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A Study of Gauge Invariance of n-Point Vertices in Higher Spin Gauge Theories

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ABSTRACT

For a long time now a central goal of theoretical physics has been to find a way to accurately describe all four fundamental interactions, preferably using one unified theory. The Standard Model describes the electromagnetic, the weak and the strong interaction with theories of spin one gauge bosons. However, it fails to include the gravitational interaction, and so some look at different ways to describe those interactions, in hope of finding a theory that does not share the same shortcomings of the Standard Model. The most famous example of such an attempt is String Theory. In the framework of String Theory, the gravitational interaction appears naturally, being mediated by a spin two particle called the graviton. While being plagued by its own flaws there is hope, that String Theory might be able to describe all fundamental interactions and be viewed as a generalization of the Standard Model one day. Amid all of this, another field of research gains interest, among other things, questioning, if it is true that only gauge fields of spin smaller or equal to two are significant for the physics of fundamental interactions. Whether out of pure mathematical curiosity or the drive to further the physical understanding of interactions, it is a goal of Higher Spin Theories to build a full non-linear Lagrangian of gauge fields with a spin larger than two. Higher Spin Gravity tries to extend gravity, mediated by a spin two field, by additional higher spin gauge fields [1–3]. In recent years, a connection between the tensionless limit of String Theory and Higher Spin Gravity has been found, further reinforcing the belief, that Higher Spin Theories might be needed to fully understand fundamental interactions.

In this light, we consider a Lagrangian of interacting massless higher spin gauge fields, obtained by a perturbative approach called the Noether procedure. We study the constraints of gauge variation on the Lagrangian and find new variables, such that it can be written as a polynomial even after imposing gauge invariance. Ultimately, we wish to be able to express such a Lagrangian as a polynomial of known variables, no matter how many of its gauge fields we vary.

ZUSAMMENFASSUNG

Seit geraumer Zeit ist das Finden einer akkuraten Beschreibung aller vier fundamentalen Wechselwirkungen, bevorzugt vereint in einer Theorie, ein grundlegendes Ziel der theoretischen Physik. Das Standardmodell beschreibt die elektromagnetische, die schwache und die starke Wechselwirkung mithilfe von Spin-Eins Eichbosonen. Es ist jedoch nicht möglich die gravitative Wechselwirkung durch das Standardmodell zu beschreiben, weshalb man auch nach anderen Beschreibungen dieser Wechselwirkungen sucht - in der Hoffnung eine Theorie zu finden, die die Nachteile des Standardmodells ausmerzt. Das berühmteste Beispiel eines solchen Versuchs ist die Stringtheorie. In dieser erscheint die gravitative Wechselwirkung ganz natürlich und wird beschrieben durch ein Spin-Zwei Teilchen, welches man das Graviton nennt. Während diese Theorie wiederum eine Vielzahl eigener Probleme mit sich trägt besteht dennoch die Hoffnung, dass sie eines Tages alle Wechselwirkungen beschreiben und somit als Verallgemeinerung des Standardmodells gesehen werden kann.
Inmitten dieser Suche wächst das Interesse an einem weiteren Forschungsgebiet, welches sich unter anderem der Frage widmet, ob es wahr ist, dass nur Eichfelder mit Spin kleiner oder gleich zwei in die Physik der fundamentellen Wechselwirkungen einfließen. Ob aus reinem mathematischen Interesse oder physikalischen Beweggründen fundamentelle Wechselwirkungen besser zu verstehen, ist es ein Ziel der Höhere-Spin-Theorien eine komplette nicht-lineare Lagrange-Funktion von Eichfeldern mit einem Spin größer als zwei zu finden. Höhere-Spin-Gravitation versucht die Gravitationstheorie, welche von einem Spin-Zwei Feld beschrieben wird, durch zusätzliche Eichfelder höheren Spins zu erweitern [1–3]. In vergangenen Jahren wurde eine potentielle Verbindung zwischen dem spannungsfreien Limes der Stringtheorie und höherer Spin Gravitation gefunden, was somit die Überzeugung, dass höhere Spin Theorien wichtig für das komplette Verständnis fundamenteller Wechselwirkungen sind, weiter bestärkt.
In diesem Sinne betrachten wir eine Lagrange-Funktion interagierender masseloser Eichfelder höheren Spins, welche wir durch einen störungstheoretischen Ansatz erhalten, der das Noetherprogramm genannt wird. Wir studieren die Einschränkungen durch Eichvariation auf die Lagrange-Funktion und finden neue Variablen, um diese, auch nachdem wir Eichinvarianz gefordert haben, als Polynom ausdrücken zu können. Schlussendlich möchten wir in der Lage sein, eine solche Lagrange-Funktion immer als Polynom uns bekannter Variablen auszudrücken, unabhängig davon wie viele Eichfelder wir variieren.

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Altogether, I immediately felt welcome in the group, which is unsurprising, as all its members are friendly and likeable people that are fun to be around. Unfortunately, due to the pandemic I only had few possibilities of meeting them face to face, but the instances I did I will remember fondly.

Last but not least I want to thank my girlfriend Lisa, my family and my friends that support me in all my endeavours and give me the confidence that I can achieve anything.

CONTENTS

1 INTRODUCTION

1.1 HIGHER SPIN GRAVITY

In the most general way, higher spin (HS) theories are field theories consisting of higher spin gauge fields. Examples for gauge theories with lower spin are widely known, such as electrodynamics, where gauge fields of spin $s = 1$ appear or linearized gravity that involves gauge fields of spin $s = 2$. Higher spin theories, however, involve gauge fields with spins $s > 2$.
General relativity is a spin-2 theory described by the Einstein field equations. Higher spin gravity tries to extend general relativity by adding massless higher spin gauge fields [1–3]. At first, this seems like a daunting task due to the existence of various no-go theorems (see e.g. [4] for a review) that hinder interactions between massless higher spin fields, especially in Minkowski background.
Examples are the Weinberg low-energy theorem [5], which states, that particles of higher spin cannot couple at low energies, or the Generalized Weinberg-Witten theorem [6], which tells us, that the energy momentum tensor of particles with a spin $s > 1$ cannot be Lorentz covariant and gauge invariant [4]. Extending this theorem it can be shown, that a massless particle with spin $s > 2$ cannot couple to gravity in Minkowski background (see e.g. [7]).

Despite these difficulties, studying higher spin theories and trying to find ways to work around these problems to build interacting theories of higher spin fields is worth it. Apart from a mathematical standpoint, higher spin theories have found their place in physics to study quantum gravity. Similarly to string theory, a correspondence has been found between HS theories in AdS spacetime and conformal field theories (CFTs) [8, 9]. Since there exists an AdS/CFT correspondence for string theory as well, this property is used to relate HS theory with the tensionless limit of string theory [10, 11].

In this thesis we build on, and wish to extend the work of [1]. The authors state, that "the aim of this work is, in particular, to obtain restrictions for all possible independent interaction vertices of order $n \geqslant 4$ for massless higher-spin fields" ([1], p.1). We study how gauge invariance restricts vertices of order $n \geqslant 4$ of a Lagrangian of a higher spin theory. The goal is, to be able to express any independent vertex that is gauge invariant under an arbitrary number of HS gauge fields as a polynomial in known variables. To this end, we will attempt to identify building blocks of vertices that are gauge invariant with respect to a varying number of HS fields. Additionally, we have to prove that these vertices can be written as a polynomial in the discovered building blocks.

1.2 METRIC-LIKE FORMULATION OF HS THEORIES

Two different approaches exist to formulate a HS theory, namely, the frame-like formulation, e.g. used in the Vasiliev HS theory, and the metric-like formulation (for a review on both see, e.g. [12]).
We are going to follow the latter, hence, let us start with a short introduction.

Note, that we are only going to consider massless bosonic fields in flat space.
The metric-like formulation was first introduced by Christian Fronsdal [13, 14] and can be viewed as a generalization of linearized gravity [12].

We describe a massless field of arbitrary spin-s using a symmetric rank-s tensor $\phi_{\mu_1 \mu_2 \dots \mu_s}$. These Fronsdal fields are gauge fields with the following gauge variation:

$$\delta^{(0)} \phi_{\mu_1 \mu_2 \dots \mu_s} = s\, \partial_{(\mu_1} \epsilon_{\mu_2 \dots \mu_s)} \,, \tag{1.1}$$

where $\delta^{(0)}$ signals that the fields are non-interacting.
It is here, that we can see the connection to linearized gravity for $s = 2$ and even electrodynamics for $s = 1$ (see, e.g. [15]).

Fronsdal fields are described by an action of the same name, the Fronsdal action [13, 14]:

$$S = \int \sqrt{g}\, \phi^{\mu_1 \dots \mu_s} \left(F_{\mu_1 \dots \mu_s} - \frac{1}{2} g_{(\mu_1 \mu_2}\, F^{\nu}{}_{\nu\, \mu_3 \dots \mu_s)} \right) \,, \tag{1.2}$$

where $F_{\mu_1 \dots \mu_s}$ is the Fronsdal tensor. It is defined to be given by the Fronsdal equation [13, 14]:

$$F_{\mu_1 \dots \mu_s} = \Box \phi_{\mu_1 \dots \mu_s} - \partial_{(\mu_1} \partial^{\rho} \phi_{\mu_2 \dots \mu_s)\rho} + \partial_{(\mu_1} \partial_{\mu_2} \phi_{\mu_3 \dots \mu_s)\rho}{}^{\rho} = 0 \,. \tag{1.3}$$

Note, that this is not the most general form of the Fronsdal equation, some terms are missing due to different assumptions: Firstly, we work in flat-space, where the cosmological constant $\Lambda = 0$. Secondly, we have already assumed $\phi_{\mu_1 \dots \mu_s}$ to be double traceless:

$$\phi^{\sigma\, \rho}{}_{\sigma\, \rho\, \mu_5 \dots \mu_s} = 0 \,. \tag{1.4}$$

Double tracelessness for $\phi_{\mu_1 \dots \mu_s}$ has to be imposed, such that we can use the Fronsdal equation to describe propagating massless spin-s fields (see, e.g. [15]).

1.3 NOETHER PROCEDURE

Given the framework of the metric-like formulation we wish to build a theory of interacting HS fields. Note, however, that as of writing this, no full non-linear Lagrangian of interacting HS fields has been found yet [1].

The Noether procedure is a perturbative approach to construct Actions or, similarly, Lagrangians of HS theories including interactions. One starts by considering HS gauge fields, in our case, described by $\phi_{\mu_1\ldots\mu_s}$. These fields are described by the free (Fronsdal) action S_2 given in equation (1.2) and the free gauge variation given in equation (1.1). Then, order by order, we expand the free action by terms involving an increasing number of fields, such that the full non-linear Action S reads [16]:

$$S = S_2 + S_3 + \ldots = S_2 + \sum_{n>2} S_n \,. \tag{1.5}$$

The index n tells us the order of the term, and therefore, the number of fields appearing. Additionally, the Action S has to be gauge invariant, not only under the free gauge transformation $\delta^{(0)}$, but also an extension involving all the different possible numbers of fields. This gauge transformation δ is obtained by deforming $\delta^{(0)}$, such that [1, 17]:

$$\delta = \delta^{(0)} + \delta^{(1)} + \ldots = \delta^{(0)} + \sum_{k\geqslant 1} \delta^{(k)} \,. \tag{1.6}$$

Again, the superscript k gives the order in the fields. Imposing gauge invariance by requiring $\delta\, S \overset{!}{=} 0$ leads to a set of differential equations:

$$\begin{aligned}
&\delta^{(0)} S_2 = 0\,,\\
&\delta^{(0)} S_3 + \delta^{(1)} S_2 = 0\,,\\
&\delta^{(0)} S_4 + \delta^{(1)} S_3 + \delta^{(2)} S_2 = 0\,,\\
&\ldots
\end{aligned} \tag{1.7}$$

To fully know S one would have to solve this set of differential equations.
To obtain a deeper understanding for the Noether procedure it is helpful to study its applications on HS theories in three dimensions (see e.g. [5, 16–19]).

Of course, when working with the Lagrangian instead of the Action, one can proceed analogously. We are going to study the constraints of gauge invariance on independent vertices of the Lagrangian. Therefore, the set of differential equations (1.7) instead reads [1, 17]:

$$\delta^{(0)} \mathcal{L}_n + \delta^{(n-2)} \mathcal{L}_2 \approx 0\,. \tag{1.8}$$

Two Lagrangians that differ only by a total derivative lead to the same equations of motion, hence, we only impose equality up to total derivatives, as marked by "$\approx$". Lastly, by imposing the equations of motion, the term involving the free Lagrangian $\mathcal{L}_2$ always vanishes, such that we obtain:

$$\delta^{(0)} \mathcal{L}_n \approx 0\,. \tag{1.9}$$

An independent vertex $\mathcal{L}_n$, therefore, has to satisfy this equation to be gauge invariant.

2 | PRELIMINARIES

In this section follows an introduction to the basic mathematical concepts and notations used throughout this thesis. Note, however, since the research in this thesis is closely tied to the works of [1, 17] and additionally being supervised by one of the authors, the notation and framework are going to follow [1, 17] closely.

2.1 HIGHER SPIN GAUGE FIELDS

As hinted above, we wish to study the restrictions gauge invariance imposes on an independent vertex $\mathcal{L}_n$ for a HS Lagrangian. Note, that throughout this work, when talking about such an independent vertex $\mathcal{L}_n$, we are assuming $n \geqslant 4$. This Lagrangian describes an arbitrary number of massless bosonic HS gauge fields $\phi_{\mu_1 \dots \mu_s}$. We can use auxiliary vectors a^μ to contract the indices of these fields $\phi_{\mu_1 \dots \mu_s}$ and express them as:

$$\phi^{(s)}(x, a) = \frac{1}{s!} \phi_{\mu_1 \, \mu_2 \dots \mu_s}(x) \, a^{\mu_1} \dots a^{\mu_s} \,. \tag{2.1}$$

This has no physical consequences, but in doing so we gain the following advantages: On the one hand, the symmetry of $\phi_{\mu_1 \, \mu_2 \dots \mu_s}$ is manifest if the a^{μ_i}'s commute. On the other hand it simplifies the notation due to all the indices already being contracted.
When working on-shell, hence when assuming the equations of motion to hold, one can always find a gauge such that a field turns traceless and transverse (TT). Therefore, we can restrict ourselves to only consider the TT part of the Lagrangian, knowing that we can expand any obtained vertex to a unique complete vertex. By inserting fields into such a vertex, we are going to obtain terms that are not TT themselves. From a physical standpoint, these terms are viewed to contain the degrees of freedom.
To sum up, we restrict ourselves to the traceless and transverse, on-shell part of the Lagrangian, meaning that the fields have to satisfy the equation of motion, taking the form of the massless Klein-Gordon equation in our case.
When expressing these constraints mathematically we can introduce another notation to ensure simplicity: Let $P^\mu = \frac{\partial}{\partial x_\mu}$ and $A^\mu = \frac{\partial}{\partial a_\mu}$. Then the gauge fields have to satisfy the following three conditions:

$$\begin{aligned} &A^2 \, \phi^{(s)} = 0 \,, \\ &A \cdot P \, \phi^{(s)} = 0 \,, \\ &P^2 \, \phi^{(s)} \big|_{\text{free e.o.m.}} = 0 \,. \end{aligned} \tag{2.2}$$

This set of conditions is known as the Fierz equations [20].

2.2 VERTEX GENERATING OPERATOR

Now that we have defined suitable notations for the gauge fields let us turn our attention to the vertices. An arbitrary independent vertex $\mathcal{L}_n$ is described by a product of various gauge fields and possibly derivatives acting on these fields. Therefore, we can express $\mathcal{L}_n$ as a product of n gauge fields as well as an operator $\mathcal{V}$ acting on this product that governs all index contractions:

$$\mathcal{L}_n(x) = \mathcal{V}\left(\prod_{i=1}^{n} \phi_i(x_i, a_i)\right)\Bigg|_{\substack{x_i = x \\ a_i = 0}} . \tag{2.3}$$

Note, that $\phi_i = \phi^{(s_i)}$ is now a function only depending on x_i and a_i. In the end all a_i are set to zero, such that no free indices are left after the contractions are performed and $\mathcal{L}_n$ is ensured to be a Lorentz scalar. Moreover, equation (2.3) only describes parity-even vertices. Therefore, we are not considering parity-odd vertices. The reason for this will be covered in a later section. The operator $\mathcal{V}$ is called the vertex generating operator and is defined as a polynomial in the variables s_{ij}, y_{ij} and z_{ij} that perform the necessary index contractions:

$$\begin{aligned} z_{ij} &= A_i \cdot A_j\big|_{1\leqslant i \leqslant j \leqslant n}\,, \\ y_{ij} &= A_i \cdot P_j\big|_{1\leqslant i, j \leqslant n}\,, \\ s_{ij} &= P_i \cdot P_j\big|_{1\leqslant i \leqslant j \leqslant n}\,. \end{aligned} \tag{2.4}$$

Note, that $P_i^\mu = \partial_{x_i^\mu}$ and $A_i^\mu = \partial_{a_i^\mu}$. The variables s_{ij}, y_{ij} and z_{ij} commute and each performs a different kind of index contraction:

- z_{ij} contracts an index of the field ϕ_i with an index of the field ϕ_j.
- y_{ij} contracts an index of the field ϕ_i with an index of a derivative acting on the field ϕ_j.
- s_{ij} contracts the indices of two derivatives, one acting on ϕ_i and the other on ϕ_j.

Using this formalism shifts the focus from finding constraints on the independent vertex $\mathcal{L}_n$ to finding constraints on the vertex generating operator $\mathcal{V}$. Additionally, due to the definitions of s_{ij}, y_{ij} and z_{ij} the vertex generating operator $\mathcal{V}$ is guaranteed to be a Lorentz scalar.

We have already established that two Lagrangians that differ only by a total derivative may be viewed as equivalent. Multiple such equivalence relations can be found for the vertex generating operator $\mathcal{V}$ as well. We define two vertex generating operators $\mathcal{V}$ and $\mathcal{V}'$ to be equivalent ($\mathcal{V} \approx \mathcal{V}'$) if they describe two equivalent Lagrangians. All transformations that satisfy this condition lead to an equivalence relation.

2.3 EQUIVALENCE RELATIONS

Due to restricting ourselves to studying gauge invariance, not many of the known equivalence relations come into play in this work. Therefore, let us briefly name the important results, such that we can continue with introducing how to impose gauge invariance on the vertex generating operator $\mathcal{V}$. For a more detailed discussion on equivalence relations see, e.g. [1].

The equivalence relation we have to consider arises due to field redefinitions. Consider a vertex generating operator $\mathcal{V}$. A non-linear field redefinition transforms a gauge field ϕ_i, such that

$$\phi_i' = \phi_i + \delta\phi_i \,, \tag{2.5}$$

where δ is non-linear in the fields. Let us now introduce a second vertex generating operator $\mathcal{V}'$ that only differs from $\mathcal{V}$ by depending on ϕ_i' instead of ϕ_i. Then it is possible for terms in the free Lagrangian $\mathcal{L}_2$ to contribute to the vertex $\mathcal{L}_n'$ generated by $\mathcal{V}'$. However, all these new terms in $\mathcal{L}_n'$ vanish when imposing the free equations of motion. From here on out, let us assume two Lagrangians to be equivalent when they only differ by such field redefinitions. From equation (2.3), where we expressed an independent n-point vertex $\mathcal{L}_n$ using $\mathcal{V}$, it then follows, that we can set $\mathcal{V}$ to be independent of $s_{ii} = P_i \cdot P_i$.
Additionally, due to requiring the fields to be traceless and divergence free, we also set $\mathcal{V}$ to be independent of $y_{ii} = A_i \cdot P_i$ and $z_{ii} = A_i \cdot A_i$. Summing up, we impose the following equivalence relations on $\mathcal{V}$:

$$s_{ii} \approx 0 \,, \quad y_{ii} \approx 0 \,, \quad z_{ii} \approx 0 \,. \tag{2.6}$$

2.4 GAUGE INVARIANCE

We wish to study invariance of an independent interacting vertex $\mathcal{L}_n$ under the free gauge variation $\delta^{(0)}$. As we have seen in the section introducing the Noether procedure, it follows that $\mathcal{L}_n$ has to satisfy:

$$\delta_k^{(0)} \mathcal{L}_n \approx 0 \,. \tag{2.7}$$

Note, that the index k got introduced to differentiate between gauge variations with respect to different gauge fields ϕ_k.
However, we wish to work in our newly introduced framework, hence, we have to find out which conditions the vertex generating operator $\mathcal{V}$ has to satisfy to describe a gauge invariant Vertex $\mathcal{L}_n$. Let us begin by writing the tensor ϵ describing the free gauge variation of the gauge fields ϕ_k, given in equation (1.1), in our new notation using auxiliary vectors a^μ:

$$\epsilon^{(s-1)}(x, a) = \frac{1}{(s-1)!} \epsilon_{\mu_1 \dots \mu_{s-1}}(x)\, a^{\mu_1} \dots a^{\mu_{s-1}} \,. \tag{2.8}$$

Then the free gauge variation of ϕ_k reads:

$$\delta_k^{(0)} \phi_k(x_k, a_k) = a_k \cdot P_k \, \epsilon_k(x_k, a_k) \,. \tag{2.9}$$

Let us now gauge vary a vertex $\mathcal{L}_n$ with respect to the field ϕ_k to find a condition for gauge invariance for the vertex generating operator $\mathcal{V}$. To this end, we are using the definition of a vertex $\mathcal{L}_n$ given in equation (2.3), as well as the new expression for free gauge variation given in equation (2.9):

$$\begin{aligned}\delta_k^{(0)}\mathcal{L}_n &= \delta_k^{(0)}\left(\mathcal{V}\left(\prod_{i=1}^{n}\phi_i(x_i,a_i)\right)\Big|_{\substack{x_i=x\\a_i=0}}\right)\\ &= \mathcal{V}\,a_k\cdot P_k\left(\epsilon_k(x_k,a_k)\prod_{1\leqslant i\leqslant n}^{i\neq k}\phi_i(x_i,a_i)\right)\Big|_{\substack{x_i=x\\a_i=0}}\approx 0\,.\end{aligned} \tag{2.10}$$

Recall, that all the auxiliary vectors a^μ are getting set to zero in the end. Therefore, we may rewrite this expression as:

$$\begin{aligned}\delta_k^{(0)}\mathcal{L}_n &= \mathcal{V}\,a_k\cdot P_k\left(\epsilon_k(x_k,a_k)\prod_{1\leqslant i\leqslant n}^{i\neq k}\phi_i(x_i,a_i)\right)\Big|_{\substack{x_i=x\\a_i=0}}\\ &\quad-\underbrace{(a_k\cdot P_k\,\mathcal{V})\left(\epsilon_k(x_k,a_k)\prod_{1\leqslant i\leqslant n}^{i\neq k}\phi_i(x_i,a_i)\right)\Big|_{\substack{x_i=x\\a_i=0}}}_{=0}\\ &= [\,\mathcal{V},a_k\cdot P_k\,]\left(\epsilon_k(x_k,a_k)\prod_{1\leqslant i\leqslant n}^{i\neq k}\phi_i(x_i,a_i)\right)\Big|_{\substack{x_i=x\\a_i=0}}\approx 0\,.\end{aligned} \tag{2.11}$$

This tells us, that, for an arbitrary vertex $\mathcal{L}_n$ to be gauge invariant under an arbitrary gauge transformation δ_k the vertex generating operator $\mathcal{V}$ has to satisfy:

$$[\,\mathcal{V},a_k\cdot P_k\,]=0\,, \tag{2.12}$$

where $k\in\{1,2,\ldots,n\}$.
Instead of working with the commutator, we can also express gauge variation as a linear first-order differential operator acting on $\mathcal{V}$.

To derive this operator, let us begin by calculating the commutator given in equation (2.12) for the variables s_{ij}, y_{ij} and z_{ij}.

$$\begin{aligned}
[\, s_{ij},\ a_k \cdot P_k \,] &= [\, P_i \cdot P_j,\ a_k \cdot P_k \,] \\
&= P_i \cdot P_j \,(a_k \cdot P_k) - a_k \cdot P_k \,(P_i \cdot P_j) \\
&= a_k \cdot P_k \,(P_i \cdot P_j) - a_k \cdot P_k \,(P_i \cdot P_j) = 0\,, \qquad (2.13)\\
[\, y_{ij},\ a_k \cdot P_k \,] &= [\, A_i \cdot P_j,\ a_k \cdot P_k \,] \\
&= A_i \cdot P_j \,(a_k \cdot P_k) - a_k \cdot P_k \,(A_i \cdot P_j) \\
&= A_i \,(a_k \cdot P_k) \cdot P_j - a_k \cdot P_k \,(A_i \cdot P_j) \\
&= \delta_{ik} P_k \cdot P_j + a_k \cdot P_k \,(A_i \cdot P_j) - a_k \cdot P_k \,(A_i \cdot P_j) \\
&= \delta_{ik} P_k \cdot P_j = \delta_{ik} s_{kj}\,, \qquad (2.14)\\
[\, z_{ij},\ a_k \cdot P_k \,] &= [\, A_i \cdot A_j,\ a_k \cdot P_k \,] \\
&= A_i \cdot A_j \,(a_k \cdot P_k) - a_k \cdot P_k \,(A_i \cdot A_j) \\
&= \delta_{jk} A_i \cdot P_k + \delta_{ik} A_j \cdot P_k = \delta_{jk} y_{ik} + \delta_{ik} y_{jk}\,. \qquad (2.15)
\end{aligned}$$

The commutator of s_{ij} vanishes, therefore, the differential operator must not differentiate with respect to s_{ij}. Next, we calculate the following:

$$\begin{aligned}
[\, y_{ij}^n,\ a_k \cdot P_k \,] &= [\, y_{ij}^{n-1} \cdot y_{ij},\ a_k \cdot P_k \,] \\
&= y_{ij}^{n-1} \,[\, y_{ij},\ a_k \cdot P_k \,] + [\, y_{ij}^{n-1},\ a_k \cdot P_k \,]\, y_{ij} \\
&= y_{ij}^{n-1} \,\delta_{ik}\, s_{kj} + [\, y_{ij}^{n-2}\, y_{ij},\ a_k \cdot P_k \,]\, y_{ij} \\
&= y_{ij}^{n-1} \,\delta_{ik}\, s_{kj} + y_{ij}^{n-2}\, \delta_{ik}\, s_{kj}\, y_{ij} + [\, y_{ij}^{n-2},\ a_k \cdot P_k \,]\, y_{ij}^2 \\
&= \sum_{m=0}^{n-1} y_{ij}^{n-1-m}\, \delta_{ik}\, s_{kj}\, y_{ij}^m = \sum_{m=0}^{n-1} y_{ij}^{n-1}\, \delta_{ik}\, s_{kj} \\
&= n \cdot y_{ij}^{n-1}\, \delta_{ik}\, s_{kj} = \left(\frac{\partial}{\partial y_{ij}} y_{ij}^n \right) \delta_{ik}\, s_{kj}\,, \qquad (2.16)\\
[\, z_{ij}^n,\ a_k \cdot P_k \,] &= \left(\frac{\partial}{\partial z_{ij}} z_{ij}^n \right) (\, \delta_{jk}\, y_{ik} + \delta_{ik}\, y_{jk} \,)\,. \qquad (2.17)
\end{aligned}$$

Using these expressions for $n = 1$, we can then write down the following linear first-order differential operator:

$$D_k = \sum_{j=1}^{n} \left(y_{jk} \frac{\partial}{\partial z_{kj}} + s_{kj} \frac{\partial}{\partial y_{kj}} \right) . \qquad (2.18)$$

Acting with D_k on $\mathcal{V}$ is equivalent to taking the commutator of $\mathcal{V}$ with $(a_k \cdot P_k)$. Therefore, from now on, we are testing gauge invariance of $\mathcal{V}$ under a field ϕ_k by checking if $\mathcal{V}$ gets annihilated by D_k:

$$[\mathcal{V}, a_k \cdot P_k] =: D_k \mathcal{V} = 0\,. \qquad (2.19)$$

Lastly, consider a vector $b = (P_1, \ldots, P_n, A_1 \ldots, A_n)$, such that we can build the following (2n x 2n)-matrix:

$$\mathcal{B} = (b_K \cdot b_L)\big|_{K,L \in (1,\ldots,2n)} = \begin{pmatrix} \mathcal{S} & \mathcal{Y}^T \\ \mathcal{Y} & \mathcal{Z} \end{pmatrix} , \tag{2.20}$$

where $\mathcal{Y} = (y_{ij})$ is a (n x n)-matrix, while $\mathcal{S} = (s_{ij})$ and $\mathcal{Z} = (z_{ij})$ are symmetric (n x n)-matrices [1].

This matrix is very convenient, since, among other things, it can be used to find gauge invariant expressions easily. Take for example a submatrix M of $\mathcal{B}$ with rows 1 and $(n+1)$ as well as two arbitrary columns 2 and 3:

$$M = \begin{pmatrix} s_{12} & s_{13} \\ y_{12} & y_{13} \end{pmatrix} . \tag{2.21}$$

Then the determinant of M is gauge invariant with respect to the field ϕ_1. This can be shown quickly. The determinant is given by:

$$det(M) = s_{12}\, y_{13} - y_{12}\, s_{13} . \tag{2.22}$$

By acting with the differential operator D_1 on this expression we find:

$$\begin{aligned} D_1\, det(M) &= \sum_{j=1}^{n} \left(y_{j1} \frac{\partial}{\partial z_{1j}} + s_{1j} \frac{\partial}{\partial y_{1j}} \right) (s_{12}\, y_{13} - y_{12}\, s_{13}) \\ &= s_{13}\, s_{12} - s_{12}\, s_{13} = 0 . \end{aligned} \tag{2.23}$$

More generally, the determinant of any submatrix M of $\mathcal{B}$ with the rows/columns k and $(n+k)$ is gauge invariant with respect to the field ϕ_k.

2.5 PARITY-ODD VERTICES

When introducing the vertex generating operator $\mathcal{V}$ in equation (2.3) to express independent vertices $\mathcal{L}_n$, we noted that this formula does not describe parity-odd vertices, meaning vertices that involve an epsilon tensor $\epsilon_{\mu_1 \ldots \mu_d}$.

To describe parity-odd vertices, we can again use equation (2.3) and simply replace $\mathcal{V}$ with:

$$\tilde{\mathcal{V}} = \sum Q_{l_1 \ldots l_d}\, \tilde{\mathcal{V}}^{l_1 \ldots l_d} . \tag{2.24}$$

The operator $\tilde{\mathcal{V}}^{l_1 \ldots l_d}$ depends on the same variables as $\mathcal{V}$ and contains the parity-even contractions. To obtain parity-odd vertices, we contract it with the operator Q, defined as:

$$Q_{l_1 \ldots l_d} = \epsilon_{\mu_1 \ldots \mu_d}\, b_{l_1}^{\mu_1} \ldots b_{l_d}^{\mu_d} , \tag{2.25}$$

for $l_k = 1, \ldots, 2n$.

We have already stated, that, when classifying independent vertices $\mathcal{L}_n$ we have to take several equivalence relations into account. Which ones, is depending on the order n of the vertex in relation to the dimension d of the space-time. In general, there are three different cases:

1. **The case $d \geqslant 2n-1$:** Due to the dimensions of the space-time being large, the only equivalence relations one has to consider are given by field redefinitions and operators that lead to a total derivative when acting on the Lagrangian.

2. **The case $d < n$:** The dimension is now smaller than the order n of the vertex. Therefore, besides the equivalence relations named in the previous case, one also has to think about equivalence due to over-antisymmetrizing of space-time indices. These dimension-dependent identities are known as Schouten identities.

3. **The case $2n-2 \geqslant d \geqslant n$:** This is an intermediate case, where one also has to allow for Schouten identities to appear, though, to a lesser degree.

Since we are only interested in studying the constraints of gauge invariance, we are working in the first case, where no Schouten identities appear.
This is important, since parity-odd vertices are also affected by relation of d and n. For $d \geqslant 2n-1$ we only have to consider two different cases:

1. **$d \geqslant 2n$:** If the space-time dimension is larger than or equal to twice the order of the vertex, we always find $Q_{l_1 \dots l_d} = 0$, since the indices of the vector b are only independent up to $(2n-1)$. Therefore, we find, that no parity-odd vertices can exist in this case.

2. **$d = 2n-1$:** As mentioned above, b only has $(2n-1)$ independent variables. Therefore, we find one possible parity-odd vertex for $d = 2n-1$. It is given by:

$$Q_{1\dots 2n-1} = \epsilon_{\mu_1 \dots \mu_{2n-1}} P_1^{\mu_1} \dots P_{n-1}^{\mu_{n-1}} A_1^{\mu_n} \dots A_n^{\mu_{2n-1}} \,. \tag{2.26}$$

2.6 APPROACH

At last, this concludes the preliminaries of this thesis.
Recall, that our aim is to study gauge invariance under a varying number of fields ϕ_i. We are trying to find all possible building blocks of the vertex generating operator $\mathcal{V}$, such that it satisfies equation (2.19). Additionally, we impose that it must be possible to write $\mathcal{V}$ as a polynomial in these building blocks.
Building blocks of a gauge invariant vertex have already been found in [1], however, in these variables the vertex generating operator $\mathcal{V}$ can contain inverse powers of the s_{ij}.
Therefore, the motivation of this thesis is to find new building blocks and prove, that any gauge invariant $\mathcal{V}$ can be written as a polynomial in these new variables.

The approach is as follows:
We start by imposing gauge invariance with respect to one field ϕ_i:

$$D_i \mathcal{V} \stackrel{!}{=} 0 \,. \tag{2.27}$$

After finding the necessary building blocks and proving that any $\mathcal{V}$ satisfying (2.27) can be written as a polynomial in them, we are going to impose gauge invariance with respect to two or more fields $\phi_i, \phi_j, \ldots, \phi_k$:

$$D_i\mathcal{V} = D_j\mathcal{V} = \ldots = D_k\mathcal{V} \stackrel{!}{=} 0\,. \tag{2.28}$$

Again it has to be proven that $\mathcal{V}$ can be written as a polynomial. Lastly, we also strive to find a solution for gauge invariance under all fields $\phi_1, \phi_2, \ldots, \phi_n$.

Two different ways on how to achieve this come to mind:

1. Take the straight forward approach and assume $\mathcal{V}$ to depend on the variables s_{ij}, y_{ij} and z_{ij} as discussed in the beginning, such that the differential operators D_k are given by equation (2.18).

2. Recall that s_{ij}, y_{ij} and z_{ij} are built using the variables A_i and P_i. Therefore, we can also take $\mathcal{V}$, as well as D_k, to directly depend on A_i and P_i.

Both of these approaches have different advantages and disadvantages, which are going to be covered in their respective sections.

3 FIRST APPROACH: $\mathcal{V}(s_{ij}, y_{ij}, z_{ij})$

Let both the vertex generating operator $\mathcal{V}$ and the differential operators D_k depend on the variables s_{ij}, y_{ij} and z_{ij}.
We assume, that all gauge invariant building blocks of $\mathcal{V}$ can be expressed as determinants of submatrices of the matrix $\mathcal{B}$. Hence, we assume that it is possible to find all the needed building blocks studying submatrices of $\mathcal{B}$.
Therefore, introduce the following notation:
Let M be a submatrix of $\mathcal{B}$ with the rows 1 and $(n+1)$ and the columns i and $(n+j)$. We are then expressing the determinant of M as:

$$det(M) = [\,1\,\bar{1}, i\,\bar{j}\,]. \tag{3.1}$$

The entries before the comma tell us the rows of M and the entries after the comma tell us its columns. Additionally, we use the bars to shorten $(n+1)$ to $\bar{1}$.

3.1 GAUGE VARYING ONE FIELD

We start by imposing gauge invariance under a single gauge transformation. Hence, without loss of generality, we can choose the transformation δ_1 under which ϕ_1 transforms at leading order. Then, the vertex generating operator $\mathcal{V}$, which is defined as a polynomial:

$$\mathcal{V} := P(s_{ij}, y_{ij}, z_{ij}), \tag{3.2}$$

has to satisfy the following equation:

$$D_1\mathcal{V} = \sum_{j=1}^{n}\left(y_{j1}\frac{\partial}{\partial z_{1j}} + s_{1j}\frac{\partial}{\partial y_{1j}}\right) P(s_{ij}, y_{ij}, z_{ij}) \overset{!}{=} 0. \tag{3.3}$$

We start by solving this differential equation to get a better idea on which building blocks we might need. Using the method of characteristics we assume:

$$P(\sigma) = P(s_{kl}(\sigma), y_{kl}(\sigma), z_{kl}(\sigma)) \tag{3.4}$$

with $k \neq l$, since we imposed the Fierz equations as well as equivalence due to field redefinitions as given in (2.6).
Hence, it follows:

$$\frac{dP(\sigma)}{d\sigma} = \frac{\partial P}{\partial s_{kl}}\frac{ds_{kl}}{d\sigma} + \frac{\partial P}{\partial y_{kl}}\frac{dy_{kl}}{d\sigma} + \frac{\partial P}{\partial z_{kl}}\frac{dz_{kl}}{d\sigma}. \tag{3.5}$$

Comparing this to (3.3), we find:

- $\frac{dy_{1j}}{d\sigma} = s_{1j}$ and $\frac{dz_{1j}}{d\sigma} = y_{j1}$ for $j \in \{2, 3, \ldots, n\}$,
- $\frac{ds_{kj}}{d\sigma} = 0$, $\frac{dy_{ij}}{d\sigma} = 0$, $\frac{dz_{ij}}{d\sigma} = 0$ for $k, j \in \{1, \ldots, n\}$ and $i \in \{2, \ldots, n\}$,
- $\frac{dP}{d\sigma} = 0$.

Due to the last point, we know that P is a function that does not directly depend on σ. Therefore, it also does not directly depend on y_{1j} and z_{1j}, since:

$$y_{1j} = s_{1j} \cdot \sigma + d_{1j} \tag{3.6}$$
$$z_{1j} = y_{j1} \cdot \sigma + k_{1j}, \tag{3.7}$$

where d_{1j}, k_{1j} are constants.
However, we can combine these expressions in three different ways to get rid of the σ dependence.

1. Combining z_{1j} and z_{1k}
Express σ in (3.7) explicitly and insert it into the expression of z_{1k}:

$$\begin{aligned} z_{1k} &= y_{k1} \cdot \sigma + k_{1k} \\ &= y_{k1} \cdot \frac{z_{1j} - k_{1j}}{y_{j1}} + k_{1k} \end{aligned} \tag{3.8}$$

By rewriting this equation we find:

$$y_{k1}k_{1j} - y_{j1}k_{1k} = y_{k1}z_{1j} - z_{1k}y_{j1} =: \alpha_{jk}. \tag{3.9}$$

2. Combining z_{1j} and y_{1k}
We proceed analogously to before to obtain:

$$y_{1k} = s_{1k} \cdot \sigma + d_{1k} \tag{3.10}$$
$$= s_{1k} \cdot \frac{z_{1j} - k_{1j}}{y_{j1}} + d_{1k}, \tag{3.11}$$

such that we find:

$$s_{1k}k_{1j} - y_{j1}d_{1k} = z_{1j}s_{1k} - y_{j1}y_{1k} =: \beta_{jk}. \tag{3.12}$$

3. Combining y_{1j} and y_{1k}
The last possibility is to combine two y_{1i}'s. In this case, we have:

$$\begin{aligned} y_{1k} &= s_{1k}\cdot\sigma + d_{1k} && (3.13)\\ &= s_{1k}\cdot\frac{y_{1j}-d_{1j}}{s_{1j}} + d_{1k}. && (3.14)\end{aligned}$$

Then, we end up with:

$$s_{1j}d_{1k} - s_{1k}d_{1j} = y_{1k}s_{1j} - s_{1k}y_{1j} =: \gamma_{jk}. \tag{3.15}$$

Therefore, for P to satisfy equation (3.3), it has to be a function in the following variables:

$$P = f\left(\alpha_{ij}\big|_{2\leqslant i,j\leqslant n}, \beta_{ij}\big|_{2\leqslant i,j\leqslant n}, \gamma_{ij}\big|_{2\leqslant i,j\leqslant n}, s_{ij}\big|_{1\leqslant i<j\leqslant n}, y_{ij}\big|_{\substack{2\leqslant i\leqslant n,\\ 1\leqslant j\leqslant n}}, z_{ij}\big|_{2\leqslant i<j\leqslant n}\right). \tag{3.16}$$

For clarity, recall that these new variables are given by:

$$\begin{aligned}\alpha_{ij} &= z_{1i}y_{j1} - y_{i1}z_{1j},\\ \beta_{ij} &= z_{1i}s_{1j} - y_{i1}y_{1j},\\ \gamma_{ij} &= s_{1i}y_{1j} - y_{1i}s_{1j}.\end{aligned} \tag{3.17}$$

Note, that these building blocks are determinants of submatrices of $\mathcal{B}$. In this case, they are 2x2 submatrices, such that:

$$\begin{aligned}\alpha_{ij} &= [\,1\,\bar{1}, \bar{j}\,\bar{i}\,],\\ \beta_{ij} &= [\,1\,\bar{1}, j\,\bar{i}\,],\\ \gamma_{ij} &= [\,1\,\bar{1}, i\,j\,].\end{aligned} \tag{3.18}$$

These are all the building blocks we need to be able to express P as a polynomial. Hence, we are going to prove the following proposition[1]:

Proposition 3.1.1. *Let $D = \sum_{i=1}^{n} u_i \frac{\partial}{\partial v_i}$. Then every polynomial $P(u_i, v_i)$ that satisfies $DP = 0$ can be written as a polynomial in the variables u_i and $c_{ij} = u_i v_j - u_j v_i$.*

Proof. The differential operator D does not change the degree of the polynomial, meaning that we can consider P to be a homogeneous polynomial of degree m.
We prove by induction over m.
The case m=0: For this case the problem is trivial.

Assume that every polynomial of degree smaller or equal m for which $DP = 0$ can be written as a polynomial in c_{ij} and u_i.

The case m+1: Let P_{m+1} be a homogeneous polynomial of degree $m+1$ that satisfies $DP_{m+1} = 0$.

1 This proof is based on arguments by Stefan Fredenhagen.

Define: $p_i = \frac{\partial}{\partial v_i} P_{m+1}$.
p_i is a polynomial of degree m that satisfies:

1. $D p_i = 0$,
2. $\frac{\partial}{\partial v_j} p_i = \frac{\partial}{\partial v_i} p_j$,
3. $\sum_{i=1}^n u_i p_i = 0$.

Next, we define two operators N_u and N_v that count the degrees in u and v:

$$N_u = \sum_{i=1}^n u_i \frac{\partial}{\partial u_i} \,, \tag{3.19}$$

$$N_v = \sum_{i=1}^n v_i \frac{\partial}{\partial v_i} \,. \tag{3.20}$$

Note, that D lowers the degree in v by one, therefore, $N_v D = D(N_v - 1)$.
We consider the object $P' = N_v^{-1}(\sum_{i=1}^n v_i p_i)$ and act on it with $\frac{\partial}{\partial v_j}$, such that:

$$\begin{aligned}
\frac{\partial}{\partial v_j} P' &= \frac{\partial}{\partial v_j} N_v^{-1} \left(\sum_{i=1}^n v_i \, p_i \right) \\
&= (N_v + 1)^{-1} \frac{\partial}{\partial v_j} \left(\sum_{i=1}^n v_i \, p_i \right) \\
&= (N_v + 1)^{-1} \cdot \left(\sum_{i=1}^n \delta_{ij} p_i + \sum_{i=1}^n v_i \underbrace{\frac{\partial}{\partial v_j} \, p_i}_{= \frac{\partial}{\partial v_i} p_j}\right) \\
&= (N_v + 1)^{-1} \cdot \left(p_j + \sum_{i=1}^n v_i \frac{\partial}{\partial v_i} \, p_j\right) \\
&= (N_v + 1)^{-1} \cdot (p_j + N_v \, p_j) \\
&= p_j = \frac{\partial}{\partial v_j} P_{m+1} \,.
\end{aligned} \tag{3.21}$$

Consequently, we have:

$$\frac{\partial}{\partial v_j} (P_{m+1} - P') = 0 \,. \tag{3.22}$$

This tells us that P_{m+1} can be written as:

$$P_{m+1} = P' + Q(u_i) \,, \tag{3.23}$$

where $Q(u_i)$ is a polynomial in u_i.

Hence, if P' is a polynomial in the desired variables, so is P_{m+1}. To show that P' satisfies our proposition, we start by applying $\frac{\partial}{\partial u_j}$ to $\sum_{i=1}^n u_i p_i$:

$$
\begin{aligned}
0 &= \frac{\partial}{\partial u_j}\left(\sum_i u_i\, p_i\right) \\
&= \sum_i \delta_{ij} p_i + \sum_i u_i \frac{\partial}{\partial u_j}\, p_i \\
&= p_j + \sum_i u_i \frac{\partial}{\partial u_j}\, p_i + \sum_i u_i \frac{\partial}{\partial u_i}\, p_j - \sum_i u_i \frac{\partial}{\partial u_i}\, p_j \\
&= p_j + N_u\, p_j + \sum_i u_i \left(\frac{\partial}{\partial u_j}\, p_i - \frac{\partial}{\partial u_i}\, p_j\right).
\end{aligned} \tag{3.24}
$$

We can rewrite this to obtain:

$$
p_j = -(N_u + 1)^{-1} \cdot \sum_i u_i \left(\frac{\partial}{\partial u_j}\, p_i - \frac{\partial}{\partial u_i}\, p_j\right), \tag{3.25}
$$

which we can insert into the definition of P':

$$
\begin{aligned}
P' &= N_v^{-1}\left(\sum_{i=1}^n v_i\, p_i\right) \\
&= -N_v^{-1}(N_u+1)^{-1} \sum_{i=1}^n v_i \sum\nolimits_j u_j \left(\frac{\partial}{\partial u_i}\, p_j - \frac{\partial}{\partial u_j}\, p_i\right) \\
&= -N_v^{-1}(N_u+1)^{-1} \sum\nolimits_{i,j} \frac{1}{2} \underbrace{(v_i\, u_j - v_j\, u_i)}_{c_{ji}} \underbrace{\left(\frac{\partial}{\partial u_i}\, p_j - \frac{\partial}{\partial u_j}\, p_i\right)}_{(*)}.
\end{aligned} \tag{3.26}
$$

Lastly, if we act with D on $(*)$, we find:

$$
\begin{aligned}
D\left(\frac{\partial}{\partial u_i}\, p_j - \frac{\partial}{\partial u_j}\, p_i\right) &= \sum_k u_k \frac{\partial}{\partial v_k}\left(\frac{\partial}{\partial u_i}\, p_j - \frac{\partial}{\partial u_j}\, p_i\right) \\
&= \frac{\partial}{\partial u_i}\left(\sum_k u_k \frac{\partial}{\partial v_k} p_j\right) - \sum_k \delta_{ki} \frac{\partial}{\partial v_k} p_j \\
&\quad - \frac{\partial}{\partial u_j}\left(\sum_k u_k \frac{\partial}{\partial v_k} p_i\right) + \sum_k \delta_{kj} \frac{\partial}{\partial v_k} p_i \\
&= \frac{\partial}{\partial u_i} \underbrace{D\, p_j}_{=0} - \frac{\partial}{\partial u_j} \underbrace{D\, p_i}_{=0} - \underbrace{\left(\frac{\partial}{\partial v_i} p_j - \frac{\partial}{\partial v_j} p_i\right)}_{=0} = 0.
\end{aligned} \tag{3.27}
$$

$(*)$ is a polynomial of degree $(m-1)$ that gets annihilated by D, therefore, due to our assumption, it can be written as a polynomial in c_{ij} and u_i. Consequently, also P' and P_{m+1} can be written as polynomials in these variables. $\square$

3.2 GAUGE VARYING TWO FIELDS

We now impose gauge invariance of $\mathcal{V}$ with respect to two fields ϕ_1 and ϕ_2. Hence, $\mathcal{V}$ has to satisfy the condition:

$$D_1\mathcal{V} = D_2\mathcal{V} = 0\,. \tag{3.28}$$

In the prior section we have already solved $D_1\mathcal{V} = 0$ and know which building blocks arise from that. So, we start by additionally solving:

$$D_2\mathcal{V} = \sum_{j=1}^{n} \left(y_{j2}\frac{\partial}{\partial z_{2j}} + s_{2j}\frac{\partial}{\partial y_{2j}} \right) P(s_{ij}, y_{ij}, z_{ij}) = 0\,. \tag{3.29}$$

Using the method of characteristics it follows that a polynomial $P(s_{ij}, y_{ij}, z_{ij})$ has to be a function:

$$P = f(\mu_{ij}\big|_{\substack{1\leqslant i,j\leqslant n,\\ i,j\neq 2}}, \nu_{ij}\big|_{\substack{1\leqslant i,j\leqslant n,\\ i,j\neq 2}}, \rho_{ij}\big|_{\substack{1\leqslant i,j\leqslant n,\\ i,j\neq 2}}, s_{ij}\big|_{1\leqslant i<j\leqslant n}, y_{ij}\big|_{\substack{1\leqslant i,j\leqslant n,\\ i\neq 2}}, z_{ij}\big|_{\substack{1\leqslant i<j\leqslant n\\ i,j\neq 2}})\,, \tag{3.30}$$

to satisfy equation (3.29). The new variables are defined as:

$$\begin{aligned} \mu_{ij} &= z_{2i}y_{j2} - y_{i2}z_{2j} = [\,2\,\bar{2}, \bar{j}\,\bar{i}\,]\,,\\ \nu_{ij} &= z_{2i}s_{2j} - y_{i2}y_{2j} = [\,2\,\bar{2}, j\,\bar{i}\,]\,,\\ \rho_{ij} &= s_{2i}y_{2j} - y_{2i}s_{2j} = [\,2\,\bar{2}, i\,j\,]\,. \end{aligned} \tag{3.31}$$

Certain variables satisfy (3.28), where we imposed gauge invariance under both fields. An example would be:

$$\alpha_{34} = z_{13}\,y_{41} - y_{31}\,z_{14}. \tag{3.32}$$

We know that α_{34} satisfies $D_1\mathcal{V} = 0$ since it is part of its solution. Additionally, all its components z_{13}, z_{14}, y_{31} and y_{41}, satisfy $D_2\mathcal{V} = 0$ and, therefore, the same is true for α_{34}. Hence, $\mathcal{V}$ may depend directly on these kinds of variables after we imposed gauge invariance under δ_1 and δ_2.

After checking all possible variables, we find, that $\mathcal{V}$ depends on the following building blocks:

$$\begin{aligned} 1\text{x}1: \quad & s_{ij}\big|_{1\leqslant i<j\leqslant n}, \quad y_{ij}\big|_{\substack{3\leqslant i\leqslant n,\\ 1\leqslant j\leqslant n}}, \quad z_{ij}\big|_{3\leqslant i<j\leqslant n}, \\ 2\text{x}2: \quad & \alpha_{ij}\big|_{3\leqslant i<j\leqslant n}, \quad \beta_{ij}\big|_{\substack{3\leqslant i\leqslant n,\\ 2\leqslant j\leqslant n}}, \quad \gamma_{ij}\big|_{2\leqslant i<j\leqslant n}, \quad \beta_{22}, \\ & \mu_{ij}\big|_{3\leqslant i<j\leqslant n}, \quad \nu_{ij}\big|_{\substack{3\leqslant i\leqslant n,\\ 1\leqslant j\leqslant n\\ j\neq 2}}, \quad \rho_{ij}\big|_{\substack{1\leqslant i<j\leqslant n.\\ i,j\neq 2}} \end{aligned} \tag{3.33}$$

Note, that (n x n) describes the size of the submatrix the variables stem from. In this case we have variables that can be written as determinants of (1x1)- and (2x2)-submatrices of $\mathcal{B}$.

Therefore, we discarded the variables:

$$\alpha_{ij}\Big|_{\substack{i=2\\3\leqslant j\leqslant n}}, \quad \beta_{ij}\Big|_{\substack{i=2\\3\leqslant j\leqslant n}}, \quad \mu_{ij}\Big|_{\substack{i=1\\3\leqslant j\leqslant n}}, \quad \nu_{ij}\Big|_{\substack{i=1\\3\leqslant j\leqslant n}}. \tag{3.34}$$

However, we do not yet know if the building blocks in (3.33) are all we need. We have to check, if all the gauge invariant building blocks of size (3x3) or larger can be expressed using these variables.

Start by calculating the determinants of all possible (3x3)-submatrices that give expressions that are gauge invariant with respect to both ϕ_1 and ϕ_2:

$$A_{ij}\Big|_{\substack{2\leqslant i\leqslant n\\1\leqslant j\leqslant n\\j\neq 2}} =: [1\,i\,\bar{1}, 2\,\bar{2}\,\bar{j}] = \color{red}{y_{2i}}\,\beta_{j2} - s_{2i}\,\color{red}{\alpha_{j2}} - y_{ji}\,\beta_{22}, \tag{3.35}$$

$$B_{ij}\Big|_{\substack{2\leqslant i\leqslant n\\1\leqslant j\leqslant n\\j\neq 2}} =: [1\,\bar{1}\,\bar{i}, 2\,\bar{2}\,\bar{j}] = z_{ij}\,\beta_{22} - \color{red}{z_{i2}}\,\beta_{j2} + y_{i2}\,\color{red}{\alpha_{j2}}, \tag{3.36}$$

$$C_{ij}\Big|_{\substack{2\leqslant i\leqslant n\\1\leqslant j\leqslant n\\j\neq 2}} =: [1\,i\,\bar{1}, 2\,j\,\bar{2}] = s_{ij}\,\beta_{22} + \color{red}{y_{2i}}\,\gamma_{j2} - s_{i2}\,\color{red}{\beta_{2j}}, \tag{3.37}$$

$$D_{ij}\Big|_{\substack{2\leqslant i\leqslant n\\1\leqslant j\leqslant n\\j\neq 2}} =: [1\,\bar{1}\,\bar{i}, 2\,j\,\bar{2}] = y_{i2}\,\color{red}{\beta_{2j}} - y_{ij}\,\beta_{22} - \color{red}{z_{i2}}\,\gamma_{j2}. \tag{3.38}$$

Coloured red are all the variables that $\mathcal{V}$ must not directly depend on. Hence, these (3x3) building blocks cannot be expressed by the variables of $\mathcal{V}$ given in (3.33). Therefore, we have found, that $\mathcal{V}$ additionally has to directly depend on the variables A_{ij}, B_{ij}, C_{ij} and D_{ij}.

We continue by calculating the possible (4x4)-submatrices, to check if we have to add further building blocks:

$$[1\,i\,j\,\bar{1}, 2\,k\,l\,\bar{2}] = s_{jl}\,C_{ik} - s_{jk}\,C_{il} - s_{il}\,C_{jk} + s_{ik}\,C_{jl} + s_{il}\,s_{jk}\,\beta_{22} - s_{ik}\,s_{jl}\,\beta_{22} + \rho_{ij}\,\gamma_{kl},$$

$$[1\,i\,j\,\bar{1}, 2\,k\,\bar{2}\,\bar{l}] = y_{li}\,C_{jk} - y_{lj}\,C_{ik} + s_{ik}\,A_{jl} - s_{jk}\,A_{il} + s_{ik}\,y_{lj}\,\beta_{22} - s_{jk}\,y_{li}\,\beta_{22} - \beta_{lk}\,\rho_{ij},$$

$$[1\,i\,j\,\bar{1}, 2\,\bar{2}\,\bar{k}\,\bar{l}] = -y_{lj}\,A_{ik} + y_{kj}\,A_{il} + y_{li}\,A_{jk} + y_{ki}\,A_{jl} + y_{li}\,y_{kj}\,\beta_{22} + y_{ki}\,y_{lj}\,\beta_{22} + \alpha_{lk}\,\rho_{ij},$$

$$[1\,i\,\bar{1}\,\bar{j}, 2\,k\,l\,\bar{2}] = y_{jl}\,C_{ik} - y_{jk}\,C_{il} + s_{il}\,D_{jk} - s_{ik}\,D_{jl} + y_{jk}\,s_{il}\,\beta_{22} - s_{ik}\,y_{jl}\,\beta_{22} + \gamma_{kl}\,\nu_{ji},$$

$$[1\,i\,\bar{1}\,\bar{j}, 2\,k\,\bar{2}\,\bar{l}] = -y_{jk}\,A_{il} - z_{jl}\,C_{ik} - s_{ik}\,B_{jl} - y_{li}\,D_{jk} + s_{ik}\,z_{jl}\beta_{22} - y_{li}\,y_{jk}\,\beta_{22} - \beta_{lk}\,\nu_{ji},$$

$$[1\,i\,\bar{1}\,\bar{j}, 2\,\bar{2}\,\bar{k}\,\bar{l}] = z_{jk}\,A_{il} - z_{jl}\,A_{ik} + y_{ki}\,B_{jl} - y_{li}\,B_{jk} + z_{jk}\,y_{li}\,\beta_{22} - y_{ki}\,z_{jl}\,\beta_{22} + \alpha_{lk}\,\nu_{ji},$$

$$[1\,\bar{1}\,\bar{i}\,\bar{j}, 2\,k\,l\,\bar{2}] = y_{il}\,D_{jk} - y_{ik}\,D_{jl} - y_{jl}\,D_{ik} + y_{jk}\,D_{il} - y_{jl}\,y_{ik}\,\beta_{22} + y_{jk}\,y_{il}\,\beta_{22} + \gamma_{kl}\,\mu_{ji},$$

$$[1\,\bar{1}\,\bar{i}\,\bar{j}, 2\,k\,\bar{2}\,\bar{l}] = z_{jl}\,D_{ik} - z_{il}\,D_{jk} + y_{jk}\,B_{il} - y_{ik}\,B_{jl} - y_{jk}\,z_{il}\,\beta_{22} + y_{ik}\,z_{jl}\,\beta_{22} - \beta_{lk}\,\mu_{ji},$$

$$[1\,\bar{1}\,\bar{i}\,\bar{j}, 2\,\bar{2}\,\bar{k}\,\bar{l}] = z_{jl}\,B_{ik} - z_{jk}\,B_{il} - z_{il}\,B_{jk} + z_{ik}\,B_{jl} + z_{il}\,z_{jk}\,\beta_{22} - z_{ik}\,z_{jl}\,\beta_{22} + \alpha_{lk}\,\mu_{ji},$$

where $i, j, k, l \in \{3, \ldots, n\}$.

All these determinants can be expressed using the building blocks of $\mathcal{V}$ as seen above. Therefore, we assume, that these (1x1), (2x2) and (3x3) building blocks suffice to express all the other gauge invariant expressions.
Furthermore, we show that any polynomial in s_{ij}, y_{ij} and z_{ij} that is annihilated by both D_1 and D_2, can be written as a polynomial in these building blocks.

Proposition 3.2.1. *Let* $D_k = \sum_{j=1}^n \left(y_{jk} \frac{\partial}{\partial z_{kj}} + s_{kj} \frac{\partial}{\partial y_{kj}} \right)$ *for* $k = 1, 2$. *Every polynomial* $P(s_{ij}, y_{ij}, z_{ij})$ *that satisfies:*

$$D_1 P = D_2 P = 0, \tag{3.39}$$

can be written as a polynomial in the following variables:

$$\begin{aligned}
\textit{1x1}: \quad & s_{ij}\big|_{1\leqslant i<j\leqslant n}\,, \quad y_{ij}\big|_{\substack{3\leqslant i\leqslant n\\ 1\leqslant j\leqslant n}}\,, \quad z_{ij}\big|_{3\leqslant i<j\leqslant n}\,, \\
\textit{2x2}: \quad & \alpha_{ij}\big|_{3\leqslant i<j\leqslant n}\,, \quad \beta_{ij}\big|_{\substack{3\leqslant i\leqslant n\\ 2\leqslant j\leqslant n}}\,, \quad \gamma_{ij}\big|_{2\leqslant i<j\leqslant n}\,, \quad \beta_{22}\,, \\
& \mu_{ij}\big|_{3\leqslant i<j\leqslant n}\,, \quad \nu_{ij}\big|_{\substack{3\leqslant i\leqslant n\\ 1\leqslant j\leqslant n\\ j\neq 2}}\,, \quad \rho_{ij}\big|_{\substack{1\leqslant i<j\leqslant n\\ i,j\neq 2}}\,, \\
\textit{3x3}: \quad & A_{ij}\big|_{\substack{2\leqslant i\leqslant n\\ 1\leqslant j\leqslant n\\ j\neq 2}}\,, \quad B_{ij}\big|_{\substack{2\leqslant i\leqslant n\\ 1\leqslant j\leqslant n\\ j\neq 2}}\,, \quad C_{ij}\big|_{\substack{2\leqslant i\leqslant n\\ 1\leqslant j\leqslant n\\ j\neq 2}}\,, \quad D_{ij}\big|_{\substack{2\leqslant i\leqslant n\\ 1\leqslant j\leqslant n\\ j\neq 2}}\,.
\end{aligned} \tag{3.40}$$

Proof. Assume that $P(s_{ij}, y_{ij}, z_{ij})$ gets annihilated by D_1. Then P can be written as a polynomial:

$$P = \tilde{P}(\alpha_{ij}\big|_{2\leqslant i,j\leqslant n}, \beta_{ij}\big|_{2\leqslant i,j\leqslant n}, \gamma_{ij}\big|_{2\leqslant i,j\leqslant n}, s_{ij}\big|_{1\leqslant i<j\leqslant n}, y_{ij}\big|_{\substack{2\leqslant i\leqslant n\\ 1\leqslant j\leqslant n}}, z_{ij}\big|_{2\leqslant i<j\leqslant n})\,. \tag{3.41}$$

Therefore, it suffices to proof that any polynomial $\tilde{P}$ that gets annihilated by D_2 can be written as a polynomial in the variables given in (3.40).
To this end, we rewrite the differential operator:

$$D_2 = \sum_{j=1}^{n} \left(y_{j2} \frac{\partial}{\partial z_{2j}} + s_{2j} \frac{\partial}{\partial y_{2j}} \right), \tag{3.42}$$

such that it directly depends on the variables of $\tilde{P}$. We find:

$$\begin{aligned}
D_2 &= \underbrace{\sum_{j=1}^{n} s_{2j} \frac{\partial}{\partial y_{2j}}}_{=:\, D_y} + \underbrace{\sum_{j=2}^{n} \gamma_{j2} \frac{\partial}{\partial \beta_{2j}}}_{=:\, D_\beta} + \underbrace{\sum_{j=3}^{n} y_{j2} \frac{\partial}{\partial z_{2j}}}_{=:\, D_z} \underbrace{- \sum_{j=3}^{n} \beta_{j2} \frac{\partial}{\partial \alpha_{2j}}}_{=:\, D_\alpha} && (3.43)\\
&= D_y + D_\beta + D_z + D_\alpha\,. && (3.44)
\end{aligned}$$

This way, D_2 is given by a sum of differential operators of the form:

$$D_v = \sum_i u_i \frac{\partial}{\partial v_i}, . \tag{3.45}$$

We have already proven, that a polynomial $P'(u_i, v_i)$ that satisfies $D_v P' = 0$ can be written as a polynomial in u_i and $c_{ij} = u_i v_j - u_j v_i$.
Hence, it suffices to show that all the possible combinations of the form c_{ij} that arise from D_y, D_z, D_α and D_β can be expressed using only the variables given in (3.40) to prove proposition 3.2.1:

$$
\begin{aligned}
s_{2i}\,y_{2j} - s_{2j}\,y_{2i} &= \rho_{ij}\big|_{\substack{1\leqslant i<j\leqslant n\\ i,j\neq 2}}\,, && (3.46)\\
y_{i2}\,z_{2j} - y_{j2}\,z_{2i} &= \mu_{ji}\big|_{3\leqslant i<j\leqslant n}\,, && (3.47)\\
\gamma_{i2}\,\beta_{2j} - \gamma_{j2}\,\beta_{2i} &= (s_{1i}\,y_{12} - y_{1i}\,s_{12})(z_{12}\,s_{1j} - y_{21}\,y_{1j})\\
&\quad - (s_{1j}\,y_{12} - y_{1j}\,s_{12})(z_{12}\,s_{1i} - y_{21}\,y_{1i})\\
&= (z_{12}\,s_{12} - y_{21}\,y_{12})(s_{1i}\,y_{1j} - y_{1i}\,s_{1j})\\
&= \beta_{22}\,\gamma_{ij}\big|_{2\leqslant i<j\leqslant n}\,, && (3.48)\\
-\beta_{i2}\,\alpha_{2j} + \beta_{j2}\,\alpha_{2i} &= -(z_{1i}\,s_{12} - y_{i1}\,y_{12})(z_{12}\,y_{j1} - y_{21}\,z_{1j})\\
&\quad + (z_{1j}\,s_{12} - y_{j1}\,y_{12})(z_{12}\,y_{i1} - y_{21}\,z_{1i})\\
&= -(z_{12}\,s_{12} - y_{21}\,y_{12})(z_{1i}\,y_{j1} - y_{i1}\,z_{1j})\\
&= \beta_{22}\,\alpha_{ji}\big|_{3\leqslant i<j\leqslant n}\,, && (3.49)\\
s_{2i}\,\beta_{2j} - \gamma_{j2}\,y_{2i} &= -\left(C_{ij} - s_{ij}\,\beta_{22}\right)\big|_{\substack{1\leqslant i\leqslant n\\ 2\leqslant j\leqslant n}}\,, && (3.50)\\
s_{2i}\,z_{2j} - y_{j2}\,y_{2i} &= \nu_{ji}\big|_{\substack{3\leqslant j\leqslant n\\ 1\leqslant i\leqslant n\\ i\neq 2}}\,, && (3.51)\\
s_{2i}\,\alpha_{2j} + \beta_{j2}\,y_{2i} &= (A_{ij} + y_{ji}\,\beta_{22})\big|_{\substack{1\leqslant i\leqslant n\\ 3\leqslant j\leqslant n}}\,, && (3.52)\\
\gamma_{i2}\,z_{2j} - y_{j2}\beta_{2i} &= -(D_{ji} + y_{ji}\,\beta_{22})\big|_{\substack{2\leqslant i\leqslant n\\ 3\leqslant j\leqslant n}}\,, && (3.53)\\
\gamma_{i2}\,\alpha_{2j} + \beta_{j2}\,\beta_{2i} &= (s_{1i}\,y_{12} - y_{1i}\,s_{12})(z_{12}\,y_{j1} - y_{21}\,z_{1j})\\
&\quad + (z_{1j}\,s_{12} - y_{j1}\,y_{12})(z_{12}\,s_{1i} - y_{21}\,y_{1i})\\
&= (z_{12}\,s_{12} - y_{21}\,y_{12})(z_{1j}\,s_{1i} - y_{j1}\,y_{1i})\\
&= \beta_{22}\,\beta_{ji}\big|_{\substack{2\leqslant i\leqslant n\\ 3\leqslant j\leqslant n}}\,, && (3.54)\\
y_{i2}\,\alpha_{2j} + \beta_{j2}\,z_{2i} &= -(B_{ij} - z_{ij}\,\beta_{22})\big|_{3\leqslant i,j\leqslant n}\,. && (3.55)
\end{aligned}
$$

We have found that all possible combinations can be expressed using the variables given in (3.40). Therefore, $\tilde{P}$ can be written as a polynomial in these variables and, furthermore, also any polynomial $P(s_{ij}, y_{ij}, z_{ij})$ that gets annihilated by D_1 and D_2. □

3.3 GAUGE VARYING THREE FIELDS

Let us now impose gauge invariance of $\mathcal{V}$ with respect to the three fields ϕ_1, ϕ_2 and ϕ_3, meaning $\mathcal{V}$ has to satisfy:

$$D_1\mathcal{V} = D_2\mathcal{V} = D_3\mathcal{V} = 0\,. \tag{3.56}$$

Introduce the following notation:

$$\alpha_{ij,k} = [\,k\,\bar{k}, \bar{j}\,\bar{i}\,]\,, \tag{3.57}$$
$$\beta_{ij,k} = [\,k\,\bar{k}, j\,\bar{i}\,]\,, \tag{3.58}$$
$$\gamma_{ij,k} = [\,k\,\bar{k}, i\,j\,]\,. \tag{3.59}$$

This simplifies the variables $\mathcal{V}$ depends on, since, e.g. $\alpha_{ij} = \alpha_{ij,1}$ and $\mu_{ij} = \alpha_{ij,2}$ are now described by the same variable.
In this case, we wish to find a generalization to be able to also solve the case of gauge invariance under an arbitrary number of gauge transformations δ_i.

Start by assuming that $\mathcal{V}$ satisfies $D_1\mathcal{V} = D_2\mathcal{V} = 0$. Then, $\mathcal{V}$ can be written as a polynomial in the variables:

$$\begin{aligned}
&1\text{x}1: && s_{ij}\big|_{1\leqslant i<j\leqslant n}\,, \quad y_{ij}\big|_{\substack{3\leqslant i\leqslant n\\ 1\leqslant j\leqslant n}}\,, \quad z_{ij}\big|_{3\leqslant i<j\leqslant n}\,,\\
&2\text{x}2: && \alpha_{ij,k}\big|_{\substack{3\leqslant i<j\leqslant n\\ 1\leqslant k\leqslant 2}}\,, \quad \beta_{ij,k}\big|_{\substack{3\leqslant i\leqslant n\\ 1\leqslant j\leqslant n\\ 1\leqslant k\leqslant 2\\ j\neq k}}\,, \quad \gamma_{ij,k}\big|_{\substack{1\leqslant i<j\leqslant n\\ 1\leqslant k\leqslant 2\\ i,j\neq k}}\,, \quad \beta_{22,1} = \beta_{11,2}\,,\\
&3\text{x}3: && A_{ij}\big|_{\substack{2\leqslant i\leqslant n\\ 1\leqslant j\leqslant n\\ j\neq 2}}\,, \quad B_{ij}\big|_{\substack{2\leqslant i\leqslant n\\ 1\leqslant j\leqslant n\\ j\neq 2}}\,, \quad C_{ij}\big|_{\substack{2\leqslant i\leqslant n\\ 1\leqslant j\leqslant n\\ j\neq 2}}\,, \quad D_{ij}\big|_{\substack{2\leqslant i\leqslant n\\ 1\leqslant j\leqslant n\\ j\neq 2}}\,.
\end{aligned} \tag{3.60}$$

By solving $D_3\,P(s_{ij}, y_{ij}, z_{ij}) = 0$, we know that P has to be a function:

$$P = f\big(\alpha_{ij,3}\big|_{\substack{1\leqslant i,j\leqslant n\\ i,j\neq 3}}, \beta_{ij,3}\big|_{\substack{1\leqslant i,j\leqslant n\\ i,j\neq 3}}, \gamma_{ij,3}\big|_{\substack{1\leqslant i,j\leqslant n\\ i,j\neq 3}}, s_{ij}\big|_{1\leqslant i<j\leqslant n}, y_{ij}\big|_{\substack{1\leqslant i,j\leqslant n\\ i\neq 3}}, z_{ij}\big|_{\substack{1 i<j\leqslant n\\ i,j\neq 3}}\big)\,. \tag{3.61}$$

We now have to find out which building blocks we have to add and which we have to get rid off, so that $\mathcal{V}$ vanishes when acted on by all three differential operators. To this end, express D_3 such that it directly depends on the variables given in (3.60):

$$
\begin{aligned}
D_3 = \sum_{j=1}^{n} y_{j3}\,\frac{\partial}{\partial z_{3j}} + s_{3j}\,\frac{\partial}{\partial y_{3j}} = \\
= \sum_{k=1}^{2}\left(-\sum_{j=4}^{n}\beta_{j3,k}\,\frac{\partial}{\partial \alpha_{3j,k}} + \sum_{\substack{j=1\\ j\neq k}}^{n}\gamma_{j3,k}\,\frac{\partial}{\partial \beta_{3j,k}}\right) + \sum_{j=4}^{n} y_{j3}\,\frac{\partial}{\partial z_{3j}} + \sum_{j=1}^{n} s_{3j}\,\frac{\partial}{\partial y_{3j}} \\
-\sum_{j=2}^{n} C_{j3}\,\frac{\partial}{\partial A_{j3}} - \sum_{j=2}^{n} D_{j3}\,\frac{\partial}{\partial B_{j3}} - \sum_{\substack{j=1\\ j\neq 2}}^{n} A_{3j}\,\frac{\partial}{\partial B_{3j}} - \sum_{\substack{j=1\\ j\neq 2}}^{n} C_{3j}\,\frac{\partial}{\partial D_{3j}}\,. \qquad (3.62)
\end{aligned}
$$

Therefore, we have expressed D_3 as a sum of differential operators of the form:

$$D_v = \sum_i u_i \frac{\partial}{\partial v_i}, \qquad (3.63)$$

as in the previous case. Using this, we can write down all the combinations of the form $c_{ij} = u_i v_j - u_j v_i$ and express them using determinants that are gauge invariant with respect to the fields ϕ_1, ϕ_2 and ϕ_3. These are then all the building blocks we need to consider.

There are many different combinations, so we divide them into different types.

Type I: (1x1)·(1x1)

$$y_{i3}\,z_{3j} - y_{j3} z_{3i} = \alpha_{ji,3} = [\,3\,\bar{3},\bar{i}\,\bar{j}\,]\big|_{4\leqslant i,j\leqslant n} \qquad (3.64)$$

$$s_{3i}\,y_{3j} - s_{3j}\,y_{3i} = \gamma_{ij,3} = [\,3\,\bar{3},\,i\,j\,]\big|_{1\leqslant i,j\leqslant n} \qquad (3.65)$$

$$y_{i3}\,y_{3j} - s_{3j}\,z_{3i} = -\beta_{ij,3} = -[\,3\,\bar{3},j\,\bar{i}\,]\big|_{\substack{4\leqslant i\leqslant n\\ 1\leqslant j\leqslant n}} \qquad (3.66)$$

We see that we have to add the building blocks $\alpha_{ij,3}$, $\beta_{ij,3}$ and $\gamma_{ij,3}$ to our solution. It is quickly checked that these variables get annihilated by all three differential operators.

Type II: (2x2)·(2x2)

$$-\beta_{i3,k}\,\alpha_{3j,k} + \beta_{j3,k}\,\alpha_{3i,k} = \beta_{33,k}\,\alpha_{ji,k} = [\,k\,\bar{k},3\,\bar{3}\,]\cdot[\,k\,\bar{k},\bar{i}\,\bar{j}\,]\big|_{\substack{4\leqslant i,j\leqslant n\\ 1\leqslant k\leqslant 2}} \qquad (3.67)$$

$$\gamma_{i3,k}\,\beta_{3j,k} - \gamma_{j3,k}\,\beta_{3i,k} = \beta_{33,k}\,\gamma_{ij,k} = [\,k\,\bar{k},3\,\bar{3}\,]\cdot[\,k\,\bar{k},\,i\,j\,]\big|_{\substack{1\leqslant i,j\leqslant n\\ 1\leqslant k\leqslant 2\\ i,j\neq k}} \qquad (3.68)$$

$$-\beta_{i3,k}\,\beta_{3j,k} - \gamma_{j3,k}\,\alpha_{3i,k} = -\beta_{33,k}\,\beta_{ij,k} = -[\,k\,\bar{k},3\,\bar{3}\,]\cdot[\,k\,\bar{k},j\,\bar{i}\,]\big|_{\substack{4\leqslant i\leqslant n\\ 1\leqslant j\leqslant n\\ 1\leqslant k\leqslant 2\\ j\neq k}} \qquad (3.69)$$

Combinations of this type can also be expressed using only (2x2) building blocks. The only new building blocks we find here are $\beta_{11,3}$ and $\beta_{22,3}$.

Type III: (2x2)·(1x1)

$$\begin{aligned}
-\beta_{i3,k}\, z_{3j} - y_{j3}\,\alpha_{3i,k} &= y_{ik}\,[\,3\,\bar{3}, \bar{k}\,\bar{j}\,] - z_{ik}\,[\,3\,\bar{3}, k\,\bar{j}\,] \\
&= ([\,3\,\bar{3}\,\bar{i}, k\,\bar{k}\,\bar{j}\,] - z_{ij}\,[\,3\,\bar{3}, k\,\bar{k}\,])\,\Big|_{\substack{4\leqslant i,j\leqslant n\\ 1\leqslant k\leqslant 2}}
\end{aligned} \tag{3.70}$$

$$\begin{aligned}
-\beta_{i3,k}\, y_{3j} - s_{3j}\,\alpha_{3i,k} &= z_{ik}\,[\,3\,\bar{3}, j\,k\,] - y_{ik}\,[\,3\,\bar{3}, j\,\bar{k}\,] \\
&= -\,([\,3\,\bar{3}\,\bar{i}, k\,j\,\bar{k}\,] + y_{ij}\,[\,3\,\bar{3}, k\,\bar{k}\,])\,\Big|_{\substack{4\leqslant i\leqslant n\\ 1\leqslant j\leqslant n\\ 1\leqslant k\leqslant 2}}
\end{aligned} \tag{3.71}$$

$$\begin{aligned}
\gamma_{i3,k}\, z_{3j} - y_{j3}\,\beta_{3i,k} &= s_{ki}\,[\,3\,\bar{3}, \bar{k}\,\bar{j}\,] - y_{ki}\,[\,3\,\bar{3}, k\,\bar{j}\,] \\
&= -\,([\,3\,i\,\bar{3}, k\,\bar{k}\,\bar{j}\,] + y_{ji}\,[\,3\,\bar{3}, k\,\bar{k}\,])\,\Big|_{\substack{1\leqslant i\leqslant n\\ 4\leqslant j\leqslant n\\ 1\leqslant k\leqslant 2\\ j\neq k}}
\end{aligned} \tag{3.72}$$

$$\begin{aligned}
\gamma_{i3,k}\, y_{3j} - s_{3j}\beta_{3i,k} &= y_{ki}\,[\,3\,\bar{3}, j\,k\,] - s_{ki}\,[\,3\,\bar{3}, j\,\bar{k}\,] \\
&= ([\,3\,i\,\bar{3}, k\,j\,\bar{k}\,] - s_{ij}\,[\,3\,\bar{3}, k\,\bar{k}\,])\,\Big|_{\substack{1\leqslant i,j\leqslant n\\ 1\leqslant k\leqslant 2\\ i\neq k}}
\end{aligned} \tag{3.73}$$

Here, we need (1x1), (2x2) and (3x3) building blocks to express the combinations in form of gauge invariant determinants. The (1x1) and (2x2) building blocks we already know, however, the (3x3) building blocks are new. They are of a similar form to $A_{ij} = [\,1\,i\,\bar{1}, 2\,\bar{2}\,\bar{j}\,]$, $B_{ij} = [\,1\,\bar{1}\,\bar{i}, 2\,\bar{2}\,\bar{j}\,]$, $C_{ij} = [\,1\,i\,\bar{1}, 2\,j\,\bar{2}\,]$ and $D_{ij} = [\,1\,\bar{1}\,\bar{i}, 2\,j\,\bar{2}\,]$.

Type IV: (3x3)·(1x1)

It is when studying these types of combinations that problems arise. Consider the combination:

$$\begin{aligned}
-\,C_{i3}\, z_{3j} - y_{j3}A_{i3} &= [\,3\,\bar{3}, 1\,\bar{j}\,]\cdot[\,2\,\bar{2}, i\,\bar{1}\,] + [\,3\,\bar{3}, \bar{1}\,\bar{j}\,]\cdot[\,2\,\bar{2}, 1\,i\,] \\
&\quad -[\,1\,\bar{1}, 2\,\bar{2}\,]\cdot[\,3\,\bar{3}, i\,\bar{j}\,] \\
&= ([\,2\,3\,\bar{2}\,\bar{3}, 1\,i\,\bar{1}\,\bar{j}\,] + y_{3i}\,[\,1\,\bar{1}\,\bar{j}, 2\,3\,\bar{2}] + s_{3i}\,[\,1\,\bar{1}\,\bar{j}, 2\,\bar{2}\,\bar{3}\,] \\
&\quad +[\,2\,\bar{2}, i\,\bar{j}\,]\cdot[\,3\,\bar{3}, 1\,\bar{1}\,] - [\,2\,\bar{2}, 1\,\bar{1}\,]\cdot[\,3\,\bar{3}, i\,\bar{j}\,])\Big|_{\substack{2\leqslant i\leqslant n\\ 4\leqslant j\leqslant n}}.
\end{aligned} \tag{3.74}$$

The terms in red do not get annihilated by either one or more differential operators. This happens analogously for other combinations of this type.
Take a closer look at the expression:

$$\Xi = [\,3\,\bar{3}, 1\,\bar{j}\,]\cdot[\,2\,\bar{2}, i\,\bar{1}\,] + [\,3\,\bar{3}, \bar{1}\,\bar{j}\,]\cdot[\,2\,\bar{2}, 1\,i\,]\,. \tag{3.75}$$

When gauge varying with respect to the field ϕ_1, we change 1 to $\bar{1}$ and vice versa. However, by doing so for Ξ, the red terms take the form of the black terms and the other way around. Hence, it is not possible to get rid of the problematic terms.
Therefore, we cannot express these combinations using determinants of submatrices of $\mathcal{B}$. Moreover, this shows, that the assumption, that all gauge invariant terms could be expressed using only determinants, is not correct.

It follows, that, to find the necessary building blocks, one needs to characterize Ξ, since the other term in equation (3.74) is already gauge invariant.
We try to achieve this using immanants, which are generalizations of the determinant. However, trying all possible immanants applicable to this problem led to no correct characterization of Ξ.

As of yet, we have not found a way to express these combinations using gauge invariant building blocks. Therefore, we cannot proceed further in this case.
However, what we have found, are generalizations for the (1x1) and (2x2) building blocks. Let $\mathcal{V}$ be gauge invariant with respect to m out of n fields $\phi_1, \ldots, \phi_m$. Then its (1x1) and (2x2) building blocks are given by:

$$\begin{aligned}
&\text{(1x1):} \quad s_{ij}\Big|_{1\leqslant i<j\leqslant n}, \quad y_{ij}\Big|_{\substack{m+1\leqslant i\leqslant n\\ 1\leqslant j\leqslant n}}, \quad z_{ij}\Big|_{m+1\leqslant i<j\leqslant n}, \\
&\text{(2x2):} \quad \alpha_{ij,k}\Big|_{\substack{m+1\leqslant i<j\leqslant n\\ 1\leqslant k\leqslant m}}, \quad \beta_{ij,k}\Big|_{\substack{m+1\leqslant i\leqslant n\\ 1\leqslant j\leqslant n\\ 1\leqslant k\leqslant m\\ j\neq k}}, \quad \gamma_{ij,k}\Big|_{\substack{1\leqslant i<j\leqslant n\\ 1\leqslant k\leqslant m\\ i,j\neq k}}, \quad \beta_{ii,k}\Big|_{\substack{1\leqslant i\leqslant m\\ 1\leqslant k\leqslant m\\ i\neq k}}.
\end{aligned} \tag{3.76}$$

4 SECOND APPROACH: $\mathcal{V}(A_i, P_i)$

The variables s_{ij}, y_{ij} and z_{ij} themselves depend on the variables A_i and P_i. Therefore, instead of taking $\mathcal{V}$ to be a polynomial in s_{ij}, y_{ij} and z_{ij}, we can also express it as a polynomial in A_i and P_i. The same can be done for the differential operators D_i.
Hence, let $\mathcal{V} = \tilde{P}(A_i^\mu|_{1\leqslant i\leqslant n}, P_i^\mu|_{1\leqslant i\leqslant n})$ be a polynomial. Using A_i and P_i the differential operators take the form:

$$D_i = P_i^\rho \frac{\partial}{\partial A_i^\rho} \,. \tag{4.1}$$

Using s_{ij}, y_{ij} and z_{ij} had the advantage, that the resulting expression was always guaranteed to be a Lorentz tensor, since all indices were already contracted in the definition of the variables.
Changing to A_i and P_i, now, this is no longer the case. However, we can still determine building blocks for a polynomial and assume that all the indices get contracted by some tensor afterwards.

4.1 GAUGE VARYING ONE FIELD

Analogously to the first approach, start by imposing gauge invariance of $\mathcal{V}$ with respect to one arbitrary field ϕ_i. Then $\mathcal{V}$ has to satisfy:

$$D_i\mathcal{V} = P_i^\rho \frac{\partial}{\partial A_i^\rho} \tilde{P}(A_i^\mu|_{1\leqslant i\leqslant n}, P_i^\mu|_{1\leqslant i\leqslant n}) \stackrel{!}{=} 0 \,. \tag{4.2}$$

By solving this equation, using the method of characteristics, we find, that $\tilde{P}$ has to be a function of the following variables:

$$\tilde{P} = f(C_i^{\mu\nu}, A_j^\mu|_{\substack{1\leqslant j\leqslant n \\ j\neq i}}, P_j^\mu|_{1\leqslant j\leqslant n}) \,. \tag{4.3}$$

The new variable C_i is an antisymmetric combination of A_i and P_i defined as:

$$C_i^{\mu\nu} = A_i^\mu P_i^\nu - A_i^\nu P_i^\mu \,. \tag{4.4}$$

Next we are going to prove that $\tilde{P}$ can always be written as a polynomial in these variables. For this, it suffices to slightly change the arguments of the proof of proposition (3.1.1), where gauge invariance under a single gauge transformation was imposed as well.

Proposition 4.1.1. *Let $D_i = P_i^\alpha \frac{\partial}{\partial A_i^\alpha}$. Then every polynomial $\tilde{P}(A_j^\mu|_{1\leqslant j\leqslant n}, P_j^\mu|_{1\leqslant j\leqslant n})$ satisfying $D_i\tilde{P} = 0$ can be written as a polynomial in $P_j^\mu|_{1\leqslant j\leqslant n}$, $A_j^\mu|_{\substack{1\leqslant j\leqslant n \\ j\neq i}}$ and $C_i^{\mu\nu} = A_i^\mu P_i^\nu - A_i^\nu P_i^\mu$.*

Proof. D_i does not change the degree of the polynomial, so one can consider $\tilde{P}$ to be a homogeneous polynomial of degree m.
We prove by induction over m.

The case m=0: For $m = 0$ the problem is trivial.

Assume that every polynomial of degree smaller or equal m for which $D_i\tilde{P} = 0$ can be written as a polynomial in $P_j^\mu|_{1\leqslant j\leqslant n}$, $A_j^\mu|_{\substack{1\leqslant j\leqslant n \\ j\neq i}}$ and $C_i^{\mu\nu}$.

The case m+1: Let $\tilde{P}_{m+1}$ be a homogeneous polynomial of degree $m + 1$ that satisfies $D_i\tilde{P}_{m+1} = 0$.
Define: $a_{i\,\rho} = \frac{\partial}{\partial A_i^\rho}\tilde{P}_{m+1}$.
These are polynomials of degree m satisfying:

1. $D_i\, a_{i\,\rho} = 0$,
2. $\frac{\partial}{\partial A_j^\mu}\, a_{i\,\nu} = \frac{\partial}{\partial A_i^\nu}\, a_{j\,\mu}$,
3. $P_i^\rho\, a_{i\,\rho} = 0$.

The first point holds, since:

$$D_i\,\frac{\partial}{\partial A_i^\rho}\,\tilde{P}_{m+1} = \frac{\partial}{\partial\, A_i^\rho} D_i\,\tilde{P}_{m+1} = 0\,. \tag{4.5}$$

Next, define N_A and N_P to be the operators counting the degrees in A_i and P_i:

$$N_A = A_i^\rho \frac{\partial}{\partial A_i^\rho}\,,$$
$$N_P = P_i^\rho \frac{\partial}{\partial P_i^\rho}\,.$$

D_i lowers the A_i degree by 1, therefore, we find that $N_A D_i = D_i(N_A - 1)$.
Now introduce: $P' = N_A^{-1}\,(A_i^\mu\,(a_i)_\mu)$.

Then:

$$\begin{aligned}
\frac{\partial P'}{\partial A_i^\rho} &= \frac{\partial}{\partial A_i^\rho} N_A^{-1} \left(A_i^\mu (a_i)_\mu\right) \\
&= (N_A+1)^{-1} \frac{\partial}{\partial A_i^\rho} \left(A_i^\mu (a_i)_\mu\right) \\
&= (N_A+1)^{-1} \cdot \Big(\delta_\rho^\mu (a_i)_\mu + A_i^\mu \underbrace{\frac{\partial}{\partial A_i^\rho} (a_i)_\mu}_{\frac{\partial}{\partial A_i^\mu} (a_i)_\rho}\Big) \\
&= (N_A+1)^{-1} \cdot \left((a_i)_\rho + A_i^\mu \frac{\partial}{\partial A_i^\mu} (a_i)_\rho\right) \\
&= (N_A+1)^{-1} \cdot (1+N_A) \cdot (a_i)_\rho \\
&= (a_i)_\rho \\
&= \frac{\partial}{\partial A_i^\rho} \tilde{P}_{m+1} \,.
\end{aligned} \tag{4.6}$$

Consequently, we have:

$$\frac{\partial}{\partial A_i^\rho} \left(\tilde{P}_{m+1} - P'\right) = 0 \,. \tag{4.7}$$

This shows, that $\tilde{P}_{m+1}$ can be written as:

$$\tilde{P}_{m+1} = P' + Q\Big(P_j^\rho\big|_{1\leqslant j\leqslant n}, A_j^\rho\big|_{\substack{1\leqslant j\leqslant n\\ j\neq i}}\Big) \,. \tag{4.8}$$

where Q is a polynomial in P_j^ρ and A_j^ρ.

Now apply $\frac{\partial}{\partial P_i^\rho}$ to $P_i^\mu (a_i)_\mu = 0$:

$$\begin{aligned}
0 &= \frac{\partial}{\partial P_i^\rho} \left(P_i^\mu (a_i)_\mu\right) \\
&= (a_i)_\rho + P_i^\mu \frac{\partial}{\partial P_i^\rho} (a_i)_\mu + \underbrace{P_i^\mu \frac{\partial}{\partial P_i^\mu}}_{N_P} (a_i)_\rho - P_i^\mu \frac{\partial}{\partial P_i^\mu} (a_i)_\rho \\
&= (N_P+1)\,(a_i)_\rho + P_i^\mu \left(\frac{\partial}{\partial P_i^\rho} (a_i)_\mu - \frac{\partial}{\partial P_i^\mu} (a_i)_\rho\right) \,.
\end{aligned} \tag{4.9}$$

Using this, the following expression for $(a_i)_\rho$ can be found:

$$(a_i)_\rho = -(N_P+1)^{-1} P_i^\mu \left(\frac{\partial}{\partial P_i^\rho} (a_i)_\mu - \frac{\partial}{\partial P_i^\mu} (a_i)_\rho\right) \,. \tag{4.10}$$

Plugging (4.10) into P' gives:

$$
\begin{aligned}
P' &= N_A^{-1} A_i^\rho (a_i)_\rho \\
&= -N_A^{-1} A_i^\rho (N_P+1)^{-1} P_i^\mu \left(\frac{\partial}{\partial P_i^\rho}(a_i)_\mu - \frac{\partial}{\partial P_i^\mu}(a_i)_\rho\right) \\
&= -N_A^{-1} (N_P+1)^{-1} \cdot \frac{1}{2} \underbrace{(A_i^\rho P_i^\mu - A_i^\mu P_i^\rho)}_{C_i^{\rho\mu}} \underbrace{\left(\frac{\partial}{\partial P_i^\rho}(a_i)_\mu - \frac{\partial}{\partial P_i^\mu}(a_i)_\rho\right)}_{(*)} .
\end{aligned} \tag{4.11}
$$

Lastly, it has to be checked if $(*)$ vanishes when acted on with D_i:

$$
\begin{aligned}
D_i \left(\frac{\partial}{\partial P_i^\rho}(a_i)_\mu - \frac{\partial}{\partial P_i^\mu}(a_i)_\rho\right) &= P_i^\sigma \frac{\partial}{\partial A_i^\sigma} \left(\frac{\partial}{\partial P_i^\rho}(a_i)_\mu - \frac{\partial}{\partial P_i^\mu}(a_i)_\rho\right) \\
&= \frac{\partial}{\partial P_i^\rho}\left(P_i^\sigma \frac{\partial}{\partial A_i^\sigma}(a_i)_\mu\right) - \frac{\partial}{\partial A_i^\rho}(a_i)_\mu \\
&\quad - \frac{\partial}{\partial P_i^\mu}\left(P_i^\sigma \frac{\partial}{\partial A_i^\sigma}(a_i)_\rho\right) + \frac{\partial}{\partial A_i^\mu}(a_i)_\rho \\
&= \frac{\partial}{\partial P_i^\rho} \underbrace{D_i (a_i)_\mu}_{=0} - \frac{\partial}{\partial P_i^\mu} \underbrace{D_i (a_i)_\rho}_{=0} - \underbrace{\left(\frac{\partial}{\partial A_i^\rho}(a_i)_\mu - \frac{\partial}{\partial A_i^\mu}(a_i)_\rho\right)}_{=0} \\
&= 0 .
\end{aligned} \tag{4.12}
$$

$(*)$ is a polynomial of degree $m-1$ and gets annihilated by D_i, as shown above. Therefore, $(*)$ can be written as a polynomial in $P_j^\mu\big|_{1\leqslant j\leqslant n}$, $A_j^\mu\big|_{\substack{1\leqslant j\leqslant n \\ j\neq i}}$ and $C_i^{\mu\nu}$. Consequently, also P' and $\tilde{P}_{m+1}$ can be written as polynomials in these variables. □

4.2 GAUGE VARYING MULTIPLE FIELDS

Let $\mathcal{V}$ be gauge invariant under the transformation δ_1. Then, $\mathcal{V}$ can be written as the polynomial:

$$
\mathcal{V} = \tilde{P}(C_1^{\mu\nu}, A_i^\mu\big|_{2\leqslant i\leqslant n}, P_i^\mu\big|_{1\leqslant i\leqslant n}) . \tag{4.13}
$$

We now wish to impose gauge invariance with respect to the field ϕ_2 as well. Hence, $\mathcal{V}$ has to satisfy:

$$
D_2\mathcal{V} = \left(P_2^\rho \frac{\partial}{\partial A_2^\rho}\right) \tilde{P}(C_1^{\mu\nu}, A_i^\mu\big|_{2\leqslant i\leqslant n}, P_i^\mu\big|_{1\leqslant i\leqslant n}) \stackrel{!}{=} 0 . \tag{4.14}
$$

All variables $\tilde{P}$ depends on, except A_2, get annihilated by D_2.

Therefore, if we solve this equation using the method of characteristics, we find that $\tilde{P}$ has to be a function of the following variables:

$$\tilde{P} = f(C_i^{\mu\nu}|_{1\leqslant i\leqslant 2}, A_i^\mu|_{3\leqslant i\leqslant n}, P_i^\mu|_{1\leqslant i\leqslant n}) \,. \tag{4.15}$$

Let us assume that we impose gauge invariance of $\mathcal{V}$ under an increasing number of gauge transformations δ_i step by step. Each time, $\tilde{P}$ will lose dependence on an A_i while gaining dependence on the corresponding C_i.
This statement holds up until having imposed gauge invariance with respect to $n-1$ out of n fields ϕ_i.

So, let us require the polynomial $\mathcal{V} = \tilde{P}(A_i^\mu|_{1\leqslant i\leqslant n}, P_i^\mu|_{1\leqslant i\leqslant n})$ to satisfy:

$$D_1\tilde{P} = D_2\tilde{P} = \ldots = D_k\tilde{P} = 0 \,, \tag{4.16}$$

for $k \in \{1, 2, \ldots, n-1\}$.
Then, $\tilde{P}$ has to take the form:

$$\tilde{P} = f(C_i^{\mu\nu}|_{1\leqslant i\leqslant k}, A_i^\mu|_{k+1\leqslant i\leqslant n}, P_i^\mu|_{1\leqslant i\leqslant n}) \,. \tag{4.17}$$

To prove that $\tilde{P}$ can also be written as a polynomial in these variables, we can use a proof analogous to the one of Proposition (4.1.1) above.

Proposition 4.2.1. *Let $\tilde{P}(A_i^\mu|_{1\leqslant i\leqslant n}, P_i^\mu|_{1\leqslant i\leqslant n})$ be a polynomial and let $D_i = P_i^\rho \frac{\partial}{\partial A_i^\rho}$ for $1 \leqslant i \leqslant n$.*
Assume $\tilde{P}$ satisfies:

$$D_1\tilde{P} = D_2\tilde{P} = \ldots = D_k\tilde{P} = 0 \,, \tag{4.18}$$

for $k \in \{1, 2, \ldots, n-1\}$. Then $\tilde{P}$ can be written as a polynomial in the variables $C_i^{\mu\nu}|_{1\leqslant i\leqslant k}$, $A_i^\mu|_{k+1\leqslant i\leqslant n}$ and $P_i^\mu|_{1\leqslant i\leqslant n}$.

Proof. Start by assuming $k = 2$. Demanding that $\tilde{P}$ satisfies $D_1\tilde{P} = D_2\tilde{P} = 0$ is similar to requiring:

$$D_2\tilde{P}(C_1^{\mu\nu}, A_i^\mu|_{2\leqslant i\leqslant n}, P_i^\mu|_{1\leqslant i\leqslant n}) = 0 \,. \tag{4.19}$$

Then, follow the proof for gauge invariance under a single transformation of proposition (4.1.1), for $i = k$. We arrive at an equation similar to equation (4.7), that reads:

$$\frac{\partial}{\partial A_2^\rho}(\tilde{P}_{m+1} - P') = 0 \,. \tag{4.20}$$

In this case, we can conclude, that $\tilde{P}_{m+1}$ can be written as:

$$\tilde{P}_{m+1} = P' + Q(C_1^{\mu\nu}, A_j^\mu|_{3\leqslant j\leqslant n}, P_j^\mu|_{1\leqslant j\leqslant n}) \,, \tag{4.21}$$

where Q is a polynomial in $C_1^{\mu\nu}$, $A_j^\mu\big|_{3\leqslant j\leqslant n}$ and $P_j^\mu\big|_{1\leqslant j\leqslant n}$. Afterwards, we can again follow the proof of proposition (4.1.1) and find that $\tilde{P}$ can be written as a polynomial in $C_j^{\mu\nu}\big|_{1\leqslant j\leqslant 2}$, $A_j^\mu\big|_{3\leqslant j\leqslant n}$ and $P_j^\mu\big|_{1\leqslant j\leqslant n}$.

This recipe can be repeated an arbitrary number of times to prove that for $k \in \{1, 2, \ldots, n-1\}$ $\tilde{P}$ can be written as a polynomial in $C_j^{\mu\nu}\big|_{1\leqslant j\leqslant k}$, $A_j^\mu\big|_{k+1\leqslant j\leqslant n}$ and $P_j^\mu\big|_{1\leqslant j\leqslant n}$. □

4.3 GAUGE VARYING ALL FIELDS

Lastly, we wish to study the case of gauge invariance with respect to all n fields ϕ_i. In this case $\mathcal{V}$ has to satisfy:

$$D_1\mathcal{V} = D_2\mathcal{V} = \ldots = D_n\mathcal{V} = 0\,. \tag{4.22}$$

A new difficulty arises due to momentum conservation. Momentum conservation has to hold, meaning that:

$$P_1 + P_2 + P_3 + \ldots + P_n \stackrel{!}{=} 0\,. \tag{4.23}$$

Therefore, since we have chosen $P_i^\mu\big|_{1\leqslant i\leqslant n-1}$ to be arbitrary momenta, we now have to require P_n^μ to satisfy:

$$P_n^\mu = -\sum_{i=1}^{n-1} P_i^\mu\,. \tag{4.24}$$

Consequently, the differential operator D_n now takes the form:

$$D_n = P_n^\rho \frac{\partial}{\partial A_n^\rho} = -\sum_{i=1}^{n-1} P_i^\rho \frac{\partial}{\partial A_n^\rho}\,. \tag{4.25}$$

If $\mathcal{V}$ is gauge invariant under the first $n-1$ transformations $\delta_1, \ldots, \delta_{n-1}$ it can be written as a polynomial $\mathcal{V} = \tilde{P}(A_n^\mu, C_i^{\mu\nu}\big|_{1\leqslant i\leqslant n-1}, P_i^\mu\big|_{1\leqslant i\leqslant n-1})$.
Knowing this to be true, we start by solving the differential equation:

$$D_n\mathcal{V} = -\left(\sum_{i=1}^{n-1} P_i^\rho \frac{\partial}{\partial A_n^\rho}\right) \tilde{P}(A_n^\mu, C_i^{\mu\nu}\big|_{1\leqslant i\leqslant n-1}, P_i^\mu\big|_{1\leqslant i\leqslant n-1}) = 0\,. \tag{4.26}$$

Using the method of characteristics we find that $\tilde{P}$ has to take the form:

$$\tilde{P} = f(D^{\mu\nu}, C_i^{\mu\nu}\big|_{1\leqslant i\leqslant n-1}, P_i^\mu\big|_{1\leqslant i\leqslant n-1})\,. \tag{4.27}$$

The new building block $D^{\mu\nu}$ is an antisymmetric combination of A_n and $\sum_{i=1}^{n-1} P_i$:

$$D^{\mu\nu} = A_n^{\mu} \sum_{i=1}^{n-1} P_i^{\nu} - A_n^{\nu} \sum_{i=1}^{n-1} P_i^{\mu} = 2\, A_n^{[\mu} \sum_{i=1}^{n-1} P_i^{\nu]} \,. \tag{4.28}$$

However, we find that such an antisymmetric combination can also be constructed in a different way making use of the dependence on the $C_i^{\mu\nu}$'s.
Consider the following:

$$\begin{aligned} A_n^{[\mu}\, C_1^{\mu_1\nu_1}\, C_2^{\mu_2\nu_2} \ldots C_{n-1}^{\mu_{n-1}\nu_{n-1}]} &= A_n^{[\mu}\, 2\, A_1^{\mu_1}\, P_1^{\nu_1}\, C_2^{\mu_2\nu_2} \ldots C_{n-1}^{\mu_{n-1}\nu_{n-1}]} \\ &= 2\, A_n^{[\mu}\, A_1^{\mu_1}\, \Big(\sum_{i=1}^{n-1} P_i^{\nu_1} - \underbrace{P_2^{\nu_1} - \ldots - P_{n-1}^{\nu_1}}_{=\,0,\ \text{due to antisymmetry}}\Big)\, C_2^{\mu_2\nu_2} \ldots C_{n-1}^{\mu_{n-1}\nu_{n-1}]} \\ &= 2\, A_n^{[\mu}\, A_1^{\mu_1} \sum_{i=1}^{n-1} P_i^{\nu_1}\, C_2^{\mu_2\nu_2} \ldots C_{n-1}^{\mu_{n-1}\nu_{n-1}]} \,. \end{aligned} \tag{4.29}$$

Therefore, such an expression leads to antisymmetric combinations of A_n and $\sum_{i=1}^{n-1} P_i$ as well. Note, however, that $\tilde{P}$ must not directly depend on $A_i\big|_{1\leqslant i\leqslant n-1}$, hence, we cannot view this as a term of the form $D^{\mu\nu}$.
It follows, that we have to add such combinations to our solution:

$$E^{\mu\,\mu_1\,\nu_1\ldots\nu_{n-1}} = A_n^{[\mu}\, C_1^{\mu_1\nu_1}\, C_2^{\mu_2\nu_2} \ldots C_{n-1}^{\mu_{n-1}\nu_{n-1}]} \,. \tag{4.30}$$

Moreover, we now have to prove, that $\tilde{P}$ can be written as a polynomial in these variables.

Proposition 4.3.1. *Let $\tilde{P}(A_n^{\mu}, C_i^{\mu\nu}\big|_{1\leqslant i\leqslant n-1}, P_i^{\mu}\big|_{1\leqslant i\leqslant n-1})$ be a polynomial and let $D_n = -\sum_{i=1}^{n-1} P_i^{\alpha} \frac{\partial}{\partial A_n^{\alpha}}$. If $D_n\tilde{P} = 0$ holds, $\tilde{P}$ can be written as a polynomial in the variables $D^{\mu\nu}$, $E^{\mu\,\mu_1\,\nu_1\ldots\nu_{n-1}}$, $C_i^{\mu\nu}\big|_{1\leqslant i\leqslant n-1}$ and $P_i^{\mu}\big|_{1\leqslant i\leqslant n-1}$, where:*

$$\begin{aligned} D^{\mu\nu} &= A_n^{[\mu} \textstyle\sum_{i=1}^{n-1} P_i^{\nu]} \,, \\ E^{\mu\,\mu_1\,\nu_1\ldots\nu_{n-1}} &= A_n^{[\mu} C_1^{\mu_1\nu_1} C_2^{\mu_2\nu_2} \ldots C_{n-1}^{\mu_{n-1}\nu_{n-1}]} \,. \end{aligned} \tag{4.31}$$

4.3.1 Proof of Proposition 4.3.1

Due to the partial derivative with respect to A_n^{μ} all terms of $\tilde{P}$ not containing A_n^{μ} fulfill the proposition trivially. Therefore, it suffices to show that A_n^{μ} can only appear as part of either $D^{\mu\nu}$ or $E^{\mu\,\mu_1\,\nu_1\ldots\nu_{n-1}}$ for $D_n\tilde{P} = 0$ to hold.

1. Step: $\tilde{P}$ linear in A_n^{μ}
Begin by showing this for $\tilde{P}$ being linear in A_n^{μ}. Consider how D_n acts on a single term $\eta_1 = A_n^{\mu}\, P'_{\mu}(C_i\big|_{1\leqslant i\leqslant n-1}, P_i\big|_{1\leqslant i\leqslant n-1})$ of $\tilde{P}$:

$$D_n\eta_1 = -\sum_{i=1}^{n-1} P_i^{\alpha} \frac{\partial}{\partial A_n^{\alpha}} (A_n^{\mu}\, P'_{\mu}(C_i\big|_{1\leqslant i\leqslant n-1}, P_i\big|_{1\leqslant i\leqslant n-1})) \,. \tag{4.32}$$

Here, $P'(C_i\big|_{1\leqslant i\leqslant n-1}, P_i\big|_{1\leqslant i\leqslant n-1})$ is an arbitrary product of the C_i's and P_i's. Require now that this expression vanishes:

$$
\begin{aligned}
-\sum_{i=1}^{n-1} P_i^{\alpha}\,\delta_{\alpha}^{\mu}\,P'_{\mu}(C_i\big|_{1\leqslant i\leqslant n-1}, P_i\big|_{1\leqslant i\leqslant n-1}) = \\
-\sum_{i=1}^{n-1} P_i^{\mu}\,P'_{\mu}(C_i\big|_{1\leqslant i\leqslant n-1}, P_i\big|_{1\leqslant i\leqslant n-1}) \stackrel{!}{=} 0\,.
\end{aligned}
\tag{4.33}
$$

Rather than directly trying to find which P'_μ satisfies equation (4.33) it turns out to be beneficial to view this as the task of finding all vectors orthogonal to $-\sum_{i=1}^{n-1} P_i^{\mu}$.

Orthogonal vectors of $\sum_{i=1}^{n-1} P_i^{\mu}$

We know that $-\sum_{i=1}^{n-1} P_i^{\mu}$ is a lightlike vector, since

$$P_n^{\mu} = -P_1^{\mu} - P_2^{\mu} - \ldots - P_{n-1}^{\mu}$$

and furthermore

$$P_n^{\mu}\,P_{n\,\mu} = 0.$$

Define the following vector:

$$W_\mu := \epsilon_{\mu\nu\rho\sigma}\left(-\sum_{i=1}^{n-1} P_i^{\nu}\right) X^{\rho} Y^{\sigma}\,. \tag{4.34}$$

It can be quickly shown, that W_μ is orthogonal to $-\sum_{i=1}^{n-1} P_i^{\mu}$, since:

$$-\sum_{i=1}^{n-1} P_i^{\mu}\,W_\mu = \underbrace{\epsilon_{\mu\nu\rho\sigma}\left(-\sum_{i=1}^{n-1} P_i^{\mu}\right)\left(-\sum_{i=1}^{n-1} P_i^{\nu}\right)}_{=0} X^{\rho} Y^{\sigma} = 0\,. \tag{4.35}$$

Next, we are going to show, that all vectors that are orthogonal to $-\sum_{i=1}^{n-1} P_i^{\mu}$ can be expressed in the form of (4.34).

Let us consider an orthonormal basis (e_0, e_1, e_2, e_3) with the following base vectors:

$$e_0 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad e_1 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad e_2 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad e_3 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}. \tag{4.36}$$

Without loss of generality we may fix $\left(-\sum_{i=1}^{n-1} P_i^\mu\right)$ to be any lightlike vector. Let us choose the following:

$$-\sum_{i=1}^{n-1} P_i^\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} . \tag{4.37}$$

We have shown W_μ to be orthogonal to $\left(-\sum_{i=1}^{n-1} P_i^\mu\right)$. Therefore, we know that, W_μ has to be either a lightlike or a spacelike vector.

Firstly, by taking $X := e_1$ and $Y := e_2$, we find W_μ to be a lightlike vector:

$$W_\mu = \epsilon_{\mu\nu\rho\sigma} \left(-\sum_{i=1}^{n-1} P_i^\nu\right) e_1^\rho e_2^\sigma = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} , \tag{4.38}$$

$$W^\mu = \eta^{\mu\nu} W_\nu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} . \tag{4.39}$$

It can be quickly shown, that any two orthogonal lightlike vectors are always multiples of each other, meaning:

$$W^\mu = k \cdot \left(-\sum_{i=1}^{n-1} P_i^\mu\right) , \tag{4.40}$$

where k is a constant. It directly follows, that all lightlike vectors that are orthogonal to $\left(-\sum_{i=1}^{n-1} P_i^\mu\right)$ can be expressed by W^μ.

Secondly, W^μ can also be a spacelike vector. However, we have to change our choices for X and Y to achieve this, since there can only exist two spacelike vectors that are orthogonal to each other as well as to a given lightlike vector. Therefore, let us choose $X := e_1$ and $Y := L$, where L is the following vector:

$$L = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} . \tag{4.41}$$

Note, that this is not the only possible choice for L to find a spacelike W^μ.
It follows that W_μ is then given by:

$$W_\mu = \epsilon_{\mu\nu\rho\sigma} \left(-\sum_{i=1}^{n-1} P_i^\nu \right) e_1^\rho L^\sigma \,. \tag{4.42}$$

Both sides of equation (4.42) transform under Lorentz transformations, so it follows that we can transform W_μ to reach any other spacelike vector that is orthogonal to $\left(-\sum_{i=1}^{n-1} P_i^\mu \right)$.

To sum up, we have found that any vector orthogonal to $\left(-\sum_{i=1}^{n-1} P_i^\mu \right)$ can be expressed using a vector W^μ of the form given in equation (4.34).

Polynomial Structure

We have shown that, if we take $\left(-\sum_{i=1}^{n-1} P_i^\mu \right)$ to be a lightlike vector in Minkowski spacetime, all vectors orthogonal to it can be expressed in terms of a vector W given in (4.34). However, we wish to prove the general case and do not want to restrict ourselves to a solution that only works if we can assign vector values to the P_i's and A_i's.
Therefore, let us assume P' to take the following form:

$$P'_\mu = \left(-\sum_{i=1}^{n-1} P_i^\nu \right) T_{[\mu\nu]} \,. \tag{4.43}$$

Here T is an arbitrary tensor that is antisymmetric in its indices μ and ν. Such an expression is the generalization of the vector W given above. Recall, that we are searching for an expression that describes all P' that satisfy:

$$\left(-\sum_{i=1}^{n-1} P_i^\mu \right) P'_\mu(C_i\big|_{1\leqslant i\leqslant n-1}, P_i\big|_{1\leqslant i\leqslant n-1}) = 0 \,. \tag{4.44}$$

It seems almost certain that all possible solutions can be expressed using (4.43). Only two other possible solutions for the equation above come to mind. The first option would be:

$$P'_\mu = \left(-\sum_{i=1}^{n-1} P_{i\mu} \right) \,. \tag{4.45}$$

In that case the term η_1 we assumed D_n to act on in the beginning, would involve an expression of the form $A_n^\mu \left(-\sum_{i=1}^{n-1} P_{i\mu} \right)$. However, we assumed all fields to be divergence free, meaning that η_1 would be zero by definition. It follows that solutions of this type cannot appear.
The second option would be the following:

$$P'_\mu = \left(-\sum_{i=1}^{n-1} P_i^\nu \right) \delta_{\mu\nu} \,. \tag{4.46}$$

Such a solution, however, leads to a similar expression as the first option and so we cannot consider these as well.

Despite being convinced that this assumption is correct, we have not yet been able to formally prove this. Therefore, until a formal proof has been found, let us assume going forward, that all P'_μ that solve equation (4.44) can be described by the following expression:

$$P'_\mu = \left(- \sum_{i=1}^{n-1} P_i^\nu \right) T_{[\mu\nu]} \,. \tag{4.47}$$

Antisymmetric Combinations

Then, it can be concluded that antisymmetrical expressions of A_n and $-\sum_{i=1}^{n-1} P_i$ are needed, such that (4.32) vanishes. Now, we have to prove that all these antisymmetric combinations can be expressed using only $D^{\mu\nu}$ and $E^{\mu\,\mu_1\,\nu_1\ldots\nu_{n-1}}$.
We start with the most general form: $A_n^{[\mu} \sum_{i=1}^{n-1} P_i^{\nu]}\,\Omega$.

1. Case: $\Omega = \xi(P_i\big|_{1\leqslant i\leqslant n-1}, C_i\big|_{1\leqslant i\leqslant n-1})$
If Ω is a product depending only on P's and C's, the expression is already in the form of $D^{\mu\nu}$.

2. Case: $\Omega = A_i \cdot \xi(P_j\big|_{1\leqslant j\leqslant n-1}, C_j\big|_{1\leqslant j\leqslant n-1})$
In this case, we assume that one C_i "lost" its P_i's to build expressions of the type $A_n^{[\mu} \sum_{i=1}^{n-1} P_i^{\nu]}$, such that the A_i's can no longer be recombined into a C_i.
Hence, we have a term Ξ of the form:

$$\begin{aligned} \Xi &= A_n^{[\mu} \sum_{i=1}^{n-1} P_i^\nu \cdot \mathcal{X}^{\alpha_1\ldots\alpha_k]} \cdot A_1^\rho \cdot \xi(P_j, C_j) \\ &= A_n^{[\mu} \left(P_1 + \ldots + P_{n-1}\right)^\nu \cdot \mathcal{X}^{\alpha_1\ldots\alpha_k]} \cdot A_1^\rho \cdot \xi(P_j, C_j) \,. \end{aligned} \tag{4.48}$$

Here $\mathcal{X}^{\alpha_1\ldots\alpha_k}$ is an aribtrary tensor that is used as a placeholder and $\xi(P_j, C_j)$ is an aribtrary product of P_j's and C_j's.
We know that an isolated A_i always has to have been part of a C_i prior to using the P_i to build an antisymmetric combination. There are two ways this A_i can gain a P_i. Firstly, the P_i can stem from $\xi(P_j, C_j)$. However, this simply leads to the case discussed above. Secondly, the P_i can be taken from $(P_1 + \ldots + P_{n-1})^\nu$ in the antisymmetric combination of (4.48). Take $i = 1$, then we can conclude, that $(P_2 + \ldots + P_{n-1})^\nu$ has to cancel.
To this end, $\mathcal{X}^{\alpha_1\ldots\alpha_k}$ has to be a product in P_i's and C_i's such that each $P_i^\nu\big|_{2\leqslant i\leqslant n-1}$ appears exactly once either independently or as part of a C_i. Assume for now, that $\mathcal{X}^{\alpha_1\ldots\alpha_k}$ consists only of C_i's:

$$\begin{aligned} \mathcal{X}^{\alpha_1\ldots\alpha_k} &= C_2^{\alpha_1\alpha_2} \ldots C_{n-1}^{\alpha_{k-1}\alpha_k} \\ &:= C_2^{\mu_2\nu_2} \ldots C_{n-1}^{\mu_{n-1}\nu_{n-1}} \,. \end{aligned} \tag{4.49}$$

Inserting this into Ξ, we obtain:

$$\Xi = A_n^{[\mu} P_1^{\nu} C_2^{\mu_2\nu_2} \dots C_{n-1}^{\mu_{n-1}\nu_{n-1}]} \cdot A_1^{\rho} \cdot \xi(P_j, C_j)\,. \tag{4.50}$$

To be able to bring this P_1 and A_1 back into the form of a C_1 we know that we have to antisymmetrize over A_1 as well. Therefore Ξ has to be of the form:

$$\Xi = A_n^{[\mu} P_1^{\nu} A_1^{\rho} C_2^{\mu_2\nu_2} \dots C_{n-1}^{\mu_{n-1}\nu_{n-1}]} \cdot \xi(P_j, C_j)\,. \tag{4.51}$$

Lastly, we need to subtract a similar term with only ν and ρ switched, such that we obtain a $C_1^{\rho\nu}$. Hence, we found, that a term of the form $A_n^{[\mu} \sum_{i=1}^{n-1} P_i^{\nu} \cdot \mathcal{X}^{\alpha_1\dots\alpha_k]} \cdot A_1^{\rho} \cdot \xi(P_j, C_j)$, can only arise from a term of the form:

$$\Xi = A_n^{[\mu} C_1^{\mu_1\nu_1} C_2^{\mu_2\nu_2} \dots C_{n-1}^{\mu_{n-1}\nu_{n-1}]}\,, \tag{4.52}$$

which is of the form $E^{\mu\,\mu_1\,\nu_1\dots\nu_{n-1}}$.

Note, by instead assuming $\mathcal{X}$ to be a product that also includes independent P_i's does not change the argument. The only difference is, that in the end, we would find a term of the form $D^{\mu\nu}$ instead of $E^{\mu\,\mu_1\,\nu_1\dots\nu_{n-1}}$.

3. Case: $\Omega = A_i \cdot A_j \cdot \xi(P_k\big|_{1\leqslant k\leqslant n-1}, C_k\big|_{1\leqslant k\leqslant n-1})$
Now we assume, that two different C_i's "lost" their respective P_i's such that an antisymmetric combination can be formed. This means that we start with the following expression:

$$\begin{aligned}\Xi &= A_n^{[\mu} \sum_{i=1}^{n-1} P_i^{\nu} \cdot \mathcal{X}^{\alpha_1\dots\alpha_k]} \cdot A_1^{\rho} A_2^{\sigma} \cdot \xi(P_k, C_k)\\ &= A_n^{[\mu} (P_1 + P_2 + \dots + P_{n-1})^{\nu} \cdot \mathcal{X}^{\alpha_1\dots\alpha_k]} \cdot A_1^{\rho} A_2^{\sigma} \cdot \xi(P_k, C_k)\,.\end{aligned} \tag{4.53}$$

Analogously to before the part $(P_3 + \dots + P_{n-1})^{\nu}$ has to vanish. Therefore, let us again assume $\mathcal{X}^{\alpha_1\dots\alpha_k}$ to be a product in the corresponding C_i's, such that Ξ reads:

$$\begin{aligned}\Xi &= A_n^{[\mu} (P_1 + P_2)^{\nu} C_3^{\mu_3\nu_3} \dots C_{n-1}^{\mu_{n-1}\nu_{n-1}]} \cdot A_1^{\rho} A_2^{\sigma} \cdot \xi(P_k, C_k)\\ &= A_n^{[\mu} P_1^{\nu} C_3^{\mu_3\nu_3} \dots C_{n-1}^{\mu_{n-1}\nu_{n-1}]} \cdot A_1^{\rho} A_2^{\sigma} \cdot \xi(P_k, C_k)\\ &\quad + A_n^{[\mu} P_2^{\nu} C_3^{\mu_3\nu_3} \dots C_{n-1}^{\mu_{n-1}\nu_{n-1}]} \cdot A_1^{\rho} A_2^{\sigma} \cdot \xi(P_k, C_k)\,.\end{aligned} \tag{4.54}$$

It follows, that for each of those two terms given in equation (4.54) it is only possible to include one of the two A_i's in an antisymmetric combination, while the other has to vanish. This means it is only possible to decompose one C_i to obtain an antisymmetric expression, which is exactly the second case.

The same arguments can be used for all cases with more than two A_i's as well, which means that it is indeed true, that all combinations of the form $A_n^{[\mu} \sum_{i=1}^{n-1} P_i^{\nu]}$ can be expressed using only $D^{\mu\nu}$ and $E^{\mu\,\mu_1\,\nu_1\ldots\nu_{n-1}}$.

Therefore, we have successfully proven our Proposition in the case of $\tilde{P}$ being linear in A_n. As a next step, we need to show that the Proposition also holds true if we allow $\tilde{P}$ to involve terms that are quadratic in A_n.

2. Step: Treating the quadratic case
Let us now assume D_n acts on a single term $\eta_2 = A_n^\mu\, A_n^\nu\, P'_{\mu\nu}(C_i\big|_{1\leqslant i\leqslant n-1}, P_i\big|_{1\leqslant i\leqslant n-1})$ of $\tilde{P}$:

$$D_n\eta_2 = -\sum_{i=1}^{n-1} P_i^\alpha \frac{\partial}{\partial A_n^\alpha}(A_n^\mu\, A_n^\nu\, P'_{\mu\nu}(C_i\big|_{1\leqslant i\leqslant n-1}, P_i\big|_{1\leqslant i\leqslant n-1}))\,. \tag{4.55}$$

Once again, we require this expression to vanish:

$$\begin{aligned} &-\sum_{i=1}^{n-1} P_i^\alpha \frac{\partial}{\partial A_n^\alpha}(A_n^\mu\, A_n^\nu\, P'_{\mu\nu}(C_i\big|_{1\leqslant i\leqslant n-1}, P_i\big|_{1\leqslant i\leqslant n-1})) = \\ &-2\cdot A_n^\mu \left(\sum_{i=1}^{n-1} P_i^\nu\right) P'_{\mu\nu}(C_i\big|_{1\leqslant i\leqslant n-1}, P_i\big|_{1\leqslant i\leqslant n-1})) \stackrel{!}{=} 0\,. \end{aligned} \tag{4.56}$$

Similarly to the first step, we now have to find the most general $P'_{\mu\nu}$ that satisfies this expression. As we have already solved the linear case, we know that we can express $P'_{\mu\nu}$ by:

$$P'_{\mu\nu} = \left(\sum_{i=1}^{n-1} P_i^{\rho_1}\right) T_{[\,\mu\rho_1\,]\,\nu}\,. \tag{4.57}$$

Note, that we can assume $P'_{\mu\nu}$ to be symmetric in its indices, as its antisymmetric part would not contribute to η_2. Therefore, we have to find a $P'_{\mu\nu}$ that satisfies:

$$\left(\sum_{i=1}^{n-1} P_i^\mu\right) P'_{\mu\nu} = \left(\sum_{i=1}^{n-1} P_i^\mu\right) P'_{\nu\mu} = 0\,. \tag{4.58}$$

By inserting our current expression for $P'_{\mu\nu}$ given in (4.57) it follows that the first equation vanishes already:

$$\left(\sum_{i=1}^{n-1} P_i^\mu\right) P'_{\mu\nu} = \left(\sum_{i=1}^{n-1} P_i^\mu\right)\left(\sum_{i=1}^{n-1} P_i^{\rho_1}\right) T_{[\,\mu\rho_1\,]\,\nu} = 0\,. \tag{4.59}$$

It remains to find an expression for $P'_{\mu\nu}$ that also satisfies the second equation, which is given by:

$$\left(\sum_{i=1}^{n-1} P_i^\mu\right)\left(\sum_{i=1}^{n-1} P_i^{\rho_1}\right) T_{[\,\nu\rho_1\,]\,\mu} \stackrel{!}{=} 0\,. \tag{4.60}$$

We are going to solve this equation in two steps.
1: We begin by solving:

$$\left(\sum_{i=1}^{n-1} P_i^{\rho_1}\right) S_{[\,\nu\rho_1\,]} = 0\,. \tag{4.61}$$

From the linear case, we already know, that S has to take the following form such that this equation vanishes:

$$S_{[\,\nu\rho_1\,]} = \left(\sum_{i=1}^{n-1} P_i^{\lambda}\right) R_{[\,\nu\rho_1\lambda\,]}\,. \tag{4.62}$$

2: For any such S we now solve:

$$\left(\sum_{i=1}^{n-1} P_i^{\mu}\right) T_{[\,\nu\rho_1\,]\,\mu} = S_{[\,\nu\rho_1\,]}\,. \tag{4.63}$$

This is an inhomogeneous equation for T, whose general solution is given by a combination of a special solution, as well as the general solution of the homogeneous equation. Therefore, let us now find these two solutions.

General solution to the homogeneous equation
We obtain this solution by solving the following equation:

$$\left(\sum_{i=1}^{n-1} P_i^{\mu}\right) T_{[\,\nu\rho_1\,]\,\mu} = 0\,. \tag{4.64}$$

Recall, that we have already solved such an equation in the linear case, hence, we know that T has to be expressed in the following way for it to be satisfied:

$$T_{[\,\nu\rho_1\,]\,\mu} = \left(\sum_{i=1}^{n-1} P_i^{\rho_2}\right) Q_{[\,\nu\rho_1\,]\,[\,\mu\rho_2\,]}\,. \tag{4.65}$$

Special solution
To complete our solution, we now have to find a special solution to equation (4.63). By inserting S we change the equation to:

$$\begin{aligned}\left(\sum_{i=1}^{n-1} P_i^{\mu}\right) T_{[\,\nu\rho_1\,]\,\mu} &= S_{[\,\nu\rho_1\,]}\\ &= \left(\sum_{i=1}^{n-1} P_i^{\lambda}\right) R_{[\,\nu\rho_1\lambda\,]}\,.\end{aligned} \tag{4.66}$$

It follows, that a possible special solution is given by:

$$T_{[\,\nu\rho_1\,]\,\mu} = R_{[\,\nu\rho_1\mu\,]}\,. \tag{4.67}$$

Therefore, the general solution to equation (4.63) takes the form:

$$T_{[\nu\rho_1]\mu} = \left(\sum_{i=1}^{n-1} P_i^{\rho_2}\right) Q_{[\nu\rho_1][\mu\rho_2]} + R_{[\nu\rho_1\mu]} \,. \tag{4.68}$$

Inserting this result back into our definition of $P'_{\nu\mu}$ we obtain:

$$P'_{\nu\mu} = \left(\sum_{i=1}^{n-1} P_i^{\rho_1}\right) \left(\left(\sum_{i=1}^{n-1} P_i^{\rho_2}\right) Q_{[\nu\rho_1][\mu\rho_2]} + R_{[\nu\rho_1\mu]}\right) \,. \tag{4.69}$$

We have now found the most general expression for $P'_{\nu\mu}$ that satisfies (4.60). However, recall, that there is another condition that has to be met, namely, that $P'_{\nu\mu}$ has to be symmetric in μ and ν, meaning that:

$$P'_{\mu\nu} \stackrel{!}{=} P'_{\nu\mu} \,. \tag{4.70}$$

Looking at the our solution it follows that we have to set $R = 0$ to satisfy this equation. This leaves us with the following expression for $P'_{\mu\nu}$:

$$P'_{\mu\nu} = \left(\sum_{i=1}^{n-1} P_i^{\rho_1}\right) \left(\sum_{i=1}^{n-1} P_i^{\rho_2}\right) Q_{[\mu\rho_1][\nu\rho_2]} \,. \tag{4.71}$$

This tells us, that both A_n's have to appear as part of antisymmetric expressions with $\left(\sum_{i=1}^{n-1} P_i\right)$. From this statement it directly follows that both A_n's may only appear as $D^{\mu\nu}$ or $E^{\mu\mu_1\nu_1\ldots\nu_{n-1}}$, as one can show using the same arguments as in the linear case.

3. Step: Treating the case of 3 A_n's

Let us now treat the case of D_n acting on a term η_3 of $\tilde{P}$ that involves three A_n's before generalizing to an arbitrary number of A_n's. Here we have to show, that:

$$\begin{aligned} D_n\eta_3 &= -\sum_{i=1}^{n-1} P_i^{\alpha} \frac{\partial}{\partial A_n^{\alpha}} (A_n^{\mu_1} A_n^{\mu_2} A_n^{\mu_3} P'_{\mu_1\mu_2\mu_3}(C_i\big|_{1\leqslant i\leqslant n-1}, P_i\big|_{1\leqslant i\leqslant n-1})) \\ &= -3 \cdot A_n^{\mu_1} A_n^{\mu_2} \left(\sum_{i=1}^{n-1} P_i^{\mu_3}\right) P'_{\mu_1\mu_2\mu_3}(C_i\big|_{1\leqslant i\leqslant n-1}, P_i\big|_{1\leqslant i\leqslant n-1})) \stackrel{!}{=} 0 \,. \end{aligned} \tag{4.72}$$

Again, we have to find the most general $P'_{\mu_1\mu_2\mu_3}$ that is symmetric in all its indices and satisfies the equation above. Having worked through the quadratic case beforehand, we may already assume $P'_{\mu_1\mu_2\mu_3}$ to take the following form:

$$P'_{\mu_1\mu_2\mu_3} = \left(\sum_{i=1}^{n-1} P_i^{\rho_1}\right) \left(\sum_{i=1}^{n-1} P_i^{\rho_2}\right) T_{[\mu_1\rho_1][\mu_2\rho_2]\mu_3} \,. \tag{4.73}$$

Due to its symmetry, a $P'_{\mu_1\mu_2\mu_3}$ has to be found such that these three equations vanish:

$$\left(\sum_{i=1}^{n-1} P_i^{\mu_1}\right) P'_{\mu_1\mu_2\mu_3} = \left(\sum_{i=1}^{n-1} P_i^{\mu_2}\right) P'_{\mu_1\mu_2\mu_3} = \left(\sum_{i=1}^{n-1} P_i^{\mu_3}\right) P'_{\mu_1\mu_2\mu_3} \stackrel{!}{=} 0\,. \tag{4.74}$$

The expression for $P'_{\mu_1\mu_2\mu_3}$ we have chosen in (4.73) already satisfies the first two equations, such that the only equation left to be solved is the following:

$$\left(\sum_{i=1}^{n-1} P_i^{\mu_3}\right)\left(\sum_{i=1}^{n-1} P_i^{\rho_1}\right)\left(\sum_{i=1}^{n-1} P_i^{\rho_2}\right) T_{[\,\mu_1\rho_1\,]\,[\,\mu_2\rho_2\,]\,\mu_3} \stackrel{!}{=} 0\,. \tag{4.75}$$

As in the previous case, we are going to solve this equation in two steps.
1: First off, we solve:

$$\left(\sum_{i=1}^{n-1} P_i^{\rho_1}\right)\left(\sum_{i=1}^{n-1} P_i^{\rho_2}\right) S_{[\,\mu_1\rho_1\,]\,[\,\mu_2\rho_2\,]} = 0\,. \tag{4.76}$$

Similarly to the previous cases, we have to express S in the following way to satisfy this equation:

$$S_{[\,\mu_1\rho_1\,]\,[\,\mu_2\rho_2\,]} = \left(\sum_{i=1}^{n-1} P_i^{\lambda}\right) R_{[\,\mu_1\rho_1\,]\,[\,\mu_2\rho_2\lambda\,]}\,. \tag{4.77}$$

2: Next, for any such S, we solve:

$$\left(\sum_{i=1}^{n-1} P_i^{\mu_3}\right) T_{[\,\mu_1\rho_1\,]\,[\,\mu_2\rho_2\,]\,\mu_3} = S_{[\,\mu_1\rho_1\,]\,[\,\mu_2\rho_2\,]}\,. \tag{4.78}$$

Again, we have to find a special solution and the general solution to the homogeneous equation to obtain the general solution for this equation.

General solution to the homogeneous equation
The homogeneous equation is given by:

$$\left(\sum_{i=1}^{n-1} P_i^{\mu_3}\right) T_{[\,\mu_1\rho_1\,]\,[\,\mu_2\rho_2\,]\,\mu_3} = 0\,. \tag{4.79}$$

We already know from the previous cases that the solution to such an equation is given by:

$$T_{[\,\mu_1\rho_1\,]\,[\,\mu_2\rho_2\,]\,\mu_3} = \left(\sum_{i=1}^{n-1} P_i^{\rho_3}\right) Q_{[\,\mu_1\rho_1\,]\,[\,\mu_2\rho_2\,]\,[\,\mu_3\rho_3\,]}\,. \tag{4.80}$$

Special solution
To find a special solution we insert our solution for S into equation (4.78):

$$\begin{aligned}\left(\sum_{i=1}^{n-1} P_i^{\mu_3}\right) T_{[\mu_1\rho_1][\mu_2\rho_2]\mu_3} &= S_{[\mu_1\rho_1][\mu_2\rho_2]} \\ &= \left(\sum_{i=1}^{n-1} P_i^{\lambda}\right) R_{[\mu_1\rho_1][\mu_2\rho_2\lambda]} \,. \end{aligned} \tag{4.81}$$

As we only need to find one solution we may choose the most obvious one, which is given by:

$$T_{[\mu_1\rho_1][\mu_2\rho_2]\mu_3} = R_{[\mu_1\rho_1][\mu_2\rho_2\mu_3]} \,. \tag{4.82}$$

This leads us to the following general solution for equation (4.78):

$$T_{[\mu_1\rho_1][\mu_2\rho_2]\mu_3} = \left(\sum_{i=1}^{n-1} P_i^{\rho_3}\right) Q_{[\mu_1\rho_1][\mu_2\rho_2][\mu_3\rho_3]} + R_{[\mu_1\rho_1][\mu_2\rho_2\mu_3]} \,. \tag{4.83}$$

Inserting this expression for T back into our definition for $P'_{\mu_1\mu_2\mu_3}$ we obtain:

$$P'_{\mu_1\mu_2\mu_3} = \prod_{j=1}^{2}\left(\sum_{i=1}^{n-1} P_i^{\rho_j}\right)\left(\left(\sum_{i=1}^{n-1} P_i^{\rho_3}\right) Q_{[\mu_1\rho_1][\mu_2\rho_2][\mu_3\rho_3]} + R_{[\mu_1\rho_1][\mu_2\rho_2\mu_3]}\right) \,. \tag{4.84}$$

Similarly to the quadratic case, we have to set $R = 0$ such that the symmetry condition of $P'_{\mu_1\mu_2\mu_3}$ is not violated. Therefore, the most general expression for $P'_{\mu_1\mu_2\mu_3}$ is given by:

$$P'_{\mu_1\mu_2\mu_3} = \prod_{j=1}^{3}\left(\sum_{i=1}^{n-1} P_i^{\rho_j}\right) Q_{[\mu_1\rho_1][\mu_2\rho_2][\mu_3\rho_3]} \,. \tag{4.85}$$

The argument that all antisymmetric combinations of A_n's with $\left(\sum_{i=1}^{n-1} P_i\right)$'s can be expressed by $D^{\mu\nu}$ or $E^{\mu\mu_1\nu_1\dots\nu_{n-1}}$ from the linear case also applies here. As we have shown that all three A_n's have to appear as part of such expressions we have proven the Proposition for this case. Seeing that all the arguments of the quadratic case can be applied to this case without problems we will generalize this for a case considering an arbitrary number of A_n's next.

4. Step: Arbitrary Number of A_n's

As a last step, it remains to be shown, that the Proposition also holds true for an arbitrary number of A_n's. Let $\tilde{P}$ be of order m in A_n. Let us now consider a term η_m of $\tilde{P}$ and act on it with D_n. Then we have:

$$\begin{aligned} D_n\eta_m &= -\sum_{i=1}^{n-1} P_i^{\alpha} \frac{\partial}{\partial A_n^{\alpha}} (A_n^{\mu_1} A_n^{\mu_2} \ldots A_n^{\mu_m} P'_{\mu_1\mu_2\ldots\mu_m}(C_i\big|_{1\leqslant i\leqslant n-1}, P_i\big|_{1\leqslant i\leqslant n-1})) \\ &= -m \cdot A_n^{\mu_1} \ldots A_n^{\mu_{m-1}} \left(\sum_{i=1}^{n-1} P_i^{\mu_m}\right) P'_{\mu_1\mu_2\ldots\mu_m}(C_i\big|_{1\leqslant i\leqslant n-1}, P_i\big|_{1\leqslant i\leqslant n-1})) \stackrel{!}{=} 0\,. \end{aligned} \tag{4.86}$$

Assume, that we have already worked through the previous $m-1$ cases using the arguments given in step 2 and 3. Then, we may choose the following form for $P'_{\mu_1\mu_2\ldots\mu_m}$:

$$P'_{\mu_1\mu_2\ldots\mu_m} = \left(\prod_{j=1}^{m-1}\left(\sum_{i=1}^{n-1} P_i^{\rho_j}\right)\right) T_{[\,\mu_1\rho_1\,]\,[\,\mu_2\rho_2\,]\ldots[\,\mu_{m-1}\rho_{m-1}\,]\,\mu_m} \tag{4.87}$$

As in the cases before, we assume $P'_{\mu_1\mu_2\ldots\mu_m}$ to be symmetric in all its indices. Again we have to find the most general $P'_{\mu_1\mu_2\ldots\mu_m}$ that solves the following m equations:

$$\left(\sum_{i=1}^{n-1} P_i^{\mu_1}\right) P'_{\mu_1\mu_2\ldots\mu_m} = \left(\sum_{i=1}^{n-1} P_i^{\mu_2}\right) P'_{\mu_1\mu_2\ldots\mu_m} = \ldots = \left(\sum_{i=1}^{n-1} P_i^{\mu_m}\right) P'_{\mu_1\mu_2\ldots\mu_m}\,. \tag{4.88}$$

One can already see, that we can work through this case analogously to the previous cases. By inserting (4.87) one can check that $P'_{\mu_1\mu_2\ldots\mu_m}$ already satisfies the first $m-1$ conditions. This means that the only equation left to solve is the following:

$$\left(\sum_{i=1}^{n-1} P_i^{\mu_m}\right)\left(\prod_{j=1}^{m-1}\left(\sum_{i=1}^{n-1} P_i^{\rho_j}\right)\right) T_{[\,\mu_1\rho_1\,]\,[\,\mu_2\rho_2\,]\ldots[\,\mu_{m-1}\rho_{m-1}\,]\,\mu_m} \stackrel{!}{=} 0\,. \tag{4.89}$$

To solve this equation we proceed similarly to the previous cases.

①: We begin by solving:

$$\left(\prod_{j=1}^{m-1}\left(\sum_{i=1}^{n-1} P_i^{\rho_j}\right)\right) S_{[\,\mu_1\rho_1\,]\,[\,\mu_2\rho_2\,]\ldots[\,\mu_{m-1}\rho_{m-1}\,]} = 0\,. \tag{4.90}$$

The solution for S then takes the form:

$$S_{[\,\mu_1\rho_1\,]\,[\,\mu_2\rho_2\,]\ldots[\,\mu_{m-1}\rho_{m-1}\,]} = \left(\sum_{i=1}^{n-1} P_i^{\lambda}\right) R_{[\,\mu_1\rho_1\,]\,[\,\mu_2\rho_2\,]\ldots[\,\mu_{m-1}\rho_{m-1}\lambda\,]}\,. \tag{4.91}$$

②: As the second step, we set up the following equation for any such S:

$$\left(\sum_{i=1}^{n-1} P_i^{\mu_m}\right) T_{[\,\mu_1\rho_1\,]\,[\,\mu_2\rho_2\,]\ldots[\,\mu_{m-1}\rho_{m-1}\,]\,\mu_m} = S_{[\,\mu_1\rho_1\,]\,[\,\mu_2\rho_2\,]\ldots[\,\mu_{m-1}\rho_{m-1}\,]}\,. \tag{4.92}$$

To obtain a general solution for T we again have to find a special solution as well as the general solution to the homogeneous equation.

General solution to the homogeneous equation
To find this solution we have to solve:

$$\left(\sum_{i=1}^{n-1} P_i^{\mu_m}\right) T_{[\mu_1\rho_1][\mu_2\rho_2]\dots[\mu_{m-1}\rho_{m-1}]\,\mu_m} = 0\,. \tag{4.93}$$

Having worked through all the previous cases already, we know that the solution is given by:

$$T_{[\mu_1\rho_1][\mu_2\rho_2]\dots[\mu_{m-1}\rho_{m-1}]\,\mu_m} = \left(\sum_{i=1}^{n-1} P_i^{\rho_m}\right) Q_{[\mu_1\rho_1][\mu_2\rho_2]\dots[\mu_{m-1}\rho_{m-1}][\mu_m\rho_m]}\,. \tag{4.94}$$

Special solution
Following the same steps as before, we insert our expression for S into equation (4.92):

$$\begin{aligned}\left(\sum_{i=1}^{n-1} P_i^{\mu_m}\right) T_{[\mu_1\rho_1][\mu_2\rho_2]\dots[\mu_{m-1}\rho_{m-1}]\,\mu_m} &= S_{[\mu_1\rho_1][\mu_2\rho_2]\dots[\mu_{m-1}\rho_{m-1}]}\\ &= \left(\sum_{i=1}^{n-1} P_i^{\lambda}\right) R_{[\mu_1\rho_1][\mu_2\rho_2]\dots[\mu_{m-1}\rho_{m-1}\lambda]}\,.\end{aligned} \tag{4.95}$$

As a special solution we then choose:

$$T_{[\mu_1\rho_1][\mu_2\rho_2]\dots[\mu_{m-1}\rho_{m-1}]\,\mu_m} = R_{[\mu_1\rho_1][\mu_2\rho_2]\dots[\mu_{m-1}\rho_{m-1}\mu_m]}\,. \tag{4.96}$$

Similarly to the previous cases, we find, that we have to set $R = 0$ after inserting our solution for T into our definition for $P'_{\mu_1\dots\mu_m}$ to preserve its symmetry condition. Then, we obtain the following expression for $P'_{\mu_1\dots\mu_m}$:

$$P'_{\mu_1\mu_2\dots\mu_m} = \left(\prod_{j=1}^{m}\left(\sum_{i=1}^{n-1} P_i^{\rho_j}\right)\right) Q_{[\mu_1\rho_1][\mu_2\rho_2]\dots[\mu_{m-1}\rho_{m-1}][\mu_m\rho_m]}\,. \tag{4.97}$$

Therefore, we have found, that even in the general case all A_n's have to appear as part of antisymmetric combinations with $\left(\sum_{i=1}^{n-1} P_i\right)$. Then as a last step one can follow the arguments made in the linear case, to show that all such combinations can be expressed using $D^{\mu\nu}$ and $E^{\mu\mu_1\nu_1\dots\nu_{n-1}}$, which proves the Proposition. □

5 DISCUSSION

5.1 SUMMARY

Our aim was to find building blocks such that we can express an arbitrary independent interaction vertex that is gauge invariant with respect to a varying number of gauge fields as a polynomial. We used two different approaches to this problem. Let us now summarize what we have achieved with each approach.

5.1.1 First Approach

We were successful in finding building blocks for the cases of gauge invariance under either one or two gauge transformations. When studying the case of imposing gauge invariance with respect to three fields, building blocks appeared that could not be expressed using determinants of submatrices of $\mathcal{B}$. This went against our assumption, that all possible gauge invariant building blocks could be expressed in determinants of this form.
Therefore, we were only able to give a generalization of the (1x1) and (2x2) building blocks that would appear when imposing gauge invariance under m out of n gauge fields $\phi_1, \ldots, \phi_m$, for $m < n$:

$$
\begin{aligned}
&\text{(1x1):} \quad s_{ij}\big|_{1\leqslant i<j\leqslant n}\,, \quad y_{ij}\big|_{\substack{m+1\leqslant i\leqslant n\\ 1\leqslant j\leqslant n}}\,, \quad z_{ij}\big|_{m+1\leqslant i<j\leqslant n}\,,\\
&\text{(2x2):} \quad \alpha_{ij,k}\big|_{\substack{m+1\leqslant i<j\leqslant n\\ 1\leqslant k\leqslant m}}\,, \quad \beta_{ij,k}\big|_{\substack{m+1\leqslant i\leqslant n\\ 1\leqslant j\leqslant n\\ 1\leqslant k\leqslant m\\ j\neq k}}\,, \quad \gamma_{ij,k}\big|_{\substack{1\leqslant i<j\leqslant n\\ 1\leqslant k\leqslant m\\ i,j\neq k}}\,, \quad \beta_{ii,k}\big|_{\substack{1\leqslant i\leqslant m\\ 1\leqslant k\leqslant m\\ i\neq k}}\,.
\end{aligned}
\tag{5.1}
$$

5.1.2 Second Approach

Using this approach, we were able to fully generalize the solution. Hence, we are able to express any vertex as a polynomial in known variables such that it is gauge invariant under an arbitrary number of gauge transformations.
In particular, by imposing the vertex generating operator $\mathcal{V}$ to be gauge invariant with respect to m out of all possible n gauge fields, it can be expressed as a polynomial:

$$\mathcal{V} = P\big(C_i^{\mu\nu}\big|_{1\leqslant i\leqslant m}, A_i^{\mu}\big|_{m+1\leqslant i\leqslant n}, P_i^{\mu}\big|_{1\leqslant i\leqslant n}\big)\,, \tag{5.2}$$

for $m < n$.
On the other hand, by requiring $\mathcal{V}$ to be gauge invariant with respect to all n gauge fields, we find:

$$\mathcal{V} = P\big(D^{\mu\nu}, E^{\mu\,\mu_1\,\nu_1\ldots\nu_{n-1}}, C_i^{\mu\nu}\big|_{1\leqslant i\leqslant n-1}, P_i^{\mu}\big|_{1\leqslant i\leqslant n-1}\big)\,. \tag{5.3}$$

5.2 CONCLUSION

Recall, that we can study three distinct cases when classifying independent n-point vertices $\mathcal{L}_n$ depending on the relation of the space-time dimension d and the order n of the vertex.

Let us look into the case $d \geqslant 2n-1$ in more detail:
Here, only two types of equivalence relations need to be taken into account. On the one hand, we have equivalence due to field redefinitions. We have already considered this, by assuming $s_{ii} \approx 0$. On the other hand, another equivalence relation arises due to total derivatives, which we have not yet discussed.

If two Lagrangians $\mathcal{L}$ and $\mathcal{L}'$ only differ by a total derivative, they lead to the same equations of motion and are therefore considered to be equal. Consider now the differential operator $D^\mu = \sum_{i=1}^n P_i^\mu$. We defined an independent vertex $\mathcal{L}_n$ in equation (2.3) as the vertex generating operator $\mathcal{V}$ acting on a product of the different fields ϕ_i.
If we act with D^μ on the product $\left(\prod_{i=1}^n \phi_i(x_i, a_i)\right)$ in the definition of $\mathcal{L}_n$, we get a total derivative. Therefore, two additional equivalence relations have to be imposed, namely [1]:

$$A_i \cdot D = \sum_{i=1}^n y_{ij} \approx 0\,, \tag{5.4}$$

$$P_i \cdot D = \sum_{i=1}^n s_{ij} \approx 0\,. \tag{5.5}$$

However, recall, that we assumed the momentum conservation to hold, meaning that $\sum_{i=1}^n P_i = 0$. Hence, the operator $D^\mu = 0$, and we have:

$$A_i \cdot D = P_i \cdot D = 0\,, \tag{5.6}$$

meaning that the equivalence relations $A_i \cdot D \approx 0$ and $P_i \cdot D \approx 0$ were naturally taken into account in our solution.

Summing up, we have therefore fully classified all independent interacting n-point vertices $\mathcal{L}_n$ for massless bosonic fields with arbitrary spin s for the case $d \geqslant 2n-1$, by finding that all gauge invariant vertex generating operators $\mathcal{V}$ can be expressed as the polynomials given in equations (5.2) and (5.3) above.

Note, that for the other two possible cases $n > d$ and $2n-2 \geqslant d \geqslant n$ one has to consider additional equivalence relations in the form of dimensionally dependent identities when working with the variables s_{ij}, y_{ij} and z_{ij}.
Recall the (2n x 2n) matrix $\mathcal{B}$ that was used in the first approach to find gauge invariant building blocks. It was given by:

$$\mathcal{B} = \begin{pmatrix} \mathcal{S} & \mathcal{Y}^T \\ \mathcal{Y} & \mathcal{Z} \end{pmatrix}\,. \tag{5.7}$$

In case the Fierz equations are imposed, it follows that the diagonal elements of the blocks $\mathcal{S}$, $\mathcal{Y}$ and $\mathcal{Z}$ vanish. Let us assume a vertex generating operator $\mathcal{V}$ that takes the form of the determinant of a (d+1 x d+1) submatrix of $\mathcal{B}$. Acting on the product of fields ϕ_i as in equation (2.3) then leads to a Lagrangian with over-antisymmetrised indices, and hence, to zero. It follows that there exist expressions of $\mathcal{V}$ that lead to a vanishing Lagrangian without being zero themselves. The set of all these Schouten identities is given by the set of all (d+1 x d+1) submatrices of $\mathcal{B}$ [1].

In our second approach we were directly working with the A_i's and P_i's. The reason Schouten identities have to be considered is, because they are non-zero terms that lead to a vanishing Lagrangian, as discussed above. However, in the framework of the A_i's and P_i's these terms vanish themselves already.
Therefore, it is expected that our results for the second approach, given in (5.2) and (5.3), also satisfy the cases $n > d$ and $2n-2 \geqslant d \geqslant n$. Hence, we have fully classified all independent interacting n-point vertices $\mathcal{L}_n$ for massless bosonic fields with arbitrary spin s with $n \geqslant 4$ for all dimensions d.

In [1] a different classification in the same framework has been found. For the case $d \geqslant 2n-1$ the authors find a generalization of the vertex generating operator $\mathcal{V}$ as a polynomial of gauge invariant building blocks multiplied by M^{-1}, where M is the set of all products of the variables s_{ij}.
When comparing the expressions for $\mathcal{V}$ found in this thesis ((5.2),(5.3)), with the one in [1] two advantages come to mind:

1. In general, vertices can contain terms whose gauge variations only vanish, once the equations of motion are imposed. Such terms give rise to deformations of gauge transformations. In [1] it is speculated that gauge invariant vertices do not induce deformations, however, strictly speaking, they can only show this after multiplying with an appropriate product of s_{ij}'s. By expressing $\mathcal{V}$ fully as a polynomial of gauge invariant building blocks, there cannot be any non-trivial terms after gauge varying. Therefore, it can be confidently stated, that no gauge invariant vertex $\mathcal{L}_n$ that is expressed using either equation (5.2) or (5.3) induces deformations.

2. Vertices $\mathcal{L}_n$ described using equations (5.2) and (5.3) are guaranteed to be local, whereas one loses locality when dividing by Mandelstam variables s_{ij}.

5.2.1 Traceless and Transverse Condition

As stated in the preliminaries, we have decided to only consider traceless and transverse vertices. Ideally, we would wish to obtain a full off-shell Lagrangian, however, such a Lagrangian is always going to involve terms that do not satisfy the Fierz equations given in (2.2). These equations involved the traceless and transverse conditions as well as the free equation of motion. Let us now shortly comment on the difficulties that arise by lifting the TT Condition.

Recall, that we described massless bosonic HS gauge fields in the following way:

$$\phi^{(s)}(x,a) = \frac{1}{s!}\phi_{\mu_1\,\mu_2\ldots\mu_s}(x)\, a^{\mu_1}\ldots a^{\mu_s}\,. \tag{5.8}$$

In general we find, that:

$$A\cdot P\,\phi^{(s)}(x,a) = \frac{1}{(s-1)!}\,\frac{\partial}{\partial x_\alpha}\phi_{\alpha\mu_2\ldots\mu_s}\, a^{\mu_2}\ldots a^{\mu_s}\,, \tag{5.9}$$

$$A\cdot A\,\phi^{(s)}(x,a) = \frac{1}{(s-2)!}\,\phi^{\alpha}{}_{\alpha\mu_3\ldots\mu_s}\, a^{\mu_3}\ldots a^{\mu_s}\,. \tag{5.10}$$

Considering these two expressions are no longer zero, the free equation of motion, given by the Fronsdal equation, no longer takes the form of the massless Klein-Gordon equation, but rather the following:

$$P^2\,\phi^{(s)}(x,a) - a\cdot P\,A\cdot P\;\phi^{(s)}(x,a) + (a\cdot P)^2\,A^2\;\phi^{(s)}(x,a) = 0\,. \tag{5.11}$$

The equation of motion now has two terms that directly involve the a's. These a's do not commute with the A's. Clearly, this makes the task of finding a classification more difficult and challenging.
One way to simplify a possible classification without requiring the gauge fields to be traceless and transverse is to use the deDonder gauge. The deDonder gauge is allowed off-shell and sets a combination of the divergence and the traceless part to be zero. It is given by:

$$\partial^\sigma\,\phi_{\mu_1\ldots\mu_{s-1}\sigma} - \frac{1}{2}\partial_{(\mu_1}\,\phi_{\mu_2\ldots\mu_{s-1})\sigma}{}^{\sigma} = 0\,. \tag{5.12}$$

5.3 OUTLOOK

5.3.1 First Approach

The progress on the first approach came to a halt, since we were unable to characterize a certain type of terms, as the one given in (3.75). It is probable, that this characterization is one of only a few missing pieces left to find the generalization of a vertex that is gauge invariant under an arbitrary amount of gauge fields. It might be interesting to look back into this despite the second approach giving better results at this moment. One reason for this is that other works in higher spin theories, for example [1, 17, 21], use variables similar to s_{ij}, y_{ij} and z_{ij}, so our results could be compared and applied to these works in the framework of our first approach more easily.

Solving this problem would lead to a classification for the case $d \geqslant 2n-1$. To obtain a classification for all possible dimensions one would have to study the cases $n > d$ and $2n-2 \geqslant d \geqslant n$ as well. Contrary to the second approach, when working with s_{ij}, y_{ij} and z_{ij} Schouten identities have to be considered as they do not vanish directly.

Let us assume, that no new vertices arise when taking Schouten identities into account, as argued in ([1], p. 13). Then our solution gets simplified due to a large number of possible Schouten identities for the case $n > d$. Therefore, one has to find which building blocks can be described by a Schouten identity and set to zero. There exists an overwhelming amount of Schouten identities for the case $n > d$, so a possible first step is to find a smaller set of Schouten identities from which all other Schouten identities can be generated, similarly to what is done in [22] for $d = 3$ and [1] for general dimensions. Then, this set of generators can be compared to our set of building blocks to obtain the possible simplifications [1].
Note, that only certain levels of Schouten identities appear in the case $2n - 2 \geqslant d \geqslant n$, so the set of generators is going to be different and has to be handeled separately.

5.3.2 AdS Spacetime

As briefly mentioned in the introduction, there exists a correspondence between HS theories on (d+1)-dimensional Anti-de-Sitter spaces and d-dimensional conformal field theories. The AdS/CFT correspondence is a famous example for holographic duality. Studying it seems essential to further our understanding on Quantum Gravity as it connects gravitational and quantum theories.
In our classification we have considered all gauge fields to be massless and living in Minkowski spacetime. Therefore, it would be interesting to figure out if our results can be carried over to AdS spacetime.

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