol{x}_j}).$$

Thus, setting $n_j := n_j^{(j)}$ for every $j \in \mathbb{N}$, there holds

$$\forall\, i \in \mathbb{N}, \quad \boldsymbol{m}_{n_j} \circ \boldsymbol{y}_{n_j} \to \boldsymbol{m} \circ \boldsymbol{y} \text{ a.e. in } B(\boldsymbol{x}_i, r_{\boldsymbol{x}_i}),$$

which, in view of (2.64), yields (2.62) for the subsequence indexed by (n_j).

Step 5 (Improved convergences). From (2.20), we obtain the first convergence in (2.21) as in (2.40). Up to subsequences, this yields $\det D\boldsymbol{y}_n \to \det D\boldsymbol{y}$ almost everywhere so that (2.21) follows by the Vitali Convergence Theorem. Indeed, the sequence $(\det D\boldsymbol{y}_n)$ is equi-integrable thanks to the second condition in (2.15) and the Dunford-Pettis Theorem [62, Theorem 2.82]. Claim (2.22) follows combining (2.20) and the second convergence in (2.21) in view of (2.1) and (2.31). To prove (2.23), observe that, by claim (ii) of Proposition 1.32 and by (2.58), there holds

$$\chi_{\mathrm{im_T}(\boldsymbol{y}_n,\Omega)} \boldsymbol{m}_n \to \chi_{\mathrm{im_T}(\boldsymbol{y},\Omega)} \boldsymbol{m} \text{ a.e. in } \mathbb{R}^N.$$

As $(\chi_{\mathrm{im_T}(\boldsymbol{y}_n,\Omega)} \boldsymbol{m}_n)$ is bounded in $L^2(\mathbb{R}^N;\mathbb{R}^N)$, by De la Vallée-Poussin Criterion [62, Theorem 2.29], the same sequence is q-equi-integrable for every $1 \leq q < 2$, so that (2.23) follows by applying the Vitali Convergence Theorem.

Step 6 (Global injectivity). If $(\boldsymbol{q}_n) \subset \mathcal{Q}^{\mathrm{inj}}$, then $\boldsymbol{q} \in \mathcal{Q}^{\mathrm{inj}}$ thanks to (2.15) and Corollary 1.43. Finally, we show (2.24). Applying Proposition 1.9 and Theorem 1.26, for every $n \in \mathbb{N}$, we compute

$$\int_{\mathrm{im_T}(\boldsymbol{y}_n,\Omega)} |\boldsymbol{m}_n|^2 \,\mathrm{d}\boldsymbol{\xi} = \int_\Omega |\boldsymbol{m}_n \circ \boldsymbol{y}_n|^2 \det D\boldsymbol{y}_n \,\mathrm{d}\boldsymbol{x} = \int_\Omega |\boldsymbol{m}_n \circ \boldsymbol{y}_n| \,\mathrm{d}\boldsymbol{x}.$$

Here, we exploited (2.31). Analogously, taking into account the injectivity of $\boldsymbol{y}$ and (2.1), we have

$$\int_{\mathrm{im_T}(\boldsymbol{y},\Omega)} |\boldsymbol{m}|^2 \,\mathrm{d}\boldsymbol{\xi} = \int_\Omega |\boldsymbol{m}\circ\boldsymbol{y}|^2 \det D\boldsymbol{y}\,\mathrm{d}\boldsymbol{x} = \int_\Omega |\boldsymbol{m}\circ\boldsymbol{y}|\,\mathrm{d}\boldsymbol{x}.$$

Thus, from (2.20), we obtain

$$\lim_{n\to\infty} \|\chi_{\mathrm{im_T}(\boldsymbol{y}_n,\Omega)}\boldsymbol{m}_n\|_{L^2(\mathbb{R}^N;\mathbb{R}^N)} = \|\chi_{\mathrm{im_T}(\boldsymbol{y},\Omega)}\boldsymbol{m}\|_{L^2(\mathbb{R}^N;\mathbb{R}^N)}$$

which, together with (2.18), yields (2.24). □

We conclude the subsection by explaining how the techniques employed here can be used to prove the strong convergence of the composition of magnetizations with deformations when the magnetic saturation constraint is formulated in the deformed configuration as in [18]. The approach is described in the following remarmk.

Remark 2.8 (Compactness for sphere-valued magnetizations). Let $(\boldsymbol{y}_n)\subset \mathcal{Y}_p(\Omega)$ and let $(\boldsymbol{m}_n)$ with $\boldsymbol{m}_n\in W^{1,2}(\mathrm{im_T}(\boldsymbol{y}_n,\Omega);\mathbb{S}^{N-1})$. Suppose that (2.15) holds for some $\boldsymbol{y}\in\mathcal{Y}_p(\Omega)$. Additionally, suppose that (2.17) holds true and that there exists $\gamma\colon(0,+\infty)\to(0,+\infty)$ Borel satisfying (2.9) such that

$$\sup_{n\in\mathbb{N}} \|\gamma(\det D\boldsymbol{y}_n)\|_{L^1(\Omega)} < +\infty. \tag{2.65}$$

Then, there exists $\boldsymbol{m}\in W^{1,2}(\mathrm{im_T}(\boldsymbol{y},\Omega);\mathbb{S}^{N-1})$ such that the convergences in (2.18)–(2.20) hold true. The identification of $\boldsymbol{m}$ as well as (2.18)–(2.19) have been achieved in [18, Proposition 7.1]. We only show how to prove the convergence of compositions. Let $\boldsymbol{x}_0\in\Omega$ be such that $\boldsymbol{y}$ is regularly approximately differentiable at $\boldsymbol{x}_0$ with $\det\nabla\boldsymbol{y}(\boldsymbol{x}_0)>0$. Also, let $r,r',r'',r_{\boldsymbol{x}_0}>0$ and $(\boldsymbol{y}_{n_k})$ be given by Lemma 2.7. Define B, B' and B'' be as in Step 2 of the proof of Theorem 2.6, so that (2.59)–(2.61) hold true. As in [18, Proposition 7.8], we introduce $\widehat{\gamma}\colon(0,+\infty)\to(0,+\infty)$ by setting $\widehat{\gamma}(z)\coloneqq z\,\gamma(1/z)$. In this case, (2.9) yields

$$\lim_{z\to+\infty}\frac{\widehat{\gamma}(z)}{z} = +\infty.$$

Applying Proposition 1.9 and Theorem 1.26, for every $k\in\mathbb{N}$, we compute

$$\begin{aligned}\int_{\mathrm{im_T}(\boldsymbol{y}_{n_k},B)} \widehat{\gamma}(\det D\boldsymbol{y}^{-1}_{n_k,B})\,\mathrm{d}\boldsymbol{\xi} &= \int_{\mathrm{im_T}(\boldsymbol{y}_{n_k},B)} \gamma\left(\frac{1}{\det D\boldsymbol{y}^{-1}_{n_k,B}}\right)\det D\boldsymbol{y}^{-1}_{n_k,B}\,\mathrm{d}\boldsymbol{\xi}\\ &= \int_B \gamma(\det D\boldsymbol{y}_{n_k})\,\mathrm{d}\boldsymbol{x},\end{aligned}$$

where the right-hand side is uniformly bounded thanks to (2.65). Here, we used the identity

$$\det D\boldsymbol{y}^{-1}_{n_k,B} = (\det D\boldsymbol{y}_{n_k})^{-1}\circ\boldsymbol{y}^{-1}_{n_k,B}\ \text{ a.e. in } \mathrm{im_T}(\boldsymbol{y}_{n_k},B).$$

In view of (2.59), there holds

$$\sup_{k\in\mathbb{N}}\int_{\mathrm{im_T}(\boldsymbol{y},B')} \widehat{\gamma}(\det D\boldsymbol{y}^{-1}_{n_k,B})\,\mathrm{d}\boldsymbol{\xi} \leq \sup_{k\in\mathbb{N}}\int_{\mathrm{im_T}(\boldsymbol{y}_{n_k},B)} \widehat{\gamma}(\det D\boldsymbol{y}^{-1}_{n_k,B})\,\mathrm{d}\boldsymbol{\xi},$$

where the sequence $(\det D\boldsymbol{y}^{-1}_{n_k,B})$ is equi-integrable on $\mathrm{im}_{\mathrm{T}}(\boldsymbol{y}, B')$ by De la Vallée-Poussin Criterion [62, Theorem 2.29]. At this point, Theorem 1.41 ensures that, up to subsequence, there holds

$$\det D\boldsymbol{y}^{-1}_{n_k,B} \rightharpoonup \det D\boldsymbol{y}^{-1}_{B} \text{ in } L^1(\mathrm{im}_{\mathrm{T}}(\boldsymbol{y}, B')). \tag{2.66}$$

Let $\boldsymbol{\varphi} \in C^0_{\mathrm{c}}(B''; \mathbb{R}^N)$. Applying Proposition 1.39, we compute

$$\begin{aligned}\int_{B''} \boldsymbol{m}_{n_k} \circ \boldsymbol{y}_{n_k} \cdot \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} &= \int_{\mathrm{im}_{\mathrm{G}}(\boldsymbol{y}_{n_k}, B'')} \boldsymbol{m}_{n_k} \cdot \boldsymbol{\varphi} \circ \boldsymbol{y}^{-1}_{n_k,B} \det D\boldsymbol{y}^{-1}_{n_k,B} \, \mathrm{d}\boldsymbol{\xi} \\ &= \int_{\mathrm{im}_{\mathrm{T}}(\boldsymbol{y}, B')} \boldsymbol{m}_{n_k} \cdot \boldsymbol{\varphi} \circ \boldsymbol{y}^{-1}_{n_k,B} \det D\boldsymbol{y}^{-1}_{n_k,B} \, \mathrm{d}\boldsymbol{\xi}.\end{aligned}$$

Recalling (2.58), (2.61) and (2.66), we pass to the limit, as $k \to \infty$, in the previous equation by employing [62, Proposition 2.61]. We obtain

$$\begin{aligned}\lim_{k\to\infty} \int_{B''} \boldsymbol{m}_{n_k} \circ \boldsymbol{y}_{n_k} \cdot \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} &= \int_{\mathrm{im}_{\mathrm{T}}(\boldsymbol{y}, B')} \boldsymbol{m} \cdot \boldsymbol{\varphi} \circ \boldsymbol{y}^{-1}_{B} \det D\boldsymbol{y}^{-1}_{B} \, \mathrm{d}\boldsymbol{\xi} \\ &= \int_{\mathrm{im}_{\mathrm{G}}(\boldsymbol{y}, B'')} \boldsymbol{m} \cdot \boldsymbol{\varphi} \circ \boldsymbol{y}^{-1}_{B} \det D\boldsymbol{y}^{-1}_{B} \, \mathrm{d}\boldsymbol{\xi} \\ &= \int_{B''} \boldsymbol{m} \circ \boldsymbol{y} \cdot \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x},\end{aligned}$$

where, in the last line, we used Proposition 1.39. This yields

$$\boldsymbol{m}_{n_k} \circ \boldsymbol{y}_{n_k} \rightharpoonup \boldsymbol{m} \circ \boldsymbol{y} \text{ in } L^2(\Omega; \mathbb{R}^N).$$

In particular, as

$$\|\boldsymbol{m}_{n_k} \circ \boldsymbol{y}_{n_k}\|_{L^2(B''; \mathbb{R}^N)} = \|\boldsymbol{m} \circ \boldsymbol{y}\|_{L^2(B''; \mathbb{R}^N)} = \sqrt{\mathscr{L}^N(B'')}$$

for every $k \in \mathbb{N}$, this actually entails the strong convergence in $L^2(B''; \mathbb{R}^N)$.
By means of a diagonal argument analogous to the one employed in Step 2 of the proof of Theorem 2.6, we select a subsequence for which the compositions converge strongly in $L^2(\Omega; \mathbb{R}^N)$ and, more generally, in $L^q(\Omega; \mathbb{R}^N)$ for every $1 \leq q < \infty$.

2.3 Proof of Theorem 2.5

The proof of Theorem 2.5 is a standard application of the Direct Method. For future reference, we present the results concerning the coercivity and the lower semicontinuity of the magnetoelastic energy functional separately. For the same reason, the coercivity result is stated in a slightly more self-contained form.

Proposition 2.9 (Coercivity). *Let $(\boldsymbol{q}_n) \subset \mathcal{Q}$ with $\boldsymbol{q}_n = (\boldsymbol{y}_n, \boldsymbol{m}_n)$ be such that*

$$(\boldsymbol{y}_n) \text{ is bounded in } W^{1,p}(\Omega; \mathbb{R}^N), \tag{2.67}$$

$$(\chi_{\mathrm{im}_{\mathrm{T}}(\boldsymbol{y}_n, \Omega)} D\boldsymbol{m}_n) \text{ is bounded in } L^2(\mathbb{R}^N; \mathbb{R}^{N\times N}), \tag{2.68}$$

$$(\gamma(\det D\boldsymbol{y}_n)) \text{ is bounded in } L^1(\Omega), \tag{2.69}$$

where $\gamma\colon (0, +\infty) \to (0, +\infty)$ is a Borel function satisfying (2.5)*. Then, there exists $\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m}) \in \mathcal{Q}$ such that, up to subsequences, the convergences in* (2.15)–(2.23) *hold true. In particular, if $(\boldsymbol{q}_n) \subset \mathcal{Q}^{\mathrm{inj}}$, then $\boldsymbol{q} \in \mathcal{Q}^{\mathrm{inj}}$ and the convergence in* (2.24) *holds true as well.*

Proof. By (2.67), there exists $\boldsymbol{y} \in W^{1,p}(\Omega;\mathbb{R}^N)$ such that, up to subsequences, there holds

$$\boldsymbol{y}_n \rightharpoonup \boldsymbol{y} \text{ in } W^{1,p}(\Omega;\mathbb{R}^N).$$

We need to prove that $\det D\boldsymbol{y} > 0$ almost everywhere. From (2.5) and (2.69), by De la Vallée-Poussin Criterion [62, Theorem 2.29] and the Dunford-Pettis Theorem [62, Theorem 2.54], there exists $h \in L^1(\Omega)$ such that

$$\det D\boldsymbol{y}_n \rightharpoonup h \text{ in } L^1(\Omega).$$

Then, by Theorem 1.24, the map $\boldsymbol{y}$ satisfies (DIV) and there holds $h = \det D\boldsymbol{y}$ almost everywhere. As $\det D\boldsymbol{y}_n > 0$ almost everywhere for every $n \in \mathbb{N}$, this yields $\det D\boldsymbol{y} \geq 0$ almost everywhere. Suppose by contradiction that $\det D\boldsymbol{y} = 0$ in A for some measurable set $A \subset \Omega$ with $\mathscr{L}^N(A) > 0$. In this case, $\det D\boldsymbol{y}_n \to 0$ in $L^1(A)$ and, up to subsequences, $\det D\boldsymbol{y}_n \to 0$ almost everywhere in A. Thus, by (2.9), $\gamma(\det D\boldsymbol{y}_n) \to +\infty$ almost everywhere in A and, by the Fatou Lemma, we have

$$\liminf_{n\to\infty} \int_\Omega \gamma(\det D\boldsymbol{y}_n)\,\mathrm{d}\boldsymbol{x} \geq \liminf_{n\to\infty} \int_A \gamma(\det D\boldsymbol{y}_n)\,\mathrm{d}\boldsymbol{x} = +\infty.$$

This contradicts (2.69) and we conclude that $\det D\boldsymbol{y} > 0$ almost everywhere. Hence, $\boldsymbol{y} \in \mathcal{Y}_p(\Omega)$.

We claim that $(1/\det D\boldsymbol{y}_n)$ is equi-integrable. To check this, we define the function $\widetilde{\gamma}\colon (0,+\infty) \to (0,+\infty)$ by setting $\widetilde{\gamma}(z) \coloneqq \gamma(1/z)$. Assumption (2.5) yields

$$\lim_{z\to+\infty} \frac{\widetilde{\gamma}(z)}{z} = +\infty.$$

Clearly, we have

$$\int_\Omega \widetilde{\gamma}(1/\det D\boldsymbol{y}_n)\,\mathrm{d}\boldsymbol{x} = \int_\Omega \gamma(\det D\boldsymbol{y}_n)\,\mathrm{d}\boldsymbol{x},$$

where the right-hand side is uniformly bounded thanks to (2.69). This gives the claim thanks to the De la Vallée-Poussin Criterion [62, Theorem 2.29]. Taking into account (2.68), the proof is concluded by applying Theorem 2.6. □

The lower semicontinuity result reads as follows.

Proposition 2.10 (Lower semicontinuity of the magnetoelastic energy). *Let* $(\boldsymbol{q}_n) \subset \mathcal{Q}$ *with* $\boldsymbol{q}_n = (\boldsymbol{y}_n, \boldsymbol{m}_n)$ *for every* $n \in \mathbb{N}$. *Suppose that there exists* $\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m}) \in \mathcal{Q}$ *such that the convergences in* (2.15) *and* (2.18)–(2.19) *as well as the almost everywhere convergences in* (2.20)–(2.21) *hold true. Then, there holds:*

$$E(\boldsymbol{q}) \leq \liminf_{n\to\infty} E(\boldsymbol{q}_n). \tag{2.70}$$

Proof. By (2.15) and by the weak continuity of Jacobian minors, we have that all Jacobian minors of $\boldsymbol{y}_n$ converge weakly in $L^1(\Omega)$ to the corresponding Jacobian minor of $\boldsymbol{y}$. At this point, we argue as in [95, Theorem 2.4]. Thanks to the weak convergence of the Jacobian minors of deformations in $L^1(\Omega)$ and to the almost everywhere convergence in (2.20)–(2.21), exploiting (2.7) and applying [14, Theorem 5.4], we obtain

$$E^{\mathrm{el}}(\boldsymbol{q}) \leq \liminf_{n\to\infty} E^{\mathrm{el}}(\boldsymbol{q}_n). \tag{2.71}$$

From (2.19), by the lower semicontinuity of the norm, we immediately get

$$E^{\mathrm{exc}}(\boldsymbol{q}) \leq \liminf_{n\to\infty} E^{\mathrm{exc}}(\boldsymbol{q}_n). \tag{2.72}$$

For simplicity, for every $n \in \mathbb{N}$, set $\psi_n \coloneqq \psi_{\boldsymbol{m}_n}$. Thus, for every $n \in \mathbb{N}$, there holds

$$\forall\, \varphi \in V^{1,2}(\mathbb{R}^N), \qquad \int_{\mathbb{R}^N} \left(D\psi_n - \chi_{\mathrm{im_T}(\boldsymbol{y}_n,\Omega)} \boldsymbol{m}_n \right) \cdot D\varphi \,\mathrm{d}\boldsymbol{\xi} = 0. \tag{2.73}$$

In view of the stability estimate (2.12), we have

$$\sup_{n\in\mathbb{N}} \|D\psi_n\|_{L^2(\mathbb{R}^N;\mathbb{R}^N)} \leq \sup_{n\in\mathbb{N}} \|\chi_{\mathrm{im_T}(\boldsymbol{y}_n,\Omega)} \boldsymbol{m}_n\|_{L^2(\mathbb{R}^N;\mathbb{R}^N)},$$

where the right-hand side is uniformly bounded by (2.18). Thus, up to subsequences, there exists $\psi \in V^{1,2}(\mathbb{R}^N)$ such that

$$D\psi_n \rightharpoonup D\psi \text{ in } L^2(\mathbb{R}^N;\mathbb{R}^N). \tag{2.74}$$

Passing to the limit, as $n \to \infty$, in (2.73) taking into account (2.18), we deduce

$$\forall\, \varphi \in V^{1,2}(\mathbb{R}^N), \qquad \int_{\mathbb{R}^N} \left(D\psi - \chi_{\mathrm{im_T}(\boldsymbol{y},\Omega)} \boldsymbol{m} \right) \cdot D\varphi \,\mathrm{d}\boldsymbol{\xi} = 0. \tag{2.75}$$

Given the uniqueness of solutions of the Maxwell equation stated in Proposition 2.2, we can assume $\psi = \psi_{\boldsymbol{m}}$. Then, by the lower semicontinuity of the norm, we obtain

$$E^{\mathrm{mag}}(\boldsymbol{q}) \leq \liminf_{n\to\infty} E^{\mathrm{mag}}(\boldsymbol{q}_n). \tag{2.76}$$

Finally, combining (2.71)–(2.72) and (2.76), we establish (2.70). □

The magnetostatic energy functional is actually continuous with respect to the notion of convergnece provided by Theorem 2.6 in the case of globally invertible deformations. This fact is the content of the following remark.

Remark 2.11 (Continuity of the magnetostatic energy). With the same notation adopted in Proposition 2.10, suppose that (2.24) and (2.74) hold true. Then, there holds

$$D\psi_{\boldsymbol{m}_n} \to D\psi_{\boldsymbol{m}} \text{ in } L^2(\mathbb{R}^N;\mathbb{R}^N)$$

and, in turn,

$$\lim_{n\to\infty} E^{\mathrm{mag}}(\boldsymbol{q}_n) = E^{\mathrm{mag}}(\boldsymbol{q}).$$

To check this, we test the weak form of the Maxwell equation with the stray field potentials to see that, for every $n \in \mathbb{N}$, there holds

$$\int_{\mathbb{R}^N} |D\psi_n|^2 \,\mathrm{d}\boldsymbol{\xi} = \int_{\mathbb{R}^N} \chi_{\mathrm{im_T}(\boldsymbol{y}_n,\Omega)} \boldsymbol{m}_n \cdot D\psi_n \,\mathrm{d}\boldsymbol{\xi}$$

and, analogously

$$\int_{\mathbb{R}^N} |D\psi|^2 \,\mathrm{d}\boldsymbol{\xi} = \int_{\mathbb{R}^N} \chi_{\mathrm{im_T}(\boldsymbol{y},\Omega)} \boldsymbol{m} \cdot D\psi \,\mathrm{d}\boldsymbol{\xi}.$$

Hence, from (2.24) and (2.74), we obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |D\psi_n|^2 \,\mathrm{d}\boldsymbol{\xi} = \int_{\mathbb{R}^N} |D\psi|^2 \,\mathrm{d}\boldsymbol{\xi},$$

which, together with (2.74), entails the claim.

We are finally ready to prove Theorem 2.5.

Proof of Theorem 2.5. Let $(\boldsymbol{q}_n) \subset \overline{\mathcal{Q}}$ be a minimizing sequence with $\boldsymbol{q}_n = (\boldsymbol{y}_n, \boldsymbol{m}_n)$ for every $n \in \mathbb{N}$. In particular, for every $n \in \mathbb{N}$, there holds $\boldsymbol{y}_n = \overline{\boldsymbol{y}}$ on Γ. The sequence $(E(\boldsymbol{q}_n))$ is uniformly bounded. Recalling (2.6), this gives the boundedness of $(D\boldsymbol{y}_n)$ in $L^p(\Omega;\mathbb{R}^{N\times N})$ and also (2.68)–(2.69). By the Poincarè inequality with boundary term, for every $n \in \mathbb{N}$, we have

$$\|\boldsymbol{y}_n\|_{L^p(\Omega;\mathbb{R}^N)} \leq C \left(\|D\boldsymbol{y}_n\|_{L^p(\Omega;\mathbb{R}^{N\times N})} + \|\overline{\boldsymbol{y}}\|_{L^p(\Gamma;\mathbb{R}^N)} \right),$$

where the constant $C > 0$ depends only on Ω, N and p. This, together with the boundedness of $(D\boldsymbol{y}_n)$, entails (2.67). Hence, by Proposition 2.9, we deduce the existence of $\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m}) \in \mathcal{Q}$ such that, up to subsequences, (2.15)–(2.21) hold true. In particular, (2.15) and the weak continuity of the trace operator ensure that $\boldsymbol{y} = \overline{\boldsymbol{y}}$ on Γ, so that $\boldsymbol{q} \in \overline{\mathcal{Q}}$. Finally, applying Proposition 2.10, we obtain

$$E(\boldsymbol{q}) \leq \liminf_{n\to\infty} E(\boldsymbol{q}_n) = \inf_{\widehat{\boldsymbol{q}} \in \overline{\mathcal{Q}}} E(\widehat{\boldsymbol{q}}),$$

so that $\boldsymbol{q}$ is a minimizer of E in $\overline{\mathcal{Q}}$. The proof under the constraint of global injectivity of deformations is totally analogous. $\square$

We conclude the chapter by presenting some possible variants of our existence result. First, in view of Remark 2.8, we realize that our results can be used to provide an alternative proof of [18, Theorem 8.9]. More precisely, our results provide a more direct proof of the lower semincontinuity of the elastic energy, while the remaining arguments to prove the existence of minimizers remain unchanged.
Other variants of Theorem 2.5 are collected in the following series of remarks.

Remark 2.12 (Applied loads). Let $\boldsymbol{f} \in L^{p'}(\Omega;\mathbb{R}^N)$, $\boldsymbol{g} \in L^{p'}(\Sigma;\mathbb{R}^N)$, where $\Sigma \subset \partial\Omega$ is measurable, and $\boldsymbol{h} \in L^2(\mathbb{R}^N;\mathbb{R}^N)$, represent applied body forces, surface forces and magnetic fields, respectively. The work of applied loads is determined by the functional $L \colon \mathcal{Q} \to \mathbb{R}$ defined by

$$L(\boldsymbol{q}) := \int_\Omega \boldsymbol{f} \cdot \boldsymbol{y} \, \mathrm{d}\boldsymbol{x} + \int_\Sigma \boldsymbol{g} \cdot \boldsymbol{y} \, \mathrm{d}\boldsymbol{a} + \int_{\mathrm{im}_{\mathrm{T}}(\boldsymbol{y},\Omega)} \boldsymbol{h} \cdot \boldsymbol{m} \, \mathrm{d}\boldsymbol{\xi}, \tag{2.77}$$

where $\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m})$. In particular, we see that the energy contribution corresponding to external magnetic fields, known as the Zeman energy, is of Eulerian type. The functional in (2.77) is continuous with respect to the convergences in (2.15) and (2.18). Therefore, the proof of Theorem 2.5 can be easily adapted to prove the existence of minimizers of the total energy functional $E - L$ in $\overline{\mathcal{Q}}$ and $\overline{\mathcal{Q}}^{\mathrm{inj}}$.

Remark 2.13 (Magnetocrystalline anisotropy). The effect of magnetocrystalline anisotropy can be accounted in the variational model by considering the functional $E^{\mathrm{ani}} \colon \mathcal{Q} \to [0, +\infty)$ defined by

$$E^{\mathrm{ani}}(\boldsymbol{q}) := \int_{\mathrm{im}_{\mathrm{T}}(\boldsymbol{y},\Omega)} \phi\left(\frac{\boldsymbol{m}}{|\boldsymbol{m}|}\right) \mathrm{d}\boldsymbol{\xi},$$

where $\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m})$. The density $\phi \colon \mathbb{S}^{N-1} \to [0, +\infty)$ is given by the restriction of a polynomial function on $\mathbb{R}^N$ to $\mathbb{S}^{N-1}$. This function is strictly positive except for a finite set of direction, known as the easy axes. These represent the preferred direction of magnetization which are determined by the crystalline structure of the sample. The case of uniaxial anisotropy corresponds to the choice

$$\phi(\boldsymbol{z}) := c_0 \left(1 - (\boldsymbol{a}_0 \cdot \boldsymbol{z})^2\right),$$

where $c_0 > 0$ and $\boldsymbol{a}_0 \in \mathbb{S}^{N-1}$, while in the case of cubic anisotropy we have

$$\begin{aligned}\phi(\boldsymbol{z}) := c_1 &\left((\boldsymbol{a}_1 \cdot \boldsymbol{z})^2 (\boldsymbol{a}_2 \cdot \boldsymbol{z})^2 + (\boldsymbol{a}_1 \cdot \boldsymbol{z})^2 (\boldsymbol{a}_3 \cdot \boldsymbol{z})^2 + (\boldsymbol{a}_2 \cdot \boldsymbol{z})^2 (\boldsymbol{a}_3 \cdot \boldsymbol{z})^2\right) \\ &+ c_2 (\boldsymbol{a}_1 \cdot \boldsymbol{z})^2 (\boldsymbol{a}_2 \cdot \boldsymbol{z})^2 (\boldsymbol{a}_3 \cdot \boldsymbol{z})^2,\end{aligned}$$

where $c_1, c_2 \in \mathbb{R}$ satisfy suitable relations and $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3 \in \mathbb{S}^{N-1}$ are mutually orthogonal.

Given the regularity of ϕ, the continuity of the functional E^{ani} with respect to the convergence provided by Theorem 2.6 follows from the almost everywhere convergence in (2.23) by applying the Dominated Convergence Theorem. Therefore, by adding the anisotropy term to the magnetoelastic energy in (2.4), the extension of Theorem 2.5 is achieved with no effort.

In particular, if $\boldsymbol{q} \in \mathcal{Q}^{\mathrm{inj}}$, then, using Proposition 1.9, we can write

$$E^{\mathrm{ani}}(\boldsymbol{q}) = \int_{\Omega} \phi(\boldsymbol{m} \circ \boldsymbol{y} \det D\boldsymbol{y}) \det D\boldsymbol{y} \, \mathrm{d}\boldsymbol{x}.$$

Looking at this expression, the continuity of E^{ani} on $\mathcal{Q}^{\mathrm{inj}}$ follows also from (2.21)–(2.22) thanks to the Dominated Convergence Theorem.

Remark 2.14 (Asymmetric exchange). Materials whose crystalline structures lacks of inversion symmetries might exhibit asymmetric exchange, also known as Dzyaloshinskii-Moriya interaction (DMI). This interaction has a fundamental role in the formation of magnetic skyrmions [108].

For $N = 3$, the asymmetric exchange is accounted by the functional $E^{\mathrm{DMI}} \colon \mathcal{Q} \to \mathbb{R}$ defined by

$$E^{\mathrm{DMI}}(\boldsymbol{q}) := A \int_{\mathrm{im_T}(\boldsymbol{y}, \Omega)} \mathrm{curl}\boldsymbol{m} \cdot \boldsymbol{m} \, \mathrm{d}\boldsymbol{\xi},$$

where $A \in \mathbb{R}$. From (2.19) and (2.24), we immediately see that E^{DMI} is continuous on $\mathcal{Q}^{\mathrm{inj}}$ with respect to the notion of convergence provided by Theorem 2.6. Note that the global injectivity of deformation is crucial for the validity of this property. Since the parameter A may be negative, the coercivity of the total energy, given by the sum of the magnetoelastic energy in (2.4) with the asymmetric exchange energy, is not obvious. This property can be deduced by employing the so-called helical derivatives [108]. For every $j \in \{1, 2, 3\}$, set

$$\partial_j^{\mathrm{hel}} \boldsymbol{m} := \partial_j \boldsymbol{m} - A \boldsymbol{e}_j \times m.$$

Also, define the helical gradient as the matrix $D^{\mathrm{hel}} \boldsymbol{m} \in \mathbb{R}^{N \times N}$ whose columns are given by $\partial_j^{\mathrm{hel}} \boldsymbol{m}$ for $j \in \{1, 2, 3\}$. Then, setting

$$E^{\mathrm{hel}}(\boldsymbol{q}) := \int_{\mathrm{im_T}(\boldsymbol{y}, \Omega)} |D^{\mathrm{hel}} \boldsymbol{m}|^2 \, \mathrm{d}\boldsymbol{\xi},$$

where $\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m})$, the following identity holds:

$$E^{\mathrm{exc}}(\boldsymbol{q}) + E^{\mathrm{DMI}}(\boldsymbol{q}) = E^{\mathrm{hel}}(\boldsymbol{q}) + \frac{A^2}{4} \int_{\mathrm{im_T}(\boldsymbol{y},\Omega)} |\boldsymbol{m}|^2 \,\mathrm{d}\boldsymbol{\xi} - \frac{3A^2}{4} \mathscr{L}^N(\Omega).$$

Exploiting this identity, it is possible to recover the coercivity of the functional $E+E^{\mathrm{DMI}}$. Therefore, in view of the previous considerations, the proof of Theorem 2.5 can be adapted to the case in which the DMI term is included in the magnetoelastic energy to prove the existence of minimizers in $\overline{\mathcal{Q}}^{\mathrm{inj}}$

Remark 2.15 (Incompressibility). The case of an incompressible material is implicitly included in our analysis. This corresponds the choice of the function γ in (2.6) which is equal to $+\infty$ whenever its argument is different from one. In that setting, the constraint (2.1) is equivalent to the requirement of magnetizations being sphere-valued. Thus, as observed in [95, Remark 2.2], the convergence of composition follows easily by employing the Change-of-variable Formula.

Remark 2.16 (Integral boundary conditions). A simple variant of Theorem 2.5 consists in replacing the boundary condition in the sense of traces with an integral boundary condition. That is, we define the functional $E^{\mathrm{Dir}} \colon \mathcal{Q} \to [0, +\infty)$ by setting

$$E^{\mathrm{Dir}}(\boldsymbol{q}) := \int_{\Gamma} |\boldsymbol{y} - \overline{\boldsymbol{y}}|^p \,\mathrm{d}\boldsymbol{a},$$

where $\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m})$, and we seek for minimizers of $E + E^{\mathrm{Dir}}$ in $\mathcal{Q}$. Obviously, the proof of the existence of such a minimizers works as the one of Theorem 2.5. From a modeling point of view, in the case in which the material is clamped, integral boundary conditions correspond to keeping track also of deformations of the clamp itself.

Chapter 3

Existence of energetic solutions

In this chapter, we address the existence of quasistatic evolutions for our variational model. Our analysis is set within the theory of rate-independent processes [110] with the notion of energetic solution. The main result of the chapter is Theorem 3.2 which claims the existence of such solutions.
In Section 3.1, we describe the quasistatic setting and we recall the definition of energetic solutions. In Section 3.2, we study the approximate incremental minimization problem and, in Section 3.3, we present the proof of Theorem 3.2. Eventually, in Section 3.4, we mention an alternative setting and we show how the analysis can be adapted according to the latter.

3.1 Setting of the problem

In this chapter, *we mantain all the assumption stated in Section 2.1.* Let $T > 0$ be the time horizon. Let $\Gamma \subset \partial\Omega$ be measurable with $\mathscr{H}^{N-1}(\Gamma) > 0$ and let $\overline{\boldsymbol{y}} \in \mathcal{Y}_p(\Omega)$ be the boundary datum. Recall the definition of $\overline{\mathcal{Q}}$ in (2.14).
Applied loads are determined by

$$\begin{gathered} \boldsymbol{f} \in AC([0,T]; L^{p'}(\Omega;\mathbb{R}^N)), \quad \boldsymbol{g} \in AC([0,T]; L^{p'}(\Sigma;\mathbb{R}^N)), \\ \boldsymbol{h} \in AC([0,T]; L^2(\mathbb{R}^N;\mathbb{R}^N)), \end{gathered} \tag{3.1}$$

representing mechanical body and surface forces, and external magnetic fields, respectively. Here, $\Sigma \subset \partial\Omega$ is a measurable subset.
Recalling (2.4), the total energy functional $\mathcal{F}\colon [0,T] \times \mathcal{Q} \to \mathbb{R}$ is defined as

$$\mathcal{F}(t,\boldsymbol{q}) := E(\boldsymbol{q}) - \mathcal{L}(t,\boldsymbol{q}). \tag{3.2}$$

Analogously to (2.77), the functional $\mathcal{L}\colon [0,T] \times \mathcal{Q} \to \mathbb{R}$ is given by

$$\mathcal{L}(t,\boldsymbol{q}) := \int_\Omega \boldsymbol{f}(t) \cdot \boldsymbol{y} \,\mathrm{d}\boldsymbol{x} + \int_\Sigma \boldsymbol{g}(t) \cdot \boldsymbol{y} \,\mathrm{d}\boldsymbol{a} + \int_{\mathrm{im}_{\mathrm{T}}(\boldsymbol{y},\Omega)} \boldsymbol{h}(t) \cdot \boldsymbol{m} \,\mathrm{d}\boldsymbol{\xi}, \tag{3.3}$$

where $\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m})$. This functional accounts for the work of applied loads. Observe that, given the regularity of the applied loads in (3.1), the map $t \mapsto \mathcal{F}(t,\boldsymbol{q})$ belongs to $AC([0,T])$ for every fixed $\boldsymbol{q} \in \mathcal{Q}$.
The dissipation distance $\mathcal{D}\colon \mathcal{Q} \times \mathcal{Q} \to [0,+\infty)$ is defined by

$$\mathcal{D}(\boldsymbol{q},\widehat{\boldsymbol{q}}) := \int_\Omega |\mathscr{Z}(\boldsymbol{q}) - \mathscr{Z}(\widehat{\boldsymbol{q}})| \,\mathrm{d}\boldsymbol{x}, \tag{3.4}$$

where, for $\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m}) \in \mathcal{Q}$, we set

$$\mathcal{Z}(\boldsymbol{q}) := \boldsymbol{m} \circ \boldsymbol{y} \det D\boldsymbol{y}. \tag{3.5}$$

From (2.1), we see that $\mathcal{Z}(\boldsymbol{q})\colon \Omega \to \mathbb{R}^N$ is sphere-valued and, in turn, the dissipation distance is always bounded.
Given a map $\boldsymbol{q}\colon [0,T] \to \mathcal{Q}$, its variation with respect to $\mathcal{D}$ on the interval $[s,t] \subset [0,T]$ is defined as

$$\mathrm{Var}_{\mathcal{D}}(\boldsymbol{q};[s,t]) := \sup\left\{ \sum_{i=1}^{l} \mathcal{D}(\boldsymbol{q}(t^{i-1}), \boldsymbol{q}(t^i)) :\ \Pi = (t^0, t^1, \dots, t^l) \right\}, \tag{3.6}$$

where the supremum is taken over all the paritions Π of $[s,t]$. By a partition of $[s,t]$, we mean any finite ordered subset $\Pi = (t^0, \dots, t^l)$ of $[0,T]$ for some $l \in \mathbb{N}$ with $s = t^0 < \cdots < t^l = t$. The quantity in (3.6) represents the amount of energy dissipated along the evolution described by $\boldsymbol{q}$ within the time interval $[s,t]$.
We present the definition of energetic solutions in our context.

Definition 3.1 (Energetic solution). A function $\boldsymbol{q}\colon [0,T] \to \overline{\mathcal{Q}}$ is termed an energetic solution of the quasistatic model if the function $t \mapsto \partial_t \mathcal{F}(t, \boldsymbol{q}(t))$ belongs to $L^1(0,T)$ and, for every $t \in [0,T]$, the following two conditions are satisfied:

(i) Global stability:

$$\forall\, t \in [0,T],\ \forall\, \widehat{\boldsymbol{q}} \in \overline{\mathcal{Q}},\quad \mathcal{F}(t, \boldsymbol{q}(t)) \leq \mathcal{F}(t, \widehat{\boldsymbol{q}}) + \mathcal{D}(\boldsymbol{q}(t), \widehat{\boldsymbol{q}}); \tag{3.7}$$

(ii) Energy-dissipation balance:

$$\forall\, t \in [0,T],\quad \mathcal{F}(t, \boldsymbol{q}(t)) + \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q};[0,t]) = \mathcal{F}(0, \boldsymbol{q}^0) + \int_0^t \partial_t \mathcal{F}(\tau, \boldsymbol{q}(\tau))\, \mathrm{d}\tau. \tag{3.8}$$

We stress that, in Definition 3.1, the Dirichlet boundary conditions have been directly incorporated into the state space.
The aim of this chapter is to establish the existence of energetic solutions for our quasistatic model. The main result reads as follows.

Theorem 3.2 (Existence of energetic solutions). *Let $\boldsymbol{q}^0 \in \overline{\mathcal{Q}}$ satisfy*

$$\forall\, \widehat{\boldsymbol{q}} \in \overline{\mathcal{Q}},\quad \mathcal{F}(0, \boldsymbol{q}^0) \leq \mathcal{F}(0, \widehat{\boldsymbol{q}}) + \mathcal{D}(\boldsymbol{q}^0, \widehat{\boldsymbol{q}}). \tag{3.9}$$

Then, there exists a map $\boldsymbol{q}\colon [0,T] \to \overline{\mathcal{Q}}$ satisfying the initial condition $\boldsymbol{q}(0) = \boldsymbol{q}^0$ which is an energetic solution of the quasistatic model. In particular, the map $\boldsymbol{q}$ is measurable and uniformly bounded. Also, the map $t \mapsto \mathcal{Z}(\boldsymbol{q}(t))$ belongs to $BV([0,T]; L^1(\Omega;\mathbb{R}^N))$, while the map $t \mapsto \mathcal{F}(t, \boldsymbol{q}(t))$ belongs to $BV([0,T])$.

In the previous theorem, the measurability of $\boldsymbol{q}$ is meant with respecto to the Borel sigma-algebra of $\mathcal{Q}$, while, setting $\boldsymbol{q}(t) = (\boldsymbol{y}(t), \boldsymbol{m}(t))$, the uniform boundedness of $\boldsymbol{q}$ is understood as the one of the map

$$t \mapsto \big(\boldsymbol{y}(t), \chi_{\mathrm{im_T}(\boldsymbol{y}(t),\Omega)} \boldsymbol{m}(t), \chi_{\mathrm{im_T}(\boldsymbol{y}(t),\Omega)} D\boldsymbol{m}(t)\big)$$

from $[0,T]$ to $W^{1,p}(\Omega;\mathbb{R}^N) \times L^2(\mathbb{R}^N;\mathbb{R}^N) \times L^2(\mathbb{R}^N;\mathbb{R}^{N\times N})$. Given the highly nonconvex character our energy functional, we cannot hope for better regularity for energetic solutions nor for their uniqueness.

Remark 3.3 (Global invertibility). For simplicity, the statement of this chapter have not been specialized for the case of of globally invertible deformations. However, it is clear that this requirement can be easily incorporated in the analysis by employing Corollary 1.43.

The remainder of the chapter is devoted to the proof of Theorem 3.2. The strategy of the proof is standard and consists of two main steps: first, for a fixed partition of the time interval, one solves an incremental minimization problem; then, one considers a sequence of partition of vanishing size and constructs the desired solution from the sequence piecewise-constant interpolants corresponding to the solutions of the incremental minimization problems by compactness arguments. These two steps will be addressed in the next two subsections.

3.2 Incremental minimization problem

Before proceeding, we show that our model fulfills the basic assumptions of the theory of rate-independent systems [110].
Denote by $Z \subset (0,T)$ the set of times for which the applied loads do not admit a pointwise derivative. Given (3.1), there holds $\mathscr{L}^1(Z)=0$.

Lemma 3.4 (Time-control of the total energy). *There exist two constants $C, L>0$ such that, defining $\kappa \in L^1(0,T)$ by setting*

$$\kappa(t) := C \left(\|\dot{\boldsymbol{f}}(t)\|_{L^{p'}(\Omega;\mathbb{R}^N)} + \|\dot{\boldsymbol{g}}(t)\|_{L^{p'}(\Sigma;\mathbb{R}^N)} + \|\dot{\boldsymbol{h}}(t)\|_{L^2(\mathbb{R}^N;\mathbb{R}^N)} \right),$$

for every $\boldsymbol{q} \in \overline{\mathcal{Q}}$, the following estimates hold:

$$\forall\, t \in (0,T)\setminus Z, \quad |\partial_t \mathcal{F}(t,\boldsymbol{q})| \leq \kappa(t)\,(\mathcal{F}(t,\boldsymbol{q})+L)\,, \tag{3.10}$$

$$\forall\, s,t \in [0,T], \quad \mathcal{F}(t,\boldsymbol{q})+L \leq (\mathcal{F}(s,\boldsymbol{q})+L)\,\mathrm{e}^{|K(t)-K(s)|}, \tag{3.11}$$

$$\forall\, t \in [0,T]\setminus Z,\ \forall s \in [0,T], \quad |\partial_t \mathcal{F}(t,\boldsymbol{q})| \leq \eta(t)\,(\mathcal{F}(s,\boldsymbol{q})+L)\,\mathrm{e}^{|K(t)-K(s)|}. \tag{3.12}$$

Here, we define the function $K \in AC([0,T])$ by setting

$$K(t) := \int_0^t \kappa(\tau)\,\mathrm{d}\tau.$$

In particular, the constant L is chosen such that

$$\inf_{(t,\boldsymbol{q})\in[0,T]\times\overline{\mathcal{Q}}} \mathcal{F}(t,\boldsymbol{q}) \geq -L. \tag{3.13}$$

Proof. First of all, we claim that there exist two constants $C_1, C_2>0$, both depending only on the given data, such that, for every $t\in[0,T]$ and $\boldsymbol{q}=(\boldsymbol{y},\boldsymbol{m})\in\overline{\mathcal{Q}}$, there holds

$$\begin{aligned}\mathcal{F}(t,\boldsymbol{q}) \geq C_1 \Big(&\|D\boldsymbol{y}\|^p_{L^p(\Omega;\mathbb{R}^{N\times N})} + \|\gamma(\det D\boldsymbol{y})\|_{L^1(\Omega)}\Big) \\ &+ \|D\boldsymbol{m}\|^2_{L^2(\mathrm{im_T}(\boldsymbol{y},\Omega);\mathbb{R}^{N\times N})} - C_2.\end{aligned} \tag{3.14}$$

Thanks to the Poincaré inequality with trace term, we have

$$\|\boldsymbol{y}\|_{W^{1,p}(\Omega;\mathbb{R}^N)} \leq C \left(\|D\boldsymbol{y}\|_{L^p(\Omega;\mathbb{R}^{N\times N})} + ||\overline{\boldsymbol{y}}||_{L^p(\Gamma;\mathbb{R}^N)} \right), \tag{3.15}$$

where the constant $C > 0$ depends only on Ω, N and p. Define $\widetilde{\gamma}$ as in the proof of Proposition 2.9 and recall that, by (2.5), this function has superlinear growth at infinity. Applying Proposition 1.9 and exploiting (2.1), we estimate

$$\begin{aligned}\int_{\mathrm{im_T}(\boldsymbol{y},\Omega)} |\boldsymbol{m}|^2 \,\mathrm{d}\boldsymbol{\xi} &\leq \int_\Omega |\boldsymbol{m}\circ\boldsymbol{y}|^2 \det D\boldsymbol{y}\,\mathrm{d}\boldsymbol{x} \\ &= \int_\Omega |\boldsymbol{m}\circ\boldsymbol{y}|\,\mathrm{d}\boldsymbol{x} = \int_\Omega \frac{1}{\det D\boldsymbol{y}}\,\mathrm{d}\boldsymbol{x} \\ &\leq C\left(1+\|\widetilde{\gamma}\,(1/\det D\boldsymbol{y})\|_{L^1(\Omega)}\right) \\ &\leq C\left(1+\|\gamma(\det D\boldsymbol{y})\|_{L^1(\Omega)}\right),\end{aligned} \tag{3.16}$$

where the constant $C > 0$ depends only on Ω and γ. Employing (3.15)–(3.16), the Hölder inequality and the trace inequality, we have

$$\begin{aligned}|\mathcal{L}(t,\boldsymbol{q})| &\leq \|\boldsymbol{f}(t)\|_{L^{p'}(\Omega;\mathbb{R}^N)}\,\|\boldsymbol{y}\|_{L^p(\Omega;\mathbb{R}^N)} + \|\boldsymbol{g}(t)\|_{L^{p'}(\Sigma;\mathbb{R}^N)}\,\|\boldsymbol{y}\|_{L^p(\Sigma;\mathbb{R}^N)} \\ &\quad + \|\boldsymbol{h}(t)\|_{L^2(\mathbb{R}^N;\mathbb{R}^N)}\,\|\boldsymbol{m}\|_{L^2(\mathrm{im_T}(\boldsymbol{y},\Omega);\mathbb{R}^N)} \\ &\leq C_3\left(\|D\boldsymbol{y}\|_{L^p(\Omega;\mathbb{R}^{N\times N})} + \|\gamma(\det D\boldsymbol{y})\|^{1/2}_{L^1(\Omega)}\right) + C_4,\end{aligned} \tag{3.17}$$

for some constants $C_3, C_4 > 0$. These constants depend on the given data. In particular, they depend on the norms of $\boldsymbol{f}$, $\boldsymbol{g}$ and $\boldsymbol{h}$ in the spaces $C^0([0,T];L^{p'}(\Omega;\mathbb{R}^N))$, $C^0([0,T];L^{p'}(\Sigma;\mathbb{R}^N))$ and $C^0([0,T];L^2(\mathbb{R}^N;\mathbb{R}^N))$, respectively, as well as on the norm of $\overline{\boldsymbol{y}}$ in $L^p(\Gamma;\mathbb{R}^N)$. Then, recalling (2.6) and using (3.17) and the Young inequality, we obtain the bound

$$\begin{aligned}\mathcal{F}(t,\boldsymbol{q}) &\geq C\|D\boldsymbol{y}\|^p_{L^p(\Omega;\mathbb{R}^{N\times N})} + \|\gamma(\det D\boldsymbol{y})\|_{L^1(\Omega)} + \|D\boldsymbol{m}\|^2_{L^2(\mathrm{im_T}(\boldsymbol{y},\Omega);\mathbb{R}^{N\times N})} \\ &\quad - C_3\left(\|D\boldsymbol{y}\|_{L^p(\Omega;\mathbb{R}^{N\times N})} + \|\gamma(\det D\boldsymbol{y})\|^{1/2}_{L^1(\Omega)}\right) - C_4 \\ &\geq C_1\left(\|D\boldsymbol{y}\|^p_{L^p(\Omega;\mathbb{R}^{N\times N})} + \|\gamma(\det D\boldsymbol{y})\|_{L^1(\Omega)}\right) \\ &\quad + \|D\boldsymbol{m}\|^2_{L^2(\mathrm{im_T}(\boldsymbol{y},\Omega);\mathbb{R}^{N\times N})} - C_2,\end{aligned}$$

where the two constants $C_1, C_2 > 0$ depend only on the diven data. This proves the claim.

Now, recall that, for every fixed $\boldsymbol{q} \in \mathcal{Q}$, the map $t \mapsto \mathcal{F}(t,\boldsymbol{q})$ belongs to $AC([0,T])$. In particular, for every $t \in (0,T)\setminus Z$, there holds

$$\partial_t \mathcal{F}(t,\boldsymbol{q}) = -\int_\Omega \dot{\boldsymbol{f}}(t)\cdot\boldsymbol{y}\,\mathrm{d}\boldsymbol{x} - \int_\Sigma \dot{\boldsymbol{g}}(t)\cdot\boldsymbol{y}\,\mathrm{d}\boldsymbol{a} - \int_{\mathrm{im_T}(\boldsymbol{y},\Omega)} \dot{\boldsymbol{h}}(t)\cdot\boldsymbol{m}\,\mathrm{d}\boldsymbol{\xi}. \tag{3.18}$$

For convenience, set

$$\widetilde{\kappa}(t) := \|\dot{\boldsymbol{f}}(t)\|_{L^{p'}(\Omega:\mathbb{R}^N)} + \|\dot{\boldsymbol{g}}(t)\|_{L^{p'}(\Sigma:\mathbb{R}^N)} + \|\dot{\boldsymbol{h}}(t)\|_{L^2(\mathbb{R}^N:\mathbb{R}^N)}.$$

Employing again Hölder and Young inequalities together with (3.14) and (3.15)–

(3.16), we estimate

$$
\begin{aligned}
|\partial_t \mathcal{F}(t,\boldsymbol{q})| &\leq \|\dot{\boldsymbol{f}}(t)\|_{L^{p'}(\Omega;\mathbb{R}^N)} \|\boldsymbol{y}\|_{L^p(\Omega;\mathbb{R}^N)} + \|\dot{\boldsymbol{g}}(t)\|_{L^{p'}(\Sigma;\mathbb{R}^N)} \|\boldsymbol{y}\|_{L^p(\Sigma;\mathbb{R}^N)} \\
&\quad + \|\dot{\boldsymbol{h}}(t)\|_{L^2(\mathbb{R}^N;\mathbb{R}^N)} \|\boldsymbol{m}\|_{L^2(\mathrm{im}_{\mathrm{T}}(\boldsymbol{y},\Omega);\mathbb{R}^N)} \\
&\leq C\,\widetilde{\kappa}(t) \left(\|D\boldsymbol{y}\|_{L^p(\Omega;\mathbb{R}^{N\times N})} + \|\gamma(\det D\boldsymbol{y})\|^{1/2}_{L^1(\Omega)} + C_5 \right) \\
&\leq C\,\widetilde{\kappa}(t) \left(\|D\boldsymbol{y}\|^p_{L^p(\Omega;\mathbb{R}^{N\times N})} + \|\gamma(\det D\boldsymbol{y})\|_{L^1(\Omega)} + C_6 \right) \\
&\leq C\,\widetilde{\kappa}(t) \left(\frac{1}{C_1} \left(\mathcal{F}(t,\boldsymbol{q}) + C_2 \right) + C_6 \right) \\
&\leq \frac{C}{C_1}\,\widetilde{\kappa}(t) \left(\mathcal{F}(t,\boldsymbol{q}) + C_2 + C_1\, C_6 \right).
\end{aligned}
$$

Setting $\kappa(t) \coloneqq C\,C_1^{-1}\widetilde{\kappa}(t)$ and $L \coloneqq C_2 + C_1\,C_6$, this proves (3.10). In particular, (3.13) follows from (3.14) as $L > C_2$. From (3.10), by applying the Gronwall inequality, we deduce (3.11). Finally, combining (3.10) –(3.11), we obtain (3.12). □

For convenience, in the next two results, we highlight the coercivity and lower semicontinuity properties of the total energy functional. These are simple consequences of the result of Chapter 2.

Lemma 3.5 (Coercivity of the total energy). *Let $(t_n) \subset [0,T]$ and $(\boldsymbol{q}_n) \subset \mathcal{Q}$ with $\boldsymbol{q}_n = (\boldsymbol{y}_n, \boldsymbol{m}_n)$. Suppose that*

$$
\sup_{n\in\mathbb{N}} \mathcal{F}(t_n,\boldsymbol{q}_n) \leq M, \tag{3.19}
$$

for some $M > 0$. Then, there exists $\boldsymbol{q} \in \mathcal{Q}$ with $\boldsymbol{q} = (\boldsymbol{y},\boldsymbol{m})$ such that, up to subsequences, the convergences in (2.15)*–*(2.23) *hold true.*

Proof. From (3.19), thanks to (3.14)–(3.15), we deduce (2.67)–(2.69). Thus, the proof is concluded by applying Proposition 2.9. □

Lemma 3.6 (Lower semicontinuity of the total energy). *Let $(t_n) \subset [0,T]$ and $(\boldsymbol{q}_n) \subset \mathcal{Q}$ with $\boldsymbol{q}_n = (\boldsymbol{y}_n, \boldsymbol{m}_n)$. Suppose that there exist $t \in [0,T]$ and $\boldsymbol{q} \in \overline{\mathcal{Q}}$ with $\boldsymbol{q} = (\boldsymbol{y},\boldsymbol{m})$ such that that $t_n \to t$ and that the convergences in* (2.15) *and* (2.18)*–*(2.19) *hold true as well as the almost everywhere convergence in* (2.20)*. Then, there holds*

$$
\mathcal{F}(t,\boldsymbol{q}) \leq \liminf_{n\to\infty} \mathcal{F}(t_n,\boldsymbol{q}_n). \tag{3.20}
$$

Proof. Observe that the functional $\widehat{\boldsymbol{q}} \mapsto \mathcal{L}(t,\widehat{\boldsymbol{q}})$ is continuous with respect to the convergences in (2.15)– (2.18). This, together with Proposition 2.10, yields

$$
\mathcal{F}(t,\boldsymbol{q}) \leq \liminf_{n\to\infty} \mathcal{F}(t,\boldsymbol{q}_n).
$$

Hence, in order to establish (3.20), it is sufficient to check that

$$
\lim_{n\to\infty} \left\{ \mathcal{F}(t_n,\boldsymbol{q}_n) - \mathcal{F}(t,\boldsymbol{q}_n) \right\} = \lim_{n\to\infty} \left\{ \mathcal{L}(t,\boldsymbol{q}_n) - \mathcal{L}(t_n,\boldsymbol{q}_n) \right\} = 0.
$$

The last equality is easily checked by exploiting the boundedness of the sequences $(\boldsymbol{y}_n)$ and $(\chi_{\mathrm{im}_{\mathrm{T}}(\boldsymbol{y}_n,\Omega)}\boldsymbol{m}_n)$ in $W^{1,p}(\Omega;\mathbb{R}^N)$ and $L^2(\mathbb{R}^N;\mathbb{R}^N)$, respectively, together with the continuity of the applied loads with respect to time ensured by (3.1). □

We now introduce the incremental minimization problem.

Definition 3.7 (Incremental minimization problem). Let $\Pi = (t^0, \dots, t^l)$ be a partition of $[0,T]$ and let $\boldsymbol{q}^0 \in \overline{\mathcal{Q}}$. The incremental minimization problem (IMP) determined by Π with initial datum $\boldsymbol{q}^0$ reads as follows: for every $i \in \{1, \dots, l\}$ find $\boldsymbol{q}^i \in \overline{\mathcal{Q}}$ such that

$$\mathcal{F}(t^i, \boldsymbol{q}^i) + \mathcal{D}(\boldsymbol{q}^{i-1}, \boldsymbol{q}^i) = \inf_{\widehat{\boldsymbol{q}} \in \overline{\mathcal{Q}}} \left\{ \mathcal{F}(t^i, \widehat{\boldsymbol{q}}) + \mathcal{D}(\boldsymbol{q}^{i-1}, \widehat{\boldsymbol{q}}) \right\}. \tag{3.21}$$

In the next result, we prove the existence of solutions of the incremental minimization problem. These turn out to satisfy time-discrete versions of the global stability condition and of the upper energy-dissipation inequality. Also, we establish some estimates that are going to be instrumental for the compactness argument adopted in the proof of Theorem 3.2.

Proposition 3.8 (Solutions of the IMP). *Let $\Pi = (t^0, \dots, t^l)$ be a partition of $[0,T]$ and let $\boldsymbol{q}^0 \in \overline{\mathcal{Q}}$. Then, the incremental minimization problem determined by Π with initial datum $\boldsymbol{q}^0$ admits a solution $(\boldsymbol{q}^1, \dots, \boldsymbol{q}^l)$. Moreover, if $\boldsymbol{q}^0$ satisfies (3.9), then, for every $i \in \{1, \dots, l\}$, the following hold:*

$$\forall \widehat{\boldsymbol{q}} \in \overline{\mathcal{Q}}, \quad \mathcal{F}(t^i, \boldsymbol{q}^i) \le \mathcal{F}(t^i, \widehat{\boldsymbol{q}}) + \mathcal{D}(\boldsymbol{q}^i, \widehat{\boldsymbol{q}}), \tag{3.22}$$

$$\mathcal{F}(t^i, \boldsymbol{q}^i) - \mathcal{F}(t^{i-1}, \boldsymbol{q}^{i-1}) + \mathcal{D}(\boldsymbol{q}^{i-1}, \boldsymbol{q}^i) \le \int_{t^{i-1}}^{t^i} \partial_t \mathcal{F}(\tau, \boldsymbol{q}^{i-1})\, \mathrm{d}\tau, \tag{3.23}$$

$$\mathcal{F}(t^i, \boldsymbol{q}^i) + L + \sum_{j=1}^{i} \mathcal{D}(\boldsymbol{q}^{j-1}, \boldsymbol{q}^j) \le \big(\mathcal{F}(0, \boldsymbol{q}^0) + L\big)\, \mathrm{e}^{K(t^i)}. \tag{3.24}$$

Moreover, for every $i \in \{1, \dots, l\}$, the following estimate holds:

$$\begin{aligned} &|\mathcal{F}(t^i, \boldsymbol{q}^i) - \mathcal{F}(t^{i-1}, \boldsymbol{q}^{i-1}) + \mathcal{D}(\boldsymbol{q}^{i-1}, \boldsymbol{q}^i)| \\ &\qquad \le (\mathcal{F}(t^{i-1}, \boldsymbol{q}^{i-1}) + L) \left(\mathrm{e}^{K(t^i) - K(t^{i-1})} - 1 \right). \end{aligned} \tag{3.25}$$

Proof. We first prove the existence of solutions. Let $i \in \{1, \dots, l\}$ and suppose that $\boldsymbol{q}^{i-1} \in \mathcal{Q}$ is given. Let $(\boldsymbol{q}_n) \subset \mathcal{Q}$ with $\boldsymbol{q}_n = (\boldsymbol{y}_n, \boldsymbol{m}_n)$ be a minimizing sequence for the functional $\boldsymbol{q} \mapsto \mathcal{F}(t^i, \boldsymbol{q}) + \mathcal{D}(\boldsymbol{q}^{i-1}, \boldsymbol{q})$. As the sequence $(\mathcal{F}(t^i, \boldsymbol{q}_n))$ is bounded with respect to $n \in \mathbb{N}$, by Lemma 3.5, there exists $\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m}) \in \overline{\mathcal{Q}}$ such that, up to subsequences, the convergence in (2.15)–(2.22) hold true. Thus, by Lemma 3.6 and by the definition of the dissipation distance, we have

$$\begin{aligned} \mathcal{F}(t^i, \boldsymbol{q}) + \mathcal{D}(\boldsymbol{q}^{i-1}, \boldsymbol{q}) &\le \liminf_{n\to\infty} \mathcal{F}(t^i, \boldsymbol{q}_n) + \lim_{n\to\infty} \mathcal{D}(\boldsymbol{q}^{i-1}, \boldsymbol{q}_n) \\ &= \liminf_{n\to\infty} \left\{ \mathcal{F}(t^i, \boldsymbol{q}_n) + \mathcal{D}(\boldsymbol{q}^{i-1}, \boldsymbol{q}_n) \right\} \\ &= \inf_{\widehat{\boldsymbol{q}} \in \mathcal{Q}} \left\{ \mathcal{F}(t^i, \widehat{\boldsymbol{q}}) + \mathcal{D}(\boldsymbol{q}^{i-1}, \widehat{\boldsymbol{q}}) \right\}. \end{aligned}$$

This proves the existence of solutions of the incremental minimization problem. The other claims are achieved by means of standard computations [110, Proposition 2.1.4]. For convenience of the reader, we provide full details. First, combining (3.21) and the triangle inequality, we obtain

$$\begin{aligned} \mathcal{F}(t^i, \boldsymbol{q}^i) &= \mathcal{F}(t^i, \boldsymbol{q}^i) + \mathcal{D}(\boldsymbol{q}^{i-1}, \boldsymbol{q}^i) - \mathcal{D}(\boldsymbol{q}^{i-1}, \boldsymbol{q}^i) \\ &\le \mathcal{F}(t^i, \widehat{\boldsymbol{q}}) + \mathcal{D}(\boldsymbol{q}^{i-1}, \widehat{\boldsymbol{q}}) - \mathcal{D}(\boldsymbol{q}^{i-1}, \boldsymbol{q}^i) \\ &\le \mathcal{F}(t^i, \widehat{\boldsymbol{q}}) + \mathcal{D}(\boldsymbol{q}^i, \widehat{\boldsymbol{q}}), \end{aligned}$$

which is (3.22). To see (3.23), we test (3.21) with $\boldsymbol{q}^{i-1}$ and we employ the Fundamental Theorem of Calculus to infer

$$\begin{aligned}\mathcal{F}(t^i,\boldsymbol{q}^i)-\mathcal{F}(t^{i-1},\boldsymbol{q}^{i-1})+\mathcal{D}(\boldsymbol{q}^{i-1},\boldsymbol{q}^i) &\leq \mathcal{F}(t^i,\boldsymbol{q}^{i-1})-\mathcal{F}(t^{i-1},\boldsymbol{q}^{i-1})\\ &= \int_{t^{i-1}}^{t^i}\partial_t\mathcal{F}(\tau,\boldsymbol{q}^{i-1})\,\mathrm{d}\tau.\end{aligned}$$

We now move to the proof of (3.24). For convenience, for every $j\in\{0,1,\ldots,l\}$, we set $f^j:=\mathcal{F}(t^j,\boldsymbol{q}^j)$ and $d^j:=\mathcal{D}(\boldsymbol{q}^{j-1},\boldsymbol{q}^j)$. From (3.24), making use of (3.12), we estimate

$$\begin{aligned}f^i+d^i &\leq f^{i-1}+\int_{t^{i-1}}^{t^i}\partial_t\mathcal{F}(\tau,\boldsymbol{q}^{i-1})\,\mathrm{d}\tau\\ &\leq f^{i-1}+\left(f^{i-1}+L\right)\int_{t^{i-1}}^{t^i}\kappa(\tau)\mathrm{e}^{K(\tau)-K(t^{i-1})}\,\mathrm{d}\tau\\ &= f^{i-1}+\left(f^{i-1}+L\right)\left(\mathrm{e}^{K(t^i)-K(t^{i-1})}-1\right)\\ &= \left(f^{i-1}+L\right)\mathrm{e}^{K(t^i)-K(t^{i-1})}-L.\end{aligned}\tag{3.26}$$

In particular,

$$f^i+L\leq\left(f^{i-1}+L\right)\mathrm{e}^{K(t^i)-K(t^{i-1})}.$$

and, by arguing by induction, we deduce

$$f^i+L\leq\left(f^0+L\right)\prod_{j=1}^{i}\mathrm{e}^{K(t^j)-K(t^{j-1})}=(f^0+L)\mathrm{e}^{K(t^i)}.\tag{3.27}$$

Summing (3.26), with j in place of i, for $j\in\{1,\ldots,i\}$, we have

$$\sum_{j=1}^{i}f^j+\sum_{j=1}^{i}d^j\leq\sum_{j=1}^{i}f^{j-1}+\sum_{j=1}^{i}(f^{j-1}+L)\left(\mathrm{e}^{K(t^j)-K(t^{j-1})}-1\right),$$

so that, employing (3.27), we obtain

$$\begin{aligned}f^i+\sum_{j=1}^{i}d^j &\leq f^0+\sum_{j=1}^{i}(f^{j-1}+L)\left(\mathrm{e}^{K(t^j)-K(t^{j-1})}-1\right)\\ &\leq f^0+(f^0+L)\sum_{j=1}^{i}\mathrm{e}^{K(t^{j-1})}\left(\mathrm{e}^{K(t^j)-K(t^{j-1})}-1\right)\\ &=\leq f^0+(f^0+L)\sum_{j=1}^{i}\left(\mathrm{e}^{K(t^j)}-\mathrm{e}^{K(t^{j-1})}\right)\\ &= (f^0+L)\mathrm{e}^{K(t^i)}.\end{aligned}$$

This gives (3.23). Eventually, we check (3.25). Testing (3.22) for $i-1$ if $i>1$ or (3.9) if $i=1$ both with $\boldsymbol{q}^i$, we have

$$f^{i-1}\leq\mathcal{F}(t^{i-1},\boldsymbol{q}^i)+d^i.$$

From this, employing the Fundamental Theorem of Calculus and (3.12), we compute

$$
\begin{aligned}
f^{i-1} - f^i - d^i &\le -\big(\mathcal{F}(t^i, \boldsymbol{q}^i) - \mathcal{F}(t^{i-1}, \boldsymbol{q}^i)\big) \\
&= -\int_{t^{i-1}}^{t^i} \partial_t \mathcal{F}(\tau, \boldsymbol{q}^i)\,\mathrm{d}\tau \\
&\le (f^{i-1} + L) \int_{t^{i-1}}^{t^i} \kappa(\tau) \mathrm{e}^{K(\tau) - K(t^{i-1})}\,\mathrm{d}\tau \\
&\le (f^{i-1} + L)\left(\mathrm{e}^{K(t^i) - K(t^{i-1})} - 1\right).
\end{aligned} \tag{3.28}
$$

Thus, combining (3.26) and (3.28), we obtain (3.25). □

3.3 Proof of Theorem 3.2

We are finally ready to proceed with the proof of Theorem 3.2.

Proof of Theorem 3.2. We rigorously follow the scheme provided in [110, Theorem 2.1.6]. The proof is subdivided into five steps.

Step 1 (A priori estimates). Let (Π_n) with $\Pi_n = (t_n^0, \dots, t_n^{l_n})$ be a sequence of partitions of $[0,T]$ such that $|\Pi_n| \to 0$, as $n \to \infty$. By Proposition 3.8, for every $n \in \mathbb{N}$, the incremental minimization problem determined by Π_n with initial datum $\boldsymbol{q}^0$ admits a solution $(\boldsymbol{q}_n^1, \dots, \boldsymbol{q}_n^{l_n})$. The corresponding right-continuous piecewise constant interpolant $\boldsymbol{q}_n \colon [0,T] \to \overline{\mathcal{Q}}$ is defined by setting

$$
\boldsymbol{q}_n(t) \coloneqq \begin{cases} \boldsymbol{q}_n^i & \text{if } t \in [t_n^{i-1}, t_n^i) \text{ for some } i \in \{1, \dots, l_n\}, \\ \boldsymbol{q}_n^{l_n} & \text{if } t = T. \end{cases}
$$

We claim that the following estimate holds:

$$
\sup_{n\in\mathbb{N}} \left\{ \sup_{t\in[0,T]} \mathcal{F}(t, \boldsymbol{q}_n(t)) + \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}_n; [0,T]) \right\} \le \big(\mathcal{F}(0, \boldsymbol{q}^0) + L\big)\, \mathrm{e}^{K(T)} =: M. \tag{3.29}
$$

To see this fix $n \in \mathbb{N}$. For every $t \in [0,T]$, let $i \in \{1, \dots, l_n\}$ be such that $t \in [t_n^{i-1}, t_n^i)$. By definition, we have

$$
\boldsymbol{q}_n(t) = \boldsymbol{q}_n^{i-1}, \quad \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}_n; [0,t]) = \sum_{j=1}^{i-1} \mathcal{D}(\boldsymbol{q}^{j-1}, \boldsymbol{q}^j).
$$

Thus, using (3.11) and (3.24), we estimate

$$
\begin{aligned}
\mathcal{F}(t, \boldsymbol{q}_n(t)) + L + \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}_n; [0,t]) &= \mathcal{F}(t, \boldsymbol{q}^{i-1}) + L + \sum_{j=1}^{i-1} \mathcal{D}(\boldsymbol{q}_n^{j-1}, \boldsymbol{q}_n^j) \\
&\le \big(\mathcal{F}(t_n^{i-1}, \boldsymbol{q}_n^{i-1}) + L\big)\, \mathrm{e}^{K(t) - K(t_n^{i-1})} + \sum_{j=1}^{i-1} \mathcal{D}(\boldsymbol{q}_n^{j-1}, \boldsymbol{q}_n^j) \\
&\le \left(\mathcal{F}(t_n^{i-1}, \boldsymbol{q}_n^{i-1}) + L + \sum_{j=1}^{i-1} \mathcal{D}(\boldsymbol{q}_n^{j-1}, \boldsymbol{q}_n^j) \right) \mathrm{e}^{K(t) - K(t_n^{i-1})} \\
&\le \big(\mathcal{F}(0, \boldsymbol{q}^0) + L\big)\, \mathrm{e}^{K(t)}.
\end{aligned}
$$

This proves (3.29).

For convenience, for every $n \in \mathbb{N}$, define $f_n \colon [0,T] \to \mathbb{R}$ by setting $f_n(t) \coloneqq \mathcal{F}(t, \boldsymbol{q}_n(t))$. By (3.29), we already know that the sequence (f_n) is uniformly bounded. We now establish a uniform bound for the total variation of the functions f_n. Denote by $[f_n]^i$ the jump of the function f_n at time t_n^i. Since the map $t \mapsto \mathcal{F}(t, \boldsymbol{q}_n^{i-1})$ is continuous, with the aid of (3.12) and (3.25), we compute

$$
\begin{aligned}
|[f_n]^i| &= \left| \lim_{s\to 0^+} \left\{ f_n(t_n^i) - f_n(t_n^i - s) \right\} \right| \\
&= \left| \lim_{s\to 0^+} \left\{ \mathcal{F}(t_n^i, \boldsymbol{q}_n^i) - \mathcal{F}(t_n^i - s, \boldsymbol{q}_n^{i-1}) \right\} \right| \\
&= |f_n^i - f_n^{i-1} - (\mathcal{F}(t_n^i, \boldsymbol{q}_n^{i-1}) - \mathcal{F}(t_n^{i-1}, \boldsymbol{q}_n^{i-1}))| \\
&\leq |f_n^i - f_n^{i-1}| + \int_{t_n^{i-1}}^{t_n^i} |\partial_t \mathcal{F}_n(\tau, \boldsymbol{q}_n^{i-1})| \,\mathrm{d}\tau \\
&\leq |f_n^i - f_n^{i-1}| + (f_n^{i-1} + L) \int_{t_n^{i-1}}^{t_n^i} \kappa(\tau) \mathrm{e}^{K(\tau) - K(t_n^{i-1})} \,\mathrm{d}\tau \\
&\leq |f_n^i - f_n^{i-1}| + (f_n^{i-1} + L) \left(\mathrm{e}^{K(t_n^i) - K(t_n^{i-1})} - 1 \right) \\
&\leq d_n^i + 2(f_n^{i-1} + L) \left(\mathrm{e}^{K(t_n^i) - K(t_n^{i-1})} - 1 \right).
\end{aligned}
\tag{3.30}
$$

Second, for $t \in (t_h^{i-1}, t_h^i)$, we have $f_n(t) = \mathcal{F}(t, \boldsymbol{q}_n^{i-1})$ and, in turn, $\dot{f}_n(t) = \partial_t \mathcal{F}(t, \boldsymbol{q}_n^{i-1})$. Making once more use of (3.12), we compute

$$
\begin{aligned}
\int_{t_n^i}^{t_n^{i-1}} |\dot{f}_n(\tau)| \,\mathrm{d}\tau &= \int_{t_n^i}^{t_n^{i-1}} |\partial_t \mathcal{F}(\tau, \boldsymbol{q}_n^{i-1})| \,\mathrm{d}\tau \\
&\leq (f_n^{i-1} + L) \int_{t_n^i}^{t_n^{i-1}} \kappa(\tau) \mathrm{e}^{K(\tau) - K(t_n^{i-1})} \,\mathrm{d}\tau \\
&\leq (f_n^{i-1} + L) \left(\mathrm{e}^{K(t_n^i) - K(t_n^{i-1})} - 1 \right).
\end{aligned}
\tag{3.31}
$$

Thus, combining (5.88)–(3.31), we obtain

$$
\begin{aligned}
\mathrm{Var}(f_n; [0,T]) &= \sum_{i=1}^{l_n} \left\{ |[f_n]^i| + \int_{t_n^i}^{t_n^{i-1}} |\dot{f}_n(\tau)| \,\mathrm{d}\tau \right\} \\
&\leq \sum_{i=1}^{l_n} \left\{ d_n^i + 3(f_n^{i-1} + L) \left(\mathrm{e}^{K(t_n^i) - K(t_n^{i-1})} - 1 \right) \right\} \\
&\leq \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}_n; [0,T]) + 3(f_n^{i-1} + L) \left(\mathrm{e}^{K(T)} - 1 \right),
\end{aligned}
\tag{3.32}
$$

where in the last line we computed

$$
\begin{aligned}
\sum_{i=1}^{l_h} \left(\mathrm{e}^{K(t_n^i) - K(t_n^{i-1})} - 1 \right) &= \sum_{i=1}^{l_h} \mathrm{e}^{-K(t_n^{i-1})} \left(\mathrm{e}^{K(t_n^i)} - \mathrm{e}^{K(t_n^{i-1})} \right) \\
&\leq \sum_{i=1}^{l_h} \left(\mathrm{e}^{K(t_n^i)} - \mathrm{e}^{K(t_n^{i-1})} \right) \\
&= \mathrm{e}^{K(T)} - 1.
\end{aligned}
$$

Thus, (5.90) together with (3.29), gives

$$\sup_{h>0} \mathrm{Var}(f_h;[0,T]) \leq C(L,M,T). \tag{3.33}$$

Step 2 (Compactness). Set

$$\mathcal{H} := \bigcup_{\widehat{t}\in[0,T]} \left\{\widehat{\boldsymbol{q}} \in \overline{\mathcal{Q}} :\ \mathcal{F}(\widehat{t},\widehat{\boldsymbol{q}}) \leq M\right\}.$$

and

$$\mathcal{K} := \bigcup_{\widehat{t}\in[0,T]} \left\{\mathcal{Z}(\widehat{\boldsymbol{q}}) :\ \widehat{\boldsymbol{q}} \in \mathcal{H}\right\}.$$

In view of (3.29), the sequence $(\boldsymbol{q}_n)$ takes values in $\mathcal{H}$. For every $n\in\mathbb{N}$, we define the functions

$$\boldsymbol{z}_n\colon [0,T]\to L^1(\Omega;\mathbb{R}^N), \quad \Delta_n\colon [0,T]\to[0,+\infty)$$

by setting

$$\boldsymbol{z}_n(t) := \mathcal{Z}(\boldsymbol{q}_n(t)), \qquad \Delta_n(t) := \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}_n;[0,t]).$$

By construction, the sequence $(\boldsymbol{z}_n)$ takes values in $\mathcal{K}$. Note that this is a compact subset of $L^1(\Omega;\mathbb{R}^N)$ thanks to Lemma 3.5 while, by (3.29), there holds

$$\sup_{n\in\mathbb{N}} \mathrm{Var}_{L^1(\Omega;\mathbb{R}^N)}(\boldsymbol{z}_n;[0,T]) = \sup_{n\in\mathbb{N}} \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}_n;[0,T]) \leq M.$$

Also, each function Δ_n is clearly increasing and the sequence (Δ_n) is uniformly bounded by (3.29). At this point, we are in a position to apply Theorem 1.55. This ensures the existence of functions

$$f\colon [0,T]\to\mathbb{R}, \qquad \boldsymbol{z}\colon[0,T]\to L^1(\Omega;\mathbb{R}^N), \quad \Delta\colon[0,T]\to[0,+\infty),$$

with $f\in BV([0,T])$, $\boldsymbol{z}\in BV([0,T];L^1(\Omega;\mathbb{R}^N))$ and Δ increasing, such that, up to subsequences, there hold

$$\forall\, t\in[0,T], \quad f_n(t)\to f(t), \tag{3.34}$$

$$\forall\, t\in[0,T], \quad \boldsymbol{z}_n(t)\to \boldsymbol{z}(t) \text{ in } L^1(\Omega;\mathbb{R}^N), \tag{3.35}$$

$$\forall\, t\in[0,T], \quad \Delta_n(t)\to\Delta(t). \tag{3.36}$$

To construct the candidate solution $\boldsymbol{q}\colon[0,T]\to\overline{\mathcal{Q}}$ we proceed as follows. First, by (3.14), every $\widehat{\boldsymbol{q}}=(\widehat{\boldsymbol{y}},\widehat{\boldsymbol{m}})\in\mathcal{H}$ satisfies

$$\|D\widehat{\boldsymbol{y}}\|_{L^p(\Omega;\mathbb{R}^{N\times N})} + \|\gamma(\det D\widehat{\boldsymbol{y}})\|_{L^1(\Omega)} + \|D\widehat{\boldsymbol{m}}\|_{L^2(\mathrm{im_T}(\widehat{\boldsymbol{y}},\Omega);\mathbb{R}^{N\times N})} \leq C(M).$$

From this, setting

$$\widehat{\boldsymbol{v}} := \chi_{\mathrm{im_T}(\widehat{\boldsymbol{y}},\Omega)}\widehat{\boldsymbol{m}}, \qquad \widehat{\boldsymbol{V}} := \chi_{\mathrm{im_T}(\widehat{\boldsymbol{y}},\Omega)} D\widehat{\boldsymbol{m}},$$

by arguing as in (3.15)–(3.16), we deduce

$$\|\widehat{\boldsymbol{y}}\|_{W^{1,p}(\Omega;\mathbb{R}^N)} + \|\widehat{\boldsymbol{v}}\|_{L^2(\mathbb{R}^N;\mathbb{R}^N)} + \|\widehat{\boldsymbol{V}}\|_{L^2(\mathbb{R}^N;\mathbb{R}^{N\times N})} \leq C(M). \tag{3.37}$$

Therefore, we define $\mathcal{X}$ as the set of triples

$$(\widehat{\boldsymbol{y}},\widehat{\boldsymbol{v}},\widehat{\boldsymbol{V}}) \in W^{1,p}(\Omega;\mathbb{R}^N)\times L^2(\mathbb{R}^N;\mathbb{R}^N)\times L^2(\mathbb{R}^N;\mathbb{R}^{N\times N})$$

satisfying (3.37). This is endowed with the product weak topology which makes it a compact metrizable space. Now, let $\boldsymbol{q}_n(t) = (\boldsymbol{y}_n(t), \boldsymbol{m}_n(t))$ for every $n \in \mathbb{N}$ and define the maps $\boldsymbol{v}_n \colon [0,T] \to L^2(\mathbb{R}^N;\mathbb{R}^N)$ and $\boldsymbol{V}_n \colon [0,T] \to L^2(\mathbb{R}^N;\mathbb{R}^{N\times N})$ by setting

$$\boldsymbol{v}_n(t) \coloneqq \chi_{\mathrm{im_T}(\boldsymbol{y}_n(t),\Omega)}\boldsymbol{m}_n(t), \qquad \boldsymbol{V}_n(t) \coloneqq \chi_{\mathrm{im_T}(\boldsymbol{y}_n(t),\Omega)}D\boldsymbol{m}_n(t).$$

In view of to the previous considerations, the map $\boldsymbol{s}_n \colon [0,T] \to \mathcal{X}$ with $\boldsymbol{s}_n(t) \coloneqq (\boldsymbol{y}(t), \boldsymbol{v}(t), \boldsymbol{V}(t))$ is well defined. We define the set-valued maps $\mathcal{S}_n \colon [0,T] \to \mathcal{P}(\mathcal{X})$ by setting $\mathcal{S}_n(t) \coloneqq \{\boldsymbol{s}_n(t)\}$. Then, we define $\mathcal{S} \colon [0,T] \to \mathcal{P}(\mathcal{X})$ where $\mathcal{S}(t)$ is given by the set of all limit points of the sequence $(\boldsymbol{s}_n(t))$ in $\mathcal{X}$. Clearly, $\mathcal{S}(t)$ is closed in $\mathcal{X}$ and nonempty for every $t \in [0,T]$. Also, by Proposition 1.58, the set-valued map $\mathcal{S}$ is measurable. Therefore, by Theorem 1.60, there exists a measurable selection of $\mathcal{S}$, namely a measurable map $\boldsymbol{s} \colon [0,T] \to \mathcal{X}$ such that $\boldsymbol{s}(t) \in \mathcal{S}(t)$ for every $t \in [0,T]$. Let $\boldsymbol{s}(t) = (\boldsymbol{y}(t), \boldsymbol{v}(t), \boldsymbol{V}(t))$. By definition of $\mathcal{S}$, for every $t \in [0,T]$, there exist a sequence of indices (n_k), possibly depending on t, such that

$$\boldsymbol{y}_{n_k}(t) \rightharpoonup \boldsymbol{y}(t) \text{ in } W^{1,p}(\Omega;\mathbb{R}^N), \tag{3.38}$$

$$\boldsymbol{v}_{n_k}(t) \rightharpoonup \boldsymbol{v}(t) \text{ in } L^2(\mathbb{R}^N;\mathbb{R}^N), \tag{3.39}$$

$$\boldsymbol{V}_{n_k}(t) \rightharpoonup \boldsymbol{V}(t) \text{ in } L^2(\mathbb{R}^N;\mathbb{R}^{N\times N}). \tag{3.40}$$

Thanks to Proposition 3.5, we deduce several facts. First, $\boldsymbol{y}(t) \in \mathcal{Y}_p(\Omega)$ satisfies $\boldsymbol{y}(t) = \overline{\boldsymbol{y}}$ on Γ and there holds

$$\det D\boldsymbol{y}_{n_k}(t) \rightharpoonup \det D\boldsymbol{y}(t) \text{ in } L^1(\Omega). \tag{3.41}$$

Second, there exists $\boldsymbol{m}(t) \in W^{1,2}(\mathrm{im_T}(\boldsymbol{y}(t),\Omega);\mathbb{R}^N)$ satisfying

$$|\boldsymbol{m}(t) \circ \boldsymbol{y}(t)| \det D\boldsymbol{y}(t) = 1 \text{ a.e. in } \Omega,$$

such that

$$\boldsymbol{v}(t) \coloneqq \chi_{\mathrm{im_T}(\boldsymbol{y}(t),\Omega)}\boldsymbol{m}(t), \qquad \boldsymbol{V}(t) \coloneqq \chi_{\mathrm{im_T}(\boldsymbol{y}(t),\Omega)}D\boldsymbol{m}(t).$$

In particular, setting $\boldsymbol{q}(t) \coloneqq (\boldsymbol{y}(t), \boldsymbol{m}(t)) \in \overline{\mathcal{Q}}$ for every $t \in [0,T]$, this defines a measurable map $\boldsymbol{q} \colon [0,T] \to \overline{\mathcal{Q}}$. Third, up to subsequences, there holds

$$\boldsymbol{z}_{n_k}(t) \to \mathcal{Z}(\boldsymbol{q}(t)) \text{ in } L^1(\Omega;\mathbb{R}^N),$$

which, together with (3.35), yields

$$\boldsymbol{z}(t) = \mathcal{Z}(\boldsymbol{q}(t)) \text{ a.e. in } \Omega.$$

For every $n \in \mathbb{N}$, define $P_n \colon [0,T] \to \mathbb{R}$ by setting $P_n(t) \coloneqq \partial_t \mathcal{E}(t, \boldsymbol{q}_n(t))$. As $\boldsymbol{q}_n$ is piecewise-constant, the function P_n is measurable. Also, by (3.10) and (3.29), we have

$$|P_n(t)| \leq \kappa(t) \left(\mathcal{F}(t, \boldsymbol{q}_n(t)) + L\right) \leq (M + L)\kappa(t) \tag{3.42}$$

for almost every $t \in (0,T)$, so that the sequence (P_n) is equi-integrable. Thus, by the Dunford-Pettis Theorem [62, Theorem 2.54], up to subsequence, there holds

$$P_n \rightharpoonup P \text{ in } L^1(0,T) \tag{3.43}$$

for some $P \in L^1(0,T)$. If we define $\widehat{P}\colon (0,T) \to \mathbb{R}$ by setting

$$\widehat{P}(t) \coloneqq \limsup_{n\to\infty} P_n(t),$$

then, by (3.42), we deduce $\widehat{P} \in L^1(0,T)$. Moreover, by the Reverse Fatou Lemma [147, Corollary 5.35], there holds $P \leq \widehat{P}$ almost everywhere.
To identify $\widehat{P}$, Let $t \in (0,T) \setminus Z$. Up to subsequences, we can assume that, for the sequence of indices (n_k) for which(3.38)–(3.39) hold true, we also have

$$\widehat{P}(t) = \lim_{k\to\infty} P_{n_k}(t).$$

In this case, recalling (3.18) and exploiting (3.38)–(3.39), we deduce that $\widehat{P}(t) = \partial_t \mathcal{F}(t, \boldsymbol{q}(t))$. Therefore, this equality holds for almost every $t \in (0,T)$.

Step 3 (Stability). We first prove that the map $\boldsymbol{q}$ satisfies the global stability condition (3.7). For every $n \in \mathbb{N}$, define $\tau_n\colon [0,T] \to \Pi_n$ by setting $\tau_n(t) \coloneqq \max\{s \in \Pi_n :\ s \leq t\}$. Since $|\Pi_n| \to 0$, as $n \to \infty$, we have $\tau_n(t) \to t$, as $n \to \infty$. Also, by definition, there holds $\boldsymbol{q}_n(t) = \boldsymbol{q}_n(\tau_n(t))$ for every $n \in \mathbb{N}$.
Fix $t \in [0,T]$ and let (n_k) be the sequence of indices for which (3.38)–(3.41) hold true. By (3.22), we have

$$\forall\, \widehat{\boldsymbol{q}} \in \overline{\mathcal{Q}}, \quad \mathcal{F}(\tau_{n_k}(t), \boldsymbol{q}_{n_k}(t)) \leq \mathcal{F}(\tau_{n_k}(t), \widehat{\boldsymbol{q}}) + \mathcal{D}(\boldsymbol{q}_{n_k}(t), \widehat{\boldsymbol{q}}).$$

Passing to the limit, as $k \to \infty$, with the aid of Lemma 3.6 taking into account (3.35) and (3.38)–(3.41), we obtain

$$\begin{aligned}
\mathcal{F}(t, \boldsymbol{q}(t)) &\leq \liminf_{k\to\infty} \mathcal{F}(\tau_{n_k}(t), \boldsymbol{q}_{n_k}(t)) \\
&\leq \liminf_{k\to\infty} \left\{ \mathcal{F}(\tau_{n_k}(t), \widehat{\boldsymbol{q}}) + \mathcal{D}(\boldsymbol{q}_{n_k}(t), \widehat{\boldsymbol{q}}) \right\} \\
&= \mathcal{F}(t, \widehat{\boldsymbol{q}}) + \mathcal{D}(\boldsymbol{q}(t), \widehat{\boldsymbol{q}}).
\end{aligned}$$

This proves (3.7) for $t \in [0,T]$ fixed.

Step 4 (Upper energy-dissipation inequality). We want to prove the following inequality

$$\forall\, t \in [0,T], \quad \mathcal{F}(t, \boldsymbol{q}(t)) + \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}; [0,t]) \leq \mathcal{F}(0, \boldsymbol{q}^0) + \int_0^t \partial_t \mathcal{F}(\tau, \boldsymbol{q}(\tau))\, \mathrm{d}\tau. \tag{3.44}$$

Fix $t \in [0,T]$ and $n \in \mathbb{N}$. Let $\tau_n(t)$ be defined as in Step 3. We compute

$$\begin{aligned}
&|\mathcal{F}(t, \boldsymbol{q}_n(t)) - \mathcal{F}(\tau_n(t), \boldsymbol{q}_n(\tau_n(t)))| \\
&\quad = |\mathcal{F}(t, \boldsymbol{q}_n(\tau_n(t))) - \mathcal{F}(\tau_n(t), \boldsymbol{q}_n(\tau_n(t)))| \\
&\quad \leq \int_{\tau_n(t)}^t |\partial_t \mathcal{F}(\tau, \boldsymbol{q}_n(\tau_n(t)))|\, \mathrm{d}\tau \\
&\quad \leq (\mathcal{F}(\tau_n(t), \boldsymbol{q}_n(\tau_n(t))) + L) \int_{\tau_n(t)}^t \kappa(\tau) \mathrm{e}^{K(\tau) - K(\tau_n(t))}\, \mathrm{d}\tau \\
&\quad \leq (M + L) \left(\mathrm{e}^{K(t) - K(\tau_n(t))} - 1 \right),
\end{aligned}$$

where we employed (3.12) and (3.29). From this, we infer

$$\mathcal{F}(t, \boldsymbol{q}_n(t)) \leq \mathcal{F}(\tau_n(t), \boldsymbol{q}_n(\tau_n(t))) + (M+L)\left(\mathrm{e}^{K(t)-K(\tau_n(t))} - 1\right). \tag{3.45}$$

Then, let $\tau_n(t) = t_n^i$ where $i \in \{1, \dots, l_n\}$. In this case, we have

$$\boldsymbol{q}_n(\tau_n(t)) = \boldsymbol{q}_h^i, \qquad \mathrm{Var}_{\mathcal{D}_h}(\boldsymbol{q}_h; [0, \tau_h(t)]) = \sum_{j=1}^{i-1} d_n^j.$$

Summing the inequality (3.23), with j in place of i, for $j \in \{1, \dots, i-1\}$ we obtain:

$$\begin{aligned} \mathcal{F}(\tau_n(t), \boldsymbol{q}_n(\tau_n(t))) &+ \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}_n; [0, \tau_n(t)]) \\ &\leq \mathcal{F}(0, \boldsymbol{q}^0) + \int_0^{\tau_n(t)} \partial_t \mathcal{F}_n(\tau, \boldsymbol{q}_n(\tau))\,\mathrm{d}\tau. \end{aligned} \tag{3.46}$$

Since

$$\mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}_n; [0,t]) = \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}_n; [0, \tau_n(t)]),$$

the previous inequality together with (3.45) yields

$$\begin{aligned} \mathcal{F}(t, \boldsymbol{q}_n(t)) + \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}_n; [0,t]) &\leq \mathcal{F}(\tau_n(t), \boldsymbol{q}_n(\tau_n(t))) + \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}_n; [0, \tau_n(t)]) \\ &\quad + (M+L)\left(\mathrm{e}^{K(t)-K(\tau_n(t))} - 1\right) \\ &\leq \mathcal{F}(0, \boldsymbol{q}^0) + \int_0^{\tau_n(t)} \partial_t \mathcal{F}(\tau, \boldsymbol{q}_n(\tau))\,\mathrm{d}\tau \\ &\quad + (M+L)\left(\mathrm{e}^{K(t)-K(\tau_n(t))} - 1\right) \\ &= \mathcal{F}(0, \boldsymbol{q}^0) + \int_0^{\tau_n(t)} P_n(\tau)\,\mathrm{d}\tau \\ &\quad + (M+L)\left(\mathrm{e}^{K(t)-K(\tau_n(t))} - 1\right). \end{aligned} \tag{3.47}$$

Now, let (n_k) be the sequence of indices for which (3.38)–(3.41) hold. Thanks to Lemma (3.6), we have

$$\mathcal{F}(t, \boldsymbol{q}(t)) \leq \liminf_{k \to \infty} \mathcal{F}(t, \boldsymbol{q}_{n_k}(t)) = f(t), \tag{3.48}$$

where in the last equality we exploited (5.45). Also, recalling (3.35)–(3.36), the lower semicontinuity of the total variation gives

$$\begin{aligned} \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}; [0,t]) &= \mathrm{Var}_{L^1(\Omega;\mathbb{R}^N)}(\boldsymbol{z}; [0,t]) \\ &\leq \liminf_{k \to \infty} \mathrm{Var}_{L^1(\Omega;\mathbb{R}^N)}(\boldsymbol{z}_{n_k}; [0,t]) \\ &= \liminf_{k \to \infty} \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}_{n_k}; [0,t]) = \Delta(t). \end{aligned} \tag{3.49}$$

Eventually, the convergence in (5.56) yields

$$\lim_{n \to \infty} \int_0^{\tau_n(t)} P_n(\tau)\,\mathrm{d}\tau = \int_0^t P(\tau)\,\mathrm{d}\tau \leq \int_0^t \widehat{P}(\tau)\,\mathrm{d}\tau.$$

Therefore, taking the inferior the inferior limit along (n_k), as $k \to \infty$, in (3.47), we obtain

$$\begin{aligned}\mathcal{F}(t, \boldsymbol{q}(t)) + \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}; [0,t]) &\leq \liminf_{k\to\infty} \left\{\mathcal{F}(t, \boldsymbol{q}_{n_k}(t)) + \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}_{n_k}; [0,t])\right\} \\ &= f(t) + \Delta(t) \\ &\leq \mathcal{F}(0, \boldsymbol{q}^0) + \int_0^t P(\tau)\,\mathrm{d}\tau \\ &\leq \mathcal{F}(0, \boldsymbol{q}^0) + \int_0^t \widehat{P}(\tau)\,\mathrm{d}\tau.\end{aligned} \tag{3.50}$$

Here, we exploited that $\tau_{n_k}(t) \to t$, as $k \to \infty$. This proves (3.44) for t fixed.

Step 5 (Lower energy-dissipation inequality). Given (3.44), in order to establish (ii), we are left to check the following inequality:

$$\forall\, t \in [0,T], \quad \mathcal{F}(t, \boldsymbol{q}(t)) + \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}; [0,t]) \geq \mathcal{F}(0, \boldsymbol{q}^0) + \int_0^t \partial_t \mathcal{F}(\tau, \boldsymbol{q}(\tau))\,\mathrm{d}\tau. \tag{3.51}$$

This is deduced from (3.7) by arguing as in [110, Proposition 2.1.23].

Step 6 (Regularity of the energy). To conclude the proof of Theorem 3.2, we have to show that the map $t \mapsto \mathcal{F}(t, \boldsymbol{q}(t))$ belongs to $BV([0,T])$. Combining (3.50)–(3.51), for every $t \in [0,T]$, we obtain

$$\mathcal{F}(t, \boldsymbol{q}(t)) + \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}; [0,t]) = f(t) + \Delta(t).$$

As $\mathcal{F}(t, \boldsymbol{q}(t)) \leq f(t)$ and $\mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}; [0,t]) \leq \Delta(t)$ by (3.48)–(3.49), we deduce the identity $\mathcal{F}(t, \boldsymbol{q}(t)) = f(t)$, which concludes the proof. $\square$

So far, we are not able to treat time-dependent boundary data in Theorem 3.2. In particular, the strategy in [64] seems not to be applicable to our problem. The obstacle is that, regardless to the regularity of the boundary datum, the magnetostatic energy is not differentiable in time when composed with it. However, the other energy terms are well behaved in this sense. Therefore, the strategy in [64] in combination with our results could be used to study quasistatic evolutions for variational models of nematic elastomers with time-dependent boundary data. For our problem, time-dependet boundary data can be treated by imposing integral-boundary conditions as described in the following remark.

Remark 3.9 (Time-dependent integral boundary conditions). For a given function $\overline{\boldsymbol{y}} \in AC([0,T]; L^p(\Gamma; \mathbb{R}^N))$, define $\mathcal{E}^{\mathrm{Dir}} \colon [0,T] \times \mathcal{Q} \to [0,+\infty)$ by setting

$$\mathcal{E}^{\mathrm{Dir}}(t, \boldsymbol{q}) := \int_\Gamma |\overline{\boldsymbol{y}}(t) - \boldsymbol{y}|^p\,\mathrm{d}\boldsymbol{a},$$

where $\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m})$. The proof of Theorem 3.2 can be easily adapted to prove the existence of energetic solutions $\boldsymbol{q} \colon [0,T] \to \mathcal{Q}$ driven by the energy $\mathcal{F} + \mathcal{E}^{\mathrm{Dir}}$. Of course, in this case, we do not prescribe any boundary condition in Definition 3.1.

Remark 3.10 (Convergence of solutions of the IMP). Theorem 3.2 can be stated as a convergence result for sequences $(\boldsymbol{q}_n)$ of piecewise-constant interpolants corresponding to a sequence (Π_n) of partitions of $[0,T]$ such that $|\Pi_n| \to 0$, as

$n \to \infty$. Apart from the convergences already proved in the theorem, we also have the convergence of the power of the applied loads. Namely, employing the notation used in the proof, we have that $P_n \to P$ in $L^1(0,T)$ and $P = \widehat{P}$ almost everywhere. Indeed, from (3.50)–(3.51), we have

$$\int_0^t P(\tau)\,\mathrm{d}\tau = \int_0^t \widehat{P}(\tau)\,\mathrm{d}\tau$$

for every $t \in [0,T]$, which yields the desired identity. Then, the strong convergence is achieved by applying [64, Lemma 3.5].

3.4 Alternative setting

In this section, we briefly describe an alternative setting for the study quasistatic evolutions for our variational model. This setting has been investigated in [29]. We refer to the same paper for more information and for the proof of the various statements.

To motivate the introduction of this alternative setting, we begin by making an observation. Let $\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m}) \in \mathcal{Q}$ and let $\boldsymbol{T} \colon \mathbb{R}^N \to \mathbb{R}^N$ be a rigid motion with $\boldsymbol{Q} := D\boldsymbol{T} \in SO(N)$. By superposition of $\boldsymbol{q}$ with $\boldsymbol{T}$, we obtain the admissible state $\widetilde{\boldsymbol{q}} = (\widetilde{\boldsymbol{y}}, \widetilde{\boldsymbol{m}}) \in \mathcal{Q}$, where $\widetilde{\boldsymbol{y}} := \boldsymbol{T} \circ \boldsymbol{y}$ and $\widetilde{\boldsymbol{m}} := \boldsymbol{Q}\boldsymbol{m} \circ \boldsymbol{T}^{-1}$. In this case, employing the notation in (3.5), we have

$$\mathcal{Z}(\widetilde{\boldsymbol{q}}) = \boldsymbol{Q}\mathcal{Z}(\boldsymbol{q})$$

so that

$$\mathcal{D}(\widetilde{\boldsymbol{q}}, \boldsymbol{q}) = \int_\Omega |(\boldsymbol{Q} - \boldsymbol{I})\mathcal{Z}(\boldsymbol{q})|\,\mathrm{d}\boldsymbol{x} > 0,$$

whenever $\boldsymbol{Q} \neq \boldsymbol{I}$. Basically, since the quantity in (3.5) is not frame-indifferent, the notion of dissipation defined in (3.6) makes possible to dissipate energy by composing with rigid motions. Although this possibility is discouraged by energy minimization, this is certainly questionable from the modeling point of view.

To overcome this issue, one can consider the map $\widetilde{\mathcal{Z}}(\boldsymbol{q}) \colon \Omega \to \mathbb{R}^N$ defined as

$$\widetilde{\mathcal{Z}}(\boldsymbol{q}) := (\mathrm{adj} D\boldsymbol{y})\boldsymbol{m} \circ \boldsymbol{y} \det D\boldsymbol{y}. \tag{3.52}$$

Notably, the quantity in (3.52) is frame-indifferent. Indeed, with $\widetilde{\boldsymbol{q}}$ as above, we have

$$\widetilde{\mathcal{Z}}(\widetilde{\boldsymbol{q}}) = \widetilde{\mathcal{Z}}(\boldsymbol{q}). \tag{3.53}$$

Also, given (2.1), there holds $\widetilde{\mathcal{Z}}(\boldsymbol{q}) \in L^{p/(N-1)}(\Omega;\mathbb{R}^N)$. Therefore, one can define an alternative dissipation distance $\widetilde{\mathcal{D}} \colon \mathcal{Q} \times \mathcal{Q} \to [0,+\infty)$ by setting

$$\widetilde{\mathcal{D}}(\boldsymbol{q}, \widehat{\boldsymbol{q}}) := \int_\Omega |\widetilde{\mathcal{Z}}(\boldsymbol{q}) - \widetilde{\mathcal{Z}}(\widehat{\boldsymbol{q}})|\,\mathrm{d}\boldsymbol{x}. \tag{3.54}$$

In this case, (3.53) trivially yields $\widetilde{\mathcal{D}}(\boldsymbol{q}, \widetilde{\boldsymbol{q}}) = 0$. That is, with the definition in (3.54), energy cannot be dissipated by means of composition with rigid motions. This property constitutes an appreciable feature of the distance in (3.54) which makes it more realistic compared with the one in (3.4). In view of this observation, this notion of dissipation appears to be selective enough to only account for the part of the magnetic reorientation that corresponds to changes of the geometry of the deformed sets.

At this point, we might aim to prove the existence of energetic solutions for our quasistatic model with $\widetilde{\mathcal{D}}$ in place of $\mathcal{D}$. First, we observe that $\widetilde{\mathcal{D}}$ is lower semicontinuous on sublevel sets of the total energy and, in turn, the incremental minimization problem is solvable. However, $\widetilde{\mathcal{D}}$ is not continuous, so that the existence of energetic solutions seems to be out-of-reach in this framework. Indeed, while the existence of mutual recovery sequences [110] is in principle still possible, the explicit construction of such sequences represents an extremely challenging task. This kind of situation is quite common in large-strain theories, see [105] for an example in finite plasticity.

A practicable way to proceed with the analyisis relies on the notion of gradient polyconvexity [22]. Recalling (1.7), we restrict the following class of deformation

$$\widetilde{\mathcal{Y}}_p^{\mathrm{inj}}(\Omega) := \left\{ \boldsymbol{y} \in \mathcal{Y}_p^{\mathrm{inj}}(\Omega) : \ \mathrm{cof} D\boldsymbol{y} \in BV(\Omega; \mathbb{R}^{N\times N}) \right\}, \tag{3.55}$$

and we regularize the energy by adding the higher-order term

$$\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m}) \mapsto |D(\mathrm{cof} D\boldsymbol{y})|(\Omega).$$

Here, we denote by $|D(\mathrm{cof} D\boldsymbol{y})|(\Omega)$ the total variation of the tensor-valued measure $D(\mathrm{cof} D\boldsymbol{y})$. Examples of deformations belonging or not to the class in (3.55) are provided in Example 3.11 and Example 3.12 below. Of course, we could equivalently restrict ourselves to admissible deformations $\boldsymbol{y} \in \mathcal{Y}_p(\Omega)$ with $\mathrm{cof} D\boldsymbol{y} \in W^{1,a}(\Omega; \mathbb{R}^{N\times N})$ for some $a > 1$ and augment the energy by the gradient term

$$\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m}) \mapsto \|D(\mathrm{cof} D\boldsymbol{y})\|^a_{L^a(\Omega;\mathbb{R}^{N\times N\times N})}.$$

In this way, by exploiting the Sobolev embedding, we recover the continuity of the dissipation distance $\widetilde{\mathcal{D}}$ on the sublevels of the regularized total energy. Therefore, the existence of energetic solutions can be proved following the usual time-discretization scheme [110].

Example 3.11. *Let $N = 3$ and let Ω, P and $\boldsymbol{y}$ be as in Example 1.50. Then, $\boldsymbol{y} \in \widetilde{\mathcal{Y}}_p(\Omega)$. To see this, for every $\boldsymbol{x} \in \Omega \setminus P$, we compute*

$$\mathrm{cof}\, D\boldsymbol{y}(\boldsymbol{x}) := \begin{pmatrix} |x_1| & 0 & -x_1\, x_3/|x_1| \\ 0 & |x_1| & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Set $f(\boldsymbol{x}) := |x_1|$ and $g(\boldsymbol{x}) := -x_1\, x_3/|x_1|$. Then $f \in W^{1,\infty}(\Omega)$, while $g \in BV(\Omega)$ since

$$D_1 g = h\, \mathscr{H}^2 \llcorner \{0\} \times (-1,1)^2,$$

where D_1 denotes the distributional derivative with respect to the first variable and we set $h(\boldsymbol{x}) := 2x_3$.

Example 3.12. *Define $f\colon [0,1] \to \mathbb{R}$ by setting $f(x) := x^2 \cos^2(\pi/x^2)$ for every $0 < x \leq 1$ and $f(0) := 0$, and let $g\colon [0,1] \to \mathbb{R}$ be given by $g(x) := \int_0^x f(z)\,\mathrm{d}z$. We have $f \in C^0([0,1]) \setminus BV([0,1])$ and $g \in C^1([0,1])$. Moreover, g is strictly increasing and, in turn, injective. Let $N = 3$ and $\Omega := (0,1)^3$. Define $\boldsymbol{y}\colon \Omega \to \mathbb{R}^3$*

by $\boldsymbol{y}(\boldsymbol{x}) := (x_1, x_2, g(x_1)x_3)$. *In this case,* $\boldsymbol{y} \in C^1(\overline{\Omega}; \mathbb{R}^3)$ *is a homeomorphism and* $\det D\boldsymbol{y} > 0$. *However*

$$\operatorname{cof} D\boldsymbol{y}(\boldsymbol{x}) := \begin{pmatrix} g(x_1) & 0 & -f(x_1)x_3 \\ 0 & g(x_1) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

so that $\operatorname{cof} D\boldsymbol{y} \notin BV(\Omega; \mathbb{R}^{3\times 3})$. *Therefore,* $\boldsymbol{y} \in \mathcal{Y}_p(\Omega) \setminus \widetilde{\mathcal{Y}}_p(\Omega)$.

We conclude the section with two observations about this regularized setting. First, by the Sobolev embedding, every $\boldsymbol{y} \in W^{1,p}(\Omega; \mathbb{R}^N)$ with $\operatorname{cof} D\boldsymbol{y} \in BV(\Omega; \mathbb{R}^{N\times N})$ satisfies

$$\operatorname{cof} D\boldsymbol{y} \in L^{N/(N-1)}(\Omega; \mathbb{R}^{N\times N}).$$

Hence, recalling Example 1.23, we have the identity

$$\widetilde{\mathcal{Y}}_p^{\mathrm{inj}}(\Omega) = \Big\{ \boldsymbol{y} \in W^{1,p}(\Omega; \mathbb{R}^N) : \ \operatorname{cof} D\boldsymbol{y} \in BV(\Omega; \mathbb{R}^{N\times N}),\ \det D\boldsymbol{y} > 0 \text{ a.e.}, \\ \boldsymbol{y} \text{ a.e. injective} \Big\}.$$

Second, in this regularized setting, we do not need to impose any polyconvexity assumption on the elastic energy density. This possibility represents one of the main advantages of gradient polyconvexity and relies on the following simple observation: if $(\boldsymbol{y}_n) \subset W^{1,p}(\Omega; \mathbb{R}^N)$ and $\boldsymbol{y} \in W^{1,p}(\Omega; \mathbb{R}^N)$ are such that $\operatorname{cof} D\boldsymbol{y}_n \to \operatorname{cof} D\boldsymbol{y}$ almost everywhere, then there holds $D\boldsymbol{y}_n \to D\boldsymbol{y}$ almost everywhere. For more information, we refer to [22, 92, 93].

Part II

Dimension reduction problems

Chapter 4

Dimension reduction in the static setting

In this chapter, we focus on the rigorous derivation of a reduced model for magnetoelastic plates by means of Γ-convergence techniques. We consider low-energy configurations by rescaling the elastic energy according to the linearized von Kármán regime [65]. The main results of the chapter are Theorem 4.3 and Theorem 4.14, describing the asymptotic behaviour of the energy of the plate, as its thickness goes to zero. Precisely, Theorem 4.3 identifies the reduced energy functional as the Γ-limit of the bulk magnetoelatic energies, while Theorem 4.14 establishes the convergences of sequence of almost minimizers of the bulk energies to minimizers of the reduced energy.
We start in Section 4.1 by describing the bulk model and the reduced model. Theorem 4.3 and Theorem 4.14 are stated and proved in Section 4.2 and Section 4.3, respectively.

Notation. In this chapter, the spatial dimension is $N = 3$. For every $\boldsymbol{a} \in \mathbb{R}^3$, we denote by $\boldsymbol{a}' \in \mathbb{R}^2$ the vector given by its first two components. The same notation is used with vector fields. For $\boldsymbol{A} \in \mathbb{R}^{3\times 3}$, we denote by $\boldsymbol{A}'' \in \mathbb{R}^{2\times 2}$ its submatrix determined by the first two rows and the first two columns. With this notation, $\boldsymbol{0}'$ is the null vector in $\mathbb{R}^2$, while $\boldsymbol{O}''$ and $\boldsymbol{I}''$ are the null vector and the identity matrix in $\mathbb{R}^{2\times 2}$. We denote by D' the gradient with respect the first two variables.

4.1 Setting of the problem

In this section, we specialize the variational model introduced in Chapter 2 to the case of plates. After stating all the assumptions, we perform a change-of-variable in order to work with functionals defined on a common domain. Then, we introduce the reduced model that will be rigorously derived in the following.
Note that, in this chapter, we are mainly interested in the asymptotic behaviour of minimizers, as the thickness of the plate goes to zero, rather than in their existence.

4.1.1 The bulk model

Let $\Omega_h := S \times hI$ represent a thin magnetoelastic plate in its reference configuration. The section $S \subset \mathbb{R}^2$ is a bounded connected Lipschitz domain, the parameter $h > 0$

gives the thickness of the plate and $I \coloneqq (-1/2, 1/2)$.
Let the exponent $p > 3$ be fixed. We consider deformations $\boldsymbol{\phi} \in W^{1,p}(\Omega_h;\mathbb{R}^3)$ that are almost everywhere injective and such $\det D\boldsymbol{\phi} > 0$ almost everywhere. In this setting, by Lemma 1.47, the topological image of Ω_h via $\boldsymbol{\phi}$ coincides with $\Omega_h^{\boldsymbol{\phi}}$. Recall that this set is defined as $\Omega_h^{\boldsymbol{\phi}} \coloneqq \boldsymbol{\phi}(\Omega_h) \setminus \boldsymbol{\phi}(\partial\Omega_h)$. The corresponding magnetizations are maps $\boldsymbol{m} \in W^{1,2}(\Omega_h^{\boldsymbol{\phi}};\mathbb{R}^3)$ satisfying the magnetic saturation constraint

$$|\boldsymbol{m} \circ \boldsymbol{\phi}| \det D\boldsymbol{\phi} = 1 \text{ a.e. in } \Omega_h. \tag{4.1}$$

We consider the following magnetoelastic energy functional:

$$\begin{aligned} G_h(\boldsymbol{\phi}, \boldsymbol{m}) \coloneqq \frac{1}{h^\beta} \int_{\Omega_h} W_h(D\boldsymbol{\phi}, \boldsymbol{m} \circ \boldsymbol{\phi})\,\mathrm{d}\boldsymbol{X} + \int_{\Omega_h^{\boldsymbol{\phi}}} |D\boldsymbol{m}|^2\,\mathrm{d}\boldsymbol{\xi} \\ + \frac{1}{2} \int_{\mathbb{R}^3} |D\psi_{\boldsymbol{m}}|^2\,\mathrm{d}\boldsymbol{\xi}. \end{aligned} \tag{4.2}$$

The first term in (4.2) accounts for the elastic energy and it is rescaled according to the linearized von Kármán regime [65]. Precisely, we assume

$$\beta > 6 \vee p.$$

The elastic energy density $W_h \colon \mathbb{R}^{3\times 3}_+ \times \mathbb{R}^3 \setminus \{\boldsymbol{0}\} \to [0, +\infty]$ is continuous and, for every $\boldsymbol{F} \in \mathbb{R}^{3\times 3}_+$ and $\boldsymbol{\lambda} \in \mathbb{R}^3$ with $|\boldsymbol{\lambda}| \det \boldsymbol{F} = 1$, takes the form

$$W_h(\boldsymbol{F}, \boldsymbol{\lambda}) \coloneqq \Phi\left(\boldsymbol{L}_h(\boldsymbol{\lambda} \det \boldsymbol{F})^{-1} \boldsymbol{F}\right), \tag{4.3}$$

for some function $\Phi \colon \mathbb{R}^{3\times 3}_+ \to [0, +\infty]$. In (4.3), we define $\boldsymbol{L}_h \colon \mathbb{S}^2 \to \mathbb{R}^{3\times 3}_+$ by setting

$$\boldsymbol{L}_h(\boldsymbol{z}) \coloneqq \boldsymbol{I} + c_h \boldsymbol{z} \otimes \boldsymbol{z}, \tag{4.4}$$

where $c_h > 0$. Regarding the asymptotic behaviour of (c_h), we assume

$$c_0 \coloneqq \lim_{h \to 0^+} \frac{c_h}{h^{\beta/2}} > 0. \tag{4.5}$$

A direct computation shows that

$$\det \boldsymbol{L}_h(\boldsymbol{z}) = 1 + c_h \tag{4.6}$$

for every $\boldsymbol{z} \in \mathbb{S}^2$. In particular, the matrix $\boldsymbol{L}_h(\boldsymbol{z})$ is invertible and its inverse is given by

$$\boldsymbol{L}_h(\boldsymbol{z})^{-1} = \boldsymbol{I} - \frac{c_h}{1 + c_h} \boldsymbol{z} \otimes \boldsymbol{z}. \tag{4.7}$$

The function Φ is assumed to satisfy the following:

(i) Normalization:

$$\Phi(\boldsymbol{I}) = 0 = \min \Phi, \tag{4.8}$$

(ii) Frame indifference:

$$\forall\, \boldsymbol{R} \in SO(3),\ \forall\, \Xi \in \mathbb{R}^{3\times 3}_+, \quad \Phi(\boldsymbol{R}\,\Xi) = \Phi(\Xi), \tag{4.9}$$

(iii) Growth: there exist $C_0, C_1, C_2 > 0$ and $a > 1$ such that

$$\forall\, \Xi \in \mathbb{R}^{3\times 3}_+, \quad \Phi(\Xi) \geq C_0 \operatorname{dist}^2(\Xi; SO(3)) \vee \operatorname{dist}^p(\Xi; SO(3)), \tag{4.10}$$

and

$$\forall\, \Xi \in \mathbb{R}^{3\times 3}_+, \quad \Phi(\Xi) \geq \frac{C_1}{|\det \Xi|^a} - C_2, \tag{4.11}$$

(iv) Regularity:

$$\Phi \text{ is continuous and of class } C^2 \text{ in a neighborhood of } SO(3), \tag{4.12}$$

Given (4.7), we have

$$\forall\, \boldsymbol{R} \in SO(3),\ \forall\, \boldsymbol{z} \in \mathbb{S}^2, \quad \boldsymbol{L}_h(\boldsymbol{R}\,\boldsymbol{z})^{-1} = \boldsymbol{R}\,\boldsymbol{L}_h(\boldsymbol{z})^{-1}\boldsymbol{R}^\top. \tag{4.13}$$

This, together with (4.9), yields the frame-indifference of W_h, namely

$$\forall\, \boldsymbol{R} \in SO(3),\ \forall\, \boldsymbol{F} \in \mathbb{R}^{3\times 3}_+,\ \forall\, \boldsymbol{\lambda} \in \mathbb{R}^3, \quad W_h(\boldsymbol{R}\boldsymbol{F}, \boldsymbol{R}\boldsymbol{\lambda}) = W_h(\boldsymbol{F}, \boldsymbol{\lambda}). \tag{4.14}$$

From (4.8) and (4.14), we realize that the function W_h is minimized on the set

$$\left\{(\boldsymbol{R}\boldsymbol{F}, \boldsymbol{R}\boldsymbol{\lambda}) :\ \boldsymbol{F} \in \mathbb{R}^{3\times 3}_+,\ \boldsymbol{\lambda} \in \mathbb{R}^3,\ \boldsymbol{R} \in SO(3),\ \boldsymbol{F} = \boldsymbol{L}_h(\boldsymbol{\lambda} \det \boldsymbol{F})\right\}.$$

Instead, by (4.10), the map Φ has global p-growth and quadratic growth close to $SO(3)$. In particular, there exist $C_1, C_2 > 0$ such that, for $q \in \{2, p\}$, there holds

$$\forall\, \Xi \in \mathbb{R}^{3\times 3}_+, \quad \Phi(\Xi) \geq C_1\, |\Xi|^q - C_2. \tag{4.15}$$

The specific form of the growth condition of Φ with respect to the determinant in (4.11) is assumed just for simplicity. More generally, the this can be replaced by the requirement

$$\forall\, \Xi \in \mathbb{R}^{3\times 3}_+, \quad \Phi(\Xi) \geq \gamma(\det \Xi),$$

where $\gamma\colon (0, +\infty) \to [0, +\infty]$ is a Borel and satisfies (2.9).

Assumptions (4.8) and (4.12) justify the second-order Taylor expansion of Φ close to the identity. Precisely, we have the following:

$$\forall\, \Xi \in \mathbb{R}^{3\times 3} :\ |\Xi| \ll 1, \quad \Phi(\boldsymbol{I} + \Xi) = \frac{1}{2} Q(\Xi) + \omega(\Xi). \tag{4.16}$$

The quadratic form Q is defined by $Q(\Xi) \coloneqq D^2\Phi(\boldsymbol{I})(\Xi, \Xi)$ while $\omega(\Xi) = o(|\Xi|^2)$, as $|\Xi| \to 0^+$. Note that Q is positive semidefinite and, in turn, convex by (4.8). Additionally, exploiting (4.10) and (4.16), and arguing as in [113, p. 927], we show that Q is positive definite on symmetric matrices. That is, there exists $C > 0$ such that

$$\forall\, \Xi \in \mathbb{R}^{3\times 3}, \quad Q(\Xi) = Q(\mathrm{sym}\Xi) \geq C|\mathrm{sym}\,\Xi|^2. \tag{4.17}$$

The last two terms in (4.2) account for the exchange energy and the magnetostatic energy. The stray-field potential $\psi_{\boldsymbol{m}}\colon \mathbb{R}^3 \to \mathbb{R}$ is defined as a weak solution to the Maxwell equation:

$$\operatorname{div}\left(D\psi_{\boldsymbol{m}} - \chi_{\Omega_h^{\boldsymbol{\phi}}}\boldsymbol{m}\right) = 0 \quad \text{in } \mathbb{R}^3. \tag{4.18}$$

4.1.2 Change of variables and rescaling

For $h > 0$, we introduce the map $\boldsymbol{\pi}_h \colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by $\boldsymbol{\pi}_h(\boldsymbol{x}) := ((\boldsymbol{x}')^\top, hx_3)^\top$. Set $\Omega := S \times I$. Given any deformation $\boldsymbol{\phi} \in W^{1,p}(\Omega_h;\mathbb{R}^3)$, we consider the map $\boldsymbol{y} := \boldsymbol{\phi} \circ \boldsymbol{\pi}_h|_\Omega \in W^{1,p}(\Omega;\mathbb{R}^3)$. Then, $\Omega^{\boldsymbol{y}} = \Omega_h^{\boldsymbol{\phi}}$, where $\Omega^{\boldsymbol{y}} := \boldsymbol{y}(\Omega) \setminus \boldsymbol{y}(\partial\Omega)$. Recalling (4.2) and applying the Change-of-variable Formula, we obtain

$$\begin{aligned}\frac{1}{h}G_h(\boldsymbol{\phi},\boldsymbol{m}) = \frac{1}{h^\beta}\int_\Omega W_h(D_h\boldsymbol{y},\boldsymbol{m}\circ\boldsymbol{y})\,\mathrm{d}\boldsymbol{x} + \frac{1}{h}\int_{\Omega^{\boldsymbol{y}}} |D\boldsymbol{m}|^2\,\mathrm{d}\boldsymbol{\xi} \\ + \frac{1}{2h}\int_{\mathbb{R}^3} |D\psi_{\boldsymbol{m}}|^2\,\mathrm{d}\boldsymbol{\xi},\end{aligned} \tag{4.19}$$

where the scaled gradient is defined as $D_h := (D', h^{-1}\partial_3)$. Also, (4.1) yields

$$|\boldsymbol{m}\circ\boldsymbol{y}|\det D_h\boldsymbol{y} = 1 \text{ a.e. in } \Omega. \tag{4.20}$$

Therefore, we define the class of admissible states as

$$\mathcal{Q}_h := \left\{(\boldsymbol{y},\boldsymbol{m}) : \ \boldsymbol{y} \in \mathcal{Y}_p^{\mathrm{inj}}(\Omega),\ \boldsymbol{m} \in W^{1,2}(\Omega^{\boldsymbol{y}};\mathbb{R}^3),\ |\boldsymbol{m}\circ\boldsymbol{y}|\det D_h\boldsymbol{y} = 1 \text{ a.e. in } \Omega\right\}.$$

Here, $\mathcal{Y}_p^{\mathrm{inj}}(\Omega)$ is defined as in (1.16). In view of (4.19), we consider the energy functional $E_h \colon \mathcal{Q}_h \to [0,+\infty)$ defined by

$$\begin{aligned}E_h(\boldsymbol{q}) := \frac{1}{h^\beta}\int_\Omega W_h(D_h\boldsymbol{y},\boldsymbol{m}\circ\boldsymbol{y})\,\mathrm{d}\boldsymbol{x} + \frac{1}{h}\int_{\Omega^{\boldsymbol{y}}} |D\boldsymbol{m}|^2\,\mathrm{d}\boldsymbol{\xi} \\ + \frac{1}{2h}\int_{\mathbb{R}^3} |D\psi_{\boldsymbol{m}}|^2\,\mathrm{d}\boldsymbol{\xi},\end{aligned} \tag{4.21}$$

where $\boldsymbol{q} = (\boldsymbol{y},\boldsymbol{m})$. We denote the three terms on the right-hand side of (4.21) by $E_h^{\mathrm{el}}(\boldsymbol{y},\boldsymbol{m})$, $E_h^{\mathrm{exc}}(\boldsymbol{y},\boldsymbol{m})$ and $E_h^{\mathrm{mag}}(\boldsymbol{y},\boldsymbol{m})$, respectively. Given (4.18), the function $\psi_{\boldsymbol{m}}$ in (4.21) is a weak solution of the equation

$$\operatorname{div}(D\psi_{\boldsymbol{m}} - \chi_{\Omega^{\boldsymbol{y}}}\boldsymbol{m}) = 0 \quad \text{in } \mathbb{R}^3. \tag{4.22}$$

Remark 4.1 (Invariance with respect to rigid motions). The functional E_h is invariant with respect to rigid motions. To see this, let $\boldsymbol{q} = (\boldsymbol{y},\boldsymbol{m}) \in \mathcal{Q}_h$ and let $\boldsymbol{T} \colon \mathbb{R}^3 \to \mathbb{R}^3$ be a rigid motion of the form $\boldsymbol{T}(\boldsymbol{\xi}) := \boldsymbol{Q}\boldsymbol{\xi} + \boldsymbol{c}$, where $\boldsymbol{Q} \in SO(3)$ and $\boldsymbol{c} \in \mathbb{R}^3$. If we set $\widetilde{\boldsymbol{q}} = (\widetilde{\boldsymbol{y}},\widetilde{\boldsymbol{m}}) \in \mathcal{Q}_h$ with $\widetilde{\boldsymbol{y}} := \boldsymbol{T}\circ\boldsymbol{y}$ and $\widetilde{\boldsymbol{m}} := \boldsymbol{Q}\boldsymbol{m}\circ\boldsymbol{T}^{-1}$, then, there holds $E_h(\widetilde{\boldsymbol{q}}) = E_h(\boldsymbol{q})$. Indeed, by (4.14), we have $E_h^{\mathrm{el}}(\widetilde{\boldsymbol{q}}) = E_h^{\mathrm{el}}(\boldsymbol{q})$ while, by Proposition 1.46, we infer $E_h^{\mathrm{exc}}(\widetilde{\boldsymbol{q}}) = E_h^{\mathrm{exc}}(\boldsymbol{q})$. Also, if ψ is a stray field potential corresponding to $\boldsymbol{q}$, then, using again Proposition 1.46 we check that $\psi\circ\boldsymbol{T}^{-1}$ is a stray field potential corresponding to $\widetilde{\boldsymbol{q}}$. Clearly, this yields $E_h^{\mathrm{mag}}(\widetilde{\boldsymbol{q}}) = E_h^{\mathrm{mag}}(\boldsymbol{q})$.

Remark 4.2 (Existence of minimizers for the bulk model). Our assumptions on the function Φ are compatible with the result obtained in Chapter 2 concerning the existence of minimizers. Precisely, if Φ has the partial polyconvexity property stated in Example 2.1, the existence of minimizers of the functional E_h for fixed $h > 0$ is guaranteed by Theorem 2.5.

4.1.3 The reduced model

We introduce some notation that is going to be used in the remainder of this work. For $h > 0$ and $\boldsymbol{q} = (\boldsymbol{y},\boldsymbol{m}) \in \mathcal{Q}_h$, we define the (scaled) horizontal and vertical averaged displacements and the (scaled) first moment, respectively

$$\mathcal{U}_h(\boldsymbol{q})\colon S \to \mathbb{R}^2, \qquad \mathcal{V}_h(\boldsymbol{q})\colon S \to \mathbb{R}, \qquad \mathcal{W}_h(\boldsymbol{q})\colon S \to \mathbb{R}^3,$$

by setting

$$\begin{aligned} \mathcal{U}_h(\boldsymbol{q})(\boldsymbol{x}') &:= \frac{1}{h^{\beta/2}} \int_I (\boldsymbol{y}'(\boldsymbol{x}', x_3) - \boldsymbol{x}')\,\mathrm{d}x_3, \\ \mathcal{V}_h(\boldsymbol{q})(\boldsymbol{x}') &:= \frac{1}{h^{\beta/2-1}} \int_I y^3(\boldsymbol{x}', x_3)\,\mathrm{d}x_3, \\ \mathcal{W}_h(\boldsymbol{q})(\boldsymbol{x}') &:= \frac{1}{h^{\beta/2}} \int_I x_3(\boldsymbol{y}(\boldsymbol{x}', x_3) - \boldsymbol{\pi}_h(\boldsymbol{x}', x_3))\,\mathrm{d}x_3, \end{aligned}$$

for every $\boldsymbol{x}' \in S$. Furthermore, we define

$$\mathcal{M}_h(\boldsymbol{q})\colon \Omega \to \mathbb{R}^3, \qquad \mathcal{N}_h(\boldsymbol{q})\colon \Omega \to \mathbb{R}^{3\times 3}, \qquad \mathcal{Z}_h(\boldsymbol{q})\colon \Omega \to \mathbb{S}^2,$$

by setting

$$\begin{aligned} \mathcal{M}_h(\boldsymbol{q}) &:= (\chi_{\Omega^{\boldsymbol{y}}}\boldsymbol{m}) \circ \boldsymbol{\pi}_h, \\ \mathcal{N}_h(\boldsymbol{q}) &:= (\chi_{\Omega^{\boldsymbol{y}}}D\boldsymbol{m}) \circ \boldsymbol{\pi}_h, \\ \mathcal{Z}_h(\boldsymbol{q}) &:= \boldsymbol{m} \circ \boldsymbol{y} \det D_h\boldsymbol{y}. \end{aligned} \tag{4.23}$$

We stress that the map $\mathcal{Z}_h(\boldsymbol{q})$ is sphere-valued as a consequence of (4.20).
Recall the quadratic form Q in (4.16). As in [65], we define the reduced quadratic by setting

$$Q_{\mathrm{red}}(\boldsymbol{\Sigma}) := \min\left\{ Q\left(\left(\begin{array}{c|c} \boldsymbol{\Sigma} & \boldsymbol{0}' \\ \hline (\boldsymbol{0}')^\top & 0 \end{array} \right) + \boldsymbol{c}\otimes\boldsymbol{e}_3 + \boldsymbol{e}_3 \otimes \boldsymbol{c} \right) : \ \boldsymbol{c} \in \mathbb{R}^3 \right\} \tag{4.24}$$

for every $\boldsymbol{\Sigma} \in \mathbb{R}^{2\times 2}$. The positive definiteness and the convexity of Q_{red} follow from that of Q. Moreover, from (4.17), we deduce that Q_{red} is also positive definite on symmetric matrices, namely, there exists $C > 0$ such that

$$\forall\, \boldsymbol{\Sigma} \in \mathbb{R}^{2\times 2}, \quad Q_{\mathrm{red}}(\boldsymbol{\Sigma}) = Q_{\mathrm{red}}(\mathrm{sym}\boldsymbol{\Sigma}) \geq C|\mathrm{sym}\,\boldsymbol{\Sigma}|^2. \tag{4.25}$$

The Γ-limit of the functionals (E_h) in (4.21), as $h \to 0^+$, is given by the functional $E_0\colon \mathcal{Q}_0 \to [0,+\infty)$, where

$$\mathcal{Q}_0 := W^{1,2}(S;\mathbb{R}^2) \times W^{2,2}(S) \times W^{1,2}(S;\mathbb{S}^2),$$

defined as

$$\begin{aligned} E_0(\boldsymbol{q}_0) := \frac{1}{2}\int_S Q_{\mathrm{red}}(\mathrm{sym}D'\boldsymbol{u} - c_0\boldsymbol{\zeta}'\otimes\boldsymbol{\zeta}')\,\mathrm{d}\boldsymbol{x}' + \frac{1}{24}\int_S Q_{\mathrm{red}}((D')^2 v)\,\mathrm{d}\boldsymbol{x}' \\ + \int_S |D'\boldsymbol{\zeta}|^2\,\mathrm{d}\boldsymbol{x}' + \frac{1}{2}\int_S |\zeta^3|^2\,\mathrm{d}\boldsymbol{x}', \end{aligned} \tag{4.26}$$

where $\boldsymbol{q}_0 = (\boldsymbol{u}, v, \boldsymbol{\zeta})$. We denote the sum of the first two terms on the right-hand side of (4.26) by $E_0^{\mathrm{el}}(\boldsymbol{q}_0)$ and the last two terms on the right-hand side of (4.26) by $E_0^{\mathrm{exc}}(\boldsymbol{q}_0)$ and $E_0^{\mathrm{mag}}(\boldsymbol{q}_0)$, respectively. With a slight abuse of notation, we will sometimes write $E_0^{\mathrm{exc}}(\boldsymbol{\zeta})$ and $E_0^{\mathrm{mag}}(\boldsymbol{\zeta})$. Note that the limiting functional E_0 is purely Lagrangian and that it trivially admits minimizers.

4.2 Γ-convergence

Our first main result asserts the Γ-convergence of (E_h) to E_0, as $h \to 0^+$, and reads as follows.

Theorem 4.3 (Γ-convergence). *The following two assertions hold.*

(i) (Compactness and lower bound). *Let $(\boldsymbol{q}_h)$ with $\boldsymbol{q}_h = (\boldsymbol{y}_h, \boldsymbol{m}_h) \in \mathcal{Q}_h$ be such that*

$$\sup_{h>0} E_h(\boldsymbol{q}_h) \leq C \tag{4.27}$$

for some $C > 0$. Then, there exist $(\boldsymbol{T}_h)$ with $\boldsymbol{T}_h \colon \mathbb{R}^3 \to \mathbb{R}^3$ rigid motions of the form $\boldsymbol{T}_h(\boldsymbol{\xi}) := \boldsymbol{Q}_h^\top \boldsymbol{\xi} - \boldsymbol{c}_h$, where $(\boldsymbol{Q}_h) \subset SO(3)$ and $(\boldsymbol{c}_h) \subset \mathbb{R}^3$, and $\widetilde{\boldsymbol{q}}_0 = (\widetilde{\boldsymbol{u}}, \widetilde{v}, \widetilde{\boldsymbol{\zeta}}) \in \mathcal{Q}_0$ such that, setting $\widetilde{\boldsymbol{q}}_h := (\widetilde{\boldsymbol{y}}_h, \widetilde{\boldsymbol{m}}_h)$ where $\widetilde{\boldsymbol{y}}_h := \boldsymbol{T}_h \circ \boldsymbol{y}_h$ and $\widetilde{\boldsymbol{m}}_h := \boldsymbol{Q}_h^\top \boldsymbol{m}_h \circ \boldsymbol{T}_h^{-1}$, up to subsequences, the following convergences hold, as $h \to 0^+$:

$$\widetilde{\boldsymbol{u}}_h := \mathcal{U}_h(\widetilde{\boldsymbol{q}}_h) \rightharpoonup \widetilde{\boldsymbol{u}} \text{ in } W^{1,2}(S;\mathbb{R}^2); \tag{4.28}$$

$$\widetilde{v}_h := \mathcal{V}_h(\widetilde{\boldsymbol{q}}_h) \to \widetilde{v} \text{ in } W^{1,2}(S); \tag{4.29}$$

$$\widetilde{\boldsymbol{z}}_h := \mathcal{Z}_h(\widetilde{\boldsymbol{q}}_h) \to \widetilde{\boldsymbol{\zeta}} \text{ in } L^q(\Omega;\mathbb{R}^3) \text{ for every } 1 \leq q < \infty. \tag{4.30}$$

Moreover, the following inequality holds:

$$E_0(\widetilde{\boldsymbol{q}}_0) \leq \liminf_{h \to 0^+} E_h(\boldsymbol{q}_h). \tag{4.31}$$

(ii) (Optimality of the lower bound). *For every $\widehat{\boldsymbol{q}}_0 = (\widehat{\boldsymbol{u}}, \widehat{v}, \widehat{\boldsymbol{\zeta}}) \in \mathcal{Q}_0$, there exists $(\widehat{\boldsymbol{q}}_h)$ with $\widehat{\boldsymbol{q}}_h \in \mathcal{Q}_h$ such that the following convergences hold, as $h \to 0^+$:*

$$\widehat{\boldsymbol{u}}_h := \mathcal{U}_h(\widehat{\boldsymbol{q}}_h) \to \widehat{\boldsymbol{u}} \text{ in } W^{1,2}(S;\mathbb{R}^2);$$

$$\widehat{v}_h := \mathcal{V}_h(\widehat{\boldsymbol{q}}_h) \to \widehat{v} \text{ in } W^{1,2}(S);$$

$$\widehat{\boldsymbol{z}}_h := \mathcal{Z}_h(\widehat{\boldsymbol{q}}_h) \to \widehat{\boldsymbol{\zeta}} \text{ in } L^q(\Omega;\mathbb{R}^3) \text{ for every } 1 \leq q < \infty.$$

Moreover, the following equality holds:

$$E_0(\widehat{\boldsymbol{q}}_0) = \lim_{h \to 0^+} E_h(\widehat{\boldsymbol{q}}_h). \tag{4.32}$$

Actually, since the functionals E_h and E_0 have different domains of definitions, the abstract definition of Γ-convergence for sequences of functionals defined on metric spaces [26, 44] does not apply. Nevertheless, it is possible to reformulate Theorem 4.3 according to the abstract framework as shown in the next remark.

Remark 4.4 (Abstract Γ-convergence). Define the functionals

$$\mathcal{E}_h, \mathcal{E}_0 \colon W^{1,p}(\Omega;\mathbb{R}^3) \times W^{1,2}(S;\mathbb{R}^2) \times W^{1,2}(S) \times L^2(\mathbb{R}^3;\mathbb{R}^3) \to [0,+\infty]$$

by setting $\mathcal{E}_h(\boldsymbol{y}, \boldsymbol{u}, v, \boldsymbol{\mu}) = E(\boldsymbol{q})$ if there exists $\boldsymbol{m} \in W^{1,2}(\Omega^{\boldsymbol{y}};\mathbb{R}^3)$ such that $\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m}) \in \mathcal{Q}_h$ satisfies

$$\boldsymbol{u} = \mathcal{U}_h(\boldsymbol{q}), \qquad v = \mathcal{V}_h(\boldsymbol{q}), \qquad \boldsymbol{\mu} = \mathcal{M}_h(\boldsymbol{q}),$$

and $\mathcal{E}_h(\boldsymbol{y},\boldsymbol{u},v,\boldsymbol{\mu})=+\infty$ otherwise, and by setting $\mathcal{E}_0(\boldsymbol{y},\boldsymbol{u},v,\boldsymbol{\mu})=E_0(\boldsymbol{q}_0)$ if there hold $\boldsymbol{y}=\boldsymbol{\pi}_0$, $v\in W^{2,2}(S)$ and there exists $\boldsymbol{\zeta}\in W^{1,2}(S;\mathbb{S}^2)$ such that $\boldsymbol{q}_0=(\boldsymbol{u},v,\boldsymbol{\zeta})\in\mathcal{Q}_0$ satisfies $\boldsymbol{\mu}=\chi_\Omega\boldsymbol{\zeta}$, and $\mathcal{E}_0(\boldsymbol{y},\boldsymbol{u},v,\boldsymbol{\mu})=+\infty$ otherwise. In this case, one proves that the functional $\mathcal{E}_0$ is the Γ-limit of the sequence $(\mathcal{E}_h)$, as $h\to0^+$, with respect to either the produc strong topology or the product weak topologies. Full details are provided in [27].
Anyway, it must be said this last result provides less information than Theorem 4.3. Indeed, we cannot deduce the convergence of almost minimizers from the abstract Γ-convergence result as the sequence $(\mathcal{E}_h)$ does not satisty suitable the compactness assumptions of the Fundamental Theorem on Γ-convergence [26, Theorem 1.21].

The remainder of the section is devoted to the proof of Theorem 4.3.

4.2.1 Compactness

For future reference, we start by collecting some preliminary compactness results which we present in a more self-contained form.
The compactness of deformations is proved by adapting the techniques in [65] to our setting. A fundamental tool in these arguments is the celebrated rigidity estimate [66, Theorem 3.1]. For convenience, given $\boldsymbol{y}\in W^{1,p}(\Omega;\mathbb{R}^3)$ and $h>0$, we set

$$\mathcal{R}_h(\boldsymbol{y}) \coloneqq \int_\Omega \mathrm{dist}^2(\boldsymbol{F}_h;SO(3))\vee\mathrm{dist}^p(\boldsymbol{F}_h;SO(3))\,\mathrm{d}\boldsymbol{x}. \tag{4.33}$$

We will use the following slight modification of [65, Theorem 6].

Lemma 4.5 (Approximation by rotations). *Let $\boldsymbol{y}\in W^{1,p}(\Omega;\mathbb{R}^3)$. For $h>0$, set $r_h\coloneqq\mathcal{R}_h(\boldsymbol{y})$ and $\boldsymbol{F}_h\coloneqq D_h\boldsymbol{y}$. Suppose that $r_h/h^2\to0$, as $h\to0^+$. Then, for $h\ll1$, there exist $\boldsymbol{R}_h\in W^{1,p}(S;SO(3))$ and $\boldsymbol{Q}_h\in SO(3)$ such that, for $q\in\{2,p\}$, the following estimates hold:*

$$\|\boldsymbol{F}_h-\boldsymbol{R}_h\|_{L^q(\Omega;\mathbb{R}^{3\times3})}\le Cr_h^{1/q},\qquad \|D'\boldsymbol{R}_h\|_{L^q(S;\mathbb{R}^{3\times3\times3})}\le Ch^{-1}r_h^{1/q}, \tag{4.34}$$

$$\|\boldsymbol{R}_h-\boldsymbol{Q}_h\|_{L^q(S;\mathbb{R}^{3\times3})}\le Ch^{-1}r_h^{1/q},\quad \|\boldsymbol{F}_h-\boldsymbol{Q}_h\|_{L^q(\Omega;\mathbb{R}^{3\times3})}\le Ch^{-1}r_h^{1/q}. \tag{4.35}$$

Proof. By [65, Theorem 6], for every $h>0$, there exists a map $\boldsymbol{P}_h\in C^\infty(S;\mathbb{R}^{3\times3})$ satisfying

$$\|\boldsymbol{F}_h-\boldsymbol{P}_h\|_{L^2(\Omega;\mathbb{R}^{3\times3})}\le C\sqrt{r_h},\qquad \|D'\boldsymbol{P}_h\|_{L^2(S;\mathbb{R}^{3\times3\times3})}\le Ch^{-1}\sqrt{r_h},$$

and

$$\sup_{\boldsymbol{x}'\in S}\mathrm{dist}(\boldsymbol{P}_h(\boldsymbol{x}');SO(3))\le Ch^{-2}r_h.$$

The construction of $\boldsymbol{P}_h$ is explicit and does not depend on the exponent $q=2$. As $r_h/h^2\to0$, as $h\to0^+$, the second estimates ensures that, for $h\ll1$, the map $\boldsymbol{P}_h$ takes values in a tubular neighborhood $\mathcal{O}$ of $SO(3)$ for which the nearest-point projection $\boldsymbol{\Pi}\colon\mathcal{O}\to SO(3)$ is defined and smooth. Therefore, we define $\boldsymbol{R}_h\coloneqq\boldsymbol{\Pi}\circ\boldsymbol{P}_h$. The estimates (4.34)–(4.35) with $q=2$ are established exactly as in [65, Theorem 6]. Moreover, by repeating the computations with $q=p$ instead of $q=2$, we prove the estimates (4.34)–(4.35) for $q=p$. Note that, at this point, we already know that $\widehat{\boldsymbol{R}}_h$ takes values in $\mathcal{O}$ for $h\ll1$, so that we do not necessarily need that (r_h/h^p) tends to zero, as $h\to0^+$. □

The next proposition provides a reformulation of the compactness results in [65].

Proposition 4.6 (Compactness of deformations). *Let $(\widehat{\boldsymbol{y}}_h) \subset W^{1,2}(\Omega;\mathbb{R}^3)$ and let $(e_h) \subset \mathbb{R}$ with $e_h > 0$ be such that $e_h/h^2 \to 0$, as $h \to 0^+$. Set $r_h := \mathcal{R}_h(\widehat{\boldsymbol{y}}_h)$ and suppose that $r_h \leq C e_h$ for every $h > 0$. Also, set $\widehat{\boldsymbol{F}}_h := D_h \widehat{\boldsymbol{y}}_h$ and suppose that there exists $(\widehat{\boldsymbol{R}}_h) \subset W^{1,2}(S;SO(3))$ such that, for every $h > 0$, the following estimates hold:*

$$\|\widehat{\boldsymbol{F}}_h - \widehat{\boldsymbol{R}}_h\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} \leq C\sqrt{r_h}, \qquad \|D'\widehat{\boldsymbol{R}}_h\|_{L^2(S;\mathbb{R}^{3\times 3\times 3})} \leq Ch^{-1}\sqrt{r_h} \tag{4.36}$$

$$\|\widehat{\boldsymbol{R}}_h - \boldsymbol{I}\|_{L^2(S;\mathbb{R}^{3\times 3})} \leq Ch^{-1}\sqrt{r_h}, \qquad \|\widehat{\boldsymbol{F}}_h - \boldsymbol{I}\|_{L^2(S;\mathbb{R}^{3\times 3})} \leq Ch^{-1}\sqrt{r_h}. \tag{4.37}$$

Eventually, assume that $\widehat{\boldsymbol{y}}_h - \boldsymbol{\pi}_h$ has null average over Ω for every $h > 0$. Define

$$\widehat{\boldsymbol{u}}_h \colon S \to \mathbb{R}^2, \qquad \widehat{v}_h \colon S \to \mathbb{R}, \qquad \widehat{\boldsymbol{w}}_h \colon S \to \mathbb{R}^3,$$

by setting

$$\begin{aligned}
\widehat{\boldsymbol{u}}_h(\boldsymbol{x}') &:= \frac{h^2}{e_h} \wedge \frac{1}{\sqrt{e_h}} \int_I \left(\widehat{\boldsymbol{y}}_h'(\boldsymbol{x}', x_3) - \boldsymbol{x}'\right) \mathrm{d}x_3, \\
\widehat{v}_h(\boldsymbol{x}') &:= \frac{h}{\sqrt{e_h}} \int_I \widehat{y}^3_h(\boldsymbol{x}', x_3)\, \mathrm{d}x_3, \\
\widehat{\boldsymbol{w}}_h(\boldsymbol{x}') &:= \frac{1}{\sqrt{e_h}} \int_I x_3 \big(\widehat{\boldsymbol{y}}_h(\boldsymbol{x}', x_3) - \boldsymbol{\pi}_h(\boldsymbol{x}', x_3)\big)\, \mathrm{d}x_3.
\end{aligned}$$

Then, the following estimates hold:

$$\|\widehat{\boldsymbol{u}}_h\|_{W^{1,2}(S;\mathbb{R}^2)} \leq C\left(\sqrt{\frac{r_h}{e_h}} + \frac{r_h}{e_h} \wedge \frac{r_h}{h^2\sqrt{e_h}}\right), \tag{4.38}$$

$$\|\widehat{v}_h\|_{L^2(S;\mathbb{R}^{3\times 3})} \leq C\sqrt{\frac{r_h}{e_h}}, \tag{4.39}$$

$$\|\widehat{\boldsymbol{w}}_h\|_{L^2(S;\mathbb{R}^{3\times 3})} \leq C\sqrt{\frac{r_h}{e_h}}. \tag{4.40}$$

Moreover, there exist $\widehat{\boldsymbol{u}} \in W^{1,2}(S;\mathbb{R}^2)$ and $\widehat{v} \in W^{2,2}(S)$ such that the following convergences hold, as $h \to 0^+$:

$$\widehat{\boldsymbol{u}}_h \rightharpoonup \widehat{\boldsymbol{u}} \text{ in } W^{1,2}(S;\mathbb{R}^2); \tag{4.41}$$

$$\widehat{v}_h \to \widehat{v} \text{ in } W^{1,2}(S); \tag{4.42}$$

$$\widehat{\boldsymbol{w}}_h \rightharpoonup -\frac{1}{12}\begin{pmatrix} D'\widehat{v} \\ 0 \end{pmatrix} \text{ in } W^{1,2}(S;\mathbb{R}^3). \tag{4.43}$$

Proof. All these results are taken from [65]. However, for convenience of the reader, we present a detailed proof. This is divided into two steps.

Step 1 (Estimates). Set

$$\widehat{\boldsymbol{A}}_h := \frac{h}{\sqrt{e_h}}(\widehat{\boldsymbol{R}}_h - \boldsymbol{I}), \qquad \widehat{\boldsymbol{B}}_h := \frac{h}{\sqrt{e_h}}(\widehat{\boldsymbol{F}}_h - \boldsymbol{I}), \qquad \widehat{\boldsymbol{C}}_h := \frac{h^2}{e_h}\mathrm{sym}(\widehat{\boldsymbol{R}}_h - \boldsymbol{I}).$$

Then, the following estimates hold:

$$\|\widehat{\boldsymbol{A}}_h\|_{W^{1,2}(S;\mathbb{R}^{3\times 3})} \leq C\sqrt{\frac{r_h}{e_h}}, \tag{4.44}$$

$$\|\widehat{\boldsymbol{B}}_h\|_{L^2(S;\mathbb{R}^{3\times 3})} \leq C\sqrt{\frac{r_h}{e_h}}, \tag{4.45}$$

$$\|\widehat{\boldsymbol{C}}_h\|_{L^2(S;\mathbb{R}^{3\times 3})} \leq C\frac{r_h}{e_h}. \tag{4.46}$$

The first two estimates follows immediately from (4.36)–(4.37). The third one follows from the identity

$$\widehat{\boldsymbol{C}}_h = -\frac{1}{2}\,\widehat{\boldsymbol{A}}_h^\top \widehat{\boldsymbol{A}}_h$$

by applying the Hölder inequality, the Sobolev embedding and (4.44). Observe that $\widehat{\boldsymbol{u}}_h$ and $\widehat{v}_h$ have both nulla average on S. First, employing the Jensen inequality, we estimate

$$\begin{aligned}
\|\mathrm{sym}D'\widehat{\boldsymbol{u}}_h\|_{L^2(S;\mathbb{R}^{2\times 2})} &\leq \frac{h^2}{e_h} \wedge \frac{1}{\sqrt{e_h}}\|\mathrm{sym}(\widehat{\boldsymbol{F}}_h'' - \boldsymbol{I}'')\|_{L^2(\Omega;\mathbb{R}^{2\times 2})}\\
&\leq \frac{h^2}{e_h} \wedge \frac{1}{\sqrt{e_h}}\|\widehat{\boldsymbol{F}}_h - \widehat{\boldsymbol{R}}_h\|_{L^2(\Omega;\mathbb{R}^{3\times 3})}\\
&+ \left(\frac{h^2}{e_h} \wedge \frac{1}{\sqrt{e_h}}\right)\frac{e_h}{h^2}\|\widehat{\boldsymbol{C}}_h\|_{L^2(S;\mathbb{R}^{3\times 3})}\\
&\leq C\left(\frac{h^2\sqrt{r_h}}{e_h} \wedge \sqrt{\frac{r_h}{e_h}} + \frac{r_h}{e_h} \wedge \frac{r_h}{h^2\sqrt{e_h}}\right),
\end{aligned}$$

where, in the last line, we employed the first estimate in (4.36) and (4.46). Then, using the Korn inequality, we obtain (4.38). Similarly, using the Jensen inequality, we see that

$$\|D'\widehat{v}_h\|_{L^2(S;\mathbb{R}^2)} \leq C\frac{h}{\sqrt{e_h}}\|\widehat{\boldsymbol{F}}_h - \boldsymbol{I}\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} = C\|\widehat{\boldsymbol{B}}_h\|_{L^2(\Omega;\mathbb{R}^{3\times 3})},$$

so that (4.39) follows from (4.45) and the Poincaré-Wirtinger inequality.

We now show (4.40). For every $h > 0$, define $\widehat{\boldsymbol{\vartheta}}_h \in W^{1,2}(\Omega;\mathbb{R}^3)$ by setting

$$\widehat{\boldsymbol{\vartheta}}_h(\boldsymbol{x}) := \widehat{\boldsymbol{y}}_h(\boldsymbol{x}) - \boldsymbol{\pi}_h(\boldsymbol{x}) - \int_I (\widehat{\boldsymbol{y}}(\boldsymbol{x}',z) - \boldsymbol{\pi}_h(\boldsymbol{x}',z))\,\mathrm{d}z\,\mathrm{d}\boldsymbol{x}'.$$

By definition, $\widehat{\boldsymbol{\vartheta}}$ has null average over Ω. Also, we have

$$h^{-1}\partial_3\widehat{\boldsymbol{\vartheta}}_h = h^{-1}\partial_3\widehat{\boldsymbol{y}}_h - \boldsymbol{e}_3 = (\widehat{\boldsymbol{F}}_h - \widehat{\boldsymbol{R}}_h)\boldsymbol{e}_3 - (\widehat{\boldsymbol{R}}_h - \boldsymbol{I})\boldsymbol{e}_3,$$

so that

$$\|h^{-1}\partial_3\widehat{\boldsymbol{\vartheta}}_h - (\widehat{\boldsymbol{R}}_h - \boldsymbol{I})\boldsymbol{e}_3\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} \leq C\sqrt{r_h}, \tag{4.47}$$

thanks to the first estimate in (4.36). Observe that, for every $\boldsymbol{x}' \in S$, the map $x_3 \mapsto h^{-1}\widehat{\boldsymbol{\vartheta}}(\boldsymbol{x}',x_3) - x_3(\widehat{\boldsymbol{R}}_h(\boldsymbol{x}') - \boldsymbol{I})\boldsymbol{e}_3$ has null average over I. Hence, by the Poincaré-Wirtinger inequality, there holds

$$\begin{aligned}
&\int_I \left|h^{-1}\widehat{\boldsymbol{\vartheta}}(\boldsymbol{x}',x_3) - x_3(\widehat{\boldsymbol{R}}_h(\boldsymbol{x}') - \boldsymbol{I})\boldsymbol{e}_3\right|^2\mathrm{d}x_3\\
&\qquad\leq C\int_I \left|h^{-1}\partial_3\widehat{\boldsymbol{\vartheta}}(\boldsymbol{x}',x_3) - (\widehat{\boldsymbol{R}}_h(\boldsymbol{x}') - \boldsymbol{I})\boldsymbol{e}_3\right|^2\mathrm{d}x_3,
\end{aligned}$$

where the constant $C > 0$ clearly does not depend on $\boldsymbol{x}' \in S$. As $\boldsymbol{x}' \in S$ is arbitrary, from this and from (4.47), we deduce the estimate

$$\|h^{-1}\widehat{\boldsymbol{\vartheta}} - x_3(\widehat{\boldsymbol{R}}_h - \boldsymbol{I})\boldsymbol{e}_3\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} \leq C\sqrt{r_h}. \tag{4.48}$$

At this point, using the Jensen inequality, we compute

$$\begin{aligned}\int_S \left|\widehat{\boldsymbol{w}}_h - \frac{1}{12}\widehat{\boldsymbol{A}}_h\boldsymbol{e}_3\right|^2 \mathrm{d}\boldsymbol{x}' &= \int_S \left|\frac{1}{\sqrt{e_h}}\int_I \Big(x_3(\widehat{\boldsymbol{y}}_h - \boldsymbol{\pi}_h) - hx_3^2(\widehat{\boldsymbol{R}}_h - \boldsymbol{I})\boldsymbol{e}_3\Big)\,\mathrm{d}x_3\right|^2 \mathrm{d}\boldsymbol{x}' \\ &= \int_S \left|\frac{h}{\sqrt{e_h}}\int_I x_3\left(h^{-1}\widehat{\boldsymbol{\vartheta}}_h - x_3\,(\widehat{\boldsymbol{R}}_h - \boldsymbol{I})\boldsymbol{e}_3\right)\mathrm{d}x_3\right|^2 \mathrm{d}\boldsymbol{x}' \\ &\leq \frac{h^2}{e_h}\int_\Omega \left|h^{-1}\widehat{\boldsymbol{\vartheta}}_h - x_3\,(\widehat{\boldsymbol{R}}_h - \boldsymbol{I})\boldsymbol{e}_3\right|^2 \mathrm{d}\boldsymbol{x} \leq C\frac{r_h}{e_h},\end{aligned}$$

where the last inequality is justified by (4.48). This yields

$$\|\widehat{\boldsymbol{w}}_h - \frac{1}{12}\widehat{\boldsymbol{A}}_h\boldsymbol{e}_3\|_{L^2(S;\mathbb{R}^3)} \leq C\sqrt{\frac{r_h}{e_h}}. \tag{4.49}$$

To estimate the norm of the gradient of $\widehat{\boldsymbol{w}}_h$, we observe that

$$\partial_j\widehat{\boldsymbol{w}}_h = \frac{1}{\sqrt{e_h}}\int_I x_3\partial_j\widehat{\boldsymbol{y}}_h\,\mathrm{d}x_3 = \frac{1}{\sqrt{e_h}}\int_I x_3(\widehat{\boldsymbol{F}}_h - \widehat{\boldsymbol{R}}_h)\boldsymbol{e}_3\,\mathrm{d}\boldsymbol{x},$$

where $j \in \{1,2\}$, which, in view of the first estimate in (4.36), gives

$$\|D'\widehat{\boldsymbol{w}}_h\|_{L^2(S;\mathbb{R}^{3\times 2})} \leq C\sqrt{\frac{r_h}{e_h}}. \tag{4.50}$$

Therefore, combining (4.44) with (4.49)–(4.50), we establish (4.40).

Step 2 (Compactness). Since (r_h/e_h) is bounded, from (4.44) we deduce that there exists $\widehat{\boldsymbol{A}} \in W^{1,2}(S;\mathbb{R}^{3\times 3})$ such that, up to subsequences, we have

$$\widehat{\boldsymbol{A}}_h \rightharpoonup \widehat{\boldsymbol{A}} \text{ in } W^{1,2}(S;\mathbb{R}^{3\times 3}). \tag{4.51}$$

Similarly, from (4.38)–(4.40) we deduce the existence of

$$\widehat{\boldsymbol{u}} \in W^{1,2}(S;\mathbb{R}^2), \quad \widehat{v} \in W^{1,2}(S), \quad \widehat{\boldsymbol{w}} \in W^{1,2}(S;\mathbb{R}^3),$$

such that , up to subsequences, the convergences in (4.41)–(4.42) hold true and also

$$\widehat{\boldsymbol{w}}_h \rightharpoonup \widehat{\boldsymbol{w}} \text{ in } W^{1,2}(S;\mathbb{R}^3). \tag{4.52}$$

Thanks to (4.51), passing to the limit, as $h \to 0^+$, in the identity

$$-\frac{\sqrt{e_h}}{h}\,\widehat{\boldsymbol{A}}_h^\top\widehat{\boldsymbol{A}}_h = \widehat{\boldsymbol{A}}_h + \widehat{\boldsymbol{A}}_h^\top$$

with the aid of the Sobolev embedding and the Dominated Convergence Theorem, we deduce that $\widehat{\boldsymbol{A}}$ takes values in Skew(3). In particular, $\widehat{A}^{33} = 0$. Then, for $j \in \{1,2\}$, we estimate

$$\|\partial_j\widehat{v}_h - \widehat{A}_h^{3j}\|_{L^2(S)} \leq \frac{h}{\sqrt{e_h}}\|\partial_j\widehat{y}_h^3 - \widehat{R}_h^{3j}\|_{L^2(S)} \leq C\frac{h\sqrt{r_h}}{\sqrt{e_h}}.$$

Here, we employed the Jensen inequality and the first estimate in (4.36). As the right-hand side of the previous inequality goes to zero, as $h \to 0^+$, this entails $\partial_j \widehat{v} = A^{3j}$. Therefore, $\widehat{v} \in W^{2,2}(S)$ and we have $\widehat{\boldsymbol{A}}\boldsymbol{e}_3 = ((D'\widehat{v})^\top, 0)^\top$. Thus, (4.49) and (4.51)–(4.52) yield (4.43). $\square$

Remark 4.7. In the setting of the previous Proposition, it is easily checked that $\widehat{\boldsymbol{B}}_h \to \widehat{\boldsymbol{A}}$ in $L^2(\Omega;\mathbb{R}^{3\times 3})$ and $\widehat{\boldsymbol{C}}_h \to -1/2\widehat{\boldsymbol{A}}^2$ in $L^2(\Omega;\mathbb{R}^{3\times 3})$. Moreover, one has

$$\widehat{\boldsymbol{A}} = \left(\begin{array}{c|c} \mathbf{O}'' & -D'\widehat{v} \\ \hline (D'v)^\top & 0 \end{array}\right).$$

These facts have been observed in [65] but will not be exploited in the following.

To identiy the limiting strain, we will employ the following result.

Lemma 4.8 ([65, Lemma 2]). *Let $(\widehat{\boldsymbol{y}}_h) \subset W^{1,2}(\Omega;\mathbb{R}^3)$ and let $(e_h) \subset \mathbb{R}$ with $e_h > 0$ be such tha $e_h/h^2 \to 0$, as $h \to 0^+$. Set $r_h := \mathcal{R}_h(\widehat{\boldsymbol{y}}_h)$ and suppose that $r_h \leq Ce_h$ for every $h > 0$. Also, set $\widehat{\boldsymbol{F}}_h := D_h\widehat{\boldsymbol{y}}_h$ and suppose that there exists $(\widehat{\boldsymbol{R}}_h) \subset W^{1,2}(S;SO(3))$ such that, for every $h > 0$, the following estimate holds:*

$$\|\widehat{\boldsymbol{F}}_h - \widehat{\boldsymbol{R}}_h\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} \leq C\sqrt{r_h}.$$

Define $\widehat{\boldsymbol{u}}_h \colon S \to \mathbb{R}^2$ and $\widehat{v}_h \colon S \to \mathbb{R}$ by setting

$$\widehat{\boldsymbol{u}}_h(\boldsymbol{x}') := \frac{h^2}{e_h} \wedge \frac{1}{\sqrt{e_h}} \int_I \left(\widehat{\boldsymbol{y}}_h'(\boldsymbol{x}', x_3) - \boldsymbol{x}'\right) \mathrm{d}x_3,$$
$$\widehat{v}_h(\boldsymbol{x}') := \frac{h}{\sqrt{e_h}} \int_I \widehat{y}_h^3(\boldsymbol{x}', x_3)\,\mathrm{d}x_3,$$

for every $\boldsymbol{x}' \in S$, and assume that there exist $\widehat{\boldsymbol{u}} \in W^{1,2}(S;\mathbb{R}^2)$ and $\widehat{v} \in W^{2,2}(S)$ such that the following convergences hold:

$$\widehat{\boldsymbol{u}}_h \rightharpoonup \widehat{\boldsymbol{u}} \text{ in } W^{1,2}(S;\mathbb{R}^2), \qquad \widehat{v}_h \to \widehat{v} \text{ in } W^{1,2}(S).$$

Then, there exists $\widehat{\boldsymbol{G}} \in L^2(\Omega;\mathbb{R}^{3\times 3})$ such that

$$\widehat{\boldsymbol{G}}_h := \frac{1}{\sqrt{e_h}}(\widehat{\boldsymbol{R}}_h^\top \widehat{\boldsymbol{F}}_h - \boldsymbol{I}) \rightharpoonup \widehat{\boldsymbol{G}} \text{ in } L^2(\Omega;\mathbb{R}^{3\times 3}).$$

Furthermore, there exists $\widehat{\boldsymbol{\Sigma}} \in L^2(\Omega;\mathbb{R}^{2\times 2})$ such that, for almost every $\boldsymbol{x} \in \Omega$, there holds

$$\widehat{\boldsymbol{G}}''(\boldsymbol{x}', x_3) = \widehat{\boldsymbol{\Sigma}}(\boldsymbol{x}') - D'\widehat{v}(\boldsymbol{x}')x_3.$$

Eventually, if $e_h/h^4 \to 0$, as $h \to 0^+$, then $\mathrm{sym}\,\widehat{\boldsymbol{\Sigma}} = \mathrm{sym} D'\widehat{\boldsymbol{u}}$.

We will use the following notation. Given $\eta > 0$, we define

$$S^\eta := \{\boldsymbol{x}' \in S : \ \mathrm{dist}(\boldsymbol{x}';\partial S) < \eta\}, \quad S^{-\eta} := \{\boldsymbol{x}' \in \mathbb{R}^2 : \ \mathrm{dist}(\boldsymbol{x}';\partial S) < \eta\}.$$

Moreover, for $\rho > 0$, we set $\Omega_\rho^\eta := S^\eta \times \rho I$ and $\Omega_\rho^{-\eta} := S^{-\eta} \times \rho I$.

Proposition 4.9 (Compactness of magnetizations). *Let* $(\widehat{\boldsymbol{q}}_h)$ *with* $\widehat{\boldsymbol{q}}_h \in \mathcal{Q}_h$ *and* $\widehat{\boldsymbol{q}}_h = (\widehat{\boldsymbol{y}}_h, \widehat{\boldsymbol{m}}_h)$ *be such that*

$$\sup_{h>0} \left\{ E_h^{\mathrm{el}}(\widehat{\boldsymbol{q}}_h) + E_h^{\mathrm{exc}}(\widehat{\boldsymbol{q}}_h) \right\} \leq C. \tag{4.53}$$

Suppose that there exists $\alpha > 1$ *such that, for every* $h > 0$*, there holds*

$$\|\widehat{\boldsymbol{y}}_h - \boldsymbol{\pi}_h\|_{C^0(\overline{\Omega};\mathbb{R}^3)} \leq Ch^\alpha, \tag{4.54}$$

$$\|\widehat{\boldsymbol{F}}_h - \boldsymbol{I}\|_{L^p(\Omega;\mathbb{R}^3)} \leq Ch^{\beta/p-1}. \tag{4.55}$$

Then, there exists $\widehat{\boldsymbol{\zeta}} \in W^{1,2}(S;\mathbb{S}^2)$ *and* $\widehat{\boldsymbol{\chi}} \in L^2(\mathbb{R}^3;\mathbb{R}^3)$ *such that, up to subsequences, the following convergences hold, as* $h \to 0^+$*:*

$$\widehat{\boldsymbol{\mu}}_h := \mathcal{M}_h(\widehat{\boldsymbol{q}}_h) \to \chi_\Omega \widehat{\boldsymbol{\zeta}} \text{ in } L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3}), \tag{4.56}$$

$$\widehat{\boldsymbol{N}}_h := \mathcal{N}_h(\widehat{\boldsymbol{q}}_h) \rightharpoonup \chi_\Omega (D'\widehat{\boldsymbol{\zeta}}|\widehat{\boldsymbol{\chi}}) \text{ in } L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3}); \tag{4.57}$$

$$\widehat{\boldsymbol{\lambda}}_h := \widehat{\boldsymbol{m}}_h \circ \widehat{\boldsymbol{y}}_h \to \widehat{\boldsymbol{\zeta}} \text{ in } L^1(\Omega;\mathbb{R}^3); \tag{4.58}$$

$$\widehat{\boldsymbol{z}}_h := \mathcal{Z}_h(\widehat{\boldsymbol{q}}_h) \to \widehat{\boldsymbol{\zeta}} \text{ in } L^q(\Omega;\mathbb{R}^{3\times 3}) \text{ for every } 1 \leq q < +\infty. \tag{4.59}$$

Proof. For convenience of the reader, the proof is subdivided into four steps.

Step 1 (Approximation of the deformed configuration). The estimate (4.54) entails the following statements (see Figure 4.1 and Figure 4.2):

$$\forall\, \eta > 0,\ \forall\, 0 < \vartheta < 1,\ \exists\, \bar{h}(\eta,\vartheta) > 0 :\ \forall\, 0 < h \leq \bar{h}(\eta,\vartheta),\ \Omega^\eta_{\vartheta h} \subset \Omega^{\widehat{\boldsymbol{y}}_h}, \tag{4.60}$$

$$\forall\, \eta > 0,\ \forall\, \ell > 1,\ \exists\, \underline{h}(\eta,\ell) > 0 :\ \forall\, 0 < h \leq \underline{h}(\eta,\ell),\ \Omega^{\widehat{\boldsymbol{y}}_h} \subset \Omega^{-\eta}_{\ell h}. \tag{4.61}$$

To see (4.60), fix $\eta > 0$ and $0 < \vartheta < 1$. Let $\boldsymbol{\xi} \in \Omega^\eta_{\vartheta h}$. As $\alpha > 1$, there exists $\bar{h}(\eta,\vartheta) > 0$ such that for every $h \leq \bar{h}(\eta,\vartheta)$ there holds

$$\mathrm{dist}(\boldsymbol{\xi};\partial\Omega_h) \geq \eta \wedge (1-\vartheta)h/2 > Ch^\alpha$$

so that, by (4.54), we obtain

$$\|\widehat{\boldsymbol{y}}_h - \boldsymbol{\pi}_h\|_{C^0(\overline{\Omega};\mathbb{R}^3)} < \mathrm{dist}(\boldsymbol{\xi};\partial\Omega_h) = \mathrm{dist}(\boldsymbol{\xi};\boldsymbol{\pi}_h(\partial\Omega)).$$

By the stability property of the topological degree, this entails $\boldsymbol{\xi} \notin \widehat{\boldsymbol{y}}_h(\partial\Omega)$ and $\deg(\widehat{\boldsymbol{y}}_h,\Omega,\boldsymbol{\xi}) = \deg(\boldsymbol{\pi}_h,\Omega,\boldsymbol{\xi}) = 1$. Thus, by the solvability property of the topological degree, we deduce $\boldsymbol{\xi} \in \Omega^{\widehat{\boldsymbol{y}}_h}$. As $\boldsymbol{\xi} \in \Omega^\eta_{\vartheta h}$ was arbitrary, this proves (4.60).
To check (4.61), fix $\eta > 0$ and $\ell > 1$. Again, as $\alpha > 1$, there exists $\underline{h}(\eta,\ell) > 0$ such that for every $h \leq \underline{h}(\eta,\ell)$ there holds

$$\mathrm{dist}(\overline{\Omega}_h;\partial\Omega^{-\eta}_{\ell h}) \geq \eta \wedge (\ell-1)h/2 > Ch^\alpha.$$

Thus, by (4.54), we have

$$\Omega^{\widehat{\boldsymbol{y}}_h} \subset \Omega_h + B(\boldsymbol{0},Ch^\alpha) \subset \Omega^{-\eta}_{\ell h}.$$

Similarly to (4.60), we also have the following:

$$\begin{gathered} \forall\, 0 < \eta_0 < \eta,\ \forall\, 0 < \vartheta < \vartheta_0 < 1,\ \exists\, \widetilde{h}(\eta,\eta_0,\vartheta,\vartheta_0) > 0 : \\ \forall\, 0 < h < \widetilde{h}(\eta,\eta_0,\vartheta,\vartheta_0), \quad \Omega^\eta_{\vartheta h} \subset \widehat{\boldsymbol{y}}_h(\Omega^{\eta_0}_{\vartheta_0}). \end{gathered} \tag{4.62}$$

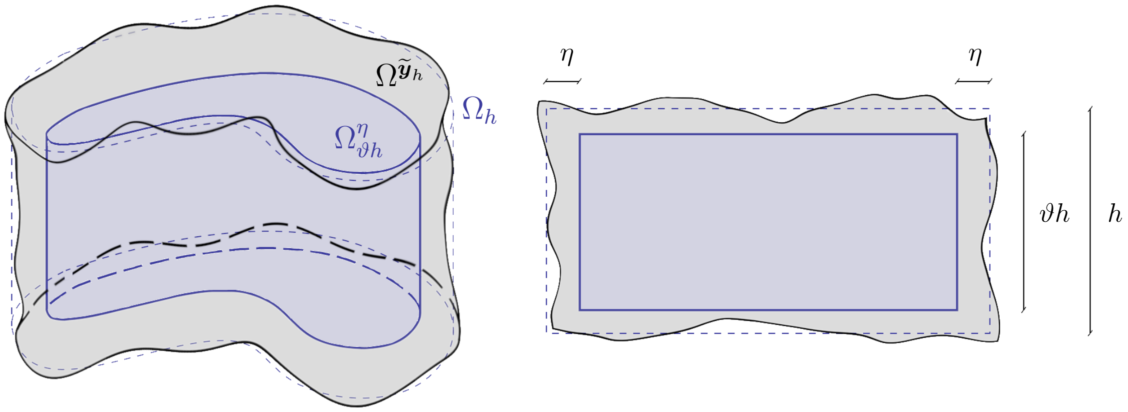

Figure 4.1: On the left, the cylinder $\Omega^{\eta}_{\vartheta h}$ is contained in the deformed configuration $\Omega^{\widehat{\boldsymbol{y}}_h}$. On the right, the corresponding section.

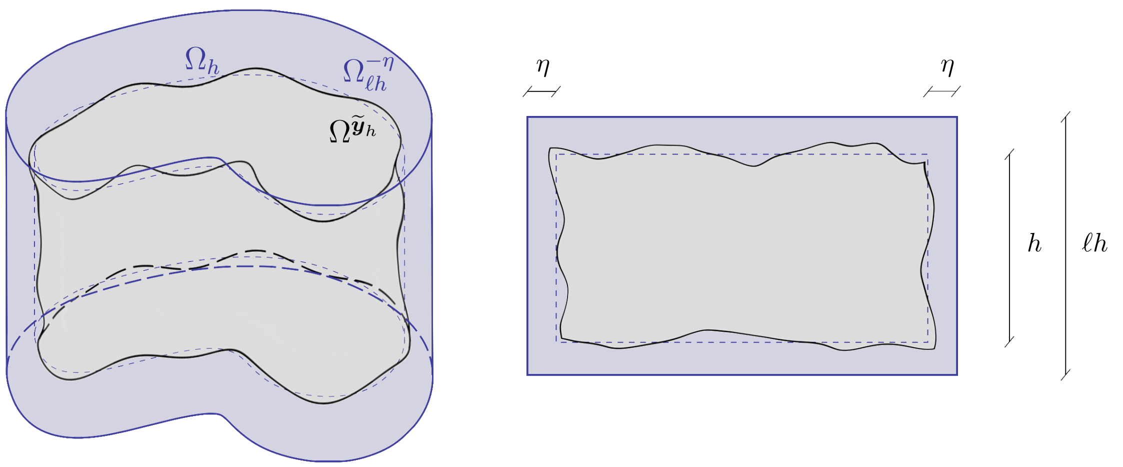

Figure 4.2: On the left, the cylinder $\Omega^{-\eta}_{\ell h}$ contains the deformed configuration $\Omega^{\widehat{\boldsymbol{y}}_h}$. On the right, the corresponding section.

Step 2 (Identification of the limiting magnetization). Employing the notation in (4.7) and (4.23), we set $\boldsymbol{F}_h := D_h\widehat{\boldsymbol{y}}_h$ and $\widehat{\boldsymbol{\Xi}}_h := \boldsymbol{L}_h(\mathcal{Z}_h(\widehat{\boldsymbol{q}}_h))^{-1}\boldsymbol{F}_h$. Thanks to (4.6) and (4.11), we have

$$\begin{aligned}\int_\Omega \frac{1}{|\det D_h\widehat{\boldsymbol{y}}_h|^a}\,\mathrm{d}\boldsymbol{x} &= (1+c_h)^{-a}\int_\Omega \frac{1}{|\det \widehat{\boldsymbol{\Xi}}_h|^a}\,\mathrm{d}\boldsymbol{x}\\ &\leq (1+c_h)^{-a}\left\{\int_\Omega \Phi(\widehat{\boldsymbol{\Xi}}_h)\,\mathrm{d}\boldsymbol{x} + C\right\}\\ &\leq (1+c_h)^{-a}\left\{h^\beta E^{\mathrm{el}}_h(\widehat{\boldsymbol{q}}_h) + C\right\} \leq C,\end{aligned} \tag{4.63}$$

where in the last line we used (4.5), (4.11) and (4.53). As $a > 1$, we infer that $(1/\det D_h\widehat{\boldsymbol{y}}_h)$ is equi-integrable by the de la Vallée-Poussin Criterion [62, Theorem 2.29].
Now, let $\eta > 0$ and $0 < \vartheta < 1$. By (4.60), for $h \leq \bar{h}(\eta,\vartheta)$, the composition $\widehat{\boldsymbol{\zeta}}_h := \widehat{\boldsymbol{m}}_h \circ \boldsymbol{\pi}_h$ is well defined as a map in $W^{1,2}(\Omega^\eta_\vartheta;\mathbb{R}^3)$. Using Proposition 1.46, we

compute

$$\begin{aligned}\int_{\Omega^\eta_\vartheta} |\widehat{\boldsymbol{\zeta}}_h|^2 \,\mathrm{d}\boldsymbol{x} &= \int_{\Omega^\eta_\vartheta} |\widehat{\boldsymbol{m}}_h \circ \boldsymbol{\pi}_h|^2 \,\mathrm{d}\boldsymbol{x} = \frac{1}{h}\int_{\Omega^\eta_{\vartheta h}} |\widehat{\boldsymbol{m}}_h|^2 \,\mathrm{d}\boldsymbol{\xi} \\ &\leq \frac{1}{h}\int_{\Omega^{\widehat{\boldsymbol{y}}_h}} |\widehat{\boldsymbol{m}}_h|^2 \,\mathrm{d}\boldsymbol{\xi} \leq \frac{1}{h}\int_{\Omega} |\widehat{\boldsymbol{m}} \circ \widehat{\boldsymbol{y}}_h|^2 \det D\widehat{\boldsymbol{y}}_h \,\mathrm{d}\boldsymbol{x} \\ &= \int_{\Omega} |\widehat{\boldsymbol{m}} \circ \widehat{\boldsymbol{y}}_h|^2 \det D_h\widehat{\boldsymbol{y}}_h \,\mathrm{d}\boldsymbol{x} = \int_\Omega \frac{1}{\det D_h \widehat{\boldsymbol{y}}_h}\,\mathrm{d}\boldsymbol{x},\end{aligned} \tag{4.64}$$

where in the last line we used (4.20). Thus, by (4.63), the sequence $(\widehat{\boldsymbol{\zeta}}_h)$ is bounded in $L^2(\Omega^\eta_\vartheta;\mathbb{R}^3)$. From (4.53) and (4.60), employing again Proposition 1.46, we estimate

$$\begin{aligned}\int_{\Omega^\eta_\vartheta} \left|D_h\widehat{\boldsymbol{\zeta}}_h\right|^2 \,\mathrm{d}\boldsymbol{x} &= \int_{\Omega^\eta_\vartheta} |D\widehat{\boldsymbol{m}}_h \circ \boldsymbol{\pi}_h|^2 \,\mathrm{d}\boldsymbol{x} = \frac{1}{h}\int_{\Omega^\eta_{\vartheta h}} |D\widehat{\boldsymbol{m}}_h|^2 \,\mathrm{d}\boldsymbol{\xi} \\ &\leq \frac{1}{h}\int_{\Omega^{\widehat{\boldsymbol{y}}_h}} |D\widehat{\boldsymbol{m}}_h|^2 \,\mathrm{d}\boldsymbol{\xi} = E^{\mathrm{exc}}_h(\widehat{\boldsymbol{q}}_h) \leq C.\end{aligned} \tag{4.65}$$

Therefore $(\widehat{\boldsymbol{\zeta}}_h)$ is bounded in $W^{1,2}(\Omega^\eta_\vartheta;\mathbb{R}^3)$, so that there exists $\widehat{\boldsymbol{\zeta}} \in W^{1,2}(\Omega^\eta_\vartheta;\mathbb{R}^3)$ such that, up to subsequences, $\widehat{\boldsymbol{\zeta}}_h \rightharpoonup \widehat{\boldsymbol{\zeta}}$ in $W^{1,2}(\Omega^\eta_\vartheta;\mathbb{R}^3)$. Also, by (4.65), there exists a map $\widehat{\boldsymbol{\chi}} \in L^2(\Omega^\eta_\vartheta;\mathbb{R}^3)$ such that, up to subsequences, $h^{-1}\partial_3\widehat{\boldsymbol{\zeta}}_h \rightharpoonup \widehat{\boldsymbol{\chi}}$ in $L^2(\Omega^\eta_\vartheta;\mathbb{R}^3)$, as $h \to 0^+$. This yields $\partial_3\widehat{\boldsymbol{\zeta}} = \boldsymbol{0}$, so that $\widehat{\boldsymbol{\zeta}} \in W^{1,2}(S^\eta;\mathbb{R}^3)$. In principle, both the subsequences and the weak limits $\widehat{\boldsymbol{\zeta}}$ and $\widehat{\boldsymbol{\chi}}$ depend on the parameters η and ϑ. However, by means of a diagonal argument, we can assume that $\widehat{\boldsymbol{\zeta}} \in W^{1,2}_{\mathrm{loc}}(S;\mathbb{R}^3)$ and $\widehat{\boldsymbol{\chi}} \in L^2_{\mathrm{loc}}(\Omega;\mathbb{R}^3)$, and that, for a not relabeled subsequence, the following holds:

$$\forall\,\eta>0, \forall\, 0<\vartheta<1, \qquad \begin{aligned}&\widehat{\boldsymbol{\zeta}}_h \rightharpoonup \widehat{\boldsymbol{\zeta}} \text{ in } W^{1,2}(\Omega^\eta_\vartheta;\mathbb{R}^3) \text{ and a.e. in } \Omega^\eta_\vartheta, \\ &h^{-1}\partial_3\widehat{\boldsymbol{\zeta}}_h \rightharpoonup \widehat{\boldsymbol{\chi}} \text{ in } L^2(\Omega^\eta_\vartheta;\mathbb{R}^3).\end{aligned} \tag{4.66}$$

Here, we exploited the Sobolev embedding, which is applicable since Ω^η_ϑ is a Lipschitz domain, at least for $\eta \ll 1$ and $1-\vartheta \ll 1$. Note that the sequences in (4.66) are defined only for $n \gg 1$ depending on η and ϑ. In view of (4.63)–(4.64) and (4.66), by lower semicontinuity, for every $\eta>0$ and $0<\vartheta<1$, there holds

$$\vartheta\int_{S^\eta} |\widehat{\boldsymbol{\zeta}}|^2 \,\mathrm{d}\boldsymbol{x}' = \int_{\Omega^\eta_\vartheta} |\widehat{\boldsymbol{\zeta}}|^2 \,\mathrm{d}\boldsymbol{x} \leq \liminf_{h\to 0^+} \int_{\Omega^\eta_\vartheta} |\widehat{\boldsymbol{\zeta}}_h|^2 \,\mathrm{d}\boldsymbol{x} \leq C. \tag{4.67}$$

Similarly, by (4.65) and (4.66), for every $\eta>0$ and $0<\vartheta<1$, there holds

$$\begin{aligned}\vartheta\int_{S^\eta} |D'\widehat{\boldsymbol{\zeta}}|^2 \,\mathrm{d}\boldsymbol{x} + \int_{\Omega^\eta_\vartheta} |\widehat{\boldsymbol{\chi}}|^2 \,\mathrm{d}\boldsymbol{x} &= \int_{\Omega^\eta_\vartheta} |D\widehat{\boldsymbol{\zeta}}|^2 \,\mathrm{d}\boldsymbol{x} + \int_{\Omega^\eta_\vartheta} |\widehat{\boldsymbol{\chi}}|^2 \,\mathrm{d}\boldsymbol{x} \\ &\leq \liminf_{h\to 0^+} \int_{\Omega^\eta_\vartheta} \left|D_h\widehat{\boldsymbol{\zeta}}_h\right|^2 \,\mathrm{d}\boldsymbol{x} \leq C.\end{aligned} \tag{4.68}$$

Letting $\eta \to 0^+$ and $\vartheta \to 1^-$ in (4.67)–(4.68), we deduce that $\widehat{\boldsymbol{\zeta}} \in W^{1,2}(S;\mathbb{R}^3)$ and $\widehat{\boldsymbol{\chi}} \in L^2(\Omega;\mathbb{R}^3)$.

Set $\widehat{\boldsymbol{\mu}}_h \coloneqq \mathcal{M}_h(\widehat{\boldsymbol{q}}_h)$ and $\widehat{\boldsymbol{N}}_h \coloneqq \mathcal{N}_h(\widehat{\boldsymbol{q}}_h)$. Note that these two maps are defined on the whole space for every $h>0$. We claim that $(\widehat{\boldsymbol{\mu}}_h)$ is bounded in $L^2(\mathbb{R}^3;\mathbb{R}^3)$. To see this, making use of Proposition 1.46 and exploiting (4.20), we compute

$$
\begin{aligned}
\int_{\mathbb{R}^3} |\widehat{\boldsymbol{\mu}}_h|^2 \,\mathrm{d}\boldsymbol{x} &= \int_{\boldsymbol{\pi}_h^{-1}(\Omega^{\widehat{\boldsymbol{y}}_h})} |\widehat{\boldsymbol{m}}_h \circ \boldsymbol{\pi}_h|^2 \,\mathrm{d}\boldsymbol{x} = \frac{1}{h}\int_{\Omega^{\widehat{\boldsymbol{y}}_h}} |\widehat{\boldsymbol{m}}_h|^2 \,\mathrm{d}\boldsymbol{\xi} \\
&= \frac{1}{h}\int_{\Omega} |\widehat{\boldsymbol{m}}_h \circ \widehat{\boldsymbol{y}}_h|^2 \,\det D\widehat{\boldsymbol{y}}_h \,\mathrm{d}\boldsymbol{x} = \int_{\Omega} |\widehat{\boldsymbol{m}}_h \circ \widehat{\boldsymbol{y}}_h|^2 \,\det D_h\widehat{\boldsymbol{y}}_h \,\mathrm{d}\boldsymbol{x} \\
&= \int_{\Omega} |\widehat{\boldsymbol{m}}_h \circ \widehat{\boldsymbol{y}}_h| \,\mathrm{d}\boldsymbol{x} = \int_{\Omega} \frac{1}{\det D_h\widehat{\boldsymbol{y}}_h} \,\mathrm{d}\boldsymbol{x}.
\end{aligned}
\tag{4.69}
$$

Then, the claim follows from (4.63). We deduce the existence of $\widehat{\boldsymbol{\mu}} \in L^2(\mathbb{R}^3;\mathbb{R}^3)$ such that, up to subsequences, $\widehat{\boldsymbol{\mu}}_h \rightharpoonup \widehat{\boldsymbol{\mu}}$ in $L^2(\mathbb{R}^3;\mathbb{R}^3)$. However, by (4.60)–(4.61), there holds $\chi_{\boldsymbol{\pi}_h^{-1}(\Omega^{\widehat{\boldsymbol{y}}_h})} \to \chi_\Omega$ almost everywhere which, together with (4.66), yields $\widehat{\boldsymbol{\mu}}_h \to \chi_\Omega \widehat{\boldsymbol{\zeta}}$ almost everywhere in $\mathbb{R}^3$, as $h \to 0^+$. Also, by (4.61), the maps $\widehat{\boldsymbol{\mu}}_h$ are supported in a common compact set containing Ω for $h \ll 1$, so that $\widehat{\boldsymbol{\mu}}_h \to \chi_\Omega \widehat{\boldsymbol{\zeta}}$ in $L^1(\mathbb{R}^3;\mathbb{R}^3)$ by the Vitali Convergence Theorem. Thus, $\widehat{\boldsymbol{\mu}} = \chi_\Omega \widehat{\boldsymbol{\zeta}}$ and this establishes the weak convergence in (4.56).
To prove (4.57), we observe that

$$
E_h^{\mathrm{exc}}(\widehat{\boldsymbol{q}}_h) = \frac{1}{h}\int_{\Omega^{\widehat{\boldsymbol{y}}_h}} |D\widehat{\boldsymbol{m}}_h|^2 \,\mathrm{d}\boldsymbol{\xi} = \int_{\boldsymbol{\pi}_h^{-1}(\Omega^{\widehat{\boldsymbol{y}}_h})} |D\widehat{\boldsymbol{m}}_h|^2 \circ \boldsymbol{\pi}_h \,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^3} |\widehat{\boldsymbol{N}}_h|^2 \,\mathrm{d}\boldsymbol{x}, \tag{4.70}
$$

where we employed Proposition 1.46. In view of (4.53), this gives the boundedness of $(\widehat{\boldsymbol{N}}_h)$ in $L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3})$. To check (4.57), let $\boldsymbol{\Phi} \in L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3})$ and set $\widehat{\boldsymbol{N}} := \chi_\Omega(D'\widehat{\boldsymbol{\zeta}}, \widehat{\boldsymbol{\chi}})$. Given $\eta > 0$ and $0 < \vartheta < 1$, we write

$$
\int_{\mathbb{R}^3} (\widehat{\boldsymbol{N}}_h - \widehat{\boldsymbol{N}}) : \boldsymbol{\Phi} \,\mathrm{d}\boldsymbol{x} = \int_{\Omega^\eta_\vartheta} (\widehat{\boldsymbol{N}}_h - \widehat{\boldsymbol{N}}) : \boldsymbol{\Phi} \,\mathrm{d}\boldsymbol{x} + \int_{\mathbb{R}^3 \setminus \Omega^\eta_\vartheta} (\widehat{\boldsymbol{N}}_h - \widehat{\boldsymbol{N}}) : \boldsymbol{\Phi} \,\mathrm{d}\boldsymbol{x}. \tag{4.71}
$$

The first integral on the right-hand side of (4.71) goes to zero, as $h \to 0^+$. Indeed, by (4.60), for every $h \leq \bar{h}(\eta, \vartheta)$, we have

$$
\begin{aligned}
\int_{\Omega^\eta_\vartheta} \left(\widehat{\boldsymbol{N}}_h - \widehat{\boldsymbol{N}}\right) : \boldsymbol{\Phi} \,\mathrm{d}\boldsymbol{x} &= \int_{\Omega^\eta_\vartheta} \left(D\widehat{\boldsymbol{m}}_h \circ \boldsymbol{\pi}_h - (D'\widehat{\boldsymbol{\zeta}}, \widehat{\boldsymbol{\chi}})\right) : \boldsymbol{\Phi} \,\mathrm{d}\boldsymbol{x} \\
&= \int_{\Omega^\eta_\vartheta} \left(D_h\widehat{\boldsymbol{\zeta}}_h - (D'\widehat{\boldsymbol{\zeta}}, \widehat{\boldsymbol{\chi}})\right) : \boldsymbol{\Phi} \,\mathrm{d}\boldsymbol{x},
\end{aligned}
$$

where the right-hand side goes to zero, as $h \to 0^+$, by (4.66). The second integral on the right-hand side of (4.71) goes as well to zero, as $h \to 0^+$. Indeed, by (4.61) and by the boundedness of $(\widehat{\boldsymbol{N}}_h)$ in $L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3})$, for every $\ell > 1$ and for every $h \leq \underline{h}(\eta, \ell)$, there holds

$$
\begin{aligned}
\left| \int_{\mathbb{R}^3 \setminus \Omega^\eta_\vartheta} \left(\widehat{\boldsymbol{N}}_h - \widehat{\boldsymbol{N}}\right) : \boldsymbol{\Phi} \,\mathrm{d}\boldsymbol{x} \right| &= \left| \int_{\Omega^{-\eta}_\ell \setminus \Omega^\eta_\vartheta} \left(\widehat{\boldsymbol{N}}_h - \widehat{\boldsymbol{N}}\right) : \boldsymbol{\Phi} \,\mathrm{d}\boldsymbol{x} \right| \\
&\leq C \, ||\boldsymbol{\Phi}||_{L^2(\Omega^{-\eta}_\ell \setminus \Omega^\eta_\vartheta;\mathbb{R}^{3\times 3})}.
\end{aligned}
$$

As the right-hand side can be made arbitrarily small by properly choosing η, ϑ and ℓ according to $\boldsymbol{\Phi}$ only, this establishes (4.57).

Step 3 (Convergence of compositions). We now prove (4.58). Recall that $(1/\det D_h\widehat{\boldsymbol{y}}_h)$ is equi-integrable. Thus, in view of (4.20), so is $(\widehat{\boldsymbol{m}}_h \circ \widehat{\boldsymbol{y}}_h)$. For this reason, in order to prove the claim, it is sufficient to show that, for every $\eta > 0$ and $0 < \vartheta < 1$, we have $\widehat{\boldsymbol{m}}_h \circ \widehat{\boldsymbol{y}}_h \to \widehat{\boldsymbol{\zeta}}$ in $L^1(\Omega^\eta_\vartheta;\mathbb{R}^3)$.
Fix $\eta > 0$ and $0 < \vartheta < 1$. Recalling (4.60), we consider $h \leq \bar{h}(\eta,\vartheta)$. We have

$$\int_{\Omega^\eta_\vartheta} |\widehat{\boldsymbol{m}}_h \circ \widehat{\boldsymbol{y}}_h - \widehat{\boldsymbol{\zeta}}|\,\mathrm{d}\boldsymbol{x} \leq \int_{\Omega^\eta_\vartheta} |\widehat{\boldsymbol{m}}_h \circ \widehat{\boldsymbol{y}}_h - \widehat{\boldsymbol{\zeta}}_h|\,\mathrm{d}\boldsymbol{x} + \int_{\Omega^\eta_\vartheta} |\widehat{\boldsymbol{\zeta}}_h - \widehat{\boldsymbol{\zeta}}|\,\mathrm{d}\boldsymbol{x}. \tag{4.72}$$

By (4.66), up to subsequences, the second integral on the rigth-hand side of (4.72) goes to zero, as $h \to 0^+$. Thus, we focus on the first one and we show that, up to subsequences, this goes as well to zero, as $h \to 0^+$.
Let $\varepsilon > 0$ be arbitrary. Let also $0 < \eta_1 < \eta$ and $\vartheta < \vartheta_1 < 1$. Observe that, for $h \leq \overline{h}(\eta_1,\vartheta_1)$, the sequence $(\widehat{\boldsymbol{\zeta}}_h)$ is equi-integrable on $\Omega^{\eta_1}_{\vartheta_1}$ by (4.66). Therefore, there exists $\delta(\eta_1,\vartheta_1,\varepsilon) > 0$ such that the following property holds:

$$\begin{gathered} \forall\, A \subset \Omega^{\eta_1}_{\vartheta_1} \text{ measurable} : \mathscr{L}^3(A) < \delta(\eta_1,\vartheta_1,\varepsilon), \\ \sup_{h \leq \overline{h}(\varepsilon_1,\vartheta_1)} \int_A |\widehat{\boldsymbol{m}}_h \circ \widehat{\boldsymbol{y}}_h - \widehat{\boldsymbol{\zeta}}_h|\,\mathrm{d}\boldsymbol{x} < \varepsilon. \end{gathered} \tag{4.73}$$

Set $A^\eta_{\vartheta,h} \coloneqq \widehat{\boldsymbol{y}}_h^{-1}(\Omega^\eta_{\vartheta h})$. Let $\eta_1 < \eta_0 < \eta$ and $\vartheta < \vartheta_0 < \vartheta_1$ be such that

$$\mathscr{L}^3(\Omega^{\eta_0}_{\vartheta_0} \setminus \Omega^\eta_\vartheta) < \delta(\eta_1,\vartheta_1,\varepsilon). \tag{4.74}$$

By (4.62), for every $h \leq \widetilde{h}(\eta,\eta_0,\vartheta,\vartheta_0)$, we have $A^\eta_{\vartheta,h} \subset \Omega^{\eta_0}_{\vartheta_0}$. We write

$$\begin{aligned} \int_{\Omega^\eta_\vartheta} |\widehat{\boldsymbol{m}}_h \circ \widehat{\boldsymbol{y}}_h - \widehat{\boldsymbol{\zeta}}_h|\,\mathrm{d}\boldsymbol{x} &= \int_{\Omega^\eta_\vartheta \cap A^\eta_{\vartheta,h}} |\widehat{\boldsymbol{m}}_h \circ \widehat{\boldsymbol{y}}_h - \widehat{\boldsymbol{\zeta}}_h|\,\mathrm{d}\boldsymbol{x} \\ &\quad + \int_{\Omega^\eta_\vartheta \setminus A^\eta_{\vartheta,h}} |\widehat{\boldsymbol{m}}_h \circ \widehat{\boldsymbol{y}}_h - \widehat{\boldsymbol{\zeta}}_h|\,\mathrm{d}\boldsymbol{x} \end{aligned} \tag{4.75}$$

We focus on the second integral on the right-hand side of the previous equation. For convenience, set $\sigma_h \coloneqq \det D_h\widehat{\boldsymbol{y}}_h - 1$. Thus, $\sigma_h \to 0$ in $L^1(\Omega)$ by (4.55) since $\beta > p$. Using Proposition 1.46, we compute

$$\mathscr{L}^3(\Omega^\eta_{\vartheta h}) = \mathscr{L}^3(\widehat{\boldsymbol{y}}_h(A^\eta_{\vartheta,h})) = \int_{A^\eta_{\vartheta,h}} \det D\widehat{\boldsymbol{y}}_h\,\mathrm{d}\boldsymbol{x} = h\mathscr{L}^3(A^\eta_{\vartheta,h}) + h\int_{A^\eta_{\vartheta,h}} \sigma_h\,\mathrm{d}\boldsymbol{x}.$$

Then

$$\mathscr{L}^3(A^\eta_{\vartheta,h}) = \mathscr{L}^3(\Omega^\eta_\vartheta) - \int_{A^\eta_{\vartheta,h}} \sigma_h\,\mathrm{d}\boldsymbol{x},$$

so that

$$\begin{aligned} \mathscr{L}^3(\Omega^\eta_\vartheta \setminus A^\eta_{\vartheta,h}) \leq \mathscr{L}^3(\Omega^{\eta_0}_{\vartheta_0} \setminus A^\eta_{\vartheta,h}) &= \mathscr{L}^3(\Omega^{\eta_0}_{\vartheta_0}) - \mathscr{L}^3(A^\eta_{\vartheta,h}) \\ &= \mathscr{L}^3(\Omega^{\eta_0}_{\vartheta_0}) - \mathscr{L}^3(\Omega^\eta_\vartheta) + \int_{A^\eta_{\vartheta,h}} \sigma_h\,\mathrm{d}\boldsymbol{x}. \end{aligned}$$

Here, we exploited the inclusion $A^\eta_{\vartheta,h} \subset \Omega^{\eta_0}_{\vartheta_0}$. This yields

$$\limsup_{h\to 0^+} \mathscr{L}^3(\Omega^\eta_\vartheta \setminus A^\eta_{\vartheta,h}) \leq \mathscr{L}^3(\Omega^{\eta_0}_{\vartheta_0} \setminus \Omega^\eta_\vartheta) < \delta(\eta_1,\vartheta_1,\varepsilon),$$

where we used (4.74). Therefore, from (4.73), we obtain

$$\limsup_{h\to 0^+} \int_{\Omega^\eta_\vartheta \setminus A^\eta_{\vartheta,h}} |\widehat{\boldsymbol{m}}_h \circ \widehat{\boldsymbol{y}}_h - \widehat{\boldsymbol{\zeta}}_h| \leq \varepsilon. \tag{4.76}$$

To estimate the first integral on the right-hand side of (4.75), we proceed as follows. Without loss of generality, we can assume that Ω^η_ϑ is a Lipschitz domain. In this case, the map $\widehat{\boldsymbol{\zeta}}_h \in W^{1,2}(\Omega^\eta_\vartheta;\mathbb{R}^3)$ admits an extension $\widehat{\boldsymbol{Z}}_h \in W^{1,2}(\mathbb{R}^3;\mathbb{R}^3)$, possibly dependent on η and ϑ, which satisfies

$$\|\widehat{\boldsymbol{Z}}_h\|_{W^{1,2}(\mathbb{R}^3;\mathbb{R}^3)} \leq C(\eta,\vartheta)\, \|\widehat{\boldsymbol{\zeta}}_h\|_{W^{1,2}(\Omega^\eta_\vartheta;\mathbb{R}^3)}.$$

In particular, recalling (4.63), (4.64) and (4.65), we have

$$\|D\widehat{\boldsymbol{Z}}_h\|_{L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3})} \leq C(\eta,\vartheta) \left(\int_{\Omega^\eta_\vartheta} |\widehat{\boldsymbol{\zeta}}_h|^2 \,\mathrm{d}\boldsymbol{x} + \int_{\Omega^\eta_\vartheta} |D\widehat{\boldsymbol{\zeta}}_h|^2 \,\mathrm{d}\boldsymbol{x} \right) \leq C(\eta,\vartheta). \tag{4.77}$$

Define $\widehat{\boldsymbol{M}}_h \coloneqq \widehat{\boldsymbol{Z}}_h \circ \boldsymbol{\pi}_h^{-1}$. By construction, $\widehat{\boldsymbol{M}}_h|_{\Omega^\eta_{\vartheta h}} = \widehat{\boldsymbol{m}}_h|_{\Omega^\eta_{\vartheta h}}$. Moreover, by (4.77) and the Change-of-variable Formula, we have

$$\begin{aligned}\int_{\mathbb{R}^3} |D\widehat{\boldsymbol{M}}_h|^2 \,\mathrm{d}\boldsymbol{\xi} &= \int_{\mathbb{R}^3} |D_h\widehat{\boldsymbol{Z}}_h \circ \boldsymbol{\pi}_h^{-1}|^2 \,\mathrm{d}\boldsymbol{\xi} \leq \frac{1}{h^2}\int_{\mathbb{R}^3} |D\widehat{\boldsymbol{Z}}_h \circ \boldsymbol{\pi}_h^{-1}|^2 \,\mathrm{d}\boldsymbol{\xi} \\ &= \frac{1}{h}\int_{\mathbb{R}^3} |D\widehat{\boldsymbol{Z}}_h|^2 \,\mathrm{d}\boldsymbol{x} \leq \frac{C(\eta,\vartheta)}{h}.\end{aligned} \tag{4.78}$$

Let $\lambda > 0$. By the Lusin-type property of Sobolev maps [1], there exists a measurable set $F_{\lambda,h} \subset \mathbb{R}^3$ such that the following estimate holds:

$$\forall\, \boldsymbol{\xi}, \widehat{\boldsymbol{\xi}} \in F_{\lambda,h}, \quad |\widehat{\boldsymbol{M}}_h(\boldsymbol{\xi}) - \widehat{\boldsymbol{M}}_h(\widehat{\boldsymbol{\xi}})| \leq C\lambda\, |\boldsymbol{\xi} - \widehat{\boldsymbol{\xi}}|. \tag{4.79}$$

Thanks to (4.78), the measure of the complement of the set $F_{\lambda,h}$ is controlled as follows

$$\mathscr{L}^3(\mathbb{R}^3 \setminus F_{\lambda,h}) \leq \frac{C}{\lambda^2}\int_{|D\widehat{\boldsymbol{M}}_h| \geq \lambda/2} |D\widehat{\boldsymbol{M}}_h|^2 \,\mathrm{d}\boldsymbol{\xi} \leq \frac{C(\eta,\vartheta)}{\lambda^2 h}. \tag{4.80}$$

Going back to the first integral on the right-hand side of (4.72), setting

$$X_{\lambda,h} \coloneqq \widehat{\boldsymbol{y}}_h^{-1}(F_{\lambda,h}), \qquad Y_{\lambda,h} \coloneqq \boldsymbol{\pi}_h^{-1}(F_{\lambda,h}),$$

we split it as

$$\begin{aligned}\int_{\Omega^\eta_\vartheta \cap A^\eta_{\vartheta,h}} |\widehat{\boldsymbol{m}}_h \circ \widehat{\boldsymbol{y}}_h - \widehat{\boldsymbol{\zeta}}_h| \,\mathrm{d}\boldsymbol{x} &= \int_{(\Omega^\eta_\vartheta \cap A^\eta_{\vartheta,h}) \cap (X_{\lambda,h} \cap Y_{\lambda,h})} |\widehat{\boldsymbol{m}}_h \circ \widehat{\boldsymbol{y}}_h - \widehat{\boldsymbol{\zeta}}_h| \,\mathrm{d}\boldsymbol{x} \\ &\quad + \int_{(\Omega^\eta_\vartheta \cap A^\eta_{\vartheta,h}) \setminus (X_{\lambda,h} \cap Y_{\lambda,h})} |\widehat{\boldsymbol{m}}_h \circ \widehat{\boldsymbol{y}}_h - \widehat{\boldsymbol{\zeta}}_h| \,\mathrm{d}\boldsymbol{x}.\end{aligned} \tag{4.81}$$

We first estimate the first integral on the right-hand side of (4.81). Note that, for every $\boldsymbol{x} \in (\Omega^\eta_\vartheta \cap A^\eta_{\vartheta,h}) \cap (X_{\lambda,h} \cap Y_{\lambda,h})$, both $\widehat{\boldsymbol{y}}_h(\boldsymbol{x})$ and $\boldsymbol{\pi}_h(\boldsymbol{x})$ belong to $\Omega^\eta_{\vartheta h} \cap F_{\lambda,h}$. Thus, by (4.54) and (4.79), we have

$$\begin{aligned}&\int_{(\Omega^\eta_\vartheta \cap A^\eta_{\vartheta,h}) \cap (X_{\lambda,h} \cap Y_{\lambda,h})} |\widehat{\boldsymbol{m}}_h \circ \widehat{\boldsymbol{y}}_h - \widehat{\boldsymbol{\zeta}}_h| \,\mathrm{d}\boldsymbol{x} \\ &\qquad = \int_{(\Omega^\eta_\vartheta \cap A^\eta_{\vartheta,h}) \cap (X_{\lambda,h} \cap Y_{\lambda,h})} |\widehat{\boldsymbol{M}}_h \circ \widehat{\boldsymbol{y}}_h - \widehat{\boldsymbol{M}}_h \circ \boldsymbol{\pi}_h| \,\mathrm{d}\boldsymbol{x} \\ &\qquad \leq C\lambda ||\widehat{\boldsymbol{y}}_h - \boldsymbol{\pi}_h||_{C^0(\overline{\Omega};\mathbb{R}^3)} \leq C\lambda h^\alpha.\end{aligned} \tag{4.82}$$

For the second integral on the right-hand side of (4.81), observe that

$$(\Omega^\eta_\vartheta \cap A^\eta_{\vartheta,h}) \setminus (X_{\lambda,h} \cap Y_{\lambda,h}) \subset (\Omega^\eta_\vartheta \setminus X_{\lambda,h}) \cup (\Omega^\eta_\vartheta \setminus Y_{\lambda,h})\,. \tag{4.83}$$

By Proposition 1.46, there holds

$$\begin{aligned}\mathscr{L}^3(\widehat{\boldsymbol{y}}_h(\Omega^\eta_\vartheta) \setminus F_{\lambda,h}) = \mathscr{L}^3(\widehat{\boldsymbol{y}}_h(\Omega^\eta_\vartheta \setminus X_{\lambda,h})) &= \int_{\Omega^\eta_\vartheta \setminus X_{\lambda,h}} \det D\widehat{\boldsymbol{y}}_h \,\mathrm{d}\boldsymbol{x}\\ &= h\mathscr{L}^3(\Omega^\eta_\vartheta \setminus X_{\lambda,h}) + h\int_{\Omega^\eta_\vartheta \setminus X_{\lambda,h}} \sigma_h \,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

From this, recalling (4.80), we obtain

$$\begin{aligned}\mathscr{L}^3(\Omega^\eta_\vartheta \setminus X_{\lambda,h}) &= \frac{1}{h}\mathscr{L}^3(\widehat{\boldsymbol{y}}_h(\Omega^\eta_\vartheta) \setminus F_{\lambda,h}) + \int_{\Omega^\eta_\vartheta \setminus X_{\lambda,h}} \sigma_h \,\mathrm{d}\boldsymbol{x}\\ &\leq \frac{C(\eta,\vartheta)}{\lambda^2 h^2} + \int_{\Omega^\eta_\vartheta \setminus X_{\lambda,h}} \sigma_h \,\mathrm{d}\boldsymbol{x}.\end{aligned} \tag{4.84}$$

Instead, using Proposition 1.46 together with (4.80), we estimate

$$\mathscr{L}^3(\Omega^\eta_\vartheta \setminus Y_{\lambda,h}) = \mathscr{L}^3(\boldsymbol{\pi}_h^{-1}(\Omega^\eta_{\vartheta h} \setminus F_{\lambda,h})) = \frac{1}{h}\mathscr{L}^3(\Omega^\eta_{\vartheta h} \setminus F_{\lambda,h}) \leq \frac{C(\eta,\vartheta)}{\lambda^2 h^2}. \tag{4.85}$$

Now, recalling (4.156), we choose $\lambda = h^{-b}$ for some $1 < b < \alpha$. In this case, the bound in (4.82) yields

$$\lim_{h\to 0^+} \int_{(\Omega^\eta_\vartheta \cap A^\eta_{\vartheta,h}) \cap (X_{\lambda,h} \cap Y_{\lambda,h})} |\widehat{\boldsymbol{m}}_h \circ \widehat{\boldsymbol{y}}_h - \widehat{\boldsymbol{\zeta}}_h| \,\mathrm{d}\boldsymbol{x} = 0. \tag{4.86}$$

Also, from (4.83)–(4.85), we obtain

$$\lim_{h\to 0^+} \mathscr{L}^3((\Omega^\eta_\vartheta \cap A^\eta_{\vartheta,h}) \setminus (X_{\lambda,h} \cap Y_{\lambda,h})) = 0,$$

which, by (4.73), entails

$$\limsup_{h\to 0^+} \int_{(\Omega^\eta_\vartheta \cap A^\eta_{\vartheta,h}) \setminus (X_{\lambda,h} \cap Y_{\lambda,h})} |\widehat{\boldsymbol{m}}_h \circ \widehat{\boldsymbol{y}}_h - \widehat{\boldsymbol{\zeta}}_h| \,\mathrm{d}\boldsymbol{x} \leq \varepsilon. \tag{4.87}$$

Therefore, combining (4.76) and (4.86)–(4.87), we obtain

$$\limsup_{h\to 0^+} \int_{\Omega^\eta_\vartheta} |\widehat{\boldsymbol{m}}_h \circ \widehat{\boldsymbol{y}}_h - \widehat{\boldsymbol{\zeta}}_h| \,\mathrm{d}\boldsymbol{x} \leq 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this shows that the first integral on the right-hand side of (4.72) goes to zero, as $h \to 0^+$.

Step 4 (Improved convergences). We are left to prove the strong convergences in (4.56) and (4.59). By (4.55), there holds $\widehat{\boldsymbol{F}}_h \to \boldsymbol{I}$ in $L^p(\Omega;\mathbb{R}^{3\times 3})$. Thus, up to subsequences, we have $\det \widehat{\boldsymbol{F}}_h \to 1$ almost everywhere. As $(1/\det D_h\widehat{\boldsymbol{y}}_h)$ is equi-integrable by (4.63), this yields

$$\frac{1}{\det D_h \widehat{\boldsymbol{y}}_h} \to 1 \text{ in } L^1(\Omega) \tag{4.88}$$

by the Vitali Convergence Theorem. By (4.58), up to subsequences, we also have $\widehat{\boldsymbol{m}}_h \circ \widehat{\boldsymbol{y}}_h \to \widehat{\boldsymbol{\zeta}}$ almost everywhere. Taking into account (4.20), this gives (4.59) by the Dominated Convergence Theorem. From (4.20), passing to the limit, as $h \to 0^+$, we deduce that $|\widehat{\boldsymbol{\zeta}}| = 1$ almost everywhere in Ω. Equivalently, $\widehat{\boldsymbol{\zeta}} \in W^{1,2}(S;\mathbb{S}^2)$. Finally, passing to the limit, as $h \to 0^+$, in (4.69) by exploiting (4.88), we obtain

$$\lim_{h \to 0^+} \int_{\mathbb{R}^3} |\widehat{\boldsymbol{\mu}}_h|^2 \,\mathrm{d}\boldsymbol{\xi} = \mathscr{L}^3(\Omega) = \int_{\mathbb{R}^3} |\chi_\Omega \widehat{\boldsymbol{\zeta}}|^2 \,\mathrm{d}\boldsymbol{\xi}.$$

Given that $\widehat{\boldsymbol{\mu}}_h \rightharpoonup \chi_\Omega \widehat{\boldsymbol{\zeta}}$ in $L^2(\mathbb{R}^3;\mathbb{R}^3)$, this proves the strong convergence in (4.56). □

4.2.2 Lower bound

We move to the the proof of the lower bound. For convenience, we highlight the results regarding the convergence of the magnetostatic energy. This is established by adapting the results in [34, 70].

Proposition 4.10 (Convergence of the magnetostatic energy). *Let $(\widehat{\boldsymbol{q}}_h)$ with $\widehat{\boldsymbol{q}}_h = (\widehat{\boldsymbol{y}}_h, \widehat{\boldsymbol{m}}_h) \in \mathcal{Q}_h$. Suppose that there exists $\widehat{\boldsymbol{\zeta}} \in W^{1,2}(S;\mathbb{S}^2)$ such that the following convergence holds, as $h \to 0^+$:*

$$\widehat{\boldsymbol{\mu}}_h \coloneqq \mathcal{M}_h(\widehat{\boldsymbol{q}}_h) \to \chi_\Omega \widehat{\boldsymbol{\zeta}} \text{ in } L^2(\mathbb{R}^3;\mathbb{R}^3). \tag{4.89}$$

Then, the following equality holds:

$$E_0^{\mathrm{mag}}(\widehat{\boldsymbol{\zeta}}) = \lim_{h \to 0^+} E_h^{\mathrm{mag}}(\widehat{\boldsymbol{q}}_h). \tag{4.90}$$

Proof. Denote by $\widehat{\psi}_h \in V^{1,2}(\mathbb{R}^3)$ a stray field potential corresponding to $\widehat{\boldsymbol{q}}_h$. Thus, we have the following:

$$\forall\, \varphi \in V^{1,2}(\mathbb{R}^3), \qquad \int_{\mathbb{R}^3} D\widehat{\psi}_h \cdot D\varphi \,\mathrm{d}\boldsymbol{\xi} = \int_{\mathbb{R}^3} \chi_{\Omega^{\widehat{\boldsymbol{y}}_h}} \widehat{\boldsymbol{m}}_h \cdot D\varphi \,\mathrm{d}\boldsymbol{\xi}. \tag{4.91}$$

By (2.12), there holds

$$\|D\widehat{\psi}_h\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \leq \|\chi_{\Omega^{\widehat{\boldsymbol{y}}_h}} \widehat{\boldsymbol{m}}_h\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}.$$

Then, taking the square at both sides and applying Proposition 1.46, we estimate

$$\int_{\mathbb{R}^3} |D\widehat{\psi}_h|^2 \,\mathrm{d}\boldsymbol{\xi} \leq \int_{\mathbb{R}^3} |\chi_{\Omega^{\widehat{\boldsymbol{y}}_h}} \widehat{\boldsymbol{m}}_h|^2 \,\mathrm{d}\boldsymbol{\xi} = \int_{\mathbb{R}^3} |\widehat{\boldsymbol{\mu}}_h|^2 \circ \boldsymbol{\pi}_h^{-1} \,\mathrm{d}\boldsymbol{\xi} = h \int_{\mathbb{R}^3} |\widehat{\boldsymbol{\mu}}_h|^2 \,\mathrm{d}\boldsymbol{\xi}. \tag{4.92}$$

Define $\widehat{\rho}_h \coloneqq \widehat{\psi}_h \circ \boldsymbol{\pi}_h \in V^{1,2}(\mathbb{R}^3)$. From (4.92), using again Proposition 1.46, we estimate

$$\int_{\mathbb{R}^3} |D_h \widehat{\rho}_h|^2 \,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^3} |D\widehat{\psi}_h|^2 \circ \boldsymbol{\pi}_h \,\mathrm{d}\boldsymbol{x} = \frac{1}{h} \int_{\mathbb{R}^3} |D\widehat{\psi}_h|^2 \,\mathrm{d}\boldsymbol{\xi} \leq \int_{\mathbb{R}^3} |\widehat{\boldsymbol{\mu}}_h|^2 \,\mathrm{d}\boldsymbol{\xi}.$$

As the right-hand side is uniformly bounded in view (4.89), we deduce that $(D_h \widehat{\rho}_h)$ is bounded in $L^2(\mathbb{R}^3;\mathbb{R}^3)$. From this, we deduce two facts. First, there exists $\widehat{\chi} \in L^2(\mathbb{R}^3)$ such that, up to subsequences, $\partial_3 \widehat{\rho}_h / h \rightharpoonup \widehat{\chi}$ in $L^2(\mathbb{R}^3)$ and, in turn, $\partial_3 \widehat{\rho}_h \to 0$ in $L^2(\mathbb{R}^3)$, as $h \to 0^+$. Second, exploiting the Hilbert space structure of the quotient

$V^{1,2}(\mathbb{R}^3)/\mathbb{R}$, we prove the existence of $\widehat{\rho} \in V^{1,2}(\mathbb{R}^3)$ such that, up to subsequences, there holds $D\widehat{\rho}_h \rightharpoonup D\widehat{\rho}$ in $L^2(\mathbb{R}^3;\mathbb{R}^3)$, as $h \to 0^+$. These two facts together imply that $\partial_3\widehat{\rho} = 0$ almost everywhere which, as $D\widehat{\rho} \in L^2(\mathbb{R}^3;\mathbb{R}^3)$, yields $D'\widehat{\rho} = \mathbf{0}'$ almost everywhere.

Now, testing again (4.91) with $\varphi = \widehat{\psi}_h$ and applying Proposition 1.46, we write

$$\begin{aligned} E_h^{\mathrm{mag}}(\widehat{\boldsymbol{q}}_h) &= \frac{1}{2h}\int_{\mathbb{R}^3} \chi_{\Omega^{\widehat{\boldsymbol{y}}_h}} \widehat{\boldsymbol{m}}_h \cdot D\widehat{\psi}_h \,\mathrm{d}\boldsymbol{\xi} \\ &= \frac{1}{2}\int_{\mathbb{R}^3} \widehat{\boldsymbol{\mu}}_h \cdot D_h\widehat{\rho}_h \,\mathrm{d}\boldsymbol{x} \\ &= \frac{1}{2}\int_{\mathbb{R}^3} \widehat{\boldsymbol{\mu}}_h' \cdot D'\widehat{\rho}_h \,\mathrm{d}\boldsymbol{x} + \frac{1}{2}\int_{\mathbb{R}^3} \widehat{\boldsymbol{\mu}}_h^3 \frac{\partial_3\widehat{\rho}_h}{h}\,\mathrm{d}\boldsymbol{x} \end{aligned}$$

Then, by (4.89), passing to the limit, we obtain

$$\lim_{h\to 0^+} E_h^{\mathrm{mag}}(\widehat{\boldsymbol{q}}_h) = \frac{1}{2}\int_\Omega \widehat{\zeta}^3\,\widehat{\chi}\,\mathrm{d}\boldsymbol{x}. \tag{4.93}$$

Thus, if we show that $\widehat{\chi} = \chi_\Omega\widehat{\zeta}^3$ almost everywhere in $\mathbb{R}^3$, then (4.90) follows from (4.93). To check this, we go back to (4.91). Using Proposition 1.46 once more, we deduce the following

$$\forall\,\varphi \in V^{1,2}(\mathbb{R}^3), \qquad \int_{\mathbb{R}^3} D_h\widehat{\rho}_h \cdot D_h\varphi\,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^3} \widehat{\boldsymbol{\mu}}_h \cdot D_h\varphi\,\mathrm{d}\boldsymbol{x}.$$

Then, multiplying by h and then passing to the limit, as $h \to 0^+$, we obtain

$$\forall\,\varphi \in V^{1,2}(\mathbb{R}^3), \qquad \int_{\mathbb{R}^3} \left(\widehat{\chi} - \chi_\Omega\widehat{\zeta}^3\right)\partial_3\varphi\,\mathrm{d}\boldsymbol{x} = 0.$$

Given the arbitrariness of φ, this entails that the function $\widehat{\chi} - \chi_\Omega\widehat{\zeta}^3$ does not depend on the third variable. However, as this function belongs to $L^2(\mathbb{R}^3)$, we necessarily have $\widehat{\chi} - \chi_\Omega\widehat{\zeta}^3 = 0$ almost everywhere. □

The next result asserts the existence of a lower bound and, for future reference, it is presented in a more self-contained form.

Proposition 4.11 (Lower bound). *Let $(\widehat{\boldsymbol{q}}_h)$ with $\widehat{\boldsymbol{q}}_h = (\widehat{\boldsymbol{y}}_h, \widehat{\boldsymbol{m}}_h) \in \mathcal{Q}_h$. Set $\widehat{\boldsymbol{F}}_h :=$ $D_h\widehat{\boldsymbol{y}}_h$. Suppose that there exist $(\widehat{\boldsymbol{R}}_h) \subset W^{1,2}(S;SO(3))$ and also $\widehat{\boldsymbol{G}} \in L^2(\Omega;\mathbb{R}^{3\times 3})$, $\widehat{\boldsymbol{u}} \in W^{1,2}(S;\mathbb{R}^2)$ and $\widehat{\boldsymbol{v}} \in W^{2,2}(S)$ such that, as $h \to 0^+$, there hold*

$$\widehat{\boldsymbol{R}}_h \to \boldsymbol{I} \ \text{in } L^q(S;\mathbb{R}^{3\times 3}) \text{ for every } 1 \le q < \infty, \tag{4.94}$$

$$\widehat{\boldsymbol{G}}_h := h^{-\beta/2}(\widehat{\boldsymbol{R}}_h^\top \widehat{\boldsymbol{F}}_h - \boldsymbol{I}) \rightharpoonup \widehat{\boldsymbol{G}} \ \text{in } L^2(\Omega;\mathbb{R}^{3\times 3}), \tag{4.95}$$

and, for almost every $\boldsymbol{x} \in \Omega$, we have

$$\widehat{\boldsymbol{G}}''(\boldsymbol{x}) = \mathrm{sym}D'\widehat{\boldsymbol{u}}(\boldsymbol{x}') + ((D')^2\widehat{v}(\boldsymbol{x}'))\,x_3. \tag{4.96}$$

Moreover, suppose that there exist $\widehat{\boldsymbol{\zeta}} \in W^{1,2}(S;\mathbb{S}^2)$ and $\widehat{\boldsymbol{\chi}} \in L^2(\mathbb{R}^3;\mathbb{R}^3)$ such that the following convergences hold, as $h \to 0^+$:

$$\widehat{\boldsymbol{\mu}}_h := \mathcal{M}_h(\widehat{\boldsymbol{q}}_h) \to \chi_\Omega\widehat{\boldsymbol{\zeta}} \ \text{in } L^2(\mathbb{R}^3;\mathbb{R}^3); \tag{4.97}$$

$$\widehat{\boldsymbol{N}}_h := \mathcal{N}_h(\widehat{\boldsymbol{q}}_h) \rightharpoonup \chi_\Omega(D'\widehat{\boldsymbol{\zeta}}|\widehat{\boldsymbol{\chi}}) \ \text{in } L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3}); \tag{4.98}$$

$$\widehat{\boldsymbol{z}}_h := \mathcal{Z}_h(\widehat{\boldsymbol{q}}_h) \to \widehat{\boldsymbol{\zeta}} \ \text{in } L^1(\Omega;\mathbb{R}^3). \tag{4.99}$$

Then, the following inequality holds:

$$E_0(\widehat{\boldsymbol{u}}, \widehat{v}, \widehat{\boldsymbol{\zeta}}) \leq \liminf_{h\to 0^+} E_h(\widehat{\boldsymbol{q}}_h). \tag{4.100}$$

Proof. It is sufficient to prove the following:

$$E_0^{\mathrm{el}}(\widehat{\boldsymbol{u}}, \widehat{v}, \widehat{\boldsymbol{\zeta}}) \leq \liminf_{h\to 0^+} E_h^{\mathrm{el}}(\widehat{\boldsymbol{q}}_h); \tag{4.101}$$

$$E_0^{\mathrm{exc}}(\widehat{\boldsymbol{\zeta}}) \leq \liminf_{h\to 0^+} E_h^{\mathrm{exc}}(\widehat{\boldsymbol{q}}_h). \tag{4.102}$$

Indeed, thanks to (4.97), the limit in (4.90) holds by Proposition 4.10. Thus, combining (4.101)–(4.102) with (4.90), we obtain (4.100).
Recalling (4.70), claim (4.102) follows immediately from (4.98). Indeed, by lower semicontinuity, we have

$$\liminf_{h\to 0^+} \int_{\mathbb{R}^3} |\widehat{\boldsymbol{N}}_h|^2 \,\mathrm{d}\boldsymbol{x} \geq \int_\Omega |D'\widehat{\boldsymbol{\zeta}}|^2 \,\mathrm{d}\boldsymbol{x} + \int_\Omega |\widehat{\boldsymbol{\chi}}|^2 \,\mathrm{d}\boldsymbol{x} \geq \int_S |D'\widehat{\boldsymbol{\zeta}}|^2 \,\mathrm{d}\boldsymbol{x}'.$$

We thus focus on (4.101). This is proved similarly to [65, Corollary 2]. Recall (4.3)–(4.7). For simplicity, set $\widehat{\boldsymbol{\lambda}}_h := \widehat{\boldsymbol{m}}_h \circ \widehat{\boldsymbol{y}}_h$and $\widehat{\boldsymbol{L}}_h := \boldsymbol{L}_h(\widehat{\boldsymbol{R}}_h^\top \widehat{\boldsymbol{z}}_h)$. By (4.94) and (4.99), recalling (4.5) and (4.20), we have

$$\widehat{\boldsymbol{K}}_h := h^{-\beta/2}(\boldsymbol{I} - \widehat{\boldsymbol{L}}_h^{-1}) \to c_0 \widehat{\boldsymbol{\zeta}} \otimes \widehat{\boldsymbol{\zeta}} \text{ in } L^q(\Omega;\mathbb{R}^{3\times 3}) \text{ for every } 1 \leq q < \infty. \tag{4.103}$$

Define $A_h := \{|\widehat{\boldsymbol{G}}_h| \leq h^{-\beta/4}\}$, so that, by (4.95), $\chi_{A_h} \to 1$ in $L^1(\Omega)$. Note that, on A_h, there holds

$$\widehat{\boldsymbol{L}}_h^{-1} \widehat{\boldsymbol{R}}_h^\top \widehat{\boldsymbol{F}}_h = \boldsymbol{I} + h^{\beta/2}(\widehat{\boldsymbol{G}}_h - \widehat{\boldsymbol{K}}_h) + O(h^{3/4\beta}). \tag{4.104}$$

Recalling (4.16), for $h \ll 1$ we have $|\widehat{\boldsymbol{L}}_h^{-1} \widehat{\boldsymbol{R}}_h^\top \widehat{\boldsymbol{F}}_h - \boldsymbol{I}| \ll 1$ on A_h. Then, exploiting (4.14) and (4.104), we write

$$\begin{aligned}
\int_\Omega W_h(\widehat{\boldsymbol{F}}_h, \widehat{\boldsymbol{\lambda}}_h) \,\mathrm{d}\boldsymbol{x} &= \int_\Omega W_h(\widehat{\boldsymbol{R}}_h^\top \widehat{\boldsymbol{F}}_h, \widehat{\boldsymbol{R}}_h^\top \widehat{\boldsymbol{\lambda}}_h) \,\mathrm{d}\boldsymbol{x} = \int_\Omega \Phi(\widehat{\boldsymbol{L}}_h^{-1} \widehat{\boldsymbol{R}}_h^\top \widehat{\boldsymbol{F}}_h) \,\mathrm{d}\boldsymbol{x} \\
&\geq \int_\Omega \chi_{A_h} \Phi(\widehat{\boldsymbol{L}}_h^{-1} \widehat{\boldsymbol{R}}_h^\top \widehat{\boldsymbol{F}}_h) \,\mathrm{d}\boldsymbol{x} \\
&= \int_\Omega \chi_{A_h} \Phi\left(\boldsymbol{I} + h^{\beta/2}(\widehat{\boldsymbol{G}}_h - \widehat{\boldsymbol{K}}_h) + O(h^{3\beta/4})\right) \,\mathrm{d}\boldsymbol{x} \\
&= \frac{h^\beta}{2} \int_\Omega Q\left(\chi_{A_h}(\widehat{\boldsymbol{G}}_h - \widehat{\boldsymbol{K}}_h) + O(h^{3\beta/4})\right) \,\mathrm{d}\boldsymbol{x} \\
&+ \int_\Omega \chi_{A_h} \omega\left(h^{\beta/2}(\widehat{\boldsymbol{G}}_h - \widehat{\boldsymbol{K}}_h) + O(h^{3\beta/4})\right) \,\mathrm{d}\boldsymbol{x}.
\end{aligned}$$

Thus

$$\begin{aligned}
E_h^{\mathrm{el}}(\widehat{\boldsymbol{q}}_h) &\geq \frac{1}{2} \int_\Omega Q\left(\chi_{A_h}(\operatorname{sym} \widehat{\boldsymbol{G}}_h - \widehat{\boldsymbol{K}}_h) + O(h^{3\beta/4})\right) \,\mathrm{d}\boldsymbol{x} \\
&+ \frac{1}{h^\beta} \int_\Omega \chi_{A_h} \omega\left(h^{\beta/2}(\operatorname{sym} \widehat{\boldsymbol{G}}_h - \widehat{\boldsymbol{K}}_h) + O(h^{3\beta/4})\right) \,\mathrm{d}\boldsymbol{x}.
\end{aligned} \tag{4.105}$$

For the first integral on the right-hand side of (4.105), we exploit the convexity of Q. By (4.95) and (4.103), we have $\chi_{A_h}\widehat{\boldsymbol{G}}_h \rightharpoonup \widehat{\boldsymbol{G}}$ in $L^2(\Omega;\mathbb{R}^{3\times 3})$ and $\chi_{A_h}\widehat{\boldsymbol{K}}_h \to c_0\widehat{\boldsymbol{\zeta}} \otimes \widehat{\boldsymbol{\zeta}}$ in $L^q(\Omega;\mathbb{R}^{3\times 3})$ for every $1 \leq q < \infty$, as $h \to 0^+$. Thus, by lower semicontinuity, we get

$$\begin{aligned}\liminf_{h\to 0^+}\int_\Omega Q\left(\chi_{A_h}(\widehat{\boldsymbol{G}}_h - \widehat{\boldsymbol{K}}_h) + O(h^{3\beta/4})\right)\mathrm{d}\boldsymbol{x} &\geq \int_\Omega Q(\widehat{\boldsymbol{G}} - c_0\widehat{\boldsymbol{\zeta}} \otimes \widehat{\boldsymbol{\zeta}})\,\mathrm{d}\boldsymbol{x}\\ &\geq \int_\Omega Q_{\mathrm{red}}(\widehat{\boldsymbol{G}}'' - c_0\widehat{\boldsymbol{\zeta}}' \otimes \widehat{\boldsymbol{\zeta}}')\,\mathrm{d}\boldsymbol{x},\end{aligned}$$

where the last inequality is justified by the definition of Q_{red}. Given (4.96), this proves (4.101) once we show that the second integral on the right-hand side of (4.105) goes to zero, as $h \to 0^+$. To prove this, for every $r > 0$, we set

$$\overline{\omega}(r) \coloneqq \sup_{\boldsymbol{F}\in\mathbb{R}^{3\times 3}\colon |\boldsymbol{F}|\leq r} \frac{|\omega(\boldsymbol{F})|}{|\boldsymbol{F}|^2}.$$

By definition, the function $\overline{\omega}$ is decreasing and satisfies $\overline{\omega}(r) \to 0$, as $r \to 0^+$. Therefore, we have

$$\begin{aligned}&\frac{1}{h^\beta}\int_\Omega \chi_{A_h}\left|\omega\left(h^{\beta/2}(\widehat{\boldsymbol{G}}_h - \widehat{\boldsymbol{K}}_h) + O(h^{3\beta/4})\right)\right|\mathrm{d}\boldsymbol{x}\\ &\leq \int_\Omega \chi_{A_h}\overline{\omega}\left(h^{\beta/2}|\widehat{\boldsymbol{G}}_h - \widehat{\boldsymbol{K}}_h| + O(h^{3\beta/4})\right)\frac{\left|h^{\beta/2}(\widehat{\boldsymbol{G}}_h - \widehat{\boldsymbol{K}}_h) + O(h^{3\beta/4})\right|^2}{h^\beta}\,\mathrm{d}\boldsymbol{x}\\ &\leq \overline{\omega}(O(h^{\beta/4}))\int_\Omega \left|\widehat{\boldsymbol{G}}_h - \widehat{\boldsymbol{K}}_h + O(h^{\beta/4})\right|^2\mathrm{d}\boldsymbol{x},\end{aligned}$$

where the right-hand side goes to zero, as $h \to 0^+$. □

4.2.3 Optimality of the lower bound

The existence of recovery sequences in ensured by the following result.

Proposition 4.12 (Recovery sequence). *Let $\widehat{\boldsymbol{q}}_0 = (\widehat{\boldsymbol{u}}, \widehat{v}, \widehat{\boldsymbol{\zeta}}) \in \mathcal{Q}_0$. Then, there exists $(\widehat{\boldsymbol{q}}_h)$ with $\widehat{\boldsymbol{q}}_h = (\widehat{\boldsymbol{y}}_h, \widehat{\boldsymbol{m}}_h) \in \mathcal{Q}_h$ such that the following convergences hold, as $h \to 0^+$:*

$$\widehat{\boldsymbol{y}}_h \to \boldsymbol{\pi}_0 \text{ in } W^{1,p}(\Omega;\mathbb{R}^3); \tag{4.106}$$

$$\widehat{\boldsymbol{F}}_h \coloneqq D_h\widehat{\boldsymbol{y}}_h \to \boldsymbol{I} \text{ in } L^p(\Omega;\mathbb{R}^{3\times 3}); \tag{4.107}$$

$$\widehat{\boldsymbol{u}}_h \coloneqq \mathcal{U}_h(\widehat{\boldsymbol{q}}_h) \to \widehat{\boldsymbol{u}} \text{ in } W^{1,2}(S;\mathbb{R}^3); \tag{4.108}$$

$$\widehat{v}_h \coloneqq \mathcal{V}_h(\widehat{\boldsymbol{q}}_h) \to \widehat{v} \text{ in } W^{2,2}(S); \tag{4.109}$$

$$\widehat{\boldsymbol{w}}_h \coloneqq \mathcal{W}_h(\widehat{\boldsymbol{q}}_h) \to -1/12\begin{pmatrix} D'\widehat{v} \\ 0 \end{pmatrix} \text{ in } W^{2,2}(S); \tag{4.110}$$

$$\widehat{\boldsymbol{\mu}}_h \coloneqq \mathcal{M}_h(\widehat{\boldsymbol{q}}_h) \to \chi_\Omega\widehat{\boldsymbol{\zeta}} \text{ in } L^2(\mathbb{R}^3;\mathbb{R}^3); \tag{4.111}$$

$$\widehat{\boldsymbol{N}}_h \coloneqq \mathcal{N}_h(\widehat{\boldsymbol{q}}_h) \to \chi_\Omega(D'\widehat{\boldsymbol{\zeta}}|\boldsymbol{0}) \text{ in } L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3}); \tag{4.112}$$

$$\widehat{\boldsymbol{\lambda}}_h \coloneqq \widehat{\boldsymbol{m}}_h \circ \widehat{\boldsymbol{y}}_h \to \widehat{\boldsymbol{\zeta}} \text{ in } L^1(\Omega;\mathbb{R}^3); \tag{4.113}$$

$$\widehat{\boldsymbol{z}}_h \coloneqq \mathcal{Z}_h(\widehat{\boldsymbol{q}}_h) \to \widehat{\boldsymbol{\zeta}} \text{ in } L^q(\Omega;\mathbb{R}^3) \text{ for every } 1 \leq q < \infty. \tag{4.114}$$

Moreover, the following equality holds:

$$E_0(\widehat{\boldsymbol{u}}, \widehat{v}, \widehat{\boldsymbol{\zeta}}) = \lim_{h\to 0^+} E_h(\widehat{\boldsymbol{q}}_h). \tag{4.115}$$

Proof. For convenience of the reader, the proof is subdivided into three steps.

Step 1 (Construction of the recovery sequence). First, we additionally assume that $\widehat{\boldsymbol{u}} \in W^{2,\infty}(S;\mathbb{R}^2)$, $\widehat{v} \in W^{3,\infty}(S)$ and $\widehat{\boldsymbol{\zeta}} \in C^1(\overline{S};\mathbb{S}^2)$. Also, let $\widehat{\boldsymbol{a}}, \widehat{\boldsymbol{b}} \in C^2_{\mathrm{c}}(S;\mathbb{R}^3)$. Deformations of the recovery sequence are constructed according to the classical ansatz of the linearized von Kármán regime [65]. That is, for every $h > 0$, we define

$$\begin{aligned} \widehat{\boldsymbol{y}}_h \coloneqq \boldsymbol{\pi}_h + h^{\beta/2} \begin{pmatrix} \widehat{\boldsymbol{u}} \\ 0 \end{pmatrix} + h^{\beta/2-1} \begin{pmatrix} \boldsymbol{0}' \\ \widehat{v} \end{pmatrix} - h^{\beta/2} x_3 \begin{pmatrix} D'\widehat{v} \\ 0 \end{pmatrix} \\ +2h^{\beta/2+1} x_3 \widehat{\boldsymbol{a}} + h^{\beta/2+1} x_3^2 \widehat{\boldsymbol{b}}. \end{aligned} \tag{4.116}$$

Thus $\widehat{\boldsymbol{y}} \in W^{2,\infty}(\Omega;\mathbb{R}^3)$ and there holds:

$$\|\widehat{\boldsymbol{y}}_h - \boldsymbol{\pi}_h\|_{C^0(\overline{\Omega};\mathbb{R}^3)} \leq C h^{\beta/2-1}. \tag{4.117}$$

From this, as $\beta > 4$, we establish (4.60)–(4.61) by arguing as in Step 1 of the proof of Proposition 4.9. We compute

$$\begin{aligned} \widehat{\boldsymbol{F}}_h = \boldsymbol{I} + h^{\beta/2} \left(\begin{array}{c|c} D'\widehat{\boldsymbol{u}} & \boldsymbol{0}' \\ \hline (\boldsymbol{0}')^\top & 0 \end{array} \right) + h^{\beta/2-1} \left(\begin{array}{c|c} \boldsymbol{O}'' & -D'\widehat{v} \\ \hline (D'v)^\top & 0 \end{array} \right) \\ - h^{\beta/2} x_3 \left(\begin{array}{c|c} (D')^2\widehat{v} & \boldsymbol{0}' \\ \hline (\boldsymbol{0}')^\top & 0 \end{array} \right) + 2h^{\beta/2} \widehat{\boldsymbol{a}} \otimes \boldsymbol{e}_3 + 2h^{\beta/2} x_3 \widehat{\boldsymbol{b}} \otimes \boldsymbol{e}_3 + O(h^{\beta/2+1}), \end{aligned} \tag{4.118}$$

so that we immediately deduce

$$\|\widehat{\boldsymbol{F}}_h - \boldsymbol{I}\|_{C^0(\overline{\Omega};\mathbb{R}^{3\times 3})} \leq C h^{\beta/2-1}. \tag{4.119}$$

Recall the identity $(\boldsymbol{I}+\boldsymbol{F})^\top(\boldsymbol{I}+\boldsymbol{F}) = \boldsymbol{I} + 2\,\mathrm{sym}\,\boldsymbol{F} + \boldsymbol{F}^\top \boldsymbol{F}$ for every $\boldsymbol{F} \in \mathbb{R}^{3\times 3}$. Thanks to the assumption $\beta > 6$, we have

$$\sqrt{\left(\widehat{\boldsymbol{F}}_h\right)^\top \widehat{\boldsymbol{F}}_h} = \boldsymbol{I} + h^{\beta/2}(\widehat{\boldsymbol{D}} + x_3\,\widehat{\boldsymbol{E}}) + O(h^{\beta/2+1}), \tag{4.120}$$

where

$$\begin{aligned} \widehat{\boldsymbol{D}} &\coloneqq \left(\begin{array}{c|c} \mathrm{sym} D'\widehat{\boldsymbol{u}} & \boldsymbol{0}' \\ \hline (\boldsymbol{0}')^\top & 0 \end{array} \right) + \widehat{\boldsymbol{a}} \otimes \boldsymbol{e}_3 + \boldsymbol{e}_3 \otimes \widehat{\boldsymbol{a}}, \\ \widehat{\boldsymbol{E}} &\coloneqq - \left(\begin{array}{c|c} (D')^2\widehat{v} & \boldsymbol{0}' \\ \hline (\boldsymbol{0}')^\top & 0 \end{array} \right) + \widehat{\boldsymbol{b}} \otimes \boldsymbol{e}_3 + \boldsymbol{e}_3 \otimes \widehat{\boldsymbol{b}}. \end{aligned}$$

Using the identity $\det(\boldsymbol{I}+\boldsymbol{F}) = 1 + \mathrm{tr}\boldsymbol{F} + \mathrm{tr}\,(\mathrm{cof}\boldsymbol{F}) + \det \boldsymbol{F}$ for every $\boldsymbol{F} \in \mathbb{R}^{3\times 3}$, we obtain

$$\det \widehat{\boldsymbol{F}}_h = 1 + O(h^{\beta/2}), \qquad D(\det \widehat{\boldsymbol{F}}_h) = O(h^{\beta/2}).$$

In particular, this ensures that

$$1/2 < \det \widehat{\boldsymbol{F}}_h < 3/2 \text{ a.e. in } \Omega, \tag{4.121}$$

at least for $h \ll 1$. We claim that the map $\widehat{\boldsymbol{y}}_h$ is injective, that is, $\widehat{\boldsymbol{y}}_h \in \mathcal{Y}_p^{\mathrm{inj}}(\Omega)$ for $h \ll 1$. To see this, we argue as in [37, Theorem 5.5-1, Claim (b)]. Fix $\boldsymbol{x}, \widehat{\boldsymbol{x}} \in \Omega$ with $\boldsymbol{x} \neq \widehat{\boldsymbol{x}}$. By [37, Exercise 1.9], there exists a finite number of distinct points $\widetilde{\boldsymbol{x}}_1, \dots, \widetilde{\boldsymbol{x}}_m \in \Omega$ with $\widetilde{\boldsymbol{x}}_1 = \boldsymbol{x}$ and $\widetilde{\boldsymbol{x}}_m = \widehat{\boldsymbol{x}}$ such that each segment connecting $\widetilde{\boldsymbol{x}}_i$ to $\widetilde{\boldsymbol{x}}_{i+1}$ is entirely containted in Ω and $\sum_{i=1}^m |\widetilde{\boldsymbol{x}}_{i+1} - \widetilde{\boldsymbol{x}}_i| \leq C|\boldsymbol{x} - \widehat{\boldsymbol{x}}|$ for some constant $C > 0$ depending on Ω only. Then, applying the Mean Value Theorem in combination with (4.119), we estimate

$$\begin{aligned}
&|\widehat{\boldsymbol{y}}_h(\boldsymbol{x}) - \widehat{\boldsymbol{y}}_h(\widehat{\boldsymbol{x}}) - (\boldsymbol{\pi}_h(\boldsymbol{x}) - \boldsymbol{\pi}_h(\widehat{\boldsymbol{x}}))| \\
&\qquad \leq \sum_{i=1}^m |(\widehat{\boldsymbol{y}}_h(\widetilde{\boldsymbol{x}}_{i+1}) - \boldsymbol{\pi}_h(\widetilde{\boldsymbol{x}}_{i+1})) - (\widehat{\boldsymbol{y}}_h(\widetilde{\boldsymbol{x}}_i) - \boldsymbol{\pi}_h(\widetilde{\boldsymbol{x}}_i))| \\
&\qquad \leq \|D\widehat{\boldsymbol{y}}_h - D\boldsymbol{\pi}_h\|_{C^0(\overline{\Omega};\mathbb{R}^{3\times 3})} \sum_{i=1}^m |\widetilde{\boldsymbol{x}}_{i+1} - \widetilde{\boldsymbol{x}}_i| \\
&\qquad \leq Ch^{\beta/2-1}|\boldsymbol{x} - \widehat{\boldsymbol{x}}| \leq Ch^{\beta/2-2}|\boldsymbol{\pi}_h(\boldsymbol{x}) - \boldsymbol{\pi}_h(\widehat{\boldsymbol{x}})|.
\end{aligned}$$

Since $\beta > 4$, for $h \ll 1$, we obtain

$$|\widehat{\boldsymbol{y}}_h(\boldsymbol{x}) - \widehat{\boldsymbol{y}}_h(\widehat{\boldsymbol{x}}) - (\boldsymbol{\pi}_h(\boldsymbol{x}) - \boldsymbol{\pi}_h(\widehat{\boldsymbol{x}}))| < |\boldsymbol{\pi}_h(\boldsymbol{x}) - \boldsymbol{\pi}_h(\widehat{\boldsymbol{x}})|.$$

As $\boldsymbol{\pi}_h(\boldsymbol{x}) \neq \boldsymbol{\pi}_h(\widehat{\boldsymbol{x}})$, we necessarily have $\widehat{\boldsymbol{y}}_h(\boldsymbol{x}) \neq \widehat{\boldsymbol{y}}_h(\widehat{\boldsymbol{x}})$, which proves the claim. By the Invariance of Domain Theorem, the map $\widehat{\boldsymbol{y}}_h$ is a homeomorphism. Thus, recalling Theorem 1.52, we have $\widehat{\boldsymbol{y}}_h^{-1} \in W^{1,1}(\Omega^{\widehat{\boldsymbol{y}}_h};\mathbb{R}^3) \cap C^0(\Omega^{\widehat{\boldsymbol{y}}_h};\mathbb{R}^3)$. This map actually belongs to $W^{1,\infty}(\Omega^{\widehat{\boldsymbol{y}}_h};\mathbb{R}^3)$. To see this, setting $\boldsymbol{I}_h := D\boldsymbol{\pi}_h$, observe that

$$\begin{aligned}
\boldsymbol{I}_h\, D\widehat{\boldsymbol{y}}_h^{-1} &= \boldsymbol{I}_h\, (D\widehat{\boldsymbol{y}}_h)^{-1} \circ \widehat{\boldsymbol{y}}_h^{-1} = \left(D\widehat{\boldsymbol{y}}_h \circ \widehat{\boldsymbol{y}}_h^{-1}\, \boldsymbol{I}_h^{-1}\right)^{-1} \\
&= \left(\widehat{\boldsymbol{F}}_h \circ \widehat{\boldsymbol{y}}_h^{-1}\right)^{-1} = \widehat{\boldsymbol{F}}_h^{-1} \circ \widehat{\boldsymbol{y}}_h^{-1}.
\end{aligned} \tag{4.122}$$

From (4.118), by approximating the matrix inverse with the Neumann series, we obtain

$$\widehat{\boldsymbol{F}}_h^{-1} = \boldsymbol{I} + O(h^{\beta/2-1}) \tag{4.123}$$

In particular, this gives $\widehat{\boldsymbol{F}}_h^{-1} \in L^\infty(\Omega;\mathbb{R}^{3\times 3})$ and proves the claim in view of (4.122). Now, define $\widehat{\boldsymbol{m}}_h \colon \Omega^{\widehat{\boldsymbol{y}}_h} \to \mathbb{R}^3$ by setting

$$\widehat{\boldsymbol{m}}_h := \frac{\widehat{\boldsymbol{\zeta}} \circ \widehat{\boldsymbol{y}}_h^{-1}}{\det D_h\widehat{\boldsymbol{y}}_h \circ \widehat{\boldsymbol{y}}_h^{-1}}.$$

Observe that $\widehat{\boldsymbol{m}}_h \in W^{1,2}(\Omega^{\widehat{\boldsymbol{y}}_h};\mathbb{R}^3)$ thanks to the regularity of $\widehat{\boldsymbol{y}}_h^{-1}$ and to (4.121). Moreover, $\widehat{\boldsymbol{m}}_h \circ \widehat{\boldsymbol{y}}_h \det D_h\widehat{\boldsymbol{y}}_h = \widehat{\boldsymbol{\zeta}}$ in Ω, so that $\widehat{\boldsymbol{q}}_h := (\widehat{\boldsymbol{y}}_h, \widehat{\boldsymbol{m}}_h) \in \mathcal{Q}_h$.

Step 2 (Convergences). The convergences in (4.106)–(4.110) follow by direct computation, while (4.114) is trivial. From (4.121) and (4.123), we also have

$$\det \widehat{\boldsymbol{F}}_h \to 1 \text{ in } L^\infty(\Omega), \qquad \frac{1}{\det \widehat{\boldsymbol{F}}_h} \to 1 \text{ in } L^\infty(\Omega), \tag{4.124}$$

and

$$D(\det \widehat{\boldsymbol{F}}_h) \to \boldsymbol{0} \text{ in } L^\infty(\Omega;\mathbb{R}^3), \qquad \widehat{\boldsymbol{F}}_h^{-1} \to \boldsymbol{I} \text{ in } L^\infty(\Omega;\mathbb{R}^{3\times 3}). \tag{4.125}$$

Thus (4.113) easily follows. We prove (4.111). We have

$$\widehat{\boldsymbol{\mu}}_h = \chi_{\boldsymbol{\pi}_h^{-1}(\Omega^{\widehat{\boldsymbol{y}}_h})} \frac{\widehat{\boldsymbol{\zeta}} \circ \widehat{\boldsymbol{y}}_h^{-1} \circ \boldsymbol{\pi}_h}{\det D_h \widehat{\boldsymbol{y}}_h \circ \widehat{\boldsymbol{y}}_h^{-1} \circ \boldsymbol{\pi}_h}.$$

Let $\boldsymbol{x} \in \mathbb{R}^3 \setminus \overline{\Omega}$, so that $\boldsymbol{x} \in \mathbb{R}^3 \setminus \Omega_\ell^{-\eta}$ for some $\eta > 0$ and $\ell > 1$. Then, (4.61) entails that $\boldsymbol{x} \in \mathbb{R}^3 \setminus \boldsymbol{\pi}_h^{-1}(\Omega^{\widehat{\boldsymbol{y}}_h})$ for $h \leq \underline{h}(\eta, \ell)$. Instead, let $\boldsymbol{x} \in \Omega$ and let $\eta > 0$ and $0 < \vartheta < 1$ be such that $\boldsymbol{x} \in \Omega_\vartheta^\eta$. By (4.60), we have $\boldsymbol{x} \in \boldsymbol{\pi}_h^{-1}(\Omega^{\widehat{\boldsymbol{y}}_h})$for $h \leq \overline{h}(\eta, \vartheta)$ and, in turn, $\boldsymbol{x}_h \coloneqq \widehat{\boldsymbol{y}}_h^{-1}(\boldsymbol{\pi}_h(\boldsymbol{x})) \in \Omega$. From (4.117), we see that

$$\|\boldsymbol{\pi}_h^{-1} \circ \widehat{\boldsymbol{y}}_h - \boldsymbol{id}\|_{C^0(\overline{\Omega};\mathbb{R}^3)} \leq \frac{1}{h} \|\widehat{\boldsymbol{y}}_h - \boldsymbol{\pi}_h\|_{C^0(\overline{\Omega};\mathbb{R}^3)} \leq C h^{\beta/2-2}.$$

Let $\varepsilon > 0$ be arbitrary. For $h \ll 1$, the right-hand side of the previous equation is smaller than ε, so that

$$|\boldsymbol{x} - \widehat{\boldsymbol{y}}_h^{-1}(\boldsymbol{\pi}_h(\boldsymbol{x}))| = |\boldsymbol{\pi}_h^{-1}(\widehat{\boldsymbol{y}}_h(\boldsymbol{x}_h)) - \boldsymbol{x}_h| \leq \|\boldsymbol{\pi}_h^{-1} \circ \widehat{\boldsymbol{y}}_h - \boldsymbol{id}\|_{C^0(\overline{\Omega};\mathbb{R}^3)} < \varepsilon.$$

This shows that $\widehat{\boldsymbol{y}}_h^{-1}(\boldsymbol{\pi}_h(\boldsymbol{x})) \to \boldsymbol{x}$, as $h \to 0^+$. Since $\widehat{\boldsymbol{\zeta}}$ is continuous and (4.124) holds, we deduce that $\widehat{\boldsymbol{\mu}}_h \to \chi_\Omega \widehat{\boldsymbol{\zeta}}$ almost everywhere in $\mathbb{R}^3$. As $(\widehat{\boldsymbol{\mu}}_h)$ is uniformly bounded and supported in a compact set containing Ω by (4.61) for $h \ll 1$, by the Dominated Convergence Theorem, there holds $\widehat{\boldsymbol{\mu}}_h \to \chi_\Omega \widehat{\boldsymbol{\zeta}}$ in $L^q(\mathbb{R}^3;\mathbb{R}^3)$ for every $1 \leq q < \infty$. In particular, this shows (4.111).
To prove (4.112), first we compute

$$D\widehat{\boldsymbol{m}}_h = \frac{D(\widehat{\boldsymbol{\zeta}} \circ \widehat{\boldsymbol{y}}_h^{-1})}{\det D_h \widehat{\boldsymbol{y}}_h \circ \widehat{\boldsymbol{y}}_h^{-1}} - \frac{\widehat{\boldsymbol{\zeta}} \circ \widehat{\boldsymbol{y}}_h^{-1}\, D(\det D_h \widehat{\boldsymbol{y}}_h \circ \widehat{\boldsymbol{y}}_h^{-1})}{(\det D_h \widehat{\boldsymbol{y}}_h \circ \widehat{\boldsymbol{y}}_h^{-1})^2}.$$

Since $\widehat{\boldsymbol{\zeta}}$ is a function of the plane variable only, there holds $\widehat{\boldsymbol{\zeta}} = \widehat{\boldsymbol{\zeta}} \circ \boldsymbol{\pi}_h$. Thus

$$D(\widehat{\boldsymbol{\zeta}} \circ \widehat{\boldsymbol{y}}_h^{-1}) = D(\widehat{\boldsymbol{\zeta}} \circ \boldsymbol{\pi}_h \circ \widehat{\boldsymbol{y}}_h^{-1}) = D\widehat{\boldsymbol{\zeta}} \circ \boldsymbol{\pi}_h \circ \widehat{\boldsymbol{y}}_h^{-1} \boldsymbol{I}_h\, D\widehat{\boldsymbol{y}}_h^{-1} = D\widehat{\boldsymbol{\zeta}} \circ \widehat{\boldsymbol{y}}_h^{-1} \widehat{\boldsymbol{F}}_h^{-1} \circ \widehat{\boldsymbol{y}}_h^{-1},$$

where, in the last line, we used (4.122). Also, we have

$$\begin{aligned}
\big(D(\det D_h \widehat{\boldsymbol{y}}_h \circ \widehat{\boldsymbol{y}}_h^{-1})\big)^\top &= D(\det D_h \widehat{\boldsymbol{y}}_h) \circ \widehat{\boldsymbol{y}}_h^{-1} D\widehat{\boldsymbol{y}}_h^{-1} \\
&= D(\det D_h \widehat{\boldsymbol{y}}_h) \circ \widehat{\boldsymbol{y}}_h^{-1} \boldsymbol{I}_h^{-1} \widehat{\boldsymbol{F}}_h^{-1} \circ \widehat{\boldsymbol{y}}_h^{-1},
\end{aligned}$$

where we employed again (4.122). Thus

$$D\widehat{\boldsymbol{m}}_h = \frac{D\widehat{\boldsymbol{\zeta}} \circ \widehat{\boldsymbol{y}}_h^{-1} \widehat{\boldsymbol{F}}_h^{-1} \circ \widehat{\boldsymbol{y}}_h^{-1}}{\det D_h \widehat{\boldsymbol{y}}_h \circ \widehat{\boldsymbol{y}}_h^{-1}} - \frac{\widehat{\boldsymbol{\zeta}} \circ \widehat{\boldsymbol{y}}_h^{-1}\, D(\det D_h \widehat{\boldsymbol{y}}_h) \circ \widehat{\boldsymbol{y}}_h^{-1} \boldsymbol{I}_h^{-1} \widehat{\boldsymbol{F}}_h^{-1} \circ \widehat{\boldsymbol{y}}_h^{-1}}{(\det D_h \widehat{\boldsymbol{y}}_h \circ \widehat{\boldsymbol{y}}_h^{-1})^2},$$

so that

$$\begin{aligned}
\widehat{\boldsymbol{N}}_h = {} & \chi_{\boldsymbol{\pi}_h^{-1}(\Omega^{\widehat{\boldsymbol{y}}_h})} \frac{D\widehat{\boldsymbol{\zeta}} \circ \widehat{\boldsymbol{y}}_h^{-1} \circ \boldsymbol{\pi}_h\, \widehat{\boldsymbol{F}}_h^{-1} \circ \widehat{\boldsymbol{y}}_h^{-1} \circ \boldsymbol{\pi}_h}{\det D_h \widehat{\boldsymbol{y}}_h \circ \widehat{\boldsymbol{y}}_h^{-1} \circ \boldsymbol{\pi}_h} \\
& - \chi_{\boldsymbol{\pi}_h^{-1}(\Omega^{\widehat{\boldsymbol{y}}_h})} \frac{\widehat{\boldsymbol{\zeta}} \circ \widehat{\boldsymbol{y}}_h^{-1} \circ \boldsymbol{\pi}_h\, D(\det D_h \widehat{\boldsymbol{y}}_h) \circ \widehat{\boldsymbol{y}}_h^{-1} \circ \boldsymbol{\pi}_h\, \boldsymbol{I}_h^{-1}\, \widehat{\boldsymbol{F}}_h^{-1} \circ \widehat{\boldsymbol{y}}_h^{-1} \circ \boldsymbol{\pi}_h}{(\det D_h \widehat{\boldsymbol{y}}_h \circ \widehat{\boldsymbol{y}}_h^{-1} \circ \boldsymbol{\pi}_h)^2}.
\end{aligned}$$

Similarly to how we did before, exploiting the continuity of $D'\widehat{\boldsymbol{\zeta}}$ together with (4.60)–(4.61), (4.121) and (4.124)–(4.125), we show that $\widehat{\boldsymbol{N}}_h \to \chi_\Omega(D'\widehat{\boldsymbol{\zeta}}, \boldsymbol{0})$ almost everywhere. As both maps are bounded and supported in a compact set containing Ω, we infer that $\widehat{\boldsymbol{N}}_h \to \chi_\Omega(D'\widehat{\boldsymbol{\zeta}}, \boldsymbol{0})$ in $L^q(\mathbb{R}^3;\mathbb{R}^{3\times 3})$ for every $1 \leq q < \infty$ by the Dominated Convergence Theorem. In particular, (4.112) is checked.
We now compute the limit of the magnetoelastic energy. Recalling (4.70) and Proposition 4.10, from (4.111)–(4.112) we obtain

$$\lim_{h\to 0^+} E_h^{\mathrm{exc}}(\widehat{\boldsymbol{q}}_h) = E_0^{\mathrm{exc}}(\widehat{\boldsymbol{\zeta}}), \qquad \lim_{h\to 0^+} E_h^{\mathrm{mag}}(\widehat{\boldsymbol{q}}_h) = E_0^{\mathrm{mag}}(\widehat{\boldsymbol{\zeta}}). \tag{4.126}$$

We move to the convergence of the elastic energy. By the Polar Decomposition Theorem, there exists $(\widehat{\boldsymbol{P}}_h) \subset W^{1,\infty}(\Omega; SO(3))$ such that $\widehat{\boldsymbol{F}}_h = \widehat{\boldsymbol{P}}_h \left(\widehat{\boldsymbol{F}}_h^\top \widehat{\boldsymbol{F}}_h\right)^{1/2}$. Thanks to (4.119)–(4.120), passing to the limit, as $h \to 0^+$, in the previous identity, we see that $\widehat{\boldsymbol{P}}_h \to \boldsymbol{I}$ uniformly in Ω. For convenience, set $\widehat{\boldsymbol{L}}_h \coloneqq \boldsymbol{L}_h(\widehat{\boldsymbol{P}}_h^\top \widehat{\boldsymbol{z}}_h)$. By (4.5) and (4.7), there holds

$$\widehat{\boldsymbol{K}}_h \coloneqq h^{-\beta/2}(\boldsymbol{I} - \widehat{\boldsymbol{L}}_h^{-1}) \to c_0\, \widehat{\boldsymbol{\zeta}} \otimes \widehat{\boldsymbol{\zeta}} \text{ in } L^1(\Omega;\mathbb{R}^{3\times 3}). \tag{4.127}$$

Recalling (4.9) and (4.13), we write

$$\begin{aligned} W_h(\widehat{\boldsymbol{F}}_h, \widehat{\boldsymbol{\lambda}}_h) &= \Phi\left(\boldsymbol{L}_h(\widehat{\boldsymbol{z}}_h)^{-1}\widehat{\boldsymbol{F}}_h\right) = \Phi\left(\boldsymbol{L}_h(\widehat{\boldsymbol{z}}_h)^{-1}\widehat{\boldsymbol{P}}_h\left(\widehat{\boldsymbol{F}}_h^\top\widehat{\boldsymbol{F}}_h\right)^{1/2}\right) \\ &= \Phi\left(\widehat{\boldsymbol{P}}_h^\top \boldsymbol{L}_h(\widehat{\boldsymbol{z}}_h)^{-1}\widehat{\boldsymbol{P}}_h\left(\widehat{\boldsymbol{F}}_h^\top\widehat{\boldsymbol{F}}_h\right)^{1/2}\right) = \Phi\left(\widehat{\boldsymbol{L}}_h^{-1}\left(\widehat{\boldsymbol{F}}_h^\top\widehat{\boldsymbol{F}}_h\right)^{1/2}\right). \end{aligned} \tag{4.128}$$

Then

$$\widehat{\boldsymbol{L}}_h^{-1}\left(\widehat{\boldsymbol{F}}_h^\top\widehat{\boldsymbol{F}}_h\right)^{1/2} = \boldsymbol{I} + h^{\beta/2}\left(\widehat{\boldsymbol{D}} - \widehat{\boldsymbol{K}}_h + x_3\widehat{\boldsymbol{E}}\right) + O(h^{\beta/2+1}). \tag{4.129}$$

Using (4.16) and (4.128)–(4.129), we compute

$$\begin{aligned} E_h^{\mathrm{el}}(\widehat{\boldsymbol{q}}_h) &= \frac{1}{h^\beta}\int_\Omega \Phi\left(\widehat{\boldsymbol{L}}_h^{-1}\left(\widehat{\boldsymbol{F}}_h^\top\widehat{\boldsymbol{F}}_h\right)^{1/2}\right)\mathrm{d}\boldsymbol{x} \\ &= \frac{1}{2}\int_\Omega Q\left(\widehat{\boldsymbol{D}} - \widehat{\boldsymbol{K}}_h + x_3\widehat{\boldsymbol{E}} + O(h)\right)\mathrm{d}\boldsymbol{x} \\ &+ \frac{1}{h^\beta}\int_\Omega \omega\left(h^{\beta/2}\left(\widehat{\boldsymbol{D}} - \widehat{\boldsymbol{K}}_h + x_3\widehat{\boldsymbol{E}}\right) + O(h^{\beta/2+1})\right)\mathrm{d}\boldsymbol{x}. \end{aligned} \tag{4.130}$$

Thanks to (4.127), applying the Dominated Convergence Theorem, we obtain

$$\lim_{h\to 0^+} E_h^{\mathrm{el}}(\widehat{\boldsymbol{q}}_h) = \frac{1}{2}\int_\Omega Q(\widehat{\boldsymbol{D}} - c_0\widehat{\boldsymbol{\zeta}} \otimes \widehat{\boldsymbol{\zeta}} + x_3\widehat{\boldsymbol{E}})\,\mathrm{d}\boldsymbol{x} \tag{4.131}$$

Therefore, combining (4.126) and (4.131), we have

$$\begin{aligned} \lim_{h\to 0^+} E_h(\widehat{\boldsymbol{q}}_h) &= \frac{1}{2}\int_S Q(\widehat{\boldsymbol{D}} - c_0\widehat{\boldsymbol{\zeta}} \otimes \widehat{\boldsymbol{\zeta}})\,\mathrm{d}\boldsymbol{x} + \frac{1}{24}\int_S Q(\widehat{\boldsymbol{E}})\,\mathrm{d}\boldsymbol{x} \\ &+ \int_S |D'\widehat{\boldsymbol{\zeta}}|^2\,\mathrm{d}\boldsymbol{x}' + \int_S |\widehat{\zeta}^3|^2\,\mathrm{d}\boldsymbol{x}'. \end{aligned} \tag{4.132}$$

Step 3 (Diagonal argument). By definition of Q_{red}, there exist $\widehat{\boldsymbol{a}}, \widehat{\boldsymbol{b}} \colon S \to \mathbb{R}^3$ such that

$$\begin{gathered} Q_{\mathrm{red}}(\operatorname{sym} D'\widehat{\boldsymbol{u}} - c_0 \widehat{\boldsymbol{\zeta}}\,' \otimes \widehat{\boldsymbol{\zeta}}\,') = Q(\widehat{\boldsymbol{D}} - c_0 \widehat{\boldsymbol{\zeta}} \otimes \widehat{\boldsymbol{\zeta}}), \\ Q_{\mathrm{red}}((D')^2 \widehat{v}) = Q_\Phi(\widehat{\boldsymbol{E}}), \end{gathered} \tag{4.133}$$

where we set

$$\begin{aligned} \widehat{\boldsymbol{D}} &:= \left(\begin{array}{c|c} \mathrm{sym} D'\widehat{\boldsymbol{u}} & \boldsymbol{0}' \\ \hline (\boldsymbol{0}')^\top & 0 \end{array}\right) + \widehat{\boldsymbol{a}} \otimes \boldsymbol{e}_3 + \boldsymbol{e}_3 \otimes \widehat{\boldsymbol{a}}, \\ \widehat{\boldsymbol{E}} &:= -\left(\begin{array}{c|c} (D')^2 \widehat{v} & \boldsymbol{0}' \\ \hline (\boldsymbol{0}')^\top & 0 \end{array}\right) + \widehat{\boldsymbol{b}} \otimes \boldsymbol{e}_3 + \boldsymbol{e}_3 \otimes \widehat{\boldsymbol{b}}. \end{aligned}$$

In particular, thanks to (4.17), we have $\widehat{\boldsymbol{a}}, \widehat{\boldsymbol{b}} \in L^2(S;\mathbb{R}^3)$.
To conclude the proof, we employ a standard diagonal argument. Let $(\widehat{\boldsymbol{u}}_j) \subset W^{2,\infty}(S;\mathbb{R}^2)$, $(\widehat{v}_j) \subset W^{3,\infty}(S)$ and $(\widehat{\boldsymbol{a}}_j), (\widehat{\boldsymbol{b}}_j) \subset C^2_{\mathrm{c}}(S;\mathbb{R}^3)$ be such that the following convergences hold, as $j \to \infty$:

$$\widehat{\boldsymbol{u}}_j \to \widehat{\boldsymbol{u}} \text{ in } W^{1,2}(S;\mathbb{R}^2), \qquad \widehat{v}_j \to \widehat{v} \text{ in } W^{2,2}(S), \tag{4.134}$$

$$\widehat{\boldsymbol{a}}_j \to \widehat{\boldsymbol{a}} \text{ in } L^2(S;\mathbb{R}^3), \qquad \widehat{\boldsymbol{b}}_j \to \widehat{\boldsymbol{b}} \text{ in } L^2(S;\mathbb{R}^3). \tag{4.135}$$

If we set

$$\begin{aligned} \widehat{\boldsymbol{D}}_j &:= \left(\begin{array}{c|c} \mathrm{sym} D'\widehat{\boldsymbol{u}}_j & \boldsymbol{0}' \\ \hline (\boldsymbol{0}')^\top & 0 \end{array}\right) + \widehat{\boldsymbol{a}}_j \otimes \boldsymbol{e}_3 + \boldsymbol{e}_3 \otimes \widehat{\boldsymbol{a}}_j, \\ \widehat{\boldsymbol{E}}_j &:= -\left(\begin{array}{c|c} (D')^2 \widehat{v}_j & \boldsymbol{0}' \\ \hline (\boldsymbol{0}')^\top & 0 \end{array}\right) + \widehat{\boldsymbol{b}}_j \otimes \boldsymbol{e}_3 + \boldsymbol{e}_3 \otimes \widehat{\boldsymbol{b}}_j, \end{aligned}$$

then, by (4.134)–(4.135), we immediately have

$$\widehat{\boldsymbol{D}}_j \to \widehat{\boldsymbol{D}} \text{ in } L^2(S;\mathbb{R}^{3\times 3}), \quad \widehat{\boldsymbol{E}}_j \to \widehat{\boldsymbol{E}} \text{ in } L^2(S;\mathbb{R}^{3\times 3}),$$

as $j \to \infty$. Moreover, by [76, Theorem 2.1], there exists $(\widehat{\boldsymbol{\zeta}}_j) \subset C^1(\overline{S};\mathbb{S}^2)$ such that

$$\widehat{\boldsymbol{\zeta}}_j \to \widehat{\boldsymbol{\zeta}} \text{ in } W^{1,2}(S;\mathbb{R}^3),$$

as $j \to \infty$, and, in turn, there holds

$$\widehat{\boldsymbol{\zeta}}_j \otimes \widehat{\boldsymbol{\zeta}}_j \to \widehat{\boldsymbol{\zeta}} \otimes \widehat{\boldsymbol{\zeta}} \text{ in } L^q(S;\mathbb{R}^{3\times 3}) \text{ for every } 1 \leq q < \infty, \tag{4.136}$$

as $j \to \infty$.
For every fixed $j \in \mathbb{N}$, by Step 1, there exists a sequence $(\widehat{\boldsymbol{q}}_h^{(j)})$ with $\widehat{\boldsymbol{q}}_h^{(j)} =$

$(\widehat{\boldsymbol{y}}_h^{(j)}, \widehat{\boldsymbol{m}}_h^{(j)}) \in \mathcal{Q}_h$ such that the following convergences hold, as $h \to 0^+$:

$$\widehat{\boldsymbol{y}}_h^{(j)} \to \boldsymbol{\pi}_0 \text{ in } W^{1,p}(\Omega;\mathbb{R}^3); \tag{4.137}$$

$$\widehat{\boldsymbol{F}}_h^{(j)} := D_h \widehat{\boldsymbol{y}}_h^{(j)} \to \boldsymbol{I} \text{ in } L^p(\Omega;\mathbb{R}^{3\times 3}); \tag{4.138}$$

$$\widehat{\boldsymbol{u}}_h^{(j)} := \mathcal{U}_h(\widehat{\boldsymbol{q}}_h^{(j)}) \to \widehat{\boldsymbol{u}}_j \text{ in } W^{1,2}(S;\mathbb{R}^3); \tag{4.139}$$

$$\widehat{v}_h^{(j)} := \mathcal{V}_h(\widehat{\boldsymbol{q}}_h^{(j)}) \to \widehat{v}_j \text{ in } W^{2,2}(S); \tag{4.140}$$

$$\widehat{\boldsymbol{w}}_h^{(j)} := \mathcal{W}_h(\widehat{\boldsymbol{q}}_h^{(j)}) \to -\frac{1}{12}\begin{pmatrix} D'\widehat{v}_j \\ 0 \end{pmatrix} \text{ in } W^{2,2}(S); \tag{4.141}$$

$$\widehat{\boldsymbol{\mu}}_h^{(j)} := \mathcal{M}_h(\widehat{\boldsymbol{q}}_h^{(j)}) \to \chi_\Omega \widehat{\boldsymbol{\zeta}}_j \text{ in } L^2(\mathbb{R}^3;\mathbb{R}^3); \tag{4.142}$$

$$\widehat{\boldsymbol{N}}_h^{(j)} := \mathcal{N}_h(\widehat{\boldsymbol{q}}_h^{(j)}) \to \chi_\Omega (D'\widehat{\boldsymbol{\zeta}}_j|\boldsymbol{0}) \text{ in } L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3}); \tag{4.143}$$

$$\widehat{\boldsymbol{\lambda}}_h^{(j)} := \widehat{\boldsymbol{m}}_h^{(j)} \circ \widehat{\boldsymbol{y}}_h^{(j)} \to \widehat{\boldsymbol{\zeta}}_j \text{ in } L^1(\Omega;\mathbb{R}^3); \tag{4.144}$$

$$\widehat{\boldsymbol{z}}_h^{(j)} := \mathcal{Z}_h(\widehat{\boldsymbol{q}}_h^{(j)}) \to \widehat{\boldsymbol{\zeta}}_j \text{ in } L^q(\Omega;\mathbb{R}^3) \text{ for every } 1 \leq q < \infty. \tag{4.145}$$

Moreover, we also have

$$\begin{aligned}\lim_{h\to 0^+} E_h(\widehat{\boldsymbol{q}}_h^{(j)}) = \frac{1}{2}\int_S Q(\widehat{\boldsymbol{D}}_j - c_0 \widehat{\boldsymbol{\zeta}}_j \otimes \widehat{\boldsymbol{\zeta}}_j)\,\mathrm{d}\boldsymbol{x} + \frac{1}{24}\int_S Q(\widehat{\boldsymbol{E}}_j)\,\mathrm{d}\boldsymbol{x} \\ + \int_S |D'\widehat{\boldsymbol{\zeta}}_j|^2\,\mathrm{d}\boldsymbol{x}' + \frac{1}{2}\int_S |\widehat{\zeta}_j^3|^2\,\mathrm{d}\boldsymbol{x}'.\end{aligned} \tag{4.146}$$

In view of (4.134)–(4.136) and (4.137)–(4.145), we select a subsequence (h_j) such that, setting $\widehat{\boldsymbol{q}}_{h_j} := (\widehat{\boldsymbol{y}}_{h_j}, \widehat{\boldsymbol{m}}_{h_j})$ with $\widehat{\boldsymbol{y}}_{h_j} := \widehat{\boldsymbol{y}}_{h_j}^{(j)}$ and $\widehat{\boldsymbol{m}}_{h_j} := \widehat{\boldsymbol{m}}_{h_j}^{(j)}$, the convergences in (4.106)–(4.114) hold for $h = h_j$, as $j \to \infty$, as well as

$$\begin{aligned}\lim_{j\to\infty} E_{h_j}^{\mathrm{el}}(\widehat{\boldsymbol{q}}_{h_j}) = \frac{1}{2}\int_S Q(\widehat{\boldsymbol{D}} - \widehat{\boldsymbol{\zeta}} \otimes \widehat{\boldsymbol{\zeta}})\,\mathrm{d}\boldsymbol{x}' + \frac{1}{24}\int_S Q(\widehat{\boldsymbol{E}})\,\mathrm{d}\boldsymbol{x}' \\ + \int_S |D'\,\widehat{\boldsymbol{\zeta}}|^2\,\mathrm{d}\boldsymbol{x}' + \frac{1}{2}\int_S |\widehat{\zeta}^3|^2\,\mathrm{d}\boldsymbol{x}'.\end{aligned} \tag{4.147}$$

As the right-hand side of (4.147) equals $E_0^{\mathrm{el}}(\widehat{\boldsymbol{q}}_0)$ by (4.133), this establishes (4.115) for the subsequences indexed by (h_j). □

4.2.4 Proof of Theorem 4.3

We are finally ready to prove our first main result.

Proof of Theorem 4.3. We have to prove just claim (i), since claim (ii) has already been proved in Proposition 4.12.
Recall (4.3)–(4.7). For simplicity, we set

$$\boldsymbol{F}_h := D_h \boldsymbol{y}_h, \quad \boldsymbol{\lambda}_h := \boldsymbol{m}_h \circ \boldsymbol{y}_h, \quad \boldsymbol{L}_h := \boldsymbol{L}_h(\boldsymbol{\lambda}_h \det \boldsymbol{F}_h), \quad \Xi_h := \boldsymbol{L}_h^{-1}\boldsymbol{F}_h.$$

First, recalling (4.4)–(4.5), we write

$$\begin{aligned}\operatorname{dist}(\boldsymbol{F}_h; SO(3)) &\leq |\boldsymbol{F}_h - \Xi_h| + \operatorname{dist}(\Xi_h; SO(3)) \\ &\leq |\boldsymbol{L}_h - \boldsymbol{I}|\,|\Xi_h| + \operatorname{dist}(\Xi_h; SO(3)) \\ &\leq Ch^{\beta/2}|\Xi_h| + \operatorname{dist}(\Xi_h; SO(3)) \\ &\leq C\left(h^{\beta/2} + \operatorname{dist}(\Xi_h; SO(3))\right).\end{aligned}$$

Thus, for $q \in \{2, p\}$, we have

$$\int_\Omega \mathrm{dist}^q(\boldsymbol{F}_h; SO(3))\,\mathrm{d}\boldsymbol{x} \leq C\left(h^{(\beta/2)q} + \int_\Omega \mathrm{dist}^q(\Xi_h; SO(3))\,\mathrm{d}\boldsymbol{x}\right).$$

Recalling (4.10) and (4.33), this shows that

$$\mathcal{R}_h(\boldsymbol{y}_h) \leq Ch^\beta \left(E_h^{\mathrm{el}}(\boldsymbol{q}_h) + 1\right). \tag{4.148}$$

Set $r_h := \mathcal{R}_h(\boldsymbol{y}_h)$. Applying Lemma 4.5 to each $\boldsymbol{y}_h$, we find $(\boldsymbol{R}_h) \subset W^{1,p}(S; SO(3))$ and $(\boldsymbol{Q}_h) \subset SO(3)$ satisfying

$$\|\boldsymbol{F}_h - \boldsymbol{R}_h\|_{L^q(\Omega;\mathbb{R}^{3\times3})} \leq Cr_h^{1/q}, \qquad \|D'\boldsymbol{R}_h\|_{L^q(S;\mathbb{R}^{3\times3\times3})} \leq Ch^{-1}r_h^{1/q}, \tag{4.149}$$

$$\|\boldsymbol{R}_h - \boldsymbol{Q}_h\|_{L^q(S;\mathbb{R}^{3\times3})} \leq Ch^{-1}r_h^{1/q}, \qquad \|\boldsymbol{F}_h - \boldsymbol{Q}_h\|_{L^q(S;\mathbb{R}^{3\times3})} \leq Ch^{-1}r_h^{1/q}. \tag{4.150}$$

Denote by $\boldsymbol{c}_h \in \mathbb{R}^3$ the average of $\boldsymbol{Q}_h^\top \boldsymbol{y}_h - \boldsymbol{\pi}_h$ over Ω and consider the rigid motion $\boldsymbol{T}_h \colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $\boldsymbol{T}_h(\boldsymbol{\xi}) := \boldsymbol{Q}_h^\top\boldsymbol{\xi} - \boldsymbol{c}_h$. Set $\widetilde{\boldsymbol{y}}_h := \boldsymbol{T}_h \circ \boldsymbol{y}_h$ and note that $\widetilde{\boldsymbol{y}}_h - \boldsymbol{\pi}_h$ has null average over Ω by our choice of $\boldsymbol{c}_h$. From (4.149)–(4.150), setting $\widetilde{\boldsymbol{R}}_h := \boldsymbol{Q}_h^\top \boldsymbol{R}_h$, we immediately deduce

$$\|\widetilde{\boldsymbol{F}}_h - \widetilde{\boldsymbol{R}}_h\|_{L^q(\Omega;\mathbb{R}^{3\times3})} \leq Cr_h^{1/q}, \qquad \|D'\widetilde{\boldsymbol{R}}_h\|_{L^q(S;\mathbb{R}^{3\times3\times3})} \leq Ch^{-1}r_h^{1/q}, \tag{4.151}$$

$$\|\widetilde{\boldsymbol{R}}_h - \boldsymbol{I}\|_{L^q(S;\mathbb{R}^{3\times3})} \leq Ch^{-1}r_h^{1/q}, \qquad \|\widetilde{\boldsymbol{F}}_h - \boldsymbol{I}\|_{L^q(S;\mathbb{R}^{3\times3})} \leq Ch^{-1}r_h^{1/q}. \tag{4.152}$$

At this point,we apply Proposition 4.6 to $\widehat{\boldsymbol{y}}_h = \widetilde{\boldsymbol{y}}_h$ with $e_h = h^\beta$. Note that, in view of (4.27) and (4.148), the sequence (r_h/e_h) is bounded. Therefore, there exist $\widetilde{\boldsymbol{u}} \in W^{1,2}(S;\mathbb{R}^2)$ and $\widetilde{v} \in W^{2,2}(S)$ such that, up to subsequences, claims (4.28)–(4.29) hold true. Thanks to these convergences and to the first estimate in (4.151), we apply Lemma 4.8 to $\widehat{\boldsymbol{y}}_h = \widetilde{\boldsymbol{y}}_h$ with $e_h = h^\beta$. Thus, there exists $\widetilde{\boldsymbol{G}} \in L^2(\Omega;\mathbb{R}^{3\times3})$ such that, up to subsequences, we have

$$\widetilde{\boldsymbol{G}}_h := h^{-\beta/2}(\widetilde{\boldsymbol{R}}_h^\top \widetilde{\boldsymbol{F}}_h - \boldsymbol{I}) \rightharpoonup \widetilde{\boldsymbol{G}} \text{ in } L^2(\Omega;\mathbb{R}^{3\times3}) \tag{4.153}$$

and, for almost every $\boldsymbol{x} \in \Omega$, there holds

$$\widetilde{\boldsymbol{G}}''(\boldsymbol{x}', x_3) = \mathrm{sym}D'\widetilde{\boldsymbol{u}}(\boldsymbol{x}') + ((D')^2\widetilde{v}(\boldsymbol{x}'))x_3. \tag{4.154}$$

Given the assumption $\beta > 6 \vee p$, we have $\beta/2 - 1 > 0$ and $\beta/p - 1 > 0$. Recall the second estimate in (4.152). Let $0 < s < 1$ and consider $2 \leq q_s \leq p$ such that $1/q_s = s/2 + (1-s)/p$. By the interpolation inequality [73, Proposition 1.1.14], there holds

$$\|\widetilde{\boldsymbol{F}}_h - \boldsymbol{I}\|_{L^{q_s}(\Omega;\mathbb{R}^{3\times3})} \leq \|\widetilde{\boldsymbol{F}}_h - \boldsymbol{I}\|^s_{L^2(\Omega;\mathbb{R}^{3\times3})}\|\widetilde{\boldsymbol{F}}_h - \boldsymbol{I}\|^{1-s}_{L^p(\Omega;\mathbb{R}^{3\times3})} \leq Ch^{\alpha_s}, \tag{4.155}$$

where we set $\alpha_s := s(\beta/2 - 1) + (1-s)(\beta/p - 1)$. We choose s in order to have

$$q_s > 3, \qquad \alpha_s > 1. \tag{4.156}$$

Note that these two conditions are equivalent to

$$s < \frac{2(p-3)}{3(p-2)}, \qquad s > \frac{2(2p-\beta)}{\beta(p-2)},$$

respectively. Therefore, as

$$\frac{2(2p-\beta)}{\beta(p-2)} < \frac{2(p-3)}{3(p-2)}$$

if and only if $\beta > 6$, such a value $0 < s < 1$ always exists. Define $\widetilde{\boldsymbol{m}}_h := \boldsymbol{Q}_h^\top \boldsymbol{m}_h \circ \boldsymbol{T}_h^{-1}$, so that $\widetilde{\boldsymbol{q}}_h := (\widetilde{\boldsymbol{y}}_h, \widetilde{\boldsymbol{m}}_h) \in \mathcal{Q}_h$. From (4.155), by applying the Poincaré-Wirtinger inequality and the Sobolev embedding, we deduce the estimate

$$\|\widetilde{\boldsymbol{y}}_h - \boldsymbol{\pi}_h\|_{C^0(\overline{\Omega};\mathbb{R}^3)} \leq C\|\widetilde{\boldsymbol{F}}_h - \boldsymbol{I}\|_{L^{q_s}(\Omega;\mathbb{R}^{3\times 3})} \leq Ch^{\alpha_s}.$$

Note that here we implicity exploited the first condition in (4.156). Hence, given the second condition in (4.156) and the second estimate in (4.152), we apply Proposition 4.9 to $\widehat{\boldsymbol{q}}_h = \widetilde{\boldsymbol{q}}_h$. This yields the existence of $\widetilde{\boldsymbol{\zeta}} \in W^{1,2}(S;\mathbb{S}^2)$ and $\widetilde{\boldsymbol{\chi}} \in L^2(\mathbb{R}^3;\mathbb{R}^3)$ such that, up to subsequences, there hold

$$\widetilde{\boldsymbol{\mu}}_h := \mathcal{M}_h(\widetilde{\boldsymbol{q}}_h) \to \chi_\Omega \widetilde{\boldsymbol{\zeta}} \text{ in } L^2(\mathbb{R}^3;\mathbb{R}^3); \tag{4.157}$$

$$\widetilde{\boldsymbol{N}}_h := \mathcal{N}_h(\widetilde{\boldsymbol{q}}_h) \rightharpoonup \chi_\Omega(D'\widetilde{\boldsymbol{\zeta}}, \widetilde{\boldsymbol{\chi}}) \text{ in } L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3}); \tag{4.158}$$

Moreover, also (4.30) holds true. From this and from (4.153)–(4.158), by applying Proposition 4.11 to $\widehat{\boldsymbol{q}}_h = \widetilde{\boldsymbol{q}}_h$, we obtain

$$E_0(\widetilde{\boldsymbol{q}}_0) \leq \liminf_{h\to 0^+} E_h(\widetilde{\boldsymbol{q}}_h),$$

where we set $\widetilde{\boldsymbol{q}}_0 := (\widetilde{\boldsymbol{u}}, \widetilde{v}, \widetilde{\boldsymbol{\zeta}})$. Thanks to Remark 4.1, this proves (4.31). □

In the next remark, we comment about the specialization of Theorem 4.3 in the case of incompressible materials.

Remark 4.13 (Incompressibility). In the case of incompressible materials, the class of admissible states for given $h > 0$ is restricted to

$$\mathcal{Q}_h^{\mathrm{inc}} := \left\{(\boldsymbol{y}, \boldsymbol{m}) : \boldsymbol{y} \in \mathcal{Y}_p^{\mathrm{inj}}(\Omega),\ \det D\boldsymbol{y} = h \text{ a.e. in } \Omega,\ \boldsymbol{m} \in W^{1,2}(\Omega^{\boldsymbol{y}};\mathbb{S}^2)\right\}.$$

Indeed, the physical deformation $\boldsymbol{\phi} := \boldsymbol{y} \circ \boldsymbol{\pi}_h^{-1} \in W^{1,p}(\Omega_h;\mathbb{R}^3)$ satisfies the incompressibility constraint $\det D\boldsymbol{\phi} = 1$ almost everywhere in Ω_h if and only if there holds $\det D\boldsymbol{y} = h$ almost everywhere in Ω. The fact that, for incompressible materials, magnetization are sphere-valued has already been observed in Remark 2.15.
In this setting, Theorem 4.3 still holds true up to replacing the reduced quadratic form Q_{red} in (4.24) with its incompressible version $Q_{\mathrm{inc}}^{\mathrm{red}} \colon \mathbb{R}^{2\times 2} \to [0,+\infty)$. This is defined by setting

$$Q_{\mathrm{red}}^{\mathrm{inc}}(\boldsymbol{\Sigma}) := \min\left\{Q\left(\left(\begin{array}{c|c} \boldsymbol{\Sigma} & \boldsymbol{0}' \\ \hline (\boldsymbol{0}')^\top & 0 \end{array}\right) + \boldsymbol{c}\otimes\boldsymbol{e}_3 + \boldsymbol{e}_3\otimes\boldsymbol{c}\right) : \boldsymbol{c}\in\mathbb{R}^3,\ c^3 = -\frac{1}{2}\mathrm{tr}\boldsymbol{\Sigma}\right\}.$$

The adaptations needed to prove Theorem 4.3 in the incompressible case rely on the techniques developed in [42, 100]. Full details are provided in [27].

4.3 Convergence of almost minimizers

Henceforth, we consider applied loads determined by body forces and by an external magnetic field. For simplicity, applied surface forces are excluded. However, these

can be easily included in the analysis. According to the assumption of dead loads, the work of mechanical forces is described by Lagrangian terms. Conversely, the energy contribution corresponding to the external magnetic field, usually called Zeeman energy, is of Eulerian type.
Given $h > 0$, let $\boldsymbol{f}_h \in L^2(S;\mathbb{R}^2)$, $\boldsymbol{g}_h \in L^2(S)$ and $\boldsymbol{h}_h \in L^2(\mathbb{R}^3;\mathbb{R}^3)$ represent an horizontal force, a vertical force and an external magnetic field, respectively. The work of applied loads is determined by the functional $L_h\colon \mathcal{Q}_h \to \mathbb{R}$ defined by

$$L_h(\boldsymbol{q}) \coloneqq \frac{1}{h^\beta}\int_\Omega \boldsymbol{f}_h \cdot (\boldsymbol{y}' - \boldsymbol{x}')\,\mathrm{d}\boldsymbol{x} + \frac{1}{h^\beta}\int_\Omega g_h\, y^3\,\mathrm{d}\boldsymbol{x} + \frac{1}{h}\int_{\Omega^{\boldsymbol{y}}} \boldsymbol{h}_h \cdot \boldsymbol{m}\,\mathrm{d}\boldsymbol{\xi}, \tag{4.159}$$

where $\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m})$, so that the total energy $F_h\colon \mathcal{Q}_h \to \mathbb{R}$ reads

$$F_h(\boldsymbol{q}) \coloneqq E_h(\boldsymbol{q}) - L_h(\boldsymbol{q}). \tag{4.160}$$

Regarding the asymptotic behaviour of the applied loads, we assume that there exist $\boldsymbol{f}_0 \in L^2(S;\mathbb{R}^2)$, $g_0 \in L^2(S)$ and $\boldsymbol{h}_0 \in L^2(\mathbb{R}^3;\mathbb{R}^2)$ such that the following convergences hold, as $h \to 0^+$:

$$h^{-\beta/2}\boldsymbol{f}_h \rightharpoonup \boldsymbol{f}_0 \text{ in } L^2(S;\mathbb{R}^2), \tag{4.161}$$

$$h^{-\beta/2-1} g_h \rightharpoonup g_0 \text{ in } L^2(S), \tag{4.162}$$

$$\boldsymbol{h}_h \circ \boldsymbol{\pi}_h \rightharpoonup \chi_I \boldsymbol{h}_0 \text{ in } L^2(\mathbb{R}^3;\mathbb{R}^3). \tag{4.163}$$

We stress that $\boldsymbol{f}_0$, g_0 and $\boldsymbol{h}_0$ are a priori assumed to be independent on the vertical variable. Also, we recall that $I \coloneqq (-1/2, 1/2)$.
The limiting total energy $F_0\colon \mathcal{Q}_0 \to \mathbb{R}$ is defined as

$$F_0(\boldsymbol{q}_0) \coloneqq E_0(\boldsymbol{q}_0) - L_0(\boldsymbol{q}_0). \tag{4.164}$$

The functional $L_0\colon \mathcal{Q}_0 \to \mathbb{R}$ is given by

$$L_0(\boldsymbol{q}_0) \coloneqq \int_S \boldsymbol{f}_0 \cdot \boldsymbol{u}\,\mathrm{d}\boldsymbol{x}' + \int_S g_0\, v\,\mathrm{d}\boldsymbol{x}' + \int_S \boldsymbol{h}_0 \cdot \boldsymbol{\zeta}\,\mathrm{d}\boldsymbol{x}', \tag{4.165}$$

where $\boldsymbol{q}_0 = (\boldsymbol{u}, v, \boldsymbol{\zeta})$. We observe that the limiting total energy is purely Lagrangian. Additionally, from now on, we impose some Dirichlet boundary conditions on deformations. Precisely, let

$$\overline{\boldsymbol{u}} \in W^{2,\infty}(S;\mathbb{R}^2), \qquad \overline{v} \in W^{3,\infty}(S). \tag{4.166}$$

Also, assume the following:

$$\begin{gathered}\Lambda \subset \partial S \text{ is a finite union of disjoint, closed and maximal connected} \\ \text{subsets of } \partial S \text{ with } \mathscr{H}^1(\Lambda) > 0 \text{ and } \Gamma \coloneqq \Lambda \times I \subset \partial\Omega.\end{gathered} \tag{4.167}$$

As in [97], we consider clamped boundary conditions by restricting ourselves to the class of admissible states

$$\overline{\mathcal{Q}}_h \coloneqq \{(\boldsymbol{y}, \boldsymbol{m}) \in \mathcal{Q}_h :\ \boldsymbol{y} = \overline{\boldsymbol{y}}_h \text{ on } \Gamma\}, \tag{4.168}$$

where we set

$$\overline{\boldsymbol{y}}_h \coloneqq \boldsymbol{\pi}_h + h^{\beta/2}\begin{pmatrix}\overline{\boldsymbol{u}}\\ 0\end{pmatrix} + h^{\beta/2-1}\begin{pmatrix}\boldsymbol{0}'\\ \overline{v}\end{pmatrix} - h^{\beta/2}x_3\begin{pmatrix}D'\overline{v}\\ 0\end{pmatrix}. \tag{4.169}$$

Accordingly, limiting admissible states belong to the class

$$\overline{\mathcal{Q}}_0 \coloneqq \{(\boldsymbol{u}, v, \boldsymbol{\zeta}) \in \mathcal{Q}_0 : \ \boldsymbol{u} = \overline{\boldsymbol{u}} \text{ on } \Lambda, \ v = \overline{v} \text{ on } \Lambda, \ D'v = D'\overline{v} \text{ on } \Lambda\}. \qquad (4.170)$$

Our second main result claims that, under the boundary conditions in (4.168), almost minimizers of the sequence (F_h) in (4.160) converge, as $h \to 0^+$, to minimizers of the energy F_0 in (4.164) in the class $\overline{\mathcal{Q}}_0$.

Theorem 4.14 (Convergence of almost minimizers). *Let $(\boldsymbol{q}_h)$ with $\boldsymbol{q}_h = (\boldsymbol{y}_h, \boldsymbol{m}_h) \in \overline{\mathcal{Q}}_h$ be such that*

$$\lim_{h \to 0^+} \left\{ F_h(\boldsymbol{q}_h) - \inf_{\overline{\mathcal{Q}}_h} F_h \right\} = 0. \qquad (4.171)$$

Then, there exist $\boldsymbol{q}_0 = (\boldsymbol{u}, v, \boldsymbol{\zeta}) \in \overline{\mathcal{Q}}_0$ such that, up to subsequences, the following convergences hold, as $h \to 0^+$:

$$\boldsymbol{u}_h \coloneqq \mathcal{U}_h(\boldsymbol{q}_h) \rightharpoonup \boldsymbol{u} \text{ in } W^{1,2}(S; \mathbb{R}^2); \qquad (4.172)$$

$$v_h \coloneqq \mathcal{V}_h(\boldsymbol{q}_h) \to v \text{ in } W^{1,2}(S); \qquad (4.173)$$

$$\boldsymbol{z}_h \coloneqq \mathcal{Z}_h(\boldsymbol{q}_h) \to \boldsymbol{\zeta} \text{ in } L^q(\Omega; \mathbb{R}^3) \text{ for every } 1 \leq q < \infty. \qquad (4.174)$$

Moreover, $\boldsymbol{q}_0$ is a minimizer of F_0 in $\overline{\mathcal{Q}}_0$.

We mention that the weak convergence in (4.172) can be improved to strong convergence by arguing as in [66, Subsection 7.2].

Remark 4.15 (Existence of minimizers for the reduced model). As a consequence of Theorem 4.14, we deduce the existence of minimizer of F_0 in $\overline{\mathcal{Q}}_0$. However, under our assumptions, this can be established directly. Indeed, the functional F_0 is lower semicontinuous with respect to the product weak topology thanks to the convexity of Q_{red}. Thus, in order to apply the Direct Method, one only has to show that the functional F_0 is coercive on $\overline{\mathcal{Q}}_0$. This is done by exploiting the positive definiteness of Q_{red} on symmetric matrices in (4.25) and applying Korn and Poincaré inequalities in view of the boundary conditions in (4.170).

Remark 4.16 (Convergence of minimizers). By Remark 2.12 and Remark 4.2, if we assume Φ to be polyconvex, then the functional F_h admits minimzers in $\overline{\mathcal{Q}}_h$ for every $h > 0$. In this case, Theorem 4.14 ensures that any sequence of such minimizers converge to a minimizer of F_0 in $\overline{\mathcal{Q}}_0$.

4.3.1 Clamped boundary conditions

Observe that in (4.172)–(4.174), unlike in (4.28)– (4.30), compactness is obtained without composing with rigid motions. This improved compactness is due to the boundary conditions imposed in (4.168). To see this, we need a preliminary result which is inspired by [97, Lemma 13].

Lemma 4.17 (Clamped boundary conditions). *Let $(\boldsymbol{y}_h) \subset W^{1,2}(\Omega; \mathbb{R}^3)$ and let $(e_h) \subset \mathbb{R}$ with $e_h > 0$ be such that $e_h/h^2 \to 0$, as $h \to 0^+$. Set $r_h \coloneqq \mathcal{R}_h(\boldsymbol{y}_n)$ and suppose that $r_h \leq Ce_h$ for every $h > 0$. Also, set $\boldsymbol{F}_h \coloneqq D_h \boldsymbol{y}_h$ and suppose that* *there exist* $(\boldsymbol{R}_h) \subset W^{1,2}(S; SO(3))$ *and* $(\boldsymbol{Q}_h) \subset SO(3)$ *such that, for every* $h > 0$, *the following estimates hold:*

$$\|\boldsymbol{F}_h - \boldsymbol{R}_h\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} \leq C\sqrt{r_h}, \qquad \|D'\boldsymbol{R}_h\|_{L^2(S;\mathbb{R}^{3\times 3\times 3})} \leq Ch^{-1}\sqrt{r_h}, \tag{4.175}$$

$$\|\boldsymbol{R}_h - \boldsymbol{Q}_h\|_{L^2(S;\mathbb{R}^{3\times 3})} \leq Ch^{-1}\sqrt{r_h}, \qquad \|\boldsymbol{F}_h - \boldsymbol{Q}_h\|_{L^2(S;\mathbb{R}^{3\times 3})} \leq Ch^{-1}\sqrt{r_h}. \tag{4.176}$$

Also, suppose that $\boldsymbol{y}_h = \overline{\boldsymbol{y}}_h$ *on* Γ *for every* $h > 0$. *Then, for* $h \ll 1$, *there holds*

$$|\boldsymbol{Q}_h - \boldsymbol{I}| \leq C\left(\frac{\sqrt{r_h}}{h} + h^{\beta/2-1}\right). \tag{4.177}$$

Moreover, denoting by $\boldsymbol{c}_h \in \mathbb{R}^3$ *the average of* $\boldsymbol{Q}_h^\top \boldsymbol{y}_h - \boldsymbol{\pi}_h$ *over* Ω, *there holds*

$$|\boldsymbol{c}_h| \leq C\left(\frac{\sqrt{r_h}}{h} + h^{\beta/2-1}\right). \tag{4.178}$$

Proof. We first prove (4.177). As in the proof of Theorem 4.3, define $\widetilde{\boldsymbol{y}}_h \coloneqq \boldsymbol{Q}_h^\top \boldsymbol{y}_h - \boldsymbol{c}_h$ with $\boldsymbol{c}_h \in \mathbb{R}^3$ chosen so that $\widetilde{\boldsymbol{y}}_h - \boldsymbol{\pi}_h$ has null average over Ω. Set $\widetilde{\boldsymbol{F}}_h \coloneqq D_h \widetilde{\boldsymbol{y}}_h$ and $\widetilde{\boldsymbol{R}}_h \coloneqq \boldsymbol{Q}_h^\top \boldsymbol{R}_h$. Assumptions (4.175)–(4.176) immediately yield

$$\|\widetilde{\boldsymbol{F}}_h - \widetilde{\boldsymbol{R}}_h\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} \leq C\sqrt{r_h}, \qquad \|D'\widetilde{\boldsymbol{R}}_h\|_{L^2(S;\mathbb{R}^{3\times 3\times 3})} \leq Ch^{-1}\sqrt{r_h},$$
$$\|\widetilde{\boldsymbol{R}}_h - \boldsymbol{I}\|_{L^2(S;\mathbb{R}^{3\times 3})} \leq Ch^{-1}\sqrt{r_h}, \qquad \|\widetilde{\boldsymbol{F}}_h - \boldsymbol{I}\|_{L^2(S;\mathbb{R}^{3\times 3})} \leq Ch^{-1}\sqrt{r_h}.$$

Define $\widetilde{\boldsymbol{u}}_h \colon S \to \mathbb{R}^2$, $\widetilde{v}_h : S \to \mathbb{R}$ and $\widetilde{\boldsymbol{w}}_h \colon S \to \mathbb{R}^3$ by setting

$$\widetilde{\boldsymbol{u}}_h(\boldsymbol{x}') \coloneqq \left(\frac{h^2}{e_h} \wedge \frac{1}{\sqrt{e_h}}\right) \int_I (\widetilde{\boldsymbol{y}}_h'(\boldsymbol{x}', x_3) - \boldsymbol{x}')\,\mathrm{d}x_3,$$
$$\widetilde{v}_h(\boldsymbol{x}') \coloneqq \frac{h}{\sqrt{e_h}} \int_I \widetilde{y}_h^3(\boldsymbol{x}', x_3)\,\mathrm{d}x_3,$$
$$\widetilde{\boldsymbol{w}}_h(\boldsymbol{x}') \coloneqq \frac{1}{\sqrt{e_h}} \int_I x_3(\widetilde{\boldsymbol{y}}_h - \boldsymbol{\pi}_h)\,\mathrm{d}x_3.$$

Applying Proposition 4.6 to $\widehat{\boldsymbol{y}}_h = \widetilde{\boldsymbol{y}}_h$ and then the trace inequality, we obtain the estimates

$$\|\widetilde{\boldsymbol{u}}_h\|_{L^2(\Lambda;\mathbb{R}^2)} \leq C\left(\frac{h^2\sqrt{r_h}}{e_h} \wedge \sqrt{\frac{r_h}{e_h}} + \frac{r_h}{e_h} \wedge \frac{r_h}{h^2\sqrt{e_h}}\right), \tag{4.179}$$

$$\|\widetilde{v}_h\|_{L^2(\Lambda)} \leq C\sqrt{\frac{r_h}{e_h}}, \tag{4.180}$$

$$\|\widetilde{\boldsymbol{w}}_h\|_{L^2(\Lambda;\mathbb{R}^3)} \leq C\left(\frac{h\sqrt{r_h}}{\sqrt{e_h}} + \sqrt{\frac{r_h}{e_h}}\right). \tag{4.181}$$

Analogously, define $\boldsymbol{u}_h \colon S \to \mathbb{R}^2$, $v_h : S \to \mathbb{R}$ and $\boldsymbol{w}_h \colon S \to \mathbb{R}^3$ by setting

$$\boldsymbol{u}_h(\boldsymbol{x}') \coloneqq \left(\frac{h^2}{e_h} \wedge \frac{1}{\sqrt{e_h}}\right) \int_I (\boldsymbol{y}_h'(\boldsymbol{x}', x_3) - \boldsymbol{x}')\,\mathrm{d}x_3,$$
$$v_h(\boldsymbol{x}') \coloneqq \frac{h}{\sqrt{e_h}} \int_I y_h^3(\boldsymbol{x}', x_3)\,\mathrm{d}x_3,$$
$$\boldsymbol{w}_h(\boldsymbol{x}') \coloneqq \frac{1}{\sqrt{e_h}} \int_I x_3(\boldsymbol{y}_h - \boldsymbol{\pi}_h)\,\mathrm{d}x_3.$$

In view of the boundary condition satisfied by $\boldsymbol{y}_h$, there hold

$$\boldsymbol{u}_h = \frac{h^{\beta/2+2}}{e_h} \wedge \frac{h^{\beta/2}}{\sqrt{e_h}}\overline{u} \quad \text{on } \Lambda, \tag{4.182}$$

$$v_h = \frac{h^{\beta/2}}{\sqrt{e_h}}\overline{v} \quad \text{on } \Lambda, \tag{4.183}$$

$$\boldsymbol{w}_h = -\frac{1}{12}\frac{h^{\beta/2}}{\sqrt{e_h}}\begin{pmatrix} D'\overline{v} \\ 0 \end{pmatrix} \quad \text{on } \Lambda. \tag{4.184}$$

For simplicity, set $\boldsymbol{d}_h \coloneqq \boldsymbol{Q}_h \boldsymbol{c}_h$. Then

$$\begin{pmatrix} h^{-2}e_h \vee \sqrt{e_h}\,\boldsymbol{u}_h \\ h^{-1}\sqrt{e_h}\,v_h \end{pmatrix} = (\boldsymbol{Q}_h - \boldsymbol{I})\,\boldsymbol{\pi}_0 + \boldsymbol{Q}_h \begin{pmatrix} h^{-2}e_h \vee \sqrt{e_h}\,\widetilde{\boldsymbol{u}}_h \\ h^{-1}\sqrt{e_h}\,\widetilde{v}_h \end{pmatrix} + \boldsymbol{d}_h, \tag{4.185}$$

$$\sqrt{e_h}\boldsymbol{w}_h = \frac{h}{12}(\boldsymbol{Q}_h - \boldsymbol{I})\boldsymbol{e}_3 + \sqrt{e_h}\boldsymbol{Q}_h\,\widetilde{\boldsymbol{w}}_h. \tag{4.186}$$

From (4.186), given (4.181) and (4.184), we obtain

$$\begin{aligned} |(\boldsymbol{Q}_h - \boldsymbol{I})\boldsymbol{e}_3| &\leq C\frac{\sqrt{e_h}}{h}\left(\|\widetilde{\boldsymbol{w}}_h\|_{L^2(\Lambda;\mathbb{R}^3)} + \frac{h^{\beta/2}}{\sqrt{e_h}}\|D'\overline{v}\|_{L^2(S;\mathbb{R}^2)}\right) \\ &\leq C\left(\frac{\sqrt{r_h}}{h} + h^{\beta/2-1}\right). \end{aligned} \tag{4.187}$$

In turn, we also have

$$|(\boldsymbol{Q}_h^\top - \boldsymbol{I})\boldsymbol{e}_3| \leq C\left(\frac{\sqrt{r_h}}{h} + h^{\beta/2-1}\right). \tag{4.188}$$

Looking at the first two components of (4.185), we estimate

$$\begin{aligned} ||(\boldsymbol{Q}_h'' - \boldsymbol{I}'')\boldsymbol{x}' + \boldsymbol{d}_h'||_{L^2(\Lambda;\mathbb{R}^2)} &\leq C\left(\frac{e_h}{h^2} \vee \sqrt{e_h}\right)\|\widetilde{\boldsymbol{u}}_h\|_{L^2(\Lambda;\mathbb{R}^2)} + C\frac{\sqrt{e_h}}{h}\|\widetilde{v}_h\|_{L^2(\Lambda)} \\ &\quad + C\left(\frac{e_h}{h^2} \vee \sqrt{e_h}\right)\|\boldsymbol{u}_h\|_{L^2(S;\mathbb{R}^2)}. \end{aligned}$$

From this, recalling (4.179)–(4.180) and (4.182)–(4.183), we obtain

$$\begin{aligned} ||(\boldsymbol{Q}_h'' - \boldsymbol{I}'')\boldsymbol{x}' + \boldsymbol{d}_h'||_{L^2(\Lambda;\mathbb{R}^2)} &\leq C\left(\frac{e_h}{h^2} \vee \sqrt{e_h}\right)\left(\sqrt{\frac{r_h}{e_h}} + \frac{r_h}{e_h}\right) + C\frac{\sqrt{e_h}}{h}\sqrt{\frac{r_h}{e_h}} \\ &\quad + C\left(\frac{e_h}{h^2} \vee \sqrt{e_h}\right)\left(\frac{h^{\beta/2+2}}{e_h} \wedge \frac{h^{\beta/2}}{\sqrt{e_h}}\right) \\ &\leq C\left(\frac{\sqrt{r_h}}{h} + h^{\beta/2}\right). \end{aligned} \tag{4.189}$$

In the last inequality, we exploited that $r_h/h^2 \leq 1$ for $h \ll 1$.
Up to translations, we can assume that

$$\int_\Lambda \boldsymbol{x}'\,\mathrm{d}\boldsymbol{l} = \boldsymbol{0}'.$$

In this case, (4.189) yields

$$|\boldsymbol{Q}_h'' - \boldsymbol{I}''| + |\boldsymbol{d}_h'| \leq C\left(\frac{\sqrt{r_h}}{h} + h^{\beta/2}\right) \tag{4.190}$$

Combining (4.187)–(4.188) with (4.190), we establish (4.177).
Similarly, looking at the third component of (4.185) and exploiting (4.179)–(4.180), (4.183) and (4.188), we estimate

$$\begin{aligned}
|d_h^3| &\leq C\left(\frac{e_h}{h^2} \vee \sqrt{e_h}\right)\|\widetilde{\boldsymbol{u}}_h\|_{L^2(\Lambda;\mathbb{R}^2)} + C\frac{\sqrt{e_h}}{h}\|\widetilde{v}_h\|_{L^2(\Lambda)} \\
&\quad + C\frac{\sqrt{e_h}}{h}\|v_h\|_{L^2(\Lambda)} + C|(\boldsymbol{Q}_h^\top - \boldsymbol{I})\boldsymbol{e}_3| \\
&\leq C\left(\frac{e_h}{h^2} \vee \sqrt{e_h}\right)\left(\sqrt{\frac{r_h}{e_h}} + \frac{r_h}{e_h}\right) + C\frac{\sqrt{e_h}}{h}\sqrt{\frac{r_h}{e_h}} \\
&\quad + C\frac{\sqrt{e_h}}{h}\frac{h^{\beta/2}}{\sqrt{e_h}} + C\left(\frac{\sqrt{r_h}}{h} + h^{\beta/2-1}\right) \\
&\leq C\left(\frac{\sqrt{r_h}}{h} + h^{\beta/2-1}\right).
\end{aligned} \tag{4.191}$$

Thus, (4.178) follows from (4.190)–(4.191). □

The next result provides the analog of Proposition 4.6 under clamped boundary conditions. For technical reasons, we need precise estimates on the norm of the averaged displacements.

Proposition 4.18 (Compactness of deformations under clamped BCs). *Let $(\widehat{\boldsymbol{y}}_h) \subset W^{1,2}(\Omega;\mathbb{R}^3)$ and let $(e_h) \subset \mathbb{R}$ with $e_h > 0$ be such that $e_h/h^2 \to 0^+$. Set $r_h \coloneqq \mathcal{R}_h(\boldsymbol{y}_h)$ and suppose that $r_h \leq Ce_h$ for every $h > 0$. Also, set $\widehat{\boldsymbol{F}}_h \coloneqq D_h\widehat{\boldsymbol{y}}_h$ and suppose that there exist $(\widehat{\boldsymbol{R}}_h) \subset W^{1,2}(S;SO(3))$ and $(\widehat{\boldsymbol{Q}}_h) \subset SO(3)$ such that, for every $h > 0$, the following estimates hold:*

$$\|\widehat{\boldsymbol{F}}_h - \widehat{\boldsymbol{R}}_h\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} \leq C\sqrt{r_h}, \qquad \|D'\widehat{\boldsymbol{R}}_h\|_{L^2(S;\mathbb{R}^{3\times3\times3})} \leq Ch^{-1}\sqrt{r_h}, \tag{4.192}$$

$$\|\widehat{\boldsymbol{R}}_h - \widehat{\boldsymbol{Q}}_h\|_{L^2(S;\mathbb{R}^{3\times 3})} \leq Ch^{-1}\sqrt{r_h}, \quad \|\widehat{\boldsymbol{F}}_h - \widehat{\boldsymbol{Q}}_h\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} \leq Ch^{-1}\sqrt{r_h}, \tag{4.193}$$

$$|\widehat{\boldsymbol{Q}}_h - \boldsymbol{I}| \leq C\left(\frac{\sqrt{r_h}}{h} + h^{\beta/2-1}\right). \tag{4.194}$$

Also, assume that $\boldsymbol{y}_h = \overline{\boldsymbol{y}}_h$ on Γ for every $h > 0$. Define

$$\widehat{\boldsymbol{u}}_h\colon S \to \mathbb{R}^2, \qquad \widehat{v}_h\colon S \to \mathbb{R}, \qquad \widehat{\boldsymbol{w}}_h\colon S \to \mathbb{R}^3,$$

by setting

$$\begin{aligned}
\widehat{\boldsymbol{u}}_h(\boldsymbol{x}') &\coloneqq \frac{h^2}{e_h} \wedge \frac{1}{\sqrt{e_h}}\int_I \left(\widehat{\boldsymbol{y}}_h'(\boldsymbol{x}', x_3) - \boldsymbol{x}'\right)\,\mathrm{d}x_3, \\
\widehat{v}_h(\boldsymbol{x}') &\coloneqq \frac{h}{\sqrt{e_h}}\int_I \widehat{y}_h^3(\boldsymbol{x}', x_3)\,\mathrm{d}x_3, \\
\widehat{\boldsymbol{w}}_h(\boldsymbol{x}') &\coloneqq \frac{1}{\sqrt{e_h}}\int_I x_3\left(\widehat{\boldsymbol{y}}_h(\boldsymbol{x}', x_3) - \boldsymbol{\pi}_h(\boldsymbol{x}', x_3)\right)\mathrm{d}x_3.
\end{aligned}$$

Then, the following estimates hold:

$$\|\widehat{\boldsymbol{u}}_h\|_{W^{1,2}(S;\mathbb{R}^2)} \leq C\left(\sqrt{\frac{r_h}{e_h}}+\frac{r_h}{e_h}\wedge\frac{r_h}{h^2\sqrt{e_h}}+\frac{h^\beta}{e_h}\right), \tag{4.195}$$

$$\|\widehat{v}_h\|_{W^{1,2}(S)} \leq C\left(\sqrt{\frac{r_h}{e_h}}+\frac{h^{\beta/2}}{\sqrt{e_h}}\right), \tag{4.196}$$

$$\|\widehat{\boldsymbol{w}}_h\|_{W^{1,2}(S;\mathbb{R}^3)} \leq C\sqrt{\frac{r_h}{e_h}}. \tag{4.197}$$

Moreover, there exist $\widehat{\boldsymbol{u}} \in W^{1,2}(S;\mathbb{R}^2)$ and $\widehat{v} \in W^{2,2}(S)$ such that, up to subsequence, the following convergences hold, as $h \to 0^+$:

$$\widehat{\boldsymbol{u}}_h \rightharpoonup \widehat{\boldsymbol{u}} \text{ in } W^{1,2}(S;\mathbb{R}^2); \tag{4.198}$$

$$\widehat{v}_h \to \widehat{v} \text{ in } W^{1,2}(S); \tag{4.199}$$

$$\widehat{\boldsymbol{w}}_h \rightharpoonup -\frac{1}{12}\begin{pmatrix} D'\widehat{v} \\ 0 \end{pmatrix} \text{ in } W^{1,2}(S;\mathbb{R}^3). \tag{4.200}$$

Proof. From (4.193)–(4.194), we immediately get the estimates

$$\|\widehat{\boldsymbol{R}}_h-\boldsymbol{I}\|_{L^2(S;\mathbb{R}^{3\times 3})} \leq C\left(\frac{\sqrt{r_h}}{h}+h^{\beta/2-1}\right),$$

$$\|\widehat{\boldsymbol{F}}_h-\boldsymbol{I}\|_{L^2(S;\mathbb{R}^{3\times 3})} \leq C\left(\frac{\sqrt{r_h}}{h}+h^{\beta/2-1}\right).$$

At this point, we proceed as in the proof of Proposition 4.6. Define

$$\widehat{\boldsymbol{A}}_h := \frac{h}{\sqrt{e_h}}(\widehat{\boldsymbol{R}}_h-\boldsymbol{I}), \qquad \widehat{\boldsymbol{B}}_h := \frac{h}{\sqrt{e_h}}(\widehat{\boldsymbol{F}}_h-\boldsymbol{I}), \qquad \widehat{\boldsymbol{C}}_h := \frac{h^2}{e_h}\mathrm{sym}(\widehat{\boldsymbol{R}}_h-\boldsymbol{I}).$$

First, we obtain the estimates

$$\|\widehat{\boldsymbol{A}}_h\|_{W^{1,2}(S;\mathbb{R}^{3\times 3})} \leq C\left(\sqrt{\frac{r_h}{e_h}}+\frac{h^{\beta/2}}{\sqrt{e_h}}\right), \tag{4.201}$$

$$\|\widehat{\boldsymbol{B}}_h\|_{L^2(S;\mathbb{R}^{3\times 3})} \leq C\left(\sqrt{\frac{r_h}{e_h}}+\frac{h^{\beta/2}}{\sqrt{e_h}}\right), \tag{4.202}$$

$$\|\widehat{\boldsymbol{C}}_h\|_{L^2(S;\mathbb{R}^{3\times 3})} \leq C\left(\frac{r_h}{e_h}++\frac{h^\beta}{e_h}\right). \tag{4.203}$$

Then, we show (4.195)–(4.197) by taking into account the boundary condition on $\boldsymbol{y}_h$ while applying the Poincaré inequality. Eventually, we prove (4.198)–(4.200). $\square$

For convenience, in the next result we register some useful estimates for the norm of the averaged displacements that follows from Proposition 4.18 in the case $e_h = h^\beta$.

Corollary 4.19 (Norm of the averaged displacements). *Let $(\widehat{\boldsymbol{q}}_h)$ with $\widehat{\boldsymbol{q}}_h = (\widehat{\boldsymbol{y}}_h,\widehat{\boldsymbol{m}}_h) \in \mathcal{Q}_h$. Set $r_h := \mathcal{R}_h(\widehat{\boldsymbol{y}}_h)$ and suppose that $r_h \leq Ch^\beta$ for every $h>0$. Also, set $\widehat{\boldsymbol{u}}_h := \mathcal{U}_h(\widehat{\boldsymbol{q}}_h)$, $\widehat{v}_h := \mathcal{V}_h(\widehat{\boldsymbol{q}}_h)$ and $\widehat{\boldsymbol{w}}_h := \mathcal{W}_h(\widehat{\boldsymbol{q}}_h)$, and suppose that the following* *estimates hold:*

$$\|\widehat{\boldsymbol{u}}_h\|_{W^{1,2}(S;\mathbb{R}^2)} \leq C\left(\frac{\sqrt{r_h}}{h^{\beta/2}} + \frac{r_h}{h^\beta} + 1\right), \tag{4.204}$$

$$\|\widehat{v}_h\|_{W^{1,2}(S)} \leq C\left(\frac{\sqrt{r_h}}{h^{\beta/2}} + 1\right), \tag{4.205}$$

$$\|\widehat{\boldsymbol{w}}_h\|_{W^{1,2}(S;\mathbb{R}^3)} \leq C\frac{\sqrt{r_h}}{h^{\beta/2}}. \tag{4.206}$$

Then, there holds

$$\|\widehat{\boldsymbol{u}}_h\|_{W^{1,2}(S;\mathbb{R}^2)} + \|\widehat{v}_h\|_{W^{1,2}(S;\mathbb{R}^2)} + \|\widehat{\boldsymbol{w}}_h\|_{W^{1,2}(S;\mathbb{R}^3)} \leq C\left(\sqrt{E_h^{\mathrm{el}}(\widehat{\boldsymbol{q}}_h)} + 1\right). \tag{4.207}$$

Proof. By (4.148), we have that $r_h/h^\beta \leq C + E_h^{\mathrm{el}}(\widehat{\boldsymbol{q}}_h)$. Thus, (4.207) immediately follows from (4.204)–(4.206). □

The major difficulty in proving Theorem 4.14 is to deduce the equi-boundedness of the elastic energy starting from the equi-boundedness of the total energy. This is accomplished by arguing by contradiction, similarly to [97, Theorem 4]. The strategy is quite involved and relies on the estiamtes provided in Proposition 4.18.

Lemma 4.20 (Energy scaling). *Let $M > 0$ and let $(\boldsymbol{q}_h)$ with $\boldsymbol{q}_h \in \overline{\mathcal{Q}}_h$ for every $h > 0$ be such that*

$$\sup_{h>0} F_h(\boldsymbol{q}_h) \leq M. \tag{4.208}$$

Then, there exist $C(M) > 0$ and $\overline{h} > 0$, where the former does not depend on $(\boldsymbol{q}_h)$, such that

$$\sup_{h\leq\overline{h}} E_h(\boldsymbol{q}_h) \leq C(M). \tag{4.209}$$

Proof. We introduce some further notation. Given $h > 0$ and $\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m}) \in \overline{\mathcal{Q}}_h$, we set

$$I_h(\boldsymbol{q}) \coloneqq \int_\Omega W_h(D_h\boldsymbol{y}, \boldsymbol{m}\circ\boldsymbol{y})\,\mathrm{d}\boldsymbol{x},$$

$$J_h(\boldsymbol{q}) = I_h(\boldsymbol{q}) - \int_\Omega \boldsymbol{f}_h\cdot(\boldsymbol{y}' - \boldsymbol{x}')\,\mathrm{d}\boldsymbol{x} + \int_\Omega g_h\, y^3\,\mathrm{d}\boldsymbol{x}.$$

For convenience of the reader, the proof is subdivided into three steps.

Step 1 (Preliminary estimates). First, we check that

$$I_h(\boldsymbol{q}_h) \leq C(M)h^{\beta/2}. \tag{4.210}$$

Let $\boldsymbol{q}_h = (\boldsymbol{y}_h, \boldsymbol{m}_h)$ and, for simplicity, set

$$\boldsymbol{F}_h \coloneqq D_h\boldsymbol{y}_h, \quad \boldsymbol{\lambda}_h \coloneqq \boldsymbol{m}_h\circ\boldsymbol{y}_h, \quad \boldsymbol{L}_h \coloneqq \boldsymbol{L}_h(\boldsymbol{\lambda}_h \det\boldsymbol{F}_h), \quad \Xi_h \coloneqq \boldsymbol{L}_h^{-1}\boldsymbol{F}_h.$$

Recalling (4.148), for every $h > 0$, we have

$$r_h \coloneqq \mathcal{R}_h(\boldsymbol{y}_h) \leq Ch^\beta + I_h(\boldsymbol{q}_h), \tag{4.211}$$

which yields

$$\begin{aligned}\|\boldsymbol{F}_h\|_{L^2(\Omega;\mathbb{R}^{3\times3})} &\leq C\left(\|\mathrm{dist}(\boldsymbol{F}_h;SO(3))\|_{L^2(\Omega)}+1\right)\\ &\leq C\left(\sqrt{r_h}+1\right)\\ &\leq C\left(\sqrt{I_h(\boldsymbol{q}_h)}+1\right).\end{aligned}$$

Then, using the Poincaré inequality with trace term, we obtain

$$\begin{aligned}\|\boldsymbol{y}_h-\boldsymbol{\pi}_h\|_{L^2(\Omega;\mathbb{R}^3)} &\leq C\left(\|\boldsymbol{F}_h-\boldsymbol{I}\|_{L^2(\Omega;\mathbb{R}^{3\times3})}+\|\boldsymbol{y}_h-\boldsymbol{\pi}_h\|_{L^2(\Lambda;\mathbb{R}^3)}\right)\\ &\leq C\left(\|\boldsymbol{F}_h\|_{L^2(\Omega;\mathbb{R}^{3\times3})}+1\right)\\ &\leq C\left(\sqrt{I_h(\boldsymbol{q}_h)}+1\right).\end{aligned}\tag{4.212}$$

Given (4.161) and (4.212), applying the Hölder inequality, we estimate

$$\begin{aligned}\left|\frac{1}{h^\beta}\int_\Omega \boldsymbol{f}_h\cdot(\boldsymbol{y}_h'-\boldsymbol{x}')\,\mathrm{d}\boldsymbol{x}\right| &\leq \frac{1}{h^\beta}\|\boldsymbol{f}_h\|_{L^2(S;\mathbb{R}^2)}\,\|\boldsymbol{y}_h'-\boldsymbol{x}'\|_{L^2(\Omega;\mathbb{R}^2)}\\ &\leq \frac{C}{h^{\beta/2}}\,\|\boldsymbol{y}_h'-\boldsymbol{x}'\|_{L^2(\Omega;\mathbb{R}^2)}\\ &\leq \frac{C}{h^{\beta/2}}\left(\sqrt{I_h(\boldsymbol{q}_h)}+1\right).\end{aligned}\tag{4.213}$$

Similarly, exploiting (4.162), we obtain

$$\begin{aligned}\left|\frac{1}{h^\beta}\int_\Omega g_h\cdot y_h^3\,\mathrm{d}\boldsymbol{x}\right| &\leq \frac{1}{h^\beta}\|g_h\|_{L^2(S)}\,\|y_h^3\|_{L^2(\Omega)}\\ &\leq \frac{C}{h^{\beta/2-1}}\left(\|y_h^3-hx_3\|_{L^2(\Omega)}+1\right)\\ &\leq \frac{C}{h^{\beta/2-1}}\left(\sqrt{I_h(\boldsymbol{q}_h)}+1\right).\end{aligned}\tag{4.214}$$

Set $\boldsymbol{\mu}_h\coloneqq\mathcal{M}_h(\boldsymbol{q}_h)$. Recalling (4.163) and using the Change-of-variable Formula, we estimate

$$\begin{aligned}\left|\frac{1}{h}\int_{\Omega^{\boldsymbol{y}_h}}\boldsymbol{h}_h\cdot\boldsymbol{m}_h\,\mathrm{d}\boldsymbol{\xi}\right| &= \left|\int_{\mathbb{R}^3}\boldsymbol{h}_h\circ\boldsymbol{\pi}_h\cdot\boldsymbol{\mu}_h\,\mathrm{d}\boldsymbol{x}\right|\\ &\leq \|\boldsymbol{h}_h\circ\boldsymbol{\pi}_h\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}\,\|\boldsymbol{\mu}_h\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}\\ &\leq C\left(\sqrt{I_h(\boldsymbol{q}_h)}+1\right).\end{aligned}\tag{4.215}$$

Indeed, as in (4.63)–(4.69), there holds

$$\int_{\mathbb{R}^3}|\boldsymbol{\mu}_h|^2\,\mathrm{d}\boldsymbol{x}=\int_\Omega\frac{1}{\det\boldsymbol{F}_h}\mathrm{d}\boldsymbol{x}\leq C\int_\Omega\frac{1}{|\det\boldsymbol{F}_h|^a}\mathrm{d}\boldsymbol{x}\leq C(I_h(\boldsymbol{q}_h)+1).$$

The combination of (4.213)–(4.215) yields

$$|L_h(\boldsymbol{q}_h)|\leq\frac{C}{h^{\beta/2}}\left(\sqrt{I_h(\boldsymbol{q}_h)}+1\right),\tag{4.216}$$

so that, employing the Young inequality, we write

$$F_h(\boldsymbol{q}_h)\geq\frac{1}{h^\beta}I_h(\boldsymbol{q}_h)-\frac{C}{h^{\beta/2}}\left(\sqrt{I_h(\boldsymbol{q}_h)}+1\right)\geq\frac{C}{h^\beta}I_h(\boldsymbol{q}_h)-\frac{C'}{h^{\beta/2}}.$$

This entails (4.210) thanks to (4.208). Recalling (4.211), we immediately get

$$r_h \leq C(M) h^{\beta/2}. \tag{4.217}$$

Also, we obtain

$$J_h(\boldsymbol{q}_h) \leq C(M) h^{\beta}, \tag{4.218}$$

since

$$h^{-\beta} J_h(\boldsymbol{q}_h) \leq F_h(\boldsymbol{q}_h) + \frac{1}{h}\int_{\Omega^{\boldsymbol{y}_h}} \boldsymbol{h}_h \cdot \boldsymbol{m}_h \,\mathrm{d}\boldsymbol{\xi} \leq M + C\left(\sqrt{I_h(\boldsymbol{q}_h)}+1\right) \leq C(M),$$

thanks to (4.208), (4.210) and (4.215).

Step 2 (Improved estimates). We claim that $I_h(\boldsymbol{q}_h)$ is actually of order h^β. By contradiction, suppose that

$$\limsup_{h\to 0^+} \frac{1}{h^\beta} I_h(\boldsymbol{q}_h) = +\infty. \tag{4.219}$$

Set $\boldsymbol{F}_h := D_h \boldsymbol{y}_h$ and note that $r_h/h^2 \to 0$, as $h \to 0^+$ by (4.217) since $\beta > 4$. By means of Lemma 4.5, we find $(\boldsymbol{R}_h) \subset W^{1,p}(S; SO(3))$ and $(\boldsymbol{Q}_h) \subset SO(3)$ such that the following estimates hold:

$$\begin{aligned}
&\|\boldsymbol{F}_h - \boldsymbol{R}_h\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} \leq C\sqrt{r_h}, &&\|D'\boldsymbol{R}_h\|_{L^2(S;\mathbb{R}^{3\times 3\times 3})} \leq Ch^{-1}\sqrt{r_h},\\
&\|\boldsymbol{R}_h - \boldsymbol{Q}_h\|_{L^2(S;\mathbb{R}^{3\times 3})} \leq Ch^{-1}\sqrt{r_h}, &&\|\boldsymbol{F}_h - \boldsymbol{Q}_h\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} \leq Ch^{-1}\sqrt{r_h}.
\end{aligned}$$

Let $e_h = I_h(\boldsymbol{q}_h)$. Given (4.219), up to subsequence, we have

$$\lim_{h\to 0^+} \frac{h^\beta}{e_h} = 0, \tag{4.220}$$

which, together with (4.211), yields

$$\limsup_{h\to 0^+} \frac{r_h}{e_h} \leq 1. \tag{4.221}$$

Also, applying Lemma 4.17, we deduce

$$|\boldsymbol{Q}_h - \boldsymbol{I}| \leq C\left(\sqrt{e_h} + \frac{\sqrt{r_h}}{h} + \frac{r_h}{h^2} + \frac{r_h}{\sqrt{e_h}} + \frac{\sqrt{r_h e_h}}{h^2} + h^{\beta/2-1}\right).$$

Now, define $\boldsymbol{U}_h \colon S \to \mathbb{R}^2$ and $V_h \colon S \to \mathbb{R}$ by setting

$$\begin{aligned}
\boldsymbol{U}_h(\boldsymbol{x}') &:= \frac{h^2}{e_h} \wedge \frac{1}{\sqrt{e_h}} \int_I \left(\boldsymbol{y}_h'(\boldsymbol{x}', x_3) - \boldsymbol{x}'\right) \mathrm{d}x_3,\\
V_h(\boldsymbol{x}') &:= \frac{h}{\sqrt{e_h}} \int_I y_h^3(\boldsymbol{x}', x_3)\,\mathrm{d}x_3.
\end{aligned}$$

As the sequences (r_h/h^2), (r_h/e_h) and (h^β/e_h) are bounded because of (4.217) and (4.220)–(4.221), Proposition 4.18 entails that $(\boldsymbol{U}_h)$ and (V_h) are bounded in $W^{1,2}(S;\mathbb{R}^2)$ and $W^{1,2}(S)$, respectively, for $h \ll 1$. Hence, taking into account (4.161)–(4.162) and (4.220), we compute

$$\begin{aligned}
&\lim_{h\to 0^+} \left\{ \frac{1}{e_h}\int_\Omega \boldsymbol{f}_h \cdot (\boldsymbol{y}_h' - \boldsymbol{x}')\,\mathrm{d}\boldsymbol{x} + \frac{1}{e_h}\int_\Omega g_h y_h^3\,\mathrm{d}\boldsymbol{x} \right\}\\
&= \lim_{h\to 0^+} \left\{ \left(h^{\beta/2-2} \vee \frac{h^{\beta/2}}{\sqrt{e_h}} \right) \int_S h^{-\beta/2} \boldsymbol{f}_h \cdot \boldsymbol{U}_h\,\mathrm{d}\boldsymbol{x}' + \frac{h^{\beta/2}}{\sqrt{e_h}} \int_S h^{-\beta/2-1} g_h\, V_h\,\mathrm{d}\boldsymbol{x}' \right\} = 0.
\end{aligned}$$

This, in turn, gives

$$\begin{aligned}1 &= \lim_{h\to 0^+} \frac{1}{e_h} I_h(\boldsymbol{q}_h)\\ &= \lim_{h\to 0^+} \left\{ \frac{1}{e_h} J_h(\boldsymbol{q}_h) + \frac{1}{e_h}\int_\Omega \boldsymbol{f}_h\cdot(\boldsymbol{y}_h' - \boldsymbol{x}')\,\mathrm{d}\boldsymbol{x} + \frac{1}{e_h}\int_\Omega g_h y_h^3\,\mathrm{d}\boldsymbol{x}\right\}\\ &= \lim_{h\to 0^+} \frac{1}{e_h} J_h(\boldsymbol{q}_h) \leq C(M) \lim_{h\to 0^+} \frac{h^\beta}{e_h} = 0,\end{aligned}$$

where the last equality is justified by (4.220). This provides a contradiction and, in turn, we necessarily have

$$L \coloneqq \limsup_{h\to 0^+} \frac{1}{h^\beta} I_h(\boldsymbol{q}_h) < +\infty. \tag{4.222}$$

Step 3 (Bound on the constant). Clearly, the constant $L \geq 0$ in (4.222) depends on the sequence $(\boldsymbol{q}_h)$. We claim that

$$L \leq C(M) \tag{4.223}$$

for some constant $C(M) > 0$ depending on M but not on $(\boldsymbol{q}_h)$. First, observe that, in view of (4.211) and (4.222), there hold

$$\limsup_{h\to 0^+} \frac{r_h}{h^\beta} \leq C + L, \qquad \lim_{h\to 0^+} \frac{r_h}{h^4} = 0. \tag{4.224}$$

Set $\boldsymbol{u}_h \coloneqq \mathcal{U}_h(\boldsymbol{q}_h)$and $v_h \coloneqq \mathcal{V}_h(\boldsymbol{q}_h)$. By applying Proposition 4.18 to $\widehat{\boldsymbol{y}}_h = \boldsymbol{y}_h$ with $e_h = h^\beta$, we deduce the two estimates:

$$\|\boldsymbol{u}_h\|_{W^{1,2}(S;\mathbb{R}^2)} \leq C\left(\frac{\sqrt{r_h}}{h^{\beta/2}} + \frac{r_h}{h^{\beta/2+2}} + 1\right), \tag{4.225}$$

$$\|v_h\|_{W^{1,2}(S)} \leq C\left(\frac{\sqrt{r_h}}{h^{\beta/2}} + 1\right). \tag{4.226}$$

Then, recalling (4.161)–(4.162) and (4.218) and employing the Hölder inequality, we estimate

$$\begin{aligned}\frac{1}{h^\beta} I_h(\boldsymbol{q}_h) &= \frac{1}{h^\beta} J_h(\boldsymbol{q}_h) + \int_S h^{-\beta/2} \boldsymbol{f}_h\cdot\boldsymbol{u}_h\,\mathrm{d}\boldsymbol{x}' + \int_S h^{-\beta/2-1} g_h\, v_h\,\mathrm{d}\boldsymbol{x}'\\ &\leq C(M) + C\left(\|\boldsymbol{u}_h\|_{L^2(S;\mathbb{R}^2)} + \|v_h\|_{L^2(S)}\right).\end{aligned}$$

Exploiting (4.225)–(4.226), we take the superior limit, as $h \to 0^+$, at both sides of the previous inequality with the aid of (4.222) and (4.224). This yields

$$L \leq C(M) + C\sqrt{L},$$

which entails (4.223) by applying the Young inequality.
Therefore, in view of (4.222), we have proved that

$$I_h(\boldsymbol{q}_h) \leq C(M) h^\beta$$

for $h \ll 1$ depending on $(\boldsymbol{q}_h)$. At this point, from (4.216), we infer $L_h(\boldsymbol{q}_h) \leq C(M)$ and, recalling (4.208), also (4.209). □

We now prove the compactness of sequences of admissible states with equi-bounded energy under clamped boundary conditions. This result is going to be instrumental also in the quasistatic setting.

Proposition 4.21 (Compactness under clamped BCs). *Let $(\boldsymbol{q}_h)$ with $\boldsymbol{q}_h = (\boldsymbol{y}_h, \boldsymbol{m}_h) \in \overline{\mathcal{Q}}_h$ be such that*

$$\sup_{h>0} \left\{ E_h^{\mathrm{el}}(\boldsymbol{q}_h) + E_h^{\mathrm{mag}}(\boldsymbol{q}_h) \right\} \leq C. \tag{4.227}$$

Then, there exist $\boldsymbol{q}_0 = (\boldsymbol{u}, v, \boldsymbol{\zeta}) \in \overline{\mathcal{Q}}_0$ and $\boldsymbol{\chi} \in L^2(\mathbb{R}^3;\mathbb{R}^3)$ such that, up to subsequences, the following convergences hold, as $h \to 0^+$:

$$\boldsymbol{u}_h := \mathcal{U}_h(\boldsymbol{q}_h) \rightharpoonup \boldsymbol{u} \text{ in } W^{1,2}(S;\mathbb{R}^2); \tag{4.228}$$

$$v_h := \mathcal{V}_h(\boldsymbol{q}_h) \to v \text{ in } W^{1,2}(S); \tag{4.229}$$

$$\boldsymbol{w}_h := \mathcal{W}_h(\boldsymbol{q}_h) \to -\frac{1}{12}\begin{pmatrix} D'v \\ 0 \end{pmatrix} \text{ in } W^{1,2}(S;\mathbb{R}^3); \tag{4.230}$$

$$\boldsymbol{\mu}_h := \mathcal{M}_h(\boldsymbol{q}_h) \to \chi_\Omega \boldsymbol{\zeta} \text{ in } L^2(\mathbb{R}^3;\mathbb{R}^3); \tag{4.231}$$

$$\boldsymbol{N}_h := \mathcal{N}_h(\boldsymbol{q}_h) \rightharpoonup \chi_\Omega \left(D'\boldsymbol{\zeta}, \boldsymbol{\chi}\right) \text{ in } L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3}); \tag{4.232}$$

$$\boldsymbol{\lambda}_h := \boldsymbol{m}_h \circ \boldsymbol{y}_h \to \boldsymbol{\zeta} \text{ in } L^1(\Omega;\mathbb{R}^3); \tag{4.233}$$

$$\boldsymbol{z}_h := \mathcal{Z}_h(\boldsymbol{q}_h) \to \boldsymbol{\zeta} \text{ in } L^q(\Omega;\mathbb{R}^3) \text{ for every } 1 \leq q < \infty. \tag{4.234}$$

Proof. Set $r_h := \mathcal{R}_h(\boldsymbol{y}_h)$. From (4.211) and (4.227), we obtain the bound $r_h \leq Ch^\beta$. By applying Lemma 4.5, we establish the existence of $(\boldsymbol{R}_h) \subset W^{1,p}(S; SO(3))$ and $(\boldsymbol{Q}_h) \subset SO(3)$ such that (4.149)–(4.150) hold for $q \in \{2, p\}$. Taking $e_h = h^\beta$ in Lemma 4.17, we obtain the estimate

$$|\boldsymbol{Q}_h - \boldsymbol{I}| \leq Ch^{\beta/2-1},$$

so that, setting $\boldsymbol{F}_h := D_h\boldsymbol{y}_h$, we deduce

$$\|\boldsymbol{F}_h - \boldsymbol{I}\|_{L^q(\Omega;\mathbb{R}^{3\times 3})} \leq Ch^{\beta/2-1}. \tag{4.235}$$

From this, by employing the interpolation inequality as in the proof of Theorem 4.3, we also find $q_0 > 3$ and $\alpha_0 > 1$ such that

$$\|\boldsymbol{F}_h - \boldsymbol{I}\|_{L^{q_0}(\Omega;\mathbb{R}^{3\times 3})} \leq Ch^{\alpha_0}. \tag{4.236}$$

Now, using (4.235) with $q = p$ and the Poincaré inequality, we infer (**??**), while, by Proposition 4.18, we deduce (4.228)–(4.230). Eventually, in view of (4.227) and (4.235)–(4.236), claims (4.231)–(4.234) follow at once by applying Proposition 4.9 to $\widehat{\boldsymbol{q}}_h = \boldsymbol{q}_h$. □

The next result provides the existence of recovery sequence under clamped boundary conditions. Although its proof is analogous to the one of Proposition 4.12, the requirement of the deformation to satisfy the boundary condition represents a technical difficulty. This difficulty is overcame by employing an approximation result which is available in the literature.

Proposition 4.22 (Recovery sequence under clamped BCs). *Let $\widehat{\boldsymbol{q}}_0 \in \overline{\mathcal{Q}}_0$ with $\widehat{\boldsymbol{q}}_0 = (\widehat{\boldsymbol{u}}, \widehat{v}, \widehat{\boldsymbol{\zeta}}) \in \overline{\mathcal{Q}}_0$. Then, there exists $(\widehat{\boldsymbol{q}}_h)$ with $\widehat{\boldsymbol{q}}_h \in \overline{\mathcal{Q}}_h$ such that the convergence in* (4.106)–(4.114) *hold true as well as* (4.115).

Proof. The proof works as the one of Proposition 4.12. In Step 1, we additionally assume that $\widehat{\boldsymbol{u}} = \overline{\boldsymbol{u}}$ on Λ, $\widehat{v} = \overline{v}$ on Λ and $D'\widehat{v} = D'\overline{v}$ on Λ. In this way, each deformation $\widehat{\boldsymbol{y}}_h$ clearly satisfies $\widehat{\boldsymbol{y}}_h = \overline{\boldsymbol{y}}_h$ on Γ. Then, in Step 3, we apply [66, Proposition A.2] to choose the sequences $(\widehat{\boldsymbol{u}}_j)$ and $(\widehat{v}_j)$ to satisfy $\widehat{\boldsymbol{u}}_j = \overline{\boldsymbol{u}}$ on Λ, $\widehat{v}_j = \overline{v}$ on Λ and $D'\widehat{v}_j = D'\overline{v}$ on Λ for every $j \in \mathbb{N}$. Here, we exploit the regularity assumption (4.167). $\square$

4.3.2 Proof of Theorem 4.14

We are now ready to prove our second main result.

Proof of Theorem 4.14. For convenience of the reader, the proof is subdivided into two steps.

Step 1 (A priori estimates). For every $h > 0$, set $\overline{\boldsymbol{F}}_h := D_h \overline{\boldsymbol{y}}_h$. Define $\overline{\boldsymbol{m}}_h \colon \Omega^{\overline{\boldsymbol{y}}_h} \to \mathbb{R}^3$ by setting

$$\overline{\boldsymbol{m}}_h := \frac{\boldsymbol{e}}{\det \overline{\boldsymbol{F}}_h \circ \overline{\boldsymbol{y}}_h^{-1}},$$

for some fixed $\boldsymbol{e} \in \mathbb{S}^2$. Thus, $\overline{\boldsymbol{q}}_h := (\overline{\boldsymbol{y}}_h, \overline{\boldsymbol{m}}_h) \in \overline{\mathcal{Q}}_h$. We claim that $F_h(\overline{\boldsymbol{q}}_h) \leq C$. First, with computations analogous to the ones in (4.118)–(4.120), we check that

$$\overline{\boldsymbol{F}}_h = \boldsymbol{I} + O(h^{\beta/2-1}), \quad \det \overline{\boldsymbol{F}}_h = 1 + O(h^{\beta/2}), \quad D(\det \overline{\boldsymbol{F}}_h) = O(h^{\beta/2}). \tag{4.237}$$

Also, we compute

$$\sqrt{\overline{\boldsymbol{F}}_h^\top \overline{\boldsymbol{F}}_h} = \boldsymbol{I} + h^{\beta/2} \left(\overline{\boldsymbol{D}} + x_3 \overline{\boldsymbol{E}} \right).$$

where, for convenience, we set

$$\overline{\boldsymbol{D}} := \left(\begin{array}{c|c} \mathrm{sym} D'\overline{\boldsymbol{u}} & \boldsymbol{0}' \\ \hline (\boldsymbol{0}')^\top & 0 \end{array} \right), \qquad \overline{\boldsymbol{E}} := \left(\begin{array}{c|c} -(D')^2 \overline{v} & \boldsymbol{0}' \\ \hline (\boldsymbol{0}')^\top & 0 \end{array} \right).$$

By the Polar Decomposition Theorem, we have $\overline{\boldsymbol{F}}_h = \overline{\boldsymbol{P}}_h \left(\overline{\boldsymbol{F}}_h^\top \overline{\boldsymbol{F}}_h \right)^{1/2}$ for some $(\overline{\boldsymbol{P}}_h) \subset W^{1,\infty}(S; SO(3))$. Setting

$$\overline{\boldsymbol{z}}_h := \mathcal{Z}_h(\overline{\boldsymbol{q}}_h), \quad \overline{\boldsymbol{L}}_h := \boldsymbol{L}_h(\overline{\boldsymbol{P}}_h^\top \overline{\boldsymbol{z}}_h), \quad \overline{\boldsymbol{K}}_h := h^{-\beta/2}(\boldsymbol{I} - \overline{\boldsymbol{L}}_h^{-1}),$$

using (4.16) as in (4.128)–(4.130), we compute

$$\begin{aligned} E_h^{\mathrm{el}}(\overline{\boldsymbol{q}}_h) &= \frac{1}{h^\beta} \int_\Omega \Phi \left(\overline{\boldsymbol{L}}_h^{-1} \left(\overline{\boldsymbol{F}}_h^\top \overline{\boldsymbol{F}}_h \right)^{1/2} \right) \mathrm{d}\boldsymbol{x} \\ &= \frac{1}{2} \int_\Omega Q \left(\overline{\boldsymbol{D}} - \overline{\boldsymbol{K}}_h + x_3 \overline{\boldsymbol{E}} \right) \mathrm{d}\boldsymbol{x} \\ &+ \frac{1}{h^\beta} \int_\Omega \omega \left(h^{\beta/2} \left(\overline{\boldsymbol{D}} - \overline{\boldsymbol{K}}_h + x_3 \overline{\boldsymbol{E}} \right) \right) \mathrm{d}\boldsymbol{x}. \end{aligned}$$

Since $(\overline{\boldsymbol{K}}_h)$ is uniformly bounded, this shows that $E_h^{\mathrm{el}}(\overline{\boldsymbol{q}}_h) \leq C$. Using Proposition

1.46, we write

$$\begin{aligned}E_h^{\mathrm{exc}}(\overline{\boldsymbol{q}}_h) &= \int_{\Omega^{\overline{\boldsymbol{y}}_h}} |D\overline{\boldsymbol{m}}_h|^2 \circ \overline{\boldsymbol{y}}_h \,\det D\overline{\boldsymbol{y}}_h \,\mathrm{d}\boldsymbol{x}\\ &= \int_\Omega \frac{|\boldsymbol{e}\otimes D(\det \overline{\boldsymbol{F}}_h|^2}{(\det \overline{\boldsymbol{F}}_h)^4} \det D\overline{\boldsymbol{y}}_h \,\mathrm{d}\boldsymbol{x}\\ &= h\int_\Omega \frac{|D(\det \overline{\boldsymbol{F}}_h)|}{(\det \overline{\boldsymbol{F}}_h)^3}\,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

Thus, $E_h^{\mathrm{exc}}(\overline{\boldsymbol{q}}_h) \leq C$ in view of (4.237). Denote by $\overline{\psi}_h$ the stray field potential corresponding to $\overline{\boldsymbol{q}}_h$. By (2.12), we have

$$\int_{\mathbb{R}^3} |D\overline{\psi}_h|^2 \,\mathrm{d}\boldsymbol{\xi} \leq \int_{\Omega^{\overline{\boldsymbol{y}}_h}} |\overline{\boldsymbol{m}}_h|^2\,\mathrm{d}\boldsymbol{\xi} = h\int_\Omega \frac{1}{\det \overline{\boldsymbol{F}}_h}\,\mathrm{d}\boldsymbol{x},$$

so that $E_h^{\mathrm{mag}}(\overline{\boldsymbol{q}}_h) \leq C$, again, thanks to (4.237). Indeed, by Proposition 1.46, we have

$$\begin{aligned}\int_{\Omega^{\overline{\boldsymbol{y}}_h}} |\overline{\boldsymbol{m}}_h|^2\,\mathrm{d}\boldsymbol{\xi} &= \int_\Omega |\overline{\boldsymbol{m}}_h\circ\overline{\boldsymbol{y}}_h|^2 \,\det D\overline{\boldsymbol{y}}_h\,\mathrm{d}\boldsymbol{x}\\ &= \int_\Omega \frac{|\boldsymbol{e}|^2}{(\det \overline{\boldsymbol{F}}_h)^2}\det D\overline{\boldsymbol{y}}_h\,\mathrm{d}\boldsymbol{x} = h\int_\Omega \frac{1}{\det \overline{\boldsymbol{F}}_h}\,\mathrm{d}\boldsymbol{x}.\end{aligned} \tag{4.238}$$

This shows that $E_h(\overline{\boldsymbol{q}}_h) \leq C$. Applying the Hölder inequality and Proposition 1.46, we estimate

$$\begin{aligned}|L_h(\overline{\boldsymbol{q}}_h)| &\leq \left|\frac{1}{h^\beta}\int_\Omega \boldsymbol{f}_h\cdot(\overline{\boldsymbol{y}}_h - \boldsymbol{x}')\,d\boldsymbol{x}\right| + \left|\frac{1}{h^\beta}\int_\Omega g_h\,\overline{y}_h^3\,\mathrm{d}\boldsymbol{x}\right| + \left|\frac{1}{h}\int_{\Omega^{\overline{\boldsymbol{y}}_h}} \boldsymbol{h}_h\cdot\overline{\boldsymbol{m}}_h\,\mathrm{d}\boldsymbol{\xi}\right|\\ &\leq \left|\int_S h^{-\beta/2}\boldsymbol{f}_h\cdot\overline{\boldsymbol{u}}\,\mathrm{d}\boldsymbol{x}'\right| + \left|\int_S h^{-\beta/2-1} g_h\cdot\overline{v}\,\mathrm{d}\boldsymbol{x}'\right|\\ &\quad+\left|\int_{\boldsymbol{\pi}_h^{-1}(\Omega^{\overline{\boldsymbol{y}}_h})} \boldsymbol{h}_h\circ\boldsymbol{\pi}_h\cdot\overline{\boldsymbol{m}}_h\circ\boldsymbol{\pi}_h\,\mathrm{d}\boldsymbol{x}\right|\\ &\leq h^{-\beta/2}\|\boldsymbol{f}_h\|_{L^2(S;\mathbb{R}^2)}\,\|\overline{\boldsymbol{u}}\|_{L^2(S;\mathbb{R}^2)} + h^{-\beta/2-1}\|g_h\|_{L^2(S)}\,\|\overline{v}\|_{L^2(S)}\\ &\quad+\|\boldsymbol{h}_h\circ\boldsymbol{\pi}_h\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}\,\|\chi_{\boldsymbol{\pi}_h^{-1}(\Omega^{\overline{\boldsymbol{y}}_h})}\overline{\boldsymbol{m}}_h\circ\boldsymbol{\pi}_h\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \leq C,\end{aligned}$$

where we used (4.161)–(4.163) and we set $\overline{\boldsymbol{\mu}}_h := \mathcal{M}_h(\overline{\boldsymbol{q}}_h)$. This because, by (4.238) and Proposition 1.46, there holds

$$\int_{\mathbb{R}^3}|\overline{\boldsymbol{\mu}}_h|^2\,\mathrm{d}\boldsymbol{\xi} = \int_{\boldsymbol{\pi}_h^{-1}(\Omega^{\overline{\boldsymbol{y}}_h})}|\overline{\boldsymbol{m}}_h\circ\boldsymbol{\pi}_h|^2\,\mathrm{d}\boldsymbol{\xi} = \frac{1}{h}\int_{\Omega^{\overline{\boldsymbol{y}}_h}}|\overline{\boldsymbol{m}}_h|^2\,\mathrm{d}\boldsymbol{\xi} = \int_\Omega\frac{1}{\det\overline{\boldsymbol{F}}_h}\,\mathrm{d}\boldsymbol{x},$$

where the right-hand side is uniformly bounded thaks to (4.237). Therefore, the claim is proved. Clearly, this gives $\inf_{\overline{\mathcal{Q}}_h} F_h \leq C$ which, in view of (4.171), yields $F_h(\boldsymbol{q}_h) \leq C$.

Step 2 (Compactness and minimality). By Lemma 4.20, we have $E_h(\boldsymbol{q}_h) \leq C$ for $h \ll 1$. Then, by Proposition 4.21, there exist $\boldsymbol{q}_0 = (\boldsymbol{u}, v, \boldsymbol{\zeta}) \in \overline{\mathcal{Q}}_0$ and $\boldsymbol{\chi} \in L^2(\mathbb{R}^3;\mathbb{R}^3)$ such that the convergences in (4.172)–(4.174) and (4.231)–(4.232) hold true. Also, set $\boldsymbol{F}_h := D_h \boldsymbol{y}_h$. By Lemma 4.5, there exists $(\boldsymbol{R}_h) \subset W^{1,2}(S;\mathbb{R}^{3\times 3})$ satisfying

$$\|\boldsymbol{F}_h - \boldsymbol{R}_h\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} \leq C h^{\beta/2}.$$

Given this and (4.172)–(4.173), we are in a position of applying Lemma 4.8 to $\widehat{\boldsymbol{y}}_h = \boldsymbol{y}_h$ with $e_h = h^\beta$. Thus, we find $\boldsymbol{G} \in L^2(\Omega;\mathbb{R}^{3\times 3})$ such that

$$\boldsymbol{G}_h := h^{-\beta/2}(\boldsymbol{R}_h^\top \boldsymbol{F}_h - \boldsymbol{I}) \rightharpoonup \boldsymbol{G} \text{ in } L^2(\Omega;\mathbb{R}^{3\times 3}), \tag{4.239}$$

and, for almost every $\boldsymbol{x} \in \Omega$, there holds

$$\boldsymbol{G}''(\boldsymbol{x}', x_3) = \mathrm{sym} D'\boldsymbol{u}(\boldsymbol{x}') + ((D')^2 v(\boldsymbol{x}'))x_3. \tag{4.240}$$

In view of (4.174), (4.231)–(4.232) and (4.239)–(4.240), we apply Proposition 4.11 to $\widehat{\boldsymbol{q}}_h = \boldsymbol{q}_h$ and we conclude that

$$F_0(\boldsymbol{q}_0) \leq \liminf_{h\to 0^+} F_h(\boldsymbol{q}_h). \tag{4.241}$$

Eventually, let $\widehat{\boldsymbol{q}}_0 = (\widehat{\boldsymbol{u}}, \widehat{v}, \widehat{\boldsymbol{\zeta}}) \in \overline{\mathcal{Q}}_0$. By Proposition 4.22, there exists $(\widehat{\boldsymbol{q}}_h)$ with $\widehat{\boldsymbol{q}}_h \in \overline{\mathcal{Q}}_h$ such that (4.115) holds true. Thus, combining (4.115), (4.171) and (4.241), we get

$$F_0(\boldsymbol{q}_0) \leq \liminf_{h\to 0^+} F_h(\boldsymbol{q}_h) \leq \liminf_{h\to 0^+} F_h(\widehat{\boldsymbol{q}}_h) = F_0(\widehat{\boldsymbol{q}}_0).$$

Since $\widehat{\boldsymbol{q}}_0$ is arbitrary, this shows that $\boldsymbol{q}_0$ is a minimizer of F_0 on $\overline{\mathcal{Q}}_0$. □

We conclude the chapter with some variants and extensions. These are collected in the following remarks.

Remark 4.23 (Integral clamped boundary conditions). Similary to Remark 2.16, we can consider integral boundary conditions. In this case, for every $h > 0$, we define the functional $E_h^{\mathrm{cl}} \colon \mathcal{Q}_h \to [0,+\infty)$ by setting

$$E_h^{\mathrm{cl}}(\boldsymbol{q}) := \frac{1}{h^\beta}\int_\Gamma |\overline{\boldsymbol{y}}_h' - \boldsymbol{y}'|^2 \,\mathrm{d}\boldsymbol{a} + \frac{1}{h^{\beta-2}}\int_\Gamma |\overline{y}_h^3 - y^3|^2 \,\mathrm{d}\boldsymbol{a} + \frac{1}{h^\beta}\int_\Gamma |x_3(\overline{\boldsymbol{y}}_h' - \boldsymbol{y}')|^2 \,\mathrm{d}\boldsymbol{a},$$

where $\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m})$. Augmenting the total energy by this functional imposes the clamped boundary condition $\boldsymbol{y} = \overline{\boldsymbol{y}}_h$ on Γ in a relaxed sense. The scalings are choosen for the corresponding reduced functional $E_0^{\mathrm{cl}} \colon \mathcal{Q}_0 \to [0,+\infty)$ to be defined as

$$E_0^{\mathrm{cl}}(\boldsymbol{q}_0) := \int_\Lambda |\overline{\boldsymbol{u}} - \boldsymbol{u}|^2 \,\mathrm{d}\boldsymbol{l} + \int_\Lambda |\overline{v} - v|^2 \,\mathrm{d}\boldsymbol{l} + \int_\Lambda |x_3\,(D'\overline{v} - D'v)\,|^2 \,\mathrm{d}\boldsymbol{l},$$

where $\boldsymbol{q}_0 = (\boldsymbol{u}, v, \boldsymbol{\zeta})$, thus recovering the limiting boundary conditions $\boldsymbol{u} = \overline{\boldsymbol{u}}$ on Λ, $v = \overline{v}$ on Λ and $D'v = D'\overline{v}$ on Λ. Indeed, whenever $(\boldsymbol{q}_h)$ with $\boldsymbol{q}_h = (\boldsymbol{y}_h, \boldsymbol{m}_h) \in \mathcal{Q}_h$ satisfies (4.228)–(4.230), we have

$$E_0^{\mathrm{cl}}(\boldsymbol{q}_0) \leq \liminf_{h\to 0^+} E_h^{\mathrm{cl}}(\boldsymbol{q}_h).$$

Therefore, with minor adaptations, is it possible to prove the Γ-convergence, as $h \to 0^+$, of the functionals $(F_h + E_h^{\mathrm{cl}})$ to the functional $F_0 + E_0^{\mathrm{cl}}$ as well as the convergence of sequence of almost minimizers to minimizers of the corresponding reduced functional.

Remark 4.24 (Incompressibility). In view of Remark 4.13, it is possible to specialize Theorem 4.14 for incompressible materials. Define $\mathcal{Q}_h^{\mathrm{inc}}$ as in Remark 4.13. On the one hand, the maps $\overline{\boldsymbol{u}}$ and $\overline{v}$ in (4.166) can be choosen in such a way that the boundary datum $\overline{\boldsymbol{y}}_h$ in (4.169) satisfies $\det D\overline{\boldsymbol{y}}_h = h$ almost everywhere in Ω. On the other hand, the approximation result employed in the proof of Proposition 4.22 does not guarantee that the deformations of the recovery sequence fulfill the incompressibility constraint. Therefore, for general clamped boundary conditions, we do not have an analog of Theorem 4.14 in the incompressible case. However, such a result holds true under the boundary condition $\boldsymbol{y}_h = \boldsymbol{\pi}_h$ on ∂S. In this case, the proof of Proposition 4.22 it is easily adapted by considering approximating sequences $(\widehat{\boldsymbol{u}}_h)$ and $(\widetilde{v}_j)$ that satisfy homogeneous Dirichlet boundary conditions on ∂S.

Chapter 5

Dimension reduction in the quasistatic setting

In this last section, we study the quasistatic evolution for the variational model studied in the previous chapter under dissipative effects. The framework of the theory of rate-independent systems [110]. In particular, we examine evolutions driven by time-dependent applied loads. Our aim is to investigate the asymptotic behaviour, as the thickness of the plate goes to zero, of quasistatic evolutions in the sense of evolutionary Γ-convergence.
The main results of the chapter are Theorem 5.4 and Theorem 5.11. The first result states the convergence of energetic solution of the bulk quasistatic model to energetic solution of the reduced model, while the second one claims an analogous results for solutions of the approximate incremental minimizations problem. These results extend the one of Chapter 4 from the static setting to the quasistatic setting.

5.1 Setting of the problem

In this chaper, *we mantain all the assumption stated in Section 4.1.* Also, we still make use of the notation introduced at the beginning of Chapter 4.
Let $T > 0$ be the time horizon. For $h > 0$, we consider time-dependent applied loads described by the functions

$$\begin{gathered} \boldsymbol{f}_h \in AC([0,T]; L^2(S;\mathbb{R}^2)), \quad g_h \in AC([0,T]; L^2(S)), \\ \boldsymbol{h}_h \in AC([0,T]; L^2(\mathbb{R}^3;\mathbb{R}^3)). \end{gathered} \tag{5.1}$$

The functional $\mathcal{L}_h \colon [0,T] \times \mathcal{Q}_h \to \mathbb{R}$ defined by

$$\mathcal{L}_h(t,\boldsymbol{q}) := \frac{1}{h^\beta} \int_\Omega \boldsymbol{f}_h(t) \cdot (\boldsymbol{y}' - \boldsymbol{x}')\,\mathrm{d}\boldsymbol{x} + \frac{1}{h^\beta} \int_\Omega g_h(t)\, y^3 \,\mathrm{d}\boldsymbol{x} + \frac{1}{h} \int_{\Omega^{\boldsymbol{y}}} \boldsymbol{h}_h(t) \cdot \boldsymbol{m}\,\mathrm{d}\boldsymbol{\xi},$$

where $\boldsymbol{q} = (\boldsymbol{y},\boldsymbol{m})$, accounts for the work of the applied loads. Thus, recalling the definition of the magnetoelastic energy in (4.21), the total energy functional $\mathcal{F}_h \colon [0,T] \times \mathcal{Q}_h \to \mathbb{R}$ is defined as

$$\mathcal{F}_h(t,\boldsymbol{q}) = E_h(\boldsymbol{q}) - \mathcal{L}_h(t,\boldsymbol{q}).$$

We define the dissipation distance $\mathcal{D}_h \colon \mathcal{Q}_h \times \mathcal{Q}_h \to [0,+\infty)$ by setting

$$\mathcal{D}_h(\boldsymbol{q},\widehat{\boldsymbol{q}}) \coloneqq \int_\Omega |\mathcal{Z}_h(\boldsymbol{q}) - \mathcal{Z}_h(\widehat{\boldsymbol{q}})| \,\mathrm{d}\boldsymbol{x},$$

where we employ the notation introduced in (4.23). Thus, the energy dissipated by an evolution $\boldsymbol{q}\colon [0,T] \to \mathcal{Q}_h$ in the time interval $[s,t] \subset [0,T]$ is given by

$$\mathrm{Var}_{\mathcal{D}_h}(\boldsymbol{q};[s,t]) \coloneqq \sup\left\{ \sum_{i=1}^{l} \mathcal{D}_h(\boldsymbol{q}(t^i),\boldsymbol{q}(t^{i-1})) : \Pi = (t^0,\dots,t^l) \text{ partition of } [s,t] \right\}.$$

For the reduced model, we also have an evolution driven by time-dependent applied loads. Precisely, we assume the existence of the functions

$$\begin{gathered} \boldsymbol{f}_0 \in AC([0,T];L^2(S;\mathbb{R}^2)), \quad g_0 \in AC([0,T];L^2(S)), \\ \boldsymbol{h}_0 \in AC([0,T];L^2(\mathbb{R}^2;\mathbb{R}^3)), \end{gathered} \tag{5.2}$$

such that the following convergences hold, as $h \to 0^+$:

$$h^{-\beta/2}\boldsymbol{f}_h \to \boldsymbol{f}_0 \text{ in } W^{1,1}(0,T;L^2(S;\mathbb{R}^2)); \tag{5.3}$$

$$h^{-\beta/2-1}g_h \to g_0 \text{ in } W^{1,1}(0,T;L^2(S)); \tag{5.4}$$

$$\boldsymbol{h}_h \circ \boldsymbol{\pi}_h \to \chi_I \boldsymbol{h}_0 \text{ in } W^{1,1}(0,T;L^2(\mathbb{R}^3;\mathbb{R}^3)). \tag{5.5}$$

As before, the functions $\boldsymbol{f}_0$, g_0 and $\boldsymbol{h}_0$ are assumed to be absolutely continuous with respect to time. Recall also that $I \coloneqq (-1/2,1/2)$. In (5.5), we trivially set $\boldsymbol{h}_h \circ \boldsymbol{\pi}_h(t) \coloneqq \boldsymbol{h}_h(t) \circ \boldsymbol{\pi}_h$ for every $t \in [0,T]$. In particular, this gives a function in $AC([0,T];L^2(\mathbb{R}^3;\mathbb{R}^3))$ whose time-derivative is defined as $\dot{\boldsymbol{h}}_h \circ \boldsymbol{\pi}_h(t) \coloneqq \dot{\boldsymbol{h}}_h(t) \circ \boldsymbol{\pi}_h$ for every $t \in [0,T]$.
Recalling (4.26), the limiting total energy $\mathcal{F}_0 \colon \mathcal{Q}_0 \to \mathbb{R}$ reads

$$\mathcal{F}_0(t,\boldsymbol{q}_0) \coloneqq E_0(\boldsymbol{q}_0) - \mathcal{L}_0(t,\boldsymbol{q}_0).$$

Here, the functional $\mathcal{L}_0 \colon [0,T] \times \mathcal{Q}_0 \to \mathbb{R}$ is defined by setting

$$\mathcal{L}_0(t,\boldsymbol{q}_0) \coloneqq \int_S \boldsymbol{f}(t)\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}' + \int_S g(t)\,v\,\mathrm{d}\boldsymbol{x}' + \int_S \boldsymbol{h}(t)\cdot\boldsymbol{\zeta}\,\mathrm{d}\boldsymbol{x}',$$

where $\boldsymbol{q}_0 = (\boldsymbol{u},v,\boldsymbol{\zeta})$. The dissipation distance $\mathcal{D}_0 \colon \mathcal{Q}_0 \times \mathcal{Q}_0 \to [0,+\infty)$ for the reduced model is defined as

$$\mathcal{D}_0(\boldsymbol{q}_0,\widehat{\boldsymbol{q}}_0) = \int_S |\boldsymbol{\zeta} - \widehat{\boldsymbol{\zeta}}| \,\mathrm{d}\boldsymbol{x}',$$

where $\boldsymbol{q}_0 = (\boldsymbol{u},v,\boldsymbol{\zeta})$ and $\widehat{\boldsymbol{q}}_0 = (\widehat{\boldsymbol{u}},\widehat{v},\widehat{\boldsymbol{\zeta}})$. Accordingly, the energy dissipated by $\boldsymbol{q}_0\colon [0,T] \to \mathcal{Q}_0$ in $[r,s] \subset [0,T]$ is given by

$$\mathrm{Var}_{\mathcal{D}_0}(\boldsymbol{q}_0;[r,s]) \coloneqq \sup\left\{ \sum_{i=1}^{m} \mathcal{D}_0(\boldsymbol{q}_0(t^i),\boldsymbol{q}_0(t^{i-1})) : (t^0,\dots,t^m) \text{ partition of } [r,s] \right\}.$$

In the quasistatic setting, we impose the same boundary conditions in (4.168) and (4.170). In particular, we assume (4.166)–(4.167) and (4.169).

5.2 Evolutionary Γ-convergence

In order to state our first main result, we give the definition of energetic solution for the bulk model and for the reduced model. Here, as in Chapter 3, the boundary conditions are included in the definition of such solutions. This applies both to the bulk and to the reduced model.

Definition 5.1 (Energetic solution of the bulk quasistatic model). Given $h > 0$, a function $\boldsymbol{q}_h\colon [0,T] \to \overline{\mathcal{Q}}_h$ is termed an energetic solution of the bulk quasistatic model at thickness h if the function $t \mapsto \partial_t \mathcal{F}_h(t, \boldsymbol{q}_h(t))$ belongs to $L^1(0,T)$ and, for every $t \in [0,T]$, the following two conditions are satisfied:

(i) Bulk global stability at thickness $\boldsymbol{h}$:

$$\forall\, \widehat{\boldsymbol{q}}_h \in \overline{\mathcal{Q}}_h, \;\; \mathcal{F}_h(t, \boldsymbol{q}_h(t)) \leq \mathcal{F}_h(t, \widehat{\boldsymbol{q}}_h) + \mathcal{D}_h(\boldsymbol{q}_h(t), \widehat{\boldsymbol{q}}_h); \tag{5.6}$$

(ii) Bulk energy-dissipation balance at thickness $\boldsymbol{h}$:

$$\mathcal{F}_h(t, \boldsymbol{q}_h(t)) + \mathrm{Var}_{\mathcal{D}_h}(\boldsymbol{q}_h; [0,t]) = \mathcal{F}_h(0, \boldsymbol{q}_h(0)) + \int_0^t \partial_t \mathcal{F}_h(\tau, \boldsymbol{q}_h(\tau))\, \mathrm{d}\tau. \tag{5.7}$$

Remark 5.2 (Existence of energetic solutions for the bulk model). The assumptions of this chapter are compatible with the existence theory for energetic solutions developed in Chapter 3. Namely, if the function Φ in (4.3) satisfies the assumption of partial polyconvexity stated in Example 2.1, then, in view of Remark 4.2 and Remark 4.16, the existence of energetic solutions of the bulk quasistatic model is ensured by Theorem 3.2.

Definition 5.3 (Energetic solution of the reduced quasistatic model). A function $\boldsymbol{q}_0\colon [0,T] \to \overline{\mathcal{Q}}_0$ is termed an energetic solution of the reduced quasistatic model if the function $t \mapsto \partial_t \mathcal{F}_0(t, \boldsymbol{q}_0(t))$ belongs to $L^1(0,T)$ and, for every $t \in [0,T]$, the following two conditions are satisfied:

(i) Reduced global stability:

$$\forall\, \widehat{\boldsymbol{q}}_0 \in \overline{\mathcal{Q}}_0, \;\; \mathcal{F}_0(t, \boldsymbol{q}_0(t)) \leq \mathcal{F}_0(t, \widehat{\boldsymbol{q}}_0) + \mathcal{D}_0(\boldsymbol{q}_0(t), \widehat{\boldsymbol{q}}_0); \tag{5.8}$$

(ii) Reduced energy-dissipation balance:

$$\mathcal{F}_0(t, \boldsymbol{q}_0(t)) + \mathrm{Var}_{\mathcal{D}_0}(\boldsymbol{q}_0; [0,t]) = \mathcal{F}_0(0, \boldsymbol{q}_0(0)) + \int_0^t \partial_t \mathcal{F}_0(\tau, \boldsymbol{q}_0(\tau))\, \mathrm{d}\tau. \tag{5.9}$$

Our first main result state the evolutionary Γ-convergence of the sequence $(\mathcal{F}_h)$ to the functional $\mathcal{F}_0$, as $h \to 0^+$. More explicitly, we prove that sequences of energetic solutions of the bulk model converge to energetic solutions of the reduced model.

Theorem 5.4 (Evolutionary Γ-convergence). *Let $(\boldsymbol{q}_h^0)$ with $\boldsymbol{q}_h^0 = (\boldsymbol{y}_h^0, \boldsymbol{m}_h^0) \in \overline{\mathcal{Q}}_h$. Suppose that, for every $h > 0$, the following condition is satisfied:*

$$\forall\, \widehat{\boldsymbol{q}}_h \in \mathcal{Q}_h, \quad \mathcal{F}_h(0, \boldsymbol{q}_h^0) \leq \mathcal{F}_h(0, \widehat{\boldsymbol{q}}_h) + \mathcal{D}_h(\boldsymbol{q}_h^0, \widehat{\boldsymbol{q}}_h). \tag{5.10}$$

Also, assume the existence of $\boldsymbol{q}_0^0 = (\boldsymbol{u}^0, v^0, \boldsymbol{\zeta}^0) \in \overline{\mathcal{Q}}_0$ *such that the following convergences hold, as* $h \to 0^+$*:*

$$\boldsymbol{u}_h^0 := \mathcal{U}_h(\boldsymbol{q}_h^0) \rightharpoonup \boldsymbol{u}^0 \text{ in } W^{1,2}(S;\mathbb{R}^2); \tag{5.11}$$

$$v_h^0 := \mathcal{V}_h(\boldsymbol{q}_h^0) \to v^0 \text{ in } W^{1,2}(S); \tag{5.12}$$

$$\boldsymbol{z}_h^0 := \mathcal{Z}_h(\boldsymbol{q}_h^0) \to \boldsymbol{\zeta}^0 \text{ in } L^1(\Omega;\mathbb{R}^3); \tag{5.13}$$

$$\mathcal{F}_h(0, \boldsymbol{q}_h^0) \to \mathcal{F}_0(0, \boldsymbol{q}_0^0). \tag{5.14}$$

Let $(\boldsymbol{q}_h)$ *with* $\boldsymbol{q}_h\colon [0,T] \to \overline{\mathcal{Q}}_h$ *an energetic solution of the bulk quasistatic model at thickness* $h > 0$*. Then, there exists a function* $\boldsymbol{q}_0\colon [0,T] \to \overline{\mathcal{Q}}_0$ *with* $\boldsymbol{q}_0(t) = (\boldsymbol{u}(t), v(t), \boldsymbol{\zeta}(t))$ *for every* $t \in [0,T]$ *satisfying the initial condition* $\boldsymbol{q}_0(0) = \boldsymbol{q}_0^0$ *which is an energetic solution of the reduced quasistatic model. Moreover, up to subsequences, the following convergences hold, as* $h \to 0^+$*:*

$$\forall\, t \in [0,T], \quad \boldsymbol{z}_h(t) := \mathcal{Z}_h(\boldsymbol{q}_h(t)) \to \boldsymbol{\zeta}(t) \text{ in } L^1(\Omega;\mathbb{R}^3); \tag{5.15}$$

$$\forall\, t \in [0,T], \quad \mathrm{Var}_{\mathcal{D}_h}(\boldsymbol{q}_h;[0,t]) \to \mathrm{Var}_{\mathcal{D}_0}(\boldsymbol{q}_0;[0,t]); \tag{5.16}$$

$$\forall\, t \in [0,T], \quad \mathcal{F}_h(t, \boldsymbol{q}_h(t)) \to \mathcal{F}_0(t, \boldsymbol{q}_0(t)); \tag{5.17}$$

$$\partial_t \mathcal{F}_h(\cdot, \boldsymbol{q}_h) \to \partial_t \mathcal{F}_0(\cdot, \boldsymbol{q}_0) \text{ in } L^1(0,T). \tag{5.18}$$

In particular, the function $\boldsymbol{q}_0$ *is measurable and bounded. Also, the function* $t \mapsto \boldsymbol{\zeta}(t)$ *belongs to* $BV([0,T];L^1(\Omega;\mathbb{R}^3))$*, while* $t \mapsto \mathcal{F}_0(t, \boldsymbol{q}_0(t))$ *belongs to* $BV([0,T])$*.*

We stress that the previous theorem is just a convergence result: the existence of energetic solutions of the bulk quasistatic model is part of the assumptions.
In the previous theorem, the measurability of $\boldsymbol{q}_0$ is meant with respect to the Borel sigma-algebra of the space $W^{1,2}(S;\mathbb{R}^2) \times W^{2,2}(S) \times W^{1,2}(S;\mathbb{S}^2)$ equipped with the weak product topology. Here, by weak topology of $W^{1,2}(S;\mathbb{S}^2)$ we refer to the topology induced on this space by the weak topology of $W^{1,2}(S;\mathbb{R}^3)$. Also, the boundedness of the map $\boldsymbol{q}_0$ is understood as the one of the function

$$t \mapsto \|\boldsymbol{u}(t)\|_{W^{1,2}(S;\mathbb{R}^2)} + \|v(t)\|_{W^{2,2}(S)} + \|\boldsymbol{\zeta}(t)\|_{W^{1,2}(S;\mathbb{S}^2)}.$$

Remark 5.5 (Existence of energetic solutions for the reduced model). From Theorem 5.4, we deduce the existence of energetic solution for the reduced quasistatic model. However, this can be established directly by means of the classical time-discretization scheme thanks to the observations made in Remark 4.15.

The rest of the section is devoted to the proof of Theorem 5.4.

5.2.1 Proof of Theorem 5.4

We start with some preliminary results. The first one constitutes the analog of Lemma 4.20 for the quasistatic setting.

Lemma 5.6 (Energy scaling). *Let* $M > 0$*. Then, there exists* $C(M) > 0$ *such that the following statement holds: given* $(\boldsymbol{q}_h)$ *with* $\boldsymbol{q}_h\colon [0,T] \to \overline{\mathcal{Q}}_h$ *satisfying*

$$\sup_{h>0} \sup_{t\in[0,T]} \mathcal{F}_h(t, \boldsymbol{q}_h(t)) \leq M, \tag{5.19}$$

there exist $C(M) > 0$ and $\overline{h} > 0$, where the former does not depend on $(\boldsymbol{q}_h)$, such that

$$\sup_{h \leq \overline{h}} \sup_{t \in [0,T]} E_h(\boldsymbol{q}_h(t)) \leq C(M). \tag{5.20}$$

Proof. The proof is analogous to the one of Lemma 4.20. For every $h > 0$, keeping the same notation for the functional I_h, we define $\mathcal{J}_h \colon [0,T] \times \mathcal{Q}_h \to \mathbb{R}$ by setting

$$\mathcal{J}_h(t, \boldsymbol{q}) \coloneqq I_h(\boldsymbol{q}) - \int_\Omega \boldsymbol{f}_h(t) \cdot (\boldsymbol{y}' - \boldsymbol{x}') \, \mathrm{d}\boldsymbol{x} - \int_\Omega g_h(t) \, y^3 \, \mathrm{d}\boldsymbol{x},$$

where $\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m})$. Let $\boldsymbol{q}_h(t) = (\boldsymbol{y}_h(t), \boldsymbol{m}_h(t))$. Similarly to (4.211), we have

$$r_h \coloneqq \sup_{t \in [0,T]} \mathcal{R}_h(\boldsymbol{y}_h(t)) \leq C h^\beta + \sup_{t \in [0,T]} I_h(\boldsymbol{q}_h(t)). \tag{5.21}$$

Arguing as in Step 1 of the proof of Lemma 4.20, we show that

$$\sup_{t \in [0,T]} I_h(\boldsymbol{q}_h(t)) \leq C(M) h^{\beta/2}, \qquad r_h \leq C(M) h^{\beta/2}. \tag{5.22}$$

Also, we deduce

$$\sup_{t \in [0,T]} \mathcal{J}_h(t, \boldsymbol{q}_h(t)) \leq C(M) h^\beta. \tag{5.23}$$

Then, we proceed as in Step 2 of the proof of Lemma 4.20. Suppose by contradiction that

$$\limsup_{h \to 0^+} \frac{1}{h^\beta} \sup_{t \in [0,T]} I_h(\boldsymbol{q}_h(t)) = +\infty. \tag{5.24}$$

Let $e_h = \sup_{t \in [0,T]} I_h(\boldsymbol{q}_h(t))$. In view of (5.24) and (5.21), there holds

$$\limsup_{h \to 0^+} \frac{r_h}{e_h} \leq 1.$$

Reasoning as in Step 2 of the proof of Lemma 4.20 and exploiting the boundedness of $(h^{-\beta/2} \boldsymbol{f}_h)$ and $(h^{-\beta/2-1} g_h)$ in $C^0([0,T]; L^2(S; \mathbb{R}^2))$ and $C^0([0,T]; L^2(S))$, respectively, we show that

$$\lim_{h \to +\infty} \frac{1}{e_h} \sup_{t \in [0,T]} \left\{ \int_\Omega \boldsymbol{f}_h(t) \cdot (\boldsymbol{y}'_h(t) - \boldsymbol{x}') \, \mathrm{d}\boldsymbol{x} + \int_\Omega g_h(t) \, y^3_h(t) \, \mathrm{d}\boldsymbol{x} \right\} = 0.$$

Hence, we have

$$\begin{aligned}
1 &= \lim_{h \to 0^+} \frac{1}{e_h} \sup_{t \in [0,T]} I_h(\boldsymbol{q}_h(t)) \\
&= \lim_{h \to 0^+} \frac{1}{e_h} \sup_{t \in [0,T]} \left\{ \mathcal{J}_h(t, \boldsymbol{q}_h(t)) + \int_\Omega \boldsymbol{f}_h(t) \cdot (\boldsymbol{y}'_h(t) - \boldsymbol{x}') \, \mathrm{d}\boldsymbol{x} + \int_\Omega g_h(t) y^3_h(t) \, \mathrm{d}\boldsymbol{x} \right\} \\
&\leq \lim_{h \to 0^+} \frac{1}{e_h} \sup_{t \in [0,T]} \mathcal{J}_h(t, \boldsymbol{q}_h(t)) \\
&\quad + \lim_{h \to 0^+} \frac{1}{e_h} \sup_{t \in [0,T]} \left\{ \int_\Omega \boldsymbol{f}_h(t) \cdot (\boldsymbol{y}'_h(t) - \boldsymbol{x}') \, \mathrm{d}\boldsymbol{x} + \int_\Omega g_h(t) \, y^3_h(t) \, \mathrm{d}\boldsymbol{x} \right\} \\
&\leq C(M) \lim_{h \to 0^+} \frac{h^\beta}{e_h} = 0,
\end{aligned}$$

where the last equality is justified by (5.23). This contradicts (5.24), so that there holds

$$L := \limsup_{h\to 0^+} \frac{1}{h^\beta} \sup_{t\in[0,T]} I_h(\boldsymbol{q}_h(t)) < +\infty. \tag{5.25}$$

At this point, we argue as in Step 3 of the proof of Lemma 4.20 to show that $L \leq C(M)$ for some constant $C(M) > 0$ depending only on M. Eventually, we establish (5.20). □

We now prove that the total energies $(\mathcal{F}_h)$ satisfy suitable controls with respect to time analogous to the ones established in Lemma 3.4. These controls hold for sequences of admissible states with equi-bounded magnetoelastic energy but involve constants depending only on their upper bound.

For convenience, we denote by Z the complement of the set of times $t \in (0,T)$ such that all the maps $\boldsymbol{f}_h$, g_h and $\boldsymbol{h}_h$ in (5.1) for every $h > 0$, as well as the maps $\boldsymbol{f}_0$, g_0 and $\boldsymbol{h}_0$ in (5.2), are differentiable at time t. Thus, there holds $\mathscr{L}^1([0,T]\setminus Z) = 0$.

Lemma 5.7 (Time-control of the total energy). *Let $M > 0$. Then, there exist two constants $C(M), L(M) > 0$ such that, defining $\kappa_h \in L^1(0,T)$ for every $h > 0$ by setting*

$$\kappa_h(t) := C(M)\left(h^{-\beta/2}\|\dot{\boldsymbol{f}}_h(t)\|_{L^2(S;\mathbb{R}^2)} + h^{-\beta/2-1}\|\dot{g}_h(t)\|_{L^2(S)} + \|\dot{\boldsymbol{h}}_h \circ \boldsymbol{\pi}_h(t)\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}\right),$$

the following statement holds: given $(\widehat{\boldsymbol{q}}_h)$ with $\widehat{\boldsymbol{q}}_h \in \overline{\mathcal{Q}}_h$ satisfying

$$\sup_{h>0} E_h(\widehat{\boldsymbol{q}}_h) \leq M, \tag{5.26}$$

there exists $\overline{h} > 0$ such that the following estimates hold for every $h \leq \overline{h}$:

$$\forall\, t \in [0,T]\setminus Z, \quad |\partial_t \mathcal{F}_h(t,\widehat{\boldsymbol{q}}_h)| \leq \kappa_h(t)\left(\mathcal{F}_h(t,\widehat{\boldsymbol{q}}_h) + L(M)\right), \tag{5.27}$$

$$\forall s,t \in [0,T], \quad \mathcal{F}_h(t,\widehat{\boldsymbol{q}}_h) + L(M) \leq \left(\mathcal{F}_h(s,\widehat{\boldsymbol{q}}_h) + L(M)\right) \mathrm{e}^{|K_h(t)-K_h(s)|}, \tag{5.28}$$

$$\begin{aligned} &\forall t \in [0,T]\setminus Z,\ \forall s \in [0,T], \\ &\qquad |\partial_t \mathcal{F}_h(t,\widehat{\boldsymbol{q}}_h)| \leq \kappa_h(t)\left(\mathcal{F}_h(s,\widehat{\boldsymbol{q}}_h) + L(M)\right) \mathrm{e}^{|K_h(t)-K_h(s)|}. \end{aligned} \tag{5.29}$$

In the previous estimates, for every $h > 0$, we define $K_h \in AC([0,T])$ by setting

$$K_h(t) := \int_0^t K_h(\tau)\,\mathrm{d}\tau.$$

In particular, the constant $L(M) > 0$ satisfies

$$\inf_{h>0} \inf_{t\in[0,T]} \mathcal{F}_h(t,\widehat{\boldsymbol{q}}_h) \geq -L(M). \tag{5.30}$$

Moreover, defining $\kappa_0 \in L^1(0,T)$ and $K_0 \in AC([0,T])$ as

$$\kappa_0(t) := C(M)\left(\|\dot{\boldsymbol{f}}_0(t)\|_{L^2(S;\mathbb{R}^2)} + \|\dot{g}_0(t)\|_{L^2(S)} + \|\chi_I \dot{\boldsymbol{h}}_0(t)\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}\right),$$

$$K_0(t) := \int_0^t \kappa_0(\tau)\,\mathrm{d}\tau,$$

there hold:

$$\kappa_h \to \kappa_0 \text{ in } L^1(0,T), \tag{5.31}$$

$$K_h \to K_0 \text{ uniformly in } [0,T]. \tag{5.32}$$

Proof. First of all, (5.31)–(5.32) come directly from (5.3)–(5.5). From Corollary 4.19 and (5.26), setting $\widehat{\boldsymbol{u}}_h \coloneqq \mathcal{U}_h(\widehat{\boldsymbol{q}}_h)$ and $\widehat{v}_h \coloneqq \mathcal{V}_h(\widehat{\boldsymbol{q}}_h)$, we obtain

$$\|\widehat{\boldsymbol{u}}_h\|_{W^{1,2}(S;\mathbb{R}^2)} + \|\widehat{v}_h\|_{W^{1,2}(S;\mathbb{R}^2)} \leq C(M)\left(\sqrt{E_h^{\mathrm{el}}(\widehat{\boldsymbol{q}}_h)}+1\right). \tag{5.33}$$

Let $\widehat{\boldsymbol{q}}_h = (\widehat{\boldsymbol{y}}_h, \widehat{\boldsymbol{m}}_h)$ and set $\widehat{\boldsymbol{\mu}}_h \coloneqq \mathcal{M}_h(\widehat{\boldsymbol{q}}_h)$. Recalling (4.11) and (4.20), with the aid of Proposition 1.46, we estimate

$$\begin{aligned}
\int_{\mathbb{R}^3} |\widehat{\boldsymbol{\mu}}_h|^2 \,\mathrm{d}\boldsymbol{x} &= \frac{1}{h}\int_{\Omega^{\widehat{\boldsymbol{y}}_h}} |\widehat{\boldsymbol{m}}_h|^2 \,\mathrm{d}\boldsymbol{\xi} = \frac{1}{h}\int_{\Omega} |\widehat{\boldsymbol{m}}_h \circ \widehat{\boldsymbol{y}}_h|^2 \det D\widehat{\boldsymbol{y}}_h \,\mathrm{d}\boldsymbol{x} \\
&= \int_{\Omega} |\widehat{\boldsymbol{m}}_h \circ \widehat{\boldsymbol{y}}_h|^2 \det D_h\widehat{\boldsymbol{y}}_h \,\mathrm{d}\boldsymbol{x} = \int_{\Omega} |\widehat{\boldsymbol{m}}_h \circ \widehat{\boldsymbol{y}}_h| \,\mathrm{d}\boldsymbol{x} \\
&= \int_{\Omega} \frac{1}{\det D_h \widehat{\boldsymbol{y}}} \,\mathrm{d}\boldsymbol{x} \leq C\left(h^\beta E_h^{\mathrm{el}}(\widehat{\boldsymbol{q}}_h)+1\right),
\end{aligned}$$

so that

$$\|\widehat{\boldsymbol{\mu}}_h\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \leq C\left(\sqrt{E_h^{\mathrm{el}}(\widehat{\boldsymbol{q}}_h)}+1\right). \tag{5.34}$$

Using (5.33)–(5.34) and the Hölder inequality, we control the work of applied loads. For $t \in (0,T) \setminus Z$, we have

$$\begin{aligned}
|\mathcal{L}_h(t,\widehat{\boldsymbol{q}}_h)| &\leq \left|\int_S h^{-\beta/2}\boldsymbol{f}_h(t)\cdot\widehat{\boldsymbol{u}}_h \,\mathrm{d}\boldsymbol{x}'\right| + \left|\int_S h^{-\beta/2-1} g_h(t)\,\widehat{v}_h \,\mathrm{d}\boldsymbol{x}'\right| \\
&\quad + \left|\int_{\mathbb{R}^3} \boldsymbol{h}_h \circ \boldsymbol{\pi}_h(t) \cdot \widehat{\boldsymbol{\mu}}_h \,\mathrm{d}\boldsymbol{x}\right| \\
&\leq C\left(\|\widehat{\boldsymbol{u}}_h\|_{W^{1,2}(S;\mathbb{R}^2)} + \|\widehat{v}_h\|_{W^{1,2}(S;\mathbb{R}^2)} + \|\widehat{\boldsymbol{\mu}}_h\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}\right) \\
&\leq C\left(\sqrt{E_h^{\mathrm{el}}(\widehat{\boldsymbol{q}}_h)}+1\right).
\end{aligned}$$

Here, we exploited the boundedness of $(h^{-\beta/2}\boldsymbol{f}_h)$, $(h^{-\beta/2-1}g_h)$ and $(\boldsymbol{h}_h \circ \boldsymbol{\pi}_h)$ in $C^0([0,T];L^2(S;\mathbb{R}^2))$, $C^0([0,T];L^2(S))$ and $C^0([0,T];L^2(\mathbb{R}^3;rt))$, respectively. Thus, using the Young inequality, we obtain

$$\begin{aligned}
\mathcal{F}_h(t,\widehat{\boldsymbol{q}}_h) &\geq E_h(\widehat{\boldsymbol{q}}_h) - |\mathcal{L}_h(t,\widehat{\boldsymbol{q}}_h)| \\
&\geq E_h(\widehat{\boldsymbol{q}}_h) - C(M)\left(\sqrt{E_h^{\mathrm{el}}(\widehat{\boldsymbol{q}}_h)}+1\right) \\
&\geq C(M)E_h(\widehat{\boldsymbol{q}}_h) - L(M),
\end{aligned} \tag{5.35}$$

for some constants $C(M), L(M) > 0$.

Now, for convenience, set

$$\widetilde{\kappa}_h(t) \coloneqq h^{-\beta/2}\|\dot{\boldsymbol{f}}_h(t)\|_{L^2(S;\mathbb{R}^2)} + h^{-\beta/2-1}\|\dot{g}_h(t)\|_{L^2(S)} + \|\dot{\boldsymbol{h}}_h \circ \boldsymbol{\pi}_h(t)\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}.$$

Making use of the Hölder inequality together with (5.33)–(5.34), we estimate

$$
\begin{aligned}
|\partial_t \mathcal{F}_h(t,\widehat{\boldsymbol{q}}_h)| &\leq \left| \int_S h^{-\beta/2} \dot{\boldsymbol{f}}_h(t) \cdot \widehat{\boldsymbol{u}}_h \,\mathrm{d}\boldsymbol{x}' \right| + \left| \int_S h^{-\beta/2-1} \dot{g}_h(t)\, \widehat{v}_h \,\mathrm{d}\boldsymbol{x}' \right| \\
&\quad + \left| \int_{\mathbb{R}^3} \dot{\boldsymbol{h}}_h \circ \boldsymbol{\pi}_h(t) \cdot \widehat{\boldsymbol{\mu}}_h \,\mathrm{d}\boldsymbol{x} \right| \\
&\leq \widetilde{\kappa}_h(t) \left(\|\widehat{\boldsymbol{u}}_h\|_{W^{1,2}(S;\mathbb{R}^2)} + \|\widehat{v}_h\|_{W^{1,2}(S;\mathbb{R}^2)} + \|\widehat{\boldsymbol{\mu}}_h\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \right) \\
&\leq C(M)\widetilde{\kappa}_h(t) \left(\sqrt{E_h^{\mathrm{el}}(\widehat{\boldsymbol{q}}_h)} + 1 \right).
\end{aligned}
$$

Combining this with (5.35), we obtain

$$
|\partial_t \mathcal{F}_h(t,\widehat{\boldsymbol{q}}_h)| \leq C(M)\widetilde{\kappa}_h(t) \left(\mathcal{F}_h(t,\widehat{\boldsymbol{q}}_h) + L(M) \right),
$$

which proves (5.27). Note that (5.30) holds in view of (5.35). From (5.27), we deduce (5.28) thanks to the Gronwall inequality. Eventually, (5.29) follows. □

To ease the expositions, in the next result we check that the reduced total energy $\mathcal{F}_0$ provides a lower bound for such sequence $(\mathcal{F}_h)$. This result is a simple reformulations of the results of Chapter 4.

Lemma 5.8 (Lower bound of the total energy). *Let $(t_h) \subset [0,T]$ and $(\widehat{\boldsymbol{q}}_h)$ with $\widehat{\boldsymbol{q}}_h = (\widehat{\boldsymbol{y}}_h, \widehat{\boldsymbol{m}}_h) \in \overline{\mathcal{Q}}_h$. Suppose that there exist $t \in [0,T]$ and $\widehat{\boldsymbol{q}}_0 = (\widehat{\boldsymbol{u}}, \widehat{v}, \widehat{\boldsymbol{\zeta}}) \in \overline{\mathcal{Q}}_0$ such that $t_h \to t$ and the convergences in* (4.228)–(4.229) *and* (4.231)–(4.232) *hold. Then, there holds*

$$
\mathcal{F}_0(t,\widehat{\boldsymbol{q}}_0) \leq \liminf_{h\to 0^+} \mathcal{F}_h(t_h,\widehat{\boldsymbol{q}}_h). \tag{5.36}
$$

Proof. Without loss of generality, we can assume that the inferior limit on the right-hand side of (5.36) is finite and it is actually a limit. In this case, we deduce that $\mathcal{F}_h(t_h,\widehat{\boldsymbol{q}}_h) \leq M$ for every $h > 0$ which, thanks to Lemma 4.20, entails

$$
E_h(\widehat{\boldsymbol{q}}_h) \leq C(M)
$$

for $h \ll 1$. From this, in view of (4.148), we deduce that

$$
r_h \coloneqq \mathcal{R}_h(\widehat{\boldsymbol{y}}_h) \leq C(M) h^{\beta}.
$$

By applying Lemma 4.5, we find $\widehat{\boldsymbol{R}}_h \in W^{1,p}(S;SO(3))$ such that, setting $\widehat{\boldsymbol{F}}_h \coloneqq D_h \widehat{\boldsymbol{y}}_h$, there holds

$$
\|\widehat{\boldsymbol{F}}_h - \widehat{\boldsymbol{R}}_h\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} \leq C(M) h^{\beta/2}.
$$

Hence, by Lemma 4.8, we deduce (4.94)–(4.96) by taking into account also (4.228)–(4.229). At this point, Proposition 4.11 yields the inequality

$$
E_0(\widehat{\boldsymbol{q}}_0) \leq \liminf_{h\to 0^+} E_h(\widehat{\boldsymbol{q}}_h). \tag{5.37}
$$

Now, given (5.3)–(5.5) and using (4.228)–(4.229) and (4.231), we see that

$$
\lim_{h\to 0^+} \mathcal{L}_h(t,\widehat{\boldsymbol{q}}_h) = \mathcal{L}_0(t,\widehat{\boldsymbol{q}}_0).
$$

Therefore, we are only left to check that

$$
\lim_{h\to 0^+} \{\mathcal{F}_h(t_h,\widehat{\boldsymbol{q}}_h) - \mathcal{F}_h(t,\widehat{\boldsymbol{q}}_h)\} = \lim_{h\to 0^+} \{\mathcal{L}_h(t_h,\widehat{\boldsymbol{q}}_h) - \mathcal{L}_h(t,\widehat{\boldsymbol{q}}_h)\} = 0.
$$

This easily follows by the continuity with respect to time of the applied loads which comes from (5.1) in view of the boundedness of $(\widehat{\boldsymbol{u}}_h)$, $(\widehat{v}_h)$ and $(\widehat{\boldsymbol{\mu}}_h)$ in $L^2(S;\mathbb{R}^2)$, $L^2(S)$ and $L^2(\mathbb{R}^3;\mathbb{R}^3)$, respectively. □

At this point, we are ready to prove the first main result of the chapter.

Proof of Theorem 5.4. . We follow the scheme introduced in [112]. Therefore, the proof is subdivided into six steps.

Step 1 (A priori estimates). Let $h>0$ and $t\in[0,T]$. Testing (5.6) with $\widehat{\boldsymbol{q}}_h=\overline{\boldsymbol{q}}_h$, where the latter is defined as in the proof of Theorem 4.14, we obtain

$$\mathcal{F}_h(t,\boldsymbol{q}_h(t))\leq \mathcal{F}_h(t,\overline{\boldsymbol{q}}_h)+\mathcal{D}_h(\boldsymbol{q}_h(t),\overline{\boldsymbol{q}}_h). \tag{5.38}$$

As the applied loads are uniformly bounded with respect to $h>0$ and $t\in[0,T]$ thanks to (5.3)–(5.5) and the Morrey embedding, by arguing as in the proof of Theorem 4.14 we check that the first term on the right-hand side of (5.38) is uniformly bounded with respect to both $h>0$ and $t\in[0,T]$. The second term on the right-hand side of (5.38) is also uniformly bounded with respect to $h>0$ and $t\in[0,T]$ because both $\boldsymbol{q}_h(t)$ and $\overline{\boldsymbol{q}}_h$ satisfy the magnetic saturation constrain. Therefore, we deduce (5.19) for some $M>0$ and, thanks to Lemma 5.6, also (5.20). Also, applying Lemma 5.7 to $\widehat{\boldsymbol{q}}_h=\boldsymbol{q}_h(T)$, we see that

$$\inf_{h\ll 1}\mathcal{F}_h(T,\boldsymbol{q}_h(T))\geq -L(M). \tag{5.39}$$

Also, for $h\ll 1$, there holds

$$\forall\, t\in[0,T]\setminus Z,\quad |\partial_t\mathcal{F}_h(t,\boldsymbol{q}_h(t))\leq \kappa_h(t)\left(\mathcal{F}_h(t,\boldsymbol{q}_h(t))+L(M)\right). \tag{5.40}$$

By (5.7), we have

$$\mathrm{Var}_{\mathcal{D}_h}(\boldsymbol{q}_h;[0,T])=\mathcal{F}_h(0,\boldsymbol{q}_h^0)-\mathcal{F}_h(T,\boldsymbol{q}_h(T))+\int_0^T\partial_t\mathcal{F}_h(\tau,\boldsymbol{q}_h(\tau))\,\mathrm{d}\tau. \tag{5.41}$$

The first term on the right-hand side of (5.41) is uniformly bounded by (5.67) and so is the second one because of (5.39). Additionally, thanks to (5.40), we estimate

$$\begin{aligned}\int_0^T|\partial_t\mathcal{F}_h(\tau,\boldsymbol{q}_h(\tau))|\,\mathrm{d}\tau&\leq\int_0^T\kappa_h(\tau)\left(\mathcal{F}_h(\tau,\boldsymbol{q}_h(\tau))+L(M)\right)\mathrm{e}^{K_h(\tau)}\mathrm{d}\tau\\&\leq (M+L(M))\left(\mathrm{e}^{K_h(T)}-1\right)\leq C(M,T),\end{aligned} \tag{5.42}$$

where in the last line we used (5.19) and the uniform boundedness of (K_h) which comes from (5.32). Therefore, we obtained the following a priori estimate:

$$\sup_{h>0}\sup_{0\leq t\leq T}\mathcal{F}_h(t,\boldsymbol{q}_h(t))+\sup_{h\ll 1}\mathrm{Var}_{\mathcal{D}_h}(\boldsymbol{q}_h;[0,T])\leq C(M). \tag{5.43}$$

For every $h>0$, we define $f_h\colon[0,T]\to\mathbb{R}$ by setting $f_h(t):=\mathcal{F}_h(t,\boldsymbol{q}_h(t))$. In view of the previous estimate, we already know that (f_h) is uniformly bounded. We claim that this sequence has also uniformly bounded total variation. Observe that, by (5.7), for every $s,t\in[0,T]$ with $s<t$, there holds

$$\mathcal{F}_h(t,\boldsymbol{q}_h(t))+\mathrm{Var}_{\mathcal{D}_h}(\boldsymbol{q}_h;[s,t])=\mathcal{F}_h(s,\boldsymbol{q}_h(s))+\int_s^t\partial_t\mathcal{F}_h(\tau,\boldsymbol{q}_h(\tau))\,\mathrm{d}\tau.$$

Let $\Pi = (t^0, \dots, t^l)$ be a partition of $[0,T]$. In this case, we estimate

$$\begin{aligned}\sum_{i=1}^{l} \left|f_h(t^i) - f_h(t^{i-1})\right| &\leq \sum_{i=1}^{l} \mathrm{Var}_{\mathcal{D}_h}(\boldsymbol{q}_h;[t^{i-1},t^i]) + \sum_{i=1}^{l} \int_{t^{i-1}}^{t^i} |\partial_t \mathcal{F}_h(\tau,\boldsymbol{q}_h(\tau))| \,\mathrm{d}\tau \\ &= \mathrm{Var}_{\mathcal{D}_h}(\boldsymbol{q}_h;[0,T]) + \int_0^T |\partial_t \mathcal{F}_h(\tau,\boldsymbol{q}_h(\tau))| \,\mathrm{d}\tau.\end{aligned}$$

As the right-hand side is uniformly by (5.42)–(5.43), given the arbitrariness of Π, we deduce

$$\sup_{h \ll 1} \mathrm{Var}\,(f_h;[0,T]) \leq C(M). \tag{5.44}$$

Step 2 (Compactness). For every $h>0$, set

$$\mathcal{H}_h := \bigcup_{0 \leq \widehat{t} \leq T} \left\{ \widehat{\boldsymbol{q}}_h \in \overline{\mathcal{Q}}_h : \, \mathcal{F}_h(\widehat{t},\widehat{\boldsymbol{q}}_h) \leq M \right\}, \quad \mathcal{K}_h := \left\{ \mathcal{Z}_h(\widehat{\boldsymbol{q}}_h) : \, \widehat{\boldsymbol{q}}_h \in \mathcal{H}_h \right\}.$$

Then, define

$$\mathcal{H} := \bigcup_{h>0} \mathcal{H}_h, \qquad \mathcal{K} := \bigcup_{h>0} \mathcal{K}_h.$$

Observe that, by Lemma 5.6, every sequence $(\widehat{\boldsymbol{q}}_h) \subset \mathcal{H}$ satisfies

$$\sup_{h \ll 1} \sup_{0 \leq t \leq T} E_h(\widehat{\boldsymbol{q}}_h) \leq C(M).$$

Therefore, by Proposition 4.21, the sequence $(\widehat{\boldsymbol{z}}_h)$, where $\widehat{\boldsymbol{z}}_h := \mathcal{Z}_h(\widehat{\boldsymbol{q}}_h)$ for every $h>0$, is compact in $L^1(\Omega;\mathbb{R}^N)$. This shows that $\mathcal{K}$ is a compact subset of $L^1(\Omega;\mathbb{R}^N)$. Now, in view of (5.19), the sequence $(\boldsymbol{q}_h)$ takes values in $\mathcal{H}$. For convenience, for every $h>0$, we define

$$\boldsymbol{z}_h \colon [0,T] \to L^1(\Omega;\mathbb{R}^N), \quad \Delta_h \colon [0,T] \to \mathbb{R}_+ \cup \{0\},$$

by setting

$$\boldsymbol{z}_h(t) := \mathcal{Z}_h(\boldsymbol{q}_h(t)), \quad \Delta_h(t) := \mathrm{Var}_{\mathcal{D}_h}(\boldsymbol{q}_h;[0,t]).$$

By (5.43)–(5.44), the sequence (f_h) is uniformly bounded in $BV([0,T])$. By construction, $(\boldsymbol{z}_h)$ takes values in $\mathcal{K}$ while, by (5.43), it has uniformly bounded variation. Eventually, by definition, each map Δ_h is increasing and the sequence (Δ_h) is uniformly bounded by (5.43). At this point, we apply Theorem 1.55. This yields the existence of three maps

$$f \colon [0,T] \to \mathbb{R}_+ \cup \{0\}, \quad \boldsymbol{z} \colon [0,T] \to L^1(\Omega;\mathbb{R}^N), \quad \Delta \colon [0,T] \to \mathbb{R}_+ \cup \{0\},$$

with $f \in BV([0,T])$, $\boldsymbol{z} \in BV(0,T;L^1(\Omega;\mathbb{R}^N))$ and Δ increasing, such that, up to subsequences, the following convergences hold:

$$\forall\, t \in [0,T], \quad f_h(t) \to f(t), \tag{5.45}$$

$$\forall\, t \in [0,T], \quad \boldsymbol{z}_h(t) \to \boldsymbol{z}(t) \text{ in } L^1(\Omega;\mathbb{R}^N), \tag{5.46}$$

$$\forall\, t \in [0,T], \quad \Delta_h(t) \to \Delta(t). \tag{5.47}$$

To construct the candidate solution $\boldsymbol{q}_0 \colon [0,T] \to \overline{\mathcal{Q}}_0$ of the reduced model we proceed as follows. For every $h>0$, define the functions

$$\boldsymbol{u}_h \colon [0,T] \to W^{1,2}(S;\mathbb{R}^2), \quad v_h \colon [0,T] \to W^{1,2}(S), \quad \boldsymbol{w}_h \colon [0,T] \to W^{1,2}(S;\mathbb{R}^3),$$
$$\boldsymbol{\mu}_h \colon [0,T] \to L^2(\mathbb{R}^3;\mathbb{R}^3), \quad \boldsymbol{N}_h \colon [0,T] \to L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3}).$$

by setting

$$\boldsymbol{u}_h(t) \coloneqq \mathcal{U}_h(\boldsymbol{q}_h(t)), \quad v_h(t) \coloneqq \mathcal{V}_h(\boldsymbol{q}_h(t)), \quad \boldsymbol{w}_h(t) \coloneqq \mathcal{W}_h(\boldsymbol{q}_h(t)),$$
$$\boldsymbol{\mu}_h(t) \coloneqq \mathcal{M}_h(\boldsymbol{q}_h(t)), \quad \boldsymbol{N}_h(t) \coloneqq \mathcal{N}_h(\boldsymbol{q}_h(t)).$$

Recalling (5.20), by Proposition 4.21, we have that

$$\begin{aligned}\sup_{h\ll 1}\sup_{t\in[0,T]} &\left\{\|\boldsymbol{u}_h(t)\|_{W^{1,2}(S;\mathbb{R}^2)} + \|v_h(t)\|_{W^{1,2}(S)} + \|\boldsymbol{w}_h(t)\|_{W^{1,2}(S;\mathbb{R}^3)}\right\} \\ &+ \sup_{h\ll 1}\sup_{t\in[0,T]} \left\{\|\boldsymbol{\mu}_h(t)\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)} + \|\boldsymbol{N}_h(t)\|_{L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3})}\right\} \leq C(M).\end{aligned} \tag{5.48}$$

Now, define $\mathcal{X}$ as the set of all quintuplets

$$(\widehat{\boldsymbol{u}}, \widehat{v}, \widehat{\boldsymbol{w}}, \widehat{\boldsymbol{\mu}}, \widehat{\boldsymbol{N}}) \in W^{1,2}(S;\mathbb{R}^2) \times W^{1,2}(S) \times W^{1,2}(S,\mathbb{R}^3) \times L^2(\mathbb{R}^3;\mathbb{R}^3) \times L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3})$$

with

$$\begin{aligned}\sup_{h>0} &\left\{\|\widehat{\boldsymbol{u}}\|_{W^{1,2}(S;\mathbb{R}^2)} + \|\widehat{v}\|_{W^{1,2}(S)} + \|\widehat{\boldsymbol{w}}\|_{W^{1,2}(S;\mathbb{R}^3)}\right\} \\ &+ \sup_{h>0}\left\{\|\widehat{\boldsymbol{\mu}}\|_{L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3})} + \|\widehat{\boldsymbol{N}}\|_{L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3})}\right\} \leq C(M),\end{aligned}$$

where $C(M) > 0$ is the same constant in (5.48). The space $\mathcal{X}$ is endowed with the product weak topology, which makes it a complete and separable metric space. For every $h>0$, we define the set-valued map $\mathcal{S}_h \colon [0,T] \to \mathcal{P}(\mathcal{X})$ by setting $\mathcal{S}_h(t) \coloneqq \{\boldsymbol{s}_h(t)\}$, where

$$\boldsymbol{s}_h(t) \coloneqq (\boldsymbol{u}_h(t), v_h(t), \boldsymbol{w}_h(t), \boldsymbol{\mu}_h(t), \boldsymbol{N}_h(t))\,.$$

In view of (5.48), this map takes indeed values in $\mathcal{X}$. Consider $\mathcal{S} \colon [0,T] \to \mathcal{P}(\mathcal{X})$ with $\mathcal{S}(t)$ defined as the set of all limit points of the sequence $(\boldsymbol{s}_h(t))$ in $\mathcal{X}$. By definition, this is a closed subset of $\mathcal{X}$ for every $t \in [0,T]$, while the map $\mathcal{S}$ is measurable thanks to Theorem 1.58. From (5.48), by weak compactness, we deduce that the set $\mathcal{S}(t)$ is nonempty for every $t \in [0,T]$. Therefore, by Theorem 1.60, there exists a measurable selection of $\mathcal{S}$, namely a measurable map $\boldsymbol{s} \colon [0,T] \to \mathcal{X}$ with $\boldsymbol{s}(t) \in \mathcal{S}(t)$ for every $t \in [0,T]$. For any such t, let $\boldsymbol{s}(t) = (\boldsymbol{u}(t), v(t), \boldsymbol{w}(t), \boldsymbol{\mu}(t), \boldsymbol{N}(t))$. By definition of $\mathcal{S}(t)$, there exists a subsequence (h_k), possibly depending on t, with $h_k \to 0$, as $k \to \infty$, such that the following convergences hold, as $k \to \infty$:

$$\boldsymbol{u}_{h_k}(t) \rightharpoonup \boldsymbol{u}(t) \text{ in } W^{1,2}(S;\mathbb{R}^2), \tag{5.49}$$
$$v_{h_k}(t) \rightharpoonup v(t) \text{ in } W^{1,2}(S), \tag{5.50}$$
$$\boldsymbol{w}_{h_k}(t) \rightharpoonup \boldsymbol{w}(t) \text{ in } W^{1,2}(S;\mathbb{R}^3), \tag{5.51}$$
$$\boldsymbol{\mu}_{h_k}(t) \rightharpoonup \boldsymbol{\mu}(t) \text{ in } L^2(\mathbb{R}^3;\mathbb{R}^3), \tag{5.52}$$
$$\boldsymbol{N}_{h_k}(t) \rightharpoonup \boldsymbol{N}(t) \text{ in } L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3}). \tag{5.53}$$

Recall (5.20). Applying Proposition 4.21 to the sequence $(\boldsymbol{q}_{h_k}(t))$ and appealing to the Urysohn property, we deduce several facts. First of all, $\boldsymbol{u}(t)$ and $v(t)$ satisfy the boundary conditions

$$\boldsymbol{u}(t) = \overline{\boldsymbol{u}} \text{ on } \Lambda, \quad v(t) = \overline{v} \text{ on } \Lambda, \quad D'v(t) = D'\overline{v} \text{ on } \Lambda.$$

Second, $v(t) \in W^{2,2}(S)$ and we have

$$\boldsymbol{w}(t) = -\frac{1}{12}\begin{pmatrix} D'v(t) \\ 0 \end{pmatrix}.$$

Third, there exist $\boldsymbol{\zeta}(t) \in W^{1,2}(S;\mathbb{S}^2)$ and $\boldsymbol{\chi}(t) \in L^2(\mathbb{R}^3;\mathbb{R}^3)$ such that

$$\boldsymbol{\mu}(t) = \chi_\Omega \boldsymbol{\zeta}(t), \quad \boldsymbol{N}(t) = \chi_\Omega \left(D'\boldsymbol{\zeta}(t) | \boldsymbol{\chi}(t) \right). \tag{5.54}$$

Eventually, recalling (5.46), we see that

$$\boldsymbol{z}(t) = \boldsymbol{\zeta}(t).$$

Hence, from the measurability of the map $t \mapsto D\boldsymbol{w}(t)$ from $[0,T]$ to $L^2(\mathbb{R}^{3\times 3})$, we infer the measurability of v as a map from $[0,T]$ to $W^{2,2}(S)$. Similarly, from the measurability of $\boldsymbol{\mu}$ and $\boldsymbol{N}$, we deduce the measurability of $t \mapsto \boldsymbol{\zeta}(t)$ as a map from $[0,T]$ to $W^{1,2}(S;\mathbb{R}^3)$. Thus, defining $\boldsymbol{q}_0 \colon [0,T] \to \overline{\mathcal{Q}}_0$ by setting $\boldsymbol{q}_0(t) := (\boldsymbol{u}(t), v(t), \boldsymbol{\zeta}(t))$ for every $t \in [0,T]$, this map results to be measurable from $[0,T]$ to $W^{1,2}(S;\mathbb{R}^2) \times W^{2,2}(S) \times W^{1,2}(S;\mathbb{R}^3)$. Also, given (5.48) and (5.49)–(5.53), by lower semicontinuity, we see that map $\boldsymbol{q}_0$ is bounded.
For every $h > 0$, let $P_h \colon [0,T] \to \mathbb{R}$ be defined by setting $P_h(t) := \partial_t \mathcal{F}_h(t, \boldsymbol{q}_h(t))$. Recall that $P_h \in L^1(0,T)$ by Definition 5.1. Thanks to (5.19) and (5.40), for every $t \in [0,T] \setminus Z$, we have

$$|P_h(t)| \leq \kappa_h(t) \left(\mathcal{F}_h(t, \boldsymbol{q}_h(t)) + L(M) \right) \leq C(M) \kappa_h(t). \tag{5.55}$$

In view of the previous identity, the equi-integrability of (κ_h), which comes from (5.31) by the Vitali Convergence Theorem, implies the one of (P_h). Thus, by the Dunford-Pettis Theorem [62, Theorem 2.54], there exists $P \in L^1(0,T)$ such that, up to subsequences, we have

$$P_h \rightharpoonup P \text{ in } L^1(0,T). \tag{5.56}$$

Define $\widehat{P} \colon [0,T] \to \mathbb{R}$ by setting

$$\widehat{P}(t) := \limsup_{h \to 0^+} P_h(t).$$

Exploiting (5.31) and(5.55), we check that $\widehat{P} \in L^1(0,T)$. Also, by the Reverse Fatou Lemma [147, Corollary 5.35], we obtain that $P \leq \widehat{P}$ almost everywhere in $(0,T)$.
We claim that $\widehat{P} = P_0$ almost everywhere in $(0,T)$, where we define $P_0 \colon [0,T] \to \mathbb{R}$ by setting $P_0(t) = \partial_t \mathcal{F}_0(t, \boldsymbol{q}_0(t))$. Recall (5.3)–(5.5). For every $t \in [0,T] \setminus Z$, we have

$$\begin{aligned} h^{-\beta/2} \dot{\boldsymbol{f}}_h(t) &\to \dot{\boldsymbol{f}}_0(t) \text{ in } L^2(S;\mathbb{R}^2), \\ h^{-\beta/2-1} \dot{g}_h(t) &\to \dot{g}_0(t) \text{ in } L^2(S), \\ \dot{\boldsymbol{h}}_h \circ \boldsymbol{\pi}_h(t) &\to \chi_I \dot{\boldsymbol{h}}_0(t) \text{ in } L^2(\mathbb{R}^3;\mathbb{R}^3), \end{aligned}$$

and

$$\begin{aligned}
P_h(t) = &-\frac{1}{h^\beta}\int_\Omega \dot{\boldsymbol{f}}_h(t)\cdot(\boldsymbol{y}_h'(t)-\boldsymbol{x}')\,\mathrm{d}\boldsymbol{x} - \frac{1}{h^\beta}\int_\Omega \dot{g}_h(t)\,y_h^3(t)\,\mathrm{d}\boldsymbol{x}' \\
&-\frac{1}{h}\int_{\mathrm{im_T}(\boldsymbol{y}_h(t),\Omega)} \dot{\boldsymbol{h}}_h(t)\cdot\boldsymbol{m}_h(t)\,\mathrm{d}\boldsymbol{\xi} \\
= &-\int_S h^{-\beta/2}\dot{\boldsymbol{f}}_h(t)\cdot\boldsymbol{u}_h(t)\,\mathrm{d}\boldsymbol{x}' - \int_S h^{-\beta/2-1}\dot{g}_h(t)\,v_h(t)\,\mathrm{d}\boldsymbol{x}' \\
&-\int_{\mathbb{R}^3} \dot{\boldsymbol{h}}_h\circ\boldsymbol{\pi}_h(t)\cdot\boldsymbol{\mu}_h(t)\,\mathrm{d}\boldsymbol{x}.
\end{aligned}$$

Without loss of generality, we can assume that the subsequence (h_k) in (5.49)–(5.53) additionally satisfies $P_{h_k}(t)\to\widehat{P}(t)$, as $k\to\infty$. Thus, taking the limit along the subsequence (h_k), as $k\to\infty$, at both sides of the previous identity, we obtain

$$\widehat{P}(t) = -\int_S \dot{\boldsymbol{f}}_0(t)\cdot\boldsymbol{u}(t)\,\mathrm{d}\boldsymbol{x} - \int_S \dot{g}_0(t)\,v(t)\,\mathrm{d}\boldsymbol{x}' - \int_S \dot{\boldsymbol{h}}_0(t)\cdot\boldsymbol{\zeta}(t)\,\mathrm{d}\boldsymbol{x}'.$$

As the right-hand side of the previous equation coincides with $P_0(t)$ for almost every $t\in[0,T]\setminus Z$, this proves the claim.

Step 3 (Reduced stability). Fix $t\in[0,T]$ and consider the subsequence (h_k) in (5.49)–(5.53). Let $\widehat{\boldsymbol{q}}_0=(\widehat{\boldsymbol{u}},\widehat{v},\widehat{\boldsymbol{\zeta}})\in\overline{\mathcal{Q}}_0$ and let $(\widehat{\boldsymbol{q}}_h)$ be the sequence provided by Proposition 4.22. Thus, for every $k\in\mathbb{N}$, condition (5.6) gives

$$\mathcal{F}_{h_k}(t,\boldsymbol{q}_{h_k}(t)) \leq \mathcal{F}_{h_k}(t,\widehat{\boldsymbol{q}}_{h_k}) + \mathcal{D}_{h_k}(\boldsymbol{q}_{h_k}(t),\widehat{\boldsymbol{q}}_{h_k}). \tag{5.57}$$

For the left-hand side, taking into account (5.3)–(5.5), (5.45) and (5.49)–(5.53), with the aid of Lemma 5.8, we have

$$\mathcal{F}_0(t,\boldsymbol{q}_0(t)) \leq \liminf_{k\to\infty}\mathcal{F}_{h_k}(t,\boldsymbol{q}_{h_k}(t)) = f(t). \tag{5.58}$$

For the right-hand side, given (5.3)–(5.5), (4.108)–(4.109) and (4.111)–(4.112), there holds

$$\lim_{k\to\infty}\mathcal{F}_{h_k}(t,\widehat{\boldsymbol{q}}_{h_k}) = \mathcal{F}_0(t,\widehat{\boldsymbol{q}}_0),$$

while (4.114) and (5.46) yield

$$\lim_{k\to\infty}\mathcal{D}_{h_k}(\boldsymbol{q}_{h_k}(t),\widehat{\boldsymbol{q}}_{h_k}) = \mathcal{D}_0(\boldsymbol{q}_0(t),\widehat{\boldsymbol{q}}_0).$$

Therefore, taking the inferior limit, as $k\to\infty$, in (5.57), we obtain

$$\mathcal{F}_0(t,\boldsymbol{q}_0(t)) \leq \mathcal{F}_0(t,\widehat{\boldsymbol{q}}_0) + \mathcal{D}_0(\boldsymbol{q}_0(t),\widehat{\boldsymbol{q}}_0).$$

As $\widehat{\boldsymbol{q}}_0$ is arbitrary, this proves (5.8) for t fixed.

Step 4 (Upper reduced energy-dissipation intequality). Fix $t\in[0,T]$ and let (h_k) be the sequence in (5.49)–(5.53). By (5.7), for every $k\in\mathbb{N}$, there holds

$$\mathcal{F}_{h_k}(t,\boldsymbol{q}_{h_k}(t)) + \mathrm{Var}_{\mathcal{D}_{h_k}}(\boldsymbol{q}_{h_k};[0,t]) = \mathcal{F}_{h_k}(0,\boldsymbol{q}_{h_k}^0) + \int_0^t P_{h_k}(\tau)\,\mathrm{d}\tau. \tag{5.59}$$

For the left-hand side, we recall (5.58). Also, (4.114) and(5.47) entails

$$\mathrm{Var}_{\mathcal{D}_0}(\boldsymbol{q}_0;[0,t]) \leq \liminf_{k\to\infty} \mathrm{Var}_{\mathcal{D}_{h_k}}(\boldsymbol{q}_{h_k};[0,t]) = \Delta(t). \tag{5.60}$$

For the right-hand side, we have (5.14). Also, recalling (5.56), there holds

$$\lim_{k\to\infty} \int_0^t P_{h_k}(\tau)\,\mathrm{d}\tau = \int_0^t P(\tau)\,\mathrm{d}\tau \leq \int_0^t P_0(\tau)\,\mathrm{d}\tau,$$

where we employed the inequality $P \leq P_0$ almost everywhere in $(0,T)$. Therefore, taking the inferior limit, as $k\to\infty$, in (5.59), we obtain

$$\begin{aligned}\mathcal{F}_0(t,\boldsymbol{q}_0(t)) + \mathrm{Var}_{\mathcal{D}_0}(\boldsymbol{q}_0;[0,t]) &\leq f(t) + \Delta(t)\\ &\leq \mathcal{F}_0(0,\boldsymbol{q}_0^0) + \int_0^t P(\tau)\,\mathrm{d}\tau,\\ &\leq \mathcal{F}_0(0,\boldsymbol{q}_0^0) + \int_0^t P_0(\tau)\,\mathrm{d}\tau,\end{aligned} \tag{5.61}$$

which is the upper reduced energy-dissipation inequality for fixed $t\in[0,T]$.

Step 5 (Lower reduced energy-dissipation inequality). We claim that, for every $t\in[0,T]$, there holds

$$\mathcal{F}_0(t,\boldsymbol{q}_0(t)) + \mathrm{Var}_{\mathcal{D}_0}(\boldsymbol{q}_0;[0,t]) \geq \mathcal{F}_0(0,\boldsymbol{q}_0^0) + \int_0^t \partial_t\mathcal{F}_0(\tau,\boldsymbol{q}_0(\tau))\,\mathrm{d}\tau.$$

Thanks to (5.8), the claims follows from [109, Proposition 5.7].

Step 6 (Improved convergences). We are left to prove (5.16)–(5.18). First, in view of (5.9) and (5.61), we have $f(t)+\Delta(t) = \mathcal{F}_0(t,\boldsymbol{q}_0(t)) + \mathrm{Var}_{\mathcal{D}_0}(\boldsymbol{q}_0;[0,t])$ for every $t\in[0,T]$. Recalling (5.58) and (5.60), this entails $f(t) = \mathcal{F}_0(t,\boldsymbol{q}_0(t))$ and $\Delta(t) = \mathrm{Var}_{\mathcal{D}_0}(\boldsymbol{q}_0;[0,t])$, so that (5.16)–(5.17) are proved. Finally, (5.9) and (5.61) yield $P = P_0$ almost everywhere on $(0,T)$. Thus, (5.18) follows by [64, Lemma 3.5]. □

We conclude the section by mentioning how to treat time-dependent boundary data in Theorem 5.4. The following remark is in the same spirit of Remark 3.9.

Remark 5.9 (Time-dependent integral BCs). Let $\Lambda\subset\partial S$ with $\mathscr{L}^1(\Lambda)>0$ and let $\overline{\boldsymbol{u}}\in AC([0,T];W^{2,\infty}(S;\mathbb{R}^2))$ and $\overline{v}\in AC([0,T];W^{3,\infty}(S))$. Similarly to (4.169), for every $h>0$, define $\overline{\boldsymbol{y}}_h\in AC([0,T];W^{2,\infty}(\Omega;\mathbb{R}^3))$ by setting

$$\overline{\boldsymbol{y}}_h(t) \coloneqq \boldsymbol{\pi}_h + h^{\beta/2}\begin{pmatrix}\overline{\boldsymbol{u}}(t)\\ 0\end{pmatrix} + h^{\beta/2-1}\begin{pmatrix}\boldsymbol{0}'\\ \overline{v}(t)\end{pmatrix} - h^{\beta/2}x_3\begin{pmatrix}D'\overline{v}(t)\\ 0\end{pmatrix}.$$

Recalling Remark 4.23, we define $\mathcal{E}_h^{\mathrm{cl}}\colon [0,T]\times\mathcal{Q}_h\to[0,+\infty)$ by setting

$$\begin{aligned}\mathcal{E}_h^{\mathrm{cl}}(t,\boldsymbol{q}) \coloneqq &\frac{1}{h^\beta}\int_\Gamma |\overline{\boldsymbol{y}}_h'(t)-\boldsymbol{y}'|^2\,\mathrm{d}\boldsymbol{a} + \frac{1}{h^{\beta-2}}\int_\Gamma |\overline{y}_h^3(t)-y^3|^2\,\mathrm{d}\boldsymbol{a}\\ &+\frac{1}{h^\beta}\int_\Gamma |x_3(\overline{\boldsymbol{y}}_h'(t)-\boldsymbol{y}')|^2\,\mathrm{d}\boldsymbol{a},\end{aligned}$$

where $\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m})$. The corresponding reduced functional $\mathcal{E}_0^{\mathrm{cl}} \colon [0,T] \times \mathcal{Q}_0 \to [0,+\infty)$ to be defined as

$$\mathcal{E}_0^{\mathrm{cl}}(t, \boldsymbol{q}_0) \coloneqq \int_\Lambda |\overline{\boldsymbol{u}}(t) - \boldsymbol{u}|^2 \,\mathrm{d}\boldsymbol{l} + \int_\Lambda |\overline{v}(t) - v|^2 \,\mathrm{d}\boldsymbol{l} + \int_\Lambda |x_3 \left(D'\overline{v}(t) - D'v\right)|^2 \,\mathrm{d}\boldsymbol{l},$$

where $\boldsymbol{q}_0 = (\boldsymbol{u}, v, \boldsymbol{\zeta})$. In this case, for every $h > 0$, there exists an energetic solution of the bulk quasitatic model at thickness h driven by the energy $\mathcal{F}_h + \mathcal{E}_h^{\mathrm{cl}}$. Of course, in this case, the definition of such solutions is given without prescribing any boundary conditions in the state space. Also, with minor modifications of the proof of Theorem 5.4, it is proved that such solutions converge, as $h \to 0^+$, to energetic solutions of the reduced quasistatic model driven by $\mathcal{F}_0 + \mathcal{E}_0^{\mathrm{cl}}$.

5.3 Convergence of solutions of the approximate incremental minimization problem

In order to state the second main result of the chapter, we introduce the approximate incremental minimization problem. This is a relaxed version of the incremental minimization problem that has been introduced in order to cope with the possible lack of energy minimizers. This is exactly our situation since, without further assumptions, minimizers of the total energy do to necessarily exist.

Definition 5.10 (Approximate incremental minimization problem). Given $h > 0$, let $\Pi_h = (t_h^0, \dots, t_h^{l_h})$ be a partition of $[0,T]$, let $\alpha_h > 0$ and let $\boldsymbol{q}_h^0 \in \overline{\mathcal{Q}}_h$. The approximate incremental minimization problem (AIMP) determined by Π_h with tolerance α_h and initial datum $\boldsymbol{q}_h^0$ reads as follows: for every $i \in \{1, \dots, l_h\}$, find $\boldsymbol{q}_h^i \in \overline{\mathcal{Q}}_h$ such that

$$\mathcal{F}_h(t_h^i, \boldsymbol{q}_h^i) + \mathcal{D}_h(\boldsymbol{q}_h^{i-1}, \boldsymbol{q}_h^i) \leq (t_h^i - t_h^{i-1})\alpha_h + \inf_{\widehat{\boldsymbol{q}}_h \in \overline{\mathcal{Q}}_h} \left\{ \mathcal{F}_h(t_h^i, \widehat{\boldsymbol{q}}_h) + \mathcal{D}_h(\boldsymbol{q}_h^{i-1}, \widehat{\boldsymbol{q}}_h) \right\}. \quad (5.62)$$

Observe that clamped boundary conditions as in (4.168) have been included in Definition 5.10 and that solutions of the AIMP always exist.

Our third main result claims that, for a sequence of partitions whose sizes vanish jointly with the thickness of the plate together with a sequence of tolerances, solutions of the approximate incremental minimization problem, or better their piecewise-constant interpolants, converge to energetic solutions for the reduced model.

Theorem 5.11 (Convergence of solutions of the AIMP). *Let (Π_h) be a sequence of partitions of $[0,T]$ such that $|\Pi_h| \to 0$, as $h \to 0^+$, and let $(\alpha_h) \subset \mathbb{R}$ with $\alpha_h > 0$ be such that $\alpha_h \to 0$, as $h \to 0^+$. Let $(\boldsymbol{q}_h^0)$ with $\boldsymbol{q}_h^0 = (\boldsymbol{y}_h^0, \boldsymbol{m}_h^0) \in \overline{\mathcal{Q}}_h$ for every $h > 0$ be such that the following holds:*

$$\forall\, \widehat{\boldsymbol{q}}_h \in \mathcal{Q}_h, \quad \mathcal{F}_h(0, \boldsymbol{q}_h^0) \leq \mathcal{F}_h(0, \widehat{\boldsymbol{q}}_h) + \mathcal{D}_h(\boldsymbol{q}_h^0, \widehat{\boldsymbol{q}}_h). \quad (5.63)$$

Moreover, assume that there exist $\boldsymbol{q}_0^0 = (\boldsymbol{u}^0, v, \boldsymbol{\zeta}^0) \in \overline{\mathcal{Q}}_0$ such that, up to subsequences, the following convergences hold, as $h \to 0^+$:

$$\boldsymbol{u}_h^0 \coloneqq \mathcal{U}_h(\boldsymbol{q}_h^0) \rightharpoonup \boldsymbol{u}^0 \ \textit{in } W^{1,2}(S;\mathbb{R}^2); \quad (5.64)$$

$$v_h^0 \coloneqq \mathcal{V}_h(\boldsymbol{q}_h^0) \to v^0 \ \textit{in } W^{1,2}(S); \quad (5.65)$$

$$\boldsymbol{z}_h^0 \coloneqq \mathcal{Z}_h(\boldsymbol{q}_h^0) \to \boldsymbol{\zeta}^0 \ \textit{in } L^1(\Omega;\mathbb{R}^3); \quad (5.66)$$

$$\mathcal{F}_h(0, \boldsymbol{q}_h^0) \to \mathcal{F}_0(0, \boldsymbol{q}_0^0). \quad (5.67)$$

Let $(\boldsymbol{q}_h)$ with $\boldsymbol{q}_h\colon [0,T] \to \overline{\mathcal{Q}}_h$ the right-continuous piecewise constant interpolant corresponding to a solution of the AIMP determined by Π_h with tolerance α_h and initial datum $\boldsymbol{q}_h^0$ for every $h>0$. Then, there exists a function $\boldsymbol{q}_0\colon [0,T] \to \overline{\mathcal{Q}}_0$ with $\boldsymbol{q}_0(t) = (\boldsymbol{u}(t), v(t), \boldsymbol{\zeta}(t))$ for every $t \in [0,T]$ satisfying the initial condition $\boldsymbol{q}_0(0) = \boldsymbol{q}_0^0$ which is an energetic solution of the reduced model. Moreover, up to subsequences, the following convergences hold, as $h \to 0^+$:

$$\forall\, t \in [0,T], \quad \boldsymbol{z}_h(t) \coloneqq \mathcal{Z}_h(\boldsymbol{q}_h(t)) \to \boldsymbol{\zeta}(t) \text{ in } L^1(\Omega;\mathbb{R}^3); \tag{5.68}$$

$$\forall\, t \in [0,T], \quad \mathrm{Var}_{\mathcal{D}_h}(\boldsymbol{q}_h;[0,t]) \to \mathrm{Var}_{\mathcal{D}_0}(\boldsymbol{q}_0;[0,t]); \tag{5.69}$$

$$\forall\, t \in [0,T], \quad \mathcal{F}_h(t,\boldsymbol{q}_h(t)) \to \mathcal{F}_0(t,\boldsymbol{q}_0(t)); \tag{5.70}$$

$$\partial_t \mathcal{F}_h(\cdot,\boldsymbol{q}_h) \to \partial_t \mathcal{F}_0(\cdot,\boldsymbol{q}_0) \text{ in } L^1(0,T). \tag{5.71}$$

In particular, the function $\boldsymbol{q}_0$ is measurable and bounded. Also, the map $t \mapsto \boldsymbol{\zeta}(t)$ belongs to $BV([0,T];L^1(\Omega;\mathbb{R}^{3\times 3}))$ while the map $t \mapsto \mathcal{F}_0(t,\boldsymbol{q}_0(t))$ belongs to $BV([0,T])$.

Here, the measurability and the boundedness of $\boldsymbol{q}_0$ are understood as in Theorem 5.4. The remainder of the chapter is devoted to the proof of Theorem 5.11.

5.3.1 Approximate incremental minimization problem

In the next result, we collect the main properties of solutions of the AIMP. This represents the analog of Proposition 3.8 in the setting of dimension reduction.

Proposition 5.12 (Solutions of the AIMP). *Let (Π_h) be a sequence of partitions of $[0,T]$ and let $(\alpha_h) \subset \mathbb{R}$ with $\alpha_h > 0$ for every $h>0$ be bounded. Also, let $(\boldsymbol{q}_h^0)$ with $\boldsymbol{q}_h^0 \in \mathcal{Q}_h$ for every $h>0$ be such that*

$$\sup_{h>0} \mathcal{F}_h(0,\boldsymbol{q}_h^0) \leq C. \tag{5.72}$$

For every $h>0$, let $\Pi_h = (t_h^0,\dots,t_h^{l_h})$ and let $(\boldsymbol{q}_h^1,\dots,\boldsymbol{q}_h^{l_h})$ be a solution of the AIMP determined by Π_h with tolerance α_h and initial datum $\boldsymbol{q}_h^0$. Then, there exists $M>0$ such that

$$\sup_{h>0} \sup_{i \in \{0,\dots,l_h\}} \mathcal{F}_h(t_h^i,\boldsymbol{q}_h^i) \leq M. \tag{5.73}$$

Moreover, for every $h \ll 1$ and $i \in \{1,\dots,l_h\}$, there hold:

$$\forall\, \widehat{\boldsymbol{q}}_h \in \mathcal{Q}_h, \quad \mathcal{F}_h(t_h^i,\boldsymbol{q}_h^i) \leq (t_h^i - t_h^{i-1})\alpha_h + \mathcal{F}_h(t_h^i,\widehat{\boldsymbol{q}}_h) + \mathcal{D}_h(\boldsymbol{q}_h^i,\widehat{\boldsymbol{q}}_h), \tag{5.74}$$

$$\begin{aligned} \mathcal{F}_h(t_h^i,\boldsymbol{q}_h^i) + \mathcal{D}_h(\boldsymbol{q}_h^{i-1},\boldsymbol{q}_h^i) &\leq (t_h^i - t_h^{i-1})\alpha_h + \mathcal{F}_h(t_h^{i-1},\boldsymbol{q}_h^{i-1}) \\ &\quad + \int_{t_h^{i-1}}^{t_h^i} \partial_t \mathcal{F}_h(\tau,\boldsymbol{q}_h^{i-1})\,\mathrm{d}\tau, \end{aligned} \tag{5.75}$$

$$\mathcal{F}_h(t_h^i,\boldsymbol{q}_h^i) + L(M) + \sum_{j=1}^{i} \mathcal{D}_h(\boldsymbol{q}_h^{j-1},\boldsymbol{q}_h^j) \leq \left(\mathcal{F}_h(0,\boldsymbol{q}_h^0) + L(M) + t_i\alpha_h\right) \mathrm{e}^{K_h(t_h^i)}. \tag{5.76}$$

Eventually, if $\boldsymbol{q}_h^0$ satisfies (5.10) for $h \ll 1$, then, for every $i \in \{1,\dots,l_h\}$, there holds

$$\begin{aligned} |\mathcal{F}_h(t_h^i,\boldsymbol{q}_h^i) - \mathcal{F}_h(t_h^{i-1},\boldsymbol{q}_h^{i-1}) + \mathcal{D}_h(\boldsymbol{q}_h^{i-1},\boldsymbol{q}_h^i)| &\leq \left(t_h^i - t_h^{i-2}\right)\alpha_h \\ &\quad + \left(\mathcal{F}_h(t_h^{i-1},\boldsymbol{q}_h^{i-1}) + 1\right)\left(\mathrm{e}^{K_h(t_h^i) - K_h(t_h^{i-1})} - 1\right), \end{aligned} \tag{5.77}$$

where we set $t_h^{-1} := 0$.

Proof. The computations that lead to (5.74)–(5.77) are similar to the ones performed in Proposition 3.8. However, we provide full details.
Let $h > 0$ and $i \in \{1, \dots, l_h\}$. For simplicity, set $\alpha_h^i := (t_h^i - t_h^{i-1})\alpha_h$. Given (5.62), for every $\widehat{\boldsymbol{q}}_h \in \mathcal{Q}_h$, we have

$$\begin{aligned}\mathcal{F}_h(t_h^i, \boldsymbol{q}_h^i) &\leq \alpha_h^i + \mathcal{F}_h(t_h^i, \widehat{\boldsymbol{q}}_h) + \mathcal{D}_h(\boldsymbol{q}_h^{i-1}, \widehat{\boldsymbol{q}}_h) - \mathcal{D}_h(\boldsymbol{q}_h^{i-1}, \boldsymbol{q}_h^i) \\ &\leq \alpha_h^i + \mathcal{F}_h(t_h^i, \widehat{\boldsymbol{q}}_h) + \mathcal{D}_h(\boldsymbol{q}_h^i, \widehat{\boldsymbol{q}}_h),\end{aligned} \tag{5.78}$$

where, in the last line, we employed the triangle inequality. This shows (5.74).
We check (5.75). For simplicity, set $f_h^i := \mathcal{F}_h(t_h^i, \boldsymbol{q}_h^i)$ and $d_h^i := \mathcal{D}_h(\boldsymbol{q}_h^{i-1}, \boldsymbol{q}_h^i)$.From (5.62), by applying the Fundamental Theorem of Calculus, we obtain

$$\begin{aligned}f_h^i - f_h^{i-1} + d_h^i &\leq \alpha_h^i - f_h^{i-1} + \mathcal{F}_h(t_h^i, \boldsymbol{q}_h^{i-1}) \\ &= \alpha_h^i + \mathcal{F}_h(t_h^i, \boldsymbol{q}_h^{i-1}) - \mathcal{F}_h(t_h^{i-1}, \boldsymbol{q}_h^{i-1}) \\ &= \alpha_h^i + \int_{t_h^{i-1}}^{t_h^i} \partial_t \mathcal{F}_h(\tau, \boldsymbol{q}_h^{i-1})\,\mathrm{d}\tau,\end{aligned} \tag{5.79}$$

which gives (5.75).
Testing (5.74) with $\widehat{\boldsymbol{q}}_h = \overline{\boldsymbol{q}}_h$, where the latter is defined as in the proof of Theorem 4.14, we have

$$\mathcal{F}_h(t_h^i, \boldsymbol{q}_h^i) \leq \alpha_h^i + \mathcal{F}_h(t_h^i, \overline{\boldsymbol{q}}_h) + \mathcal{D}_h(\boldsymbol{q}_h^i, \overline{\boldsymbol{q}}_h).$$

Exploting the uniform boundedness of the applied loads with respect to time, which follows from (5.3)–(5.5) by the Morrey embedding, with computations analogous to the one in the proof of Theorem 4.14 we check that the right-hand side of the previous inequality is uniformly bounded with respect to $i \in \{0, \dots, l_h\}$ and $h > 0$. Here, we also take advantage of (5.72) and of the boundedness of (α_h). Therefore, (5.73) is proved.
Now, for the sake of clarity, we specify the sequence (h_n) such that $h_n \to 0^+$, as $n \to \infty$, in place of $h > 0$. Thus, $(\boldsymbol{q}_{h_n}^1, \dots, \boldsymbol{q}_{h_n}^{l_{h_n}})$ is a solution of the AIMP determined by Π_{h_n} with tolerance α_{h_n} and initial datum $\boldsymbol{q}_{h_n}^0$ for every $n \in \mathbb{N}$. By applying Lemma 5.7 with $(\widehat{\boldsymbol{q}}_h)$ given by the sequence

$$\boldsymbol{q}_{h_1}^0, \boldsymbol{q}_{h_1}^1, \dots, \boldsymbol{q}_{h_1}^{l_{h_1}}, \boldsymbol{q}_{h_2}^0, \boldsymbol{q}_{h_2}^1, \dots, \boldsymbol{q}_{h_2}^{l_{h_2}}, \boldsymbol{q}_{h_3}^0, \boldsymbol{q}_{h_3}^1, \dots, \boldsymbol{q}_{h_3}^{l_{h_3}}, \dots$$

we deduce that there exists $\overline{n} \in \mathbb{N}$ such that, for every $n \in \mathbb{N}$ with $n \geq \overline{n}$ and for every $i \in \{0, 1, \dots, l_{h_n}\}$, the following estimates hold:

$$\begin{aligned}\forall\, t \in [0,T] \setminus Z, \quad & |\partial_t \mathcal{F}_{h_n}(t, \boldsymbol{q}_{h_n}^i)| \leq \kappa_{h_n}(t) \left(\mathcal{F}_{h_n}(t, \boldsymbol{q}_{h_n}^i) + L(M)\right), \\ \forall s, t \in [0,T], \quad & \mathcal{F}_{h_n}(t, \boldsymbol{q}_{h_n}^i) + L(M) \leq \left(\mathcal{F}_{h_n}(s, \boldsymbol{q}_{h_n}^i) + L(M)\right) \mathrm{e}^{|K_{h_n}(t) - K_{h_n}(s)|}, \\ \forall t \in [0,T] \setminus Z,\ & \forall s \in [0,T], \\ & |\partial_t \mathcal{F}_{h_n}(t, \boldsymbol{q}_{h_n}^i)| \leq \kappa_{h_n}(t) \left(\mathcal{F}_{h_n}(s, \boldsymbol{q}_{h_n}^i) + L(M)\right) \mathrm{e}^{|K_{h_n}(t) - K_{h_n}(s)|}.\end{aligned}$$

Henceforth, for simplicity, we will go back writing h as a subscript without specifying the sequence of thicknesses. Accordingly, there exists $\overline{h} > 0$ such that, for every

$h \leq \overline{h}$ and $i \in \{0, 1, \dots, l_h\}$, there hold:

$$\forall\, t \in [0,T] \setminus Z, \quad |\partial_t \mathcal{F}_h(t, \boldsymbol{q}_h^i)| \leq \kappa_h(t) \left(\mathcal{F}_h(t, \boldsymbol{q}_h^i) + L(M)\right), \tag{5.80}$$

$$\forall s, t \in [0,T], \quad \mathcal{F}_h(t, \boldsymbol{q}_h^i) + L(M) \leq \left(\mathcal{F}_h(s, \boldsymbol{q}_h^i) + L(M)\right) \mathrm{e}^{|K_h(t) - K_h(s)|}, \tag{5.81}$$

$$\begin{aligned} &\forall t \in [0,T] \setminus Z, \, \forall s \in [0,T], \\ &\qquad |\partial_t \mathcal{F}_h(t, \boldsymbol{q}_h^i)| \leq \kappa_h(t) \left(\mathcal{F}_h(s, \boldsymbol{q}_h^i) + L(M)\right) \mathrm{e}^{|K_h(t) - K_h(s)|}. \end{aligned} \tag{5.82}$$

Going back to proof of (5.76), let $h \leq \overline{h}$ and $i \in \{1, \dots, l_h\}$. For convenience, set $K_h^i \coloneqq K_h(t_h^i)$. Combining (5.75) with (5.82), we compute

$$\begin{aligned} f_h^i \leq f_h^i + d_h^i &\leq \alpha_h^i + f_h^{i-1} + \int_{t_h^{i-1}}^{t_h^i} \partial_t \mathcal{F}_h(\tau, \boldsymbol{q}_h^{i-1}) \, \mathrm{d}\tau \\ &\leq \alpha_h^i + f_h^{i-1} + \left(f_h^{i-1} + L(M)\right) \int_{t_h^{i-1}}^{t_h^i} \kappa_h(\tau) \mathrm{e}^{K_h(\tau) - K_h^{i-1}} \, \mathrm{d}\tau \\ &= \alpha_h^i + f_h^{i-1} + \left(f_h^{i-1} + L(M)\right) \left(\mathrm{e}^{K_h^i - K_h^{i-1}} - 1\right) \\ &= \alpha_h^i - L(M) + \left(f_h^{i-1} + L(M)\right) \mathrm{e}^{K_h^i - K_h^{i-1}}. \end{aligned}$$

Thus

$$f_h^i + L(M) \leq \alpha_h^i + \left(f_h^{i-1} + L(M)\right) \mathrm{e}^{K_h^i - K_h^{i-1}},$$

from which, by induction, we obtain

$$f_h^i + L(M) \leq \left(f_h^0 + L(M) + \sum_{j=1}^{i} \mathrm{e}^{-K_h^j} \alpha_h^j \right) \mathrm{e}^{K_h^i}. \tag{5.83}$$

Here, we set $f_h^0 \coloneqq \mathcal{F}_h(0, \boldsymbol{q}_h^0)$. From (5.75), using (5.82), we estimate

$$\begin{aligned} f_h^i - f_h^{i-1} + d_h^i &\leq \alpha_h^i + \int_{t_h^{i-1}}^{t_h^i} \partial_h \mathcal{F}_h(\tau, \boldsymbol{q}_h^{i-1}) \, \mathrm{d}\tau \\ &\leq \alpha_h^i + (f_h^{i-1} + c) \int_{t_h^{i-1}}^{t_h^i} \kappa_h(\tau) \mathrm{e}^{K_h(\tau) - K_h(t_h^{i-1})} \, \mathrm{d}\tau \\ &\leq \alpha_h^i + (f_h^{i-1} + c) \left(\mathrm{e}^{K_h(t_h^i) - K_h(t_h^{i-1})} - 1\right). \end{aligned} \tag{5.84}$$

Then, summing (5.84), with j in place of i, for $j \in \{1, \dots, i\}$ and employing (5.83), with $j-1$ in place of i, we obtain

$$
\begin{aligned}
f_h^i + \sum_{j=1}^{i} d_h^j + L(M) &\le f_h^0 + L(M) + \sum_{j=1}^{i} \alpha_h^j + \sum_{j=1}^{i} (f_h^{j-1} + L(M)) \left(\mathrm{e}^{K_h^j - K_h^{j-1}} - 1 \right) \\
&\le f_h^0 + L(M) + \sum_{j=1}^{i} \alpha_h^j \\
&\quad + \sum_{j=1}^{i} \left(f_h^0 + L(M) + \sum_{k=1}^{j-1} \mathrm{e}^{-K_h^k} \alpha_h^k \right) \left(\mathrm{e}^{K_h^j} - \mathrm{e}^{K_h^{j-1}} \right) \\
&= \sum_{j=1}^{i} \alpha_h^j + (f_h^0 + c)\mathrm{e}^{K_h^i} + \sum_{j=1}^{i} \left(\mathrm{e}^{K_h^j} - \mathrm{e}^{K_h^{j-1}} \right) \sum_{k=1}^{j-1} \mathrm{e}^{-K_h^k} \alpha_h^k \\
&\le \sum_{j=1}^{i} \alpha_h^j + (f_h^0 + L(M))\mathrm{e}^{K_h^i} + \sum_{j=1}^{i} \left(\mathrm{e}^{K_h^j} - \mathrm{e}^{K_h^{j-1}} \right) \sum_{k=1}^{i} \alpha_h^k \\
&= \left(f_h^0 + L(M) + \sum_{j=1}^{i} \alpha_h^i \right) \mathrm{e}^{K_h^i},
\end{aligned}
$$

which yields (5.76).
Finally, we prove (5.77). Testing (5.74) for $i-1$ if $i>1$ or (5.63) if $i=1$ both with $\widehat{\boldsymbol{q}}_h = \boldsymbol{q}_h^i$, we have

$$
f_h^{i-1} \le \alpha_h^{i-1} + \mathcal{F}_h(t_h^{i-1}, \boldsymbol{q}_h^i) + d_h^i.
$$

Here, in the second case, consider $\alpha_h^0 := 0$. From this, employing the Fundamental Theorem of Calculus and (5.82), we compute

$$
\begin{aligned}
f_h^{i-1} - f_h^i - d_h^i &\le \alpha_h^{i-1} - \left(\mathcal{F}_h(t_h^i, \boldsymbol{q}_h^i) - \mathcal{F}_h(t_h^{i-1}, \boldsymbol{q}_h^i) \right) \\
&= \alpha_h^{i-1} - \int_{t_h^{i-1}}^{t_h^i} \partial_t \mathcal{F}_h(\tau, \boldsymbol{q}_h^i) \, \mathrm{d}\tau \\
&\le \alpha_h^{i-1} + (f_h^{i-1} + c) \int_{t_h^{i-1}}^{t_h^i} \kappa_h(\tau) \mathrm{e}^{K_h(\tau) - K_h^{i-1}} \, \mathrm{d}\tau \\
&\le \alpha_h^{i-1} + (f_h^{i-1} + c) \left(\mathrm{e}^{K_h^i - K_h^{i-1}} - 1 \right).
\end{aligned}
\tag{5.85}
$$

Combining (5.84)–(5.85), since $\alpha_h^{i-1} + \alpha_h^i = (t_h^i - t_h^{i-2})\alpha_h$, we obtain (5.77). □

5.3.2 Proof of Theorem 5.11

We now move to the proof of the second main result of the chapter. This is formally similar to the one of Theorem 3.2 in view of the observation made in Remark 3.10.

Proof of Theorem 5.11. We follow the scheme introduced in [112]. In turn, the proof is subdivided into six steps.

Step 1 (A priori estimates). For every $h>0$, let $(\boldsymbol{q}_h^1, \dots, \boldsymbol{q}_h^{l_h})$ be a solution of the AIMP determined by $\Pi_h = (t_h^0, \dots, t_h^{l_h})$ with tolerance $\alpha_h > 0$ and initial datum

$\boldsymbol{q}_h^0$. We introduce the corresponding right-continuous piecewise-constant interpolant $\boldsymbol{q}_h\colon [0,T] \to \overline{\mathcal{Q}}_h$ by setting

$$\boldsymbol{q}_h(t) \coloneqq \begin{cases} \boldsymbol{q}_h^{i-1} & \text{if } t \in [t_h^{i-1}, t_h^i) \text{ for some } i \in \{1,\dots,l_h\}, \\ \boldsymbol{q}_h^{l_h} & \text{if } t = T. \end{cases}$$

Let $t \in [0,T]$ and let $i \in \{1,\dots,l_h\}$ be such that $t \in [t_h^{i-1}, t_h^i)$. By definition, we have

$$\boldsymbol{q}_h(t) = \boldsymbol{q}_h^{i-1}, \qquad \mathrm{Var}_{\mathcal{D}_h}(\boldsymbol{q}_h; [0,t]) = \sum_{j=1}^{i-1} \mathcal{D}_h(\boldsymbol{q}_h^{j-1}, \boldsymbol{q}_h^j). \tag{5.86}$$

Thus, using (5.76) and (5.81), we infer

$$\begin{aligned} &\mathcal{F}_h(t, \boldsymbol{q}_h(t)) + L(M) + \mathrm{Var}_{\mathcal{D}_h}(\boldsymbol{q}_h; [0,t]) \\ &\qquad \leq (\mathcal{F}_h(t_h^{i-1}, \boldsymbol{q}_h^{i-1}) + L(M)) \mathrm{e}^{K_h(t) - K_h(t_h^{i-1})} + \sum_{j=1}^{i-1} \mathcal{D}_h(\boldsymbol{q}_h^{j-1}, \boldsymbol{q}_h^j) \\ &\qquad \leq \left(\mathcal{F}_h(t_h^{i-1}, \boldsymbol{q}_h^{i-1}) + L(M) + \sum_{j=1}^{i-1} \mathcal{D}_h(\boldsymbol{q}_h^{j-1}, \boldsymbol{q}_h^j) \right) \mathrm{e}^{K_h(t) - K_h(t_h^{i-1})} \\ &\qquad \leq (\mathcal{F}_h(0, \boldsymbol{q}_h^0) + L(M) + t_h^{i-1}\alpha_h) \left(\mathrm{e}^{K_h(t)} - 1\right) \\ &\qquad \leq (\mathcal{F}_h(0, \boldsymbol{q}_h^0) + L(M) + T\alpha_h) \left(\mathrm{e}^{K_h(T)} - 1\right). \end{aligned}$$

Since the sequences $(\mathcal{F}_h(0, \boldsymbol{q}_h^0))$, (α_h) and $(K_h(T))$ are all bounded, we obtain the estimate

$$\sup_{h>0} \sup_{t\in[0,T]} \mathcal{F}_h(t, \boldsymbol{q}_h(t)) + \sup_{h>0} \mathrm{Var}_{\mathcal{D}_h}(\boldsymbol{q}_h; [0,T]) \leq C(M,T). \tag{5.87}$$

For simplicity, for every $h > 0$, define $f_h\colon [0,T] \to \mathbb{R}$ by setting $f_h(t) \coloneqq \mathcal{F}_h(t, \boldsymbol{q}_h(t))$. We aim to establish a uniform bound for the total variation of f_h. For every $i \in \{1,\dots,l_h\}$, we set

$$f_h^i \coloneqq f_h(t_h^i) = \mathcal{F}_h(t_h^i, \boldsymbol{q}_h^i), \quad d_h^i \coloneqq \mathcal{D}_h(\boldsymbol{q}_h^{i-1}, \boldsymbol{q}_h^i), \quad K_h^i \coloneqq K_h(t_h^i).$$

Also, we recall the notation $t_h^{-1} \coloneqq 0$. First, denote by $[f_h]^i$ the jump of f_h at time t_h^i. Exploiting the continuity of $t \mapsto \mathcal{F}_h(t, \boldsymbol{q}_h^{i-1})$ and employing (5.77) and (5.81), we compute

$$
\begin{aligned}
|[f_h]^i| &= \left| \lim_{s\to 0^+} \left\{ f_h(t_h^i) - f_h(t_h^i - s) \right\} \right| \\
&= \left| \lim_{s\to 0^+} \left\{ \mathcal{F}_h(t_h^i, \boldsymbol{q}_h^i) - \mathcal{F}_h(t_h^i - s, \boldsymbol{q}_h^{i-1}) \right\} \right| \\
&= |f_h^i - f_h^{i-1} - (\mathcal{F}_h(t_h^i, \boldsymbol{q}_h^{i-1}) - \mathcal{F}_h(t_h^{i-1}, \boldsymbol{q}_h^{i-1}))| \\
&\leq |f_h^i - f_h^{i-1}| + \int_{t_h^{i-1}}^{t_h^i} |\partial_t \mathcal{F}_h(\tau, \boldsymbol{q}_h^{i-1})| \,\mathrm{d}\tau \\
&\leq |f_h^i - f_h^{i-1}| + (f_h^{i-1} + L(M)) \int_{t_h^{i-1}}^{t_h^i} \kappa_h(\tau) \mathrm{e}^{K_h(\tau) - K_h^{i-1}} \,\mathrm{d}\tau \\
&\leq |f_h^i - f_h^{i-1}| + (M + L(M)) \left(\mathrm{e}^{K_h^i - K_h^{i-1}} - 1 \right) \\
&\leq d_h^i + \alpha_h^{i-1} + \alpha_h^i + 2(M + L(M)) \left(\mathrm{e}^{K_h^i - K_h^{i-1}} - 1 \right).
\end{aligned}
\tag{5.88}
$$

Second, for $t \in (t_h^{i-1}, t_h^i)$, we have $f_h(t) = \mathcal{F}_h(t, \boldsymbol{q}_h^{i-1})$ and, in turn, $\dot{f}_h(t) = \partial_t \mathcal{F}_h(t, \boldsymbol{q}_h^{i-1})$. Making once more use of (5.81), we compute

$$
\begin{aligned}
\int_{t_h^i}^{t_h^{i-1}} |\dot{f}_h(\tau)| \,\mathrm{d}\tau &= \int_{t_h^i}^{t_h^{i-1}} |\partial_t \mathcal{F}_h(\tau, \boldsymbol{q}_h^{i-1})| \,\mathrm{d}\tau \\
&\leq (f_h^{i-1} + L(M)) \int_{t_h^i}^{t_h^{i-1}} \kappa_h(\tau) \mathrm{e}^{K_h(\tau) - K_h^{i-1}} \,\mathrm{d}\tau \\
&\leq (M + L(M)) \left(\mathrm{e}^{K_h^i - K_h^{i-1}} - 1 \right).
\end{aligned}
\tag{5.89}
$$

Thus, combining (5.88)–(5.89), we obtain

$$
\begin{aligned}
\mathrm{Var}(f_h; [0,T]) &= \sum_{i=1}^{l_h} \left\{ |[f_h]^i| + \int_{t_h^i}^{t_h^{i-1}} |\dot{f}_h(\tau)| \,\mathrm{d}\tau \right\} \\
&\leq \sum_{i=1}^{N_h} \left\{ d_h^i + \alpha_h^i + \alpha_h^{i-1} + 3(M + L(M)) \left(\mathrm{e}^{K_h^i - K_h^{i-1}} - 1 \right) \right\} \\
&\leq \mathrm{Var}_{\mathcal{D}_h}(\boldsymbol{q}_h; [0,T]) + 2T\alpha_h + 3(M + L(M)) \left(\mathrm{e}^{K_h(T)} - 1 \right),
\end{aligned}
\tag{5.90}
$$

where in the last line we computed

$$
\begin{aligned}
\sum_{i=1}^{l_h} \left(\mathrm{e}^{K_h^i - K_h^{i-1}} - 1 \right) &= \sum_{i=1}^{l_h} \mathrm{e}^{-K_h^{i-1}} \left(\mathrm{e}^{K_h^i} - \mathrm{e}^{K_h^{i-1}} \right) \\
&\leq \sum_{i=1}^{l_h} \left(\mathrm{e}^{K_h^i} - \mathrm{e}^{K_h^{i-1}} \right) \\
&= \mathrm{e}^{K_h(T)} - 1.
\end{aligned}
$$

Thus, recalling the boundedness of (α_h) and $(K_h(T))$, where the latter follows from (5.32), the inequality (5.90) together with (5.87) gives

$$
\sup_{h>0} \mathrm{Var}(f_h; [0,T]) \leq C(M,T). \tag{5.91}
$$

Step 2 (Compactness). For every $h > 0$, define the functions

$$\boldsymbol{z}_h\colon [0,T] \to L^1(\Omega;\mathbb{R}^N), \quad \Delta_h\colon [0,T] \to \mathbb{R}_+ \cup \{0\},$$

and

$$\boldsymbol{u}_h\colon [0,T] \to W^{1,2}(S;\mathbb{R}^2), \quad v_h\colon [0,T] \to W^{1,2}(S), \quad \boldsymbol{w}_h\colon [0,T] \to W^{1,2}(S;\mathbb{R}^3),$$
$$\boldsymbol{\mu}_h\colon [0,T] \to L^2(\mathbb{R}^3;\mathbb{R}^3), \quad \boldsymbol{N}_h\colon [0,T] \to L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3}).$$

as in in Step 2 of the proof of Theorem 5.4. Exploiting the a priori estimates (5.87) and (5.91), with the aid of Theorem 1.55, we establish the existence of functions $f \in BV([0,T])$, $\boldsymbol{z} \in BV([0,T];L^1(\Omega;\mathbb{R}^3))$ and $\Delta\colon [0,T] \to [0,+\infty)$ such that, up to subsequences, the convergences in (5.45)–(5.47) hold true. Moreover, employing Theorem 1.60, we identify a function $\boldsymbol{q}_0\colon [0,T] \to \overline{\mathcal{Q}}_0$ with $\boldsymbol{q}_0(t) = (\boldsymbol{u}(t), v(t), \boldsymbol{\zeta}(t))$ for every $t \in [0,T]$ which is measurable and bounded. This function is characterized by the property that, for every $t \in [0,T]$, there exists a subsequence (h_k), possibly depending on t, such that (5.49)–(5.50) hold true and also

$$\boldsymbol{\mu}_{h_k}(t) \rightharpoonup \chi_\Omega \boldsymbol{\zeta}(t) \text{ in } L^2(\mathbb{R}^3;\mathbb{R}^3), \tag{5.92}$$
$$\boldsymbol{N}_h(t) \rightharpoonup \chi_\Omega (D'\boldsymbol{\zeta}(t), \boldsymbol{\chi}(t)) \text{ in } L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3}), \tag{5.93}$$

for some $\boldsymbol{\chi}\colon [0,T] \to L^2(\mathbb{R}^3;\mathbb{R}^3)$ measurable and bounded.
Then, for every $h > 0$, we define $P_h\colon [0,T] \to \mathbb{R}$ as in Step 2 of the proof of Theorem 5.4 and, with the same arguments, we prove that (5.56) holds true for some $P \in L^1(0,T)$. Eventually, defining $P_0\colon [0,T] \to \mathbb{R}$ and $\widehat{P}\colon [0,T] \to \mathbb{R}$ as in Step 2 of the proof of Theorem 5.4, we show that $P \le \widehat{P}$ and $\widehat{P} = P_0$ almost everywhere in $(0,T)$.

Step 3 (Reduced stability). In view of (5.74), for every $h > 0$, there holds

$$\begin{aligned}\forall\, t_h \in \Pi_h,\ \forall\, \widehat{\boldsymbol{q}}_h \in \mathcal{Q}_h, \quad \mathcal{F}_h(t_h, \boldsymbol{q}_h(t_h)) \le{}& |\Pi_h|\alpha_h \\ &+ \mathcal{F}_h(t_h, \widehat{\boldsymbol{q}}_h) + \mathcal{D}_h(\boldsymbol{q}_h(t_h), \widehat{\boldsymbol{q}}_h).\end{aligned} \tag{5.94}$$

Define $\tau_h\colon [0,T] \to \Pi_h$ by setting $\tau_h(t) := \{s \in \Pi_h : s \le t\}$. Since $|\Pi_h| \to 0$, as $h \to 0^+$, we have $\tau_h(t) \to t$, as $h \to 0^+$. Fix $t \in [0,T]$ and let (h_k) be the subsequence such that (5.49)–(5.53) hold. By definition, we have $\boldsymbol{q}_{h_k}(t) = \boldsymbol{q}_{h_k}(\tau_{h_k}(t))$. Let $\widehat{\boldsymbol{q}}_0 \in \overline{\mathcal{Q}}_0$ and let $(\widehat{\boldsymbol{q}}_h)$ a corresponding recovery sequence given by Proposition 4.22. By (5.94), there holds

$$\mathcal{F}_{h_k}(\tau_{h_k}(t), \boldsymbol{q}_{h_k}(t)) \le |\Pi_{h_k}|\alpha_{h_k} + \mathcal{F}_{h_k}(t, \widehat{\boldsymbol{q}}_{h_k}) + \mathcal{D}_{h_k}(\boldsymbol{q}_{h_k}(t), \widehat{\boldsymbol{q}}_{h_k}). \tag{5.95}$$

Applying Lemma 5.8, we obtain

$$\mathcal{F}_0(t, \boldsymbol{q}_0(t)) \le \liminf_{k\to\infty} \mathcal{F}_{h_k}(\tau_{h_k}(t), \boldsymbol{q}_{h_k}(t)) = f(t). \tag{5.96}$$

On the other hand, we have

$$\mathcal{F}_0(t, \widehat{\boldsymbol{q}}_0) + \mathcal{D}_0(\boldsymbol{q}_0(t), \widehat{\boldsymbol{q}}_0) = \lim_{k\to\infty} \left\{\mathcal{F}_{h_k}(t, \widehat{\boldsymbol{q}}_{h_k}) + \mathcal{D}_{h_k}(\boldsymbol{q}_{h_k}(t), \widehat{\boldsymbol{q}}_{h_k})\right\}. \tag{5.97}$$

Thus, combining (5.95)–(5.97), we get

$$\begin{aligned}\mathcal{F}_0(t, \boldsymbol{q}_0(t)) &\le \liminf_{k\to\infty} \mathcal{F}_{h_k}(\tau_{h_k}(t), \boldsymbol{q}_{h_k}(t)) \\ &\le \liminf_{k\to\infty} \left\{|\Pi_{h_k}|\,\alpha_{h_k} + \mathcal{F}_{h_k}(\tau_{h_k}(t), \widehat{\boldsymbol{q}}_{h_k}) + \mathcal{D}_{h_k}(\boldsymbol{q}_{h_k}(t), \widehat{\boldsymbol{q}}_{h_k})\right\} \\ &= \mathcal{F}_0(t, \widehat{\boldsymbol{q}}_0) + \mathcal{D}_0(\boldsymbol{q}_0(t), \widehat{\boldsymbol{q}}_0),\end{aligned}$$

which proves (5.8) for t fixed.

Step 4 (Upper energy-dissipation inequality). We prove the following:

$$\forall\, t \in [0,T], \quad \mathcal{F}_0(t, \boldsymbol{q}_0(t)) + \mathrm{Var}_{\mathcal{D}_0}(\boldsymbol{q}_0; [0,t]) \le \mathcal{F}_0(0, \boldsymbol{q}_0^0) + \int_0^t \partial_t \mathcal{F}_0(\tau, \boldsymbol{q}_0(\tau))\,\mathrm{d}\tau. \tag{5.98}$$

First, fix $t \in [0,T]$ and let $h > 0$. Note that $\tau_h(t) \le t$. Applying the Fundamental Theorem of Calculus and employing (5.82) and (5.87), we estimate

$$\begin{aligned} |\mathcal{F}_h(t, \boldsymbol{q}_h(t)) - \mathcal{F}_h(\tau_h(t), \boldsymbol{q}_h(\tau_h(t)))| &\le \int_{\tau_h(t)}^t |\partial_t \mathcal{F}_h(\tau, \boldsymbol{q}_h(\tau_h(t)))|\,\mathrm{d}\tau \\ &\le (\mathcal{F}_h(\tau_h(t), \boldsymbol{q}_h(\tau_h(t))) + L(M)) \int_{\tau_h(t)}^t \kappa_h(\tau) \mathrm{e}^{K_h(\tau) - K_h(\tau_h(t))}\,\mathrm{d}\tau \\ &= (M + L(M)) \left(\mathrm{e}^{K_h(t) - K_h(\tau_h(t))} - 1\right). \end{aligned} \tag{5.99}$$

Then, let $\tau_h(t) = t_h^i$ where $i \in \{1, \dots, l_h\}$ and recall (5.86). Summing the inequality (5.75), with j in place of i, for $j \in \{1, \dots, i-1\}$ we obtain:

$$\mathcal{F}_h(\tau_h(t), \boldsymbol{q}_h(\tau_h(t))) + \mathrm{Var}_{\mathcal{D}_h}(\boldsymbol{q}_h; [0, \tau_h(t)]) \le \tau_h(t)\alpha_h + \mathcal{F}_h(0, \boldsymbol{q}_h^0) + \int_0^{\tau_h(t)} \partial_t \mathcal{F}_h(\tau, \boldsymbol{q}_h(\tau))\,\mathrm{d}\tau. \tag{5.100}$$

Also, by definition

$$\mathrm{Var}_{\mathcal{D}_h}(\boldsymbol{q}_h; [0,t]) = \mathrm{Var}_{\mathcal{D}_h}(\boldsymbol{q}_h; [0, \tau_h(t)]).$$

Thus, (5.99)–(5.100) yield

$$\begin{aligned} \mathcal{F}_h(t, \boldsymbol{q}_h(t)) + \mathrm{Var}_{\mathcal{D}_h}(\boldsymbol{q}_h; [0,t]) &\le \mathcal{F}_h(\tau_h(t), \boldsymbol{q}_h(\tau_h(t))) + \mathrm{Var}_{\mathcal{D}_h}(\boldsymbol{q}_h; [0, \tau_h(t)]) \\ &\quad + (M + L(M)) \left(\mathrm{e}^{K_h(t) - K_h(\tau_h(t))} - 1\right) \\ &\le \tau_h(t)\alpha_h + \mathcal{F}_h(0, \boldsymbol{q}_h^0) + \int_0^{\tau_h(t)} P_h(\tau)\,\mathrm{d}\tau \\ &\quad + (M + L(M)) \left(\mathrm{e}^{K_h(t) - K_h(\tau_h(t))} - 1\right). \end{aligned} \tag{5.101}$$

Now, let (h_k) be the subsequence such that (5.49)–(5.53) hold. For the first term while, by lower semicontinuity, (5.46) yields

$$\begin{aligned} \mathrm{Var}_{\mathcal{D}_0}(\boldsymbol{q}_0; [0,t]) &= \mathrm{Var}_{L^1(\Omega;\mathbb{R}^3)}(\boldsymbol{z}; [0,t]) \\ &\le \liminf_{k\to\infty} \mathrm{Var}_{L^1(\Omega;\mathbb{R}^3)}(\boldsymbol{z}_{h_k}; [0,t]) \\ &= \liminf_{k\to\infty} \mathrm{Var}_{\mathcal{D}_{h_k}}(\boldsymbol{q}_{h_k}; [0,t]) = \Delta(t). \end{aligned} \tag{5.102}$$

Thus, taking the inferior limit along the subsequnece (h_k) , as $k \to \infty$, at both sides of (5.101), we obtain

$$\begin{aligned} \mathcal{F}_0(t, \boldsymbol{q}_0(t)) + \mathrm{Var}_{\mathcal{D}_0}(\boldsymbol{q}_0; [0,t]) &\le f(t) + \Delta(t) \\ &\le \mathcal{F}_0(0, \boldsymbol{q}_0^0) + \int_0^t P(\tau)\,\mathrm{d}\tau \\ &\le \mathcal{F}_0(0, \boldsymbol{q}_0^0) + \int_0^t P_0(\tau)\,\mathrm{d}\tau. \end{aligned} \tag{5.103}$$

This proves (5.98).

Step 5 (Lower energy-dissipation inequality). As in Step 5 of the proof of Theorem 5.4, the lower energy-dissipation inequality is deduced from the reduced stability (5.8) by applying [109, Proposition 5.7].

Step 6 (Improved estimates). To check (5.69)–(5.70), we identify the functions f and Δ in (5.45)–(5.46) by arguing as in Step 6 of the proof of Theorem 5.4. Eventually, to establish (5.71), we show that $P = P_0$ almost everywhere in $(0,T)$ again by arguing as in Step 6 of the proof of Theorem 5.4. □

We mention that it si possible to adapt Theorem 5.11 to treat time-dependent integral boundary conditions as described in Remark 5.9 in place of time-independent boundary conditions in the sense of traces.
We conclude the chapter by briefly describing possible variants of Theorem 5.4 and Theorem 5.11 involving an alternative notion of dissipation, similar to the one examined in Section 3.4, which is frame-indifferent.

Remark 5.13 (Alternative dissipation). Similarly to (3.52), for every $h>0$, we define

$$\widetilde{\mathcal{Z}}_h(\boldsymbol{q}) \coloneqq (\mathrm{adj} D_h \boldsymbol{y})\boldsymbol{m}\circ\boldsymbol{y}\det D_h\boldsymbol{y}, \tag{5.104}$$

where $\boldsymbol{q}=(\boldsymbol{y},\boldsymbol{m})\in\mathcal{Q}_h$. As in Section 3.4, we see that the quantity in (5.104) is frame-indifferent. Thus, the dissipation distance $\widetilde{\mathcal{D}}_h\colon \mathcal{Q}_h\times\mathcal{Q}_h\to[0,+\infty)$ defined by setting

$$\widetilde{\mathcal{D}}_h(\boldsymbol{q},\widehat{\boldsymbol{q}}) \coloneqq \int_\Omega |\widetilde{\mathcal{Z}}_h(\boldsymbol{q})-\widetilde{\mathcal{Z}}_h(\widehat{\boldsymbol{q}})|\,\mathrm{d}\boldsymbol{x}, \tag{5.105}$$

excludes the dissipation of energy by means of composition with rigid motions and hence appears to be physically more realistic.
In view of the observation made in Section 3.4, the existence of energetic solutions for the bulk quasistatic model with the dissipation distance defined in (5.105) cannot be ensured without resorting to any regularization. Nevertheless, one might aim to prove the analogous of Theorem 5.4 and Theorem 5.11 in this alternative setting. In this case, for the reduced model, one expects to recover the dissipation distance $\mathcal{D}_0$. On the one hand, the notion of dissipation in (5.105) is compatible with out framework. Indeed, $\mathcal{D}_0$ constitutes the Γ-limit of $(\widetilde{\mathcal{D}}_h)$, as $h\to0^+$, with respect to the convergence in the sense of dimension reduction [112]. On the other hand, the fact that $\widetilde{\mathcal{D}}_h$ takes values in a possibly unbounded interval represents a substantial obstacle in proving the compactness of sequences of energetic solutions for the bulk quasistatic model and of sequences of solutions of the AIMP. Therefore, it is not possible to establish convergence results like Theorem 5.4 and Theorem 5.11 having $\widetilde{\mathcal{D}}_h$ in place of $\mathcal{D}_h$.
There are two possible way to overcome this issue. The first one consists in imposing a locking constraint [22] that forces the quantity in (5.104) to satisfy a given bound in $L^\infty(\Omega;\mathbb{R}^N)$. The second one is to replace the dissipation distance in (5.105) with $\widehat{\mathcal{D}}_h\colon\mathcal{Q}_h\times\mathcal{Q}_h\to[0,+\infty)$ defined by

$$\widehat{\mathcal{D}}_h(\boldsymbol{q},\widehat{\boldsymbol{q}}) \coloneqq \int_\Omega \left|\frac{\widetilde{\mathcal{Z}}_h(\boldsymbol{q})}{|\widetilde{\mathcal{Z}}_h(\boldsymbol{q})|}-\frac{\widetilde{\mathcal{Z}}_h(\widehat{\boldsymbol{q}})}{|\widetilde{\mathcal{Z}}_h(\widehat{\boldsymbol{q}})|}\right|\mathrm{d}\boldsymbol{x}.$$

Observe that $\widehat{\mathcal{D}}_h$ is also frame-indifferent and that $\mathcal{D}_0$ is also the Γ-limit, as $h \to 0^+$, of $(\widehat{\mathcal{D}}_h)$ with respect to the convergence in the sense of dimension reduction. As $\widehat{\mathcal{D}}_h$ takes clearly values in a bounded interval, it is possible to adapt the proofs of Theorem 5.4 and Theorem 5.11 in order to prove analogous convergence results with $\widehat{\mathcal{D}}_h$ in place of $\mathcal{D}_h$.

Appendix

Sobolev maps on the boundary of smooth domains

In this appendix, we recall the definition of Sobolev maps on the boundary of smooth domains given in terms of an integration-by-parts formula involving the mean curvature of the boundary. Although this definition is quite natural in view of the Divergence Theorem on surfaces, the definition given in term of charts is more popular becauses it requires only the Lipschitz reguliarity of the boundary. The main advantage of the definition given here is that it makes the notion of the tangential gradient more explicit.

Geometric preliminaries

Henceforth, $U \subset\subset \mathbb{R}^N$ denotes a domain of class at least C^2 and we adopt the notation introduced in Section 1.4. For every $\boldsymbol{x}_0 \in \partial U$, there exist a local chart at $\boldsymbol{x}_0$, namely an injective immersion $\boldsymbol{\eta} \in C^2(\overline{A}; \mathbb{R}^N)$, where $A \subset \mathbb{R}^{N-1}$ is a bounded domain, such that $\boldsymbol{\eta}(A)$ is an relatively open set of ∂U containing $\boldsymbol{x}_0$. In this case, the tangent space $T_{\boldsymbol{x}_0}\partial U$ is spanned by the unit vectors $(\boldsymbol{t}_U^{(1)}(\boldsymbol{x}_0), \ldots, \boldsymbol{t}_U^{(N-1)}(\boldsymbol{x}_0))$, where, for every $i \in \{1, \ldots, N-1\}$, we set

$$\boldsymbol{t}_U^{(i)}(\boldsymbol{x}_0) := \frac{D\boldsymbol{\eta}(\boldsymbol{\eta}^{-1}(\boldsymbol{x}_0))\boldsymbol{e}'_i}{|D\boldsymbol{\eta}(\boldsymbol{\eta}^{-1}(\boldsymbol{x}_0))\boldsymbol{e}'_i|}.$$

Here, $(\boldsymbol{e}'_1, \ldots, \boldsymbol{e}'_{N-1})$ is the canonical basis of $\mathbb{R}^{N-1}$. We denote the projection operator onto the tangent space by $\boldsymbol{\Pi}_{T_{\boldsymbol{x}_0}\partial U} \colon \mathbb{R}^N \to T_{\boldsymbol{x}_0}\partial U$. Given the map from $\mathbb{R}^N$ to $\mathbb{R}^{N-1}$ defined by

$$\boldsymbol{x} \mapsto \sum_{i=1}^{N-1} \left(\boldsymbol{x} \cdot \boldsymbol{t}_U^{(i)}(\boldsymbol{x}_0) \right) \boldsymbol{e}'_i,$$

we denote by $\boldsymbol{\Lambda}_U(\boldsymbol{x}_0) \in \mathbb{R}^{(N-1)\times N}$ the matrix representing it with respect to the canonical basis of $\mathbb{R}^N$ and $\mathbb{R}^{N-1}$. In this case, $\boldsymbol{\Lambda}_U \in C^1(\partial U; \mathbb{R}^{(N-1)\times N})$. Moreover, there holds $\boldsymbol{\Lambda}_U(\boldsymbol{\eta}(\boldsymbol{x}_0))D\boldsymbol{\eta}(\boldsymbol{\eta}^{-1}(\boldsymbol{x}_0)) = \boldsymbol{I}''$, where $\boldsymbol{I}'' \in \mathbb{R}^{(N-1)\times(N-1)}$ is the identity matrix.

Tangential differentiability

We now recall the classical definition of tangential differentiability. We give the definition for vector-valued maps which are the ones of interest for our purposes.

Definition A.1 (Tangential differentiability). A map $\boldsymbol{y}\colon \partial U \to \mathbb{R}^N$ is termed to be of class C^1 if there exists a map $\boldsymbol{Y} \in C^1(\mathbb{R}^N;\mathbb{R}^N)$ such that $\boldsymbol{Y}|_{\partial U} = \boldsymbol{y}$. In this case, we write $\boldsymbol{y} \in C^1(\partial U;\mathbb{R}^N)$ and we define the tangential gradient of $\boldsymbol{y}$ at $\boldsymbol{x}_0 \in \partial U$ as $D^{\partial U}\boldsymbol{y}(\boldsymbol{x}_0) \coloneqq D\boldsymbol{y}(\boldsymbol{x}_0)\,(\boldsymbol{I} - \boldsymbol{n}_U(\boldsymbol{x}_0) \otimes \boldsymbol{n}_U(\boldsymbol{x}_0))$.

Note that $D^{\partial U}\boldsymbol{y}(\boldsymbol{x}_0) \in \mathbb{R}^{N\times N}$. Basically, the rows of this matrix are given by $\boldsymbol{\Pi}_{T_{\boldsymbol{x}_0}\partial U}(DY^i(\boldsymbol{x}_0))$ for $i \in \{1,\ldots,N\}$. It is easily shown that the previous definiton is well posed in the sense that does not depend on the choice of the extension $\boldsymbol{Y}$. Clearly, for $\boldsymbol{y} \in C^1(\mathbb{R}^N;\mathbb{R}^N)$, we have

$$D^{\partial U}\boldsymbol{y}|_{\partial U}(\boldsymbol{x}_0) = D\boldsymbol{y}(\boldsymbol{x}_0)\,(\boldsymbol{I} - \boldsymbol{n}_U(\boldsymbol{x}_0) \otimes \boldsymbol{n}_U(\boldsymbol{x}_0))\,. \tag{A.1}$$

Definition A.1 is equivalent to the one generally used in differential geometry: a map $\boldsymbol{y}\colon \partial U \to \mathbb{R}^N$ belongs to $C^1(\partial U;\mathbb{R}^N)$ if and only if, for every $\boldsymbol{x}_0 \in \partial U$, there exists a local chart $\boldsymbol{\eta} \in C^2(\overline{A};\mathbb{R}^N)$ at $\boldsymbol{x}_0$ such that $\boldsymbol{v} \coloneqq \boldsymbol{y} \circ \boldsymbol{\eta} \in C^1(A;\mathbb{R}^N)$. In this case, we have the identity

$$D^{\partial U}\boldsymbol{y}(\boldsymbol{x}_0) = D\boldsymbol{v}(\boldsymbol{\eta}^{-1}(\boldsymbol{x}_0))\boldsymbol{\Lambda}_U(\boldsymbol{x}_0). \tag{A.2}$$

Given $\boldsymbol{y} \in C^1(\partial U;\mathbb{R}^N)$, its tangential divergence at $\boldsymbol{x}_0 \in \partial U$ is simply defined as

$$\mathrm{div}^{\partial U}\boldsymbol{y}(\boldsymbol{x}_0) \coloneqq \mathrm{tr}D^{\partial U}\boldsymbol{y}(\boldsymbol{x}_0).$$

Divergence Theorem and Integration-by-parts Formula

We know recall the Divergence Theorem on surfaces. Here, we limit ourselves to the case which is of interest for us, namely the case of boundaries of smooth domains. However, these results hold true for more general smooth hypersurfaces possibly with boundary.
Before proceeding, we need to introduce the mean curvature. Observe that, given the regularity of U, we have $\boldsymbol{n}_U \in C^1(\partial U;\mathbb{R}^N)$. The scalar mean curvature of ∂U is defined as

$$H_U \coloneqq \mathrm{div}^{\partial U}\boldsymbol{n}_U.$$

The reader must be warned that different conventions of sign in the previous definition are also considered in the literature.
We now state the Divergence Theorem.

Theorem A.2 (Divergence Theorem on boundaries). *Let $U \subset\subset \mathbb{R}^N$ be a domain of class C^2 and let $\boldsymbol{y} \in C^1(\partial U;\mathbb{R}^N)$. Then, there holds:*

$$\int_{\partial U} \mathrm{div}^{\partial U}\boldsymbol{y}\,\mathrm{d}\boldsymbol{a} = \int_{\partial U} (\boldsymbol{y}\cdot\boldsymbol{n}_U)\,H_U\,\mathrm{d}\boldsymbol{a}.$$

Proof. We refet to [3, Chapter 3], [69, Theorem 2, Chapter 2, Section 1.5] or [104, Theorem 11.8 and Remark 11.12]. □

From Theorem A.2, we easily deduce the following identities.

Corollary A.3 (Integration-by-parts Formula on boundaries). *Let $U \subset\subset \mathbb{R}^N$ be a domain of class C^2 and let $\boldsymbol{y} \in C^1(\partial U; \mathbb{R}^N)$. Then, for every $\boldsymbol{\varphi} \in C^1(\partial U; \mathbb{R}^N)$, the following formula holds:*

$$\int_{\partial U} (D^{\partial U}\boldsymbol{y})\boldsymbol{\varphi}\, \mathrm{d}\boldsymbol{a} = \int_{\partial U} \boldsymbol{y}(\boldsymbol{\varphi}\cdot \boldsymbol{n}_U)\, H_U \, \mathrm{d}\boldsymbol{a} - \int_{\partial U} \boldsymbol{y}\, \mathrm{div}^{\partial U}\boldsymbol{\varphi}\, \mathrm{d}\boldsymbol{a}.$$

Equivalently, for every $\boldsymbol{\Phi} \in C^1(\partial U; \mathbb{R}^{N\times N})$, there holds

$$\int_{\partial U} D^{\partial U}\boldsymbol{y} : \boldsymbol{\Phi}\, \mathrm{d}\boldsymbol{a} = \int_{\partial U} (\boldsymbol{\Phi}\boldsymbol{n}_U)\cdot \boldsymbol{y}\, H_U \, \mathrm{d}\boldsymbol{a} - \int_{\partial U} \boldsymbol{y}\cdot \mathrm{div}^{\partial U}\boldsymbol{\Phi}\, \mathrm{d}\boldsymbol{a}.$$

In the last equation, the tangential divergence operator is suppose to act row-wise on matrix-valued maps. Namely, the rows of the vector field $\mathrm{div}^{\partial U}\boldsymbol{\Phi}$ are given by $\mathrm{div}^{\partial U}\boldsymbol{\Phi}^i$ for $i \in \{1,\dots,N\}$.

Weak tangential differentiability

The notion of weak tangetial differentiability is conceptually analogous to the one of weak differentiability. In this case, one the classical Integration-by-parts Formula is replaced by the one provided by Corollary A.3. Again, we focus on the case of vector-valued maps.

Definition A.4 (Weak tangential differentiability). A map $\boldsymbol{y} \in L^1(\partial U; \mathbb{R}^N)$ is termed weakly tangentially differentiable if there exists $\boldsymbol{F} \in L^1(\partial U; \mathbb{R}^{N\times N})$ such that, for every $\boldsymbol{\varphi} \in C^1(\partial U; \mathbb{R}^N)$, there holds

$$-\int_{\partial U} \boldsymbol{y}\, \mathrm{div}^{\partial U}\boldsymbol{\varphi}\, \mathrm{d}\boldsymbol{a} = \int_{\partial U} \boldsymbol{F}\boldsymbol{\varphi}\, \mathrm{d}\boldsymbol{a} - \int_{\partial U} \boldsymbol{y}(\boldsymbol{\varphi}\cdot \boldsymbol{n}_U)\, H_U \, \mathrm{d}\boldsymbol{a},$$

or, equivalently, for every $\boldsymbol{\Phi} \in C^1(\partial U; \mathbb{R}^{N\times N})$, there holds

$$-\int_{\partial U} \boldsymbol{y}\cdot \mathrm{div}^{\partial U}\boldsymbol{\Phi}\, \mathrm{d}\boldsymbol{a} = \int_{\partial U} \boldsymbol{F} : \boldsymbol{\Phi}\, \mathrm{d}\boldsymbol{a} - \int_{\partial U} (\boldsymbol{\Phi}\boldsymbol{n}_U)\cdot \boldsymbol{y}\, H_U \, \mathrm{d}\boldsymbol{a}.$$

In this case, $D^{\partial U}\boldsymbol{y} \coloneqq \boldsymbol{F}$ is termed the weak tangential gradient of $\boldsymbol{y}$.

The weak tangential gradient is unique up to sets of measure zero and, in view of Corollay A.3, coincides with the tangential gradient in the case of tangentially differentiable maps.
At this point, for every $1 \leq p < \infty$, we define the Sobolev space $W^{1,p}(\partial U; \mathbb{R}^N)$ as the class of maps $\boldsymbol{y} \in L^p(\partial U; \mathbb{R}^N)$ admitting weak tangential gradient $D^{\partial U}\boldsymbol{y}$ which belongs to $L^p(\partial U; \mathbb{R}^{N\times N})$. Up to considering equivalence classes of maps that coincide almost everywhere, this is Banach space equipped with the norm

$$\boldsymbol{y} \mapsto \left\{ \|\boldsymbol{y}\|^p_{L^p(\partial U;\mathbb{R}^N)} + \|D^{\partial U}\boldsymbol{y}\|^p_{L^p(\partial U;\mathbb{R}^{N\times N})} \right\}^{1/p}. \tag{A.3}$$

By exploiting the natural embedding of $W^{1,p}(\partial U; \mathbb{R}^N)$ in $L^p(\partial U; \mathbb{R}^N) \times L^p(\partial U; \mathbb{R}^{N\times N})$ given by $\boldsymbol{y} \mapsto (\boldsymbol{y}, D^{\partial U}\boldsymbol{y})$, we see that elements of the dual of $W^{1,p}(\partial U; \mathbb{R}^N)$ admit a representation analogous to the one valid in the case of domains. Given $(\boldsymbol{y}_n) \subset W^{1,p}(\partial U; \mathbb{R}^N)$ and $\boldsymbol{y} \in W^{1,p}(\partial U; \mathbb{R}^N)$, we have that $\boldsymbol{y}_n \rightharpoonup \boldsymbol{y}$ in $W^{1,p}(\partial U; \mathbb{R}^N)$ if and only if $\boldsymbol{y}_n \rightharpoonup \boldsymbol{y}$ in $L^p(\partial U; \mathbb{R}^N)$ and $D^{\partial U}\boldsymbol{y}_n \rightharpoonup D^{\partial U}\boldsymbol{y}$ in $L^p(\partial U; \mathbb{R}^{N\times N})$.

The following result shows that, in the case of domains of class C^2, the definition of Sobolev maps on the boundary coincides with the usual definition of Sobolev maps on the boundary given by means of local charts [123]. This can be proved using standard techniques involving partitions of unity and the density of smooth functions in Sobolev spaces by exploiting (A.1)–(A.2).

Proposition A.5 (Sobolev maps on the boundary). *Let $U \subset\subset \mathbb{R}^N$ be a domain of class C^2 and let $\boldsymbol{y} \in L^1(\partial U;\mathbb{R}^N)$. Then, $\boldsymbol{y} \in W^{1,p}(\partial U;\mathbb{R}^N)$ if and only if for every $\boldsymbol{x}_0$ and for every local chart $\boldsymbol{\eta} \in C^2(\overline{A};\mathbb{R}^N)$ at $\boldsymbol{x}_0$, there holds $\boldsymbol{v} := \boldsymbol{y} \circ \boldsymbol{\eta} \in W^{1,p}(A;\mathbb{R}^N)$.*

This proposition allows us to extend various results about Sobolev spaces on domain to Sobolev spaces on boundaries of smooth domains. Among these, we have the density of $C^1(\partial U;\mathbb{R}^N)$ in $W^{1,p}(\partial U;\mathbb{R}^N)$, the embeddings of Sobolev and Morrey, and the Poincaré inequality. In particular, the density of smooth maps shows that the approach presented here is equivalent to the one usually adopted in the theory of Sobolev spaces on Riemannian manifolds [77] which are defined as the closure of smooth functions with respect to the norm in (A.3). Finally, we stress that the validity of the Morrey embedding in $W^{1,p}(\partial U;\mathbb{R}^N)$ is essential for the analysis carried out in the present work.

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