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Abstract

This thesis aims to show, how to include double-soft gluon emissions in already existing parton shower algorithms at leading colour ($N \rightarrow \infty$) and beyond. Soft refers in this context to a energy scale, which is much smaller than the typical hard scale of the perturbative QCD process considered. To do this consistently, an evolution equation, of the on the colour space defined renormalized operator valued amplitude, can be developed. The solution of this evolution equation then provides a numerical resummation of the soft-gluons. For concrete calculations a basis for the colour space has to be chosen. In this work, the basis of choice is the so called colour-flow basis, which has a physical interpretation in terms of flow of colour and allows one to compute the colour structure of the amplitude in the colour space of many quarks and gluons efficiently. For the numerical solutions Monte-Carlo methods are used, both for the colour contribution beyond leading colour and for the phase space integral of the resummation. The necessary ingredients to calculate the colour structure beyond leading order for double gluon emission with Monte Carlo methods are presented in this work. To simulate the amplitude evolution, furthermore, Sudakov veto algorithms and extensions of it are presented and used in this work. Another aspect of this work is to study resampling procedures such that the statistical convergence of the resulting weighted parton showers gets improved.

Zusammenfassung

Diese Arbeit beschäftigt sich damit, wie man die doppelte Emission von weichen Gluonen in bereits vorhandenen Parton Kaskaden inkludieren kann, auf führender Farbbildung ($N \rightarrow \infty$) und darüber hinaus. Weiche Gluonen haben eine charakteristische Energie, die sich unter jener des betrachteten harten perturbativen QCD-Prozesses befindet. Um die Parton Kaskade konsistent zu erweitern, wird eine Evolutionsgleichung für die am Farbraum definierte renormalisierte operatorwertige Amplitude entwickelt. Deren numerische Lösung liefert dann eine numerische Resummation der weichen Gluonen. Für konkrete Rechnungen muss dann eine Basis für den Farbraum gewählt werden. In dieser Arbeit wurde die Farbflussbasis gewählt, welche eine physikalische Interpretation bezüglich des Farbflusses besitzt und es möglich macht Farbraumstrukturen im Farbraum von vielen Quarks und Gluonen effizient zu berechnen. Um die numerischen Lösungen der Evolutionsgleichung zu berechnen, werden Monte-Carlo Methoden angewandt, sowohl für den Farbanteil als auch für das Phasenraumintegral der Resummation. Die nötigen Bestandteile für die Berechnung der Farbstrukturen, über die führenden Farbe hinaus, werden in die Arbeit präsentiert. Um die Amplitudenevolution zu simulieren werden außerdem Sudakov-Veto Algorithmen und Erweiterungen von diesen, vorgestellt und verwendet. Ein anderer Aspekt dieser Arbeit ist es, Resampling Strategien zu untersuchen, um die statistische Konvergenz der resultierenden gewichteten Parton Kaskaden zu verbessern.

1 Motivation and overview

1.1 Motivation

The aim of this thesis is to investigate the evolution of non-global observables, by systematically including double soft-gluon emissions and to simulate these processes with Monte Carlo methods. Soft denotes here an energy scale, which is much lower than the hard energy scale. The observables in perturbative QCD are of the form [1]

$$\Sigma(\tau) = \int_0^\tau \sum_n d\sigma_n u_n(k_1, k_1, \dots), \quad (1.1)$$

which contains possible large logarithms L . The leading order refers within parton showers to the $\mathcal{O}(\alpha_s^n L_{SC}^{2n})$ contribution, where α_s is the strong interaction constant and S and C refer to soft and collinear singularities, respectively. The problem with conventional parton showers, which only take single soft-gluons into account, is that the necessary factorization formula is derived from the strongly-ordered approximation [2]. This limit describes only the terms $\alpha_s^n (L_{SC}^{2n}, L_C^n, L_S^n)$ correctly, while sub-leading terms $\alpha_s^n L_S^{n-1}$ are not consistently accounted for. To circumvent this problem, the more general double soft-gluon limit has to be considered, which describes correctly the double- and single-logarithmic contribution. This is especially important when non-global observables [3, 4] are calculated, where the cancellation of the infrared (IR) divergences only takes place, when summing over all emissions, due to the Bloch-Nordsieck theorem/Kinoshita-Lee-Nauenberg theorem. This resummation can be achieved by systematically solving the soft-gluon evolution equations [4, 5] in colour space. Why Monte Carlo methods are required can already be seen at the example $q\bar{q} \rightarrow q\bar{q}$, the required colour factor for the squared amplitude is given by

$$\frac{1}{N} \sum_{a=1}^N \frac{1}{N} \sum_{b=1}^N \sum_{c=1}^N \sum_{d=1}^N \left| \sum_{g=1}^N (t^e)_b^a (t^g)_d^c \right|^2, \quad (1.2)$$

where t^a are fundamental representation matrices of the generators and t^g the structure constant of the Lie group $SU(N)$. For $N = 3$ these are already $3^4 \cdot 8 = 648$ terms to evaluate, which showcases why for more complex processes Monte Carlo methods are needed.

1.2 Overview

The thesis is structured in the following way:

In the first section 2, basic facts about quantum chromodynamics (QCD) and its Lagrangian density get recapitulated, especially about their underlying Lie Group $SU(N)$ and their Lie algebra $su(N)$. Then the later needed renormalization program will be shortly reviewed. Furthermore, the basic ideas of the needed Monte-Carlo (MC) methods get introduced.

The second section 3 is then about soft-gluon evolution equations, which provide a resummation of soft-gluons. To do so, the necessary QCD factorization formulas, for the single and double emission case, will be briefly revised, including a short discussion about the relation between the two underlying limits, from which the factorization formulas follow.

In the first part of section 4, an extensive discussion about the colour space and the colour operator(s) defined on it is given, different possible basis choices are shortly presented. The second part of this section is then dedicated to the colour flow basis, and the illustration how the necessary calculation can be executed.

The part of this work concerned with the necessary numerical methods starts in section 5, here the Sudakov veto, the weighted Sudakov and a new possible extension of the latter are presented. The former two are then supplemented with the competition algorithm, such that splitting rates can be added efficiently, which is needed when simulating the amplitude evolution. Furthermore, resampling methods are presented, for the case of only positive weights and the case where there are also negative weights and lastly a possible resampling algorithm for complex weights gets discussed. The resampling procedures allow one to improve the statistical convergence of the weighted parton shower. Throughout this chapter, different studies of application of the presented algorithm to different collinear cutoffs are given, for example, at the end the differential cross section for the gaps-between jets observables is presented.

The next section 6 provides then the necessary ingredients to simulate double gluon emission at leading and beyond leading colour order. For the beyond leading colour order calculation, also the relevant ingredients for the necessary colour MC simulation, for including double gluon emission, are presented. Furthermore, a detailed analysis of the leading colour contribution is given. An algorithm to include double-gluon emission in a parton shower gets discussed.

The last section 7 then provides a short summary of this work, including a short presentation for future application. This section concludes the main part of the work.

The appendix 8 provides details, skipped in the main part, such as the Fierz identity, some useful integral formulas, the used colour-flow Feynman rules and its conventions, or some algorithmic aspects for treating permutations algorithmically. Most importantly, the calculated necessary function for double soft-gluon emission is listed, including the ordered limit. Also Lorentz transformations in d dimensions are presented.

The diagrams, illustrations and flow charts in this work were either prepared with the tikz package [6] or with JaxoDraw [7].

2 Preliminaries

2.1 Quantum chromodynamics

Quantum Chromodynamics (QCD) is a quantum field theory which is invariant under a non-abelian local gauge transformation and describes the dynamics of coloured quarks and gluons. The central and defining object of the theory is the Lagrangian density [8, 9],

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} + \sum_{i=1}^{n_f} \bar{\psi}_f (i\not{D} - m_i) \psi_f, \quad (2.1)$$

where the sum runs over all possible flavours. The quark field ψ_f with flavour f transforms under the local gauge group $SU(N)$ in the following way

$$SU(N) : \psi(x) \rightarrow \psi'(x) = U(x)\psi(x) \quad \text{with} \quad U(x) \in SU(N), \quad (2.2)$$

where every element $U(x) \in SU(N)$ could be written as

$$U(x) = \exp(i\theta(x)_a t^a). \quad (2.3)$$

The $\theta(x)_a$ are local functions of the space-time point x and t^a are the generators of the Lie algebra $su(n)$ of the Lie group $SU(N)$ in the fundamental representation. The Lie algebra $su(N)$ is given by the following set of $N \times N$ matrices

$$su(N) = \{M \in gl(n, \mathbb{C}); M^\dagger = -M, \text{Tr}(M) = 0\}. \quad (2.4)$$

In general the dimension of a Lie group G is defined as the dimension of its Lie algebra $\mathfrak{g}$. From the definition (2.4) of $su(N)$ one could deduce that $\dim su(N) = N^2 - 1$, hence

$$\dim SU(N) = N^2 - 1. \quad (2.5)$$

The generators t^a of $SU(N)$ satisfy the following relations

$$[t^a, t^b] = if^{abc}t^c \quad \text{Tr}(t^a) = 0 \quad \text{Tr}(t^a t^b) = T_R \delta^{ab}, \quad (2.6)$$

where f^{abc} is the structure constant of $su(n)$ and T_R a normalization constant. If not otherwise stated, the choice throughout this work is $T_R = 1/2$. The Casimir operators are

$$f^{ade} f^{bde} = C_A \delta^{ab} \quad t^a t^a = C_F \mathbf{1} \quad \text{where} \quad C_A = N \quad C_F = \frac{N^2 - 1}{2N}. \quad (2.7)$$

Furthermore, one can establish the following frequently used relation

$$t_{ij}^a t_{kl}^a = \frac{1}{2} \left(\delta_{il} \delta_{jk} - \frac{1}{N} \delta_{ij} \delta_{kl} \right), \quad (2.8)$$

which is called Fierz identity. A proof of this relation is presented in appendix C. The first term in the Lagrange density (2.1) is the kinetic term for the quark fields A_μ^a , where the field-strength

tensor $F_{\mu\nu}^a$ is defined through

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc} A_\mu^b A_\nu^c. \quad (2.9)$$

Furthermore, the so called covariant derivative $\mathcal{D} = \gamma^\mu D_\mu$ (2.1) is given by

$$D_\mu = \partial_\mu - igA^\mu, \quad (2.10)$$

where g is the gauge coupling and $A_\mu = A_\mu^a t^a$, where $a = 1, \dots, \dim SU(N)$. The transformation law of the gluon field is chosen such that Lagrangian density (2.1) is invariant under the local gauge transformation (2.2), particularly A_μ transforms as

$$A'^\mu = U(x)A^\mu U^{-1}(x) + \frac{i}{g_s} (\partial^\mu U(x)) U(x)^{-1}. \quad (2.11)$$

As a consequence, the transformation law of the covariant derivative D_μ is given by

$$D'^\mu = U(x)D^\mu(A)U(x)^{-1}. \quad (2.12)$$

As already mentioned the first term in (2.1) is the kinematic term of the gluon field A_μ^a , which is also called gauge term. Note that other terms, which are also gauge invariant, would have been possible, terms like

$$\bar{q}(x) (D_\mu D^\mu)^2 q(x), \quad [\bar{q}(x) F_{\mu\nu} D^\mu D^\nu q(x)]^3, \quad (F_{\mu\nu} F^{\mu\nu})^4 \quad \text{etc.}, \quad (2.13)$$

but all these terms require coefficients with negative mass dimension to ensure that the action is dimensionless quantity, which would lead to non renormalizable theory. A possible term would be

$$\text{Tr}[F\tilde{F}], \quad (2.14)$$

where $\tilde{F}$ is the dual tensor with $\tilde{F} = F$, but this term implies CP-violation, which is up to a high accuracy not observed in QCD [8]. In addition to that, the term 2.14 is a surface term and would not give rise to perturbative physics and will therefore not be included [8].

To quantize the theory, gauge fixing and ghost fields have to be introduced. This is necessary due the degeneracy of the gluon fields, which comes from the gluon field configurations, which are connected via a gauge transformation. More concretely [10] every physical configuration A_{phys} is identified with a gauge orbit A' through A in the so called Faddeev-Popov procedure [11]. An orbit A' consists of all configurations which can be obtained from acting with the respective gauge transformation U on A . Configurations of A , which are connected through gauge transformations are called gauge-equivalent, hence $A_{\text{phys}} := \{A' | A' = UA\}$. The physical configuration space $\mathcal{P}$ is then

$$\mathcal{P} = \{A_{\text{phys}}\}. \quad (2.15)$$

This situation is sometimes described by the statement that $\mathcal{P}$ is the quotient space of the space of configurations $\mathcal{A} = \{A\}$, modulo the group of local gauge transformations $\mathcal{G} = \{U\}$,

$$\mathcal{P} = \mathcal{A}/\mathcal{G}. \quad (2.16)$$

A complete gauge fixing is therefore a parametrization of the quotient space by choosing a single representative from each equivalence class.

There are gauge conditions, which are not ideal (for example the Landau gauge), ideal in the sense, that the gauge conditions intersect with every gauge orbit exactly once. In principle there could be three possibilities: A gauge orbit can intersect with the gauge condition only once (L), more than once (L'), or it can have no intersection (L''), this is the so called Gribov problem [10]. In the case of situation (L') the literature refers to this by saying that Gribov copies are present. In QED no Gribov copies are present, but in non-abelian theories they can occur in certain gauges.

The whole quantization procedure of QCD is for example presented in [9]. In summary it is necessary to add a gauge fixing term, here the covariant gauge, and in order to preserve unitary ghost fields (2.1)

$$\mathcal{L}_{\text{fix+ghost}} = -\frac{1}{2\xi} (\partial_\mu A^{a\mu}) (\partial_\nu A^{a\nu}) + \partial_\mu \eta^{a\dagger} \left(\partial^\mu \delta^{ab} + g_s f_{abc} A^{c\mu} \right) \eta^b, \quad (2.17)$$

where η^a are the ghost fields which are unphysical complex valued Lorentz scalars which obey Fermi-Dirac statistics and furthermore transform under the adjoint representation of the gauge group. Different gauges are then chosen by assigning different values to ξ , often the calculations are most easily arrived in the Feynman-t'Hoft gauge where $\xi = 1$.

2.2 Renormalization

The following section aims to give a brief overview over the multiplicative renormalization program needed to develop the amplitude evolution equations. The section is based on the respective chapters about renormalization in [12], [9] and the lecture notes of "Particle Physics, Standard Model and Phenomenology" [13].

In relativistic quantum field theories complications arise due to the occurrence of UV-divergencies and IR-divergencies in the calculations of loop integrals. In order to circumvent the problem of UV divergencies a suitable regularization prescription for the gauge theory has to be found, such that the divergences can be eliminated by absorbing them into renormalization constants of unphysical quantities, such as constants and fields.

The traditional way to regularize UV divergencies was to introduce a cutoff scale Λ . The problem with this approach is that the resulting expressions are no longer Lorentz and gauge invariant. A technique to parametrize the arising infinities, while keeping the quantities Lorentz and gauge invariant is dimensional regularization (dimreg). The idea is that some integrals $I(d)$ converge in certain dimensions, while they are divergent in others, therefore to parametrize the divergences one analytically continues the integral $I(d)$ to a meromorphic function $I(d) : \mathbb{C} \rightarrow \mathbb{C}$. The analytical continuation is in general not unique, since in order to achieve uniqueness a meromorphic function must be defined at least on an open neighbourhood of a point $p \in \mathbb{C}$. The

non-uniqueness poses no problems in concrete calculations, since only the correct limits $\lim_{d \rightarrow d^*} I(d)$, where $d^* \in \mathbb{N}$, are of interest. Due to the generalization of $d \in \mathbb{C}$, one has to multiply the loop integrals by a parameter μ^{4-d} such that the correct dimension is persevered. Furthermore one chooses $d = 4 - 2\epsilon$ and $\epsilon \ll 1$. The encountered loop-integrals are therefore of the form,

$$I(d) = \mu^{4-d} \int d^d k f(k^2). \quad (2.18)$$

Extra care has to be taken with respect of the numerator which is typically proportional to a product of gamma matrices, hence the gamma matrices and their algebra also has to be generalized to d dimensions. A popular choice to do so is the so called *t'Hooft Veltman-scheme* [13, 9]. To perform the calculations of the loop integrals the arising integrals can be brought in the standard form such that following formula could be used for the evaluation

$$\int \frac{d^d k}{i\pi^{d/2}} \frac{(k^2)^n}{(k^2 - A)^m} = (-1)^{-n-m} A^{n-m+d/2} \frac{\Gamma(n + d/2) \Gamma(m - n - d/2)}{\Gamma(m) \Gamma(d/2)}. \quad (2.19)$$

The divergencies of the theory are then parametrized as $(\mu^2/Q^2)^\epsilon \cdot (\text{poles in } \epsilon)$, where Q^2 is the outer momentum scale.

Renormalization then subtracts terms of the form $-(\mu^2/\mu_R^2)^\epsilon \cdot (\text{poles in } \epsilon)$, which changes the dependence from μ to the renormalization scale μ_R , since for example

$$\left(\frac{\mu^2}{Q^2}\right)^\epsilon \frac{1}{\epsilon} - \left(\frac{\mu^2}{\mu_R^2}\right)^\epsilon \frac{1}{\epsilon} = \ln\left(\frac{\mu_R^2}{Q^2}\right) + \mathcal{O}(\epsilon). \quad (2.20)$$

Defining the structure of the divergent part is however a non-unique procedure. Different possible definitions are referred to as subtraction schemes. For example, the scheme, where only $1/\epsilon$ is subtracted is called minimal subtraction scheme. Another popular choice is the $\overline{\text{MS}}$ -scheme, where also the constant $\ln(4\pi) - \gamma_E$ gets subtracted. The $\overline{\text{MS}}$ -scheme has the advantage that it could improve the speed of convergence of the perturbative series.

A general quantity $X(\mu, \epsilon)$ in dimreg is then renormalized through the following relation

$$X(\mu, \epsilon) = Z_X(\mu, \mu_R, \epsilon) X_R(\mu_R), \quad (2.21)$$

where the left hand side of the equation shows the un-renormalized (bare) quantity and the right hand side the divergent renormalization constant Z_X and the finite renormalized quantity $X_R(\mu_R)$. The renormalized quantity can therefore be obtained by calculating the limit $X_R(\mu_R) = \lim_{\epsilon \rightarrow 0} Z_X^{-1} X_0(\mu, \epsilon)$, which implies an evolution equation in μ_R ,

$$\frac{d}{d\mu_R} X_R(\mu_R) = \frac{d}{d\mu_R} \lim_{\epsilon \rightarrow 0} Z_X^{-1} X_0(\mu, \epsilon). \quad (2.22)$$

If a cutoff Λ is used as regulator then instead of 2.21 one would start from $X(\mu, \Lambda) = Z_X(\mu, \mu_{R,\Lambda}) X_R(\mu)$ and subsequently, the limit $\Lambda \rightarrow \infty$ has to be calculated to obtain the renormalized quantity.

Expanding 2.21 in the renormalized coupling $g_R < 1$ allows one to calculate Z_X in a perturbative

manner,

$$Z_X(\mu, \mu_R, \epsilon) = 1 + \left(\frac{\mu}{\mu_R}\right)^{2\epsilon} g_R Z_X^{(1)} + \left(\frac{\mu}{\mu_R}\right)^{4\epsilon} g_R^2 Z_X^{(2)} + \dots \quad (2.23)$$

The renormalized coupling g_R itself is related to the bare coupling g through (2.21) and (2.23)

$$\begin{aligned} g(\mu, \mu_R, \epsilon) &= Z_g(\mu, \mu_R, \epsilon) g_R(\mu_R) \\ &= \left(1 + \left(\frac{\mu}{\mu_R}\right)^{2\epsilon} g_R Z_g^{(1)} + \left(\frac{\mu}{\mu_R}\right)^{4\epsilon} g_R^2 Z_g^{(2)} + \dots\right) g_R(\mu_R). \end{aligned} \quad (2.24)$$

Redefining $g_R \rightarrow (\mu_R/\mu)^{2\epsilon} g_R$ gives with the appropriate $\tilde{Z}_g$ that

$$g(\mu, \mu_R, \epsilon) \mu^{2\epsilon} = \tilde{Z}_g(\mu, \mu_R, \epsilon) \mu_R^{2\epsilon} g_R(\mu_R), \quad (2.25)$$

which is the reason why in the literature $g(\mu, \mu_R, \epsilon) \mu^{2\epsilon}$ is sometimes referred to as the bare coupling and $\mu_R^{2\epsilon} g_R(\mu_R)$ as the renormalized coupling in dimensional regularisation. Through the expansion in 2.23 the Lagrange density could be written as a sum of the renormalized Lagrangian $\mathcal{L}_{\text{ren}}$ and the counter term $\mathcal{L}_{\text{ct}}$

$$\mathcal{L}_{\text{bare}} = \mathcal{L}_{\text{ren}} + \mathcal{L}_{\text{ct}}. \quad (2.26)$$

The $\mathcal{L}_{\text{ct}}$ term then generates the counter-term Feynman rules which allow one to obtain finite results. In general, a theory is called renormalizable if only a finite number of counter terms is needed for the renormalization procedure.

QCD is such a renormalizable theory, where the necessary renormalization transformations are

$$\begin{aligned} A_\mu^a &= Z_3^{1/2} A_{\mu,R}^a, \\ \psi &= Z_2^{1/2} \psi_R, \\ \eta_\mu &= Z_\eta^{1/2} \eta_{\mu,R}, \\ g &= Z_g g, \\ m &= Z_m m_R, \end{aligned} \quad (2.27)$$

which allows, through the expansion in the renormalized coupling to derive $\mathcal{L}_{\text{ren}}$ and $\mathcal{L}_{\text{ct}}$ and therefore all Feynman rules, including the counter term Feynman rules [12].

To relate the renormalized quantity $X_R(\mu)$ to different energy-scales μ_R , its renormalization group equation (RGE) 2.22 has to be solved. How to do so will be demonstrated in the following for the n-point Greens function G . Note that the same principle will be used to solve the RGE for the cross section.

The bare Green's function G is related to the renormalized Greens function G_R as

$$G(g_0, m_0, \{p_i\}_{i=1}^n) = Z_\varphi^{-n/2} G_R(g_R(\mu_R), m_R(\mu_R), \mu_R, \{p_i\}_{i=1}^n), \quad (2.28)$$

where μ is the mass regulator (*t'Hooft*) and $Z_\varphi^{-n/2}$ the field renormalization. The evolution equation of the Green's function is

$$\left(\mu_R \frac{\partial}{\partial \mu_R} + \beta_g \frac{\partial}{\partial g_R} + m_R \gamma_m \frac{\partial}{\partial m_R} - n \gamma_\varphi \right) G_R(g_R(\mu_R), m_R(\mu_R), \mu_R, \{p_i\}_{i=1}^n) = 0, \quad (2.29)$$

which is called the Callan-Symanzyk equation. The introduced quantities are defined through

$$\begin{aligned} \beta_g &= \mu_R \frac{\partial g_R}{\partial \mu_R}, \\ \gamma_m &= \frac{\mu_R}{m_R} \frac{\partial m_R}{\partial \mu_R}, \\ \gamma_\varphi &= \frac{1}{2} Z_\varphi^{-1} \mu_R \frac{\partial}{\partial \mu_R} Z_\varphi. \end{aligned} \quad (2.30)$$

The expression β_g is the beta function of g , γ_m the anomalous dimension and γ_φ the field anomalous dimension. Performing a variable transformation such that

$$d t = \frac{d \bar{\mu}_R}{\bar{\mu}} = \frac{d \bar{g}_R}{\beta(\bar{g}_R, \bar{m}_R/\bar{\mu}_R, \epsilon)} = \frac{d \bar{m}_R}{\bar{m}_R \gamma_m(\bar{g}_R, \bar{m}_R, \epsilon)}, \quad (2.31)$$

with the initial conditions $\bar{\mu}_R(t=0) = \mu_R$, $\bar{g}_R(0) = g_R$, $\bar{m}_R(t=0) = m_R$ simplifies the Callan-Symanzyk equation into

$$\left(\frac{d}{d t} - n \gamma_\varphi \right) G_R(\bar{g}_R, \bar{m}_R, \bar{\mu}_R, \{p_i\}_{i=1}^n) = 0. \quad (2.32)$$

Integrating the differential equation then gives the solution

$$G_R(g_R, m_R, \mu_R, \{p_i\}_{i=1}^n) = \exp \left(-n \int_{g_R}^{\bar{g}_R} d \xi \frac{\gamma_\varphi(x)}{\beta_g(x)} \right) G_R(\bar{g}_R, \bar{m}_R, \bar{\mu}_R, \{p_i\}_{i=1}^n). \quad (2.33)$$

2.3 Generating Random Variables

In this chapter the basic techniques and notions to generate random variables, that are distributed according to a general probability distribution π , will be introduced. At first, one has to be able to generate random variables distributed uniformly on the interval $[0, 1]$. To generate a sequence of true random numbers on a deterministic machine is of course not possible, but what the computer can do is to generate pseudo-random variables. Such a program is called a uniform pseudo-random number generator and is an algorithm that starts from an initial value, often referred to as seed and generates, with a large periodicity, a sequence of numbers in the interval $[0, 1]$, which mimics, a sample of independent identically distributed (i.i.d.) uniformly distributed random variables $(X_1, X_2, \dots)$. Such programs are widely available and will be taken for granted. The following chapter is mainly based on the book [14], chapter two and on the book [15], chapter four.

The simplest techniques to produce non-uniform random variables is the inversion method, which is in general applicable for one-dimensional distribution function f if its cumulative distribution function (cdf) F is known and invertible.

Theorem 1. Let F be the one-dimensional invertible cdf and $U \sim \text{Uniform}[0, 1]$ a random variable, then the random variable $X = F^{-1}(U)$ is distributed according to F . The inverse is defined as $F^{-1}(u) = \inf\{x \in \mathbf{R}; F(x) \geq u\}$.

Proof 1. Let $U(0, 1)$ be the uniform distribution on $[0, 1]$ and $f = F'$ the distribution function of the cdf, then

$$f(x)dx = U(0, 1)du. \quad (2.34)$$

Integrating both sides gives

$$\int_{-\infty}^x f(z)dz = F(x) = u. \quad (2.35)$$

Due to the assumption that F is invertible, it is possible to solve for x ,

$$x = F^{-1}(u). \quad (2.36)$$

□

Example. Consider the probability density $f(x) : (0, 1) \rightarrow \mathbb{R}$, $f(x) = 2x$, then $u = \int_0^x 2zdz = x^2$, hence $x = \sqrt{u}$. Another example is the exponential distribution $g(x, \lambda) = \lambda \exp(-\lambda x)$, then $u = \int_0^x \lambda \exp(-\lambda x) = 1 - \exp(-\lambda x)$. Solving for x gives $x = -\frac{1}{\lambda} \ln(1 - u)$. When the sequence u_j is evenly distributed, then $1 - u_j$ as well, hence the exponentially distributed sequence is given by $x_j = -\frac{1}{\lambda} \ln(u_j)$.

In practice the assumption that the cdf exists or is invertible does not always hold. In such cases the so called rejection method could be applied. The method is at least in principle applicable for any density function, known up to a normalization constant.

Theorem 2. Let $f(x) = c\pi(x)$ be computable and $\pi(x)$ a probability density function, whereas the constant c could be in principle unknown. Then, if there exists a probability distribution $g(x)$ and a constant $M > 0$, where it is possible to draw samples according to $g(x)$ and $Mg(x) \geq f(x)$ for all x , then the following algorithm allows to sample according to the target probability distribution $\pi(x)$.

(I) Propose a sample x according to $g(x)$ and compute the ratio r ,

$$r = \frac{f(x)}{Mg(x)}. \quad (2.37)$$

(II) Accept the proposal x with probability r , if the proposal gets rejected return to (I).

Proof 2. To show the correctness of the algorithm, let's define that $I = 1$, if the proposed event was excepted and $I = 0$ otherwise, then the probability to accept the sample is

$$P(I = 1) = \int P(I = 1|X = x) g(x) dx = \int \frac{f(x)}{Mg(x)} g(x) dx = \frac{c}{M}. \quad (2.38)$$

The distribution of the accepted events $p(x|I = 1)$ can then be deduced by noting

$$p(x|I = 1)P(I = 1) = \frac{f(x)}{Mg(x)} g(x), \quad (2.39)$$

hence

$$p(x|I = 1) = \pi(x). \quad (2.40)$$

□

The algorithm is 100% efficient in the sense that it terminates with probability one. From (2.38), where of course $M \geq c$, it is evident that for an efficient implementation of the algorithm it is beneficial to find a trial distribution $g(x)$ such that the constant M is small and furthermore $g(x)$ is similar to the target distribution. In most of the practical implementations the rejection method is rather difficult to implement, at least for high-dimensional distributions.

2.4 Markov Chains and Monte Carlo integration

In this chapter Markov chain Monte Carlo (MCMC) methods will be introduced. In its essence MCMC is Monte Carlo integration while using the framework of Markov chains. The necessary definitions are give in the following. The section is based on the introductory chapters of [16] and [17].

Definition 1. Let T be an index set, where T is usually $\mathbb{N}$ or $\mathbb{R}$, representing time, then a sequence of random variables $\{X_t\}_{t \in T}$ is called a stochastic process. A stochastic process $X = (X_n)_{n \in \mathbb{N}}$ is then called a Markov chain, if the probability to transition from state X_n^* to X_{n+1}^* , denoted by $P(X_{n+1} = X_{n+1}^* | X_n = X_n^*)$ satisfies

$$P(X_{n+1} = X_{n+1}^* | X_n = X_n^*) = P(X_{n+1} = X_{n+1}^* | X_n = X_n^*, \dots, X_0). \quad (2.41)$$

A more intuitive picture to the mathematical definition is that given a state X_n^* , the probability to be in the next state X_{n+1}^* does not depend on the full history of the chain. Note that the given definition covers discrete Markov chains as well as continuous Markov chains. In the literature $P(\cdot|\cdot)$ is also referred to as transition probability of the chain, for all $n \in \mathbb{N}$ the transition probabilities satisfies

$$0 \leq P(X_{n+1} = X_{n+1}^* | X_n = X_n^*) \leq 1 \quad \text{and} \quad \int dX_{n+1}^* P(X_{n+1} = X_{n+1}^* | X_n = X_n^*) = 1. \quad (2.42)$$

The probability to be in step n in the state X_n^* , while starting from state X_0^* , is then given by

$$P(X_n^*) = P(X_n^* | X_0 = X_0^*). \quad (2.43)$$

Then, the probability to be in step $n + 1$ in state X_{n+1}^* is given by the following recursion relation

$$P_{n+1}(X_{n+1}^*) = \int dX_n^* P(X_{n+1}^* | X_n^*) P(X_n^*), \quad (2.44)$$

where the integral becomes a sum in the case of discrete Markov chains.

Now with some knowledge of random variables and Markov chains the notion of Monte Carlo integration can be given. The expectation value $E_\pi[f(x)]$ of some integrable function $f(x)$, with respect to the probability density $\pi(x)$, is defined as

$$E_\pi[f(x)] = \int_V f(x) \pi(x) dx, \quad (2.45)$$

where V denotes the domain of the probability density π , therefore $\int_V \pi(x) dx = 1$. Then, with the shortcut notation $\text{Vol}(V) = \int_V dx$, the integral I could be rewritten as

$$I = \int_V f(x) dx = \text{Vol}(V) \int_V \frac{1}{\text{Vol}(V)} f(x) dx = \text{Vol}(V) E_\pi[f(x)], \quad (2.46)$$

where the probability density $\pi(x)$ is the uniform distribution on the integration volume V . Monte Carlo integration techniques then approximate the expectation value $E_\pi[f(x)]$, from (2.45), by drawing samples $\{X_i\}_{i=1}^n$, with $X_i \sim \pi(x)$, by

$$E_\pi[f(x)] \approx \frac{1}{n} \sum_{i=1}^n f(X_i). \quad (2.47)$$

In the following the estimate for the integral I will be denoted by $\hat{I}_n$ and is, as argued before, given by

$$I \approx \hat{I}_n := \text{Vol}(V) \frac{1}{n} \sum_{i=1}^n f(X_i). \quad (2.48)$$

To quantify the quality of the estimate I_{MC} the variance σ_I^2 , can be calculated. Using the property that if $X = \sum_{i=1}^n a_i X_i$ then $\text{Var}[X] = \sum_{i=1}^n a_i^2 \text{Var}[X_i]$, and the assumption that for the identical distributed random variables X_i the variance is finite, it follows that

$$\text{Var}[I_{\text{MC}}] = \left(\frac{\text{Vol}(V)}{n} \right)^2 \sum_{i=1}^n \text{Var}[f(X_i)] = \frac{\text{Vol}(V)^2}{n} \text{Var}[f(X_i)]. \quad (2.49)$$

The variance is defined through $\text{Var}[X] = E[(X - \bar{X})^2]$, therefore an estimate for $\text{Var}[f(X_i)]$ is given by

$$\text{Var}[f(X_i)] = E\left[\left(f(X_i) - \overline{f(X_i)}\right)^2\right] \approx \left[\frac{\sum f(X_i)^2}{n} - \left(\frac{\sum f(X_i)}{n}\right)^2\right]. \quad (2.50)$$

In total the variance of I_{MC} reads

$$\text{Var}[I_{\text{MC}}] = \frac{\text{Vol}(V)^2}{n} \left[\frac{\sum f(X_i)^2}{n} - \left(\frac{\sum f(X_i)}{n} \right)^2 \right] = \frac{1}{n} \left[\text{Vol}(V)^2 \frac{\sum f(X_i)^2}{n} - I_{\text{MC}}^2 \right] \xrightarrow{n \rightarrow \infty} 0. \quad (2.51)$$

This is one particular case of the law of large numbers, a stronger version states that the average of random variables, denoted by $\hat{\mu}_n$, with identical mean μ and finite variances tends to μ as n goes to infinity

$$\lim_{n \rightarrow \infty} \hat{\mu}_n = \mu \quad \text{with probability 1.} \quad (2.52)$$

From the central limit theorem it follows that

$$\sqrt{n} (\hat{I}_n - I) \xrightarrow{n \rightarrow \infty} N(0, \sigma^2) \quad \text{in distribution.} \quad (2.53)$$

The error of the Monte Carlo approximation is therefore $\mathcal{O}(\sqrt{n})$, independent of the dimensionality of the integration domain, provided that one samples from the Gaussian regime and n large enough. A classical counterexample, where the law of large number is not applicable, is $f(x) = \frac{1}{\sqrt{x}}$ integrated on the interval $[0, 1]$. Since $f(x)$ is not square integrable on this interval, the variance is not finite. The un-applicability of the law of large numbers does then not mean that one could not use Monte Carlo integration successfully to approximate the integral of $f(x)$ on $[0, 1]$.

Although the presented Monte Carlo integration method converges already faster than conventional Riemann sums in high dimensions ($d \geq 4$) one could easily come up with examples, where computation time is wasted due to the evaluation of points on which the function is nearly zero. The solution to that problem is to sample from regions, which contribute to the integral more often, this technique is called importance sampling. The procedure goes as follows, consider again problem (2.45) and rewrite it with the probability density $g(x)$ as

$$I = E_\pi[f(x)] = \int_V f(x) \frac{\pi(x)}{g(x)} g(x) \, dx = E_g \left[f(x) \frac{\pi(x)}{g(x)} \right], \quad (2.54)$$

where $g(x)$ has the same support V as $\pi(x)$. Then draw $\{X_i\}_{i=1}^n$, where $X_i \sim g(x)$ and calculate the weights $w^{(i)}$, defined as

$$w^{(i)} := \frac{\pi(X_i)}{g(X_i)}, \quad \text{for } i = 1, \dots, n \quad (2.55)$$

and approximate the expectation value (2.45) with

$$E_\pi[f(x)] \approx \frac{1}{n} \sum_{i=1}^n w^{(i)} f(X_i). \quad (2.56)$$

Importance sampling is of course only beneficial if it is computationally cheap to sample from the trial distribution $g(x)$. Another advantage of that method is that the variance of the estimation could be significantly reduced if the shape of $g(x)$ is close to the integrand $f(x)\pi(x)$. This is also the reason why in the literature this method is sometimes referred to as variance reduction method.

3 Soft gluon evolution

3.1 QCD-Factorization

QCD factorization formulas appear naturally in soft or collinear limits and provide a tool to describe realistic initial or final states iteratively. They play, not only, a crucial role in the development of parton showers but also in the development of soft-collinear effective theory (SCET). At first, the eikonal approximation will be discussed, then the resulting factorization formulas for the single and double gluon emission case. The main sources for this section are [8] and [2].

The eikonal approximation and eikonal current are the main ingredient to describe the factorization property of the tree-level matrix element $\mathcal{M}_{g,a_1,\dots,a_n}(q,p_1,\dots,p_n)$ in the soft limit $g \rightarrow 0$ and are consequently the main ingredient to describe the evolution of the amplitude in the case of gluon emission only, see section 3.2, at leading order. The arising singularities of the soft limits are so called infrared (IR) singularities, the other type of IR singularities are collinear singularities. In sufficiently inclusive observables singularities, arising from soft gluon emission, will cancel each other. In not inclusive measurements the cancellation is incomplete, which then leads, due to logarithmic enhancement, to a larger emission probability of the soft gluon. Soft gluons are, due to their large wavelength only sensitive to the global structure of the event and consequently only dependent on the hard momenta of the final partons and their colour connections. In the case of soft photons this phenomenon is described by Low's theorem [18] with the difference that in that case the factorization is exact, due to the lack of charge correlations. To demonstrate the described property in the QCD case let's consider N hard final state partons with the set of momenta $\{p_i^\mu\}_{i=1}^N$, where in addition also all relative transverse momenta are hard. With these conditions the arising soft singularities can only have their origin from the soft gluon, emitted from the i -th external hard parton. The situation is depicted in figure 1, the grey blob represents the process with $N - 1$ hard final partons, where each parton does not emit a soft gluon, whereas the line, labelled with i represents the hard parton with hard momenta $q + k$, which emits the soft gluon with momenta k and colour label a .

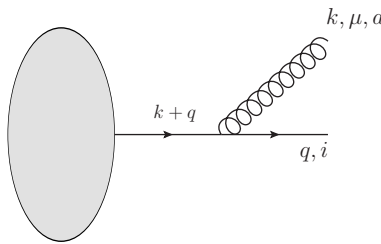


Figure 1: Soft gluon emission of a final state, consisting of N hard partons.

Applying the Feynman rules from appendix E to the described situation gives for the amplitude of $N + 1$ partons

$$i\mathcal{M}_i^{N+1}(q, k) = ig_s t_{ij}^a \epsilon^*(k)_\mu \bar{u}(q) \gamma^\mu \frac{(\not{k} + \not{q} + m)}{(q + k)^2 - m^2 + i0} \mathcal{M}_j^N(q + k). \quad (3.1)$$

Using the commutator relations of the gamma matrices to interchange γ^μ with the slashed four momenta in order to be able to apply the Dirac equation $(\not{p} - m)u(p) = 0$ results in

$$i\mathcal{M}_i^{N+1}(q, k) = ig_s t_{ij}^a \epsilon^*(k)_\mu \frac{(2q^\mu + \gamma^\mu \not{k})}{2k \cdot q} \mathcal{M}_j^N(q + k). \quad (3.2)$$

The soft gluon approximation is then performed by replacing the four momenta of the gluon k by λk , where $\lambda > 0$ and letting $\lambda \rightarrow 0$. Furthermore, Taylor expanding the amplitude of the N partons $\mathcal{M}_j^N(q + \lambda k) = \mathcal{M}_j^N(q) + \mathcal{O}(\lambda)$ gives with the previous result

$$i\mathcal{M}_i^{N+1}(q, k) = ig_s t_{ij}^a \frac{1}{\lambda} \frac{\epsilon^*(k) \cdot q}{k \cdot q} \mathcal{M}_j^N(q) + \mathcal{O}(\lambda^0). \quad (3.3)$$

The eikonal approximation is then the leading term in the particular limit $\lambda \rightarrow 0$. Summing over all external hard partons in the eikonal approximation and with the definition of the colour charge operator $\mathbf{T}_i$, introduced in section 4, gives

$$i\mathcal{M}^{N+1}(q, k) = ig_s \epsilon^*(k)_\mu \left(\sum_{i=1}^N \mathbf{T}_i^a \frac{q^\mu}{k \cdot q} \right) \mathcal{M}_j^N(q). \quad (3.4)$$

The factorization formula 3.4 was here derived in a concrete basis, concretely in the trace-basis, but the statement is of course basis independent. Keeping this in mind and generalizing to $d = 4 - 2\epsilon$ space-time dimension, the factorization formula can be brought in the following form [2]

$$\langle c; \mu | \mathcal{M}_{g, a_1, \dots, a_n}(q, p_1, \dots, p_n) \rangle \simeq g_s \mu^\epsilon J^{c; \mu}(q) | \mathcal{M}_{a_1, \dots, a_n}(p_1, \dots, p_n) \rangle, \quad (3.5)$$

where c and μ a colour and spin index respectively and $\simeq$ means equality up to terms which are less singular than $1/q$. The factor $\mathbf{J}^\mu(q)$ is then referred to as eikonal current, defined as

$$\mathbf{J}^\mu(q) = \sum_{i=1}^n \mathbf{T}_i \frac{p_i^\mu}{p_i \cdot q}, \quad (3.6)$$

which depends on the momenta and the colour charges, through the already introduced colour operator $\mathbf{T}_i$ of the hard partons in the considered matrix element. An important property of the eikonal current is that for a colour-singlet state $| \mathcal{M}_{a_1, \dots, a_n}(p_1, \dots, p_n) \rangle$ colour is conserved

$$q_\mu \mathbf{J}^\mu(q) | \mathcal{M}_{a_1, \dots, a_n}(p_1, \dots, p_n) \rangle = \sum_{i=1}^n \mathbf{T}_i | \mathcal{M}_{a_1, \dots, a_n}(p_1, \dots, p_n) \rangle = 0. \quad (3.7)$$

This is an equivalent way to say that the amplitudes are gauge invariant, which then also implies that the sum over all soft gluon polarizations of the squared amplitude is gauge invariant. Although the squared amplitude is strictly speaking not particularly important for this work, since this work aims to show how the amplitude could be simulated directly, the most important fact will be summarized in the following for completeness. For that it is useful to introduce the gluon polarization tensor $d_{\mu\nu}(q) = (-\eta_{\mu\nu} + \text{gauge terms})$, which yields

$$(\mathbf{J}^\mu(q))^\dagger d_{\mu\nu}(q) \mathbf{J}^\nu(q) = - \sum_{i,j=1}^n \mathbf{T}_i \cdot \mathbf{T}_j \frac{p_i \cdot p_j}{(p_i \cdot q)(p_j \cdot q)} + \dots, \quad (3.8)$$

where in the dots terms are only proportional to q^μ or q^ν , due to longitudinal polarizations, hence these terms are proportional to the total charge $\sum_i \mathbf{T}_i$ and when they act on a colour singlet state they vanish. Therefore, the squared amplitude factorizes in the following way

$$|\mathcal{M}_{g,a_1,\dots,a_n}(q,p_1,\dots,p_n)|^2 \simeq -4\pi\alpha_s\mu^{2\epsilon} \sum_{i,j=1}^n \mathcal{S}_{ij}(q) |\mathcal{M}_{a_1,\dots,a_n}^{i,j}(p_1,\dots,p_n)|^2, \quad (3.9)$$

where the scalar eikonal function for the emission of a single gluon $\mathcal{S}_{ij}(q)$ was introduced and is defined as

$$\mathcal{S}_{ij}(q) = \frac{p_i \cdot p_j}{(p_i \cdot q)(p_j \cdot q)}, \quad (3.10)$$

or in terms of the sub-energies $s_{ij} = (p_i + p_j)^2$

$$\mathcal{S}_{ij}(q) = \frac{2s_{ij}}{s_{iq}s_{jq}}. \quad (3.11)$$

The $|\mathcal{M}_{a_1,\dots,a_n}^{i,j}(p_1,\dots,p_n)|^2$ are the square of the colour-correlated tree amplitudes and defined as

$$\begin{aligned} |\mathcal{M}_{a_1,\dots,a_n}^{i,j}(p_1,\dots,p_n)|^2 &= \langle \mathcal{M}_{a_1,\dots,a_n}(p_1,\dots,p_n) | \mathbf{T}_i \cdot \mathbf{T}_j | \mathcal{M}_{a_1,\dots,a_n}(p_1,\dots,p_n) \rangle \\ &= \left(\mathcal{M}_{a_1,\dots,a_n}^{c_1,\dots,b_i\dots b_j\dots c_n}(p_1,\dots,p_n) \right)^* t_{b_i d_i}^c t_{b_j d_j}^c \mathcal{M}_{a_1,\dots,a_n}^{c_1,\dots,d_i\dots d_j\dots c_n}(p_1,\dots,p_n), \end{aligned} \quad (3.12)$$

they take, as the name suggests, the colour correlation, produced by the emitted soft-gluon into account.

Now the double soft gluon case will be discussed, for that the double soft gluon emission current $J_{\mu\nu}^{a_1 a_2}$ gets introduced. The double soft gluon emission current describes the situation when the momenta of two gluons become simultaneously soft. This particular limit was for the first time independently studied in [19] and [20]. Concretely, let's consider the QCD tree-amplitude $\mathcal{M}_{g,g,a_1,\dots,a_n}(q_1,q_2,p_1,\dots,p_n)$, where the momenta of the gluons become simultaneously soft. The limit is obtained by rescaling the momenta q_1 and q_2 , of the two gluons

$$q_1 \rightarrow \lambda q_1, \quad q_2 \rightarrow \lambda q_2, \quad (3.13)$$

and then letting $\lambda \rightarrow 0$. In that limit the matrix elements then fulfil [2] the following factorization formula

$$\langle a_1, a_2; \mu_1, \mu_2 | \mathcal{M}_{g,g,a_1,\dots,a_n}(q_1, q_2, p_1, \dots, p_n) \rangle \simeq g_S^2 \mu^{2\epsilon} J_{\mu_1 \mu_2}^{a_1 a_2}(q_1, q_2) |\mathcal{M}_{a_1,\dots,a_n}(p_1, \dots, p_n)\rangle, \quad (3.14)$$

where $J_{\mu_1 \mu_2}^{a_1 a_2}$ is the double soft gluon emission current, a_1, a_2 colour indices and μ_1, μ_2 are the Lorentz indices. The two-gluon soft current reads

$$\begin{aligned}
J_{\mu_1 \mu_2}^{a_1 a_2} = & \sum_{i \neq j} \mathbf{T}_i^{a_1} \frac{p_i^{\mu_1}}{p_i \cdot q_1} \mathbf{T}_j^{a_2} \frac{p_j^{\mu_2}}{p_j \cdot q_2} \\
& + \sum_i \left[\left(\delta^{a_1 a} \mathbf{T}_i^{a_2} \frac{p_i^{\mu_2}}{p_i \cdot q_2} - i f^{a_2 a_1 a} \frac{q_1^{\mu_2}}{q_1 \cdot q_2} \right) \mathbf{T}_i^a \frac{p_i^{\mu_1}}{p_i \cdot (q_1 + q_2)} + \right. \\
& \left(\delta^{a_2 a} \mathbf{T}_i^{a_1} \frac{p_i^{\mu_1}}{p_i \cdot q_1} - i f^{a_1 a_2 a} \frac{q_2^{\mu_1}}{q_1 \cdot q_2} \right) \mathbf{T}_i^a \frac{p_i^{\mu_2}}{p_i \cdot (q_1 + q_2)} + \\
& \left. + \frac{1}{2} i f^{a a_1 a_2} \mathbf{T}_i^a \frac{g^{\mu_1 \mu_2}}{q_1 \cdot q_2} \frac{p_i \cdot (q_2 - q_1)}{p_i \cdot (q_2 + q_1)} \right]. \tag{3.15}
\end{aligned}$$

The double soft limit (3.13) is more general than the previously considered strongly ordered limit, where $p_i \gg q_1 \gg q_2 \quad \forall i \in \{1, \dots, n\}$, and reproduces correctly the single- and double-logarithmic soft singularities, whereas the iterated and therefore strongly ordered single gluon emission current only reproduces the double-logarithmic soft singularities correctly. The current (3.15) could be for example derived while using a physical gauge and by applying soft-gluon insertion technique, described in [2]. The contributions, produced from the current $J_{\mu\nu}^{a_1 a_2}$, are depicted in figure 2.

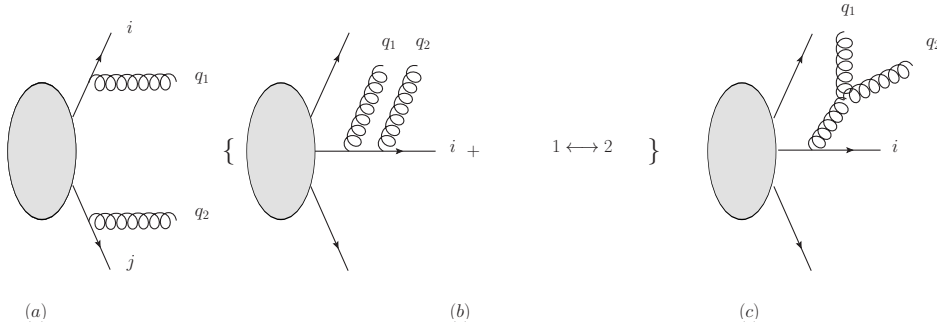


Figure 2: Diagrams produced by the double soft gluon emission current $J_{\mu_1 \mu_2}^{a_1 a_2}$.

Diagram (a) originates from the first line in (3.15), (b) from the second and third line and (c) from the fourth line, which has its origin from the non-abelian character of the colour algebra. From (3.15) the following three properties [2] of the current $J_{\mu_1 \mu_2}^{a_1 a_2}$ could be derived:

- The current $J_{\mu_1 \mu_2}^{a_1 a_2}$ is invariant under the exchange of the two soft gluons

$$J_{\mu_1 \mu_2}^{a_1 a_2}(q_1, q_2) = J_{\mu_2 \mu_1}^{a_2 a_1}(q_2, q_1), \tag{3.16}$$

which is an expected property, since in this particular limit the emitted gluons have no ordering.

- The divergences of $J_{\mu_1 \mu_2}^{a_1 a_2}$ are proportional to the total colour charge

$$q_1^{\mu_1} J_{\mu_1 \mu_2}^{a_1 a_2} = \left(J_{\mu_2}^{a_2}(q_2) \delta_{a_1 a} + \frac{i}{2} f_{a_1 a_2 a} \frac{q_1^{\mu_2}}{q_1 \cdot q_2} \right) \sum_{i=1}^{n-2} \mathbf{T}_i^a \tag{3.17}$$

$$q_2^{\mu_2} J_{\mu_1 \mu_2}^{a_1 a_2} = \left(J_{\mu_1}^{a_1}(q_1) \delta_{a_2 a} + \frac{i}{2} f_{a_2 a_1 a} \frac{q_2^{\mu_1}}{q_1 \cdot q_2} \right) \sum_{i=1}^{n-2} \mathbf{T}_i^a, \tag{3.18}$$

from which it follows that current $J_{\mu_1 \mu_2}^{a_1 a_2}$ is conserved and also that the resulting amplitudes are gauge invariant.

- In the ordered limit $q_2 \ll q_1$ the current $J_{\mu_1 \mu_2}^{a_1 a_2}$ becomes

$$J_{\mu_1 \mu_2}^{a_1 a_2} \xrightarrow{q_2 \ll q_1} \left(J_{\mu_2}^{a_2} \delta^{a_1 a} + i f_{a_1 a_2 a} \frac{q_1^{\mu_2}}{q_1 \cdot q_2} \right) J_{\mu_1}^a(q_1), \quad (3.19)$$

which is a product of two eikonal currents, that match the result if the leading-order factorization formula gets applied. Note that 3.19 is only the leading contribution in the limit $\lambda \rightarrow 0$ after replacing q_2 with $\lambda^2 q_2$ and q_1 with λq_1 . The last term in 3.15 vanishes in the strongly ordered limit due to colour conservation.

In [2] also the resulting factorization formulae for the squared amplitude are given, they could also be derived in the same fashion as in the case of single soft gluon emission. But for the latter applied algorithm, described in chapter 3.2, the formula for the squared amplitude is not necessary, since the algorithm describes the evolution of the amplitude $\mathcal{M}$ rather than from the absolute value $|\mathcal{M}|^2$.

3.2 Soft gluon evolution equations

In the following the algorithm to calculate the cross section for the emission of n soft partons, at leading order, is presented [4]. The algorithm assumes energy ordering in successive emission energies $E_1 \geq E_2 \geq \dots E_n$ of the emitted soft partons. The algorithm in this section is the basis to generate partonic events within a Monte Carlo simulation.

The idea is to start from a hard scattering operator [5]

$$\mathbf{H}(Q) \in (|\text{colour}\rangle \otimes |\text{spin}\rangle) \otimes (\langle \text{spin}| \otimes \langle \text{colour}|), \quad (3.20)$$

which is the initial condition of the evolution and Q is the total external momentum, and then to dress up the initial state with soft partons, either with 1-loop virtual exchanges or 1-real emissions. The process of dressing up the hard scattering matrix is then described, as suggested by the soft factorization formula, by applying successfully the Sudakov operator $\mathbf{V}_{ab}$, which describes virtual exchanges at 1-loop and then the real 1-emission operator $\mathbf{D}_i^\mu$.

More concretely the algorithm goes as following, the operator $\mathbf{A}_n = |\mathcal{M}_n\rangle \langle \mathcal{M}_n|$ denotes the state with n emitted soft partons and is defined recursively through

$$\mathbf{A}_n(E) = \mathbf{V}_{E, E_n} \mathbf{D}_n^\mu \mathbf{A}_{n-1}(E_n) \mathbf{D}_{n\mu}^\dagger \mathbf{V}_{E, E_n}^\dagger \Theta(E_n - E), \quad (3.21)$$

with the initial condition

$$\mathbf{A}_0(\mu) = \mathbf{V}_{\mu, Q} \mathbf{H}(Q) \mathbf{V}_{\mu, Q}^\dagger, \quad (3.22)$$

which is the initial hard scattering matrix $\mathbf{H}(Q)$ dressed up with one virtual exchange.

The $\Theta(E_n - E)$ in 3.21 denotes a Heaviside function, which ensures energy ordering in successive

emissions of the soft partons. The differential cross section is then given by

$$\begin{aligned}\sigma_0 &= \text{Tr}[\mathbf{A}_0(\mu)], \\ d\sigma_n &= \text{Tr}[\mathbf{A}_n(\mu)] \prod_{i=1}^n d\Pi_i.\end{aligned}\tag{3.23}$$

In the eikonal approximation the real emission operator $\mathbf{D}_i^\mu$ and the Sudakov operator $\mathbf{V}_{a,b}$ are given by

$$\begin{aligned}\mathbf{D}_i^\mu &= \sum_j \mathbf{T}_j E_i \frac{p_j^\mu}{p_j \cdot q_i}, \\ \mathbf{V}_{a,b} &= \mathcal{P} \left\{ \exp \left(- \int_a^b \frac{dE}{E}, \mathbf{\Gamma}_n(E) \right) \right\}\end{aligned}\tag{3.24}$$

whereas the sum in the real emission operator goes over all partons, $\mathbf{T}_j$ denotes the colour charge operator and $\mathbf{\Gamma}_n$ the anomalous dimension matrix. By differentiating the recursion relation (3.21) with respect to the energy E and multiplying with E one obtains the following evolution equation

$$E \frac{\partial \mathbf{A}_n(E)}{\partial E} = \mathbf{\Gamma}_n(E) \mathbf{A}_n(E) + \mathbf{A}_n(E) \mathbf{\Gamma}_n^\dagger(E) - \mathbf{D}_n^\mu \mathbf{A}_{n-1}(E) \mathbf{D}_{n\mu}^\dagger E \delta(E - E_n).\tag{3.25}$$

The evolution equation could then be generalized to higher orders and more emission, this was done in [21], the generalization reads

$$E \frac{\partial \mathbf{A}_n(E)}{\partial E} = \mathbf{\Gamma}_n(E) \mathbf{A}_n(E) + \mathbf{A}_n(E) \mathbf{\Gamma}_n^\dagger(E) - \sum_k \mathbf{R}_n^{(k)}(E) \mathbf{A}_{n-k}(E) \mathbf{R}_n^{(k)\dagger}(E).\tag{3.26}$$

Here $\mathbf{R}_n^{(k)}$ is the generalized emission operator, describing how from a $n - k$ parton state k partons get emitted. One major goal of this work is to find the explicit expression for the double emission operator $\mathbf{R}_n^{(2)}$.

To start with the postulation of the evolution equation of the operator $\mathbf{A}_n$ would have been equivalent to the path that was taken in this section, since the recursive definition of the operator with the chosen initial condition solves by construction the evolution equation.

The operator $\mathbf{A}_n$ itself has still IR singularities and therefore the multiplicative renormalization procedure, as discussed in section 2.2, has to be applied. How this can be done is shortly presented in the following and based on unpublished notes [22] of Simon Plätzer.

Let's denote the renormalized operator of $\mathbf{A}_n$ with $\mathbf{M}_n$, they are related through

$$\mathbf{M}_n = \mathbf{Z}_n \mathbf{A}_n \mathbf{Z}_n^\dagger + \sum_{k=1}^n \alpha_S^k \mathbf{E}_n^{(k)} \mathbf{A}_{n-k} \mathbf{E}_n^{(k)\dagger} \Theta_{n,k},\tag{3.27}$$

where $\mathbf{Z}_n$ is the renormalization constant. The operator $\mathbf{M}_n$ itself is independent of the renormalization scale μ_R and the resolution scale S , since it is by construction a renormalization group invariant. The operators $\mathbf{Z}_n$, $\mathbf{E}_n^{(k)}$ have the following expansions

$$\mathbf{Z}_n = 1 + \sum_{k \geq 1} \alpha_S^k \mathbf{Z}_n^k \quad \mathbf{E}_n^k = \sum_{j \geq 0} \alpha_S^j \mathbf{E}_n^{(k,j)}.\tag{3.28}$$

Similarly for the anomalous dimension operator and the emission operator

$$\mathbf{\Gamma}_{S,n} = \sum_{k \geq 1} \alpha_S^k \mathbf{\Gamma}_{S,n} \quad \mathbf{R}_{S,n}^{(s)} \circ \mathbf{R}_{S,n}^{(s)\dagger} = \sum_{k \geq 0} \alpha_S^k \mathbf{R}_{S,n}^{(s,k)} \circ \mathbf{R}_{S,n}^{(s,k)\dagger}. \quad (3.29)$$

The d -dimensional β -function has the expansion

$$\partial_S \alpha_S = \alpha_S \sum_{k \geq 0} \beta_{S,k-1} \alpha_S^k \quad \text{where} \quad \beta_{R,-1} = -2\epsilon. \quad (3.30)$$

In this work only the evolution with respect to the resolution scale is of interest, hence the appearing generalized β -function could be regarded as zero, since $\partial_S \alpha_S = 0$ if $S \neq R$. Differentiating 3.27 and comparing with the generalized evolution equation 3.26, while keeping in mind that $\mathbf{E}_n^{(\cdot,\cdot)}$ is independent of the resolution scale, gives for the renormalization constant

$$\mathbf{T}_n = -\mathbf{Z}_n^{-1} \partial_S \mathbf{Z}_n = \partial_S \ln \mathbf{Z}_n^{-1}. \quad (3.31)$$

Furthermore, when differentiating with respect to the resolution scale S ,

$$\begin{aligned} \mathbf{R}_{S,n}^{(1,0)} \circ \mathbf{R}_{S,n}^{(1,0)\dagger} &= \mathbf{E}_n^{(1,0)} \circ \mathbf{E}_n^{(1,0)\dagger} \partial_S \Theta_{n,1} \\ &+ \mathbf{E}_n^{(1,0)} \circ \partial_S \mathbf{E}_n^{(1,0)\dagger} \Theta_{n,1} + \partial_S \mathbf{E}_n^{(1,0)} \circ \mathbf{E}_n^{(1,0)\dagger} \Theta_{n,1} \\ &+ \mathbf{E}_n^{(1,0)} \circ \mathbf{E}_n^{(1,0)\dagger} \beta_{S,-1} \Theta_{n,1} \\ &= \mathbf{E}_n^{(1,0)} \circ \mathbf{E}_n^{(1,0)\dagger} \partial_S \Theta_{n,1}. \end{aligned} \quad (3.32)$$

$$\begin{aligned} \mathbf{R}_{S,n}^{(1,1)} \circ \mathbf{R}_{S,n}^{(1,1)\dagger} &= \left(\mathbf{E}_n^{(1,1)} \circ \mathbf{E}_n^{(1,0)\dagger} + \mathbf{E}_n^{(1,0)} \circ \mathbf{E}_n^{(1,1)\dagger} \right) \partial_S \Theta_{n,1} \\ &+ \mathbf{E}_n^{(1,1)} \circ \partial_S \mathbf{E}_n^{(1,0)\dagger} \Theta_{n,1} + \mathbf{E}_n^{(1,0)} \circ \partial_S \mathbf{E}_n^{(1,1)\dagger} \Theta_{n,1} \\ &+ \partial_S \mathbf{E}_n^{(1,1)} \circ \mathbf{E}_n^{(1,0)\dagger} \Theta_{n,1} + \partial_S \mathbf{E}_n^{(1,0)} \circ \mathbf{E}_n^{(1,1)\dagger} \Theta_{n,1} \\ &+ 2\mathbf{E}_n^{(1,0)} \circ \mathbf{E}_n^{(1,0)\dagger} \beta_{S,-1} \Theta_{n,1} \\ &- \mathbf{Z}_n^{(1)} \mathbf{R}_{S,n}^{(1,0)} \circ \mathbf{R}_{S,n}^{(1,0)\dagger} + \mathbf{R}_{S,n}^{(1,0)} \mathbf{\Gamma}_{S,n}^{(1)} \circ \mathbf{R}_{S,n}^{(1,0)\dagger} \\ &- \mathbf{R}_{S,n}^{(1,0)} \circ \mathbf{R}_{S,n}^{(1,0)\dagger} \mathbf{Z}_n^{(1)\dagger} + \mathbf{R}_{S,n}^{(1,0)} \circ \mathbf{\Gamma}_{S,n}^{(1)\dagger} \mathbf{R}_{S,n}^{(1,0)\dagger} \\ &+ \mathbf{R}_{S,n}^{(1,0)} \circ \mathbf{R}_{S,n}^{(1,0)\dagger} \beta_{S,0} \end{aligned} \quad (3.33)$$

$$\begin{aligned} \mathbf{R}_{S,n}^{(2,0)} \circ \mathbf{R}_{S,n}^{(2,0)\dagger} &= \mathbf{E}_n^{(2,0)} \circ \mathbf{E}_n^{(2,0)\dagger} \partial_S \Theta_{n,2} \\ &+ \mathbf{E}_n^{(2,0)} \circ \partial_S \mathbf{E}_n^{(2,0)\dagger} \Theta_{n,2} + \partial_S \mathbf{E}_n^{(2,0)} \circ \mathbf{E}_n^{(2,0)\dagger} \Theta_{n,2} \\ &+ 2\mathbf{E}_n^{(2,0)} \circ \mathbf{E}_n^{(2,0)\dagger} \beta_{S,-1} \Theta_{n,2} \\ &- \mathbf{E}_n^{(1,0)} \mathbf{R}_{S,n-1}^{(1,0)} \circ \mathbf{R}_{S,n-1}^{(1,0)\dagger} \mathbf{E}_n^{(1,0)} \Theta_{n,1} \\ &= \mathbf{E}_n^{(2,0)} \circ \mathbf{E}_n^{(2,0)\dagger} \partial_S \Theta_{n,2} - \mathbf{E}_n^{(1,0)} \mathbf{E}_{n-1}^{(1,0)} \circ \mathbf{E}_{n-1}^{(1,0)\dagger} \mathbf{E}_n^{(1,0)\dagger} \Theta_{n,1} \partial_S \Theta_{n-1,1} \end{aligned} \quad (3.34)$$

The expression $\mathbf{R}_{S,n}^{(2,0)} \circ \mathbf{R}_{S,n}^{(2,0)\dagger}$ is the key ingredient for simulating double gluon emission and will be further discussed in section 6.3.

The rest of the section is concerned with the notion of observables, which reflect for example the special geometry of a certain detector setup within a collider experiment. In general, an

observable Σ is formally represented by

$$\begin{aligned}\Sigma(\mu) &= \int \sum_{n \geq 0} d\sigma_n u_n(p_1, p_2, \dots, p_n), \\ &= \sum_n \left(\prod_{i=1}^n \int d\Pi_i \right) \text{Tr}[\mathbf{A}_n] u_n(p_1, p_2, \dots, p_n),\end{aligned}\quad (3.35)$$

where the cutoff dependency of the observable was explicitly written down, the functions u_n are the measurement functions and the p_i are the momenta of the emitted soft gluons.

In especially simple cases the measurement function is factorizable, in the sense that

$$u_n(p_1, p_2, \dots, p_n) = \prod_{i=1}^n u(q_i). \quad (3.36)$$

This is the case, for example, for the hemisphere jet mass and the gaps-between jets, where

$$\begin{aligned}u(q_i) &= \Theta(\rho - q_i) + \Theta(q_i - \rho)\Theta(\Omega_{\text{out}}) \\ &= \Theta(\rho - q_i)(\Theta(\Omega_{\text{in}}) + \Theta(\Omega_{\text{out}})) + \Theta(q_i - \rho)\Theta(\Omega_{\text{out}}) \\ &= \Theta(\Omega_{\text{out}}) + \Theta(\rho - q_i)\Theta(\Omega_{\text{in}}),\end{aligned}\quad (3.37)$$

and the scale ρ satisfies $\mu \leq \rho \leq Q$.

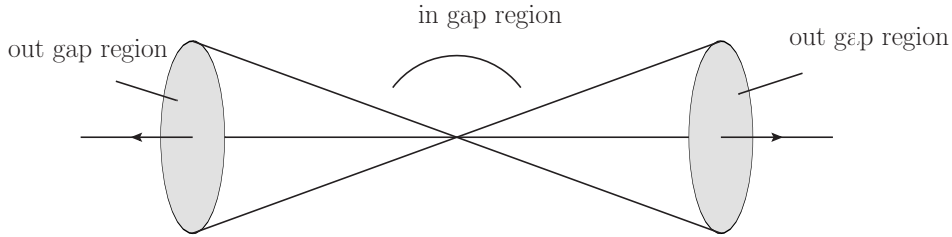


Figure 3: Gaps between jets observable in the back to back frame for a quark and antiquark pair.

In general, there are two types of classes of observables, global and non-global observables. Global observables are everywhere restricted in the phase space, whereas non-global observables are restricted in certain patches of phase space. The hemisphere jet mass and the gaps-between jets observables are for example such non-global observables. For global observables a fixed order algorithm to calculate the cross section would have been sufficient for a precise prediction, since then the infrared divergences completely cancel between the virtual and real emission contribution. In the case of non-global observables the problem is that large logarithms appear [3] and a summation over all possible emissions is necessary, such that the IR divergences cancel sufficiently. This is guaranteed by the Bloch-Nordsieck/Kinoshita Lee-Nauenberg theorem. The algorithm presented in this section provides such a resummation procedure.

4 Colour space and colour-flow basis

4.1 Colour space

The following section aims to introduce the colour-space formalism. The colour formalism is a framework, where amplitudes involving many partons can be treated in a systematic and transparent way. The formalism was mainly developed to describe the soft behaviour of tree-level amplitudes, but it is more general and could be extended to other regimes. Despite the possible generalization the following primarily aims to describe the evolution of amplitudes in the soft limit. Perturbative QCD amplitudes have, as suggested by the QCD Feynman rules (see appendix E), the following structure

$$\text{amplitude} = \text{colour} \otimes \text{kinematic}, \quad (4.1)$$

where the colour part has its origin from the non-abelian character of the theory. A formal definition of the colour space is then given by [23].

Definition 2. Let's n_q denote the number of quarks and n_g the number of gluons in the considered process. A colour structure or colour state is then a tensor $c \in (\bar{V} \otimes V)^{\otimes n_q} \otimes A^{\otimes n_g}$, due to the fact that only colourless states can be observed, are singlets and therefore transform under the trivial representation of $SU(N)$. The colour space $\mathcal{C}_n$, with $n = 2n_q + n_g$ partons, is then the colour singlet-subspace of $(\bar{V} \otimes V)^{\otimes n_q} \otimes A^{\otimes n_g}$.

So the defining representation of the gauge group $SU(N)$ is carried by the vector space $V \simeq \mathbb{C}^N$, where $\mathbb{C}^N$ could be equipped with a scalar product. The scalar product on $\mathbb{C}^N$ then induces also a scalar product on $\bar{V}$ and A , or on any tensor product of $V, \bar{V}, A$. For the colour states $a, b \in (\bar{V} \otimes V)^{\otimes n_q} \otimes A^{\otimes n_g}$ the scalar product is then given by

$$\langle a, b \rangle = \text{Tr}[a^\dagger b] = \left(((a)^{ab\dots})^{ij\dots}_{kl\dots} \right)^\dagger \left((b)^{ab\dots} \right)^{ij\dots}_{kl\dots}. \quad (4.2)$$

To describe the colour structure an abstract orthonormal basis for the n^2 dimensional colour + helicity space has to be introduced [2], the basis is denoted by

$$\{|c_1, c_2, \dots, c_n\rangle \otimes |s_1, s_2, \dots, s_n\rangle\}, \quad (4.3)$$

where one may notice that the usual rule was applied that Hilbert spaces, describing different degrees of freedoms, get combined via the tensor product. Later in this work a concrete basis will be chosen in order to describe how concrete computations could be done.

Then, by construction, the tree-level matrix element for n final-state partons, denoted by

$$\mathcal{M}_{a_1, a_2, \dots, a_n}^{c_1, c_2, \dots, c_n; s_1, s_2, \dots, s_n}, \quad (4.4)$$

where $\{c_1, c_2, \dots, c_n\}$, $\{s_1, s_2, \dots, s_n\}$ and $\{a_1, a_2, \dots, a_n\}$ are colour, spin and flavour indices, can be written as

$$\mathcal{M}_{a_1, \dots, a_n}^{c_1, \dots, c_n; s_1, \dots, s_n}(p_1, \dots, p_n) := (\langle c_1, c_2, \dots, c_n | \otimes \langle s_1, s_2, \dots, s_n |) | \mathcal{M}_{a_1, \dots, a_n}(p_1, \dots, p_n) \rangle, \quad (4.5)$$

hence the amplitude $|\mathcal{M}_{a_1, \dots, a_n}(p_1, \dots, p_n)\rangle$ itself is a state in the colour+helicity space. The matrix element squared $|\mathcal{M}|^2$ is then given by

$$\begin{aligned} |\mathcal{M}|^2 &= \sum_{\text{colour spin}} \sum \mathcal{M}_{a_1, \dots, a_n}^{c_1, \dots, c_n; s_1, \dots, s_n; i, \dagger}(p_1, \dots, p_n) \mathcal{M}_{a_1, \dots, a_n}^{c_1, \dots, c_n; s_1, \dots, s_n}(p_1, \dots, p_n) \\ &= \langle \mathcal{M}_{a_1, \dots, a_n}(p_1, \dots, p_n) | \mathcal{M}_{a_1, \dots, a_n}(p_1, \dots, p_n) \rangle \\ &= |\mathcal{M}_{a_1, \dots, a_n}(p_1, \dots, p_n)|^2, \end{aligned} \quad (4.6)$$

where in the first step the completeness relation for the colour + helicity space was used. In the soft limit the emission of a gluon does not affect the momenta and spins of the radiating partons, however, it always carries colour charge away, which leads to colour correlations. So in order to describe soft-gluon emission and therefore the colour correlation in the soft factorization formula, it is convenient to introduce the so called colour charge operator $\mathbf{T}_i$, which contains the relevant information of the colour charge of the emitted gluon, from the parton labelled by the index i . The colour charge operator appears naturally when one develops factorization formulas in the soft gluon emission limit, but can then also be used to describe the colour structure of amplitudes outside the soft limit.

On the spin subspace the charge operator acts like the identity operator, so it is sufficient to define the colour charge operator $\mathbf{T}_i$ via the action on the colour space. As already previously indicated, the orthonormal basis for the colour space $\mathcal{C}_n$ for n partons is denoted by

$$\{|c_1, \dots, c_n\rangle\}, \quad (4.7)$$

where the numbers c_i label the colour charge of parton i hence $i \in \{1, 2, \dots, N^2 - 1\}$, N denotes the dimension of the gauge group $SU(N)$. The colour charge operator is then a map from the colour space of n partons $\mathcal{C}_n$ to the colour space for $n + 1$ partons $\mathcal{C}_{n+1}$

$$\begin{aligned} \mathbf{T}_i : \mathcal{C}_n &\rightarrow \mathcal{C}_{n+1} \\ \mathbf{T}_i : |b_1, \dots, b_n\rangle &\rightarrow \mathbf{T}_i |b_1, \dots, b_n\rangle, \end{aligned} \quad (4.8)$$

where the action [2] is defined as

$$\langle c_1, \dots, c_i, \dots, c_n, c | \mathbf{T}_i | b_1, \dots, b_i, \dots, b_n \rangle = \delta_{c_1 b_1} \dots T_{c_i b_i}^c \dots \delta_{c_n b_n} \quad (4.9)$$

and $T_{cb}^a := if_{cab}$ if the parton with label i is a gluon and $T_{\alpha\beta}^a = t_{\alpha\beta}^a$ if the parton with label i is a quark. It is then possible to decompose $\mathbf{T}_i$ in the following way

$$\mathbf{T}_i = \mathbf{C}^a \mathbf{T}_i^a = \mathbf{T}_i^a \mathbf{C}^a, \quad (4.10)$$

where on the right side the sum goes over a and the operator $\mathbf{C}^a$ is a map between the n parton colour space $\mathcal{C}_n$ and the $n + 1$ parton colour space $\mathcal{C}_{n+1}$, defined as

$$\begin{aligned} \mathbf{C}^a : \mathcal{C}_n &\rightarrow \mathcal{C}_{n+1} \\ \mathbf{C}^a : |b_1, \dots, b_n\rangle &\rightarrow \mathbf{C}^a |b_1, \dots, b_n\rangle = |b_1, \dots, b_n, a\rangle, \end{aligned} \quad (4.11)$$

and the operator $\mathbf{T}_i^a$ a map between $\mathcal{C}_n$ and $\mathcal{C}_n$

$$\begin{aligned}\mathbf{T}_i^a &: \mathcal{C}_n \rightarrow \mathcal{C}_n \\ \mathbf{T}_i^a &: |b_1, \dots, b_n\rangle \rightarrow \mathbf{T}_i^a |b_1, \dots, b_n\rangle\end{aligned}\quad (4.12)$$

with the action defined through

$$\langle c_1, \dots, c_i, \dots, c_n | \mathbf{T}_i^a | b_1, \dots, b_i, \dots, b_n \rangle = \delta_{c_1 b_1} \dots T_{c_i b_i}^a \dots \delta_{c_n b_n}, \quad (4.13)$$

where, again, $T_{cb}^a := if_{cab}$ if index i labelled a gluon and $T_{\alpha\beta}^a = t_{\alpha\beta}^a$ if index i labelled a quark. From the respective definitions of the operators $\mathbf{T}_i^a$ and $\mathbf{C}^a$ it becomes clear that they commute, since $\mathbf{T}_i^a$ only acts on $\mathcal{C}_n \subset \mathcal{C}_{n+1}$, while $\mathbf{C}^a$, loosely speaking, simply enlarges a state from $\mathcal{C}_n$ to a state in $\mathcal{C}_{n+1}$, without changing the state on $\mathcal{C}_n$.

That the decomposition in (4.10) indeed reproduces the colour charge $\mathbf{T}_i$ operator can then be shown in the following way

$$\begin{aligned}\langle c_1, \dots, c_n, c | \mathbf{C}^a \cdot \mathbf{T}_i^a | b_1, \dots, b_n \rangle &= \langle c_1, \dots, c_n, c | \mathbf{T}_i^a \cdot \mathbf{C}^a | b_1, \dots, b_n \rangle \\ &= \langle c_1, \dots, c_n, c | \mathbf{T}_i^a | b_1, \dots, b_n, a \rangle \\ &= \langle c_1, \dots, c_n | \mathbf{T}_i^a | b_1, \dots, b_n \rangle \delta_{ca} \\ &= \delta_{c_1 b_1} \dots T_{c_i b_i}^a \dots \delta_{c_n b_n} \delta_c^a \\ &= \delta_{c_1 b_1} \dots T_{c_i b_i}^c \dots \delta_{c_n b_n},\end{aligned}\quad (4.14)$$

which is in agreement with the definition of $\mathbf{T}_i$ in (4.9). With the action of $\mathbf{T}_i^a$ and the completeness relation of the basis and expression for $\mathbf{T}_i^a |b_1, \dots, b_n\rangle$ can be derived

$$\begin{aligned}\mathbf{T}_i^a |b_1, \dots, b_n\rangle &= \sum_{\{f\}} |f_1, \dots, f_n\rangle \langle f_1, \dots, f_n | \mathbf{T}_i^a | b_1, \dots, b_n \rangle \\ &= \sum_{f_i} T_{f_i b_i}^a |b_1, \dots, b_{i-1}, f_i, b_{i+1}, \dots, b_n\rangle.\end{aligned}\quad (4.15)$$

Now the product $\mathbf{T}_i^\dagger \cdot \mathbf{T}_j$ could be investigated, which mimics the colour flow of a gluon exchange and therefore is a map from the $\mathcal{C}_n$ to the $\mathcal{C}_n$ colour space, with the decomposition it follows

$$\mathbf{T}_i^\dagger \cdot \mathbf{T}_j = \mathbf{T}_i^{a\dagger} \mathbf{C}^{a\dagger} \mathbf{C}^b \cdot \mathbf{T}_j^b. \quad (4.16)$$

Explicitly calculating at the matrix elements of $\mathbf{C}^{a\dagger} \cdot \mathbf{C}^b$ gives

$$\begin{aligned}\langle c_1, \dots, c_n | \mathbf{C}^{a\dagger} \cdot \mathbf{C}^b | d_1, \dots, d_n \rangle &= \langle c_1, \dots, c_n, a | d_1, \dots, d_n, b \rangle \\ &= \langle c_1, \dots, c_n | d_1, \dots, d_n \rangle \delta_{ab},\end{aligned}\quad (4.17)$$

therefore

$$\begin{aligned}\langle c_1, \dots, c_n | \mathbf{T}_i^\dagger \cdot \mathbf{T}_j | d_1, \dots, d_n \rangle &= \langle c_1, \dots, c_n | \mathbf{T}_i^{a\dagger} \cdot \mathbf{C}^{a\dagger} \cdot \mathbf{C}^b \cdot \mathbf{T}_j^b | d_1, \dots, d_n \rangle \\ &= \langle c_1, \dots, c_n | \mathbf{T}_i^{a\dagger} \cdot \mathbf{T}_j^b | d_1, \dots, d_n \rangle \delta_{ab} \\ &= \langle c_1, \dots, c_n | \mathbf{T}_i^{a\dagger} \cdot \mathbf{T}_j^a | d_1, \dots, d_n \rangle.\end{aligned}\quad (4.18)$$

Now, wlog $1 \leq i \leq j \leq n$, then

$$\begin{aligned}
& \langle c_1, \dots, c_i, \dots, c_j, \dots, c_n | \mathbf{T}_i^a \mathbf{T}_j^b | b_1, \dots, b_i, \dots, b_j, \dots, b_n \rangle \\
&= \langle c_1, \dots, c_i, \dots, c_j, \dots, c_n | \mathbf{T}_i^a \sum_{f_i} T_{f_i b_i}^b | b_1, \dots, b_i, \dots, b_{j-1}, f_j, b_{j+1}, \dots, b_n \rangle \\
&= \langle c_1, \dots, c_i, \dots, c_j, \dots, c_n | \sum_{f_i, f_j} T_{f_i b_i}^a T_{f_j b_j}^b | b_1, \dots, b_{i-1}, f_i, b_{i+1}, \dots, b_{j-1}, f_j, b_{j+1}, \dots, b_n \rangle \\
&= \delta_{c_1 b_1} \dots T_{c_i b_i}^a \dots T_{c_j b_j}^b \dots \delta_{c_n b_n}.
\end{aligned} \tag{4.19}$$

From (4.18) it follows then that

$$\mathbf{T}_i \cdot \mathbf{T}_j := \mathbf{T}_i^a \mathbf{T}_j^a, \tag{4.20}$$

where, on the right side, the sum goes over a and therefore from (4.19) the colour-charge algebra follows

$$\mathbf{T}_i \cdot \mathbf{T}_j = \mathbf{T}_j \cdot \mathbf{T}_i \quad \text{for } i \neq j \quad \text{and} \quad \mathbf{T}_i^2 = C_i, \tag{4.21}$$

the symbol C_i (one should write $C_i \cdot \mathbf{1}$) denotes the Casimir operator, which depends on the type of parton line i . If i labels a gluon then $C_i = C_A$ and if i labels a quark/anti-quark then $C_i = C_F$. A physical state $|\mathcal{M}_{a_1, \dots, a_n}(p_1, \dots, p_n)\rangle$ is a colour singlet state, which is reflected by the relation

$$\sum_{i=1}^n \mathbf{T}_i |\mathcal{M}_{a_1, \dots, a_n}(p_1, \dots, p_n)\rangle = 0. \tag{4.22}$$

In order to get some intuition for the colour algebra it is insightful to look at the simplest cases possible [24]. The trivial case is the one parton case, where there is only one colour charge operator $\mathbf{T}_1$ and the square $\mathbf{T}_1^2$ equals due to the colour algebra (4.21) to C_1 . The first interesting case occurs when one considers two partons, then due to the colour algebra 4.21 and colour conservation

$$\mathbf{T}_1 \cdot \mathbf{T}_2 |1, 2\rangle = \mathbf{T}_2 \cdot \mathbf{T}_1 |1, 2\rangle = -\mathbf{T}_1^2 |1, 2\rangle = -\mathbf{T}_2^2 |1, 2\rangle. \tag{4.23}$$

Hence all squares of charge operators $\{\mathbf{T}_1^2, \mathbf{T}_2^2, \mathbf{T}_1 \cdot \mathbf{T}_2\}$ could, as in the previous example, be written in terms of linear combinations of the Casimir operator $C_1 = C_2$. In the same way the squares of the colour operator can be written in terms of linear combinations of the Casimir operators C_1, C_2, C_3 in the three partons $|1, 2, 3\rangle$ case. The natural question to ask is if this is possible in general? The answer is no. To show this the possible relations between the different squares of colour charge operators have to be counted. To reduce the charge operators to Casimir operators the trivial relation $\mathbf{T}_i^2 = C_i$ and colour conservation 4.22 was used, hence there are $2n$ relations for $n(n-1)/2$ charge operators, so if there are more than 3 partons ($n > 3$), the rewriting in terms of linear combinations of Casimir invariants is not possible any longer.

For concrete calculation a (matrix-) representation for the $\mathbf{T}_i$'s has to be chosen and therefore a basis for the colour space. The most common choices are the trace bases, the adjoint bases and the colour flow basis. The last one is the basis of choice and will be properly introduced in section 4.2. In the trace basis the colour factors are represented as traces of products of fundamental

representation generators $\{t^{a_i}\}$. As a concrete example let's consider the amplitude $\mathcal{M}$ for n gluons, with colour labels $a_1, a_2, \dots, a_n$, where $a_i \in \{1, \dots, N^2 - 1\}$. Then such an amplitude can be written at tree level as [25]

$$\mathcal{M}(ng) = \sum_{\sigma \in S_{n-1}} \text{Tr}(t^{a_1} t^{a_{\sigma_2}} \dots t^{a_{\sigma_n}}) A(1, \sigma_2, \dots, \sigma_n), \quad (4.24)$$

where the sum goes over all $(n-1)!$ permutations of $(2, 3, \dots, n)$ and A the partial amplitudes, which are dependent on the four-momenta p_i and polarization vectors ϵ_i of the n gluons. Note that in [25] the normalization convention $T_R = 1$ was used, the same convention holds for the rest of this section as well. The partial amplitudes A are not linear independent, there are linear relations between them, the so called Kleiss-Kuijff relations, which reduce the number of linear independent amplitudes to $(n-2)!$. How to insert the Kleiss-Kuijff relations was explicitly shown in [26]. In the adjoint basis the n gluon amplitude with the colour labels $a_1, \dots, a_n$ reads then

$$\mathcal{M}(ng) = \sum_{\sigma \in S_{n-2}} (F^{a_{\sigma_2}} \dots F^{a_{\sigma_{n-1}}})_{a_1 a_n} A(\sigma_1, \sigma_2, \dots, \sigma_{n-1}, n), \quad (4.25)$$

where $(F^a)_{bc} := if^{bac}$ are the adjoint matrix representations of $SU(N)$. Note that the sum in (4.25) only goes over $(n-2)!$ permutations of the set of indices $(2, \dots, n-1)$, whereas the partial amplitudes are the same as in (4.24), this is possible due to the already mentioned Kleiss-Kuijff relations. One drawback of this adjoint-representation is that it is only applicable for n -gluon amplitudes. To change from the trace basis to the adjoint representation the following identity could be used

$$f^{a_1 a_2 x_1} f^{x_1 a_3 x_2} \dots f^{x_{n-3} a_{n-1} a_n} = (-i)^{n-2} \text{Tr}(t^{a_1} [t^{a_2}, [t^{a_3}, \dots, [t^{a_{n-1}}, t^{a_n}] \dots]]) , \quad (4.26)$$

which becomes clear when looking at the base case $n = 4$, inverting the commutation relation $[t^a, t^b] = if^{abc} t^c$ with respect to the structure constant $f^{abc} = -i \text{Tr}(t^a [t^b, t^c])$ and plugging this into the right hand side of (4.26) gives

$$\begin{aligned} (-i)^{4-2} \text{Tr}(t^{a_1} [t^{a_2}, [t^{a_3}, t^{a_4}]]) &= (-i)^{4-2} \text{Tr}(t^{a_1} [t^{a_2}, if^{a_3 a_4 c} t^c]) = (-i)^{4-2} if^{a_3 a_4 c} \text{Tr}(t^{a_1} [t^{a_2}, t^c]) \\ &= (-i)^{4-2} (i)^2 f^{a_3 a_4 c} f^{a_2 c d} \text{Tr}(t^{a_1} t^d) = f^{a_1 a_2 c} f^{c a_3 a_4}. \end{aligned} \quad (4.27)$$

With induction the relation (4.26) follows. The colour flow basis could be derived by treating the gauge group $SU(N)$ as a $N \times N$ matrix field $(A_\mu)^i_j$, where $i, j \in \{1, \dots, N\}$ instead of a one-index field A_μ^a , with $a \in \{1, \dots, N^2 - 1\}$. Then the n -gluon amplitude is of the following form in the colour flow basis

$$\mathcal{M}(ng) = \sum_{P(2, \dots, n)} \delta^{i_1}_{j_2} \delta^{i_2}_{j_3} \dots \delta^{i_n}_{j_1} A(1, 2, \dots, n), \quad (4.28)$$

where the sum is over all $(n-1)!$ permutations of $(2, \dots, n)$ and the partial amplitudes are the same as before.

4.2 Colour-flow basis

In this section the colour-flow basis, which is a non-orthogonal and over-complete spanning set, will be introduced. Despite the over-completeness the colour-flow basis provides an advantage in comparison to the usually used trace or adjoint basis, due to its physical interpretation in terms of flow of colour. Furthermore, the basis tensors can be uniquely labelled by elements of the permutation group S_n . The elements of the permutation group can then be ordered lexicographically (see appendix G) and algorithms to generate the $k + 1$ -th element, given the k -th element in that lexicographical order are widely available, for example in the standard C++ library. This bijection and the easy to organize set S_n makes the colour-flow basis ideal for colour Monte Carlo simulations.

In principle, there are at least two ways to introduce the colour-flow basis and the resulting Feynman rules. One way [25] is to start again at the QCD Lagrange density (2.1), then instead of decomposing the gluon field in terms of the fundamental-representation matrices $\{t^a\}$ to express the Lagrangian density in terms of A_μ^a , where $a = 1, \dots, N^2 - 1$ to obtain subsequently the conventional Feynman rules, listed in appendix E, it is also possible to work directly with the $N \times N$ matrix fields $(\mathcal{A}_\mu)^i{}_j := \sqrt{2} (A_\mu^a t^a)^i{}_j$, where the factor $\sqrt{2}$ is chosen such that the Lagrange density is canonically normalized. The Lagrange density reads then

$$\mathcal{L} = -\frac{1}{4} (\mathcal{F}^{\mu\nu})^i{}_j (\mathcal{F}_{\mu\nu})^j{}_i + i\bar{\psi}_i \gamma^\mu \left(\delta^i{}_j \partial_\mu + i\frac{g}{\sqrt{2}} (\mathcal{A}_\mu)^i{}_j \right) \psi^j - m\bar{\psi}_i \psi^i, \quad (4.29)$$

whereas the field strength tensor reads

$$(\mathcal{F})^i{}_j = \partial_\mu (\mathcal{A}_\nu)^i{}_j - \partial_\nu (\mathcal{A}_\mu)^i{}_j + i\frac{g}{\sqrt{2}} (\mathcal{A}_\mu)^i{}_k (\mathcal{A}_\nu)^k{}_j - i\frac{g}{\sqrt{2}} (\mathcal{A}_\nu)^i{}_k (\mathcal{A}_\mu)^k{}_j. \quad (4.30)$$

The resulting Feynman rules include strings of Kronecker deltas instead of the fundamental-representation matrices, they can be found in appendix F. In [25] they also introduced the convention that upper/lower indices transform under the fundamental/anti-fundamental representation of $SU(N)$ respectively, the same convention will be used throughout this work. Note that through the global $SU(N)$ symmetry each vertex preserves colour conservation. The gluon quark-vertex is in the colour basis especially simple, since the vertex is proportional to the product of two Kronecker deltas [25], whereas for the gluon propagator

$$\langle (\mathcal{A}_\mu)^{i_1}{}_{j_1} (\mathcal{A}_\nu)^{i_2}{}_{j_2} \rangle \propto \delta^{i_1}{}_{j_2} \delta^{i_2}{}_{j_1} - \frac{1}{N} \delta^{i_1}{}_{j_1} \delta^{i_2}{}_{j_2}. \quad (4.31)$$

In the trace basis the gluon propagator is proportional to

$$\langle A_\mu^a A_\nu^b \rangle \propto \delta^{ab}. \quad (4.32)$$

The simple colour structure of the vertices comes therefore with a trade-off, a more complicated colour structure for the gluon propagator. Another way to obtain the colour-flow Feynman rules is to start from the conventional Feynman rules and rewrite them such that the final expressions are proportional to Kronecker deltas. This can be done with the Fierz identity (2.8) and the

following two relations

$$\begin{aligned} \text{Tr}[t^a t^b] &= T_R \delta^{ab} \\ M_a N^a &= M_b N_a \delta^{ab} = \frac{1}{\sqrt{T_R}} (M^a t^a)^i_j \frac{1}{\sqrt{T_R}} (N^b t^b)^j_i, \end{aligned} \quad (4.33)$$

where M^a and N^a are objects which carry an adjoint index a . How this is exactly done is demonstrated in [27] and in appendix F. In a general QCD process with arbitrary many incoming and outgoing quarks, anti-quarks and gluons, the colour state is then a product of quark-vertices, 3-gluon and 4-gluon vertices, which are all proportional to a linear combination of products of Kronecker deltas in the colour flow basis. Therefore, the colour state of a general QCD process can be written as a linear combination of products of Kronecker deltas. The set of all possible strings of Kronecker deltas of a fixed length then form the over-complete, non-orthogonal basis tensors of the colour space. For the basis tensors the following notation is used

$$\left\langle \begin{matrix} c_1 & \dots & c_m \\ \bar{c}_1 & \dots & \bar{c}_m \end{matrix} \middle| \sigma \right\rangle = \Delta_{\bar{c}_1, \dots, \bar{c}_m; \sigma}^{c_1, \dots, c_m} = \delta^{c_1}_{\bar{c}_{\sigma(1)}} \dots \delta^{c_m}_{\bar{c}_{\sigma(m)}}, \quad (4.34)$$

where $m = n_q + n_g = n_{\bar{q}} + n_g$ is the number of colour lines, c_i are fundamental and $\bar{c}_i$ are anti-fundamental indices, with $c_i, \bar{c}_i = 1, \dots, N$, $i = 1, \dots, m$ and σ the basis tensor labels. Since only colourless states are considered, a quark and anti-quark pair is only one colour line and furthermore, due to the colourlessness each colour state has the same number of fundamental and anti-fundamental indices. For a fixed integer m there are $m!$ possible basis tensors, from which it follows that this set of tensors can be uniquely labelled with elements of the permutation group S_m . The elements of the permutation group $\sigma \in S_m$ are completely determined through their images, so an element gets abbreviated through

$$\sigma = (\sigma(1), \sigma(2), \dots, \sigma(m)), \quad (4.35)$$

which should not be confused with the commonly used cycle notation.

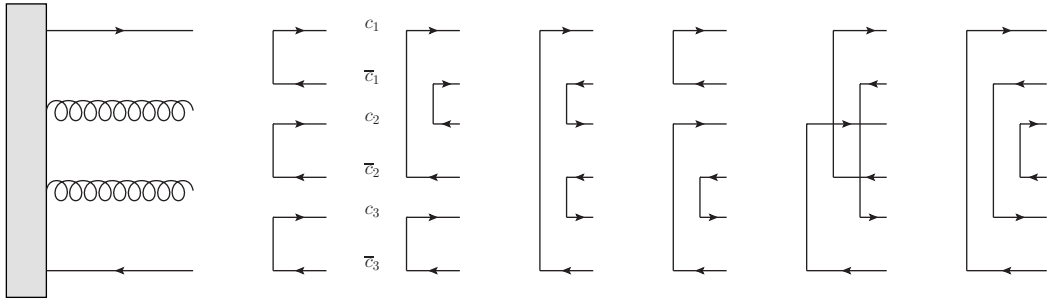


Figure 4: Basis tensors for $m = 3$ colour lines, inspired and adapted from [4].

Conjugation of the basis states (4.34) gives

$$\left\langle \sigma \middle| \begin{matrix} c_1 & \dots & c_m \\ \bar{c}_1 & \dots & \bar{c}_m \end{matrix} \right\rangle = \Delta_{c_1, \dots, c_m}^{\bar{c}_1, \dots, \bar{c}_m; \sigma} = \delta_{c_1}^{\bar{c}_{\sigma(1)}} \dots \delta_{c_m}^{\bar{c}_{\sigma(m)}}. \quad (4.36)$$

Diagrammatically conjugation corresponds in this framework to mirroring the diagram along the vertical axis and simultaneously reversing all arrows. The scalar product is then given through

(4.2). Additionally, there is the notion of an abstract colour state with label σ , denoted with

$$|\sigma\rangle. \quad (4.37)$$

The scalar product between two abstract colour states with label $\sigma \in S_n$ and label $\tau \in S_n$, where n is the number of colour lines can then be obtained by simply inserting the completeness relation

$$\langle\sigma|\tau\rangle = \sum_{\{c_i, \bar{c}_i\}} \left\langle\sigma \left| \begin{smallmatrix} c_1 & \cdots & c_n \\ \bar{c}_1 & \cdots & \bar{c}_n \end{smallmatrix} \right. \right\rangle \left\langle \begin{smallmatrix} c_1 & \cdots & c_n \\ \bar{c}_1 & \cdots & \bar{c}_n \end{smallmatrix} \right| \tau \rangle, \quad (4.38)$$

which then equals the scalar product in (4.2). The scalar product in (4.38) between two abstract colour states could then be calculated with the formula

$$\langle\sigma|\tau\rangle = N^{n-\#\text{transpositions}(\sigma,\tau)}, \quad (4.39)$$

where with $\#(\sigma, \tau)$ the minimal number of transpositions, by which the permutations σ, τ differ, is denoted. The scalar product of two colour states is therefore fully determined by the permutation, by which they are labelled with. In order to get an intuition for formula (4.39) let's consider the simplest case, the case of one colour line and therefore only one basis tensor and only one scalar-product to calculate. Applying formula (4.2) gives

$$\text{Tr} \left[\left\langle (1) \left| \begin{smallmatrix} c_1 \\ \bar{c}_1 \end{smallmatrix} \right. \right\rangle \left\langle \begin{smallmatrix} c_1 \\ \bar{c}_1 \end{smallmatrix} \right| (1) \right\rangle \right] = \delta^{\bar{c}_1}_{c_1} \delta^{c_1}_{\bar{c}_1} = N. \quad (4.40)$$

Now the case for two colour lines, which could be two gluons, two quark-antiquark pairs or one quark-antiquark pair and one gluon. For two colour lines there are $2! = 2$ colour basis states, the respective scalar-products are

$$\begin{aligned} \text{Tr} \left[\left\langle (12) \left| \begin{smallmatrix} c_1 & c_2 \\ \bar{c}_1 & \bar{c}_2 \end{smallmatrix} \right. \right\rangle \left\langle \begin{smallmatrix} c_1 & c_2 \\ \bar{c}_1 & \bar{c}_2 \end{smallmatrix} \right| (12) \right\rangle \right] &= \delta^{\bar{c}_1}_{c_1} \delta^{\bar{c}_2}_{c_2} \delta^{c_1}_{\bar{c}_1} \delta^{c_2}_{\bar{c}_2} = N^2, \\ \text{Tr} \left[\left\langle (21) \left| \begin{smallmatrix} c_1 & c_2 \\ \bar{c}_1 & \bar{c}_2 \end{smallmatrix} \right. \right\rangle \left\langle \begin{smallmatrix} c_1 & c_2 \\ \bar{c}_1 & \bar{c}_2 \end{smallmatrix} \right| (21) \right\rangle \right] &= \delta^{\bar{c}_2}_{c_1} \delta^{\bar{c}_1}_{c_2} \delta^{c_1}_{\bar{c}_2} \delta^{c_2}_{\bar{c}_1} = N^2, \end{aligned} \quad (4.41)$$

and

$$\text{Tr} \left[\left\langle (12) \left| \begin{smallmatrix} c_1 & c_2 \\ \bar{c}_1 & \bar{c}_2 \end{smallmatrix} \right. \right\rangle \left\langle \begin{smallmatrix} c_1 & c_2 \\ \bar{c}_1 & \bar{c}_2 \end{smallmatrix} \right| (21) \right\rangle \right] = \delta^{\bar{c}_1}_{c_1} \delta^{\bar{c}_2}_{c_2} \delta^{c_1}_{\bar{c}_2} \delta^{c_2}_{\bar{c}_1} = N. \quad (4.42)$$

The next step is to prove formula (4.39) by induction. The base case for the induction proof is that σ and τ differ by one permutation, hence there exists a transposition operator $\mathbf{s}_{ij}$ such that $|\tau\rangle = \mathbf{s}_{ij}|\sigma\rangle$, then without loss of generality $i < j$ it follows

$$\begin{aligned} \langle\sigma|\tau\rangle &= \langle\sigma|\mathbf{s}_{ij}|\sigma\rangle \\ &= \left\langle \begin{smallmatrix} c_1 1 & \cdots & c_i & \cdots & c_j & \cdots & c_n \\ \bar{c}_1 & \cdots & \bar{c}_i & \cdots & \bar{c}_j & \cdots & \bar{c}_n \end{smallmatrix} ; \sigma \left| \begin{smallmatrix} c_1 & \cdots & c_i & \cdots & c_j & \cdots & c_n \\ \bar{c}_1 & \cdots & \bar{c}_i & \cdots & \bar{c}_j & \cdots & \bar{c}_n \end{smallmatrix} ; \mathbf{s}_{ij} \circ \sigma \right. \right\rangle \\ &= \delta^{\bar{c}_{\sigma(1)}}_{c_1} \cdots \delta^{\bar{c}_{\sigma(i)}}_{c_i} \cdots \delta^{\bar{c}_{\sigma(j)}}_{c_j} \cdots \delta^{\bar{c}_{\sigma(n)}}_{c_n} \delta^{c_1}_{\bar{c}_i} \cdots \delta^{c_i}_{\bar{c}_{\sigma(j)}} \cdots \delta^{c_j}_{\bar{c}_{\sigma(i)}} \cdots \delta^{c_n}_{\bar{c}_{\sigma(n)}} = N^{n-1}. \end{aligned} \quad (4.43)$$

Now for the case when σ and τ differ by $m + 1 \leq n!$ transpositions, while assuming that formula (4.39) is right. When they differ by $m + 1$ transpositions, then there are $m + 1$ transposition operators $\{\mathbf{s}_{i_k j_k}\}_{k=1}^{m+1}$ such that

$$\langle \sigma | \tau \rangle = \langle \sigma | \mathbf{s}_{i_1 j_1} \cdots \mathbf{s}_{i_{m+1} j_{m+1}} | \sigma \rangle. \quad (4.44)$$

Then one has to recognize that the colour state $\mathbf{s}_{i_2 j_2} \cdots \mathbf{s}_{i_{m+1} j_{m+1}} | \sigma \rangle$ differs from $| \sigma \rangle$ by one transposition and since it was assumed that (4.39) holds, when the states differ by m transpositions the result follows with the base case (4.43). For the following it is useful to introduce the scalar-product matrix S , the components $S_{\sigma\tau}$ of the matrix are defined through

$$S_{\sigma\tau} = \langle \sigma | \tau \rangle, \quad (4.45)$$

in the literature the matrix is also referred to as inner product matrix or metric. From (4.39) it follows that S has full rank and is therefore invertable. With the formula (4.39) it is also clear that neither the colour states nor the abstract colour states are mutual orthogonal to each other. To fix the issue of calculating in a non-orthogonal basis it is possible to introduce the so called dual basis spanning set of the colour-flow basis, as follows

$$\left\langle \begin{matrix} c_1 & \cdots & c_m \\ \bar{c}_1 & \cdots & \bar{c}_m \end{matrix} \middle| \sigma \right\rangle = \sum_{\tau} S_{\sigma\tau}^{-1} \left\langle \begin{matrix} c_1 & \cdots & c_m \\ \bar{c}_1 & \cdots & \bar{c}_m \end{matrix} \middle| \tau \right\rangle, \quad (4.46)$$

such that

$$\text{Tr} \left[\left\langle \begin{matrix} c_1 & \cdots & c_m \\ \bar{c}_1 & \cdots & \bar{c}_m \end{matrix} \middle| \sigma \right\rangle \left\langle \begin{matrix} c_1 & \cdots & c_m \\ \bar{c}_1 & \cdots & \bar{c}_m \end{matrix} \middle| \tau \right\rangle \right] = \delta_{\sigma\tau}. \quad (4.47)$$

Multiplying equation (4.46) with $S_{\alpha\sigma}$ and summing over index σ gives

$$\left| \begin{matrix} c_1 & \cdots & c_m \\ \bar{c}_1 & \cdots & \bar{c}_m \end{matrix} ; \alpha \right\rangle = \sum_{\rho} S_{\alpha\rho} \left| \begin{matrix} c_1 & \cdots & c_m \\ \bar{c}_1 & \cdots & \bar{c}_m \end{matrix} ; \rho \right\rangle. \quad (4.48)$$

The abstract dual tensor is then simply denoted by

$$| \tau], \quad (4.49)$$

where again no choice of colour was made and the label $\tau \in S_n$. With the definition of the scalar-product matrix the identity operator $\mathbb{I}$ of the colour space in the colour-flow basis is then given by

$$\mathbb{I} = \sum_{\sigma, \tau} S_{\sigma\tau}^{-1} \left| \begin{matrix} c_1 & \cdots & c_m \\ \bar{c}_1 & \cdots & \bar{c}_m \end{matrix} ; \sigma \right\rangle \left\langle \begin{matrix} c_1 & \cdots & c_m \\ \bar{c}_1 & \cdots & \bar{c}_m \end{matrix} ; \tau \right|, \quad (4.50)$$

since for an arbitrary basis state with label ρ

$$\begin{aligned}\mathbb{I} \left| \begin{smallmatrix} c_1 & \dots & c_m \\ \bar{c}_1 & \dots & \bar{c}_m \end{smallmatrix} ; \rho \right\rangle &= \sum_{\sigma, \tau} S_{\sigma\tau}^{-1} \left| \begin{smallmatrix} c_1 & \dots & c_m \\ \bar{c}_1 & \dots & \bar{c}_m \end{smallmatrix} ; \sigma \right\rangle \left\langle \begin{smallmatrix} c_1 & \dots & c_m \\ \bar{c}_1 & \dots & \bar{c}_m \end{smallmatrix} ; \tau \right| \left| \begin{smallmatrix} c_1 & \dots & c_m \\ \bar{c}_1 & \dots & \bar{c}_m \end{smallmatrix} ; \rho \right\rangle \\ &= \sum_{\sigma, \tau} S_{\sigma\tau}^{-1} S_{\tau\rho} \left| \begin{smallmatrix} c_1 & \dots & c_m \\ \bar{c}_1 & \dots & \bar{c}_m \end{smallmatrix} ; \sigma \right\rangle = \left| \begin{smallmatrix} c_1 & \dots & c_m \\ \bar{c}_1 & \dots & \bar{c}_m \end{smallmatrix} ; \rho \right\rangle.\end{aligned}\quad (4.51)$$

The identity operator $\mathbb{I}$ can also, with the definition of the dual tensor, be written as

$$\mathbb{I} = \sum_{\rho} \left| \begin{smallmatrix} c_1 & \dots & c_m \\ \bar{c}_1 & \dots & \bar{c}_m \end{smallmatrix} ; \rho \right\rangle \left[\begin{smallmatrix} c_1 & \dots & c_m \\ \bar{c}_1 & \dots & \bar{c}_m \end{smallmatrix} ; \rho \right] = \sum_{\rho} \left| \begin{smallmatrix} c_1 & \dots & c_m \\ \bar{c}_1 & \dots & \bar{c}_m \end{smallmatrix} ; \rho \right\rangle \left\langle \begin{smallmatrix} c_1 & \dots & c_m \\ \bar{c}_1 & \dots & \bar{c}_m \end{smallmatrix} ; \rho \right|.\quad (4.52)$$

This representation of the identity is quite useful and will be used quite regularly, for example by inserting this representation one could derive from

$$\delta_{\sigma\tau} = [\sigma|\tau] = \sum_{\rho} [\sigma|\rho] \langle \rho|\tau \rangle = \sum_{\rho} [\sigma|\rho] S_{\rho\tau},\quad (4.53)$$

that

$$[\sigma|\rho] = S_{\sigma\rho}^{-1}.\quad (4.54)$$

A linear operator $\mathbf{A}$, defined on the colour space, can then be decomposed as a linear combination of the abstract colour states

$$\mathbf{A} = \sum_{\tau, \sigma} \mathcal{A}_{\tau, \sigma} |\tau\rangle \langle \sigma|,\quad (4.55)$$

where the coefficients $\mathcal{A}_{\tau, \sigma}$ are basis dependent. Given the operator $\mathbf{A}$, the coefficients can be calculated with the dual basis as

$$[\alpha|\mathbf{A}|\beta] = \sum_{\tau, \sigma} \mathcal{A}_{\tau\sigma} [\alpha|\tau] \langle \sigma|\beta \rangle = \mathcal{A}_{\alpha\beta}.\quad (4.56)$$

The coefficients $\mathcal{A}_{\alpha\beta}$ are then the matrix representation of the colour space operator $\mathbf{A}$ [28]. The trace of the operator $\mathbf{A}$ in the non-orthogonal colour-flow basis is then given by

$$\text{Tr}[\mathbf{A}] = \sum_{\sigma} [\sigma|\mathbf{A}|\sigma] = \sum_{\sigma, \tau} [\sigma|\mathbf{A}|\tau] \langle \tau|\sigma \rangle = \sum_{\sigma, \tau} \mathcal{A}_{\sigma\tau} S_{\tau\sigma} = \text{Tr}[\mathcal{A}S].\quad (4.57)$$

This relation is important, since this work aims to describe how the amplitude can be simulated on the amplitude level in the colour-flow basis and with the following relation it follows that

$$|\mathcal{M}|^2 = \langle \mathcal{M}|\mathcal{M} \rangle = \text{Tr}[\mathcal{M}]\langle \mathcal{M} \rangle = \text{Tr}[\mathbf{M}S],\quad (4.58)$$

where $\mathbf{M}_{\alpha\beta} = \mathcal{M}_{\alpha} \mathcal{M}_{\beta}^*$ is the representation matrix of the amplitude, interpreted as an operator defined on the colour space. A general linear operator $\mathbf{A}$ in the colour-flow basis then acts on

the abstract colour states as

$$\mathbf{A}|\sigma\rangle = \sum_{\alpha,\beta} |\alpha\rangle [\alpha|\mathbf{A}|\beta] \langle\beta|\sigma\rangle = \sum_{\alpha,\beta} \mathcal{A}_{\alpha\beta} S_{\beta\sigma} |\alpha\rangle. \quad (4.59)$$

A concrete example for such a linear operator is the anomalous dimension operator $\mathbf{\Gamma}$. The operator can be written as an expansion in 't Hooft coupling [21],

$$[\tau|\mathbf{T}|\sigma\rangle = (\alpha_s N) [\tau|\mathbf{T}^{(1)}|\sigma\rangle + (\alpha_s N)^2 [\tau|\mathbf{T}^{(2)}|\sigma\rangle + \dots, \quad (4.60)$$

whereas the leading contribution $\mathbf{T}^{(1)}$ is given by

$$\bar{\Gamma}_{\tau\sigma} \equiv [\tau|\mathbf{T}|\sigma\rangle = \left(\Gamma_{\sigma}\delta_{\sigma\tau} + \frac{1}{N}\Sigma_{\tau\sigma} + \frac{1}{N^2}\rho\delta_{\tau\sigma} \right). \quad (4.61)$$

Here the coefficients $\Sigma_{\tau\sigma}$ give the off-diagonal components and the terms with the $\delta_{\sigma\tau}$ denote diagonal elements. The matrix elements of the anomalous dimension in the colour-flow basis can then be obtained by determining

$$\begin{aligned} \langle\alpha|\mathbf{\Gamma}|\beta\rangle &= \sum_{\lambda} S_{\alpha\lambda} \bar{\Gamma}_{\lambda\beta}, \\ [\alpha|\mathbf{\Gamma}|\beta] &= \sum_{\lambda} \bar{\Gamma}_{\alpha\lambda} S_{\lambda\beta}^{-1}. \end{aligned} \quad (4.62)$$

Iterativley inserting the identity operator gives

$$\begin{aligned} \langle\alpha|\mathbf{\Gamma}^n|\beta\rangle &= \sum_{\lambda} S_{\alpha\lambda} \bar{\Gamma}_{\lambda\beta}^n, \\ [\alpha|\mathbf{\Gamma}^n|\beta] &= \sum_{\lambda} \bar{\Gamma}_{\alpha\lambda}^n S_{\lambda\beta}^{-1}, \end{aligned} \quad (4.63)$$

for any non-negative integer n . Consequently, the anomalous dimension operator $\mathbf{\Gamma}$ exponentiates in the following way

$$\begin{aligned} \langle\alpha|e^{\mathbf{\Gamma}}|\beta\rangle &= \sum_{\lambda} S_{\alpha\lambda} (e^{\mathbf{\Gamma}})_{\lambda\beta}, \\ [\alpha|e^{\mathbf{\Gamma}}|\beta] &= \sum_{\lambda} (e^{\mathbf{\Gamma}})_{\alpha\lambda} S_{\lambda\beta}^{-1}. \end{aligned} \quad (4.64)$$

Calculating the powers of a general linear operator $\mathbf{A}$ would work in the same way as it was demonstrated here with the anomalous dimension operator. The following aims to investigate how the emission operator $\mathbf{T}_i$ acts on the colour-flow basis states, for that consider the emission of a soft gluon from a hard quark line i , recall that

$$\epsilon_a^*(t^a)^{c_i}_{\bar{c}_i} = \epsilon_a^* \delta^{ab} (t^a)^{c_i}_{\bar{c}_i} = \frac{1}{\sqrt{T_R}} (\epsilon^*)^{\bar{c}_n}_{c_n} ((t^a)^{c_n}_{\bar{c}_n} (t^a)^{c_i}_{\bar{c}_i}), \quad (4.65)$$

where the polarisation tensor is written as a linear combination of generators $\sqrt{T_R}(\epsilon^*)^{\bar{c}_n}_{c_n} = \epsilon_a^*(t^a)^{\bar{c}_n}_{c_n}$. The colour amplitude of soft gluon emission from a hard quark line i is therefore the expression in (4.65) without the polarisation tensor. For a simple illustration of the action of the colour space operator a concrete example is chosen, where $i = 2$ is a hard quark and that there are $n = 3$ colour lines and the state has label $\sigma = (213)$, then, by comparing with (4.15) and

(4.65) it follows

$$\begin{aligned}
\mathbf{T}_i \Big|_{\bar{c}_1 \bar{c}_2 \bar{c}_3}^{c_1 c_2 c_3}; \sigma \Big\rangle &= (\mathbf{C}_i^a \circ \mathbf{T}^a_i) \Big|_{\bar{c}_1 \bar{c}_2 \bar{c}_3}^{c_1 c_2 c_3}; \sigma \Big\rangle \\
&= \frac{1}{\sqrt{T_R}} \sum_{b_2} (t^a)^{c_4}_{\bar{c}_4} (t^a)^{c_2}_{b_2} \Big|_{\bar{c}_1 \bar{c}_2 \bar{c}_3}^{c_1 b_2 c_3}; (2, 1, 3) \Big\rangle \\
&= \frac{1}{\sqrt{T_R}} \sum_{b_2} (t^a)^{c_4}_{\bar{c}_4} (t^a)^{c_2}_{b_2} \delta_{\bar{c}_2}^{c_1} \delta_{\bar{c}_1}^{b_2} \delta_{\bar{c}_3}^{c_3} \\
&= \sqrt{T_R} \left(\delta_{\bar{c}_1}^{c_4} \delta_{\bar{c}_4}^{c_2} \delta_{\bar{c}_3}^{c_1} \delta_{\bar{c}_3}^{c_3} - \frac{1}{N} \delta_{\bar{c}_4}^{c_4} \delta_{\bar{c}_1}^{c_2} \delta_{\bar{c}_2}^{c_1} \delta_{\bar{c}_3}^{c_3} \right) \\
&= \sqrt{T_R} \left(\Big|_{\bar{c}_1 \bar{c}_2 \bar{c}_3 \bar{c}_4}^{c_1 c_2 c_3 c_4}; (2, 4, 3, 1) \Big\rangle - \frac{1}{N} \Big|_{\bar{c}_1 \bar{c}_2 \bar{c}_3 \bar{c}_4}^{c_1 c_2 c_3 c_4}; (2, 1, 3, 4) \Big\rangle \right),
\end{aligned}$$

where the Fierz identity was used to obtain the result from the second to the third line. The above example can easily be generalized and therefore, if i is hard quark line

$$\mathbf{C}_i^a \Big|_{\bar{c}_1 \dots \bar{c}_m}^{c_1 \dots c_m}; \sigma \Big\rangle = (t^a)^{c_{m+1}}_{\bar{c}_{m+1}} \Big|_{\bar{c}_1 \dots \bar{c}_m}^{c_1 \dots c_m}; \sigma \Big\rangle, \quad (4.66)$$

and

$$\mathbf{T}^a_i \Big|_{\bar{c}_1 \dots \bar{c}_i \dots \bar{c}_m}^{c_1 \dots c_i \dots c_m}; \sigma \Big\rangle = \frac{1}{\sqrt{T_R}} \sum_{f_i} (t^a)^{c_i}_{f_i} \Big|_{\bar{c}_1 \dots \bar{c}_i \dots \bar{c}_m}^{c_1 \dots f_i \dots c_m}; \sigma \Big\rangle. \quad (4.67)$$

So, in total if m is the number of colour lines, $\sigma \in S_m$ the label of the basis tensor and i is a hard quark line, the action of the colour operator reads

$$\begin{aligned}
\mathbf{T}_i \Big|_{\bar{c}_1 \dots \bar{c}_i \dots \bar{c}_m}^{c_1 \dots c_i \dots c_m}; \sigma \Big\rangle &= \frac{1}{\sqrt{T_R}} \sum_{f_i} (t^a)^{c_{m+1}}_{\bar{c}_{m+1}} (t^a)^{c_i}_{f_i} \Big|_{\bar{c}_1 \dots \bar{c}_i \dots \bar{c}_m}^{c_1 \dots f_i \dots c_m}; \sigma \Big\rangle \\
&= \sqrt{T_R} \sum_{f_i} \left(\delta^{c_{m+1}}_{f_i} \delta^{c_i}_{\bar{c}_{m+1}} - \frac{1}{N} \delta^{c_{m+1}}_{\bar{c}_{m+1}} \delta^{c_i}_{f_i} \right) \Big|_{\bar{c}_1 \dots \bar{c}_i \dots \bar{c}_m}^{c_1 \dots f_i \dots c_m}; \sigma \Big\rangle \\
&= \sqrt{T_R} \Big|_{\bar{c}_1 \dots \bar{c}_m \bar{c}_{m+1}}^{c_1 \dots c_m c_{m+1}}; s_{i,m+1} \circ (\sigma(1) \dots \sigma(m) m+1) \Big\rangle \\
&\quad - \frac{\sqrt{T_R}}{N} \Big|_{\bar{c}_1 \dots \bar{c}_m \bar{c}_{m+1}}^{c_1 \dots c_m c_{m+1}}; (\sigma(1) \dots \sigma(m) m+1) \Big\rangle. \quad (4.68)
\end{aligned}$$

Similar calculations can be done when i is an anti-quark line or a gluon line. The action of the colour operator on the colour basis states can then be summarized with the following expression [4]

$$\mathbf{T}_i = \lambda_i \mathbf{t}_{c_i} - \bar{\lambda}_i \bar{\mathbf{t}}_{\bar{c}_i} - \frac{1}{N} (\lambda_i - \bar{\lambda}_i) \mathbf{s}, \quad (4.69)$$

where i denotes the colour line on which the operator acts. The quantities λ_i and $\bar{\lambda}$ are dependent on the kind of parton line on which the colour charge operator is acting

$$\lambda_i = \begin{cases} \sqrt{T_R} & \text{i is a quark or gluon line} \\ 0 & \text{else} \end{cases} \quad (4.70)$$

and

$$\bar{\lambda}_i = \begin{cases} \sqrt{T_R} & \text{i is an anti-quark or gluon line} \\ 0 & \text{else.} \end{cases} \quad (4.71)$$

The colour-line operators $\mathbf{t}_{c_i}$, $\bar{\mathbf{t}}_{\bar{c}_i}$ and $\mathbf{s}$ are defined through

$$\begin{aligned} \mathbf{t}_i|\sigma\rangle &= \mathbf{t}_i \left| \begin{array}{cccccc} 1 & 2 & \cdots & i & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(i) & \cdots & \sigma(n) \end{array} \right\rangle \\ &= \left| \begin{array}{cccccc} 1 & 2 & \cdots & i & \cdots & n & n+1 \\ \sigma(1) & \sigma(2) & \cdots & n+1 & \cdots & \sigma(n) & \sigma(i) \end{array} \right\rangle, \\ \bar{\mathbf{t}}_i|\sigma\rangle &= \mathbf{t}_{\sigma^{-1}(i)}|\sigma\rangle, \\ \mathbf{s}|\sigma\rangle &= \mathbf{s} \left| \begin{array}{ccc} 1 & \cdots & n \\ \sigma(1) & \cdots & \sigma(n) \end{array} \right\rangle \\ &= \left| \begin{array}{cccc} 1 & \cdots & n & n+1 \\ \sigma(1) & \cdots & \sigma(n) & n+1 \end{array} \right\rangle. \end{aligned} \quad (4.72)$$

Furthermore, it is useful to define a transposition operator $\mathbf{s}_{ij}$ through

$$\begin{aligned} \mathbf{s}_{ij}|\sigma\rangle &= \mathbf{s}_{ij} \left| \begin{array}{cccccc} 1 & \cdots & i & \cdots & j & n \\ \sigma(1) & \cdots & \sigma(i) & \cdots & \sigma(j) & \cdots & \sigma(n) \end{array} \right\rangle \\ &= \left| \begin{array}{cccccc} 1 & \cdots & i & \cdots & j & n \\ \sigma(1) & \cdots & \sigma(j) & \cdots & \sigma(i) & \cdots & \sigma(n) \end{array} \right\rangle. \end{aligned} \quad (4.73)$$

Then, the operator $\mathbf{t}_i$ can be decomposed as

$$\mathbf{t}_i|\tau\rangle = \mathbf{s}_{i,n+1}\mathbf{s}|\tau\rangle, \quad (4.74)$$

which is quite useful in various calculations. Let's consider two distinct permutations $\sigma, \tau \in S_n$, then the following is true [4],

- The permutations, which label the basis tensors $\mathbf{t}_\alpha|\sigma\rangle$ and $\mathbf{s}|\tau\rangle$ differ by $n+1$ transpositions.
- The permutations, which label the basis tensors $\mathbf{s}|\sigma\rangle$ and $\mathbf{s}|\tau\rangle$ differ by the same n transpositions than before the operator $\mathbf{s}$.

In [4], they also state that if $\alpha \neq \beta$, then the labels of $\mathbf{t}_\alpha|\sigma\rangle$ and $\mathbf{t}_\beta|\tau\rangle$ differ by $n+2$ permutations if $\sigma(\alpha) \neq \tau(\beta)$ and by n transpositions if $\sigma(\alpha) = \tau(\beta)$. This is in general not true, since a counter example could be found.

It is important to note that $\mathbf{t}_i$, $\bar{\mathbf{t}}_i$ and $\mathbf{s}$ are surjective maps.

5 Numerical resummation

In order to be able to simulate soft-gluon evolution, a representation of the emission operators has to be chosen. The evolution of amplitude then gets mapped to a purely probabilistic system by modelling the amplitude through a Markov chain

$$X_{\text{initial}} \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_{\text{final}}, \quad (5.1)$$

where the system starts from a chosen initial configuration $X_{\text{initial}} = X_{\text{initial}}(Q)$, including an initial scale Q and evolves until a certain chosen cutoff scale μ is reached $X_{\text{final}} = X_{\text{final}}(\mu)$, whereas the intermediate evolution scales are ordered $Q \geq q_1 \geq q_2 \geq \cdots \geq \mu$. In principle, the type of evolution scale depends on the type of the constructed shower, in an angular shower the evolution parameter is the angle of the dipoles. In this work the evolution scale is the energy of the emitted soft gluon/gluons. In this chapter the different Sudakov algorithms get reviewed. These algorithms allow to sample the evolution scale q , which is distributed according to a density $dS(q, \mu|Q)$. Furthermore it will be discussed how to add density distribution dS_i such that a shower algorithm could be constructed. Also resampling techniques will be discussed, which allow to improve the convergence of the numerical resummation. The density $dS(q, \mu|Q)$ is constructed from the Sudakov operator and by introducing the cutoff scale μ . The Sudakov operators for a dipole with label (i, j) is then of the following form,

$$\Delta_{(ij)}(q|Q) = \exp \left(-\frac{\alpha_s N}{\pi} \int_q^Q \frac{dk}{k} \int d\Omega \omega_{ij}(\Omega) \right) \quad (5.2)$$

with $\omega_{ij}(\Omega)$ defined as

$$\omega_{ij}(\Omega) = \frac{n_i \cdot n_j}{n_i \cdot n n_j \cdot n} \Theta_{ij}(\Omega), \quad (5.3)$$

where the four vectors are parametrized as $n = (1, \vec{n})$, $n_{i,j} = (1, \vec{n}_{i,j})$ and $\Theta_{ij}(\Omega)$ is the respective cutoff description of the collinear singularity of the parton pair with label (i, j) . The function in the exponent is sometimes referred to as a splitting rate $P_x(k)$, where x labels the type of splitting produced by the parton pair with label (i, j) . The four angle cutoff descriptions, which are investigated in this work are given in the following, the cutoff parameter $\lambda > 0$ should be understood as small $\lambda \ll 1$.

- (i). $\Theta_{ij}(\Omega) = \frac{n_j \cdot n}{n_i \cdot n + n_j \cdot n} \Theta(n_i \cdot n - \lambda) + (i \leftrightarrow j)$ (local dipole)
- (ii). $\Theta_{ij}(\Omega) = \Theta(n_i \cdot n - \lambda) \Theta(n_j \cdot n - \lambda)$ (local dipole)
- (iii). $\Theta_{ij}(\Omega) = \prod_k \Theta(n_k \cdot n - \lambda)$ (global cutoff)
- (iv). $\Theta_{ij}(\Omega) = 1 - \frac{n_i \cdot n n_j \cdot n}{n_i \cdot n_j} \left(\frac{1}{n_i \cdot n} + \frac{1}{n_j \cdot n} \right)$ (local dipole)

Figure 5: List of collinear cutoffs.

The densities constructed from the Sudakov operators with the first three cutoff prescriptions will be generated with the Sudakov algorithm, which uses an overestimate. The fourth cutoff description poses an additional problem in the sense that the splitting rate is possibly negative,

therefore, as it will be discussed later, a generalization of the Sudakov algorithm is needed, which results in the weighted Sudakov algorithm. Note that the integral in the Sudakov factor (5.2) is invariant under rotation of the spatial coordinates but not invariant under general Lorentz transformations, since one could easily come up for example with a situation where two frames are connected through a Lorentz transformation, which includes a boost where in the one frame the cones, the area where the theta function is zero, do not intersect and in the other frame they do intersect. A careful analysis of the arising integrals where done in appendix D.

5.1 Sudakov veto algorithm

In a more general shower it is necessary to generate the dynamic variables (q, z, x) , where q is the scale at which the splitting occurs, z is the splitting variable and x is a label for the type of branching. The main problem to generate these dynamic variables is that q is distributed according to the density $dS_{P_x}(q, z|Q)$, where Q is the hard scale at which the process starts. The input for the construction of the density $dS_{P_x}(q, z|Q)$ from pQCD are the branching rates $P_x(q, z)$, which then fully determine the probability density. In principle, the branching rate $P_x(q, z)$ could also be dependent for example on the direction or directions of the emitted parton/partons, but this possible dependency will be suppressed in order to keep the construction as transparent as possible.

The central object for the construction of the density is the so-called Sudakov form factor[29] $\Delta_{P_x}(q|Q)$, defined as

$$\Delta_{P_x}(q|Q) = \exp \left(- \int_q^Q d\xi \int dz P_x(\xi, z) \right), \quad (5.4)$$

which gives the probability that splitting of type x does not occur in the interval $[q, Q]$. The probability that none of the possible splitting occurs in the interval $[q, Q]$ is then given by

$$\Delta_P(q|Q) = \prod_x \Delta_{P_x}(q|Q), \quad (5.5)$$

where on the left hand side $P = \sum_x P_x$ is the sum of the splitting rates. From the definition it follows that the Sudakov form factor $\Delta_{P_x}(q|Q)$ could also be defined through a differential equation

$$\frac{d\Delta_{P_x}(q|Q)}{dq} = - \int dz P_x(q, z) \Delta_{P_x}(q|Q), \quad (5.6)$$

satisfying the initial condition $\Delta_{P_x}(Q|Q) = 1$. Now with a simple unitary argument, namely that the probability for emission and non-emission has to add up to one, it follows that the density for the dynamical branching variables (q, z, x) is given by [30]

$$\frac{dS_{P_x}(q, z, \mu|Q)}{dq dz} = \Delta_{P_x}(\mu|Q) \delta(q - \mu) \delta(z - z_{\min}) + P_x(q, z) \Delta_P(q|Q, x) \theta(Q - q) \theta(q - \mu), \quad (5.7)$$

where $\Delta_{P_x}(\mu|Q) \delta(q - \mu) \delta(z - z_{\min})$ accounts for the non-emission events until the chosen cut-off scale μ , the parameter $z_{\min}$ associates fixed branching variables to μ , the scale where there are no branching events. It is important to mention that despite the fact that for the splitting

rates $P_x(q, z)$ only real emissions were considered, one also takes one loop corrections into account through the above unitary argument. The question which arises at this point is how to generate dynamic variables (q, z, x) distributed according to the distribution in (5.7). If $P_x(q, z)$ is integrable to an invertible function, then sampling by inversion can be used. For simplicity lets assume $\int dz P_x(q, z)$ is integrable to an invertible function, then the following equation can be solved with respect to q

$$\int_{-\infty}^q dk \int dz \frac{dS_{P_x}(k, z, \mu|Q)}{dk} = \Delta_{P_x}(q|Q) \theta(q - \mu) = \mathbf{rnd}, \quad (5.8)$$

where $\mathbf{rnd} \sim U(0, 1)$. In this particular simple case the Sudakov veto algorithm to draw according to the density $dS_{P_x}(q, \mu|Q)$ goes as follows [30, 31, 32].

Algorithm 1. The Sudakov algorithm, to generate the density $dS_{P_x}(q, \mu|Q)$ for a splitting rate $P_x(q) = \int dz P_x(q, z)$, which is integrable to a invertible function with respect to q .

Select a starting scale $Q' \leftarrow Q$ and a cutoff μ .

Sample $r \leftarrow \mathbf{rnd} \sim U(0, 1)$.

if $r \leq \Delta_{P_x}(\mu|Q', x)$.

return cutoff μ

else invert $\Delta_{P_x}(q|Q', x)$ such that $q = \Delta_{P_x}^{-1}(r|Q', x)$

return emission scale q

end if

The correctness of the algorithm follows directly from the correctness of the sample by the inversion algorithm, already presented in section 2.3. In general, the splitting rate $P_x(q, z)$ is not integrable to an invertible function, so the presented algorithm could not be applied. In these situations it is possible to use the following Sudakov algorithm, where one samples the dynamic variables (q, z, x) according to an positive splitting rate $R_x(q, z)$, which is an overestimate of the target splitting rate $P_x(q, z)$ such that $R_x(q, z) \geq P_x(q, z)$ for all (q, z, x) and also integrable to an invertible function such that algorithm 1 could be applied. The proposed dynamic variables (q, z, x) are then accepted with probability $P_x(q, z)/R_x(q, z)$. The algorithm is described in 2.

Algorithm 2. The Sudakov algorithm to generate dynamical variables (q, z, x) according to the density $dS_{P_x}(q, z, \mu|Q)$ by using an overestimate such that $R_x(q, z) \geq P_x(q, z)$ for all (q, z, x) .

Select a starting scale $Q' \leftarrow Q$

loop

Sample $r \leftarrow \text{rnd}$

if $r \leq \Delta_R(\mu|Q', x)$

return cutoff μ

else invert $\Delta_R(q|Q', x)$ such that $q = \Delta_R^{-1}(r|Q', x)$

and select z in proportion to $R(q, z, x)$

return emission scale q with probability $P(q, z, x)/R(q, z, x)$

end if

$Q' \leftarrow q$

end loop

The efficiency of the algorithm 2 is strongly dependent on quality of the overestimate, which means that if the ration $P_x(q, z)/R_x(q, z)$ is close to one then nearly all proposed dynamic variables (q, z, x) get accepted. In order to proof that the algorithm produces the correct density $dS_{P_x}(q, z, \mu|Q)$, the following theorem [30] has to be proven.

Theorem 1. The density in the variable q , generated by the Sudakov veto algorithm after the n -th rejection and one final acceptance step, is given by

$$\begin{aligned} \frac{dS_{\text{veto}}^{(n)}(q, z, \mu|Q)}{dq dz} &= \Delta_{R_x}(\mu|Q) \delta(q - \mu) \delta(z - z_{\min}) \Delta_{P_x - R_x}^{(n)}(\mu|Q) \\ &\quad + \theta(Q - q) \theta(q - \mu) P_x(q, z) \Delta_{R_x}(q|Q) \Delta_{P_x - R_x}^{(n)}(q|Q), \end{aligned} \quad (5.9)$$

where

$$\Delta_{P_x - R_x}^{(n)}(q|Q) = \frac{1}{n!} \left(- \int_q^Q d\xi \int dz (P_x(\xi, z) - R_x(\xi, z)) \right)^n. \quad (5.10)$$

Correctness of the algorithm follows then immediately by noting that one has to sum over all possible rejection steps and by exploiting the relation $\Delta_{R_x}(q|Q) \Delta_{P_x - R_x}(q|Q) = \Delta_{P_x}(q|Q)$. To show theorem 1, induction will be used. One has to note that the probability to accept $dS^{\text{accept}}(q, z, \mu|k, Q)$, an event, starting from an arbitrary intermediate scale $k \in [\mu, Q]$, is given by the probability that the event did not radiate plus the probability that the event radiates

$$dS^{\text{accept}}(q, z, \mu|k) = \Delta_{R_x}(\mu|k) \delta(q - \mu) \delta(z - z_{\min}) + \theta(k - q) \theta(q - \mu) P(q, z, x) \Delta_{R_x}(q|k), \quad (5.11)$$

where radiating events are accepted with probability $P(q, z, x)/R(q, z, x)$. This shows already

the base case $n = 0$ for the induction proof, when setting the intermediate scale k equal to the starting scale Q

$$d S_{\text{veto}}^{(0)}(q, z, \mu|Q) = d S^{\text{accept}}(q, z, \mu|Q). \quad (5.12)$$

Now the induction step $n \rightarrow n+1$, the q density for $n+1$ rejection steps and one final acceptance step is

$$\begin{aligned} d S_{\text{veto}}^{(n+1)}(q, z, \mu|Q) &= \int_{\mu}^Q dk d S_{\text{veto}}^{(n)}(q, z, \mu|k) \int dz (1 - P_x(k, z)/R_x(k, z)) R_x(k, z) \Delta_{R_x}(k|Q) \\ &= \int_{\mu}^Q dk d S_{\text{veto}}^{(n)}(q, z, \mu|k) \int dz (R_x(k, z) - P_x(k, z)) \Delta_{R_x}(k|Q), \end{aligned} \quad (5.13)$$

where $(1 - P_x(k, z)/R_x(k, z))$ is the probability that the event was rejected, also called the veto probability and $R_x(k, z)$ the splitting rate according the events were proposed during the algorithm. The expression for $d S_{\text{veto}}^{(n)}(q, z, \mu|Q,)$ in (5.13) reflects a proposed event at scale $k \in [\mu, Q]$, proposed with density $R(k, z, x) \Delta_{R_x}(k|Q)$, which was rejected with the veto probability $(1 - P_x(k, z)/R_x(k, z))$ and then enters a sequence of n veto steps and a final acceptance step. To obtain the final result in theorem 1 and therefore conclude the induction proof it is useful to use

$$\frac{1}{n!} \int_q^Q \left(\int_q^k f(k') dk' \right)^n f(k) dk = \frac{1}{(n+1)!} \left(\int_q^Q f(k) dk \right)^{n+1}, \quad (5.14)$$

which was shown in appendix B and the chain property of $\Delta_{R_x}(q|Q)$ for an intermediate scale $k \in [q, Q]$

$$\Delta_{R_x}(q|Q) = \Delta_{R_x}(q|k) \Delta_{R_x}(k|Q). \quad (5.15)$$

So in order to apply the Sudakov algorithm 2 to the first collinear cutoff description 5, an overestimate R has to be developed,

$$\begin{aligned} \int d\Omega \omega_{ij}(\Omega) &= \int d\Omega \left(\frac{n_i \cdot n_j}{n_i \cdot n} \frac{1}{n_i \cdot n + n_j \cdot n} \Theta(n_i \cdot n - \lambda) + (i \leftrightarrow j) \right) \\ &\leq \int d\Omega \left(\frac{n_i \cdot n_j}{\lambda} \frac{1}{\lambda + n_j \cdot n} + (i \leftrightarrow j) \right) \\ &= \frac{4\pi n_i \cdot n_j}{\lambda} \ln \left(\frac{2 + \lambda}{\lambda} \right). \end{aligned} \quad (5.16)$$

The differential of the overestimating rate R is therefore given by

$$dR(q, \Omega) = \frac{dq}{q} \frac{\alpha_s N}{\pi} d\Omega \frac{n_i \cdot n_j}{\lambda} \left(\frac{1}{\lambda + n_j \cdot n} + \frac{1}{\lambda + n_i \cdot n} \right) \quad (5.17)$$

and the Sudakov factor $\Delta_R(q|Q)$ reads

$$\Delta_R(q|Q) = \exp \left(-\frac{\alpha_s N}{\pi} \ln \left(\frac{Q}{q} \right) \frac{4\pi n_i \cdot n_j}{\lambda} \ln \left(\frac{2 + \lambda}{\lambda} \right) \right). \quad (5.18)$$

To propose a scale q , it is necessary to solve $rnd = \Delta_R(q|Q)$ with respect to q in the case that

$$rnd \leq \Delta_R(q|Q)$$

$$q = Q \exp \left(\frac{\lambda}{4\alpha_s N n_i \cdot n_j \ln \left(\frac{2+\lambda}{\lambda} \right)} \ln(rnd) \right). \quad (5.19)$$

The probability to accept the proposed scale q is then given by

$$\frac{dP(q, \Omega)}{dR(q, \Omega)} = \frac{\left(\frac{1}{n_i \cdot n} \frac{\Theta(n_i \cdot n - \lambda)}{n_i \cdot n + n_j \cdot n} + \frac{1}{n_j \cdot n} \frac{\Theta(n_j \cdot n - \lambda)}{n_i \cdot n + n_j \cdot n} \right)}{\frac{1}{\lambda} \left(\frac{1}{\lambda + n_j \cdot n} + \frac{1}{\lambda + n_i \cdot n} \right)}. \quad (5.20)$$

The same procedure can be applied for the second collinear cutoff 5. The overestimating splitting rate $R(q)$ can then be deduced from,

$$\begin{aligned} \int d\Omega \omega_{ij}(\Omega) &= \int d\Omega \frac{n_i \cdot n_j}{n_i \cdot n n_j \cdot n} \Theta(n \cdot n_i - \lambda) \Theta(n \cdot n_j - \lambda) \\ &\leq \frac{n_i \cdot n_j}{\lambda^2} \int d\Omega \Theta(n \cdot n_i - \lambda) \Theta(n \cdot n_j - \lambda) \\ &\leq \frac{n_i \cdot n_j}{\lambda^2} \int d\Omega (1 - \Theta(\lambda - n_i \cdot n) - \Theta(\lambda - n_j \cdot n)) \\ &= \frac{4\pi n_i \cdot n_j}{\lambda^2} (1 - \lambda). \end{aligned} \quad (5.21)$$

Then the differential of the overestimating rate R for the second cutoff is given by

$$dR(q, \Omega) = \frac{dq}{q} \frac{\alpha_s N}{\pi} d\Omega \frac{n_i \cdot n_j}{\lambda^2} (1 - \Theta(\lambda - n_i \cdot n) - \Theta(\lambda - n_j \cdot n)), \quad (5.22)$$

which could be negative if the direction is in the intersection of the cones, but for the algorithm only the fraction $dP(q, \Omega)/dR(q, \Omega)$ matters and for this configuration the ration is simply zero. Furthermore, the angle cutoff λ can always be chosen so small that there is no intersection of the cones. The Sudakov factor $\Delta_R(q|Q)$ for the splitting rate $R(q)$ reads

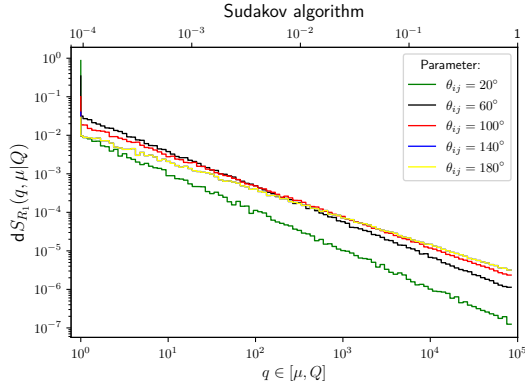
$$\Delta_R(q|Q) = \exp \left(-\frac{\alpha_s N}{\pi} \ln \left(\frac{Q}{q} \right) \frac{4\pi n_i \cdot n_j (1 - \lambda)}{\lambda^2} \right). \quad (5.23)$$

Furthermore, solving $rnd = \Delta_R(q|Q)$ for q , if $rnd > \Delta_R(q|Q)$, gives

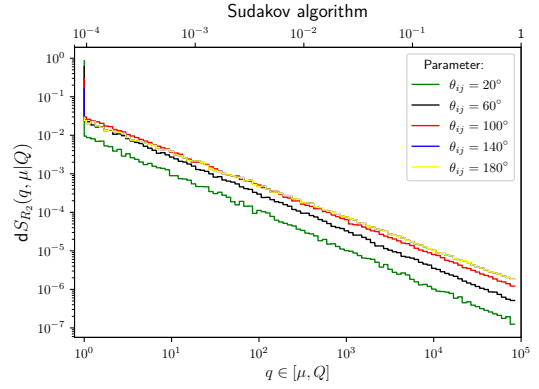
$$q = Q \exp \left(\frac{\ln(rnd) \lambda^2}{4\alpha_s N n_i \cdot n_j (1 - \lambda)} \right). \quad (5.24)$$

The acceptance probability for the proposed scale q is then given by

$$\frac{dP(q, \Omega)}{dR(q, \Omega)} = \frac{\lambda^2}{n_i \cdot n n_j \cdot n} \Theta(n \cdot n_j - \lambda) \Theta(n \cdot n_i - \lambda). \quad (5.25)$$



(a) Cutoff 1



(b) Cutoff 2

Figure 6: The parameters, on which the densities depend, are chosen as following: $N = 3$, $\mu = 1$, $Q = 10^5$ and $\lambda = 0.3$.

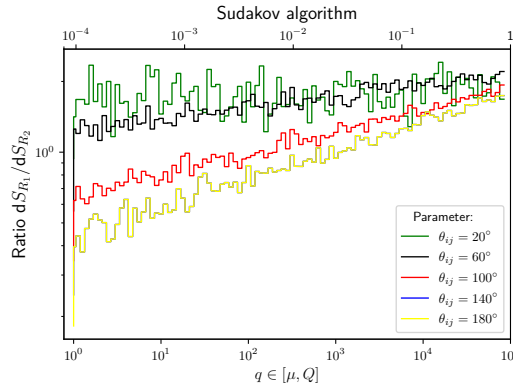


Figure 7: Comparison between cutoff 1 and 2 for the Sudakov algorithm, with $N = 3$, $\mu = 1$, $Q = 10^5$ and $\lambda = 0.3$.

In figure 6,7 numerical result from the implementation of the Sudakov algorithm are shown, the zero bin gives the no-emission probability.

5.2 Weights and the reweighing algorithm

In the following the weighted Sudakov veto algorithm will be discussed. This algorithm is a generalization of the previously introduced Sudakov veto algorithm, it has the advantage that it relaxes certain requirements previously needed. The first relaxation is that it is now not required anymore that $P_x(q, z)$ is non-negative and the second is that $R_x(q, z)$ is not necessarily a majorant of $P_x(q, z)$. The trade-off for this relaxation is that one has to assign to each event (q_i, z_i, x_i) a weight w_i . Non-negative splitting rates appear when correction beyond leading order or sub leading effects in N are taken into account in the evolution of the amplitude.

If it is necessary to use such an algorithm, the Monte Carlo simulation generates besides the phase space points, denoted for simplicity with $\{\xi_i\}_{i=1}^N$ where $\xi_i = (q_i, z_i, x_i)$, also a set of associated normalized weights $\{w_i\}_i^N$. The phase space points are sampled according to the random variable X and the weights according to the random variable W , then the estimation of the expectation

value, of an observable $\mathcal{O}$, is given through the weighted sum [33],

$$\langle \mathcal{O} \rangle \approx \sum_{i=1}^N w_i \mathcal{O}(\xi_i). \quad (5.26)$$

In the case of an unweighted sample the formula is still applicable, then set $w_i = 1/N$ for all i . For example the value of the observable $\mathcal{O}(\xi_i)$ is $\mathcal{O}(\xi_i) = \delta(\xi_i \in \text{bin})$ for a histogram bin. The uncertainty of the estimation (5.26) is then given by the variance of the observable $\mathcal{O}$

$$\text{Var} [\mathcal{O}] = N \text{Var} [W\mathcal{O}(X)]. \quad (5.27)$$

In practice the numerical value of $\text{Var} [W\mathcal{O}(X)]$ is not known and therefore the sample variance is used to estimate the error [33]

$$\text{Var} [\mathcal{O}] \approx \frac{1}{N} \sum_{i=1}^N (w_i \mathcal{O}(\xi_i))^2 - \left(\frac{1}{N} \sum_{i=1}^N w_i \mathcal{O}(\xi_i) \right)^2. \quad (5.28)$$

For the weighted Sudakov veto algorithm [34],[32] it is necessary to introduce an acceptance probability $\epsilon(q, z, x)$, satisfying

$$0 \leq \epsilon(q, z, x) < 1 \quad \forall q, z, x. \quad (5.29)$$

As indicated, the acceptance probability could be in principal depending on the splitting variables (q, x, z) , but often it is sufficient to introduce a constant acceptance probability. In principle, ϵ could also depend on the previously vetoed point [34], which gives rise to the possibility that some sequences get more often sampled than others. If this is the case the algorithm does not produce independent events and is therefore biased. The weighted Sudakov algorithm goes as follows.

Algorithm 3. To start with the weighted Sudakov veto algorithm, start with an initial non-zero weight w_0 and a starting scale Q . Choose a trial density $dS_{R_x}(q, z, \mu|Q,)$, where $R_x(q, z)$ is not necessarily an overestimate of the target splitting rate $P_x(q, z)$, hence $dS_{R_x}(q, z, \mu|Q)$ could be any density. Select the acceptance probability ϵ , as described in (5.29).

Select a starting scale and an initial weight $Q' \leftarrow Q, w \leftarrow w_0$.

loop

Generate trial splitting variables (q, z, x) according to the density $dS_{R_x}(q, z, \mu|Q)$.

if $q = \mu$ **then**

No emission, the cutoff scale μ is returned while the weight w of the event is kept unchanged.

else

if $\text{rnd} \leq \epsilon$ **then**

The variables (q, z, x) get accepted, and the weight w gets changed in the following way

$$w \leftarrow w \times \frac{1}{\epsilon} \times \frac{P_x(q, z)}{R_x(q, z)} \quad (5.30)$$

else

The variables (q, z, x) get rejected, the weight w changes and the algorithm continues, by starting again with a new scale q .

$$w \leftarrow w \times \frac{1}{1 - \epsilon} \times \left(1 - \frac{P_x(q, z)}{R_x(q, z)}\right)$$

$$Q' \leftarrow q. \quad (5.31)$$

end if

end if

end loop

One major issue with this algorithm, or in general with reweighing algorithms, is that it could produce a broad weight distribution, which leads to even broader weight distributions in the simulation of an event, since weights get multiplied. The problem of large weight distribution will be further discussed in chapter 5.5. In the case that the target splitting rate P is non-negative and the chosen rate R is an overestimate, it is possible to reduce the weighted Sudakov algorithm to the Sudakov algorithm which uses an overestimate by choosing ϵ as

$$\epsilon = \frac{P_x(q, z)}{R_x(q, z)} \quad (5.32)$$

and choosing unit initial weights. If the target splitting rate P is not non-negative, the initial

weights are $w_0 > 0$ and the ϵ is chosen as,

$$\epsilon = \left| \frac{P_x(q, z)}{R_x(q, z)} \right|, \quad (5.33)$$

then the weights of the produced events are either w_0 or $-w_0$.

To show the correctness of the algorithm [34], it is necessary to note that the algorithm generates the following distribution after n rejection steps and one acceptance step if the starting scale is $q_1 = Q$

$$\begin{aligned} \frac{d S_{R_x, \epsilon}^{(n)}(q, z, \mu | Q)}{d q d z} &= (\epsilon(q, z | q_n, z_n) R_x(q, z) \Delta_{R_x}(q | Q) \theta(q_n - q) \theta(q - \mu) + \Delta_{R_x}(\mu | Q) \delta(q - \mu)) \\ &\times \prod_{i=2}^n R_x(q_i, z_i) (1 - \epsilon(q_i, z_i | q_{i-1}, z_{i-1})) \theta(q_{i-1} - q_i) \theta(q_i - \mu), \end{aligned} \quad (5.34)$$

where in the $n - 1$ veto steps the sequence $(q, z | q_n, z_n | \dots | q_1, z_1)$ of length n was produced. The argument that this probability density gets produced is similar as in the proof for theorem 1. In the sequence of $n - 1$ veto steps an event accumulates the following weight

$$w_{P_x, R_x, \epsilon}^{(n)}(q, z, \mu | Q) = \prod_{i=2}^n \frac{1}{1 - \epsilon(q_i, z_i)} \left(1 - \frac{P_x(q_i, z_i)}{R_x(q_i, z_i)} \right) \times \begin{cases} \frac{1}{\epsilon(q, z | q_n, z_n)} \cdot \frac{P_x(q, z)}{R_x(q, z)} & q > \mu \\ 1 & q = \mu. \end{cases} \quad (5.35)$$

Again the density generated from the algorithm is the sum over all possible rejection steps of the product of the produced density times the respective weight

$$d S_{P_x, R_x, \epsilon}(q, z, \mu | Q) = \sum_{n=0}^{\infty} \left(\prod_{i=2}^n \int d q_i d z_i \right) d S_{R_x, \epsilon}^{(n)}(q, z, \mu | Q) w_{P_x, R_x, \epsilon}^{(n)}(q, z, \mu | Q). \quad (5.36)$$

The product of the produced density and the accumulated weight is then given by

$$\begin{aligned} d S_{R_x, \epsilon}^{(n)}(q, z, \mu | Q) w_{P_x, R_x, \epsilon}^{(n)}(q, z, \mu | Q) &= \left(\Delta_{R_x}(\mu | Q) \delta(q - \mu) \delta(z - z_{\min}) \right. \\ &\quad \left. + P_x(q, z) \theta(Q - q) \theta(q - \mu) \Delta_{R_x}(q | Q) \right) \\ &\quad \times \prod_{i=2}^n \theta(q_{i-1} - q_i) \theta(q_i - \mu) (R_x(q_i, z_i) - P_x(q_i, z_i)). \end{aligned} \quad (5.37)$$

When performing the sum in 5.36, then the difference $R_x(q_i, z_i) - P_x(q_i, z_i)$ gets exponentiated, which can again be seen with the integral identity from appendix B, similar as in the proof for the Sudakov veto algorithm 2. This shows that

$$d S_{P_x, R_x, \epsilon}(q, z, \mu | Q) = d S_{P_x}(q, z, \mu | Q), \quad (5.38)$$

hence the algorithm produces indeed the correct density. The weighted Sudakov algorithm is in principle only necessary for cutoff four from the list of cutoffs in 5, since the splitting rate $P(q, \Omega)$ is not non-negative, for the others the algorithm 2 could be used. In the case that the

overestimate of cutoff one is used to propose q , the necessary factor to reweight is

$$\frac{dP(q, \Omega)}{dR(q, \Omega)} = \frac{\left(\frac{n_i \cdot n_j}{n \cdot n_i n_j \cdot n} - \frac{1}{n \cdot n_i} - \frac{1}{n_j \cdot n} \right)}{\frac{n_i \cdot n_j}{\lambda} \left(\frac{1}{\lambda + n_j \cdot n} + \frac{1}{\lambda + n_i \cdot n} \right)}. \quad (5.39)$$

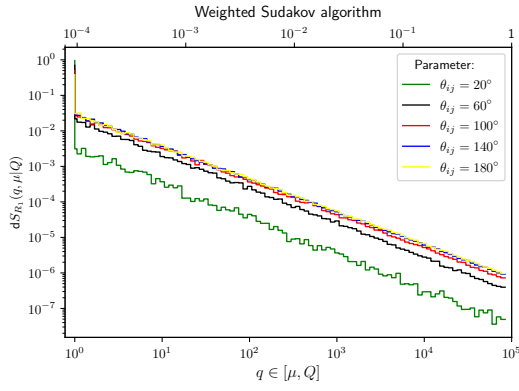
In order to obtain the same density as in cutoff i-iii, the density produced from the following Sudakov factor has to be added back

$$\Delta(q|Q) = \exp \left(-\frac{\alpha_s N}{\pi} \int_q^Q \frac{dk}{k} \int d\Omega \frac{2}{n_i \cdot n} \Theta(n_i \cdot n - \lambda) \right). \quad (5.40)$$

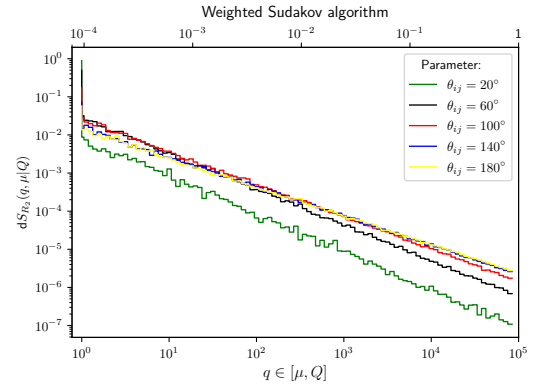
How to add densities, produced from possible different Sudakov factors, will be discussed in section 5.4. If the density produced from the Sudakov factor in 5.40 should be produced with the reweighting algorithm, the relevant Sudakov factor for the overestimate and reweighting factor is

$$\Delta_R(q|R) = \exp \left(-8\alpha_s N \ln \left(\frac{Q}{q} \right) \right), \quad (5.41)$$

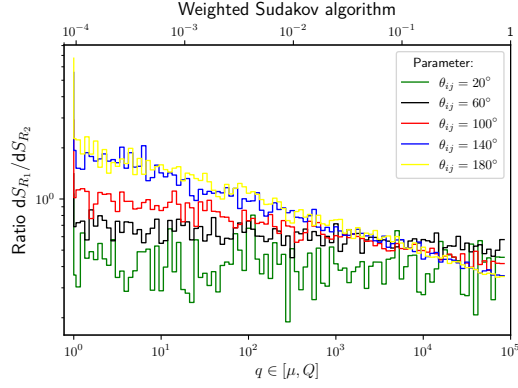
$$\frac{dP(q, \Omega)}{dR(q, \Omega)} = \frac{\lambda}{2} \left(\frac{\Theta(n \cdot n_i - \lambda)}{n \cdot n_i} + \frac{\Theta(n \cdot n_j - \lambda)}{n \cdot n} \right). \quad (5.42)$$



(a) Cutoff 1

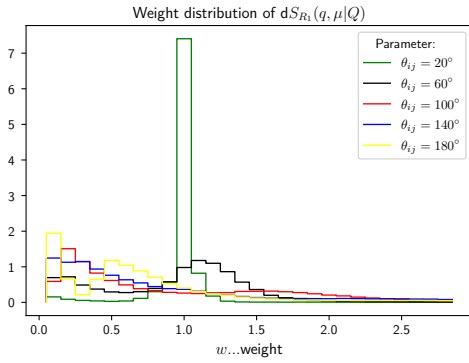


(b) Cutoff 2

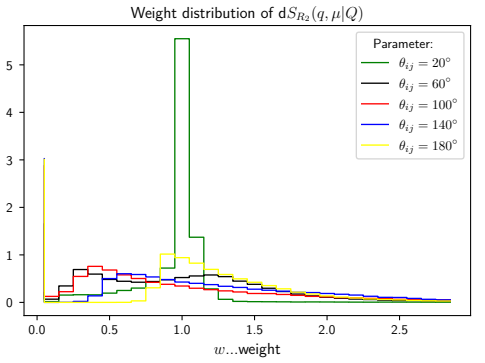


(c) Comparison cutoff 1 and 2.

Figure 8: Weighted Sudakov algorithm for cutoff 1 and 2. Parameters: $N = 3$, $\mu = 1$, $Q = 10^5$ and $\lambda = 0.3$, number of events 10^5 .



(a) Cutoff 1



(b) Cutoff 2

Figure 9: Weight distribution, produced by the weighted Sudakov algorithm for cutoff 1 and 2. Parameters: $N = 3$, $\mu = 1$, $Q = 10^5$ and $\lambda = 0.3$, number of events 10^5 .

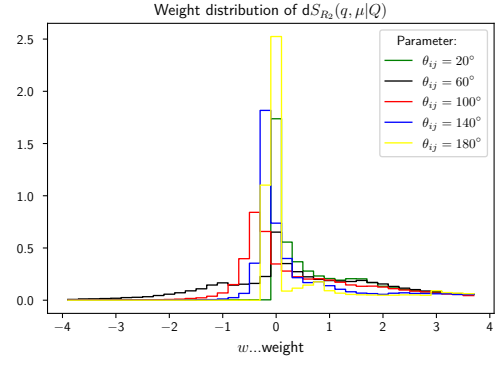
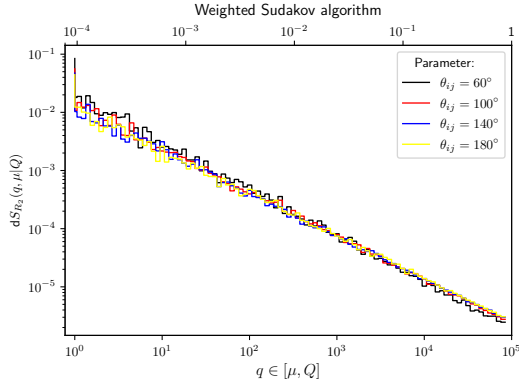
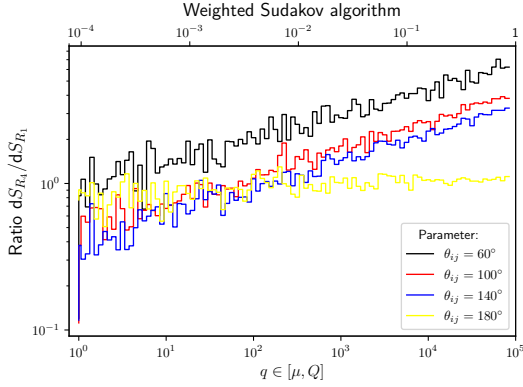
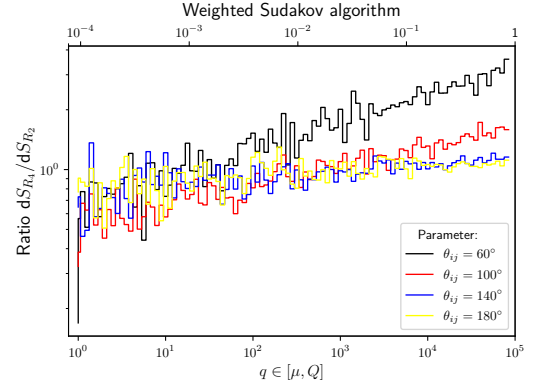


Figure 10: Weighted Sudakov algorithm for cutoff 4 and their weight distribution. Parameters: $N = 3$, $\mu = 1$, $Q = 10^5$ and $\lambda = 0.3$.



(a) Cutoff 4 vs. cutoff 1.



(b) Cutoff 4 vs cutoff 2.

Figure 11: Comparison of cutoffs, produced by the weighted Sudakov algorithm. Parameters: $N = 3$, $\mu = 1$, $Q = 10^5$ and $\lambda = 0.3$, number of events 10^5 .

5.3 Sudakov algorithm matrix exponential

The aim of this section is to present an algorithm, which generates the density, constructed from the Sudakov factor

$$\Delta_{\Gamma,S}(q|Q) = \frac{1}{\text{Tr}[S]} \text{Tr} \left[\mathcal{T} \left\{ \exp \left(- \int_q^Q \frac{dk}{k} \Gamma(k) \right) \right\} S \right], \quad (5.43)$$

where $\Gamma(k)$ and S are $n \times n$ matrices, possible complex valued, and $\text{Tr}[S] \neq 0$. The matrix S is independent of the evolution scale q and represents a measurement in the colour space. For simplicity it is assumed in the following that $\text{Tr}[S] = 1$. Here $\mathcal{T}\{\exp(\cdot)\}$ denotes the time ordered exponential, defined as

$$\mathcal{T} \left\{ \exp \left(\int_{t_0}^t A(t') dt' \right) \right\} (t) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{t_0}^t \cdots \int_{t_0}^t \mathcal{T}\{A(t'_1) \cdots A(t'_n)\} dt'_1 \cdots dt'_n, \quad (5.44)$$

where $\mathcal{T}\{\cdot\}$ ensures that the product is time ordered. The time ordered exponential simplifies to $\exp(\int_{t_0}^t A(t') dt')$ if $[A(t_1), A(t_2)] = 0$ for all t_1, t_2 . The construction of the density from the Sudakov factor $\Delta_{\Gamma,S}$ follows the same principle as in section 5. By applying the property of the ordered exponential it follows that the derivative of the Sudakov factor $\Delta_{\Gamma,S}(q|Q)$ is given by

$$\frac{d}{dq} \Delta_{\Gamma,S}(q|Q) = \frac{1}{q} \text{Tr} \left[\Gamma(q) \mathcal{T} \left\{ \exp \left(- \int_q^Q \frac{dk}{k} \Gamma(k) \right) \right\} S \right]. \quad (5.45)$$

The density for the dynamical variable (q) is therefore

$$\begin{aligned} \frac{d S_{\Gamma,S}(q, \mu|Q)}{dq} &= \Delta_{\Gamma,S}(\mu|Q) \delta(q - \mu) \\ &+ \frac{1}{q} \text{Tr} \left[\Gamma(q) \mathcal{T} \left\{ \exp \left(- \int_q^Q \frac{dk}{k} \Gamma(k) \right) \right\} S \right] \Theta(Q - q) \Theta(q - \mu). \end{aligned} \quad (5.46)$$

The factor $1/\text{Tr}[S]$ in (5.43) follows from the requirement that the density $d S_{\Gamma,S}(q, \mu|Q)$ is normalized on the interval $[\mu, Q]$. The later presented algorithm then makes use of the fact that trace of the product of two square matrices A, B can be approximated with Monte Carlo methods, as introduced in section 2.4. To illustrate this, consider the trace of the product of the two $n \times n$ matrices A and B

$$\text{Tr}[AB] = \sum_{i,j=1}^n A_{ij} B_{ji} = n^2 \sum_{i,j=1}^n \frac{1}{n^2} A_{ij} B_{ji} \quad (5.47)$$

and interpret $1/n^2$ as the probability $p(i, j)$ to generate the index pair (i, j) . The trace can then be, through the law of large number, approximated through

$$\text{Tr}[AB] \approx n^2 \langle A_{ij} B_{ji} \rangle_{p(i,j)}, \quad (5.48)$$

where the index pair (i, j) is uniformly distributed on $\{1, \dots, n\} \times \{1, \dots, n\}$, hence $p(i, j) = 1/n^2$. To improve the convergence, the approximation in (5.48) could be supplemented with important

sampling, then

$$\text{Tr}[AB] \approx n^2 \langle w_{ij} A_{ij} B_{ji} \rangle_{q(i,j)}, \quad (5.49)$$

whereas $q(i, j)$ is again a discrete distribution for the index pair (i, j) with the same support as $p(ij)$. The weight w_{ij} is given by

$$w_{ij} = \frac{p(i, j)}{q(i, j)}, \quad (i, j) > 0 \quad \forall i, j = 1, 2, \dots, n. \quad (5.50)$$

For simplicity let's define the following two quantities,

$$P(k) = \frac{1}{k} \Gamma(k) \quad (5.51)$$

and

$$\Delta_{P,ij}(q|Q) = \mathcal{T}_{ij} \left\{ \exp \left(- \int_q^Q dk P(k) \right) \right\}. \quad (5.52)$$

From 5.46 and the linearity of the trace it follows that inside the trace there is a matrix multiplication between S and

$$\frac{d S_{ij}(q, \mu|Q)}{d q} = \Delta_{P,ij}(\mu|Q) \delta(q - \mu) + \sum_k P_{ik}(q) \Delta_{P,kj}(q|Q) \Theta(Q - q) \Theta(q - \mu), \quad (5.53)$$

which forms a density for every index pair (i, j) . In general, $P(k)$ is not positive nor can the integral $\int dk P(k)$ be calculated analytically, furthermore, it is not guaranteed that the exponential can be calculated efficiently. A possible solution to the problem of generating such a dynamical variable is to write down a generalization of the weighted Sudakov algorithm. As in the weighed Sudakov algorithm a scale has to be proposed. Here a suitable matrix $R(q)$ has to be chosen such that the matrix exponential can be calculated analytically and such that the rate $R(q)$ cancels in the exponent after summing the produced density over all veto steps, after exponentiating the difference $R(k) - P(k)$. Hence it is necessary, due to the properties of the time ordered exponential 5.44, that $R(k)$ commutes with $P(k)$. The easiest choice for $R(k)$ is a diagonal matrix proportional to the identity matrix

$$R(k) = \hat{R}(k) \mathbf{1}, \quad (5.54)$$

where $\hat{R}(k)$ is a positive valued scalar function. With that choice the Sudakov factor, for the proposed matrix valued rate $R(k)$, becomes

$$\Delta_{R,ij}(q|Q) = \exp \left(- \int_q^Q dk \hat{R}(k) \right) \delta_{ij}. \quad (5.55)$$

The algorithm to generate 5.46 then reads.

Algorithm 4. Generate a index triple (i, j, k) with probability $p(i, j, k)$.
Choose starting values Q_0 and w_0 : $Q \leftarrow Q_0$, $w_{ijk} \leftarrow w_0$.

loop

Propose a scale k , which is distributed according to the density $dS_{\hat{R}}(k|Q)$.

if $k = \mu$ **then** multiply the weight with δ_{ik} ,

$$w_{ijk} \leftarrow w_{ijk} \delta_{ik} \quad (5.56)$$

else

if $\text{rnd} \leq \epsilon_{ij}(k)$ **then**,

$$w_{ijk} \leftarrow w_{ijk} \frac{1}{\epsilon(k)} \frac{P_{ij}(k)}{R(k)} \quad (5.57)$$

else

$$w_{ijk} \leftarrow w_{ijk} \frac{\delta_{kj} - \frac{P_{kj}(k)}{R(k)}}{(1 - \epsilon(k))} \quad (5.58)$$

end if

end if

end loop

To perform the trace with importance sampling multiply the resulting weight with $\frac{S_{ji}}{p(i, j, k)}$

$$w_{ij} \leftarrow \frac{S_{ji}}{p(i, j, k)} w_{ijk}. \quad (5.59)$$

The notation w_{ijk} should be understood such that the weights are dependent on the generated indices. The first two indices are necessary to generate since the trace of the product of two square matrices has to be performed, the third index comes from the necessity to perform matrix multiplication within the trace. The efficiency of the algorithm is again largely dependent on the structure of the acceptance probability ϵ , which could be dependent on the evolution scale or other splitting variables. The proof is similar as the proof for the weighted Sudakov algorithm. The probability density, produced by the algorithm, after $n - 1$ veto steps, which starts from the initial scale Q and produces the sequence $(q_n, q_{n-1}, \dots, q_2, Q)$ and one final acceptance step, with

the accepted scale q , reads

$$\begin{aligned} \frac{d S_{\hat{R}, \epsilon}^{(n)}(q, \mu|Q)}{d q} &= \left(\epsilon(q) \hat{R}(q) \Delta_{\hat{R}}(q|Q) \Theta(q_n - q) \Theta(q - \mu) + \Delta_{\hat{R}}(\mu|Q) \delta(q - \mu) \right) \\ &\times \prod_{i=2}^n \hat{R}(q_i) (1 - \epsilon(q_i)) \Theta(q_{i-1} - q_i) \Theta(q_i - \mu). \end{aligned} \quad (5.60)$$

The accumulated weight in this sequence of $n - 1$ veto steps and one final acceptance step is given by

$$w_{P, \hat{R}, \epsilon, i, j, k}^{(n)}(q, \mu|Q) = \frac{S_{ji}}{p(i, j)} \prod_{i=2}^n \frac{1}{1 - \epsilon(q_i)} \left(\delta_{kj} - \frac{P_{kj}(q_i)}{\hat{R}(q_i)} \right) \times \begin{cases} \frac{1}{\epsilon(q|q_n)} \frac{P_{ik}(q)}{\hat{R}(q)} & q > \mu \\ 1 & q = \mu. \end{cases} \quad (5.61)$$

The density generated by the algorithm is then the weighted sum over all possible veto steps and the sum over the index triple (i, j, k)

$$d S_{P, \hat{R}, \epsilon}(q, z, \mu|Q) = \sum_{(i, j, k)} \sum_{n=0}^{\infty} \left(\prod_{i=2}^n \int d q_i \right) d S_{\hat{R}, \epsilon}^{(n)}(q, \mu|Q) w_{P, \hat{R}, \epsilon, i, j, k}^{(n)}(q, \mu|Q), \quad (5.62)$$

where the product of the generated weights and density is given by

$$\begin{aligned} d S_{\hat{R}, \epsilon}^{(n)}(q, \mu|Q) w_{P, \hat{R}, \epsilon, i, j, k}^{(n)}(q, \mu|Q) &= \frac{S_{ji}}{p(i, j, k)} (P_{ik}(q) \Delta_{\hat{R}}(q|Q) \Theta(q_n - q) \Theta(q - \mu) + \Delta_{\hat{R}}(\mu|Q) \delta_{ik} \delta(q - \mu)) \\ &\times \prod_{i=2}^n \left(\hat{R} \delta_{kj} - P_{kj}(q_i) \right) \Theta(q_{i-1} - q_i) \Theta(q_i - \mu). \end{aligned} \quad (5.63)$$

The difference exponentiates then in the same way as in the Sudakov algorithm, which uses a overestimate. In the case of matrices the difference exponentiates if $R(k)$ and $P(k)$ commute, which is the case since for $R(k)$ a diagonal matrix was chosen. The exponentiation and the fact that $\exp(A + B) = \exp(A) \exp(B)$ if the matrices A and B commute then gives the desired result.

To test the algorithm the following matrices could be chosen, for Γ_1, Γ_2

$$\Gamma_1 = \begin{pmatrix} 1 & \frac{1}{N} & \frac{1}{N^2} & \cdots \\ \frac{1}{N} & 1 & \frac{1}{N} & \cdots \\ \frac{1}{N^2} & \frac{1}{N} & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 1 & \frac{1}{N} & \frac{1}{N^2} & \cdots \\ 0 & 1 & \frac{1}{N} & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (5.64)$$

and for S , if all colour states have the same probability

$$S_{ij} = \frac{1}{\text{Tr}[S]}, \quad \forall i, j = 1, \dots, n. \quad (5.65)$$

The matrices Γ_1 and Γ_2 are of interest since they mimic full colour evolution, in which off-diagonal processes are colour suppressed. To implement the algorithm, it is necessary to generate three indices (i, j, k) with probability $p(i, j, k)$, where $(i, j, k) \in \{1, \dots, n\}^3$. A probability distribution

for the case of S and Γ_1 could be constructed in the following way. In the product of the generated density and generated weights factors of $\Gamma_{1,ik}$ and $\Gamma_{1,kj}$ appear, therefore, one could impose the following form on the discrete probability distribution $p(i, j, k)$,

$$p(i, j, k) \propto p_1(i, k)p_2(k, j). \quad (5.66)$$

Due to the structure of Γ_2 , the probability distributions $p_{1,2}(k, j)$ could be chosen such that they depend on the difference of first and second argument,

$$p_{1,2}(k, j) \propto \begin{cases} \frac{1}{N^{j-k}} & j - k \geq 0 \\ 0 & j - k < 0. \end{cases} \quad (5.67)$$

For Γ_1 in principle the same probability distributions, as for Γ_2 , can be used for $p_{1,2}(k, j)$, with the difference that the entries below the main diagonal are not zero anymore, hence

$$p_{1,2}(k, j) \propto \frac{1}{N^{|j-k|}}. \quad (5.68)$$

5.4 Competition algorithm

In a parton shower multiple branchings could occur, where each possible branching is described by the density dS_{R_x} constructed from the rate R_x . The overall Sudakov factor is then the product of the Sudakov factors of all possible branchings 5.5 of the considered processes. In section 5.1 and section 5.2 algorithms to generate splitting variables distributed according to dS_{R_x} for one type of branching were discussed. In this section these algorithms will be supplemented with the competition algorithm such that the density of the multiple competing branching rates can be generated.

The algorithm assumes that the individual branchings are independent from each other. The competition algorithm reads.

Algorithm 5. Generate the splitting variables $\{q_1, q_2, \dots, q_n\}$, where q_i is distributed according to dS_{P_i} , $\forall i \in \{1, 2, \dots, n\}$ with a Sudakov algorithm.

Calculate $q = \max(q_1, q_2, \dots, q_n)$

return q

The evolution continues then with the scale q produced from the competition algorithm. To see that the algorithm produces the right distribution, consider the case $n=2$, where in the evolution two different types of branching can occur, the first branching with the rate R_1 and the second with the rate R_2 , the first produces scale q_1 and the second scale q_2 . The densities constructed through the Sudakov factors dS_{R_1} and dS_{R_2} both have the same cutoff scale μ , both processes start at scale Q . The probability that the competition algorithm returns the scale μ is then given by

$$\Delta_{R_1+R_2}(\mu|Q) = \Delta_{R_1}(\mu|Q) \Delta_{R_2}(\mu|Q). \quad (5.69)$$

If the algorithm returns a scale $q > \mu$, then at least one of the two generated scales was larger than the cutoff scale μ , the returned scale q is then distributed according to

$$\begin{aligned} q &\sim R_1(q) \Delta_{R_1}(q|Q) \Delta_{R_2}(q|Q) + R_2(q) \Delta_{R_2}(q|Q) \Delta_{R_1}(q|Q) \\ &= (R_1(q) + R_2(2)) \Delta_{R_1+R_2}(q|Q), \end{aligned} \quad (5.70)$$

which is the desired distribution for $q > \mu$. In the first line the first term represents the case where $q_1 \geq q_2$ and the second term the case where $q_2 > q_1$. Integrating (5.70) from μ to Q and adding the no emission probability in the interval $[\mu, Q]$ from (5.69) shows that the produced density is properly normalized. With an induction proof in the number of spitting variables n the correctness of the competition algorithm follows.

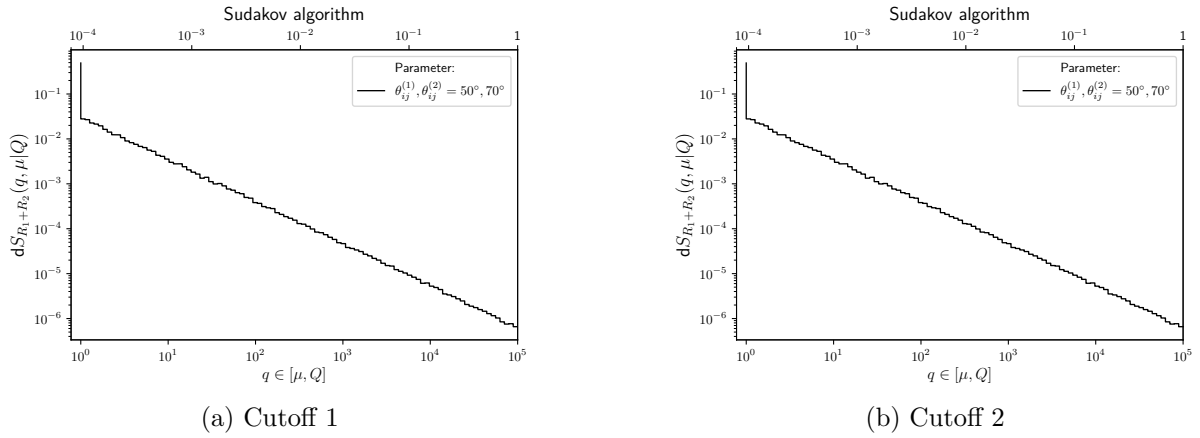


Figure 12: Two competing dipoles for cutoff 1 and 2, using the Sudakov algorithm. Parameters: $N = 3$, $\mu = 1$, $Q = 10^5$ and $\lambda = 0.3$.

To consider the collinear cutoff description number three 5 becomes interesting when considering a system with $m \geq 3$ partons. In the case of $m = 2$ the cutoff description simply equals to cutoff two. The overestimate $R(q)$ for cutoff three can then be constructed in the same way as in the case of cutoff description two

$$\begin{aligned} \int d\Omega \omega_{ij} &= \int d\Omega \frac{n_i \cdot n_j}{n_i \cdot n \, n_j \cdot n} \prod_{k=1}^m \Theta(n_k \cdot n - \lambda) \\ &\leq \frac{n_i \cdot n_j}{\lambda^2} \int d\Omega \prod_{k=1}^m \Theta(n_k \cdot n - \lambda) \\ &\leq \frac{n_i \cdot n_j}{\lambda^2} \int d\Omega \left(1 - \sum_{k=1}^m \Theta(\lambda - n_k \cdot n) \right) \\ &= \frac{4\pi n_i \cdot n_j}{\lambda^2} \left(1 - \frac{m\lambda}{2} \right), \end{aligned} \quad (5.71)$$

which gives an upper bound for λ , dependent on the number of partons $\lambda < \frac{2}{m}$. The differential of the overestimate $R(q)$ reads,

$$dR(q, \Omega) = \frac{\alpha_s N}{\pi} \frac{dq}{q} d\Omega \frac{4\pi n_i \cdot n_j}{\lambda^2} \left(1 - \sum_{k=1}^m \Theta(\lambda - n_k \cdot n) \right), \quad (5.72)$$

again this could be negative, as in the case of cutoff two, this doesn't matter since for the algorithm only the fraction $P(q, \Omega)/R(q, \Omega)$ is of interest, which is then always greater or equal to zero. The Sudakov factor $\Delta_R(q|Q)$ is then given by

$$\Delta_R(q|Q) = \exp \left(-\ln \left(\frac{Q}{q} \right) \frac{4\alpha_s N n_i \cdot n_j}{\lambda^2} \left(1 - \frac{m\lambda}{2} \right) \right). \quad (5.73)$$

Solving $rnd = \Delta_r(q|Q)$, if $rnd > \Delta_R(q|Q)$ with respect to the scale q gives

$$q = Q \exp \left(\frac{\lambda^2 \ln(rnd)}{4\alpha_s N n_i \cdot n_j (1 - \frac{\lambda m}{2})} \right). \quad (5.74)$$

For reweighing in the reweighed Sudakov algorithm the following ration is needed

$$\frac{dP(q, \Omega)}{dR(q, \Omega)} = \frac{\lambda^2}{n_i \cdot n_j \cdot n} \prod_{k=1}^m \Theta(n_i \cdot n - \lambda). \quad (5.75)$$

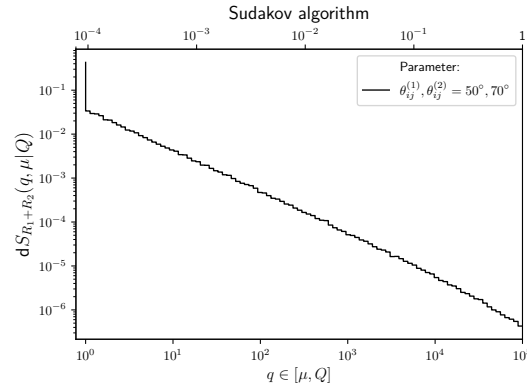


Figure 13: Two competing dipoles for cutoff 3, using the Sudakov algorithm. Parameters: $N = 3$, $\mu = 1$, $Q = 10^5$ and $\lambda = 0.3$.

5.5 Resampling

In this chapter the notion of resampling gets introduced. Broadly speaking resampling is a technique to increase the stability of a Monte Carlo simulation, when running over a large time horizon. In the last decades it became one of the standard tools when it comes to weighted Monte Carlo simulations, for example in the development of parton showers and for QCD lattice calculations, when dealing with the infamous sign problem. The section is mainly based on [33, 35] and [32].

Let's consider the total cross section σ , estimated by

$$\sigma \approx \sum_{i=1}^N w_i, \quad (5.76)$$

which directly follows from (5.26). Since only the total cross section σ , hence the sum of all weights, is the object of interest, it becomes clear that it is possible to modify the weights of the generated events $\{x_i\}_{i=1}^N$ as long as the total sum of weights in (5.76) stays the same. The motivation to change the weights is, as already indicated, to reduce the error of the estimation, which could become rather large since in a Monte Carlo simulation the distribution of the set of generated weights $\{w_i\}_{i=1}^N$ could become rather broad which, leads to a relatively large sample variance

$$\text{Var}[\sigma] \approx \frac{1}{N} \sum_{i=1}^N w_i^2 - \left(\frac{1}{N} \sum_{i=1}^N w_i \right)^2. \quad (5.77)$$

To reduce the variance and therefore the error of the simulation it is beneficial to keep the weight distribution as narrow as possible. For simplicity let's assume that the Monte Carlo simulation generates only positive weights, then the resampling procedure goes as following [32]. Select $M > 0$ events from the N generated events in proportion to their weight, where $M \leq N$. More concretely choose the event x_i with weight w_i with probability $p(i)$,

$$p(i) = \frac{w_i}{\sum_{i=1}^N w_i}. \quad (5.78)$$

After choosing the set of M events, assign each event of the set the new weight w'_i ,

$$w'_i = \frac{\sum_{i=1}^N w_i}{M}. \quad (5.79)$$

Note that with this choice for the new set of weights $\{w'_i\}_{i=1}^M$ the weight sum stays the same

$$\sum_{i=1}^N w_i = \sum_{i=1}^M w'_i. \quad (5.80)$$

The selection process of the M events in proportion to their weight could be implemented [32] by noting that this problem is equivalent to generating M identical distributed indices $\{i_l\}_{l=1}^M$,

where $i_l \in \{1, \dots, M\}$ such that

$$P(i_l = j) = \frac{w_j}{\sum_{k=1}^N w_k}. \quad (5.81)$$

The naive method to generate such an index is to generate a uniformly distributed number $u_l \in (0, 1)$ and find a $k \in \{1, \dots, M\}$, which satisfies

$$\sum_{i=1}^{k-1} \frac{w_i}{\sum_{j=1}^N w_j} \leq u_l \leq \sum_{i=1}^k \frac{w_i}{\sum_{j=1}^N w_j}, \quad (5.82)$$

then $i_l = k$ is such an index. For simplicity let $M = N$, then if one implements the above search in the naive way, then in the worst case one would need N comparisons to find such an index, hence the overall complexity would be N^2 . The complexity could be reduced by turning (5.82) into a balanced tree, for example with the AVL algorithm. In a balanced tree the maximal depth is $\log_2 N$, hence the overall complexity to choose N events becomes $N \log_2 N$, which is already significantly better than the naive approach. However, if the generated uniforms $\{u_i\}_{i=1}^N$ are ordered in the sense that $u_1 \leq u_2 \leq \dots \leq u_N$, then it requires only N comparisons to find the set of indices $\{i_l\}_{l=1}^N$ in (5.82). Luckily algorithms, which generate such an ordered statistics are already available, three of them are presented in [36], chapter five. The best suited algorithm from [36] is the method, which uses exponential spacing and is based on the observation that the following sequence of numbers coincides with a sequence of ordered independent uniforms on $(0, 1)$,

$$\left\{ \prod_{i=1}^N \sqrt[i]{v_i}, \prod_{i=2}^N \sqrt[i]{v_i}, \dots, \sqrt[n]{v_n} \right\}, \quad (5.83)$$

where $\{v_i\}_{i=1}^N$ is again a sequence of uniforms over $(0, 1)$. That the sequence in (5.83) and a sequence of ordered uniforms indeed coincide was shown in [36]. An algorithm to generate such a sequence in linear time $\mathcal{O}(N)$ is presented in the following.

Algorithm 6. Algorithm to generate an ordered statistics $\{u_i\}_{i=1}^N$ of independent uniforms.

```

Sample  $v_n \sim U(0, 1)$ .
Set  $u_N = \sqrt[n]{v_N}$ .
for  $i = n - 1 \rightarrow 1$  do
    Sample  $v_i \sim U(0, 1)$ .
    Set  $u_i = u_{i+1} \sqrt[i]{v_i}$ .
end for
```

One of the disadvantages of the presented algorithm is that it requires to calculate powers in each step, which could potentially slow down the algorithm. Note that the algorithm is an on-pass method. The so far described resampling procedure requires that all weights are positive. In the case, where also negative weights are allowed, the resampling procedure goes as following.

Let Ω be the set of all events, then the set Ω could be split up into two regions of importance $\Omega = \Omega_+ \cup \Omega_-$, where the set Ω_+ contains all positive and Ω_- all negative weights of Ω . Then, as before, M events from the N generated events, where $M \leq N$, get selected. The difference is that if an event gets selected, the region Ω gets selected with probability p_+ and the region Ω_- with probability p_- , whereas $p_+ + p_- = 1$. Within a region of importance an event gets selected in proportion to the absolute value of its weight. After the selection of the M events, from the region of importance Ω_+ get the new weight w'_+ and events from the region of importance Ω_- the new weight w'_- assigned,

$$w'_\pm = \frac{W_\pm}{N_\pm \cdot p_\pm}, \quad (5.84)$$

where $W_\pm$ is given by

$$W_+ = \sum_{w_i \in \Omega_+} w_i, \quad W_- = \sum_{w_i \in \Omega_-} |w_i| \quad (5.85)$$

and N_+ is the number of events in Ω_+ and N_- is the number of events in Ω_- . From the requirement that the sum of weights has to be the same after resampling it follows that

$$p_\pm = \frac{\sqrt{W_\pm}}{\sqrt{W_+} + \sqrt{W_-}}. \quad (5.86)$$

In order to keep the distribution of the weights as narrow as possible the resampling procedure could be applied at each intermediate step. It is important to recognize that this technique produces events which have an identical or partially identical evolution history, hence not all events are independent from each other. The two obvious ways to reduce the number of events with identical or partially identical evolution history are either to increase the sample size M or by simply reducing the number of resampling steps [32]. A criterion to decide if the generated events are independent enough from each other is the coefficient of variation C_V^2 , defined as [32],[37]

$$\begin{aligned} C_V^2 &= N \sum_{i=1}^N \left(\frac{w_i}{\sum_{j=1}^N w_j} - \frac{1}{N} \right)^2 \\ &= N \sum_{i=1}^N \left(\frac{w_i}{\sum_{j=1}^N w_j} \right)^2 - 1. \end{aligned} \quad (5.87)$$

The coefficient of variation is a measure of skewness and takes the minimal value zero if all weights are the same and the maximal value $N - 1$ if only one weight is not zero.

With the coefficient of variation one could also define the so called effective sample size

$$\text{ESS} = \frac{1}{1 + C_V^2}, \quad (5.88)$$

which is equal to N if all weights are the same and one if only one weight is not zero. It is important to mention that the coefficient of variation and the effective sample size could only be applied heuristically, which means that concrete values for C_V^2 and ESS at which the sample gets resampled or not had to be chosen accordingly from case to case. Nevertheless, resampling provides a suitable tool to rejuvenate the sample [37], produced by the weighted parton shower.

At which states the resampling could be integrated in the Sudakov algorithm is indicated by the red dots in figure 14.

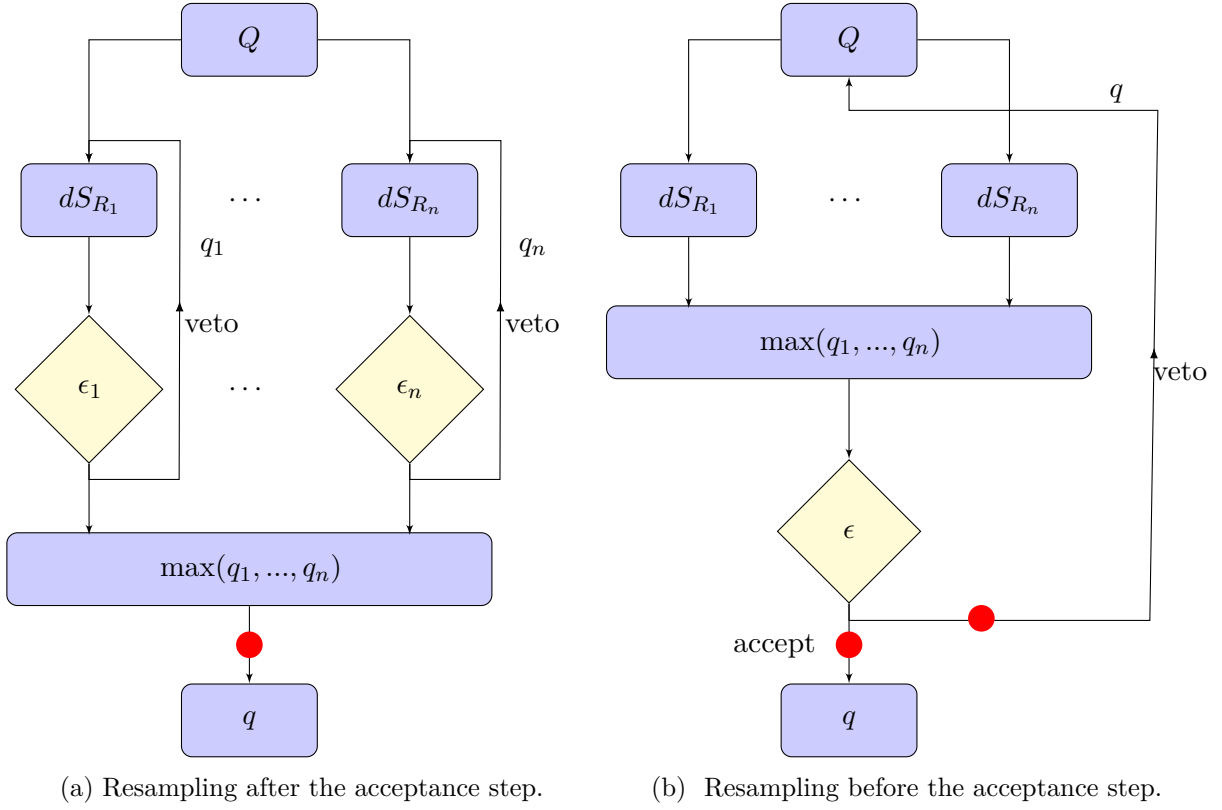


Figure 14: Two possible implementations of resampling procedure within the Sudakov veto algorithm [32].

In a general weighted parton shower the produced weights of the events could be complex $w_i \in \mathbb{C}$. How to resample in that case will be presented in the following and is based on unpublished notes and discussions from/with Simon Plätzer [38] and refined in the following. Let's consider such a set of complex weights $\Omega \{w_i\}_{i=1}^N$, then each of these weights is of the form $w_i = |w_i|e^{i\varphi_i}$, where $\varphi_i \in \mathbb{R}$. Then the set Ω could be split into n regions of importance

$$\Omega = \bigcup_{\alpha=1}^n \Omega_{\alpha}, \quad (5.89)$$

where Ω_{α} is the region of importance such that the phases φ_i of all elements w_i , which are contained in Ω_{α} , satisfy

$$(\alpha - 1) \frac{2\pi}{n} \leq \varphi_i < \alpha \frac{2\pi}{n}. \quad (5.90)$$

To simplify things it is assumed that the number of regions of importance n is that large such that

$$\varphi_i \approx \varphi_{\alpha} \quad \forall i, \quad \text{for which } w_i \in \Omega_{\alpha}, \quad (5.91)$$

where $\varphi_{\alpha} = (\alpha - 1) \frac{2\pi}{n} + \frac{2\pi}{2n} = \frac{2\pi}{n} (2\alpha - 1)$. The real part of the sum of all weights could then be

approximated in the following way

$$\operatorname{Re} \left(\sum_i w_i \right) = \sum_i \operatorname{Re} (w_i) \approx \sum_{\alpha=1}^n \operatorname{Re} (e^{i\varphi_\alpha}) \delta(\Omega_\alpha) \sum_{w_i \in \Omega_\alpha} |w_i|. \quad (5.92)$$

Note that the approximation improves the large n and tends to equality since only a finite set $|\Omega| < \infty$ of weights is considered. The function $\delta(\Omega_\alpha)$ indicates if the region of importance Ω_α is empty or not, hence

$$\delta(\Omega_\alpha) = \begin{cases} 1 & |\Omega_\alpha| \neq 0 \\ 0 & |\Omega_\alpha| = 0. \end{cases} \quad (5.93)$$

In the actual resampling procedure M events out of the N generated events get selected, where $M \leq N$, at first a region of importance gets selected with probability p_α , whereas $p_\alpha = 0$ if the region Ω_α is empty and nonzero otherwise. Then within a region of importance an event gets selected with probability $p_i^{(\alpha)}$

$$p_i^{(\alpha)} = \frac{|w_i|}{W_\alpha}, \quad (5.94)$$

where

$$W_\alpha = \sum_{w_i \in \Omega_\alpha} |w_i|. \quad (5.95)$$

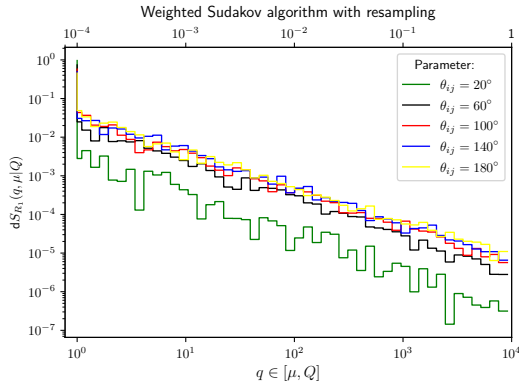
After the selection of an event of the region of importance Ω_α , the old weight w_i gets replaced by the new weight $w_i^{(\alpha')}$,

$$w_i^{(\alpha')} = \frac{W_\alpha}{N_\alpha p_\alpha}, \quad (5.96)$$

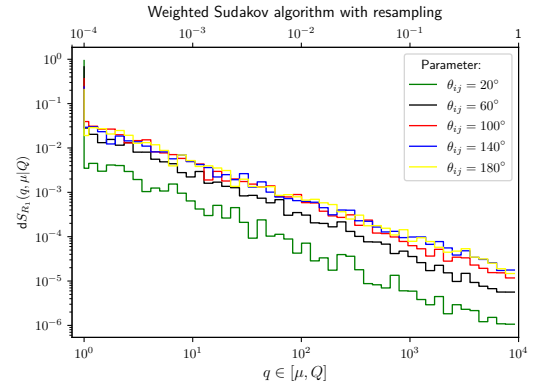
whereas N_α is the number of events in Ω_α . The probabilities are again determined by requiring that the sum of weights stays the same

$$\sum_{\alpha=1}^n \delta(\Omega_\alpha) \operatorname{Re} (e^{i\varphi_\alpha}) \frac{W_\alpha}{p_\alpha} = \sum_{\alpha=1}^n \operatorname{Re} (e^{i\varphi_\alpha}) \delta(\Omega_\alpha) W_\alpha. \quad (5.97)$$

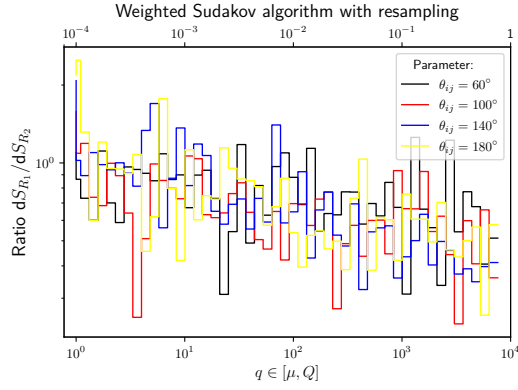
Regarding the approximation made in 5.92, although it is at first sight tempting to choose n rather large, one should keep in mind that the larger the number of region of importance, the smaller is the number of events within the region, which could make the resampling procedure beneficial. A way to improve the approximation in 5.92 is to take the phase distribution of the weights into account and partition the complex plane in 5.90 accordingly. Furthermore the phases $\varphi_\alpha \in \mathbb{R}$ could be chosen differently, for example, one could assign the mean phase of the events within the respective region of importance.



(a) Cutoff 1

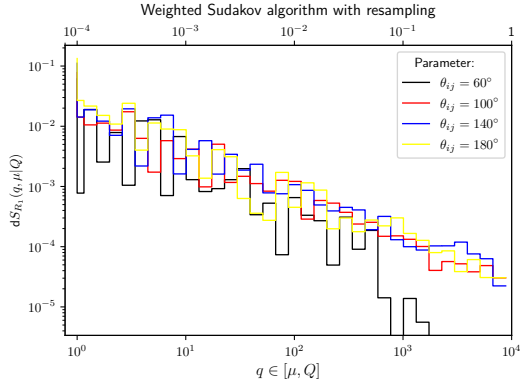


(b) Cutoff 2

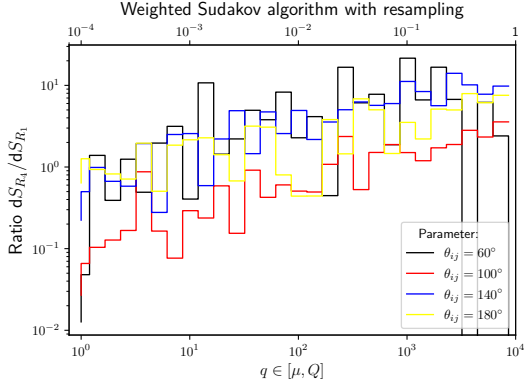


(c) Comparison cutoff 1 and 2.

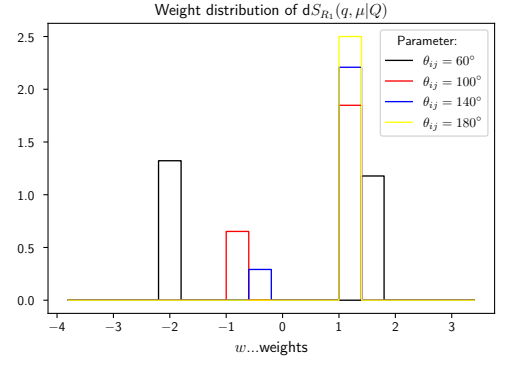
Figure 15: Weighted Sudakov algorithm for cutoff 1 and 2 with resampling, second implementation from figure 14. Parameters: $N = 3$, $\mu = 1$, $Q = 10^4$ and $\lambda = 0.3$, number of events 10^4 .



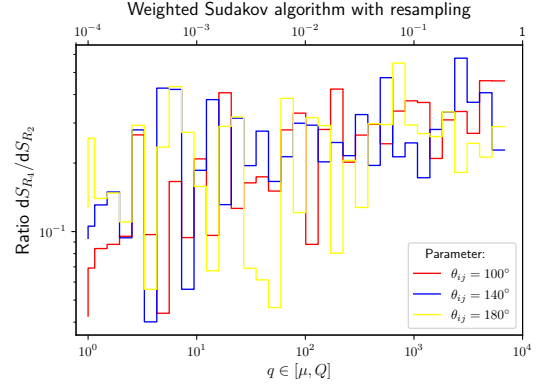
(a) Cutoff 4



(c) Comparison cutoff 4 and 1.



(b) Resulting weights, cutoff 4



(d) Comparison cutoff 4 and 2.

Figure 16: Weighted Sudakov algorithm for cutoff 4 with resampling, second implementation from figure 14. Parameters: $N = 3$, $\mu = 1$, $Q = 10^4$ and $\lambda = 0.3$, number of events 10^4 .

In figure 17 we observe that resampling with negative weights produces indeed the same statistic. Note that our implementation of resampling was still computational expensive, because of that the number of events was about a factor of ten smaller than the number of events used in the weighted Sudakov algorithm. The finding is therefore, that with lesser events we found the same statistic, which shows that resampling indeed could be used to improve the uncertainty of parton showers.

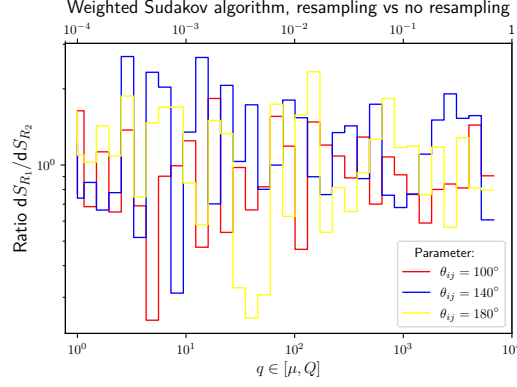


Figure 17: Weighted Sudakov algorithm with resampling, second implementation from figure 14. Comparison between resampled and not resampled densities. Parameters: $N = 3$, $\mu = 1$, $Q = 10^4$ and $\lambda = 0.3$, number of events 10^4 .

5.6 Shower algorithm

In this section the result for the shower algorithm are listed for the gaps-between jets observable, see section 3.2. The implemented shower algorithm goes as following 7.

Algorithm 7. Start with $n > 0$ dipoles at starting scale Q .

loop

Sample $(E_i, \vec{q}_i)$ according to $dS_{P_i}(E_i|Q)$ for $i = 1, \dots, n$.

Calculate $\xi = \max\{E_1, \dots, E_n\}$.

if $\xi = \mu$ **then**

there is no emission and the algorithm terminates.

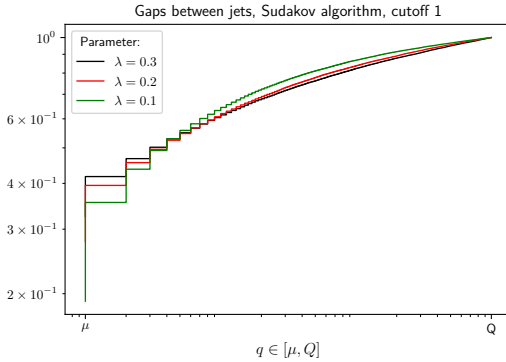
else

Update the list of n dipoles to a list of $n + 1$ dipoles with starting scale ξ .

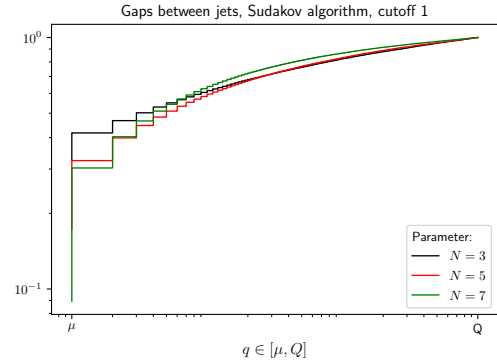
end if

end loop

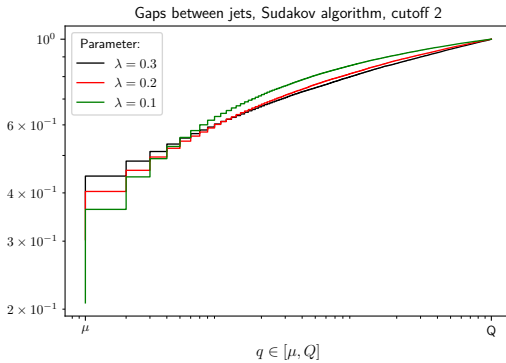
The algorithm is chosen such that there is no fixed number of possible emissions.



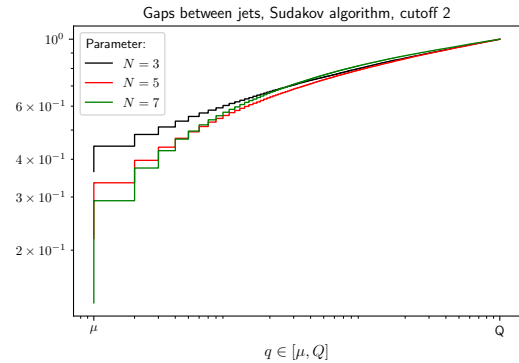
(a) Cutoff 1



(b) Cutoff 1

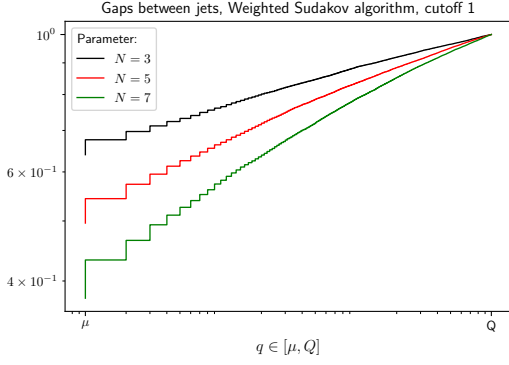


(c) Cutoff 2

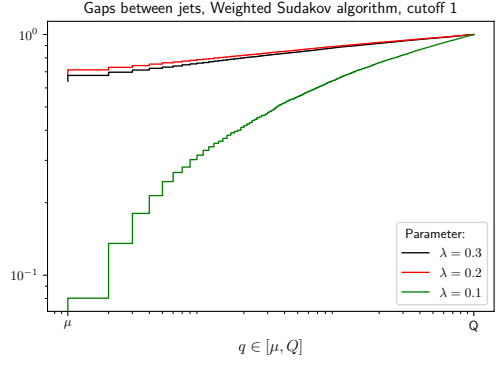


(d) Cutoff 2

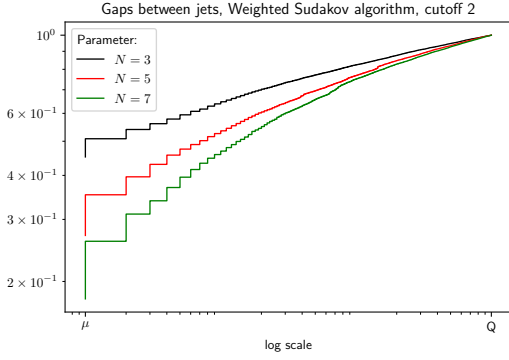
Figure 18: Gaps-between jets observable, 45° . The parameters: $N = 3$, $\mu = 1$, $Q = 10^4$.



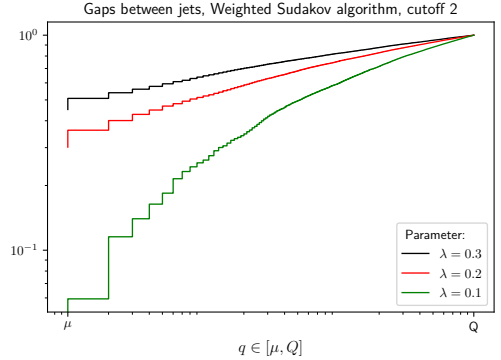
(a) Cutoff 1



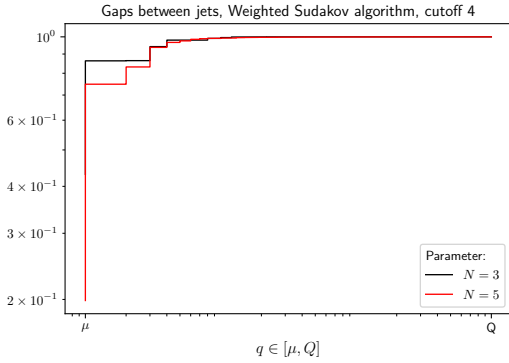
(b) Cutoff 1



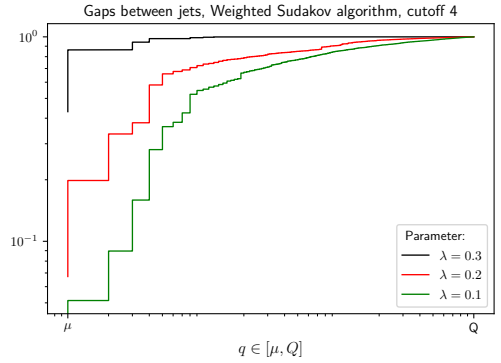
(c) Cutoff 2



(d) Cutoff 2



(e) Cutoff 4



(f) Cutoff 4

Figure 19: Gaps-between jets observable with weighted Sudakov, 45° . The parameters: $\mu = 1$, $Q = 10^4$.

Figure 19 showcases the problem with negative weights, which are needed for cutoff description four. The plan to solve the problem with negative weights in the future, is to apply resampling strategies. Both figure 18 and 19 show also the cutoff dependents of the shower, this was already observed in [1]. From the presented differential cross section one can also observe that the chosen collinear cutoffs differ in their sensibility with respect to the cutoff parameter λ .

The differential cross-sections were calculated in the back to back frame, to obtain the results in different frames, they can be boosted with the transformations, given in appendix A.

6 Ingredients for simulating double soft gluon emission

6.1 Dipoles, rings and strings

The following section aims to give a short introduction to rings, strings and dipoles, which are used to decompose the density matrix $\mathbf{A}_n$ at tree level. Then a possible extension of this framework for the double emission processes is presented. The main source for the introductory part is [28].

The main motivation for developing the ring function is the well known fact that off-diagonal density matrix elements are free of collinear singularities, although they are written as linear combination of dipole functions $w_{ij}(q)$, defined as

$$\omega_{ij}(q_n) = \frac{p_i \cdot p_j}{p_i \cdot q_n p_j \cdot q_n}, \quad (6.1)$$

which have the symmetry $w_{ij}(q_n) = w_{ji}(q_n)$. These functions are also often referred to as antenna function in the literature. Collinear singularities arise if either p_i or p_j is collinear to the emitted soft gluon n with momenta q_n . The simplest linear combination of dipoles, which is free of singularities, is the ring function $R_{k,l}^{i,l}(q_n)$ [28],

$$R_{k,l}^{i,l}(q_n) = \omega_{ij}(q_n) - \omega_{ik}(q_n) - \omega_{jl}(q_n) + \omega_{kl}(q_n), \quad (6.2)$$

where $i, j, k, l \in \{1, 2, \dots, n-1\}$ are pairwise different and q_n is again the momentum associated to the n th gluon, which is soft. That this is indeed the simplest linear combination of dipoles, which is collinear singularities free, can be seen by noting that if one starts from the dipole $w_{ij}(q_n)$, then to remove the collinear dipoles of $q_n || p_i$ or $q_n || p_j$, two dipoles have to be subtracted $w_{ik}(q_n)$ and $w_{jl}(q_n)$. But then there are collinear singularities with negative sign if $q_n || p_k$ or $q_n || p_l$, hence the dipole $w_{kl}(q_n)$ has to be added. The ring functions have the following symmetries

$$R_{k,l}^{i,j}(q_n) = R_{l,k}^{j,i}(q_n) \quad R_{k,l}^{i,j}(q_n) = -R_{k,i}^{l,j}(q_n) \quad R_{k,l}^{i,j}(q_n) = -R_{j,l}^{i,k}(q_n). \quad (6.3)$$

In [28], they also presented a graphical representation of the rings, strings and dipoles, the strings will be introduced in moment. To count the number of linearly independent rings, let's first count the number of possible dipole functions. If $i, j \in \{1, \dots, n-1\}$, then for a fixed i there are $n-2$ possible choices for j , since $i \neq j$. Furthermore, with $w_{ij}(q_n) = w_{ji}(q_n)$ and that there are $n-1$ possible choices for i , it follows that the number of possible dipoles is

$$\#w_{ij}(q_n) = \frac{(n-1)(n-2)}{2}. \quad (6.4)$$

With the unique identification of the dipole function with the vector [28]

$$\omega_{ij}(q_n) \equiv \begin{pmatrix} \delta_{1,i} + \delta_{1,j} \\ \delta_{2,i} + \delta_{2,j} \\ \vdots \\ \delta_{n-1,i} + \delta_{n-1,j} \end{pmatrix} \in \mathbb{R}^{n-1}, \quad (6.5)$$

it follows that the set of dipole functions $\omega_{ij}(q_n)$ span an $n - 1$ dimensional vector space. The representation of the dipole functions in 6.5 is called pole representation. By construction, the ring functions are collinear singularities free, therefore, the number of rings that could be constructed from the spanning set $\{\omega_{ij}(q_n)\}_{i,j=1,i \neq j}^{n-1}$, is the number of linear independent null vectors, which can be constructed from the pole representation 6.5, which is given by

$$\#\omega_{ij}(q_n) - \dim(\mathbb{R}^{n-1}) = \frac{(n-1)(n-4)}{2}. \quad (6.6)$$

An example for complete a basis of ring functions is then given by [28]

$$\left\{ R_{3,i}^{1,2}, R_{i,3}^{1,2}, R_{i,j}^{1,2} \middle| (i > 3) \wedge (j > i) \right\}. \quad (6.7)$$

The string functions are defined as the linear combinations of dipoles, which carry only one collinear singularity $p_i || q_n$. For convenience, the shortest linear combination will be chosen for the string basis

$$S_i^{jk}(q_n) = \frac{1}{2} (w_{ij}(q_n) + w_{ik}(q_n) - w_{jk}(q_n)), \quad (6.8)$$

which is independent of the indices j, k . The identification through the pole representation of the dipoles gives

$$S_i^{jk}(q_n) = \begin{pmatrix} \delta_{1,i} \\ \delta_{2,i} \\ \vdots \\ \delta_{n-1,i} \end{pmatrix} \in \mathbb{R}^{n-1}. \quad (6.9)$$

Therefore, the set of string functions $\{S_i^{jk}(q_n)\}_{i=1}^{n-1}$ forms a basis for the space spanned by the set of dipoles $\{\omega_{ij}(q_n)\}_{i,j=1,i \neq j}^{n-1}$. It is possible to write every dipole as

$$\omega_{ij}(q_n) = S_i^{k,l}(q_n) + S_j^{m,n}(q_n) + \frac{1}{2} (R_{k,l}^{i,j}(q_n) + R_{m,i}^{j,l}(q_n) + R_{m,n}^{i,j}(q_n)). \quad (6.10)$$

It follows [28],

$$\begin{aligned} \mathbf{A}_n &= -\frac{\alpha_s}{\pi} \sum_{i \neq j} \mathbf{T}_i |M_{n-1}\rangle \langle M_{n-1}| \mathbf{T}_j^\dagger \omega_{ij}(q_n) \\ &= \frac{2\alpha_s}{\pi} \sum_{i=1}^{n-1} \mathbf{T}_i |M_{n-1}\rangle \langle M_{n-1}| \mathbf{T}_i^\dagger S_i^{k,l}(q_n) \\ &\quad - \frac{\alpha_s}{2\pi} \sum_{\substack{i,j=1 \\ i \neq j}}^{n-1} \mathbf{T}_i |M_{n-1}\rangle \langle M_{n-1}| \mathbf{T}_j^\dagger (R_{k,l}^{i,j}(q_n) + R_{m,i}^{j,l}(q_n) + R_{m,n}^{i,j}(q_n)). \end{aligned} \quad (6.11)$$

This shows that the contributions of the tree level density operator $\mathbf{A}_n$, which carry a collinear pole, are colour diagonal. Note that some care has to be taken with the second sum in 6.11, since the sum goes over more terms than there are linear independent rings, for more detail see [28]. It was shown in [28], that the decomposition in 6.11 improves the numeral evaluation of the gaps-between jets observables of the soft gluon evolution within the CVolver [1] [39] framework.

Why this is the case is intuitively clear, since the dipole decomposition relies on the effect that for large samples the divergences cancel statistically, while in the rings and strings decomposition the cancellation was incorporated intrinsically.

For the rest of the work, the following shorthand notation will be used

$$\mathbf{T}_i^a \mathbf{T}_j^b \circ \mathbf{T}_k^c \mathbf{T}_l^{d\dagger} \equiv \mathbf{T}_i^a \mathbf{T}_j^b |M_{n-2}\rangle \langle M_{n-2}| \mathbf{T}_k^c \mathbf{T}_l^{d\dagger}. \quad (6.12)$$

The rest of the section aims to generalize the notion of rings and strings to the double emission case, the first step was to identify all possible colour topologies for the double emission. The arising topologies are listed in figure 20.

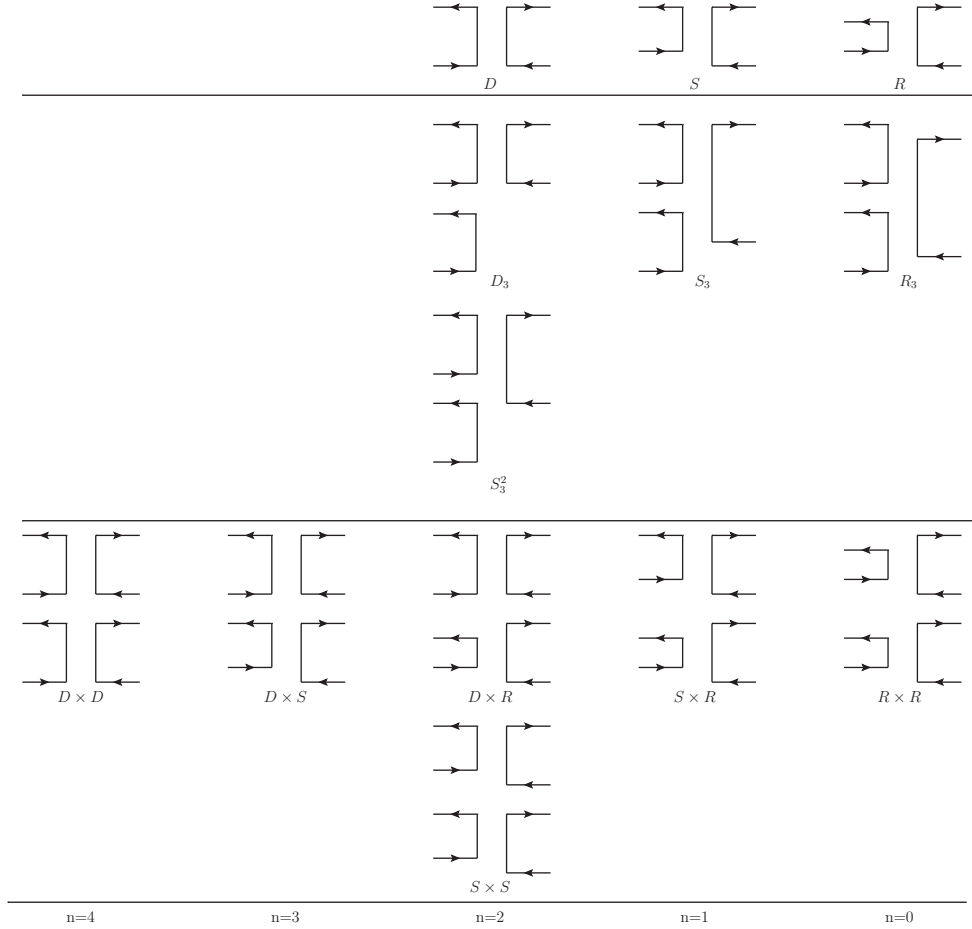


Figure 20: In the first row, the colour topologies for the dipole (D), ring (R) and string (S) in the single emission are listed. The second and third line give the colour topologies, up to complex conjugation, for the double emission case.

Figure 20 shows the possible colour topologies for double gluon emission. The topologies are labelled such that they are a generalization from the single emission case. The dipole D is the topology, where the gluon gets emitted from the same dipole on the amplitude and conjugated amplitude, that have therefore two indices in common, the string S is the topology where amplitude and conjugated amplitude have one index in common and the ring R is the topology where they have no index in common.

Applying the double-soft gluon emission current 3.15 to the amplitude $|M_{n-2}\rangle\langle M_{n-2}|$ gives rise to the following pattern

$$J_{\mu,\sigma}^{a_2 a_1}(q_2, q_1) \circ J^{\mu,\sigma,a_2 a_1 \dagger}(q_2, q_1) = \sum_{\substack{(a,b),(c,d) \\ =(a_1,a_2),(a_2,a_1)}} \sum_{i,j,k,l=1}^{n-2} \mathbf{T}_i^{(a)} \mathbf{T}_j^{(b)} \circ \mathbf{T}_k^{(c),\dagger} \mathbf{T}_l^{(d),\dagger} \omega_{ijkl}^{abcd}(q_1, q_2), \quad (6.13)$$

where the commutator relation of the colour operator was used

$$\mathbf{T}_g^{(12)} \mathbf{T}_i^{(g)} = \mathbf{T}_i^{(1)} \mathbf{T}_i^{(2)} - \mathbf{T}_i^{(2)} \mathbf{T}_i^{(1)}. \quad (6.14)$$

$\mathbf{T}_g$ is the emission operator for gluons. The full list of the functions w_{ijkl}^{abcd} is given in appendix H. In the following the possible combinations and the symmetries in lower and upper indices are listed in table 1, 2 and 3.

Indices	Conditions
$ijkl$	$i \neq j$
$ijil$	$i \neq j, i \neq l, j \neq l$
$ijkj$	$i \neq j, i \neq k, j \neq k$
$ijij$	$i \neq j$
$iikl$	$i \neq k, i \neq l, k \neq l$
$iiki$	$i \neq k$
$iiil$	$i \neq l$
$ijkk$	$k \neq i, k \neq j, i \neq j$
$kjkk$	$k \neq j$
$ikkk$	$k \neq i$
$iikk$	$i \neq k$
$iiii$	

Table 1: Possible combinations of lower indices.

Symmetries	Conditions
$w_{ijij}^{a_1 a_2 a_2 a_1} = w_{jiji}^{a_1 a_2 a_2 a_1}$	$i \neq j, k \neq l, j \neq k, i \neq l$
$w_{ijkk}^{abcd} = w_{kkji}^{abcd}$	$i \neq k, i \neq l, k \neq l$
$w_{iiik}^{abcd} = w_{kiii}^{abcd}$	$i \neq k$
$w_{iiki}^{abcd} = w_{ikii}^{abcd}$	$i \neq k$

Table 2: Symmetries in lower indices.

The symmetries can be used to reduce the number of kernels from which one has to sample in the parton shower. Concretely, they will be used for that in section 6.3, there the leading colour contribution of the double emission process will be analysed.

Symmetries	Conditions
$\omega_{kkki}^{a_1 a_2 a_2 a_1} = -\omega_{kkki}^{a_2 a_1 a_2 a_1}$	$k \neq i$
$\omega_{kkjk}^{a_1 a_2 a_2 a_1} = -\omega_{kkjk}^{a_2 a_1 a_2 a_1}$	$k \neq j$
$\omega_{ikkk}^{a_1 a_2 a_2 a_1} = -\omega_{ikkk}^{a_1 a_2 a_1 a_2}$	$i \neq k$
$\omega_{kjkk}^{a_1 a_2 a_2 a_1} = -\omega_{kjkk}^{a_1 a_2 a_1 a_2}$	$j \neq k$
$w_{kkl}^{a_2 a_1 a_1 a_2} = w_{kkl}^{a_1 a_2 a_2 a_1}$	$k \neq l$
$\omega_{llll}^{a_2 a_1 a_1 a_2} = \omega_{llll}^{a_1 a_2 a_2 a_1} = -\omega_{llll}^{a_1 a_2 a_1 a_2} = -\omega_{llll}^{a_2 a_1 a_2 a_2}$	

Table 3: Symmetries in upper indices.

A notable finding is that some functions descend from each other by setting indices equal and using the on shell conditions. The cases where this is true are

$$w_{ijkk}^{abcd} \longrightarrow w_{ikkk}^{abcd}, w_{kjkk}^{abcd}, \quad (6.15)$$

and

$$w_{kkl}^{abcd} \longrightarrow w_{kkkk}^{abcd}. \quad (6.16)$$

To recover the ordered limit $q_2 \ll q_1$, one has to replace q_1 with λq_1 and $\lambda^2 q_2$ and let $\lambda \rightarrow 0$, the conditions on lower indices are not affected by performing this particular limit. For example

$$\omega_{ijkl}^{a_1 a_2 a_2 a_1} \xrightarrow{q_2 \ll q_1} = \frac{1}{\lambda^6} \omega_{il}^{(1)} \omega_{jk}^{(2)}, \quad (6.17)$$

where the notation $\omega_{ij}^n = \omega_{ij}(q_n)$ was used. A more interesting case is

$$\omega_{ijkk}^{a_2 a_1 a_1 a_2} \xrightarrow{q_2 \ll q_1} = \frac{1}{\lambda^6} \omega_{ik}^{(2)} \omega_{jk}^{(1)} - \frac{1}{\lambda^6} \omega_{i1}^{(2)} \omega_{jk}^{(1)} + \mathcal{O}\left(\frac{1}{\lambda^5}\right). \quad (6.18)$$

The full list can be found in appendix I. The obtained result is reinsuring in the sense that only terms appear, where the second gluon q_2 can be emitted either from the $n - 2$ hard partons and the first gluon q_1 , while the first gluon will only be emitted from the $n - 2$ hard partons, never from the second gluon q_2 . Of course, the choice that $q_2 \ll q_1$ was arbitrary, one can do a similar calculation with $q_1 \ll q_2$.

6.2 Colour Monte Carlo simulation

This chapter aims to show how to simulate the evolution of colour states beyond leading order, expressed in the colour-flow basis, in the case of real emissions with Monte Carlo methods. At first the necessary ingredients for the single emission case are presented, then the ingredients for double emission.

It is useful to introduce the notation $\underline{T}_j$ for the emission operator in the colour-flow basis representation, the components are defined through [28, 1]

$$(\underline{T}_i)_{\sigma\tau} = [\sigma|\mathbf{T}_i|\tau], \quad (6.19)$$

where it is important to remember that the colour operator maps the colour flow state $|\sigma\rangle$, describing the colour state of n partons to a colour state which describes $n + 1$ partons, hence adds a colour line in the colour-flow basis. More generally, when considering a sequence of real emissions, the representation of correlators of many colour charge operators, in the colour-flow basis representation, are denoted with [28]

$$\left(\underline{T}_{i_m, \dots, i_1, j_1, \dots, j_m}\right)_{\sigma\tau} = [\sigma|\mathbf{T}_{i_m}^\dagger \cdots \mathbf{T}_{i_1}^\dagger \mathbf{T}_{j_1} \cdots \mathbf{T}_{j_m}|\tau]. \quad (6.20)$$

Inserting the identity at different positions and by using that $\mathbf{T}_i \cdot \mathbf{T}_j$ commutes if $i \neq j$, it follows that [28]

$$\underline{T}_j^\dagger S_{n+1} \underline{T}_i = S_n \underline{T}_{ij} = \underline{T}_i^\dagger S_{n+1} \underline{T}_j. \quad (6.21)$$

With the cyclicity of the trace

$$\text{Tr}[\mathbf{T}_i \mathbf{A}_n \mathbf{T}_j^\dagger] = \text{Tr}[\mathbf{T}_j^\dagger \mathbf{T}_i \mathbf{A}_n], \quad (6.22)$$

note that the identity $\text{Tr}[AB] = \text{Tr}[BA]$ also holds if A and B are rectangular matrices of the same dimensionality. The resulting contribution of the evolving amplitude in the colour space to the cross section through a sequence of m real emissions is in the colour-flow basis given by

$$\text{Tr} \left[S_{n+m} \underline{T}_{j_1} \cdots \underline{T}_{j_m} \mathcal{A}_n \underline{T}_{i_m}^\dagger \cdots \underline{T}_{i_1}^\dagger \right] = \text{Tr} \left[S_n \underline{T}_{i_m, \dots, i_1, j_1, \dots, j_m} \mathcal{A}_n \right], \quad (6.23)$$

where $\mathcal{A}_n$ is the initial hard density operator of n partons in the colour-flow representation. A recursive definition of the trace in terms of single colour charge correlations 6.23 could then be obtained by inserting identity operators 4.52

$$\frac{\left(\underline{T}_{i_m, \dots, i_1, j_1, \dots, j_m}\right)_{\sigma\bar{\sigma}}}{\langle\sigma|\bar{\sigma}\rangle} = \sum_{\tau, \bar{\tau}} \xi_{i_m, j_m}^{\sigma\tau\bar{\tau}\bar{\sigma}} \frac{\left(\underline{T}_{i_{m-1}, \dots, i_1, j_1, \dots, j_{m-1}}\right)_{\tau\bar{\tau}}}{\langle\tau|\bar{\tau}\rangle}, \quad (6.24)$$

where

$$\xi_{ij}^{(\sigma\tau\bar{\tau}\bar{\sigma})} = \frac{\langle\tau|\bar{\tau}\rangle}{\langle\sigma|\bar{\sigma}\rangle} [\sigma|\mathbf{T}_i^\dagger|\tau][\bar{\tau}|\mathbf{T}_j|\bar{\sigma}]. \quad (6.25)$$

For the recursive calculation of 6.24 the initial condition $\underline{T}_{i_1 j_1}$ has to be given. In practice, 6.24 will never be calculated exactly, since the number of terms scale as $\mathcal{O}((m-1)!)$, if m denotes the number of emitted partons. In order to avoid the factorial scaling, MC methods can be used to evaluate 6.24. Furthermore, it is then important to note that for a given colour state $|\tau\rangle$ only a few colour states, after emitting a parton, can be reached. Hence, $\xi_{ij}^{(\tau\sigma\bar{\sigma}\tau)}$ is mostly zero, which decreases the computational cost in the evaluation. The object 6.25 is therefore a key ingredient for colour-flow MC simulations and is for example used in *CVolver* [1] to sample colour flows of real emissions in the soft-gluon evolution. An important question to clarify is how ξ depends on N , this will be discussed in the following for the single emission case.

Due to the structure 4.69 of the emission operator $\mathbf{T}_i$, the quantity $\xi_{i_1, j_1}^{\sigma\bar{\sigma}, \sigma\bar{\sigma}}$ could be further decomposed as

$$\begin{aligned} \xi_{i_1, j_1}^{\sigma\tau\bar{\tau}\bar{\sigma}} = & \lambda_{i_1} \lambda_{j_1} \xi_{i_1, j_1}^{\sigma\tau\bar{\tau}\bar{\sigma}, \mathbf{t}_{i_1}, \mathbf{t}_{j_1}} - \lambda_{i_1} \bar{\lambda}_{j_1} \xi_{i_1, j_1}^{\sigma\tau\bar{\tau}\bar{\sigma}, \mathbf{t}_{i_1}, \bar{\mathbf{t}}_{j_1}} - \bar{\lambda}_{i_1} \lambda_{j_1} \xi_{i_1, j_1}^{\sigma\tau\bar{\tau}\bar{\sigma}, \bar{\mathbf{t}}_{i_1}, \mathbf{t}_{j_1}} + \bar{\lambda}_{i_1} \bar{\lambda}_{j_1} \xi_{i_1, j_1}^{\sigma\tau\bar{\tau}\bar{\sigma}, \bar{\mathbf{t}}_{i_1}, \bar{\mathbf{t}}_{j_1}} \\ & - \lambda_{i_1} (\lambda_{i_1} - \lambda_{j_1}) \xi_{i_1, j_1}^{\sigma\tau\bar{\tau}\bar{\sigma}, \mathbf{t}_{i_1}, \mathbf{s}} - (\lambda_{i_1} - \lambda_{j_1}) \lambda_{j_1} \xi_{i_1, j_1}^{\sigma\tau\bar{\tau}\bar{\sigma}, \mathbf{s}, \mathbf{t}_{j_1}} + \bar{\lambda}_{i_1} (\lambda_{i_1} - \lambda_{j_1}) \xi_{i_1, j_1}^{\sigma\tau\bar{\tau}\bar{\sigma}, \bar{\mathbf{t}}_{i_1}, \mathbf{s}} \\ & + (\lambda_{i_1} - \lambda_{j_1}) \lambda_{j_1} \xi_{i_1, j_1}^{\sigma\tau\bar{\tau}\bar{\sigma}, \mathbf{s}, \bar{\mathbf{t}}_{j_1}} + (\lambda_{i_1} - \lambda_{j_1})^2 \xi_{i_1, j_1}^{\sigma\tau\bar{\tau}\bar{\sigma}, \mathbf{s}, \mathbf{s}}, \end{aligned} \quad (6.26)$$

where the first line is the contribution from the case, where only gluon emission is considered. Furthermore the newly introduced quantities are, with $\mathbf{A}, \mathbf{B} = \mathbf{t}_i, \bar{\mathbf{t}}_j, \mathbf{s}$, defined as

$$\xi_{i_1, j_1}^{\sigma\tau\bar{\tau}\bar{\sigma}, \mathbf{A}, \mathbf{B}} = \frac{1}{N^{\#\mathbf{s}}} \frac{\langle \tau | \bar{\tau} \rangle}{\langle \sigma | \bar{\sigma} \rangle} [\sigma | \mathbf{A}^\dagger | \tau] [\bar{\tau} | \mathbf{B} | \bar{\sigma}]. \quad (6.27)$$

These quantities can then be partially calculated with the definitions of the $\mathbf{t}_i, \bar{\mathbf{t}}_i$ and $\mathbf{s}$ operators in the following way

$$\frac{\langle \sigma | \mathbf{s}^\dagger \mathbf{s} | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} = \frac{\langle \sigma(1) \cdots \sigma(n) n+1 | \bar{\sigma}(1) \cdots \bar{\sigma}(n) n+1 \rangle}{\langle \sigma | \bar{\sigma} \rangle} = \frac{N^{n+1 - \#\text{transpositions}(\sigma, \bar{\sigma})}}{N^{n - \#\text{transpositions}(\sigma, \bar{\sigma})}} = N. \quad (6.28)$$

So, if the permutations σ and $\bar{\sigma}$ differ by m transpositions, then also the permutation, which labels the colour states $\mathbf{s} | \sigma \rangle$ and $\mathbf{s} | \bar{\sigma} \rangle$, differs by m transpositions. The next case

$$\begin{aligned} \frac{\langle \sigma | \mathbf{t}_i^\dagger \mathbf{s} | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} &= \frac{\langle \sigma(1) \cdots n+1 \cdots \sigma(n) \sigma(i) | \bar{\sigma}(1) \cdots \bar{\sigma}(n) n+1 \rangle}{\langle \sigma(1) \cdots \sigma(n) | \bar{\sigma}(1) \cdots \bar{\sigma}(n) \rangle} \\ &= \frac{N^{n+1 - (\#\text{transpositions}(\sigma, \bar{\sigma}) + 1)}}{N^{n - \#\text{transpositions}(\sigma, \bar{\sigma})}} = 1. \end{aligned} \quad (6.29)$$

The permutations, which label the states $\mathbf{t}_i | \sigma \rangle$ and $\mathbf{s} | \bar{\sigma} \rangle$ differ therefore by $m+1$ transpositions if σ and $\bar{\sigma}$ differ by m transpositions.

Now the most difficult expression $\langle \sigma | \mathbf{t}_i^\dagger \mathbf{t}_j | \bar{\sigma} \rangle / \langle \sigma | \bar{\sigma} \rangle$, where two cases could be distinguished $i = j$ and $i \neq j$. In the first case, it is usefule to decomposed $\mathbf{t}_i$ as $\mathbf{t}_i = \mathbf{s}_{in+1} \mathbf{s}$, then

$$\mathbf{t}_i^\dagger \mathbf{t}_i = (\mathbf{s}_{in+1} \mathbf{s})^\dagger \mathbf{s}_{in+1} \mathbf{s} = \mathbf{s}^\dagger \mathbf{s}_{in+1}^\dagger \mathbf{s}_{in+1} \mathbf{s} = \mathbf{s}^\dagger \mathbf{s}. \quad (6.30)$$

Therefore,

$$\frac{\langle \sigma | \mathbf{t}_i^\dagger \mathbf{t}_i | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} = \frac{\langle \sigma | \mathbf{s}^\dagger \mathbf{s} | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} = N. \quad (6.31)$$

The resulting statement is that, given that the permutation τ and $\bar{\tau}$ differ by m transpositions, also the permutations, which label $\mathbf{t}_i|\tau\rangle$ and $\mathbf{t}_i|\bar{\tau}\rangle$, differ by m transpositions. The second case, where $i \neq j$ can in general not be calculated and depends on σ and $\bar{\sigma}$, hence

$$\frac{\langle \sigma | \mathbf{t}_i^\dagger \mathbf{t}_j | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} = N \text{ or } \frac{1}{N}. \quad (6.32)$$

In total if $[\sigma | \mathbf{A}_i^\dagger | \tau] [\bar{\sigma} | \mathbf{B}_j | \bar{\sigma}] = 1$, where $A_i = \mathbf{t}_i, \bar{\mathbf{t}}_i, \mathbf{s}$ and $B_j = \mathbf{t}_j, \bar{\mathbf{t}}_j, \mathbf{s}$, the quantity $\xi_{i,j}^{\sigma\tau\bar{\tau}\bar{\sigma}}$ becomes for the dipole, string and ring evolution:

Dipole $i = j$ and $k = l$:

$$\xi_{i,j}^{\sigma\tau\bar{\tau}\bar{\sigma}} = \lambda^2 N - \lambda \bar{\lambda} \frac{\langle \sigma | \mathbf{t}_i^\dagger \mathbf{t}_{\bar{\sigma}^{-1}(i)} | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} - \lambda \bar{\lambda} \frac{\langle \sigma | \mathbf{t}_{\tau^{-1}(i)}^\dagger \mathbf{t}_i | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} + \bar{\lambda}^2 N - \frac{1}{N} (\lambda - \bar{\lambda})^2. \quad (6.33)$$

String $i = j$ and $k \neq l$:

$$\xi_{i,j}^{\sigma\tau\bar{\tau}\bar{\sigma}} = \lambda^2 N - \lambda \bar{\lambda} \frac{\langle \sigma | \mathbf{t}_i^\dagger \mathbf{t}_{\bar{\sigma}^{-1}(i)} | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} - \lambda \bar{\lambda} \frac{\langle \sigma | \mathbf{t}_{\sigma^{-1}(i)}^\dagger \mathbf{t}_i | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} + \bar{\lambda}^2 N - \frac{1}{N} (\lambda - \bar{\lambda})^2. \quad (6.34)$$

String if $i \neq j$ and $k = l$:

$$\xi_{i,j}^{\sigma\tau\bar{\tau}\bar{\sigma}} = \lambda^2 N - \lambda \bar{\lambda} \frac{\langle \sigma | \mathbf{t}_i^\dagger \mathbf{t}_{\bar{\sigma}^{-1}(j)} | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} - \lambda \bar{\lambda} \frac{\langle \sigma | \mathbf{t}_{\sigma^{-1}(i)}^\dagger \mathbf{t}_j | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} + \bar{\lambda}^2 N - \frac{1}{N} (\lambda - \bar{\lambda})^2. \quad (6.35)$$

Ring $i \neq j$ and $j \neq k$:

$$\begin{aligned} \xi_{i,j}^{\sigma\tau\bar{\tau}\bar{\sigma}} = & \lambda^2 \frac{\langle \sigma | \mathbf{t}_i^\dagger \mathbf{t}_j | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} - \lambda \bar{\lambda} \frac{\langle \sigma | \mathbf{t}_i^\dagger \mathbf{t}_{\bar{\sigma}^{-1}(j)} | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} - \lambda \bar{\lambda} \frac{\langle \sigma | \mathbf{t}_{\sigma^{-1}(i)}^\dagger \mathbf{t}_j | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} + \bar{\lambda}^2 \frac{\langle \sigma | \mathbf{t}_{\sigma^{-1}(i)}^\dagger \mathbf{t}_{\bar{\sigma}^{-1}(j)} | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} \\ & - \frac{1}{N} (\lambda - \bar{\lambda})^2. \end{aligned} \quad (6.36)$$

If double emission gets included in a parton shower, then there are two competing processes in each evolution step, the one emission process and the double emission process. The previous recursive definition of the multi-colour correlator 6.24 was suitable to sample the colour for a single emission process, so in order to sample the colour flow for the double emission case, the multi-colour correlator can be recursively written as

$$\frac{(T_{i_m, \dots, i_1, j_1, \dots, j_m})_{\sigma\bar{\sigma}}}{\langle \sigma | \bar{\sigma} \rangle} = \sum_{\tau, \bar{\tau}} \xi_{i_m, i_{m-1}, j_{m-1}, j_m}^{\sigma\tau\bar{\tau}\bar{\sigma}} \frac{(T_{i_{m-2}, \dots, i_1, j_1, \dots, j_{m-2}})_{\tau\bar{\tau}}}{\langle \tau | \bar{\tau} \rangle}, \quad (6.37)$$

where

$$\xi_{i_m, i_{m-1}, j_{m-1}, j_m}^{\sigma\tau\bar{\tau}\bar{\sigma}} = \frac{\langle \tau | \bar{\tau} \rangle}{\langle \sigma | \bar{\sigma} \rangle} [\sigma | \mathbf{T}_{i_m}^\dagger \mathbf{T}_{i_{m-1}}^\dagger | \tau] [\bar{\tau} | \mathbf{T}_{j_{m-1}} \mathbf{T}_{j_m} | \bar{\sigma}]. \quad (6.38)$$

Therefore, the following quantity has to be investigated

$$\frac{\langle \sigma | \mathbf{t}_i^\dagger \mathbf{t}_j^\dagger \mathbf{t}_k \mathbf{t}_l | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle}. \quad (6.39)$$

The easiest case is where $i = j = k = l$, then

$$\mathbf{t}_i^\dagger \mathbf{t}_i^\dagger \mathbf{t}_i \mathbf{t}_i = \mathbf{t}_i^\dagger \mathbf{s}^{n+1} \mathbf{s}^{n+1} \mathbf{t}_i = \mathbf{s}^n \mathbf{s}^{n+1} \mathbf{s}^{n+1} \mathbf{s}^n, \quad (6.40)$$

where n indicates the number of partons contained in the colour state on which $\mathbf{s}$ is acting. With that

$$\frac{\langle \sigma | \mathbf{t}_i^\dagger \mathbf{t}_i^\dagger \mathbf{t}_i \mathbf{t}_i | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} = \frac{N^{n+2} - \# \text{transposition}(\sigma, \bar{\sigma})}{N^n - \# \text{transpositions}(\sigma, \bar{\sigma})} = N^2. \quad (6.41)$$

A similar calculation can be done for $i = l$ and $j = k$ but $i \neq j$, hence

$$\frac{\langle \sigma | \mathbf{t}_i^\dagger \mathbf{t}_j^\dagger \mathbf{t}_j \mathbf{t}_i | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} = N^2. \quad (6.42)$$

Also, the case where $i \neq l$ and $j = k$ can be simplified with the same considerations

$$\frac{\langle \sigma | \mathbf{t}_i^\dagger \mathbf{t}_j^\dagger \mathbf{t}_j \mathbf{t}_l | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} = N \frac{\langle \sigma | \mathbf{t}_i^\dagger \mathbf{t}_l | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle}. \quad (6.43)$$

The most difficult case, where $i \neq l$ and $j \neq k$, can not be calculated in general

$$\frac{\langle \sigma | \mathbf{t}_i^\dagger \mathbf{t}_k^\dagger \mathbf{t}_j \mathbf{t}_l | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} = \frac{1}{N^2}, 1 \quad \text{or} \quad N^2. \quad (6.44)$$

The resulting $\xi_{i_m, i_{m-1}, j_{m-1}, j_m}^{\sigma \tau \bar{\tau} \bar{\sigma}}$ can be found in appendix J, furthermore, double emission colour correlators, in the case that quark and anti-quark emissions are considered, are listed.

The for the colour sampling required algorithms are listed in appendix G.

6.3 Leading colour contribution for double gluon emission

In the following, the leading colour contribution for double soft-gluon emission is presented. The leading colour contribution is the part of the amplitude, where the amplitude and the conjugated amplitude is described by the same permutation σ and is consequently not colour suppressed.

At leading colour the amplitude could evolve, in the case of double gluon emission, through one of the four possible transitions. Two transitions start at two colour lines and end at four colour lines and the other two transitions start at one colour line and end at three colour lines.

The possible evolutions for single gluon emission for the amplitude, starting at one colour line is shown in figure 21.

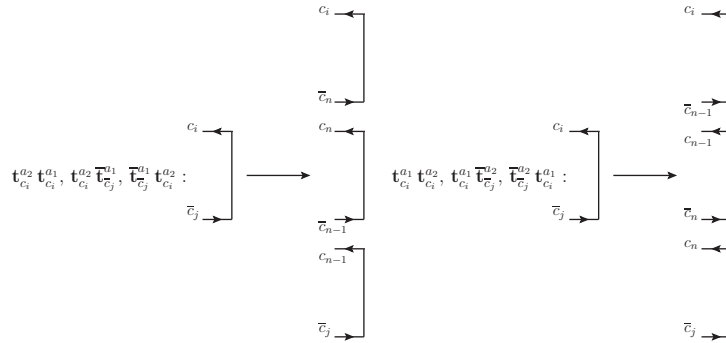


Figure 21: Transitions of the amplitude, starting from one colour line for single gluon emission.

The double gluon emission transitions at leading colour are depicted in the following in figure 22, 28, 23 and 24. That there are two transitions to start from one or two colour lines respectively, arises from freedom of labeling the two gluons with colour charge a_1 and a_2 or vice versa.

All possible transitions are then listed in table 4.

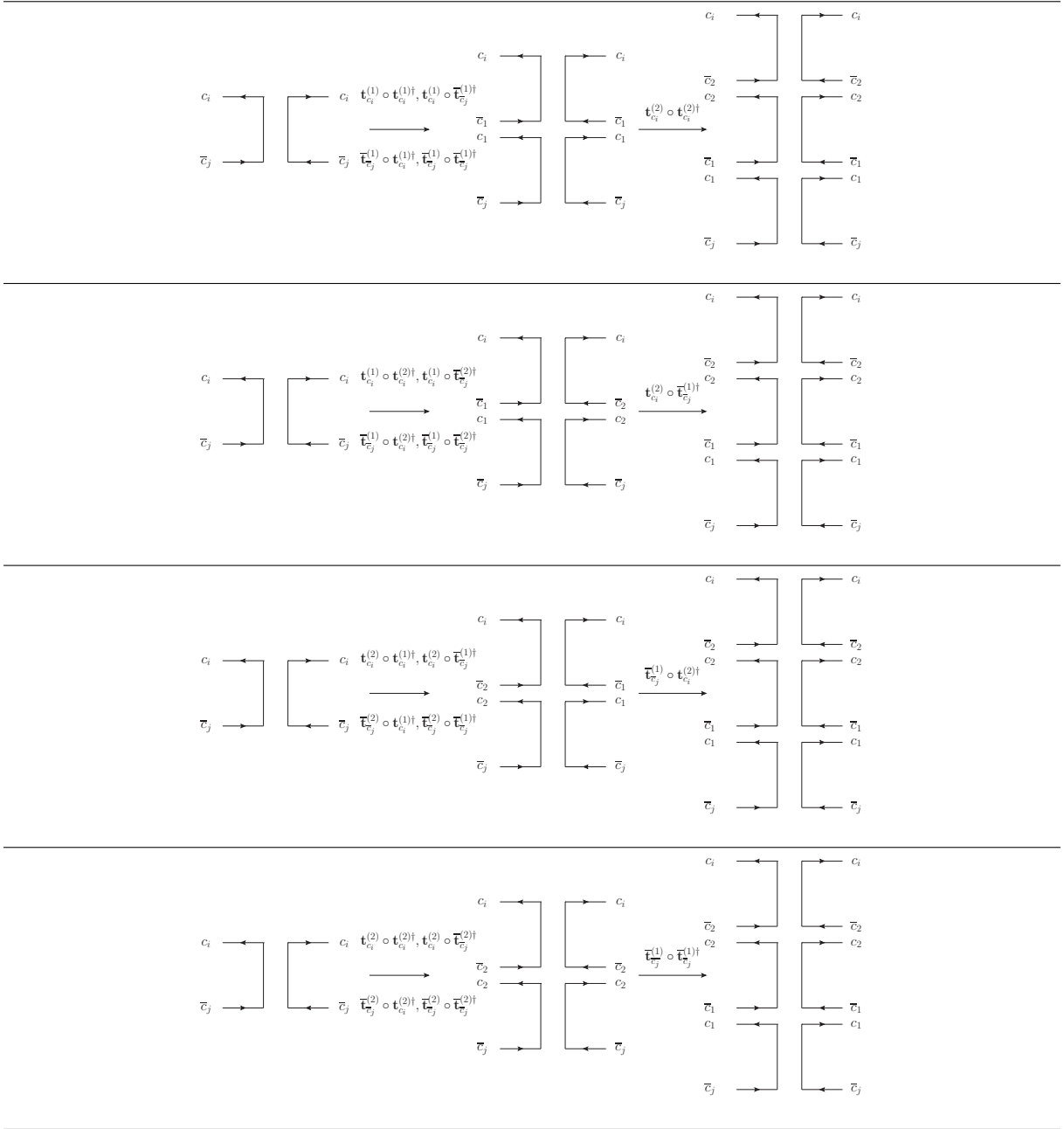


Figure 22: Leading colour transitions, starting from one colour line, the resulting transition kernel is denoted with K_1 .

The transitions, where the colour labels a_1 and a_2 are interchanged in comparison to the transitions in figure 22, can be found in appendix K, figure 28.

With table 4, one can write down the splitting kernels, for the first two possible transitions. The first 16 entries of table 4 contribute to transition 1 and the next 16 entries contribute to transition 2, kernel K_1 reads

$$\begin{aligned}
K_1 = & \omega_{iii}^{2112} - \omega_{iij}^{2112} - \omega_{iji}^{2112} + \omega_{ijj}^{2112} \\
& - \omega_{iij}^{2121} + \omega_{iij}^{2121} + \omega_{ijj}^{2121} - \omega_{ijj}^{2121} \\
& - \omega_{jii}^{1212} + \omega_{jii}^{1212} + \omega_{jji}^{1212} - \omega_{jji}^{1212} \\
& + \omega_{jij}^{1221} - \omega_{jij}^{1221} - \omega_{jjj}^{1221} + \omega_{jjj}^{1221} \\
= & \omega_{iii}^{2112} + \omega_{jjj}^{1221} \\
& + \omega_{iij}^{2112} + \omega_{iij}^{2112} - \omega_{iij}^{2112} - \omega_{iij}^{2112} \\
& + \omega_{jji}^{1212} + \omega_{jji}^{1212}.
\end{aligned} \tag{6.45}$$

Equality in 6.45 comes from comparing the calculated functions ω_{ijkl}^{abcd} , listed in appendix H, and applying the symmetries listed in section 6.1. Kernel K_2 can be obtained by simply interchanging index i with index j . By taking the ordered limit $q_2 \ll q_1$, we find

$$\lim_{q_2 \ll q_1} K_1 = 4\omega_{1i}^{(2)}\omega_{ij}^{(1)}. \tag{6.46}$$

The limits of the double emission kernels K_1 and K_2 therefore recover, up to the number 4, the expected result for the evolution from one colour line to three colour lines by iterating single soft-gluons.

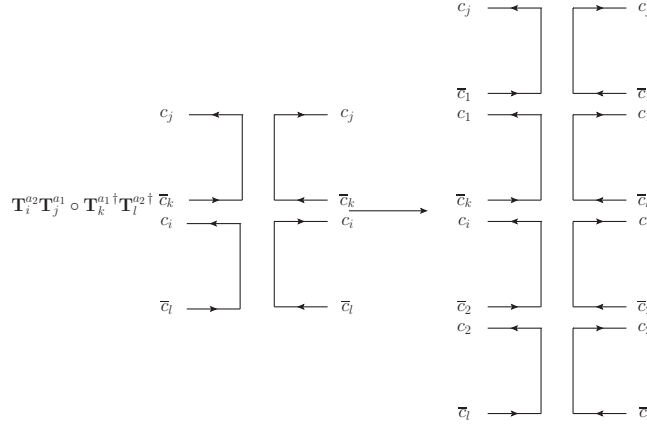


Figure 23: Leading colour transitions, starting from two colour lines, the resulting transition kernel is denoted by K_3 . The indices i, j, k, l are pairwise different or, $i = k$ and $j \neq l$, $j = l$ and $i \neq k$.

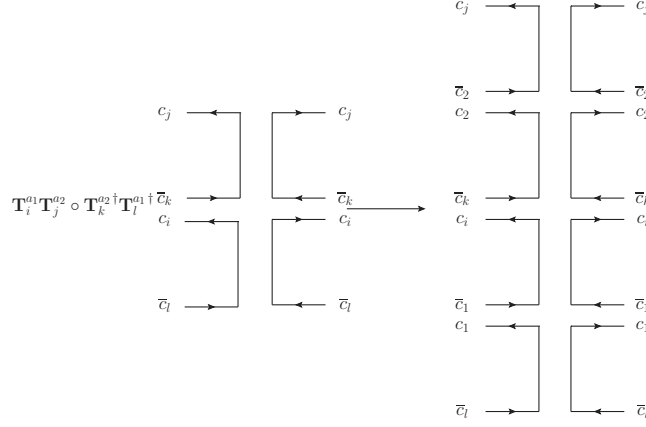


Figure 24: Leading colour transitions, starting from two colour lines, the resulting transition kernel is denoted by K_4 . The indices i, j, k, l are pairwise different or, $i = k$ and $j \neq l$, $j = l$ and $i \neq k$.

Transition kernel of the leading colour process number 3, depicted in figure 23, for pairwise different indices i, j, k, l :

$$K_3 = \sum_{i,j,k,l=1}^{n-2} \omega_{ijkl}^{a_2 a_1 a_1 a_2} + \sum_{i,j,l=1}^{n-2} \omega_{ijil}^{a_2 a_1 a_1 a_2} + \sum_{i,j,k=1}^{n-2} \omega_{ijkj}^{a_2 a_1 a_1 a_2}. \quad (6.47)$$

Similar for the transition K_4 , depicted in figure 24, then only the colour indices a_1 and a_2 have to be swapped.

Applying the double soft-gluon current, while recovering the ordered limit and using colour conservation, gives the following result, which is needed for the subtraction in the renormalization procedure described in 3.34.

$$\begin{aligned} J_{\mu,\sigma}^{a_2 a_1}(q_2, q_1) \circ J^{\mu,\sigma,a_2 a_1 \dagger}(q_2, q_1) &= \sum_{\substack{i,l=1 \\ i \neq l}}^{n-2} \sum_{\substack{j,k=1 \\ j \neq k}}^{n-2} \mathbf{T}_i^{a_2} \mathbf{T}_j^{a_1} \circ \mathbf{T}_k^{a_1 \dagger} \mathbf{T}_l^{a_2 \dagger} \omega_{il}^{(2)} \omega_{jk}^{(1)} \\ &- \sum_{i=1}^{n-2} \sum_{\substack{j,k=1 \\ j \neq k}}^{n-2} \mathbf{T}_i^{a_2} \mathbf{T}_j^{a_1} \circ \mathbf{T}_k^g \mathbf{T}_g^{a_1 a_2} \omega_{jk}^{(1)} \omega_{i1}^{(2)} \\ &- \sum_{i=1}^{n-2} \sum_{\substack{k,l=1 \\ i \neq k}}^{n-2} \mathbf{T}_g^{a_2 a_1} \mathbf{T}_i^g \circ \mathbf{T}_k^{a_1} \mathbf{T}_l^{a_2} \omega_{ik}^{(1)} \omega_{1l}^{(2)}. \end{aligned} \quad (6.48)$$

It is important to note that 6.48 includes contributions, which are not leading colour, for example the following term is included

$$\mathbf{T}_i^{a_2} \mathbf{T}_j^{a_1} \circ \mathbf{T}_i^{a_1 \dagger} \mathbf{T}_j^{a_2 \dagger} \omega_{ij}^{(2)} \omega_{ji}^{(1)}, \quad (6.49)$$

which starts at one colour line, and ends at three colour lines but is not leading colour. To see this, compare with table 4.

Now we have the necessary ingredients to write down the renormalized emission operator $\mathbf{R}_{S,n}^{(2,0)} \circ \mathbf{R}_{S,n}^{(2,0)\dagger}$ 3.34 for the respective transitions at leading colour order.

For transition 1 we find

$$\begin{aligned} \mathbf{R}_{S,n}^{(2,0)} \circ \mathbf{R}_{S,n}^{(2,0)\dagger} &= \mathbf{E}_n^{(2,0)} \circ \mathbf{E}_n^{(2,0)\dagger} \partial_S \Theta_{n,2} - \mathbf{E}_n^{(1,0)} \mathbf{E}_{n-1}^{(1,0)} \circ \mathbf{E}_{n-1}^{(1,0)\dagger} \mathbf{E}_n^{(1,0)\dagger} \Theta_{n,1} \partial_S \Theta_{n-1,1} \\ &= \left(\omega_{iiii}^{2112} + \omega_{jjjj}^{1221} + \omega_{iijj}^{2112} + \omega_{ijjj}^{2112} - \omega_{iiji}^{2112} - \omega_{ijii}^{2112} + \omega_{jjii}^{1212} + \omega_{iijj}^{2121} \right) \Theta_{n,2} \\ &\quad - 4\omega_{1i}^{(2)} \omega_{ji}^{(2)} \Theta_{n,1} \partial_S \Theta_{n-1,1}, \end{aligned} \quad (6.50)$$

where for the subtraction the functions ω_{ijkl}^{abcd} for the ordered limit, listed in appendix I, were used for the terms proportional to the resolution function $\Theta_{n,1} \partial_S \Theta_{n-1,1}$.

With 6.46 and the assumption, that the resolution functions are bounded in the limit we find for transition 1,

$$\lim_{q_2 \ll q_1} \mathbf{R}_{S,n}^{(2,0)} \circ \mathbf{R}_{S,n}^{(2,0)\dagger} = 4\omega_{1i}^{(2)} \omega_{ji}^{(1)} (\Theta_{n,2} - \Theta_{n,1} \partial_S \Theta_{n-1,1}). \quad (6.51)$$

The same calculation can be done for transition 2. The result 6.51 then gives, how to choose the resolution function $\Theta_{n,2}$ consistently, since this limit of the two emission operator must vanish.

Now the same procedure for transition 3. The renormalized emission operator $\mathbf{R}_{S,n}^{(2,0)} \circ \mathbf{R}_{S,n}^{(2,0)\dagger}$ reads in that case at leading colour:

$$\begin{aligned} \mathbf{R}_{S,n}^{(2,0)} \circ \mathbf{R}_{S,n}^{(2,0)\dagger} &= \mathbf{E}_n^{(2,0)} \circ \mathbf{E}_n^{(2,0)\dagger} \partial_S \Theta_{n,2} \\ &= \mathbf{E}_n^{(2,0)} \circ \mathbf{E}_n^{(2,0)\dagger} \partial_S \Theta_{n,2} - \mathbf{E}_n^{(1,0)} \mathbf{E}_{n-1}^{(1,0)} \circ \mathbf{E}_{n-1}^{(1,0)\dagger} \mathbf{E}_n^{(1,0)\dagger} \Theta_{n,1} \partial_S \Theta_{n-1,1} \\ &= \left(\sum_{i,j,k,l} \omega_{ijkl}^{a_2 a_1 a_1 a_2} + \sum_{i,j,i,l} \omega_{ijil}^{a_2 a_1 a_1 a_2} + \sum_{i,j,k} \omega_{ijkj}^{a_2 a_1 a_1 a_2} \right) \Theta_{n,2} - \\ &\quad \left(\sum_{i,j,k,l} \omega_{il}^{(2)} \omega_{jk}^{(1)} + \sum_{i,j,l} \omega_{il}^{(2)} \omega_{ji}^{(1)} + \sum_{i,j,k} \omega_{ij}^{(2)} \omega_{ji}^{(1)} \right) \Theta_{n,1} \partial_S \Theta_{n-1,1}. \\ &= \left(\sum_{i,j,k,l} \omega_{il}^{(2)} \omega_{jk}^{(1)} + \sum_{i,j,l} \omega_{il}^{(2)} \omega_{ji}^{(1)} + \sum_{i,j,k} \omega_{ij}^{(2)} \omega_{ji}^{(1)} \right) (\Theta_{n,2} - \Theta_{n,1} \partial_S \Theta_{n-1,1}). \end{aligned} \quad (6.52)$$

Similar for transition 4, with the difference that a_1 and a_2 get swapped. The result in 6.52 is reassuring in the sense that it shows that the iterated single emission is included in the double emission current, which should be the case since the double emission case is a generalization of the single emission case and should therefore recover the result correctly.

In figure 25 we sketched out the algorithm needed to sample the distribution which is needed, when double emission gets included. The competing rates resemble the two possible transitions, single and double emission. In the case of double emission both soft-gluon get emitted at a comparable $E_{g_1} \approx E_{g_2}$ energy scale, which gets determined by the Sudakov algorithm, to get the energy for the second gluon the ratio $x = E_{g_1}/E_{g_2}$ must be sampled from a chosen interval. With Ω_1 and Ω_2 the directions of the emitted gluons are denoted.

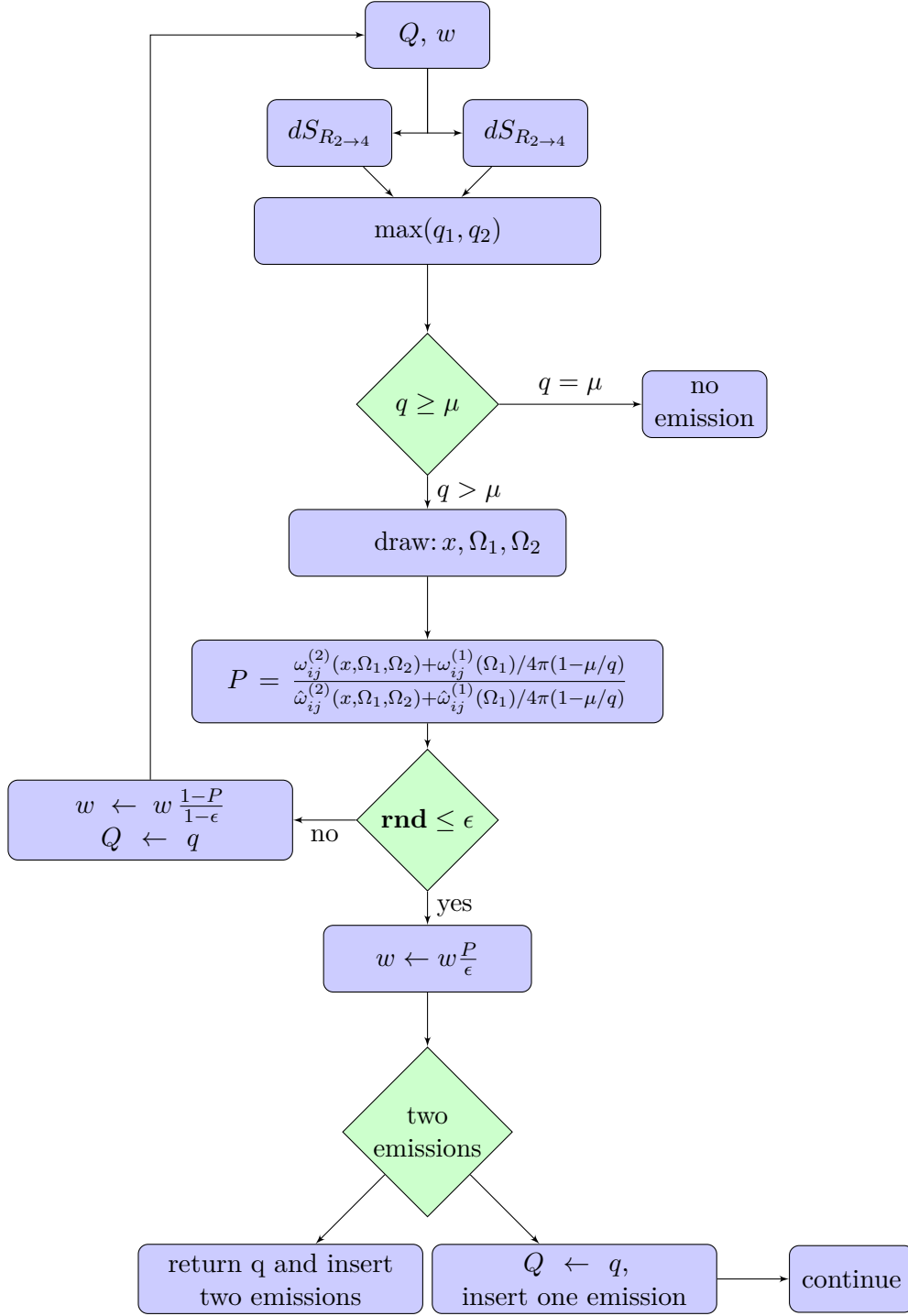


Figure 25: Algorithm for double emission.

7 Conclusion and outlook

In summary, the main objective of this work was to determine the necessary ingredients to consistently include double soft-gluons within parton shower algorithms. To achieve that, the soft-gluon evolution equation at amplitude level, beyond leading order was renormalized and discussed. After that, the colour-flow basis was chosen, with the colour-flow basis we were then able to identify the leading and sub-leading colour contributions for the double soft-gluon emission process, while also bringing the possible evolutions into context with the ring and string basis. In the leading colour contribution we identified four distinct transitions, for which we wrote down the respective transition kernels and analysed the behavior, when recovering the strongly ordered limit. Taking the limit showed that the independent emissions are correctly reproduced and gives necessary constraints for the resolution function. The result for the resolution function also shows that the additional collinear divergencies lead to no further problems. For the sub-leading colour contribution we also presented the necessary ingredients for the colour MC simulation. Furthermore, we presented an algorithm for a parton shower, which includes double emission. Also some numerical aspects were discussed, like sampling the dynamical variables of the energy ordered parton shower from densities, generated by different collinear cutoff descriptions with the Sudakov and weighted Sudakov algorithm. In order to improve the convergence, resampling methods were used and refined such that the resampling strategy could also be applied to negative weights. A generalized version of the weighted Sudakov algorithm was presented, this algorithm is able to generate densities, generated from matrix valued splitting rates.

In future efforts, the presented ingredients for double soft-gluon emission will be combined for an improved parton shower. Other investigations concerning the generalized weighted Sudakov algorithm and the shortly discussed resampling algorithm with complex valued weights will be done. Other ideas for the future would be to use modern machine learning algorithms for faster sampling.

8 Appendix

A Lorentz transformation in d dimensions

This chapter aims to describe how to find Lorentz transformations $\Lambda = \Lambda(Q, \hat{Q})$ in $d \geq 4$ dimensions such that

$$Q = \Lambda \hat{Q}, \quad (\text{A.1})$$

where Q and $\hat{Q}$ are vectors in d dimensional Lorentz space-time. In this problem, only the vectors Q , $\hat{Q}$ and the metric g are available to carry the necessary Lorentz indices, hence the Lorentz transformation $\Lambda^\mu{}_\nu$ has to be of the following form

$$\Lambda^\mu{}_\nu(Q, \hat{Q}) = Ag^\mu{}_\nu + B\hat{Q}^\mu Q_\nu + CQ^\mu \hat{Q}_\nu + D\hat{Q}^\mu \hat{Q}_\nu + EQ^\mu Q_\nu, \quad (\text{A.2})$$

where $A, B, C, D, E \in \mathbb{R}$. In practice, only the proper orthochronous Lorentz transformations are of interest, those transformations form a subgroup of the Lorentz group $\mathcal{L}$, denoted by $\mathcal{L}_+^\uparrow$. The subgroup is defined through the conditions $\det(\Lambda) = 1$ and $\Lambda^0{}_0 \geq 1$. From these conditions it is clear that the set of proper orthochronous Lorentz transformations is smoothly connected to the identity and therefore there exists a matrix L for every $A \in \mathcal{L}_+^\uparrow$ such that

$$A = e^L. \quad (\text{A.3})$$

Expanding A.3 up to linear order and requiring that up to this order the requirement that $A^T \eta A = \eta$ is satisfied, leads to

$$L_{\nu\mu} + L_{\mu\nu} = 0, \quad (\text{A.4})$$

hence $L_{\mu\nu}$ is antisymmetric and has therefore $\frac{d(d-1)}{2}$ free parameters. For $d = 4$ this gives, as expected, six parameters, three for boost and three for rotation. Comparing the ansatz A.2 with A.3 and the requirement that L is antisymmetric A.4, leads to $A = 1$. Therefore, to solve the problem, the ansatz A.2 with $A = 1$ could be used, while requiring that A.1 and

$$\Lambda^T g \Lambda = g \quad (\text{A.5})$$

is satisfied. Solving this system of equations, with the on shell conditions that $Q^2 = \hat{Q}^2 = m^2 \neq 0$ and $Q \cdot \hat{Q} \neq 0$ gives the following two solutions

$$\Lambda^\mu{}_\nu(Q, \hat{Q}) = g^\mu{}_\nu + \frac{1}{m^2 - Q \cdot \hat{Q}} \left(\hat{Q}^\mu Q_\nu + Q^\mu \hat{Q}_\nu - \hat{Q}^\mu \hat{Q}_\nu - Q^\mu Q_\nu \right) \quad (\text{A.6})$$

and

$$\begin{aligned} \Lambda^\mu{}_\nu(Q, \hat{Q}) = g^\mu{}_\nu - \frac{1}{m^2 + Q \cdot \hat{Q}} \hat{Q}^\mu Q_\nu + \frac{m^2 + 2Q \cdot \hat{Q}}{m^2 (m^2 + Q \cdot \hat{Q})} Q^\mu \hat{Q}_\nu \\ - \frac{1}{m^2 + Q \cdot \hat{Q}} \hat{Q}^\mu \hat{Q}_\nu - \frac{1}{m^2 + Q \cdot \hat{Q}} Q^\mu Q_\nu. \end{aligned} \quad (\text{A.7})$$

In the case of $Q^2 = \hat{Q}^2 = 0$ and $Q \cdot \hat{Q} \neq 0$ there exists only one solution, that is given by

$$\Lambda^\mu{}_\nu(Q, \hat{Q}) = g^\mu{}_\nu + \frac{1}{Q \cdot \hat{Q}} (Q^\mu Q_\nu - \hat{Q}^\mu Q_\nu - Q^\mu \hat{Q}_\nu + \hat{Q}^\mu \hat{Q}_\nu). \quad (\text{A.8})$$

In the last case, where $Q^2 = \hat{Q}^2 = m^2 \neq 0$ but $Q \cdot \hat{Q} = 0$ there are again two solutions

$$\Lambda^\mu{}_\nu(Q, \hat{Q}) = g^\mu{}_\nu - \frac{1}{m^2} (Q^\mu Q_\nu \pm \hat{Q}^\mu Q_\nu - Q^\mu \hat{Q}_\nu + \hat{Q}^\mu \hat{Q}_\nu). \quad (\text{A.9})$$

For the implementation of the simulation, the case of interest is, where $Q^2 = \hat{Q}^2 = m^2 \neq 0$ and $Q \cdot \hat{Q} \neq 0$. Specialising to $d = 4$ and the following four vectors

$$\begin{aligned} \hat{Q}^T &= (m, 0, 0, 0) \\ Q^T &= (\sqrt{m^2 + p_z^2}, 0, 0, p_z) \end{aligned} \quad (\text{A.10})$$

gives the following

$$\begin{pmatrix} \frac{\sqrt{m^2 + p_z^2}}{m} & 0 & 0 & \frac{p_z}{m} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \pm \frac{p_z}{m} & 0 & 0 & \frac{m^2 \pm \sqrt{m^2 + p_z^2} + p_z^2}{m^2 \pm m\sqrt{m^2 + p_z^2}} \end{pmatrix}. \quad (\text{A.11})$$

Note that it is always possible to supplement the boost $\Lambda(Q, \hat{Q})$ with a pure rotation $\mathcal{R} \in \mathcal{L}_+^\uparrow$, such that

$$\begin{pmatrix} \sqrt{m^2 + p_z^2} \\ p'_x \\ p'_y \\ p'_z \end{pmatrix} = \mathcal{R} \Lambda(Q, \hat{Q}) \hat{Q}, \quad (\text{A.12})$$

where $\vec{p}' = R p_z \vec{e}_z$ and R the spatial part of $\mathcal{R}$.

B Integral Identity

The proof of correctness of the Sudakov veto algorithm in chapter 5 requires the following corollary.

Corollary 1. Let $f(x)$ be a real valued continuous function on the interval $[q, b]$, then for all $Q \in [q, b]$ and for integers $n \geq 0$ the following holds

$$\frac{1}{n!} \int_q^Q \left(\int_q^k f(k') \, dk' \right)^n f(k) \, dk = \frac{1}{(n+1)!} \left(\int_q^Q f(k) \, dk \right)^{n+1}. \quad (\text{B.1})$$

For the proof, let's define the function $g(Q) = \left(\int_q^Q f(k) \, dk \right)^{n+1}$ on the interval $[q, b]$. Since $f(x)$ is a real valued continuous function on $[q, b]$, then also $g(Q)$ is a real valued continuous function on $[q, b]$, therefore, the fundamental theorem of calculus could be applied to $g(Q)$, which states that $g(Q) = g(q) + \int_q^Q \frac{d}{dt} g(t) \, dt$, note that $g(q) = 0$. In total,

$$\begin{aligned} \frac{1}{(n+1)!} \left(\int_q^Q f(k) \, dk \right)^{n+1} &= \frac{1}{(n+1)!} g(Q) = \frac{1}{(n+1)!} \left(g(q) + \int_q^Q \frac{d}{dk} g(k) \, dk \right) \\ &= \frac{1}{(n+1)!} \int_q^Q \frac{d}{dk} g(k) \, dk = \frac{1}{n!} \int_q^Q \left(\int_q^k f(k') \, dk' \right)^n f(k) \, dk. \end{aligned} \quad (\text{B.2})$$

C Proof of Fierz identity

In this appendix the so called Fierz identity will be proven

$$t^a_{ij} t^a_{kl} = \frac{1}{2} \left(\delta_{il} \delta_{jk} - \frac{1}{N} \delta_{ij} \delta_{kl} \right). \quad (\text{C.1})$$

For the proof, an arbitrary element of $su(N)$ will be decomposed in the basis $\{\mathbf{1}, t^a\}$

$$M = m_0 \mathbf{1} + m_a t^a, \quad (\text{C.2})$$

where on the right side of the equation the sum goes over a . Since the matrices $\{t^a\}$ are traceless, one obtains the following from tracing the relation for the coefficient m_0

$$m_0 = \frac{\text{Tr}(M)}{N}. \quad (\text{C.3})$$

Multiplying with t^a and then tracing gives with the commutator relation

$$m^a = 2 \text{Tr}(M t^a). \quad (\text{C.4})$$

Plugging this relation in for the coefficients gives

$$M = \frac{1}{N} \text{Tr}(M) \mathbf{1} + 2 \text{Tr}(M t^a) t^a. \quad (\text{C.5})$$

Component-wise this relation reads

$$M_{ij} = M_{kl} \left(\frac{1}{N} \delta_{ij} \delta_{kl} + 2 t^a_{lk} t^a_{ij} \right). \quad (\text{C.6})$$

Hence, in total

$$M_{kl} t^a_{ij} t^b_{lk} = M_{kl} \frac{1}{2} \left(\delta_{ik} \delta_{jl} - \frac{1}{N} \delta_{ij} \delta_{kl} \right), \quad (\text{C.7})$$

and since the matrix M was arbitrary, the claimed result follows.

D A contour integral

In this section the following integral with $a, b \in \mathbb{R}$ will be calculated

$$\int_0^{2\pi} \frac{d\varphi}{a + b \cos(\varphi)}. \quad (\text{D.1})$$

The first step is to factor out b and set $c = \frac{a}{b}$, the next one is to rewrite the cosine in terms of the exponential function

$$\begin{aligned} \frac{1}{b} \int_0^{2\pi} \frac{d\varphi}{c + \cos(\varphi)} &= \frac{1}{b} \int_0^{2\pi} \frac{d\varphi}{c + \frac{1}{2}(e^{i\varphi} + e^{-i\varphi})} = \frac{1}{b} \int_0^{2\pi} \frac{2e^{i\varphi} d\varphi}{2ce^{i\varphi} + e^{2i\varphi} + 1} \\ &= \frac{2}{ib} \int_0^{2\pi} \frac{ie^{i\varphi} d\varphi}{2ce^{i\varphi} + e^{2i\varphi} + 1}. \end{aligned} \quad (\text{D.2})$$

Then the above expression can be brought into a form such that the Cauchy's integral formula can be applied

$$\frac{2}{ib} \int_{\gamma} \frac{dz}{z^2 + 2cz + 1}, \quad (\text{D.3})$$

where γ is the oriented curve anticlockwise along the unit circle at the origin $z = 0$. The roots of the complex valued polynomial $z^2 + 2cz + 1$ are

$$\begin{aligned} z_1 &= -c + \sqrt{c^2 - 1} \\ z_2 &= -c - \sqrt{c^2 - 1}, \end{aligned} \quad (\text{D.4})$$

where one may notice that $z_1 \cdot z_2 = 1 \quad \forall c \in \mathbb{R}$. Now there are different cases, which should be considered, the first case is where $c > 1$, then

$$z_2 = -c - \sqrt{c^2 - 1} < -c < -1, \quad (\text{D.5})$$

so z_2 is outside of the unit circle, hence z_1 inside. By Cauchy's integral formula, one obtains

$$\int_{\gamma} \frac{\frac{2}{ib} \frac{1}{z - z_2}}{z - z_1} dz = 2\pi i \frac{2}{ib} \frac{1}{z_1 - z_2} = \frac{2\pi}{b} \frac{1}{\sqrt{c^2 - 1}}. \quad (\text{D.6})$$

So for $\frac{a}{b} > 1$, one finds the following result

$$\int_0^{2\pi} \frac{d\varphi}{a + b \cos(\varphi)} = \frac{2\pi}{b} \frac{1}{\sqrt{\left(\frac{a}{b}\right)^2 - 1}}. \quad (\text{D.7})$$

Now the case where $c < -1$, then

$$z_1 = -c + \sqrt{c^2 - 1} > 1. \quad (\text{D.8})$$

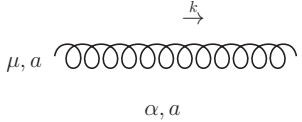
So z_1 is outside of the unit circle, and therefore z_2 is inside. In that case, the integral becomes for $\frac{a}{b} < -1$,

$$\int_{\gamma} \frac{\frac{2}{ib} \frac{1}{z-z_1}}{z-z_2} dz = 2\pi i \frac{2}{ib} \frac{1}{z_2 - z_1} = -\frac{2\pi}{b} \frac{1}{\sqrt{c^2 - 1}}. \quad (\text{D.9})$$

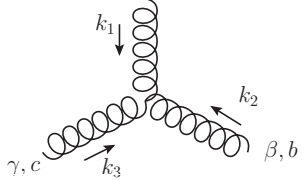
The last case is where $|c| < 1$, then the roots have a non-zero imaginary part and z_1 is the complex conjugate of z_2 , or vice versa. Since $z_1 \cdot z_2 = 1$, the roots are on the unit circle and therefore, the previous method cannot be applied. Furthermore, if $|c| < 1$ the integral does not exist.

E QCD-Feynman rules

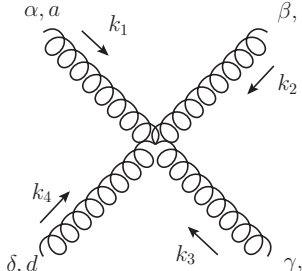
The QCD-Feynman rules [8], in a covariant gauge, are given in this appendix. The momenta are always chosen such that they are ingoing, due to momentum conservation, they sum up to zero. The constant g denotes the coupling constant.



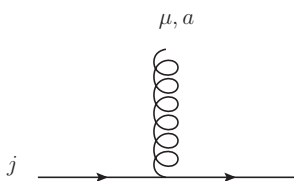
$$= \frac{i(-g^{\mu\nu} + (1-\xi)\frac{k^\mu k^\nu}{k^2})}{k^2 + i0} \delta^{ab}$$



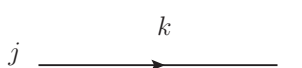
$$= -gf^{abc} [g^{\alpha\beta}(k_1 - k_2) + g^{\beta\gamma}(k_2 - k_3) + g^{\gamma\alpha}(k_3 - k_1)]$$



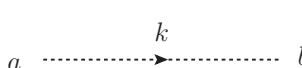
$$= ig^2 [f^{abc}f^{cde}(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\delta}g^{\beta\gamma}) + f^{ace}f^{dbe}(g^{\alpha\delta}g^{\beta\gamma} - g^{\alpha\beta}g^{\gamma\delta}) + f^{ade}f^{bce}(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta})]$$



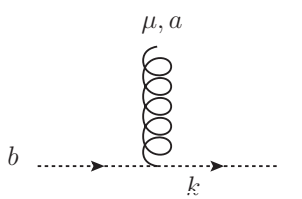
$$= ig\gamma^\mu (t^a)^i_j$$



$$= \frac{i(\gamma^\nu k_\nu + m)}{k^2 - m^2 + i0} \delta^i_j$$



$$= \frac{i}{k^2 + i0} \delta^{ab}$$



$$= gf^{abc}k^\mu$$

Figure 26: QCD-Feynman rules.

F Colour-flow Feynman rules

The QCD colour-flow Feynman rules are listed in this section. As mentioned in section 4.2, in order to obtain the colour-flow Feynman rules it is convenient to start from the usual Feynman rules, and reformulate the colour factors in terms of Kronecker deltas. This can be done with the replacement rule [27],

$$A^{a_1 \dots a_n} \rightarrow (A^{a_1, \dots, a_n})^{i_1, \dots, i_n}_{j_1, \dots, j_n} = \frac{1}{T_R^{n/2}} A^{a_1, \dots, a_n} (t^{a_1})^{i_1}_{j_1} \dots (t^{a_n})^{i_n}_{j_n}, \quad (\text{F.1})$$

the Fierz identity 2.8 and $\text{Tr}[t^a t^b] = T_R \delta^{ab}$.

The figure displays five Feynman diagrams for gluon interactions, each with its corresponding colour factor expressed in terms of Kronecker deltas and the number of colours N .

- Propagator:** A horizontal gluon line with momentum k and indices i_1, j_1 on the left and i_2, j_2 on the right. The colour factor is $\propto \delta^{i_1}_{j_2} \delta^{i_2}_{j_1} - \frac{1}{N} \delta^{i_1}_{j_1} \delta^{i_2}_{j_2}$.
- Three-gluon vertex:** Three gluon lines meeting at a central vertex. The top line has momentum k_1 and indices j_1, i_1 . The bottom-left line has momentum k_3 and indices i_3, j_3 . The bottom-right line has momentum k_2 and indices j_2, i_2 . The colour factor is $\propto \delta^{i_3}_{j_1} \delta^{i_1}_{j_2} \delta^{i_2}_{j_3} - \delta^{i_3}_{j_2} \delta^{i_1}_{j_3} \delta^{i_2}_{j_1}$.
- Four-gluon vertex:** Four gluon lines meeting at a central vertex. The top-left line has momentum k_1 and indices i_1, j_1 . The top-right line has momentum k_2 and indices j_2, i_2 . The bottom-left line has momentum k_4 and indices i_4, j_4 . The bottom-right line has momentum k_3 and indices j_3, i_3 . The colour factor is $\propto (\delta^{i_4}_{j_1} \delta^{i_1}_{j_2} \delta^{i_2}_{j_3} \delta^{i_3}_{j_4} + \delta^{i_4}_{j_3} \delta^{i_3}_{j_4} \delta^{i_1}_{j_2} \delta^{i_2}_{j_1} + \delta^{i_4}_{j_1} \delta^{i_1}_{j_3} \delta^{i_2}_{j_4} \delta^{i_3}_{j_2} - \delta^{i_4}_{j_2} \delta^{i_1}_{j_4} \delta^{i_2}_{j_3} \delta^{i_3}_{j_1} - \delta^{i_4}_{j_2} \delta^{i_1}_{j_3} \delta^{i_2}_{j_1} \delta^{i_3}_{j_4} - \delta^{i_4}_{j_3} \delta^{i_1}_{j_2} \delta^{i_2}_{j_4} \delta^{i_3}_{j_1})$.
- Ghost-gluon vertex:** A horizontal ghost line with momentum k and indices i_1, j_1 on the left and i_2, j_2 on the right, with a vertical gluon line attached to it. The colour factor is $\propto \delta^{i_2}_{j_1} \delta^{j_1}_{i_1}$.
- Quark-gluon vertex:** A horizontal quark line with momentum k and indices i_1, j_1 on the left and i_2, j_2 on the right, with a vertical gluon line attached to it. The colour factor is $\propto \delta^{i_2}_{j_1} \delta^{j_1}_{i_1}$.

Figure 27: Colour QCD-Feynman rules.

The three gluon vertex is proportional to the sum of products of Kronecker deltas, where the sum goes over the two non-cyclic maps (permutations) of the set $\{i_1, i_2, i_3\}$ to $\{j_1, j_2, j_3\}$, the first non-cyclic permutation is $(2, 3, 1)$ and the second non-cyclic permutation of S_3 is $(3, 2, 1)$. Similar for the four gluon vertex, where the sum goes over the six non-cyclic maps (permutations) of the set $\{i_1, i_2, i_3, i_4\}$ to $\{j_1, j_2, j_3, j_4\}$, the six non-cyclic permutations are $(2, 3, 4, 1), (4, 1, 2, 3), (3, 4, 2, 1), (4, 3, 1, 2), (3, 1, 4, 2)$ and $(2, 4, 1, 3)$. For more details see [25]. A more graphical derivation of the colour-flow Feynman rules can be found in the appendix of [40].

G Lexical ordering of permutations

The following section aims to show how to lexical order the elements of the permutation group S_n for a fixed n [41] and how to algorithmically find the position of a given permutation in the lexicon of permutations. Furthermore, an algorithm to generate all permutations in lexicographical order [41] is presented.

Notice that a permutation $\sigma \in S_n$ is fully specified by the sequence $(\sigma_1, \sigma_2, \dots, \sigma_n)$ of its images of $\{1, 2, \dots\}$, where of course not all sequences are allowed, since every element of $\{1, \dots, n\}$ appears exactly once. In the lexicon of permutations, the first $(n-1)!$ entries start with 1 and then the next $(n-1)!$ entries start with 2 and so on. So the list has n blocks with $(n-1)!$ entries each. Let's consider the case that a permutation $\sigma \in S_n$ is given and denote its position in the lexical ordered list of S_n with k_0 . The k_0 -th entry of the list can be written as

$$k_0 = q_1(n-1)! + r_1 \quad \text{with} \quad 0 \leq r_1 \leq (n-1)! \quad \text{and} \quad 0 \leq q_1 \leq n-1. \quad (\text{G.1})$$

The euclidean division theorem ensures that for given k_0 and $(n-1)!$ there exists a unique pair (q_1, r_1) such that the above relation holds. Furthermore, one observes that

$$\sigma_1 = q_1 + 1. \quad (\text{G.2})$$

In the block starting with σ_1 , the first $(n-2)!$ entries start with $\min(\{1, \dots, n\} \setminus \{\sigma_1\})$, the next $(n-2)!$ entries start with $\min([n] \setminus \{\sigma_1\}) + 1$ and so on. The remainder can then be decomposed as

$$r_1 = q_2(n-2)! + r_2 \quad \text{with} \quad 0 \leq r_2 \leq (n-2)! \quad \text{and} \quad 0 \leq q_2 \leq n-2, \quad (\text{G.3})$$

again, uniqueness of the pair (q_2, r_2) is ensured by the euclidean theorem, furthermore, $\sigma_2 = q_2 + 1$. The continuation of this pattern gives

$$r_k = q_{k+1}(n-k-1)! + r_{k+1} \quad \text{with} \quad 0 \leq r_{k+1} \leq (n-k-1)! \quad \text{and} \quad 0 \leq q_{k+1} \leq n-k, \quad (\text{G.4})$$

and $\sigma_{k+1} = q_{k+1} + 1$, hence for a given permutation $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ the sequences $(q_1, q_2, \dots, q_n)$ and $(r_0, r_1, \dots, r_{n-1})$ can be recursively calculated, from which the position k_0 follows.

The algorithm to generate the next permutation in lexicographical order, given a permutation, goes as following. Consider an element $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n) \in S_n$, then

- Search for the largest index $i \in \{1, 2, \dots, n\}$ such that $\sigma_i < \sigma_{i+1}$. In case that there is no such index, the considered permutation σ is already the last permutation in the lexicographically ordered list, namely $\sigma = (n, n-1, \dots, 1)$.
- Swap the entries σ_j and σ_i , where j is the largest index greater than i , where $\sigma_i < \sigma_j$ holds.
- Change the sequence from $(\sigma_1, \sigma_2, \dots, \sigma_i, \sigma_{i+1}, \dots, \sigma_{n-1}, \sigma_n)$ to $(\sigma_1, \sigma_2, \dots, \sigma_i, \sigma_{n-1}, \dots, \sigma_{i+1}, \sigma_n)$.

So, starting from the sequence $(1, 2, \dots, n)$ and applying the above algorithm iteratively generates all permutations in lexicographical order of length n .

H Double soft gluon emission basis

The ω functions are derived from calculating

$$J_{\mu\rho}^{a_1 a_2}(q_1, q_2) M_{n-2}[J_{\nu\sigma}^{a_1 a_2}(q_1, q_2) d^{\mu\nu}(q_1)]^\dagger d^{\rho\sigma}(q_2) \Big|_{\text{Feynman gauge}} = J_{\mu\rho}^{a_1 a_2}(q_1, q_2) M_{n-2}[J_{a_1 a_2}^{\mu\rho}(q_1, q_2)]^\dagger, \quad (\text{H.1})$$

applying

$$[\mathbf{T}_k^a, \mathbf{T}_k^b] = i f^{abc} \mathbf{T}_k^c \equiv \mathbf{T}_g^{(ab)} \mathbf{T}_k^{(g)}, \quad (\text{H.2})$$

and comparing with 6.13. The appearing lower indices are always pairwise different in the following list. The following four weights descend from the iterated part of the double soft-gluon emission.

$$\omega_{ijkl}^{a_2 a_1 a_1 a_2} = \omega_{il}^{(2)} \omega_{jk}^{(1)} \quad (\text{H.3})$$

$$\omega_{ijil}^{a_2 a_1 a_1 a_2} = \omega_{il}^{(2)} \omega_{ji}^{(1)} \quad (\text{H.4})$$

$$\omega_{ijkj}^{a_2 a_1 a_1 a_2} = \omega_{ij}^{(2)} \omega_{jk}^{(1)} \quad (\text{H.5})$$

$$\omega_{ijij}^{a_2 a_1 a_1 a_2} = \omega_{ij}^{(2)} \omega_{ji}^{(1)} \quad (\text{H.6})$$

The next ω 's descend from the case, where on the amplitude side two gluons get emitted from different legs and on the conjugated amplitude they get emitted from the same leg.

$$\omega_{ijkk}^{a_2 a_1 a_1 a_2} = \omega_{ik}^{(2)} \omega_{jk}^{(1,1+2)} - \omega_{i1}^{(2)} \omega_{jk}^{(1,1+2)} + \omega_{ik}^{(2,1+2)} \omega_{jk}^{(1)} + \omega_{ik}^{(2,1+2)} \omega_{j2}^{(1)} + \omega_{ij}^{(2,1)} \frac{1}{2} (\omega_{k1}^{(1+2),2} - \omega_{k2}^{(1+2,1)}) \quad (\text{H.7})$$

$$\omega_{ijkk}^{a_2 a_1 a_2 a_1} = \omega_{i1}^{(2)} \omega_{jk}^{(1,1+2)} - \omega_{ik}^{(2,1+2)} \omega_{j2}^{(1)} - \omega_{ij}^{(2,1)} \frac{1}{2} (\omega_{k1}^{(1+2),2} - \omega_{k2}^{(1+2,1)}) \quad (\text{H.8})$$

$$\omega_{ijii}^{a_2 a_1 a_1 a_2} = -\omega_{i1}^{(2)} \omega_{ji}^{(1,1+2)} + \omega_{ij}^{(2,1)} \frac{1}{2} (\omega_{i1}^{(1+2),2} - \omega_{i2}^{(1+2,1)}) \quad (\text{H.9})$$

$$\omega_{ijii}^{a_2 a_1 a_2 a_1} = \omega_{i1}^{(2)} \omega_{ji}^{(1,1+2)} - \omega_{ij}^{(2,1)} \frac{1}{2} (\omega_{i1}^{(1+2),2} - \omega_{i2}^{(1+2,1)}) \quad (\text{H.10})$$

$$\omega_{ijjj}^{a_2 a_1 a_2 a_1} = -\omega_{ij}^{(2,1+2)} \omega_{j2}^{(1)} - \omega_{ij}^{(2,1)} \frac{1}{2} (\omega_{j1}^{(1+2),2} - \omega_{j2}^{(1+2,1)}) \quad (\text{H.11})$$

$$\omega_{ijjj}^{a_2 a_1 a_1 a_2} = \omega_{ij}^{(2,1+2)} \omega_{j2}^{(1)} + \omega_{ij}^{(2,1)} \frac{1}{2} (\omega_{j1}^{(1+2),2} - \omega_{j2}^{(1+2,1)}) \quad (\text{H.12})$$

Now the contributions, which descend from the diagrams where on the amplitude side two gluons get emitted from the same leg and on the conjugated amplitude from different legs.

$$\omega_{iikl}^{a_2 a_1 a_1 a_2} = \omega_{il}^{(2)} \omega_{ki}^{(1,1+2)} - \omega_{1l}^{(2)} \omega_{ki}^{(1,1+2)} + \omega_{ik}^{(1)} \omega_{il}^{(2,1+2)} - \omega_{2k}^{(1)} \omega_{il}^{(1+2,2)} + \omega_{lk}^{(2,1)} \frac{1}{2} (\omega_{k1}^{(1+2,2)} - \omega_{k2}^{(1+2,1)}) \quad (\text{H.13})$$

$$\omega_{iikl}^{a_1 a_2 a_1 a_2} = \omega_{1l}^{(2)} \omega_{ki}^{(1,1+2)} - \omega_{2k}^{(1)} \omega_{il}^{(1+2,2)} - \omega_{lk}^{(1,2)} \frac{1}{2} (\omega_{k1}^{(1+2,2)} - \omega_{k2}^{(1+2,1)}) \quad (\text{H.14})$$

$$\omega_{iiki}^{a_2 a_1 a_1 a_2} = -\omega_{1i}^{(2)} \omega_{ki}^{(1,1+2)} + \omega_{ki}^{(1,2)} \frac{1}{2} (\omega_{i1}^{(1+2,2)} - \omega_{i2}^{(1+2,1)}) \quad (\text{H.15})$$

$$\omega_{iiki}^{a_1 a_2 a_1 a_2} = \omega_{1i}^{(2)} \omega_{ki}^{(1,1+2)} - \omega_{ik}^{(1,2)} \frac{1}{2} (\omega_{i1}^{(1+2,2)} - \omega_{i2}^{(1+2,1)}) \quad (\text{H.16})$$

$$\omega_{iiil}^{a_1 a_2 a_1 a_2} = -\omega_{2i}^{(1)} \omega_{il}^{(1+2,2)} - \omega_{li}^{(1,2)} \frac{1}{2} (\omega_{i1}^{(1+2,2)} - \omega_{i2}^{(1+2,1)}) \quad (\text{H.17})$$

$$\omega_{iiil}^{a_2 a_1 a_1 a_2} = \omega_{2i}^{(1)} \omega_{il}^{(1+2,2)} + \omega_{li}^{(1,2)} \frac{1}{2} (\omega_{i1}^{(1+2,2)} - \omega_{i2}^{(1+2,1)}) \quad (\text{H.18})$$

The remaining contributions come from the case, where on the amplitude and the conjugated amplitude the gluons get emitted from the same leg.

$$\begin{aligned} \omega_{iijj}^{a_2 a_1 a_1 a_2} &= \omega_{ij}^{(2)} \omega_{ij}^{(1+2)} - \omega_{i1}^{(2)} \omega_{ij}^{(1+2)} - \omega_{j1}^{(2)} \omega_{ij}^{(1+2)} \\ &+ \frac{d}{4} (\omega_{i1}^{(1+2,2)} - \omega_{i2}^{(1+2,1)}) (\omega_{j1}^{(1+2,2)} - \omega_{j2}^{(1+2,1)}) \\ &- \omega_{1i}^{(2,1+2)} \frac{1}{2} (\omega_{j1}^{(1+2,2)} - \omega_{j2}^{(1+2,1)}) \\ &- \omega_{1j}^{(2,1+2)} \frac{1}{2} (\omega_{i1}^{(1+2,2)} - \omega_{i2}^{(1+2,1)}) \\ &+ \omega_{ij}^{(1+2,2)} \omega_{2j}^{(1,1+2)} - \omega_{2j}^{(1,1+2)} \omega_{i1}^{(1+2,2)} + \omega_{ij}^{(2,1+2)} \omega_{i2}^{(1+2,1)} \\ &+ \omega_{2,i}^{(1,1+2)} \frac{1}{2} (\omega_{j,1}^{(1+2,2)} - \omega_{j2}^{(1+2,2)}) \\ &+ \omega_{2j}^{(1,1+2)} \frac{1}{2} (\omega_{i,1}^{(1+2,2)} - \omega_{i2}^{(1+2,2)}) \end{aligned} \quad (\text{H.19})$$

$$\begin{aligned}
\omega_{iijj}^{a_2 a_1 a_2 a_1} &= \omega_{i1}^{(2)} \omega_{ij}^{(1+2)} - \frac{d}{4} (\omega_{i1}^{(1+2,2)} - \omega_{i2}^{(1+2,1)}) (\omega_{j1}^{(1+2,2)} - \omega_{j2}^{(1+2,1)}) \\
&+ \omega_{1i}^{(2,1+2)} \frac{1}{2} (\omega_{j1}^{(1+2,2)} - \omega_{j2}^{(1+2,1)}) \\
&+ \omega_{1j}^{(2,1+2)} \frac{1}{2} (\omega_{i1}^{(1+2,2)} - \omega_{i2}^{(1+2,1)}) \\
&+ \omega_{ij}^{(1+2)} \omega_{j2}^{(1,2)} + \omega_{ij}^{(2,1+2)} \omega_{ij}^{(1+2,1)} \\
&- \omega_{ij}^{(2,1+2)} \omega_{i2}^{(1+2,1)} - \omega_{1j}^{(2,1+2)} \omega_{ij}^{(1+2,1)} \\
&+ \omega_{1j}^{(2,1+2)} \omega_{i2}^{(1+2,1)} + \omega_{2j}^{(1,1+2)} \omega_{i1}^{(1+2,2)} \\
&- \omega_{2,i}^{(1,1+2)} \frac{1}{2} (\omega_{j,1}^{(1+2,2)} - \omega_{j2}^{(1+2,2)}) \\
&- \omega_{2j}^{(1,1+2)} \frac{1}{2} (\omega_{i,1}^{(1+2,2)} - \omega_{i2}^{(1+2,2)})
\end{aligned} \tag{H.20}$$

$$\begin{aligned}
\omega_{iijj}^{a_1 a_2 a_1 a_2} &= \omega_{1j}^{(2)} \omega_{ij}^{(1+2)} - \frac{d}{4} (\omega_{i1}^{(1+2,2)} - \omega_{i2}^{(1+2,1)}) (\omega_{j1}^{(1+2,2)} - \omega_{j2}^{(1+2,1)}) \\
&+ \omega_{1i}^{(2,1+2)} \frac{1}{2} (\omega_{j1}^{(1+2,2)} - \omega_{j2}^{(1+2,1)}) \\
&+ \omega_{1j}^{(2,1+2)} \frac{1}{2} (\omega_{i1}^{(1+2,2)} - \omega_{i2}^{(1+2,1)}) \\
&+ \omega_{ij}^{(1+2)} \omega_{i2}^{(1,2)} + \omega_{1j}^{(2,1+2)} \omega_{i2}^{(1+2,1)} \\
&+ \omega_{ij}^{(1,1+2)} \omega_{ij}^{(1+2,2)} - \omega_{ij}^{(1,1+2)} \omega_{i1}^{(1+2,2)} \\
&- \omega_{ij}^{(1+2,2)} \omega_{2j}^{(1,1+2)} + \omega_{2j}^{(1,1+2)} \omega_{i1}^{(1+2,2)} \\
&- \omega_{2,i}^{(1,1+2)} \frac{1}{2} (\omega_{j,1}^{(1+2,2)} - \omega_{j2}^{(1+2,2)}) \\
&- \omega_{2j}^{(1,1+2)} \frac{1}{2} (\omega_{i,1}^{(1+2,2)} - \omega_{i2}^{(1+2,2)})
\end{aligned} \tag{H.21}$$

$$\begin{aligned}
\omega_{iijj}^{a_1 a_2 a_2 a_1} &= \frac{d}{4} (\omega_{i1}^{(1+2,2)} - \omega_{i2}^{(1+2,1)}) (\omega_{j1}^{(1+2,2)} - \omega_{j2}^{(1+2,1)}) \\
&- \omega_{1i}^{(2,1+2)} \frac{1}{2} (\omega_{j1}^{(1+2,2)} - \omega_{j2}^{(1+2,1)}) \\
&- \omega_{1j}^{(2,1+2)} \frac{1}{2} (\omega_{i1}^{(1+2,2)} - \omega_{i2}^{(1+2,1)}) \\
&+ \omega_{ij}^{(1)} \omega_{ij}^{(1+2)} - \omega_{ij}^{(1+2)} \omega_{j2}^{(1,2)} - \omega_{ij}^{(1+2)} \omega_{i2}^{(1,2)} \\
&+ \omega_{ij}^{(1,1+2)} \omega_{i1}^{(1+2,2)} - \omega_{2j}^{(1,1+2)} \omega_{i1}^{(1+2,2)} + \omega_{1j}^{(2,1+2)} \omega_{ij}^{(1+2,1)} \\
&+ \omega_{2,i}^{(1,1+2)} \frac{1}{2} (\omega_{j,1}^{(1+2,2)} - \omega_{j2}^{(1+2,2)}) \\
&+ \omega_{2j}^{(1,1+2)} \frac{1}{2} (\omega_{i,1}^{(1+2,2)} - \omega_{i2}^{(1+2,2)})
\end{aligned} \tag{H.22}$$

Lastly, the cases, where all lower indices are equal.

$$\begin{aligned}
\omega_{iiii}^{a_2 a_1 a_1 a_2} &= \frac{d}{4} (\omega_{i1}^{(1+2,2)} - \omega_{i2}^{(1+2,1)})^2 \\
&- \omega_{1i}^{(2,1+2)} (\omega_{i1}^{(1+2,2)} - \omega_{i2}^{(1+2,1)}) \\
&- \omega_{2i}^{(1,1+2)} \omega_{i1}^{(1+2,2)} - \omega_{1i}^{(2,1+2)} \omega_{i2}^{(1+2,1)} \\
&+ \omega_{2i}^{(1,1+2)} (\omega_{i1}^{(1+2,2)} - \omega_{i2}^{(1+2,1)})
\end{aligned} \tag{H.23}$$

$$\begin{aligned}
\omega_{iii}^{a_1 a_2 a_2 a_1} &= \frac{d}{4} (\omega_{i1}^{(1+2,2)} - \omega_{i2}^{(1+2,1)})^2 \\
&\quad + \omega_{1i}^{(2,1+2)} (\omega_{i1}^{(1+2,2)} - \omega_{i2}^{(1+2,1)}) \\
&\quad - \omega_{2i}^{1,1+2} \omega_{i1}^{(1+2,2)} - \omega_{1i}^{(2,1+2)} \omega_{i2}^{(1+2,1)} \\
&\quad + \omega_{2i}^{(1,1+2)} (\omega_{i1}^{1+2,2} - \omega_{i2}^{(1+2,1)})
\end{aligned} \tag{H.24}$$

$$\begin{aligned}
\omega_{iii}^{a_1 a_2 a_1 a_2} &= -\frac{d}{4} (\omega_{i1}^{(1+2,2)} - \omega_{i2}^{(1+2,1)})^2 \\
&\quad - \omega_{1i}^{(2,1+2)} (\omega_{i1}^{(1+2,2)} - \omega_{i2}^{(1+2,1)}) \\
&\quad + \omega_{2i}^{1,1+2} \omega_{i1}^{(1+2,2)} + \omega_{1i}^{(2,1+2)} \omega_{i2}^{(1+2,1)} \\
&\quad - \omega_{2i}^{(1,1+2)} (\omega_{i1}^{1+2,2} - \omega_{i2}^{(1+2,1)})
\end{aligned} \tag{H.25}$$

$$\begin{aligned}
\omega_{iii}^{a_2 a_1 a_2 a_1} &= -\frac{d}{4} (\omega_{i1}^{(1+2,2)} - \omega_{i2}^{(1+2,1)})^2 \\
&\quad - \omega_{1i}^{(2,1+2)} (\omega_{i1}^{(1+2,2)} - \omega_{i2}^{(1+2,1)}) \\
&\quad + \omega_{2i}^{1,1+2} \omega_{i1}^{(1+2,2)} + \omega_{1i}^{(2,1+2)} \omega_{i2}^{(1+2,1)} \\
&\quad - \omega_{2i}^{(1,1+2)} (\omega_{i1}^{1+2,2} - \omega_{i2}^{(1+2,1)})
\end{aligned} \tag{H.26}$$

I Ordered limit of the double soft gluon emission basis

In this appendix, the leading contribution of the strongly ordered limit ($q_2 \ll q_1$) of the double soft-gluon emission current is listed for two cases. In the first case, the functions ω 's are derived from the strongly ordered limit of the double soft-gluon, while also performing the sum over all possible colour flows and consequently using colour conservation 3.19 and in the second case only the strongly ordered limit of the current, while not performing the sum. The reason for considering the two cases that is that the procedure of selecting the leading colour contribution and taking the strongly ordered limit does not commute in general. The appearing lower indices are always pairwise different. For the first case the ω 's are,

$$\omega_{ijkl}^{a_2 a_1 a_1 a_2} \xrightarrow{q_2 \ll q_1} \frac{1}{\lambda^6} \omega_{il}^{(2)} \omega_{jk}^{(1)} \quad (\text{I.1})$$

$$\omega_{ijil}^{a_2 a_1 a_1 a_2} \xrightarrow{q_2 \ll q_1} \frac{1}{\lambda^6} \omega_{il}^{(2)} \omega_{ji}^{(1)} \quad (\text{I.2})$$

$$\omega_{ijkj}^{a_2 a_1 a_1 a_2} \xrightarrow{q_2 \ll q_1} \frac{1}{\lambda^6} \omega_{ij}^{(2)} \omega_{jk}^{(1)} \quad (\text{I.3})$$

$$\omega_{ijij}^{a_2 a_1 a_1 a_2} \xrightarrow{q_2 \ll q_1} \frac{1}{\lambda^6} \omega_{ij}^{(2)} \omega_{ji}^{(1)} \quad (\text{I.4})$$

The next ω 's descend from the case, where on the amplitude side two gluons get emitted from different legs and on the conjugated amplitude they get emitted from the same leg.

$$\omega_{ijkk}^{a_2 a_1 a_1 a_2} \xrightarrow{q_2 \ll q_1} \frac{1}{\lambda^6} \omega_{ik}^{(2)} \omega_{jk}^{(1)} - \frac{1}{\lambda^6} \omega_{i1}^{(2)} \omega_{jk}^{(1)} + \mathcal{O}\left(\frac{1}{\lambda^5}\right) \quad (\text{I.5})$$

$$\omega_{ijkk}^{a_2 a_1 a_2 a_1} \xrightarrow{q_2 \ll q_1} \frac{1}{\lambda^6} \omega_{i1}^{(2)} \omega_{jk}^{(1)} + \mathcal{O}\left(\frac{1}{\lambda^5}\right) \quad (\text{I.6})$$

$$\omega_{ijii}^{a_2 a_1 a_1 a_2} \xrightarrow{q_2 \ll q_1} -\frac{1}{\lambda^6} \omega_{i1}^{(2)} \omega_{ji}^{(1)} + \mathcal{O}\left(\frac{1}{\lambda^5}\right) \quad (\text{I.7})$$

$$\omega_{ijii}^{a_2 a_1 a_2 a_1} \xrightarrow{q_2 \ll q_1} \frac{1}{\lambda^6} \omega_{i1}^{(2)} \omega_{ji}^{(1)} + \mathcal{O}\left(\frac{1}{\lambda^5}\right) \quad (\text{I.8})$$

Now the leading order contributions, which descend from the diagrams, where on the amplitude side two gluons get emitted from the same leg and on the conjugated amplitude from different legs.

$$\omega_{iikl}^{a_2 a_1 a_1 a_2} \xrightarrow{q_2 \ll q_1} \frac{1}{\lambda^6} \omega_{il}^{(2)} \omega_{ki}^{(1)} - \frac{1}{\lambda^6} \omega_{1l}^{(2)} \omega_{ki}^{(1)} + \mathcal{O}\left(\frac{1}{\lambda^5}\right) \quad (\text{I.9})$$

$$\omega_{iikl}^{a_1 a_2 a_1 a_2} \xrightarrow{q_2 \ll q_1} \frac{1}{\lambda^6} \omega_{1l}^{(2)} \omega_{ki}^{(1)} + \mathcal{O}\left(\frac{1}{\lambda^5}\right) \quad (\text{I.10})$$

$$\omega_{iiki}^{a_2 a_1 a_1 a_2} \xrightarrow{q_2 \ll q_1} -\frac{1}{\lambda^6} \omega_{1i}^{(2)} \omega_{ki}^{(1)} + \mathcal{O}\left(\frac{1}{\lambda^5}\right) \quad (\text{I.11})$$

$$\omega_{iiki}^{a_1 a_2 a_1 a_2} \xrightarrow{q_2 \ll q_1} \frac{1}{\lambda^6} \omega_{1i}^{(2)} \omega_{ki}^{(1)} + \mathcal{O}\left(\frac{1}{\lambda^5}\right) \quad (\text{I.12})$$

The remaining leading contributions come from the case, where on the amplitude and the conjugated amplitude the gluons get emitted from the same leg.

$$\omega_{iijj}^{a_2 a_1 a_1 a_2} \xrightarrow{q_2 \ll q_1} \frac{1}{\lambda^6} \omega_{ij}^{(2)} \omega_{ij}^{(1)} - \frac{1}{\lambda^6} \omega_{i1}^{(2)} \omega_{ij}^{(1)} - \frac{1}{\lambda^6} \omega_{j1}^{(2)} \omega_{ij}^{(1)} + \mathcal{O}\left(\frac{1}{\lambda^5}\right) \quad (\text{I.13})$$

$$\omega_{iijj}^{a_2 a_1 a_2 a_1} \xrightarrow{q_2 \ll q_1} \frac{1}{\lambda^6} \omega_{i1}^{(2)} \omega_{ij}^{(1)} + \mathcal{O}\left(\frac{1}{\lambda^5}\right) \quad (\text{I.14})$$

$$\omega_{iijj}^{a_1 a_2 a_1 a_2} \xrightarrow{q_2 \ll q_1} \frac{1}{\lambda^6} \omega_{1i}^{(2)} \omega_{ij}^{(1)} + \mathcal{O}\left(\frac{1}{\lambda^5}\right) \quad (\text{I.15})$$

Now the second case, the strongly ordered current then reads

$$\begin{aligned} J_{a_2 a_1}^{\mu\nu}(q_2, q_1) &\xrightarrow{q_2 \ll q_1} \sum_{i,j=1}^{n-2} \mathbf{T}_i^{a_2} \mathbf{T}_j^{a_1} \frac{p_i^\mu}{p_i \cdot q_2} \frac{p_j^\nu}{p_j \cdot q_1} - \sum_{i=1}^{n-2} \mathbf{T}_g^{(a_2 a_1)} \mathbf{T}_i^g \frac{q_1^\mu}{q_1 \cdot q_2} \frac{p_i^\nu}{p_i \cdot q_1} \\ &+ \sum_{i=1}^{n-2} \mathbf{T}_g^{(a_2 a_1)} \mathbf{T}_i^g \frac{g^{\mu\nu}}{2} \frac{1}{q_1 \cdot q_2}. \end{aligned} \quad (\text{I.16})$$

To make the distinction clear between the two sets of functions, they functions are now denoted with $\hat{\omega}$. The first four functions are unchanged in comparison to the previous case

$$\hat{\omega}_{ijkl}^{a_2 a_1 a_1 a_2} \xrightarrow{q_2 \ll q_1} \frac{1}{\lambda^6} \omega_{il}^{(2)} \omega_{jk}^{(1)} \quad (\text{I.17})$$

$$\hat{\omega}_{ijil}^{a_2 a_1 a_1 a_2} \xrightarrow{q_2 \ll q_1} \frac{1}{\lambda^6} \omega_{il}^{(2)} \omega_{ji}^{(1)} \quad (\text{I.18})$$

$$\hat{\omega}_{ijkj}^{a_2 a_1 a_1 a_2} \xrightarrow{q_2 \ll q_1} \frac{1}{\lambda^6} \omega_{ij}^{(2)} \omega_{jk}^{(1)} \quad (\text{I.19})$$

$$\hat{\omega}_{ijij}^{a_2 a_1 a_1 a_2} \xrightarrow{q_2 \ll q_1} \frac{1}{\lambda^6} \omega_{ij}^{(2)} \omega_{ji}^{(1)} \quad (\text{I.20})$$

The next $\hat{\omega}$'s descend again from the case, where on the amplitude side two gluons get emitted from different legs and on the conjugated amplitude they get emitted from the same leg. Now the limits read.

$$\hat{\omega}_{ijkk}^{a_2 a_1 a_1 a_2} \xrightarrow{q_2 \ll q_1} \frac{1}{\lambda^6} \omega_{ik}^{(2)} \omega_{jk}^{(1)} - \frac{1}{\lambda^6} \omega_{i1}^{(2)} \omega_{jk}^{(1)} + \frac{1}{\lambda^6} \omega_{ij}^{(2,1)} \frac{1}{2} \frac{1}{q_1 \cdot q_2} + \mathcal{O}\left(\frac{1}{\lambda^5}\right) \quad (\text{I.21})$$

$$\hat{\omega}_{ijkk}^{a_2 a_1 a_2 a_1} \xrightarrow{q_2 \ll q_1} \frac{1}{\lambda^6} \omega_{i1}^{(2)} \omega_{jk}^{(1)} - \frac{1}{\lambda^6} \omega_{ij}^{(2,1)} \frac{1}{2} \frac{1}{q_1 \cdot q_2} + \mathcal{O}\left(\frac{1}{\lambda^5}\right) \quad (\text{I.22})$$

$$\omega_{ijii}^{a_2 a_1 a_1 a_2} \xrightarrow{q_2 \ll q_1} -\frac{1}{\lambda^6} \omega_{i1}^{(2)} \omega_{ji}^{(1)} + \frac{1}{\lambda^6} \omega_{ij}^{(2,1)} \frac{1}{2} \frac{1}{q_1 \cdot q_2} + \mathcal{O}\left(\frac{1}{\lambda^5}\right) \quad (\text{I.23})$$

$$\hat{\omega}_{ijii}^{a_2 a_1 a_2 a_1} \xrightarrow{q_2 \ll q_1} \frac{1}{\lambda^6} \omega_{i1}^{(2)} \omega_{ji}^{(1)} - \frac{1}{\lambda^6} \omega_{ij}^{(2,1)} \frac{1}{2} \frac{1}{q_1 \cdot q_2} + \mathcal{O}\left(\frac{1}{\lambda^5}\right) \quad (\text{I.24})$$

$$\omega_{ijjj}^{a_2 a_1 a_1 a_2} \xrightarrow{q_2 \ll q_1} \frac{1}{\lambda^6} \omega_{ij}^{(2,1)} \frac{1}{2} \frac{1}{q_1 \cdot q_2} + \mathcal{O}\left(\frac{1}{\lambda^5}\right) \quad (\text{I.25})$$

$$\omega_{ijjj}^{a_2 a_1 a_2 a_1} \xrightarrow{q_2 \ll q_1} -\frac{1}{\lambda^6} \omega_{ij}^{(2,1)} \frac{1}{2} \frac{1}{q_1 \cdot q_2} + \mathcal{O}\left(\frac{1}{\lambda^5}\right) \quad (\text{I.26})$$

Now the leading order contributions, which descend from the diagrams, where on the amplitude side two gluons get emitted from the same leg and on the conjugated amplitude from different legs.

$$\hat{\omega}_{iikl}^{a_2 a_1 a_1 a_2} \xrightarrow{q_2 \ll q_1} \frac{1}{\lambda^6} \omega_{il}^{(2)} \omega_{ki}^{(1)} - \frac{1}{\lambda^6} \omega_{1l}^{(2)} \omega_{ki}^{(1)} + \frac{1}{\lambda^6} \omega_{lk}^{(2,1)} \frac{1}{2} \frac{1}{q_1 \cdot q_2} + \mathcal{O}\left(\frac{1}{\lambda^5}\right) \quad (\text{I.27})$$

$$\hat{\omega}_{iikl}^{a_1 a_2 a_1 a_2} \xrightarrow{q_2 \ll q_1} \frac{1}{\lambda^6} \omega_{1l}^{(2)} \omega_{ki}^{(1)} - \frac{1}{\lambda^6} \omega_{lk}^{(2,1)} \frac{1}{2} \frac{1}{q_1 \cdot q_2} + \mathcal{O}\left(\frac{1}{\lambda^5}\right) \quad (\text{I.28})$$

$$\hat{\omega}_{iiki}^{a_2 a_1 a_1 a_2} \xrightarrow{q_2 \ll q_1} -\frac{1}{\lambda^6} \omega_{1i}^{(2)} \omega_{ki}^{(1)} + \frac{1}{\lambda^6} \omega_{ik}^{(2,1)} \frac{1}{2} \frac{1}{q_1 \cdot q_2} + \mathcal{O}\left(\frac{1}{\lambda^5}\right) \quad (\text{I.29})$$

$$\hat{\omega}_{iiki}^{a_1 a_2 a_1 a_2} \xrightarrow{q_2 \ll q_1} \frac{1}{\lambda^6} \omega_{1i}^{(2)} \omega_{ki}^{(1)} - \frac{1}{\lambda^6} \omega_{ik}^{(2,1)} \frac{1}{2} \frac{1}{q_1 \cdot q_2} + \mathcal{O}\left(\frac{1}{\lambda^5}\right) \quad (\text{I.30})$$

$$\omega_{iiil}^{a_2 a_1 a_1 a_2} \xrightarrow{q_2 \ll q_1} \frac{1}{\lambda^6} \omega_{li}^{(2,1)} \frac{1}{2} \frac{1}{q_1 \cdot q_2} + \mathcal{O}\left(\frac{1}{\lambda^5}\right) \quad (\text{I.31})$$

$$\omega_{iiil}^{a_1 a_2 a_1 a_2} \xrightarrow{q_2 \ll q_1} -\frac{1}{\lambda^6} \omega_{li}^{(2,1)} \frac{1}{2} \frac{1}{q_1 \cdot q_2} + \mathcal{O}\left(\frac{1}{\lambda^5}\right) \quad (\text{I.32})$$

The remaining leading contributions come from the case, where on the amplitude and the conjugated amplitude the gluons get emitted from the same leg.

$$\begin{aligned} \hat{\omega}_{ijjj}^{a_2 a_1 a_1 a_2} \xrightarrow{q_2 \ll q_1} & \frac{1}{\lambda^6} \omega_{ij}^{(2)} \omega_{ij}^{(1)} - \frac{1}{\lambda^6} \omega_{i1}^{(2)} \omega_{ij}^{(1)} - \frac{1}{\lambda^6} \omega_{j1}^{(2)} \omega_{ij}^{(1)} + \frac{1}{\lambda^6} \frac{d}{4} \frac{1}{(q_1 \cdot q_2)^2} \\ & - \frac{1}{\lambda^6} \omega_{1i}^{(2,1)} \frac{1}{2} \frac{1}{q_1 \cdot q_2} - \frac{1}{\lambda^6} \omega_{1j}^{(2,1)} \frac{1}{2} \frac{1}{q_1 \cdot q_2} + \mathcal{O}\left(\frac{1}{\lambda^5}\right) \end{aligned} \quad (\text{I.33})$$

$$\hat{\omega}_{ijj}^{a_2 a_1 a_2 a_1} \xrightarrow{q_2 \ll q_1} \frac{1}{\lambda^6} \omega_{i1}^{(2)} \omega_{ij}^{(1)} - \frac{1}{\lambda^6} \frac{d}{4} \frac{1}{(q_1 \cdot q_2)^2} + \frac{1}{\lambda^6} \omega_{1i}^{(2,1)} \frac{1}{2} \frac{1}{q_1 \cdot q_2} + \frac{1}{\lambda^6} \omega_{1j}^{(2,1)} \frac{1}{2} \frac{1}{q_1 \cdot q_2} + \mathcal{O}\left(\frac{1}{\lambda^5}\right) \quad (\text{I.34})$$

$$\hat{\omega}_{ijj}^{a_1 a_2 a_1 a_2} \xrightarrow{q_2 \ll q_1} \frac{1}{\lambda^6} \omega_{1k}^{(2)} \omega_{ij}^{(1)} - \frac{1}{\lambda^6} \frac{d}{4} \frac{1}{(q_1 \cdot q_2)^2} + \frac{1}{\lambda^6} \omega_{1i}^{(2,1)} \frac{1}{2} \frac{1}{q_1 \cdot q_2} + \frac{1}{\lambda^6} \omega_{1j}^{(2,1)} \frac{1}{2} \frac{1}{q_1 \cdot q_2} + \mathcal{O}\left(\frac{1}{\lambda^5}\right) \quad (\text{I.35})$$

$$\hat{\omega}_{ijj}^{a_1 a_2 a_2 a_1} \xrightarrow{q_2 \ll q_1} \frac{1}{\lambda^6} \frac{d}{4} \frac{1}{(q_1 \cdot q_2)^2} - \frac{1}{\lambda^6} \omega_{1i}^{(2,1)} \frac{1}{2} \frac{1}{q_1 \cdot q_2} - \frac{1}{\lambda^6} \omega_{1j}^{(2,1)} \frac{1}{2} \frac{1}{q_1 \cdot q_2} + \mathcal{O}\left(\frac{1}{\lambda^5}\right) \quad (\text{I.36})$$

$$\hat{\omega}_{iii}^{a_2 a_1 a_1 a_2} = \hat{\omega}_{iii}^{a_1 a_2 a_2 a_1} \xrightarrow{q_2 \ll q_1} \frac{1}{\lambda^6} \frac{d}{4} \frac{1}{(q_1 \cdot q_2)^2} - \frac{1}{\lambda^6} \omega_{1i}^{(2,1)} \frac{1}{q_1 \cdot q_2} + \mathcal{O}\left(\frac{1}{\lambda^5}\right) \quad (\text{I.37})$$

$$\hat{\omega}_{iii}^{a_1 a_2 a_1 a_2} = \hat{\omega}_{iii}^{a_2 a_1 a_2 a_1} \xrightarrow{q_2 \ll q_1} - \frac{1}{\lambda^6} \frac{d}{4} \frac{1}{(q_1 \cdot q_2)^2} + \frac{1}{\lambda^6} \omega_{1i}^{(2,1)} \frac{1}{q_1 \cdot q_2} + \mathcal{O}\left(\frac{1}{\lambda^5}\right) \quad (\text{I.38})$$

Now the string, where $i \neq l$ and $j \neq k$ but $m \neq p$ and $n = o$:

Now the string, where $i \neq l$ and $j \neq k$ but $m = p$ and $n \neq o$:

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Strings $i \neq l$ and $j \neq k$ but $m = p$ and $n = o$ is the same as in the dipole case.

For the ring there are no simplifications:

$$\begin{aligned}
\xi_{i,j,k,l}^{\sigma\tau\bar{\tau}\bar{\sigma}} = & \lambda^4 \frac{\langle \sigma | \mathbf{t}_i^\dagger \mathbf{t}_j^\dagger \mathbf{t}_k \mathbf{t}_l | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} \\
& - \lambda^3 \bar{\lambda} \frac{\langle \sigma | \mathbf{t}_i^\dagger \mathbf{t}_j^\dagger \mathbf{t}_k \bar{\mathbf{t}}_i | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} - \lambda^3 \bar{\lambda} \frac{\langle \sigma | \mathbf{t}_i^\dagger \mathbf{t}_j^\dagger \mathbf{t}_k \bar{\mathbf{t}}_l | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} - \lambda^3 \bar{\lambda} \frac{\langle \sigma | \mathbf{t}_i^\dagger \mathbf{t}_j^\dagger \mathbf{t}_k \mathbf{t}_i | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} - \lambda^3 \bar{\lambda} \frac{\langle \sigma | \bar{\mathbf{t}}_i^\dagger \mathbf{t}_j^\dagger \mathbf{t}_k \mathbf{t}_i | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} \\
& + \lambda^2 \bar{\lambda}^2 \frac{\langle \sigma | \mathbf{t}_i^\dagger \mathbf{t}_j^\dagger \mathbf{t}_k \bar{\mathbf{t}}_i | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} + \lambda^2 \bar{\lambda}^2 \frac{\langle \sigma | \mathbf{t}_i^\dagger \mathbf{t}_j^\dagger \mathbf{t}_k \bar{\mathbf{t}}_l | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} + \lambda^2 \bar{\lambda}^2 \frac{\langle \sigma | \mathbf{t}_i^\dagger \mathbf{t}_j^\dagger \mathbf{t}_k \mathbf{t}_l | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} + \lambda^2 \bar{\lambda}^2 \frac{\langle \sigma | \bar{\mathbf{t}}_i^\dagger \mathbf{t}_j^\dagger \mathbf{t}_k \bar{\mathbf{t}}_l | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} \\
& + \lambda^2 \bar{\lambda}^2 \frac{\langle \sigma | \bar{\mathbf{t}}_i^\dagger \mathbf{t}_j^\dagger \mathbf{t}_k \mathbf{t}_i | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} + \lambda^2 \bar{\lambda}^2 \frac{\langle \sigma | \bar{\mathbf{t}}_i^\dagger \mathbf{t}_j^\dagger \mathbf{t}_k \mathbf{t}_l | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} \\
& - \lambda \bar{\lambda}^3 \frac{\langle \sigma | \mathbf{t}_i^\dagger \bar{\mathbf{t}}_j^\dagger \mathbf{t}_k \bar{\mathbf{t}}_i | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} - \lambda \bar{\lambda}^3 \frac{\langle \sigma | \bar{\mathbf{t}}_i^\dagger \mathbf{t}_j^\dagger \mathbf{t}_k \bar{\mathbf{t}}_l | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} - \lambda \bar{\lambda}^3 \frac{\langle \sigma | \bar{\mathbf{t}}_i^\dagger \mathbf{t}_j^\dagger \mathbf{t}_k \mathbf{t}_l | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} - \lambda \bar{\lambda}^3 \frac{\langle \sigma | \bar{\mathbf{t}}_i^\dagger \bar{\mathbf{t}}_j^\dagger \mathbf{t}_k \mathbf{t}_i | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} \\
& + \bar{\lambda}^4 N \frac{\langle \sigma | \bar{\mathbf{t}}_j^\dagger \mathbf{t}_k | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle}.
\end{aligned} \tag{J.6}$$

If quark and anti-quark emissions are included in the evolution of the shower, then colour correlators, which include at least one $\mathbf{s}$ operator, have to be evaluated, some of them can be calculated independently of σ and $\bar{\sigma}$. These easy to calculate colour correlators are listed in the following.

$$\frac{\langle \sigma | \mathbf{s}^\dagger \mathbf{s}^\dagger \mathbf{s} \mathbf{s} | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} = \frac{N^{n+2-\#(\sigma, \bar{\sigma})}}{N^{n-\#(\sigma, \bar{\sigma})}} = N^2 \tag{J.7}$$

Now the cases including one $\mathbf{s}$, for all i :

$$\frac{\langle \sigma | \mathbf{s}^\dagger \mathbf{s}^\dagger \mathbf{t}_i \mathbf{s} | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} = \frac{\langle \sigma | \mathbf{s}^\dagger \mathbf{t}_i^\dagger \mathbf{s} \mathbf{s} | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} = \frac{\langle \sigma | \mathbf{s}^\dagger \mathbf{s}^\dagger \mathbf{t}_i \mathbf{t}_i | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} = \frac{\langle \sigma | \mathbf{t}_i^\dagger \mathbf{s}^\dagger \mathbf{s} \mathbf{s} | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} = N. \tag{J.8}$$

Next the cases with two $\mathbf{s}$ operators.

For $i \neq j$ and wlog $i < j$:

$$\frac{\langle \sigma | \mathbf{t}_i^\dagger \mathbf{t}_j^\dagger \mathbf{s} \mathbf{s} | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} = \frac{\langle \sigma | \mathbf{s}^\dagger \mathbf{s}^\dagger \mathbf{t}_i \mathbf{t}_j | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} = 1. \tag{J.9}$$

For $i = j$:

$$\frac{\langle \sigma | \mathbf{t}_i^\dagger \mathbf{t}_i^\dagger \mathbf{s} \mathbf{s} | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} = \frac{\langle \sigma | \mathbf{s}^\dagger \mathbf{s}^\dagger \mathbf{t}_i \mathbf{t}_i | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} = 1. \tag{J.10}$$

For $i \neq j$:

$$\frac{\langle \sigma | \mathbf{t}_i^\dagger \mathbf{s}^\dagger \mathbf{s} \mathbf{t}_j | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} = N \frac{\langle \sigma | \mathbf{t}_i^\dagger \mathbf{t}_j | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle}, \tag{J.11}$$

and if $i = j$:

$$\frac{\langle \sigma | \mathbf{t}_i^\dagger \mathbf{s}^\dagger \mathbf{s} \mathbf{t}_i | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} = N \frac{\langle \sigma | \mathbf{t}_i^\dagger \mathbf{t}_i | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} = N \frac{\langle \sigma | \mathbf{s}^\dagger \mathbf{s} | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} = N^2. \tag{J.12}$$

Furthermore, if $i = j$, then:

$$\frac{\langle \sigma | \mathbf{s}^\dagger \mathbf{t}_i^\dagger \mathbf{t}_i \mathbf{s} | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} = \frac{\langle \sigma | \mathbf{s}^\dagger \mathbf{s}^\dagger \mathbf{s} \mathbf{s} | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} = N^2. \quad (\text{J.13})$$

If $i \neq j$ and $i < n + 1$, and $j < n + 1$, then

$$\frac{\langle \sigma | \mathbf{s}^\dagger \mathbf{t}_i^\dagger \mathbf{t}_j \mathbf{s} | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} = N \frac{\langle \sigma | \mathbf{t}_i^\dagger \mathbf{t}_j | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle}. \quad (\text{J.14})$$

If i or j equals $n + 1$ then the result is not independent of σ and $\bar{\sigma}$ anymore.

For $i \neq j$ and $i = j$:

$$\frac{\langle \sigma | \mathbf{t}_i^\dagger \mathbf{s}^\dagger \mathbf{t}_j \mathbf{s} | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} = 1. \quad (\text{J.15})$$

For $i \neq j$:

$$\frac{\langle \sigma | \mathbf{s}^\dagger \mathbf{t}_i^\dagger \mathbf{s} \mathbf{t}_j | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} = 1. \quad (\text{J.16})$$

The next case $i = j$:

$$\frac{\langle \sigma | \mathbf{s}^\dagger \mathbf{t}_i^\dagger \mathbf{s} \mathbf{t}_i | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} = 1. \quad (\text{J.17})$$

Now the cases with one $\mathbf{s}$. If $j = k$, then:

$$\frac{\langle \sigma | \mathbf{s}^\dagger \mathbf{t}_j^\dagger \mathbf{t}_j \mathbf{t}_l | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} = \frac{\langle \sigma | \mathbf{s}^\dagger \mathbf{s}^\dagger \mathbf{s} \mathbf{t}_l | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} = N. \quad (\text{J.18})$$

For $i = l$ and $i \neq k$:

$$\frac{\langle \sigma | \mathbf{t}_i^\dagger \mathbf{s}^\dagger \mathbf{t}_k \mathbf{t}_l | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} = N. \quad (\text{J.19})$$

If $j = k$, then

$$\frac{\langle \sigma | \mathbf{t}_i^\dagger \mathbf{t}_j^\dagger \mathbf{t}_k \mathbf{s} | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} = N. \quad (\text{J.20})$$

For $i = l$ and $i \neq j$:

$$\frac{\langle \sigma | \mathbf{t}_i^\dagger \mathbf{t}_j^\dagger \mathbf{s} \mathbf{t}_l | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} = N. \quad (\text{J.21})$$

Lastly, if all colour operators act on the same colour line:

$$\frac{\langle \sigma | \mathbf{s}^\dagger \mathbf{t}_i^\dagger \mathbf{t}_i \mathbf{t}_i | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} = \frac{\langle \sigma | \mathbf{t}_i^\dagger \mathbf{s}^\dagger \mathbf{t}_i \mathbf{t}_i | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} = \frac{\langle \sigma | \mathbf{t}_i^\dagger \mathbf{t}_i^\dagger \mathbf{s} \mathbf{t}_i | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} = \frac{\langle \sigma | \mathbf{t}_i^\dagger \mathbf{t}_i^\dagger \mathbf{t}_i \mathbf{s} | \bar{\sigma} \rangle}{\langle \sigma | \bar{\sigma} \rangle} = N. \quad (\text{J.22})$$

K Supplementary material for double emission at leading order

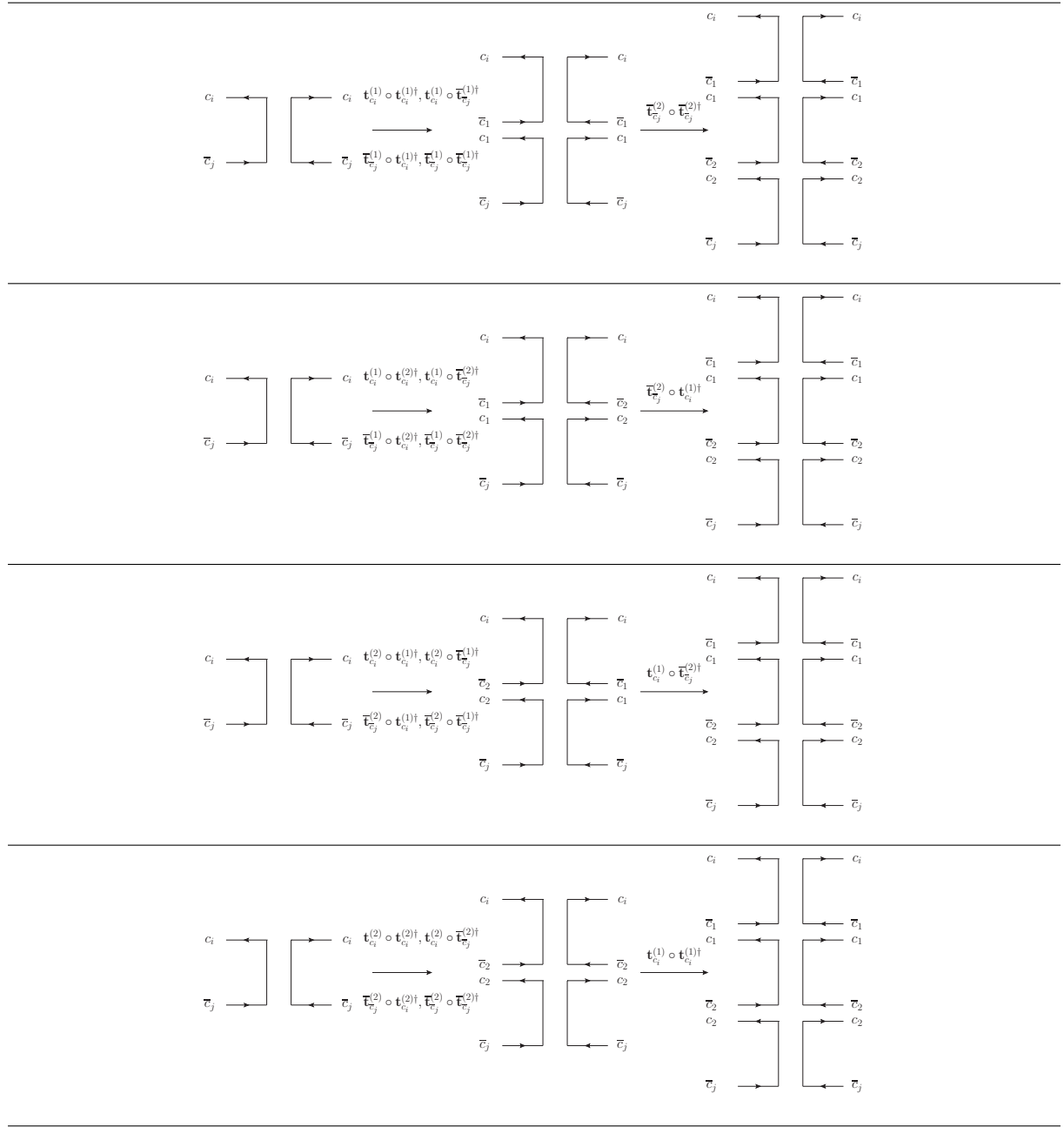


Figure 28: Leading colour transitions, starting from one colour line, the resulting transition kernel is denoted with K_2 .

Reading off the possible transitions in terms of emission operators $\mathbf{t}_{c_i}, \bar{\mathbf{t}}_{\bar{c}_i}$ from figure 22 and 28 gives then the following list of leading colour transtions, which start at one colour line.

1.	$\mathbf{t}_{c_i}^{a_2} \mathbf{t}_{c_i}^{a_1} \circ \mathbf{t}_{c_i}^{a_1 \dagger} \mathbf{t}_{c_i}^{a_2 \dagger}$
2.	$\mathbf{t}_{c_i}^{a_2} \mathbf{t}_{c_i}^{a_1} \circ \bar{\mathbf{t}}_{\bar{c}_j}^{a_1 \dagger} \mathbf{t}_{c_i}^{a_2 \dagger}$
3.	$\mathbf{t}_{c_i}^{a_2} \bar{\mathbf{t}}_{\bar{c}_j}^{a_1} \circ \mathbf{t}_{c_i}^{a_1 \dagger} \mathbf{t}_{c_i}^{a_2 \dagger}$
4.	$\mathbf{t}_{c_i}^{a_2} \bar{\mathbf{t}}_{\bar{c}_i}^{a_1} \circ \bar{\mathbf{t}}_{\bar{c}_j}^{a_1 \dagger} \mathbf{t}_{c_i}^{a_2 \dagger}$
5.	$\mathbf{t}_{c_i}^{a_2} \mathbf{t}_{c_i}^{a_1} \circ \mathbf{t}_{c_i}^{a_2 \dagger} \bar{\mathbf{t}}_{\bar{c}_j}^{a_1 \dagger}$
6.	$\mathbf{t}_{c_i}^{a_2} \mathbf{t}_{c_i}^{a_1} \circ \bar{\mathbf{t}}_{\bar{c}_j}^{a_2 \dagger} \bar{\mathbf{t}}_{\bar{c}_j}^{a_1 \dagger}$
7.	$\mathbf{t}_{c_i}^{a_2} \bar{\mathbf{t}}_{\bar{c}_j}^{a_1} \circ \mathbf{t}_{c_i}^{a_2 \dagger} \bar{\mathbf{t}}_{\bar{c}_j}^{a_1 \dagger}$
8.	$\mathbf{t}_{c_i}^{a_2} \bar{\mathbf{t}}_{\bar{c}_j}^{a_1} \circ \bar{\mathbf{t}}_{\bar{c}_j}^{a_2 \dagger} \bar{\mathbf{t}}_{\bar{c}_j}^{a_1 \dagger}$
9.	$\bar{\mathbf{t}}_{\bar{c}_j}^{a_1} \mathbf{t}_{c_i}^{a_2} \circ \mathbf{t}_{c_i}^{a_1 \dagger} \mathbf{t}_{c_i}^{a_2 \dagger}$
10.	$\bar{\mathbf{t}}_{\bar{c}_j}^{a_1} \mathbf{t}_{c_i}^{a_2} \circ \bar{\mathbf{t}}_{\bar{c}_j}^{a_1 \dagger} \mathbf{t}_{c_i}^{a_2 \dagger}$
11.	$\bar{\mathbf{t}}_{\bar{c}_j}^{a_1} \bar{\mathbf{t}}_{\bar{c}_j}^{a_2} \circ \mathbf{t}_{c_i}^{a_1 \dagger} \mathbf{t}_{c_i}^{a_2 \dagger}$
12.	$\bar{\mathbf{t}}_{\bar{c}_j}^{a_1} \bar{\mathbf{t}}_{\bar{c}_j}^{a_2} \circ \bar{\mathbf{t}}_{\bar{c}_j}^{a_1 \dagger} \mathbf{t}_{c_i}^{a_2 \dagger}$
13.	$\bar{\mathbf{t}}_{\bar{c}_j}^{a_1} \mathbf{t}_{c_i}^{a_2} \circ \mathbf{t}_{c_i}^{a_2 \dagger} \bar{\mathbf{t}}_{\bar{c}_j}^{a_1 \dagger}$
14.	$\bar{\mathbf{t}}_{\bar{c}_j}^{a_1} \mathbf{t}_{c_i}^{a_2} \circ \bar{\mathbf{t}}_{\bar{c}_j}^{a_2 \dagger} \bar{\mathbf{t}}_{\bar{c}_j}^{a_1 \dagger}$
15.	$\bar{\mathbf{t}}_{\bar{c}_j}^{a_1} \bar{\mathbf{t}}_{\bar{c}_j}^{a_2} \circ \mathbf{t}_{c_i}^{a_2 \dagger} \bar{\mathbf{t}}_{\bar{c}_j}^{a_1 \dagger}$
16.	$\bar{\mathbf{t}}_{\bar{c}_j}^{a_1} \bar{\mathbf{t}}_{\bar{c}_j}^{a_2} \circ \bar{\mathbf{t}}_{\bar{c}_i}^{a_2 \dagger} \bar{\mathbf{t}}_{\bar{c}_j}^{a_1 \dagger}$
17.	$\bar{\mathbf{t}}_{\bar{c}_j}^{a_2} \mathbf{t}_{c_i}^{a_1} \circ \mathbf{t}_{c_i}^{a_1 \dagger} \bar{\mathbf{t}}_{\bar{c}_j}^{a_2 \dagger}$
18.	$\bar{\mathbf{t}}_{\bar{c}_j}^{a_2} \mathbf{t}_{c_i}^{a_1} \circ \bar{\mathbf{t}}_{\bar{c}_j}^{a_1 \dagger} \bar{\mathbf{t}}_{\bar{c}_j}^{a_2 \dagger}$
19.	$\bar{\mathbf{t}}_{\bar{c}_j}^{a_2} \bar{\mathbf{t}}_{\bar{c}_j}^{a_1} \circ \mathbf{t}_{c_i}^{a_1 \dagger} \bar{\mathbf{t}}_{\bar{c}_j}^{a_2 \dagger}$
20.	$\bar{\mathbf{t}}_{\bar{c}_j}^{a_2} \bar{\mathbf{t}}_{\bar{c}_j}^{a_1} \circ \bar{\mathbf{t}}_{\bar{c}_j}^{a_1 \dagger} \bar{\mathbf{t}}_{\bar{c}_j}^{a_2 \dagger}$
21.	$\bar{\mathbf{t}}_{\bar{c}_j}^{a_2} \mathbf{t}_{c_i}^{a_1} \circ \mathbf{t}_{c_i}^{a_2 \dagger} \mathbf{t}_{c_i}^{a_1 \dagger}$
22.	$\bar{\mathbf{t}}_{\bar{c}_j}^{a_2} \mathbf{t}_{c_i}^{a_1} \circ \bar{\mathbf{t}}_{\bar{c}_j}^{a_2 \dagger} \mathbf{t}_{c_i}^{a_1 \dagger}$
23.	$\bar{\mathbf{t}}_{\bar{c}_j}^{a_2} \bar{\mathbf{t}}_{\bar{c}_j}^{a_1} \circ \mathbf{t}_{c_i}^{a_2 \dagger} \mathbf{t}_{c_i}^{a_1 \dagger}$
24.	$\bar{\mathbf{t}}_{\bar{c}_j}^{a_2} \bar{\mathbf{t}}_{\bar{c}_j}^{a_1} \circ \bar{\mathbf{t}}_{\bar{c}_j}^{a_2 \dagger} \mathbf{t}_{c_i}^{a_1 \dagger}$
25.	$\mathbf{t}_{c_i}^{a_1} \mathbf{t}_{c_i}^{a_2} \circ \mathbf{t}_{c_j}^{a_1 \dagger} \bar{\mathbf{t}}_{\bar{c}_j}^{a_2 \dagger}$
26.	$\mathbf{t}_{c_i}^{a_1} \mathbf{t}_{c_i}^{a_2} \circ \bar{\mathbf{t}}_{\bar{c}_j}^{a_1 \dagger} \bar{\mathbf{t}}_{\bar{c}_j}^{a_2 \dagger}$
27.	$\mathbf{t}_{c_i}^{a_1} \bar{\mathbf{t}}_{\bar{c}_j}^{a_2} \circ \mathbf{t}_{c_i}^{a_1 \dagger} \bar{\mathbf{t}}_{\bar{c}_j}^{a_2 \dagger}$
28.	$\mathbf{t}_{c_i}^{a_1} \bar{\mathbf{t}}_{\bar{c}_j}^{a_2} \circ \bar{\mathbf{t}}_{\bar{c}_j}^{a_1 \dagger} \bar{\mathbf{t}}_{\bar{c}_j}^{a_2 \dagger}$
29.	$\mathbf{t}_{c_i}^{a_1} \mathbf{t}_{c_i}^{a_2} \circ \mathbf{t}_{c_i}^{a_2 \dagger} \mathbf{t}_{c_i}^{a_1 \dagger}$
30.	$\mathbf{t}_{c_i}^{a_1} \mathbf{t}_{c_i}^{a_2} \circ \bar{\mathbf{t}}_{\bar{c}_j}^{a_2 \dagger} \mathbf{t}_{c_i}^{a_1 \dagger}$
31.	$\mathbf{t}_{c_i}^{a_1} \bar{\mathbf{t}}_{\bar{c}_j}^{a_2} \circ \mathbf{t}_{c_i}^{a_2 \dagger} \mathbf{t}_{c_i}^{a_1 \dagger}$
32.	$\mathbf{t}_{c_i}^{a_1} \bar{\mathbf{t}}_{\bar{c}_j}^{a_2} \circ \bar{\mathbf{t}}_{\bar{c}_j}^{a_2 \dagger} \mathbf{t}_{c_i}^{a_1 \dagger}$

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