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„Static solutions to the mass-less Einstein-Vlasov system with a non-vanishing cosmological constant“

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Contents

Kurzfassung

Gegenstand der hier vorgestellten Arbeit ist das masselose Einstein-Vlasov System mit einer positiven beziehungsweise negativen kosmologischen Konstante Λ.
Es wird gezeigt, dass statische, sphärisch symmetrische Lösungen zu beiden Systemen existieren. Diese haben die Form von Photonenschalen und zeigen im Falle einer positiven kosmlogischen Konstante ein Verhältnis $\frac{2m(r)}{r}$, welches sich an $\frac{8}{9}$ annähert. Das bedeutet, dass diese Lösungen höchst relativistisch sind. Die Vakumregion außerhalb der Photonen Schale entspricht im Falle einer positiven kosmologischen Konstanten, dem Schwarzschild-deSitter Raum, im Falle einer negativen Konstanten dem Schwarzschild-Anti-deSitter Raum. Eine grundlegende Bedingung in dieser Arbeit ist, dass der innere Radius der Photonenschale, beliebig klein gewählt werden kann. Diese Arbeit präsentiert eine Erweiterung der Arbeit von Andréasson. H, et al., [14], in welcher das statische, sphärisch symmetrische Einstein-Vlasov System ohne kosmologische Konstante betrachtet wurde.

Abstract

In this paper we investigate the mass-less Einstein-Vlasov system with a positive and a negative cosmological constant Λ. We show that static, spherically symmetric, photon shell solutions of this system exist. For these solutions with a positive cosmological constant it can be shown that $\frac{2m(r)}{r}$ approximates $\frac{8}{9}$, i.e. the solutions are highly relativistic.
The exterior vacuum region of these results coincides with the Schwarzschild-deSitter and Schwarzschild-anti-deSitter space, for a positive or a negative constant, respectively. The most important constraint we impose is that the inner radius of the matter region can be chosen arbitrary small. This work represents an extension of the paper from Andréasson. H, et al., [14], in which the static, spherically symmetric Einstein-Vlasov system without the cosmological constant was investigated.

1 Introduction

The gravitational force in the universe arises due to the curvature of space time. Moreover the curvature is influenced by the energy and matter present.
The Einstein-field-equations describe how energy and matter are connected to the curvature of space time, [16]. One solution of these equations is the Schwarzschild solution, which is the only static, spherically symmetric vacuum solution, [22].
One important question is how to describe the matter in the universe. In this paper we choose the Vlasov equation for this. It describes collision-less particles, which are all of the same type, i.e. they all have the same mass. More precisely it shows a system of particles within a force field, depending on their position x and velocity v, as well as the time t [27]. In our case this force field will be the gravitational field caused by the particles themselves.
The particles we are interested in could be stars in a galaxy, galaxies in a galaxy cluster or even larger cosmological objects. Since interactions between stars or galaxies are scarce, the Vlasov equation is a well-suited approximation. Furthermore, an advantage of Vlasov matter, in comparison to, e.g. perfect fluids, is that no singularities arise, when small initial data is considered [21]. The Einstein-Vlasov system describes a system, where the Einstein-field-equations are coupled to the Vlasov-equation. The mathematical description, i.e. the system of equations will be shown in section 2.

The first to study the massive Einstein-Vlasov system, were Rein and Rendall in the early 1990s. In [21] it is shown that this system has global solutions, for spherical symmetry and sufficiently small initial data. Rein G., et al also showed that solutions of the Einstein-Vlasov system can only develop a singularity, if this singularity is at the center, [23].
In [22] and [20] the existence of static, spherically symmetric solutions was shown, which have finite total mass as well as finite extensions of the matter. Furthermore, they showed that a finite total mass implies, that the boundary conditions for the metric coefficients, see section 2, have to hold. This corresponds to an

asymptotically flat space time. The coordinates they used, do not only lead to global solutions, but also show that these solutions are singularity-free, i.e. the resulting space time is timelike and null geodesically complete. They considered both, solutions with a regular center and ones with a singularity, i.e. a Schwarzschild black hole at the center.
A good overview over the global properties of solutions of the Einstein-Vlasov system gives [8]. There the author looks at the found results from the point of kinetic theory.
A paper with a similar approach to this work was done by Andréasson H. et al. in 2015. They studied the massive, static, spherically symmetric Einstein-Vlasov system with a negative and a sufficiently small positive cosmological constant. The obtained solutions coincide with the Schwarzschild-anti-deSitter and the Schwarzschild-deSitter solution in the exterior vacuum region, respectively, [13].

There are also numerous papers, which deal with the Einstein-Vlasov system under different constraints, than the ones we considered. As mentioned earlier, in [20] a spherically symmetric system with non-isotropic pressure was studied. Several papers analyze systems with different symmetry conditions. See [11] and [12] for solutions of an axially symmetric system. Here solutions are found for the non-rotating and the rotating case. [17] proves global existence for the cylindrically symmetric Einstein system, not only coupled to Vlasov matter, but also to perfect fluids. The analysis covers both time-independent and time-dependent space times. Andersson et al. showed that if one couples the Einstein equations to elastic matter, then solutions can be found without any symmetric assumptions, [5]. Furthermore they found solutions to a rotating system with elastic matter, [6]. More recent results for non-spherically symmetric space-times were found by Ames E., et al, showing the existence of solutions for the axisymmetric and toroidal Vlasov system, [4, 2, 3].

The aim of this work is to investigate the mass-less Einstein-Vlasov system with a non-vanishing cosmological constant. Therefore it is a continuation of the work from Andréasson H., et al., [14] and the outline of the proof will closely follow their approach. In their paper the existence of static, spherically symmetric solutions for the mass-less Einstein-Vlasov system is shown. A similar approach was used in [9], where the mass-less system with a Schwarzschild singularity at the center was considered. The most significant difference between the system with and without singularity is the treatment of the inner radius of the matter region R_0. If there is a black hole at the center, this radius has to be sufficiently large. Whereas if we have a regular center, we want it to be arbitrarily small, as shown later in the proof. The first to study the mass-less system was Rendall in 1996, [25]. He investigated the dynamics of solutions both for the massive and mass-less case, showing that the mass-less case behaves differently near the singularity than the already known results for perfect fluids showed. In 2014 Akbarian A., et al. solved the mass-less Einstein-Vlasov system by numerical integration and showed that solutions with compact support exist, [1].

Implications of the cosmological constant Λ

An extra term with the cosmological constant on the geometry side of the Einstein-field-equations was already added by Einstein in his original paper on general relativity. His idea was to add a repulsive term to balance the attractive gravitational force in order to obtain a static universe, [15]. The form then proposed was:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{1}$$

Here $G_{\mu\nu}$ is the Einstein tensor, $g_{\mu\nu}$ the metric of the space time, $T_{\mu\nu}$ the energy-momentum tensor and κ is a constant. This alteration however changed the non-relativistic limit of the Einstein-field-equations and it did no longer agree with Newtons equations. Another problem of this idea is that experiments showed that our universe is not static but in fact expanding. [19].
A different approach is to add an extra source term on the right-hand side of the Einstein-field-equations. This extension would then be called the vacuum-energy-momentum tensor $T^{\Lambda}_{\mu\nu}$, which describes the energy-density ρ_Λ and the pressure p_Λ of the vacuum [15].

$$\rho_\Lambda = \frac{\Lambda c^2}{8\pi G_N} \ \ \& \ \ p_\Lambda = -\rho_\Lambda c^2 \tag{2}$$

These equations correspond to a positive cosmological constant Λ. We see that depending on the sign of Λ, the vacuum pressure can be negative, leading to a repulsive force or positive, leading to an attractive one.

In modern physics the cosmological constant is used as an explanation for two fundamental problems. The first one deals with the missing energy in the universe. Experimental data suggests that the universe is flat at very large scales. Furthermore, the Friedmann equation states that the energy-density of the universe has to be equal to a critical value. However, the observable matter and energy only make up about 25 percent of this critical value [15]. Hence, a different type of energy, often referred to as dark energy could fill this gap and the cosmological constant, i.e. the vacuum energy, could be a possible explanation.

The second problem is called the cosmic age problem. The calculated time of an expanding universe is younger than the measured age of some stars. This implies that the universe is in fact older, which can be explained if its expansion is accelerating. Since a positive Λ is associated with an accelerated expansion of the universe, it could be a suitable explanation [19].

Outline of the paper

This work is organized in the following manner. In section 2 there will be given a short review of the static, spherically symmetric Einstein-Vlasov system as well as known result and an overview of the used notation. The ansatz we use for the distribution function $f(x,v)$, see eq. (15) reduces to an integro-differential equation, due to the spherical symmetry, [13]. This equation is essential for the proof, since it is used to calculate the matter quantities, $\rho(r), p(r)$ and $p_T(r)$. Furthermore, we introduce a new variable $\gamma(r)$, see eq. (21), called the characteristic function. We use this function to control the distribution function $f(x,v)$ and therefore the matter quantities.

In section 3 the main theorem for the existence of globally regular solutions is stated.

Section 4 shows the proof of the main theorem with a negative cosmological constant and section 5 will show a similar proof, but with a positive constant. This proof is divided into several steps. It will start by showing, that estimates on the lower and upper bounds of the matter quantities $\rho(r)$, $p(r)$ and $p_T(r)$ can be found for a certain interval $[R_0, 2R_0]$, see section 4.1. This interval margins the photon shells, i.e. within it matter appears. Then we will continue to show that the characteristic function $\gamma(r)$ and its derivative are bounded. In particular we need to show that the inner region is vacuum, i.e. $\gamma \leq 0$ for $r < R_0$, because then all the matter functions become zero as well. In order for the characteristic function to become positive at R_0, its derivative has to be positive for $r \leq R_0$.
In section 4.2 we will see that the characteristic function is indeed increasing and becomes positive on the interval $[R_0, 2R_0]$. Furthermore, it is bounded and can be estimated by a power of r.
In section 4.3 we take a look at the behavior of the characteristic function as r approaches the outer radius R_1. In order for matter to disappear at this outer radius, $\gamma(r)$ has to decrease and we will show that it approaches a certain value γ_*. Furthermore, we find an upper bound for this radius r_2, in particular it is bounded by powers of R_0. The next step will be to proof proposition 1, which shows the following bound $\frac{m(r_2)}{r_2} \geq \frac{4}{9} - C$. Here C indicates a constant, which depends on the cosmological constant Λ and the inner radius R_0.
This then leaves the last step, section 4.5, which is to show that the distribution function $f(x, v)$ vanishes again and that this happens within the interval $[R_1, R_2]$. In order to do this we introduce a new variable x, dependent on the characteristic function and the radius. If this variable x goes to infinity, it means that the characteristic function goes to zero. Thus the idea is to show that $x \to \infty$ in this interval $[R_1, R_2]$.
The last section, section 6 provides a resumé of the proof and a discussion on the implications of the results.

2 Preliminaries

2.1 The Einstein-Vlasov System

The Einstein equation

Here we will use the metric in Schwarzschild coordinates.

$$g = -e^{2\mu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2 d\Theta^2 + r^2 \sin(\Theta)^2 d\phi^2 \tag{3}$$

$$r \in [0, \infty), \quad \Theta \in [0, \pi), \quad \phi \in [0, 2\pi)$$

In order to ensure a regular center at $r = 0$, we use the following boundary condition, [20].

$$\lambda(0) = 0 \tag{4}$$

The Einstein-field-equation have the form, see [13]:

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = 8\pi T_{ab} \tag{5}$$

with R_{ab} the Ricci tensor, R the Ricci scalar, g_{ab} the metric and T_{ab} the energy-momentum tensor in the three spatial dimensions. Λ denotes the cosmological constant. The gravitational constant G and the velocity of light c are set to 1. This system in spherical symmetry has two non trivial components and one extra solutions, which is compounded of the other two. Thus the static mass-less Einstein equations reduce to:

$$e^{-2\lambda}(2r\lambda' - 1) + 1 - r^2\Lambda = 8\pi r^2 \rho(r) \tag{6}$$

$$e^{-2\lambda}(2r\mu' + 1) - 1 + r^2\Lambda = 8\pi r^2 p(r) \tag{7}$$

$$e^{-2\lambda}\left(\mu''(r)\left(\mu + \frac{1}{r}\right)(\mu' - \lambda')\right) = 8\pi p_T \tag{8}$$

Here ρ(r) is the energy density, p(r) is the radial pressure and p_T is the tangential pressure, respectively. The prime stands for a derivative, with respect to the radius r, [13].

The Vlasov Equation

The Vlasov equation describes a system of mass-less, identical particles which are non-interacting, i.e. free falling particles, [14]. It is fulfilled by the matter distribution $f(x, v)$. In this paper $f(x, v)$ is static, i.e. does not depend on the time as well as spherically symmetric, [20]. Spherical symmetry is ensured if we set $f(x, v) = f(Ax, Av)$, with A being a matrix of the type $A \in SO(3)$, [14]. Hence, the Vlasov equation takes the following form:

$$\frac{e^{\mu-\lambda}}{|v|} v^a \frac{\partial f(x,v)}{\partial x^a} - e^{\mu-\lambda} \frac{x^a}{r} |v| \mu' \frac{\partial f(x,v)}{\partial v^a} = 0 \tag{9}$$

The connection between the Einstein equation, eq. (5) and the Vlasov equation, eq. (9), are the field equations for the Einstein-Vlasov system. In order to find a complete system of equations, a definition of the energy momentum tensor in terms of the distribution function $f(x, v)$ and the metric g_{ab} is necessary, [24]. In the coordinates (x^a, p^a) the tensor T_{ab} takes the form of an integral over the distribution function and the determinant of the metric. When we introduce intrinsic initial data, we can find expressions for the matter quantities:

$$\rho(r) = \int_{\mathbb{R}^3} f(x,v)|v| \, dv^1 \, dv^2 \, dv^3 \tag{10}$$

$$p(r) = \int_{\mathbb{R}^3} \frac{f(x,v)}{|v|} \left[\frac{\delta_{ab} x^a v^b}{r} \right]^2 dv^1 \, dv^2 \, dv^3 \tag{11}$$

$$p_T = \frac{1}{2} \int_{\mathbb{R}^3 \setminus 0} \left| \frac{x \times v}{r} \right|^2 \frac{f(x,v)}{|v|} \, dv^1 \, dv^2 \, dv^3 \tag{12}$$

For a more detailed derivation of this equation see [24].

Furthermore we want the matter distribution $f(x,v)$ to satisfy the Vlasov equation automatically. We can ensure this by defining a $f(x,v)$ that takes the form of a sufficiently regular function $\Phi(E,L)$. This function should only depend on the particle energy E and the square of the angular momentum L. Those two functions are given by:

$$E = e^{\mu}|v| \qquad L = x^2v^2 - (xv)^2 \tag{13}$$

With those we can define the matter function as follows:

$$f(x,v) = \Phi(E,L) \tag{14}$$

In this paper we use $\Phi(E,L)$ of the form:

$$\Phi(E,L) = \left[1 - \frac{E}{E_0}\right]_+^k [L - L_0]_+^l \tag{15}$$

where E_0 is the cut-off energy and L_0 is the minimal value of the angular momentum, with the following boundary conditions:

$$L_0 > 0, \quad E_0 > 0, \quad k \geq 0, \quad l \geq -\frac{1}{2} \tag{16}$$

When we insert the ansatz, eq. (15) into the equations for the matter quantities and transform the variables, we find the following expressions:

$$\rho(r) = c_l r^{2l} \int_{\sqrt{\frac{L_0}{r^2}}}^{e^y} (1 - \epsilon e^{-y})^k \epsilon^2 \left(\epsilon^2 - \frac{L_0}{r^2}\right)^{l+\frac{1}{2}} d\epsilon \tag{17}$$

$$p(r) = \frac{c_l}{2l+3} r^{2l} \int_{\sqrt{\frac{L_0}{r^2}}}^{e^y} (1 - \epsilon e^{-y})^k \left(\epsilon^2 - \frac{L_0}{r^2}\right)^{l+\frac{3}{2}} d\epsilon \tag{18}$$

Here the constant c_l is defined as :

$$c_l = 2\pi \int_0^1 \frac{s^l}{\sqrt{1-s}} ds \tag{19}$$

A more detailed derivation of these equations can be found in [20], where the calculations for the massive case are shown.

It is convenient to use a different variable instead of the metric coefficient $\mu(r)$. This new variable $y(r)$, allows us to eliminate the cut-off energy E_0 from the equations. The value of E_0 is fixed and dependent on the initial value of μ_0, but usually not known. Thus it is possible to find a solution $y(r)$ of the system by using μ_0 instead.

$$y(r) = ln(E_0) - \mu(r) \tag{20}$$

Furthermore by solving for $y(r)$, it is possible to determine E_0 in such a manner, that the solution is asymptotically flat, [14].

In order to be able to control the matter quantities, we will use the characteristic function $\gamma(r)$, defined by the equation:

$$\gamma(r) = y(r) - \frac{1}{2} ln \left(\frac{L_0}{r^2} \right) \tag{21}$$

It is important to note, that matter is present, i.e. the matter distribution is positive, **if and only if the characteristic function is positive**. This can be seen from the integration limits in eq. (17) and eq. (18). This implies that:

$$\gamma(r) \leq 0 \leftrightarrow f(x, v) = 0 \tag{22}$$

We want to obtain a solution of the form of a static, spherically symmetric photon shell. On the inside of this shell, there is the inner vacuum region bounded by the radius R_0. We define this radius as:

$$R_0 = \sqrt{\frac{L_0}{e^{2y_0}}} \tag{23}$$

with $y(R_0) = y_0$.

$$\gamma(R_0) = y_0 - \frac{1}{2} ln \left(\frac{L_0}{R_0^2} \right) = y_0 - \frac{1}{2} ln \left(e^{2y_0} \right) = 0 \tag{24}$$

This shows that the characteristic function will be zero for the first time at R_0.

An important equation, which will be used in this proof is the Hawking mass. It links the energy-density $\rho(r)$ to the mass m, [14].

$$m(r) = 4\pi \int_0^r s^2 \rho(s)\, ds \tag{25}$$

Finally we also need an equation that describes a spherically symmetric object in gravitational equilibrium. The analytical background of finding static solutions of the Einstein field equations for fluid spheres was given by Tolman in 1939 [26]. The combined equation for an isotropic, spherically symmetric object is called the generalized Tolman-Oppenheimer-Volkoff (TOV) equation [18]. For the the massive Einstein-Vlasov system a derivation of the TOV equation was shown, [20], which was adapted for the mass-less case in [14]. It reads:

$$p' = y'(\rho + p) - \frac{2}{r}(p - p_T) \tag{26}$$

.

3 Main theorem

As mentioned earlier we want to show that there exist static, spherically symmetric solutions of the Einstein-Vlasov system for mass-less particles and with a cosmological constant. In order to define these solutions with compactly supported matter quantities, we use a tool from Andréasson. H, et al., the definition of a quasi shell solution. A quasi shell solution is defined as a set of the functions $\mu(r)$, $\lambda(r) \in C^1([0,\infty))$ and $f(x,v) \in L^1(\mathbb{R}^6)$. These have to fulfill the Einstein-Vlasov system. Further, the matter quantities are differentiable functions and their support and the support of $f(x,v)$ is contained in (R_0,∞), for a radius $0 < R_0 < \infty$, [14]. We will start with a theorem for the Einstein-Vlasov system with a negative cosmological constant and continue with a theorem for the system with a positive one.

Theorem 3.1. *For an ansatz of the matter function $f(x,v)$ of the form, eq.* (14)*, with the corresponding regular center condition $\lambda(0) = 0$, we can find solutions to the Einstein-Vlasov system with a negative cosmological constant, which are static, spherically symmetric and asymptotically flat. Furthermore the matter quantities $\rho(r)$ and $p(r)$ are compactly supported on the interval $[R_0, R_1]$. These solutions have the following property, for R_0 sufficiently small and for $R_1 > R_0$ finite:*

$$\frac{4}{5} < \underset{r\in[0,\infty)}{sup} \frac{2m(r)}{r} \tag{27}$$

In case of a negative cosmological constant, the upper bound of the ratio $\frac{m(r)}{r}$ has not yet been investigated. Therefore we leave the ratio $\frac{m(r)}{r}$ unbounded from above. The lower bound was investigated numerically and is the same for all constructed solutions, [14].

The solution can be extended to a global solution, by gluing the Schwarzschild-Anti-deSitter metric to it, for $r > R_2$. The resulting metric has the following coefficients, $\lambda_A(r)$ and $\mu_A(r)$:

$$e^{-\lambda_A} = e^{\mu_A(r)} = 1 + \frac{r^2|\Lambda|}{3} - \frac{2M}{r} \tag{28}$$

Here M denotes the mass parameter of the photon shell.

Theorem 3.2. *By choosing an ansatz of the form eq.* (14) *for the matter function $f(x,v)$ and ensuring the corresponding regular center condition $\lambda(0) = 0$, we can find solutions to the Einstein-Vlasov system with a positive cosmological constant which are static, spherically symmetric and asymptotically flat. Furthermore the matter quantities $\rho(r)$ and $p(r)$ are compactly supported on the interval $[R_0, R_1]$. For R_0 sufficiently small and for $R_1 > R_0$ finite, we can show that the solutions have the following property:*

$$\frac{4}{5} < \sup_{r\in[0,\infty)} \frac{2m(r)}{r} < \frac{8}{9} \tag{29}$$

The upper bound is a result of the Buchdahl inequality for isotropic pressure and non-increasing energy density. For a positive cosmological constant it is modified to the following equation, [10]:

$$\frac{m(r)}{r} \leq \frac{2}{9} - \frac{\Lambda r^2}{3} + \frac{2}{9}\sqrt{1 + 3\Lambda r^2} \tag{30}$$

This inequality holds for $r \in [R_0, R_1]$, which is the interval of the static photon shell, i.e. here matter is present. When we investigate its derivative in order to find extreme values, we see that the maximum of this ratio corresponds to the minimal radius R_0. Furthermore, this expression approaches the Buchdahl limit, $\frac{m(r)}{r} \leq \frac{4}{9}$, for $\Lambda r^2 \to 0$. This agrees with our constraint on the inner radius, i.e. we can choose R_0 as small as necessary, to let this ratio converge towards the Buchdahl limit. The lower bound in the ratio is the same for all constructed solutions and was found numerically, [14].

For a Einstein-Vlasov system with a positive cosmological constant, we can find global solutions by glueing the Schwarzschild-deSitter metric to our solution. This metric has the following Schwarzschild-deSitter coefficients $\lambda_S(r)$, $\mu_S(r)$:

$$e^{-\lambda_S} = e^{\mu_S(r)} = 1 - \frac{r^2\Lambda}{3} - \frac{2M}{r} \tag{31}$$

with the mass parameter M.

4 Outline of the proof for a negative cosmological constant Λ

The outline of this part will follow the one from [14]. The similar sections are included for completeness. and the sections, which are influenced by the additional cosmological constant are shown is more detail.

4.1 Bounds on the matter quantities

We begin with the proof by showing that the matter quantities ρ, p and p_T are bounded from below and above.

Lemma 1. *Let $k \geq 0$ and $l \geq -\frac{1}{2}$. Moreover let R_0 have the form as defined in eq.* (23) *and $\gamma(r)$ as in eq.* (21)*. Then for $r \in [R_0, 2R_0]$, two positive constants C_u and C_l can be found, which depend on k, l and L_0 such that when $\gamma \geq 0$ and $z = \rho - p - 2p_T$,*

$$C_l \frac{\gamma^{l+k+\frac{3}{2}}}{r^4} \leq \rho(r) \leq C_u \frac{\gamma^{l+k+\frac{3}{2}}}{r^4} \tag{32}$$

$$C_l \frac{\gamma^{l+k+\frac{5}{2}}}{r^4} \leq p(r) \leq C_u \frac{\gamma^{l+k+\frac{5}{2}}}{r^4} \tag{33}$$

$$C_l \frac{\gamma^{l+k+\frac{3}{2}}}{r^2} \leq z(r) \leq C_u \frac{\gamma^{l+k+\frac{3}{2}}}{r^2} \tag{34}$$

Proof of eq. (32)

Proof. We start with the expression for $\rho(r)$ as stated in eq. (17). In order to rewrite it in a more understandable form, we first use,

$$e^y = e^\gamma \sqrt{\frac{L_0}{r^2}} \tag{35}$$

which follows from the definition of γ given in eq. (21).
This gives us an integral of the form:

$$\rho(r) = c_l r^{2l} \int_{\sqrt{\frac{L_0}{r^2}}}^{e^\gamma \sqrt{\frac{L_0}{r^2}}} \left(1 - \epsilon e^{-\gamma}\sqrt{\frac{r^2}{L_0}}\right)^k \epsilon^2 \left[\epsilon^2 - \frac{L_0}{r^2}\right]^{l+\frac{1}{2}} d\epsilon \tag{36}$$

In the next step we substitute:

$$a = \sqrt{\frac{L_0}{r^2}}, \quad b = e^\gamma \tag{37}$$

This changes the integral to:

$$\rho(r) = c_l r^{2l} \int_a^{ab} \left(1 - \frac{\epsilon}{ab}\right)^k \epsilon^2 \left[\epsilon^2 - a^2\right]^{l+\frac{1}{2}} d\epsilon \tag{38}$$

With a further change of variables given by $t = \frac{\frac{\epsilon}{a}-1}{b-1}$ and some transformations, we find:

$$\rho(r) = c_l \frac{L_0^{l+2}}{r^4} b^{-k}(b-1)^{k+l+\frac{3}{2}} \quad I_{k,l,\rho}(e^\gamma) \tag{39}$$

$$I_{k,l,\rho}(e^\gamma) = \int_0^1 t^{l+\frac{1}{2}}(1-t)^k (t(b-1)+1)^2 (t(b-1)+2)^{l+\frac{1}{2}}\, dt \tag{40}$$

where $I_{k,l,\rho}(e^\gamma)$ is always positive due to the assumption $\gamma \geq 0$. We can find a lower and upper bound of $I_{k,l,\rho}(e^\gamma)$, by using the bounds on k and l from lemma 1 and the assumption $\gamma \geq 0$.

$$I_{k,l,\rho} \geq \int_{\frac{1}{4}}^{\frac{1}{2}} \left(\frac{1}{4}\right)^{l+\frac{1}{2}} \left(\frac{1}{2}\right)^k 2^{l+\frac{1}{2}}\, dt = 2^{-(k+l+3)} \tag{41}$$

$$I_{k,l,\rho} \leq \int_0^1 b^2(b+1)^{l+\frac{1}{2}}\, dt = b^2(b+1)^{l+\frac{1}{2}} \tag{42}$$

We can see that the upper bound only depends on b, which means that it is bounded if b is bounded, which is true for $r \in [R_0, 2R_0]$. In order to further simplify the expression for $\rho(r)$, we combine all the constants to a new positive constant. Here $I_{l,\rho}$ and $I_{u,\rho}$ are the lower and upper bound of the integral $I_{k,l,\rho}$ respectively.

$$c_l \frac{L_0^{l+2}}{r^4} I_{l,\rho} e^{-\gamma k}(e^\gamma - 1)^{k+l+\frac{3}{2}} \leq \rho \leq c_l \frac{L_0^{l+2}}{r^4} I_{u,\rho} e^{-\gamma k}(e^\gamma - 1)^{k+l+\frac{3}{2}} \tag{43}$$

$$C_l e^{-\gamma k} \frac{(e^\gamma - 1)^{k+l+\frac{3}{2}}}{r^4} \leq \rho(r) \leq C_u e^{-\gamma k} \frac{(e^\gamma - 1)^{k+l+\frac{3}{2}}}{r^4} \tag{44}$$

The last step of the proof is to rewrite $(e^\gamma - 1)$ and $e^{-\gamma k}$. For this we use the derivative of $\gamma(r)$.

$$\mu'(r) = e^{2\lambda}\left(4\pi r p + \frac{m(r)}{r^2} - \frac{r\Lambda}{3}\right) \tag{45}$$

$$y'(r) = -\mu' = -e^{2\lambda}\left(4\pi r p + \frac{m(r)}{r^2} - \frac{r\Lambda}{3}\right) \tag{46}$$

$$\gamma'(r) = y' + \frac{1}{r} = -\mu' + \frac{1}{r} = -e^{2\lambda}\left(4\pi r p + \frac{m(r)}{r^2} - \frac{r\Lambda}{3}\right) + \frac{1}{r} \tag{47}$$

For $\Lambda < 0$ is takes the form:

$$\gamma'(r) = -e^{2\lambda}\left(4\pi r p(r) + \frac{m(r)}{r^2}\right) - e^{2\lambda}\frac{r|\Lambda|}{3} + \frac{1}{r} \tag{48}$$

This shows for the last equation that $\gamma' \leq \frac{1}{r}$, because the first two terms drop out due to their sign.

From eq. (24) we know that $\gamma(R_0) = 0$. This gives:

$$\gamma(r) = y(r) - \frac{1}{2} ln\left(\frac{L_0}{r^2}\right) \tag{49}$$

$$\gamma(r) \leq -\frac{1}{2}ln\left(\frac{L_0}{r}\right) = -\frac{1}{2}ln\left(\frac{R_0^2 e^{2y_0}}{r}\right) = -ln\left(\frac{R_0}{r}\right) - ln(e^{y_0}) = ln\left(\frac{r}{R_0}\right) - y_0 \tag{50}$$

If we consider that y_0 is a positive constant and use our constraint for $r \in [R_0, 2R_0]$, we get:

$$\gamma(r) \leq ln\left(\frac{2R_0}{R_0}\right) = ln(2) < 1 \tag{51}$$

So $\gamma(r)$ is bounded from below and above. This gives the following bounds on $e^{-\gamma k}$:

$$e^{-k} \leq e^{-\gamma k} \leq e^0 = 1 \tag{52}$$

In order to evaluate e^γ we will use its Taylor expansion

$$e^\gamma = \sum_{j=0}^{\infty} \frac{\gamma^j}{j!} \leq 1 + \gamma + \frac{1}{2}\sum_{j=2}^{\infty} \gamma^j = 1 + \gamma + \frac{\gamma^2}{2(1-\gamma)} \tag{53}$$

$$\gamma(r) \leq e^\gamma - 1 \leq \gamma \frac{2-\gamma}{2-2\gamma} = \gamma(r)C \tag{54}$$

When we insert the last expression into eq. (44), we get:

$$C_l e^{-k} \frac{\gamma^{k+l+\frac{3}{2}}}{r^4} \leq \rho(r) \leq C_u C^{k+l+\frac{3}{2}} \frac{\gamma^{k+l+\frac{3}{2}}}{r^4} \tag{55}$$

$$C_l \frac{\gamma^{k+l+\frac{3}{2}}}{r^4} \leq \rho(r) \leq C_u \frac{\gamma^{k+l+\frac{3}{2}}}{r^4} \tag{56}$$

This concludes the proof for $\rho(r)$. □

Proof for eq. (33)

Proof. We will continue with the proof for p(r), be rewriting eq. (18) in the same manner as we did with $\rho(r)$.

$$p(r) = \frac{c_l}{2l+3} r^{2l} \int_{\sqrt{\frac{L_0}{r^2}}}^{e^\gamma \sqrt{\frac{L_0}{r^2}}} \left(1 - \epsilon e^{-\gamma}\sqrt{\frac{r^2}{L_0}}\right)^k \left[\epsilon^2 - \frac{L_0}{r^2}\right]^{l+\frac{3}{2}} d\epsilon \tag{57}$$

$$p(r) = \frac{c_l}{2l+3} r^{2l} \int_a^{ab} \left(1 - \frac{\epsilon}{ab}\right)^k \epsilon^2 \left[\epsilon^2 - a^2\right]^{l+\frac{3}{2}} \, d\epsilon \tag{58}$$

$$p(r) = \frac{c_l}{2l+3} r^{2l} \frac{L_0^{l+2}}{r^4} b^{-k} (b-1)^{k+l+\frac{5}{2}} \quad I_{k,l,p}(e^\gamma) \tag{59}$$

$$I_{k,l,p}(e^\gamma) = \int_0^1 t^{l+\frac{3}{2}} (1-t)^k (t(b-1)+2)^{l+\frac{3}{2}} \, dt \tag{60}$$

where $I_{k,l,p}(e^\gamma)$ is always positive due to the assumption $\gamma \geq 0$. Similar to the proof before we can find bounds on p(r).

$$\frac{c_l L_0^{l+2}}{2l+3} I_{l,p} \frac{e^{-\gamma k}(e^\gamma - 1)^{k+l+\frac{5}{2}}}{r^4} \leq p(r) \leq \frac{c_l L_0^{l+2}}{2l+3} I_{u,p} \frac{e^{-\gamma k}(e^\gamma - 1)^{k+l+\frac{5}{2}}}{r^4} \tag{61}$$

Here again $I_{l,p}$ and $I_{u,p}$ are the lower and upper bound of the integral $I_{k,l,p}$ respectively.

With the results from eq. (54) we can further simplify the last expression to find the result from lemma 1.

$$C_l \frac{\gamma^{k+l+\frac{5}{2}}}{r^4} \leq p(r) \leq C_u \frac{\gamma^{k+l+\frac{5}{2}}}{r^4} \tag{62}$$

The last part of this section is the proof of eq. (34) from lemma 1.

□

4.2 Bounds on the characteristic function

In this chapter we want to find bounds on $\gamma(r)$ and its derivative. We will start with the upper bound.

Lemma 2. *There exists a positive constant C_γ such that*

$$\gamma(r) \leq C_\gamma r^{\frac{2}{q+1}} \tag{63}$$

for all $r \in [R_0, 2R_0]$, if R_0 is sufficiently small. Here we used $k \geq 0$, $l \geq -\frac{1}{2}$ and $q = l + k + \frac{3}{2} \geq 1$.

Proof. First we rewrite the expression eq. (195).

$$\gamma(r)^{q+1} \leq C_\gamma^{q+1} r^2 = \frac{r^2}{C_l 4\pi} \tag{64}$$

Here we used $C_\gamma = \left(\frac{1}{4\pi C_L}\right)$ and C_l is the lower bound constant from lemma 1. We now find a sub interval $[r_1, r_2] \subset [R_0, 2R_0]$, such that the lowest value of γ is given by,

$$\gamma(r_1)^{q+1} = \frac{r_1^2}{4\pi C_l} \tag{65}$$

In this interval $\gamma(r)$ is increasing. This implies:

$$\gamma(r)^{q+1} > \frac{r^2}{4\pi C_l} \tag{66}$$

In the next step we use the last expression and the lower bound of p(r) as stated in lemma 1.

$$p(r) \geq C_l \frac{\gamma(r)^{k+l+\frac{5}{2}}}{r^4} \tag{67}$$

$$p(r_1) \geq C_l \frac{\gamma(r_1)^{q+1}}{r^4} = C_l \frac{\frac{r_1^2}{4\pi C_l}}{r_1^4} = \frac{1}{4\pi r_1^2} \tag{68}$$

We will insert the expression into eq. (47) for $\gamma'(r_1)$.

$$\gamma'(r_1) \leq -e^{2\lambda}\left(\frac{1}{r_1} - \frac{r_1\Lambda}{3} + \frac{m(r_1)}{r_1^2}\right) + \frac{1}{r_1} \tag{69}$$

$$\gamma'(r_1) \leq \frac{1}{r_1}\left(1 - e^{2\lambda}\right) + \frac{e^{2\lambda} r_1 \Lambda}{3} - \frac{e^{2\lambda} m(r_1)}{r_1^2} \tag{70}$$

The first term is smaller than zero, because $e^{2\lambda} \geq 1$, which follows from the definition of λ. The second term becomes negative for $\Lambda < 0$. Thus we can show for the derivative of γ,

$$\gamma'(r_1) \leq -\frac{e^{2\lambda} r_1 |\Lambda|}{3} - \frac{e^{2\lambda} m(r_1)}{r_1^2} \leq 0 \tag{71}$$

With the last expression we have shown, that the characteristic function is decreasing at r_1.

However in the beginning of the proof we showed that the characteristic function should be increasing to fulfill eq. (66). This yields a contradiction, which means that lemma 6 is fulfilled on $[R_0, 2R_0]$ and that an upper bound of the characteristic function can be estimated by a power of r.

□

We will continue, by looking for a lower bound on the derivative of γ.

Lemma 3. *On an interval $r \in [R_0, R_0 + \delta]$ with R_0 sufficiently small, let*

$$C_m = max[1, C_u, 4\pi C_u] \tag{72}$$

$$\delta \leq \frac{R_0^{\frac{q+3}{q+1}}}{C_m 8^{\frac{1}{q+1}}} \tag{73}$$

Then

$$\gamma'(r) \geq \frac{1}{2r}. \tag{74}$$

Proof. In the proof we can use the results from lemma 1, as long as R_0 is sufficiently small, because then $\delta < 2R_0$. In the next step we define a new variable $\sigma \in [0, \delta]$. With this, the mean value theorem and the first part from eq. (47) we find:

$$\gamma(R_0 + \sigma) = \gamma(R_0 + \sigma) + \gamma(R_0) = (\gamma(R_0 + \sigma) + \gamma(R_0))\frac{\gamma'(R_0)}{\gamma'(R_0)} \tag{75}$$

$$= (\gamma(R_0 + \sigma) + \gamma(R_0))\frac{\sigma\gamma'(R_0)}{(\gamma(R_0 + \sigma) + \gamma(R_0)} \leq \sigma\gamma'(\xi) \leq \frac{\delta}{R_0} \tag{76}$$

In the last step we used, that $\sigma < \delta$ and that $\gamma'(R_0)$ is the minimum of $\gamma'(\xi)$, with $\xi \in [R_0, R_0 + \delta]$. This implies that $\gamma'(r) \leq \frac{\delta}{R_0}$. We insert this result into the upper bound of eq. (32):

$$\rho \leq C_u \frac{\delta^q}{R_0^q r^4} \tag{77}$$

In order to get an estimate on $\gamma'(r)$, we need estimates on $m(r), p(r)$ and $e^{2\lambda(r)}$. We start with the Hawking mass, eq. (25) on the interval $[R_0, R_0 + \sigma]$ and insert the last expression for ρ.

$$m(R_0+\sigma) \leq 4\pi \int_{R_0}^{R_0+\sigma} s^2 \frac{C_u \delta^q}{R_0^q s^4} \tag{78}$$

$$= 8\pi \frac{\sigma}{R_0(R_0+\sigma)} \left(-\frac{1}{R_0+\sigma} + \frac{1}{R_0} \right) = 8\pi \frac{C_u \delta^q}{R_0^q} \frac{\sigma}{R_0(R_0+\sigma)} \tag{79}$$

$$\frac{m(R_0+\sigma)}{(R_0+\sigma)} \leq \frac{C_u \delta^q}{R_0^q} \frac{\delta}{R_0(R_0+\sigma)^2} \leq \frac{C_u \delta^{q+1}}{R_0^{q+1}(R_0+\delta)^2} \leq \frac{C_u \delta^{q+1}}{R_0^{q+3}} \tag{80}$$

Into the last expression we insert the estimate for δ from lemma 3 and bring it to the following form:

$$\delta \leq \frac{R_0^{\frac{q+3}{q+1}}}{C_m 8^{\frac{q}{q+1}}} \rightarrow \delta^{q+1} \leq \frac{R_0^{q+3}}{8C_m} \tag{81}$$

$$\frac{m(R_0+\sigma)}{(R_0+\sigma)} \leq \frac{1}{8} \tag{82}$$

With this bound we can find an estimate for $e^{2\lambda(r)}$. Here we use the bound on σ, which is by definition smaller than $2R_0$ and we insert a negative Λ.

$$e^{2\lambda(R_0+\sigma)} = \frac{1}{1 - \frac{2m(R_0+\sigma)}{(R_0+\sigma)} - \frac{(R_0+\sigma)^2\Lambda}{3}} \leq \frac{1}{\frac{3}{4} + 3R_0^2|\Lambda|} \leq \frac{4}{3} \tag{83}$$

We will continue with the upper bound of the pressure, for which we use eq. (33) from lemma 1 and the upper limit of C_m, given in lemma 3.

$$4\pi r^2 p(r) \leq 4\pi r^2 C_u \frac{\delta^{q+1}}{r^4 R_0^{q+1}} \leq \frac{R_0^2}{8r^2} \leq \frac{1}{8} \tag{84}$$

$$ry'(r) = -re^{2\lambda} \left(4\pi r p + \frac{m(r)}{r^2} - \frac{r\Lambda}{3} \right) = -e^{2\lambda} \left(4\pi r^2 p + \frac{m(r)}{r} - \frac{r^2\Lambda}{3} \right) \tag{85}$$

After inserting the estimations from above, this equation simplifies to:

$$ry'(r) \geq -\frac{4}{3} \left(\frac{1}{4} - \frac{r^2\Lambda}{3} \right) = -\frac{1}{3} + \frac{4}{9} r^2 \Lambda \tag{86}$$

$$\gamma'(r) = y'(r) + \frac{1}{r} = \frac{1}{r}(ry'(r) + 1) \tag{87}$$

$$\gamma'(r) \geq \frac{2}{3r} + \frac{4r\Lambda}{9} = \frac{2}{3}\left(\frac{1}{r} + \frac{2}{3}r\Lambda\right) \geq \frac{1}{2}\left(\frac{1}{r} + \frac{2}{3}r\Lambda\right) \tag{88}$$

When we insert a negative Λ, the second term becomes negative:

$$\gamma'(r) \geq \frac{1}{2r} - \frac{r|\Lambda|}{3} \tag{89}$$

In order for $\gamma'(r)$ to be positive, the first term has to be larger than the second one, giving the following inequality:

$$\frac{1}{2r} > \frac{r|\Lambda|}{3} \tag{90}$$

As stated in lemma 1, R_0 [1]is to be sufficiently small and for $r \in R_0, 2R_0$ we conclude that r is also small. Since we can choose the value of R_0 freely, we fix it in such a way, that the inequality in fulfilled.

□

4.3 Existence of characteristic radii

As stated earlier we want to show that the matter region is restricted to a certain interval. Therefore the characteristic function becomes positive at a radius R_0 and should then increase. This was shown in the last section, by proving that the derivative of $\gamma(r)$ is positive, and bounded from below, by $\gamma'(r) \geq \frac{1}{2r}$. At the outer radius of this interval, the characteristic function should then reach zero again. In the following section we want to show how $\gamma(r)$ decreases and reaches zero.

This section is not affected by an extra term with the cosmological constant. We use the result form lemma 3, which shows that a similar estimate for the deriva-

[1] R_0 is the inner radius of the matter shell. The shell we construct has to be sufficiently small, in order to fulfill the inequality.

tive of $\gamma(R)$ as shown in [14], can be found in the case with non-vanishing cosmological constant. The result from [14] holds if we choose R_0 to be sufficiently small. Similarly we use the upper bound on $\rho(r)$ from lemma 1, which is also equal to the corresponding lemma in [14]. Thus we can use the same proof and it is added here for completeness and better understanding.
We will start by defining some quantities:

$$\sigma_* = C_0 R_0^{1+\frac{2q+1}{q(q+1)}} \tag{91}$$

$$C_0 = min\left[\frac{1}{4}, (C_m 8^{\frac{1}{1+q}})^{-1}\right] \tag{92}$$

Thus we can estimate σ_* with:

$$\sigma_* = C_0 R_0^{1+\frac{2q+1}{q(q+1)}} \leq \frac{1}{C_m 8^{\frac{1}{1+q}}} R_0^{1+\frac{2q+1}{q(q+1)}} \leq \frac{1}{C_m 8^{\frac{1}{1+q}}} R_0^{\frac{q+3}{q+1}} \tag{93}$$

The last step is possible for an R_0 which is sufficiently small, because then:

$$\frac{3+q}{1+q} \leq 1 + \frac{2q+1}{q(q+1)} \rightarrow R_0^{\frac{3+q}{1+q}} \geq R_0^{1+\frac{2q+1}{q(q+1)}} \tag{94}$$

After comparing this to lemma 3, we see that:

$$\sigma_* \leq \delta \tag{95}$$

Furthermore we define:

$$\gamma_* = \frac{\sigma_*}{2(R_0 + \sigma_*)} \tag{96}$$

We want to call the radius at which $\gamma(r)$ reaches γ_* for the first time, r_1. Thus $\gamma(r_1) = \gamma_*$ It should reach this value again, and the following lemma shows that it will for $r \geq R_0 + \sigma_*$.

Lemma 4. *For* $K = max[C_0, \kappa]$, *with* $\kappa = \frac{9*2^{2q}}{C_l C_0^q}$ *and find a solution with* R_0 *such that,*

$$R_0^{\frac{1}{1+q}} K \leq 1 \tag{97}$$

Then there exists a radius r_2, *such that* $\gamma(r) \searrow \gamma_*$ *as* $r \nearrow r_2$ *and the radius is bounded by,*

$$r_2 \leq R_0 + \sigma_* + \kappa R_0^{\frac{q+2}{q+1}} \tag{98}$$

Proof. We will start by rewriting the derivative of $\gamma(r)$ with the formula for the difference quotient.

$$\gamma'(\xi) = \frac{\gamma(R_0 + \sigma) - \gamma(R_0)}{R_0 + \sigma - R_0} \tag{99}$$

Here we used the $\gamma(r)'$ from lemma 3, with $\sigma \in [0, \delta]$.

$$\gamma(R_0 + \sigma) = \gamma'(\xi) \cdot \sigma + \gamma(R_0) \geq \gamma(R_0) + \sigma \cdot inf \gamma'(R_0 + \sigma) \tag{100}$$

The result from lemma 3 gives $\gamma'(r) \geq \frac{1}{2r}$ and $\gamma(R_0) = 0$ by construction. If we insert this into the last expression we find:

$$\gamma(R_0 + \sigma) \geq \frac{\sigma}{2(R_0 + \sigma)} \tag{101}$$

For σ_* instead of σ this result shows:

$$\gamma(R_0 + \sigma_*) \geq \frac{\sigma_*}{2(R_0 + \sigma_*)} = \gamma_* \tag{102}$$

Since the derivative is positive, then $\gamma \geq \gamma_*$ for an interval after r_1. We will define this interval as follows, $[R_0 + \sigma_*, R_0 + \sigma_* + \Delta] \subset [R_0, 2R_0]$, where Δ is a positive constant. The estimate eq. (98) from lemma 4 can be proven, by showing that:

$$\Delta \leq \kappa R_0^{\frac{q+2}{q+1}} \tag{103}$$

From lemma 1 we know that $\rho \geq C_l \frac{\gamma^q}{r^4}$. Inserting this into eq. (25) for the Hawking mass, yields:

$$m(r) = \int_{R_0}^{r} \frac{\gamma^q(s)}{2} ds \geq 4\pi C_l \left(\int_{R_0}^{R_0+\sigma_*} \frac{\gamma^q(s)}{2} ds + \int_{R_0+\sigma_*}^{R_0+\sigma_*+\Delta} \frac{\gamma^q(s)}{2} ds \right) \tag{104}$$

Both integrals in this expression are positive. With this and eq. (102) we can further estimate:

$$m(r) \geq 4\pi C_l \int_{R_0+\sigma_*}^{R_0+\sigma_*+\Delta} \frac{\gamma_*^q}{2}\, ds = 4\pi C_l \int_{R_0+\sigma_*}^{R_0+\sigma_*+\Delta} \frac{\sigma_*^q}{2^q (R_0+\sigma_*)^q s^2}\, ds \tag{105}$$

Solving the last integral gives:

$$m(r) = 4\pi C_l \frac{\sigma_*^q \Delta}{2^q (R_0+\sigma_*)^{q+1}(R_0+\sigma_*+\Delta)} \tag{106}$$

Both eq. (93) and eq. (97) combined, guarantee that $\sigma_* \leq R_0$. Thus we will insert for the σ_* in the denominator, the largest possible value for it, i.e. R_0 and for the σ_* in the numerator, the definition for eq. (93).

$$\frac{m(R_0+\sigma_*+\Delta)}{R_0+\sigma_*+\Delta} \geq 4\pi C_l \frac{C_0^q R_0^{q(1+\frac{2q+1}{q(q+1)})} \Delta}{2^q 2^{q+1} R_0^{q+1} (R_0+\sigma_*+\Delta)^2} = \frac{4\pi C_l C_0^q R_0^{\frac{q}{q+1}} \Delta}{2^{2q+1}(R_0+\sigma_*+\Delta)^2} \tag{107}$$

It is known that all static solutions must fulfill $\frac{m(r)}{r} \leq \frac{4}{9}$. With this in mind, we are going to verify eq. (103) with a proof by contradiction.
Assuming $\Delta \geq \kappa R_0^{\frac{q+2}{q+1}}$ and inserting in into the last expression gives:

$$\frac{m(R_0+\sigma_*+\Delta)}{R_0+\sigma_*+\Delta} \geq \frac{4\pi C_l C_0^q \kappa R_0^2}{2^{2q+1}(3R_0)^2} = \frac{4\pi C_l C_0^q \kappa}{2^{2q+1} 3^2} \tag{108}$$

With the definition of κ from lemma 4 we find:

$$\frac{m(R_0+\sigma_*+\Delta)}{R_0+\sigma_*+\Delta} \geq \frac{9 * 4\pi * 2^{2q} C_l C_0^q}{2^{2q+1} 3^2 C_l C_0^q} \geq 2\pi \geq \frac{4}{9} \tag{109}$$

Thus we see, that $\Delta \geq \kappa R_0^{\frac{q+2}{q+1}}$ leads to a contradiction and we have shown that:

$$\Delta < \kappa R_0^{\frac{q+2}{q+1}} \tag{110}$$

From this follows that: $r_2 \leq R_0 + \sigma_* + \kappa R_0^{\frac{q+2}{q+1}}$.

□

4.4 Lower bound for $\frac{m(r)}{r}$

Proposition 1. *For r_2 as defined in the previous section, the solution will satisfy:*

$$\frac{m(r_2)}{r_2} \geq \frac{2}{5} \tag{111}$$

In the last sections we have found some constraints for the characteristic function $\gamma(r)$. In particular we have shown that at the radius r_2, the characteristic function is decreasing and reaches γ_* for the second time. The proposition 1 ensures that $\frac{m(r)}{r}$ is bounded from below as it is a necessary condition for all constructed solutions, see [14].

Proof. We start the proof, with the expression $r_2 \leq 4m(r_2)$ and prove this, by showing that:

$$\frac{m(r_2)}{r_2} \leq \frac{1}{4} \to r_2 \geq 4m(r_2) \tag{112}$$

leads to a contradiction. Recall from lemma 1, that:

$$p(r) \leq \frac{C_u \gamma q + 1}{r^4} \tag{113}$$

At the radius r_2, and for $R_0 < r_2$ with R_0 sufficiently small, this gives:

$$r_2 p(r_2) \leq \frac{C_u \gamma_* q + 1}{r_2^3} \leq \frac{C_u \gamma_* q + 1}{R_0^3} \leq \frac{C_u \gamma_* q + 1}{R_0^{\frac{q-1}{q}}} \leq \frac{C}{R_0^{\frac{q-1}{q}}} \tag{114}$$

We will now insert this estimate into the expression for the derivative of y(r). Furthermore we will use $\frac{m(r_2)}{r_2} \leq \frac{1}{4}$.

$$e^{2\lambda(r_2)} = \frac{1}{1 - \frac{2m(r_2)}{r_2} - \frac{r_2^2 \Lambda}{3}} \leq \frac{1}{1 - \frac{1}{2} + \frac{r_2^2 |\Lambda|}{3}} \leq 2 \tag{115}$$

The last inequality holds for $\Lambda < 0$.

$$y'(r_2) = -e^{2\lambda}(r_2) \left(4\pi r_2 p(r_2) + \frac{m(r_2)}{r_2^2} - \frac{r_2 \Lambda}{3} \right) \geq -2 \left(\frac{4\pi C}{R_0^{\frac{q-1}{q}}} + \frac{r_2}{4r_2^2} - \frac{r_2 \Lambda}{3} \right) \tag{116}$$

$$y'(r_2) \geq -\frac{8\pi C}{R_0^{\frac{q-1}{q}}} - \frac{1}{2r_2} + \frac{2}{3}r_2\Lambda \tag{117}$$

We want to show that the last expression is larger than $-\frac{1}{r_2}$. Since $-\frac{1}{2r_2} \geq -\frac{1}{r_2}$, it simplifies to:

$$-\left(\frac{8\pi C}{R_0^{\frac{q-1}{q}}} - \frac{2}{3}r_2\Lambda\right) \leq -\frac{1}{2r_2} \tag{118}$$

We will now analyse this expression for $\Lambda < 0$. Here it is clear that eq. (118) is automatically satisfied.

$$\frac{8\pi C}{R_0^{\frac{q-1}{q}}} + \frac{2}{3}r_2|\Lambda| \geq \frac{1}{2r_2} \tag{119}$$

From this follows:

$$y'(r_2) \geq -\frac{1}{r_2} \tag{120}$$

Now we want to analyse if this result is reasonable. The lemma 4 tells us that $\gamma(r)$ reaches the value γ_* two times, the first time from below and the second time from above. The second time is close to $r = r_2$. Thus the characteristic function is decreasing close to this radius. With eq. (47) we find:

$$\gamma'(r_2) = y'(r_2) + \frac{1}{r_2} \leq 0 \tag{121}$$

$$y'(r_2) \leq -\frac{1}{r_2} \tag{122}$$

Now we have a discrepancy between eq. (117) and eq. (122). Thus eq. (112) is wrong and we have shown that $r_2 \leq 4m(r_2)$.

We will continue by calculating the r-derivative of the following expression:

$$e^{\lambda-y}\left(m(r)-\frac{r^3\Lambda}{3}+4\pi r^3 p(r)\right) \tag{123}$$

In order to do that we will use the TOV-Equation eq. (26) and the Einstein equations eq. (6) and eq. (7). From section 4.1 we know that $y'(r)=-\mu'(r)$ and inserting this into the second Einstein equation will give:

$$-y'(r)=\frac{(8\pi r^2 p+1-r^2\Lambda)e^{2\lambda}-1}{2r} \tag{124}$$

Rewriting the first Einstein equation in a similar manner, gives:

$$\lambda'(r)=\frac{(8\pi r^2\rho-1+r^2\Lambda)e^{2\lambda}-1}{2r} \tag{125}$$

Combining the last two expressions leads to:

$$\lambda'(r)-y'(r)=4\pi r e^{2\lambda}(\rho+p) \tag{126}$$

We will use this and for all terms with $p'(r)$ we will insert the TOV equation.

$$\frac{d}{dr}(e^{\lambda-y}\left(m(r)-\frac{r^3\Lambda}{3}+4\pi r^3 p(r))\right)=e^{\lambda-y}(4\pi r^2(\rho+p(r)+2p_T(r))-r^2\Lambda) \tag{127}$$

Integrating both sides, gives:

$$e^{\lambda-y}\left(m(r)-\frac{r^3\Lambda}{3}+4\pi r^3 p(r)\right)=\int_0^r e^{(\lambda-y)(s)}(4\pi s^2(\rho(s)+p(s)+2p_T(s))-s^2\Lambda)\,ds \tag{128}$$

In order to simplify this expression, we use the definition for z from lemma 1 and evaluate the expression at $r=r_2$. Recall that there is no matter before the radius R_0 and after r_2. This makes it possible to adjust the integral boundaries.

$$
\begin{aligned}
m(r_2)e^{(\lambda-y)(r_2)} =& \\
= \int_{R_0}^{r_2} e^{(\lambda-y)(s)} \left(4\pi s^2(2\rho - z) - s^2\Lambda\right) ds &- e^{(\lambda-y)(r_2)} \left(4\pi r_2^3 p(r_2) - \frac{r_2^3\Lambda}{3}\right) \\
= \int_{R_0}^{r_2} 4\pi s^2 e^{(\lambda-y)(s)} 2\rho(s)\, ds &- \int_{R_0}^{r_2} 4\pi s^2 e^{(\lambda-y)(s)} z(s)\, ds \\
- \int_{R_0}^{r_2} e^{(\lambda-y)(s)} s^2\Lambda\, ds &- e^{(\lambda-y)(r_2)} 4\pi r_2^3 p(r_2) + e^{(\lambda-y)(r_2)} \frac{r_2^3\Lambda}{3} \qquad (129)
\end{aligned}
$$

The first integral of this expression can be simplified because y(r) is monotonically decreasing, determining that $-y(R_0)$ is the lowest possible value. Thus we can pull the factor with y out of the integral.

$$
2\int_{R_0}^{r_2} 4\pi s^2 e^{(\lambda-y)(s)} \rho(s)\, ds \geq 2e^{-y(R_0)} R_0 \int_{R_0}^{r_2} 4\pi s e^{\lambda(s)} \rho(s)\, ds \qquad (130)
$$

By rewriting the first Einstein equation, we find:

$$
4\pi r e^{\lambda(r)} \rho(r) = -\frac{d}{dr} e^{-\lambda} + \frac{m(r)}{r^2} e^{\lambda} - \frac{r\Lambda}{3} \qquad (131)
$$

We insert this expression into the integral eq. (130).

$$
\int_{R_0}^{r_2} 4\pi s e^{\lambda(s)} \rho(s)\, ds = \int_{R_0}^{r_2} \left(-\frac{d}{ds} e^{-\lambda(s)}\right) ds + \int_{R_0}^{r_2} \frac{m(s)}{s^2} e^{\lambda(s)}\, ds - \int_{R_0}^{r_2} \frac{s\Lambda}{3}\, ds \qquad (132)
$$

The first two terms give the same result as seen in [14], so only the last term has to be evaluated. The upper limit of this integral can be estimated as follows:

$$
\int_{R_0}^{r_2} 4\pi s e^{\lambda(s)} \rho(s)\, ds \leq 1 - \sqrt{1 - \frac{2m(r_2)}{r_2}} + Cm(r_2) R_0^{\frac{q+2}{q+1}-2} - \left. \frac{s^2}{2} \right|_{r_2}^{R_0} \frac{\Lambda}{3}
$$

$$
\leq \frac{Cm(r_2)}{R_0} - \frac{\Lambda}{6}(r_2^2 - R_0^2) \leq \frac{Cm(r_2)}{R_0} - \frac{\Lambda}{6}(r_2^2 - R_0^2) \leq \frac{Cm(r_2)}{r_2} - \frac{\Lambda}{6}(r_2^2 - R_0^2) \qquad (133)
$$

In the next step we will estimate the lower limit of the integral. Again we start with the expression from the first Einstein equation and the term with Λ can again be omitted for R_0 sufficiently small. Thus we find:

$$\int_{R_0}^{r_2} 4\pi s e^{\lambda(s)} \rho(s)\, ds \geq \frac{2m(r_2)}{r_2 \left(1 + \sqrt{1 - \frac{2m(r_2}{r_2}}\right)} \tag{134}$$

We can rewrite the eq. (130), by using the lower limit, i.e. eq. (134). This gives:

$$2\int_{R_0}^{r_2} 4\pi s^2 e^{(\lambda - y)(s)} \rho(s)\, ds \geq e^{-y(R_0)} \frac{4R_0 m(r_2)}{r_2 \left(1 + \sqrt{1 - \frac{2m(r_2}{r_2}}\right)} \tag{135}$$

The integral part with z(r) from eq. (129) can further be estimated, by combining lemma 1 and the lower limit from eq. (34). We know that the function $y(r)$ is monotonically decreasing. Thus $-y(r_2)$ is the largest value, and the term can be pulled out of the integral.

$$z \leq C_u \frac{\gamma^{l+k+\frac{3}{2}}}{r^2} = \frac{C_u}{C_l} r^2 C_l \frac{\gamma^{l+k+\frac{3}{2}}}{r^4} \leq \frac{C_u}{C_l} r^2 \rho(r) \tag{136}$$

$$\int_{R_0}^{r_2} 4\pi s^2 e^{(\lambda - y)(s)} z(s)\, ds \leq \frac{C_u}{C_l} r_2^3 e^{-y(r_2)} \int_{R_0}^{r_2} 4\pi s e^{\lambda(s)} \rho(s)\, ds \tag{137}$$

To further estimate the last expression, we use the upper limit, i.e. eq. (133), which gives:

$$\int_{R_0}^{r_2} 4\pi s^2 e^{(\lambda - y)(s)} z(s)\, ds \leq C e^{-y(r_2)} \left(r_2^2 m(r_2) - r_2^3 \frac{\Lambda}{6}(r_2^2 - R_0^2)\right) \tag{138}$$

The variable C here is a arbitrary positive constant. The last term from eq. (129) can be rewritten with $r_2 \leq 4m(r_2)$. We recall that $R_0 < r_2$.

$$4\pi e^{(\lambda - y)} r_2^2 r_2 p(r_2) \leq 4\pi e^{(\lambda - y)} r_2^2 4m(r_2) p(r_2) = C e^{(\lambda - y)} R_0 r_2 m(r_2) p(r_2) \tag{139}$$

The third integral form eq. (129) can be simplified, in a similar manner like the others.

Here we see that $e^{-y(r_2)}$ can again be pulled out of the integral.

$$-\int_{R_0}^{r_2} e^{(\lambda-y)(s)} s^2 \Lambda \, ds \geq -\Lambda e^{-y(r_2)} \int_{R_0}^{r_2} e^{\lambda(s)} s^2 \, ds = C_\Lambda \tag{140}$$

Both functions in the integral are positive, thus the whole expression remains negative and we can estimate it with a constant C_Λ. This constant will stay negative, but its value may change from line to line during the remainder of the proof.
With these estimates we return to eq. (129).

$$\begin{aligned}
&m(r_2)e^{(\lambda-y)(r_2)} \\
&\geq e^{-y(R_0)} \frac{4R_0 m(r_2)}{r_2\left(1+\sqrt{1-\frac{2m(r_2)}{r_2}}\right)} + C_\Lambda - Ce^{-y(r_2)}\left(r_2^2 m(r_2) - r_2^3\frac{\Lambda}{6}(r_2^2 - R_0^2)\right) \\
&\qquad + Ce^{(\lambda-y)(r_2)} R_0 r_2 m(r_2) p(r_2) - e^{(\lambda-y)(r_2)}\frac{R_0^3\Lambda}{3} \\
&\geq e^{-y(R_0)} \frac{4R_0 m(r_2)}{r_2\left(1+\sqrt{1-\frac{2m(r_2}{r_2}}\right)} + C_\Lambda - Ce^{-y(r_2)}\left(R_0^2 m(r_2) - r_2^3\frac{\Lambda}{6}(r_2^2 - R_0^2)\right) \\
&\qquad + Ce^{(\lambda-y)(r_2)} R_0 r_2 m(r_2) p(r_2) - e^{(\lambda-y)(r_2)}\frac{R_0^3\Lambda}{3}
\end{aligned} \tag{141}$$

We replace $p(r_2)$ in the fourth term, with eq. (114).

$$\begin{aligned}
&e^{(\lambda-y)(r_2)} \\
&\geq e^{-y(R_0)} \frac{4R_0}{r_2\left(1+\sqrt{1-\frac{2m(r_2}{r_2}}\right)} - Ce^{-y(r_2)}\left(R_0^2 - r_2^3\frac{\Lambda}{6m(r_2)}(r_2^2 - R_0^2)\right) \\
&\qquad + C_\Lambda + Ce^{(\lambda-y)(r_2)} R_0 R_0^{-\frac{q-1}{q}} - e^{(\lambda-y)(r_2)}\frac{R_0^3\Lambda}{3m(r_2)} \\
&= e^{-y(R_0)} \frac{4R_0}{r_2\left(1+\sqrt{1-\frac{2m(r_2}{r_2}}\right)} - Ce^{-y(r_2)}\left(R_0^2 - r_2^3\frac{\Lambda}{6m(r_2)}(r_2^2 - R_0^2)\right) \\
&\qquad + C_\Lambda + Ce^{(\lambda-y)(r_2)} R_0^{\frac{1}{q}} - e^{(\lambda-y)(r_2)}\frac{R_0^3\Lambda}{3m(r_2)}
\end{aligned} \tag{142}$$

$$1 \geq e^{y(r_2)-y(R_0)} e^{-\lambda(r_2)} \frac{4R_0}{r_2\left(1+\sqrt{1-\frac{2m(r_2}{r_2}}\right)} + C_\Lambda$$
$$- Ce^{-\lambda(r_2)}\left(R_0^2 - r_2^3 \frac{\Lambda}{6m(r_2)}(r_2^2 - R_0^2)\right) + CR_0^{\frac{1}{q}} - \frac{R_0^3\Lambda}{3m(r_2)} \quad (143)$$

The next step is to find an estimate for $e^{y(r_2)-y(R_0)}$. For this we recall eq. (46).

$$y(r_2) - y(R_0) = -\int_{R_0}^{r_2} e^{2\lambda(s)}\left(4\pi sp + \frac{m(s)}{s^2} - \frac{s\Lambda}{3}\right) ds \quad (144)$$

In order to estimate the first term of this integral we combine lemma 1 and eq. (65).

$$p(r) \leq C_u \frac{\gamma^{q+1}}{r^4} \leq \frac{C_u r^2}{4\pi C_l r^4} = \frac{C}{r^2} \quad (145)$$

$$4\pi r p(r) \leq \frac{4\pi C}{r} \quad (146)$$

[7] tell us that all static solutions of the Einstein-Vlasov system have to fulfill the inequality $\frac{2m(r)}{r} \leq \frac{8}{9}$. We can use this to rewrite the second term in the integral:

$$\frac{m(r)}{r^2} \leq \frac{4}{9r} \quad (147)$$

Furthermore, this gives:

$$e^{2\lambda(r)} = \frac{1}{1 - \frac{2m(r)}{r} - \frac{r^2\Lambda}{3}} \leq 9 \quad (148)$$

The inequality above is fulfilled for a negative cosmological constant. Hence the last term from eq. (144) can be solved as follows:

$$\int_{R_0}^{r_2} \frac{9s\Lambda}{3}\, ds = \frac{9\Lambda}{6}(r_2^2 - R_0^2) \quad (149)$$

All these estimates together give:

$$y(r_2)-y(R_0)\geq -\int_{R_0}^{r_2} 9\left(\frac{4\pi C}{s}+\frac{4}{9s}\right)\,ds+\frac{9\Lambda}{6}(r_2^2-R_0^2)$$
$$=-\int_{R_0}^{r_2}\frac{C}{s}\,ds+\frac{9\Lambda}{6}(r_2^2-R_0^2)=-C\ln\frac{r_2}{R_0}+\frac{9\Lambda}{6}(r_2^2-R_0^2) \quad (150)$$

We insert the last expression into eq. (143).

$$1\geq e^{-C\ln\frac{r_2}{R_0}}e^{\frac{9\Lambda}{6}(r_2^2-R_0^2)}e^{-\lambda(r_2)}\frac{4R_0}{r_2\left(1+\sqrt{1-\frac{2m(r_2}{r_2}}\right)}+C_\Lambda$$
$$-Ce^{-\lambda(r_2)}\left(R_0^2-r_2^3\frac{\Lambda}{6m(r_2)}(r_2^2-R_0^2)\right)+CR_0^{\frac{1}{q}}-\frac{R_0^3\Lambda}{3m(r_2)}$$
$$=\left(\frac{R_0}{r_2}\right)^{C+1}e^{\frac{9\Lambda}{6}(r_2^2-R_0^2)}\frac{4\left(\sqrt{\frac{1}{1-\frac{2m(r_2)}{r_2}}}\right)}{1+\sqrt{1-\frac{2m(r_2)}{r_2}}}+C_\Lambda$$
$$-Ce^{-\lambda(r_2)}\left(R_0^2-r_2^3\frac{\Lambda}{6m(r_2)}(r_2^2-R_0^2)\right)+CR_0^{\frac{1}{q}}-\frac{R_0^3\Lambda}{3m(r_2)} \quad (151)$$

In order to evaluate this expression for R_0 sufficiently small, we recall lemma 4. We know that r_2 depends on R_0.
Therefore if R_0 is arbitrary small, then $\left(\frac{R_0}{r_2}\right)^{C+1}\to 1$. Furthermore, the difference $(r_2^2-R_0^2)$ becomes negligible. Thus $e^{\frac{9\Lambda}{6}(r_2^2-R_0^2)}\to 1$, independent of the sign of Λ. The last three terms from eq. (151), will be small enough to satisfy the following estimate.

$$1+\sqrt{1-\frac{2m(r_2)}{r_2}}\geq 4\sqrt{1-\frac{2m(r_2)}{r_2}}+C_\Lambda \quad (152)$$

$$1\geq 3\sqrt{1-\frac{2m(r_2)}{r_2}}+C_\Lambda\to 1\geq 9\left(1-\frac{2m(r_2)}{r_2}\right)+C_\Lambda \quad (153)$$

$$\to\frac{m(r_2)}{r_2}\geq\frac{4}{9}+C_\Lambda \quad (154)$$

In order to estimate C_Λ, we recall $\frac{m(r_2)}{r_2} \geq \frac{1}{4}$ and that $\frac{m(r)}{r} \leq \frac{8}{9}$ has to be fulfilled by all static solutions. The constant with the modifications from the calculation above is given by:

$$C_\Lambda \geq -\Lambda \frac{r_2}{m(r_2)} e^{-\lambda(r_2)} \int_{R_0}^{r_2} e^{\lambda(s)} s \, ds \geq -\Lambda \frac{9}{4} \int_{R_0}^{r_2} e^{\lambda(s)} s \, ds \tag{155}$$

In order to simplify the integral, we insert eq. (148) and then solve the integral.

$$C_\Lambda \geq -\Lambda \frac{9}{4} \int_{R_0}^{r_2} 3s \, ds = -\frac{27}{8} \Lambda (r_2^2 - R_0^2) \tag{156}$$

With this estimate we return to eq. (154) and insert $\Lambda < 0$.

$$\frac{m(r_2)}{r_2} \geq \frac{4}{9} + \frac{27}{8} |\Lambda| (r_2^2 - R_0^2) \tag{157}$$

Thus we have shown, that $\frac{m(r_2)}{r_2} \geq \frac{2}{5}$ if we choose R_0 small enough. This concludes the proof of proposition 1.

□

4.5 Proof of Proposition 2

Proposition 2. *If one considers a quasi shell solution of the mass-less Einstein-Vlasov system and chooses the inner radius R_0 to be sufficiently small, then it can be shown that the distribution function $f(x, v)$ vanishes within an interval $[R_1, R_2]$, with:*

$$R_1 = R_0 \left(1 + B_0 R_0^{\frac{2}{q+1}} \right) \tag{158}$$

$$R_2 = R_0 \frac{1 + B_2 R_0^{\frac{1}{q+1}}}{1 - B_1 R_0^{\frac{2}{q+1}}} \tag{159}$$

The constants B_0, B_1, B_2 and q are positive and should only depend on the Ansatz function for the matter distribution eq. (14).

Proposition 2 states that there exists an interval I in which the distribution function $f(x, v)$ vanishes.

Proof. In the first part, we will concentrate on the lower bound of the interval. We want to show that $f(x, v)$ will not vanish until until the radius reaches R_1. From section 2 we know that the matter quantities ρ, p, p_T are non-zero if and only if the characteristic function $\gamma(r)$ is positive. Thus we want $\gamma(r) > 0$ for:

$$r < R_0 + B_0 R_0^{\frac{3+q}{q+1}} \tag{160}$$

Per definition $\gamma(R_0) = 0$ and in lemma 3 we have shown that on the interval $[R_0, R_0 + \delta]$ the derivative fulfills the inequality: $\gamma'(r) \geq \frac{1}{2r}$. We see that the maximum value for which γ is still increasing lies at:

$$R_0 + \delta \leq R_0 + \frac{R_0^{\frac{q+3}{q+1}}}{C_m 8^{\frac{1}{q+1}}} \tag{161}$$

Here we used the estimate for δ from eq. (73). Furthermore we see that:

$$\frac{1}{C_m 8^{\frac{1}{q+1}}} = B_0 \tag{162}$$

Thus we have shown that the characteristic function $\gamma(r)$ is increasing and can in fact not vanish until the radius $R_1 > R_0$, with:

$$R_1 = R_0 \left(1 + B_0 R_0^{\frac{2}{q+1}}\right) \tag{163}$$

We will continue with the proof of the upper bound, i.e. we need to show that the function $\gamma(r)$ vanishes before the radius reaches R_2.
The first step is to introduce a new variable:

$$x(r) := \frac{m(r)}{r\gamma(r)} \tag{164}$$

We can see that $\gamma(r)$ goes to zero if $x \to \infty$ and we will see that this happens close to r_2, as long as R_0 is sufficiently small. When we take the derivative of $x(r)$ with respect to r, we find:

$$rx = \frac{m(r)}{\gamma(r)} \to rx' = \frac{m'\gamma - m\gamma'}{\gamma^2} - x \tag{165}$$

After inserting eq. (25) and eq. (47) as well as rewriting the exponential as $e^{2\lambda} = \frac{1}{1-2x\gamma-\frac{r^2\Lambda}{3}}$ we find:

$$\begin{aligned} &rx' = \\ &\frac{4\pi r^2\rho}{\gamma} + \frac{4\pi rpm}{\gamma^2(1-2x\gamma-\frac{r^2\Lambda}{3})} + \frac{m^2}{r^2\gamma^2(1-2x\gamma-\frac{r^2\Lambda}{3})} - \frac{m}{r\gamma^2} - \frac{mr\Lambda}{3\gamma^2(1-2x\gamma-\frac{r^2\Lambda}{3})} - x \\ &\quad = \frac{4\pi r^2\rho}{\gamma} + \frac{4\pi xr^2p}{\gamma(1-2x\gamma-\frac{r^2\Lambda}{3})} + \frac{x^2}{(1-2x\gamma-\frac{r^2\Lambda}{3})} - \frac{x}{\gamma} - \frac{xr^2\Lambda}{3\gamma(1-2x\gamma-\frac{r^2\Lambda}{3})} - x \end{aligned} \tag{166}$$

We are currently looking at the interval $[R_0, r_2]$ in which matter is present. Hence ρ, $p \geq 0$, i.e. the first two terms are non-negative and the last expression can be rewritten as:

$$rx' \geq -\frac{xr^2\Lambda}{3\gamma(1-2x\gamma-\frac{r^2\Lambda}{3})} - x + \frac{2x^2}{3(1-2x\gamma-\frac{r^2\Lambda}{3})} - \frac{x}{\gamma} + \frac{x^2}{3(1-2x\gamma-\frac{r^2\Lambda}{3})} \tag{167}$$

The next step is to show that the last two terms are also non-negative and can be omitted. For this we use the result from proposition 1: $\frac{m(r_2)}{r_2} \geq \frac{2}{5}$, and consider it on a new interval $[r_2, r_2\frac{16}{15}]$, which is subsequent to the interval with matter from before. If we use it as an upper bound for r this gives us the estimate:

$$\frac{m(r)}{r} \geq \frac{m(r_2)}{r} = \frac{m(r_2)}{r_2}\frac{r_2}{r} \geq \frac{r_2}{r} * \frac{15}{16} * \frac{2}{5} = \frac{3}{8} \tag{168}$$

We will further insert this result to estimate:

$$\frac{x}{(1-2x\gamma-\frac{r^2\Lambda}{3})} = e^{2\lambda}\frac{m(r)}{r\gamma} \geq \frac{3}{2\gamma} \tag{169}$$

$$\frac{2x^2}{3(1-2x\gamma-\frac{r^2\Lambda}{3})} \geq \frac{2x}{3} * \frac{3}{2\gamma} = \frac{x}{\gamma} \tag{170}$$

$$\frac{2x^2}{3(1-2x\gamma-\frac{r^2\Lambda}{3})} - \frac{x}{\gamma} \geq 0 \tag{171}$$

The last term of eq. (167) can be rewritten in the same manner:

$$\frac{x^2}{3(1-2x\gamma-\frac{r^2\Lambda}{3})} = \frac{x}{3} * \frac{3}{2\gamma} = \frac{3rx^2}{6m} \geq \frac{4}{3}x^2 \tag{172}$$

Lastly we need an estimate for the term with Λ.

$$-\frac{\Lambda r^2}{3\gamma}\frac{x}{(1-2x\gamma-\frac{r^2\Lambda}{3})} \geq -\frac{\Lambda r^2}{3\gamma}\frac{3}{2\gamma} = -\frac{3\Lambda r^2}{6\gamma^2} = -\frac{3\Lambda r^2}{6\frac{m(r)^2}{x^2r^2}} \geq -4\Lambda r^2x^2 \tag{173}$$

For Λ negative, this term becomes positive as long as $\gamma(r)$ is positive, which is the case in this interval because matter is present. Thus we can drop the term due to its sign.

$$rx' \geq \frac{4}{3}x^2 - x \tag{174}$$

The next step is to solve this differential equation. This gives:

$$x(r) \geq \frac{3}{4}\frac{1}{1-\frac{r(\frac{4}{3}x(r_2)-1)}{\frac{4}{3}r_2x(r_2)}} \tag{175}$$

It becomes clear that the expression for $x(r)$ diverges if

$$r \to \frac{\frac{4}{3}r_2x(r_2)}{\frac{4}{3}x(r_2)-1} = \frac{x(r_2)r_2}{x(r_2)-\frac{3}{4}} \tag{176}$$

In order to further simplify the last expression, we need the result from lemma 6.

$$\gamma \leq C_\gamma r^{\frac{2}{q+1}} \tag{177}$$

On the interval $r \in [R_0, 2R_0]$ we can use it to estimate $x(r_2)$:

$$x(r_2) = \frac{m(r_2)}{r_2}\frac{1}{\gamma(r_2)} \geq \frac{2}{5}\frac{C}{r_2^{\frac{2}{q+1}}} \tag{178}$$

$$\frac{x(r_2)}{x(r_2) - \frac{3}{4}} \leq \frac{\frac{2}{5}\frac{C}{r_2^{\frac{2}{q+1}}}}{\frac{2}{5}\frac{C}{r_2^{\frac{2}{q+1}}} - \frac{3}{4}} = \frac{1}{1 - \frac{15}{8}\frac{r_2^{\frac{2}{q+1}}}{C}} \tag{179}$$

From earlier we know that $x \to \infty$ as long as R_0 is sufficiently small. In order to show that $x \to \infty$ for $R_0 \to 0$, we will use the following condition:

$$\frac{x(r_2)}{x(r_2) - \frac{3}{4}} \leq \frac{16}{15} \tag{180}$$

From this follows, that $x(r_2) \geq 12$.
Furthermore lemma 4 tells us that:

$$r_2 \leq R_0 + \sigma_* + \kappa R_0^{\frac{q+2}{q+1}} \tag{181}$$

$$with\ \sigma_* \leq \kappa R_0^{\frac{q+2}{q+1}} \tag{182}$$

For R_0 sufficiently small, $R_0 > R_0^{\frac{q+2}{q+1}}$ and thus:

$$r_2 \leq C_* R_0 \tag{183}$$

If we insert this into eq. (179) and we find:

$$\frac{x(r_2)}{x(r_2) - \frac{3}{4}} \leq \frac{1}{1 - B_1 R_0^{\frac{2}{q+1}}} \tag{184}$$

With B_1 a positive constant. This expression goes to 1 if $R_0 \to 0$, following that $x \to \infty$. We can also rewrite expression eq. (98) with a positive constant B_2, such that:

$$r_2 \leq R_0 + B_2 R_0^{\frac{q+2}{q+1}} \tag{185}$$

The expression for $x(r)$ has to diverge in order for $\gamma \to 0$. This is the case if r fulfills eq. (176) and we will rewrite this by combining the last two expressions:

$$\frac{x(r_2)}{x(r_2) - \frac{3}{4}} r_2 \leq \frac{R_0 + B_2 R_0^{\frac{q+2}{q+1}}}{1 - B_1 R_0^{\frac{2}{q+1}}} \tag{186}$$

We will call this radius, at which $x \to \infty$ and therefore $\gamma \to 0$, R_2.

$$R_2 \leq \frac{R_0 + B_2 R_0^{\frac{q+2}{q+1}}}{1 - B_1 R_0^{\frac{2}{q+1}}} \tag{187}$$

From the condition eq. (180), we can follow that:

$$R_2 \leq \frac{16}{15} r_2 \tag{188}$$

This shows that the matter distribution reaches zero, hence the matter quantities will vanish within the interval $[r_2, \frac{16}{15} r_2]$, before the radius R_1 as stated in the proposition.

□

5 Outline of the proof for a positive cosmological constant Λ

For the predominant part, the proof with a positive cosmological constant Λ is similar to the section above with a negative one. For straightforwardness only the passages which differ will be explicitly described here.

5.1 Bounds on the matter quantities

In this first section we want to show that lemma 1 is also fulfilled if we choose a positive cosmological constant Λ. Since it is a part of the Einstein equations, it directly affects the derivative of $y(r)$ and therefore the derivative of the characteristic function $\gamma(r)$. The calculation, which lead to eq. (44) are omitted, because they give the same equations as seen in section 4.1.

Lemma 5. *Let $k \geq 0$ and $l \geq -\frac{1}{2}$. Moreover let R_0 have the form as defined in eq. (23) and $\gamma(r)$ as in eq. (21). Then for $r \in [R_0, 2R_0]$, two positive constants C_u and C_l can be found, which depend on k, l and L_0 such that when $\gamma \geq 0$ and $z = \rho - p - 2p_T$,*

$$C_l \frac{\gamma^{l+k+\frac{3}{2}}}{r^4} \leq \rho(r) \leq C_u \frac{\gamma^{l+k+\frac{3}{2}}}{r^4} \tag{189}$$

$$C_l \frac{\gamma^{l+k+\frac{5}{2}}}{r^4} \leq p(r) \leq C_u \frac{\gamma^{l+k+\frac{5}{2}}}{r^4} \tag{190}$$

$$C_l \frac{\gamma^{l+k+\frac{3}{2}}}{r^2} \leq z(r) \leq C_u \frac{\gamma^{l+k+\frac{3}{2}}}{r^2} \tag{191}$$

Proof of eq. (189)

Proof. In order find bounds in this case, we start with eq. (17) and simplify it in the same manner as seen in section 4.1.

$$C_l e^{-\gamma k}\frac{(e^\gamma-1)^{k+l+\frac{3}{2}}}{r^4} \leq \rho(r) \leq C_u e^{-\gamma k}\frac{(e^\gamma-1)^{k+l+\frac{3}{2}}}{r^4} \tag{192}$$

The last step of the proof was to rewrite $(e^\gamma - 1)$ and $e^{-\gamma k}$. For this we use the derivative of $\gamma(r)$. With a positive Λ, this takes the form:

$$\gamma'(r) = -e^{2\lambda}\left(4\pi r p(r) + \frac{m(r)}{r^2}\right) + e^{2\lambda}\frac{r\Lambda}{3} + \frac{1}{r} \tag{193}$$

The first term drops out due to its sign, which leaves:

$$\gamma'(r) \leq e^{2\lambda}\frac{r\Lambda}{3} + \frac{1}{r}$$

In order to find a bound on the characteristic function, we need to know that its derivative is positive, but bounded from above. The last expression shows that this is the case for a positive Λ, because clearly the radius and the expression $e^{2\lambda}$ are positive as well.

The bound of y(r), follows from eq. (24), which then gives us bounds $e^{-\gamma k}$.
Following the same calculations as in section 4.1, we find the bounds on $\rho(r)$ as stated in lemma 1.

$$C_l \frac{\gamma^{k+l+\frac{3}{2}}}{r^4} \leq \rho(r) \leq C_u \frac{\gamma^{k+l+\frac{3}{2}}}{r^4} \tag{194}$$

This concludes the proof for $\rho(r)$. $\square$

The proof for p(r) follows the same manner as in section 4.1 with the same alterations for a positive constant seen above.

5.2 Bounds on the characteristic function

Lemma 6. *There exists a positive constant C_γ such that*

$$\gamma(r) \leq C_\gamma r^{\frac{2}{q+1}} \tag{195}$$

for all $r \in [R_0, 2R_0]$, if R_0 is sufficiently small. Here we used $k \geq 0$, $l \geq -\frac{1}{2}$ and $q = l + k + \frac{3}{2} \geq 1$.

The proof for lemma 6 starts in the same way as seen in section 4.2 and remains the same until eq. (69), which changes to the following expression:

Proof.

$$\gamma'(r_1) \leq \frac{1}{r_1}\left(1 - e^{2\lambda}\right) + \frac{e^{2\lambda} r_1 \Lambda}{3} - \frac{e^{2\lambda} m(r_1)}{r_1^2} \tag{196}$$

The first term is smaller than zero and drops out due to its sign. That leaves:

$$\gamma'(r_1) \leq e^{2\lambda}\left(\frac{r_1\Lambda}{3} - \frac{m(r_1)}{r_1^2}\right) \tag{197}$$

In order for the derivative of the characteristic function to be lower or equal to zero, the following estimate has to hold.

$$\frac{r_1\Lambda}{3} < \frac{m(r_1)}{r_1^2} \tag{198}$$

If we choose R_0 to be small, then also r_1 will be small. Thus $r_1 < \frac{1}{r_1^2}$, which fulfills the estimate. Thus we have shown that $\gamma'(r_1)$ becomes negative and the characteristic function is therefore decreasing.

However in the beginning of the proof we showed that the characteristic function should be increasing to fulfill eq. (66), which states that:

$$\gamma(r)^{q+1} > \frac{r^2}{4\pi C_l} \tag{199}$$

This yields a contradiction, which means that lemma 6 is fulfilled on $[R_0, 2R_0]$ and that an upper bound of the characteristic function can be estimated by a power of r. □

In order to find a lower bound of the characteristic function we use the same approach here, as we did for section 4.2.

Lemma 7. *On an interval* $r \in [R_0, R_0 + \delta]$ *with* R_0 *sufficiently small, let*

$$C_m = max[1, C_u, 4\pi C_u] \tag{200}$$

$$\delta \leq \frac{R_0^{\frac{q+3}{q+1}}}{C_m 8^{\frac{1}{q+1}}} \tag{201}$$

Then

$$\gamma'(r) \geq \frac{1}{2r} + \frac{r\Lambda}{3} \tag{202}$$

Proof. The change of the sign of Λ does not change the most part of this section, except for the lower bound of $y'(r)$, eq. (88) at the end of section 4.2.

$$\gamma'(r) \geq \frac{2}{3r} + \frac{4r\Lambda}{9} = \frac{2}{3}\left(\frac{1}{r} + \frac{2}{3}r\Lambda\right) \geq \frac{1}{2}\left(\frac{1}{r} + \frac{2}{3}r\Lambda\right) \tag{203}$$

For a positive Λ we see that both terms are positive and hence $\gamma'(r)$ is bounded from below by:

$$\gamma'(r) \geq \frac{1}{2r} + \frac{r\Lambda}{3} \tag{204}$$

□

5.3 Existence of characteristic radii

In section 4.3 we argued that the this part of the proof is the same as in the case with a vanishing cosmological constant, [14].
Hence, the results of lemma 3 are not affected by the cosmological constant, negative or positive. The proof was already shown in section 4.2 and will not be written down again.

5.4 Lower bound for $\frac{m(r)}{r}$

Proposition 3. *For r_2 as defined in the previous section, the solution will satisfy:*

$$\frac{m(r_2)}{r_2} \geq \frac{2}{5} \tag{205}$$

Proof. Analogous to section 4.4 we start the proof of proposition (3) by showing that $r_2 \leq 4m(r_2)$ holds, by contradiction. We again find the estimate for the lower bound of $y'(r_2)$:

$$y'(r_2) \geq -\frac{8\pi C}{R_0^{\frac{q-1}{q}}} - \frac{1}{2r_2} + \frac{2}{3} r_2 \Lambda \tag{206}$$

We want to show that the last expression is larger than $-\frac{1}{r_2}$. Since $-\frac{1}{2r_2} \geq -\frac{1}{r_2}$, it simplifies to:

$$-\left(\frac{8\pi C}{R_0^{\frac{q-1}{q}}} - \frac{2}{3} r_2 \Lambda \right) \leq -\frac{1}{2r_2} \tag{207}$$

For a positive Λ the second term on the left-hand side remains negative.

$$\frac{8\pi C}{R_0^{\frac{q-1}{q}}} - \frac{2}{3} r_2 \Lambda \geq \frac{1}{2r_2} \tag{208}$$

From lemma 6 we know that $q \geq 1$ and since R_0 is sufficiently small, we find that R_0^0

is the largest value. On the interval, which we are looking at $[r_0, 2R_0]$, the following holds: $r_2 \leq 2R_0$. We insert this estimate for r_2.

$$8\pi C - \frac{4}{3}R_0\Lambda \geq \frac{1}{4R_0} \tag{209}$$

$$8\pi C R_0 - \frac{4}{3}R_0^2\Lambda \geq \frac{1}{4} \tag{210}$$

Since we can choose R_0 freely, we choose it as small as necessary to ensure that the last expression is fulfilled.
From this follows:

$$y'(r_2) \geq -\frac{1}{r_2} \tag{211}$$

However from eq. (47) we know that the following holds:

$$\gamma'(r_2) = y'(r_2) + \frac{1}{r_2} \leq 0 \tag{212}$$

$$y'(r_2) \leq -\frac{1}{r_2} \tag{213}$$

This shows that eq. (112) is wrong and we have shown that $r_2 \leq 4m(r_2)$.

□

Proof. The following part, the calculation of the r-derivative of the following expression eq. (123) is analogous to the one in section 4.4. Since we did not determine the sign of the cosmological constant until eq. (154), we will start the proof from there.

$$\frac{m(r_2)}{r_2} \geq \frac{4}{9} + C_\Lambda \tag{214}$$

We insert the following estimates into eq. (140) to find an estimate for C_Λ.

$$\frac{m(r_2)}{r_2} \geq \frac{1}{4} \;\; \& \;\; \frac{m(r)}{r} \leq \frac{8}{9} \tag{215}$$

$$C_\Lambda \geq -\Lambda \frac{r_2}{m(r_2)} e^{-\lambda(r_2)} \int_{R_0}^{r_2} e^{\lambda(s)} s\, ds \geq -\Lambda \frac{9}{4} \int_{R_0}^{r_2} e^{\lambda(s)} s\, ds \tag{216}$$

Further simplifications and a positive Λ give:

$$C_\Lambda \geq -\frac{27}{8} \Lambda (r_2^2 - R_0^2) \tag{217}$$

With this estimate we return to eq. (154).

$$\frac{m(r_2)}{r_2} \geq \frac{4}{9} - \frac{27}{8} \Lambda (r_2^2 - R_0^2) \tag{218}$$

Thus we have shown, that $\frac{m(r_2)}{r_2} \geq \frac{2}{5}$ if we choose R_0 small enough. This concludes the proof from proposition (3). □

5.5 Proof of Proposition 2

Proposition 4. *If one considers a quasi shell solution of the mass-less Einstein-Vlasov system and chooses the inner radius R_0 to be sufficiently small, then it can be shown that the distribution function $f(x, v)$ vanishes within an interval $[R_1, R_2]$, where*

$$R_1 = R_0 \left(1 + B_0 R_0^{\frac{2}{q+1}}\right) \tag{219}$$

$$R_2 = R_0 \frac{1 + B_2 R_0^{\frac{1}{q+1}}}{1 - B_1 R_0^{\frac{2}{q+1}}} \tag{220}$$

The constants B_0, B_1, B_2 and q are positive and should only depend on the Ansatz function for the matter distribution eq. (14)*. Moreover we need $R_0 < R_1$ to ensure vacuum at the center.*

Proof. In order to find the lower bound of this interval we use the same approach as in section 4.5. Since the proof does not include the cosmological constant we find the same result as before, independent of the sign of Λ. We know from lemma 3 that the derivative of $\gamma(r)$ is positive on the interval $[R_0, R_0 + \delta]$.

With the estimate for δ from eq. (73), we find:

$$\delta \leq \frac{R_0^{\frac{q+3}{q+1}}}{C_m 8^{\frac{1}{q+1}}} \rightarrow \frac{1}{C_m 8^{\frac{1}{q+1}}} = B_0 \tag{221}$$

Thus we have shown that the characteristic function $\gamma(r)$ remains increasing and can in fact not vanish until the radius R_1.

In order to find an estimate for the upper bound, we need to show that the function $\gamma(r)$ vanishes before the radius reaches R_2.
Again we start by introducing a new variable:

$$x(r) := \frac{m(r)}{r\gamma(r)} \tag{222}$$

We can see that $\gamma(r)$ goes to zero if $x \rightarrow \infty$ and we will see that this happens close to r_2, as long as R_0 is sufficiently small.
For a positive cosmological constant and the same estimates as in section 4.5 we find a different differential equation, which has the form of a Bernoulli equation.

$$rx' \geq \frac{4}{3}x^2 - x - 4\Lambda r^2 x^2 \tag{223}$$

The solution of this equation is of the form:

$$x(r) = \frac{3}{c_1 r + 12\Lambda r^2 + 4} \tag{224}$$

To determine the constant c_1, we use the boundary condition at $r = r_2$.

$$c_1 = \frac{3}{x(r_2) r_2} - 12\Lambda r_2 - \frac{4}{r_2}$$

$$x(r) = \frac{3}{4} * \frac{1}{1 - \frac{r}{\frac{4}{3} r_2 x(r_2)} \left(\frac{4}{3}x(r_2) - 1 - 3\Lambda r_2 + 3\Lambda r\right)} \tag{225}$$

We observe that x(r) has a similar form to eq. (175), except for two extra terms in the denominator. Similar to before the function x(r) diverges if:

$$r \to \frac{1}{\frac{1}{\frac{4}{3}r_2 x(r_2)}\left(\frac{4}{3}x(r_2) - 1 - 3\Lambda r_2 + 3\Lambda r\right)} \tag{226}$$

$$\leq r_2 * \frac{1}{1 - 3\Lambda r_2^2 + 3\Lambda r r_2 - \frac{15}{8}\frac{r_2^{\frac{2}{q+1}}}{C}} \leq \frac{16}{15} r_2 \tag{227}$$

Into this expression we insert the same estimates for r_2 as seen above in section 4.5, to find a similar estimate for R_2.

$$R_2 \leq \frac{R_0 + B_2 R_0^{\frac{q+2}{q+1}}}{1 - 3\Lambda C_* R_0^2 + 3\Lambda C_* r R_0 - B_1 R_0^{\frac{2}{q+1}}} \tag{228}$$

In order to get the desired form from proposition 2, we need to show that the two terms in the middle are either sufficiently small, or cancel out.

$$3\Lambda C_* R_0^2 = 3\Lambda C_* r R_0 \tag{229}$$

This equation is fulfilled for $r = R_0$, which holds if R_0 is sufficiently small. Thus we find the same result for R_2, which shows that the matter distribution reaches zero, hence the matter quantities will vanish within the interval $[r_2, \frac{16}{15}r_2]$.

$$R_2 \leq \frac{R_0 + B_2 R_0^{\frac{q+2}{q+1}}}{1 - B_1 R_0^{\frac{2}{q+1}}} \leq \frac{16}{15} r_2 \tag{230}$$

□

6 Proof of main theorem and discussion

Following section 4.5 as well as section 5.5, for a negative and positive cosmological constant respectively, and the proof of propositions 2 and 4 we have shown that the matter region is restricted to a region within the interval $[R_0, R_2]$. Furthermore we have shown that a radius R exists, which is smaller than R_2. At this radius we find:

$$\rho(R) = 0 \qquad\qquad p(R) = 0 \tag{231}$$

Hence matter disappears and the distribution function f(x,v) becomes zero and can then be continued by constant zero for all $r > R$. If we would not set the matter to zero at this radius, we would see matter would reappear again. This is due to the form of the characteristic function $\gamma(r)$ in the case of mass-less particles. The ansatz, eq. (21), shows a function which can become positive again for a radius $r > R_2$. By setting all the matter quantities to zero, outside of the first photon shell, we guarantee asymtotical flatness.
The auxiliary variable y(r), which we defined in section 2 as $y(r) = ln(E_0) - \mu$ will take the following form for $r > R$:

$$y_{out}(r) = ln(E_0) - \frac{1}{2}ln\left(1 - \frac{2M}{r} - \frac{r^2\Lambda}{3}\right) \tag{232}$$

The second term corresponds to the Schwarzschild parameter, with the mass parameter M, [14].

$$M = m(R) = 4\pi \int_{R_0}^{R_2} s^2\rho(s)ds \tag{233}$$

This investigation shows that globally static, spherically symmetric solutions can be found for a positive and negative cosmological constant Λ. Furthermore, there are no restriction on the the value of $\Lambda > 0$, i.e. it does not have to be sufficiently small. The only necessary requirement for the proof is, that the inner radius of the photon shell R_0 can be chosen arbitrarily small. This is due to the fact, that the terms with the cosmological constant can mostly be neglected, as long as R_0 is sufficiently small.

Continuation beyond the non-vacuum region

In the previous proof we have shown that there exists a unique solution $\gamma(r)$ for the system, see eq. (6), eq. (7) and eq. (9) on the interval $r \in [0, R_2]$ for a non-vanishing cosmological constant Λ. A solution for $\gamma(r)$ then also induces a solution for the metric coefficients $\lambda(r)$ and $\mu(r)$. In order to find a global solution to the static, spherically symmetric Einstein-Vlasov system with a negative cosmological constant, we need to glue the Schwarzschild-Anti-deSitter metric, [13] to our solution, for $r > R_2$, i.e. the vacuum region outside of the shell solution. The coefficients for this metric, $\lambda_A(r)$ and $\mu_A(r)$ will be given by:

$$e^{-\lambda_A} = e^{\mu_A(r)} = 1 + \frac{r^2|\Lambda|}{3} - \frac{2M}{r} \tag{234}$$

Here M denotes the mass parameter of the photon shell, see eq. (233).
Similarly for a positive cosmological constant, we can find the Schwarzschild-deSitter coefficients $\lambda_S(r)$, $\mu_S(r)$, which are given by:

$$e^{-\lambda_S} = e^{\mu_S(r)} = 1 - \frac{r^2\Lambda}{3} - \frac{2M}{r} \tag{235}$$

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