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Probabilistic solutions of the supercooled Stefan problem

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Abstract

Stefan problems describe the evolution of the interface between two phases of a substance undergoing a phase transition. They were first studied in 1831 by Lamé and Clapeyron, who studied the formation of the earth's crust, and later by the eponymous physicist Josef Stefan. In the case of solidification, i.e., considering the liquid and solid phase of a material and the liquid is initially cooled below its freezing point, one refers to the corresponding problem as a *supercooled Stefan problem*.

Traditionally, Stefan problems are formulated as systems of partial differential equations with free boundaries. Here, we consider so-called *McKean–Vlasov* equations, a class of stochastic differential equations in which the dynamics depend on the law of the solution.

The McKean–Vlasov problems considered here arise as mean-field limits of particle systems with singular interactions that have natural interpretations in the field of systemic risk in mathematical finance and the modeling of neurobiological phenomena. A common feature of these applications is the occurrence of discontinuous threshold effects: from the economic point of view, this corresponds to the triggering of the default of a bank once its equity value goes below a certain threshold, from the neurobiological point of view this corresponds to the firing of a neuron once its membrane potential exceeds the threshold and triggers the action potential.

A recent idea, put forth by François Delarue, Sergey Nadtochiy and Mykhaylo Shkolnikov, is to define solutions to the supercooled Stefan problem through the associated McKean–Vlasov equation, which is treated as a probabilistic reformulation that allows to define solutions *globally in time* even when finite-time blow-up occurs in the corresponding system of partial differential equations.

In this work, we expand on this idea and study the solution concepts of *minimal* and *physical* solutions, proving that minimal solutions are physical. Next, we prove the convergence of an efficient and robust numerical scheme that allows us to reliably approximate the minimal solution even in the presence of blow-ups. Finally, we illustrate how the preceding insights might be applied to the problem faced by a central bank that aims to prevent the default of a given proportion of banks with minimal costs.

Zusammenfassung

Stefan-Probleme beschreiben die zeitliche Entwicklung des Grenzbereichs zwischen zwei Phasen einer Substanz während eines Phasenüberganges. Das Studium solcher Probleme geht auf Lamé und Clapeyron zurück, die im Jahr 1831 die Bildung der Erdkruste zu beschreiben versuchten, sowie auf den namensgebenden Physiker Josef Stefan. Im Fall eines Gefrierprozesses, bei dem die flüssige und feste Phase eines Materials betrachtet wird und die flüssige Phase anfangs unter den Gefrierpunkt abgekühlt ist, wird das entsprechende Problem *supercooled Stefan problem* genannt.

Traditionell wird ein Stefan-Problem als System von partiellen Differentialgleichungen mit freiem Rand formuliert. In der vorliegenden Arbeit betrachten wir sogenannte *McKean–Vlasov–Gleichungen*, eine Klasse von stochastischen Differentialgleichungen die eine Abhängigkeit der Dynamik von der Verteilung der Lösung selbst aufweisen.

Die hier betrachteten McKean–Vlasov-Gleichungen treten als Limiten von Partikelsystemen mit singulären Interaktionen auf, die sowohl im Gebiet der systemischen Risikobewertung der Finanzmathematik als auch in der Modellierung neurobiologischer Phänomene natürliche Interpretationen zulassen. Eine Gemeinsamkeit dieser Anwend- ungsgebiete liegt im Vorkommen unstetiger Schwelleneffekte: aus dem wirtschaftlichen Blickwinkel treten solche Effekte bei Banken auf, deren Kapitalisierungen eine Schwelle unterschreiten und die dadurch zahlungsunfähig werden, aus neurobiologischer Sicht beim Feuern eines Neurons, dessen Membranpotential das Schwellenpotential übersteigt und dadurch ein Aktionspotential auslöst.

Eine neue Idee, zurückgehend auf François Delarue, Sergey Nadtochiy und Mykhaylo Shkolnikov, ist es Lösungen des supercooled Stefan problem durch die zugehörige McKean–Vlasov–Gleichung zu definieren, die eine probabilistische Umformulierung des Problems darstellt. Dies lässt es zu, Lösungen *global in der Zeit* zu definieren, obwohl das zugehörige System von partiellen Differential- gleichungen in endlicher Zeit explodiert.

In der vorliegenden Arbeit bauen wir auf diese Idee auf und untersuchen die Lösungskonzepte der *minimalen* und *physikalischen* Lösungen, und beweisen dass minimale Lösungen physikalisch sind. Danach beweisen wir die Konvergenz eines effizienten und robusten numerischen Schemas, das es ermöglicht die minimale Lösung im Fall der Explosion der partiellen Differentialgleichung zu approximieren. Abschließend illustrieren wir, wie die vorangehenden Erkenntnisse auf ein Kontrollproblem angewendet werden können, bei welchem die Insolvenz einer bestimmten Proportion von Banken bei minimalen Kosten verhindert werden soll.

Contents

Introduction and summary of results

An analogy

Consider the (probabilist's) heat equation on the real line, i.e.,

$$\partial_t u = \frac{1}{2}\partial_{xx} u, \qquad x \in \mathbb{R}, \quad t > 0, \tag{1a}$$

$$u(0,x) = f(x), \qquad x \in \mathbb{R}, \tag{1b}$$

where u is the temperature and f is a given function. Problem (1) is also referred as *Cauchy problem* or *initial value problem* and describes the evolution of the temperature u of a substance whose initial temperature profile is given by f. Here, we set the heat capacity equal to 2 and thermal conductivity equal to 1. Even though in the case of a bounded domain, problem (1) is well-posed and possesses remarkable smoothing properties, the situation is markedly different on an unbounded domain: indeed, it is well-known that Problem (1) is ill-posed in the sense that it allows for multiple distinct solutions even if f is arbitrarily smooth. To wit, consider the so-called *Tychonoff example*: define

$$g(t) = \begin{cases} e^{-1/t^2} & t > 0, \\ 0 & t \leq 0, \end{cases} \tag{2}$$

and set

$$w(t,x) = \sum_{n=0}^{\infty} \frac{2^n g^{(n)}(t)}{(2n)!} x^{2n}, \quad t \geq 0, \quad x \in \mathbb{R}. \tag{3}$$

Then, w is a classical solution of (1) with $f \equiv 0$ (see [7, Chapter 7] for details), and so is the trivial solution $u \equiv 0$. In this case, $u \equiv 0$ is physically meaningful, corresponding to perfect thermal equilibrium, while the Tychonoff example violates Fourier's law of heat conductance, since it does not preserve the initial thermal equilibrium. For an interpretation of this phenomenon, we quote [9, p. 342f] verbatim:

> "The new solutions that we have found consist of a great blast of heat from infinity, and a little thought suggests that the possibility of such solutions springs from the fact that the equation used gives no limiting velocity for the propagation of heat. Thus, provided only it is large enough, a very distant disturbance can produce a rapid change in temperature near the origin. We may suspect that the kind of solution described in [the Tychonoff example] cannot occur if we have an equation like the wave equation where disturbances propagate with a finite velocity, and that even with the heat equation we can exclude such solutions by imposing a condition of slow growth at infinity. To the applied mathematician [the Tychonoff example] is simply an embarrassment reminding her of the defects of a model which allows an unbounded speed of propagation. To the numerical analyst it is just a mild warning that the heat equation may present problems which the wave equation does not. But the pure mathematician looks at it with the same simple pleasure with which a child looks at a rose which has just been produced from the mouth of a respectable uncle by a passing magician."

It seems desirable to introduce an additional selection principle that allows us to choose the physically meaningful solution from the set of all possible solutions. This can be done for example by restricting the growth of the solution as $|x| \to \infty$ (see [5, Section 2.3, Theorem 7]), or in the case where $f \geq 0$ by requiring that $u \geq 0$ (see [7, Chapter 7, Section (d)]). Specializing to the case where f is a density, i.e. $f \geq 0$ and $\int_0^\infty f(x)\mathrm{d}x = 1$, we may consider the process $X_t := X_0 + B_t$, where B is a standard Brownian motion independent of X_0, and f is the density of X_0. Then, if we let $u(t, \cdot)$ be the continuous representative of the density of X_t, we obtain that u is the unique bounded solution of Problem (1). This principle of constructing a physically meaningful solution to an ill-posed partial differential equation through a probabilistic reformulation is one of the key ideas that we wish to explore in this thesis.

The supercooled Stefan problem

Stefan problems are models for the evolution of the boundary between two phases of a material in the process of a phase transition. They were first studied by Lamé and Clapeyron [10], who sought to describe the formation of the earth's crust, and later by the eponymous physicist Josef Stefan (Jožef Štefan) [14], who was interested in describing the growth of ice. The one-dimensional, one-phase supercooled Stefan problem is an archetypal free boundary problem and is a simple model for the freezing of a supercooled liquid on the semi-infinite strip $[0, \infty)$. As a system of partial differential equations, it can be described as

$$\partial_t u = \frac{1}{2}\partial_{xx} u, \qquad \Lambda_t < x < \infty, \quad t > 0, \tag{4a}$$

$$u(t, \Lambda_t) = 0, \qquad t > 0, \tag{4b}$$

$$\frac{\alpha}{2}\partial_x u(t, \Lambda_t) = \dot{\Lambda}_t, \qquad t > 0, \tag{4c}$$

$$u(0, x) = f(x), \qquad x \geq 0, \tag{4d}$$

where $\alpha > 0$ is a given constant, f is a given probability density function and u, Λ are to be determined. Here, u is the number of degrees below the equilibrium freezing point and Λ is the location of the interface between liquid and solid phase, also called the *freezing front*. The solid phase occupies $[0, \Lambda_t]$, while the liquid phase occupies (Λ_t, ∞). The freezing point of the liquid is $u = 0$. Condition (4a) says that the heat transport in the fluid is governed by the heat equation, while condition (4b) amounts to assuming that the phase change is isothermal and the temperature of the solid phase is uniform and equal to zero. Equation (4c) is called a *Stefan condition*, and it amounts to the assumption that the discontinuity in heat flux across the phase interface is equal to the release of latent heat during freezing. Finally, (4d) says that the liquid is initially cooled below its freezing point everywhere, i.e., it is *supercooled*.

Since the Stefan problem (without supercooling) has been studied since the year 1831, the literature on the problem is so vast that we do not attempt to give an exhaustive account of it here. Instead, we refer the interested reader to [12, Part Two, Chapter I] for some classical well-posedness results and to [15, Section IV.9] for a historical overview. In contrast, the problem with supercooling is not as well understood, since it is notoriously ill-posed and may exhibit finite-time blow-up of the freezing rate $\dot{\Lambda}$, as first demonstrated by Sherman [13] in the 70s. This was viewed as an inadequacy of the model and led to efforts to refine the physics behind either the isothermal phase change condition (4b) or the Stefan condition (4c) and thus obtain a well-posed formulation of the problem (see [4]). A recent idea, pioneered by François Delarue, Sergey Nadtochiy and Mykhaylo Shkolnikov (see [3]), is to define solutions globally in time, even in the presence of blow-ups, through an associated probabilistic reformulation. We explore this idea in the next section.

Probabilistic reformulation

Let X_{0-} be a random variable with density f and B be an independent standard Brownian motion. Then, the probabilistic reformulation of (4) reads

$$\begin{cases} X_t = X_{0-} + B_t - \Lambda_t \\ \tau = \inf\{t \geq 0 : X_t \leq 0\} \\ \Lambda_t = \alpha \mathbb{P}\left(\tau \leq t\right). \end{cases} \tag{5}$$

A solution to this equation is a triple (X, τ, Λ), where $\Lambda : [0, \infty) \to [0, \alpha]$ is a deterministic, increasing càdlàg function such that (5) holds for any Brownian motion B, and X and τ are the resulting process and stopping time. Problem (5) is a *McKean–Vlasov equation*, since the coefficient Λ in the dynamics of X depends on the law of X itself. The connection between the supercooled Stefan problem (4) and the McKean–Vlasov problem (5) was first noticed by Ben Hambly, Sean Ledger and Andreas Sojmark in [6] and later François Delarue, Sergey Nadtochiy and Mykhaylo Shkolnikov proposed to treat (5) as a reformulation of (4) in [3]. For a heuristic derivation of the connection between (4) and (5) we refer to Section 1.2 of Paper 1.

An obvious advantage of this idea is that for (5) to make sense we require only very little regularity, since the problem still makes sense even if Λ exhibits jump discontinuities. However, a difficulty encountered here is that solutions to (5) are not unique in general (see Example 2.4 in Paper 1). We therefore have to face the problem of how to select the (hopefully unique) physically meaningful solution from the set of all solutions. This motivates the concept of a so-called *physical solution*, introduced in [1], and defined as a solution Λ to (5) satisfying

$$\Delta\Lambda_t := \Lambda_t - \Lambda_{t-} = \alpha \inf\{x > 0 \colon \mathbb{P}\left(\tau \geq t, X_{t-} \in [0, \alpha x]\right) < x\}, \quad t \geq 0. \tag{6}$$

Physically, condition (6) amounts to choosing the minimal jump size among all solutions which respect an integrated version of the Stefan condition (4c), ensuring conservation of energy. Mathematically, condition (6) encodes that at any instant, the jump size of the physical solution is minimal among all solutions with the same past [6, Proposition 1.2]. At this point it will presumably not shock the reader to learn that solutions to (5) may have jumps; indeed, a remarkably simple argument shows that any solution to (5) must be discontinuous if only $2\mathbb{E}[X_{0-}] < \alpha$ holds [6, Theorem 1.1]. At such discontinuities, the freezing rate $\dot{\Lambda}$ blows up and classical solutions to the supercooled Stefan problem fail to exist, however condition (6) pins down the size of the jump uniquely[1] and hence allows us to define a (notion of) solution globally nevertheless.

An examination of the examples of nonuniqueness (cf. Example 2.4 in Paper 1) seems to suggest that nonuniqueness is somehow connected to the occurrence of jumps. This intuition is further strengthened by the fact that in the case where we know that all solutions are necessarily continuous, uniqueness also follows [11, Theorem 2.2]. Developing this train of thought further we might then conclude that by pinning down the jump sizes through (6), we should obtain a unique physical solution. This question, the uniqueness of physical solutions, first posed in 2014 in [1], remained open until 2019, where it was resolved in [3] in the case where X_{0-} admits a bounded, non-oscillatory density. We hope to have convinced the reader to some extent that it is natural to expect this result, even though the rigorous proof of this fact is quite difficult and technical.

Another solution concept is the so-called *minimal solution*, introduced in [3] as an auxiliary device[2]. We call $\underline{\Lambda}$ minimal if

$$\underline{\Lambda}_t \leq \Lambda_t, \quad t \geq 0, \tag{7}$$

for every solution Λ to (5). In Paper 1, we investigate this solution concept, and prove that the minimal solution is in fact physical whenever $\mathbb{E}[X_{0-}] < \infty$. We are interested in this question because the minimal solution is unique by definition, and therefore its well-posedness is not as difficult to establish as it is for the physical solution. In the case where we know the physical solution to be unique, we can therefore treat the minimal solution as an alternative characterization, and we will leverage property (7) in Papers 2-4 to prove the convergence of certain approximations to the minimal (and thus physical) solution. Before we expand on this, we first discuss the relevance of Problem (5) in the context of mathematical finance and other applications.

[1] Here we mean that given the past of the solution, i.e., $(\Lambda_s)_{0 \leq s < t}$, the jump size at time t is uniquely determined by (6).

[2] Note that the concept of minimal solution is only mentioned in the first preprint version of this paper and was removed from later versions.

The particle system and systemic risk

Let $N \in \mathbb{N}$, and consider the system of stochastic processes given by

$$\begin{cases} X_t^{i,N} = X_{0-}^i + B_t^i - \Lambda_t^N \\ \tau^{i,N} = \inf\{t \geq 0 : X_t^{i,N} \leq 0\} \\ \Lambda_t^N = \dfrac{\alpha}{N} \displaystyle\sum_{i=1}^N \mathbb{1}_{[\tau_{i,N} \leq t]}, \end{cases} \tag{8}$$

where $(B^i)_{i\geq 1}$ are independent standard Brownian motions independent of X_{0-}^N. This *particle system* can be viewed as a finite-dimensional version of the McKean-Vlasov problem (5). Indeed, in the case where the physical solution to (5) is unique, it can be shown that the physical solution arises as a limit point of (8) as $N \to \infty$ in a certain sense, i.e., *propagation of chaos* holds true [1, 3]. In Paper 1, we show that the notions of minimal and physical solution, defined in a similar way as for the McKean–Vlasov equation (see Section 3 of Paper 1 for details) are equivalent in the particle system and derive a similar propagation of chaos type result for the minimal solution, where we do not need to pose any requirements on the initial condition X_{0-} if we allow for a perturbation of the initial condition in the particle system.

The particle system (8) admits a financial interpretation: Suppose that we are interested in analyzing the default probabilities of a system of N banks, and we let the distance-to-default of the i-th bank be represented by the process $X^{i,N}$, such that bank i defaults when $X^{i,N}$ crosses the origin for the first time. Suppose further that the banks form a perfectly homogeneous lending network, meaning that each bank lends a quantity of α/N to each bank. In this setup, whenever one bank defaults, all the other banks experience an instant loss of α/N to their distance-to-default, potentially triggering further defaults, which leads to *default cascades*. Here, $X^{i,N}$ evolves as a Brownian motion and interacts with the other banks only through the hitting times of zero, corresponding to the times when at least one bank defaults. In this framework, up to a multiplicative constant, Λ_t^N is the proportion of banks that already have defaulted up to time t.

This induces a financial interpretation for the McKean–Vlasov problem: Λ_t corresponds to the losses a representative bank incurs through default contagion in a large, homogeneous network of banks where the exposure (the cumulative amount of all debts) amounts to α. Simultaneously, up to a multiplicative constant, Λ_t is the proportion of banks that have already defaulted up to time t. Jump discontinuities of Λ then can be interpreted as *systemic events*, which we might also think of as financial crises. It is then of interest to derive conditions guaranteeing that systemic events occur or do not occur, even though from the financial perspective, (8) is somewhat of a laboratory example, relying on some rather artificial assumptions. Nevertheless, we hope that the insights gathered from this laboratory example will prove useful for future research on this subject involving more realistic structural assumptions. Finally, let us also mention that similar models play a role in neurobiology [1, 2], where the contagious interaction corresponds to the triggering of action potentials in neurons, and jump discontinuities in the limit correspond to a phenomenon called *neural synchronization*.

Approximation schemes and numerics

At this point, we hope to have sufficiently motivated why we believe that the physical/minimal solution of (5) is interesting from a theoretical point of view. It might however be less clear that the previously mentioned ideas are also feasible in practice, since it is not obvious how to apply standard numerical techniques to solve (4) globally in the presence of blow-ups. In Paper 2, we discuss numerical schemes and (globally) prove their convergence to the minimal solution of (5). Besides an extension of the time-stepping scheme considered in [8], we propose a scheme based on a Donsker approximation of Brownian motion, given for $h > 0$ by

$$\begin{cases} X_t^h = X_{0-}^h + B_t^h - \Lambda_t^h \\ \tau^h = \inf\{t \geq 0 : X_t^h \leq 0\} \\ \Lambda_t^h = \alpha \mathbb{P}\left(\tau^h \leq t\right), \end{cases} \tag{9}$$

where

$$B_t^h = \sqrt{h} \sum_{k=0}^{\lfloor t/h \rfloor} Y_k, \tag{10}$$

and $(Y_i)_{i \in \mathbb{N}}$ is an iid sequence of random variables satisfying $\mathbb{E}[Y_1] = 0$ and $\mathbb{E}[Y_1^2] = 1$ with nondegenerate moment generating function. Assuming again that X_{0-} admits a bounded density, we then prove that $\underline{\Lambda}^h \to \underline{\Lambda}$ as $h \to 0$, where $\underline{\Lambda}^h$ is the minimal solution of (9) with a perturbed initial condition. Specializing to the case where the $(Y_i)_{i \in \mathbb{N}}$ are Rademacher variables, i.e., $\mathbb{P}(Y_i = 1) = \mathbb{P}(Y_i = -1) = 1/2$, we then derive an efficient numerical scheme that allows to compute $\underline{\Lambda}^h$ through a deterministic recursion. This scheme is robust in the sense that we empirically obtain a convergence rate of $\mathcal{O}(\sqrt{h})$ for any choice of X_{0-} and α. In particular, we also (empirically) obtain this convergence rate in the blow-up regime. In the weak feedback regime, where $\alpha \|f\|_\infty < 1$ (recall that f is the density of X_{0-}), we also prove a convergence rate of order $\mathcal{O}(\sqrt{h} \log(h^2))$.

Optimal bailout strategies

Coming back to the financial point of view, in Paper 3, we study the problem faced by a central bank that bails out distressed financial institutions in order to reduce systemic risk in the banking sector. We assume here that the system without intervention of the central agent is given by (8) and then pass to the mean-field limit to obtain a tractable problem formulation. Denoting the capital injection from the central bank to a representative agent at time t by β_t, we then obtain the McKean–Vlasov problem

$$\begin{cases} X_t = X_{0-} + \int_0^t \beta_s \, \mathrm{d}s + B_t - \Lambda_t \\ \tau = \inf\{t \geq 0 : X_t \leq 0\} \\ \Lambda_t = \alpha \mathbb{P}(\tau \leq t). \end{cases} \tag{11}$$

We assume the constraint $0 \leq \beta_t \leq b_{\max}$, which amounts to the assumption that the resources the central agent disposes of at any instant are limited. The central bank then aims to keep the proportion of defaults that occur before time T, given by

$$L_{T-}(\beta) := \mathbb{P}\left(\inf_{0\le s<T} X_s \le 0\right) = \alpha^{-1}\Lambda_{T-}(\beta),$$

below a certain threshold δ, while minimizing the expected total cost

$$C_T(\beta) = \mathbb{E}\left[\int_0^T \beta_t \; dt\right].$$

Under some technical assumptions (see the introduction of Paper 3), this problem can be tackled by considering the associated Lagrange function, i.e., for some $\gamma > 0$ we minimize the objective function

$$\begin{aligned} J(\beta) &= \mathbb{E}\left[\int_0^T \beta_t \, dt\right] + \gamma\,\mathbb{P}\Big(\inf_{0\le s<T} \underline{X}_s(\beta) \le 0\Big) \\ &= \mathbb{E}\left[\int_0^T \beta_t \, dt + \gamma\,\mathbb{1}_{\{\widehat{\underline{X}}_{T-}=0\}}\right], \end{aligned} \tag{12}$$

where $\underline{X}(\beta)$ is the minimal solution of (11) with drift β and $\widehat{\underline{X}} = \underline{X}_t \mathbb{1}_{\{\tau>t\}}$ is the absorbed minimal solution process. In Paper 3, we prove the existence of optimizers of (12), and prove that they are ϵ-optimal for the analogous optimization problem formulated at the level of a controlled particle system similar to (8).

Since (12) is in the form of a mean-field stochastic optimal control problem, we can (at least heuristically) apply the Hamilton-Jacobi-Bellman formalism and derive the structure of an optimal strategy. Based on this, we then propose a numerical scheme and test its convergence empirically. Finally, we study a linear-quadratic version of the model for which we are able to provide semi-analytic solutions which we again illustrate numerically.

Bibliographic information on papers contained in this thesis

Paper 1.

Propagation of minimality in the supercooled Stefan problem
Christa Cuchiero, Stefan Rigger, Sara Svaluto-Ferro.
To appear in Annals of Applied Probability.
Contribution of the author of this dissertation:
The paper was worked out in close collaboration.
Online versions available at: https://arxiv.org/abs/2010.03580

Paper 2.

Global implicit approximation of the supercooled Stefan problem in the presence of blow-ups.
Christa Cuchiero, Christoph Reisinger, Stefan Rigger.
Not published elsewhere yet.
Contribution of the author of this dissertation:
The paper was worked out in close collaboration.
Online versions available at:

Paper 3.

Optimal bailouts resulting from the drift-controlled supercooled Stefan problem.
Christa Cuchiero, Christoph Reisinger, Stefan Rigger.
Not published elsewhere yet.
Contribution of the author of this dissertation:
The paper was worked out in close collaboration.
Online versions available at: https://arxiv.org/abs/2111.01783

Bibliography for the introduction

[1] F. Delarue, J. Inglis, S. Rubenthaler, and E. Tanré. Particle systems with a singular mean-field self-excitation. Application to neuronal networks. *Stochastic Processes and their Applications*, 125(6):2451–2492, 2015.

[2] F. Delarue, J. Inglis, S. Rubenthaler, E. Tanré, et al. Global solvability of a networked integrate-and-fire model of McKean–Vlasov type. *The Annals of Applied Probability*, 25(4):2096–2133, 2015.

[3] F. Delarue, S. Nadtochiy, and M. Shkolnikov. Global solutions to the supercooled Stefan problem with blow-ups: regularity and uniqueness. 2019. To appear in Probab. Math. Phys., arXiv:1902.05174.

[4] J. N. Dewynne. A survey of supercooled Stefan problems. In *Mini-Conference on Free and Moving Boundary and Diffusion Problems*, pages 42–56, Canberra

AUS, 1992. Centre for Mathematics and its Applications, Mathematical Sciences Institute, The Australian National University.

[5] L. Evans. *Partial Differential Equations.* Graduate studies in mathematics. American Mathematical Society, 2010.

[6] B. Hambly, S. Ledger, and A. Søjmark. A McKean–Vlasov equation with positive feedback and blow-ups. *Ann. Appl. Probab.*, 29(4):2338–2373, 08 2019.

[7] F. John. *Partial Differential Equations.* Applied Mathematical Sciences. Springer New York, 1991.

[8] V. Kaushansky, C. Reisinger, M. Shkolnikov, and Z. Q. Song. Convergence of a time-stepping scheme to the free boundary in the supercooled Stefan problem. *arXiv preprint arXiv:2010.05281*, 2020.

[9] T. W. Körner. *Fourier analysis.* Cambridge university press, 1988.

[10] G. Lamé and B. P. E. Clapeyron. Mémoire sur la solidification par refroidissement d'un globe liquide. In *Annales Chimie Physique*, volume 47, pages 250–256, 1831.

[11] S. Ledger and A. Søjmark. Uniqueness for contagious McKean–Vlasov systems in the weak feedback regime. *Bulletin of the London Mathematical Society*, 52(3):448–463, 2020.

[12] L. Rubinšteĭn. *The Stefan Problem*, volume 8. American Mathematical Soc., 2000.

[13] B. Sherman. A general one-phase Stefan problem. *Quarterly of Applied Mathematics*, 28(3):377–382, 1970.

[14] J. Stefan. Über die Theorie der Eisbildung, insbesondere über die Eisbildung im Polarmeere. *Annalen der Physik und Chemie*, 42:269–286, 1891.

[15] A. Visintin. *Models of phase transitions*, volume 26. Springer Science & Business Media, 1996.

Propagation of minimality in the supercooled Stefan problem

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Abstract

Supercooled Stefan problems describe the evolution of the boundary between the solid and liquid phases of a substance, where the liquid is assumed to be cooled below its freezing point. Following the methodology of Delarue, Nadtochiy and Shkolnikov, we construct solutions to the one-phase one-dimensional supercooled Stefan problem through a certain McKean–Vlasov equation, which allows to define global solutions even in the presence of blow-ups. Solutions to the McKean–Vlasov equation arise as mean-field limits of particle systems interacting through hitting times, which is important for systemic risk modeling. Our main contributions are: (i) A general tightness theorem for the Skorokhod M_1-topology which applies to processes that can be decomposed into a continuous and a monotone part. (ii) A propagation of chaos result for a perturbed version of the particle system for general initial conditions. (iii) The proof of a conjecture of Delarue, Nadtochiy and Shkolnikov, relating the solution concepts of so-called minimal and physical solutions, showing that minimal solutions of the McKean–Vlasov equation are physical whenever the initial condition is integrable.

Keywords: supercooled Stefan problem, McKean–Vlasov equations, singular interactions, propagation of chaos, systemic risk
MSC (2020) Classification: 60H30, 60K35, 35Q84

1 Introduction

1.1 The classical supercooled Stefan problem

Stefan problems are models for the evolution of the interface between two phases of a substance undergoing a phase transition. Historically, the study of these problems goes back to the eponymous physicist Stefan (1891) studying the growth of ice, and to

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Lamé and Clapeyron (1831), studying the formation of the earth's crust. The classical one-dimensional supercooled Stefan problem is a simple model for the freezing of a supercooled liquid on the semi-infinite strip $[0, \infty)$. It can be formulated using the set of equations

$$\partial_t u = \frac{1}{2}\partial_{xx} u, \qquad \Lambda_t < x < \infty, \quad t > 0, \tag{1.1a}$$

$$u(t, \Lambda_t) = 0, \qquad t > 0, \tag{1.1b}$$

$$\frac{\alpha}{2}\partial_x u(t, \Lambda_t) = -\dot{\Lambda}_t, \qquad t > 0, \tag{1.1c}$$

$$u(0, x) = -f(x), \qquad x > 0. \tag{1.1d}$$

Here, we interpret t, x and $u = u(t, x)$ as time, position and temperature, respectively. The freezing point is at $u = 0$, and the freezing front (the interface between solid and liquid) at time t is located at position $x = \Lambda_t$. We assume that $\alpha > 0$ is a known constant and that f is a known probability density function. Equation (1.1d) then implies that the liquid is initially below or at its freezing point - which is precisely the reason why we refer to the problem as *supercooled*. Equation (1.1a) describes the heat transport in the fluid phase. Condition (1.1b) asserts that the phase change is isothermal, and condition (1.1c) is a so-called *Stefan condition*, which balances the discontinuity in heat flux across the freezing front with the release of latent heat during freezing. The set of equations (1.1) describes a *one-phase* model, which means that we do not account for heat transport in the solid phase, which amounts to assuming that the temperature in the solid is constant and equal to $u = 0$. Solving the supercooled Stefan problem then amounts to finding functions u and Λ such that (1.1) is satisfied.

It is well-known that for certain initial conditions, problem (1.1) exhibits blow-ups in finite time (see Sherman (1970)). The classical remedy for this issue is a modification of the boundary condition, see Dewynne (1992) for a survey and Baker and Shkolnikov (2020) for recent developments. We follow the approach introduced in Delarue et al. (2019), which allows us to globally define solutions to (1.1) in the presence of blow-ups by virtue of a probabilistic reformulation.

1.2 Probabilistic reformulation

Following Delarue et al. (2019), we consider the probabilistic reformulation of the supercooled Stefan problem (1.1), given by the following McKean–Vlasov equation

$$\begin{cases} X_t = X_{0-} + B_t - \Lambda_t \\ \tau = \inf\{t \geqslant 0 : X_t \leqslant 0\} \\ \Lambda_t = \alpha \mathbb{P}\left(\tau \leqslant t\right), \end{cases} \tag{1.2}$$

where the initial condition X_{0-} is supported in $[0, \infty)$, and B is an independent Brownian motion started at 0. (The assumption about the support of X_{0-} is not needed, but natural for the applications we have in mind). A solution to this equation is a triple (X, τ, Λ), where $\Lambda : [0, \infty) \to [0, \alpha]$ is a deterministic, (not necessarily strictly-) increasing càdlàg function such that (1.2) holds for any Brownian motion B, and X and τ are the resulting process and stopping time, respectively.

We heuristically motivate the connection between (1.1) and (1.2) by means of a formal calculation. Suppose that (X, τ, Λ) solves (1.2) for some continuously differentiable loss function Λ, let $\varphi \in C^2$ be a test function with $\varphi(0) = 0$ and let X_{0-} admit the density f. Applying Itô's formula and taking expectations yields

$$\mathbb{E}\left[\varphi(X_t)\mathbb{1}_{[\tau>t]}\right] = \mathbb{E}\left[\varphi(X_{0-})\right] + \frac{1}{2}\int_0^t \mathbb{E}\left[\varphi''(X_s)\mathbb{1}_{[\tau>s]}\right]\, \mathrm{d}s - \int_0^t \mathbb{E}\left[\varphi'(X_s)\mathbb{1}_{[\tau>s]}\right]\, \mathrm{d}\Lambda_s.$$

Denoting the subdensity of $X_t\mathbb{1}_{[\tau>t]}$ on $(0,\infty)$ by $p(t,\cdot)$, integrating by parts yields

$$\partial_t p = \frac{1}{2}\partial_{xx}p + \dot{\Lambda}_t\partial_x p, \quad p(0,\cdot) = f, \quad p(\cdot, 0) = 0.$$

Taking the derivative with respect to time of the equation $\Lambda_t = \alpha(1 - \int_0^\infty p(t,x)\,\mathrm{d}x)$, we see that

$$\dot{\Lambda}_t = -\alpha\int_0^\infty \partial_t p(t,x)\,\mathrm{d}x = -\alpha\int_0^\infty \frac{1}{2}\partial_{xx}p(t,x)\,\mathrm{d}x - \alpha\int_0^\infty \dot{\Lambda}_t\partial_x p(t,x)\,\mathrm{d}x = \frac{\alpha}{2}\partial_x p(t,0).$$

Therefore, setting $u(t,x) := -p(t, x - \Lambda_t)$, we (formally) obtain a solution to (1.1). As (Delarue et al., 2019, Theorem 1.1) shows, this argument can be made rigorous for so-called *physical solutions* to the McKean–Vlasov problem (1.2) in between jump times.

Apart from the physical significance of this equation, systemic risk and neuro-science applications have recently sparked considerable interest in the McKean–Vlasov problem (1.2) and variants thereof, see e.g. Delarue et al. (2015a); Nadtochiy and Shkolnikov (2019); Hambly et al. (2019); Delarue et al. (2019); Ledger and Søjmark (2020). The existence question could be clarified in Delarue et al. (2015a), where (for a slightly different equation) it is shown that global solutions to (1.2) can be obtained as limit points of the following particle system

$$\begin{cases} X_t^{i,N} = X_{0-}^i + B_t^i - \Lambda_t^N \\ \tau_{i,N} = \inf\{t \geqslant 0 : X_t^{i,N} \leqslant 0\} \\ \Lambda_t^N = \dfrac{\alpha}{N}\displaystyle\sum_{i=1}^N \mathbb{1}_{[\tau_{i,N}\leqslant t]}, \end{cases} \tag{1.3}$$

with $N \in \mathbb{N}$ and $i = 1, \ldots, N$. Here, the initial conditions are iid with the same law as X_{0-} in (1.2), and B^i are independent Brownian motions independent of $(X_{0-}^i)_{i=1}^N$. A solution to this equation is a triple (X^N, τ_N, Λ^N), where Λ^N is a $[0,\alpha]$-valued, increasing, càdlàg process such that (1.2) holds for the Brownian motions $(B^i)_{i=1}^N$, and X^N and τ_N are the resulting N-dimensional processes and stopping times, respectively.

1.3 Connections to systemic risk

From the particle system (1.3) the connection to systemic risk in finance becomes apparent. Indeed, consider N banks and assume that $X^{i,N}$ stands for the evolution of bank i's equity value (or rather the distance-to-default, see Feinstein and Søjmark (2021) and the introduction of Hambly et al. (2019) regarding possible economic interpretations). The dynamics described by (1.3) then correspond to a perfectly homogeneous lending network, where every bank lends a quantity of α/N to every other bank in the system. Whenever one $X^{i,N}$ hits 0, then – due to the exposure to all other banks – a contagion effect occurs. It results in a loss of α/N for all other banks, which in turn can cause further defaults leading to default cascades. Such default cascades can trigger so-called *systemic events,* whose definition, introduced in Nadtochiy and Shkolnikov (2019), is based on the limiting McKean–Vlasov equation (1.2). Indeed, for a solution (X, τ, Λ) to (1.2), the function Λ can be interpreted as the aggregate loss of a typical bank in a large banking system caused by defaults from other banks. Mathematically, a *systemic event* is then defined as a jump-discontinuity of Λ, meaning that a considerable fraction of banks defaults in an instant. For more on the relevance of models such as (1.2) to the topic of systemic risk, we refer the reader to Hambly and Søjmark (2019).

A natural question at this point is under which conditions jump-discontinuities of Λ occur. Not surprisingly, the smoothness of Λ depends on the tuple (X_{0-}, α). A remarkably short argument (Hambly et al., 2019, Theorem 1.1) shows that for any solution (X, τ, Λ) to the McKean–Vlasov problem (1.2), the function Λ must be discontinuous whenever $2\mathbb{E}\left[X_{0-}\right] < \alpha$. Conversely, in the weak feedback regime (if the feedback parameter α is "small" relative to the the initial condition), global uniqueness and continuity of Λ is proved in Ledger and Søjmark (2020). A similar result is obtained in Delarue et al. (2015b) for deterministic initial conditions.

1.4 Solution concepts for the McKean–Vlasov problem

The preceding definition of a systemic event only makes sense when a suitable *propagation of chaos-result* for the sequence of empirical measures that corresponds to the particle system (1.3) can be established. By the results of Delarue et al. (2015a), this holds true provided that there is uniqueness among the limit points. But this is exactly (one of) the crucial and still unresolved issue(s): it is open whether uniqueness of limit points of the particle system and uniqueness of solutions to (1.2) hold within the class of so-called *physical solutions* for general initial conditions. Following Delarue et al. (2015a), we call a solution (X, τ, Λ) to (1.2) *physical*, if

$$\Delta\Lambda_t = \alpha \inf\{x > 0\colon \mathbb{P}\left(\tau \geqslant t, X_{t-} \in [0, \alpha x]\right) < x\}, \quad t \geqslant 0. \tag{1.4}$$

Proposition 1.2 in Hambly et al. (2019) reveals that for each instant in time, (1.4) amounts to choosing the smallest possible jump size of Λ that allows for a càdlàg solution of (1.2). In particular, solutions whose loss function Λ is continuous are physical.

There is an analogue of the notion of physical solution for the particle system; we provide more details in Section 3.2. In the particle system, restricting to physical solutions is economically and physically meaningful, and allows to preclude economically elusive solutions and to conclude uniqueness of (1.3) (for every fixed N) among the set of physical solutions. Moreover, it turns out that the physical solution coincides with another meaningful type of solution, namely the *minimal solution* (see Lemma 3.3).

In the current paper we shall focus on this solution concept and illustrate that this is in several respects the right way to look at the problem. We call a solution $(\underline{X}^N, \underline{\tau}_N, \underline{\Lambda}^N)$ to (1.3) *minimal* if, almost surely, for any other solution (X^N, τ_N, Λ^N) to (1.3) coupled to the same Brownian motions $(B^i)_{i=1}^N$, we have

$$\underline{\Lambda}_t^N \leqslant \Lambda_t^N, \quad t \geqslant 0. \tag{1.5}$$

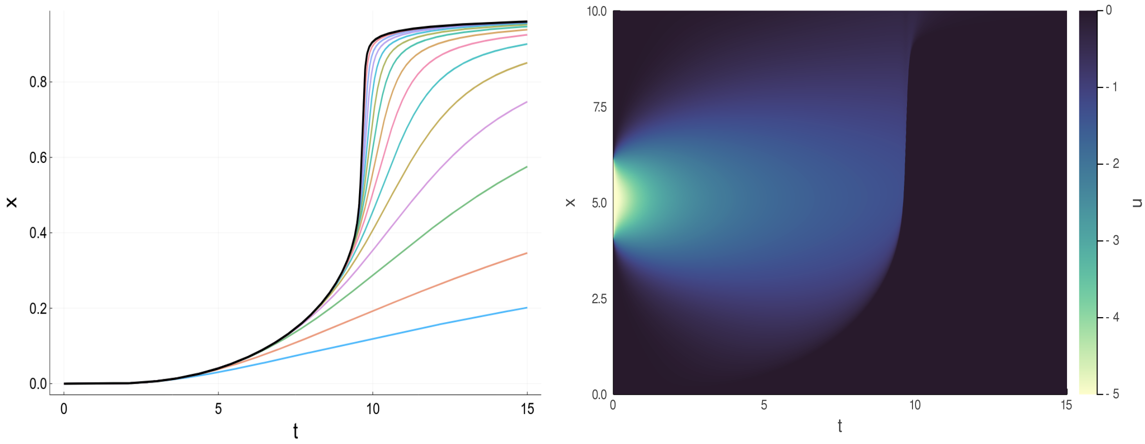

Figure 1: The left-hand side shows the iterates $\Gamma^{(k)}[0]$. The right-hand side shows the solution $u(t,x)$ to the supercooled Stefan problem. The parameters were $\alpha = 10$, and X_{0-} was chosen to be uniformly distributed on (4,6).

In an analogous manner, we call a solution $(\underline{X}, \underline{\tau}, \underline{\Lambda})$ to the McKean–Vlasov problem (1.2) *minimal*, if for every solution (X, τ, Λ) to (1.2) we have

$$\underline{\Lambda}_t \leqslant \Lambda_t, \quad t \geqslant 0. \tag{1.6}$$

Note that this condition is deterministic as $\underline{\Lambda}$ and Λ are necessarily deterministic.

Significant progress regarding the question of uniqueness of physical solutions has been made recently, going beyond the weak feedback regime. In Delarue et al. (2019), uniqueness of the physical solution to the McKean–Vlasov problem (1.2) is established for initial conditions with a bounded and non-oscillatory density for any $\alpha > 0$. However, as mentioned above uniqueness within the class of physical solutions for general X_{0-} and the question whether the (unique) minimal solution coincides with (one of) the physical solution(s), are open problems.

1.5 Main results

One of the main contributions of the current paper is to answer the latter question raised by Delarue et al. (2019) with "yes":[1] *the minimal solution is a physical solution* (see Theorem 6.5). The key for proving this is a propagation of chaos-result for a perturbed particle system. Indeed, consider (1.3) with an initial condition that is slightly larger, namely $X_{0-}^i + N^{-\gamma}$, for $\gamma \in (0, \frac{1}{2})$. Then we can show that the empirical distribution of the minimal (and thus physical) solution to the perturbed particle system converges in probability to the law of the minimal solution to (1.2). This is what we call *perturbed propagation of minimality* (see Theorem 6.2). Combining this with the (known) fact that physical solutions of the (perturbed) particle system converge to physical solutions of the McKean–Vlasov equation, we can thus conclude.

Part of the motivation for studying the concept of minimal solutions was the question whether one could more easily prove propagation of chaos using this concept, as the

[1]Note that the conjecture was stated on p.55 in v1 of the arXiv preprint of Delarue et al. (2019) and has been removed from the updated version.

methods employed in Delarue et al. (2019) are not easily applicable to more general driving processes and initial conditions. While we believe that the methods presented here can be generalized in many ways, we only prove propagation of chaos with a perturbed initial condition (see Theorem 6.2), and still rely on the uniqueness result of Delarue et al. (2019) to obtain "propagation of minimality" for the unperturbed system. However, if the minimal solution of the McKean–Vlasov equation were stable under additive perturbations of the initial condition, our methods would yield full propagation of chaos. We formulate this as a conjecture at the end of the paper.

Our results also allow to exploit iteration (2.9) for numerical purposes (for more on numerical methods for this problem, see Kaushansky and Reisinger (2019), Kaushansky et al. (2020) and Lipton et al. (2019)). Applying a time-discretziation and iteratively solving first-passage time problems as in (Peskir and Shiryaev, 2006, p.230), one can calculate the minimal (and thus physical) solution numerically. Figure 1 illustrates an implementation of this method. Rigorously establishing convergence of this scheme is however nontrivial and beyond the scope of this article.

In the following let us summarize the main contributions of the current article.

- A general tightness result in the M_1 topology for stochastic processes that can be decomposed into a continuous and an increasing càdlàg process (Theorem 4.3).
- (Perturbed) Propagation of minimality (Theorem 6.2 and Theorem 6.6) via the trilogy of arguments consisting of tightness with respect to Skorokhod's M_1 topology (Corollary 4.5), convergence of solutions (Proposition 5.6) and identification of the limit (Section 6).
- The physical nature of the minimal solution and hence global existence of physical solutions whenever the initial condition is integrable (Theorem 6.5).

The remainder of the article is structured as follows. In Section 2, we construct the minimal solution through a fixed-point iteration. In Section 3, we discuss the notions of physical and minimal solutions for the particle system and show that they are equivalent. Section 4 is dedicated to proving tightness of the empirical measures associated to the particle system, in Section 5 we show that limit points of such empirical measures correspond to solutions of the McKean–Vlasov problem in a certain sense. Finally, in Section 6, we prove propagation of minimality under various perturbations and deduce that the minimal solution of the McKean–Vlasov problem is physical whenever the initial condition is integrable.

2 The minimal solution of the McKean–Vlasov problem

In this section, we follow the same strategy employed in Hambly et al. (2019), Nadtochiy and Shkolnikov (2020) and Delarue et al. (2015b), which is to decouple system (1.2) by rewriting it as a fixed-point problem for a certain operator. In Delarue et al. (2015b), global well-posedness results are shown in the weak feedback regime (i.e., for small α) for deterministic initial conditions. In the work Nadtochiy and Shkolnikov (2020) more general networks are considered and a version of the Schauder-Tychonoff theorem is proved for the Skorokhod M_1-topology, however these results do not prove the existence of a minimal or physical solution. In Hambly et al. (2019), existence (and uniqueness) of solutions is shown up to the time of the first discontinuity under some regularity assumptions on the initial condition. The results presented in the current article are global in time and show the global existence of minimal solutions without any restrictions on the initial condition X_{0-} and the feedback parameter α. Note that global existence of minimal solutions was first shown in a preprint version of Delarue et al. (2019), however this result was removed from subsequent versions of said paper and we follow a somewhat different proof strategy here.

Define the operator Γ for a càdlàg function ℓ as

$$\begin{cases} X_t^\ell = X_{0-} + B_t - \alpha \ell_t \\ \tau^\ell = \inf\{t \geqslant 0 : X_t^\ell \leqslant 0\} \\ \Gamma[\ell]_t = \mathbb{P}\left(\tau^\ell \leqslant t\right). \end{cases} \tag{2.1}$$

Note here that $(X^\ell, \tau^\ell, \alpha\ell)$ solves (1.2) if and only if ℓ is a fixed-point of Γ.

It is straightforward to see that Γ is monotone in the sense that

$$\ell_t^1 \leqslant \ell_t^2, \quad t \geqslant 0 \qquad \Longrightarrow \qquad \Gamma[\ell^1]_t \leqslant \Gamma[\ell^2]_t, \quad t \geqslant 0. \tag{2.2}$$

We are now interested in finding a space on which the operator Γ stabilizes. Let $\overline{\mathbb{R}}$ denote the extended real line (i.e., the two-point compactification of $\mathbb{R}$). As a consequence of $\overline{\mathbb{R}}$ being a compact metric space and $[0,\infty]$ being a closed subset thereof, the space of probability measures on $[0,\infty]$, denoted as $\mathcal{P}([0,\infty])$, endowed with the topology of weak convergence of probability measures is a compact Polish space (for more details see e.g. Klenke (2013))[2]. Set

$$M := \{\ell\colon \overline{\mathbb{R}} \to [0,1] \mid \ell \text{ càdlàg and increasing}, \ \ell_{0-} = 0, \ \ell_\infty = 1\}, \tag{2.3}$$

then we may identify the elements of M with distribution functions of measures in $\mathcal{P}([0,\infty])$ via the map $\ell \mapsto \mu_\ell$, where we set $\mu_\ell([0,t]) := \ell_t$ for $t \geqslant 0$. Convergence in M is then equivalent to weak convergence of probability measures in $\mathcal{P}([0,\infty])$. In particular, we have that $\ell^n \to \ell$ in M if and only if $\mu_{\ell^n} \to \mu_\ell$ weakly in $\mathcal{P}([0,\infty])$, if and only if $\ell_t^n \to \ell_t$ for all $t \in [0,\infty]$ that are continuity points of ℓ. Equipped with this topology, M is a compact Polish space. We now show that Γ is a continuous operator on M.

Proposition 2.1. *The operator $\Gamma : M \to M$ is continuous.*

Proof. Let $\ell^n \to \ell$ in M. We show that $\Gamma[\ell^n]_t \to \Gamma[\ell]_t$ for all $t \in [0,\infty]$ that are continuity points of $\Gamma[\ell]$. We begin by proving

$$\limsup_{n\to\infty} \Gamma[\ell^n]_t \leqslant \Gamma[\ell]_t, \quad t \geqslant 0. \tag{2.4}$$

To prove (2.4), note that since $\Gamma[\ell]_t = \mathbb{P}\left(\exists s \in [0,t] : X_{0-} + B_s \leqslant \alpha\ell_s\right)$, by the reverse Fatou lemma it is enough to show that

$$\limsup_{n\to\infty} \mathbb{1}_{\{\exists s\in[0,t]:X_{0-}+B_s\leqslant\alpha\ell_s^n\}} \leqslant \mathbb{1}_{\{\exists s\in[0,t]:X_{0-}+B_s\leqslant\alpha\ell_s\}} \tag{2.5}$$

[2]Note that here we use the language of probabilists, this mode of convergence corresponds to the weak*-convergence of measures in the language of functional analysis.

holds almost surely. If the left-hand side of (2.5) is equal to zero there is nothing to prove, so suppose that $\omega \in \Omega$ is such that the left-hand side of (2.5) is equal to one. Then, after passing to subsequences if necessary, we can find a sequence $s_n \to s$ with $s \in [0,t]$ such that $X_{0-}(\omega) + B_{s_n}(\omega) \leqslant \alpha \ell^n_{s_n}$. Let $s' > s$ be a continuity point of ℓ, then since the ℓ^n are increasing we find $X_{0-}(\omega) + B_s(\omega) \leqslant \alpha \limsup_{n\to\infty} \ell^n_{s^n} \leqslant \limsup_{n\to\infty} \alpha \ell^n_{s'} = \alpha \ell_{s'}$. Letting s' tend to s we obtain $X_{0-}(\omega) + B_s(\omega) \leqslant \alpha \ell_s$, which yields (2.5) and hence (2.4).

Let $t = 0$. If zero is not a point of continuity for $\Gamma[\ell]$, there is nothing to prove. If zero is a point of continuity, then $\Gamma[\ell]_0 = 0$. Inequality (2.4) shows that $\Gamma[\ell^n]_0$ goes to 0 as n goes to infinity.

Now let $t > 0$ be a point of continuity for $\Gamma[\ell]$. We claim that

$$\lim_{n\to\infty} (\Gamma[\ell]_t - \Gamma[\ell^n]_t)^+ = 0. \tag{2.6}$$

We may write

$$(\Gamma[\ell]_t - \Gamma[\ell^n]_t)^+ \leqslant \mathbb{P}\left(\tau^{\ell^n} > t, \tau^\ell \leqslant t\right) = \int_{[0,t]} \mathbb{P}\left(\tau^{\ell^n} > t \mid \tau^\ell = s\right) d\Gamma[\ell]_s.$$

We now split up the integrand in its continuous and jump part, writing $\Gamma[\ell]^c_s$ for the continuous part. Then, following the proof of Proposition 3.1 in Hambly et al. (2019), we find

$$\int_0^t \mathbb{P}\left(\tau^{\ell^n} > t \mid \tau^\ell = s\right) d\Gamma[\ell]^c_s \leqslant \int_0^t \left(2\Phi\left(\alpha \frac{\ell_s - \ell^n_s}{\sqrt{t-s}}\right) - 1\right) d\Gamma[\ell]^c_s, \tag{2.7}$$

where Φ denotes the cumulative distribution function of a standard normal random variable. Because the set of discontinuity times of ℓ is at most countable, the integrand in (2.7) converges $\Gamma[\ell]^c$-almost everywhere to 0. Consequently, the integral in (2.7) vanishes as $n \to \infty$ by the dominated convergence theorem.

Regarding now the integral with respect to the jump part of $\Gamma[\ell]$, we obtain

$$\begin{aligned}
\int_0^t \mathbb{P}\left(\tau^{\ell^n} > t \mid \tau^\ell = s\right) d(\Gamma[\ell] - \Gamma[\ell]^c)_s &= \sum_{s\leqslant t} \mathbb{P}\left(\tau^{\ell^n} > t \mid \tau^\ell = s\right) \Delta\Gamma[\ell]_s \\
&= \sum_{s<t} \mathbb{P}\left(\tau^{\ell^n} > t \mid \tau^\ell = s\right) \mathbb{P}\left(\tau^\ell = s\right) \\
&= \sum_{s<t} \mathbb{P}\left(\tau^{\ell^n} > t, \tau^\ell = s\right), \qquad (2.8)
\end{aligned}$$

where we may take the sum over $s < t$ because t was assumed to be a continuity point of $\Gamma[\ell]$. Again, by the reverse Fatou lemma and the Portmanteau theorem, we find

$$\begin{aligned}
\limsup_{n\to\infty} \mathbb{P}\left(\tau^{\ell^n} > t, \tau^\ell = s\right) &\leqslant \mathbb{P}\left(\forall \varepsilon \in (0, t-s) : X_{0-} + B_{s+\varepsilon} \geqslant \alpha \ell_{s+\varepsilon-}, \tau^\ell = s\right) \\
&\leqslant \mathbb{P}\left(X_{0-} + B_s \geqslant \alpha \ell_s, \tau^\ell = s\right) \\
&\leqslant \mathbb{P}\left(X_{0-} + B_s = \alpha \ell_s\right) \\
&= 0.
\end{aligned}$$

By the dominated convergence theorem we find that the sum in (2.8) converges to zero as n goes to infinity. Together with (2.7) this yields claim (2.6), which, combined with (2.4), proves $\lim_{n\to\infty} \Gamma[\ell^n]_t = \Gamma[\ell]_t$. □

Using the monotonicity of the operator Γ, one can iteratively construct the minimal solution to the McKean–Vlasov equation, an idea due to (Delarue et al., 2019). Figure 1 illustrates this iteration.

Remark 2.2. Note that we did not define $\Gamma[\ell]_\infty$, but by definition, if $\Gamma[\ell] \in M$ should hold, we need to set $\Gamma[\ell]_\infty := 1$. We will continue to only define elements of M on $[0,\infty)$, as the value at infinity necessarily needs to be equal to one. In particular, we will often consider the function $t \mapsto 0$ as an element of M, and really mean $t \mapsto \mathbb{1}_{\{\infty\}}(t)$ on $[0,\infty]$.

Proposition 2.3. *For any initial condition X_{0-} and $\alpha > 0$, there is a minimal solution to (1.2), which we denote by $(\underline{X}, \underline{\Lambda})$. It holds that*

$$\alpha \lim_{k\to\infty} \Gamma^{(k)}[0] = \underline{\Lambda}, \tag{2.9}$$

in M, where $\Gamma^{(k)}$ denotes the k-th iterate of the operator Γ as defined in (2.1).

Proof. By definition, we have that $0 \leqslant \Gamma[0]$. Using the monotonicity of Γ, this implies that

$$\Gamma[0] \leqslant \Gamma[\Gamma[0]] = \Gamma^{(2)}[0]$$

and a straightforward induction shows that $\Gamma^{(k)}[0] \leqslant \Gamma^{(k+1)}[0]$, for each $k \in \mathbb{N}$. The sequence $(\Gamma^{(k)}[0]_t)_{k\in\mathbb{N}}$ is therefore increasing and bounded by 1 for every $t \geqslant 0$, which implies that we may define $\tilde{\Lambda}$ to be the pointwise limit

$$\tilde{\Lambda}_t := \alpha \lim_{k\to\infty} \Gamma^{(k)}[0]_t, \quad t \geqslant 0.$$

Clearly, $\tilde{\Lambda}$ is increasing with $\tilde{\Lambda}_{0-} = 0$ and $\tilde{\Lambda}_\infty = \alpha$, so its càdlàg modification $\underline{\Lambda}_t := \tilde{\Lambda}_{t+}$ lies in αM. By construction, we have that $\lim_{k\to\infty} \Gamma^{(k)}[0] = \frac{1}{\alpha}\underline{\Lambda}$ in M. By continuity of Γ on M (Proposition 2.1), we obtain

$$\alpha\Gamma\left[\frac{1}{\alpha}\underline{\Lambda}\right] = \alpha\Gamma[\lim_{k\to\infty} \Gamma^{(k)}[0]] = \alpha \lim_{k\to\infty} \Gamma^{(k+1)}[0] = \underline{\Lambda},$$

so $\underline{\Lambda}$ solves (1.2). Now suppose that Λ is another solution to (1.2). By definition, it holds that $\Lambda \geqslant 0$, and using the monotonicity of Γ this leads to

$$\alpha\Gamma[0] \leqslant \alpha\Gamma\left[\frac{1}{\alpha}\Lambda\right] = \Lambda.$$

A straightforward induction shows that $\alpha\Gamma^{(k)}[0] \leqslant \Lambda$, for each $k \in \mathbb{N}$. If t is a continuity point of $\underline{\Lambda}$, this implies

$$\underline{\Lambda}_t = \alpha \lim_{k\to\infty} \Gamma^{(k)}[0]_t \leqslant \Lambda_t, \tag{2.10}$$

which by right-continuity implies $\underline{\Lambda} \leqslant \Lambda$. As Λ was an arbitrary solution to (1.2), this proves that $\underline{\Lambda}$ is in fact the minimal solution. $\square$

The next example illustrates how the solution to the McKean–Vlasov problem (1.2) can fail to be unique.

Example 2.4 (Several solutions to the McKean–Vlasov problem)**.** Let $X_{0-} \equiv 1$ and set $\alpha = 1$. Then, $\overline{\Lambda} \equiv 1$ solves the McKean–Vlasov problem (1.2). Later on, we will prove (see Theorem 6.5) that the minimal solution is physical if the initial condition is integrable, which implies by definition that

$$\underline{\Lambda}_0 = \underline{\Lambda}_0 - \underline{\Lambda}_{0-} = \alpha \inf\{x \geqslant 0 : \mathbb{P}\left(X_{0-} \in [0,x]\right) < x\} = 0.$$

Therefore, clearly we have $\underline{\Lambda} \neq \overline{\Lambda}$, and hence nonuniqueness of the McKean–Vlasov problem (1.2).

3 Solutions of the particle system

3.1 Minimal solutions

Although it may seem intuitively obvious, we have not shown yet that there is a minimal solution to the particle system (1.3). A natural question at this point is whether the same construction outlined in Lemma 2.3 works, and a moment's reflection shows that it does. Define the operator Γ_N via

$$\begin{cases} X_t^{i,N}[\mathsf{L}] = X_{0-}^i + B_t^i - \alpha \mathsf{L}_t \\ \tau_{i,N}[\mathsf{L}] = \inf\{t \geqslant 0 : X_t^{i,N}[\mathsf{L}] \leqslant 0\} \\ \Gamma_N[\mathsf{L}]_t = \dfrac{1}{N}\displaystyle\sum_{i=1}^N \mathbb{1}_{\left[\tau_{i,N}[\mathsf{L}] \leqslant t\right]}, \end{cases} \tag{3.1}$$

where L is some càdlàg process. In analogy to (2.2), Γ_N is monotone in the sense that

$$\mathsf{L}_t^1 \leqslant \mathsf{L}_t^2, \quad t \geqslant 0 \quad \Longrightarrow \quad \Gamma[\mathsf{L}^1]_t \leqslant \Gamma[\mathsf{L}^2]_t, \quad t \geqslant 0.$$

Making use of this montonicity, we readily see by straightforward induction arguments that

$$\alpha \Gamma_N^{(k)}[0] \leqslant \Lambda^N, \quad \Gamma_N^{(k)}[0] \leqslant \Gamma_N^{(k+1)}[0], \quad k \in \mathbb{N}, \tag{3.2}$$

holds almost surely, where Λ^N is any solution to the particle system and $\Gamma_N^{(k)}$ denotes the k-th iterate of Γ_N. We did not show any suitable continuity properties of Γ_N to conclude in the same way as in Lemma 2.3, but as we prove in the next lemma, the iteration $(\Gamma_N^{(k)}[0])_{k \in \mathbb{N}}$ is constant after at most N steps.

Lemma 3.1. *For $N \in \mathbb{N}$, let Γ_N be defined as in (3.1). Then $\underline{\Lambda}^N := \alpha \Gamma_N^{(N)}[0]$ is the minimal solution to the particle system (1.3) and the error bound*

$$\|\alpha \Gamma_N^{(k)}[0] - \underline{\Lambda}^N\|_\infty \leqslant \alpha \frac{(N-k)^+}{N} \tag{3.3}$$

holds almost surely.

Proof. For $k \in \mathbb{N}$, define the stopping times

$$\sigma_k := \inf\left\{t \geqslant 0 \colon \Gamma_N^{(k)}[0]_t \geqslant k/N\right\}.$$

First, we show by induction on k that

$$\Gamma_N^{(k-1)}[0]_t = \Gamma_N^{(k)}[0]_t, \quad t < \sigma_k, \tag{3.4}$$

for each $k \in \mathbb{N}$. For the base case, let $k = 1$ (we define $\Gamma_N^{(0)}$ to be the identity operator). Observe that the first time that $\Gamma_N[0]$ jumps coincides with the first time any of the Brownian motions $(X_{0-}^i + B^i)_{i\in\mathbb{N}}$ hit 0, and $\Gamma_N[0]$ is equal to 0 before that time. This means that we have $0 = \Gamma_N^{(0)}[0]_t = \Gamma_N^{(1)}[0]_t$ for each $t < \sigma_1$. For the inductive step, assume the claim holds for all natural numbers up to k. Applying Γ_N to both sides of (3.4), we obtain

$$\Gamma_N^{(k)}[0]_t = \Gamma_N^{(k+1)}[0]_t, \quad t < \sigma_k. \tag{3.5}$$

We distinguish two cases: In the first case, suppose that $\Gamma_N^{(k)}[0]_{\sigma_k} > \frac{k}{N}$. Due to (3.2), we then must have $\Gamma_N^{(k+1)}[0]_{\sigma_k} \geqslant \frac{k+1}{N}$ and hence $\sigma_k = \sigma_{k+1}$, completing the inductive step. In the second case, we have $\Gamma_N^{(k)}[0]_{\sigma_k} = \frac{k}{N}$, and using (3.2) we find for $t \in (\sigma_k, \sigma_{k+1})$

$$\frac{k}{N} = \Gamma_N^{(k)}[0]_{\sigma_k} \leqslant \Gamma_N^{(k)}[0]_t \leqslant \Gamma_N^{(k+1)}[0]_t < \frac{k+1}{N}$$

which shows that $\Gamma_N^{(k)}[0]$ and $\Gamma_N^{(k+1)}[0]$ agree on all of $[0, \sigma_{k+1})$, completing the inductive step.

Having established (3.4), we show that $\underline{\Lambda}^N = \alpha\Gamma_N^{(N)}[0]$ solves the particle system (1.3). For $k \in \mathbb{N}$, repeatedly applying Γ_N to (3.4), we find

$$\Gamma_N^{(k)}[0]_t = \Gamma_N^{(k+1)}[0]_t = \ldots = \frac{1}{\alpha}\underline{\Lambda}_t^N = \Gamma_N\left[\frac{1}{\alpha}\underline{\Lambda}^N\right]_t, \quad t < \sigma_k. \tag{3.6}$$

Choosing $k = N$, we obtain $\underline{\Lambda}^N = \alpha\Gamma_N\left[\frac{1}{\alpha}\underline{\Lambda}^N\right]$ on $[0, \sigma_N)$. Recall that by (3.2) it holds that $\underline{\Lambda}^N \leqslant \alpha\Gamma_N\left[\frac{1}{\alpha}\underline{\Lambda}^N\right]$, so we see that $\alpha = \underline{\Lambda}_{\sigma_N}^N \leqslant \alpha\Gamma_N\left[\frac{1}{\alpha}\underline{\Lambda}^N\right]_{\sigma_N} \leqslant \alpha$ and therefore $\underline{\Lambda}^N = \alpha\Gamma_N[\frac{1}{\alpha}\underline{\Lambda}^N]$. By (3.2), it follows that $\underline{\Lambda}^N$ is indeed the minimal solution.

The error bound (3.3) is now straightforward: By (3.6), $\alpha\Gamma_N^{(k)}[0]$ agrees with $\underline{\Lambda}^N$ for $t < \sigma_k$ and is increasing in t, and therefore

$$\sup_{t\geqslant 0} |\underline{\Lambda}_t^N - \alpha\Gamma_N^{(k)}[0]_t| = \sup_{t\geqslant\sigma_k} |\underline{\Lambda}_t^N - \alpha\Gamma_N^{(k)}[0]_t| \leqslant \alpha - \alpha\Gamma_N^{(k)}[0]_{\sigma_k} \leqslant \alpha\left(1 - \frac{k}{N}\right),$$

for each $k \leqslant N$. □

As mentioned in the introduction, in the weak feedback regime the solution to the McKean–Vlasov problem (1.2) is unique. As the examples in Section 3.1.1 in Delarue et al. (2015a) show, no such condition can guarantee uniqueness of solutions for the particle system.

3.2 Physical solutions

Definition 3.2. If X^N is a solution process to the particle system (1.3), define the random subprobability measure

$$\nu_{t-}^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_{t-}^{i,N}} \mathbb{1}_{[\tau^{i,N} \geqslant t]}$$

for $t \geqslant 0$. We call a solution $(\hat{X}^N, \hat{\Lambda}^N)$ of (1.3) *physical*, if we have

$$\Delta\hat{\Lambda}_t^N = \alpha \inf\left\{\frac{k}{N} \geqslant 0 : \; k \in \mathbb{N}, \; \nu_{t-}^N\left(\left[0, \alpha\frac{k}{N}\right]\right) \leqslant \frac{k}{N}\right\}, \quad t \geqslant 0. \tag{3.7}$$

Note that $N\nu_{t-}^N([0, \alpha\frac{k}{N}])$ is the number of particles surviving up to time t which would not survive a kick of $\alpha\frac{k}{N}$. Therefore we see that (3.7) corresponds to choosing the smallest possible jump size at any $t \geqslant 0$. We recognize formula (3.7) as the discrete analogue of the physical jump condition.

Lemma 3.3. *The physical solution to* (1.3) *is equal to the minimal solution to* (1.3)*.*

Proof. Note that the physical solution $(\hat{X}^N, \hat{\Lambda}^N)$ to (1.3) is pathwise unique, as it is unique between jump times and (3.7) uniquely specifies the size of the jump at any given time. Since the minimal solution $(\underline{X}^N, \underline{\Lambda}^N)$ is unique by definition, it is sufficient to show that the physical solution is minimal.

Let σ be the first time the physical solution $(\hat{X}^N, \hat{\Lambda}^N)$ jumps, then for all $t < \sigma$ we have $\hat{\Lambda}_t^N = \underline{\Lambda}_t^N = 0$, because if the first jump of the minimal solution would happen before σ this would contradict the minimality of $(\underline{X}^N, \underline{\Lambda}^N)$. As physical solutions have minimal jumps, we find $\hat{\Lambda}_\sigma^N \leqslant \underline{\Lambda}_\sigma^N$, which due to the minimality of $\underline{\Lambda}^N$ implies $\hat{\Lambda}_\sigma^N = \underline{\Lambda}_\sigma^N$. Repeating this argument for each jump of the physical solution proves the claim. $\square$

4 Tightness

As in the previous works on problem at hand, we will deal with the Skorokhod M_1-topology in this paper. We explain this choice and collect some fundamental results regarding the M_1-topology in Section A in the appendix.

Definition 4.1. Let $w^n, w \in C([0, \infty))$. We say that w^n converges to w with respect to the topology of compact convergence if $w^n \to w$ in $C([0, T])$ for every $T > 0$. We also define d_L to be the Lévy-metric on M, that is if $\ell^1, \ell^2 \in M$ we have

$$d_L(\ell^1, \ell^2) = \inf\{\varepsilon > 0 : \ell_{t+\varepsilon}^1 + \varepsilon \geqslant \ell_t^2 \geqslant \ell_{t-\varepsilon}^1 - \varepsilon, \text{ for all } t \geqslant 0\}.$$

It is well-known that the Lévy-metric metrizes weak convergence. In order to avoid having to work directly with the unwieldy M_1-metric, the following theorem comes in handy. Recall that $D([T_0, \infty))$ denotes the space of càdlàg paths from $[T_0, \infty)$ to $\mathbb{R}$ furnished with the M_1-topology (see Definition A.6 for more detail).

Theorem 4.2. *Define the space $\bar{E}$ as*

$$\bar{E} := C([0,\infty)) \times M,$$

where M is defined as in (2.3). Endowed with the product topology induced by compact convergence on $C([0,\infty))$ and the Lévy-metric on M, the space $\bar{E}$ is Polish. For $w \in C([0,\infty))$ and $\ell \in M$, define

$$\hat{w}_t := \begin{cases} w_0 & t \in [-1,0) \\ w_t & t \in [0,\infty) \end{cases} \qquad \check{\ell}_t = \begin{cases} 0 & t \in [-1,0) \\ \ell_t & t \in [0,\infty). \end{cases}$$

Then, for any $\alpha \in \mathbb{R}$, the embedding $\iota_\alpha \colon \bar{E} \to D([-1,\infty))$ defined via

$$\iota_\alpha(w,\ell) = \hat{w} - \alpha\check{\ell}$$

is continuous.

Proof. We show the continuity of ι_α. Let $(w^n,\ell^n) \to (w,\ell)$ in $\bar{E}$. Let $T > 0$ be a continuity point of ℓ. By assumption, we have that $w^n \to w$ in $C([0,T])$, which implies that $\widehat{w^n} \to \hat{w}$ in $C([-1,T])$ and thus in $D([-1,T])$. Again, by assumption, ℓ^n_t converges to ℓ_t for all $t > 0$ that are continuity points of ℓ, from which we deduce (using Lemma A.3) that $-\alpha\check{\ell^n} \to -\alpha\check{\ell}$ in $D([-1,T])$. By Lemma A.5, it follows that

$$\lim_{n\to\infty} \iota_\alpha(w^n,\ell^n) = \lim_{n\to\infty} (\widehat{w^n} - \alpha\check{\ell^n}) = \hat{w} - \alpha\check{\ell} = \iota_\alpha(w,\ell)$$

in $D([-1,T])$. The conclusion now follows from Lemma A.7. □

It is necessary to extend the domain artificially to the left to obtain a continuous embedding of $\bar{E}$ into the càdlàg functions equipped with the M_1-topology as in Theorem 4.2. The reason lies in the requirements of Lemma A.3, more precisely the requirement of pointwise convergence in the left interval endpoint. The extension procedure might now look like a cheap trick to circumvent having to show pointwise convergence in 0 (which it is), but indeed, one easily constructs examples of sequences $(\ell^n)_{n\in\mathbb{N}}$ that converge in M, but do not converge pointwise in 0 against their càdlàg limit, such as $\ell^n_t := \mathbb{1}_{[1/n \leqslant t]}$.

Theorem 4.3. *Let $(X^N)_{N\in\mathbb{N}}$ be a sequence of stochastic processes with paths in $D([0,\infty))$. Suppose that X^N almost surely admits a decomposition*

$$X^N = Z^N - \alpha_N \mathsf{L}^N \tag{4.1}$$

where α_N is a (possibly random) real number, Z^N is continuous, and $\mathsf{L}^N \in M$. Suppose that $(Z^N)_{N\in\mathbb{N}}$ is tight on $C([0,T])$ for each $T > 0$ and that $(\alpha_N)_{N\in\mathbb{N}}$ is tight on $\mathbb{R}$. Set

$$\hat{X}^N_t := \begin{cases} Z^N_0, & t \in [-1,0), \\ X^N_t, & t \in [0,\infty). \end{cases} \tag{4.2}$$

Then, the random variables $((Z^N,\mathsf{L}^N))_{N\in\mathbb{N}}$ are tight on $\bar{E}$ and random variables $(\hat{X}^N)_{N\in\mathbb{N}}$ are tight on $D([-1,\infty))$.

Proof. Let $\varepsilon > 0$ and let $r > 0$ be such that $\mathbb{P}(|\alpha_N| > r) < \varepsilon/2$ or each $N \in \mathbb{N}$. Choose a sequence of positive numbers $(T_k)_{k\in\mathbb{N}}$ such that $T_k \nearrow \infty$. Because $(Z^N)_{N\in\mathbb{N}}$ is tight on $C([0, T_k])$, we may pick $K_\varepsilon^k \subseteq C([0, T_k])$ compact, such that

$$\mathbb{P}\left(Z^N \notin K_\varepsilon^k\right) < \varepsilon 2^{-(k+1)}, \quad N \in \mathbb{N}.$$

Set $K_\varepsilon := \{w \in C([0,\infty)) : w \in K_\varepsilon^k,\ k \in \mathbb{N}\}$. Then K_ε is compact in the topology of compact convergence (which follows e.g. from Tychonoff's theorem). We obtain that

$$\mathbb{P}\left((Z^N, \mathsf{L}^N) \notin K_\varepsilon \times M\right) = \mathbb{P}\left(Z^N \notin K_\varepsilon\right) \leqslant \sum_{k=1}^{\infty} \mathbb{P}\left(Z^N \notin K_\varepsilon^k\right) \leqslant \varepsilon/2,$$

and as M is compact, this shows that $((Z^N, \mathsf{L}^N))_{N\in\mathbb{N}}$ is tight on $\bar{E}$. Theorem 4.2 now shows that

$$K := (\hat{K}_\varepsilon - r\check{M}) \cup (\hat{K}_\varepsilon + r\check{M})$$

is compact in $D([-1,\infty))$. We obtain, uniformly in $N \in \mathbb{N}$,

$$\mathbb{P}\left(\hat{X}^N \notin K\right) \leqslant \mathbb{P}\left(Z^N \notin K_\varepsilon\right) + \mathbb{P}\left(|\alpha_N| > r\right) < \varepsilon.$$

So $(\hat{X}^N)_{N\in\mathbb{N}}$ is tight on $D([-1,\infty))$. □

When we are concerned with a process admitting a decomposition of the form (4.1) in $D([0,\infty))$, from now on we will not distinguish between the process and its extension as given in (4.2) to $D([-1,\infty))$ in our notation.

Definition 4.4. Set $X^N := (X^{1,N}, \ldots, X^{N,N})$. We say that X^N is N-exchangeable, if

$$\operatorname{law}(X^N) = \operatorname{law}((X^{\sigma(1),N}, X^{\sigma(2),N}, \ldots, X^{\sigma(N),N})),$$

for any permutation σ of $\{1, \ldots, N\}$.

Corollary 4.5. *Suppose that X^N satisfies the dynamics*

$$X_t^{i,N} = X_{0-}^{i,N} + B_t^i - \alpha \mathsf{L}_t^N$$

where $\alpha > 0$, $(X_{0-}^N)_{N\in\mathbb{N}}$ are N-exchangeable random vectors, $(B_i)_{i\in\mathbb{N}}$ are independent Brownian motions, and $L^N \in M$. If $(X_{0-}^{1,N})_{N\in\mathbb{N}}$ is tight on $\mathbb{R}$, then the empirical measures

$$\mu_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{X^{i,N}} \qquad \text{and} \qquad \xi_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{(X_{0-}^{i,N} + B^i, \mathsf{L}^N)} \tag{4.3}$$

are tight on $\mathcal{P}(D([-1,\infty)))$ and $\mathcal{P}(\bar{E})$, respectively.

Proof. Fix $\varepsilon > 0$ and let $T > 0$. As $C([0,T])$ is a Polish space, there is a compact set $K \subseteq C([0,T])$ such that $\mathbb{P}(B^1 \in K) > 1 - \varepsilon/2$. By assumption, there is a compact set $K_{0-} \subseteq \mathbb{R}$ such that $\mathbb{P}(X_{0-}^{1,N} \in K_{0-}) > 1 - \varepsilon/2$, for each $N \in \mathbb{N}$. As $K_{0-} + K$ is compact in $C([0,T])$, and

$$\mathbb{P}\left(X_{0-}^{1,N} + B^1 \notin K_{0-} + K\right) \leqslant \varepsilon, \quad N \in \mathbb{N},$$

we find that $(X_{0-}^{1,N} + B^1)_{N\in\mathbb{N}}$ is tight on $C([0,T])$ for every $T > 0$. The claim now follows from Theorem 4.3 and Proposition 2.2 in Sznitman (1991). □

We state a technical result for future reference.

Corollary 4.6. *Let $(\mu_N)_{N\in\mathbb{N}}$ be given as in Corollary 4.5. Then there are $\mathcal{P}(\bar{E})$-valued random variables ξ, ξ_N such that, after passing to subsequences if necessary,* $\mathrm{law}(\mu_N) = \mathrm{law}(\iota_\alpha(\xi_N))$, $\xi_N \to \xi$, *and* $\iota_\alpha(\xi_N) \to \iota_\alpha(\xi)$ *almost surely. Moreover,*

$$\mathrm{law}(\xi_N) = \mathrm{law}\left(\frac{1}{N}\sum_{i=1}^{N} \delta_{(X_{0-}^{i,N}+B^i,\mathsf{L}^N)}\right). \tag{4.4}$$

Proof. Set ξ_N as in Corollary 4.5 and note that $\mu_N = \iota_\alpha(\xi_N)$ for every $N \in \mathbb{N}$. After passing to subsequences if necessary, we may assume that $\mathrm{law}(\xi_N) \to \mathrm{law}(\xi)$ by virtue of Corollary 4.5 for some random variable ξ. By the Skorokhod representation theorem, we may assume as well that $\lim_{N\to\infty} \xi_N = \xi$ holds in $\mathcal{P}(\bar{E})$ for a representation sequence. Since ι_α is continuous by Theorem 4.2, this implies $\lim_{N\to\infty} \iota_\alpha(\xi_N) = \iota_\alpha(\xi)$ almost surely. □

5 Convergence of solutions

Consider the setting described in Corollary 4.5 where

$$X_t^{i,N} = X_{0-}^{i,N} + B_t^i - \alpha \mathsf{L}_t^N$$

for $\alpha > 0$, some N-exchangeable random vectors $(X_{0-}^N)_{N\in\mathbb{N}}$, some independent Brownian motions $(B_i)_{i\in\mathbb{N}}$, and some $\mathsf{L}^N \in M$ adapted to the filtration $(\mathcal{F}_t^N)_{t\geqslant 0}$ generated by $(X_{0-}^N, B^1, \ldots, B^N)$. Assume that $(X_{0-}^N)_{N\in\mathbb{N}}$ is independent of $(B_i)_{i\in\mathbb{N}}$ as well as $(X_{0-}^{1,N})_{N\in\mathbb{N}}$ is tight on $\mathbb{R}$. Setting again

$$\mu_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{X^{i,N}} \tag{5.1}$$

we can then make use of Corollary 4.6 to deduce that there are $\mathcal{P}(\bar{E})$-valued random variables ξ, ξ_N such that ξ_N satisfies (4.4) and, after passing to subsequences if necessary,

$$\mathrm{law}(\mu_N) = \mathrm{law}(\iota_\alpha(\xi_N)), \quad \xi_N \to \xi, \quad \text{and} \quad \iota_\alpha(\xi_N) \to \iota_\alpha(\xi) \tag{5.2}$$

almost surely. The goal of this section is now to study the properties of the random measure ξ. This analysis will lead to the conclusion that solutions to the particle systems converge to solutions of the McKean–Vlasov problem (1.2).

This result might not seem new, but the fact that we are working with the space $\overline{E}$ embedded into $D([-1,\infty))$ via the map ι_α implies that the involved results in the literature cannot be applied directly. When significant simplifications due to the different setting are possible, we include them in the (sometimes alternative) proofs

Definition 5.1. For $t \in \mathbb{R}$ and $x \in D([-1,\infty))$, define the path functionals

$$\tau_0(x) := \inf\{s \geqslant 0 : x_s \leqslant 0\} \quad \text{and} \quad \lambda_t(x) := \mathbb{1}_{[\tau_0(x)\leqslant t]}.$$

Considering a sequence of constant positive functions converging to zero shows that λ_t is not continuous on $D([-1,\infty))$ with the M_1-topology. However, it turns out that λ_t is continuous at paths that satisfy a certain crossing property (going back to Delarue et al. (2015a)), which ensures that the path actually dips below the x-axis at the first hitting time of zero (see Lemma 5.3).

Lemma 5.2. *Let $x^n, x \in \iota_\alpha(\bar{E})$ and suppose that $x^n \to x$ in $D([-1,\infty))$. Then, if $t \geqslant 0$ is any continuity point of x, it holds that*

$$\lim_{n\to\infty} \inf_{0\leqslant s\leqslant t} x^n_s = \inf_{0\leqslant s\leqslant t} x_s.$$

Proof. The result follows by Lemma A.9 and the fact that for each $y \in \iota_\alpha(\bar{E})$ and $t \geqslant 0$ we have $\inf_{-1\leqslant s\leqslant t} y_s = \inf_{0\leqslant s\leqslant t} y_s$. □

Lemma 5.3. *Assume that $(\xi^n)_{n\in\mathbb{N}}$ is a convergent sequence of probability measures on $\bar{E}$ with limit ξ. Define $\mu^n := \iota_\alpha(\xi^n)$ and $\mu := \iota_\alpha(\xi)$. If μ-almost every path satisfies the crossing property, i.e., we have*

$$\mu\Big(\Big\{x \in D([-1,\infty)) : \inf_{0\leqslant s\leqslant h}(x_{\tau_0+s} - x_{\tau_0}) = 0\Big\}\Big) = 0, \quad h > 0, \tag{5.3}$$

then $\lim_{n\to\infty}\langle \mu^n, \lambda\rangle = \langle \mu, \lambda\rangle$ holds in M.

Proof. The proof is analogous to the proof of Proposition 5.8 in Delarue et al. (2015a), making use of Lemma 5.2. □

The next lemma can be proven by translating the last part of the proof of Lemma 5.9 in Delarue et al. (2015a) to the current setting. As our setup allows for a simpler proof using martingale convergence arguments[3], we provide a proof sketch.

Lemma 5.4. *For almost every realization ω, if $\mathrm{law}((W,\mathsf{L})) = \xi(\omega)$, then $W - W_0$ is a Brownian motion with respect to the filtration generated by (W,L). In particular, $W - W_0$ is independent of W_0.*

Proof Sketch. We employ Lévy's characterization of Brownian motion. Let $g_i \in C_b(\mathbb{R}^2)$ for $i = 1,\dots,n$ and let $0 \leqslant s_1 < \dots < s_n \leqslant s < t$, and set $G(w,\ell) := \prod_{i=1}^n g(w_{s_i}, \ell_{s_i})$. It holds that

$$\begin{aligned}\mathbb{E}\bigg[\bigg(\int_{\bar{E}}(w_t - w_s)G(w,\ell)\; d\xi(w,\ell)\bigg)^2\bigg] &= \lim_{N\to\infty}\mathbb{E}\left[\bigg(\int_{\bar{E}}(w_t - w_s)G(w,\ell)\; d\xi_N(w,\ell)\bigg)^2\right]\\ &= \lim_{N\to\infty}\frac{1}{N}\mathbb{E}\left[\Big((B^1_t - B^1_s)G(X^{1,N}_{0-} + B^1, \mathsf{L}^N)\Big)^2\right] = 0,\end{aligned}$$

Using (right-)continuity and the fact that the Borel σ-algebra on $\bar{E}$ is generated by the evaluation mappings, we conclude that for almost every realization ω, if $\mathrm{law}((W,\mathsf{L})) = \xi(\omega)$, then $W - W_0$ is a martingale with respect to the filtration generated by (W,L). Analogously, we can show that $(W_t - W_0)^2 - t$ is a martingale with respect to the filtration generated by (W,L). By Lévy's characterization of Brownian motion the claim follows. □

The next lemma shows that the limiting measures of particle systems such as (1.3) satisfy the crossing property, rendering the loss functional continuous. This result is Lemma 5.9 in Delarue et al. (2015a).

[3]Note that in Ledger and Søjmark (2021), martingale convergence arguments are employed to characterize limit points of the particle system with more general dynamics, also allowing for common noise.

Lemma 5.5. *Suppose that* $\operatorname{law}(\mu_N) \to \operatorname{law}(\mu)$ *for some random variable* μ*. Then* μ *satisfies the crossing property* (5.3) *almost surely.*

Proof. Recall that by (5.2) we have $\operatorname{law}(\mu) = \operatorname{law}(\iota_\alpha(\xi))$.

Fixing $h > 0$ we can then compute

$$
\begin{aligned}
&\mathbb{E}\big[\mu\big(\big\{x \in D([-1,\infty))\colon \inf_{0\leqslant s\leqslant h}(x_{\tau_0+s} - x_{\tau_0}) = 0\big\}\big)\big] \\
&\quad = \mathbb{E}\big[\xi\big(\big\{(w,\ell) \in \bar{E}\colon \inf_{0\leqslant s\leqslant h}[(w_{\tau_0+s} - w_{\tau_0}) - \alpha(\ell_{\tau_0+s} - \ell_{\tau_0})] = 0\big\}\big)\big] \\
&\quad \leqslant \mathbb{E}\big[\xi\big(\big\{(w,\ell) \in \bar{E}\colon \inf_{0\leqslant s\leqslant h}(w_{\tau_0+s} - w_{\tau_0}) = 0\big\}\big)\big],
\end{aligned}
$$

where the inequality is due to the fact that ℓ is increasing. Observe that for almost every realization ω, if $\operatorname{law}((W,L)) = \xi(\omega)$, then $\tau_0 = \tau_0(\iota_\alpha(W,L))$ is a stopping time with respect to the filtration generated by (W,L). Since $W - W_0$ is a Brownian motion with respect to the same filtration by Lemma 5.4, the strong Markov property yields

$$
\mathbb{E}\big[\xi\big(\big\{(w,\ell) \in \bar{E}\colon \inf_{0\leqslant s\leqslant h}(w_{\tau_0+s} - w_{\tau_0}) = 0\big\}\big)\big] = \mathbb{P}\big(\inf_{0\leqslant s\leqslant h} B_s = 0\big) = 0.
$$

We have shown that

$$
\mu\big(\big\{x \in D([-1,\infty)) : \inf_{0\leqslant s\leqslant h}(x_{\tau_0+s} - x_{\tau_0}) = 0\big\}\big) = 0
$$

holds almost surely. Repeating this reasoning for $h = h_n$ for a sequence $(h_n)_{n\in\mathbb{N}}$ such that $h_n > 0$ and $\lim_{n\to\infty} h_n = 0$ yields the result. □

The next result shows that weak limits of laws pertaining to the particle system (1.3) correspond to laws of solution processes to the McKean–Vlasov problem (1.2) and was first shown as Theorem 4.4 in Delarue et al. (2015a). We present an alternative proof.

Proposition 5.6. *For* $N \in \mathbb{N}$*, let* (X^N, Λ^N) *be a solution to the particle system*

$$
\begin{cases}
X_t^{i,N} = X_{0-}^{i,N} + B_t^i - \Lambda_t^N \\
\tau_{i,N} = \inf\{t \geqslant 0 : X_t^{i,N} \leqslant 0\} \\
\Lambda_t^N = \dfrac{\alpha}{N}\displaystyle\sum_{i=1}^N \mathbb{1}_{[\tau_{i,N}\leqslant t]},
\end{cases}
\tag{5.4}
$$

and let $(\mu_N)_{N\in\mathbb{N}}$ *denote the corresponding empirical measures* (5.1)*. Suppose that for some random variable* μ *and some measure* $\nu_{0-} \in \mathcal{P}(\mathbb{R})$ *we have*

$$
\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N \delta_{X_{0-}^{i,N}} = \nu_{0-},
$$

and $\operatorname{law}(\mu_N) \to \operatorname{law}(\mu)$ *along some subsequence. Then* μ *coincides almost surely with the law of a solution process to the McKean–Vlasov problem* (1.2) *for* $\operatorname{law}(X_{0-}) = \nu_{0-}$.

Proof. Without loss of generality we may assume that $\mu_N = \iota_\alpha(\xi_N)$ (and thus (4.3)) and set $\mu = \iota_\alpha(\xi)$. Observe that the map $t \mapsto \mathbb{E}\left[\langle \mu, \lambda_t \rangle\right]$ is increasing, and therefore has at most countably many discontinuities, and the same holds for the map $t \mapsto \mathbb{E}\left[\int \ell_t \, d\xi(w,\ell)\right]$. Let J be the set of discontinuities of these maps and fix $t \notin J$. Then $\langle \mu, \lambda_{t-} \rangle = \langle \mu, \lambda_t \rangle$ almost surely. By Lemma 5.5 we can apply Lemma 5.3 to obtain that

$$\lim_{N\to\infty} \langle \mu_N, \lambda_t \rangle = \langle \mu, \lambda_t \rangle \tag{5.5}$$

holds almost surely. In the next three steps we prove that μ coincides almost surely with the law of a solution process to (1.2).

Step 1: We show that for almost every realization ω, if $\mathrm{law}((W, L)) = \xi(\omega)$, then $L \equiv \langle \mu(\omega), \lambda \rangle$ almost surely. Note that we have

$$\mathbb{E}\left[\int_{\bar{E}} \big| |\ell_t - \langle \mu, \lambda_t \rangle| - |\ell_t - \langle \mu_N, \lambda_t \rangle| \big| \; \mathrm{d}\xi_N(w,\ell)\right] \leqslant \mathbb{E}\left[|\langle \mu, \lambda_t \rangle - \langle \mu_N, \lambda_t \rangle|\right]$$

which vanishes as $N \to \infty$ by (5.5) and the dominated convergence theorem. By choice of $t \notin J$, the map $\ell \to \ell_t$ is continuous from M to $\mathbb{R}$ for ξ-almost every ℓ, and it follows that

$$\begin{aligned}
\mathbb{E}\left[\int_{\bar{E}} |\ell_t - \langle \mu, \lambda_t \rangle| \; \mathrm{d}\xi(w,\ell)\right] &= \lim_{N\to\infty} \mathbb{E}\left[\int_{\bar{E}} |\ell_t - \langle \mu, \lambda_t \rangle| \; \mathrm{d}\xi_N(w,\ell)\right] \\
&= \lim_{N\to\infty} \mathbb{E}\left[\int_{\bar{E}} |\ell_t - \langle \mu_N, \lambda_t \rangle| \; \mathrm{d}\xi_N(w,\ell)\right].
\end{aligned}$$

Since $\langle \mu_N, \lambda \rangle = \ell$ holds for ξ_N-almost every $\ell \in M$, almost surely, we can conclude that this expression is equal to 0. The claim follows by letting t range through a countable dense subset of $[0, \infty) \backslash J$ and using right-continuity.

Step 2: We show that for almost every realization ω, if $\mathrm{law}((W, \mathsf{L})) = \xi(\omega)$, then $\mathrm{law}(W_0) = \nu_{0-}$. To that end, let that the evaluation map $\overline{\pi}_0$ defined on $\bar{E}$ be given by $(w, \ell) \mapsto w_0$. Since $\overline{\pi}_0$ is a continuous map, letting $\overline{\pi}_0(\xi)$ denote the pushforward of ξ by $\overline{\pi}_0$, the continuous mapping theorem shows

$$\overline{\pi}_0(\xi) = \lim_{N\to\infty} \overline{\pi}_0(\xi_N) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \delta_{X_{0-}^{i,N}} = \nu_{0-}.$$

Step 3: We conclude by showing that for almost every realization ω, if $\mathrm{law}(X) = \mu(\omega)$, then $\mathrm{law}(X_{0-}) = \nu_{0-}$ and $X - X_{0-} + \alpha\langle \mu(\omega), \lambda \rangle$ is a Brownian motion independent of X_{0-}. Since $\iota_\alpha(\xi) = \mu$, by Step 1 we know that if $\mathrm{law}((W, \mathsf{L})) = \xi(\omega)$, then

$$\mathrm{law}((W_0, W - W_0)) = \mathrm{law}((X_{0-}, X - X_{0-} + \alpha\langle \mu(\omega), \lambda \rangle)).$$

The claim now follows by Step 2 and Lemma 5.4. $\square$

6 Identification of the limit

Having proved convergence of the particle system to solutions of the McKean–Vlasov problem, we are now interested in characterizing the limit points of minimal and physical solutions of the particle system. We shed some light on this intricate topic by either perturbing the initial condition of the particle system (Section 6.1) or of the McKean–Vlasov problem (Section 6.2).

6.1 The perturbed particle system

The next result is a key technical result for the theorems in this section.

Proposition 6.1. *Fix $\alpha > 0$, $\gamma \in (0, 1/2)$, and let $\underline{\tilde{\Lambda}}^N$ denote the perturbed minimal solution to the particle system, which for $N \in \mathbb{N}$ is the minimal solution to the perturbed particle system*

$$\begin{cases} X_t^{i,N} = X_{0-}^i + N^{-\gamma} + B_t^i - \Lambda_t^N \\ \tau_{i,N} = \inf\{t \geqslant 0 : X_t^{i,N} \leqslant 0\} \\ \Lambda_t^N = \dfrac{\alpha}{N} \displaystyle\sum_{i=1}^N \mathbb{1}_{[\tau_{i,N} \leqslant t]}. \end{cases} \tag{6.1}$$

If $\underline{\Lambda}$ is the minimal solution to the McKean–Vlasov problem (1.2), then

$$\limsup_{N\to\infty} \mathbb{E}\left[\underline{\tilde{\Lambda}}_t^N\right] \leqslant \underline{\Lambda}_t, \tag{6.2}$$

holds for every $t > 0$.

Proof. The key idea of this proof is to compare both $\underline{\Lambda}$ and $(\underline{\tilde{\Lambda}}^N)_{N\in\mathbb{N}}$ to a sequence of optimizers of suitable optimization problems. To that end, let $\varepsilon > 0$ be such that $\beta := \gamma + \varepsilon < 1/2$. Set $\mathcal{R}$ to be the set of all random variables $\mu : \Omega \to \mathcal{P}(E)$. Fix $t > 0$ and define the cost functional

$$c_N(\mu) := \langle \mu, \lambda_t \rangle + N^\beta \|\Gamma_N[\langle \mu, \lambda \rangle] - \langle \mu, \lambda \rangle\|_\infty,$$

where Γ_N is the operator introduced in (3.1) and $\|\cdot\|_\infty$ is the uniform norm on $[0, \infty)$. For $t \geqslant 0$, let V_t^N be the optimal value

$$V_t^N := \inf_{\mu\in\mathcal{R}} \mathbb{E}\left[c_N(\mu)\right]. \tag{6.3}$$

The definition (6.2) depends on the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. However, our comparison argument works for any choice of $(\Omega, \mathcal{F}, \mathbb{P})$ as long as $\mu_N \in \mathcal{R}$ for all $N \in \mathbb{N}$ and for any $\mu \in \mathcal{R}$, the map $\omega \mapsto c_N(\mu(\omega))$ is measurable. As (6.2) does not depend on the choice of probability space, it is sufficient to carry out the argument for a specific choice of $(\Omega, \mathcal{F}, \mathbb{P})$. For the construction of a probability space satisfying the required properties see Example C.1 in Section C in the appendix.

Step 1: We show that asymptotically, $\underline{\Lambda}$ is an upper bound for αV_t^N. Set $\underline{\mathsf{L}}_t := \frac{1}{\alpha}\underline{\Lambda}_t$. Taking μ to be constant and equal to the law of the minimal solution process of (1.2), that is $\mu(\omega) = \underline{\mu}$, we find

$$V_t^N \leqslant \underline{\mathsf{L}}_t + N^\beta \mathbb{E}\left[\|\Gamma_N[\underline{\mathsf{L}}] - \underline{\mathsf{L}}\|_\infty\right].$$

Note that by definition, $\Gamma_N[\underline{\mathsf{L}}]$ is the N-th empirical distribution function associated to the iid sequence $\tau^1[\underline{\mathsf{L}}], \tau^2[\underline{\mathsf{L}}], \dots$, where

$$\tau^i[\underline{\mathsf{L}}] := \inf\{t \geqslant 0 : X_{0-}^i + B_t^i - \underline{\Lambda}_t \leqslant 0\}, \quad i \in \mathbb{N},$$

and that $\mathbb{P}\left(\tau^i[\underline{\mathsf{L}}] \leqslant t\right) = \underline{\mathsf{L}}_t$ for each t. By the Dvoretzky-Kiefer-Wolfowitz inequality (cf Corollary 1 in Massart (1990)), it holds for $x > 0$ that

$$\mathbb{P}\left(\sqrt{N}\|\Gamma_N[\underline{\mathsf{L}}] - \underline{\mathsf{L}}\|_\infty > x\right) \leqslant 2\exp(-2x^2).$$

It therefore follows that

$$\begin{aligned}
\mathbb{E}\left[N^\beta\|\Gamma_N[\underline{\mathsf{L}}] - \underline{\mathsf{L}}\|_\infty\right] &= \int_0^\infty \mathbb{P}\left(N^\beta\|\Gamma_N[\underline{\mathsf{L}}] - \underline{\mathsf{L}}\|_\infty > x\right)\,\mathrm{d}x \\
&= \int_0^\infty \mathbb{P}\left(N^{1/2}\|\Gamma_N[\underline{\mathsf{L}}] - \underline{\mathsf{L}}\|_\infty > N^{1/2-\beta}x\right)\,\mathrm{d}x \\
&\leqslant 2\int_0^\infty \exp(-2N^{1-2\beta}x^2)\,\mathrm{d}x \\
&= \sqrt{\frac{\pi}{2}}N^{\beta-1/2}.
\end{aligned}$$

As $\beta < 1/2$, this quantity vanishes as $N \to \infty$, which proves

$$\limsup_{N\to\infty} \alpha V_t^N \leqslant \underline{\Lambda}_t.$$

Step 2: We show that asymptotically, $\limsup_{n\to\infty} \mathbb{E}[\underline{\tilde{\Lambda}}_t^N] \leqslant \limsup_{n\to\infty} \alpha V_t^N$. Consider a sequence of $\mathcal{P}(E)$-valued random variables $(\zeta_N)_{N\in\mathbb{N}}$ such that

$$\mathbb{E}\left[c_N(\zeta_N)\right] - V_t^N \leqslant \frac{1}{N}, \quad N \in \mathbb{N}.$$

Plugging the empirical measure associated to the solution of the N-particle system into the objective function in (6.3) yields $V_t^N \leqslant 1$ and consequently

$$\mathbb{E}\left[N^\beta\|\Gamma_N[\langle\zeta_N, \lambda\rangle] - \langle\zeta_N, \lambda\rangle\|_\infty\right] \leqslant 1 + \frac{1}{N}, \quad N \in \mathbb{N}. \tag{6.4}$$

Let $A_N := [\|\Gamma_N[\langle\zeta_N, \lambda\rangle] - \langle\zeta_N, \lambda\rangle\|_\infty \leqslant N^{-\gamma}]$, then the Markov inequality together with (6.4) yield

$$\mathbb{P}\left(A_N^c\right) \leqslant N^{\gamma-\beta}(1 + \frac{1}{N}) \leqslant 2N^{-\varepsilon}. \tag{6.5}$$

For the last step, in analogy to (3.1), define the operator $\tilde{\Gamma}_N$ via

$$\begin{cases} \tilde{X}^{i,N}_t[\ell] := X^i_{0-} + \alpha N^{-\gamma} + B^i_t - \alpha \ell_t \\ \tilde{\tau}_{i,N}[\ell] := \inf\{t \geqslant 0 : \tilde{X}^{i,N}_t[\ell] \leqslant 0\} \\ \tilde{\Gamma}_N[\ell]_t := \dfrac{1}{N}\displaystyle\sum_{i=1}^N \mathbb{1}_{[\tilde{\tau}_{i,N}[\ell] \leqslant t]}. \end{cases}$$

On the set A_N, we have by definition

$$\langle \zeta_N, \lambda_t \rangle \geqslant \Gamma_N[\langle \zeta_N, \lambda \rangle]_t - N^{-\gamma}, \quad t \geqslant 0. \tag{6.6}$$

As $\langle \zeta_N, \lambda \rangle$ is nonnegative and Γ_N is monotone, this implies for all $t \geqslant 0$

$$\langle \zeta_N, \lambda_t \rangle \geqslant \Gamma_N[-N^{-\gamma}]_t - N^{-\gamma} = \tilde{\Gamma}_N[0]_t - N^{-\gamma}$$

Using the monotonicity of Γ_N and applying (6.6) again, this leads to

$$\langle \zeta_N, \lambda_t \rangle \geqslant \Gamma_N[\tilde{\Gamma}_N[0] - N^{-\gamma}]_t - N^{-\gamma} = \tilde{\Gamma}^{(2)}_N[0]_t - N^{-\gamma}$$

for all $t \geqslant 0$. A straightforward induction shows that

$$\langle \zeta_N, \lambda_t \rangle \geqslant \tilde{\Gamma}^{(k)}_N[0]_t - N^{-\gamma}, \quad t \geqslant 0$$

holds for all $k \in \mathbb{N}$ on the event A_N. Lemma 3.1 shows that $\alpha \tilde{\Gamma}^{(N)}_N[0] = \underline{\tilde{\Lambda}}^N$ and so, finally we obtain that on A_N we have

$$\alpha \langle \zeta_N, \lambda \rangle \geqslant \underline{\tilde{\Lambda}}^N - \alpha N^{-\gamma}. \tag{6.7}$$

As $\underline{\tilde{\Lambda}}^N$ is bounded by α, we find

$$0 \leqslant \mathbb{E}\big[\underline{\tilde{\Lambda}}^N\big] - \mathbb{E}\big[\underline{\tilde{\Lambda}}^N \mathbb{1}_{A_N}\big] = \mathbb{E}\big[\underline{\tilde{\Lambda}}^N \mathbb{1}_{A_N^c}\big] \leqslant \alpha \mathbb{P}\left(A_N^c\right) \tag{6.8}$$

which goes to zero as $N \to \infty$ by (6.5). Conditions (6.8) and (6.7) yield

$$\begin{aligned} \limsup_{N\to\infty} \mathbb{E}\big[\underline{\tilde{\Lambda}}^N_t\big] &= \limsup_{N\to\infty} \mathbb{E}\big[\underline{\tilde{\Lambda}}^N_t \mathbb{1}_{A_N}\big] \leqslant \limsup_{N\to\infty} \mathbb{E}\big[\alpha \langle \zeta_N, \lambda_t \rangle \mathbb{1}_{A_N}\big] \\ &\leqslant \limsup_{N\to\infty} \alpha \mathbb{E}\left[c_N(\zeta_N)\right] = \limsup_{N\to\infty} \alpha V^N_t. \end{aligned}$$

Combining Step 1 and Step 2 then yields the result. □

By virtue of Proposition 6.1, we are now in a position to prove a propagation of chaos result for the perturbed particle system. We obtain this result without imposing any restrictions on the distribution of the initial condition X_{0-}.

Theorem 6.2 (Propagation of chaos, perturbed)**.** *Fix $\gamma \in (0, 1/2)$ and define $(\underline{\tilde{X}}^N, \underline{\tilde{\Lambda}}^N)$ to be the minimal solution to the perturbed particle system (6.1). Let $\underline{\mu}$ be the law of the minimal solution process $\underline{X}$ to the McKean–Vlasov problem (1.2). Then, it holds that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^N \delta_{\underline{\tilde{X}}^{i,N}} = \underline{\mu}$$

in probability on $\mathcal{P}(D([-1,\infty))$. Furthermore, the sequence of loss functions $(\underline{\tilde{\Lambda}}^N)_{N\in\mathbb{N}}$ converges to $\underline{\Lambda}$ in probability with respect to the Lévy-metric, i.e., for every $\varepsilon > 0$ it holds that

$$\lim_{N\to\infty} \mathbb{P}\left(d_L(\underline{\tilde{\Lambda}}^N, \underline{\Lambda}) > \varepsilon\right) = 0,$$

where d_L denotes the Lévy-metric.

Proof. We follow the classical sequence of arguments, showing tightness of the empirical measures first, then their convergence to the law of a solution's process to the McKean–Vlasov problem, and finally we check that the corresponding solution is minimal.

Step 1: Since the sequence $(X^1_{0-}+N^{-\gamma})_{N\in\mathbb{N}}$ is tight, tightness of the empirical measures follows from Corollary 4.5.

Step 2: Let $\underline{\tilde{\mu}}_N$ be the empirical measures associated to the minimal solution to (6.1). Then by Step 1 there is a random variable μ such that, after passing to a subsequence if necessary, $\mathrm{law}(\underline{\tilde{\mu}}_N) \to \mathrm{law}(\mu)$. Moreover, because of Varadarajan's theorem (cf (Dudley, 2018, Theorem 11.4.1)) and the fact that uniformly continuous functions are convergence determining (cf. Proposition 3.4.4 in Ethier and Kurtz (2009)) we can establish that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \delta_{X^i_{0-}+N^{-\gamma}} = \nu_{0-}$$

holds almost surely in $\mathcal{P}(\mathbb{R})$. Proposition 5.6 then shows that μ coincides almost surely with the law of a solution process to the McKean–Vlasov problem (1.2).

Step 3: We now show that $\mu = \underline{\mu}$ almost surely, i.e. that μ coincides with law of the minimal solution process to (1.2). Let $J \subseteq [0,\infty)$ be the countable set of discontinuities of the increasing map $t \mapsto \mathbb{E}\left[\langle \mu, \lambda_t\rangle\right]$ and fix $t \notin J$. Then, arguing as in the proof of Proposition 5.6, we obtain that $\lim_{N\to\infty}\langle \underline{\tilde{\mu}}_N, \lambda_t\rangle = \langle \mu, \lambda_t\rangle$ almost surely. To be precise the convergence holds for a subsequence of a representative sequence of $(\underline{\tilde{\mu}}_N)_{n\in\mathbb{N}}$ but we can assume without loss of generality that it coincides with the original one. Letting D be a countable, dense subset of $[0,\infty)$ with $D \cap J = \varnothing$, we then have that

$$\lim_{N\to\infty} \langle \underline{\tilde{\mu}}_N, \lambda_t\rangle = \langle \mu, \lambda_t\rangle, \quad t \in D \tag{6.9}$$

holds almost surely. By Step 2 and the definition of minimal solution, we have $\langle \mu, \lambda\rangle \geqslant \underline{\Lambda}$ almost surely. The dominated convergence theorem and Proposition 6.1 imply

$$\mathbb{E}\left[\alpha\langle \mu, \lambda_t\rangle\right] = \lim_{N\to\infty} \mathbb{E}\left[\alpha\langle \underline{\tilde{\mu}}_N, \lambda_t\rangle\right] = \lim_{N\to\infty} \mathbb{E}\big[\underline{\tilde{\Lambda}}^N_t\big] \leqslant \underline{\Lambda}_t, \quad t \in D.$$

We conclude that $\mathbb{P}\left(\underline{\Lambda}_t = \alpha\langle \mu, \lambda_t\rangle,\ t \in D\right) = 1$ and therefore we have $\underline{\Lambda} = \alpha\langle \mu, \lambda\rangle$ almost surely by right-continuity. It follows that $\mu = \underline{\mu}$ almost surely.

Step 4: By (6.9) and right continuity we know that $\mathrm{law}(\underline{\tilde{\Lambda}}^N) \to \mathrm{law}(\alpha\langle \mu, \lambda\rangle)$. Since $\mathrm{law}(\langle \mu, \lambda\rangle) = \delta_{\underline{\Lambda}/\alpha}$ by Step 3, we can conclude that $\underline{\tilde{\Lambda}}^N \to \underline{\Lambda}$ in probability on M. $\square$

Similarly to the situation in the particle system, it turns out that physical solutions to the McKean–Vlasov problem have minimal jumps, which is the content of the following proposition.

Proposition 6.3. *Suppose* (X, τ, Λ) *solves* (1.2). *Then it holds that*

$$\Delta\Lambda_t \geqslant \alpha \inf\{x > 0 \colon \mathbb{P}\left(\tau \geqslant t, X_{t-} \in [0, \alpha x]\right) < x\}$$

for any $t \geqslant 0$.

Proof. See Proposition 1.2 in Hambly et al. (2019). □

It has been established in previous works that weak limits of physical solutions of the particle system (1.3) correspond to physical solutions to the McKean–Vlasov problem (1.2). Recall that we call a solution (X, Λ) to (1.2) physical, if it satisfies the physical jump condition (1.4).

Theorem 6.4 (Physical solutions converge to physical solutions)**.** *Let* $(\hat{X}^N, \hat{\Lambda}^N)$ *be a physical solution to the particle system* (5.4), *where* $X_{0-}^{i,N} = X_{0-}^i + a_N$ *for some deterministic sequence* $(a_N)_{N\in\mathbb{N}}$ *converging to* 0 *and some iid sequence* $(X_{0-}^i)_{i\in\mathbb{N}}$ *such that* $\mathbb{E}[X_{0-}^i] < \infty$. *Then, if for some* $\hat{\Lambda} \in M$ *we almost surely have* $\hat{\Lambda}^N \to \hat{\Lambda}$ *along some subsequence in* M, *it follows that* $\hat{\Lambda}$ *is a physical solution to the McKean–Vlasov problem* (1.2).

Proof. See Section B in the appendix. □

The next theorem gives a positive answer to a conjecture of Delarue, Nadtochiy and Shkolnikov.

Theorem 6.5. *Suppose that* $\mathbb{E}\left[X_{0-}\right] < \infty$. *Then, the minimal solution* $(\underline{X}, \underline{\Lambda})$ *to the McKean–Vlasov problem* (1.2) *is physical.*

Proof. Let $(\underline{\tilde{X}}^N, \underline{\tilde{\Lambda}}^N)$ be the minimal solution to the perturbed particle system (6.1). By Theorem 6.2 we know that $\underline{\tilde{\Lambda}}^N \to \underline{\Lambda}$ in probability on M. We also know by Lemma 3.3 that $\underline{\tilde{\Lambda}}^N$ is a physical solution to the perturbed particle system. After passing to a subsequence if necessary, we may assume that $\underline{\tilde{\Lambda}}^N \to \underline{\Lambda}$ almost surely on M. Theorem 6.4 then shows that $\underline{\Lambda}$ is a physical solution to the McKean–Vlasov problem (1.2). □

Theorem 6.6 (Propagation of minimality)**.** *Suppose that* $\mathbb{E}\left[X_{0-}\right] < \infty$ *and that the physical solution to the McKean–Vlasov problem* (1.2) *is unique. Denote by* $(\underline{X}^N, \underline{\Lambda}^N)$ *the minimal solution to the particle system* (1.3). *Let* $\underline{\mu}$ *be the law of the minimal solution process* $\underline{X}$ *to the McKean–Vlasov problem* (1.2). *Then, it holds that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \delta_{\underline{X}^{i,N}} = \underline{\mu} \tag{6.10}$$

in probability on $\mathcal{P}(D([-1, \infty))$. *Furthermore, the sequence of loss functions* $(\underline{\Lambda}^N)_{N\in\mathbb{N}}$ *converges to* $\underline{\Lambda}$ *in probability with respect to the Lévy-metric, i.e., for every* $\varepsilon > 0$ *it holds that*

$$\lim_{N\to\infty} \mathbb{P}\left(d_L(\underline{\Lambda}^N, \underline{\Lambda}) > \varepsilon\right) = 0,$$

where d_L *denotes the Lévy-metric.*

Proof. We argue again in the classical way. Tightness of the empirical measures (6.10) follows from Corollary 4.5 and their convergence to the law of a solution process of the McKean–Vlasov problem (1.2) follows by Proposition 5.6. We know from Lemma 3.3 that minimal solutions of the particle system are physical, which allows us to apply Theorem 6.4, telling us that the limit must be the unique physical solution to the McKean–Vlasov problem. By Theorem 6.5, the physical solution must be equal to the minimal solution. □

Remark 6.7. By (Delarue et al., 2019, Theorem 1.4), if the initial condition X_{0-} admits a bounded Lebesgue density that changes monotonicity finitely often on compact subsets of $[0,\infty)$, the physical solution to the McKean–Vlasov problem (1.2) is unique.

6.2 The perturbed McKean–Vlasov problem

A modification of the proof of Proposition 6.1 allows us to "shift" the perturbation from the particle system to the McKean–Vlasov problem.

Proposition 6.8. *Fix $\alpha > 0$, $\gamma \in (0, 1/2)$, and for $x \in \mathbb{R}$ let $\underline{\Lambda}(x)$ denote the minimal solution of the perturbed McKean–Vlasov problem*

$$\begin{cases} X_t = X_{0-} - x + B_t - \Lambda_t \\ \tau = \inf\{t \geqslant 0 : X_t \leqslant 0\} \\ \Lambda_t = \alpha \mathbb{P}(\tau \leqslant t) . \end{cases} \tag{6.11}$$

If $\underline{\Lambda}^N$ is the minimal solution to the particle system (1.3), then

$$\limsup_{N\to\infty} \mathbb{E}\left[\underline{\Lambda}_t^N\right] \leqslant \limsup_{N\to\infty} \underline{\Lambda}_t(N^{-\gamma}), \tag{6.12}$$

for every $t > 0$.

Proof. The proof is analogous to the proof of Proposition 6.1, only that now we consider the sequence of cost functionals

$$c_N(\mu) := \langle \mu, \lambda_t \rangle + N^\beta \| \Gamma_N[\langle \mu, \lambda \rangle + N^{-\gamma}] - \langle \mu, \lambda \rangle \|_\infty$$

where $\beta = \gamma + \varepsilon$. □

Shifting the perturbation to the McKean–Vlasov problem allows us to show that propagation of minimality holds true for Lebesgue almost every (fixed) additive perturbation of the initial condition.

Theorem 6.9 (Propagation of chaos, almost everywhere)**.** *For $x \in \mathbb{R}$, let $(\underline{X}^N(x), \underline{\Lambda}^N(x))$ be the minimal solution to the particle system*

$$\begin{cases} X_t^{i,N} = X_{0-}^i - x + B_t^i - \Lambda_t^N \\ \tau_{i,N} = \inf\{t \geqslant 0 : X_t^{i,N} \leqslant 0\} \\ \Lambda_t^N = \dfrac{\alpha}{N} \displaystyle\sum_{i=1}^N \mathbb{1}_{[\tau_{i,N} \leqslant t]}, \end{cases}$$

Furthermore, let $\underline{\mu}(x)$ be the law of the minimal solution process $\underline{X}(x)$ to the perturbed McKean–Vlasov problem (6.11). Then, there is a co-countable set $D \subseteq \mathbb{R}$ such that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \delta_{\underline{X}^{i,N}(x)} = \underline{\mu}(x)$$

in probability on $\mathcal{P}(D([-1,\infty))$ for $x \in D$. Furthermore, the sequence of loss functions $(\underline{\Lambda}^N(x))_{N\in\mathbb{N}}$ converges to $\underline{\Lambda}(x)$ in probability with respect to the Lévy-metric for every $x \in D$, i.e., for every $\varepsilon > 0$ it holds that

$$\lim_{N\to\infty} \mathbb{P}\left(d_L(\underline{\Lambda}^N(x), \underline{\Lambda}(x)) > \varepsilon\right) = 0,$$

where d_L denotes the Lévy-metric.

Proof. Let D_0 be a countable dense subset of $[0,\infty)$. Fix $t \in D_0$ and note that the map $x \mapsto \underline{\Lambda}_t(x)$ is increasing and therefore has at most countably many discontinuities. Let the set of all such discontinuities be denoted by J_t. Then, $J := \bigcup_{t\in D_0} J_t$ is countable, and we set $D := [0,\infty)\backslash J$. Then, for all $x \in D$ it follows from Proposition 6.8 that we have

$$\limsup_{N\to\infty} \mathbb{E}\left[\underline{\Lambda}_t^N(x)\right] \leqslant \limsup_{N\to\infty} \underline{\Lambda}_t(x + N^{-\gamma}) = \underline{\Lambda}_t(x), \quad t \in D_0.$$

The remainder of the proof is analogous to the proof of Theorem 6.2. $\square$

Following the proof of Theorem 6.9 we can see that if we would have stability of the minimal solution to the McKean–Vlasov problem under additive perturbations of the initial condition, we would obtain propagation of minimality as in Theorem 6.6 without having to assume that the physical solution to the McKean–Vlasov problem is unique. We conjecture that such a stability result holds true.

Conjecture 6.10. The map $x \mapsto \underline{\Lambda}(x)$ is continuous from $\mathbb{R}$ to M.

Appendices

A On the M_1- and J_1-topologies

A.1 Why the J_1-topology is ill-suited to the problem

Of the four topologies initally proposed by Skorokhod, the J_1-topology is the most popular, and often simply referred to as “the Skorokhod topology”. However, for the present purpose, the J_1-topology seems to be simply too strong - in particular, local accumulations of small jumps can obstruct convergence in the J_1-topology. We illustrate this point with an example, for which we need the following theorem.

Theorem A.1. *Let ℓ^n, ℓ be increasing càdlàg functions on $[0,\infty)$. Then, $\ell^n \to \ell$ as $n \to \infty$ in the J_1-topology if and only if there is a dense subset $D \subseteq [0,\infty)$ consisting of continuity points of ℓ such that for all $t \in D$*

(i) $\ell_t^n \to \ell_t$

(ii) $\sum_{s \leqslant t} |\Delta \ell_s^n|^2 \to \sum_{s \leqslant t} |\Delta \ell_s|^2$.

Proof. See Theorem VI.2.15 in Jacod and Shiryaev (2013) □

Example A.2. For $n \in \mathbb{N}$, let ℓ^n be an increasing step function such that ℓ^n has n jumps of size $1/n$ in the interval $[1/2 - 1/n, 1/2 + 1/n]$ and is constant otherwise. Interpreting ℓ_t^n as the proportion of banks that defaults up to time t, this would mean that for all $n \in \mathbb{N}$, all of the banks in the system default after time $1/2 - 1/n$ and before time $1/2 + 1/n$. Certainly, as n goes to infinity, we would expect the limit to be the function $\ell_t = \mathbb{1}_{[1/2,\infty]}(t)$, which corresponds to all the banks defaulting at time $1/2$. Assuming that ℓ^n converges to some function g in the J_1-topology, condition (i) in Theorem A.1 yields that $g(t) = \mathbb{1}_{[1/2,\infty]}(t)$. However, as $\sum_{s \leqslant t} |\Delta \ell_s^n|^2 = \frac{1}{n}$, condition (ii) of Theorem A.1 yields that g must be continuous, a contradiction.

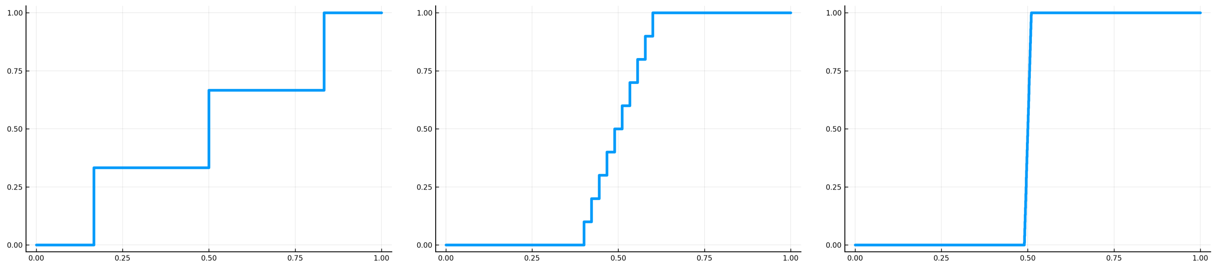

Figure 2: The functions ℓ^n as defined in Example A.2 for $n = 3, 10, 100$.

Roughly speaking, the J_1-topology allows for some flexibility in the location of jumps in convergent sequences, while requiring that the size of the jumps in the approximating sequence remains close to the size of the jumps in the limit in a certain sense. With this intuition in mind, it is not surprising that the space of continuous functions is closed in the space of càdlàg functions endowed with the J_1-topology. In contrast, continuous functions are dense in the space of càdlàg functions endowed with the M_1-topology, and the sequence given in Example A.2 is convergent in the M_1-topology.

A.2 Some properties of the M_1-topology

The M_1-topology is strictly weaker than the J_1-topology (Example A.2 serves as an example of a sequence which converges in M_1 but not in J_1 in view of Lemma A.3). This is not necessarily a weakness: whenever we want to show tightness, a weaker topology is favorable because the conditions for compactness are less strict. We will see that the M_1-topology is particularly well-suited to deal with monotone functions in this context.

We mention some fundamental properties of the M_1-topology in the following. For a path $x \in D([T_0, T])$, the space of càdlàg paths defined on $[T_0, T]$ taking values in $\mathbb{R}$ that are left-continuous at T, we define the completed graph

$$\mathscr{G}_x := \{(t, y) \in [T_0, T] \times \mathbb{R} : y \in [x_{t-}, x_t]\},$$

where $[x_{t-}, x_t]$ is the non-ordered segment between x_{t-} and x_t (this takes into account that x_{t-} might be larger than x_t). The set $\mathscr{G}_x$ can be ordered in the following way:

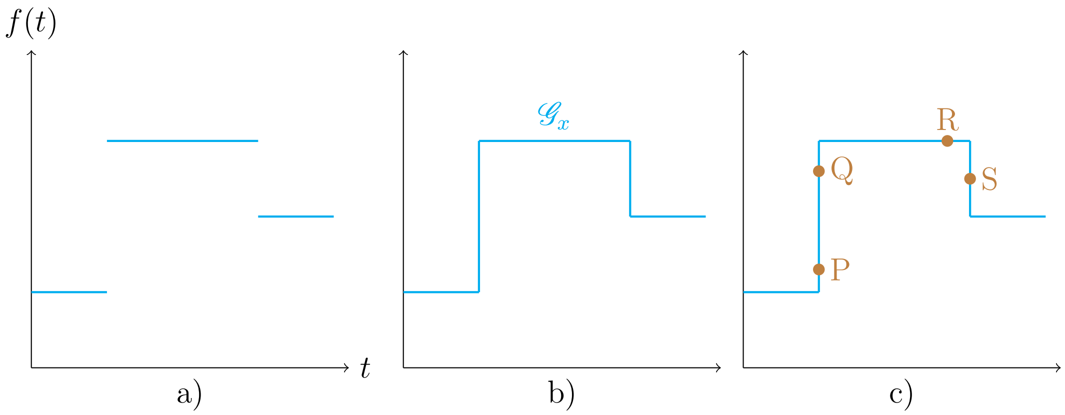

Figure 3: In a), the graph of a piecewise constant path x is plotted, b) shows the completion of its graph $\mathscr{G}_x$. In c), four points on the completed graph are marked to illustrate the order on the graph: For $A, B \in \mathscr{G}_x$, we have $A \preceq B$ iff A is reached before B when $\mathscr{G}_x$ is traced out from left to right. In the situation depicted in c), we find $P \prec Q \prec R \prec S$.

for $(t_1, y_1), (t_2, y_2) \in \mathscr{G}_x$, we say that $(t_1, y_1) \preceq (t_2, y_2)$ if either $t_1 < t_2$, or $t_1 = t_2$ and $|x_{t_1-} - y_1| \leqslant |x_{t_2-} - y_2|$. This order can be conceptualized more easily in the following way. The completion $\mathscr{G}_x$ can be imagined as a path in 2-dimensional space, where we complete the graph of x by connecting the discontinuities with straight lines going through the points (t, x_{t-}) and (t, x_t). We can imagine a particle traveling along $\mathscr{G}_x$ from left to right; for $A, B \in \mathscr{G}_x$, we have $A \preceq B$ iff the particle reaches A before it reaches B. This is illustrated in Figure A.2.

We then define a *parametric representation* of $\mathscr{G}_x$ to be a continuous function (r, u) that maps $[T_0, T]$ onto $\mathscr{G}_x$ such that $t \mapsto (r(t), u(t))$ is increasing with respect to $(\mathscr{G}_x, \preceq)$. Denote the set of all parametric representations of x as R_x.

For $x^1, x^2 \in D([T_0, T])$, we define the M_1-metric as

$$d_T^{M_1}(x^1, x^2) = \inf_{\substack{(r^j, u^j) \in R_{x^j} \\ j=1,2}} \max(\|r^1 - r^2\|_\infty, \|u^1 - u^2\|_\infty), \tag{A.1}$$

where $\|.\|_\infty$ is the supremum norm on $C([T_0, T])$. If we want to relate this to the picture with particles described earlier, we let the particles travel along the respective completed graphs and are allowed some freedom in choosing the velocities of the particles (albeit, due to the requirement that the parametric representations are increasing with respect to $(\mathscr{G}_x, \preceq)$, the velocities can never be negative). Then, two functions are close to each other in the M_1-topology, if there are velocity profiles for the particles such that the vertical and horizontal distance between the particles remains uniformly small.

We provide some fundamental results regarding the M_1-topology which play a crucial role in many proofs of this paper.

The space $D([T_0, T])$ endowed with the M_1-topology is a Polish space, even though the metric defined in (A.1) is incomplete. It is generally not very pleasant to work directly with the metric defined above, and we will seek to avoid doing so whenever possible. The following theorem will prove to be expedient in this endeavor, as it allows us to relate convergence in the M_1-topology to pointwise convergence when every path in the sequence is monotone. Let M be the space defined in (2.3).

Lemma A.3. *Let $(\ell^n)_{n\in\mathbb{N}}, \ell \subset M$. Then $\ell^n \to \ell$ in the M_1-topology if and only if $\ell^n_t \to \ell_t$ for each t in a subset of full Lebesgue measure of $[T_0, T]$ that includes T_0 and T.*

Proof. See (Whitt, 2002, Theorem 12.5.1). □

Convergence in the uniform norm implies convergence in the M_1-topology. The next theorem shows that if the limit path is continuous, the converse holds as well.

Lemma A.4. *Suppose that $x^n \to x$ in $D([T_0, T])$ equipped with the M_1-topology. Then we have locally uniform convergence at all continuity point of x. In particular, for all points t at which x is continuous it holds that $x^n_t \to x_t$.*

Proof. See (Whitt, 2002, Theorem 12.4.1). □

Similarly to the J_1-topology, the M_1-topology does not turn $D([T_0, T])$ into a topological vector space. In particular, addition is not continuous in general. However, the following result holds.

Lemma A.5. *Assume that $x^n \to x$ and $y^n \to y$ in $D([T_0, T])$ equipped with the M_1-topology. If x and y have no common jumps of opposite sign, that is*

$$\Delta x_t \cdot \Delta y_t \geqslant 0, \quad t \in [T_0, T],$$

then $x^n + y^n \to x + y$ in $D([T_0, T])$.

Proof. See (Whitt, 2002, Theorem 12.7.3). □

If $(\ell_n)_{n\in\mathbb{N}}$ is a sequence of distribution functions such that $\ell_n \to \ell$ in M, Lemma A.3 tells us that $\ell_n \to \ell$ in the M_1-topology if and only if ℓ^n converges pointwise to ℓ in the interval endpoints. To remove this restriction of convergence in the interval endpoints, we will consider processes on all of $[0, \infty)$, and (somewhat artifically) extend the domain to $[-1, \infty)$, where we let the processes stay constant for all negative times. Following Whitt (2002), we need to define the M_1-metric on noncompact domains.

Definition A.6. We say that $x^n \to x$ in $D([T_0, \infty))$ if x^n converges to x in $D([T_0, T_k])$ for each T_k in some sequence $(T_k)_{k\in\mathbb{N}}$ with $T_k \to \infty$, where the sequence $(T_k)_{k\in\mathbb{N}}$ may depend on x. For $t > 0$, let $\hat{d}_t$ be a metric that makes $D([T_0, t])$ complete, then $D([T_0, \infty))$ is metrized by

$$d^{M1}_{[T_0,\infty]}(x^1, x^2) := \int_{T_0}^{\infty} e^{-t}(\hat{d}_t(x^1, x^2) \wedge 1)\ \mathrm{d}t.$$

Equipped with this metric, $D([T_0, \infty))$ is a Polish space by construction. We have the following equivalence.

Lemma A.7. *For $x^n, x \in D([T_0, \infty))$, convergence of x^n against x with respect to $d^{M_1}_{[T_0,\infty]}$ is equivalent to convergence of x^n against x with respect to $\hat{d}_t$ for all $t > T_0$ where x is continuous.*

Proof. See (Whitt, 2002, Theorem 12.9.3). □

Lemma A.8. *The Borel σ-field of $D([T_0, \infty))$ endowed with the M_1-topology is generated by the evaluation mappings.*

Proof. By (Jacod and Shiryaev, 2013, Theorem 1.14c), the claim holds for the Borel σ-field on $D([T_0, \infty))$ generated by the J_1-topology. By definition, the J_1-topology is stronger than the M_1-topology, and as any two comparable Lusin spaces have the same Borel sets (Schwartz, 1973, p.101), the claim follows. □

Lemma A.9. *Assume that $x^n \to x$ in $D([T_0, \infty))$. Then if $t \in [T_0, \infty)$ is a continuity point of x, it holds that $\lim_{n\to\infty} \inf_{s\in[T_0,t]} x^n_s = \inf_{s\in[T_0,t]} x_s$.*

Proof. Fix a continuity point t of x and choose $T > t$ as a continuity point of x as well. Then by Lemma A.7 we have $x^n \to x$ in $D([T_0, T])$. By Theorem 13.4.1 in Whitt (2002), the map $x \mapsto \inf_{s\in[T_0,\cdot]} x_s$ is continuous from $D([T_0, T])$ to $D([T_0, T])$, and therefore the claim follows from Lemma A.4. □

B Proofs regarding physical solutions

We start by introducing some useful notation.

Definition B.1. If (X, τ, Λ) is a solution to the McKean–Vlasov problem (1.2), we denote by ν_{t-} the marginal subprobability distribution at time $t-$ of the particles surviving up to time t, i.e.,

$$\nu_{t-}(A) := \mathbb{P}\left(\tau \geqslant t, X_{t-} \in A\right), \quad t \geqslant 0$$

and we denote the measure corresponding to the minimal solution as $\underline{\nu}$.

The next technical lemma is a key result when it comes to showing that physical solutions of the particle system converge to physical solutions of the McKean–Vlasov problem. Roughly speaking, it says that there is a very small chance of observing more than one jump of a macroscopic proportion of particles in a small interval and can be seen as an extension of (3.7) to small intervals. This result in its original form is due to (Delarue et al., 2015a, Proposition 5.3). We follow the proof given in (Ledger and Søjmark, 2021, Lemma 3.10) here.

Lemma B.2. *Suppose that $\mathbb{E}\left[X_{0-}\right] < \infty$ and fix $T > 0$. Let $(\hat{X}^N, \hat{\Lambda}^N)$ be a physical solution to the particle system (1.3) and let ν^N_{t-} be the corresponding subprobabilty measure as defined in Definition 3.2. Then there is a constant $C > 0$ such that for every (sufficiently small) $\varepsilon > 0$*

$$\mathbb{P}\left(\nu^N_{t-}([0, \alpha z + C\varepsilon^{1/3}]) \geqslant z \quad \forall z \leqslant \frac{1}{\alpha}(\hat{\Lambda}^N_{t+\varepsilon} - \hat{\Lambda}^N_{t-}) - C\varepsilon^{1/3}\right) \geqslant 1 - C\varepsilon^{1/3}, \quad t < T,$$

whenever $N \geqslant \varepsilon^{-1/3}$.

Proof of Lemma B.2. Note that $\frac{N}{\alpha}(\hat{\Lambda}^N_{t+\varepsilon} - \hat{\Lambda}^N_{t-})$ equals the number of particles defaulting in the interval $[t, t+\varepsilon]$. By definition, as $\hat{\Lambda}^N$ is a physical solution, if t_0 is any jump time in $[t, t+\varepsilon]$, we must have

$$\nu^N_{t_0-}\left(\left[0, \alpha\frac{k}{N}\right]\right) \geqslant \frac{k}{N} \quad \text{for} \quad k = 0, 1, \ldots, \frac{N}{\alpha}\Delta\hat{\Lambda}^N_{t_0}.$$

Let $t_1, \ldots, t_m$ be the jump times of $\hat{\Lambda}^N$ in $[t, t+\varepsilon]$. If $k \leqslant \frac{N}{\alpha}(\hat{\Lambda}^N_{t+\varepsilon} - \hat{\Lambda}^N_{t-})$, then there are numbers $k_1, \ldots, k_m$ such that $k_i \leqslant \frac{N}{\alpha}\Delta\hat{\Lambda}^N_{t_i}$ and $k_1 + \cdots + k_m = k$, which then shows that

$$\sum_{i=1}^m \nu^N_{t_i-}\left(\left[0, \alpha\frac{k_i}{N}\right]\right) \geqslant \frac{1}{N}\sum_{i=1}^m k_i = \frac{k}{N}.$$

Now by definition, the number on the left-hand side of the above inequality is dominated by $\frac{1}{N}\sum_{i=1}^N \mathbb{1}_{E_1^{i,k}}$ where

$$E_1^{i,k} := \big\{\hat{X}^{i,N}_{t-} - \alpha\frac{k}{N} - \varepsilon(1 + \sup_{s\leqslant t+\varepsilon}|\hat{X}^{i,N}_s|) - \sup_{h\leqslant\varepsilon}|B^i_{t+h} - B^i_t| \leqslant 0,\ \tau^{i,N} \geqslant t\big\}.$$

We have obtained that for $k = 0, 1, \ldots, \frac{N}{\alpha}(\hat{\Lambda}^N_{t+\varepsilon} - \hat{\Lambda}^N_{t-})$

$$\frac{1}{N}\sum_{i=1}^N \mathbb{1}_{E_1^{i,k}} \geqslant \frac{k}{N}. \tag{B.1}$$

Now fix $z \in \mathbb{R}$ such that $z \leqslant \frac{1}{\alpha}(\hat{\Lambda}^N_{t+\varepsilon} - \hat{\Lambda}^N_{t-}) - 2\varepsilon^{1/3}$ and set $k_0 := \lfloor z + 2\varepsilon^{1/3}\rfloor \leqslant \frac{N}{\alpha}(\hat{\Lambda}^N_{t+\varepsilon} - \hat{\Lambda}^N_{t-})$. This choice implies that $z \geqslant \frac{k_0}{N} - 2\varepsilon^{1/3}$ as well as $\frac{k_0}{N} \geqslant z + 2\varepsilon^{1/3} - \frac{1}{N}$ by definition, and we also have that (B.1) holds for $k = k_0$. Define

$$E_2^i := \big\{\varepsilon(1 + \sup_{s\leqslant t+\varepsilon}|\hat{X}^{i,N}_s|) + \sup_{h\leqslant\varepsilon}|B^i_{t+h} - B^i_t| \geqslant \varepsilon^{1/3}\big\}.$$

On the event $E_1^{i,k_0} \cap (E_2^i)^c$, we have $\hat{X}^{i,N}_{t-} - \alpha\frac{k_0}{N} \leqslant \varepsilon^{1/3}$, and hence

$$\hat{X}^{i,N}_{t-} - \alpha z \leqslant \hat{X}^{i,N}_{t-} - \alpha\left(\frac{k_0}{N} - 2\varepsilon^{1/3}\right) \leqslant (1 + 2\alpha)\varepsilon^{1/3}.$$

It follows that

$$\nu^N_{t-}([0, \alpha z + (1 + 2\alpha)\varepsilon^{1/3}]) \geqslant \frac{1}{N}\sum_{i=1}^N \mathbb{1}_{E_1^{i,k_0}}\mathbb{1}_{(E_2^i)^c}.$$

Finally, letting $E := \big\{\frac{1}{N}\sum_{i=1}^N \mathbb{1}_{E_2^i} \leqslant \varepsilon^{1/3}\big\}$, we find that on the event E we have

$$\begin{aligned}\nu^N_{t-}([0, \alpha z + (1 + 2\alpha)\varepsilon^{1/3}]) &\geqslant \frac{1}{N}\sum_{i=1}^N \mathbb{1}_{E_1^{i,k_0}} - \frac{1}{N}\sum_{i=1}^N \mathbb{1}_{E_2^i} \geqslant \frac{k_0}{N} - \varepsilon^{1/3}\\ &\geqslant (z + 2\varepsilon^{1/3} - 1/N) - \varepsilon^{1/3} = z + \varepsilon^{1/3} - 1/N.\end{aligned}$$

Taking $N \geqslant \varepsilon^{-1/3}$ implies $\varepsilon^{1/3} - 1/N \geqslant 0$, so we conclude that

$$\nu_{t-}^N([0, \alpha z + (1+2\alpha)\varepsilon^{1/3}]) \geqslant z \quad \text{on} \quad E,$$

for any $z \leqslant \frac{1}{\alpha}(\hat{\Lambda}_{t+\varepsilon}^N - \hat{\Lambda}_{t-}^N) - 2\varepsilon^{1/3}$. It remains to prove that there is a $C > 0$ such that $\mathbb{P}(E^c) \leqslant C\varepsilon^{1/3}$. To do that, note that $\mathbb{E}\big[\sup_{h\leqslant\varepsilon} |B_{t+h}^i - B_t^i|\big] \leqslant \sqrt{\pi/2\varepsilon}$ and

$$\mathbb{E}\big[1 + \sup_{s\leqslant T} |\hat{X}_s^{i,N}|\big] \leqslant 1 + \mathbb{E}[X_{0-}] + \mathbb{E}\big[\sup_{s\leqslant T} |B_s|\big] + \alpha \leqslant 1 + \alpha + \mathbb{E}[X_{0-}] + \sqrt{\pi/2T}.$$

Now letting $c \geqslant 1 + \alpha + \mathbb{E}[X_{0-}] + \sqrt{\frac{\pi}{2}}(1+T)$ we find, by applying the Markov inequality twice

$$\begin{aligned}\mathbb{P}(E^c) = \mathbb{P}\Big(\frac{1}{N}\sum_{i=1}^N \mathbb{1}_{E_2^i} > \varepsilon^{1/3}\Big) &\leqslant \varepsilon^{-1/3}\frac{1}{N}\sum_{i=1}^N \mathbb{P}(E_2^i) \\ &\leqslant \varepsilon^{1/3}(\mathbb{E}\big[1 + \sup_{s\leqslant T}|\hat{X}_s^{i,N}|\big] + \mathbb{E}\big[\sup_{h\leqslant\varepsilon}|B_{t+h}^i - B_t^i|\big]) \leqslant 2c\varepsilon^{1/3}.\end{aligned}$$

Now $C := \max(2c, (1+2\alpha))$ satisfies the requirements of the lemma. □

Proof of Theorem 6.4. Let $\hat{\mu}_N$ be the empirical measure corresponding to $\hat{X}^N$. From Corollary 4.6 we know that there are random variables $\hat{\xi}, \hat{\xi}_N$ such that, after passing to subsequences if necessary, $\text{law}(\hat{\mu}_N) = \text{law}(\iota_\alpha(\hat{\xi}_N))$, $\hat{\xi}_N \to \hat{\xi}$, and $\iota_\alpha(\hat{\xi}_N) \to \iota_\alpha(\hat{\xi})$ almost surely. Without loss of generality we may assume that $\hat{\mu}_N = \iota_\alpha(\hat{\xi}_N)$ and set $\hat{\mu} = \iota_\alpha(\hat{\xi})$. Note that this implies $\hat{\Lambda} = \alpha\langle\hat{\mu}, \lambda\rangle$ and thus, by Proposition 5.6, that $\hat{\Lambda}$ solves the McKean–Vlasov problem (1.2). In the following three steps we prove that $\hat{\Lambda}$ is physical by verifying Condition (1.4).

Step 1: We show convergence of the laws of the subprobability measures $\hat{\nu}_{t-}^N$. For $t \geqslant 0$, let $\pi_{t-}(x)$ denote the map $x \mapsto x_{t-}$ and define the transformation $S\colon D([-1,\infty)) \to \mathbb{R}$ through $S_{t-}(x) = \pi_{t-}(x)(1 - \lambda_{t-}(x))$. Note that for $f \in C_b(\mathbb{R})$ and $\mu \in \mathcal{P}(D([-1,\infty)))$ it holds that

$$\langle S_{t-}(\mu), f\rangle = \int f(x_{t-}\mathbb{1}_{[\tau_0(x)\geqslant t]})\,\mathrm{d}\mu(x) = \int f(x_{t-})\mathbb{1}_{[\tau_0(x)\geqslant t]}\,\mathrm{d}\mu(x) + f(0)\langle\mu, \lambda_{t-}\rangle.$$

It follows that

$$\langle\hat{\nu}_{t-}^N, f\rangle = \langle S_{t-}(\hat{\mu}_N), f\rangle - \frac{f(0)}{\alpha}\hat{\Lambda}_t^N, \quad \langle\hat{\nu}_{t-}, f\rangle = \langle S_{t-}(\hat{\mu}), f\rangle - \frac{f(0)}{\alpha}\hat{\Lambda}_t.$$

Let J be the set of discontinuity points of $\hat{\Lambda}$. We claim that $S_{t-} \circ \iota_\alpha$ is continuous at $\hat{\xi}$-almost every $(w, \ell) \in \bar{E}$ whenever $t \notin J$. To see this, consider that π_{t-} is continuous at all paths $x \in D([-1,\infty))$ such that t is a continuity point of x by Lemma A.4. Secondly, suppose that for $x = \iota_\alpha(w, \ell)$, the measure δ_x satisfies the crossing property (5.3). Then, applying Lemma 5.3 for $\xi = \delta_{(w,\ell)}$ and for a sequence $(\xi^n)_{n\in\mathbb{N}}$ converging to ξ with $\xi^n = \delta_{(w^n,\ell^n)}$, it follows that $\lambda_t \circ \iota_\alpha$ is continuous at (w, ℓ) except possibly for $t = \tau_0(x)$. The same holds for $\lambda_{t-} \circ \iota_\alpha$. We have

$$\hat{\xi}(\{(w,\ell) \in \bar{E} : \tau_0(\iota_\alpha(w,\ell)) = t\}) = \hat{\mu}(\{x \in D([-1,\infty))\colon \tau_0(x) = t\}) = \mathbb{P}(\hat{\tau} = t)$$

and by definition, $\mathbb{P}(\hat{\tau} = t) > 0$ if and only if $t \in J$. Since for $x = \iota_\alpha(w, \ell)$, the measure δ_x satisfies the crossing property for $\hat{\xi}$-almost every $(w, \ell) \in \bar{E}$, we have thus proved the aforementioned continuity property of $S_{t-} \circ \iota_\alpha$. It follows from the Portmanteau theorem that for every $t \notin J$

$$\lim_{N\to\infty} S_{t-}(\hat{\mu}_N) = \lim_{N\to\infty} (S_{t-} \circ \iota_\alpha)(\hat{\xi}_N) = (S_{t-} \circ \iota_\alpha)(\hat{\xi}) = S_{t-}(\hat{\mu}),$$

almost surely on $\mathcal{P}(\mathbb{R})$. Therefore we obtain, for $t \notin J$,

$$\lim_{N\to\infty} \hat{\nu}^N_{t-} = \hat{\nu}_{t-},$$

almost surely on $\mathcal{S}(\mathbb{R})$, where $\mathcal{S}(\mathbb{R})$ is the space of subprobability measures on $\mathbb{R}$ endowed with the topology of weak convergence.

Step 2: Fix $T > 0$ and a sufficiently small $\varepsilon > 0$. We take the limit as $N \to \infty$ of

$$\mathbb{P}\left(\hat{\nu}^N_{t-}([0, \alpha z + C\varepsilon^{1/3}]) \geqslant z \quad \forall z \leqslant \frac{1}{\alpha}(\hat{\Lambda}^N_{t+\varepsilon} - \hat{\Lambda}^N_{t-}) - C\varepsilon^{1/3}\right) \geqslant 1 - C\varepsilon^{1/3}, \qquad t < T. \tag{B.2}$$

The above equation holds due to Lemma B.2, where we make use of the assumption $\mathbb{E}[X_{0-}] < \infty$. Fix $t, t+\varepsilon \in [0,T)\backslash J$ and $z \in \mathbb{R}$ such that $z < \frac{1}{\alpha}(\hat{\Lambda}_{t+\varepsilon} - \hat{\Lambda}_{t-}) - C\varepsilon^{1/3}$. Introduce the events

$$A^N_z := \left\{z \leqslant \frac{1}{\alpha}(\hat{\Lambda}^N_{t+\varepsilon} - \hat{\Lambda}^N_{t-}) - C\varepsilon^{1/3}\right\}.$$

By assumption, we know that $\hat{\Lambda}^N_{t+\varepsilon} - \hat{\Lambda}^N_{t-} \to \hat{\Lambda}_{t+\varepsilon} - \hat{\Lambda}_{t-}$, and therefore it holds that $\lim_{N\to\infty} \mathbb{P}\left(A^N_z\right) = 1$. Recalling Step 1 and (B.2), on applying the Portmanteau theorem and the reverse Fatou lemma we find

$$\begin{aligned}\hat{\nu}_{t-}[0, \alpha z + C\varepsilon^{1/3}] &\geqslant \mathbb{E}\big[\limsup_{N\to\infty} \hat{\nu}^N_{t-}[0, \alpha z + C\varepsilon^{1/3}]\big] \geqslant \limsup_{N\to\infty} \mathbb{E}\left[\hat{\nu}^N_{t-}[0, \alpha z + C\varepsilon^{1/3}]\right] \\ &\geqslant \limsup_{N\to\infty} \mathbb{E}\left[\hat{\nu}^N_{t-}[0, \alpha z + C\varepsilon^{1/3}]\mathbb{1}_{A^N_z}\right] \geqslant z(1 - C\varepsilon^{1/3}).\end{aligned}$$

We have established

$$\hat{\nu}_{t-}[0, \alpha z + C\varepsilon^{1/3}] \geqslant z(1 - C\varepsilon^{1/3}), \quad z < \frac{1}{\alpha}(\hat{\Lambda}_{t+\varepsilon} - \hat{\Lambda}_{t-}) - C\varepsilon^{1/3}, \quad t, t+\varepsilon \in [0,T)\backslash J.$$

Step 3: We take the limit as $\varepsilon \to 0$. To that end, consider that for $f \in C_b(\mathbb{R})$, by the dominated convergence theorem, the map

$$t \mapsto \int f(x_{t-})\mathbb{1}_{[\tau_0(x) \geqslant t]}\, \mathrm{d}\hat{\mu}(x) = \langle \hat{\nu}_{t-}, f\rangle$$

is left-continuous, and hence we have $\lim_{s\nearrow t} \hat{\nu}_{s-} = \hat{\nu}_{t-}$ in $\mathcal{S}(\mathbb{R})$. Now fix $t \in [0,T)$ and let t_n, ε_n be such that $t_n, t_n + \varepsilon_n \notin J$ with $t_n < t < t_n + \varepsilon_n$, $t_n \nearrow t$ and $\varepsilon_n \searrow 0$. Let $0 \leqslant z <$

$\frac{1}{\alpha}\Delta\hat{\Lambda}_t$, then as $\hat{\Lambda}$ is càdlàg , for all sufficiently large n we have $z < \frac{1}{\alpha}(\hat{\Lambda}_{t_n+\varepsilon_n} - \hat{\Lambda}_{t_n-}) - C\varepsilon_n^{1/3}$. Then, for any $\delta > 0$, by the Portmanteau theorem and Step 2, we obtain

$$\begin{aligned}\hat{\nu}_{t-}[0, \alpha z + \delta] &\geqslant \limsup_{n\to\infty} \hat{\nu}_{t_n-}[0, \alpha z + \delta] \geqslant \limsup_{n\to\infty} \hat{\nu}_{t_n-}[0, \alpha z + C\varepsilon_n^{1/3}] \\ &\geqslant \limsup_{n\to\infty} z(1 - C\varepsilon_n^{1/3}) = z.\end{aligned}$$

Letting $\delta \to 0$, as $T > 0$ was arbitrary we finally see that $\hat{\nu}_{t-}[0, \alpha z] \geqslant z$ for each $z < \frac{1}{\alpha}\Delta\hat{\Lambda}_t$ and $t \geqslant 0$, which implies

$$\Delta\hat{\Lambda}_t \leqslant \alpha \inf\{z > 0 \colon \hat{\nu}_{t-}[0, \alpha z] < z\}.$$

The reverse inequality follows from Proposition 6.3. $\square$

C Supplements

Example C.1. Define

$$\Omega := C([0,\infty))^{\mathbb{N}}$$

and let ω be the the canonical process on $C([0,\infty))^{\mathbb{N}}$. We take $\mathcal{F}$ to be the product σ-field induced by the Borel σ-field on $C([0,\infty))$. Let $\mathbb{P} = \bigotimes_{n\in\mathbb{N}} \eta$ for a probability measure η on $C([0,\infty))$, such that the law of ω^i under η coincides with the law of $X_{0-} + B$ for a Brownian motion B and a random variable X_{0-}. We write $E := D([-1,\infty))$ in the following.

Let $\mathcal{R}$ be the space of all random variables $\mu\colon \Omega \to \mathcal{P}(E)$ and let $\mu \in \mathcal{R}$. As the Borel σ-field in $\mathcal{P}(E)$ is generated by the mappings $\mu \mapsto \mu(A)$ where A is a Borel set in E (cf. (Carmona and Delarue, 2018, Proposition 5.7)), any such map is measurable. In addition, Lemma A.9 shows that the map $x \mapsto \inf_{s\leqslant\cdot} x_s$ is continuous on E, and as the evaluation mappings $\pi_t := x \mapsto x_t$ are measurable, it follows that the set

$$A_t := \Big\{x \in E \colon \pi_t\big(\inf_{s\leqslant\cdot} x_s\big) \in (-\infty, 0]\Big\}$$

is a Borel set in E. We find that $\langle\mu, \lambda_t\rangle = \mu(A_t)$, so we obtain that $\omega \mapsto \langle\mu(\omega), \lambda_t\rangle$ is measurable for every $t \geqslant 0$. As the Borel σ-field on E is generated by the evaluation mappings, it follows that $\omega \mapsto \langle\mu(\omega), \lambda\rangle$ is measurable as a map into E.

By Lemma A.5, for $N \in \mathbb{N}$ the map

$$\begin{aligned}\Psi^N \colon E \times C([0,\infty))^{\mathbb{N}} &\to \mathcal{P}(E) \\ (x, \omega) &\mapsto \frac{1}{N}\sum_{i=1}^{N} \delta_{\omega^i - \alpha x}\end{aligned}$$

is continuous. It follows that

$$\Gamma_N[\langle\mu, \lambda\rangle](\omega) = \langle\Psi^N(\langle\mu(\omega), \lambda\rangle, \omega), \lambda\rangle = \Psi^N(\langle\mu(\omega), \lambda\rangle, \omega)(A_t)$$

and is thus measurable. This shows that the map $\omega \mapsto c_N(\mu(\omega))$ is measurable for every $\mu \in \mathcal{R}$. Defining

$$\mu_N^{(1)} := \Psi^N(0, \omega), \quad \mu_N^{(k)} := \Psi^N(\langle\mu_N^{(k-1)}(\omega), \lambda\rangle, \omega), \quad k \in \mathbb{N},$$

by Lemma 3.1 we find that $\mu_N = \mu_N^{(N)} \in \mathcal{R}$.

References

Graeme Baker and Mykhaylo Shkolnikov. Zero kinetic undercooling limit in the supercooled Stefan problem. 2020. To appear in Ann. Inst. Henri Poincaré Probab. Stat, arXiv preprint arXiv:2003.07239.

René Carmona and François Delarue. *Probabilistic Theory of Mean Field Games with Applications I*. Springer, 2018.

François Delarue, James Inglis, Sylvain Rubenthaler, and Etienne Tanré. Particle systems with a singular mean-field self-excitation. Application to neuronal networks. *Stochastic Processes and their Applications*, 125(6):2451–2492, 2015a.

François Delarue, James Inglis, Sylvain Rubenthaler, Etienne Tanré, et al. Global solvability of a networked integrate-and-fire model of McKean–Vlasov type. *The Annals of Applied Probability*, 25(4):2096–2133, 2015b.

François Delarue, Sergey Nadtochiy, and Mykhaylo Shkolnikov. Global solutions to the supercooled Stefan problem with blow-ups: regularity and uniqueness. 2019. To appear in Probab. Math. Phys., arXiv:1902.05174.

Jeffrey N. Dewynne. A survey of supercooled Stefan problems. In *Mini-Conference on Free and Moving Boundary and Diffusion Problems*, pages 42–56, Canberra AUS, 1992. Centre for Mathematics and its Applications, Mathematical Sciences Institute, The Australian National University. URL https://projecteuclid.org/euclid.pcma/1416323070.

Richard M. Dudley. *Real Analysis and Probability: 0*. Chapman and Hall/CRC, 2018.

Stewart N. Ethier and Thomas G. Kurtz. *Markov processes: characterization and convergence*, volume 282. John Wiley & Sons, 2009.

Zachary Feinstein and Andreas Søjmark. Dynamic default contagion in heterogeneous interbank systems. *SIAM Journal on Financial Mathematics*, 12(4):SC83–SC97, 2021.

Ben Hambly and Andreas Søjmark. An SPDE model for systemic risk with endogenous contagion. *Finance and Stochastics*, 23(3):535–594, 2019.

Ben Hambly, Sean Ledger, and Andreas Søjmark. A McKean–Vlasov equation with positive feedback and blow-ups. *Ann. Appl. Probab.*, 29(4):2338–2373, 08 2019. doi: 10.1214/18-AAP1455. URL https://doi.org/10.1214/18-AAP1455.

Jean Jacod and Albert Shiryaev. *Limit theorems for stochastic processes*, volume 288. Springer Science & Business Media, 2013.

Vadim Kaushansky and Christoph Reisinger. Simulation of a simple particle system interacting through hitting times. *Discrete and Continuous Dynamical Systems - B*, 24(10):5481–5502, 2019.

Vadim Kaushansky, Christoph Reisinger, Mykhaylo Shkolnikov, and Zhuo Qun Song. Convergence of a time-stepping scheme to the free boundary in the supercooled Stefan problem. *arXiv preprint arXiv:2010.05281*, 2020.

Achim Klenke. *Probability theory: a comprehensive course*. Springer Science & Business Media, 2013.

Gabriel Lamé and Benoît P. E. Clapeyron. Mémoire sur la solidification par refroidissement d'un globe liquide. In *Annales Chimie Physique*, volume 47, pages 250–256, 1831.

Sean Ledger and Andreas Søjmark. Uniqueness for contagious McKean–Vlasov systems in the weak feedback regime. *Bulletin of the London Mathematical Society*, 52(3):448–463, 2020. doi: 10.1112/blms.12337. URL https://londmathsoc.onlinelibrary.wiley.com/doi/abs/10.1112/blms.12337.

Sean Ledger and Andreas Søjmark. At the mercy of the common noise: blow-ups in a conditional McKean–Vlasov Problem. *Electronic Journal of Probability*, 26(none):1 – 39, 2021. doi: 10.1214/21-EJP597. URL https://doi.org/10.1214/21-EJP597.

Alexander Lipton, Vadim Kaushansky, and Christoph Reisinger. Semi-analytical solution of a McKean–Vlasov equation with feedback through hitting a boundary. *European Journal of Applied Mathematics*, pages 1–34, 2019.

Pascal Massart. The tight constant in the Dvoretzky-Kiefer-Wolfowitz inequality. *The Annals of Probability*, 18(3):1269–1283, 1990. ISSN 00911798. URL http://www.jstor.org/stable/2244426.

Sergey Nadtochiy and Mykhaylo Shkolnikov. Particle systems with singular interaction through hitting times: Application in systemic risk modeling. *Ann. Appl. Probab.*, 29 (1):89–129, 02 2019. doi: 10.1214/18-AAP1403. URL https://doi.org/10.1214/18-AAP1403.

Sergey Nadtochiy and Mykhaylo Shkolnikov. Mean field systems on networks, with singular interaction through hitting times. *Ann. Probab.*, 48(3):1520–1556, 05 2020. doi: 10.1214/19-AOP1403. URL https://doi.org/10.1214/19-AOP1403.

Goran Peskir and Albert Shiryaev. *Optimal stopping and free-boundary problems*. Springer, 2006.

Laurent Schwartz. Radon measures on arbitrary topological spaces and cylindrical measures. *Tata. Inst. Fund. Res.*, 1973.

Bernard Sherman. A general one-phase Stefan problem. *Quarterly of Applied Mathematics*, 28(3):377–382, 1970.

Josef Stefan. Über die Theorie der Eisbildung, insbesondere über die Eisbildung im Polarmeere. *Annalen der Physik und Chemie*, 42:269–286, 1891.

Alain-Sol Sznitman. Topics in propagation of chaos. In *Ecole d'été de probabilités de Saint-Flour XIX—1989*, pages 165–251. Springer, 1991.

Ward Whitt. *Stochastic-process limits: an introduction to stochastic-process limits and their application to queues*. Springer Science & Business Media, 2002.

Global implicit approximation of the supercooled Stefan problem in the presence of blow-ups

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Abstract

We consider two global implicit approximation schemes of the supercooled Stefan problem and prove their convergence, even in the presence of finite time blow-ups. All proofs are based on a probabilistic reformulation of the supercooled Stefan problem recently considered in the literature. The first scheme is a version of the time-stepping scheme considered in *V. Kaushansky, C. Reisinger, M. Shkolnikov, and Z. Q. Song, arXiv:2010.05281, 2020*, but here the flux over the free boundary and its movement are coupled implicitly. Moreover, we can extend the scheme to more general driving processes than Brownian motion, including fractional Brownian motion. The second scheme is an implicit treatment of a Donsker-type approximation for which global convergence holds under minor technical conditions, allowing in particular for regimes with blow-up. Under a strengthening of the conditions, which applies in cases without blow-ups, we obtain additionally a convergence rate. Our numerical results suggest that a similar rate also holds in the case of less regular solutions. The results illustrate a main advantage of implicit treatments, namely the sharper resolution of the jump of the free boundary in the blow-up regime.

1 Introduction

The classical PDE formulation of the one-dimensional one-phase Stefan problem is

$$
\begin{aligned}
&\partial_t u = \frac{1}{2}\partial_{xx} u, \quad x > \Lambda_t, \quad t \geqslant 0,\\
&u(t,\Lambda_t) = 0, \quad t \geqslant 0,\\
&\dot{\Lambda}_t = \frac{\alpha}{2}\partial_x u(t,\Lambda_t), \quad t \geqslant 0,\\
&u(0,x) = f(x), \quad x \geqslant 0 \quad \text{and} \quad \Lambda_0 = 0.
\end{aligned} \tag{1}
$$

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In physics terminology, $-f$ is the initial temperature in a liquid, where we will consider $f \geqslant 0$ corresponding to the supercooled regime; $-u(t,\cdot)$ is the temperature in the liquid phase at time t; and Λ_t is the location of the liquid-solid boundary at time t. The above relationship between the flux $\partial_x u$ at the free boundary Λ_t and the growth rate $\dot{\Lambda}_t$ of the solid phase is known as *Stefan condition* and expresses the energy conservation at the interface. In integrated form, this is expressed by

$$\Lambda_t = \alpha \left(1 - \int_{\Lambda_t}^{\infty} u(t,x)\, dx \right).$$

Variants of the Stefan problem, originally proposed in [38], have been studied in great detail in the applied mathematics literature. In particular, it has been established in a string of works starting in the 1970s (see, e.g., [37, 22, 16, 17, 12, 24, 28, 18, 39, 11, 23, 41]) that the supercooled Stefan problem may exhibit finite time *blow-ups*, whereby continuous solutions $t \mapsto \Lambda_t$ cease to exist.

As closed-form solutions are rarely available for this type of free boundary problems, one typically has to resort to numerical methods. We refer to [4, 33] for a survey of numerical schemes. These articles treat classical cases with different boundary conditions where sign requirements (related to well-posedness without blow-ups) on the initial and/or boundary data, usually that the temperature of the material in the liquid phase is positive, are imposed. In the supercooled regime, where such conditions are not satisfied, regularization techniques to prevent the formation of blow-ups in finite time can be applied. For instance, in [27] the effect of kinetic undercooling as a regularizing mechanism is analyzed and it is shown how the (unregularized) supercooled Stefan problem can be recovered asymptotically. These asymptotics are also compared to numerical solutions of the unregularized problem. The authors follow here (for both the regularized and the unregularized case) the so-called method of lines considered in [15]. More precisely, the continuous time PDE systems are discretized and solved at successive times as a sequence of free boundary problems for ordinary differential equations. They can in turn be tackled via the so-called method of invariant embedding described in [32], which gives rise to explicitly solvable Riccati equations. Note that in the numerical studies of the unregularized case the blow-up behavior cannot be fully reproduced due to truncation errors. Further alternative (less physical) regularization approaches are discussed in [19], albeit without a thorough numerical analysis. In [2], the ill-posed Stefan problem for melting a superheated solid, which is mathematically identical to the supercooled Stefan problem, is analyzed. There, similarly as in [27], the method of lines is applied and existing numerical results from the literature in the blow-up regime are reproduced. In [1], blow-up points to one-phase Stefan problems, however with Dirichlet boundary conditions, are treated and numerically studied. Let us finally mention the recent paper [31], which analyses the Fisher–Stefan model, a generalization of the well-known Fisher–KPP model, in the context of biological invasion, where the speed of the moving boundary is related to the flux of population there. By rescaling this problem it can be compared to the supercooled Stefan problem. In this context, [31] provides new links between these models and a numerical scheme for the Fisher–Stefan model.

Despite this plethora of articles on ill-posed Stefan problems, to our knowledge, with the exception of [26], no provably convergent algorithm is known for the blow-up scenario. We contribute to this literature by proposing a class of numerical schemes for which we can prove global convergence, more specifically, that the discrete approximation of the free boundary converges to the true free boundary at all continuity points.

The following simple finite difference scheme is a canonical example of the schemes we will consider for (1), where u_i^k is an approximation to $u(k\Delta t, i\Delta x)$ for $i, k > 0$ and some fixed mesh widths $\Delta x, \Delta t > 0$:

$$
\begin{aligned}
&\frac{u_i^k - u_i^{k-1}}{\Delta t} = \frac{1}{2}\frac{u_{i+1}^{k-1} - 2u_i^{k-1} + u_{i+1}^{k-1}}{\Delta x^2}, \quad k > 0,\ i > i_k, \\
&i_k = \left\lfloor \frac{\alpha}{\Delta x}\left(1 - \sum_{i=i_k+1}^{\infty} u_i^k\right)\right\rfloor, \quad k > 0, \\
&u_i^0 = \int_{i\Delta x}^{(i+1)\Delta x} f(x)\,dx, \quad i \geqslant 0, \quad \text{and} \quad u_i^k = 0, \quad k > 0, i \leqslant i_k.
\end{aligned} \tag{2}
$$

We will analyze and implement a scheme of type (2) in Section 4. with a slightly perturbed initial condition, as motivated later. Note that (2) is reminiscent of the forward Euler approximation of the heat equation, but is nonetheless an implicit nonlinear equation through the dependence of the discrete free boundary $i_k\Delta x$ on the solution $(u_i^k)_i$ at time $k\Delta t$.

Our analysis is based on the following probabilistic reformulation of the problem:

$$
\begin{cases}
X_t = X_{0-} + B_t - \Lambda_t, \\
\tau = \inf\{t \geqslant 0 : X_t \leqslant 0\}, \\
\Lambda_t = \alpha\mathbb{P}\left(\tau \leqslant t\right),
\end{cases} \tag{3}
$$

where X_{0-} is a real-valued non-negative random variable, $\alpha > 0$, and B is a standard Brownian motion independent of X_{0-}. We will also study extensions where the Brownian motion B is replaced by a more general continuous-time process.

The link between (3) and (1) is found by noticing that over sufficiently small times the forward density $p(t,\cdot)$, $t \in [0,T]$ of the absorbed process $X_t\,\mathbf{1}_{\{\tau>t\}}$, $t \in [0,T]$ on $(0,\infty)$ satisfies the initial-boundary value problem

$$
\begin{aligned}
&\partial_t p = \frac{1}{2}\partial_{xx} p + \dot{\Lambda}_t \partial_x p, \quad x \geqslant 0, \quad t \in [0,T], \\
&p(0,x) = f(x), \quad x \geqslant 0 \quad \text{and} \quad p(t,0) = 0, \quad t \in [0,T], \\
&\Lambda_t = \alpha\left(1 - \int_0^\infty p(t,x)dx\right), \quad t \in [0,T],
\end{aligned} \tag{4}
$$

as long as X_{0-} has a sufficiently regular density f supported on $[0,\infty)$. Applying the transformation $u(t,x) := p(t, x - \Lambda_t)$, $x \geqslant \Lambda_t$ then leads to (1) (see [10] for a rigorous proof).

The problem (3) and its variants have recently been used to model systemic risk of interconnected financial systems. Here, Λ_t models the proportion of banks defaulted at a certain time t, and α the strength of interbank borrowing (see e.g. [20, 34, 21, 7] for details and further references). Equations with similar mean-field effects through thresholding derive from integrate-and-fire models of neurons as considered in [9].

Beside the applicability of (3) to systemic risk modeling but also in the context of neuro-science [9], which is justified by certain propagation of chaos results (see e.g.

[9, 8]), the probabilistic formulation (3) has also several mathematical advantages. Most notably, it allows for a rigorous definition of global solutions to (1) even in the presence of blow-ups. For this it is however necessary to first establish appropriate solution concepts for the probabilistic version of (3). Indeed, due to the singular interaction, (3) does not have unique solutions in general. Therefore, two physically and economically meaningful solution concepts have been developed: so-called *physical solutions* and *minimal solutions*. A rigorous characterization of unique physical solutions was given under certain technical assumptions on the initial conditions in [10]. The solution concept of *minimal solutions*, which are by definition unique, was in detail treated in [8]. In particular, it was shown there that the minimal solution is also a physical one. It is still an open question if uniqueness of physical solutions holds in general, but it is clear that the minimal solution coincides with the minimal physical one.

Among the numerical schemes introduced recently for the approximation of (3) are those using time-stepping (see [26]), iterations with heat kernels (see [30]) and particle systems (see [25]), where in the latter the numerical schemes are based on the corresponding propagation of chaos result.

Motivated by promising numerical experiments based on implicit instead of explicit schemes, we here first prove a global convergence result for an *implicit version of the Euler time-stepping scheme*, as considered in [26]. Furthermore we extend this result to more general drivers than Brownian motion and allow for processes that satisfy the so-called crossing property (see (8)), which holds e.g. for fractional Brownian motion with arbitrary Hurst parameter in $(0,1)$.

To further reduce numerical complexity we then consider a scheme that – in contrast to the time-stepping algorithm – does not require a (time-consuming) Monte Carlo simulation of the corresponding particle system. Indeed, the main result in the second part of the paper is that convergence in the Brownian case still holds after a Donsker approximation of the driving Brownian motion, i.e. the increments over single time-steps are approximated by random variables matching the first two moments of the Brownian increments.

The computational significance of this result is that by use of suitable discrete random variables, the discrete measure is supported on a recombining binomial tree and can be computed deterministically by a recursion over time-steps without simulation. The resulting discrete equations, of which (2) is a special case, are then interpretable as approximation schemes for (1).

We observe the following advantage in computational complexity. If the step size in time is h, the spacing of the tree nodes is $h^{1/2}$. On a given time horizon and finite spatial domain (by truncation of the real line), the total number of nodes, and hence computational complexity, is therefore of order $O(h^{-3/2})$. Although we only prove an order of convergence in the case of no blow-ups, we find empirically that in all regimes studied (jumps, no jumps, different regularities) the error is of order $h^{1/2}$; see Section 4 for details. Hence, to achieve an error of ϵ, the computational complexity is $O(\epsilon^{-3})$. In contrast to that, for the Monte Carlo time-stepping scheme of [25, 26], the simulation error using N particles is $O(N^{-1/2})$, at cost $O(N)$, and the time stepping error in the most regular case is $h^{1/2}$. Setting $N = \lfloor h^{-1} \rfloor$, the computational complexity is then $O(\epsilon^{-4})$ to achieve an error of ϵ. Moreover, for low regularity, the convergence of the (explicit) time-stepping scheme is found to be arbitrarily slow in [26], and hence the computational complexity for prescribed accuracy arbitrarily high. For the Donsker scheme we do not observe such a phenomenon, but rather similar convergence rates in regular and irregular regimes. At the same time the computational time is also considerably lower.

The remainder of the article is organized as follows. In Section 1.1 we start by introducing relevant notation applied throughout the paper. Section 2 is dedicated to the convergence result of the implicit time-stepping scheme extended to more general driving processes than Brownian motion. In Section 3 we then introduce the Donsker-type approximation scheme which has lower complexity and for which we can prove a rate of convergence in the regular case without blow-up. Section 4 concludes with numerical tests showing promising convergence rates of the Donsker scheme even in the case of blow-ups.

1.1 Notation

Throughout the paper, $D([-1,\infty))$ denotes the space of càdlàg functions on $[-1,\infty)$ endowed with the Skorokhod M_1-topology. For properties of the M_1-topology we refer to [40] and [8, Appendix A]. For $x \in D([-1,\infty))$ we denote by $\mathrm{Disc}(x)$ the set of discontinuity points of x. We also define the path functional $\lambda_t(x) := \mathbb{1}_{(0,\infty)}(\inf_{0\leqslant s\leqslant t} x_s)$, such that $\lambda_t(x) = 0$ if $x_s > 0$ on $[0,t]$ and $\lambda_t(x) = 1$ otherwise. The space of continuous functions on $[0,\infty)$ is denoted by $C([0,\infty))$ and endowed with the topology of compact convergence, i.e., $w_n \to w$ in $C([0,\infty))$ if and only if $w_n|_K \to w|_K$ uniformly for every compact $K \subseteq [0,\infty)$. We define the space of cumulative distribution functions on $[0,\infty]$ by

$$M := \{\ell\colon \overline{\mathbb{R}} \to [0,1] \mid \ell \text{ càdlàg and increasing}, \;\; \ell_{0-} = 0,\; \ell_\infty = 1\}, \tag{5}$$

where $\overline{\mathbb{R}}$ is the two-point compactification of $\mathbb{R}$. We endow M with the topology induced by the Lévy metric, i.e., $\ell^n \to \ell$ in M if and only if $\ell^n_t \to \ell_t$ for every $t \in [0,\infty]\setminus\mathrm{Disc}(\ell)$. This topology turns M into a compact Polish space. Furthermore, we define $\bar{E} := C([0,\infty)) \times M$ and endow it with the product topology. For $\alpha > 0$, we define $\iota_\alpha\colon \bar{E} \to D([-1,\infty))$ for $w \in C([0,\infty))$ and $\ell \in M$ as

$$\iota_\alpha(w,\ell)_t := \begin{cases} w_0 & t \in [-1,0) \\ w_t - \alpha\ell_t & t \in [0,\infty). \end{cases} \tag{6}$$

Throughout the paper, we denote by x a generic element of $D([-1,\infty))$, by w a generic element of $C([0,\infty))$ and by ℓ a generic element of M. If S is a Polish space, we denote the space of probability measures on S by $\mathcal{P}(S)$ and endow it with the topology of weak convergence, i.e., we say that $\mu_n \to \mu$ in $\mathcal{P}(S)$ iff $\int_S F(x)\mathrm{d}\mu_n(x) \to \int_S F(x)\mathrm{d}\mu(x)$ for all $F \in C_b(S;\mathbb{R})$, where $C_b(S;\mathbb{R})$ denotes the space of continuous bounded functions from S to $\mathbb{R}$. If $\mu \in \mathcal{P}(S)$ and $F\colon S \to \mathbb{R}$, we denote the integral of F with respect to μ also with brackets, i.e., we write $\int_S F(x)\ \mathrm{d}\mu(x) = \langle \mu, F\rangle$. Furthermore, if ν is the pushforward of the measure μ with respect to the map T, we denote this by $T(\mu) = \nu$. In particular, with a slight abuse of notation, if ξ denotes a measure on $\bar{E}$, $\iota_\alpha(\xi)$ denotes its pushforward on $D([-1,\infty))$.

2 Convergence of the time-stepping scheme

In this section we prove convergence of an implicit version of the time-stepping scheme considered in [26] for more general driving processes than Brownian motion.

Indeed, consider the following McKean–Vlasov problem

$$\begin{cases} X_t = X_{0-} + Z_t - \Lambda_t \\ \tau = \inf\{t \geqslant 0 : X_t \leqslant 0\} \\ \Lambda_t = \alpha \mathbb{P}\left(\tau \leqslant t\right), \end{cases} \tag{7}$$

where $X_{0-} \in \mathbb{R}$, $\alpha > 0$, and Z is a continuous stochastic process with $Z_0 = 0$ independent of X_{0-} satisfying the crossing property, i.e., we have

$$\mathbb{P}\left(\tau < \infty, \inf_{0 \leqslant s \leqslant h} (Z_{\tau+s} - Z_\tau) = 0\right) = 0, \quad h > 0, \tag{8}$$

for every stopping time τ with respect to the natural filtration of $X_{0-} + Z$. Note that the crossing property we use here is slightly more general than the one defined in [9, 8, 35] since we require (8) to hold for any stopping time, not just for the hitting time of zero. Note also that (8) holds for every stopping time if and only if it holds for every bounded stopping time.

Example 2.1. a) Let $Z := M+Y$, where M is a continuous local martingale and Y is 1/2-Hölder continuous on compacts. Suppose additionally that for every $K \in \mathbb{N}$ there is a strictly positive random variable ϵ_K such that $\langle M\rangle_t - \langle M\rangle_s \geqslant \epsilon_K(t-s)$ for $0 \leqslant s \leqslant t \leqslant K$, where $\langle M\rangle$ is the quadratic variation of M. Then, the proof of Theorem 3.5 in [3] shows that Z satisfies the crossing property (8). In particular we may choose Z to be Brownian motion.

b) The process $Z := B^H$, where B^H is fractional Brownian motion with Hurst parameter $H \in (0,1)$ satisfies the crossing property (8) by [36, Theorem 1.1].

We start by establishing existence (and uniqueness) of minimal solutions which are defined as follows. We call a solution $(\underline{X}, \underline{\tau}, \underline{\Lambda})$ to the McKean–Vlasov problem (7) *minimal*, if for every solution (X, τ, Λ) to (7) we have

$$\underline{\Lambda}_t \leqslant \Lambda_t, \quad t \geqslant 0. \tag{9}$$

We introduce the fixed-point operator Γ, defined for $\ell \in M$,

$$\begin{cases} X_t[\ell] = X_{0-} + Z_t - \alpha \ell_t \\ \tau[\ell] = \inf\{t \geqslant 0 : X_t[\ell] \leqslant 0\} \\ \Gamma[\ell]_t = \mathbb{P}\left(\tau[\ell] \leqslant t\right). \end{cases} \tag{10}$$

As the next proposition shows, Γ is continuous on M.

Proposition 2.2. *The operator* $\Gamma \colon M \to M$ *is continuous.*

Proof. Suppose that $\ell^n \to \ell$ in M. Define

$$\xi^n := \mathrm{law}((X_{0-} + Z, \ell^n)), \quad \xi := \mathrm{law}((X_{0-} + Z, \ell)).$$

Then, $\xi^n \to \xi$ in $\mathcal{P}(\bar{E})$. Set $\mu_n := \iota_\alpha(\xi^n)$, $\mu := \iota_\alpha(\xi)$ and note that $\langle \mu_n, \lambda_t\rangle = \Gamma[\ell^n]_t$, $\langle \mu, \lambda_t\rangle = \Gamma[\ell]_t$, where λ was defined in Section 1.1.

By Theorem 4.2 in [8] and the continuous mapping theorem, $\mu_n \to \mu$ in $\mathcal{P}(D([-1,\infty)))$. For $x \in D([-1,\infty))$ set $\tau_0(x) := \inf\{t \geqslant 0 \colon x_t \leqslant 0\}$ and define $\tau_\ell(w) := \inf\{t \geqslant$

$0\colon w_t - \alpha\ell_t \leqslant 0\}$. Fixing $h > 0$, we compute

$$\begin{aligned}
&\mu\big(\big\{x \in D([-1,\infty))\colon \tau_0(x) < \infty, \inf_{0\leqslant s\leqslant h}(x_{\tau_0+s} - x_{\tau_0}) = 0\big\}\big) \qquad (11)\\
&= \xi\big(\big\{(w,\bar\ell) \in \bar E\colon \tau_0(w-\alpha\bar\ell) < \infty, \inf_{0\leqslant s\leqslant h}[(w_{\tau_0+s} - w_{\tau_0}) - \alpha(\bar\ell_{\tau_0+s} - \bar\ell_{\tau_0})] = 0\big\}\big)\\
&\leqslant \xi\big(\big\{(w,\bar\ell) \in \bar E\colon \tau_\ell < \infty, \inf_{0\leqslant s\leqslant h}(w_{\tau_\ell+s} - w_{\tau_\ell}) = 0\big\}\big)\\
&= \mathbb{P}\left(\tau_\ell < \infty, \inf_{0\leqslant s\leqslant h}(Z_{\tau_\ell+s} - Z_{\tau_\ell}) = 0\right) = 0,
\end{aligned}$$

where the inequality is due to the fact that $\bar\ell \in M$ is increasing. Noting that the analogue of Lemma 5.5 in [8] holds with the assumption that (11) vanishes for every $h > 0$, it follows that

$$\lim_{n\to\infty}\Gamma[\ell^n] = \lim_{n\to\infty}\langle \mu_n, \lambda.\rangle = \langle \mu, \lambda.\rangle = \Gamma[\ell].$$

□

Theorem 2.3. *There is a (unique) minimal solution to* (7)*, and it is given by*

$$\underline{\Lambda} = \alpha \lim_{k\to\infty}\Gamma^{(k)}[0], \qquad (12)$$

where the convergence means $\Gamma^{(k)}[0] \to \alpha^{-1}\underline{\Lambda}$ *in* M.

Proof. This is analogous to the proof of Proposition 2.3 in [8], making use of Proposition 2.2 above. □

For $\Delta > 0$, define a time-discretized version Z^Δ by $Z^\Delta_t := Z_{\lfloor \frac{t}{\Delta}\rfloor\Delta}$ for $t \geqslant 0$. In analogy to the situation in continuous time, we define a fixed-point operator which we will use to construct the minimal solution in this time-discretized version of the McKean–Vlasov problem and prove that the operator is continuous in a suitable sense.

Lemma 2.4. *For* $\ell \in M$*, set*

$$\begin{cases} X^\Delta_t[\ell] = X_{0-} + Z^\Delta_t - \alpha\ell_{\lfloor \frac{t}{\Delta}\rfloor\Delta}\\ \tau^\Delta[\ell] = \inf\{t \geqslant 0 : X^\Delta_t[\ell] \leqslant 0\}\\ \Gamma_\Delta[\ell]_t = \mathbb{P}\left(\tau^\Delta[\ell] \leqslant t\right). \end{cases} \qquad (13)$$

Suppose that $\ell^n_{\lfloor \frac{\cdot}{\Delta}\rfloor\Delta} \to \ell_{\lfloor \frac{\cdot}{\Delta}\rfloor\Delta}$ *in* M *and that* $\ell^n_0 = \ell_0$*. Assume in addition that either* law(X_{0-}) *is atomless or that* law(Z_t) *is atomless for every* $t > 0$*. Then,* $\lim_{n\to\infty}\Gamma_\Delta[\ell^n] = \Gamma_\Delta[\ell]$ *in* M.

Proof. Note that the condition $\ell^n_{\lfloor \frac{\cdot}{\Delta}\rfloor\Delta} \to \ell_{\lfloor \frac{\cdot}{\Delta}\rfloor\Delta}$ in M implies that $\ell^n_{k\Delta} \to \ell_{k\Delta}$ for $k \in \mathbb{N}$. The assumption $\ell^n_0 = \ell_0$ implies that $\mathbb{P}\left(\tau^\Delta[\ell^n] = 0, \tau^\Delta[\ell] > 0\right) = 0$. For any

$t \geqslant 0$,

$$
\begin{aligned}
\Gamma_\Delta[\ell^n]_t - \Gamma_\Delta[\ell]_t &\leqslant \mathbb{P}\left(\tau^\Delta[\ell^n] \leqslant t,\ \tau^\Delta[\ell] > t\right), \\
&= \sum_{k=1}^{\lfloor \frac{t}{\Delta} \rfloor} \mathbb{P}\left(\tau^\Delta[\ell^n] = k\Delta, \tau^\Delta[\ell] > t\right) \\
&\leqslant \sum_{k=1}^{\lfloor \frac{t}{\Delta} \rfloor} \mathbb{P}\left(X^\Delta_{k\Delta}[\ell^n] \leqslant 0,\ X^\Delta_{k\Delta}[\ell] > 0\right), \\
&= \sum_{k=1}^{\lfloor \frac{t}{\Delta} \rfloor} \mathbb{P}\left(X_{0-} + Z_{k\Delta} - \alpha \ell^n_{k\Delta} \leqslant 0,\ X_{0-} + Z_{k\Delta} - \alpha \ell_{k\Delta} > 0\right), \\
&= \sum_{k=1}^{\lfloor \frac{t}{\Delta} \rfloor} \mathbb{P}\left(X_{0-} + Z_{k\Delta} \in (\alpha \ell_{k\Delta}, \alpha \ell^n_{k\Delta}]\right).
\end{aligned}
$$

Now if $\mathrm{law}(Z_t)$ is atomless for every $t > 0$, using the independence of Z and X_{0-} we may rewrite this as

$$
\sum_{k=1}^{\lfloor \frac{t-\Delta}{\Delta} \rfloor} \mathbb{E}\left[\mathbb{P}\left(x + Z_{k\Delta} \in (\alpha \ell_{k\Delta}, \alpha \ell^n_{k\Delta}]\right)\big|_{x = X_{0-}}\right]
$$

and the dominated convergence theorem yields that $\limsup_{n\to\infty} \Gamma_\Delta[\ell^n]_t \leqslant \Gamma_\Delta[\ell]_t$. The same argument works if we assume that $\mathrm{law}(X_{0-})$ is atomless, as we see by noticing that $\mathbb{P}\left(X_{0-} + Z_{k\Delta} \in (\alpha \ell_{k\Delta}, \alpha \ell^n_{k\Delta}]\right) = \mathbb{E}\left[\mathbb{P}\left(X_{0-} + z \in (\alpha \ell_{k\Delta}, \alpha \ell^n_{k\Delta}]\right)|_{z = Z_{k\Delta}}\right]$. Interchanging the roles of ℓ^n and ℓ in the estimates then yields the claim. □

Definition 2.5. We say that Λ^Δ solves the *discretized McKean–Vlasov problem* if $\alpha \Gamma_\Delta[\alpha^{-1}\Lambda^\Delta] = \Lambda^\Delta$. If $\underline{\Lambda}^\Delta$ is such that $\underline{\Lambda}^\Delta \leqslant \Lambda^\Delta$ for every Λ^Δ that solves the discretized McKean–Vlasov problem, we call $\underline{\Lambda}^\Delta$ *minimal* and shall refer to it as *implicit time-stepping scheme.*

Remark 2.6. With Definition (2.5), Λ^Δ is a solution if and only if

$$
\Lambda^\Delta_{k\Delta} = \alpha \mathbb{P}\left(\min_{0 \leqslant i \leqslant k} \{X_{0-} + Z_{i\Delta} - \Lambda^\Delta_{i\Delta}\} \leqslant 0\right), \tag{14}
$$

for every $k \in \mathbb{N}$. Since $\Lambda^\Delta_{k\Delta}$ appears on both sides of (14), this is an implicit equation for Λ^Δ (the solution of which is not unique in general), therefore we call our time-stepping scheme implicit. If we were to define a notion of solution through (14) and would take the minimum on the right-hand side over $\{0, \ldots, k-1\}$ instead of $\{0, \ldots, k\}$, this would give a recursion for Λ^Δ, and we would obtain the time-stepping scheme of [26], which we refer to as explicit in the following.

Proposition 2.7. *There is a (unique) minimal solution of the discretized McKean–Vlasov problem and it is given by*

$$
\underline{\Lambda}^\Delta = \alpha \lim_{k\to\infty} \Gamma^{(k)}_\Delta[0]. \tag{15}
$$

Proof. Γ_Δ is monotone in the sense that if $\ell^1 \leqslant \ell^2$ then $\Gamma_\Delta[\ell^1] \leqslant \Gamma_\Delta[\ell^2]$. Therefore,

$$
0 \leqslant \Gamma_\Delta[0] \leqslant \Gamma^{(2)}_\Delta[0] \leqslant \ldots
$$

This allows us to define $\hat{\ell}^{\Delta}_t := \lim_{k\to\infty} \Gamma^{(k)}_{\Delta}[0]_t$, then $\tilde{\ell}^{\Delta}_t := \hat{\ell}^{\Delta}_{t+}$ lies in M and by construction $\lim_{k\to\infty} \Gamma^{(k)}_{\Delta}[0] = \tilde{\ell}^{\Delta}$ in M. Note that $\Gamma^{(k)}_{\Delta}[0]$ is a step function with jumps at times $\Delta\mathbb{N}$, and so the same holds for $\tilde{\ell}^{\Delta}$. This implies that $\Gamma^{(k)}_{\Delta}[0]_{\lfloor \frac{t}{\Delta}\rfloor\Delta} = \Gamma^{(k)}_{\Delta}[0]_t$ and $\tilde{\ell}^{\Delta}_{\lfloor \frac{t}{\Delta}\rfloor\Delta} = \tilde{\ell}^{\Delta}_t$ for $t \geqslant 0$. As $\Gamma^{(k)}_{\Delta}[0]_s = \Gamma_{\Delta}[0]_s$ for $0 \leqslant s < \Delta$, it follows that $\tilde{\ell}^{\Delta}_0 = \Gamma^{(k)}_{\Delta}[0]_0$ for every $k \in \mathbb{N}$, and therefore we may apply Lemma 2.4 to find

$$\Gamma_{\Delta}[\tilde{\ell}^{\Delta}] = \Gamma_{\Delta}[\lim_{k\to\infty} \Gamma^{(k)}_{\Delta}[0]] = \lim_{k\to\infty} \Gamma_{\Delta}[\Gamma^{(k)}_{\Delta}[0]] = \tilde{\ell}^{\Delta},$$

so $\tilde{\Lambda} := \alpha\tilde{\ell}^{\Delta}$ is a solution of the discretized McKean–Vlasov problem. If Λ^{Δ} is another solution, then since $0 \leqslant \alpha^{-1}\Lambda^{\Delta}$, and Γ^{Δ} is monotone we have $\alpha\Gamma[0] \leqslant \alpha\Gamma[\alpha^{-1}\Lambda^{\Delta}] = \Lambda^{\Delta}$. Proceeding inductively we obtain $\alpha\Gamma^{(k)}[0] \leqslant \Lambda^{\Delta}$ and therefore $\tilde{\Lambda}^{\Delta}_t \leqslant \Lambda^{\Delta}_t$ for every continuity point of $\tilde{\Lambda}^{\Delta}$. By right-continuity, it follows that $\tilde{\Lambda}^{\Delta}$ is minimal. ☐

Remark 2.8. In contrast to the explicit scheme in [26], the time-stepping scheme we propose to solve here requires us to solve the implicit condition that $\underline{\Lambda}^{\Delta}$ is the minimal solution of (14) through the iteration (15). The analogous results we prove here hold as well for the explicit version of the scheme, but we expect that the explicit version smooths out the jumps while the implicit scheme should not, and therefore the implicit scheme should lead to a more accurate approximation of jump sizes and jump times. We observe this empirically in the case of the Donsker-type approximation, see Section 4.

We are now ready to state the main result of this section.

Theorem 2.9. *Choose a sequence $\Delta_n > 0$ such that the resulting discretizations are nested, i.e., $\Delta_n\mathbb{N} \subseteq \Delta_{n+1}\mathbb{N}$ for all $n \in \mathbb{N}$, and such that $\lim_{n\to\infty} \Delta_n = 0$. Assume in addition that either $\mathrm{law}(X_{0-})$ is atomless or that $\mathrm{law}(Z_t)$ is atomless for every $t > 0$. Then $\lim_{n\to\infty} \underline{\Lambda}^{\Delta_n} = \underline{\Lambda}$ in M.*

We collect some properties of the space M in the next lemma.

Lemma 2.10. *Let $\ell^n, \ell \in M$. The following statements are equivalent.*

(a) $\ell^n \to \ell$ in M.

(b) Let t be a continuity point of ℓ. Then, for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$\limsup_{n\to\infty} \sup_{s\in[t-\delta,t+\delta]} |\ell^n_s - \ell_s| \leqslant \epsilon. \tag{16}$$

(c) There is a co-countable set $D \subseteq [0,\infty)$ such that $\ell^n_t \to \ell_t$ for $t \in D$.

Proof. $[(a) \implies (b)]$: Fix $\epsilon > 0$. As ℓ is continuous at t and ℓ has at most countably many discontinuity points, there is $\delta > 0$ such that both $t+\delta$ and $t-\delta$ are continuity points and $|\ell_s - \ell_r| < \frac{\epsilon}{2}$ for $s, r \in [t-\delta, t+\delta]$. Choose $n_0 \in \mathbb{N}$ large enough such that $|\ell^n_{t+\delta} - \ell_{t+\delta}| < \frac{\epsilon}{2}$ and $|\ell^n_{t-\delta} - \ell_{t-\delta}| < \frac{\epsilon}{2}$ for all $n \geqslant n_0$. Then, for $s \in [t-\delta, t+\delta]$ we have

$$\begin{aligned}
\ell^n_s - \ell_s &\leqslant \ell^n_{t+\delta} - \ell_s < \ell_{t+\delta} - \ell_s + \frac{\epsilon}{2} < \epsilon,\\
\ell^n_s - \ell_s &\geqslant \ell^n_{t-\delta} - \ell_s > \ell_{t-\delta} - \ell_s - \frac{\epsilon}{2} > -\epsilon,
\end{aligned}$$

which proves that $\sup_{s\in[t-\delta,t+\delta]}|\ell^n_s - \ell_s| \leqslant \epsilon$ for $n \geqslant n_0$. The reverse implication is clear.

$[(c) \implies (a)]$: By compactness of M, after passing to a subsequence if necessary, we may assume that $\ell^n \to \bar{\ell}$ for some $\bar{\ell} \in M$. Since D is co-countable, the set $D\setminus \mathrm{Disc}(\bar{\ell})$ is co-countable as well, and therefore dense in $[0,\infty)$. For $t \in D\setminus \mathrm{Disc}(\bar{\ell})$, we obtain $\ell_t = \lim_{n\to\infty} \ell^n_t = \bar{\ell}_t$, and by right-continuity it follows that $\bar{\ell} = \ell$. Hence $\ell^n \to \ell$ in M. The reverse implication is clear. $\square$

Proposition 2.11. *Let $\Delta_n \to 0$ and suppose that $\ell^n \to \ell$ in M. Then,*

$$\lim_{n\to\infty} \Gamma_{\Delta_n}[\ell^n] = \Gamma[\ell]. \tag{17}$$

Proof. Step 1: We first show that $(X_{0-} + Z^{\Delta_n}, \ell^n_{\lfloor \frac{\cdot}{\Delta_n} \rfloor \Delta_n}) \to (X_{0-} + Z, \ell)$ in $\bar{E}$. Fix $T > 0$. For $\omega \in \Omega$ and $\epsilon > 0$, by uniform continuity there almost surely is $\delta > 0$ such that $|Z_t(\omega) - Z_s(\omega)| < \epsilon$ whenever $|s - t| < \delta$ and $s, t \in [0,T]$. For $\Delta_n < \delta$, we then have

$$|Z_t(\omega) - Z^{\Delta_n}_t(\omega)| < \epsilon, \quad t \in [0,T],$$

and hence $\lim_{n\to\infty} \|Z - Z^{\Delta_n}\|_{C([0,T])} = 0$ almost surely. Let t be a continuity point of ℓ. Write $g^n := \ell^n_{\lfloor \frac{\cdot}{\Delta_n} \rfloor \Delta_n}$. Fix $\epsilon > 0$, and choose $\delta > 0$ as in Lemma 2.10.(b). We then have

$$\limsup_{n\to\infty} |g^n_t - \ell_t| \leqslant \limsup_{n\to\infty} \sup_{s\in[t-\delta,t+\delta]} |\ell^n_s - \ell_s| + \limsup_{n\to\infty} |\ell_{\lfloor \frac{t}{\Delta_n} \rfloor \Delta_n} - \ell_t| \leqslant \epsilon.$$

As $\epsilon > 0$ was arbitrary, we see that $g^n \to \ell$ in M.

Step 2: Setting $\xi_n := \mathrm{law}((X_{0-} + Z^{\Delta_n}, g^n))$ and $\xi := \mathrm{law}((X_{0-} + Z, \ell))$, by Step 1 we have $\xi_n \to \xi$ in $\mathcal{P}(\bar{E})$. By the same reasoning as in the proof of Proposition 2.2 we obtain that $\langle \iota_\alpha(\xi^n), \lambda.\rangle \to \langle \iota_\alpha(\xi), \lambda.\rangle$ in M with λ being defined in Section 1.1. Since $\Gamma_{\Delta_n}[\ell^n]_t = \langle \iota_\alpha(\xi^n), \lambda_t\rangle$ and $\Gamma[\ell]_t = \langle \iota_\alpha(\xi), \lambda_t\rangle$, this yields the claim. $\square$

We are now prepared to prove Theorem 2.9.

Proof of Theorem 2.9. A straightforward induction using Proposition 2.11 shows that $\lim_{n\to\infty} \Gamma^{(k)}_{\Delta_n}[0] = \Gamma^{(k)}[0]$. Note that since $\Delta_n\mathbb{N} \subseteq \Delta_{n+1}\mathbb{N}$, it holds that $\Gamma_{\Delta_n}[\ell] \leqslant \Gamma_{\Delta_{n+1}}[\ell]$ for any $\ell \in M$. Set $J := \mathrm{Disc}(\underline{\Lambda}) \cup \{\mathrm{Disc}(\Gamma^{(k)}[0]) \mid k \in \mathbb{N}\} \cup \bigcup_{n\geqslant 0} \Delta_n\mathbb{N}$. Then J is countable, and for $t \notin J$ we have by Proposition 2.7

$$\underline{\Lambda}_t = \alpha \lim_{k\to\infty} \Gamma^{(k)}[0]_t = \alpha \lim_{k\to\infty} \lim_{n\to\infty} \Gamma^{(k)}_{\Delta_n}[0]_t = \alpha \lim_{n\to\infty} \lim_{k\to\infty} \Gamma^{(k)}_{\Delta_n}[0]_t = \lim_{n\to\infty} \underline{\Lambda}^{\Delta_n},$$

where we may exchange the order of the limits because $\Gamma^{(k)}_{\Delta_n}[0]_t$ is increasing both in n and in k. Lemma 2.10.(c) yields the claim. $\square$

3 A Donsker-type approximation

In this section we consider the numerical approximation of a special case of the McKean–Vlasov problem (7), where we assume $Z = B$, with B a Brownian motion independent of X_{0-}. The idea is to replace the Brownian motion by its Donsker approximation, i.e., for $h>0$ we consider

$$\begin{cases} X_t^h &= X_{0-}^h + B_t^h - \Lambda_t^h \\ \tau^h &= \inf\{t \geqslant 0 : X_t^h \leqslant 0\} \\ \Lambda_t^h &= \alpha \mathbb{P}\left(\tau^h \leqslant t\right), \end{cases} \tag{18}$$

where

$$B_t^h = \sqrt{h} \sum_{k=0}^{\lfloor t/h \rfloor} Y_k, \tag{19}$$

and $(Y_i)_{i\in\mathbb{N}}$ is an iid sequence of random variables satisfying $\mathbb{E}[Y_1]=0$ and $\mathbb{E}[Y_1^2]=1$. We repeat the same reasoning as previously, obtaining the minimal solution of (18) through a fixed-point iteration. We call a solution $\underline{\Lambda}^h$ of (18) minimal, if $\underline{\Lambda}^h \leqslant \Lambda^h$ for every Λ^h that solves (7). The minimal solution will then give rise to the implicit Donsker approximation scheme presented in Section 4.

Lemma 3.1. *Assume that X_{0-}^h is atomless. Define the operator Γ_h via*

$$\begin{aligned} X_t^h[\ell] &= X_{0-}^h + B_t^h - \alpha \ell_{\lfloor \frac{t}{h} \rfloor h} \\ \tau^h[\ell] &= \inf\{t \geqslant 0 : X_t^h[\ell] \leqslant 0\} \\ \Gamma_h[\ell]_t &:= \mathbb{P}\left(\tau^h[\ell] \leqslant t\right). \end{aligned}$$

Then, it holds that

$$\underline{\Lambda}^h = \lim_{k\to\infty} \Gamma_h^{(k)}[0]. \tag{20}$$

Proof. Note that by the same arguments given in Lemma 2.4, if $\ell^n_{\lfloor \frac{\cdot}{h} \rfloor h} \to \ell_{\lfloor \frac{\cdot}{h} \rfloor h}$ in M, then $\lim_{n\to\infty} \Gamma_h[\ell^n] = \lim_{n\to\infty} \Gamma_h[\ell]$ in M. Reasoning as in the proof of Theorem 2.9 using Step 1 of Proposition 2.11, we obtain (20). $\square$

Remark 3.2. If we were to pose problem (18) on a finite time horizon $[0,T]$, the analogous result to Lemma (3.1) would hold by the same reasoning. In particular, since the solution operator for the finite time horizon is the restriction to $[0,T]$ of Γ_h, the representation (20) shows that the minimal solution for the problem on $[0,T]$ is simply the restriction of the global minimal solution.

From here on, we will denote by $\underline{\Lambda}^h$ the minimal solution in the Donsker approximation with initial condition

$$X_{0-}^h = \left(\left\lfloor \frac{X_{0-}}{\sqrt{h}} \right\rfloor + \left\lfloor \log(h)^2 \right\rfloor \right) \sqrt{h}. \tag{21}$$

The reasons for this choice are twofold: first to make the theoretical results applicable to the numerical scheme of Section 4, we replace X_{0-} by a space-discretized version of itself, given by $\left\lfloor \frac{X_{0-}}{\sqrt{h}} \right\rfloor \sqrt{h}$; second for technical reasons, we require a perturbation term of order $\mathcal{O}(\sqrt{h}\log(h)^2)$ in the initial condition.

The goal of this section is to show that, with the above choice of initial condition (21), the default probabilities corresponding to the minimal solution of the Donsker approximation (18) converge to the default probability corresponding to the minimal solution of the McKean–Vlasov problem (7), i.e., we intend to show that $\underline{\Lambda}^h \to \underline{\Lambda}$ in M as $h \to 0$. We require some additional assumptions to prove this.

Assumptions 3.3. We assume that

(i) X_{0-} has a bounded density, which we denote by V_{0-}.

(ii) The moment generating function of Y_1 exists in a neighborhood of 0, i.e., $\mathbb{E}[\exp(uY_1)] < \infty$ for some $\delta > 0$ and $|u| < \delta$.

3.1 Convergence results

On an abstract level, the proof of convergence has two main parts: (i) Proving that limit points of solutions to the Donsker problem are solutions to the McKean–Vlasov problem, (ii) identifying limit points of minimal solutions of the Donsker problem as the minimal solution of the McKean–Vlasov problem. Typically, part (ii) is the more challenging one. Fortunately, a similar approach as in [8] works. The central idea of this approach is to find a sequence of optimization problems that allows us to relate the minimal solution of the (perturbed) Donsker problem to the minimal solution of the McKean–Vlasov equation, showing that the latter dominates the former asymptotically as $h \to 0$. The following lemma constitutes the main technical ingredient of this proof. As a rough intuition, by setting $\ell = \underline{\Lambda}$, the lemma allows us to quantify how far away the minimal solution of the McKean–Vlasov problem is from being a solution to the Donsker problem when h is small.

Lemma 3.4. *Suppose that Assumptions 3.3 are satisfied. Then, there is a constant $C > 0$ depending on T and $\|V_{0-}\|_\infty$ such that for all sufficiently small $h > 0$ it holds that*

$$\sup_{\ell \in M} \sup_{t \in [0,T]} |\Gamma_h[\ell]_t - \Gamma[\ell]_t| \leqslant C\sqrt{h}\log(T/h), \tag{22}$$

where Γ_h is defined as in Lemma 3.1 with $X_{0-}^h = \left\lfloor \frac{X_{0-}}{\sqrt{h}} \right\rfloor \sqrt{h}$.

The proof of the above lemma requires some preparations. The convergence rate in (22) is essentially the convergence rate of the Donsker approximation with respect to the Prokhorov metric, which we transfer to the estimate above through a Lipschitz mapping theorem.

Definition 3.5. Let (S, d) be a Polish space. For $\mu, \nu \in \mathcal{P}(S)$ define the Prokhorov distance between μ and ν as

$$\rho_P^S(\mu, \nu) = \inf\{\epsilon > 0 : \mu(B) \leqslant \nu(B^\epsilon) + \epsilon \text{ for all Borel sets } B\}, \tag{23}$$

where $B^\epsilon = \{x \in S \colon d_S(x, B) < \epsilon\}$. If X, Y are random variables on S we will write $\rho_P^S(X, Y)$ in place of $\rho_P(\mathrm{law}(X), \mathrm{law}(Y))$.

It is well-known that the Prokhorov metric metrizes weak convergence (in the probabilistic sense) and $(\mathcal{P}(S), \rho_P^S)$ is a Polish space. By [40, Theorem 3.4.2], Lipschitz maps preserve the Prokhorov metric in the sense that $\rho_P^S(g(X), g(Y)) \leqslant (1 \vee K)\rho_P^S(X, Y)$ for any K-Lipschitz map g. We will need a slight generalization of this result, which allows g to depend on an additional random variable that is independent of the Lipschitz component.

Theorem 3.6. *Let $(S,d),(S',d'),(R,d_R)$ be Polish spaces and suppose that $g\colon S'\times S\to R$ is K-Lipschitz in the second variable, i.e.,*

$$d_R(g(x,y),g(x,z))\leqslant Kd_S(y,z),\quad x\in S',\ y,z\in S. \tag{24}$$

Suppose furthermore that the random variables X,Y,Z satisfy $X\perp Y$, $X\perp Z$. Then,

$$\rho_P^R(g(X,Y),g(X,Z))\leqslant(1\vee K)\rho_P^S(Y,Z). \tag{25}$$

Proof. We may assume wlog that $K\geqslant 1$. By the Strassen-Dudley representation theorem [13, Corollary 11.6.4], there is a probability space $(\tilde{\Omega},\tilde{\mathbb{P}})$ with random variables $\tilde{Y},\tilde{Z}$ satisfying $\operatorname{law}(\tilde{Y})=\operatorname{law}(Y),\operatorname{law}(\tilde{Z})=\operatorname{law}(Z)$ such that $\rho_P^S(Y,Z)=\rho_K(\tilde{Y},\tilde{Z})$, where ρ_K is the Ky-Fan metric, i.e.,

$$\rho_K(\tilde{Y},\tilde{Z})=\inf\{\epsilon>0\colon \tilde{\mathbb{P}}(d(\tilde{Y},\tilde{Z})>\epsilon)\leqslant\epsilon\}.$$

Now set $\overline{\Omega}=\Omega\times\tilde{\Omega}$ and $\overline{\mathbb{P}}=\mathbb{P}\otimes\tilde{\mathbb{P}}$, then $\operatorname{law}((X,\tilde{Y}))=\operatorname{law}((X,Y))$ and $\operatorname{law}((X,\tilde{Z}))=\operatorname{law}((X,Z))$ by construction of $(\overline{\Omega},\overline{\mathbb{P}})$ and the independence assumption. In particular, we have $\rho_P^R(g(X,Y),g(X,Z))=\rho_P^R(g(X,\tilde{Y}),g(X,\tilde{Z}))$. Noting the general fact that $\rho_P\leqslant\rho_K$, we consider $\epsilon>0$ such that $\tilde{\mathbb{P}}(d(\tilde{Y},\tilde{Z})>\epsilon/K)\leqslant\epsilon/K$. Then,

$$\overline{\mathbb{P}}(d_R(g(X,\tilde{Y}),g(X,\tilde{Z}))>\epsilon)\leqslant\overline{\mathbb{P}}(d(\tilde{Y},\tilde{Z})>\epsilon/K)\leqslant\epsilon/K\leqslant\epsilon.$$

It follows that

$$\rho_K(g(X,\tilde{Y}),g(X,\tilde{Z}))\leqslant\inf\{\epsilon>0\colon \tilde{\mathbb{P}}(d(\tilde{Y},\tilde{Z})>\epsilon/K)\leqslant\epsilon/K\}=K\rho_K(\tilde{Y},\tilde{Z}).$$

We have obtained that

$$\rho_P^R(g(X,Y),g(X,Z))\leqslant\rho_K(g(X,\tilde{Y}),g(X,\tilde{Z}))\leqslant K\rho_K(\tilde{Y},\tilde{Z})=K\rho_P^S(Y,Z).$$

□

Example 3.7. We illustrate with an example that Theorem 3.6 does not hold in general without the independence assumption. We consider $S=S'=R=\mathbb{R}$, we set $\operatorname{law}((X,Y))=\frac{1}{2}\delta_{(0,0)}+\frac{1}{2}\delta_{(1,1)}$, and $\operatorname{law}((X,Z))=\frac{1}{2}\delta_{(0,1)}+\frac{1}{2}\delta_{(1,0)}$. Then, $\operatorname{law}(X)=\operatorname{law}(Y)=\operatorname{law}(Z)=\frac{1}{2}\delta_0+\frac{1}{2}\delta_1$, so in particular $\rho_P^S(Y,Z)=0$. However, taking $g(x,y):=|x-y|$, we see that $\operatorname{law}(g(X,Y))=\delta_0$ and $\operatorname{law}(g(X,Z))=\delta_1$, so $\rho_P^R(g(X,Y),g(X,Z))=1$.

Corollary 3.8. *Suppose $g:S'\times S\to\mathbb{R}$ is K-Lipschitz in the second variable. Suppose that $X\perp Y,X\perp Z$ and that the cdf of $g(X,Y)$ is Lipschitz with constant L. Then, for all $x\in\mathbb{R}$ we have*

$$|\mathbb{P}(g(X,Y)\leqslant x)-\mathbb{P}(g(X,Z)\leqslant x)|\leqslant(1+L)(1\vee K)\rho_P^R(Y,Z). \tag{26}$$

Proof. Denote the cdf of $g(X,Y)$ by F and that of $g(X,Z)$ by G. It holds that

$$\begin{aligned}\sup_{x\in\mathbb{R}}|G(x)-F(x)|&\leqslant(1+L)\inf\{\epsilon>0\colon F(x-\epsilon)-\epsilon\leqslant G(x)\leqslant F(x+\epsilon)+\epsilon\ \ \forall x\in\mathbb{R}\}\\&\leqslant(1+L)\rho_P^R(g(X,Y),g(X,Z)).\end{aligned}$$

The claim now follows from Theorem 3.6. □

We state two technical lemmata for future reference.

Lemma 3.9. *Let x^1, x^2 be càdlàg paths. Then,*

$$|\inf_{0\leqslant s\leqslant t} x^1_s - \inf_{0\leqslant s\leqslant t} x^2_s| \leqslant \sup_{0\leqslant s\leqslant t} |x^1_s - x^2_s|$$

Proof. Let $\epsilon > 0$ and choose $r \in [0,t]$ such that $x^2_r \leqslant \inf_{0\leqslant s\leqslant t} x^2_s + \epsilon$. Then,

$$\inf_{0\leqslant s\leqslant t} x^1_s - \inf_{0\leqslant s\leqslant t} x^2_s \leqslant x^1_r - x^2_r + \epsilon \leqslant \sup_{0\leqslant s\leqslant t} |x^1_s - x^2_s| + \epsilon.$$

Reversing the roles and noting that $\epsilon > 0$ was arbitrary proves the claim. □

Lemma 3.10. *Let X_{0-} admit a bounded density, denoted by V_{0-} and let Y^1, Y^2 be càdlàg and such that X_{0-} is independent of (Y^1, Y^2). Then, it holds that*

$$\left|\mathbb{P}\left(\inf_{0\leqslant s\leqslant t} X_{0-} + Y^1_s \leqslant 0\right) - \mathbb{P}\left(\inf_{0\leqslant s\leqslant t} X_{0-} + Y^2_s \leqslant 0\right)\right| \leqslant \|V_{0-}\|_\infty \mathbb{E}\left[\sup_{0\leqslant s\leqslant t} |Y^1_s - Y^2_s|\right].$$

Proof. Set $I := [-\inf_{s\leqslant t} Y^1_s, -\inf_{s\leqslant t}, Y^2_s] \cup [-\inf_{s\leqslant t} Y^1_s, -\inf_{s\leqslant t}, Y^2_s]$ and note that by Lemma 3.9, the length of I is bounded by $\sup_{0\leqslant s\leqslant t} |Y^1_s - Y^2_s|$. We argue as in the proof of Lemma 2.1 in [29], using the assumption on X_{0-} and the general inequality $\mathbb{P}(A) - \mathbb{P}(B) \leqslant \mathbb{P}(A \backslash B)$,

$$\begin{aligned}
&\left|\mathbb{P}\left(\inf_{0\leqslant s\leqslant t} X_{0-} + Y^1_s \leqslant 0\right) - \mathbb{P}\left(\inf_{0\leqslant s\leqslant t} X_{0-} + Y^2_s \leqslant 0\right)\right| \\
&\leqslant \mathbb{P}\left(X_{0-} \in I\right) \\
&= \mathbb{E}\left[\int_0^\infty \mathbb{1}_I(x) V_{0-}(x)\, dx\right] \\
&\leqslant \|V_{0-}\|_\infty \mathbb{E}\left[\sup_{0\leqslant s\leqslant t} |Y^1_s - Y^2_s|\right].
\end{aligned}$$

□

Proof of Lemma 3.4. Step 1: We transfer the convergence rate on $[0,1]$ to $[0,T]$ for $T > 0$. Set

$$S := \{x \in D[0,1] \colon \operatorname{Disc}(x) \subseteq h\mathbb{Q}\}, \quad R := \left\{x \in D[0,T] \colon \operatorname{Disc}(x) \subseteq \frac{h}{T}\mathbb{Q}\right\}. \tag{27}$$

In words, S is the set of càdlàg paths on $[0,1]$ that can only jump at times that are rational multiples of h. Note that if we endow the sets S and R with the uniform metric, they are separable and hence Polish spaces. By Theorem 1.16 in [6][1], we have $\rho^S_P(B^{h'}, W) \leqslant C\sqrt{h'}\log(1/h')$ for sufficiently small $h' > 0$. Define $f : S \to R$ by $f(x) = \sqrt{T}x_{\cdot/T}$, then f is $\sqrt{T}$-Lipschitz. Theorem 3.6 and the self-similarity of the Donsker approximation and Brownian motion give

$$\begin{aligned}
\rho^R_P(B^h, W) &= \rho^R_P(\sqrt{T}B^{h/T}_{\cdot/T}, \sqrt{T}W_{\cdot/T}) = \rho^R_P(f(B^{h/T}), f(W)) \\
&\leqslant C\frac{(1 \vee \sqrt{T})}{\sqrt{T}}\sqrt{h}\log(T/h),
\end{aligned}$$

[1]Note that in [6], the result is stated to hold for all $h > 0$, but the proof reveals that it only holds for sufficiently small $h > 0$

for $h > 0$ sufficiently small, which concludes Step 1.

Step 2: Fix $\ell \in M$ and define the map $g_t : \mathbb{R}\times R \to \mathbb{R}$ by $g_t(x_0, x) := \inf_{0\leqslant s\leqslant t} x_0 + x_s - \alpha\ell_s$. Note that $g(x_0, \cdot)$ is 1-Lipschitz for every $x_0 \in \mathbb{R}$ by Lemma 3.9. Denote the cdf of $g(X_{0-}, B)$ by F. We check that F is Lipschitz continuous: for $x, y \in \mathbb{R}$, setting $Y^1 := B - \alpha\ell - x$ and $Y^2 := B - \alpha\ell - y$ in Lemma 3.10, we obtain

$$|F(x) - F(y)| \leqslant \|V_{0-}\|_\infty |x - y|. \tag{28}$$

By Corollary 3.8, we therefore obtain for $t \in [0, T]$ and $h > 0$ small enough

$$|\mathbb{P}(g_t(X_{0-}, B^h) \leqslant 0) - \mathbb{P}(g_t(X_{0-}, B) \leqslant 0)| \leqslant C(1 + \|V_{0-}\|_\infty)\frac{(1 \vee \sqrt{T})}{\sqrt{T}}\sqrt{h}\log(T/h).$$

Since with $X^h_{0-} := \left\lfloor \frac{X_{0-}}{\sqrt{h}} \right\rfloor \sqrt{h}$ it holds that $|X_{0-} - X^h_{0-}| \leqslant \sqrt{h}$ almost surely, setting $Y^1 := -\sqrt{h} + B^h - \alpha\ell$ and $Y^2 := B^h - \alpha\ell$ in Lemma 3.10, we find

$$\begin{aligned}
&\mathbb{P}(g_t(X^h_{0-}, B^h) \leqslant 0) - \mathbb{P}(g_t(X_{0-}, B^h) \leqslant 0) \\
&\leqslant \mathbb{P}(g_t(X_{0-} - \sqrt{h}, B^h) \leqslant 0) - \mathbb{P}(g_t(X_{0-}, B^h) \leqslant 0) \\
&\leqslant \|V_{0-}\|_\infty \sqrt{h},
\end{aligned}$$

and arguing analogously for the downwards estimate we find

$$|\mathbb{P}(g_t(X^h_{0-}, B^h) \leqslant 0) - \mathbb{P}(g_t(X_{0-}, B^h) \leqslant 0)| \leqslant \|V_{0-}\|_\infty \sqrt{h}. \tag{29}$$

Noting that $\Gamma_h[\ell]_t = \mathbb{P}(g_t(X^h_{0-}, B^h) \leqslant 0)$ and $\Gamma[\ell]_t = \mathbb{P}(g_t(X_{0-}, B) \leqslant 0)$, taking the supremum over $t \in [0, T]$ and absorbing some terms into the constant C, the claim follows. $\square$

Lemma 3.11. *Let $\underline{\Lambda}^h$ be the minimal solution to* (18) *with initial condition given by* (21) *and suppose that Assumptions 3.3 are satisfied. Then for any $t > 0$, it holds that*

$$\limsup_{h\to 0} \underline{\Lambda}^h_t \leqslant \underline{\Lambda}_t. \tag{30}$$

Proof. Define the sequence of optimal values

$$V^h_t = \inf_{\ell\in M} c_h(\ell), \quad c_h(\ell) = \alpha\ell_t + \frac{\alpha}{\sqrt{h}\lfloor\log(h)^2\rfloor} \sup_{s\in[0,t]} |\Gamma_h[\ell]_s - \ell_s|, \tag{31}$$

where Γ_h is defined as in Lemma 3.1.

Step 1: We show that V^h_t is asymptotically dominated by $\underline{\Lambda}$ as $h \to 0$. By Lemma 3.4, for small enough $h > 0$ we have

$$V^h_t \leqslant c_h(\alpha^{-1}\underline{\Lambda}) = \underline{\Lambda}_t + \frac{\alpha}{\sqrt{h}\lfloor\log(h)^2\rfloor} \sup_{s\in[0,t]} |\Gamma_h[\underline{\Lambda}]_s - \underline{\Lambda}_s| \leqslant \underline{\Lambda}_t + \alpha C\frac{\log(t/h)}{\lfloor\log(h)^2\rfloor} \tag{32}$$

and therefore $\limsup_{h\to 0} V^h_t \leqslant \underline{\Lambda}_t$.

Step 2: We show that V^h_t asymptotically dominates $\underline{\Lambda}^h_t$. Choose a sequence $(L^h)_{h\geqslant 0}$ with $L^h \in M$ such that $c_h(L^h) - V^h_t \to 0$ as $h \to 0$. Suppose that

$\sup_{s\in[0,t]} |\Gamma_h[L^h]_s - L^h_s| > \alpha^{-1}h^{1/2}\lfloor\log(h)^2\rfloor$ for some h, then we have $c_h(L^h) > c_h(\alpha^{-1}\tilde{\Lambda}^h)$, where $\tilde{\Lambda}^h$ is a solution to (18) with $X^h_{0-} := \left\lfloor \frac{X_{0-}}{\sqrt{h}} \right\rfloor \sqrt{h}$. We may therefore assume wlog that $\sup_{s\in[0,t]} |\Gamma_h[L^h]_s - L^h_s| \leqslant \alpha^{-1}h^{1/2}\lfloor\log(h)^2\rfloor$ for all $h > 0$. For the sake of brevity, we write $d_h := \alpha^{-1}h^{1/2}\lfloor\log(h)^2\rfloor$ in the following. We then have

$$L^h_s \geqslant \Gamma_h[L^h]_s - d_h \quad s \in [0, t]. \tag{33}$$

For $\ell \in M$ define the operator

$$\tilde{\Gamma}_h[\ell]_t := \mathbb{P}\left(\inf_{0\leqslant s\leqslant t}\left[\left(\left\lfloor \frac{X_{0-}}{\sqrt{h}} \right\rfloor + \lfloor\log(h)^2\rfloor\right)\sqrt{h} + B^h_s - \alpha\ell_s\right] \leqslant 0\right). \tag{34}$$

Inequality (33) and the monotonicity of Γ_h imply

$$L^h_s \geqslant \Gamma_h[-d_h]_s - d_h = \tilde{\Gamma}_h[0]_s - d_h, \quad s \in [0, t]. \tag{35}$$

Substituting this again into inequality (33) yields

$$L^h_s \geqslant \Gamma_h[\tilde{\Gamma}_h[0] - d_h]_s - d_h = \tilde{\Gamma}^{(2)}_h[0]_s - d_h, \quad s \in [0, t].$$

Proceeding inductively, we obtain $L^h_s \geqslant \tilde{\Gamma}^{(k)}_h[0]_s - d_h$ for every $s \in [0, t]$ and $k \in \mathbb{N}$, and taking the limit as $k \to \infty$, by Lemma 3.1 we obtain $\alpha L^h_s \geqslant \underline{\Lambda}^h_s - \alpha d_h$ for $s \in [0, t]$. Since $d_h \to 0$ as $h \to 0$, combined with the definition of c_h and Step 1, this yields

$$\limsup_{h\to 0} \underline{\Lambda}^h_t \leqslant \limsup_{h\to 0} \alpha L^h_t \leqslant \limsup_{h\to 0} c_h(L^h) = \limsup_{h\to 0} V^h_t \leqslant \underline{\Lambda}_t.$$

□

Theorem 3.12. *Suppose that Assumptions 3.3 are satisfied. Then, the sequence of (perturbed) minimal solutions to the Donsker problem converges to the minimal solution to the McKean–Vlasov problem, i.e., it holds that*

$$\lim_{h\to 0} \underline{\Lambda}^h_t = \underline{\Lambda}_t, \tag{36}$$

for every $t \geqslant 0$ that is a continuity point of $\underline{\Lambda}$.

Proof. Choose a sequence $(h_n)_{n\geqslant 1}$ such that $h_n > 0$ and $\lim_{n\to\infty} h_n = 0$. By compactness of M, after passing to a subsequence if necessary, we may assume that $\underline{\Lambda}^{h_n} \to \Lambda$ for some Λ with $\alpha^{-1}\Lambda \in M$. If x is a càdlàg path, we introduce the notation $\Gamma^\alpha[x] := \alpha\Gamma[\alpha^{-1}x]$, such that Λ solves (7) iff $\Gamma^\alpha[\Lambda] = \Lambda$. We use the notation Γ^α_h in the analogous way. We show that Λ solves the McKean-Vlasov problem. By Theorem 2.2, Γ is continuous, so for $t \notin \mathrm{Disc}(\Lambda) \cup \mathrm{Disc}(\Gamma[\Lambda])$ we have

$$\begin{aligned}|\Gamma^\alpha[\Lambda]_t - \Lambda_t| &= \lim_{n\to\infty} |\Gamma^\alpha[\underline{\Lambda}^{h_n}]_t - \underline{\Lambda}^{h_n}_t| \\ &\leqslant \limsup_{n\to\infty}\left[|\Gamma^\alpha[\underline{\Lambda}^{h_n}]_t - \Gamma^\alpha_{h_n}[\underline{\Lambda}^{h_n}]_t| + |\Gamma^\alpha_{h_n}[\underline{\Lambda}^{h_n}]_t - \tilde{\Gamma}^\alpha_{h_n}[\underline{\Lambda}^{h_n}]_t|\right].\end{aligned}$$

where $\tilde{\Gamma}^\alpha[x]$ is defined as $\alpha\tilde{\Gamma}_{h_n}[\alpha^{-1}x]$ with $\tilde{\Gamma}_h$ defined in (34) and we used that $\underline{\Lambda}^{h_n} = \tilde{\Gamma}^\alpha[\underline{\Lambda}^{h_n}]$ since $\underline{\Lambda}^{h_n}$ is a solution to the perturbed Donsker problem. Due to

Lemma 3.4, the first term above vanishes as $n \to \infty$. For the second term, setting $Y^1 := \sqrt{h_n}(1 + \lfloor \log(h_n)^2 \rfloor) + B^h - \underline{\Lambda}^{h_n}$ and $Y^2 := -\sqrt{h_n} + B^h - \underline{\Lambda}^{h_n}$, we have

$$\alpha \mathbb{P}\left(X_{0-} + \inf_{s \leqslant t} Y^1_s \leqslant 0\right) \leqslant \tilde{\Gamma}^\alpha_{h_n}[\underline{\Lambda}^{h_n}]_t \leqslant \Gamma^\alpha_{h_n}[\underline{\Lambda}^{h_n}]_t \leqslant \alpha \mathbb{P}\left(X_{0-} + \inf_{s \leqslant t} Y^2_s \leqslant 0\right),$$

so by Lemma 3.10 it follows that

$$|\Gamma^\alpha_{h_n}[\underline{\Lambda}^{h_n}]_t - \tilde{\Gamma}^\alpha_{h_n}[\underline{\Lambda}^{h_n}]_t| \leqslant \alpha \|V_{0-}\|_\infty \sqrt{h_n}(2 + \lfloor \log(h_n)^2 \rfloor).$$

Therefore, $\Gamma^\alpha[\Lambda]_t = \Lambda_t$. By right-continuity of Λ and $\Gamma^\alpha[\Lambda]$, we find $\Lambda = \Gamma^\alpha[\Lambda]$. By definition, this implies $\Lambda \geqslant \underline{\Lambda}$. On the other hand, for $t \notin \mathrm{Disc}(\Lambda)$, Lemma 3.11 yields

$$\Lambda_t = \limsup_{n \to \infty} \underline{\Lambda}^{h_n}_t \leqslant \underline{\Lambda}_t.$$

By right-continuity, we find $\Lambda \leqslant \underline{\Lambda}$ and hence $\Lambda = \underline{\Lambda}$. □

In the case $\alpha \|V_{0-}\|_\infty < 1$, we also get a rate of convergence as shown in the next theorem. By [29, Theorem 2.2 and the comment below], this corresponds to the weak feedback regime, where uniqueness and in particular continuity of solutions holds true.

Theorem 3.13. *Suppose that Assumptions 3.3 are satisfied and that additionally $\alpha \|V_{0-}\|_\infty < 1$. Then, there is a constant C depending on T, α and $\|V_{0-}\|_\infty$ such that*

$$\sup_{t \in [0,T]} |\underline{\Lambda}_t - \underline{\Lambda}^h_t| \leqslant C \log(h)^2 \sqrt{h}.$$

for $h > 0$ sufficiently small.

Proof. Applying Lemma 3.10 with $Y^1 = B - \alpha \ell^1$ and $Y^2 = B - \alpha \ell^2$ shows

$$\sup_{t \in [0,T]} |\Gamma[\ell^1]_t - \Gamma[\ell^2]_t| \leqslant \alpha \|V_{0-}\|_\infty \sup_{t \in [0,T]} |\ell^1_t - \ell^2_t|. \tag{37}$$

Recalling the notation of the proof of Theorem 3.12, we have $\underline{\Lambda}^{h_n} = \tilde{\Gamma}^\alpha_{h_n}[\underline{\Lambda}^{h_n}]$, since $\underline{\Lambda}^{h_n}$ is a solution to the perturbed Donsker problem and $\Gamma^\alpha[\underline{\Lambda}] = \underline{\Lambda}$. Moreover,

$$\begin{aligned}
\sup_{t \in [0,T]} |\underline{\Lambda}_t - \underline{\Lambda}^h_t| &= \sup_{t \in [0,T]} |\Gamma^\alpha[\underline{\Lambda}]_t - \tilde{\Gamma}^\alpha_{h_n}[\underline{\Lambda}^h]_t| \\
&\leqslant \sup_{t \in [0,T]} |\Gamma^\alpha[\underline{\Lambda}]_t - \Gamma^\alpha[\underline{\Lambda}^h]_t| + \sup_{t \in [0,T]} |\Gamma^\alpha[\underline{\Lambda}^h]_t - \Gamma^\alpha_h[\underline{\Lambda}^h]_t| \\
&\quad + \sup_{t \in [0,T]} |\Gamma^\alpha_h[\underline{\Lambda}^h]_t - \tilde{\Gamma}^\alpha_h[\underline{\Lambda}^h]_t| \\
&\leqslant \alpha \|V_{0-}\| \sup_{t \in [0,T]} |\underline{\Lambda}_t - \underline{\Lambda}^h_t| + \alpha C \sqrt{h} \log(T/h) \\
&\quad + \alpha \|V_{0-}\|_\infty \sqrt{h}(2 + \lfloor \log(h)^2 \rfloor),
\end{aligned}$$

where the last estimate follows from (37), Lemma 3.4 and the third term is estimated as in the proof of Theorem 3.12. Rearranging terms and absorbing some into the constant C, the claim follows. □

Remark 3.14. Note that the choice $\sqrt{h}\lfloor\log(h)^2\rfloor$ for the perturbation of the initial condition was somewhat arbitrary, since the proofs of the previous theorems work for any function $\delta(h)$ that converges to zero slowly enough such that

$$\lim_{h\to 0}\frac{\sqrt{h}\log(T/h)}{\delta(h)}=0$$

for every $T>0$. In particular, we could consider $\sqrt{h}\lfloor\epsilon\log(h)^2\rfloor$ for any $\epsilon>0$, which suggests that the perturbation can be ignored in practice, and we will indeed ignore it in the numerical tests in the next section. It seems natural to conjecture that the result should also hold with $\epsilon=0$, and this would indeed follow from Conjecture 6.10 in [8], however we do not have a proof at this point.

4 Numerical tests

In this section, we first describe our implementation of the Donsker scheme from Section 3 and then present the results from numerical tests.

4.1 A tree-type scheme

We consider (18) and specify Y_k to be Rademacher random variables, i.e. $\mathbb{P}(Y_k=-1)=\mathbb{P}(Y_k=1)=1/2$.

Then we introduce a time mesh $t_k=kh$, $k\geqslant 0$ an integer, and

$$p_i^k \quad = \quad \mathbb{P}(X_{t_k}^h=i\sqrt{h}-\underline{\Lambda}_{t_k}^h,\tau^h>t_k),$$

for $i\in\mathbb{Z}$, using the notation defined in Section 3, from which it follows

$$\underline{\Lambda}_{t_k}^h=\alpha\left(1-\sum_{i=i_k+1}^{\infty}p_i^k\right),\qquad i_k=\left\lfloor\frac{\underline{\Lambda}_{t_k}^h}{\sqrt{h}}\right\rfloor.$$

We have that

$$p_i^k \quad = \quad \begin{cases} \frac{1}{2}p_{i-1}^{k-1}+\frac{1}{2}p_{i+1}^{k-1}, & i>i_k+1,\\ \frac{1}{2}p_{i+1}^{k-1}, & i=i_k+1,\\ 0, & i<i_k+1,\end{cases} \tag{38}$$

for $k>0$. This is an implicit scheme as p^k and $\underline{\Lambda}^h$ are implicitly coupled. The recurrence relation (38) has the resemblance of a binomial tree shifted by the interaction term. It can also be interpreted as a special type of semi-Lagrangian scheme (see [5, 14]) for the forward equation (4).

Note that (38) is a non-linear equation through the dependence of i_k on p^k via $\underline{\Lambda}_{t_k}^h$. Lemma 3.1 suggests a fixed-point iteration to solve simultaneously for $\underline{\Lambda}_{t_k}^h$ and the vector p^k for each k. We assume that we know the cdf of X_{0-} exactly, and therefore we can calculate,

$$\mathbb{P}(X_{0-}^h=i\sqrt{h})=\mathbb{P}(X_{0-}\in[i\sqrt{h},(i+1)\sqrt{h})),$$

for all $i\in\mathbb{Z}$. To calculate p_i^0 for $i\in\mathbb{Z}$, we need to determine $\underline{\Lambda}_0^h$ first, which we obtain as in Lemma 3.1 through the iteration initialized with $\lambda^0=0$ and

$$\lambda^{n+1}=\alpha\sum_{j=0}^{\iota^n}\mathbb{P}(X_{0-}^h=j\sqrt{h}),\quad \iota^n=\left\lfloor\frac{\lambda^n}{\sqrt{h}}\right\rfloor.$$

The iteration terminates when $\iota^{n+1} = \iota^n$, which happens after at most $\lfloor \alpha/\sqrt{h} \rfloor$ iterations, since $\lambda^n \leqslant \lambda^{n+1}$ and hence $\iota^n \leqslant \iota^{n+1}$ as well. This yields $\underline{\Lambda}_0^h$. We then calculate p_i^0 via

$$p_i^0 = 0 \text{ for } i \leqslant \left\lfloor \frac{\underline{\Lambda}_0^h}{\sqrt{h}} \right\rfloor \quad \text{and} \quad p_i^0 = \mathbb{P}(X_{0-}^h = i\sqrt{h}) \text{ for } i > \left\lfloor \frac{\underline{\Lambda}_0^h}{\sqrt{h}} \right\rfloor.$$

The vectors $p^k = (p_i^k)_i$ are then calculated recursively through (38), using a local in time version of the iteration in Lemma 3.1 that iterates only over the scalar loss at each time point. Set $\lambda^0 = \underline{\Lambda}_{t_{k-1}}^h$, $\iota^0 = \lfloor \lambda^0/\sqrt{h} \rfloor$, and for $n \geqslant 0$,

$$\begin{aligned} p_i^{k,n+1} &= \begin{cases} \frac{1}{2} p_{i-1}^{k-1} + \frac{1}{2} p_{i+1}^{k-1}, & i > \iota^n + 1, \\ \frac{1}{2} p_{i+1}^{k-1}, & i = \iota^n + 1, \\ 0, & i < \iota^n + 1, \end{cases} \\ \lambda^{n+1} &= \alpha \left(1 - \sum_{i=\iota^n+1}^{\infty} p_i^{k,n+1} \right), \qquad \iota^{n+1} = \left\lfloor \frac{\lambda^{n+1}}{\sqrt{h}} \right\rfloor. \end{aligned} \tag{39}$$

Expressing this in terms of Γ_h, if we define

$$\hat{\Lambda}^0 := \begin{cases} \underline{\Lambda}_t^h, & t \in [0, t_{k-1}] \\ \underline{\Lambda}_{t_{k-1}}^h, & t \in [t_{k-1}, t_k] \end{cases} \tag{40}$$

and set $\hat{\Lambda}^{(n+1)} = \Gamma_h[\hat{\Lambda}^{(n)}]$, then λ^{n+1} above is equal to $\Gamma_h[\hat{\Lambda}^{(n)}]_{t_k}$. We convince ourselves that this computes the minimal solution: As in the case of the initial condition, the iteration is increasing in n for the losses, $\lambda^{n+1} \geqslant \lambda^n$ and $\iota^{n+1} \geqslant \iota^n$, and the iteration terminates in at most $\lfloor \alpha/\sqrt{h} \rfloor$ iterations (because ι^n can only take values in $\{i_{k-1}, \ldots, \lfloor \alpha/\sqrt{h} \rfloor\}$). If n_0 is the smallest n such that $\lambda^{n+1} = \lambda^n$, then $\hat{\Lambda}^{(n_0)}$ solves (18) on $[0, t_k]$, and therefore $\Lambda_t^{(n_0)} \geqslant \underline{\Lambda}_t^h$ for $t \in [0, t_k]$ by Remark 3.2. On the other hand, since $\underline{\Lambda}^h$ is increasing, it holds that $\hat{\Lambda}^{(0)} \leqslant \underline{\Lambda}^h$, and by the monotonicity of Γ_h and a straightforward induction it follows that $\hat{\Lambda}^{(n)} \leqslant \underline{\Lambda}^h$ for $n \in \mathbb{N}$ and hence $\hat{\Lambda}_t^{(n_0)} = \underline{\Lambda}_t^h$ for $t \in [0, t_k]$.

4.2 Results

In this section, we analyse computational aspects and, especially, the numerical accuracy of the scheme.

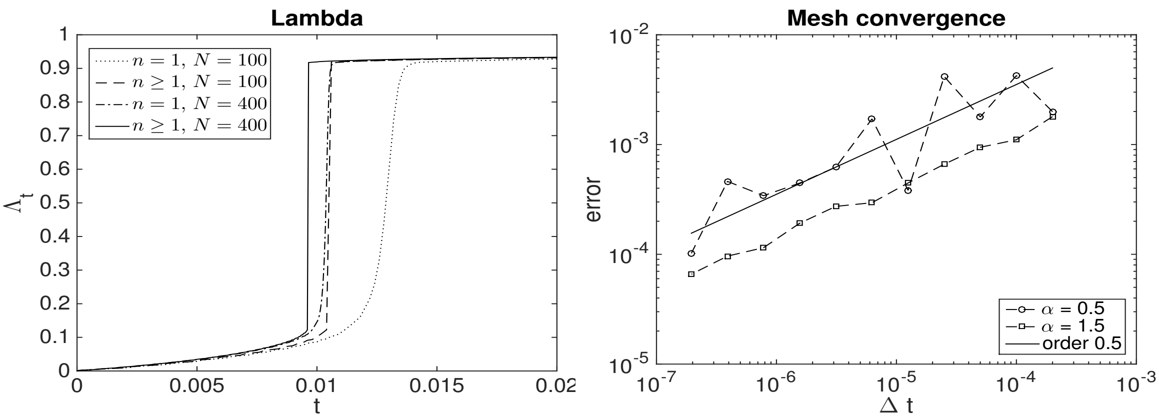
Initially, we consider a $\Gamma(k, \theta)$ distribution for X_{0-} with $k = 2$ and $\theta = 1/2$ and note that the initial density V_{0-} is globally Lipschitz in this case. For simplicity, we here set $p_i^0 = V_{0-}(i\sqrt{h})$. Furthermore, we fix the time interval $[0, T] = [0, 0.02]$.

We first examine the iteration (39). For two different values of α and an increasing number of time points N, Table 1 gives the number of iterations before termination, first averaged over all time steps, and then the maximum number of iterations for any time step.

In the regular case without a jump ($\alpha = 0.5$), there are never more than 2 iterations needed, while the average number is close to 1. This is explained by the fact that $i_k \neq i_{k-1}$ only if $\underline{\Lambda}_{t_k}^h - \underline{\Lambda}_{t_{k-1}}^h > \alpha\sqrt{h}$, so in the regular regime, where $\underline{\Lambda}_t$ is differentiable, a change of i_k will only happen every $O(1/\sqrt{h})$ time steps, and usually by only 1. Put differently, because of the monotonicity of $\underline{\Lambda}_{t_k}^h$ in k and that

	time points N	100	200	400	800	1600	3200
$\alpha = 0.5$,	av. iter.	1.0200	1.0150	1.0125	1.0088	1.0063	1.0047
	max. iter.	2	2	2	2	2	2
$\alpha = 1.5$,	av. iter.	1.2800	1.1800	1.1175	1.0775	1.0513	1.0341
	max. iter.	16	18	20	20	22	23

Table 1: Average and maximum number of iterations over all N time points for $\alpha = 0.5$ (no jump) and $\alpha = 1.5$ (jump).



(a) Explicit and implicit.

(b) $\alpha = 0.5$ and $\alpha = 1.5$.

Figure 1: Time step convergence.

of the iteration, the total number of iterations summed up over all time steps is bounded by $\alpha/\sqrt{h}$. So the average number of iterations is $1 + O(\sqrt{h})$.

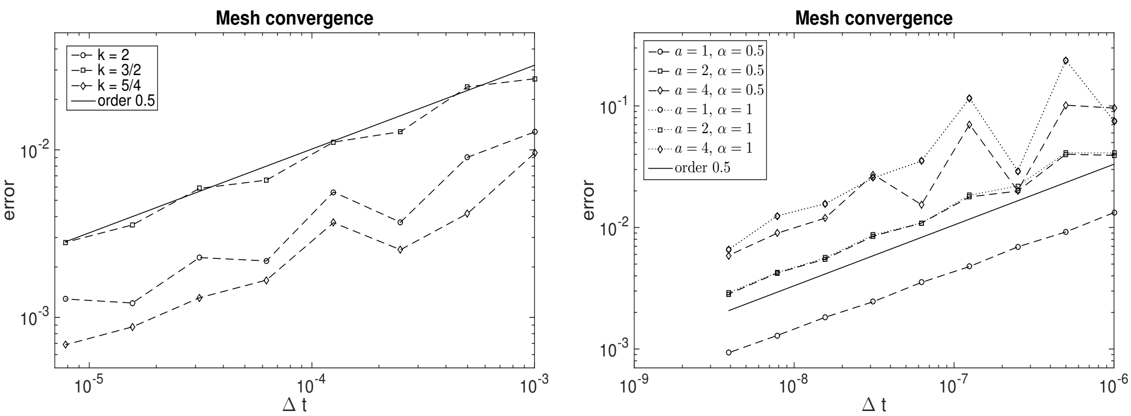
In the presence of a jump ($\alpha = 1.5$), a larger number of iterations is needed at the time of the jump, but this number only grows mildly under mesh refinement, and on average the number of iterations is still close to 1.

In either case, therefore, the computational cost of the iteration amounts to less than 10% of the overall cost for reasonable fine time meshes.

Figure 1a compares the converged numerical loss process $(t_k, \underline{\Lambda}^h_{t_k})$, labeled '$n \geqslant 1$', to the one computed with a simpler scheme where the iteration is stopped after the first iteration, labeled '$n = 1$'. The latter can be considered an explicit treatment where the loss computed at each time point is used to compute the density at the following point, in contrast to scheme (38) where p^k and $\underline{\Lambda}^h_{t_k}$ are implicitly coupled. This reveals that the explicit scheme smooths out the jump and takes a significant time for the losses to 'catch up', in a way already seen for the Euler-type time-stepping particle scheme of [26] (see in particular Figure 3 there). In contrast, the implicit scheme (38) reproduces a sharp jump, i.e., for sufficiently small h the increment $\underline{\Lambda}^h_{t_{k+1}} - \underline{\Lambda}^h_{t_k}$ is small for all but one k, whereas this largest increase does not go to 0 as h diminishes, but converges to the true jump size.[2]

We now turn to the convergence of $\underline{\Lambda}^h$ as the step size h goes to 0. Figure 1b shows the error estimator $2(\underline{\Lambda}^h_T - \underline{\Lambda}^{2h}_T)$ for decreasing h, for $\alpha = 0.5$ and $\alpha = 1.5$, and otherwise the same parameters as earlier. In both cases the order of convergence appears to be 0.5, with the more regular convergence happening in the case $\alpha = 1.5$,

[2]We thank Andreas Søjmark for a discussion on the implicit treatment of jumps.



(a) Different $\Gamma(k, 1/2)$ initial values. (b) Initial densities as in (41).

Figure 2: time step convergence.

where a jump happens. In spite of the discretization of the driving Brownian motion, convergence is faster than that of the explicit time-stepping scheme in [25, 26], where no positive order of convergence was observed in the jump case. This appears to be a result of the implicit treatment of the jump as explained above.

We now investigate two further cases of intermediate regularity, taken from [26]. First, we vary k in the $\Gamma(k, 1/2)$ initial distribution to $k = 3/2$ and $5/4$, such that the density is only Hölder $1/2$ and $1/4$, respectively. The empirical convergence order is still $1/2$ in all cases.

Second, we consider the initial density

$$V_{0-}(x) = \begin{cases} \frac{1}{\alpha} - cx^a, & 0 \leqslant x \leqslant A, \\ 0, & x > A, \end{cases} \tag{41}$$

for $\alpha > 0$ and $a > 0$, which we vary in the tests, and where $A > 0$ is determined by $\int_0^\infty V_{0-}(x)\,\mathrm{d}x = 1$ for given $c > 0$, the latter being sufficiently small. Moreover, we let $T = 10^{-4}$ be small enough to precede a possible discontinuity. Here, the convergence order of the explicit time-stepping scheme in [26] is $1/(2(a+1))$ (see Theorem 1.5 and Table 1 there). Remarkably, the asymptotic order of our scheme appears to be $1/2$ irrespective of a.

References

[1] T. Aiki and H. Imai. Blow-up points to one phase Stefan problems with Dirichlet boundary conditions. In *Modelling and Optimization of Distributed Parameter Systems Applications to Engineering*, pages 83–89. Springer, 1996.

[2] J. M. Back, S.W. McCue, and T.J. Moroney. Numerical study of two ill-posed one phase Stefan problems. *ANZIAM Journal*, 52:C430–C446, 2010.

[3] C. Bender. Simple arbitrage. *The Annals of Applied Probability*, 22(5):2067 – 2085, 2012.

[4] J. Caldwell and Y.Y. Kwan. Numerical methods for one-dimensional Stefan problems. *Communications in Numerical Methods in Engineering*, 20(7):535–545, 2004.

[5] F. Camilli and M. Falcone. An approximation scheme for the optimal control of diffusion processes. *ESAIM: Mathematical Modelling and Numerical Analysis*, 29(1):97–122, 1995.

[6] M. Csörgö and L. Horváth. *Weighted Approximations in Probability and Statistics*. Wiley Series in Probability and Statistics. Wiley, 1993.

[7] C. Cuchiero, C. Reisinger, and S. Rigger. Optimal bailout strategies resulting from the drift controlled supercooled Stefan problem. *arXiv preprint arXiv:2111.01783*, 2021.

[8] C. Cuchiero, S. Rigger, and S. Svaluto-Ferro. Propagation of minimality in the supercooled Stefan problem. arXiv:2010.03580.

[9] F. Delarue, J. Inglis, S. Rubenthaler, and E. Tanré. Particle systems with a singular mean-field self-excitation. Application to neuronal networks. *Stochastic Processes and their Applications*, 125(6):2451–2492, 2015.

[10] F. Delarue, S. Nadtochiy, and M. Shkolnikov. Global solutions to the supercooled Stefan problem with blow-ups: regularity and uniqueness. *arXiv preprint arXiv:1902.05174*, 2019.

[11] J. N. Dewynne, S. D. Howison, J. R. Ockendon, and W. Q. Xie. Asymptotic behavior of solutions to the Stefan problem with a kinetic condition at the free boundary. *J. Austral. Math. Soc. Ser. B*, 31(1):81–96, 1989.

[12] E. DiBenedetto and A. Friedman. The ill-posed Hele–Shaw model and the Stefan problem for supercooled water. *Trans. Amer. Math. Soc.*, 282(1):183–204, 1984.

[13] R. M. Dudley. *Real Analysis and Probability: 0*. Chapman and Hall/CRC, 2018.

[14] M. Falcone and R. Ferretti. *Semi-Lagrangian approximation schemes for linear and Hamilton–Jacobi equations*. SIAM, 2013.

[15] A. Fasano, G.H. Meyer, and M. Primicerio. On a problem in the polymer industry: theoretical and numerical investigation of swelling. *SIAM Journal on Mathematical Analysis*, 17(4):945–960, 1986.

[16] A. Fasano and M. Primicerio. New results on some classical parabolic free-boundary problems. *Quart. Appl. Math.*, 38(4):439–460, 1980/81.

[17] A. Fasano and M. Primicerio. A critical case for the solvability of Stefan-like problems. *Math. Methods Appl. Sci.*, 5(1):84–96, 1983.

[18] A. Fasano, M. Primicerio, S. D. Howison, and J. R. Ockendon. On the singularities of one-dimensional Stefan problems with supercooling. In *Mathematical models for phase change problems (Óbidos, 1988)*, volume 88 of *Internat. Ser. Numer. Math.*, pages 215–226. Birkhäuser, Basel, 1989.

[19] A. Fasano, M. Primicerio, S.D. Howison, and J.R. Ockendon. Some remarks on the regularization of supercooled one-phase Stefan problems in one dimension. *Quarterly of applied mathematics*, 48(1):153–168, 1990.

[20] B. Hambly, S. Ledger, and A. Søjmark. A McKean–Vlasov equation with positive feedback and blow-ups. *Ann. Appl. Probab.*, 29(4):2338–2373, 08 2019.

[21] B. Hambly and A. Søjmark. An SPDE model for systemic risk with endogenous contagion. *Finance and Stochastics*, 23(3):535–594, 2019.

[22] M.A. Herrero and J.L. Velázquez. Singularity formation in the one-dimensional supercooled Stefan problem. *European Journal of Applied Mathematics*, 7(2):119–150, 1996.

[23] S. Howison and W. Q. Xie. Kinetic undercooling regularization of supercooled Stefan problems. In *Mathematical models for phase change problems (Óbidos, 1988)*, volume 88 of *Internat. Ser. Numer. Math.*, pages 227–237. Birkhäuser, Basel, 1989.

[24] S.D. Howison, J.R. Ockendon, and A.A. Lacey. Singularity development in moving-boundary problems. *Quart. J. Mech. Appl. Math.*, 38(3):343–360, 1985.

[25] V. Kaushansky and C. Reisinger. Simulation of a simple particle systems interacting through hitting times. *Disc. Cont. Dyn. Sys. B*, (10), 2019.

[26] V. Kaushansky, C. Reisinger, M. Shkolnikov, and Z. Q. Song. Convergence of a time-stepping scheme to the free boundary in the supercooled Stefan problem. *arXiv preprint arXiv:2010.05281*, 2020.

[27] J.R. King and J.D. Evans. Regularization by kinetic undercooling of blow-up in the ill-posed Stefan problem. *SIAM Journal on Applied Mathematics*, 65(5):1677–1707, 2005.

[28] A.A. Lacey and J.R. Ockendon. Ill-posed free boundary problems. *Control Cybernet.*, 14(1-3):275–296 (1986), 1985.

[29] S. Ledger and A. Søjmark. Uniqueness for contagious McKean–Vlasov systems in the weak feedback regime. *Bulletin of the London Mathematical Society*, 52(3):448–463, 2020.

[30] A. Lipton, V. Kaushansky, and C. Reisinger. Semi-analytical solution of a McKean–Vlasov equation with feedback through hitting a boundary. *European Journal of Applied Mathematics*, pages 1–34, 2019.

[31] S.W. McCue, M. El-Hachem, and M.J. Simpson. Traveling waves, blow-up, and extinction in the Fisher–Stefan model. *Studies in Applied Mathematics*, 148(2):964–986, 2022.

[32] G.H. Meyer. One-dimensional parabolic free boundary problems. *SIAM Review*, 19(1):17–34, 1977.

[33] S.L. Mitchell and M. Vynnycky. Finite-difference methods with increased accuracy and correct initialization for one-dimensional Stefan problems. *Applied Mathematics and Computation*, 215(4):1609–1621, 2009.

[34] S. Nadtochiy and M. Shkolnikov. Particle systems with singular interaction through hitting times: Application in systemic risk modeling. *Ann. Appl. Probab.*, 29(1):89–129, 02 2019.

[35] S. Nadtochiy and M. Shkolnikov. Mean field systems on networks, with singular interaction through hitting times. *Ann. Probab.*, 48(3):1520–1556, 05 2020.

[36] R. Peyre. Fractional Brownian motion satisfies two-way crossing. *Bernoulli*, 23(4B):3571 – 3597, 2017.

[37] B. Sherman. A general one-phase stefan problem. *Quarterly of Applied Mathematics*, 28(3):377–382, 1970.

[38] J. Stefan. Über einige Probleme der Theorie der Wärmeleitung. *Sitzungber., Wien, Akad. Mat. Natur.*, 98:473–484, 1889.

[39] A. Visintin. Stefan problem with a kinetic condition at the free boundary. *Ann. Mat. Pura Appl. (4)*, 146:97–122, 1987.

[40] W. Whitt. *Stochastic-process limits: an introduction to stochastic-process limits and their application to queues.* Springer Science & Business Media, 2002.

[41] W. Q. Xie. The Stefan problem with a kinetic condition at the free boundary. *SIAM J. Math. Anal.*, 21(2):362–373, 1990.

Optimal bailout strategies resulting from the drift controlled supercooled Stefan problem

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Abstract

We consider the problem faced by a central bank which bails out distressed financial institutions that pose systemic risk to the banking sector. In a structural default model with mutual obligations, the central agent seeks to inject a minimum amount of cash in order to limit defaults to a given proportion of entities. We prove that the value of the central agent's control problem converges as the number of defaultable institutions goes to infinity, and that it satisfies a drift controlled version of the supercooled Stefan problem. We compute optimal strategies in feedback form by solving numerically a forward-backward coupled system of PDEs. Our simulations show that the central agent's optimal strategy is to subsidise banks whose asset values lie in a non-trivial time-dependent region. Finally, we study a linear-quadratic version of the model where instead of the terminal losses, the agent optimises a terminal cost function of the equity values. In this case, we are able to give semi-analytic strategies, which we again illustrate numerically.

Keywords: Systemic risk, Mean-field control, Supercooled Stefan problem, Bail-outs

1 Introduction

In this paper, we analyse a simple mathematical model for a central bank that optimally injects cash into a banking system with interbank lending in order to prevent systemic default events. By way of introduction, we first review known results on the dynamics without intervention and its relation to the classical Stefan problem. We then describe the optimisation problem faced by the central agent and discuss its setting within the literature on mean-field control problems together with this paper's contributions.

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1.1 Interbank lending and the supercooled Stefan problem

We study a market with N financial institutions and their equity value process $X = (X_t^i)$ for $t \in [0, T]$ with finite time horizon $T > 0$ and $i = 1, \dots, N$. We interpret X^i in the spirit of structural credit risk models as the value of assets minus liabilities. Hence, we consider an institution to be defaulted if its equity value hits 0. We refer the reader to [41] for the classical treatment as well as to [35] and the references therein for a discussion of such models in the present multivariate context with mutual obligations.

We consider specifically a stylised model of interbank lending where all firms are exchangeable, their equity values are driven by Brownian motion, and where the default of one firm leads to a uniform downward jump in the equity value of the surviving entities. The latter effect is the crucial mechanism for credit contagion in our model as it describes how the default of one firm affects the balance sheet of others. Here, we follow [33, 42, 40]) to assume that the X^i satisfy

$$X_t^i = X_{0-}^i + B_t^i - \alpha \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{\{\tau^i \le t\}}, \tag{1}$$

where $\tau^i = \inf\{t : X_t^i \le 0\}$, X_{0-}^i are non-negative i.i.d. random variables, $(B^i)_{1 \le i \le N}$ is an N-dimensional standard Brownian motion, independent of $X_{0-} = (X_{0-}^i)_{1 \le i \le N}$, and $\alpha \ge 0$ is a given parameter measuring the interconnectedness in the banking system. The initial condition reflects the current state of the banking system. This might include minimal capital requirements prescribed by the regulator as conditions to enter the banking system, but we do not consider this question explicitly.

Even this highly stylised simple system produces complex behaviour for large pools of firms, including systemic events where cascades of defaults caused by interbank lending instantaneously wipe out significant proportions of the firm pool (see [33, 42, 27]).

One way of analysing this is to pass to the mean-field limit for $N \to \infty$. It is known (see, e.g., [26]) that the interaction (contagion) term in (1) converges (in an appropriate sense) to a deterministic function $\Lambda : [0, T] \to [0, \alpha]$ as $N \to \infty$, i.e.,

$$\alpha \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{\{\tau^i \le t\}} \;\to\; \Lambda_t.$$

Moreover, the X^i are asymptotically independent with the same law as a process X which together with Λ satisfies a probabilisitic version of the *supercooled Stefan problem*, namely

$$X_t = X_{0-} + B_t - \Lambda_t, \quad t \ge 0, \tag{2}$$

where Λ is subject to the constraint

$$\Lambda_t = \alpha \mathbb{P}\Big(\inf_{0 \le s \le t} X_s \le 0 \Big), \quad t \ge 0. \tag{3}$$

Here, B is a standard Brownian motion independent of the random variable X_{0-}, which has the same law as all X_{0-}^i. We refer to [27] for a discussion on how this probabilistic formulation relates to the classical PDE version of the supercooled Stefan problem.

From a large pool perspective (see [33, 42]), X_t may be viewed as the equity value of a representative bank and $\tau = \inf\{t \ge 0 : X_t \le 0\}$ as its default time, while Λ_t

describes the interaction with other institutions under the assumption of uniform lending and exchangeable dynamics. In particular, $\mathbb{P}(\inf_{0\le s\le t} X_s \le 0)$ can be interpreted as the fraction of defaulted banks at time t and consequently Λ_t as the loss that the default of these entities has caused for the survivors.

It is known that solutions to (2), (3) are not unique in general (see [26, 23]), which explains the need to single out so-called *physical* solutions that are meaningful from an economic and physical perspective. Under appropriate conditions on X_{0-}, these physical solutions are characterised by open intervals with smooth $t \mapsto \Lambda_t$, separated by points at which this dependence may only be Hölder continuous or even exhibit a discontinuity, an event frequently referred to as *blow-up* (see [27]). If the mean of the initial values is close enough to zero relative to the interaction parameter α, a jump necessarily happens (see [33]).

In case a discontinuity does occur at some $t \ge 0$, the following restriction on the jump size defines such a *physical solution*:

$$\Lambda_t - \Lambda_{t-} = \inf\Big\{x > 0 : \mathbb{P}\big(\tau \ge t,\, X_{t-} \in (0, x]\big) < \frac{x}{\alpha}\Big\}, \quad t \ge 0, \tag{4}$$

with $\Lambda_{t-} := \lim_{s\uparrow t} \Lambda_s$ and $X_{t-} := \lim_{s\uparrow t} X_s$. By the results of [27], the above definition uniquely defines a solution to (2), (3) under some restrictions on the initial condition X_{0-}. For future reference, we also introduce the concept of *minimal solutions*, which we know to be physical whenever the initial condition is integrable (see [23]). We call a solution $\underline{\Lambda}$ *minimal*, if for any other X that satisfies (2) with loss process Λ, we have that

$$\underline{\Lambda}_t \le \Lambda_t, \quad t \ge 0. \tag{5}$$

1.2 The central agent's optimisation problem

The purpose of this paper is to analyse strategies that a central bank (*central agent*) can take to limit the number of defaults. They achieve this by controlling the rate of capital injected to distressed institutions. That is to say, rather than bailing out firms which are already defaulted, the central agent intervenes already ahead of the time their equity values become critical. This rate of capital[1] received by bank i is determined by processes β^i and added to (1). In the finite dimensional situation, the X^i then satisfy

$$X_t^i = X_{0-}^i + \int_0^t \beta_s^i \, ds + B_t^i - \alpha \frac{1}{N} \sum_{i=1}^N \mathbb{1}_{\{\tau^i \le t\}}. \tag{6}$$

A mathematically similar problem has been studied in [50]. There the question of finding an optimal drift in order to maximize the number of Brownian particles that stay above 0 is treated, however without the singular interaction term appearing in (6).

In anticipation of a propagation of chaos result (proved in Section 2), we therefore

[1] As we are working in continuous time, we also assume that the central agent is able to inject money continuously. At first sight this is a bold approximation of reality but allows us to work without an a-priori fixed time grid determining when the central agent reacts.

consider an extension of (2) and (3) with a drift process β, i.e.,

$$X_t = X_{0-} + \int_0^t \beta_s \, ds + B_t - \Lambda_t, \tag{7}$$

$$\Lambda_t = \alpha \mathbb{P}\Big(\inf_{0 \le s \le t} X_s \le 0 \Big). \tag{8}$$

For most of the paper, we will consider a constraint $0 \le \beta_t \le b_{\max}$, which amounts to the assumption that at any point in time the central agent has limited resources for bail-outs. We will specify further technical conditions on β later, which allow us to show that indeed the finite system converges in a suitable sense to this mean-field problem. We will then analyse the resulting singular mean-field control problem and provide numerical illustrations of the resulting control strategies.

Specifically, we consider now a central agent who injects capital into a representative bank at rate β_t at time t in order to keep

$$L_{T-}(\beta) = \mathbb{P}\Big(\inf_{0 \le s < T} X_s \le 0 \Big) = \Lambda_{T-}(\beta)/\alpha,$$

that is the number of defaults that occur before[2] a given time T, below a specified threshold δ, while minimising the expected total cost

$$C_T(\beta) = \mathbb{E}\Big[\int_0^T \beta_t \, dt \Big].$$

We therefore consider the following constrained optimisation problem: For given δ, the central agent solves

$$C_T(\beta) \longrightarrow \min_\beta \qquad \text{subject to} \qquad L_{T-}(\beta) \le \delta. \tag{9}$$

Define now for $\lambda \in \mathbb{R}_+$ the Lagrange function $\mathcal{L}(\beta, \lambda) = C_T(\beta) + \lambda(L_{T-}(\beta) - \delta)$ and suppose that a saddlepoint $(\beta^\star, \gamma)$ of $\mathcal{L}$ exists, i.e. $\mathcal{L}(\beta^\star, \lambda) \le \mathcal{L}(\beta^\star, \gamma) \le \mathcal{L}(\beta, \gamma)$ for all β, λ. Then $\mathcal{L}(\beta^\star, \gamma) = C_T(\beta^\star) = \inf\{C_T(\beta) : L_{T-}(\beta) \le \delta\}$ and instead of solving (9), the central agent can equivalently solve $\min_\beta \mathcal{L}(\beta, \gamma)$ for the optimal Lagrange multiplier γ. Hence, we shall from now on consider this optimisation problem assuming that $\gamma > 0$ (which can – due to the complementary slackness condition – only be true if the constraint is binding, i.e. $L_{T-}(\beta) = \delta$).

Writing $\underline{X}(\beta)$ for the solution process associated with the minimal solution $\underline{\Lambda}(\beta)$, analogously defined as in (5) but now for (7), we thus minimise the following objective function

$$\begin{aligned} J(\beta) &= \mathbb{E}\Big[\int_0^T \beta_t \, dt \Big] + \gamma \, \mathbb{P}\Big(\inf_{0 \le s < T} \underline{X}_s(\beta) \le 0 \Big) \\ &= \mathbb{E}\Big[\int_0^T \beta_t \, dt + \gamma \, \mathbb{1}_{\{\underline{\hat{X}}_{T-} = 0\}} \Big], \end{aligned} \tag{10}$$

[2]We consider L_{T-} rather than L_T in the constraint because Λ may have a jump discontinuity precisely at T, which would considerably complicate the analysis. However, by [39, Corollary 2.3], we know that solutions to (1) cannot have discontinuities after time $\alpha^2/(2\pi)$. We can derive an analogous result for the controlled case using Girsanov's theorem if, in addition to the pointwise bound on β, we assume a bound on the total cost over the infinite horizon, i.e. $\int_0^\infty \beta_s \, \mathrm{d}s \le c_{\max}$ for some $c_{\max} > 0$. In this setting, by choosing T sufficiently large, we then do not need to distinguish between $L_{T-}(\beta)$ and $L_T(\beta)$.

where $\widehat{X} = \underline{X}_t \mathbb{1}_{\{\tau>t\}}$ is the absorbed minimal solution process and τ the default time. Note that the only difference between $J(\beta)$ and $\mathcal{L}(\beta,\gamma)$ is the constant $-\gamma\delta$, which however does not play a role in the optimisation over β. By varying γ, we can therefore trace out pairs of costs and losses which are solutions to (9) for different δ. The Lagrange multiplier γ (as a function of δ) can then be interpreted as shadow price of preventing further defaults. Indeed, as for usual constrained optimization problems, the optimal cost $C_T^\star$ seen as a function of the loss level δ satisfies under certain technical conditions

$$\partial_\delta C_T^\star(\delta) = \lim_{h\to 0} \frac{C_T^\star(\delta+h) - C_T^\star(\delta)}{h} = -\gamma(\delta).$$

As we show numerically in Section 3, the optimal loss $L_{T-}^\star$ as a fuction of γ is monotone decreasing, so that for large enough γ, the threshold δ becomes small enough to avoid systemic events.

We shall analyse the objective function (10) together with the dynamics (7), which is a *mean-field control problem with a singular interaction* through hitting the boundary, and its relation to existing literature. As we show in Section 2, in particular Theorem 2.7, optimisation of the McKean–Vlasov equation (7) yields the same result as first optimising in the N-particle system and then passing to the limit. In particular, by Theorem 2.9, optimizers of the McKean–Vlasov equation (7) are ϵ-optimal for the N-particle system. In this sense, the mean-field and McKean–Vlasov control problems coincide in this setting. We refer to [18] for a discussion on the differences in general set-ups.

1.3 Relation to the literature

Mean-field and McKean–Vlasov control problems

Due to the vast amount of literature on mean-field control and McKean–Vlasov control problems we focus here on relatively recent works.

For McKean–Vlasov control problems and the corresponding dynamic programing principles we refer to [46, 28]. A formulation using a Bellman equation on the space of probability measures for closed-loop controls can be found in [47, 37]. For a dynamic programming principle for open-loop controls we refer to [10], and to [17, 1] for a characterisation by a stochastic maximum principle. In [4] a characterisation as a coupled system of nonlinear forward and backward PDEs is provided, following the pioneering work on mean-field games (MFGs) in [36] and [34] (see also [11]).

In the latter formulation, which our work mimics to solve (10), a Kolmogorov forward PDE governs the evolution of a probability density describing the mean-field distribution of agents under a feedback control, and a backward HJB equation results from a central planner's control problem given the distribution of all agents' states.[3] The system is hence coupled in two ways, the dependence of the forward equation on the feedback control obtained from the backward equation, and that of the backward equation on the probability density obtained from the forward equation.

As a matter of fact, we deal with subprobability densities here describing the marginal distributions of the absorbed process $\widehat{X} = X_t \mathbb{1}_{\{\tau>t\}}$. We note that the total mass of these

[3]In the case of mean-field games, where each player controls their own actions, the central planner's control problem is replaced by one of a representative player.

subprobability measures as well as the underlying dynamics (7) can exhibit jumps if Λ is discontinuous, and that these jumps emerge endogenously from the feedback mechanism.

This is in contrast to some other recent papers where jumps are exogenously given. For instance, the very recent article [7] considers the control of McKean–Vlasov dynamics with jumps and associated HJB-PIDEs, while in [8] a stochastic maximum principle is derived to anaylse a mean-field game with random jump time penalty and interaction on the control.

In the context of systemic risk and contagion via singular interactions through hitting times the paper [43] is especially relevant. There a game in which the banks determine their (inhomogeneous) connections strategically is analysed. It turns out that by a reduction of lending to critical institutions in equilibrium systemic events can be avoided. Of particular interest for the current work is also [20], which considers the mean-field limit of a system of banks who optimise a quadratic target by controlling their borrowing from, and lending to, a central bank, and where the interaction comes from interbank lending. For general linear-quadratic mean-field control problems see [52], as well as [9] and the references therein for more recent extensions and applications to economics.

We here take inspiration from the model in [20], incorporating however interaction via defaults and considering a central agent who optimises a quadratic target with terminal costs that depend on the firm's final equity value. As it is typical for linear-quadratic optimal control problems, we are able to derive a semi-analytical[4] control strategy based on Riccati ODEs in this set-up. This is not the case in our original model where the objective involves the minimisation of losses. For this first model we therefore rely on numerical methods.

Numerics for mean-field control problems

Among the numerical methods proposed for mean-field control problems, we refer to [21] for a policy gradient-type method where feedback controls are approximated by neural networks and optimised for a given objective function; to [32] and again to [21] for a mean-field FBSDE method, generalising the deep BSDE method to mean-field dependence and in the former case to delayed effects; to [48] for a hybrid model where the mean-field distribution is approximated by a particle system and the control is obtained by numerical approximation of a PDE; and to [6] for a survey of methods for the coupled PDE systems, mainly in the spirit of the seminal works [3, 2, 5]; see also a related semi-Lagrangian scheme in [16]; a gradient method and penalisation approach in [44]; and a recent analysis of policy iteration in [38].

Our modeling and numerical approach is most closely related to these works on PDE systems. What distinguishes the model considered here from all those discussed above is the singular interaction through boundary absorption.

A few works on mean-field games with absorption have appeared recently. In [14], the players' dynamics depends on the empirical measure representing players who have not exited a domain. This is extended to smooth dependence on boundary losses prior to the present time in [15], and to the presence of common noise in [13]. The economic motivation for these models are, among others, systemic risk and bank runs.

[4]The control strategy is a solution of a Riccati ODE that depends on Λ, which is not known explicitely. Therefore we speak of a semi-analytic solution.

1.4 Contributions and findings

Our model differs in a number of fundamental points from the aforementioned works: first, while [14, 15, 13] study mean-field game solutions and equilibria of the N-player game, where each player maximises their own objective, we study the problem of a central planner who specifically seeks to control the number of defaults. Second, in contrast to [15], where the coefficients of the players' processes may depend on the loss process and to [13], which further includes a driver which is a smoothed version of L (hence modeling delayed effects of hitting the boundary), we consider the firm values driven by L directly, resulting in an instantaneous effect of defaults and the emergence of systemic events.

The techniques we use are also entirely different from those in these preceding works. Instead of relying on techniques for MFGs as in [19], we extend the method from [23] to show the convergence of the finite system to the mean-field limit. Moreover, we formulate a coupled PDE system to characterise the value of the control problem, as already discussed earlier. In particular, the forward problem is given by the Stefan problem with a drift term determined by the feedback control.

From an economic point of view, our findings indicate a high sensitivity on the parameter γ in (7). As shown in Figure 5b, for a certain value of γ and in a regime where α triggers jump discontinuities in the uncontrolled regime, the optimal control strategy switches from not avoiding a jump to avoiding a jump. Moreover, our numerical experiments suggest that it is not possible to vary the capital injection to control the *size* of the jump continuously, since the possible jump size is restricted by the physical jump constraint (4). Viewed differently, a large systemic event can happen if the central agent withdraws a small amount of capital from a scenario without jumps. We find similar results also in the linear-quadratic situation with regard to the parametrisation of the terminal cost. There also the potential occurrence of systemic events can depend in a highly sensitive way on the corresponding parameter, which is visible from the jump discontinuity in Figure 12a.

Summarizing, the main contributions of the present paper are as follows:

- We show convergence of the system with N agents to the mean-field limit (see Section 2), including well-posedness of the central agent's optimisation problem, i.e. the existence of unique minimal solutions to the Stefan problem for any suitably regular control process and the existence of an optimal control which minimises (7).

- We provide a formal derivation of the coupled optimality system of forward and backward PDEs satisfied by the density of the equity value process and the central agent's value function, respectively, and propose a numerical scheme (see Section 3).

- We analyse by way of detailed numerical studies the structure of the central agent's optimal strategy in different market environments, and the minimal losses that are attained under optimal strategies with varying cost (see also Section 3).

- We examine a variant where the central agent minimises a quadratic target with terminal costs depending on the final state of the equity process. In this case we are able to derive a semi-analytic solution which we analyse also numerically (see Section 4).

2 Convergence to a mean-field limit

In this section, we show the existence of a minimising strategy for the central agent's objective function in the mean-field limit, as well as convergence of the N-agent control problem.

2.1 The model setup

We define a reference probability space to be a tuple $\mathscr{S} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ such that $\mathscr{S}$ supports a Brownian motion that is adapted to $(\mathcal{F}_t)_{t\geq 0}$ and there is a $\mathcal{F}_0$-measurable random variable X_{0-} whose law is fixed. Note that with this definition, X_{0-} is independent of B by construction. We endow the space

$$S_T := \{f \in L^2([0,\infty)) \mid 0 \leq f \leq b_{\max} \text{ a.e., } f_{|_{(T,\infty)}} = 0\} \tag{11}$$

with the topology of weak convergence in $L^2([0,\infty))$. Since S_T is bounded in the $L^2([0,\infty))$-norm and weakly closed, S_T is a compact Polish space. We then define the space of admissible controls

$$\mathcal{B}_T := \{\beta \text{ is } (\mathcal{F}_t)_{t\geq 0} - \text{progressively measurable} \mid \mathbb{P}(\beta \in S_T) = 1\}. \tag{12}$$

Note that the space of admissible controls $\mathcal{B}_T$ as well as the objective functional J as defined in (10) depend implicitly on the choice of stochastic basis $\mathscr{S}$. We will sometimes write $\mathcal{B}_T(\mathscr{S})$ or $J^{(\mathscr{S})}$ when we wish to emphasize this dependence. To be able to guarantee existence of optimizers and to make the optimization problem independent of the choice of stochastic basis $\mathscr{S}$, we will consider the relaxed optimization problem

$$V_\infty := \inf_{\mathscr{S}} \inf_{\beta \in \mathcal{B}_T(\mathscr{S})} J^{(\mathscr{S})}(\beta), \tag{13}$$

as is standard in the stochastic optimal control literature (see e.g. [30])

Note that it is not clear a priori that the process X given in (7) is well-defined. Indeed, it is known that the McKean–Vlasov problem (2) and (3) may admit more than one solution, and it is not known that physical solutions exist for general $\beta \in \mathcal{B}_T$, although it is known for β of the special form $b(t, X_t)$, where b is Lipschitz (see e.g. [26, 39]). To pin down a meaningful solution concept, we therefore rely on the notion of *minimal solutions* as defined in (5). By the results of [23], we know that minimal solutions of the uncontrolled system are physical whenever the initial condition X_{0-} is integrable.

Throughout the following sections $\mathcal{P}(E)$ always denotes the set of probability measures on a Polish space E. For function spaces etc. we apply rather standard notation and refer to Section C for more details.

2.2 Well-posedness of minimal solutions for general drift

We now show that minimal solutions exist for general $\beta \in \mathcal{B}$. Define the operator Γ for a càdlàg function ℓ and $\beta \in \mathcal{B}_T$ as

$$\begin{cases} X_t^\ell(\beta) = X_{0-} + \int_0^t \beta_s \, ds + B_t - \alpha \ell_t, \\ \tau_\beta^\ell = \inf\{t \geq 0 : X_t^\ell(\beta) \leq 0\}, \\ \Gamma[\ell, \beta]_t = \mathbb{P}\left(\tau_\beta^\ell \leq t\right). \end{cases} \tag{14}$$

Note here that $(X^\ell(\beta), \tau_\beta^\ell, \ell)$ solves (7) if and only if ℓ is a fixed-point of $\Gamma[\cdot, \beta]$. We next introduce a function space that is mapped to itself by $\Gamma[\cdot, \beta]$: Set

$$M := \{\ell \colon \overline{\mathbb{R}} \to [0,1] \mid \ell \text{ càdlàg and increasing}, \;\; \ell_{0-} = 0, \;\; \ell_\infty = 1\}, \tag{15}$$

where $\overline{\mathbb{R}}$ is the extended real line. Note that for $\ell \in M$, ℓ defines a cumulative distribution function of a probability measure on $[0, \infty]$. Therefore, equipping M with the topology of weak convergence, i.e., we have that $\ell^n \to \ell$ in M if and only if $\ell_t^n \to \ell_t$ for all $t \in [0, \infty]$ that are continuity points of ℓ, we obtain that M is a compact Polish space. As in the uncontrolled case, $\Gamma[\cdot, \beta]$ is continuous on M.

Theorem 2.1. *For any $\beta \in \mathcal{B}_T$, the operator $\Gamma[\cdot, \beta] \colon M \to M$ is continuous. Furthermore, there is a (unique) minimal solution to* (7)*, and it is given by*

$$\underline{\Lambda}(\beta) = \alpha \lim_{k \to \infty} \Gamma^{(k)}[0, \beta]. \tag{16}$$

Proof. Using Lemma C.1, this follows from Theorem 2.3 in [22]. □

2.3 Existence of an optimal control

We say that $\text{law}(\beta^n) \to \text{law}(\beta)$ in $\mathcal{P}(S_T)$, if $\text{law}(\beta^n) \to \text{law}(\beta)$ in the weak topology[5], i.e., if for any $F \in C_b(S_T; \mathbb{R})$ we have $\mathbb{E}(F(\beta^n)) \to \mathbb{E}(F(\beta))$. A key step in proving existence of an optimizer is to show that if $\text{law}(\beta_n) \to \text{law}(\beta)$ in $\mathcal{P}(S_T)$, then the sequence of solutions $(\Lambda(\beta_n))$ associated to the McKean–Vlasov problem converges to a solution of the McKean–Vlasov problem with drift β. This is the content of the next theorem.

Theorem 2.2. *Let $\text{law}(\beta^n) \to \text{law}(\beta)$ in $\mathcal{P}(S_T)$. Then, after passing to a subsequence if necessary, there is $\Lambda \in \alpha M$ such that $\frac{1}{\alpha}\underline{\Lambda}(\beta^n) \to \frac{1}{\alpha}\Lambda$ in M, and Λ solves the McKean–Vlasov problem* (7) *with drift β.*

Proof. See Section C.2 in the Appendix. □

Next, we prove that the infinite-dimensional problem (13) admits an optimizer.

Theorem 2.3. *There is an optimizer of* (13)*, i.e., there is a stochastic basis $\mathscr{S}^\star$ and $\beta^\star \in \mathcal{B}_T(\mathscr{S}^\star)$ such that*

$$V_\infty = \inf_{\mathscr{S}} \inf_{\beta \in \mathcal{B}_T(\mathscr{S})} J^{(\mathscr{S})}(\beta) = J^{(\mathscr{S}^\star)}(\beta^\star).$$

[5] in the sense of probability theory

Proof. Let $\mathscr{S}^n = (\Omega^n, \mathcal{F}^n, \mathbb{P}^n)$ and $\beta^n \in \mathcal{B}_T(\mathscr{S}^n)$ be such that $J^{(\mathscr{S}^n)}(\beta^n) \leq V_\infty + \frac{1}{n}$, and let B^n be a Brownian motion with respect to $\mathscr{S}^n$. Set $W^n := X^n_{0-} + B^n$. Since $C([0,\infty))$ is a Polish space, for any $\epsilon > 0$, there is $K_c \subseteq C([0,\infty))$ compact such that $\mathbb{P}^n(W^n \in K_c) > 1-\epsilon$ for all $n \in \mathbb{N}$. It then follows that $\mathbb{P}^n((W^n, \beta^n) \in K_c \times S_T) = \mathbb{P}^n(W^n \in K_c) > 1-\epsilon$, and since S_T is compact, the sequence (W^n, β^n) is tight on $C([0,\infty)) \times S_T$. By Prokhorov's theorem, after passing to a subsequence if necessary, we may assume that $\mathrm{law}((W^n, \beta^n)) \to \mathrm{law}((W, \beta^\star))$ in $\mathcal{P}(C([0,\infty)) \times S_T)$. By the Skorokhod representation theorem, we may without loss of generality assume that $(W^n, \beta^n) \to (W, \beta^\star)$ holds almost surely on some probability space, which we denote by $\mathscr{S}^\star = (\Omega^\star, \mathcal{F}^\star, \mathbb{P}^\star)$. We then define $\mathcal{F}^\star_t$ to be $\sigma(\{(W_s, \int_0^s \beta^\star_u\, ds), s \leq t\})$ for $t \geq 0$. Since $\beta^\star$ is measurable and $(\mathcal{F}^\star_t)_{t\geq 0}$-adapted, it admits a progressively measurable modification, which we again denote by $\beta^\star$, and we see that $\beta^\star \in \mathcal{B}_T(\mathscr{S}^\star)$. We next show that $B := W - W_0$ is a $(\mathcal{F}^\star_t)_{t\geq 0}$-Brownian motion: by the continuous mapping theorem and the continuity of the coordinate projections, it follows that $\mathrm{law}(B)$ is the Wiener measure. Let $0 < r_1 < \cdots < r_k < s < t$ and let $f_i \in C_b(\mathbb{R}^2; \mathbb{R}), i = 1, \ldots, k$ and $g \in C_b(\mathbb{R}; \mathbb{R})$ and set $F(w^1, w^2) := \prod_{i=1}^k f_i(w^1_{r_i}, w^2_{r_i})$, then we have by dominated convergence

$$\begin{aligned}
\mathbb{E}^\star\left[F\left(W, \int_0^\cdot \beta^\star_u\, du\right) g(B_t - B_s)\right] &= \lim_{n\to\infty} \mathbb{E}^\star\left[F\left(W^n, \int_0^\cdot \beta^n_u\, du\right) g(B^n_t - B^n_s)\right] \\
&= \lim_{n\to\infty} \mathbb{E}^\star\left[F\left(W^n, \int_0^\cdot \beta^n_u\, du\right)\right] \mathbb{E}^\star\left[g(B^n_t - B^n_s)\right] \\
&= \mathbb{E}^\star\left[F\left(W, \int_0^\cdot \beta^\star_u\, du\right)\right] \mathbb{E}^\star\left[g(B_t - B_s)\right].
\end{aligned}$$

Since the Borel σ-algebra on $C([0,\infty))$ is generated by the evaluation mappings, we obtain that $B_t - B_s$ is independent of $\mathcal{F}^\star_s$ for $0 < s < t$, and therefore B is an $(\mathcal{F}^\star_t)_{t\geq 0}$-Brownian motion. The continuous mapping theorem applied to $(w, b) \mapsto w_0$ for $w \in C([0,\infty), b \in S_T$ shows that $\mathrm{law}(W_0) = \mathrm{law}(X_{0-})$. We have shown that $\mathscr{S}^\star$ is an admissible reference space.

In the following, we simply write J instead of $J^{(\mathscr{S}^\star)}$. By construction, we have $J(\beta^n) \leq V_\infty + \frac{1}{n}$ and hence $\liminf_{n\to\infty} J(\beta^n) \leq V_T$. We proceed to show that $J(\beta^\star) \leq \liminf_{n\to\infty} J(\beta^n)$.

By dominated convergence, it holds that

$$\lim_{n\to\infty} \mathbb{E}^\star\left[\int_0^T \beta^n_s\, ds\right] = \mathbb{E}^\star\left[\int_0^T \beta^\star_s\, ds\right]. \tag{17}$$

By Theorem 2.2, we may pass to subsequences and assume without loss of generality that $\frac{1}{\alpha}\underline{\Lambda}(\beta^n) \to \frac{1}{\alpha}\Lambda$ in M, where Λ solves the McKean–Vlasov problem (7) with drift $\beta^\star$. The Portmanteau theorem implies that

$$\liminf_{n\to\infty} \underline{\Lambda}_{T-}(\beta^n) \geq \Lambda_{T-}, \tag{18}$$

and since Λ solves (7) with drift β and $\underline{\Lambda}(\beta)$ is the minimal solution with drift β, it follows that $\underline{\Lambda}_{T-}(\beta) \leq \Lambda_{T-}$ which is equivalen to $\underline{L}_{T-}(\beta) \leq L_{T-}$, which concludes the proof. $\square$

2.4 Properties of the controlled N-particle system

Definition 2.4. Set $X^N := (X^{1,N}, \dots, X^{N,N})$, where the $X^{i,N}$ are real-valued random variables. We say that X^N is N-exchangeable, if

$$\mathrm{law}(X^N) = \mathrm{law}((X^{\sigma(1),N}, X^{\sigma(2),N}, \dots, X^{\sigma(N),N})),$$

for any permutation σ of $\{1, \dots, N\}$.

We describe the controlled N-particle system mentioned in the introduction in more detail. We consider a stochastic basis $\mathscr{S}_N = (\Omega_N, \mathcal{F}^N, (\mathcal{F}_t^N)_{t\geq 0}, \mathbb{P}_N)$ supporting an N-dimensional Brownian motion B^N and an N-exchangeable, $\mathcal{F}_0^N$-measurable random vector X_{0-}^N. The particles in the system then satisfy the dynamics

$$X_t^{i,N} := X_{0-}^{i,N} + \int_0^t \beta_s^{i,N}\, ds + B_t^{i,N} - \Lambda_t^N, \tag{19}$$

where β^N is N-exchangeable, and $\Lambda_t^N = \frac{\alpha}{N}\sum_{i=1}^N \mathbb{1}_{\{\tau^{i,N}\leq t\}}$, where $\tau_{i,N} := \inf\{t \geq 0 : X_t^{i,N} \leq 0\}$. In analogy to the infinite-dimensional case, we denote $L^N := \frac{1}{\alpha}\Lambda^N$. We then consider the set of admissible controls

$$\mathcal{B}_T^N := \{\beta^N \text{ is } N\text{-exchangeable, } (\mathcal{F}_t^N)_{t\geq 0} - \text{progressively measurable} \mid \mathbb{P}(\beta^{1,N} \in S_T) = 1\}. \tag{20}$$

The same examples as in the uncontrolled case show that solutions to (19) are not unique in general (cf. Section 3.1.1 in [26]). Therefore, in [25], so-called physical solutions are introduced. We call a solution $\underline{\Lambda}^N$ to (19) minimal, if for every solution Λ^N to (19)

$$\underline{\Lambda}_t^N \leq \Lambda_t^N, \quad t \geq 0,$$

holds almost surely. The same argument as in [23, Lemma 3.3] shows that the notions of physical and minimal solution are equivalent in the controlled N-particle system. In analogy to the infinite-dimensional case, we introduce the operator

$$\begin{cases} X_t^{i,N}[\mathsf{L}, \beta^N] = X_{0-}^{i,N} + \int_0^t \beta_s^{i,N}\, ds + B_t^{i,N} - \alpha \mathsf{L}_t \\ \tau_{i,N}[\mathsf{L}, \beta^N] = \inf\{t \geq 0 : X_t^{i,N}[\mathsf{L}, \beta^N] \leq 0\} \\ \Gamma_N[\mathsf{L}, \beta^N]_t = \dfrac{1}{N}\displaystyle\sum_{i=1}^N \mathbb{1}_{\left\{\tau_{i,N}[\mathsf{L},\beta^N]\leq t\right\}}, \end{cases} \tag{21}$$

where L is some càdlàg process adapted to the filtration generated by B^N. We will often simply write $\Gamma_N[\mathsf{L}]$ instead of $\Gamma_N[\mathsf{L}, \beta^N]$. The statements are then meant to hold for arbitrary, fixed β^N. An important property is that $\Gamma_N[\cdot, \beta^N]$ is monotone in the sense that

$$\mathsf{L}_t^1 \leq \mathsf{L}_t^2, \quad t \geq 0 \quad \Longrightarrow \quad \Gamma_N[\mathsf{L}^1, \beta^N]_t \leq \Gamma_N[\mathsf{L}^2, \beta^N]_t, \quad t \geq 0.$$

We then readily see by straightforward induction arguments that

$$\alpha\Gamma_N^{(k)}[0] \leq \Lambda^N, \quad \Gamma_N^{(k)}[0] \leq \Gamma_N^{(k+1)}[0], \quad k \in \mathbb{N}, \tag{22}$$

holds almost surely, where Λ^N is any solution to the particle system and $\Gamma_N^{(k)}$ denotes the k-th iterate of Γ_N. A similar argument as for the system without drift in [23] shows that the iteration $(\Gamma_N^{(k)}[0])_{k\in\mathbb{N}}$ converges to the minimal solution after at most N iterations.

Lemma 2.5. *For $N \in \mathbb{N}$, let Γ_N be defined as in* (21)*. Then $\underline{\Lambda}^N := \alpha\Gamma_N^{(N)}[0, \beta^N]$ is the minimal solution to the particle system with drift β^N and the error bound*

$$\|\alpha\Gamma_N^{(k)}[0, \beta^N] - \underline{\Lambda}^N\|_\infty \leq \alpha\frac{(N-k)^+}{N} \tag{23}$$

holds almost surely.

Proof. Analogous to the proof of Lemma 3.1 in [23]. □

Roughly speaking, the next result says that limit points (in distribution) of solutions to the controlled N-particle system converge (along subsequences) to solutions to the controlled McKean–Vlasov equation (7). By $D[-1, \infty)$ we denote here the space of càdlàg paths on $[-1, \infty)$ equipped with the M_1-topology.

Theorem 2.6. *For $N \in \mathbb{N}$, let $(X^N, \beta^N, \Lambda^N)$ be a solution to the particle system* (19) *on the stochastic basis $\mathscr{S}_N$ and define $\mu_N := \frac{1}{N}\sum_{i=1}^N \delta_{X^{i,N}}$. Suppose that for some measure $\nu_{0-} \in \mathcal{P}(\mathbb{R})$ we have*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N \delta_{X_{0-}^{i,N}} = \nu_{0-}.$$

Then there is a subsequence (again denoted by N) such that $\mathrm{law}(\mu_N) \to \mathrm{law}(\mu)$ *in $\mathcal{P}(\mathcal{P}(D([-1, \infty))))$, where μ coincides almost surely with the law of a solution process X to the McKean–Vlasov problem* (7) *satisfying* $\mathrm{law}(X_{0-}) = \nu_{0-}$.

Proof. See Section C.4 □

The next theorem shows that when we optimize the policy in the particle system and then take the limit of the resulting optimal values, we obtain the same value that we find by optimizing the inifinite-dimensional version of the problem. In the context of the distinction between mean field games and controlled McKean–Vlasov problems, we show that the diagram in Section 2.4 in [18] is commutative in this case.

Theorem 2.7. *For $\kappa \in (0, 1/2)$, define the value of a perturbed controlled N-particle system as*

$$V_N := \inf_{\mathscr{S}_N} \inf_{\beta^N \in \mathcal{B}(\mathscr{S}_N)} J_N(\beta^N), \quad J_N(\beta^N) := \mathbb{E}^N\left[\int_0^T \beta_s^{1,N}\, ds + \gamma\underline{\tilde{L}}_{T-}^N(\beta^N)\right],$$

where $\underline{\tilde{L}}^N(\beta^N) := \frac{1}{\alpha}\underline{\tilde{\Lambda}}^N(\beta^N)$ and $\underline{\tilde{\Lambda}}^N(\beta^N)$ is the minimal solution of the controlled N-particle system with drift β^N as introduced in Lemma 2.5 and perturbed initial condition $\tilde{X}_{0-}^{i,N} = X_{0-}^i + N^{-\kappa}$ for $\kappa \in (0, 1/2)$ and all $i = 1, \ldots, N$. Then it holds that

$$\lim_{N\to\infty} V_N = V_\infty. \tag{24}$$

Proof. Step 1: We show the inequality $\liminf_{N\to\infty} V_N \geq V_\infty$. To that end, choose $\mathscr{S}_N$ and $\beta^N \in \mathcal{B}_T(\mathscr{S}_N)$ such that

$$\mathbb{E}^N\left[\int_0^T \beta_s^{1,N}\, ds + \gamma \underline{\tilde{L}}_{T-}^N(\beta^N)\right] \leq V_N + \frac{1}{N}.$$

Arguing as in the proof of Theorem 2.6, we see that $\xi_N = \frac{1}{N}\sum_{i=1}^N \delta_{(X_{0-}^i + N^{-\gamma} + B, \beta^N, \underline{\tilde{L}}^N(\beta^N))}$ is tight, and by Theorem 2.6 converges to a limit ξ that is supported on the set of solutions to the McKean–Vlasov problem (7). By Skorokhod representation, we may assume that this happens almost surely on some stochastic basis $\mathscr{S}$. Since the map $(w, b, \ell) \mapsto \int_0^T b_s\, ds + \gamma \ell_{T-}$ is bounded and lower semicontinuous on $C([0,\infty)) \times S_T \times M$, Fatou's Lemma and the Portmanteau theorem imply

$$\begin{aligned}\liminf_{N\to\infty} \mathbb{E}^N\left[\int_0^T \beta_s^{1,N}\, ds + \gamma \underline{\tilde{L}}_{T-}^N(\beta^N)\right] &= \liminf_{N\to\infty} \mathbb{E}\left[\int\left(\int_0^T b_s\, ds + \gamma \ell_{T-}\right)\, \mathrm{d}\xi_N(w,b,\ell)\right] \\ &\geq \mathbb{E}\left[\liminf_{N\to\infty} \int\left(\int_0^T b_s\, ds + \gamma \ell_{T-}\right)\, \mathrm{d}\xi_N(w,b,\ell)\right] \\ &\geq \mathbb{E}\left[\int\left(\int_0^T b_s\, ds + \gamma \ell_{T-}\right)\, \mathrm{d}\xi(w,b,\ell)\right].\end{aligned}$$

Defining $\mathscr{S}(\omega) = (C([0,\infty)) \times S_T \times M, \mathscr{B}(C([0,\infty)) \times S_T \times M), \xi(\omega))$, let (w, b, ℓ) denote the canoncial process on $\mathscr{S}$. By the arguments in the proof of Theorem 2.6 we have that w is Brownian motion under $\xi(\omega)$ with respect to the filtration generated by (w, b, ℓ), and we see that $\mathscr{S}(\omega)$ is an admissible reference space for almost every ω. Since $\xi(\omega)$ corresponds to the law of a solution to the McKean–Vlasov problem (7), it follows that $\int\left(\int_0^T b_s\, ds + \gamma \ell_{T-}\right)\, \mathrm{d}\xi(w,b,\ell) \geq V_\infty$ almost surely. We have therefore obtained

$$\liminf_{N\to\infty} V_N \geq \liminf_{N\to\infty} \mathbb{E}^N\left[\int_0^T \beta_s^{1,N}\, ds + \gamma \underline{\tilde{L}}_{T-}^N(\beta^N)\right] \geq V_\infty. \tag{25}$$

Step 2: We show that $\limsup_{N\to\infty} V_N \leq V_\infty$. Let $\mathscr{S}^\star$ be a probability space and $\beta^\star \in \mathcal{B}(\mathscr{S}^\star)$ be an optimizer attaining V_∞, whose existence was shown in Theorem 2.3. Let $\mathscr{S}_N^\star$ be the product space obtained by taking N copies of $\mathscr{S}^\star$, and consider the (random) cost functional

$$c_N(b^N, \ell) = \int_0^T b_s^{1,N}\, ds + \gamma \ell_{T-} + N^\kappa \|\Gamma_N[\ell, b^N] - \ell\|_\infty, \quad b^N \in S_T^N,\ \ell \in M, \tag{26}$$

where $\|\cdot\|_\infty$ is the supremum norm on $[0,\infty)$. Let $\mathsf{M}(\mathscr{S})$ be the set of all $(\mathcal{F}_t)$-adapted processes in M on $\mathscr{S}$ and consider the problem

$$\hat{V}_N := \inf_{\substack{\beta^N \in \mathcal{B}(\mathscr{S}_N^\star) \\ \mathsf{L} \in \mathsf{M}(\mathscr{S}_N^\star)}} \mathbb{E}_\star^N\left[c_N(\beta^N, \mathsf{L})\right]. \tag{27}$$

Letting $(\beta^\star)^N$ be the vector obtained by taking N i.i.d. copies of $\beta^\star$, and choosing $\mathsf{L} \equiv \underline{L}(\beta^\star)$, which we abbreviate in the following with $\underline{L} := \underline{L}(\beta^\star)$, we obtain

$$\hat{V}_N \leq \mathbb{E}_\star^N\left[\int_0^T \beta_s^\star\, ds + \gamma \underline{L}_{T-} + N^\kappa \|\Gamma_N[\underline{L}, (\beta^\star)^N] - \underline{L}\|_\infty\right].$$

Noting that $\Gamma_N[\underline{L}, (\beta^\star)^N]$ is the empirical cumulative distribution function of the i.i.d. random variables $\tau^i_{\beta^\star} := \inf\{t \geq 0 : X^i_{0-} + \int_0^t \beta^{\star,i}_s \mathrm{d}s + B^i_t - \underline{\Lambda}_t \leq 0\}$, and that $\mathbb{P}(\tau^i_{\beta^\star} \leq t) = \underline{L}_t(\beta^\star)$, the same estimates as in Step 1 of the proof of Proposition 6.1 in [23] show that

$$\lim_{N\to\infty} \mathbb{E}\left[N^\kappa \|\Gamma_N[\underline{L}, (\beta^\star)^N] - \underline{L}\|_\infty\right] = 0.$$

We have therefore shown that

$$\limsup_{N\to\infty} \hat{V}_N \leq \mathbb{E}^N_\star \left[\int_0^T \beta^\star_s \, ds + \gamma \underline{L}_{T-}(\beta^\star)\right] = V_\infty.$$

Now choose a sequence $\hat{\beta}^N \in \mathcal{B}(\mathscr{S}^\star_N), \hat{\mathsf{L}}^N \in \mathsf{M}(\mathscr{S}^\star)$ such that $\mathbb{E}^N_\star[c(\hat{\beta}^N, \hat{\mathsf{L}}^N)] \leq \hat{V}_N + \frac{1}{N}$. Note that we may assume without loss of generality that $\|\Gamma_N[\hat{\mathsf{L}}^N, \hat{\beta}^N] - \hat{\mathsf{L}}^N\|_\infty \leq N^{-\kappa}$ holds almost surely, since otherwise we can decrease the value of c by redefining $\hat{\mathsf{L}}^N$ to be equal to $\underline{L}^N(\hat{\beta}^N)$ on the set where the inequality is violated. In particular, this implies

$$\hat{\mathsf{L}}^N \geq \Gamma_N[\hat{\mathsf{L}}^N, \hat{\beta}^N] - N^{-\kappa}. \tag{28}$$

Since $\hat{\mathsf{L}}^N \geq -N^{-\kappa}$, the monotonicity of Γ_N implies that

$$\hat{\mathsf{L}}^N \geq \Gamma_N[-N^{-\kappa}, \hat{\beta}^N] - N^{-\kappa} = \tilde{\Gamma}_N[0, \hat{\beta}^N] - N^{-\kappa} \tag{29}$$

where $\tilde{\Gamma}_N$ is defined as in (21) with initial condition $\tilde{X}^{i,N}_{0-} := X^i_{0-} + \alpha N^{-\kappa}$. Combining (29) with (28) and again using the monotonicity of Γ_N, we obtain

$$\hat{\mathsf{L}}^N \geq \Gamma_N[\tilde{\Gamma}_N[0, \hat{\beta}^N] - N^{-\kappa}, \hat{\beta}^N] - N^{-\kappa} = \tilde{\Gamma}^{(2)}_N[0, \hat{\beta}^N] - N^{-\kappa}.$$

A straightforward induction then shows that $\hat{\mathsf{L}}^N \geq \tilde{\Gamma}^{(k)}_N[0, \hat{\beta}^N] - N^{-\kappa}$ for all $k \in \mathbb{N}$, and Lemma 2.5 then yields that we have

$$\hat{\mathsf{L}}^N \geq \underline{\tilde{L}}^N(\hat{\beta}^N) - N^{-\kappa},$$

where $\underline{\tilde{L}}^N(\hat{\beta}^N)$ corresponds to the loss process associated to the particle system with initial condition $\tilde{X}^N_{0-}$. This yields

$$\begin{aligned} V_N - \gamma N^{-\kappa} &\leq \mathbb{E}^N_\star \left[\int_0^T \hat{\beta}^{1,N}_s \, ds + \gamma \underline{\tilde{L}}^N_{T-}(\hat{\beta}^N)\right] - \gamma N^{-\kappa} \leq \mathbb{E}^N_\star \left[\int_0^T \hat{\beta}^{1,N}_s \, ds + \gamma \hat{\mathsf{L}}^N_{T-}\right] \\ &\leq \mathbb{E}^N_\star \left[c(\hat{\beta}^N, \hat{\mathsf{L}}^N)\right] \leq \hat{V}_N + \frac{1}{N}. \end{aligned}$$

Since we have already shown that $\limsup_{N\to\infty} \hat{V}_N \leq V_\infty$, this shows $\limsup_{N\to\infty} V_N \leq V_\infty$, which concludes the proof. $\square$

Remark 2.8. We conjecture that the perturbation in the initial condition of the particle system in Theorem 2.7 is an artefact of our proof technique rather than a necessity.

Theorem 2.9. *Let $\mathscr{S}^\star$ be a probability space and $\beta^\star \in \mathcal{B}(\mathscr{S}^\star)$ be an optimizer attaining V_∞. Let $\mathscr{S}_N^\star$ be the product space obtained by taking N copies of $\mathscr{S}^\star$ and let $(\beta^\star)^N$ be the vector obtained by taking N i.i.d. copies of $\beta^\star$. Then, $(\beta^\star)^N$ is ϵ-optimal for the particle system, i.e., for every $\epsilon > 0$, it holds that*

$$J_N((\beta^\star)^N) \leq V_N + \epsilon$$

for N sufficiently large.

Proof. Let $\epsilon > 0$ be given. Recall the notation introduced in Step 2 of the proof of Theorem 2.7. Consider the problem

$$\bar{V}_N := \inf_{\mathsf{L} \in \mathsf{M}(\mathscr{S}_N^\star)} \mathbb{E}_\star^N \left[c_N((\beta^\star)^N, \mathsf{L}) \right]. \tag{30}$$

Proceeding as in Step 2 of the proof of Theorem 2.7, it follows that $\limsup_{N\to\infty} \bar{V}_N \leq V_\infty$. By Theorem 2.7, we have $\lim_{N\to\infty} V_N = V_\infty$, and therefore $\bar{V}_N \leq V_N + \epsilon/3$ for N large enough. Arguing as in Step 2 of the proof of Theorem 2.7, we can find $\mathsf{L}^N \in \mathsf{M}(\mathscr{S}_N^\star)$ such that $\mathbb{E}[c_N((\beta^\star)^N, \mathsf{L}^N)] \leq \bar{V}_N + \epsilon/3$ and $\mathsf{L}^N \geq \underline{\tilde{L}}^N((\beta^\star)^N) - N^{-\kappa}$ holds. Choosing N large enough such that $\gamma N^{-\kappa} < \epsilon/3$, we obtain

$$J_N((\beta^\star)^N) \leq \mathbb{E}_\star^N \left[\int_0^T \beta_s^\star \, ds + \gamma \hat{\mathsf{L}}_{T-}^N \right] + \gamma N^{-\kappa} \leq \bar{V}_N + 2\epsilon/3 \leq V_N + \epsilon.$$

□

3 Computational analysis of central agent's strategy

In this section, we state the optimality conditions for the mean-field control problem in the form of a forward equation for the density of the absorbed process and a backward equation of HJB type for the central agent's value function. We propose an iterative procedure for the solution of the coupled system, with details of the finite difference schemes used given in Appendix B. Detailed numerical studies illustrate the structure of the optimal controls and resulting behaviour of the controlled system.

Throughout this and the next section, $\widehat{X}$ stands for the absorbed minimal solution of (7), where we omit for notational convenience the underscore used in the previous sections. Indeed, the results of [22] in the uncontrolled setting suggest that the numerical scheme proposed below converges to this minimal (and physical) solution.

3.1 Optimality conditions and coupled forward-backward system

We assume the optimal control to be of feedback form, $\beta_t^\star = \beta^\star(t, X_t)$, where we slightly abuse notation to denote by $\beta^\star$ also a sufficiently regular function $\beta^\star : [0,T] \times [0,\infty) \to \mathbb{R}$. Let $\mathbb{T}$ be the set of all $t \in [0,T]$ where $t \to L_t$ is differentiable. Then the density p of the absorbed process $\widehat{X}$ associated with (7) satisfies the forward Kolmogorov equation

$$\begin{aligned} &\partial_t p + \partial_x(\beta^\star p) = \frac{1}{2}\partial_{xx} p + \dot{\Lambda}_t \partial_x p, \quad x \geq 0, \quad t \in \mathbb{T}, \\ &p(0,x) = f(x), \quad x \geq 0 \quad \text{and} \quad p(t,0) = 0, \quad t \in \mathbb{T}, \end{aligned} \tag{31}$$

assuming a sufficiently regular initial state, where

$$\Lambda_t = \alpha\Big(1 - \int_0^\infty p(t,x)\,dx\Big), \qquad t \in \mathbb{T}. \tag{32}$$

In the event of a blow-up at t, we use the following jump condition for the solution of (31), $p(t-,x) = p(t, x - \Lambda_t + \Lambda_{t-})$, otherwise p is continuous in t.

Conversely, for a given loss function Λ, the value function associated with (10) is

$$v(t,x) \quad = \quad \inf_{\beta\in\mathcal{B}} \mathbb{E}\Big[\int_t^T \beta_s\,ds + \gamma\,\mathbb{1}_{\{\widehat{X}_T=0\}} \Big| \widehat{X}_t = x \Big], \tag{33}$$

and satisfies the HJB equation

$$\begin{aligned} &\partial_t v + \min_{b\in[0,b_{\max}]} \{b\partial_x v + b\} - \dot{\Lambda}_t \partial_x v + \frac{1}{2}\partial_{xx} v = 0, \quad x \geq 0, \quad t \in \mathbb{T}, \\ &v(T,x) = \gamma \mathbb{1}_{x=0}, \quad x \geq 0 \quad \text{and} \quad v(t,0) = \gamma, \quad t \in \mathbb{T}. \end{aligned} \tag{34}$$

In blow-up points we use again a jump condition, $v(t-,x) = v(t, x - \Lambda_t + \Lambda_{t-})$.

Under optimality, the forward equation (31) and backward equation (34) are coupled both ways: the optimal feedback control $\beta^\star$ to be inserted in (31) has to satisfy

$$\beta^\star(t,x) \in \operatorname{argmin}_{b\in[0,b_{\max}]} \{ b\,(\partial_x v(t,x) + 1)\} \tag{35}$$

for the solution v of (34); and the term $\dot{\Lambda}_t$ in (34) depends on the solution p of (31) via (32). Having found the value function v and the terminal loss $L_T^\star$ under the optimal control $\beta^\star$,

$$L_T^\star = 1 - \int_0^\infty p(T,x)\,dx, \tag{36}$$

i.e. the fraction of firms that have defaulted, we can use (33) to work out the expected cost to the central planner,

$$C_T^\star = \mathbb{E}\Big[\int_0^T \beta_s^\star\,ds \Big] \quad = \quad \int_0^\infty v(0,x) f(x)\,dx - \gamma L_T^\star, \tag{37}$$

with $L_T^\star$ from (36).

The optimizer in (35) is given by

$$\beta^\star(t,x) = \begin{cases} 0 & \partial_x v(t,x) > -1, \\ b_{\max} & \partial_x v(t,x) < -1, \end{cases} \tag{38}$$

determining an action region and a no-action region. Hence, one might expect the following structure of the control: there exists a critical value $\xi_+(t)$ such that the controller does nothing if $X_t > \xi_+(t)$ and subsidises at the maximum rate for $X_t < \xi_+(t)$ until X is back in the no-action region. We will see in the numerical experiments that the action region is indeed often, but not always, of this form.

3.2 Numerical solution

There is a sizeable literature on the numerical solution of mean-field control problems and mean-field games. We refer the reader to the references given in the introduction and specifically to [6] for a recent survey of methods related to our broad approach. What makes our model specific and different from the classical mean-field setting is the singular interaction through hitting times. We will therefore propose our own numerical approach exploiting these features.

We will work with the equivalent system of PDEs for $q(t,x) = p(t, x - \Lambda_t)$, $u(t,x) = v(t, x - \Lambda_t)$, $\nu(t,x) = \beta(t, x - \Lambda_t)$:

$$\begin{aligned} &\partial_t q + \partial_x(\nu(t,x)q) = \frac{1}{2}\partial_{xx} q, \quad x \geq \Lambda_t, \quad t \in (0,T], \\ &q(0,x) = f(x), \quad x \geq 0 \quad \text{and} \quad q(t,\Lambda_t) = 0, \quad t \in [0,T], \\ &\Lambda_t = \alpha\Big(1 - \int_{\Lambda_t}^{\infty} q(t,y)\,dy\Big), \quad t \in [0,T], \end{aligned} \tag{39}$$

and

$$\begin{aligned} &\partial_t u + \min_{b \in [0, b_{\max}]} \{b\partial_x u + b\} + \frac{1}{2}\partial_{xx} u = 0, \quad x \geq 0, \quad t \in [0,T], \\ &u(T,x) = \gamma \mathbb{1}_{x=\Lambda_t}, \quad x \geq \Lambda_t \quad \text{and} \quad u(t,\Lambda_t) = \gamma, \quad t \in [0,T]. \end{aligned} \tag{40}$$

This is closer to the formulation of the original (physical) Stefan problem and has the advantage that we do not need to approximate derivatives of Λ_t.

We use an iterative procedure starting with a given $\nu^{(0)}$. Then, inductively, for any $m \geq 1$, we first solve

$$\begin{aligned} &\partial_t q^{(m)} + \partial_x(\nu^{(m-1)}(t,x)q^{(m)}) = \frac{1}{2}\partial_{xx} q^{(m)}, \quad x \geq \Lambda_t^{(m)}, \quad t \in (0,T], \\ &q^{(m)}(0,x) = f(x), \quad x \geq 0 \quad \text{and} \quad q^{(m)}(t,\Lambda_t^{(m)}) = 0, \quad t \in [0,T], \\ &\Lambda_t^{(m)} = \alpha\Big(1 - \int_{\Lambda_t^{(m)}}^{\infty} q^{(m)}(t,y)\,dy\Big), \quad t \in [0,T], \end{aligned} \tag{41}$$

for the pair $q^{(m)}$, $\Lambda^{(m)}$, and then

$$\begin{aligned} &\partial_t u^{(m)} + \min_{b \in [0, b_{\max}]} \big\{b\partial_x u^{(m)} + b\big\} + \frac{1}{2}\partial_{xx} u^{(m)} = 0, \quad x \geq \Lambda_t^{(m)}, \quad t \in [0,T), \\ &u^{(m)}(T,x) = \gamma \mathbb{1}_{x=\Lambda_t^{(m)}}, \quad x \geq \Lambda_t^{(m)} \quad \text{and} \quad u^{(m)}(t,\Lambda_t^{(m)}) = \gamma, \quad t \in [0,T], \end{aligned} \tag{42}$$

for $u^{(m)}$ and set $\nu^{(m)}(t,x) = \arg\min_{b \in [0,b_{\max}]} \big\{b\partial_x u^{(m)} + b\big\}$.

Note that both (41) and (42) are themselves non-linear PDEs. We will use finite difference approaches with implicit time stepping for the diffusive and drift components for the sake of unconditional stability, but with explicit treatment of the control, in order to avoid 'inner' iterations.

The numerical solution of moving boundary problems is classical (see, e.g., [29]), but a rigorous treatment of PDEs in the supercooled case with discontinuity is to our knowledge lacking. Similarly, approximation schemes for HJB equations are well-studied, and we refer the reader to [31] for a discussion of applications in finance, including an example with bang-bang drift control similar to that in the model above.

We specify the finite difference schemes for (41) and (42) used within the iterative process in Appendix B.

A similar iterative procedure has been analysed concurrently with this work in [38] for the classical mean-field game setting. The authors there propose a policy iteration where iterative approximation of the optimal control, as in the context of standard control problems, is intertwined with the iteration over the forward density, and prove convergence locally in time.

A Newton iteration with continuation strategy for similarly coupled systems is discussed in [6] and earlier works by the first author.

We found here that the number of iterations required by our procedure is relatively small (see Table 1 below). Moreover, the solution of both the forward and backward equations is constructed without inner iterations. This allows us to compute sufficiently accurate approximations quickly for the purpose of this study, and we did not investigate alternative algorithms. Theoretical properties of the iterative solution and finite difference approximation of the present system with singular interaction is an interesting question for future research. In the following section, we briefly demonstrate the reasonable practical performance of the numerical strategy, before moving on to the main goal of this section, the computational analysis of the central agent's control problem.

Convergence of iteration for loss and value functions

In the following, we verify the accuracy and stability of the chosen computational method. Parameters are chosen as detailed at the start of Section 3.3 below, unless stated otherwise.

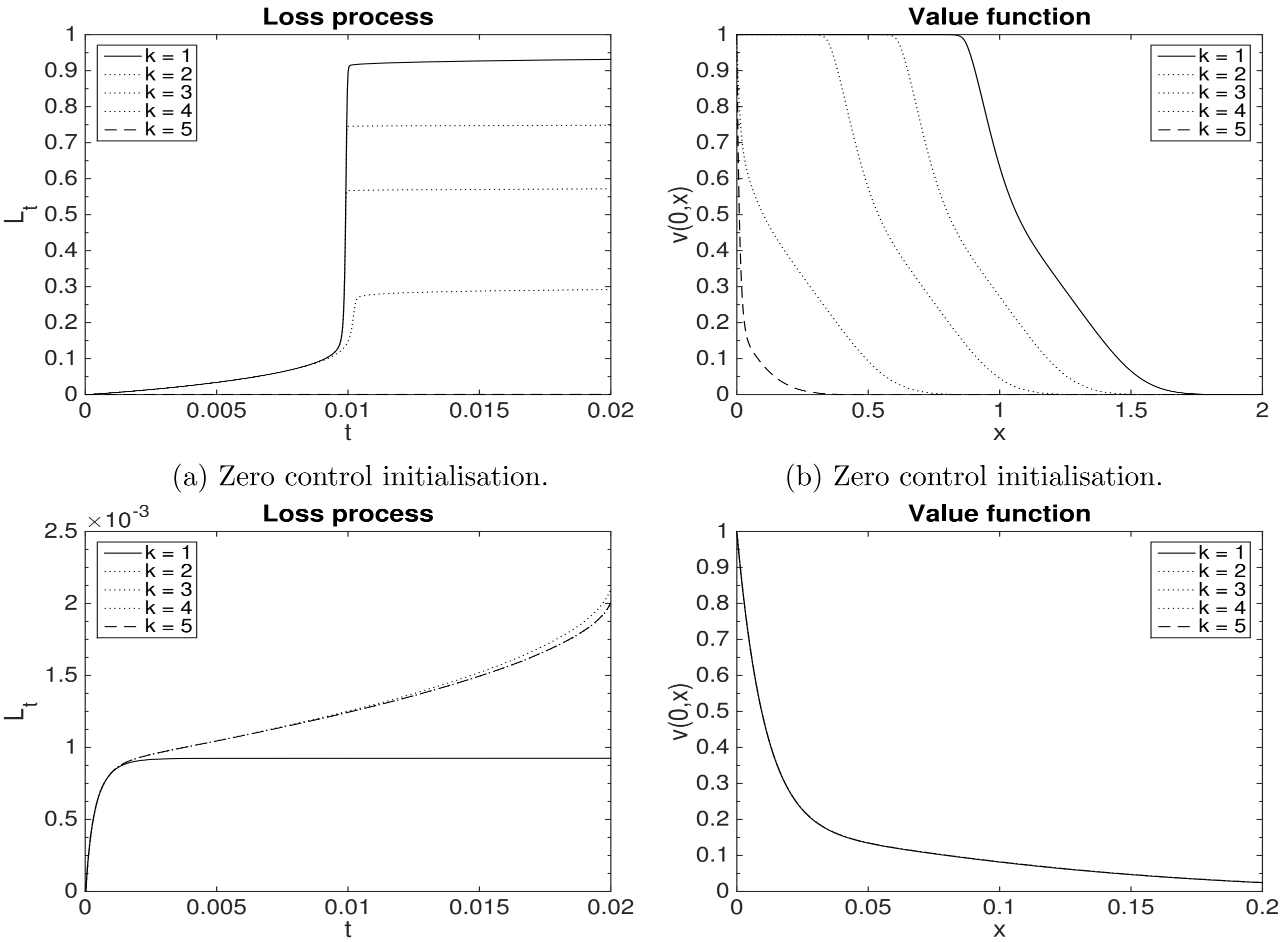
We illustrate in Figure 1 for a particular parameter setting iterates of L (derived from p as in (36)) and v as defined by the iteration described in Section 3.2, where the PDE solutions are approximated by finite differences as detailed in Appendix B.

Plots 1a and 1b used an initial control of $\nu^{(0)} = 0$. In this example, it appears that the loss function decreases from the first iterate (the loss function of the uncontrolled process) and similarly the value function. From iteration 4 onwards, the jump has disappeared and in the fifth iteration the loss is indistinguishable from 0 on this scale. It is indeed approximately $2 \cdot 10^{-3}$.

This monotonicity of the iterates is not observed universally, however. If we choose $\nu^{(0)} = b_{\max}$, illustrated in plots 1c and 1d, the loss behaves non-monotonically (over the iterations), and converges to around $2 \cdot 10^{-3}$ at T, as above. The value function (note the different x scale from 1b above) does not change visually between iterations, and gives the same cost as above.

Due to the finite number of possible feedback controls in the discretised system, the iteration terminates in an exact fixed point after finitely many iterations. This motivates the simple stopping criterion that the vector of controls on the mesh does not change from the previous to the current iteration. Then we have found an exact solution of the discretised system.

We have observed for certain intermediate ranges of γ and relatively coarse meshes that two different solutions to the discretised system exist and can be found by varying the initial control $\nu^{(0)}$, usually one indicating a jump and one without. Only the latter



(a) Zero control initialisation.

(b) Zero control initialisation.

(c) Non-zero control initialisation.

(d) Non-zero control initialisation.

Figure 1: Loss function (left) and value function (right) for the first five iterations, where $\nu^{(0)} = 0$ (top) and $\nu^{(0)} = b_{\max}$ (bottom); $\alpha = 1.5$, $\gamma = 1$.

approximates a true minimiser of the cost function. In all cases considered, this non-uniqueness did not happen for sufficiently fine meshes[6], and therefore we conjecture this to be a non-uniqueness property of the discretisation scheme and not of the continuous model.

Mesh convergence

We analyse here the speed of convergence of quantities of interest under mesh refinement. In particular, we consider approximations by numerical quadrature to $C_T^\star$ and $L_T^\star$ from (36) and (37), respectively, derived from the finite difference approximations to p and v in (31) and (34).

Figure 2 shows the estimated[7] error for both the losses and costs for $\gamma = 1$, $b_{\max} = 50$, and $\alpha \in \{0.5, 1.5\}$, chosen such that the first choice for α never exhibits a jump, whereas the second choice has a jump in the uncontrolled case. The number of time steps and

[6]The results in Section 3.3 are therefore not affected by this.

[7]We estimate the error by extrapolation from the numerical solutions with different discretisation parameters.

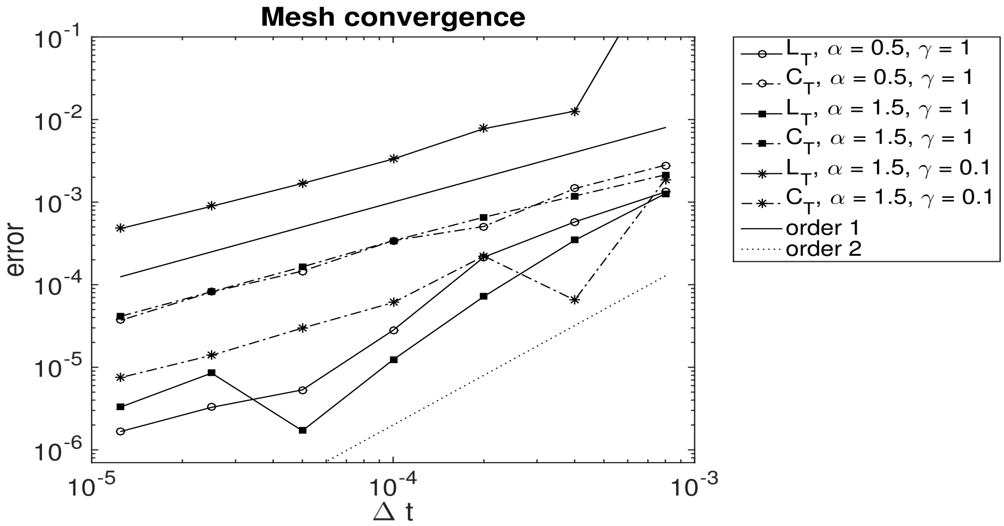


Figure 2: Mesh convergence of $C_T^\star$ and $L_T^\star$ for different α and γ.

mesh points is given in the top two rows of Table 1.

For all parameters considered, the convergence is approximately of first order, as expected from the first order consistency of the finite difference scheme. In cases without jump, and hence small losses (i.e., $\gamma = 1$), the loss has an error of an order of magnitude smaller, but converges less regularly than the cost. In the jump case (i.e., $\alpha = 1.5$, $\gamma = 0.1$), the absolute error of the loss is relatively large, but converges smoothly with first order and has a relative error below 10^{-3} (given a jump of about 1.2) on the two finest meshes considered.

Lastly, we give in Table 1 the number of iterations needed for different meshes, and the computational times. We initialise the strategy such that $b^{(0)}(t,x) = b_{\max}$ for $x \leq 0.15$ and 0 otherwise. The number of iterations increases mildly as the mesh is refined, but

	time points N mesh points N_x	25 469	50 938	100 1875	200 3750	400 7500	800 15000	1600 30000	3200 60000
$\alpha = 0.5$,	iterations	3	3	3	3	3	3	3	4
$\gamma = 1$	CPU time (sec)	0.03	0.10	0.34	1.1	4.8	22	90	577
$\alpha = 1.5$,	iterations	3	3	3	4	4	4	4	4
$\gamma = 1$	CPU time (sec)	0.04	0.09	0.34	1.7	6.7	29	123	578
$\alpha = 1.5$,	iterations	4	5	7	7	6	9	8	8
$\gamma = 0.1$	CPU time (sec)	0.04	0.15	0.74	2.6	8.8	56	211	1043

Table 1: Number of iterations and CPU time using Matlab on a 2.8 GHz Intel Core i7 with 16 GB 1600 MHz DDR3, for different mesh refinements, different α and γ.

settles at an asymptotic value for fine meshes. Slightly more steps are required in the presence of a jump (i.e., $\alpha = 1.5$, $\gamma = 0.1$), which has to be detected by the iteration. We note that acceptable accuracy is obtained after a much smaller number of iterations in many cases. For larger scale computations, one may stop the iteration when the difference between subsequent iterations is comparable to the finite difference error.

The computational time is asymptotically roughly quadratic, due to the simultaneous linear increase in both the number of time steps and mesh points.

3.3 Parameter studies

In this section, we give illustrations of the model's suggested strategies and resulting loss behaviour in different market scenarios, influenced by the interaction parameter α, the risk aversion γ, the initial state f, and maximum cash injection rate $b_{\max}$.

In all examples, we choose a gamma initial density,

$$f(x) = 1/\Gamma(k)\theta^{-k}x^{k-1}\mathrm{e}^{-x/\theta}, \qquad x \geq 0. \tag{43}$$

The parameters of the initial distribution could be calibrated to CDS spreads if they are traded (see [12]). The default parameters we use are $k = 2$, $\theta = 1/3$, chosen to give a range of different behaviours by varying the other parameters.

In this case, f is differentiable with $f(0+) = 0$. This choice implies that there are smooth solutions for a short enough time interval (see [33, 27]). It also implies (see [33]) that a blow-up is guaranteed to happen at some time for $\alpha > k\theta = 2/3$, this being the expectation of the initial value. We will consider various values of α around 1. A simple estimation of meaningful α from typical asset volatilities, recovery rates, and mutual lending as proportion of overall debt is found in [40], suggesting possible values from 0.3 to possibly higher than 5. We shall here consider $\alpha \in \{0.5, 1, 1.5\}$.

The terminal time is typically $T = 0.02$, and we find that the uncontrolled system does not jump in this interval for $\alpha = 0.5$ and $\alpha = 1$ (although we know there is a jump eventually for $\alpha = 1$), and does jump halfway through the interval for $\alpha = 1.5$. We have chosen an initial distribution where blow-ups can happen at such relatively short time scales. We fix $b_{\max} = 30$ at first, but investigate changes later on.

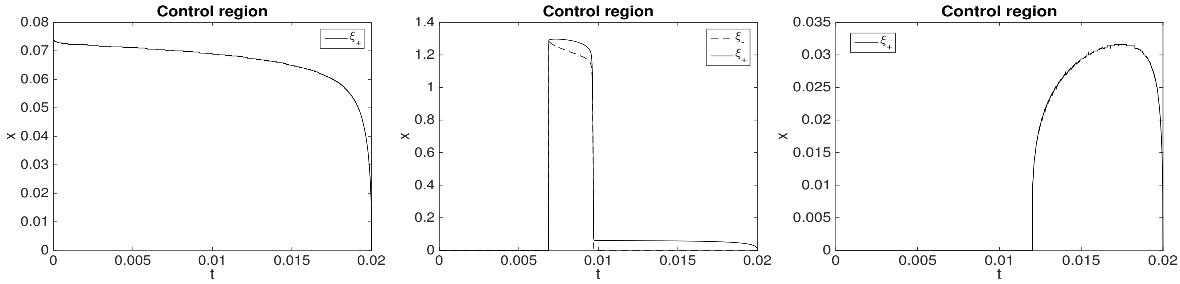
In the following, when not stated otherwise, all solutions are computed with $N = 800$ timesteps and $N_x = 3750$ mesh points on $[0, x_{\max}]$ with $x_{\max} = 6$.

Optimal strategies

We first show the structure of optimal strategies β in two typical settings. The optimal control given in (38) is of 'bang-bang' type, that is to say at any point in time, depending on an individual player's state X_t, the central agent either injects cash at the maximum rate $b_{\max}$ or does nothing.

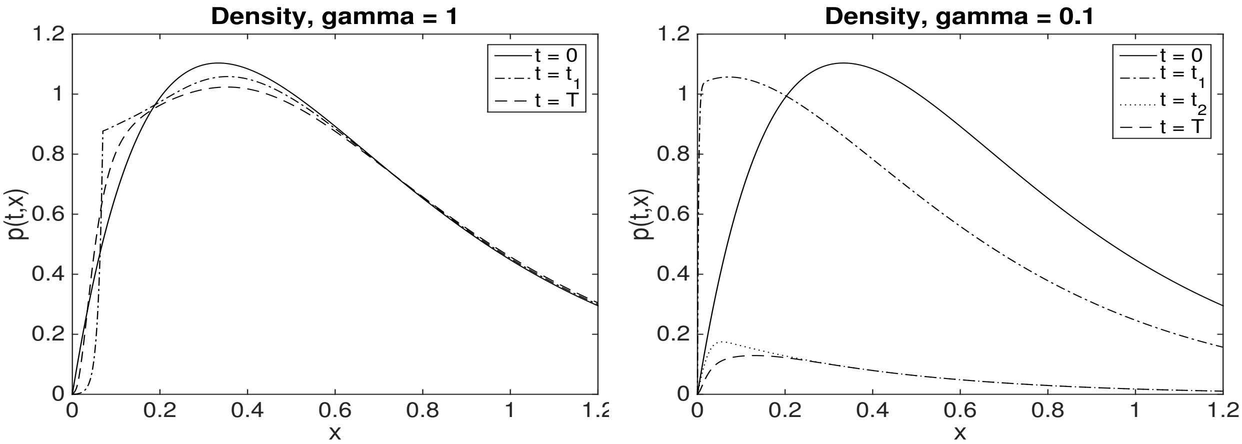
We consider $\alpha = 1.5$ and two values of the loss penalty γ, i.e. different prioritisations of minimising defaults versus costs. For $\gamma = 1$, Figure 3a shows that the region where the central agent acts is for states X_t between 0 and the upper boundary $\xi_+(t)$, which is slightly decreasing (and around 0.07) for most of the period and goes down to 0 rapidly close to the end point, when the chance of defaults before the target date goes to 0 naturally. This suggests that for large γ a constant in time strategy is close to optimal (see Figure 6 below for a more detailed analysis of this point). A jump can be prevented in this case, at a cost of approximately $8 \cdot 10^{-3}$.

For the same $\alpha = 1.5$, but $\gamma = 0.2$, shown in Figure 3b, a jump cannot be prevented by the central agent and occurs around time $t = 0.01$. The optimally acting agent anticipates the jump, finds no benefit in rescuing firms which are destined to default at the systemic



(a) $\gamma = 1$. (b) $\gamma = 0.2$. (c) $\gamma = 0.1$.

Figure 3: The action region of the central agent at time t is in the interval $(0, \xi_+(t))$ for $\gamma \in \{0.1, 1\}$ and $(\xi_-(t), \xi_+(t))$ for $\gamma = 0.2$; $\alpha = 1.5$.



(a) $\gamma = 1$, without jump. (b) $\gamma = 0.1$, with jump.

Figure 4: The density $p(t, \cdot)$ of $\widehat{X}_t$ under optimal control for $t \in \{0, t_1 = 0.01, t_2 = 0.011, T = 0.02\}$, where $\alpha = 1.5$ and varying γ.

event, and instead injects cash into agents which survive the shock, before reverting to a qualitatively similar strategy (after the jump) to the one we found in the jump-free case.

For even smaller risk aversion, $\gamma = 0.1$ in 3c, the central agent only acts after the jump time.

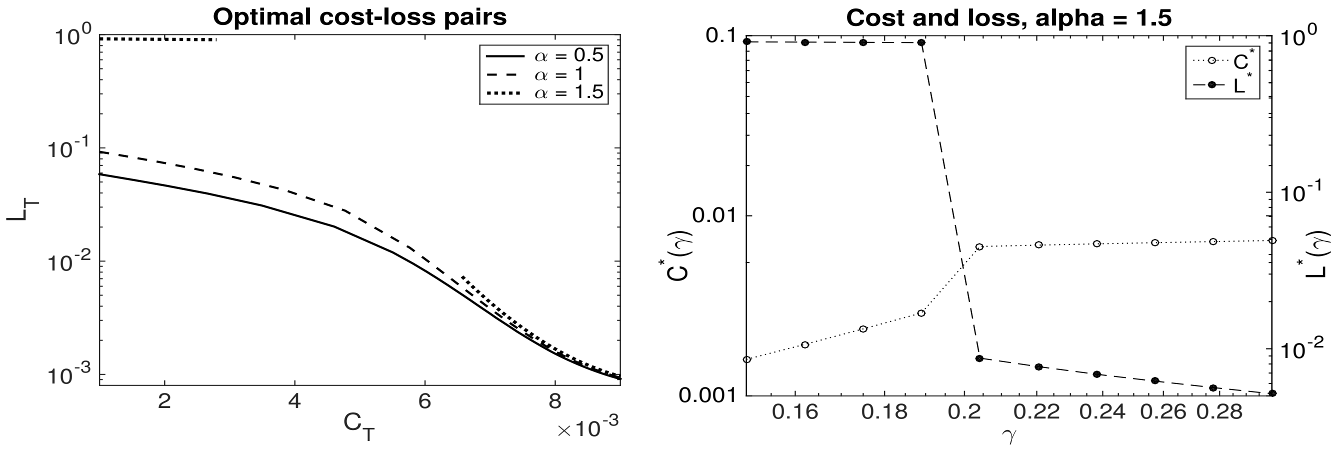
Densities

Here, in Figures 4a and 4b, we analyse the densities p of X under the optimal strategies from above. The initial gamma density is shown in both plots.[8]

For $\gamma = 1$, the central agent's aggressive bailout strategy for small X_t pushes the process away from 0 so that the density at the intermediate time $t_1 = 0.01 = T/2$ in particular has a flat slope at 0.

For $\gamma = 0.35$, minimising losses is much less of a priority to the central agent, which is visible from the much reduced area under the density graphs for $t = T$. The density

[8]We truncate at $x = 1.2$ for clarity of the plot. The computational domain was $x \in [0, 6]$ with a zero boundary condition for p at $x = 6$.



(a) Pairs $(C_T^\star, L_T^\star)$ for different α.

(b) $C_T^\star$ and $L_T^\star$ as function of γ, $\alpha = 1.5$.

Figure 5: Cost $C_T^\star$ and loss $L_T^\star$ in the optimal regime for logarithmically spaced $\gamma \in [0.03, 30]$ and different α in (a) and the dependence on γ over a small range in (b).

immediately after the jump, at $t = t_2$, equals the density immediately before the jump, at $t = t_1$, translated to the left by the jump size (≈ 1.2), i.e., $p(t_2, x) \approx p(t_1 + \alpha(L_{t_2} - L_{t_1}))$.

We illustrate the effect of the initial densities in Appendix A.1.

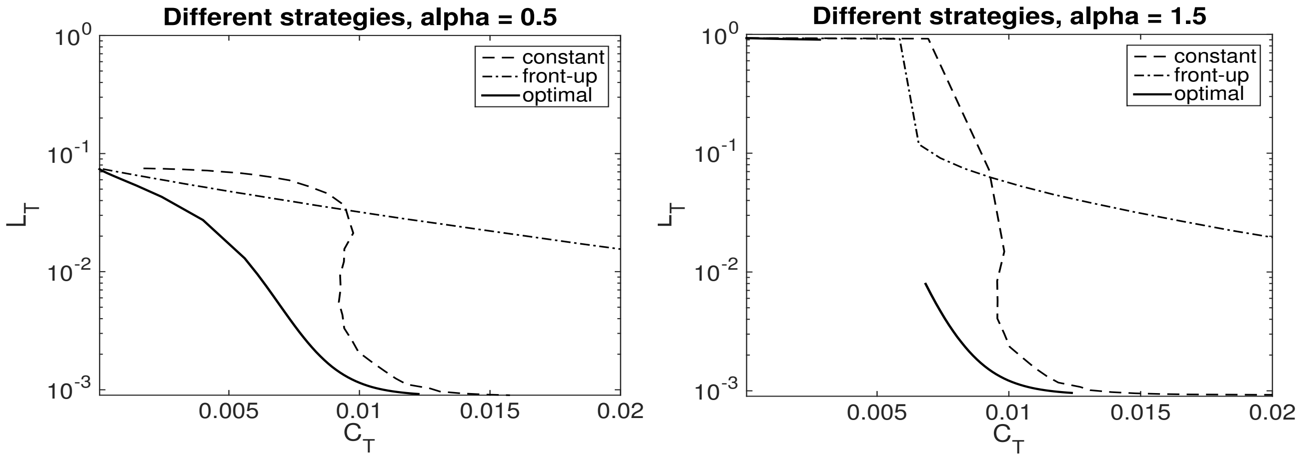
Costs versus losses

In Figure 5a, we vary γ to show the optimal pairs $(C_T^\star(\gamma), L_T^\star(\gamma))$ under different interaction strengths α. We trace out the curve $(C_T^\star(\gamma), L_T^\star(\gamma))$, where $C_T^\star(\gamma)$ and $L_T^\star(\gamma)$ are the costs and losses given by (36) and (37), and derived from (31) and (34), for the chosen γ. For given cost, the graph gives the loss achievable under the optimal strategy. To achieve a smaller loss, a higher cost is generally incurred.

We focus first on the data for $\alpha = 0.5$ and $\alpha = 1$. In these cases, the uncontrolled system exhibits no jumps and cash injection simply reduces the losses. For small γ, minimising the cost is the priority and the losses approach those of the uncontrolled system. For growing γ, it becomes favourable to increase the cash injection and a significant reduction of losses can be achieved. This levels off for large γ as the cap $b_{\max}$ on the cash injection rate limits the overall effect of bail-outs.

For strong interaction, here exemplified by $\alpha = 1.5$ with the dotted curve, we observe a discontinuity, which is further analysed in Figure 5b. For γ around 0.2, the optimal strategy switches from not preventing a jump to preventing a jump. This is manifested in Figure 5b by an downward discontinuity in the number of losses, and an upward discontinuity in the cost. The optimal value of the central agent's control problem is also discontinuous in γ at this point. In other words, it is not possible to vary the capital injection to control the *size* of the jump continuously. Rather, the possible jump size is restricted by the constraint (4) on physical solutions. Conversely, withdrawal of a small amount of cash by the central agent from a scenario with low losses can trigger a large systemic event.

We proceed by comparing the costs and losses under the optimal strategy to some other heuristic strategies. As first benchmark, we consider a uniform strategy by which



(a) $\alpha = 0.5$, no jumps.

(b) $\alpha = 1.5$, jump possible.

Figure 6: Cost-loss pairs $(C^\star_T, L^\star_T)$ under optimal strategy compared to those for a constant strategy, $(C^{\mathrm{u}}_T, L^{\mathrm{u}}_T)$, and front-up strategy, $(C^{\mathrm{f}}_T, L^{\mathrm{f}}_T)$, for two values of α.

the central agent injects cash at a constant rate $b_{\max}$ whenever an agent's value $X_t \le c$ for a constant c, which we vary, resulting in pairs $(C^{\mathrm{u}}_T(c), L^{\mathrm{u}}_T(c))$. The total cost here can be computed as $C^{\mathrm{u}}_T(c) = b_{\max} \cdot \int_0^T \int_0^c p^{\mathrm{u}}(t,x)\,dxdt$, where p^{u} is the density of the absorbed process with uniform control.

We also consider a 'front loaded' strategy whereby at the outset, for some chosen 'floor' $d > 0$, the central agent injects a lump sum of $d - X_{0-}$ into all players with $X_{0-} < d$, hence lifting their reserves up to d. Again, we vary d to obtain a parametrised curve $(C^{\mathrm{f}}_T(d), L^{\mathrm{f}}_T(d))$. The total cost in this case is found as $C^{\mathrm{f}}_T(d) = \int_0^d (d-x) f(x)\,dx$.

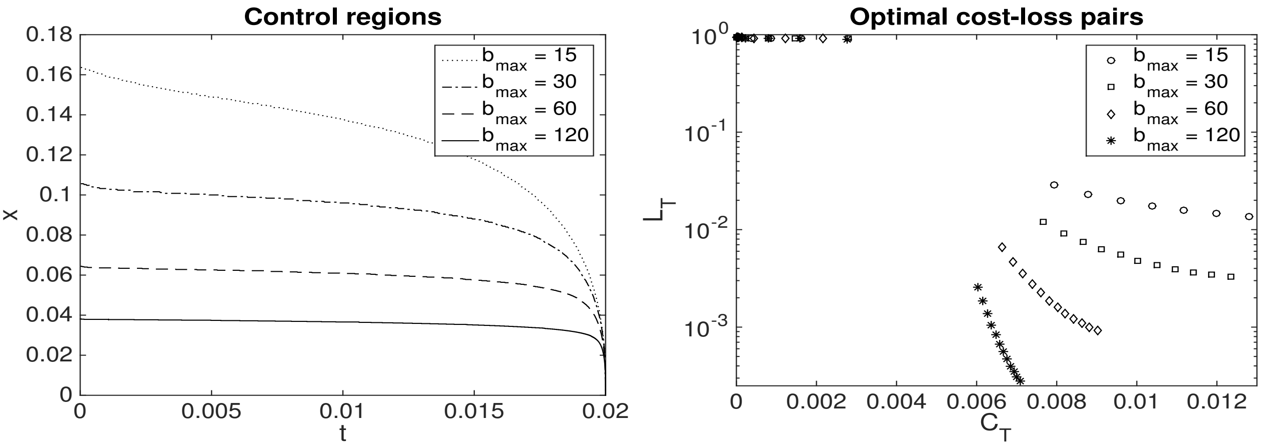
The pairs of cost and loss are shown in Figure 6. In particular, Figure 6a illustrates the case without jump for $\alpha = 0.5$, whereas in the situation of 6b with $\alpha = 1.5$ there is a jump in the uncontrolled system, which can be avoided with sufficiently large control. By construction, the optimal strategy gives lower losses than the heuristic strategies for the same fixed cost. Conversely, less cash injection is required for a given loss tolerance. For a uniform strategy without optimisation, higher cost does not necessarily lead to less loss. The upfront strategy behaves reasonably for small costs, but is far from optimal if small losses are required.

Varying $b_{\max}$ and T

Next, we discuss the effect of increasing $b_{\max}$, the maximal rate of cash injection, on the optimal control strategies and achievable cost-loss pairs.

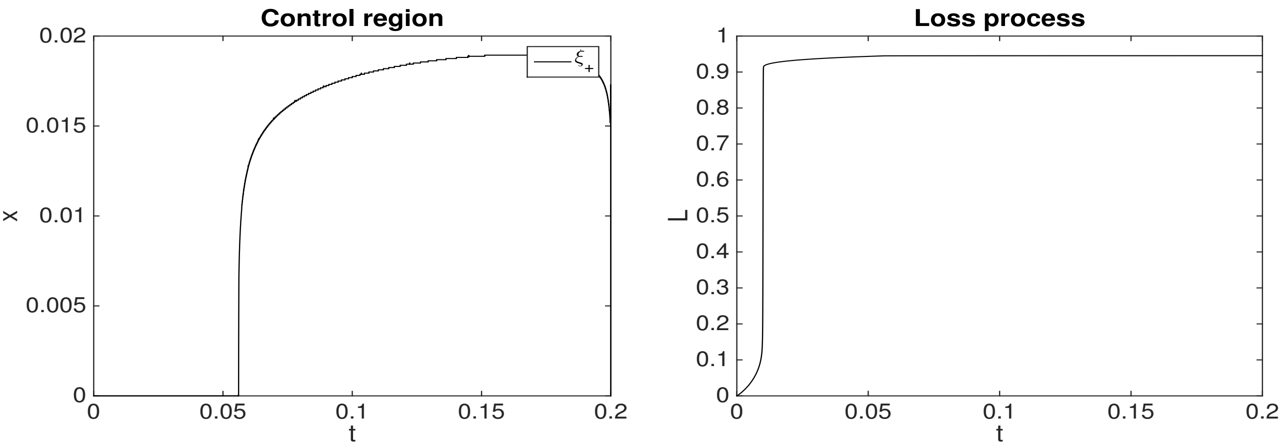
Figure 7a shows that the firm values where the central agent intervenes become more confined to the default boundary. The control of losses for a given cost improves significantly for increased peak injection rate, as is demonstrated in Figure 7b by the pairs of expected cost and resulting loss under the optimal strategies for different $b_{\max}$.

To explore this further, we study a longer time horizon $T = 0.2$ in Figure 8. The value of α is still 1.5 and $\gamma = 0.32$ is chosen to have a jump early on, which allows us to illustrate a more complex setting where the optimal control (see 8a) kicks in a certain time after the jump (see 8b). For high numerical accuracy, we chose $N = 6400$, $N_x = 96000$

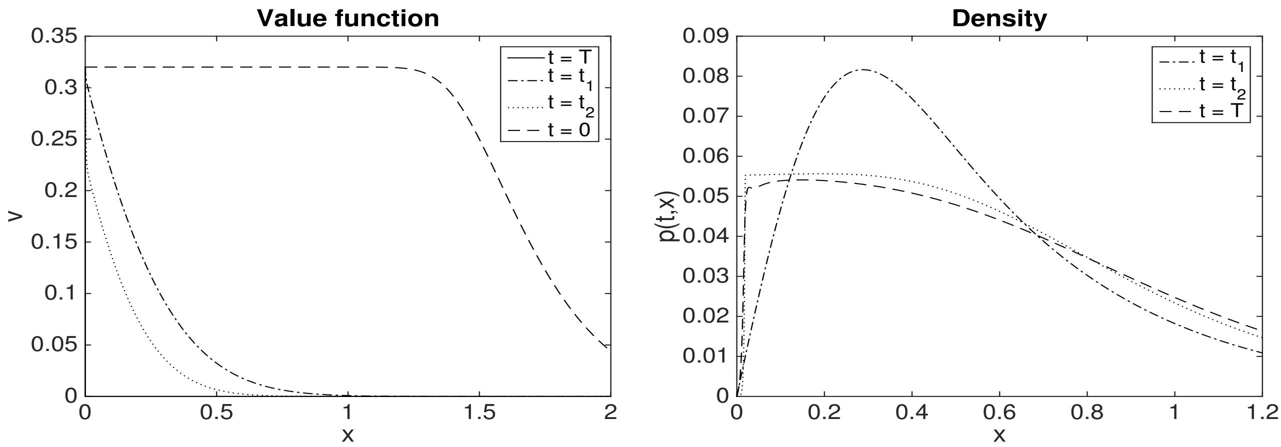


(a) Effect of $b_{\max}$ on control region. (b) Effect of $b_{\max}$ on cost-loss pairs.

Figure 7: Effect of $b_{\max}$ on control region and cost-loss pairs for $\alpha = 1.5$ and $\gamma = 1$.



(a) The control starts after the jump. (b) A systemic event happens immediately.



(c) The value function is discontinuous at 0. (d) The density is discontinuous at 0.

Figure 8: Model outputs for $T = 0.2$, $\alpha = 1.5$, $\gamma = 0.32$, $t_1 = T/4$, $t_2 = 3T/4$, illustrating the limiting case for $b_{\max} \to \infty$ (here, $b_{\max} = 240$).

in these runs.

We expect that for $b_{\max} \to \infty$, the value function satisfies a variational inequality

$$\min\Big(\partial_t v - \dot{\Lambda}_t \partial_x v + \frac{1}{2}\partial_{xx} v, \partial_x v + 1\Big) = 0, \;\; x \geq 0, \;\; t \in [0, T]. \tag{44}$$

The control is applied singularly when the process X_t reaches the boundary of a no-action region to prevent it from exiting. This behaviour is known, e.g., from the literature on portfolio selection under transaction costs, such as in the seminal work [24].

At time intervals where the control is applied, the boundary value γ is not attained continuously in the limit $b_{\max} \to \infty$, as suggested in Figure 8c for $t = t_2 = 3T/4$. Moreover, we have $\partial_x v(t, 0+) = -1$. For large but finite $b_{\max}$, there appears to be a boundary layer near $x = 0$. The HJB equation (34) can be seen as a penalised version of (44).

A related effect is observed for the density in 8d. The boundary condition $p(t, 0) = 0$ is retained for $t = t_1 = T/4$, i.e., before the control starts. At $t = t_2 = 3T/4$, during the period of maximum control, $p(t_1, x)$ approaches a finite non-zero value for $x \to 0$ (for $b_{\max} \to \infty$). Moreover, we have $\partial_x p(t, 0+) = 0$. The behaviour of X_t is locally that of reflected Brownian motion.

4 A central agent minimizing a quadratic target

We consider here again the stochastic process

$$X_t = X_{0-} + \int_0^t \beta_s \, ds + W_t - \Lambda_t, \;\; t \geq 0, \tag{45}$$

as equity value process of a typical financial institution, where a central bank or the tax payer injects[9] capital at a rate β with the goal to avoid systemic events. We here deal with the non-absorbed process X but still assume that the interaction is through

$$\Lambda_t = \alpha \mathbb{P}(\inf_{0 \leq s \leq t} X_s \leq 0) = \alpha L_t, \;\; t \geq 0,$$

i.e., only the first default time matters. This means that even after reaching 0 the bank stays in the system until the terminal time T and continues to receive central bank money at rate β. This is similar to the model framework considered in [20]. In the spirit of [20], we also choose a quadratic cost function, namely namely $f(b) = b^2/2$ for the running cost and

$$g(x) = \left(\sqrt{\frac{G}{2}}x - \sqrt{\gamma}\right)^2 = \frac{1}{2}Gx^2 - \sqrt{2G\gamma}x + \gamma \tag{46}$$

for the terminal cost with constants $G, \gamma \geq 0$, so that the objective function becomes

$$J(\beta) \;\; = \;\; \mathbb{E}\Big[\int_0^T \frac{\beta_t^2}{2} \, dt + \frac{1}{2}GX_T^2 - \sqrt{2G\gamma}X_T + \gamma\Big]. \tag{47}$$

[9]As shown below, β can here also become negative in certain situations meaning that the central bank receives money.

Together with the linear structure of (45), this quadratic cost functional allows us to obtain explicit solutions for the control β in terms of the interaction function Λ. In other words, assuming that the interaction function Λ is known, (45) together with (47) reduces to a *linear-quadratic* optimal control problem, which can be solved explicitly. We refer to the recent book [49] and the references therein for an overview of linear-quadratic stochastic optimal control problems and their relation to mean-field games. In the context of McKean–Vlasov equations with common noise, and coefficients as well as costs depending on the conditional expectation, linear-quadratic optimal control has also been considered in [45]. We here rather pursue the mean-field game approach and refer to [18] for a comparision of these two approaches.

Note that here the loss L corresponding to the interaction term does not explicitly enter in the objective function (47), but is implicitely regulated via the terminal cost. We refer to Remark 4.1 for an extension where the loss enters explicitely in the running costs. The choice of the terminal cost (46) gives an incentive to have a terminal equity value close to the zero of the parabola (46), which is $\sqrt{2\gamma/G}$. Indeed, social costs decrease on $[-\infty, \sqrt{2\gamma/G}]$, in particular as firms with small positive equity move further from default, and increase on $[\sqrt{2\gamma/G}, \infty)$ if a financial institution becomes too rich.

4.1 Solution via Riccati ODEs

In the following we derive the explicit solution of (47) subject to (45) for a given interaction function Λ. The value function associated with (47) is

$$v(t,x) \quad = \quad \inf_{\beta\in\mathcal{B}} \mathbb{E}\left[\int_t^T \frac{\beta_t^2}{2}\,dt + \frac{1}{2}GX_T^2 - \sqrt{2G\gamma}X_T + \gamma \,\middle|\, X_t = x \right]$$

and satisfies the following HJB equation

$$\begin{aligned} &-\partial_t v(t,x) + \sup_{b\in\mathbb{R}}\left\{-\frac{1}{2}b^2 - b\partial_x v(t,x)\right\} + \dot{\Lambda}_t\partial_x v(t,x) - \frac{1}{2}\partial_{xx}v(t,x) = 0,\\ &\text{for } x \geq 0, \quad t \in \mathbb{T},\\ &v(T,x) = \frac{1}{2}Gx^2 - \sqrt{2G\gamma}x + \gamma, \end{aligned}$$

where we denote by $\mathbb{T}$ again the set of all $t \in [0,T]$ where $t \mapsto L_t$ is differentiable. Since $\operatorname{argmax}_{b\in\mathbb{R}}[-\frac{1}{2}b^2 - b\partial_x v(t,x)]$ is given by $b = -\partial_x v(t,x)$, the PDE can be written as

$$-\partial_t v(t,x) + \frac{1}{2}(\partial_x v(t,x))^2 + \dot{\Lambda}\partial_x v(t,x) - \frac{1}{2}\partial_{xx}v(t,x) = 0, \quad x \geq 0, \quad t \in \mathbb{T}. \tag{48}$$

We now make the following ansatz

$$v(t,x) = \frac{1}{2}A_t x^2 + B_t x + C_t.$$

Inserting and separating powers of x, this leads to

$$\begin{aligned} \partial_t A_t &= A_t^2, \quad A_T = G,\\ \partial_t B_t &= A_t B_t + A_t\dot{\Lambda}_t, \quad B_T = -\sqrt{2G\gamma},\\ \partial_t C_t &= \frac{1}{2}B_t^2 - \frac{1}{2}A_t + B_t\dot{\Lambda}_t, \quad C_T = \gamma. \end{aligned}$$

This system can be solved explicitly and the solution is given by

$$\begin{aligned}
A_t &= \frac{G}{1+(T-t)G},\\
B_t &= -\exp\Big(-\int_t^T A_s ds\Big)\sqrt{2G\gamma} - \int_t^T \exp\Big(-\int_t^s A_u\,du\Big)\dot{\Lambda}_s A_s\,ds\\
&= \frac{-\sqrt{2G\gamma}}{1+(T-t)G} - \int_t^T \frac{1+(T-s)G}{1+(T-t)G}\frac{G}{1+(T-s)G}\dot{\Lambda}_s\,ds\\
&= \frac{-\sqrt{2G\gamma} - G(\Lambda_T-\Lambda_t)}{1+(T-t)G},\\
C_t &= \gamma - \int_t^T (\frac{1}{2}B_s^2 - \frac{1}{2}A_s + B_s\dot{\Lambda}_s)\,ds.
\end{aligned}$$

Note that we have for all $t \in [0,T]$, $A_t \geq 0$ and $B_t \leq 0$.

Denoting the optimal values associated with the quadratic target (47) still via *, the optimal strategy $\beta^\star(t,x) = -\partial_x v(t,x)$ is now given by

$$\beta^\star(t,x) = -A_t x - B_t = \frac{-Gx + \sqrt{2G\gamma} + G(\Lambda_T - \Lambda_t)}{1+(T-t)G}. \tag{49}$$

This becomes negative whenever

$$x \geq -\frac{B_t}{A_t} = \frac{\sqrt{2G\gamma} + G(\Lambda_T-\Lambda_t)}{G}, \tag{50}$$

which means that the financial institution has to pay money to the central agent if it becomes too rich. From (50) it is clear that this treshold can be regulated via the ratio $\sqrt{2\gamma/G}$ and it thus equals the zero of the terminal cost. If $x=0$, $-B_t \geq 0$ is invested and this amount increases by $-A_t x \geq 0$ if $x<0$.

Inserting now the form of $\beta^\star$ into (45) yields

$$X_t = X_{0-} + \int_0^t \frac{1}{1+(T-s)G}(-GX_s + \sqrt{2G\gamma} + G(\Lambda_T - \Lambda_s))\,ds + W_t - \Lambda_t, \quad t \geq 0. \tag{51}$$

Moreover, as in (31), under the assumption of a sufficiently regular initial state, the density of the absorbed process $\widehat{X}$ satisfies the following forward Kolmogorov equation,

$$\begin{aligned}
&\partial_t p = A_t p + \Big(\frac{Gx - \sqrt{2G\gamma} - G(\Lambda_T-\Lambda_t)}{1+(T-t)G} + \dot{\Lambda}_t\Big)\partial_x p + \frac{1}{2}\partial_{xx}p, \quad x \geq 0, \quad t \in \mathbb{T},\\
&p(0,x) = f(x), \quad x \geq 0 \quad \text{and} \quad p(t,0) = 0, \quad t \in \mathbb{T},
\end{aligned} \tag{52}$$

where

$$\Lambda_t = \alpha\Big(1 - \int_0^\infty p(t,x)\,dx\Big), \qquad t \in \mathbb{T}. \tag{53}$$

As for (31), we use the following jump condition for the solution of $p(t-,x) = p(t, x - \Lambda_t + \Lambda_{t-})$ in the event of a blow-up at t, otherwise p is continuous in t. Note that the forward Kolmogorov equation (52) depends on the terminal value Λ_T which is an additional complication in comparison to the forward equation of the standard Stefan problem. However, once we have solved (52) and thus get the function $t \mapsto \Lambda_t$ via (53), the control β and the value function v is fully explicit. Note that we here only need to solve a single PDE instead of the coupled system of Section 3.

Remark 4.1. It is possible to modify the cost functional to include the loss explicitely into the running costs. Indeed, we can consider running costs of the form

$$\frac{\beta_t^2}{2} - \delta\beta_t\dot{\Lambda}_t + \frac{\eta^2}{2}\dot{\Lambda}_t^2 \tag{54}$$

with $\eta \in \mathbb{R}$ and $\delta \geq 0$ such that $\delta^2 \leq \eta^2$. The effect of this modification is that in (48) $\dot{\Lambda}_t$ is replaced by $(1-\delta)\dot{\Lambda}_t$ and additionally the term $-\frac{(\eta^2-\delta^2)\dot{\Lambda}_t^2}{2}$ is added. This in turn means that in all subsequent equations $\dot{\Lambda}_t$ is replaced by $(1-\delta)\dot{\Lambda}_t$ and $-\frac{(\eta^2-\delta^2)\dot{\Lambda}_t^2}{2}$ is added to the right-hand side of $\partial_t C$. The optimal strategy is then given by

$$\beta^\star(t,x) = -A_t x - B_t + \delta\dot{\Lambda}_t$$

with solutions for A and B modified in the obvious way. This has the effect that also in (51) all Λ-terms are replaced by $(1-\delta)\Lambda$. In other words, including the loss in the running cost via (54) just means changing the contagion parameter from α to $\alpha(1-\delta)$ via the central planner's strategy that additionally injects $\delta\dot{\Lambda}_t \geq 0$ at any time.

Note that we here use the mean-field game approach, i.e., to search for an optimum for any given interaction function Λ. The McKean-Vlasov control approach could potentially lead to a different solution in the current linear-quadratic case.

Moreover, it is crucial to the solution ansatz that we consider the non-absorbed process as state variable in the control problem, as otherwise a boundary condition at 0 would appear in the HJB equation. This has the counter-intuitive effect that for $\delta = 0$ the quadratic loss rate term in the objective function has no effect on the central planner's strategy.

4.2 Numerical solution

We now explain our numerical procedure to solve (52). As in Section 3.2 we work with the equivalent PDE (39) and use the iterative procedure (41). We start with the uncontrolled system, i.e. $\nu^{(0)} = 0$, and define inductively

$$\begin{aligned}\nu^{(m)}(t,x) &= \beta^{(m)}(t, x - \Lambda_t^{(m)}) \\ &= \frac{-G(x-\Lambda_t^{(m)}) + \sqrt{2G\gamma} + G(\Lambda_T^{(m)} - \Lambda_t^{(m)})}{1+(T-t)G} = \frac{-Gx + \sqrt{2G\gamma} + G\Lambda_T^{(m)}}{1+(T-t)G}.\end{aligned}$$

We then iterate (41) until we find a fix point Λ_T. In contrast to the setting in Section 3.2 with a finite number of control states, the iteration does not terminate in the exact solution after finitely many steps. Thus, we use a termination criterion based on $|\Lambda_T^{(m)} - \Lambda_T^{(m-1)}|$. It turned out experimentally that the number of iterations required for a reasonable error tolerance to be reached is small, usually smaller than 10 (except in some special situations where convergence fails, as explained below). The finite difference scheme to solve (41) is explained in Appendix B.1 (see the last paragraph). We do not discuss theoretical properties of this iterative solution procedure but graphically demonstrate the convergence for certain parameter settings.

Convergence of iteration for loss and value functions

We illustrate in Figure 9 for certain parameter settings iterates of the loss function L (derived from p as in (36)) and the value function $v(0,\cdot)$. More precisely, we here parametrize the terminal cost (46) just by γ and choose $G = \frac{\gamma}{2}$ so that

$$g(x) = \gamma \left(\frac{x}{2} - 1\right)^2 . \tag{55}$$

We then analyze the cases $\gamma = 0.2$ and $\gamma = 1$ with contagion parameter $\alpha = 1.3$. In these examples, the loss function and the value function decrease from the first iteration (corresponding to the uncontrolled process) and converge very fast. In the case $\gamma = 0.2$ (see Figure 9a and Figure 9b), the jump does not disappear but is only postponed to a slightly later time, which has also a rather small impact on the value function. For $\gamma = 1$ (see Figure 9c and Figure 9d), the jump disappears after the first iteration and the value function decreases much more substantially due to the avoidance of the systemic event.

Similar to Section 3.2, for certain parameter combinations, here e.g. for $\alpha = 1.3$ and $\gamma = 0.5$, it seems that two different solutions to the discretised system exist, one with a jump that occurs at the very end of the time horizon and the other one without. The numerical iteration scheme may then oscillate between these two solutions. This phenomenon can be explained via (49). From there it is clear that in the case of a jump, β^* has a higher value than without, which means that due to the higher capital injection we can switch from a regime with a jump to one without. In the next iteration, the value of β^* is however smaller, more precisely too small to prevent a jump, implying that we are back to the jump regime. Another observation is that with growing γ, the jump is postponed further and further and this non-uniqueness occurs exactly when it is about to leave the fixed time horizon, meaning that for a slightly higher γ, no jumps appear anymore up to time T. We leave it as an open question if this non-uniqueness property is a feature of the continuous model, or rather due to the discretisation scheme.

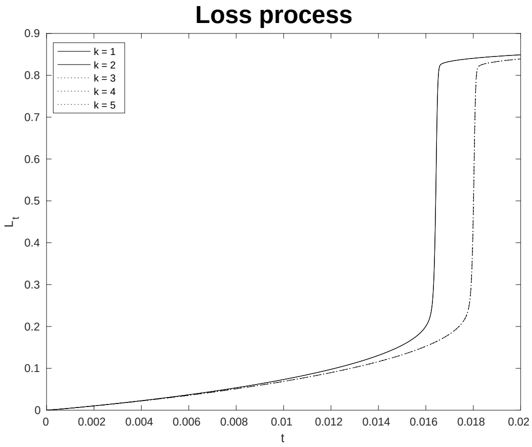
4.3 Parameter studies

In this section we illustrate the control strategies and resulting loss and cost behaviour in dependence of α and γ, where we use again a terminal cost of the form (55). The terminal time is $T = 0.02$.

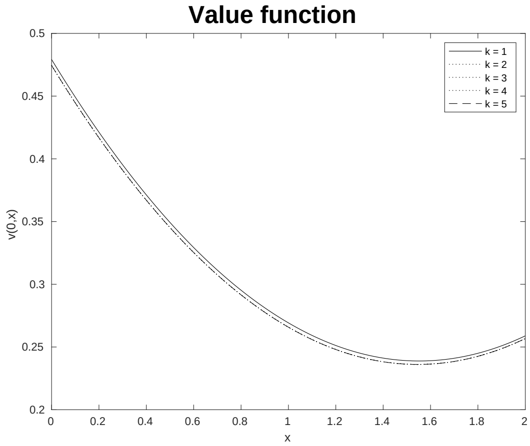
We choose again a gamma initial density as given in (43) with parameters $k = 2$, $\theta = 1/3$. All PDE solutions are computed with $N = 1000$ timesteps and $N_x = 18750$ mesh points on $[0, x_{\max}]$ with $x_{\max} = 6$.

Optimal strategies

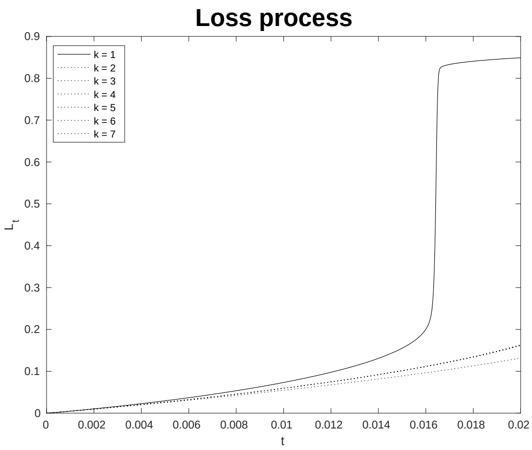
We here show the structure of the optimal strategies β for the situation when a jump in the loss function cannot be avoided. From (49) it is clear that also the control β jumps at this time. This is illustrated in Figure 10 for $\gamma = 0.2$ and $\alpha = 1.3$. Due to our



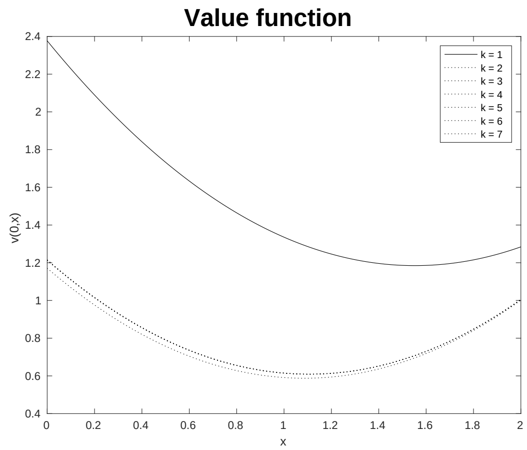
(a) $\gamma = 0.2$ and $\alpha = 1.3$



(b) $\gamma = 0.2$ and $\alpha = 1.3$



(c) $\gamma = 1$ and $\alpha = 1.3$



(d) $\gamma = 1$ and $\alpha = 1.3$.

Figure 9: Loss function (left) and value function (right) for all iterations until $|\Lambda_T^{(k)} - \Lambda_T^{(k-1)}| \leq 10^{-8}$, where we start with $\nu^{(0)} = 0$.

parametrization, β is always nonnegative for all $x \leq 2$; for $x > 2$, however, it can become negative, which is also visible in Figure 10.

Densities

In Figures 11a and 11b, we analyse the densities of the absorbed process $\widehat{X}$ under the optimal control. In both plots the initial gamma density is shown, as well as the resulting

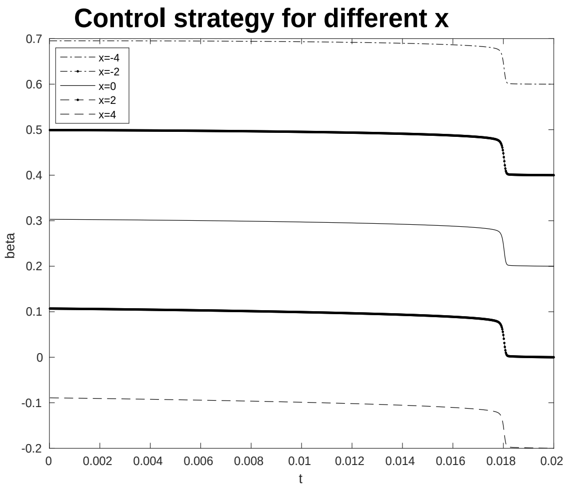


Figure 10: $t \mapsto \beta(t, x)$ for different x; $\gamma = 0.2$ and $\alpha = 1.3$

densities at time $t_1 = 0.01 = T/2$ and time T.

For $\gamma = 1$, the evolution of the density is minor since the systemic event can be avoided. For $\gamma = 0.2$ this is not case and a substantial drop in the density becomes visible at the terminal time T.

We study the densities of the absorbed and non-absorbed processes via Monte Carlo simulations further in Appendix A.2.

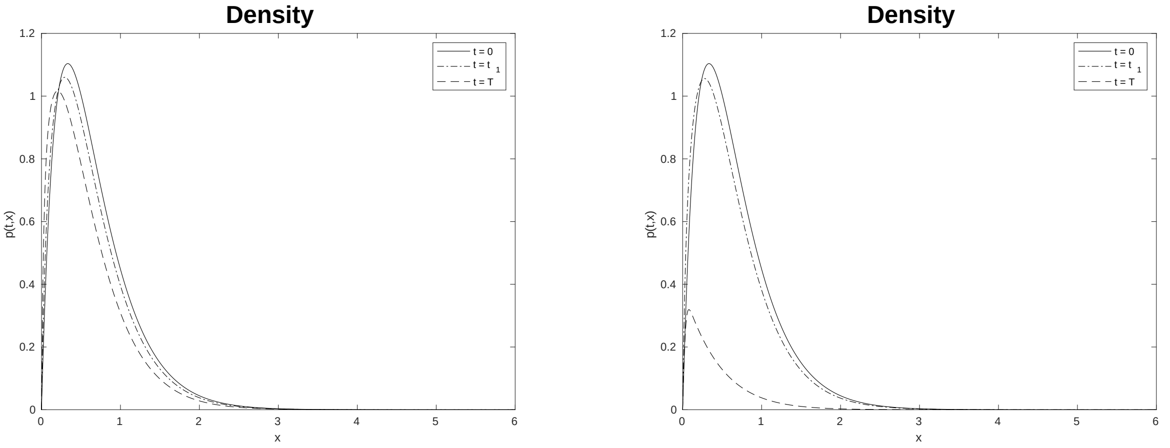
Costs versus losses

In Figure 12a and 12b we vary γ to show the behaviour of $L_T^\star(\gamma)$ and the objective

$$J^*(\gamma) = \mathbb{E}\left[\int_0^T \frac{1}{2}(\beta^\star)^2(t, X_t)dt + \gamma\left(\frac{1}{2}X_T^2 - 1\right)^2\right] \tag{56}$$

in dependence of γ, which parametrises the terminal cost as specified in (55). We denote again all quantities related to the objective function (56) via *. The shape of $\gamma \mapsto L_T^\star(\gamma)$ in Figure 12a is comparable to the one of the bang-bang strategy shown in Figure 5b. In particular, we see a downward discontinuity in the number of losses. The value for $\gamma = 0.5$ is marked in red to indicate that our numerical scheme oscillates between two solutions, one with a jump an the other without, as described above. For all graphs here we choose the one *with* the jump.

At this discontinuity point, the objective function shown in Figure 12b exhibits a decrease, which is then followed by a (linear) increase. This can be explained by the fact that the terminal value X_T is much more concentrated around 1 when the systemic event is avoided, so that the terminal cost becomes lower (see (56)).



(a) $\gamma = 1$ and $\alpha = 1.3$, without jump

(b) $\gamma = 0.2$ and $\alpha = 1.3$, with jump

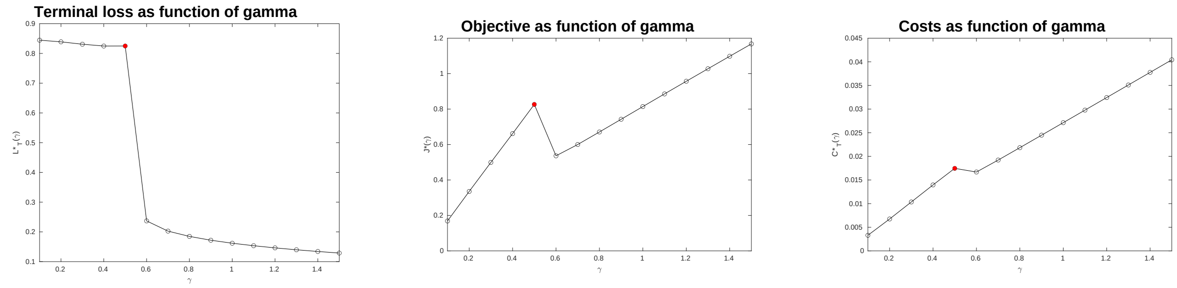
Figure 11: The density $p(t, \cdot)$ of $\widehat{X}_t$ under the optimal control for $t \in \{0, t_1 = 0.01, T = 0.02\}$, where $\alpha = 1.3$ and varying γ.

Even though the loss L and the expected costs

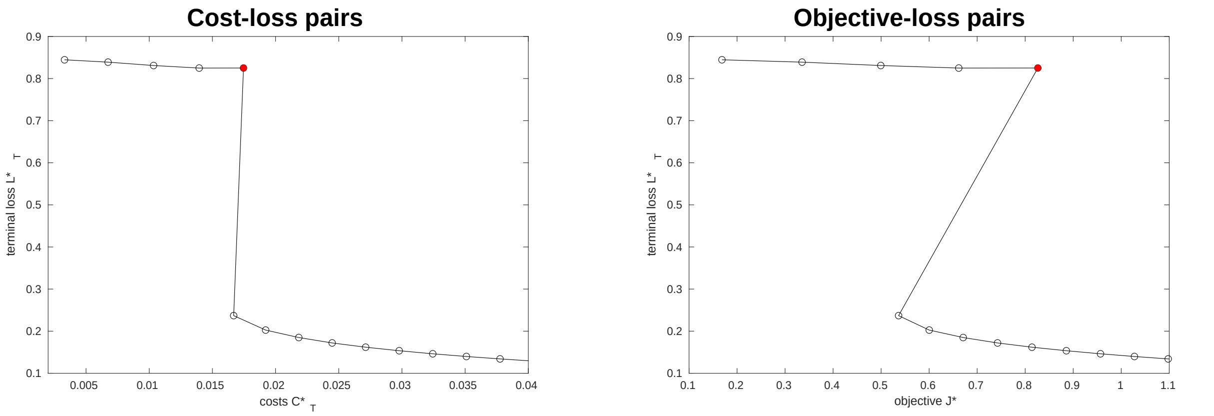
$$C_T = \mathbb{E}\left[\int_0^T \beta(t, X_t)\, dt\right]$$

do not enter explicitely in the objective function, we can nevertheless analyse optimal cost-loss pairs $(C_T^\star(\gamma), L_T^\star(\gamma))$ by varying γ. Since β is linear in X, and as the expected value of X can be calculated from (51) by variation of constants, we can compute $C_T^\star$ easily by a numerical integration over time. We plot $L_T^\star(\gamma)$ in Figure 12a and $C_T^\star(\gamma)$ in 12c. In contrast to $J^\star(\gamma)$ in 12b, the costs nearly stay constant at the discontinuity point since the terminal cost does not enter here. In Figure 12d we then trace out the curve $(C_T^\star(\gamma), L_T^\star(\gamma))$ for the chosen range of γ, which is at least qualitatively comparable to the one corresponding to the optimal bang-bang strategy of Figure 6b. As already described above, for γ around 0.5, the optimal strategy switches from not preventing a jump to preventing a jump. This means also in the current model that a withdrawal of a small amount of cash by the central agent from a scenario with low losses can trigger a large systemic event.

In Figure 12e we plot the curve $(J_T^\star(\gamma), L_T^\star(\gamma))$, which demonstrates that the objective function is not optimally designed to control the loss, as a lower value of the objective function does not necessarily lead to less losses. Nevertheless, several implications of this model to prevent systemic events are qualitatively comparable to the ones with direct loss minimisation, even though the structure of the control strategy is very different from the bang-bang strategy there.



(a) $L_T^\star$ as function of γ. (b) $J^\star$ as function of γ. (c) $C_T^\star$ as function of γ.



(d) Cost $C_T^\star$ and loss $L_T^\star$ for $\gamma \in [0.1, 1.5]$. (e) Objective $J^\star$ and loss $L_T^\star$ for $\gamma \in [0.1, 1.5]$.

Figure 12: Losses, costs, and objective under the optimal linear-quadratic strategy for $\alpha = 1.3$.

5 Conclusions and further work

This paper provides the first investigation of a controlled version of the classical Stefan problem. We draw conclusions for a central agent's optimal actions to stabilize interbank lending. The work opens a number of questions of mathematical and economical interest.

A first obvious question is whether the assumption of a feedback control is justified. The well-posedness of the forward-backward coupled PDE system is another open question. Related to this question, the convergence of our numerical scheme to the minimal solution of the controlled McKean–Vlasov problem is also open. Another intriguing question arises from the numerical tests for large $b_{\max}$, which led us to conjectures on the limiting behaviour. We are planning to pursue a rigorous characterisation of the problem without a bound on β_t in future work. Further plausible deviations from continuous strategies include impulse strategies, which would result from fixed costs of intervention, or also if any intervention itself signals a crisis to the market and causes a downward shock to the banks' assets.

A Further parameter studies

A.1 Different initial densities

In this section, we show the effect of changing the initial density on the control and evolution of the system from Section 3.

In Figure 13, we fix all parameters as previously, except for k and θ in the initial gamma density (43). We pick two pairs (k, θ) such that the expectation $k\theta = 2/3$ as in our earlier examples. In particular, as $k\theta < \alpha = 1.5$, a jump occurs for the uncontrolled system (see [33]).

For $k = 4$, $\theta = 1/6$, the initial density has less mass near $x = 0$ initially, whereas for $k = 1$, $\theta = 2/3$, we have $f(0+) = 3/2 > 1/\alpha$ and a jump is immediately triggered (see [27]).

Figures 13a and 13b show the corresponding control regions, whereby in the case of an immediate jump the qualitative strategy immediately reverts to the strategy without jump.

Figures 13c and 13d show the corresponding densities of X_t, where little losses occur in the regular case, while most of the mass is wiped out instantly for the irregular case.

A.2 Path simulation and densities for the linear-quadratic model

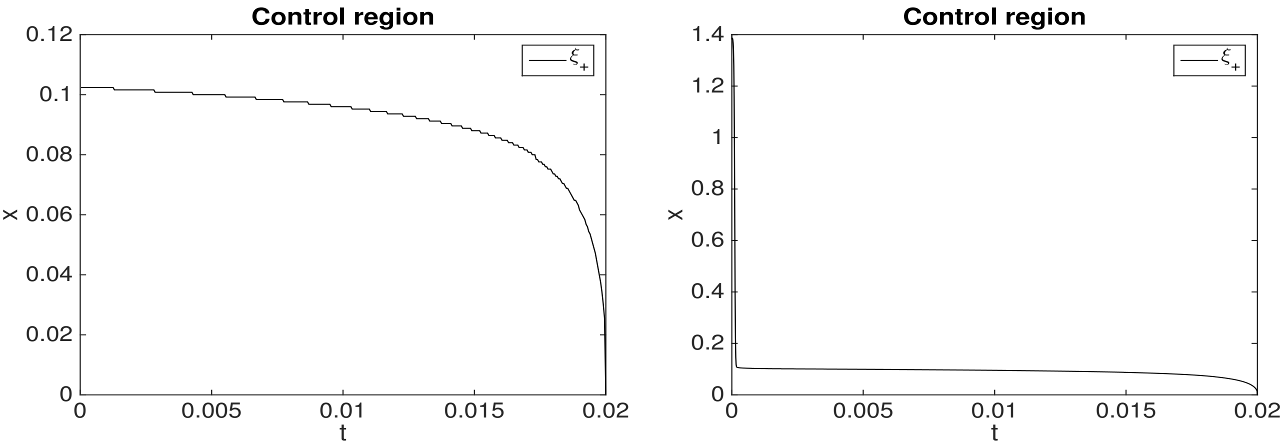
We verified the density plots in Section 4.3 by sampling 10000 trajectories from (51) by solving this equation via variation of constants, given the optimal Λ. Plots of several trajectories are shown in Figure 14a and Figure 14b. The corresponding histograms compared with the above densities for the initial and terminal time are displayed in Figure 15a and Figure 15b.

Since we here deal with the non-absorbed process, meaning financial institutions stay in the system after hitting 0, we also analysed the histogram of the non-absorbed process at time T. It appears that its positive part is more or less idential to the histogram of the absorbed process, which means that financial institutions are not able to recover, at least in this short period. This is illustrated in Figure 16a and Figure 16b.

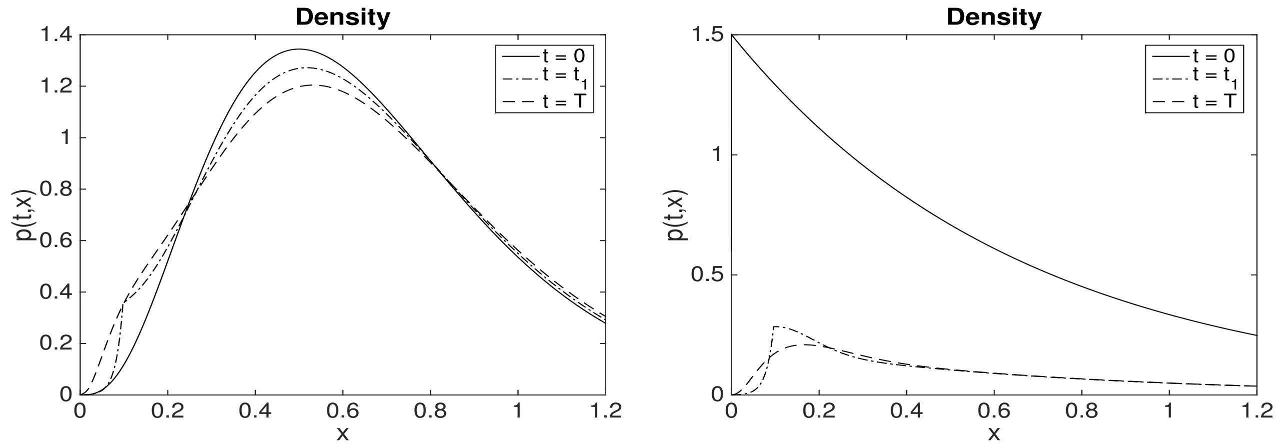
B Numerical schemes

We define a numerical approximation on a time mesh $t_i = i\Delta t$, $i \in \mathbb{I} = \{0, 1, \ldots, N\}$, $\Delta t = T/N$ for a positive integer N.

We also define a spatial mesh $(x_j)_{j\in\mathbb{J}}$ with mesh size Δx, $x_j = j\Delta x$, $j \in \mathbb{J} = \{0, 1, \ldots, J\}$, for $x_{\max} = J\Delta x$ for a positive integer J. We set $\Delta x = \kappa\Delta t$ for some fixed κ.



(a) Gamma initial density with $k = 4$, $\theta = 1/6$. (b) Gamma initial density with $k = 1$, $\theta = 2/3$.



(c) Gamma initial density with $k = 4$, $\theta = 1/6$. (d) Gamma initial density with $k = 1$, $\theta = 2/3$.

Figure 13: Control regions and densities for $\alpha = 1.5$, $\gamma = 1$, $b_{\max} = 30$, and varying k and θ in the initial gamma density.

B.1 Finite difference approximation of Kolmogorov equation

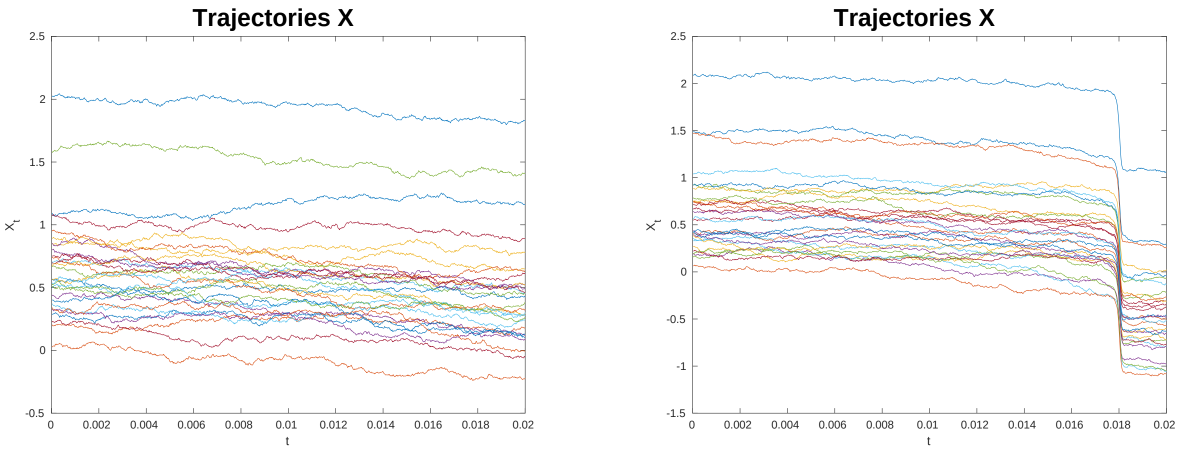
Given is an input control matrix $(\nu_j^i) \in \{0, b_{\max}\}^{\mathbb{I}\times\mathbb{J}}$. We define a numerical approximation (q_j^i) to $(q(t_i, x_j))$ as follows. Set the initial condition

$$\begin{aligned} q_j^0 &= f(x_j), \qquad 0 \le j \le J, \\ L^0 &= 0. \end{aligned}$$

For future use, we also introduce an nonnegative integer vector $\ell = (\ell^i)$ where ℓ^i the index of the largest mesh point below or equal to αL^i. So $\ell^0 = L^0$.

At each time point t_i with $0 < i \le N$, then follow the steps below. First, for given $i > 0$, compute the accumulated loss and index of displacement,

$$\begin{aligned} L^i &= 1 - \sum_{j=0}^{J} q_j^{i-1} \Delta x, \\ \ell^i &= \lfloor \alpha L^i / \Delta x \rfloor, \end{aligned}$$



(a) $\gamma = 1$ and $\alpha = 1.3$, without jump

(b) $\gamma = 0.2$ and $\alpha = 1.3$, with jump

Figure 14: Trajectories of X given by (51) for $\alpha = 1.3$ and varying γ.

and cascade the defaults,

$$q_j^{i-1} \leftarrow \begin{cases} 0, & 0 \le j \le \ell^i, \\ q_j^{i-1}, & \ell^i < j \le J. \end{cases}$$

In the case of non-negative control $\nu_j^i \in \mathbb{R}_{\not{V},+}$, then apply an implicit upwind finite difference scheme

$$\begin{aligned} \frac{q_j^i - q_j^{i-1}}{\Delta t} + \frac{\nu_j^i q_j^i - \nu_{j-1}^i q_{j-1}^i}{\Delta x} = \frac{1}{2}\frac{q_{j-1}^i - 2q_j^i + q_{j+1}^i}{\Delta x^2}, \qquad & \ell^i < j < J, \\ q_j^i = 0, \qquad & \text{else.} \end{aligned}$$

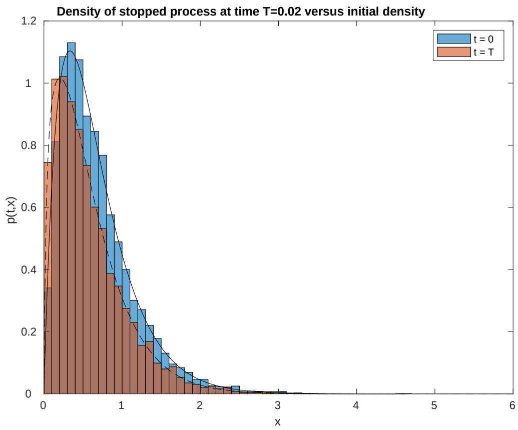
This is an implicit Euler step which results in the tridiagonal linear system for $\ell^i < j < J$

$$-\left(\frac{1}{2}\frac{\Delta t}{\Delta x^2} + \nu_{j-1}^i \frac{\Delta t}{\Delta x}\right) q_{j-1}^i + \left(1 + \frac{\Delta t}{\Delta x^2} + \nu_j^i \frac{\Delta t}{\Delta x}\right) q_j^i - \frac{1}{2}\frac{\Delta t}{\Delta x^2} q_{j+1}^i = q_j^{i-1}$$

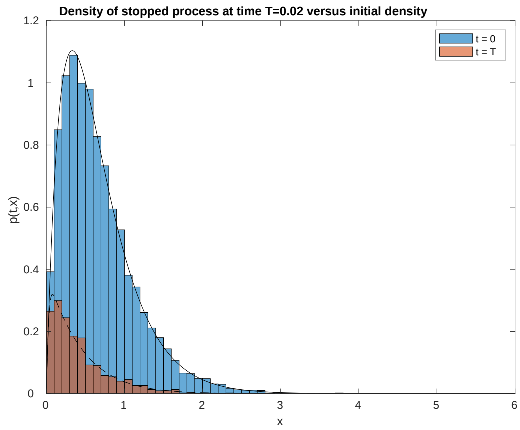
with $q_{\ell^i}^i = q_J^i = 0$. In the linear-quadratic case, where $\nu_j^i \in \mathbb{R}$, the tridiagonal linear system that we solve is slightly different. Indeed, for $\ell^i < j < J$, it reads as

$$\begin{aligned} -\left(\frac{1}{2}\frac{\Delta t}{\Delta x^2} + \max\left(\nu_{j-1}^i \frac{\Delta t}{\Delta x}, 0\right)\right) q_{j-1}^i + \left(1 + \frac{\Delta t}{\Delta x^2} + \left|\nu_j^i \frac{\Delta t}{\Delta x}\right|\right) q_j^i \\ - \left(\frac{1}{2}\frac{\Delta t}{\Delta x^2} + \min\left(\nu_{j-1}^i \frac{\Delta t}{\Delta x}, 0\right)\right) q_{j+1}^i = q_j^{i-1} \end{aligned}$$

and $q_{\ell^i}^i = q_J^i = 0$. Finally, we obtain an approximation p_j^i to $p(t_i, x_j)$ by $p_j^i = q_{j+\ell^i}^i$.



(a) $\gamma = 1$ and $\alpha = 1.3$, without jump



(b) $\gamma = 0.2$ and $\alpha = 1.3$, with jump

Figure 15: The histogram of $\widehat{X}_t$ versus its density under the optimal control for $t \in \{0, T = 0.02\}$, where $\alpha = 1.3$ and varying γ.

B.2 Finite difference approximation of HJB equation

Given is an input loss index vector $(\ell^i) \in \mathbb{R}^{\mathbb{I}}$. Set the terminal condition

$$u_j^N = \begin{cases} \gamma, & 0 \leq j \leq \ell^N, \\ 0, & \ell^N < j \leq J. \end{cases}$$

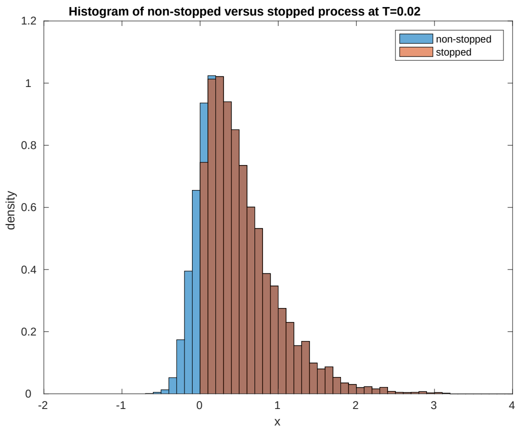
At each time point t_i with $0 \leq i < N$, then follow the steps below, running backwards. First, determine the optimal control based on u^{i+1}, for $0 \leq j < J$,

$$\nu_j^i = \begin{cases} b_{\max}, & u_{j+1}^{i+1} - u_j^{i+1} < -\Delta x, \\ 0, & \text{else}, \end{cases}$$

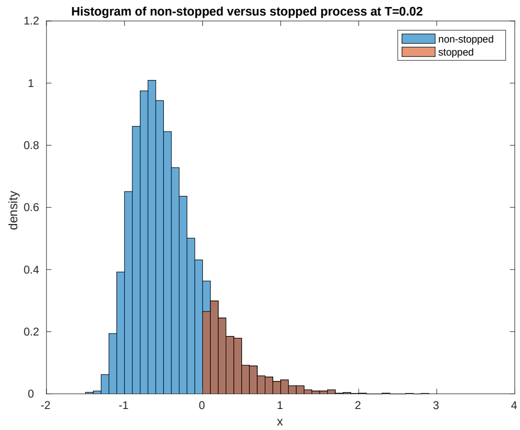
and $\nu_J^i = 0$. This is a discrete version of the optimality condition embedded in the HJB equation. Viewed differently, this step determines whether the reduction in the value function achieved by moving from x_j to x_{j+1} is worth the cost $\Delta x = b_{\max}\Delta t$. The reason we use u^{i+1} and not u^i to determine the control at t_i is simply to allow for an explicit computation.

Then, apply an implicit upwind finite difference scheme

$$\begin{aligned} \frac{u_j^i - u_j^{i+1}}{\Delta t} = \nu_j^i u_j^i + \nu_j^i \frac{u_{j+1}^i - u_j^i}{\Delta x} + \frac{1}{2}\frac{u_{j-1}^i - 2u_j^i + u_{j+1}^i}{\Delta x^2}, &\qquad \ell^i < j < J, \\ u_j^i = \gamma, &\qquad 0 \leq j \leq \ell^i, \\ u_J^i = 0. & \end{aligned}$$



(a) $\gamma = 1$ and $\alpha = 1.3$, without jump



(b) $\gamma = 0.2$ and $\alpha = 1.3$, with jump

Figure 16: The histogram of X_T versus $\widehat{X}_T$ where $\alpha = 1.3$ and varying γ.

Finally, we obtain an approximation v_j^i to $v(t_i, x_j)$ by $v_j^i = u_{j+\ell^i}^i$, and an approximation β_j^i to $\beta(t_i, x_j)$ by $\beta_j^i = \nu_{j+\ell^i}^i$.

C Proofs

C.1 Notation

Throughout the paper, $D([-1,\infty))$ denotes the space of càdlàg functions on $[-1,\infty)$ endowed with the M_1-topology, $C([0,\infty))$ denotes the space of continuous functions on $[0,\infty)$ endowed with the topology of compact convergence, i.e., $f_n \to f$ in $C([0,\infty))$ if and only if $f_n|_K \to f|_K$ uniformly for every compact $K \subseteq [0,\infty)$. If S is a polish space, we denote the space of probability measures on S by $\mathcal{P}(S)$ and endow it with the topology of weak convergece, i.e., we say that $\mu_n \to \mu$ in $\mathcal{P}(S)$ iff $\int_S F(x)\mathrm{d}\mu_n(x) \to \int_S F(x)\mathrm{d}\mu(x)$ for all $F \in C_b(S;\mathbb{R})$. If $\mu \in \mathcal{P}(S)$ and $F\colon S \to \mathbb{R}$, we denote the integral of F with respect to μ also with brackets, i.e., we write $\int_S F(x)\,\mathrm{d}\mu(x) = \langle \mu, F\rangle$. Furthermore, if ν is the pushforward of the measure μ with respect to the map T, we denote this by $T(\mu) = \nu$.

C.2 Existence of minimal solutions and optimizers

Lemma C.1. *For any $\beta \in \mathcal{B}$, define the process*

$$Z_t = X_{0-} + \int_0^t \beta_s \, ds + B_t, \quad t \geq 0. \tag{57}$$

Then, $\xi = \mathrm{law}(Z) \in \mathcal{P}(C([0,\infty)))$ satisfies the extended crossing property, i.e.,

$$\mathbb{P}\left(\inf_{0 \leq s \leq h} (Z_{\tau+s} - Z_\tau) = 0\right) = 0, \quad h > 0, \tag{58}$$

for any stopping time τ with respect to $(\mathcal{F}_t)_{t\geq 0}$.

Proof. Let τ be a $(\mathcal{F}_t)_{t\geq 0}$–stopping time. Since $\beta \in \mathcal{B}_T$ is almost surely bounded, by Novikov's condition and Girsanov's theorem we may find an equivalent probability measure $\mathbb{Q}$ such that Z is a $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion under $\mathbb{Q}$. The strong Markov property of Brownian motion then yields

$$\mathbb{Q}\left(\inf_{0 \leq s \leq h} (Z_{\tau+s} - Z_\tau) = 0\right) = \mathbb{Q}\left(\inf_{0 \leq s \leq h} Z_s = 0\right) = 0, \quad h > 0.$$

the equivalence of $\mathbb{P}$ and $\mathbb{Q}$ then yields the claim. □

C.3 Existence of solutions

Lemma C.2. *Suppose that $f_n \to f$ in S. Then, $\int^\cdot f_n(s)\, ds \to \int^\cdot f(s)\, ds$ uniformly in t on any compact subset of $[0,\infty)$.*

Proof. Let $T' > 0$. By weak $L^2([0,\infty))$-convergence, we have $\int_0^t f_n(s)\, ds \to \int_0^t f(s)\, ds$ for any $t \in [0, T']$. Let $\epsilon > 0$ and choose t_k with $0 = t_0 \leq t_1 \leq \cdots \leq t_m = T'$ such that

$$\int_{t_i}^{t_{i+1}} f(s)\, ds < \epsilon/2, \quad i \in \{0, \ldots, m\}. \tag{59}$$

Choose n large enough such that $\left|\int_0^{t_i} f_n(s) - f(s)\, ds\right| < \epsilon/2$ for all $i \in \{0, \ldots, m\}$. We obtain, for $t \in [t_i, t_{i+1}]$,

$$\begin{aligned}
\int_0^t f_n(s)\, ds - \int_0^t f(s)\, ds &\leq \int_0^{t_{i+1}} f_n(s)\, ds - \int_0^{t_{i+1}} f(s)\, ds + \epsilon/2 \leq \epsilon, \\
\int_0^t f_n(s)\, ds - \int_0^t f(s)\, ds &\geq \int_0^{t_i} f_n(s)\, ds - \int_0^{t_i} f(s)\, ds - \epsilon/2 \geq -\epsilon,
\end{aligned}$$

which yields the claim. □

Proof of Theorem 2.2. Set $\overline{E} := C([0,\infty)) \times M$. Let $Z^n := X_{0-} + B + \int^\cdot \beta^n_s\, ds$ and define $\xi^n := \mathrm{law}(Z^n, \frac{1}{\alpha}\underline{\Lambda}(\beta^n)) \in \mathcal{P}(\overline{E})$.

Step 1: We show that $(Z^n, \frac{1}{\alpha}\Lambda(\beta^n))$ converges in distribution to a limit $(Z, \frac{1}{\alpha}\Lambda)$ with $\Lambda \in \alpha M$ and $Z := X_{0-} + B + \int^\cdot \beta_s\, ds$. Let $F \in C_b(C([0,\infty)); \mathbb{R})$. By the Skorokhod representation theorem, we may assume without loss of generality that $\beta^n \to \beta$ holds almost surely on S_T. Together with Lemma C.2 and the dominated convergence theorem, this shows

$$\lim_{n\to\infty} \mathbb{E}\left[F\left(X_{0-} + B + \int^{\cdot} \beta^n_s \, ds\right)\right] = \mathbb{E}\left[F\left(X_{0-} + B + \int^{\cdot} \beta_s \, ds\right)\right] \tag{60}$$

which proves that $Z^n \to Z$ in distribution on $C([0,\infty))$. Since M is compact, after passing to a subsequence if necessary, we may assume that $\frac{1}{\alpha}\underline{\Lambda}(\beta^n) \to \frac{1}{\alpha}\Lambda$ for some $\Lambda \in \alpha M$. Since Λ is deterministic, it follows that $(Z^n, \frac{1}{\alpha}\underline{\Lambda}(\beta^n)) \to (Z, \frac{1}{\alpha}\Lambda)$ in distribution on $\bar{E}$.

Step 2: We show that Λ solves the McKean–Vlasov problem (7) with drift β. We introduce some notation: Set $\xi := \mathrm{law}(Z, \frac{1}{\alpha}\Lambda)$. Define $\iota\colon \bar{E} \to D([-1,\infty))$ for $w \in C([0,\infty))$ and $\ell \in M$ as

$$\iota(w,\ell)_t := \begin{cases} w_0, & t \in [-1,0), \\ w_t - \alpha\ell_t, & t \in [0,\infty). \end{cases} \tag{61}$$

For $t \in \mathbb{R}$ and $x \in D([-1,\infty))$, define the path functionals $\tau_0(x) := \inf\{s \geq 0 : x_s \leq 0\}$ and $\lambda_t(x) := \mathbb{1}_{\{\tau_0(x)\leq t\}}$. Then, Λ is a solution to (7) if and only if $\alpha\langle \iota(\xi), \lambda_t\rangle = \Lambda_t$ for $t \geq 0$. Since Z satisfies the extended crossing property by Lemma C.1, $\iota(\xi)$ satisfies the crossing property. Lemma 5.3 in [23] and Step 1 imply that

$$\Lambda = \lim_{n\to\infty} \underline{\Lambda}(\beta^n) = \lim_{n\to\infty} \alpha\langle \iota(\xi^n), \lambda\rangle = \alpha\langle \iota(\xi), \lambda\rangle. \tag{62}$$

□

C.4 Proof of Theorem 2.6

Proof. The proof is largely analogous to the proof of Proposition 5.6 in [23]. Define

$$\xi_N := \frac{1}{N}\sum_{i=1}^{N} \delta_{(X^{i,N}_{0-} + B^{i,N}, \beta^{i,N}, L^N)}.$$

Since ξ_N is a random probability measure on $C([0,\infty)) \times S_T \times M$ and the spaces S_T and M are compact, ξ_N is tight by the same reasoning as in Corollary 4.5 in [23]. Therefore, after passing to a subsequence if necessary, we may assume that $\mathrm{law}(\xi_N) \to \mathrm{law}(\xi)$ for some random probability measure ξ on $C([0,\infty)) \times S_T \times M$. By Skorokhod representation, we may assume without loss of generality that the convergence happens almost surely on the same probability space $\mathscr{S}$. Arguing in the same fashion as in the proof of Lemma 5.4 in [23], we see that for almost every $\omega \in \Omega$, if $\mathrm{law}((W, \beta, \mathsf{L})) = \xi(\omega)$, then $W - W_0$ is a Brownian motion with respect to the filtration generated by $(W, \int_0^{\cdot} \beta_s \, ds, \mathsf{L})$. For $(w, b, \ell) \in C([0,\infty)) \times S_T \times M$, set $\hat{\iota}(w,b,\ell)_t := \iota(w,\ell)_t + \int_0^t b_s \, ds$, where ι is defined as in (61). By Lemma C.2, the map $b \mapsto \int_0^{\cdot} b_s \, ds$ is continuous from S_T to $C([0,\infty))$, and therefore also as a map from S_T to $D([-1,\infty))$. Theorem 4.2 in [23] together with Corollary 12.7.4 in [51] then shows that $\hat{\iota}$ is continuous. Since $\hat{\iota}(\xi_N) = \mu_N$, the continuous mapping theorem implies that $\hat{\iota}(\xi) = \mu$. Applying the continuous mapping theorem to

$(w, b, \ell) \mapsto w_0$, we see that $\mathrm{law}(W_0) = \nu_0$ holds almost surely. It remains to check that if $\mathrm{law}(W, \beta, \mathsf{L}) = \xi(\omega)$, then $\mathsf{L}_t \equiv \langle \mu(\omega), \lambda_t \rangle$ for $t \geq 0$ holds $\xi(\omega)$-almost surely for almost every $\omega \in \Omega$. This can be checked as in Step 1 of the proof of Proposition 5.6 in [23], making use of Lemma C.1 to show that $\mu(\omega)$ satisfies the crossing property (almost surely). $\square$

Declaration

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References

[1] B. Acciaio, J. Backhoff-Veraguas, and R. Carmona, *Extended mean field control problems: stochastic maximum principle and transport perspective*, SIAM J. Control Optim. **57** (2019), no. 6, 3666–3693.

[2] Y. Achdou, F. Camilli, and I. Capuzzo-Dolcetta, *Mean field games: convergence of a finite difference method*, SIAM J. Numer. Anal. **51** (2013), no. 5, 2585–2612.

[3] Y. Achdou and I. Capuzzo-Dolcetta, *Mean field games: numerical methods*, SIAM J. Numer. Anal. **48** (2010), no. 3, 1136–1162.

[4] Y. Achdou and M. Laurière, *On the system of partial differential equations arising in mean field type control*, Discr. Contin. Dyn. Sys. **35** (2015), no. 9, 3879–3900.

[5] ———, *Mean field type control with congestion (II): An augmented Lagrangian method*, Appl. Math. Optim. **74** (2016), no. 3, 535–578.

[6] ———, *Mean field games and applications: Numerical aspects*, Mean Field Games (2020), 249–307.

[7] N. Agram and B. Øksendal, *Fokker–Planck PIDE for McKean–Vlasov diffusions with jumps and applications to HJB equations and mean-field games*, arXiv:2110.02193 (2021).

[8] Clémence Alasseur, Luciano Campi, Roxana Dumitrescu, and Jia Zeng, *Mfg model with a long-lived penalty at random jump times: application to demand side management for electricity contracts*, arXiv:2101.06031 (2021).

[9] M. Basei and H. Pham, *Linear-quadratic McKean–Vlasov stochastic control problems with random coefficients on finite and infinite horizon, and applications*, arXiv:1711.09390 (2017).

[10] E. Bayraktar, A. Cosso, and H. Pham, *Randomized dynamic programming principle and Feynman–Kac representation for optimal control of McKean–Vlasov dynamics*, Transactions Amer. Math. Soc. **370** (2018), no. 3, 2115–2160.

[11] A. Bensoussan, J. Frehse, and P. Yam, *Mean field games and mean field type control theory*, Springer, 2013.

[12] K. Bujok and C. Reisinger, *Numerical valuation of basket credit derivatives in structural jump-diffusion models*, J. Comp. Finance **15** (2012), no. 4, 115.

[13] M. Burzoni and L. Campi, *Mean field games with absorption and common noise with a model of bank run*, arXiv:2107.00603 (2021).

[14] L. Campi and M. Fischer, *N-player games and mean-field games with absorption*, Ann. Appl. Probab. **28** (2018), no. 4, 2188–2242.

[15] L. Campi, M. Ghio, and G. Livieri, *N-player games and mean-field games with smooth dependence on past absorptions*, Available at SSRN 3329456 (2019).

[16] E. Carlini and F. J. Silva, *A semi-Lagrangian scheme for a degenerate second order mean field game system*, Discr. Contin. Dyn. Sys. **35** (2015), no. 9, 4269–4292.

[17] R. Carmona and F. Delarue, *Forward–backward stochastic differential equations and controlled McKean–Vlasov dynamics*, Ann. Probab. **43** (2015), no. 5, 2647–2700.

[18] R. Carmona, F. Delarue, and A. Lachapelle, *Control of McKean–Vlasov dynamics versus mean field games*, Math. Financ. Econ. **7** (2013), no. 2, 131–166.

[19] R. Carmona, F. Delarue, and D. Lacker, *Mean field games with common noise*, Ann. Probab. **44** (2016), no. 6, 3740–3803.

[20] R. Carmona, J.-P. Fouque, and L. H. Sun, *Mean field games and systemic risk*, Comm. Math. Sci. **13** (2015), no. 4, 911–933.

[21] R. Carmona and M. Laurière, *Convergence analysis of machine learning algorithms for the numerical solution of mean field control and games: II – the finite horizon case*, arXiv:1908.01613 (2019).

[22] C. Cuchiero, C. Reisinger, and S. Rigger, *Approximation schemes for the supercooled Stefan problem*, Working paper (November 2021).

[23] C. Cuchiero, S. Rigger, and S. Svaluto-Ferro, *Propagation of minimality in the supercooled Stefan problem*, arXiv:2010.03580.

[24] M. H. A. Davis and A. R. Norman, *Portfolio selection with transaction costs*, Math. Oper. Res. **15** (1990), no. 4, 676–713.

[25] F. Delarue, J. Inglis, S. Rubenthaler, and E. Tanré, *Global solvability of a networked integrate-and-fire model of McKean-Vlasov type*, Ann. Appl. Probab. **25** (2015), no. 4, 2096–2133. MR 3349003

[26] ———, *Particle systems with a singular mean-field self-excitation. Application to neuronal networks*, Stoch. Proc. Appl. **125** (2015), no. 6, 2451–2492.

[27] F. Delarue, S. Nadtochiy, and M. Shkolnikov, *Global solutions to the supercooled Stefan problem with blow-ups: regularity and uniqueness*, to appear in Probab. Math. Phys., arXiv:1902.05174.

[28] M. Djete, D. Possamaï, and X. Tan, *McKean–Vlasov optimal control: the dynamic programming principle*, arXiv:1907.08860 (2019).

[29] C. M. Elliott and J. R. Ockendon, *Weak and variational methods for moving boundary problems*, vol. 59, Pitman, 1982.

[30] W. H. Fleming and H. M. Soner, *Controlled Markov processes and viscosity solutions*, vol. 25, Springer, 2006.

[31] P. A. Forsyth and G. Labahn, *Numerical methods for controlled Hamilton–Jacobi–Bellman PDEs in finance*, Journal of Computational Finance **11** (2007), no. 2.

[32] J.-P. Fouque and Z. Zhang, *Deep learning methods for mean field control problems with delay*, Frontiers Appl. Math. Stat. **6** (2020), 11.

[33] B. Hambly, S. Ledger, and A. Søjmark, *A McKean–Vlasov equation with positive feedback and blow-ups*, Ann. Appl. Probab. **29** (2019), no. 4, 2338–2373.

[34] M. Huang, R. P. Malhamé, and P. E. Caines, *Large population stochastic dynamic games: closed-loop McKean–Vlasov systems and the nash certainty equivalence principle*, Comm. Info. Sys. **6** (2006), no. 3, 221–252.

[35] A. Itkin and A. Lipton, *Structural default model with mutual obligations*, Rev. Deriv. Res. **20** (2017), no. 1, 15–46.

[36] J.-M. Lasry and P.-L. Lions, *Mean field games*, Japanese J. Math. **2** (2007), no. 1, 229–260.

[37] M. Laurière and O. Pironneau, *Dynamic programming for mean-field type control*, Comptes Rendus Mathematique **352** (2014), no. 9, 707–713.

[38] M. Laurière, J. Song, and Q. Tang, *Policy iteration method for time-dependent mean field games systems with non-separable Hamiltonians*, arXiv:2110.02552 (2021).

[39] S. Ledger and A. Søjmark, *At the mercy of the common noise: blow-ups in a conditional McKean–Vlasov Problem*, Electr. J. Probab. **26** (2021), no. none, 1 – 39.

[40] A. Lipton, V. Kaushansky, and C. Reisinger, *Semi-analytical solution of a McKean–Vlasov equation with feedback through hitting a boundary*, Europ. J. Appl. Math. (2019), 1–34.

[41] R. C. Merton, *On the pricing of corporate debt: The risk structure of interest rates*, J. Finance **29** (1974), no. 2, 449–470.

[42] S. Nadtochiy and M. Shkolnikov, *Particle systems with singular interaction through hitting times: application in systemic risk modeling*, Ann. Appl. Probab. **29** (2019), no. 1, 89–129. MR 3910001

[43] ______, *Mean field systems on networks, with singular interaction through hitting times*, Ann. Probab. **48** (2020), no. 3, 1520–1556.

[44] L. Pfeiffer, *Numerical methods for mean-field-type optimal control problems*, Pure Appl. Funct. Anal. **1** (2016), no. 4, 629–655.

[45] H. Pham, *Linear quadratic optimal control of conditional McKean–Vlasov equation with random coefficients and applications*, Probab., Uncert. Quant. Risk **1** (2016), no. 1, 1–26.

[46] H. Pham and X. Wei, *Dynamic programming for optimal control of stochastic McKean–Vlasov dynamics*, SIAM Journal on Control and Optimization **55** (2017), no. 2, 1069–1101.

[47] ______, *Bellman equation and viscosity solutions for mean-field stochastic control problem*, ESAIM: Contr. Optim. Calculus. Var. **24** (2018), no. 1, 437–461.

[48] C. Reisinger, W. Stockinger, and Y. Zhang, *A fast iterative PDE-based algorithm for feedback controls of nonsmooth mean-field control problems*, arXiv preprint arXiv:2108.06740 (2021).

[49] J. Sun and J. Yong, *Stochastic linear-quadratic optimal control theory: Differential games and mean-field problems*, Springer Nature, 2020.

[50] W. Tang and L. Tsai, *Optimal surviving strategy for drifted Brownian motions with absorption*, The Annals of Probability **46** (2018), no. 3, 1597–1650.

[51] W. Whitt, *Stochastic-process limits*, Springer Series in Operations Research, Springer-Verlag, New York, 2002, An introduction to stochastic-process limits and their application to queues. MR 1876437

[52] J. Yong, *Linear-quadratic optimal control problems for mean-field stochastic differential equations*, SIAM J. Control Optim. **51** (2013), no. 4, 2809–2838.

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- Under the supervision of Prof. Christa Cuchiero, I investigated interacting particle systems with singular, "contagious" interactions, motivated by questions arising in Mathematical Finance .

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Aix-Marseille Université, France.

Bachelor Studies BSc, TU Wien — 2012-2015

Personal Data

- Citizenship: Austrian
- Language skills: German (*mother tongue*), English (*fluent*), French (*advanced*), Hungarian (*basic*)

Software Proficiencies

Julia, Python3, Matlab, C/C++, LaTeX, git.

Publications and Preprints

C. Cuchiero, *C. Reisinger* and **S. Rigger** 2021
Global implicit approximation of the supercooled Stefan problem.
(*preprint*)

C. Cuchiero, *C. Reisinger* and **S. Rigger** 2021
Optimal bailout strategies resulting from the drift controlled supercooled Stefan problem
(*preprint, available on arXiv*)

C. Cuchiero, **S. Rigger** and *S. Svaluto-Ferro* 2020
Propagation of Minimality in the Supercooled Stefan Problem
(*to appear in Annals of Applied Probability*)

C. Heitzinger, *G. Pammer* and **S. Rigger** 2018
Cubature Formulas for Multisymmetric Functions - Applications to Stochastic Partial Differential Equations
SIAM, Journal of Uncertainty Quantification

C. Heitzinger, *M. Leumüller*, *G. Pammer* and **S. Rigger** 2018
Existence, Uniqueness and Comparison of Non-intrusive Methods for the Stochastic Nonlinear Poisson-Boltzmann Equation
SIAM, Journal of Uncertainty Quantification

Conference presentations

XXII Workshop on Quantitative Finance, Verona	January 2021
Conference on High-dimensional Stochastics, Vienna	September 2020
13th European Summer School in Financial Mathematics, Vienna	August 2020
Vienna-Zurich Symposium, Vienna	November 2019
OeMG Conference, Dornbirn	September 2019
Vienna Congress on Mathematical Finance, Vienna	September 2019
Freiburg-Wien-Padova Seminar, Padova	April 2019
Austrian Numerical Analysis Day, Salzburg	May 2017