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„Fermionic interaction Vertices in Higher-Spin Theory with a Focus on quartic interaction Terms"

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Filipp Lausch, BSc

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Abstract

The main focus of this thesis is to extend the results of S. Fredenhagen's, O. Krüger's and K. Mkrtchyan's paper *"Vertex-Constraints in 3D Higher Spin Theories"* [1] – which examines the Lagrangian formulation of higher-spin gravities in three dimensions for orders $N \geq 4$ [16] – to higher spin interaction vertices of order $N = 4$ with fermions in three dimensions. The outcome of it will further be used to describe the vertex constrains for orders $N \geq 4$.

In three dimensions the additional class of Schouten-identities – alongside Lorentz-invariance, gauge invariance and other invariances of the lagrangian – has to be taken into account when evaluating the constraints for interaction vertices of high order with respect to the Noether-Fronsdal program [1][3][6]. Thus, the results differ from the general form of interaction vertices in four dimensions. In fact, Schouten-identities provide a powerful tool to simplify the calculations and also as a result constrain the outcome immensely.

Eventually, the quartic vertices are constrained to only contain fields of spin-1 or lower. This also translates to the generalization of the vertex constraints for higher order vertices $N \geq 4$.

Contents

Introduction

1.1 Motivation: the AdS-CFT Correspondence

Among others, a very well known motivation for studying Higher Spin systems is given by the AdS/CFT-correspondence [1].

The AdS/CFT-correspondence is a conjecture which states that a d-dimensional conformal field theory is dual to a gravitational theory on a (d+1)-dimensional asymptotic anti-de-Sitter spacetime [1][7]. It is specifically important for three-dimensional higher spin gravities solely for the reason that two-dimensional conformal field theories yield a lot of exact results and thus it is a very powerful tool to examine them. In this section the correspondence will very shortly be introduced with the example of a scalar field [1][7].

The metric of anti-de-Sitter spacetime can be written as

$$\mathrm{d}s^2_{AdS} = L^2\left(-(1+r^2)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1+r^2} + r^2\mathrm{d}\Omega^2_{d-1}\right)$$

where $\mathrm{d}\Omega^2_{d-1}$ corresponds to the (d-1)-dimensional unit sphere S^{d-1}[7]. It is important to note that at $r = 0$ it corresponds to Minkowski-space whilst for rising $r > 0$ the spacetime corresponds to a rising gravitational potential,

which can clearly be seen by looking at $V \approx \sqrt{-g_{00}}$. For massless geodesics it is possible to reach infinity and come back in a finite amount of time. Geodesics of massive objects will at some point get repelled and thus cannot reach that far [7].
The full symmetry group of AdS spacetime is given through SO(2,d). This can be seen by the fact that AdS spacetime can be described as a hyperboloid in $\mathbb{R}^{2,d}$ given through

$$-X_0^2 - X_1^2 + X_2^2 + ... + X_d^2 = -L^2$$

which clearly yields SO(2,d) to be the group of isometries [7].

The asymptotic boundary of AdS spacetime is given through $\mathbb{R} \times S^{d-1}$. As a result the isometries of it act on the asymptotic boundary in a way that points on the boundary are mapped to points on the boundary. This is the action of the conformal group in d dimensions, which yields that the quantum field theories that asymptotically AdS spacetimes can be dual to are given through conformal field theories [7].

To demonstrate the way this correspondence manifests we will look at the example of the scalar field given through the action [7]

$$S[\phi] = \int d^{d+1}x \sqrt{-\det(g)} \left(\nabla_\mu \phi \nabla^\mu \phi + m^2 \phi^2 \right) \ .$$

First of all there exists a coordinate transformation such that the line element can be conveniently rewritten as:

$$ds^2_{AdS} \to ds^2 = \frac{L^2}{z^2} \left(-dt^2 + dz^2 + ds^2_{d-1} \right)$$

where ds^2_{d-1} is just euclidean space in d-1 dimensions. These coordinates are useful whenever it is necessary to look at a conformal field theory in d-dimensional Minkowski-spacetime [7].

Given some fixed boundary condition $\phi(x,z)|_{z=\epsilon} = \phi_0(x)\epsilon^{d-\Delta}$ one finds that the solution for ϕ at the asymptotic boundary looks as follows in Fourier space [7]:

$$\tilde{\phi}(k,z) = \tilde{\phi}_0(k)\epsilon^{d-\Delta}e^{ik(z-\epsilon)}\frac{z^{d/2}K_\nu(kz)}{\epsilon^{d/2}K_\nu(k\epsilon)}$$
$$\nu = \sqrt{\frac{d^2}{4}+m^2L^2}$$

where $K_\nu(kz)$ is a modified BesselK function of the second kind. After some more computation of the action $S[\phi]$ one finds a non-local contribution given by

$$S_{\text{non-local}} = -\frac{2\nu\Gamma(\Delta)}{\pi^{d/2}\Gamma(\nu)}L^{d-1}\int\int d^dxd^dy\frac{\phi_0(x)\phi_0(y)}{|x-y|^{2\Delta}}$$

which corresponds to the generating function regarding the correlation functions for a corresponding operator in the dual conformal field theory. For a field ϕ that corresponds to a single-trace operator $\mathcal{O}$ we find

$$Z_{\text{Gravity}}[\phi_0] = Z_{\text{CFT}}[\phi_0] = \langle e^{\int d^dx\phi_0(x)\mathcal{O}(x)}\rangle$$

for the partition functions of the respective theories. This beautifully demonstrates the AdS/CFT-correspondence.

As a remark it is important to note that an argument can be made that results in higher spin theory in its metric-like formulation [14][15] in Minkowski spacetime also apply to AdS backgrounds [1][3]. This simplifies the methods of this thesis to the framework of Minkowski spacetime, which will be used going forward from here. According to [3] that is the case because the most relevant obstacles that prevent it in higher dimensions are not applying in three dimensions, which are Weinberg's S-matrix argument [9] as well as the

Aragone-Deser problem [10].

1.2 Construction of massless higher-spin fermionic Fields

1.2.1 Representation Theory of the Lorentz-Group

The Lorentz-group $\mathrm{SO}(1, d-1)$ is the group of linear homogeneous transformations that leave the metric of Minkowski-spacetime invariant[11].

$$\Lambda^{\mu}{}_{\alpha}\eta_{\mu\nu}\Lambda^{\nu}{}_{\beta} = \eta_{\alpha\beta}$$

for the metric tensor

$$\eta = diag(1, -1, ..., -1)$$

as well as the Lorentz-transformations Λ^{μ}_{ν}.
For practical reasons, in physics we exclude parity transformations as well as transformations that reverse time. We call the resulting group the proper orthochronous Lorentz-group $\mathrm{SO}^{\uparrow}_{+}(1, d-1)$ [12]. The Lie-algebra identified with $\mathrm{SO}^{\uparrow}_{+}(1, d-1)$ is $\mathrm{Lie}(\mathrm{SO}^{\uparrow}_{+}(1, d-1)) := \mathfrak{so}^{\uparrow}_{+}(1, d-1)$, which can be constructed by the matrix exponential as follows:

For simplicity, let us look at the infinitesimal Lorentz-transformations (which are sufficient for this construction)[2][11]. Start with an infinitesimal variation ρ^{μ}_{ν} of the Kronecker-delta δ^{μ}_{ν}, such that

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \rho^{\mu}{}_{\nu}$$
$$\left(\delta^{\mu}{}_{\alpha} + \rho^{\mu}{}_{\alpha}\right)\eta_{\mu\nu}\left(\delta^{\nu}{}_{\beta} + \rho^{\nu}{}_{\beta}\right) = \eta_{\alpha\beta} \ .$$

Solving this for $\rho_{\mu\nu}$ results in

$$\rho_{\mu\nu} = -\rho_{\nu\mu}$$

which means the Lie-group has $\frac{d(d-1)}{2}$ free parameters, since ρ is antisymmetric. Now assume that the antisymmetric part of an infinitesimal Lorentz-transformation is generated by generators $(M_{\alpha\beta})^{\mu}_{\nu} = -(M_{\beta\alpha})^{\mu}_{\nu}$ with corresponding coefficients $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$. Specifically, a general infinitesimal Lorentz-transformation can be written as

$$\Lambda^{\mu}_{\ \nu} = \delta^{\mu}_{\nu} + \frac{1}{2}\omega^{\alpha\beta}\left(M_{\alpha\beta}\right)^{\mu}_{\ \nu} \ .$$

We find

$$\left[\exp\left(\frac{1}{2}\omega^{\alpha\beta}M_{\alpha\beta}\right)\right]^{\mu}_{\ \nu} = \delta^{\mu}_{\ \nu} + \frac{1}{2}\omega^{\alpha\beta}\left(M_{\alpha\beta}\right)^{\mu}_{\ \nu}$$

for the infinitesimal case. However, for appropriate $\omega^{\alpha\beta}$ this yields that a general Lorentz-transformation can be written as

$$\Lambda^{\mu}_{\ \nu} = \left[\exp\left(\frac{1}{2}\omega^{\alpha\beta}M_{\alpha\beta}\right)\right]^{\mu}_{\ \nu}$$

and thus the Lie-algebra $\mathfrak{so}^{\uparrow}_{+}(1, d-1)$ of the Lorentz-group is spanned by the generators

$$\boxed{M_{\alpha\beta} = -M_{\beta\alpha}} \ .$$

Now assume $\rho_{\mu\nu} = \omega_{\mu\nu}$. This results in the following equation

$$\omega^{\mu}_{\ \nu} = \frac{1}{2}\omega^{\alpha\beta}\left(M_{\alpha\beta}\right)^{\mu}_{\ \nu} \ .$$

It can be solved by

$$\left(M_{\alpha\beta}\right)^{\mu}_{\ \nu} = \delta^{\mu}_{\ \alpha}\eta_{\nu\beta} - \delta^{\mu}_{\ \beta}\eta_{\nu\alpha} \ .$$

This is a **fundamental representation** [19] of the Lorentz-algebra. Using it we can compute the commutator of two arbitrary generators [11]

$$\boxed{[M_{\alpha\beta}, M_{\gamma\delta}] = \eta_{\beta\gamma}M_{\alpha\delta} + \eta_{\alpha\delta}M_{\beta\gamma} - \eta_{\beta\delta}M_{\alpha\gamma} - \eta_{\alpha\gamma}M_{\beta\delta}}$$

which ultimately concludes our definition of the Lorentz-algebra.

1.2.1.1 Bosonic Representations

The generators of the fundamental representation can now be used to express Lorentz-transformations. Let us start by looking at the action of M_{12} for $\mathfrak{so}^{\uparrow}_{+}(1,2)$ on $\mathbb{R}^{1,2}$:

$$\exp(\theta M_{12}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

which is a rotation that obeys $\exp(2\pi M_{12}) = \mathbb{1}_{3\times 3}$. That means that the Lorentz-transformations induced by the generators $M_{\alpha\beta}$ of the fundamental representation can be attributed to bosons [12].

A bosonic field $\Phi(x)$ can for instance be defined by the group action on it [12]:

$$\Phi(x) \to \exp\left(\frac{1}{2}\omega^{\alpha\beta}M_{\alpha\beta}\right)\Phi(\Lambda^{-1}x) \ .$$

This is a spin-1 field. Because of the transformation-law we can easily see that it carries a spacetime-index and can be written as

$$\Phi^{\mu}(x) \to \Phi'^{\mu}(x) = \left[\exp\left(\frac{1}{2}\omega^{\alpha\beta}M_{\alpha\beta}\right)\right]^{\mu}_{\ \nu}\Phi^{\nu}(\Lambda^{-1}x)$$

The representation given by the fundamental representation and the spin-1 field, $(M_{\alpha\beta}, \Phi(x))$, will from now on be denoted by $\mathsf{R}(1)$.

It is possible to construct higher integer spin representations by combining multiple spin-1 fields via a tensor product [5]. The corresponding product space is given by:

$$\mathcal{R}_s := \bigotimes_{i=1}^{s} \mathsf{R}(1)$$

It consists of the tensors

$$\tilde{\Phi}(x) := \bigotimes_{i=1}^{s} \Phi^{(i)}(x)$$

where (i) is just a dummy-index to indicate difference between the individual fields Φ.

A spin-s field transforms as

$$\tilde{\Phi}(x) \to \left[\bigotimes_{i=1}^{s} \Lambda\right] \tilde{\Phi}(\Lambda^{-1}x)$$

where $\tilde{\Lambda} := \bigotimes_{i=1}^{s} \Lambda$ defines the way a Lorentz-transformation acts on the product space. One immediately finds that $\tilde{\Lambda}$ is symmetric with respect to the tensor product, so $\tilde{\Lambda} = \bigodot_{i=1}^{s} \Lambda$. This results in the following equation

$$\tilde{\Lambda}\tilde{\Phi}(\Lambda^{-1}x) = \left[\bigodot_{j=1}^{s} \Lambda\right] \bigotimes_{i=1}^{s} \Phi^{(i)}(\Lambda^{-1}x)$$

This suggests that the symmetric (respective antisymmetric) tensors $\mathrm{Sym}(\mathcal{R}_s)$ of $\mathcal{R}_s$ yield a representation of $\mathcal{R}_s$. In order to get an irreducible spin-s representation $\mathsf{R}(s)$ from $\mathcal{R}_s$ it is not only necessary that $\tilde{\Phi}$ is symmetric $\tilde{\Phi}_{(\mu_1 \dots \mu_s)}(x) = \tilde{\Phi}_{\mu_1 \dots \mu_s}(x)$ but also that it is η-traceless $\eta^{\mu_1\mu_2}\tilde{\Phi}_{\mu_1\mu_2\dots\mu_s} = 0$ as given in [5]. The η-trace is a gauge condition that can always be achieved

on-shell.
So we find that

$$\boxed{\mathsf{R}(s) = \Big(\mathrm{Sym}(\mathcal{R}_s); \eta^{\mu_1\mu_2}\tilde{\Phi}_{\mu_1\mu_2\ldots\mu_s} = 0\Big)}$$

is a proper way to represent a spin-s field [5].

1.2.1.2 Spin-1/2 Representations

Clifford-algebra and gamma-matrices:

The Clifford-algebra [11] $\mathfrak{Cl}(\mathbb{R}^{1,d-1}, \eta)$ over the Minkowski-spacetime $\mathbb{R}^{1,d-1}$ with the quadratic form $\eta(v, w) := \eta_{\mu\nu} v^\mu w^\nu$ can be constructed with elements γ^μ that obey

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1} \ .$$

With those generators, the full algebra can be obtained as follows [2]:

$$\begin{aligned} &\mathbb{1} \\ &\gamma^\mu \\ \gamma^{\mu\nu} &:= \gamma^{[\mu}\gamma^{\nu]} \\ \gamma^{\mu\nu\rho} &:= \gamma^{[\mu}\gamma^\nu\gamma^{\rho]} \\ &\vdots \\ \gamma^{\mu_1\ldots\mu_d} &:= \gamma^{[\mu_1} \cdots \gamma^{\mu_d]} \ . \end{aligned}$$

Upon counting the number of elements, one finds that there are 2^d unique elements contained in $\mathfrak{Cl}(\mathbb{R}^{1,d-1}, \eta)$.

One way of representing $\mathfrak{Cl}(\mathbb{R}^{1,d-1}, \eta)$ is given by using the Pauli-matrices, which results in the so-called γ-matrices [2][11]. They will play a big role in constructing a fermionic theory. As an example, let us represent the first three γ-matrices with a choice of Pauli-matrices for $d = 2k$ even dimension [18]:

$$\gamma^0 = \sigma_1 \otimes \left(\bigotimes_{i=1}^{k-1} \mathbb{1}_{2\times 2} \right)$$
$$\gamma^1 = i\sigma_2 \otimes \left(\bigotimes_{i=1}^{k-1} \mathbb{1}_{2\times 2} \right)$$
$$\gamma^2 = i\sigma_3 \otimes \sigma_1 \otimes \left(\bigotimes_{i=1}^{k-2} \mathbb{1}_{2\times 2} \right)$$
$$\cdots$$

In the case of odd dimension $d = 2k+1$, we can construct the matrices with the generators of $\mathfrak{Cl}(\mathbb{R}^{1,d-2}, \eta)$ and one additional generator, given by [18]

$$\pm\gamma^{d-1} = \pm(-1)^{k+1}\gamma^0\gamma^1 \cdots \gamma^{d-2}$$

with

$$(\gamma^{d-1})^2 = \bigotimes_{i=1}^{k} \mathbb{1}_{2\times 2}$$

and

$$\{\gamma^\mu, \gamma^{d-1}\} = 0$$

for $\mu = 0, 1, ..., d-2$, where the choice of $\pm$ in front of γ^d yields two non-equivalent representations of $\mathfrak{Cl}(\mathbb{R}^{1,d-1}, \eta)$.

It is possible to construct the generators in the following way [2]:

$$\tilde{M}_{\alpha\beta} = \frac{1}{2}\gamma_{\alpha\beta}$$

which fulfil the commutation relations of the Lorentz-algebra and thus yield a representation of it:

$$\left[\tilde{M}_{\alpha\beta}, \tilde{M}_{\gamma\delta}\right] = \eta_{\beta\gamma}\tilde{M}_{\alpha\delta} + \eta_{\alpha\delta}\tilde{M}_{\beta\gamma} - \eta_{\beta\delta}\tilde{M}_{\alpha\gamma} - \eta_{\alpha\gamma}\tilde{M}_{\beta\delta} \ .$$

In order to see how this representation acts on a corresponding vector space, let us look at the example of $\mathfrak{so}^+(1,2)$ acting on $\mathbb{C}^2$. Choose the exponential

$$\exp\left(\theta\tilde{M}_{12}\right) = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & i\sin\left(\frac{\theta}{2}\right) \\ i\sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{pmatrix} \ .$$

It is immediately apparent that $\exp\left(2\pi\tilde{M}_{12}\right) = -\mathbb{1}_{2\times 2}$ and as a result $\exp\left(4\pi\tilde{M}_{12}\right) = \mathbb{1}_{2\times 2}$. This means this representation is a **projective representation** [19]. The Lorentz-transformations induced by the generators $\tilde{M}_{\alpha\beta}$ can be attributed to Fermions.

All this leads to the definition of a spinor $\Psi(x)$, which transforms under this representation of the Lorentz-algebra as [12]

$$\Psi(x) \to \exp\left(\frac{1}{2}\omega^{\alpha\beta}\tilde{M}_{\alpha\beta}\right)\Psi(\Lambda^{-1}x) \ .$$

They form a projective representation with respect to the $\tilde{M}_{\alpha\beta}$. Spinors constructed this way are called **Dirac Spinors** [2][12]. Dirac Spinors carry a so-called **Spinor-index** which is defined by the transformation-laws as follows [12]:

$$\Psi^a(x) \to \left[\exp\left(\frac{1}{2}\omega^{\alpha\beta}\tilde{M}_{\alpha\beta}\right)\right]^a_{\ b}\Psi^b(\Lambda^{-1}x) \ .$$

Looking at the $(\tilde{M}_{\alpha\beta}, \Psi(x))$ representation's generators of $\mathfrak{so}^+(1,2)$

$$\tilde{M}_{12} = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \frac{i}{2}\sigma_1$$
$$\tilde{M}_{01} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{2}\sigma_3$$
$$\tilde{M}_{02} = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{1}{2}\sigma_2$$

it quickly becomes clear that they are not simultaneously diagonizable and thus $(\tilde{M}_{\alpha\beta}, \Psi(x))$ is an irreducible representation of $\mathfrak{so}^+(1,2)$ [19]. Without further ado this representation will be denoted by $\mathsf{R}^+(\frac{1}{2})$. The irreducible representation $\mathsf{R}^-(\frac{1}{2})$ corresponds to the choice of $-\gamma^2$ as one of the generators of $\mathfrak{Cl}(\mathbb{R}^{1,2}, \eta)$. See the appendix (A1) for the Pauli- and γ-matrices in this setting.

1.2.2 Higher-spin fermionic Fields in three Dimensions

Now that we have constructed irreducible representations for both bosonic spin-s fields as well as fermionic spin-1/2 fields in three dimensions, it is possible to construct fields of arbitrary half-integer spin $s = l + 1/2$ (where $l \in \mathbb{N}$) with the tools we are already provided with. Corresponding to the construction of spin-l bosonic fields, for fermionic spin-$(l + 1/2)$ fields we make an Ansatz by forming the tensor product of $\mathsf{R}(l)$ and $\mathsf{R}^+(\frac{1}{2})$:

$$\mathcal{R}_{l+1/2} := \mathsf{R}(l) \otimes \mathsf{R}^+\left(\frac{1}{2}\right) \ .$$

The fields are then given by

$$\tilde{\Psi}(x) := \tilde{\Phi}(x) \otimes \Psi(x) \ .$$

However, according to the Clebsch-Gordon decomposition, $\mathcal{R}_{l+1/2}$ decomposes to [12]

$$\mathcal{R}_{l+1/2} \cong \mathsf{R}(l+1/2) \oplus \mathsf{R}(l-1/2)\ .$$

So in order to construct a proper spin-s fermionic representation $\mathsf{R}(s)$ for $s > 1/2$ we need to divide away the $s-1$ contribution in $\mathcal{R}_s$. The best way to do so is by contracting one of the Lorentz-indices with the γ-matrices, because this equals to projecting $\tilde{\Psi}(x)$ onto the representation $\mathsf{R}(s-1)$. The $s-1$ contribution is then divided out by identifying the projection with zero:

$$\gamma^{\mu_1}\tilde{\Psi}_{\mu_1\ldots\mu_l} = 0$$

which yields the fermionic spin-s representation $\mathsf{R}(s)$

$$\boxed{\mathsf{R}(s) = \left(\mathcal{R}_s; \gamma^{\mu_1}\tilde{\Psi}_{\mu_1\ldots\mu_l} = 0\right)}\ .$$

1.3 Fronsdal Field Equations for Bosons

From now on we will omit the $\sim$ in $\tilde{\Phi}(x)$ and only use $\Phi(x)$ in order to describe the spin-s bosonic field. When constructing the free Lagrangian for free massless spin-s fields it is already possible to exploit the condition of the η-tracelessness. First of all, terms of quadratic order in a single spin-s field have $2s$ indices that need to be contracted. Since a quadratic term without derivatives would in one way or another equal a mass-term, we have to include two derivatives into this scheme in order to construct the appropriate free Lagrangian. Define an operator $\Omega^{(\nu_1 \ldots \nu_s)\alpha}_{(\mu_1 \ldots \mu_s)\beta}$ without loss of generality such that

$$\mathcal{L}_s = \Omega^{(\nu_1 \ldots \nu_s)\alpha}_{(\mu_1 \ldots \mu_s)\beta} \partial_\alpha \Phi_{\nu_1 \ldots \nu_s} \partial^\beta \Phi^{\mu_1 \ldots \mu_s} \ .$$

Because of the η-trace condition no contraction between two space-time indices of the same field, i.e. terms like $\eta^{\nu_i \nu_j}$ or $\eta_{\mu_i \mu_j}$, are possible. This simplifies the problem quite a lot, however let us first write down the most general form of the operator Ω:

$$\begin{aligned} \Omega^{(\nu_1 \ldots \nu_s)\alpha}_{(\mu_1 \ldots \mu_s)\beta} &= \epsilon_1 \eta^{\alpha(\nu_1} \eta^{\nu_2}_{(\mu_2} \cdots \eta^{\nu_s)}_{\mu_s} \eta_{\mu_1)\beta} + \text{trace-terms of that type} \\ &\quad + \epsilon_2 \eta^\alpha_\beta \eta^{(\nu_1}_{(\mu_1} \cdots \eta^{\nu_s)}_{\mu_s)} + \text{trace-terms of that type} \ . \end{aligned}$$

This is the most general form because the equivalence class given through total derivatives allows us to switch around the differentials up to signs that eventually will get compensated by the $\epsilon_i \in \mathbb{R}$. Because the trace-terms vanish for massless bosonic fields, we get the simple case of

$$\mathcal{L}_s = \epsilon_1 \left(\partial_\alpha \Phi_{\mu_1 \ldots \mu_s} \right) \partial^\alpha \Phi^{\mu_1 \ldots \mu_s} + \epsilon_2 \left(\partial^\alpha \Phi_{\alpha \mu_2 \ldots \mu_s} \right) \partial_\beta \Phi^{\beta \mu_2 \ldots \mu_s}$$

where $\epsilon_1 = 1/2 = -\epsilon_2$ [5] is given through convention with the relative sign being a result of demanding gauge invariance. Calculating the equations of motion this results in the so called Fronsdal tensor [5] for massless fields:

$$\boxed{\mathcal{F}_{\mu_1\ldots\mu_s} = \Box\Phi_{\mu_1\ldots\mu_s} - \partial_{(\mu_1}\partial^\alpha\Phi_{\mu_2\ldots\mu_s)\alpha} = 0} \ .$$

1.3.1 Gauge Invariance of the Fronsdal Tensor

The gauge field of the spin-s bosonic field is given by a spin-$(s-1)$ field. This can beautifully be demonstrated upon analysing gauge invariance of the Fronsdal tensor [5]. Let the gauge transformation of zeroth order in the fields itself be given in its most general form as:

$$\Phi \to \Phi + \delta^{(0)}\Phi$$

with $\delta^{(0)}$ yielding the gauge variation with respect to order zero in the fields. Then, given a gauge field ξ, the following holds [5]:

$$\delta^{(0)}\Phi_{\mu_1\ldots\mu_s} = \alpha\partial_{(\mu_1}\xi_{\mu_2\ldots\mu_s)}$$

Per convention we can set $\alpha = 1$ [5].
This results in

$$0 = \delta^{(0)}\mathcal{F}_{\mu_1\ldots\mu_s} = -\partial_{(\mu_1}\partial_{\mu_2}\partial^\alpha\xi_{\mu_3\ldots\mu_s)\alpha} - \partial_{(\mu_1}\partial_{\mu 2}\partial_{\mu_3}\xi_{\mu_4\ldots\mu_s)\alpha}{}^\alpha \ .$$

By gauge fixing via $\partial^{\mu_1}\Phi_{\mu_1\mu_2\ldots\mu_s} = 0$ [5] we obtain the transversality condition for the gauge field $\partial^{\mu_1}\xi_{\mu_1\ldots\mu_{s-1}} = 0$. Combining the gauge fix with the η-trace condition of Φ one also obtains:

$$\eta^{\mu_1\mu_2}\xi_{\mu_1\mu_2\ldots\mu_{s-1}} = 0$$

which yields $\delta^{(0)}\mathcal{F}_{\mu_1\ldots\mu_s} = 0$ as anticipated. Of course without loss of generality $\xi_{\mu_1\mu_2\ldots\mu_{s-1}} = \xi_{(\mu_1\mu_2\ldots\mu_{s-1})}$. For the remaining gauge field, with all constraints imposed, we also have $\Box\xi = 0$.

With this the complete set of equations for the free and massless spin-s theory is fixed.

1.3.2 Complete Set of Equations for the free massless Theory

The complete set of equations of the free massless theory for bosonic spin-s fields can be summed up as follows. In the first and second row we have the conditions given through the construction of the fields. The transversality condition in the third row is achieved by gauge fixing, which results in the fourth row that gives the free equations of motion. The last row describes the zeroth order gauge variation in the fields. Likewise for the second column, in which the spin-$(s-1)$ gauge-field constraints are listed [5].

$$\boxed{\begin{array}{cc}
\Phi_{(\mu_1\ldots\mu_s)} = \Phi_{\mu_1\ldots\mu_s} & \xi_{(\mu_1\ldots\mu_{s-1})} = \xi_{\mu_1\ldots\mu_{s-1}} \\[2em]
\eta^{\mu_1\mu_2}\Phi_{\mu_1\mu_2\ldots\mu_s} = 0 & \eta^{\mu_1\mu_2}\xi_{\mu_1\mu_2\ldots\mu_{s-1}} = 0 \\[2em]
\partial^{\mu_1}\Phi_{\mu_1\mu_2\ldots\mu_s} = 0 & \partial^{\mu_1}\xi_{\mu_1\mu_2\ldots\mu_{s-1}} = 0 \\[2em]
\Box\Phi = 0 & \Box\xi = 0 \\[2em]
\delta^{(0)}\Phi_{\mu_1\ldots\mu_s} = \partial_{(\mu_1}\xi_{\mu_2\ldots\mu_s)} &
\end{array}}$$

1.4 Fang-Fronsdal Field Equations for Fermions

From now on we will also omit the $\sim$ in $\tilde{\Psi}(x)$ and only use $\Psi(x)$ to describe the spin-$(s+1/2)$ fermionic field. In order to construct the free Lagrangian for an arbitrary spin-$(s+1/2)$ fermionic massless field $\Psi(x)$ one first needs to find its conjugate field with respect to Lorentz-invariance. Naively we can assume that $\Psi^\dagger(x)$ is the conjugate field of $\Psi(x)$, however, $\Psi^\dagger\Psi$ is not Lorentz-invariant. The complex conjugate of $\Psi(x)$ transforms under Lorentz-transformations as [2]

$$\Psi^\dagger(x) \to \left(\left[\tilde{\Lambda}\otimes\lambda\right]\Psi(\Lambda^{-1}x)\right)^\dagger = \Psi^\dagger(\Lambda^{-1}x)\left[\tilde{\Lambda}^T\otimes\lambda^\dagger\right]$$

where $\lambda := \exp\left(\frac{1}{2}\omega^{\alpha\beta}\tilde{M}_{\alpha\beta}\right)$. However,

$$\tilde{\Lambda}^T\otimes\lambda^\dagger = \left(\tilde{\eta}\tilde{\Lambda}^{-1}\tilde{\eta}\right)\otimes\left(\gamma^0\lambda^{-1}\gamma^0\right) \neq \tilde{\Lambda}^{-1}\otimes\lambda^{-1}$$

where $\tilde{\eta} := \bigodot_{i=1}^{s}\eta$. This leads to the definition of the conjugate field given by [2]

$$\bar{\Psi}(x) := \Psi^\dagger(x)\left[\tilde{\eta}\otimes\gamma^0\right]$$

for which the expression $\bar{\Psi}\Psi$ is now Lorentz-invariant. If we look at this equation in index notation, we get

$$\bar{\Psi}^{\mu_1\ldots\mu_s} = \eta^{\mu_1\nu_1}\cdots\eta^{\mu_s\nu_s}\Psi^\dagger_{\nu_1\ldots\nu_s}\gamma^0\ .$$

In order to construct the free massless Lagrangian for a spin-$(s+1/2)$ fermionic field we make the Ansatz of introducing a tensor Γ that also contracts the spinor indices ($\bar{\Psi}\Gamma\Psi := \bar{\Psi}_a\Gamma^{ab}\Psi_b$), to have

$$\mathcal{L}_{(s+1/2)} = \bar{\Psi}^{\mu_1\ldots\mu_s}\Gamma^{\alpha\nu_1\ldots\nu_s}_{\mu_1\ldots\mu_s}\partial_\alpha\Psi_{\nu_1\ldots\nu_s}\ .$$

Due to the fields Ψ being totally symmetric in their spacetime indices, Γ can also be chosen to be totally symmetric in the μ_i- and ν_j-indices. We can expand Γ in the basis of the γ-matrices. The identity-term is zero from the get-go, because the expression has an odd number of Lorentz-indices, and thus if we expand $\Gamma^{\alpha\nu_1\ldots\nu_s}_{\mu_1\ldots\mu_s}$ with the γ-matrices, we get

$$\Gamma^{\alpha\nu_1\ldots\nu_s}_{\mu_1\ldots\mu_s} = \left[\Gamma^{\alpha\nu_1\ldots\nu_s}_{\mu_1\ldots\mu_s}\right]_\sigma \gamma^\sigma \ .$$

For the same reason of the identity-term being zero, the γ^σ have to contract with one of the indices of either Ψ or ∂. Then, after imposing Lorentz-invariance one finds

$$\Gamma^{\alpha\nu_1\ldots\nu_s}_{\mu_1\ldots\mu_s} = \epsilon_1 \tilde{\eta}^{\nu_1\ldots\nu_s}_{\mu_1\ldots\mu_s}\gamma^\alpha + s\cdot\epsilon_2\eta^\alpha_{(\mu_1}\tilde{\eta}^{(\nu_1\ldots\nu_{s-1}}_{\mu_2\ldots\mu_s)}\gamma^{\nu_s)}$$

where $\epsilon_1, \epsilon_2 \in \mathbb{C}$ are complex numbers in order to ensure that the Lagrangian is hermitian later. The Lagrangian now takes on the form:

$$\begin{aligned}\mathcal{L}_{(s+1/2)} &= \epsilon_1\bar{\Psi}^{\mu_1\ldots\mu_s}\gamma^\alpha\partial_\alpha\Psi_{\mu_1\ldots\mu_s} + s\cdot\epsilon_2\bar{\Psi}^{\mu_1\ldots\mu_{s-1}\alpha}\gamma^\sigma\partial_{(\alpha}\Psi_{\mu_1\ldots\mu_{s-1})\sigma} \\ &= \epsilon_1\bar{\Psi}\partial\!\!\!/\Psi + s\cdot\epsilon_2\bar{\Psi}^{\mu_1\ldots\mu_{s-1}\alpha}\partial_{(\alpha}\Psi\!\!\!/_{\mu_1\ldots\mu_{s-1})}\end{aligned}$$

Imposing that the Lagrangian has to be hermitian leads to:

$$\mathcal{L}_{(s+1/2)} = i|\epsilon_1|\bar{\Psi}\partial\!\!\!/\Psi - i|\epsilon_2|s\cdot\bar{\Psi}^{\mu_1\ldots\mu_{s-1}\alpha}\partial_{(\alpha}\Psi\!\!\!/_{\mu_1\ldots\mu_{s-1})}$$

where $|\epsilon_1| = |\epsilon_2| = 1$, again due to convention. The relative sign between the two terms is given through imposing gauge invariance.

This results in the **Fang-Fronsdal field equations** given by the **Fang-Fronsdal tensor** [13][17]

$$\boxed{S_{\mu_1\ldots\mu_s} = i\left(\partial\!\!\!/\Psi_{\mu_1\ldots\mu_s} - s\cdot\partial_{(\mu_1}\Psi\!\!\!/_{\mu_2\ldots\mu_s)}\right) = 0} \ .$$

1.4.1 Gauge Invariance of the Fang-Fronsdal Tensor

Upon analysing the gauge-invariance of the Fang-Fronsdal tensor one finds the gauge field to be of spin-$(s-1/2)$. In order to do so, one has to assume a gauge transformation of Ψ given by

$$\Psi \rightarrow \Psi + \delta^{(0)}\Psi$$

where $\delta^{(0)}$ implicates gauge variation with 0-th order in the Ψ-field itself. So given a gauge field ϵ, $\delta^{(0)}\Psi$ can be expressed as [5][17]

$$\delta^{(0)}\Psi_{\mu_1\ldots\mu_s} = \alpha\partial_{(\mu_1}\epsilon_{\mu_2\ldots\mu_s)} \ .$$

We can find constraints for ϵ by simply substituting this into $\delta^{(0)}S_{\mu_1\ldots\mu_s}$. This results in

$$\begin{aligned}\delta^{(0)}\Psi_{\mu_1\ldots\mu_s} &= s\cdot\partial_{(\mu_1}\epsilon_{\mu_2\ldots\mu_s)}\\ \gamma^{\mu_1}\epsilon_{\mu_1\ldots\mu_{s-1}} &= 0\end{aligned}$$

The γ-trace condition also has to be imposed on the ϵ-gauge field, because without it the Fang-Fronsdal tensor is not gauge invariant, but instead

$$\delta^{(0)}S_{\mu_1\ldots\mu_s} = -is\cdot\partial_{(\mu_1}\partial_{\mu_2}\not{\epsilon}_{\mu_3\ldots\mu_s)} \ .$$

Moreover, we can make a choice of gauge fixing by

$$\not{\partial}\delta^{(0)}\Psi_{\mu_1\ldots\mu_s} = s\cdot\not{\partial}\partial_{(\mu_1}\epsilon_{\mu_2\ldots\mu_s)} = 0 \ \Rightarrow\ \not{\partial}\epsilon = 0 \ .$$

Furthermore $\not{\partial}\epsilon = 0 \Rightarrow \Box\epsilon = 0$, which is relevant for

$$\partial^{\mu_1}\delta^{(0)}\Psi_{\mu_1\ldots\mu_s} = 0 \ \wedge\ \Box\epsilon = 0 \ \Rightarrow\ \partial^{\mu_1}\epsilon_{\mu_1\ldots\mu_{s-1}} = 0 \quad ,$$

where we chose to gauge fix with $\partial^{\mu_1}\delta^{(0)}\Psi_{\mu_1 \dots \mu_s} = 0$.
Last but not least, the η-trace condition of Ψ combined with $\partial^{\mu_1}\epsilon_{\mu_1 \dots \mu_{s-1}} = 0$ results in

$$\eta^{\mu_1 \mu_2}\epsilon_{\mu_1 \mu_2 \dots \mu_{s-1}} = 0 \ .$$

In order to fulfil these gauge conditions, clearly

$$\epsilon_{(\mu_1 \dots \mu_{s-1})} = \epsilon_{\mu_1 \dots \mu_{s-1}} \ .$$

This means the ϵ-gauge field is a representative of the irreducible spin-$(s-1/2)$ representation $\mathsf{R}(s-1/2)$ of $\mathfrak{so}^+(2,1)$.

1.4.2 Complete Set of Equations for the free massless Theory

With all the preliminary calculations, we have now arrived at combining everything together in order to define the complete set of equations that are necessary to describe the free theory of a massless spin-$(s+1/2)$ particle [5][13][17]. The first two rows stand for the conditions a spin$-s$ field has to fulfil, whilst the third row represents the annihilation of the spin-$(s-1/2)$ contribution of the field. The fourth row represents gauge fixing. Resulting from that, in the fifth row are the field equations and in the last one is the gauge-transformation of Ψ. Likewise for the second column, in which the spin-$(s-1/2)$ gauge-field constraints are listed.

$\Psi_{(\mu_1\ldots\mu_s)} = \Psi_{\mu_1\ldots\mu_s}$	$\epsilon_{(\mu_1\ldots\mu_{s-1})} = \epsilon_{\mu_1\ldots\mu_{s-1}}$
$(\eta^{\mu_1\mu_2}\Psi_{\mu_1\mu_2\ldots\mu_s} = 0)$	$(\eta^{\mu_1\mu_2}\epsilon_{\mu_1\mu_2\ldots\mu_{s-1}} = 0)$
$\gamma^{\mu_1}\Psi_{\mu_1\ldots\mu_s} = 0$	$\gamma^{\mu_1}\epsilon_{\mu_1\ldots\mu_{s-1}} = 0$
$\partial^{\mu_1}\Psi_{\mu_1\ldots\mu_s} = 0$	$\partial^{\mu_1}\epsilon_{\mu_1\ldots\mu_{s-1}} = 0$
$\not{\partial}\Psi = 0$	$\not{\partial}\epsilon = 0$
$\delta^{(0)}\Psi_{\mu_1\ldots\mu_s} = s \cdot \partial_{(\mu_1}\epsilon_{\mu_2\ldots\mu_s)}$	

A little remark about the η-trace condition: Because $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}$, for any fermionic higher-spin field the condition $\gamma^{\mu_1}\Psi_{\mu_1\ldots\mu_s} = 0$ already implies the η-trace condition. This means that only the γ-trace condition has to be imposed in order to yield the result for both of them.

1.5 The Noether-Fronsdal Program

The Noether Fronsdal program is a perturbative approach to solving Lagrangians of different orders in the fields iteratively with respect to gauge invariance. The perturbative expansion of the Lagrangian for small parameters g and $g_{s_1 \dots s_N}$ is given as follows [1][3][4][6]

$$\mathcal{L} = \mathcal{L}_{(2)} + \sum_{N \geq 3} g^{N-2} \mathcal{L}_{(N)}$$
$$\mathcal{L}_{(N)} = \sum_{s_1, \dots, s_N} g_{s_1 \dots s_N} \mathcal{L}_{s_1 \dots s_N}$$

where $\mathcal{L}_{(2)}$ is the free Lagrangian which is quadratic in the fields and s_i is the spin of the $i-th$ field. Now let $\delta^{(i)}$ be the gauge variation of the fields with respect to the $i-th$ order in the fields, then a general gauge variation is given perturbatively through $\delta = \delta^{(0)} + g\delta^{(1)} + g^2\delta^{(2)} + \dots$
Then imposing $\delta\mathcal{L} = 0$ yields the following system of equations with respect to the corresponding orders in g:

$$\delta^{(0)} \mathcal{L}_{(2)} = 0 \tag{1.1}$$
$$\delta^{(1)} \mathcal{L}_{(2)} + \delta^{(0)} \mathcal{L}_{(3)} = 0 \tag{1.2}$$
$$\delta^{(2)} \mathcal{L}_{(2)} + \delta^{(1)} \mathcal{L}_{(3)} + \delta^{(0)} \mathcal{L}_{(4)} = 0 \tag{1.3}$$
$$\cdots \tag{1.4}$$

In this thesis this system will be approached with a very specific set of assumptions that need to be stated in the following. First of all, as previously derived it will be assumed that for massless fermionic as well as bosonic fields the free equations of motion are given. As a result this will yield on-shell solutions to the system of equations. Moreover, $\delta^{(0)}$ is thus assumed to be solved through gauge invariance of the free system combined with the fact that we constructed the fields to be traceless, which yields the condition that they also are transversal (so all the fields are "TT-fields"=traceless-transverse-fields [1][3][4]).

Another aspect that needs to be analysed is the equivalences that leave the action and as a result the equations of motion unchanged, given we assume TT-fields. With gauge invariance of the free system, one of them was already implemented. However, there are three more that occur in this setting [1]:

1) If one adds a total derivative to the Lagrangian, the corresponding action remains unchanged because the total derivative does not contribute. So even though the Lagrangian changes by a non-zero term, the resulting system does not change. This yields that a Lagrangian $\tilde{\mathcal{L}} = \mathcal{L} + \partial_\mu f^\mu$ is equivalent to $\mathcal{L}$ with respect to the equivalence class given through total derivatives [1].

2) It is possible to redefine the fields $\Phi^{(i)}$, $\Psi^{(i)}$ with appropriate differential operators Δ_i and $\tilde{\Delta}_i$ as follows

$$\tilde{\Phi}^{(i)} = \Phi^{(i)} + g\Delta_1\Phi^{(i)} + g^2\Delta_2\Phi^{(i)} + ...$$
$$\tilde{\Psi}^{(i)} = \Psi^{(i)} + g\tilde{\Delta}_1\Psi^{(i)} + g^2\tilde{\Delta}_2\Psi^{(i)} + ...$$

which yields a change of $\mathcal{L}_{(N)}$ only by terms that are proportional to the terms occurring in the free equations of motion. In this case this will yield the identification of operators $\eta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} \approx_F 0$ (as an example for the massless bosonic fields), where $\approx_F$ is to be understood as an equivalence relation [1][3].

3) Last but not least, the **Schouten identities** [1], which are a class of equivalences that become relevant for vertices with lower orders in three dimensions than in four dimensions. They are given through antisymmetrization followed up by contraction of a number of space-time indices larger than the dimension of the spacetime. In three dimensions they start to contribute

at lower orders in the fields and thus are very significant for finding the correct solutions to our system of equations given through the Noether-Fronsdal program.

A more precise description of 1) - 3) will be given in the next section, because it is necessary to define a little bit more precisely how the Lagrangians $\mathcal{L}_{(N)}$ are going to be constructed in order to define the correct equivalence relations. For now assume that $\approx$ is an equivalence relation that takes everything that was mentioned into account. Then clearly

$$\delta^{(0)}\mathcal{L}_{(2)} \approx 0 \tag{1.5}$$

$$\delta^{(1)}\mathcal{L}_{(2)} + \delta^{(0)}\mathcal{L}_{(3)} \approx 0 \tag{1.6}$$

$$\delta^{(2)}\mathcal{L}_{(2)} + \delta^{(1)}\mathcal{L}_{(3)} + \delta^{(0)}\mathcal{L}_{(4)} \approx 0 \tag{1.7}$$

$$\cdots \tag{1.8}$$

Because of imposing the free equations of motions with respect to $\approx$ we also know $\delta^{(1)}\mathcal{L}_{(2)} \approx 0$ and thus find that (1.6) reduces to finding a $\mathcal{L}_{(3)}$ that solves $\mathcal{L}_{(3)} \approx 0$.

The next step is to solve equation (1.7) by finding a solution to the equation $\delta^{(1)}\mathcal{L}_{(3)} + \delta^{(0)}\mathcal{L}_{(4)} \approx 0$ due to $\delta^{(2)}\mathcal{L}_{(2)} \approx 0$ already being given through imposing the equations of motion. In order to find the most general solution of (1.7), one first needs to find the homogeneous equation $\delta^{(0)}\mathcal{L}_{(4)} \approx 0$ whenever the equations of motion are imposed.

This procedure can iteratively be continued to any order, getting exceedingly more elaborate order by order. In this thesis the focus will be on solving $\delta^{(0)}\mathcal{L}_{(4)} \approx 0$ specifically under the assumption that the free equations of motion are imposed [1].

1.6 Properties of massless fermionic Fields in the Bulk Theory

The solutions of the fields far away from the asymptotic boundary of a given spacetime are called solutions of the bulk theory [1][7]. In this section I will specifically look at the quantization of the spin-1/2 as well as spin-3/2 fields in three-dimensional Minkowski spacetime and draw a conclusion after having discussed the latter. This will yield a useful insight to the behaviour of the fields in three spacetime dimensions that intends to complete the picture a little bit more.

1.6.1 Quantization of the Spin-1/2 Field

The case for spin-1/2 is particularly simple, because it is just the usual Dirac-field only in 3 dimensions. In classical field theory, the solutions to the Klein-Gordon equation are sums of plane waves [2]. This can be used to find the quantized solutions of the field equations by expanding them in a basis of plane-wave solutions combined with the corresponding quantized operators. Given the Lorentz-invariant measure $d\mu(p) = \frac{d^2p}{2E(\vec{p})}$ (see Appendix (A3)), one can make the following Ansatz for the Dirac-field [2]:

$$\Psi(x) = \int \frac{d\mu(p)}{(2\pi)^2} \left[a_p u(\vec{p}) e^{-ip\cdot x} + b_p^\dagger v(\vec{p}) e^{ip\cdot x} \right]$$

with

$$\{a_p, a_{p'}^\dagger\} = \{b_p, b_{p'}^\dagger\} = f(\vec{p})\delta^{(2)}(\vec{p} - \vec{p}')$$
$$\{a_p, a_{p'}\} = \{b_p, b_{p'}\} = \{a_p, b_{p'}\} = \{a_p, b_{p'}^\dagger\} = 0$$

where $f(\vec{p})$ has to be determined, and the conditions for the spinors are

$$\not{p} u(\vec{p}) = \not{p} v(\vec{p}) = 0 \ .$$

In 3 spacetime dimensions this equation needs to be treated very cautiously. Not only because of the massless set-up, but also because it therefore is a Weyl-equation. To demonstrate this, let us first look at u. Following the derivation in [2], it is appropriate to start off with assuming the frame $(p^\mu) = (p^0, p^1, 0)^T$ for which clearly $p^0 = p^1$. This immediately results in

$$u(\vec{p}) = \zeta \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

where ζ is going to be to be determined shortly. With this solution for u the general solution can be obtained by boosting along p^2, which is given through applying $\lambda = e^{\eta M_{02}}$:

$$u(\vec{p}) \to u'(\vec{p}') = \lambda u(\Lambda_I^{-1}\vec{p}) = \zeta \begin{pmatrix} \cosh(\eta/2) \\ i\sinh(\eta/2) \end{pmatrix}$$

where Λ_I^{-1} is the induced Lorentz-transformation on $\vec{p}$ and η can be determined by the transformation of $p^\mu \to p'^\mu = \Lambda^\mu_\nu p^\nu$, which is given by $\not{p}' = \lambda \not{p} \lambda^{-1}$. Because we only boost along p^2, in the new coordinates this results in $p'^1 = p^1$ which yields

$$\eta(\vec{p}) = \operatorname{arcsinh}\left(\frac{p_2}{p_1}\right)$$
$$\Rightarrow u(\vec{p}) = \frac{\zeta}{\sqrt{2}} \begin{pmatrix} \sqrt{1 + \frac{E(\vec{p})}{p_1}} \\ i\sqrt{\frac{E(\vec{p})}{p_1} - 1} \end{pmatrix}$$

as the solution.
After some more calculation one finds

$$u(\vec{p})\bar{u}(\vec{p}) = \frac{\zeta\zeta^*}{2p_1}\not{p}$$
$$\bar{u}(\vec{p})u(\vec{p}) = 0$$

from which $\zeta = \sqrt{2p_1}$ follows without loss of generality. So

$$u(\vec{p}) = \begin{pmatrix} \sqrt{E(\vec{p}) + p_1} \\ i\sqrt{E(\vec{p}) - p_1} \end{pmatrix} .$$

Clearly, because this solution has no more degrees of freedom except for a global phase-factor it also yields that $v(\vec{p}) = e^{i\varphi}u(\vec{p})$ (where $\varphi \in [0, 2\pi)$) which simplifies the field to

$$\boxed{\Psi(x) = \int \frac{d\mu(p)}{(2\pi)^2} \left[a_p e^{-ip\cdot x} + e^{i\varphi} b_p^\dagger e^{ip\cdot x}\right] u(\vec{p})} .$$

The fact that $u(\vec{p}) = e^{i\varphi}v(\vec{p})$ is actually very intuitive if we compare this case to the massive field. In the field equations for the massive field the minus sign generated when differentiating creates a difference in the equations for the two spinors, but when $m = 0$ this sign is just a global factor that simply cancels out. This yields the following if we take $f(\vec{p}) = (2\pi)^2 E(\vec{p})$

$$\begin{aligned} \{\Psi_1(x), \Psi_2^\dagger(y)\}\big|_{x^0=y^0} &= -i\int \frac{d^2p}{2} p_2 \left[e^{i\vec{p}\cdot(\vec{x}-\vec{y})} + e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\right] \\ &= -\{\Psi_2(x), \Psi_1^\dagger(y)\}\big|_{x^0=y^0} \\ \{\Psi_a(x), \Psi_a^\dagger(y)\}\big|_{x^0=y^0} &= \delta^{(2)}(\vec{x}-\vec{y}) + (-1)^{a+1}\int \frac{d^2p}{2} p_1 \left[e^{i\vec{p}\cdot(\vec{x}-\vec{y})} + e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\right] . \end{aligned}$$

The integrands in the anticommutators are odd functions in either p_1 or p_2, so they vanish and we end up with the desired result:

$$\begin{aligned} \{\Psi_a(x), \Psi_b^\dagger(y)\}\big|_{x^0=y^0} &= \delta_{ab}\delta^{(2)}(\vec{x}-\vec{y}) \\ \Rightarrow \{\Psi_a(x), \Pi_b(y)\}\big|_{x^0=y^0} &= i\delta_{ab}\delta^{(2)}(\vec{x}-\vec{y}) . \end{aligned}$$

1.6.2 Quantization of the Spin-3/2 Field

Because of the field equations $\partial\!\!\!/\Psi_\mu = 0$ it is immediately clear that the spin-3/2 field admits Dirac-spinor solutions component-wise. Thus we can again follow the solutions for the Dirac field in [2]. The additional conditions that need to be watched out for are given by

$$\begin{aligned} \gamma^\mu \Psi_\mu &= 0 \\ \partial^\mu \Psi_\mu &= 0 \\ \{\Psi_{\mu a}(x), \Pi_{\nu b}(y)\}\big|_{x^0=y^0} &= i\eta_{\mu\nu}\delta_{ab}\delta^{(2)}(\vec{x}-\vec{y}) \ . \end{aligned} \tag{1.9}$$

Each component will have the same form as the previously derived spin-1/2 field. The big difference is given through the one spacetime index for which it is necessary to analyse the quantization process in a little bit more detail. In that sense constructing the one-particle Hilbert-space from scratch will be necessary.

Since it is already specified how the field transforms
($\Psi^\mu(x) \to \Lambda^\mu_\nu \lambda(\Lambda)\Psi^\nu(\Lambda^{-1}x)$), the only objects inheriting the spacetime index can be $a_{\vec{p}} \to a^\mu_{\vec{p}}$ as well as $b_{\vec{p}} \to b^\mu_{\vec{p}}$, because the spinor already transforms with $\lambda(\Lambda)$ and has per construction (corresponding to the spin-1/2 solution) no degree of freedom left other than a global phase-factor. It can easily be seen that the only solution would be the trivial one if the spinor would be endowed with spacetime indices, because otherwise (1.9) would not be fulfilled. The instance regarding the raising/lowering operators isn't very trivial, so following up the Hilbert-space will be constructed more specifically.

Assume a vacuum state given by $|\Omega\rangle$, $\langle\Omega|\Omega\rangle = 1$, such that :

$$\begin{aligned}
a_p^\mu |\Omega\rangle &= b_p^\mu |\Omega\rangle = 0 \\
|p,\mu,+\rangle &:= \sqrt{2} b_p^{\mu\dagger} |\Omega\rangle \\
|p,\mu,-\rangle &:= \sqrt{2} a_p^{\mu\dagger} |\Omega\rangle \\
|p_1,\mu_1,-;p_2,\mu_2,+\rangle &:= 2 a_{p_1}^{\mu_1\dagger} b_{p_2}^{\mu_2\dagger} |\Omega\rangle = |p_1,\mu_1,-\rangle \wedge |p_2,\mu_2,+\rangle \\
&\cdots
\end{aligned} \tag{1.10}$$

So because of the anticommutators the annihilation and creation operators clearly yield a Hilbert-/Fock-space with respect to the structure of an exterior algebra. Take the momentum states $|p,\mu,\pm\rangle$ to be normalised as follows:

$$\langle p,\mu,\pm|q,\nu,\pm\rangle = 2E(\vec{p})(2\pi)^2 \eta^{\mu\nu} \delta^{(2)}(\vec{p}-\vec{q})$$

(See appendix (A2) for Lorentz-invariance of the delta distribution). The generating states for Ψ^μ should then be constructed such that componentwise for the spinor

$$\langle p,\mu,+|\Psi,\nu,+\rangle = \eta^{\mu\nu}\Psi_+(p)$$

where $\Psi^\mu(p) = \Psi_-(p)a_p^\mu + \Psi_+(p)b_p^{\mu\dagger}$. This can be achieved by the following

$$|\Psi,\mu,+\rangle := \int \frac{d\mu(p)}{(2\pi)^2}\Psi^\mu(p)\,|\Omega\rangle = \int \frac{d\mu(p)}{(2\pi)^2}\Psi_+(p)\,|p,\mu,+\rangle$$

which results in

$$|\Psi,\nu,+\rangle = \sum_\mu \int \frac{d\mu(p)}{(2\pi)^2} \langle p,\mu,+|\Psi,\nu,+\rangle\,|p,\mu,+\rangle$$

and thus yields the completeness relation for the Hilbert-space generated through $b_p^{\dagger\mu}$ given through

$$\mathbb{1}_{\mathcal{H}_+} = \sum_{\mu} \int \frac{d\mu(p)}{(2\pi)^2} \left|p, \mu, +\right\rangle \left\langle p, \mu, +\right| \ .$$

The space $\mathcal{H}_+$ is the one-particle Hilbert-space with respect to the generators $b_p^{\dagger\mu}$. If $\mathcal{H}_-$ is defined as the space generated by the $a_p^{\mu\dagger}$, then we find that the multi-particle spaces can be constructed via the appropriate exterior products of those two spaces.

If one now analyses the behaviour of $\Psi^\mu(x) \rightarrow \Lambda^\mu_\nu \lambda(\Lambda)\Psi^\nu(\Lambda^{-1}x)$ under Lorentz-transformations the following holds:

$$\begin{aligned}
\Lambda^\mu_\nu \lambda(\Lambda)\Psi^\nu(\Lambda^{-1}x) = & \\
&= \Lambda^\mu{}_\nu \lambda(\Lambda) \int \frac{d\mu(p)}{(2\pi)^2} \left[a_p^\nu e^{-ip\cdot(\Lambda^{-1}x)} + e^{i\varphi} b_p^{\nu\dagger} e^{ip\cdot(\Lambda^{-1}x)} \right] u(\vec{p}) \\
&= \Lambda^\mu{}_\nu \lambda(\Lambda) \int \frac{d\mu(p)}{(2\pi)^2} \left[a^\nu_{\Lambda^{-1}p} e^{-ip\cdot x} + e^{i\varphi} b^{\nu\dagger}_{\Lambda^{-1}p} e^{ip\cdot x} \right] u(\Lambda_I^{-1}\vec{p})
\end{aligned}$$

$$\Rightarrow \Psi^\mu(p) \rightarrow \left(\lambda(\Lambda)\Psi_-(\Lambda^{-1}p)\right) \left(\Lambda^\mu{}_\nu a^{\nu\dagger}_{\Lambda^{-1}p}\right) + \left(\lambda(\Lambda)\Psi_+(\Lambda^{-1}p)\right) \left(\Lambda^\mu{}_\nu b^{\nu\dagger}_{\Lambda^{-1}p}\right) \ .$$

And because of that $\langle p, \mu, + | \Psi, \nu, + \rangle \rightarrow \langle q, \mu, + | \left(\sum_\alpha \Lambda^\nu{}_\alpha \lambda(\Lambda)\right) | \Psi, \alpha, + \rangle$ such that

$$\boxed{\left|p, \mu, +\right\rangle \xrightarrow{\Lambda} \sum_\nu \Lambda^\mu{}_\nu \left|\Lambda^{-1}p, \nu, +\right\rangle} \ .$$

With exactly the same procedure one could do the calculation for the $a_p^{\mu\dagger}$-sector and end up with the corresponding result. Thus with this we found a set of Hilbert-spaces and the corresponding transformation rules with respect to Lorentz-transformations in order to describe the spin-3/2 field with quantized operators that carry a spacetime index. This yields a complete description of the representation. In order to make it irreducible (1.9) has to be imposed. It basically translates to

$$\begin{pmatrix} -ia_p^2 u_1(\vec{p}) + (a_p^0 + a_p^1)u_2(\vec{p}) \\ (a_p^0 - a_p^1)u_1(\vec{p}) + ia_p^2 u_2(\vec{p}) \end{pmatrix} = 0$$

$$\begin{pmatrix} -ib_p^{2\dagger} u_1(\vec{p}) + (b_p^{0\dagger} + b_p^{1\dagger})u_2(\vec{p}) \\ (b_p^{0\dagger} - b_p^{1\dagger})u_1(\vec{p}) + ib_p^{2\dagger} u_2(\vec{p}) \end{pmatrix} = 0 \ .$$

Now let us apply this to the vacuum states $\langle\Omega|$ and $|\Omega\rangle$ to understand how to solve these equations efficiently

$$\begin{pmatrix} \langle p,2,-| \, (-iu_1(\vec{p})) + (\langle p,0,-| + \langle p,1,-|)u_2(\vec{p}) \\ (\langle p,0,-| - \langle p,1,-|)u_1(\vec{p}) + \langle p,2,-| \, iu_2(\vec{p}) \end{pmatrix} = 0$$

$$\begin{pmatrix} -iu_1(\vec{p}) \, |p,2,+\rangle + u_2(\vec{p})(|p,0,+\rangle + |p,1,+\rangle) \\ u_1(\vec{p})(|p,0,+\rangle - |p,1,+\rangle) + iu_2(\vec{p}) \, |p,2,+\rangle \end{pmatrix} = 0 \ .$$

This is clearly problematic. By the completeness of $\mathcal{H}_+$ and $\mathcal{H}_-$ it yields that because of the orthogonality of the states $|p,\mu,\pm\rangle$

$$|p,0,\pm\rangle = |p,1,\pm\rangle = |p,2,\pm\rangle = 0$$

which as a result is a contradiction to the construction of $\mathcal{H}_+$ as well as $\mathcal{H}_-$ and thus it is only possible that

$$\begin{aligned} \mathcal{H}_+ &= \mathcal{H}_- = \emptyset \\ a_p^\mu &= b_p^\mu = 0 \ \forall\mu \end{aligned}$$

$$\Rightarrow \boxed{\Psi^\mu = 0 \ \forall\mu} \ .$$

$\Psi^\mu = 0$ means that the spin-3/2 field does not have any propagating degrees of freedom in the bulk theory. Looking at the process of the calculation one might already see that the fact that fields with fermionic spin higher than 3/2 are even more restricted due to having more spacetime indices hints at them also not having propagating degrees of freedom. In fact, according to [1] this is the case for any field with spin higher than 2 for bosons. With respect to that we find that any higher spin field with $s \geq 3/2$ has no propagating degrees of freedom in the bulk theory.

Constraints for quartic Vertices

2.1 Fierz System for Bosons and Fermions

If we introduce auxiliary vector variables $\{a^{\mu_i}\}$, it is possible to conveniently rewrite the fields as [1]

$$\Phi^{(s)}(x,a) = \frac{1}{s!}\Phi_{\mu_1,\ldots,\mu_s}(x)a^{\mu_1}\cdots a^{\mu_s}$$
$$\Psi^{(s)}(x,a) = \frac{1}{s!}\Psi_{\mu_1,\ldots,\mu_s}(x)a^{\mu_1}\cdots a^{\mu_s} \ .$$

In writing the fields that way, we gain a new way to represent the free equations of motion as well as the conditions for bosonic fields and the equations we derived from gauge invariance at 0-th order.

First of all we define the operators [1]:

$$A_\mu := \frac{\partial}{\partial a^\mu}$$

$$P_\mu := \frac{\partial}{\partial x^\mu}$$

as well as

$$B \cdot D := B_\mu D^\mu \ .$$

This yields the Fierz-system for bosons [1]:

$$
\begin{aligned}
P \cdot P\Phi^{(s)} &= 0 \\
A \cdot A\Phi^{(s)} &= 0 \\
A \cdot P\Phi^{(s)} &= 0 \ ,
\end{aligned}
$$

as well as for fermions:

$$
\begin{aligned}
P \cdot P\Psi^{(s)} &= 0 \\
A \cdot A\Psi^{(s)} &= 0 \\
A \cdot P\Psi^{(s)} &= 0 \\
\gamma \cdot P\Psi^{(s)} &= 0 \\
\gamma \cdot A\Psi^{(s)} &= 0 \ .
\end{aligned}
$$

Throughout this thesis this will be a powerful tool to simplify the computation of the different cases of vertices. For further simplicity the fields will be denoted as $\Phi^{(s_i)}(x_i, a_i) = \Phi_i(x_i, a_i)$ from now on.

The first part of the analysis of the vertices will focus on higher-spin fields and their gauge fields exclusively (3.2 - 3.5). Afterwards the case differentiations for Spin-0, Spin-1/2 and Spin-1 fields will be calculated in 3.7.

2.2 Vertices with four HS Bosons (even parity)

The simplest possible quartic vertex consists of four bosonic fields. Strictly bosonic vertices in particular are much simpler than fermionic vertices because contractions of spinor-indices make the calculations a little bit more complicated as more case differentiations have to be accounted for. A vertex with four bosons and $s_4 \geq s_3 \geq s_2 \geq s_1 \geq 1$ can generally be written as follows [1]

$$\mathcal{L}_{s_1,s_2,s_3,s_4} = \mathcal{V}_{s_1,s_2,s_3,s_4}\left(\prod_{i=1}^{4}\Phi_i(x_i,a_i)\right)\Big|_{\substack{x_i=x\\ a_i=0}}$$

where the vertex operator $\mathcal{V}_{s_1,s_2,s_3,s_4}$ is a polynomial in $A_i \cdot A_j$, $P_i \cdot P_j$ and $A_i \cdot P_j$. All of those operators commute with each other. Also, each of them contracts two Lorentz-indices. For convenience, let us define matrices of the operators by [1]

$$\begin{aligned} s_{ij} &= P_i \cdot P_j \\ y_{ij} &= A_i \cdot P_j \\ z_{ij} &= A_i \cdot A_j \ . \end{aligned}$$

Because of the Fierz system we can set

$$\begin{aligned} s_{ii} &\approx_F 0 \\ y_{ii} &\approx_F 0 \\ z_{ii} &\approx_F 0 \end{aligned}$$

because neither of them contributes to the vertex operator. We define $\approx_F$ as an equivalence relation with respect to the Fierz system. Furthermore, because $s_{ij} = s_{ji}$ and $z_{ij} = z_{ji}$, we find that

$$\mathcal{V}_{s_1,s_2,s_3,s_4} \in \mathbb{R}\left[s_{ij}|_{i<j}, z_{ij}|_{i<j}, y_{ij}|_{i\neq j}\right] \ .$$

The conditions given by the Fierz system are called the traceless and transverse (TT) conditions of the fields [1]. They are are the initial conditions we impose on the system. However, looking at the symmetries of the Lagrangian we can find another condition that has to be imposed on the vertex. A Lagrangian of the form $\tilde{\mathcal{L}} = \mathcal{L} + \partial_\mu f^\mu$ does not change the theory. In that sense we call the two Lagrangians equivalent $\tilde{\mathcal{L}} \approx \mathcal{L}$. This means that we can define an equivalence relation such that $\partial_\mu f^\mu \approx_D 0$. Imposing this relation on our operators of our Fierz system we find

$$\sum_{j=1}^{4} s_{ij} \approx_D 0$$
$$\sum_{j=1}^{4} y_{ij} \approx_D 0 \ .$$

These two equivalence relations are the generators of an ideal we henceforth denote by $\mathcal{I}_D$ [1].

If we look at the expression $A_{[\mu}B_\nu C_\delta D_{\sigma]}E^\mu F^\nu G^\delta H^\sigma$ in 3 spacetime dimensions we quickly find that it has to be zero due to over-antisymmetrization. This relation produces the Schouten-identities, which form another class of equivalence relations that need to be accounted for in the construction of the vertex operator. These identities generate an ideal $\mathcal{I}_S$ [1]. With both the ideals $\mathcal{I}_S$ and $\mathcal{I}_D$ to be divided out of the polynomial ring, the vertex is an element of [1]:

$$\boxed{\mathcal{V}_{s_1,s_2,s_3,s_4} \in \frac{\mathbb{R}\left[s_{ij}|_{i<j}, z_{ij}|_{i<j}, y_{ij}|_{i\neq j}\right]}{\mathcal{I}_S + \mathcal{I}_D}} \ .$$

To be more precise, the vertex is a representative of a whole equivalence class of vertices $[\mathcal{V}_{s_1,s_2,s_3,s_4}]$ that lead to the same theory.

The Schouten identities can be calculated as follows.
We first define $\vec{b}^T := (P_1, P_2, P_3, P_4, A_1, A_2, A_3, A_4)$ and then construct the matrix

$$\mathcal{B} = \vec{b} \cdot \vec{b}^T = \begin{bmatrix} \mathcal{S} & \mathcal{Y}^T \\ \mathcal{Y} & \mathcal{Z} \end{bmatrix} \ .$$

$\mathcal{B}$ is a 8x8-matrix. If we take the determinants of all 4x4-minors and set them equal to zero, we find the complete set of Schouten identities for the used operators [1].

Because using the Schouten identities to calculate the vertex operator is very complicated, in order to simplify the computation of it we introduce an appropriate product of Mandelstan variables s_{ij} with $\Delta(s_{ij})$ [1] and apply it on $\mathcal{V}_{s_1,s_2,s_3,s_4}$. This will simplify the process of finding a proper representative a lot.

The vertex operator can be expanded with respect to its degree in the P_i as follows

$$\mathcal{V}_{s_1,s_2,s_3,s_4}(P_i, A_i) = \sum_{j=0}^{M} \left(\mathcal{V}_{s_1,s_2,s_3,s_4}(P_i, A_i)\Big|_{\deg_P(\mathcal{V}_{s_1,s_2,s_3,s_4})=j} \right) \ .$$

If we define $\tilde{\mathcal{L}}_{s_1,s_2,s_3,s_4}$ to be the vertex operator $\mathcal{V}_{s_1,s_2,s_3,s_4}(P_i, A_i)$ applied to the fields without setting $x_i = x$ as well as $a_i = 0$ for $i = 1, 2, 3, 4$, the Fourier transformation (with an appropriate choice of fields as well as space of integration) of $\Delta(s_{ij})\tilde{\mathcal{L}}_{s_1,s_2,s_3,s_4}(P_i, A_i)$ can be written as follows:

$$\begin{aligned}\mathcal{F}\left[\Delta(s_{ij})\tilde{\mathcal{L}}_{s_1,s_2,s_3,s_4}\right](k_l) =& \Delta(-k_i \cdot k_j) \\ & \times \sum_{t=0}^{M} \left((i)^t \mathcal{V}_{s_1,s_2,s_3,s_4}(k_j, A_j)\Big|_{\deg_k(\mathcal{V}_{s_1,s_2,s_3,s_4})=t} \right) \\ & \times \mathcal{F}\left[\prod_{r=1}^{4} \Phi_r(x_r, a_r)\right](k_l) \ .\end{aligned}$$

Now assume $\Delta(s_{ij})\mathcal{V}_{s_1,s_2,s_3,s_4}(P_i, A_i) \approx 0$, which means that it is equivalent to a trivial vertex operator [1]. In Fourier space $\Delta(-k_i \cdot k_j)$ is not equal to zero on the subvariety $k_i \cdot k_i = 0$ and $\sum_{i=1}^{N} k_i = 0$ almost everywhere [1]. However, since $\Delta(s_{ij})\mathcal{V}_{s_1,s_2,s_3,s_4}(P_i, A_i) \approx 0$ applied to the fields vanishes, this yields that $\sum_{l=0}^{M} \left((i)^l \mathcal{V}_{s_1,s_2,s_3,s_4}(k_j, A_j)\Big|_{\deg_k(\mathcal{V}_{s_1,s_2,s_3,s_4})=l} \right)$ vanishes in Fourier space almost everywhere [1]. Because trivially $\mathcal{V}_{s_1,s_2,s_3,s_4} \approx 0$ implies $\Delta\mathcal{V}_{s_1,s_2,s_3,s_4} \approx$

0, we conclude [1]:

$$\boxed{\Delta\mathcal{V}_{s_1,s_2,s_3,s_4} \approx 0 \Leftrightarrow \mathcal{V}_{s_1,s_2,s_3,s_4} \approx 0}\,. \tag{2.1}$$

This relation will become especially useful when imposing gauge invariance on the vertex operator.

Following [1] there is another way to simplify $\Delta\mathcal{V}_{s_1,s_2,s_3,s_4}$ according to three steps that can be done:

(1) Take a 4x4-submatrix of $\mathcal{B}$ consisting of its first three rows as well as columns and also take the $(4+i)$-th row and $(4+j)$-th column such that $i \neq j$. The Schouten identity that results from that 4x4 submatrix M is given by $\det(M) = 0$ and it yields that we can replace z_{ij} by both y_{mn} and s_{mn}. Doing this for all possible cases where $i \neq j$ shows that $\Delta\mathcal{V}_{s_1,s_2,s_3,s_4}$ is completely independent of the z_{ij}.

(2) Another set of variables can be substituted by others by taking the columns i (mod 4), $i+1$ (mod 4), $i+2$ (mod 4) as well as $4+i$ such that $4+i$ contains y_{ii}, y_{ii+1} and y_{ii+2} as well as for example y_{ij}. The corresponding Schouten identity can now be used to replace y_{ij} by y_{ii+1}, y_{ii+2} and a set of Mandelstan variables.

(3) Finally, we make a change of variables and replace y_{ii+2} with a linear combination of $Y_i := s_{ii+2}y_{ii+1} - s_{ii+1}y_{ii+2}$ as well as y_{ii+1}. Because of the Schouten identities we have $Y_i^2 \approx_S 0$.

Applying those steps to $\Delta\mathcal{V}_{s_1,s_2,s_3,s_4}$ leads to the conclusion that for any vertex operator $\mathcal{V}_{s_1,s_2,s_3,s_4}$ there exists a polynomial $Q_{\mathcal{V}_{s_1,s_2,s_3,s_4}}(y_{ii+1}, Y_i, s_{ij})$ which is at most linear in each of the Y_i such that $\Delta\mathcal{V}_{s_1,s_2,s_3,s_4} = Q_{\mathcal{V}_{s_1,s_2,s_3,s_4}}(y_{ii+1}, Y_i, s_{ij})$

[1]. Define $\mathcal{I}_R$ to be the remaining set of equivalence classes [1]. Then

$$\boxed{\left[Q_{\mathcal{V}_{s_1,s_2,s_3,s_4}}(y_{ii+1}, Y_i, s_{ij})\right] \in \frac{\mathbb{R}[y_{ii+1}, Y_i, s_{ij}]}{\mathcal{I}_R + \langle Y_i^2 \rangle}} \ . \tag{2.2}$$

2.2.1 Constraints from Gauge Invariance

Another important class of constraints stems from the gauge invariance imposed on the Lagrangian via the Noether-Fronsdal program [1][6]. If we demand $\delta^{(0)}\mathcal{L}_4 = 0$, new conditions emerge for the vertex operators $\mathcal{V}$. In order to find them for the strictly bosonic vertex, first of all we have to translate the gauge field of the bosonic field into the system of auxiliary vector variables. Let $\xi_j := \xi^{(s_j-1)}$ be the bosonic gauge field for Φ_j. Then

$$\delta_l^{(0)}\Phi_j = \delta_{lj} s_j (a_j \cdot P_j)\xi_j$$

if $\delta_k^{(0)}$ acts on the k-th bosonic field. Without loss of generality choose $\delta_1^{(0)}$. This yields

$$\delta_1^{(0)}\mathcal{L}_{s_1,s_2,s_3,s_4} = s_1 \mathcal{V}_{s_1,s_2,s_3,s_4}(a_1 \cdot P_1)\xi_1(x_1, a_1) \left(\prod_{i=2}^{4} \Phi_i(x_i, a_i) \right) \Bigg|_{\substack{x_i = x \\ a_i = 0}} \ .$$

Since each of the gauge transformations for the fields are independent of one another, this results in [1]

$$\boxed{[\mathcal{V}_{s_1,s_2,s_3,s_4}, a_i \cdot P_i] \approx 0}$$

for $i = 1, 2, 3, 4$.

Because of $[\Delta(s_{ij}), a_i \cdot P_i] \approx 0$ we find

$$\begin{aligned} &[\Delta\mathcal{V}_{s_1,s_2,s_3,s_4}, a_i \cdot P_i] \approx 0 \\ \Rightarrow &\left[Q_{\mathcal{V}_{s_1,s_2,s_3,s_4}}, a_i \cdot P_i\right] \approx 0 \ . \end{aligned} \tag{2.3}$$

In order to compute the commutator in (2.3) we first need to look at the commutators of y_{ii+1} and Y_i with $a_k \cdot P_k$:

$$[y_{ii+1}, a_k \cdot P_k] = \delta_{ik} s_{kk+1}$$

and thus

$$[Y_i, a_k \cdot P_k] = s_{ii+2}\delta_{ik}s_{kk+1} - s_{ii+1}\delta_{ik}s_{kk+2} = 0 \ .$$

This result means that Y_i has to be treated as an independent variable[1]. Now assume we expand $Q_{\mathcal{V}_{s_1,s_2,s_3,s_4}}$ in y_{kk+1} with coefficient operators $a_n(y_{ii+1}|_{i\neq k}, Y_i, s_{ij})$

$$Q_{\mathcal{V}_{s_1,s_2,s_3,s_4}} = \sum_{n=0}^{M} a_n(y_{ii+1}|_{i\neq k}, Y_i, s_{ij}) y^n_{kk+1} \ .$$

Plugging this expansion into (2.3) immediately yields:

$$\begin{aligned}
\left[Q_{\mathcal{V}_{s_1,s_2,s_3,s_4}}, a_k \cdot P_k\right] =& \sum_{n=0}^{M} a_n(y_{ii+1}|_{i\neq k}, Y_i, s_{ij}) \left[y^n_{kk+1}, a_k \cdot P_k\right] \\
=& s_{kk+1} \sum_{n=1}^{M} n a_n(y_{ii+1}|_{i\neq k}, Y_i, s_{ij}) y^{n-1}_{kk+1} \\
=& s_{kk+1} \frac{\partial Q_{\mathcal{V}_{s_1,s_2,s_3,s_4}}}{\partial y_{kk+1}} \\
\approx& 0 \ .
\end{aligned}$$

This is true for all k which leads us to the conclusion that we can choose $Q_{\mathcal{V}_{s_1,s_2,s_3,s_4}}$ to be completely independent of all y_{kk+1} [1].

$$\Rightarrow \boxed{\Delta\mathcal{V}_{s_1,s_2,s_3,s_4} = Q_{\mathcal{V}_{s_1,s_2,s_3,s_4}}(Y_i, s_{ij})} \ . \tag{2.4}$$

2.2.2 Summary of the Constraints

In (2.4) we found a very constrained polynomial $Q_{\mathcal{V}_{s_1,s_2,s_3,s_4}}(Y_i, s_{ij})$ that depends on s_{ij} and linearly on each Y_i (terms with Y_iY_j for $i \neq j$ and higher orders of that type are still possible). Since s_{ij} does not depend on any A_i, this yields that $\Delta\mathcal{V}_{s_1,s_2,s_3,s_4}$ does not include terms with $A_i \cdot A_j$ or $A_i^\mu A_i^\nu$. Because of (2.1), $\mathcal{V}_{s_1,s_2,s_3,s_4}$ inherits those characteristics of $Q_{\mathcal{V}_{s_1,s_2,s_3,s_4}}$ and **we can come to the conclusion that the vertex depends only on fields with spin less or equal to one** [1].

Because the ideal $\mathcal{I}_R$ is gauge invariant, its representatives can also be chosen to not depend on the y_{kk+1}. However, if we want to calculate an explicit expression for $\mathcal{V}_{s_1,s_2,s_3,s_4}$ up to some degree, we still need to evaluate all remaining equivalences in $\mathcal{I}_R$ and impose them on the polynomial expansion of the vertex operator.

2.3 Vertices with two HS Bosons and two HS Fermions

Introducing fermions to the big picture makes the circumstances a lot more complicated. Fermions always come in pairs due to the necessity to covariantly contract the spinor-indices of the fermionic fields [2]. This means the simplest quartic fermionic vertex consists of a pair of fermions as well as bosons. Generally, a vertex of this type will be constructed by the following two cases: Either the Spinor indices are contracted by δ^{ab} -which will be called Type I- or the γ-matrices $(\gamma^\mu)^{ab}$ -which will be called Type II-. Fortunately, beyond that it is possible to compute the vertex constraints very analogously to the bosonic case when it comes to differentiating the parities of the vertices. The method for parity even purely bosonic vertices results in a parity odd type I and a parity even type II vertex. Later, when applying the method of extending to parity odd vertices for bosons, it will result in a parity even type I as well as parity odd type II vertex, which will yield all possible cases.

$$\begin{aligned}&\mathcal{L}^{TypeI}_{s_1,s_2,s_3+\frac{1}{2},s_4+\frac{1}{2}} = \\ &\quad \mathcal{V}^{TypeI}_{s_1,s_2,s_3+\frac{1}{2},s_4+\frac{1}{2}}\left(\Phi_1(x_1,a_1)\Phi_2(x_2,a_2)\bar{\Psi}_3(x_3,a_3)\Psi_4(x_4,a_4)\right)\Bigg|_{\substack{x_i=x\\a_i=0}} + c.c.\end{aligned}$$

$$\begin{aligned}&\mathcal{L}^{TypeII}_{s_1,s_2,s_3+\frac{1}{2},s_4+\frac{1}{2}} = \\ &\quad c\mathcal{V}^{TypeII}_{s_1,s_2,s_3+\frac{1}{2},s_4+\frac{1}{2}}\left(\Phi_1(x_1,a_1)\Phi_2(x_2,a_2)\bar{\Psi}_3(x_3,a_3)\gamma(a_5)\Psi_4(x_4,a_4)\right)\Bigg|_{\substack{x_i=x\\a_i=0}} + c.c.\end{aligned}$$

where $c \in \mathbb{C}$ is an appropriate complex constant to ensure that the Lagrangian is real. The reason for the γ-matrices only appearing linearly in the vertices is given by $\gamma^\mu\gamma^\nu = \eta^{\mu\nu}\mathbb{1} + i\epsilon^{\mu\nu\rho}\gamma_\rho$, which yields that every product of γ-matrices can always be reduced to a linear combination of the identity and the γ-matrices themselves. For example, if we choose to put a quadratic expression with γ-matrices in-between a pair of fermionic fields, the vertex can also be described by a linear combination of a type I and a type II vertex.

Here, we introduce $\gamma(a_5) := \gamma_\mu a_5^\mu$. Contracting the γ-matrices with yet another auxiliary vector variable is very convenient for computing the Schouten identities later on and a result of trial and error attempts of finding them in other ways.

2.3.1 Type I Vertices

In case of a contraction with δ^{ab} the vertex operator simply corresponds to the vertex operator for the quartic vertex with four bosons. We find that the vertex class can easily be constructed by a representative of the vertex class $[\mathcal{V}_{s_1,s_2,s_3,s_4}]$ in the following way

$$[\mathcal{V}^{TypeI}_{s_1,s_2,s_3+\frac{1}{2},s_4+\frac{1}{2}}] = [\mathcal{V}_{s_1,s_2,s_3,s_4}]^* \ .$$

The definition of $[\cdots] \to [\cdots]^*$ is given in appendix (A3).
In the same way we found $Q_{\mathcal{V}_{s_1,s_2,s_3,s_4}}$ for $\mathcal{V}_{s_1,s_2,s_3,s_4}$ we can now find $Q^{TypeI}_{\mathcal{V}_{s_1,s_2,s_3+\frac{1}{1},s_4+\frac{1}{2}}}$ for $\mathcal{V}_{s_1,s_2,s_3+\frac{1}{1},s_4+\frac{1}{2}}$, which results in [1]

$$\Delta\mathcal{V}_{s_1,s_2,s_3+\frac{1}{1},s_4+\frac{1}{2}} = Q^{TypeI}_{\mathcal{V}_{s_1,s_2,s_3+\frac{1}{1},s_4+\frac{1}{2}}} = Q_{\mathcal{V}_{s_1,s_2,s_3,s_4}} \ . \tag{2.5}$$

Furthermore, following the same three steps that lead to (2.2), we find that

$$\boxed{\left[Q^{TypeI}_{\mathcal{V}_{s_1,s_2,s_3+\frac{1}{1},s_4+\frac{1}{2}}}(y_{ii+1}, Y_i, s_{ij})\right] \in \frac{\mathbb{R}[y_{ii+1}, Y_i, s_{ij}]}{\mathcal{I}_R + \langle Y_i^2\rangle}}$$

as well as obviously (2.1) is also true for $\mathcal{V}^{TypeI}_{s_1,s_2,s_3+\frac{1}{1},s_4+\frac{1}{2}}$. Combined with the constraints from gauge invariance this will eventually lead to a very similar result like in 2.2.2 [1].

2.3.2 Type II Vertices

The type II vertex is defined by a contraction of the spinor indices via the γ-matrices. Since the bosonic fields do not obey any equation with the operators $\gamma \cdot P_i$ and $\gamma \cdot A_i$ $(i = 1, 2)$, they can be used on the bosonic part of the Lagrangian in order to contract the spinor indices. Translated for the introduction of $\gamma(a_5)$, the operators $\gamma \cdot P_i$ and $\gamma \cdot A_i$ become

$$\begin{aligned}
\gamma \cdot P_i(\Phi_i(x_i, a_i)) &= A_5 \cdot P_i(\gamma(a_5)\Phi_i(x_i, a_i)) \\
\gamma \cdot A_i(\Phi_i(x_i, a_i)) &= A_5 \cdot A_i(\gamma(a_5)\Phi_i(x_i, a_i)) \ .
\end{aligned}$$

Because of the Fierz system and the invariance of the theory under total derivatives we find

$$\begin{aligned}
\gamma \cdot (P_1 + P_2 + P_3 + P_4) &\approx_D 0 \\
\text{and } \gamma \cdot (P_3 + P_4) &\approx_F 0 \\
\Rightarrow \gamma \cdot P_1 &\approx_D -\gamma \cdot P_2
\end{aligned}$$

which yields

$$y'_{51} := A_5 \cdot P_1 \approx_D -A_5 \cdot P_2 =: -y'_{52} \ .$$

Furthermore, we also have

$$\begin{aligned}
z'_{53} &:= A_5 \cdot A_3 \approx_F 0 \\
z'_{54} &:= A_5 \cdot A_4 \approx_F 0 \\
y'_{53} &:= A_5 \cdot P_3 \approx_F 0 \\
y'_{54} &:= A_5 \cdot P_4 \approx_F 0 \ .
\end{aligned}$$

Because $\gamma(a_5)$ only depends linearly on a_5, we can define $A_5 \cdot A_5 = 0$. This yields the definition of $\vec{b}'^T := (P_1, P_2, P_3, P_4, A_1, A_2, A_3, A_4, A_5)$ [1], where

$$\mathcal{B}' = \vec{b}' \cdot \vec{b}'^T = \begin{bmatrix} \mathcal{S}' & \mathcal{Y}'^T \\ \mathcal{Y}' & \mathcal{Z}' \end{bmatrix}$$

which is a 9x9-matrix that will be used to compute the Schouten identities. The principle is the same as before: take all 4x4-minors of $\mathcal{B}'$ and set their determinants equal to zero in order to retrieve the complete set of Schouten identities for the system.

Because $\gamma(a_5)$ is only linear in a_5, the vertex operator only contributes to the vertex if it is linear in y'_{5i} and z'_{5i} for $i = 1, 2$. Thus there exists an expansion in those variables given by

$$\mathcal{V}^{TypeII}_{s_1,s_2,s_3+\frac{1}{2},4+\frac{1}{2}} \approx \sum_{i=1}^{2} \left(\mathcal{V}^{TypeII}_{s_1,s_2,s_3+\frac{1}{2},s_4+\frac{1}{2}}\bigg|_{\mathcal{O}(y'_{5i})} + \mathcal{V}^{TypeII}_{s_1,s_2,s_3+\frac{1}{2},s_4+\frac{1}{2}}\bigg|_{\mathcal{O}(z'_{5i})} \right) .$$

When looking a little bit more closely at the Schouten identities one finds the following: Take the second to fifth row and the second to fourth column. Then, taking the 9^{th} column we created a matrix M whose determinant only depends on $y_{52} \approx_D -y_{51}$, z_{51} and the Mandelstan variables in the following way:

$$\det(M) = y'_{51} q_1(y'_{1i}, s'_{ij}) + z'_{51} q_2(s'_{ij}) = 0$$

where the q_i are appropriate polynomials in the corresponding Mandelstan variables. Likewise, instead of the fifth row take the sixth row to find:

$$\det(M) = y'_{51} q_3(y'_{2i}, s'_{ij}) + z'_{52} q_4(s'_{ij}) = 0 .$$

Using this one gets

$$
\begin{aligned}
q_2(s_{mn}) \left(\mathcal{V}^{TypeII}_{s_1,s_2,s_3+\frac{1}{2},s_4+\frac{1}{2}} \Big|_{\mathcal{O}(z'_{51})} \right) &\approx_S -q_1(s_{mn}) \left(\mathcal{V}^{TypeII}_{s_1,s_2,s_3+\frac{1}{2},s_4+\frac{1}{2}} \Big|_{\mathcal{O}(z'_{51})} \right) \Big|_{z'_{51}=1} y'_{51} \\
q_4(s_{mn}) \left(\mathcal{V}^{TypeII}_{s_1,s_2,s_3+\frac{1}{2},s_4+\frac{1}{2}} \Big|_{\mathcal{O}(z'_{52})} \right) &\approx_S -q_3(s_{mn}) \left(\mathcal{V}^{TypeII}_{s_1,s_2,s_3+\frac{1}{2},s_4+\frac{1}{2}} \Big|_{\mathcal{O}(z'_{52})} \right) \Big|_{z'_{52}=1} y'_{51} \ .
\end{aligned}
$$

This yields that the vertex operator $\mathcal{V}^{TypeII}_{s_1,s_2,s_3+\frac{1}{2},s_4+\frac{1}{2}}$ can be chosen to be completely independent of z'_{51} and z'_{52}.

Now take a 4x4-submatrix of $\mathcal{B}'$ consisting of its first three rows as well as columns and also take the $(4+i)$-th row and $(4+j)$-th column such that $i \neq j$ and $4+i \neq 9$ as well as $4+j \neq 9$. The Schouten identity that results from that 4x4 submatrix M yields that we can replace z'_{ij} $(i \neq 5)$ by both y'_{mn} and s'_{mn}. Doing this for all possible cases where $i \neq j$ shows that $\Delta\mathcal{V}_{s_1,s_2,s_3,s_4}$ is completely independent of the z'_{ij}.

With this the vertex operator can be chosen to be completely independent of all z'_{ij}. Because it also has to be linear in y'_{51} we conclude that the remaining vertex can be represented by a Type I vertex, which yields

$$
\boxed{\mathcal{V}^{TypeII}_{s_1,s_2,s_3+\frac{1}{2},s_4+\frac{1}{2}} \approx y'_{51} \mathcal{V}^{TypeI}_{s_1,s_2,s_3+\frac{1}{2},s_4+\frac{1}{2}}} \ . \tag{2.6}
$$

2.3.3 Constraints from Gauge Invariance

First of all, let us take a look at the combined quartic Lagrangian for the case of two bosons and two fermions:

$$
\mathcal{L}_{s_1,s_2,s_3+\frac{1}{2},s_4+\frac{1}{2}} = \mathcal{L}^{TypeI}_{s_1,s_2,s_3+\frac{1}{2},s_4+\frac{1}{2}} + \mathcal{L}^{TypeII}_{s_1,s_2,s_3+\frac{1}{2},s_4+\frac{1}{2}} \ .
$$

Acting with a gauge transformation with respect to the k-th field yields two case differentiations, one for the bosonic and one for the fermionic fields. But first of all, we will translate the gauge field for the fermionic fields according to our formalism. Let $\epsilon_i := \epsilon^{(s_i-1)}$ be the fermions gauge field for Ψ_i. Then

$$\delta_k^{(0)}\Psi_i = \delta_{ki}s_i(a_i\cdot P_i)\epsilon_i$$

Without loss of generality choose $\delta_1^{(0)}$ and $\delta_4^{(0)}$. This yields

$$\begin{aligned}0 =&\delta_4^{(0)}\mathcal{L}_{s_1,s_2,s_3+\frac{1}{2},s_4+\frac{1}{2}} = \\ =&s_4\mathcal{V}^{TypeI}_{s_1,s_2,s_3+\frac{1}{2},s_4+\frac{1}{2}}(a_4\cdot P_4)\left(\Phi_1(x_1,a_1)\Phi_2(x_2,a_2)\bar{\Psi}_3(x_3,a_3)\epsilon_4(x_4,a_4)\right)\Big|_{\substack{x_i=x\\a_i=0}} \\ &+ s_4c\mathcal{V}^{TypeII}_{s_1,s_2,s_3+\frac{1}{2},s_4+\frac{1}{2}}(a_4\cdot P_4)\left(\Phi_1(x_1,a_1)\Phi_2(x_2,a_2)\bar{\Psi}_3(x_3,a_3)\gamma(a_5)\epsilon_4(x_4,a_4)\right)\Big|_{\substack{x_i=x\\a_i=0}} \\ &+ c.c.\end{aligned}$$

as well as

$$\begin{aligned}0 =&\delta_1^{(0)}\mathcal{L}_{s_1,s_2,s_3+\frac{1}{2},s_4+\frac{1}{2}} = \\ =&s_1\mathcal{V}^{TypeI}_{s_1,s_2,s_3+\frac{1}{2},s_4+\frac{1}{2}}(a_1\cdot P_1)\left(\xi_1(x_1,a_1)\Phi_2(x_2,a_2)\bar{\Psi}_3(x_3,a_3)\Psi_4(x_4,a_4)\right)\Big|_{\substack{x_i=x\\a_i=0}} \\ &+ s_1c\mathcal{V}^{TypeII}_{s_1,s_2,s_3+\frac{1}{2},s_4+\frac{1}{2}}(a_1\cdot P_1)\left(\xi_1(x_1,a_1)\Phi_2(x_2,a_2)\bar{\Psi}_3(x_3,a_3)\gamma(a_5)\Psi_4(x_4,a_4)\right)\Big|_{\substack{x_i=x\\a_i=0}} \\ &+ c.c.\end{aligned}$$

which for $i = 1, 2, 3, 4$ immediately implies [1]

$$\begin{aligned}\left[\mathcal{V}^{TypeI}_{s_1,s_2,s_3+\frac{1}{2},s_4+\frac{1}{2}}, a_i\cdot P_i\right] &\approx 0 \\ \left[\mathcal{V}^{TypeII}_{s_1,s_2,s_3+\frac{1}{2},s_4+\frac{1}{2}}, a_i\cdot P_i\right] &\approx 0\ .\end{aligned}$$

Because $\mathcal{V}^{TypeII}_{s_1,s_2,s_3+\frac{1}{2},s_4+\frac{1}{2}} \approx y'_{51}\mathcal{V}^{TypeI}_{s_1,s_2,s_3+\frac{1}{2},s_4+\frac{1}{2}}$ and the fact that $[y'_{51}, a_i \cdot P_i] = 0$ the problem reduces completely to gauge constraining a type I vertex. For appropriate Δ the vertex $\Delta\mathcal{V}^{TypeI}_{s_1,s_2,s_3+\frac{1}{2},s_4+\frac{1}{2}}$ obeys (2.5) and thus we get the same results for it as in (2.4):

$$\boxed{\Delta\mathcal{V}^{TypeI}_{s_1,s_2,s_3+\frac{1}{2},s_4+\frac{1}{2}} = Q_{\mathcal{V}_{s_1,s_2,s_3,s_4}}(Y_i, s_{ij})} \tag{2.7}$$

$\Rightarrow \mathcal{V}^{typeII}_{s_1,s_2,s_3+\frac{1}{2},s_4+\frac{1}{2}}$ only depends on s_{ij}, linearly on y'_{51} and at most linearly on Y_i [1].

2.3.4 Summary of the Constraints

In (2.7) we found that $\mathcal{V}^{TypeI}_{s_1,s_2,s_3+\frac{1}{2},s_4+\frac{1}{2}}$ only depends on the variables Y_i and s_{ij} if (2.1) is applied [1]. For the vertex type II vertex operator we also conclude that because of it not depending on terms with $A_i \cdot A_j$ or $A_i^{\mu} \cdot A_i^{\nu}$ **it can only depend on fields with spin less or equal to** $3/2$.

2.4 Vertices with four HS Fermions

A vertex with four fermions consists of two pairs of fermions. This results in even more case differentiations compared to the last section, as we now have two contractions of spinor indices. The possible vertices can be classified as follows. Again, Type I consists purely of contractions with δ^{ab}, Type II has one contraction with the γ-matrices and Type III is purely contracted with γ-matrices.

$$
\begin{aligned}
&\mathcal{L}^{TypeI}_{s_1+\frac{1}{2},s_2+\frac{1}{2},s_3+\frac{1}{2},s_4+\frac{1}{2}} = \\
&\mathcal{V}^{TypeI}_{s_1+\frac{1}{2},s_2+\frac{1}{2},s_3+\frac{1}{2},s_4+\frac{1}{2}} \left(\bar{\Psi}_1(x_1,a_1)\Psi_2(x_2,a_2)\bar{\Psi}_3(x_3,a_3)\Psi_4(x_4,a_4)\right)\Big|_{\substack{x_i=x\\a_i=0}} \\
&+ c.c. \\
&\mathcal{L}^{TypeII}_{s_1+\frac{1}{2},s_2+\frac{1}{2},s_3+\frac{1}{2},s_4+\frac{1}{2}} = \\
&c_1\mathcal{V}^{TypeII}_{s_1+\frac{1}{2},s_2+\frac{1}{2},s_3+\frac{1}{2},s_4+\frac{1}{2}} \left(\bar{\Psi}_1(x_1,a_1)\Psi_2(x_2,a_2)\bar{\Psi}_3(x_3,a_3)\gamma(a_5)\Psi_4(x_4,a_4)\right)\Big|_{\substack{x_i=x\\a_i=0}} \\
&+ c.c. \\
&\mathcal{L}^{TypeIII}_{s_1+\frac{1}{2},s_2+\frac{1}{2},s_3+\frac{1}{2},s_4+\frac{1}{2}} = \\
&c_2\mathcal{V}^{TypeIII}_{s_1+\frac{1}{2},s_2+\frac{1}{2},s_3+\frac{1}{2},s_4+\frac{1}{2}} \left(\bar{\Psi}_1(x_1,a_1)\gamma(a_6)\Psi_2(x_2,a_2)\bar{\Psi}_3(x_3,a_3)\gamma(a_5)\Psi_4(x_4,a_4)\right)\Big|_{\substack{x_i=x\\a_i=0}} \\
&+ c.c.
\end{aligned}
$$

where $c_1, c_2 \in \mathbb{C}$ are appropriate complex constants to ensure that the Lagrangian is real.

2.4.1 Type I Vertices

As before, if all fermionic spinor indices are contracted with δ^{ab} the vertex operator directly corresponds to the purely bosonic quartic vertex operator.

We again take a representative of $[\mathcal{V}_{s_1,s_2,s_3,s_4}]$ and find

$$[\mathcal{V}^{TypeI}_{s_1+\frac{1}{2},s_2+\frac{1}{2},s_3+\frac{1}{2},s_4+\frac{1}{2}}] = [\mathcal{V}_{s_1,s_2,s_3,s_4}]^* \ .$$

2.4.2 Type II Vertices

If two fermions are contracted with δ^{ab} in terms of the vertex operator they can be calculated with as if they were bosonic fields. This yields that the Type II vertex operator with four fermions is related to the Type II vertex with two fermions and two bosons. Thus because of (2.6)

$$[\mathcal{V}^{TypeII}_{s_1+\frac{1}{2},s_2+\frac{1}{2},s_3+\frac{1}{2},s_4+\frac{1}{2}}] = [\mathcal{V}^{TypeII}_{s_1,s_2,s_3+\frac{1}{2},s_4+\frac{1}{2}}]^* \ . \tag{2.8}$$

2.4.3 Type III Vertices

The only new case differentiation is given by the type III vertices. Although it is quite similar to what we have done for the type II vertices, some new equivalences arise as well as the computation of the Schouten identities changes significantly. Again, because of the Fierz equations only $A_5 \cdot A_i$ and $A_5 \cdot P_i$ for $i = 1, 2$ as well as $A_6 \cdot A_j$ and $A_6 \cdot P_j$ for $j = 3, 4$ contribute to the vertex operator. Thus we now have the two conditions

$$\begin{aligned} y''_{51} &:= A_5 \cdot P_1 \approx_D -A_5 \cdot P_2 =: -y''_{52} \\ y''_{63} &:= A_6 \cdot P_3 \approx_D -A_6 \cdot P_4 =: -y''_{64} \end{aligned}$$

and also

$$\begin{aligned}
z''_{53} &:= A_5 \cdot A_3 \approx_F 0\\
z''_{54} &:= A_5 \cdot A_4 \approx_F 0\\
y''_{53} &:= A_5 \cdot P_3 \approx_F 0\\
y''_{54} &:= A_5 \cdot P_4 \approx_F 0\\
z''_{61} &:= A_6 \cdot A_1 \approx_F 0\\
z''_{62} &:= A_6 \cdot A_2 \approx_F 0\\
y''_{61} &:= A_6 \cdot P_1 \approx_F 0\\
y''_{62} &:= A_6 \cdot P_2 \approx_F 0 \;.
\end{aligned}$$

In order to find the generating matrix for the Schouten identities we define $\vec{b}''^T := (P_1, P_2, P_3, P_4, A_1, A_2, A_3, A_4, A_5, A_6)$ [1] in order to construct

$$\mathcal{B}'' = \vec{b}'' \cdot \vec{b}''^T = \begin{bmatrix} \mathcal{S}'' & \mathcal{Y}''^T \\ \mathcal{Y}'' & \mathcal{Z}'' \end{bmatrix} \;.$$

Moreover the choice of $z''_{55} = z''_{66} = 0$ is made, due to $\gamma(a_i)$ being only linear in the corresponding a_i.

The vertex operator has to contract $a_5^\mu a_6^\nu$ and thus an expansion of it can be made according to all possible combinations of $\{z''_{51}, z''_{52}, y''_{51}\}$ multiplied with $\{z''_{63}, z''_{64}, y''_{63}\}$ as well as z_{56}. Let $Z := (\{z''_{51}, z''_{52}, y''_{51}\} \times \{z''_{63}, z''_{64}, y''_{63}\}) \cup \{z_{56}\}$, then

$$\mathcal{V}^{TypeIII}_{s_1+\frac{1}{2},s_2+\frac{1}{2},s_3+\frac{1}{2},s_4+\frac{1}{2}} = \sum_{a\in Z} \mathcal{V}^{TypeIII}_{s_1+\frac{1}{2},s_2+\frac{1}{2},s_3+\frac{1}{2},s_4+\frac{1}{2}}\Big|_{\mathcal{O}(a)} \;.$$

With regards to this expansion one needs to examine three different types of terms in $\mathcal{V}^{TypeIII}_{s_1+\frac{1}{2},s_2+\frac{1}{2},s_3+\frac{1}{2},s_4+\frac{1}{2}}$: $y''_{ij}y''_{kl}$, $y''_{ij}z''_{kl}$ and $z''_{ij}z''_{kl}$.

For every expansion term that is at least linear in either z''_{5i} or z''_{6i} we can find $q_i(s''_{ij})$ such that the following is true: Again take the second to fifth row (resp. $2^{nd}-4^{th}$ and 6^{th}) and the second to fourth as well as the 9^{th} column. Repeat the process for the second to fourth as well as sixth (resp. 7^{th}) row and second to fourth as well as 10^{th} column. The determinants of those matrices yield the following Schouten identities:

$$\begin{aligned}
y''_{51}q_5(y''_{1i}, s''_{ij}) + z''_{51}q_6(s''_{ij}) &= 0 \\
y''_{51}q_7(y''_{2i}, s''_{ij}) + z''_{52}q_8(s''_{ij}) &= 0 \\
y''_{63}q_9(y''_{3i}, s''_{ij}) + z''_{63}q_{10}(s''_{ij}) &= 0 \\
y''_{63}q_{11}(y''_{4i}, s''_{ij}) + z''_{64}q_{12}(s''_{ij}) &= 0 \ .
\end{aligned}$$

Which as before allows us to find a product of Mandelstan variables $\Delta = q_6q_8q_{10}q_{12}$ such that

$$\Delta\mathcal{V}^{TypeIII}_{s_1+\frac{1}{2},s_2+\frac{1}{2},s_3+\frac{1}{2},s_4+\frac{1}{2}} = y''_{51}y''_{63}\tilde{Q}_{\mathcal{V}}(s_{ij}, y_{ij}) + \Delta\mathcal{V}^{TypeIII}_{s_1+\frac{1}{2},s_2+\frac{1}{2},s_3+\frac{1}{2},s_4+\frac{1}{2}}\Big|_{\mathcal{O}(z_{56})} \tag{2.9}$$

In the same manner as before there exist Schouten identities that replace z_{56} with the y_{ij}. Take the $1^{st}, 2^{nd}, 4^{th}$ and 9^{th} row as well as $2^{nd}-4^{th}$ and 10^{th} column to find

$$y''_{51}y''_{63}q_{13}(s''_{ij}) + z_{56}q_{14}(s''_{ij}) = 0$$

and thus combined with (2.9) results in

$$\boxed{\mathcal{V}^{TypeIII}_{s_1+\frac{1}{2},s_2+\frac{1}{2},s_3+\frac{1}{2},s_4+\frac{1}{2}} \approx y''_{51}y''_{63}\mathcal{V}^{TypeI}_{s_1+\frac{1}{2},s_2+\frac{1}{2},s_3+\frac{1}{2},s_4+\frac{1}{2}}} \ .$$

2.4.4 Constraints from Gauge Invariance

The computation of the constraints does not differ much from the last two times and it simply yields [1]

$$\begin{aligned}
\left[\mathcal{V}^{TypeI}_{s_1+\frac{1}{2},s_2+\frac{1}{2},s_3+\frac{1}{2},s_4+\frac{1}{2}}, a_i\cdot P_i\right] &\approx 0\\
\left[\mathcal{V}^{TypeII}_{s_1+\frac{1}{2},s_2+\frac{1}{2},s_3+\frac{1}{2},s_4+\frac{1}{2}}, a_i\cdot P_i\right] &\approx 0\\
\left[\mathcal{V}^{TypeIII}_{s_1+\frac{1}{2},s_2+\frac{1}{2},s_3+\frac{1}{2},s_4+\frac{1}{2}}, a_i\cdot P_i\right] &\approx 0
\end{aligned}$$

for $i = 1, 2, 3, 4$. Generally speaking, because of the structure of the bosonic as well as fermionic gauge field the constraints from gauging will always result in the commutator of $a_i \cdot P_i$ with the corresponding linearly independent vertex operators of a particular order. The exact generalization of that statement will be covered at a later point in this thesis.

The equations for $\mathcal{V}^{TypeII}_{s_1+\frac{1}{2},s_2+\frac{1}{2},s_3+\frac{1}{2},s_4+\frac{1}{2}}$ and $\mathcal{V}^{TypeIII}_{s_1+\frac{1}{2},s_2+\frac{1}{2},s_3+\frac{1}{2},s_4+\frac{1}{2}}$ again reduce to the problem for a type I vertex because both y''_{51} and y''_{63} are gauge invariant. Moreover, similar to the way we derived it for the last section, because $\Delta\mathcal{V}^{TypeI}_{s_1+\frac{1}{2},s_2+\frac{1}{2},s_3+\frac{1}{2},s_4+\frac{1}{2}}$ obeys (2.5), we get the same results for it as in (2.4) [1]:

$$\boxed{\Delta\mathcal{V}^{TypeI}_{s_1+\frac{1}{2},s_2+\frac{1}{2},s_3+\frac{1}{2},s_4+\frac{1}{2}} = Q_{\mathcal{V}_{s_1,s_2,s_3,s_4}}(Y_i, s_{ij})} \tag{2.10}$$

$\Rightarrow \mathcal{V}^{TypeI}_{s_1+\frac{1}{2},s_2+\frac{1}{2},s_3+\frac{1}{2},s_4+\frac{1}{2}}$ only depends on s_{ij} and at most linearly Y_i. Thus the same applies for the type I part of the type II and type III vertices.

2.4.5 Summary of the Constraints

For the type I vertex we find (2.10)[1] and thus it does not depend on $A_i \cdot A_j$ or $A_i^\mu A_j^\nu$ which yields that **the type II and type III vertices can only depend on fields with spin less or equal to** $3/2$.

2.5 Vertex Constraints for the remaining Cases of Parity

According to [1] there is a very simple way to extend the bosonic vertex operator for TT-fields from even parity to odd parity for any order higher or equal to four. Take the class of operators given through $B_{IJK} := \epsilon_{\alpha\beta\gamma} b_I^\alpha b_J^\beta b_K^\gamma$. Then, as already hinted, the remaining cases of parities for the vertex operators can be constructed by taking a representative of the preceding operators and taking the linear combination

$$\mathcal{V}_{s_1,\ldots,s_4} = \sum_{I,J,K=1}^{8} c_{IJK} \mathcal{V}_{s_1,\ldots,s_4} \cdot B_{IJK}$$

where the $c_{IJK} \in \mathbb{R}$ are constants. Then, when taking the matrix consisting of the first three rows and columns of $\mathcal{S}$, one finds according to [1] after using a property of the product of two triple products that $det(M) = (B_{123})^2$ applied to B_{IJK} yields

$$det(M) \cdot B_{IJK} = \frac{1}{6} \left(\mathcal{B}_{1I} \mathcal{B}_{2J} \mathcal{B}_{3J} \pm 5 \text{ terms} \right) \cdot B_{123} \ .$$

This property yields that there exists a product of Mandelstan variables $\Delta(s_{ij})$ for which the vertex operator of odd parity can be simplified as follows:

$$\Delta(s_{ij}) \mathcal{V}_{s_1,\ldots,s_4} \approx Q_{\mathcal{V}_4}(Y_i, s_{ij}) \cdot B_{123}$$

where $Q_{\mathcal{V}_4}$ is an appropriate polynomial as described in (2.4).

While this immediately applies to the type I vertices of any of the above case differentiations, type II and type III need a little bit closer inspection. Firstly, assuming the same Ansatz as for the type I vertices for type II of the case with two bosons and two fermions, we want to examine if B'_{IJK} has to include b'_9 or if the problem simplifies. Assuming that b'_9 can possibly be included in B'_{IJK} yields the case differentiations where it is and where it is not. In the second case the problem reduces to simply extending a type I vertex due to y'_{51} commuting with everything else. However, in case of B'_{IJ9} the vertex to be extended can only be a type I vertex per default since y'_{51} is not allowed in it ($(y'_{51})^2$ would yield 0 in the end). We find

$$\begin{aligned} det(M)B'_{IJ9} =& \frac{B_{123}}{6}(\mathcal{B}'_{1I}\mathcal{B}'_{2J}\mathcal{B}'_{39} - \mathcal{B}'_{1J}\mathcal{B}'_{2I}\mathcal{B}'_{39} \\ &+ \mathcal{B}'_{1J}\mathcal{B}'_{29}\mathcal{B}'_{3I} - \mathcal{B}'_{1I}\mathcal{B}'_{29}\mathcal{B}'_{3J} \\ &+ \mathcal{B}'_{19}\mathcal{B}'_{2I}\mathcal{B}'_{3J} - \mathcal{B}'_{19}\mathcal{B}'_{2J}\mathcal{B}'_{3I}) \\ \approx_F& \frac{B_{123}\mathcal{B}'_{19}}{6}(\mathcal{B}'_{1I}\mathcal{B}'_{3J} - \mathcal{B}'_{1J}\mathcal{B}'_{3I} + \mathcal{B}'_{2I}\mathcal{B}'_{3J} - \mathcal{B}'_{2J}\mathcal{B}'_{3I}) \end{aligned}$$

where $I \neq 9$ and $J \neq 9$ and $\mathcal{B}'_{19} = y'_{51}$. Since this result is an even expression except for the B_{123} part (with respect to bosonic vertices), without calculating further one can immediately conclude that it is zero or yields B_{123} multiplied with a representative of the even vertex operators. Thus, without loss of generality one can extend the type II vertex with the same procedure that is used for the type I vertices to achieve the opposite parity. This result can easily be extended to the case of a vertex with four fermions, for which the vertex operator can also be extended through this trick.

This yields a product of Mandelstan variables such that each type of vertex with opposite parity can be constructed by a respective counterpart of the initial vertices. Moreover, because the product of Mandelstan variables is gauge invariant we conclude that for those vertices the following holds:

$$\boxed{\begin{gathered}\Delta(s_{ij})\mathcal{V}^{TypeI}_{s_1,\ldots,s_4} \approx Q_1(Y_i, s_{ij}) \cdot B_{123} \\ \\ \Delta(s_{ij})\mathcal{V}^{TypeII}_{s_1,\ldots,s_4} \approx y'_{51} Q_2(Y_i, s_{ij}) \cdot B_{123} \\ \\ \Delta(s_{ij})\mathcal{V}^{TypeIII}_{s_1,\ldots,s_4} \approx y''_{51} y''_{63} Q_3(Y_i, s_{ij}) \cdot B_{123}\end{gathered}}$$

2.6 Vertices with Spin-0 and Spin-1/2 Fields

While the first part of the vertex constraints discussion strictly followed the procedure of [1], the second part will follow the solutions given by [3], because including scalar (spin-0) and pseudo-scalar (spin-1/2) fields yields a reduction in Lorentz-indices in the vertex. In this part I follow a much more brute force approach of solving the constraints in order to verify that the results of the previous sections indeed hold when including spin-0 as well as spin-1/2 fields in the vertices. This part will start off with the simple case were no field with spin greater than 1/2 as well as no γ-matrices are included. Afterwards we will proceed with type I vertices and finish off the discussion by making an argument how the results of this translate to type II and type III vertices.

2.6.1 Type I Vertices with only Spin-0 and Spin-1/2 Fields

There are three different cases for quartic vertices with only (pseudo) scalar fields as well as no γ-matrices in them:

$$\begin{aligned}
\mathcal{L}_{0,0,0,0} &= \mathcal{V}_{0,0,0,0}\left(\Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3)\Phi_4(x_4)\right)\Big|_{x_i=x} \\
\mathcal{L}^{TypeI}_{0,0,\frac{1}{2},\frac{1}{2}} &= \mathcal{V}^{TypeI}_{0,0,\frac{1}{2},\frac{1}{2}}\left(\Phi_1(x_1)\Phi_2(x_2)\bar{\Psi}_3(x_3)\Psi_4(x_4)\right)\Big|_{x_i=x} + c.c. \\
\mathcal{L}^{TypeI}_{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}} &= \mathcal{V}^{TypeI}_{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}}\left(\bar{\Psi}_1(x_1)\Psi_2(x_2)\bar{\Psi}_3(x_3)\Psi_4(x_4)\right)\Big|_{x_i=x} + c.c.
\end{aligned}$$

For the case that corresponds to even parity in the purely bosonic setting all of them can only depend on the Mandelstan variables s_{ij}. So whenever $s_1, s_2, s_3, s_4 \in \{0, \frac{1}{2}\}$ the vertex operator is a polynomial in the s_{ij} that is only restricted by the Schouten identities given through the ideal $\mathcal{I}_S$ as well as the ideal $\mathcal{I}_D$ with respect to total derivatives [1][3]:

$$[\mathcal{V}_{s_1,s_2,s_3,s_4}] \in \frac{\mathbb{R}[s_{ij}|_{i<j}]}{\mathcal{I}_S + \mathcal{I}_D}$$

In this case the ideal $\mathcal{I}_S$ is particularly simple and given through the equation

$$s_{14}^2 s_{23}^2 + s_{13}^2 s_{24}^2 + s_{12}^2 s_{34}^2 - 2\left[s_{13}s_{14}s_{23}s_{24} + s_{12}s_{14}s_{23}s_{34} + s_{12}s_{13}s_{24}s_{34}\right] = 0 \ . \tag{2.11}$$

This equation does not allow the elimination of the dependence on a single variable, it only provides another rule to exchange variables besides the ones implied by the total derivatives given through

$$\sum_{i=1}^{4} s_{ij} \approx_D 0 \ .$$

Vertices of opposite parity can be achieved by applying $\epsilon^{\mu\nu\rho}\frac{\partial}{\partial x_i^\mu}\frac{\partial}{\partial x_j^\nu}\frac{\partial}{\partial x_k^\rho}$ to the initial vertices and combining them linearly [1].

2.6.2 Type I Vertices with arbitrary HS Fields and at least one Spin-0 or Spin-1/2 Field

Because an interaction term with $\bar{\Psi}\Psi$ contributes the same way as two spin-0 fields Φ^2, there is a direct analogy between vertices of type $\mathcal{V}_{s_1,s_2,\frac{1}{2},\frac{1}{2}}$ and $\mathcal{V}_{s_1,s_2,0,0}$. So the problem reduces to describing vertices $\mathcal{V}_{s_1,s_2,s_3,0}$ with bosonic spins $s_1 \geq s_2 \geq s_3 \geq 0$ and $s_i \geq 1$ for at least one of them. For simplicity, let's first focus on vertex operators that correspond to the even cases of the bosonic setting. That way one can construct the Schouten identities the same way as in the previous sections and directly apply them to the vertices.

Let's first look at the case $\mathcal{V}_{s_1,0,0,0}$. With three scalar fields the most general

Ansatz for the even parity vertex operator is given by

$$\mathcal{V}_{s_1,0,0,0} = \sum_{k=0}^{s_1} \alpha_k y_{13}^k y_{14}^{s_1-k}$$

where $\alpha_k \in \frac{\mathbb{R}[s_{ij}]}{\mathcal{I}_S+\mathcal{I}_D}$. For this vertex the gauge invariance given through $[\mathcal{V}_{s_1,0,0,0}, a_1 \cdot P_1] = 0$ [1] yields that:

$$\begin{aligned} \alpha_{k+1}(k+1)s_{13} + \alpha_k(s_1-k)s_{14} = 0 \\ \forall k:\ 0 \leq k \leq s_1 - 1\ . \end{aligned}$$

Because there is none of the y_{1j} to be eliminated through the one contributing Schouten identity for them ($\mathcal{I}_D$ already does the job), the solutions for the condition on gauge invariance are not restricted any further and thus it admits an infinite set of solutions. Solving the recursion relation is very difficult, because assuming an appropriately fixed α_0 yields that one has to apply the Schouten identity (2.11) at every step of the iteration. This problem will be adressed shortly.

Additionaly, clearly it is true that

$$\begin{aligned} \left[\mathcal{V}^{TypeI}_{s_1,0,\frac{1}{2},\frac{1}{2}}\right] = [\mathcal{V}_{s_1,0,0,0}]^* \\ \left[\mathcal{V}^{TypeI}_{s_1+\frac{1}{2},\frac{1}{2},0,0}\right] = [\mathcal{V}_{s_1,0,0,0}]^*\ . \end{aligned}$$

The big difference for this class of vertices is that because of one more field it is a two-derivative vertex opposed to the one-derivative solutions in [3].

As an example let us take a look at the case for $s_1 = 1$. For any combination of Δ and $\tilde{\Delta}$ such that (2.11) yields $\Delta s_{13} \approx_S \tilde{\Delta} s_{14}$ we find that

$$\mathcal{V}_{1,0,0,0} = \tilde{\Delta} y_{14} - \Delta y_{13}\ .$$

Because $\Delta s_{13} \approx_S \tilde{\Delta} s_{14} \Rightarrow \Delta y_{13} \approx_S \tilde{\Delta} y_{14}$ the only possible solutions are $\Delta = \alpha s_{14}$ and $\tilde{\Delta} = \alpha s_{13}$. Thus $\mathcal{V}_{1,0,0,0} = \alpha\,(s_{13}y_{14} - s_{14}y_{13})$. One might recognize that this looks very similar to the Y_i from before. However, one needs to redefine them a little bit in this case. Take $\tilde{Y}_i = s_{ii+2}y_{ii+3} - s_{ii+3}y_{ii+2}$ with every index (mod 4). In fact $\tilde{Y}_i \approx_D Y_i$. Then

$$\boxed{\mathcal{V}_{1,0,0,0} = \alpha \tilde{Y}_1}\ .$$

However, things are different when solving the recursion relation for $s_2 = 2$. For the vertex with coupling to gravity whenever $\Delta s_{13} \approx \tilde{\Delta} s_{14}$ one finds

$$\alpha_2 s_{13} \approx_S \Delta s_{14}\ .$$

There is only one way to properly solve this, which is given through: $\Delta \to \alpha s_{13}s_{14}$ with α arbitrary. Then $\tilde{\Delta} \to \alpha s_{13}^2$ and thus

$$\alpha_2 s_{13} \approx \alpha s_{13} s_{14}^2$$

yields $\alpha_2 = \alpha s_{14}^2$. All this results for arbitrary $s_1 \geq 2$ in

$$\begin{aligned} \alpha_k &= (-1)^k \frac{s_1^k}{k!} \alpha s_{13}^{s_1-k} s_{14}^k = (-1)^k \binom{s_1}{k} \alpha s_{13}^{s_1-k} s_{14}^k \\ \Rightarrow \mathcal{V}_{s_1,0,0,0} &= \alpha\,(s_{13}y_{14} - s_{14}y_{13})^{s_1} = \alpha \tilde{Y}_1^{s_1} \end{aligned}$$

Because of the Schouten-identities we know $\tilde{Y}_i^2 \approx_S 0$ so it is clear that for $s_1 \geq 2$ we have $\mathcal{V}_{s_1,0,0,0} = \alpha \tilde{Y}_1^{s_1} \approx_S 0$ and thus

$$\boxed{[\mathcal{V}_{s_1,0,0,0}] = [0]}\ .$$

This procedure can beautifully be continued to the vertex operator with two arbitrary higher-spins $\mathcal{V}_{s_1,s_2,0,0}$. The problem that has to be solved specifically is how to cope with the additional z_{12} terms that occur in the most general form of the vertex operator, which is given through:

$$\mathcal{V}_{s_1,s_2,0,0} = \sum_{k=0}^{s_1}\sum_{m=0}^{s_2} \alpha_{km} y_{13}^k y_{14}^{s_1-k} y_{21}^m y_{24}^{s_2-m} +$$
$$\sum_{A(n,l,r,s)=0}^{\min(s_1,s_2)} \begin{pmatrix} \text{All possible combinations in } y_{13}, y_{14}, y_{21} \text{ and } y_{24} \\ \text{such that the sum of exponents} = s_1 + s_2 - A \end{pmatrix} \times z_{12}^A \ .$$

The Schouten-identities yield that there exists a Δ such that

$$\Delta\mathcal{V}_{s_1,s_2,0,0} \approx_S \sum_{k=0}^{s_1}\sum_{m=0}^{s_2} \tilde{\alpha}_{km} y_{13}^k y_{14}^{s_1-k} y_{21}^m y_{24}^{s_2-m}$$

for some redefinition $\alpha_{km} \to \tilde{\alpha}_{km}$. As previously done, it is now necessary to analyse how gauge invariance constrains $\Delta\mathcal{V}_{s_1,s_2,0,0}$. $[\Delta\mathcal{V}_{s_1,s_2,0,0}, a_i \cdot P_i] \approx 0$ for $i = 1, 2$ results in the following system of equations:

$$\begin{aligned}
&\tilde{\alpha}_{k+1m}(k+1)s_{13} + \tilde{\alpha}_{km}(s_1 - k)s_{14} \approx 0 \\
&\forall k : 0 \leq k \leq s_1 - 1 \\
&\tilde{\alpha}_{km+1}(m+1)s_{12} + \tilde{\alpha}_{km}(s_2 - m)s_{24} \approx 0 \\
&\forall m : 0 \leq m \leq s_2 - 1 \ .
\end{aligned}$$

Like before it is useful to look at what happens at $s_1 = s_2 = 1$. In this case the coefficient matrix $\tilde{\alpha}$ can be solved for any Δ_i and $\tilde{\Delta}_i$ $(i = 1, 2)$ with $\Delta_1 s_{13} \approx \tilde{\Delta}_1 s_{14}$ and $\Delta_2 s_{12} \approx \tilde{\Delta}_2 s_{24}$ as follows:

$$\tilde{\alpha} = \begin{pmatrix} \tilde{\Delta}_1\tilde{\Delta}_2 & -\tilde{\Delta}_1\Delta_2 \\ -\Delta_1\tilde{\Delta}_2 & \Delta_1\Delta_2 \end{pmatrix} \ .$$

This time around, for the purpose of completeness, we will not neglect the z_{12} as a choice of representative which is why it is necessary to again analyse the constraints from gauge invariance for the following general Ansatz:

$$\mathcal{V}_{1,1,0,0} = \alpha_{00} y_{14} y_{24} + \alpha_{01} y_{14} y_{21} + \alpha_{10} y_{13} y_{24} + \alpha_{11} y_{13} y_{21} + \beta z_{12}$$

The βz_{12} adds an extra $\pm\beta$ to three of the four equations stemming from the commutator with $a_i \cdot P_i$:

$$\begin{aligned} \alpha_{10} s_{13} + \alpha_{00} s_{14} &\approx 0 \\ \alpha_{11} s_{13} + \alpha_{01} s_{14} + \beta &\approx 0 \\ \alpha_{k1} s_{21} + \alpha_{k0} s_{24} - \beta &\approx 0 \ . \end{aligned}$$

The simplest one is whenever $\beta \approx 0$, then:

$$\mathcal{V}_{1,1,0,0} = \tilde{\Delta}_1 \tilde{\Delta}_2 y_{14} y_{24} - \tilde{\Delta}_1 \Delta_2 y_{14} y_{21} - \Delta_1 \tilde{\Delta}_2 y_{13} y_{24} + \Delta_1 \Delta_2 y_{13} y_{21} = \alpha \tilde{Y}_1 \tilde{Y}_2 \ .$$

If $\beta \neq 0$ one finds for some $\alpha' s_{12} \approx \beta$:

$$\mathcal{V}_{1,1,0,0} = -\alpha' y_{12} y_{21} + \beta z_{12} \approx \alpha' \left(s_{12} z_{12} - y_{12} y_{21} \right)$$

which because of $[\mathcal{V}_{1,1,0,0}]^* = \left[\mathcal{V}^{TypeI}_{1,1,\frac{1}{2},\frac{1}{2}}\right]$ also gives a solution for the fermionic vertex with two spin-1 fields. Those are the only two solutions up to equivalences. Of course, the complete vertex is the superposition of those two vertices. A very interesting observation that can be made here is that the solution for the system with z_{12} yields a gauge invariant quantity $Z_{ij} := s_{ij} z_{ij} - y_{ij} y_{ji}$. Moreover it is clear by now that one finds for $s_2 \geq s_1 \geq 2$

$$\begin{aligned} &\Delta \mathcal{V}_{s_1,s_2,0,0} = \tilde{\alpha} \tilde{Y}_1^{s_1} \tilde{Y}_2^{s_2} \approx 0 \\ &\Rightarrow \boxed{[\mathcal{V}_{s_1,s_2,0,0}] = [0]} \end{aligned}$$

and for $s_1 = s_2 = 1$

$$\mathcal{V}_{1,1,0,0} = \alpha \tilde{Y}_1 \tilde{Y}_2 + \alpha' Z_{12} \ .$$

However, gauge invariance does not imply that the restriction $\mathcal{V}_{1,1,0,0} \to \Delta\mathcal{V}_{1,1,0,0} \approx \tilde{\alpha}\tilde{Y}_1\tilde{Y}_2$ holds for all possible Schouten-identities combined with $\mathcal{I}_D$, as they might change something that is gauge invariant into something else that is not. It turns out to be a key feature of $\mathcal{I}_S + \mathcal{I}_D$ that it does not necessarily preserve gauge invariance. As a side note, the gauge variation of a Schouten identity is zero up to Schouten identities. Applying Δ to the vertex operator results in:

$$\Delta\mathcal{V}_{1,1,0,0} = \alpha\Delta\tilde{Y}_1\tilde{Y}_2 + \alpha'\Delta Z_{12} \approx \tilde{\alpha}\tilde{Y}_1\tilde{Y}_2 \ .$$

(after implementing gauge invariance) which can only be true for all α' whenever $\Delta Z_{12} \approx \tilde{\Delta}\tilde{Y}_1\tilde{Y}_2$ for every possible Δ such that $\Delta z_{12} \approx_S q(s_{ij}, y_{ij})$. However, it turns out that there are Schouten-identities such that this is not true, which yields that the only possible solution is given by $\alpha = \tilde{\alpha}$ and $\alpha' = 0$, which ultimately contradicts the assumption that $\beta \neq 0$. Another way to see this is to specifically analyse how the β changes the α_{km} to $\tilde{\alpha}_{km}$ (before implementing gauge invariance). Upon solving the equations for the $\tilde{\alpha}_{km}$ one finds that they only admit a solution whenever $\beta = 0$. Thus the $\alpha' Z_{12}$ term cannot contribute. Generally speaking this is true for any Z_{ij}, which puts a big restriction on the vertices. In retrospective it is also the reason why the vertex operators in 3.2-3.5 can be chosen to be independent of the z_{ij} after changing them to y_{ij}-terms with the Schouten-identities.

$$\Rightarrow \boxed{\mathcal{V}_{1,1,0,0} = \alpha\tilde{Y}_1\tilde{Y}_2} \ .$$

After having calculated the vertices with one and two arbitrary higher spins, it is clear that the same strategy will be effective to solve the vertex $\mathcal{V}_{s_1,s_2,s_3,0}$. The assumption is that only $\mathcal{V}_{1,1,1,0}$ will survive. Thus we again start by writing down the most general Ansatz for the parity even vertex operator

$$\mathcal{V}_{s_1,s_2,s_3,0} = \sum_{k=0}^{s_1}\sum_{m=0}^{s_2}\sum_{l=0}^{s_3} \alpha_{kml} y_{13}^{k} y_{14}^{s_1-k} y_{23}^{m} y_{24}^{s_2-m} y_{32}^{l} y_{34}^{s_3-l} + \sum_{A=1}^{\min(s_1,s_2)} \sum_{B=1}^{\min(s_1,s_3)} \sum_{C=1}^{\min(s_2,s_3)} \beta_{ABC}(s_{ij}, y_{ij}) z_{12}^{A} z_{13}^{B} z_{23}^{C}$$

where A, B and C can depend on several indices of summation and $\beta_{ABC} = 0$ whenever $A + B > s_1$, $A + C > s_2$ or $B + C > s_3$. Similar to the case with two arbitrary higher spins, we can find a Schouten identity for each of the z_{ij} terms such that there exists a Δ for which

$$\Delta\mathcal{V}_{s_1,s_2,s_3,0} = \sum_{k=0}^{s_1}\sum_{m=0}^{s_2}\sum_{l=0}^{s_3} \tilde{\alpha}_{kml} y_{13}^{k} y_{14}^{s_1-k} y_{23}^{m} y_{24}^{s_2-m} y_{32}^{l} y_{34}^{s_3-l} \ .$$

Imposing the condition of gauge invariance this yields the following system of equations

$$\begin{aligned}
&\tilde{\alpha}_{k+1ml}(k+1)s_{13} + \tilde{\alpha}_{kml}(s_1 - k)s_{14} \approx 0 \\
&\forall k : 0 \leq k \leq s_1 - 1 \\
&\tilde{\alpha}_{km+1l}(m+1)s_{23} + \tilde{\alpha}_{kml}(s_2 - m)s_{24} \approx 0 \\
&\forall m : 0 \leq m \leq s_2 - 1 \\
&\tilde{\alpha}_{kml+1}(l+1)s_{32} + \tilde{\alpha}_{kml}(s_2 - l)s_{34} \approx 0 \\
&\forall l : 0 \leq m \leq s_3 - 1 \ .
\end{aligned}$$

The solution to this can analogously to the previous two cases be calculated for $s_1 = s_2 = s_3 = 1$ whenever $\Delta_1 s_{13} \approx \tilde{\Delta}_1 s_{14}$, $\Delta_2 s_{23} \approx \tilde{\Delta}_2 s_{24}$ and $\Delta_3 s_{32} \approx \tilde{\Delta}_3 s_{34}$:

$$
\begin{aligned}
(\tilde{\alpha}_{km0}) &= \begin{pmatrix} \tilde{\Delta}_1\tilde{\Delta}_2\tilde{\Delta}_3 & -\tilde{\Delta}_1\Delta_2\tilde{\Delta}_3 \\ -\Delta_1\tilde{\Delta}_2\tilde{\Delta}_3 & \Delta_1\Delta_2\tilde{\Delta}_3 \end{pmatrix} \\
(\tilde{\alpha}_{km1}) &= \begin{pmatrix} -\tilde{\Delta}_1\tilde{\Delta}_2\Delta_3 & \tilde{\Delta}_1\Delta_2\Delta_3 \\ \Delta_1\tilde{\Delta}_2\Delta_3 & -\Delta_1\Delta_2\Delta_3 \end{pmatrix} .
\end{aligned}
$$

Again there is no solution other than the trivial one whenever at least one $s_i \geq 2$. However, the z_{ij}-terms are a little bit more complicated for the $\mathcal{V}_{1,1,1,0}$ vertex. The system of equations gets very large (15 equations that need to be solved simultaneously) which makes it very hard to solve keeping the Schouten-identities as well as $\mathcal{I}_D$ in mind. However, the vertex includes only a sum over $z_{ij}y_{km}$. Previously, gauge invariant correspondents to the y_{ij} and z_{ij} were found through $\tilde{Y}_i$ as well as Z_{ij}. Therefore

$$
\frac{1}{2}\sum_{\sigma\in S_3} \tilde{\beta}_{\sigma(1)\sigma(2)\sigma(3)} Z_{\sigma(1)\sigma(2)} \tilde{Y}_{\sigma(3)}
$$

is the only possible term that can be added [3] to the homogeneous solution in α. Here, clearly $\tilde{\beta}_{ijk} = \tilde{\beta}_{jik}$ can be assumed without loss of generality. However, since the $\tilde{Y}_i$ are all linearly independent of one another it can only be that $\beta_{ijk} = 0 \ \forall i, j, k$. Thus one finds

$$
\boxed{\mathcal{V}_{1,1,1,0} = \alpha \tilde{Y}_1 \tilde{Y}_2 \tilde{Y}_3}
$$

$$
\boxed{[\mathcal{V}_{s_1,s_2,s_3,0}] = [0] \ \forall s_3 \geq s_2 \geq s_1 \geq 2}
$$

Finally, clearly the same approach as used in the previous section 3.5 with respect to [1] in order to find a simplified expression of the parity odd vertices through the parity even ones holds in the scenario of vertices with spin-0 and spin-1/2 fields.

2.6.3 Type II and Type III Vertices

In case of the Type II and Type III vertices everything simplifies to solving the type I vertices just as it happened in 3.3 and 3.4. For type I vertices due to $\mathcal{I}_D$ we find that the vertices always linearly depend on y'_{51} such that $\mathcal{V}^{TypeII}_{s_1,\ldots,s_N} \approx y'_{51}\mathcal{V}^{TypeI}_{s_1,\ldots,s_N}$ and for type II vertices z''_{56} can be replaced by a product of y''_{51} and y''_{63} due to the Schouten identities such that $\mathcal{V}^{TypeIII}_{s_1,\ldots,s_N} \approx y''_{51}y''_{63}\mathcal{V}^{TypeI}_{s_1,\ldots,s_N}$. Because y'_{51}, y''_{51} and y''_{63} (as well as z''_{56}) are gauge invariant, every calculation in 3.6.3 applies to the two higher type vertices.

The result of all this is that everything that was found in 3.1 - 3.5 is already the general solution for the vertex constraints and thus can be applied whenever necessary [1].

Generalization of the Vertex Constraints

3.1 Overview of the general Constraints for bosonic Fields

In [1] S. Fredenhagen, O. Krüger and K. Mkrtchyan use the Ansatz that was as a result used for the vertex operator with four bosons in order to find generalized vertex constraints for bosonic vertices of any order $N \geq 4$. They assume free massless transverse and traceless Fronsdal fields and the remaining conditions on the fields given in section 2.3.2 in order to construct the vertex with

$$\mathcal{L}_{s_1,\ldots,s_N} = \mathcal{V}_{s_1,\ldots,s_N} \left(\prod_{i=1}^{N} \Phi(x_i, a_i) \right) \Bigg|_{\substack{x_i = x \\ a_i = 0}} .$$

The problem of finding the generalized vertex operator simplifies when first investigating vertices of even order and then using a mathematical trick to derive the parity odd vertices from the preceding even vertices. Assuming $N \geq 4$, using the same procedure as for the $N = 4$ vertex by imposing the equivalence relations given through the ideals $\mathcal{I}_S$ and $\mathcal{I}_D$ as well as gauge invariance $\delta^{(0)}\mathcal{L}_N = 0$ there exists a polynomial $Q_{\mathcal{V}}(Y_i, s_{ij})$ such that

$$[Q_{\mathcal{V}}] \in \frac{\mathbb{R}[Y_i, s_{ij}]}{\mathcal{I}_R + \langle Y_i^2 \rangle}$$

yields a representative of the vertex operator equivalence class $[\mathcal{V}_{s_1,...,s_N}]$ due to $\mathcal{V}_{s_1,...,s_N}$ inheriting all features of $Q_{\mathcal{V}}$. [1]

Following up with the vertices with odd parity, there exists an extension to the vertex of even parity given through a linear combination of $\mathcal{V}_{s_1,...,s_N}$ with a class of operators for $I, J, K = 1, ..., 2N$ defined by

$$B_{IJK} := \epsilon_{\alpha\beta\gamma} b_I^\alpha b_J^\beta b_K^\gamma$$

(where $\epsilon_{\alpha\beta\gamma}$ is the Levi-Civita symbol) such that

$$\mathcal{V}_{s_1,...,s_N} = \sum_{I,J,K=1}^{2N} c_{IJK} \mathcal{V}_{s_1,...,s_N} \cdot B_{IJK}$$

for some appropriate constants $c_{IJK} \in \mathbb{R}$. Now define a matrix M consisting of the first 3 rows and columns of $\mathcal{S}$ such that $det(M) = 2s_{12}s_{13}s_{23}$. Then according to [1] this applied to B_{IJK} yields

$$det(M) \cdot B_{IJK} = \frac{1}{6} \left(\mathcal{B}_{1I}\mathcal{B}_{2J}\mathcal{B}_{3K} \pm 5 \text{ terms} \right) \cdot B_{123} \ .$$

Using the technique of applying products of Mandelstan variables to the vertex operator it can be shown that for the correct choice of $\Delta(s_{ij})$ the following is true:

$$\Delta(s_{ij})\mathcal{V}_{s_1,...,s_N} \approx Q_{\mathcal{V}_N}(Y_i, s_{ij}) \cdot B_{123}$$

where clearly B_{123} is gauge invariant such that the statement holds.
Ultimately this yields that the vertices for $N \geq 4$ do not contain any mass-

less higher-spin field with spin greater than $s = 1$. When comparing this to the result that there are no massless higher-spin fields with spin higher than $s = 3/2$ in the vertices of quartic order that include fermions this result is not surprising. Because of the linearity in the Y_i, clearly only ever one space-time index per individual quantum field can be contracted which strongly restricts the allowed spins. This result will later also be generalized to arbitrary higher order in the fermionic vertices.

3.2 Overview of the general Constraints for Vertices including fermionic Fields

Following up the generalization of the vertex constraints for strictly bosonic vertices [1], we will now generalize the fermionic vertices. Again, this will be done assuming free massless transverse and traceless Fronsdal fields as well as the additional conditions given in section 2.4.2. For order $N \geq 4$ the general form of the Lagrangian with $2p \geq 2$ fermions is as follows:

$$\mathcal{L}^{2p\text{ fermions}}_{s_1,\ldots,s_N} = \sum_{i\in\{2k\}_{k\leq p}} c_i \mathcal{L}^{Type\ \frac{i}{2}+1}_{s_1,\ldots,s_N} =$$

$$\sum_{i\in\{2k\}_{k\leq p}} c_i \mathcal{V}^{Type\ \frac{i}{2}+1}_{s_1,\ldots,s_N} \left(\prod_{l=1}^{N-2p} \Phi(x_l, a_l)\right) \cdot \bar{\Psi}(x_{N-2p+1}, a_{N-2p+1})\Psi(x_{N-2p+2}, a_{N-2p+2}) \cdot$$

$$\cdots \bar{\Psi}(x_{N-2p+i+1}, a_{N-2p+i+1})\gamma(a_{N+1})\Psi(x_{N-2p+i+2}, a_{N-2p+i+2}) \cdots$$

$$\bar{\Psi}(x_{N-1}, a_{N-1})\gamma(a_{N+i})\Psi(x_N, a_N)\Big|_{\substack{x_j=x\\ a_j=0\\ \forall j}} + c.c.$$

where $c_i \in \mathbb{C}$ are appropriate complex coefficients to ensure that the Lagrangian is real.

In accordance with the preceding calculations the problem of finding the general set of constraints for the vertices is given by dividing them into the different types and calculating each of them separately. Using the same procedure as used for the $N = 4$ vertex types one can find through imposing the equivalence relations given through the ideals $\mathcal{I}_S$ and $\mathcal{I}_D$ as well as gauge invariance that each type $k+1$ vertex reduces to polynomials

$$Q_{\mathcal{V}}^{Type\ k+1}(y_{(N+1)i}, ..., y_{(N+k)i}, Y_i, s_{ij}) \in \frac{\mathbb{R}[Y_i, s_{ij}, y_{(N+1)i}, ..., y_{(N+k)i}]}{\mathcal{I}_R + \langle Y_i^2 \rangle + \left\langle y_{(N+1)i} y_{(N+1)j} \right\rangle + ... + \left\langle y_{(N+k)i} y_{(N+k)j} \right\rangle} \tag{3.1}$$

To demonstrate this I will provide a short example for an $N = 5$ type III vertex. The simplest way to find a representative for its equivalence class is given through

$$(y_{61} + y_{64} + y_{65})(y_{71} + y_{72} + y_{73}) \approx_D 0\ .$$

Whenever the sum on the left is completely split up, upon adding up terms of it in combination with some polynomials $Q_k(Y_i, s_{ij})$ such that the sum is not equivalent to zero due to the equivalence relation given through total derivatives (or any other condition from $\mathcal{I}_R$ that applies), we get a representative of the corresponding vertex class. For example

$$\mathcal{V}_{s_1,...,s_5}^{TypeIII} = y_{61} y_{72} Q_1(Y_i, s_{ij}) + y_{64} y_{73} Q_2(Y_i, s_{ij})$$

is a permitted vertex operator in this case for appropriate Q_1 and Q_2.

As already discussed for $N = 4$, there exists a way of extending those vertices to vertices of opposite parity that is similar to the procedure given for bosonic operators [1]. The fact that including the terms b_{N+i} for any type-$(k+1)$ vertex in B_{IJK} does not change the structure of the resulting vertex with respect to the corresponding $y_{(N+i)j}$ holds for any given order and type of vertex. Setting any of the indices equal to $N+i$ and applying the product of Mandelstan variables $det(M)$ to it yields terms with $y_{(N+i)j}$ times a parity odd polynomial (parity odd with respect to a purely bosonic vertex) in the other operators. This means that for a given such vertex we can find an appropriate polynomial $Q_{\mathcal{V}}^{Type\ k+1}(y_{(N+1)i}, ..., y_{(N+k)i}, Y_i, s_{ij})$ such that applying the correct product of Mandelstan variables yields

$$\Delta(s_{ij})\mathcal{V}_{s_1,\ldots,s_N}^{Type\ k+1} \approx Q_{\mathcal{V}}^{Type\ k+1}(y_{(N+1)i}, ..., y_{(N+k)i}, Y_i, s_{ij}) \cdot B_{123}$$

Again, due to B_{123} being gauge invariant this provides us with a very defined class of polynomials that can be used to construct the vertices. Eventually, one comes to the conclusion that vertices of order $N \geq 4$ can not contain any massless higher spin fields with spin greater than $s = 3/2$, which stems from the fact that a given operator Y_i is only allowed at most linearly in the vertex operators.

3.3 Further Constraints and the Spin-3/2 Contribution

Whilst we already found the general constraints for the vertices, following the procedures of [1] there is one more relation that will be discussed here shortly and will significantly contribute to the conclusion. With respect to [1], the contribution of spin-3/2 fields is expected to not occur in higher order interaction vertices. This will be addressed in the following few passages. As already used when defining the different possible types of fermionic vertices, $\gamma^{\mu_1}\psi_{\mu_1 \dots \mu_s} = 0$ as well as $\gamma^\mu\gamma^\nu = \eta^{\mu\nu}\mathbb{1} + i\epsilon^{\mu\nu\rho}\gamma_\rho$ reduce the possibilities with respect to the placement of γ-matrices immensely.

Due to $\gamma^\alpha\gamma^{\mu_1}\Psi_{\mu_1\dots\mu_s} \approx_F 0$ and $\gamma^\mu\gamma^\nu = \eta^{\mu\nu}\mathbb{1} + i\epsilon^{\mu\nu\rho}\gamma_\rho$ we find

$$\begin{aligned} &\Psi^\alpha_{\ \mu_2\dots\mu_s} + i\epsilon^{\alpha\mu_1\rho}\gamma_\rho\Psi_{\mu_1\dots\mu_s} \approx_F 0 \\ &\qquad\qquad \Rightarrow \Psi^\alpha_{\ \mu_2\dots\mu_s} \approx_F -i\epsilon^{\alpha\mu_1\rho}\gamma_\rho\Psi_{\mu_1\dots\mu_s} \ . \end{aligned} \tag{3.2}$$

As a side note, this means that any type I vertex with fields that have spin higher or equal to 3/2 can always be turned into a type II vertex. In fact, this relation can be applied as often as we like starting from any given type of vertex with at least one pair of fermions in it. Going the direction of creating higher type vertices from lower type vertices, for a vertex with $k \geq 1$ pairs of spin-3/2 fields, at most type $k+1$ can be reached with respect to those fields.

From this we deduce that for a vertex with k pairs of fermionic fields, which all have $s \geq 3/2$, the problem of describing all possibilities for this scenario reduces to describing the class of type $k+1$ vertices with respect to those fields.

A question that needs to be asked is whether or not it is possible to go from higher type vertices to those of lower type. Since not every vertex of type $k \geq 2$ will have $-i\epsilon^{\alpha\mu_1\rho}\gamma_\rho\Psi_{\mu_1\dots\mu_s}$ as a part of it, it is necessary to examine what kinds of vertices can be reduced to one of lower type.

At first, let us look at ways to reduce the Levi-Civita symbol down to some sort of convenient linear combination of Kronecker deltas. The first equation that comes to mind is $\epsilon_{\alpha\beta\gamma}\epsilon^{\alpha\omega\rho} = \delta^\omega_\beta\delta^\rho_\gamma - \delta^\rho_\beta\delta^\omega_\gamma$, which results in

$$\begin{aligned} -i\epsilon_{\alpha\beta\delta}\epsilon^{\alpha\mu_1\rho}\gamma_\rho\Psi_{\mu_1\dots\mu_s} &= -i\gamma_\beta\Psi_{\delta\mu_2\dots\mu_s} + i\gamma_\delta\Psi_{\beta\mu_2\dots\mu_s} \\ \Rightarrow -i\gamma_\beta\Psi_{\delta\mu_2\dots\mu_s} + i\gamma_\delta\Psi_{\beta\mu_2\dots\mu_s} &\approx_F \epsilon_{\alpha\beta\delta}\Psi^\alpha{}_{\mu_2\dots\mu_s} \ . \end{aligned}$$

This means type II vertices with $-i\gamma_\beta\Psi_{\delta\mu_2\dots\mu_s} + i\gamma_\delta\Psi_{\beta\mu_2\dots\mu_s}$ are equivalent to a specific class of type I vertices. In general, reducing vertices of higher type to those of lower type would be a complicated process, but exploring it will not be necessary due to the next consequence resulting from (3.2).

It is very important to note that this discussion does not apply to spin-1/2 fields, since (3.2) cannot be applied to them.

Equation (3.2) and the previous paragraph hint that finding further constraints might have something to do with antisymmetry. Starting with the generalized constraints for the vertex operators, there is specifically with respect to spin-3/2 one set of operators remaining that are closely correlated to antisymmetric terms: $\{Y_i\}$.
First of all, (3.2) yields the following:

$$\begin{aligned} \epsilon_{\mu\nu\rho}\partial^\mu\Psi^\nu &\approx_F \epsilon_{\mu\nu\rho}\partial^\mu\left(-i\epsilon^{\nu\alpha\beta}\gamma_\alpha\Psi_\beta\right) \\ &= i\left(\delta^\alpha_\mu\delta^\beta_\rho - \delta^\beta_\mu\delta^\alpha_\rho\right)\gamma_\alpha\partial^\mu\Psi_\beta \\ &= i\partial\!\!\!/\Psi_\rho - i\gamma_\rho\partial^\mu\Psi_\mu \\ &\approx_F 0 \end{aligned}$$

which implies $\partial_{[\mu}\Psi_{\nu]} \approx 0$. In order to see how this correlates to the Y_i, let us look at the example of the following simple type I vertex with two spin-0 bosons as well as two spin-3/2 fermions:

$$\begin{aligned}
\mathcal{L}^{TypeI}_{0,0,\frac{3}{2},\frac{3}{2}} &= Y_3 Y_4 \left(\Phi_1(x_1)\Phi_2(x_2)\bar{\Psi}_3(x_3,a_3)\Psi_4(x_4,a_4)\right)\Big|_{\substack{x_i=x\\ a_i=0}} \\
&= (\partial^\mu\Phi)(\partial^\alpha\Phi)(\partial^\beta\partial_{[\mu}\bar{\Psi}_{\nu]})(\partial^\nu\partial_{[\alpha}\Psi_{\beta]}) \\
&\approx 0 \ .
\end{aligned}$$

This implies that applying the Y_i to the fields will result in terms involving $\partial_{[\alpha}\Psi_{\beta]} \approx 0$, which leads to the vertex vanishing. As a result the Y_i can be identified with zero for all fermions

$$\boxed{Y_i^{\text{fermion}} \approx 0} \ .$$

Thus, fields with spin-3/2 cannot contribute to vertices of order $N \geq 4$. Generalizing this statement for any field we observe that no field with spin $s > 1$ can contribute in vertices of order $N \geq 4$.

Looking back at equation (3.1) the only expression that changes is $\langle Y_i^2 \rangle$, which now only includes the operators with respect to bosonic fields. With regards to spin-1/2 fields there do not seem to be any further constraints than already established.

Conclusion

In this thesis we have shown that in three dimensions the higher order interactions of massless higher-spin fields are severely constrained by gauge invariance as well as the Schouten-identities and the equivalence relations given through total derivatives. For that reason fields with spin $s \geq 3/2$ can not occur in interaction vertices of order $N \geq 4$. This result is meeting the expectations with respect to a speculation in [1] that states the following:

"...the HS fields are known to admit a Chern- Simons description in three dimensions, in a first-order formulation. The findings of this paper imply that all the fields that do not carry bulk propagating d.o.f., cannot participate in independent higher-order interactions. This is consistent with the statement on the absence of higher-order self-interactions of Chern-Simons fields [8] and may indicate an exact equivalence of the Chern- Simons and metric descriptions of HS fields in the gauge sector. It is therefore tempting to speculate that any nonlinear action of massless HS fields without matter can be written in a Chern-Simons form." ([1]; p.4)

In fact, in 2.6.2 we have stated that fermionic fields of spin $s \geq 3/2$ do not carry bulk propagating degrees of freedom, which supports the speculation with respect to fermions.

For the remaining vertices with spin $s \in \{0; 1/2; 1\}$ fields we have found a simple formulation for constructing interaction vertices of arbitrary order $N \geq 4$, following the procedures of [1] and in accordance with the resulting constraints.

Appendix

5.1 Mathematical Additions

(A1) Pauli- and Gamma-matrices

The set of Pauli-matrices looks as follows [2]:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ .$$

The γ-matrices where constructed with respect to the $\mathsf{R}^+(\frac{1}{2})$-representation regarding $\mathfrak{Cl}(\mathbb{R}^{1,2}, \eta)$:

$$\gamma^0 = \sigma_1$$
$$\gamma^1 = i\sigma_2$$
$$\gamma^2 = -i\sigma_3 \ .$$

A very useful result from this is

$$\not{p} = \begin{pmatrix} -ip_2 & p_0 + p_1 \\ p_0 - p_1 & ip_2 \end{pmatrix} .$$

(A2) Lorentz-invariant integral measure and delta distribution

The Lorentz-invariant integral measure $d\mu(p)$ is a result of the conditions on the four-momenta of massless particles. They are given by $p_\mu p^\mu = 0$ and $p^0 > 0$. These are the conditions that yield the future oriented part of the mass-shell (minus the point $p^\mu = 0$, which has to be added in order to close the set). Whenever one integrates an integrable scalar function $f : \mathbb{R}^{1,2} \to \mathbb{R}$, it is necessary to do so on-shell. For $\theta(x)$ being the Heaviside function and $E(\vec{p}) = |\vec{p}|$, this yields the following [11]:

$$\begin{aligned} \int d^3p f(p)\theta(p^0)\delta(p_\mu p^\mu) &= \int d^3p f(p)\theta(p^0)\delta((p^0)^2 - \vec{p}\cdot\vec{p}) \\ &= \int d^3p f(p)\theta(p^0)\frac{1}{2|p^0|}\left[\delta(p^0 + E(\vec{p})) + \delta(p^0 - E(\vec{p}))\right] \\ &= \int \frac{d^2p}{2E(\vec{p})} f(p) =: \int d\mu(p) f(p) \end{aligned}$$

and so one recovers the Lorentz-invariant integral measure $d\mu(p) = \frac{d^2p}{2E(\vec{p})}$, which is invariant because $\int d^3p f(p)\theta(p^0)\delta(p_\mu p^\mu)$ is invariant. This can easily be seen by the fact that the measure d^3p does not change under proper orthochronous Lorentz-transformations:

$$d^3(\Lambda p) = |\det(\Lambda)| d^3p = d^3p \ .$$

It is also useful to analyse how $\delta^{(2)}(\vec{p}-\vec{q})$ transforms under Lorentz-transformations. In [2] an Ansatz is made where the spatial momenta is directed along one axis to simplify the calculation. Therefore, without loss of generality $p^\mu =$

$(p,p,0)^T$. For $p' = \Lambda p$ one finds that $\frac{dp'_1}{dp_1} = \frac{p'_0}{p_0} = \frac{E'(\vec{p})}{E(\vec{p})}$, which yields:

$$\delta^{(2)}(\vec{p}' - \vec{q}') = \frac{E}{E'}\delta^{(2)}(\vec{p} - \vec{q})$$
$$\Rightarrow E'\delta^{(2)}(\vec{p}' - \vec{q}') = E\delta^{(2)}(\vec{p} - \vec{q})$$

which is the reason why the coefficient $E(\vec{p})$ was chosen in the anti-commutator relations.

(A3) Correlation between bosonic and fermionic vertex classes

At several points throughout this thesis $[\cdot]^*$ was used to describe an extension of the bosonic vertices to the corresponding fermionic vertices. In the following, this extension will be defined more rigorously. Because Type I vertices with at least two spin-$(s_i + \frac{1}{2})$ fields are the only non-trivial fermionic vertices, the vertex operators can be described by the vertex operators that correspond to a purely bosonic system with spin-s_i replacing the spin-$(s_i + \frac{1}{2})$. The reason is that the Fierz-equations involving γ-matrices do not contribute to type I vertex operators.

First of all we will classify the bosonic Schouten ideal for a set of bosonic fields with spins $\{s_1, ..., s_N\}$ as well as ideal of total derivatives as $\mathcal{I}_S^b$ and $\mathcal{I}_D^b$. Then clearly it will be useful to denote the fermionic ideals corresponding to the fermionic counterpart $\{s_1 + \frac{1}{2}, ..., s_{2k} + \frac{1}{2}, s_{2k+1}, ..., s_N\}$ as $\mathcal{I}_S^f$ and $\mathcal{I}_D^f$. Because there are more Fierz-equations for fermions than there are for bosons it is true that $\mathcal{I}_S^b \subseteq \mathcal{I}_S^f$ and $\mathcal{I}_D^b \subseteq \mathcal{I}_D^f$. This leads to the definition of the additional set of identifications given through:

$$\mathcal{I}_S^A := \frac{\mathcal{I}_S^f}{\mathcal{I}_S^b}$$
$$\mathcal{I}_D^A := \frac{\mathcal{I}_D^f}{\mathcal{I}_D^b} \; .$$

Then whenever $[\mathcal{V}_{s_1,\ldots,s_N}] \in \frac{\mathbb{R}[s_{ij},y_{ij},z_{ij}]}{\mathcal{I}_S^b+\mathcal{I}_D^b}$ we define $[\cdot]^*$ through dividing out $\mathcal{I}_S^A+\mathcal{I}_D^A$:

$$[\mathcal{V}_{s_1,\ldots,s_N}] \to \left[\mathcal{V}_{s_1+\frac{1}{2},\ldots,s_{2k}+\frac{1}{2},s_{2k+1},\ldots,s_N}\right]^* \in \frac{\mathbb{R}[s_{ij},y_{ij},z_{ij}]}{\mathcal{I}_S^b+\mathcal{I}_D^b+\mathcal{I}_S^A+\mathcal{I}_D^A}$$

where the representatives for the vertex classes are identical in $\frac{\mathbb{R}[s_{ij},y_{ij},z_{ij}]}{\mathcal{I}_S^b+\mathcal{I}_D^b}$

$$\mathcal{V}_{s_1,\ldots,s_N} = \mathcal{V}_{s_1+\frac{1}{2},\ldots,s_{2k}+\frac{1}{2},s_{2k+1},\ldots,s_N} \ .$$

5.2 Zusammenfassung (Abstract in German)

Der Fokus dieser Arbeit liegt darin auf S. Fredenhagens, O. Krügers und K. Mkrtchyans Paper *"Vertex-Constraints in 3D Higher Spin Theories"* [1] – welches die Lagrange Formulierung von höheren Spin Gravitationstheorien in drei Dimensionen in den Ordnungen $N \geq 4$ untersucht [16] – auf zu bauen und dessen Ergebnisse auf Interaktions-Vertices der Ordnung $N = 4$ mit Fermionen in drei Dimensionen zu erweitern. Das daraus resultierende Ergebnis wird weiterhin benutzt um Einschränkungen an Vertices der Ordnung $N \geq 4$ zu erörtern.

Beim Erarbeiten der Einschränkungen an die Interaktions-Vertices höherer Ordnung müssen in drei Dimensionen die Schouten-Identitäten – neben Lorentz-Invarianz, Eichinvarianz und anderen Invarianzen der Lagrangian – in Betracht gezogen werden, was gemäß des sogenannten Noether-Fronsdal Programms [1][3][6] bewerkstelligt wird. In diesem Sinne weichen die Resultate von der Form der entsprechenden Interaktionsvertices in vier Dimensionen ab, da die Schouten Identitäten ein sehr starkes Werkzeug zum Vereinfachen der Berechnungen sowie zum Einschränken der Ergebnisse darstellen.

Letztendlich stellt sich heraus, dass die quartischen Vertices nur Felder mit Spin-1 oder niedriger beinhalten können. Dieses Ergebnis lässt sich ebenso auf die Vertices der Ordnungen $N \geq 4$ erweitern.

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