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Abstract

In 2012, numerical evidence was given for the non-linear instability of a portion of $(3+1)$ dimensional Minkowski spacetime enclosed inside a timelike worldtube $\mathbb{R} \times B^3$ with a perfectly reflecting wall ('flat space enclosed in a cavity') under arbitrarily small perturbations [21]. The numerical evidence was corroborated by a weakly non-linear perturbation analysis which revealed the existence of secular terms, i.e. terms that grow without bounds, in the perturbative series, which were conjectured to be progenitors of a resonant mode mixing which causes the transfer of energy from lower to higher modes. In light of this analysis, we derive the *resonant approximation* for this system using 1) the method of multiple scales [3] and 2) the renormalization group method [8] with the aim of resumming the secular terms in the perturbative series in order to get a description of the relevant dynamics that is valid at long times. We solve the resulting system of coupled first order differential equations, called the resonant system, numerically and illustrate in detail its solution. Furthermore, we analyze the mode amplitude spectrum using the analyticity strip method [27]. We find that the amplitude spectrum evolves to a power law in finite time, which corresponds to the solution of the resonant system becoming singular. Moreover, we find that the value for the power law exponent determined in our analysis is in agreement with the corresponding value observed in the fully non-linear evolution of the system [21].

Zusammenfassung

Im Jahr 2012 wurde numerische Evidenz für die nichtlineare Instabilität des Minkowskiraumes, eingesperrt in einer zeitartigen Weltröhre $\mathbb{R} \times B^3$ mit einer perfekt reflektierenden Wand, vorgelegt [21]. Die numerischen Erkenntnisse wurden durch eine schwach nichtlineare störungstheoretische Analyse ergänzt, welche die Existenz von säkularen Termen, also Termen, die mit der Zeit anwachsen, in der Störungsreihe aufzeigte. Die säkularen Terme wurden als Ursache für das Auftreten einer resonanten Modenmischung interpretiert, welche das Übertragen von Energie von niedrigeren zu höheren Moden bedingte. Im Lichte dieser Erkenntnisse wird in der vorliegenden Arbeit die *Resonanznäherung* für dieses System 1) mithilfe der Mehrskalen-Methode [3], 2) mithilfe der Renormierungsgruppenmethode [8] hergeleitet. Dies geschieht mit dem Ziel, die säkularen Terme in der Störungsreihe zu resummieren und somit eine auf langen Zeitskalen gültige Beschreibung der wesentlichen dynamischen Elemente des Systems zu erreichen. Das sich ergebende sogenannte resonante System, ein System gekoppelter Differentialgleichungen erster Ordnung, wird mithilfe eines numerischen Verfahrens gelöst, und folglich eine detaillierte Beschreibung seiner Lösung vorgelegt. Des Weiteren wird eine Analyse des Amplitudenspektrums mithilfe der Analytizätsstreifenmethode [27] durchgeführt. Es wird gezeigt, dass das Amplitudenspektrum nach einer endlichen Entwicklungszeit durch ein Potenzgesetz gegeben ist. Dies entspricht der Entstehung einer Singularität in der Lösung des resonanten Systems. Außerdem wird aufgezeigt, dass der in der vorliegenden Analyse ermittelte Exponent des Potenzgesetzes mit dem entsprechenden, in der vollständig nichtlinearen Entwicklung des Systems ermittelten Wert [21] übereinstimmt.

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Contents

Chapter 1

Introduction

This thesis is concerned with the derivation and solution of the resonant system for flat space enclosed in a cavity. The choice of this system is motivated as follows. In 2011, Bizoń and Rostworowski studied the non-linear evolution of weakly perturbed Anti-de Sitter (AdS) space in spherical symmetry by solving the Einstein-massless-scalar field equations with negative cosmological constant numerically [5]. The results obtained suggested the instability of AdS space under arbitrarily small perturbations and have led to an increase of interest in the topic of non-linear stability of AdS space [22].

The interest in spacetimes asymptotically approaching AdS space is mainly sparked by the formulation of the AdS/CFT correspondence, a widely researched topic in theoretical high-energy physics [16]. Furthermore, AdS space is the unique maximally symmetric solution to the vacuum Einstein equations with a negative cosmological constant [5]. Given that the other two maximally symmetric solutions to the vacuum Einstein equations, Minkowski ('flat') space and de Sitter (dS) space, for a zero and positive cosmological constant, respectively, have been proven to be stable under small perturbations [9, 15], it remains to shed further light on AdS.

It was argued in [5] that two effects are responsible for the dynamics causing the instability. The first one is the existence of a timelike boundary at infinity in AdS space, which precludes the escape of energy and acts as a mirror for waves propagating outwards. This behavior is dependent on the specific 'reflective' Dirichlet boundary conditions used in [5]. The second one is the phenomenon of weak turbulence due to resonant interactions, which involves a shift of energy from lower to higher frequencies in the wave modes. In [5], the existence of secular terms in the perturbative series was shown using a weakly non-linear perturbation analysis. It was further argued that the secular terms are progenitors of the observed turbulent cascade of energy from low to high frequencies.

The results of [5] have raised the question: Do other 'confined' general-relativistic models exhibit similar behavior? In 2012, Maliborski considered the Einstein equations for a spherically symmetric self-gravitating massless scalar field inside a timelike worldtube $\mathbb{R} \times B^3$ with reflective boundary conditions ('flat space enclosed in a cavity') and gave numerical evidence that it evolves into a black hole for arbitrarily small generic initial data [21]. Similarly to AdS, the system exhibits a fully resonant spectrum of linearized modes, and hence features the same turbulent shift of energy to increasingly high frequencies which engenders instability. A weakly non-linear perturbation analysis, in analogy to [5], revealed the existence of secular terms in the perturbative solution [21]. Furthermore, the formation of a power law spectrum in the distribution of energy among the linear modes was observed just before the time of collapse.

In response to [5], in 2014, Craps, Evnin, and Vanhoof proposed a resonant approximation for the system studied in [5] based on the renormalization group as a means to resum the secular terms in the perturbative series [11]. In a following paper [12], the technique was shown to be equivalent to time averaging, a standard procedure in non-linear perturbation theory [24]. In [2], a technique based on the method of multiple scales ('Two-Time-Framework') was used to derive the same system. Given the full system starting from small initial data of order ϵ, it can be shown that the resonant approximation accurately captures the weakly non-linear dynamics on a time scale up to order ϵ^{-2} [4], which is the same time scale at which secular terms start to invalidate the naive perturbation theory for the model in [5].

In 2015, Bizoń, Maliborski and Rostworowski [4] solved the resonant system for AdS space [11, 12] numerically and gave evidence of the formation of an oscillatory singularity in finite time. In the paper [4], the blowup of the solution of the resonant system was analyzed using the analyticity strip method [27] for the mode amplitude spectrum, as well as asymptotic methods which established the logarithmic blowup of the mode phases. It was argued that the blowup of the resonant system corresponds to the formation of a black hole in the full system. The results of [4] were interpreted as providing further evidence in favor of the AdS instability conjecture.

In this thesis, we derive the resonant system for flat space enclosed in a cavity [21] and analyze it following a similar approach as [4]. In Chapter 2, we state essential preliminaries from general relativity as well as the theory of scalar fields. In Chapter 3, we study the notion of resonance in oscillatory phenomena, show how it generates secular terms in perturbative expansions, and review the method of multiple scales as well as the renormalization group method as techniques for resumming such terms. In Chapter 4, we review the model from [21] ('flat space enclosed in a cavity'), derive its equations of motion, repeat the weakly non-linear perturbation analysis, derive the resonant system, discuss its symmetries, and compare the system to AdS. In Chapter 5, we study the resonant system numerically and analyze its solution using the analyticity strip method [4, 27]. In Chapter 6, we provide a summary and give conclusive remarks. In Appendix A, we give details on the eigenvalue problem that arises as part of the perturbation analysis in Chapter 3. In Appendix B, we consider the cubic radial wave equation, a model whose resonant system shares several similarities with the resonant system derived in Chapter 4.

Chapter 2

Preliminaries

In this chapter, we state essential preliminaries from the theory of general relativity and the theory of scalar fields.

2.1 General relativity

We recall some basic facts and results from general relativity based on [10], [7], [23].

2.1.1 Minkowski space

In units where the speed of light $c = 1$, the Minkowski ('flat') metric in Cartesian coordinates (t, x, y, z) is given by

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \,. \tag{2.1}$$

We introduce $(x^\mu) = (t, x, y, z)$, where $\mu = 0, 1, 2, 3$. Then the line element (2.1) can be written as

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \,, \tag{2.2}$$

where

$$(\eta_{\mu\nu}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{2.3}$$

We write the Minkowski metric in spherical coordinates (t, r, θ, φ) as

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_2^2 \,, \tag{2.4}$$

where $d\Omega_2^2 = d\theta^2 + \sin^2\theta d\varphi^2$ is the round metric on the unit two-sphere S^2.

2.1.2 Curvature

We consider a (pseudo-)Riemannian metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \,. \tag{2.5}$$

The inverse of $g_{\mu\nu}$ is $g^{\mu\nu}$ and is defined by the relation

$$g^{\mu\nu} g_{\nu\sigma} = g_{\lambda\sigma} g^{\lambda\mu} = \delta^{\mu}_{\sigma} \,, \tag{2.6}$$

where δ is the Kronecker delta. The Christoffel symbols are given by

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu} \right) \,. \tag{2.7}$$

The formula for the Riemann curvature tensor is

$$R^{\rho}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda} \Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda} \Gamma^{\lambda}_{\mu\sigma} \,. \tag{2.8}$$

The Ricci tensor is given by the contraction of the Riemann tensor

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu} \,. \tag{2.9}$$

The Ricci scalar is the trace of the Ricci tensor,

$$R = R^{\mu}_{\mu} \equiv g^{\mu\nu} R_{\mu\nu} \,. \tag{2.10}$$

The Einstein tensor is given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \,. \tag{2.11}$$

2.1.3 Gravitation

The Einstein equations are

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \,, \tag{2.12}$$

where Λ is the cosmological constant, G is Newton's constant and $T_{\mu\nu}$ is the energy-momentum tensor. The Hilbert energy-momentum tensor is defined as

$$T_{\mu\nu} = -2 \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}} \,, \tag{2.13}$$

where g refers to the metric determinant, S_M is the action for the matter source, and $\frac{\delta S_M}{\delta g^{\mu\nu}}$ refers to the variation of the action S_M with respect to the inverse metric $g^{\mu\nu}$.

2.2 Scalar fields

We list some basic properties of a scalar field and derive its energy-momentum tensor based on [7]. The action for a scalar field ϕ is given by

$$S_\phi = \int d^4x \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi - V(\phi) \right] \,, \tag{2.14}$$

where $V(\phi)$ is the potential. For $V(\phi) = \frac{1}{2} m^2 \phi^2$ we get the action for a free Klein-Gordon field, i.e. a free scalar field with mass m. The Euler-Lagrange equations for the action (2.14) are

$$\Box \phi - \frac{dV}{d\phi} = 0 \,, \tag{2.15}$$

where $\Box\phi \equiv \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi)$ refers to the d'Alembertian acting on ϕ.

We calculate the energy-momentum tensor using formula (2.13). We have

$$\delta S_\varphi = (\delta S)_A + (\delta S)_B \,, \tag{2.16}$$

where

$$(\delta S)_A = -\int d^4x \frac{1}{2}\sqrt{-g}\delta g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi \,, \tag{2.17}$$

$$(\delta S)_B = \int d^4x \left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V(\phi)\right]\delta\sqrt{-g} \,. \tag{2.18}$$

We write

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta g \,. \tag{2.19}$$

Taking the variation of the identity [7]

$$\ln(\det M) = \mathrm{Tr}(\ln M) \,, \tag{2.20}$$

where M is a matrix with non-vanishing determinant and Tr refers to the trace, yields

$$\frac{1}{\det M} = \mathrm{Tr}(M^{-1}\delta M) \,, \tag{2.21}$$

where we have used the cyclic property of the trace

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA) \,. \tag{2.22}$$

Using (2.21) with $M = g_{\mu\nu}$ and $\det M = \det g_{\mu\nu} \equiv g$, we have

$$\delta g = g(g^{\mu\nu}\delta g_{\mu\nu}) = -g(g_{\mu\nu}\delta g^{\mu\nu}) \,, \tag{2.23}$$

where we have used

$$0 = \delta\delta^\mu_\nu = \delta(g^{\mu\lambda}g_{\lambda\nu}) = g_{\lambda\nu}\delta g^{\mu\lambda} + g^{\mu\lambda}\delta g_{\lambda\nu} \,. \tag{2.24}$$

Inserting (2.23) back into (2.19), and (2.19) into (2.18), we have for the variation (2.16)

$$\delta S_\varphi = \int d^4x\sqrt{-g}\left[-\frac{1}{2}\partial_\mu\phi\,\partial_\nu\phi + \left(\frac{1}{2}g_{\mu\nu}\right)\left(\frac{1}{2}g^{\rho\sigma}\partial_\rho\phi\,\partial_\sigma\phi + V(\phi)\right)\right]\delta g^{\mu\nu} \tag{2.25}$$

$$\equiv \int d^4x \frac{\delta S_\varphi}{\delta g^{\mu\nu}}\delta g^{\mu\nu} \,. \tag{2.26}$$

Thus

$$T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\partial_\rho\phi\,\partial_\sigma\phi - g_{\mu\nu}V(\phi) \,. \tag{2.27}$$

For a self-gravitating massless scalar field, $V(\phi) = 0$, this reduces to

$$T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\partial_\rho\phi\,\partial_\sigma\phi \,. \tag{2.28}$$

Chapter 3

Resonant perturbations, secular terms, and secular term resummation

In this chapter, we introduce some important concepts underlying the motivation for the resonant approximation. We introduce the concept of resonance in oscillatory phenomena, where a perturbation matching the frequency of a harmonic oscillator causes the solution to become unbounded. Next, we show that resonance in the context of perturbation theory leads to secular terms, i.e. terms that grow with time, which invalidate the theory on a certain time scale. Finally, we review two powerful and established resummation techniques, namely the method of multiple scales [3] and the renormalization group method [8], which can be used to remove secular terms from perturbative expansions. We treat the *Duffing* equation as an example.

Historically, the study of 'secular' terms (deriving from the Latin word for 'century') in perturbation theory was motivated by celestial mechanics [18, 25]. In particular, it was studied how small perturbations caused by the presence of the other planets would affect, say, the orbit of the earth around the sun at long time scales.

3.1 Resonance in oscillatory phenomena

We introduce the notion of resonance by considering the equation

$$\ddot{y} + y = -\cos t\,, \quad y(0) = 1\,, \quad \dot{y}(0) = 0\,, \tag{3.1}$$

which describes a harmonic oscillator with natural frequency 1 under the influence of a periodic driving force matching its natural frequency. We show that as a result of this, (3.1) has the unbounded term $\sim t \sin t$ in its solution. We use the method of variation of parameters.

We seek a general solution of the form

$$y = y_h + y_p\,, \tag{3.2}$$

where y_h is the solution to the homogeneous equation

$$\ddot{y}_h + y_h = 0 \tag{3.3}$$

and y_p is a particular solution. We write down y_h as

$$y_h(t) = Au_1(t) + Bu_2(t) \equiv A\cos t + B\sin t\,. \tag{3.4}$$

For y_p we write

$$y_p(t) = C_1(t)u_1(t) + C_2(t)u_2(t)\,, \tag{3.5}$$
$$\dot{y}_p(t) = C_1(t)\dot{u}_1(t) + C_2(t)\dot{u}_2(t)\,, \tag{3.6}$$

where we have made the ansatz

$$\dot{C}_1(t)u_1(t) + \dot{C}_2(t)u_2(t) = 0\,. \tag{3.7}$$

Calculating the second derivative of y_p and using $\ddot{u}_{1,2} = -u_{1,2}$, we get

$$\ddot{y}_p + y_p = \dot{C}_1\dot{u}_1 + \dot{C}_2\dot{u}_2 = f(t) \equiv -\cos t\,. \tag{3.8}$$

Solving the system of equations (3.7)-(3.8) for $\dot{C}_1$ and $\dot{C}_2$ and integrating, we find

$$C_1(t) = -\int_0^t dt' \frac{f(t')u_2(t')}{W}\,, \quad C_2(t) = \int_0^t dt' \frac{f(t')u_1(t')}{W}\,, \tag{3.9}$$

where $W = u_1\dot{u}_2 - \dot{u}_1 u_2 = 1$ is the Wronskian determinant. Calculating (3.9), we find

$$C_1(t) = \frac{1}{2}\sin^2 t \tag{3.10}$$
$$C_2(t) = -\frac{1}{2}(t + \sin t\cos t)\,. \tag{3.11}$$

We insert (3.10)-(3.11) into (3.5) and find

$$y_p(t) = -\frac{1}{2}t\sin t\,. \tag{3.12}$$

Thus we find for the general solution

$$y(t) = A\cos t + B\sin t - \frac{1}{2}t\sin t\,. \tag{3.13}$$

Imposing the initial conditions $y(0) = 1$, $\dot{y}(0) = 0$ yields

$$y(t) = \cos t - \frac{1}{2}t\sin t\,. \tag{3.14}$$

A treatment of the method of variation of parameters can be found in [3]. We learn from (3.14) that a resonant term on the right-hand side of the oscillator equation (3.1) generates a term proportional to $t\sin t$ in the solution. We note that this term is unbounded for $t \to \infty$.

3.2 Secular terms in perturbative expansions

We consider how the emergence of secular terms, i.e. terms that grow with t, affect perturbative expansions. We show that the perturbative solution to the Duffing equation contains secular terms. The derivation is based on [3].

We consider the undamped and unforced Duffing equation

$$\ddot{y} + y + \epsilon y^3 = 0\,, \quad y(0) = 1, \quad \dot{y}(0) = 0\,, \tag{3.15}$$

where y is a real function of t and $0 < \epsilon \ll 1$. We expand

$$y(t) = \sum_{j=0}^{\infty} \epsilon^j y_j(t) = y_0 + \epsilon y_1 + \mathcal{O}(\epsilon^2)\,, \tag{3.16}$$

and write down the equations at the respective powers of ϵ.

At zero order, we get the homogeneous equation

$$\ddot{y}_0 + y_0 = 0 \quad y_0(0) = 1\,, \quad \dot{y}_0 = 0\,, \tag{3.17}$$

which has the solution

$$y_0(t) = \cos t\,. \tag{3.18}$$

At first order, we find the equation

$$\ddot{y}_1 + y_1 = -y_0^3 = -\cos^3 t \quad y_1(0) = 0\,, \quad \dot{y}_1 = 0\,. \tag{3.19}$$

We use the identity $2\cos\theta\cos\varphi = \cos(\theta-\varphi) + \cos(\theta+\varphi)$ to rewrite the right-hand side of (3.19) as

$$-\cos^3 t = -\frac{3}{4}\cos t - \frac{1}{4}\cos 3t\,. \tag{3.20}$$

We note that $-\frac{3}{4}\cos t$ is a resonant term. We solve (3.19) using the variation of parameters method. We find for the general solution

$$y_1(t) = A\cos t + B\sin t + \frac{1}{32}\cos 3t - \frac{3}{8}t\sin t\,. \tag{3.21}$$

Imposing the initial conditions (3.19), we find

$$y_1(t) = \frac{1}{32}\cos 3t - \frac{1}{32}\cos t - \frac{3}{8}t\sin t\,. \tag{3.22}$$

Inserting (3.22) into (3.16), we find the solution of (3.15) up to first order

$$y(t) = \cos t + \epsilon\left[\frac{1}{32}\cos 3t - \frac{1}{32}\cos t - \frac{3}{8}t\sin t\right] + \mathcal{O}(\epsilon^2)\,. \tag{3.23}$$

We note that the perturbative solution (3.23) contains a secular term $\sim t\sin t$. We note that the ansatz (3.16) for the perturbation theory breaks down when t is of the order ϵ^{-1}. We find that because of the term $\frac{3}{8}t\sin t$, the expression is unbounded for $t \to \infty$. In contrast, it can be shown that the exact solution to (3.15) is bounded for all t [3]. In Figure 3.1, we compare the perturbative solution (3.23) with the exact solution[1] $y(t)$.

We have thus highlighted the need to resum secular terms in perturbative expansions.

[1] The exact solution was computed numerically using an explicit Runge-Kutta method [26] in *Python*.

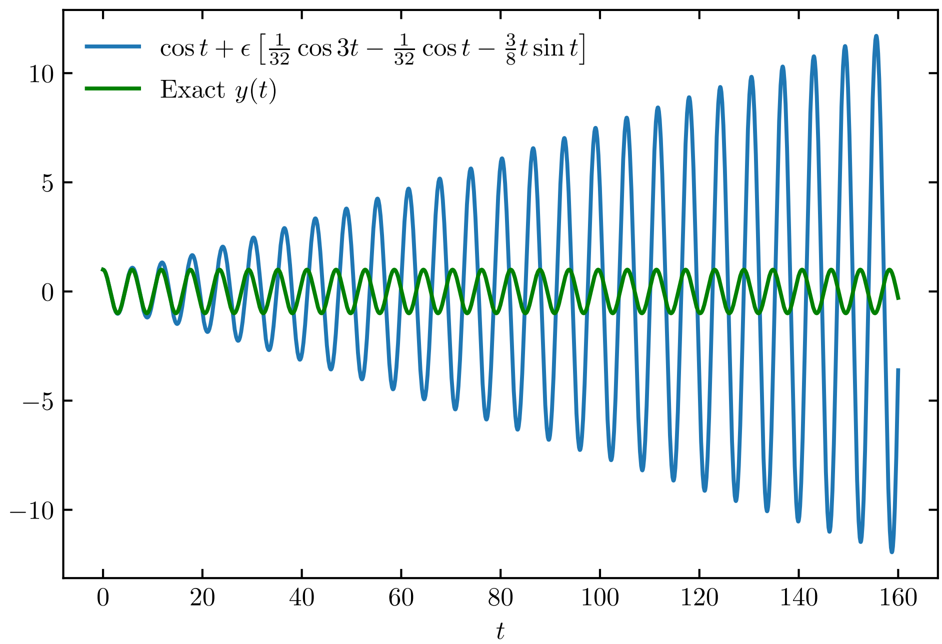

Figure 3.1: Comparison of the perturbative solution at first order (3.23) with the (numerically computed) exact solution to the Duffing equation (3.15) $y(t)$ for $\epsilon = 0.2$. We find that the amplitude of the perturbative solution increases with time, while the amplitude of the exact solution remains bounded.

3.3 Resummation techniques

In this section, we review two resummation techniques which achieve the removal of secular terms of the type seen in (3.23) for the Duffing equation. We will show how to construct perturbative solutions to the Duffing equation that are free of secular terms up to first order using both the method of multiple scales [3] and the renormalization group method [8]. In particular, we will show that both techniques lead to the same solution.

3.3.1 Multiple-scale analysis

We introduce the method of multiple scales, or *multiple-scale analysis*. In applying the method of multiple scales, we introduce a slow time dependence in the integration constants of the unperturbed solution. We may use this slow time dependence to remove secular terms from the solution at higher order. We show how to construct a solution to the Duffing equation that is bounded, i.e. free of secular terms, based on [3].

We consider, again,

$$\ddot{y} + y + \epsilon y^3 = 0\,, \quad y(0) = 1, \quad \dot{y}(0) = 0\,, \tag{3.24}$$

where dots refer to derivatives with respect to t. We construct a bounded solution, free of secular terms, up to first order of ϵ. We introduce the slow time-scale $\tau = \epsilon t$. We treat τ as an independent variable and write

$$y(t) = y_0(t, \tau) + \epsilon y_1(t, \tau) + \mathcal{O}(\epsilon^2)\,. \tag{3.25}$$

We compute

$$\begin{aligned}\frac{dy}{dt} &= \left(\frac{\partial y_0}{\partial t} + \frac{\partial y_0}{\partial \tau}\frac{d\tau}{dt}\right) + \epsilon\left(\frac{\partial y_1}{\partial t} + \frac{\partial y_1}{\partial \tau}\frac{d\tau}{dt}\right) + \mathcal{O}(\epsilon^2) \\ &= \frac{\partial y_0}{\partial t} + \epsilon\left(\frac{\partial y_0}{\partial \tau} + \frac{\partial y_1}{\partial t}\right) + \mathcal{O}(\epsilon^2)\,, \end{aligned} \tag{3.26}$$

where have used $\tau = \epsilon t$ in the second line. Calculating the second time derivative yields

$$\frac{d^2y}{dt^2} = \frac{\partial^2 y_0}{\partial t^2} + \epsilon\left(2\frac{\partial^2 y_0}{\partial\tau\partial t} + \frac{\partial^2 y_1}{\partial t^2}\right) + \mathcal{O}(\epsilon^2)\,. \tag{3.27}$$

We insert (3.27) and (3.25) into (3.24). At zero order, we find

$$\frac{\partial^2 y_0}{\partial t^2} + y_0 = 0\,. \tag{3.28}$$

For a real $y(t)$, we write down the general solution to (3.28) as

$$y_0(t,\tau) = A(\tau)e^{it} + \bar{A}(\tau)e^{-it}\,, \tag{3.29}$$

where $\bar{A}(\tau)$ is the complex conjugate of $A(\tau)$. At first order of ϵ, we have

$$\frac{\partial^2 y_1}{\partial t^2} + y_1 = -y_0^3 - 2\frac{\partial^2 y_0}{\partial\tau\partial t}\,. \tag{3.30}$$

We calculate

$$\begin{aligned} y_0^3 &= \left(Ae^{it} + \bar{A}e^{-it}\right)^3 \\ &= 3A^2\bar{A}e^{it} + 3A\bar{A}^2e^{-it} + A^3e^{3it} + \bar{A}^3e^{-3it}\,, \end{aligned} \tag{3.31}$$

$$2\frac{\partial^2 y_0}{\partial\tau\partial t} = 2i\left(\frac{dA}{d\tau}e^{it} - \frac{d\bar{A}}{d\tau}e^{-it}\right)\,, \tag{3.32}$$

thus

$$\frac{\partial^2 y_1}{\partial t^2} + y_1 = \left[-3A^2\bar{A} - 2i\frac{dA}{d\tau}\right]e^{it} + \left[-3A\bar{A}^2 - 2i\frac{d\bar{A}}{d\tau}\right]e^{-it} - A^3e^{3it} - \bar{A}^3e^{-3it}\,. \tag{3.33}$$

We note that e^{it} and e^{-it} are resonant terms which generate secular terms in the solution $y_1(t,\tau)$. We seek to eliminate them by requiring that their coefficients vanish. We find the equation

$$2i\frac{dA}{d\tau} = -3A^2\bar{A}\,, \tag{3.34}$$

where we note that the equation for $\bar{A}(\tau)$ is the complex conjugate of the equation for $A(\tau)$. We represent $A(\tau)$ as

$$A(\tau) = R(\tau)e^{i\theta(\tau)}\,, \tag{3.35}$$

where $R(\tau)$ and $\theta(\tau)$ are real. Inserting (3.35) into (3.34) yields

$$\frac{dR}{d\tau} = 0\,, \tag{3.36}$$

$$\frac{d\theta}{d\tau} = \frac{3}{2}R^2\,. \tag{3.37}$$

The solutions are

$$R(\tau) = R_0\,, \tag{3.38}$$
$$\theta(\tau) = \frac{3}{2}R_0^2\tau + \theta_0\,, \tag{3.39}$$

where R_0 and θ_0 are constants, thus

$$A(\tau) = R_0 e^{i(\theta_0 + \frac{3}{2}R_0^2\tau)}\,. \tag{3.40}$$

Inserting (3.40) into (3.29) yields

$$y_0(t,\tau) = 2R_0 \cos\left[\theta_0 + \frac{3}{2}R_0^2\tau + t\right]\,. \tag{3.41}$$

Imposing the initial conditions $y_0(0) = 1$, $\dot{y}_0(0) = 0$ sets $R_0 = \frac{1}{2}$, $\theta_0 = 0$, thus

$$y_0(t,\tau) = \cos\left[t + \frac{3}{8}\tau\right] = \cos\left[\left(1 + \frac{3}{8}\epsilon\right)t\right]\,, \tag{3.42}$$

where we have set $\tau = \epsilon t$. We find

$$y(t) = \cos\left[\left(1 + \frac{3}{8}\epsilon\right)t\right] + \mathcal{O}(\epsilon)\,. \tag{3.43}$$

We have thus, by applying the method of multiple scales and requiring that the right-hand side of (3.30) be free of resonant terms, introduced a small frequency shift in the solution at zero order. The solution obtained in this way is free of secular terms of the type (3.23) up to first order of ϵ.

3.3.2 Renormalization group method

We review the renormalization group method as a means to resum secular terms. The method was introduced in [8] by Chen, Goldenfeld and Oono, and is based on the renormalization group treatment of ultraviolet divergences in quantum field theory. Craps, Evnin and Vanhoof applied the method in the context of AdS in [11]. Considering, again, the Duffing equation, we check whether we obtain the same result for the resummed perturbation series as in the multiple-scale analysis.

We construct, again, the naive perturbation series for the Duffing equation. We have

$$\ddot{y} + y + \epsilon y^3 = 0\,, \quad y(0) = 1, \quad \dot{y}(0) = 0\,. \tag{3.44}$$

We expand

$$y(t) = \sum_{j=0}^{\infty} \epsilon^j y_j(t)\,. \tag{3.45}$$

At zero order, we find

$$\ddot{y}_0 + y_0 = 0\,. \tag{3.46}$$

We write down the general solution to (3.46) as

$$y_0(t) = a\cos(t + b)\,, \tag{3.47}$$

where a and b are real. At first order of ϵ, we find

$$\ddot{y}_1 + y_1 = -y_0^3 = -a^3 \cos^3(t+b)\,, \tag{3.48}$$

We use the identity $2\cos\theta\cos\varphi = \cos(\theta-\varphi) + \cos(\theta+\varphi)$ rewrite the right-hand side of (3.48) as

$$-\frac{3}{4}a^3\cos(t+b) - \frac{1}{4}a^3\cos(3(t+b))\,. \tag{3.49}$$

We note that $-\frac{3}{4}a^3\cos(t+b)$ is a resonant term. We solve (3.48) using the variation of parameters method and find

$$y_1(t) = -\frac{3}{8}a^3 t\sin(t+b) + (\dots)\,, \tag{3.50}$$

where the ellipsis denotes non-secular terms at first order of ϵ. Thus we write the perturbative expansion up to first order as

$$y(t) = a\cos(t+b) + \epsilon\left[-\frac{3}{8}a^3 t\sin(t+b) + (\dots)\right] + \mathcal{O}(\epsilon^2)\,. \tag{3.51}$$

We identify $-\frac{3}{8}a^3 t\sin(t+b)$ as a secular term. We renormalize the integration constants a and b in a way that eliminates the secular term at the arbitrary moment in time τ. Using (3.51), we introduce

$$a = a_R\,, \tag{3.52}$$

$$b = b_R - \frac{\epsilon}{a}\frac{3}{8}a_R^3\tau\,, \tag{3.53}$$

where a_R and b_R denote the renormalized integration constants. The renormalized perturbation expansion, expressed in terms of a_R and b_R, is

$$y(t) = a_R\cos(t+b_R) + \epsilon\left[-\frac{3}{8}a^3(t-\tau)\sin(t+b_R) + (\dots)\right] + \mathcal{O}(\epsilon^2)\,. \tag{3.54}$$

We note that in (3.54), the secular term $-\frac{3}{8}a^3 t\sin(t+b)$ is cancelled in the exact moment where $t=\tau$. We demand that the expansion (3.54) should be independent of τ. We thus apply the condition

$$\left.\frac{\partial y}{\partial\tau}\right|_{t=\tau} = 0\,. \tag{3.55}$$

We find the set of first order differential equations

$$\frac{da_R}{d\tau} = 0\,, \tag{3.56}$$

$$\frac{db_R}{d\tau} = \epsilon\frac{3}{8}a_R^3\,. \tag{3.57}$$

We solve (3.56)-(3.57) and find

$$a_R(\tau) = a_R(0)\,, \tag{3.58}$$

$$b_R(\tau) = \epsilon\frac{3}{8}a_R^2(0)\tau + b_R(0)\,, \tag{3.59}$$

We set $\tau = t$, insert (3.58)-(3.59) back into the renormalized perturbation expansion (3.54). We find

$$y(t) = a_R(0) \cos \left[t + \epsilon \frac{3}{8} a_R^2(0) t + b_R(0) \right] + \mathcal{O}(\epsilon) \,. \tag{3.60}$$

Applying the initial conditions $y(0) = 1$, $\dot{y}(0) = 0$ sets $a_R(0) = 1$, $b_R(0) = 0$. Thus

$$y(t) = \cos \left[\left(1 + \frac{3}{8} \epsilon \right) t \right] + \mathcal{O}(\epsilon) \,, \tag{3.61}$$

which is the same result as in (3.43). We have thus found a perturbation expansion at first order of ϵ that is free of secular terms. Furthermore, we have shown that at first order, the method of multiple scales and the renormalization group method yield the same result.

In Figure 3.2, we compare the resummed result to the naive solution at zero order as well as the exact solution.

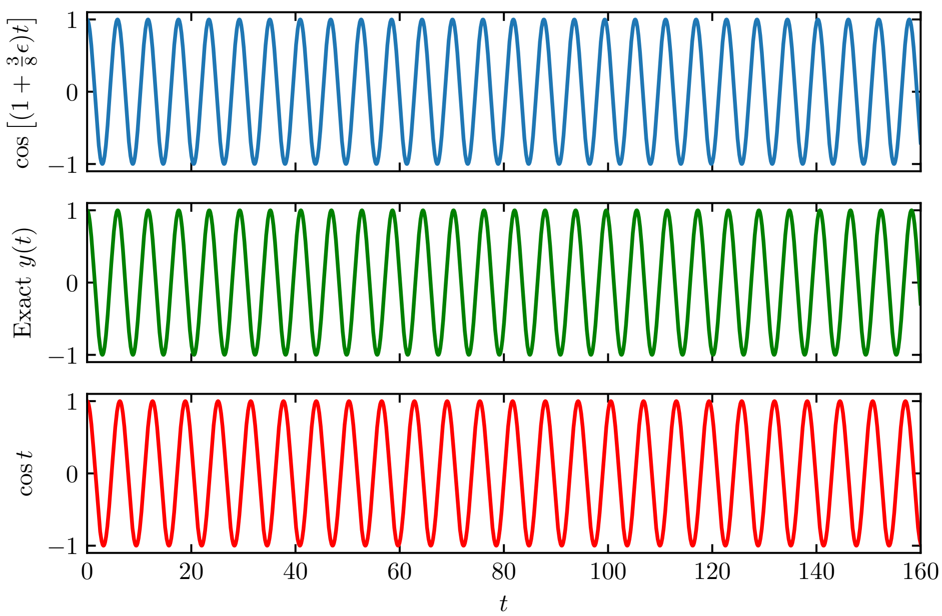

Figure 3.2: Comparison of the resummed result at zero order $\cos\left[\left(1 + \frac{3}{8}\epsilon\right) t\right]$ with the exact solution to the Duffing equation (3.44) $y(t)$, $\epsilon = 0.2$, as well as the naive perturbative result at zero order $\cos t$ (compare to Figure 11.3 in [3]). We note that at $t = 160$, $\cos t$ is approximately two cycles out of phase with $y(t)$.

Chapter 4

The resonant approximation for flat space enclosed in a cavity

In this chapter, we derive the resonant system for flat space enclosed in a cavity. First, we introduce the model, based on [21]. Second, we derive both the wave equation and the Einstein equations. Third, we repeat the weakly non-linear perturbation analysis from [21], in analogy to [5]. In the rest of the chapter, we give a detailed derivation of the resonant approximation for flat space enclosed in a cavity, using 1) multiple-scale analysis, and 2) the renormalization group method. In particular, we show that the method of multiple scales leads to the same system as the renormalization group method. Finally, we discuss symmetries of the system as well as compare it to the resonant system for AdS [4, 11].

4.1 The model

We consider the model from [21], i.e. a spherically symmetric self-gravitating massless scalar field enclosed inside a timelike worldtube $\mathbb{R} \times B^3$ with reflective boundary conditions ('flat space enclosed in a cavity'). Here B^3 refers to a 3-ball of some finite radius $R > 0$. In spherical symmetry, the metric can be written as

$$ds^2 = -Ae^{-2\delta}dt^2 + A^{-1}dr^2 + r^2 d\Omega_2^2\,, \tag{4.1}$$

where $A = A(t, r)$, $\delta = \delta(t, r)$ and $d\Omega_2^2$ is the round metric on S^2,

$$d\Omega_2^2 = d\theta^2 + \sin^2\theta\, d\varphi^2\,. \tag{4.2}$$

Unperturbed Minkowski spacetime corresponds to $A = 1$, $\delta = 0$. The spherically symmetric scalar field, $\phi = \phi(t, r)$, functions as matter source and obeys the wave equation

$$\Box\phi \equiv \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right) = 0\,, \tag{4.3}$$

where $g = \det g_{\mu\nu}$.

We note that it is necessary to add a matter source to the system in order to make the system dynamical. The reason is Birkhoff's theorem, which states that any spherically symmetric solution to the vacuum Einstein equations can locally be written as the Schwarzschild solution. Since the Schwarzschild solution is static, matter is needed to observe any dynamics. For an exposition of Birkhoff's theorem, see e.g. [23].

We impose the 'reflecting' Dirichlet boundary conditions[1] at $r = R = 1$

$$\phi(t, 1) = 0\,, \tag{4.4}$$

where we note that we may choose $R = 1$ without loss of generality due to scale invariance. The Einstein equations[2] are

$$G_{\mu\nu} = 8\pi G \left[\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\partial_\rho\phi\,\partial_\sigma\phi\right]. \tag{4.5}$$

4.2 Equations of motion

We derive the equations of motion for the model introduced in Section 4.1.

We start with the wave equation (4.3). The metric (4.1) in matrix form is

$$(g_{\mu\nu}) = \begin{pmatrix} -Ae^{-2\delta} & 0 & 0 & 0 \\ 0 & A^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}, \tag{4.6}$$

its determinant

$$\det g_{\mu\nu} = -e^{-2\delta}r^4\sin^2\theta\,, \tag{4.7}$$

hence

$$\sqrt{-g} = e^{-\delta}r^2\sin\theta\,. \tag{4.8}$$

The inverse of the metric (4.6) is

$$(g^{\mu\nu}) = \begin{pmatrix} -A^{-1}e^{2\delta} & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix}. \tag{4.9}$$

Inserting the metric components of (4.9) as well as (4.8) into (4.3) yields

$$\begin{aligned} 0 &= \frac{1}{\sqrt{-g}}\partial_t\left(\sqrt{-g}g^{tt}\partial_t\phi\right) + \frac{1}{\sqrt{-g}}\partial_r\left(\sqrt{-g}g^{rr}\partial_r\phi\right) \\ &= \partial_t\left(A^{-1}e^{\delta}\partial_t\phi\right) - \frac{1}{r^2}\partial_r\left(r^2Ae^{-\delta}\partial_r\phi\right). \end{aligned} \tag{4.10}$$

Introducing the auxiliary variables

$$\Phi = \phi'\,, \quad \Pi = A^{-1}e^{\delta}\dot{\phi}\,, \tag{4.11}$$

where the prime and the dot refer to ∂_r and ∂_t, respectively, we rewrite (4.10) as the first order (in time) wave equation

$$\dot{\Phi} = \left(Ae^{-\delta}\Pi\right)'\,, \quad \dot{\Pi} = \frac{1}{r^2}\left(r^2Ae^{-\delta}\Phi\right)'\,. \tag{4.12}$$

[1]The choice of these boundary conditions corresponds to the positioning of a perfectly reflecting mirror, i.e. cavity, at the boundary $r = R$. The purpose of the cavity here is to mimic the timelike boundary of AdS [21].

[2]We set the cosmological constant to zero.

We proceed with the Einstein equations. We compute the Christoffel symbols $\Gamma^{\lambda}_{\mu\nu}$ for the metric (4.1) as

$$\begin{aligned}\Gamma^t_{tt} = \frac{1}{2}\frac{\dot{A}}{A} - \dot{\delta}\,, \quad \Gamma^t_{tr} = \frac{1}{2}\frac{A'}{A} - \delta'\,, \quad \Gamma^t_{rr} = -\frac{1}{2}e^{2\delta}\frac{\dot{A}}{A^3}\,,\\ \Gamma^r_{tt} = \frac{1}{2}e^{-2\delta}A\left(A' - 2A\delta'\right)\,, \quad \Gamma^r_{tr} = -\frac{1}{2}\frac{\dot{A}}{A}\,, \quad \Gamma^r_{rr} = -\frac{1}{2}\frac{A'}{A}\,,\\ \Gamma^r_{\theta\theta} = -rA\,, \quad \Gamma^r_{\varphi\varphi} = -rA\sin^2\theta\,, \quad \Gamma^\theta_{r\theta} = \frac{1}{r}\,,\\ \Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta\,, \quad \Gamma^\varphi_{r\varphi} = \frac{1}{r}\,, \quad \Gamma^\varphi_{\theta\varphi} = \cot\theta\,,\end{aligned} \tag{4.13}$$

where all other symbols are either zero or related by the symmetry $\Gamma^\sigma_{\mu\nu} = \Gamma^\sigma_{\nu\mu}$.

We compute the components of the Einstein tensor[3] $G_{\mu\nu}$ as

$$\begin{aligned}&G_{tt} = -\frac{1}{r^2}e^{-2\delta}A\left(-1 + A + rA'\right)\,,\\ &G_{tr} = -\frac{1}{r}\frac{\dot{A}}{A}\,,\\ &G_{rr} = \frac{1}{r^2A}\left(-1 + rA' + A\left(1 - 2r\delta'\right)\right)\,,\\ &G_{\theta\theta} = \frac{1}{2}r\left(A'\left(2 - 3r\delta'\right) + rA'' + 2A\left(\delta'\left(-1 + r\delta'\right) - r\delta''\right) + \frac{e^{2\delta}r}{A^3}\left(-2\dot{A}^2 + A\dot{A}\dot{\delta} + A\ddot{A}\right)\right)\,,\\ &G_{\varphi\varphi} = \sin^2\theta\, G_{\theta\theta}\,.\end{aligned} \tag{4.14}$$

The components of the energy momentum tensor for a massless scalar field,

$$T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\partial_\rho\phi\,\partial_\sigma\phi\,, \tag{4.15}$$

in terms of the auxiliary variables (4.11), are given by

$$\begin{aligned}T_{tt} = \frac{1}{2}A^2e^{-2\delta}\left(\Phi^2 + \Pi^2\right)\,, \quad T_{tr} = e^{-\delta}A\Phi\Pi\,, \quad T_{rr} = \frac{1}{2}\left(\Phi^2 + \Pi^2\right)\,,\\ T_{\theta\theta} = -\frac{1}{2}r^2A\left(\Phi^2 - \Pi^2\right)\,, \quad T_{\varphi\varphi} = \sin^2\theta\, T_{\theta\theta}\,.\end{aligned} \tag{4.16}$$

Choosing units where $4\pi G = 1 \Longleftrightarrow 8\pi G = 2$, we get the Einstein equations (4.5)

$$tt \text{ component}: \quad A' = \frac{1-A}{r} - Ar\left(\Phi^2 + \Pi^2\right)\,, \tag{4.17}$$

$$tr \text{ component}: \quad \dot{A} = -2re^{-\delta}A^2\Phi\Pi\,, \tag{4.18}$$

$$rr \text{ component}: \quad \delta' = -r\left(\Phi^2 + \Pi^2\right)\,, \tag{4.19}$$

where we note that the $\theta\theta$ and $\varphi\varphi$ components are redundant because they are linear combinations of the other components. Introducing the mass function

$$m := \frac{1}{2}r(1 - A)\,, \tag{4.20}$$

we use (4.17) to write the total energy [21] of the system as

$$M = \frac{1}{2}\int_0^1 dr r^2 A\left(\Phi^2 + \Pi^2\right)\,. \tag{4.21}$$

[3]The components of $G_{\mu\nu}$ were computed using the *Mathematica* [17] package *GRQUICK* [28].

4.3 Weakly non-linear perturbation theory

We repeat the weakly non-linear perturbation analysis of [21]. We seek a perturbative solution to the system of equations

$$\dot{\Phi} = \left(Ae^{-\delta}\Pi\right)' , \quad \dot{\Pi} = \frac{1}{r^2}\left(r^2 Ae^{-\delta}\Phi\right)' , \tag{4.22}$$

$$A' = \frac{1-A}{r} - Ar\left(\Phi^2 + \Pi^2\right) , \tag{4.23}$$

$$\dot{A} = -2re^{-\delta}A^2\Phi\Pi , \tag{4.24}$$

$$\delta' = -r\left(\Phi^2 + \Pi^2\right) \tag{4.25}$$

with initial data $\phi(0,r) = \epsilon f(r)$, $\dot{\phi}(0,r) = \epsilon g(r)$, $\epsilon \ll 1$, where $\phi = \phi(t,r)$, $A = A(t,r)$, $\delta = \delta(t,r)$, $\Phi = \phi'$, $\Pi = A^{-1}e^{\delta}\dot{\phi}$, and primes and dots refer to ∂_r and ∂_t, respectively. We impose the Dirichlet boundary condition $\phi(t,1) = 0$. We construct a perturbative solution to the equations by using the ansatz

$$\phi(t,r) = \sum_{k=0}^{\infty} \epsilon^{2k+1}\phi_{2k+1}(t,r) , \quad A(t,r) = 1 - \sum_{k=1}^{\infty} \epsilon^{2k} A_{2k}(t,r) , \quad \delta(t,r) = \sum_{k=1}^{\infty} \epsilon^{2k}\delta_{2k}(t,r) . \tag{4.26}$$

At first order of ϵ, we find

$$\ddot{\phi}_1 + L\phi_1 = 0 , \quad L = -\frac{1}{r^2}\partial_r\left(r^2\partial_r\right) , \tag{4.27}$$

with initial data $\phi_1(0,r) = f(r)$, $\dot{\phi}_1(0,r) = g(r)$. We solve the eigenvalue problem

$$Le_j(r) = \omega_j^2 e_j(r) \tag{4.28}$$

for the linear operator L. Imposing the Dirichlet boundary conditions $\phi(t,1) = 0$, we get the eigenfunctions and eigenvalues

$$e_j(r) = \sqrt{2}\,\frac{\sin\omega_j r}{r}, \quad \omega_j^2 = (j+1)^2\pi^2 , \quad j \in \mathbb{N}_0 , \tag{4.29}$$

where $\mathbb{N}_0$ denotes the natural numbers including zero. We note that the frequencies ω_j are equidistant, i.e. the spectrum is ‘fully resonant’. The eigenmodes e_j form an orthonormal basis on the space $L^2([0,1], r^2dr)$ with respect to the inner product

$$\langle e_i, e_j\rangle := \int_0^1 drr^2 e_i(r)e_j(r) , \tag{4.30}$$

i.e. $\langle e_i, e_j\rangle = \delta_{ij}$, where δ_{ij} is the Kronecker delta. For more information on the eigenvalue problem for L, see Appendix A. For information on the cubic radial wave equation, a system that, too, features this eigenvalue problem at the first order of its perturbation theory, see Appendix B. We expand ϕ_1 in the orthonormal basis of eigenfunctions e_j

$$\phi_1(t,r) = \sum_{j=0}^{\infty} c_j(t)e_j(r) , \tag{4.31}$$

and insert into (4.27). We project onto the eigenbasis $\{e_n\}$ and get

$$\ddot{c}_n(t) + \omega_n^2 c_n(t) = 0\,. \tag{4.32}$$

We write down the solution to (4.32) as

$$c_n(t) = \alpha_n e^{i\omega_n t} + \bar{\alpha}_n e^{-i\omega_n t}\,, \tag{4.33}$$

where the α_n are determined by the initial conditions $f(r)$ and $g(r)$. Thus

$$\phi_1(t,r) = \sum_{n=0}^{\infty} \left(\alpha_n e^{i\omega_n t} + \bar{\alpha}_n e^{-i\omega_n t}\right) e_n(r)\,. \tag{4.34}$$

At second order of ϵ, we find

$$A_2' = -\frac{A_2}{r} + r\left((\phi_1')^2 + (\dot{\phi}_1)^2\right)\,, \tag{4.35}$$

$$\delta_2' = -r\left((\phi_1')^2 + (\dot{\phi}_1)^2\right)\,, \tag{4.36}$$

$$\dot{A}_2 = -2r\phi_1'\dot{\phi}_1\,. \tag{4.37}$$

We integrate (4.36) and get

$$\delta_2(t,r) = -\int_0^r dss\left(\phi_1'(t,s)^2 + \dot{\phi}_1(t,s)^2\right)\,. \tag{4.38}$$

We note that in (4.38), we have used the interior time gauge fixing condition $\delta(t,0) = 0$. We give details on the derivation in the boundary time gauge $\delta(t,1) = 0$ in Section 4.4.2. We rewrite (4.35) as

$$A_2' = -\frac{A_2}{r} - \delta_2'\,, \tag{4.39}$$

which is equivalent to

$$(rA_2)' \equiv rA_2' + A_2 = -r\delta_2'\,, \tag{4.40}$$

hence

$$rA_2 = -\int_0^r dss\delta_2'(t,s)\,. \tag{4.41}$$

Thus

$$A_2(t,r) = \frac{1}{r}\int_0^r dss^2\left(\phi_1'(t,s)^2 + \dot{\phi}_1(t,s)^2\right)\,. \tag{4.42}$$

We rewrite (4.39) to get the formula

$$A_2' + \delta_2' = -\frac{A_2}{r} = -\frac{1}{r^2}\int_0^r dss^2\left(\phi_1'(t,s)^2 + \dot{\phi}_1(t,s)^2\right)\,. \tag{4.43}$$

At third order of ϵ, we find

$$\ddot{\phi}_3 + L\phi_3 = S \equiv -(A_2' + \delta_2')\phi_1' - (\dot{A}_2 + \dot{\delta}_2)\dot{\phi}_1 - 2(A_2 + \delta_2)\ddot{\phi}_1\,. \tag{4.44}$$

We expand

$$\phi_3(t,r) = \sum_{n=0}^{\infty} c_n^{(3)}(t) e_n(r) \tag{4.45}$$

and project (4.44) onto the eigenbasis $\{e_l\}$ and get the system of equations

$$\ddot{c}_l^{(3)} + \omega_l^2 c_l^{(3)} = S_l \equiv \langle S, e_l \rangle \,, \tag{4.46}$$

where $\langle f, g \rangle = \int_0^1 drr^2 f(r)g(r)$. We have

$$\begin{aligned}
\langle (A_2' + \delta_2') \, \phi_1', e_l \rangle &= -\int_0^1 drr^2 e_l(r)\phi_1'(t,r)\frac{1}{r^2}\int_0^r dss^2 \left(\phi_1'(t,s)^2 + \dot{\phi}_1(t,s)^2\right) \\
&= -\sum_{i,j,k=0}^{\infty} c_k(t)\int_0^1 dre_l(r)e_k'(r) \\
&\quad \int_0^r dss^2 \left\{c_i(t)c_j(t)e_i'(s)e_j'(s) + \dot{c}_i(t)\dot{c}_j(t)e_i(s)e_j(s)\right\} \,,
\end{aligned} \tag{4.47}$$

$$\begin{aligned}
\langle \dot{A}_2\dot{\phi}_1, e_l \rangle &= \int_0^1 drr^2 e_l(r)\dot{\phi}_1(t,r)\frac{1}{r}\int_0^r dss^2 \left(\frac{\partial}{\partial t}\phi_1'(t,s)^2 + \frac{\partial}{\partial t}\dot{\phi}_1(t,s)^2\right) \\
&= \sum_{i,j,k=0}^{\infty} \dot{c}_k(t)\int_0^1 drre_l(r)e_k(r) \\
&\quad \int_0^r dss^2 \left\{\frac{\partial}{\partial t}\left(c_i(t)c_j(t)\right)e_i'(s)e_j'(s) + \frac{\partial}{\partial t}\left(\dot{c}_i(t)\dot{c}_j(t)\right)e_i(s)e_j(s)\right\} \,,
\end{aligned} \tag{4.48}$$

$$\begin{aligned}
\langle \dot{\delta}_2\dot{\phi}_1, e_l \rangle &= -\int_0^1 drr^2 e_l(r)\dot{\phi}_1(t,r)\int_0^r dss \left(\frac{\partial}{\partial t}\phi_1'(t,s)^2 + \frac{\partial}{\partial t}\dot{\phi}_1(t,s)^2\right) \\
&= -\sum_{i,j,k=0}^{\infty} \dot{c}_k(t)\int_0^1 drr^2 e_l(r)e_k(r) \\
&\quad \int_0^r dss \left\{\frac{\partial}{\partial t}\left(c_i(t)c_j(t)\right)e_i'(s)e_j'(s) + \frac{\partial}{\partial t}\left(\dot{c}_i(t)\dot{c}_j(t)\right)e_i(s)e_j(s)\right\} \,,
\end{aligned} \tag{4.49}$$

$$\begin{aligned}
\langle A_2\ddot{\phi}_1, e_l \rangle &= \int_0^1 drr^2 e_l(r)\ddot{\phi}_1(t,r)\frac{1}{r}\int_0^r dss^2 \left(\phi_1'(t,s)^2 + \dot{\phi}_1(t,s)^2\right) \\
&= \sum_{i,j,k=0}^{\infty} \ddot{c}_k(t)\int_0^1 drre_l(r)e_k(r) \\
&\quad \int_0^r dss^2 \left\{c_i(t)c_j(t)e_i'(s)e_j'(s) + \dot{c}_i(t)\dot{c}_j(t)e_i(s)e_j(s)\right\} \,,
\end{aligned} \tag{4.50}$$

$$\begin{aligned}
\langle \delta_2\ddot{\phi}_1, e_l \rangle &= -\int_0^1 drr^2 e_l(r)\ddot{\phi}_1(t,r)\int_0^r dss \left(\phi_1'(t,s)^2 + \dot{\phi}_1(t,s)^2\right) \\
&= -\sum_{i,j,k=0}^{\infty} \ddot{c}_k(t)\int_0^1 drr^2 e_l(r)e_k(r) \\
&\quad \int_0^r dss \left\{c_i(t)c_j(t)e_i'(s)e_j'(s) + \dot{c}_i(t)\dot{c}_j(t)e_i(s)e_j(s)\right\} \,.
\end{aligned} \tag{4.51}$$

We define different types of integrals appearing in (4.47)-(4.51) as

$$\int_0^1 dre_k' e_l \int_0^r dss^2 e_i' e_j' = M_{ijkl}^* \,, \tag{4.52}$$

$$\int_0^1 dre_k' e_l \int_0^r dss^2 e_i e_j = M_{ijkl} \,, \tag{4.53}$$

$$\int_0^1 drre_k e_l \int_0^r dss^2 e_i' e_j' = K_{ijkl}^* \,, \tag{4.54}$$

$$\int_0^1 drre_k e_l \int_0^r dss^2 e_i e_j = K_{ijkl} \,, \tag{4.55}$$

$$\int_0^1 drr^2 e_k e_l \int_0^r dsse_i' e_j' = L_{ijkl}^* \,, \tag{4.56}$$

$$\int_0^1 drr^2 e_k e_l \int_0^r dsse_i e_j = L_{ijkl} \,. \tag{4.57}$$

Using (4.47)-(4.57), we have

$$\begin{aligned} S_l = -\sum_{ijk} \Big[& -c_i c_j c_k M_{ijkl}^* - \dot{c}_i \dot{c}_i c_k M_{ijkl} \\ & + \frac{\partial}{\partial t} (c_i c_j) \, \dot{c}_k \left(K_{ijkl}^* - L_{ijkl}^* \right) + \frac{\partial}{\partial t} (\dot{c}_i \dot{c}_j) \, \dot{c}_k \left(K_{ijkl} - L_{ijkl} \right) \\ & + 2 c_i c_j \ddot{c}_k \left(K_{ijkl}^* - L_{ijkl}^* \right) + 2 \dot{c}_i \dot{c}_j \ddot{c}_k \left(K_{ijkl} - L_{ijkl} \right) \Big] \,, \end{aligned} \tag{4.58}$$

where we have introduced the notation

$$\sum_{ijk} \equiv \sum_{i,j,k=0}^{\infty} \,. \tag{4.59}$$

Using (4.32) as well as the product rule, we rewrite (4.58) as

$$\begin{aligned} S_l = \sum_{ijk} \Big[& c_i c_j c_k \left\{ M_{ijkl}^* + 2\omega_k^2 W_{ijkl}^* \right\} + \dot{c}_i \dot{c}_i c_k \left\{ M_{ijkl} + 2\omega_k^2 W_{ijkl} \right\} \\ & - \dot{c}_i c_j \dot{c}_k \left\{ W_{ijkl}^* - \omega_j^2 W_{ijkl} \right\} - c_i \dot{c}_j \dot{c}_k \left\{ W_{ijkl}^* - \omega_i^2 W_{ijkl} \right\} \Big] \,, \end{aligned} \tag{4.60}$$

where we have introduced

$$W_{ijkl}^{(*)} = K_{ijkl}^{(*)} - L_{ijkl}^{(*)} \,. \tag{4.61}$$

Inserting (4.33) and its derivatives into (4.60), and collecting terms, we find

$$\begin{aligned} S_l = \sum_{ijk} \Big[& \alpha_i \alpha_j \alpha_k e^{i(\omega_i + \omega_j + \omega_k)t} R_{ijkl} \\ & + \bar{\alpha}_i \alpha_j \alpha_k e^{i(-\omega_i + \omega_j + \omega_k)t} O_{ijkl} \\ & + \alpha_i \bar{\alpha}_j \alpha_k e^{i(\omega_i - \omega_j + \omega_k)t} O_{jikl} \\ & + \alpha_i \alpha_j \bar{\alpha}_k e^{i(\omega_i + \omega_j - \omega_k)t} P_{ijkl} \\ & + \alpha_i \bar{\alpha}_j \bar{\alpha}_k e^{i(\omega_i - \omega_j - \omega_k)t} O_{ijkl} \\ & + \bar{\alpha}_i \alpha_j \bar{\alpha}_k e^{i(-\omega_i + \omega_j - \omega_k)t} O_{jikl} \\ & + \bar{\alpha}_i \bar{\alpha}_j \alpha_k e^{i(-\omega_i - \omega_j + \omega_k)t} P_{ijkl} \\ & + \bar{\alpha}_i \bar{\alpha}_j \bar{\alpha}_k e^{i(-\omega_i - \omega_j - \omega_k)t} R_{ijkl} \Big] \,, \end{aligned} \tag{4.62}$$

where

$$R_{ijkl} = X^*_{ijkl} - \omega_i\omega_j X_{ijkl} + \omega_i\omega_k \left(W^*_{ijkl} - \omega_j^2 W_{ijkl}\right) + \omega_j\omega_k \left(W^*_{ijkl} - \omega_i^2 W_{ijkl}\right) , \tag{4.63}$$

$$O_{ijkl} = X^*_{ijkl} + \omega_i\omega_j X_{ijkl} - \omega_i\omega_k \left(W^*_{ijkl} - \omega_j^2 W_{ijkl}\right) + \omega_j\omega_k \left(W^*_{ijkl} - \omega_i^2 W_{ijkl}\right) , \tag{4.64}$$

$$P_{ijkl} = X^*_{ijkl} - \omega_i\omega_j X_{ijkl} - \omega_i\omega_k \left(W^*_{ijkl} - \omega_j^2 W_{ijkl}\right) - \omega_j\omega_k \left(W^*_{ijkl} - \omega_i^2 W_{ijkl}\right) , \tag{4.65}$$

and where we have defined

$$X^{(*)}_{ijkl} = M^{(*)}_{ijkl} + 2\omega_k^2 W^{(*)}_{ijkl} . \tag{4.66}$$

Renaming the dummy indices i and j in the third and sixth term, and i and k in the fourth and seventh term, we rewrite (4.62) as

$$\begin{aligned} S_l = \sum_{ijk} \Big[& \alpha_i\alpha_j\alpha_k e^{i(\omega_i+\omega_j+\omega_k)t} R_{ijkl} \\ & + \bar{\alpha}_i\alpha_j\alpha_k e^{i(-\omega_i+\omega_j+\omega_k)t} \left(2O_{ijkl} + P_{kjil}\right) \\ & + \alpha_i\bar{\alpha}_j\bar{\alpha}_k e^{i(\omega_i-\omega_j-\omega_k)t} \left(2O_{ijkl} + P_{kjil}\right) \\ & + \bar{\alpha}_i\bar{\alpha}_j\bar{\alpha}_k e^{i(-\omega_i-\omega_j-\omega_k)t} R_{ijkl} \Big] \end{aligned} \tag{4.67}$$

$$\begin{aligned} = \sum_{ijk} \Big[& \alpha_i\alpha_j\alpha_k e^{i(\omega_i+\omega_j+\omega_k)t} R_{ijkl} \\ & + \bar{\alpha}_i\alpha_j\alpha_k e^{i(-\omega_i+\omega_j+\omega_k)t} S_{ijkl} \\ & + \alpha_i\bar{\alpha}_j\bar{\alpha}_k e^{i(\omega_i-\omega_j-\omega_k)t} S_{ijkl} \\ & + \bar{\alpha}_i\bar{\alpha}_j\bar{\alpha}_k e^{i(-\omega_i-\omega_j-\omega_k)t} R_{ijkl} \Big] , \end{aligned} \tag{4.68}$$

where we have introduced

$$S_{ijkl} = 2O_{ijkl} + P_{kjil} . \tag{4.69}$$

We find that in accordance with the analysis in Section 3.1, the solution to (4.46) will have a secular term whenever there is a resonant term, i.e. a term for which the resonance condition $\pm\omega_l = \omega_i \pm \omega_j \pm \omega_k$ is satisfied, in the source (4.68). Hence, if the right-hand side of (4.46) is $\sim \mathcal{A}e^{i\omega_l t} + (\dots) + c.c.$, where the ellipsis denotes non-resonant terms and $c.c.$ denotes the complex conjugate, then (4.46) will have a solution of the form

$$c_l^{(3)}(t) = -\frac{i\mathcal{A}}{2\omega_l} t e^{i\omega_l t} + (\dots) + c.c. , \tag{4.70}$$

thus we conclude that due to the presence of the term (4.70), the perturbation theory is invalidated at $t \sim \epsilon^{-2}$.

4.4 The resonant system

We derive the resonant system for flat space enclosed in a cavity. Our aim is to find a resummed perturbation theory that is free from secular terms of the type (4.70). While there are several ways to do this, we focus here on two approaches: 1) the method of multiple scales [3], outlined in Section 3.3.1, and 2) the renormalization group method [8, 11], outlined in Section 3.3.2. Both methods involve introducing a slow time dependence in the integration constants of the solution at linear order in a way that eliminates secular terms from the perturbative expansion. We show that both approaches lead to the same result, namely a system of (infinitely many) coupled first order differential equations that serves as a condition for the perturbative expansion being free of secular terms. We call this system the resonant system.

In addition to the pictures based on multiple scales, the resonant system can heuristically be understood in the context of time averaging [4]. For an oscillating system involving resonant terms in the source as in (4.69), we note that introducing the slow time scale $\tau = \epsilon^2 t$ means that non-resonant terms $\sim e^{i\Omega\tau/\epsilon^2}$, $\Omega \neq 0$, will be highly oscillatory. In the context of slowly drifting integration constants, the contributions of such oscillatory terms can be neglected. By discarding them and keeping only the resonant terms, one is left with an independent, *time averaged* system that approximates the dynamics of the original system on a time-scale $1/\epsilon^2$ [4]. For a detailed exposition of the time averaging method as well as theorems regarding its accuracy, see [24]. Furthermore, it can be proven that the multiscale methods mentioned above and the time averaging method are equivalent at lowest order, which means that the standard accuracy theorems for the averaging method also apply for the multiscale methods [12].

In the remainder of the chapter, we focus on the derivation of the resonant system for flat space enclosed in a cavity using the two methods outlined in Section 3.3.

4.4.1 Derivation via multiple-scale analysis

We use multiple-scale analysis, outlined in Section 3.3.1, to derive the resonant system for flat space enclosed in a cavity. We repeat our perturbation analysis while introducing a new slow time scale $\tau = \epsilon^2 t$ so that

$$\phi = \phi(t, \tau, r)\,, \quad A = A(t, \tau, r)\,, \quad \delta = \delta(t, \tau, r)\,. \tag{4.71}$$

We use the same expansions as in (4.26) and note that in particular,

$$\partial_t^2 \phi(t, r) \to \partial_t^2 \phi(t, \tau, r) + 2\epsilon^2 \partial_t \partial_\tau \phi(t, \tau, r) + \epsilon^4 \partial_\tau^2 \phi(t, \tau, r)\,. \tag{4.72}$$

Using the notation where dots refer to partial derivatives with respect to t, and denoting partial derivatives with respect to τ explicitly, we get the same equations as above for the first and second order of ϵ, in analogy to (4.27) and (4.35)-(4.37), respectively. We write down the solution at first order as

$$\phi_1(t, \tau, r) = \sum_{n=0}^{\infty} c_n(t, \tau) e_n(r) \equiv \sum_{n=0}^{\infty} \left(\alpha_n(\tau) e^{i\omega_n t} + \bar{\alpha}_n(\tau) e^{-i\omega_n t}\right) e_n(r)\,, \tag{4.73}$$

where the coefficients α_n and $\bar{\alpha}_n$ and hence c_n now depend on the new slow time scale τ. The solutions to the equations at second order are

$$\delta_2(t, \tau, r) = -\int_0^r dss \left(\phi_1'(t, \tau, s)^2 + \dot{\phi}_1(t, \tau, s)^2\right)\,, \tag{4.74}$$

$$A_2(t, \tau, r) = \frac{1}{r} \int_0^r dss^2 \left(\phi_1'(t, \tau, s)^2 + \dot{\phi}_1(t, \tau, s)^2\right)\,, \tag{4.75}$$

and we have

$$A_2' + \delta_2' = -\frac{1}{r^2} \int_0^r dss^2 \left(\phi_1'(t, \tau, s)^2 + \dot{\phi}_1(t, \tau, s)^2\right)\,, \tag{4.76}$$

in complete analogy to (4.38), (4.42) and (4.43).

At third order, we find

$$\ddot{\phi}_3 + L\phi_3 = -2\partial_\tau \dot{\phi}_1 + S\,, \tag{4.77}$$

where

$$S = -(A_2' + \delta_2')\phi_1' - (\dot{A}_2 + \dot{\delta}_2)\dot{\phi}_1 - 2(A_2 + \delta_2)\ddot{\phi}_1\,, \tag{4.78}$$

in analogy to (4.44). We note the presence of a new term $-2\partial_\tau \dot{\phi}_1$ in (4.77). In analogy to (4.45) and (4.46), we expand

$$\phi_3(t, r) = \sum_{n=0}^{\infty} c_n^{(3)}(t, \tau) e_n(r)\,, \tag{4.79}$$

and project (4.77) onto the eigenbasis $\{e_l\}$ and get the system of equations

$$\ddot{c}_l^{(3)} + \omega_l^2 c_l^{(3)} = \langle -2\partial_\tau \dot{\phi}_1, e_l \rangle + S_l\,. \tag{4.80}$$

The calculation for the projection $S_l \equiv \langle S, e_l \rangle$ can be repeated in full analogy to (4.47)-(4.68) and yields

$$\begin{aligned} S_l = \sum_{ijk} \big[&\alpha_i \alpha_j \alpha_k e^{i(\omega_i + \omega_j + \omega_k)t} R_{ijkl} \\ &+ \bar{\alpha}_i \alpha_j \alpha_k e^{i(-\omega_i + \omega_j + \omega_k)t} S_{ijkl} \\ &+ \alpha_i \bar{\alpha}_j \bar{\alpha}_k e^{i(\omega_i - \omega_j - \omega_k)t} S_{ijkl} \\ &+ \bar{\alpha}_i \bar{\alpha}_j \bar{\alpha}_k e^{i(-\omega_i - \omega_j - \omega_k)t} R_{ijkl} \big]\,, \end{aligned} \tag{4.81}$$

where R_{ijkl} and S_{ijkl} are as above and where the coefficients α_n and their complex conjugates are now functions of τ.

We derive a condition that removes resonant terms $\sim e^{i\omega_l t} + c.c.$ from the right-hand side of (4.80). To this end, we project the right-hand side of (4.80) onto $e^{i\omega_l t}$, and set this projection[4] to zero, which is equivalent to

$$\left(\langle -2\partial_\tau \dot{\phi}_1, e_l \rangle + S_l, e^{i\omega_l t} \right) = 0\,, \tag{4.82}$$

where we have defined

$$(f, g) \coloneqq \frac{1}{2} \int_0^2 dt f(t) \bar{g}(t)\,. \tag{4.83}$$

We note that

$$\left(e^{i\omega_m t}, e^{i\omega_n t} \right) = \delta_{mn}\,, \quad \omega_m = (m+1)\pi, \quad m, n \in \mathbb{N}_0\,, \tag{4.84}$$

and calculate

$$\begin{aligned} \left(\langle -2\partial_\tau \dot{\phi}_1, e_l \rangle, e^{i\omega_l t} \right) &= -2i\omega_l \left[\frac{d\alpha_l}{d\tau} \left(e^{i\omega_l t}, e^{i\omega_l t} \right) - \frac{d\bar{\alpha}_l}{d\tau} \left(e^{-i\omega_l t}, e^{i\omega_l t} \right) \right] \\ &= -2i\omega_l \frac{d\alpha_l}{d\tau}\,. \end{aligned} \tag{4.85}$$

[4]We note that the projection onto the complex conjugate of $e^{i\omega_l t}$ is redundant.

For the projection of S_l onto $e^{i\omega_l t}$ we get

$$\begin{aligned}
\left(S_l, e^{i\omega_l t}\right) = \sum_{ijk} \Big[&\alpha_i\alpha_j\alpha_k \left(e^{i(\omega_i+\omega_j+\omega_k)t}, e^{i\omega_l t}\right) R_{ijkl} \\
&+\bar{\alpha}_i\alpha_j\alpha_k \left(e^{i(-\omega_i+\omega_j+\omega_k)t}, e^{i\omega_l t}\right) S_{ijkl} \\
&+\alpha_i\bar{\alpha}_j\bar{\alpha}_k \left(e^{i(\omega_i-\omega_j-\omega_k)t}, e^{i\omega_l t}\right) S_{ijkl} \\
&+\bar{\alpha}_i\bar{\alpha}_j\bar{\alpha}_k \left(e^{i(-\omega_i-\omega_j-\omega_k)t}, e^{i\omega_l t}\right) R_{ijkl}\Big] \qquad (4.86) \\
= \sum_{ijk} \Big[&\alpha_i\alpha_j\alpha_k R_{ijkl}\delta_{l+1,i+1+j+1+k+1} \\
&+\bar{\alpha}_i\alpha_j\alpha_k S_{ijkl}\delta_{l+1,-(i+1)+j+1+k+1} \\
&+\alpha_i\bar{\alpha}_j\bar{\alpha}_k S_{ijkl}\delta_{l+1,i+1-(j+1)-(k+1)} \\
&+\bar{\alpha}_i\bar{\alpha}_j\bar{\alpha}_k R_{ijkl}\delta_{l+1,-(i+1)-(j+1)-(k+1)}\Big] \,. \qquad (4.87)
\end{aligned}$$

Given that $\delta_{l+1,-(i+1)-(j+1)-(k+1)} = 0$ for $i, j, k, l \in \mathbb{N}_0$, we find that

$$\left(S_l, e^{i\omega_l t}\right) = \sum_{ijk}^{+++} R_{ijkl}\alpha_i\alpha_j\alpha_k + \sum_{ijk}^{-++} S_{ijkl}\bar{\alpha}_i\alpha_j\alpha_k + \sum_{ijk}^{+--} S_{ijkl}\alpha_i\bar{\alpha}_j\bar{\alpha}_k \,, \tag{4.88}$$

where

$$\sum_{ijk}^{+++} = \sum_{\substack{i,j,k=0 \\ l+1=i+1+j+1+k+1}}^{\infty} \,, \qquad \sum_{ijk}^{-++} = \sum_{\substack{i,j,k=0 \\ l+1=-(i+1)+j+1+k+1}}^{\infty} \,, \qquad \sum_{ijk}^{+--} = \sum_{\substack{i,j,k=0 \\ l+1=i+1-(j+1)-(k+1)}}^{\infty} \,. \tag{4.89}$$

Using (4.85) and (4.88), (4.82) transforms into

$$\frac{d\alpha_l}{d\tau} = -\frac{i}{2\omega_l}\left[\sum_{ijk}^{+++} R_{ijkl}\alpha_i\alpha_j\alpha_k + \sum_{ijk}^{-++} S_{ijkl}\bar{\alpha}_i\alpha_j\alpha_k + \sum_{ijk}^{+--} S_{ijkl}\alpha_i\bar{\alpha}_j\bar{\alpha}_k\right] , \tag{4.90}$$

which is the resonant system for flat space enclosed in a cavity.

4.4.2 Boundary time gauge $\delta(t, 1) = 0$

We redo our calculation of the resonant system in the boundary time gauge $\delta(t, 1) = 0$. At second order, we get the equation

$$\delta_2' = -r\left((\phi_1')^2 + (\dot{\phi}_1)^2\right) . \tag{4.91}$$

We write the solution as

$$\delta_2(t, r) = -\int_0^r dss\left(\phi_1'(t, s)^2 + \dot{\phi}_1(t, s)^2\right) + f(t) \,, \tag{4.92}$$

and require that $\delta_2(t, 1) = 0$, in agreement with the gauge condition $\delta(t, 1) = 0$. This yields

$$f(t) = \int_0^1 dss\left(\phi_1'(t, s)^2 + \dot{\phi}_1(t, s)^2\right) . \tag{4.93}$$

Using (4.93), we write the solution of (4.91) as

$$\delta_2(t,r) = -\int_1^r dss\left(\phi_1'(t,s)^2 + \dot{\phi}_1(t,s)^2\right) \tag{4.94}$$

$$= \int_r^1 dss\left(\phi_1'(t,s)^2 + \dot{\phi}_1(t,s)^2\right) . \tag{4.95}$$

Repeating our calculations (4.49), (4.51), we find

$$\begin{aligned}\langle \dot{\delta}_2\dot{\phi}_1, e_l\rangle &= \int_0^1 drr^2 e_l(r)\dot{\phi}_1(t,r)\int_r^1 dss\left(\frac{\partial}{\partial t}\phi_1'(t,s)^2 + \frac{\partial}{\partial t}\dot{\phi}_1(t,s)^2\right)\\ &= \sum_{i,j,k=0}^{\infty}\dot{c}_k(t)\int_0^1 drr^2 e_l(r)e_k(r)\\ &\int_r^1 dss\left\{\frac{\partial}{\partial t}\left(c_i(t)c_j(t)\right)e_i'(s)e_j'(s) + \frac{\partial}{\partial t}\left(\dot{c}_i(t)\dot{c}_j(t)\right)e_i(s)e_j(s)\right\} ,\end{aligned} \tag{4.96}$$

$$\begin{aligned}\langle \delta_2\ddot{\phi}_1, e_l\rangle &= \int_0^1 drr^2 e_l(r)\ddot{\phi}_1(t,r)\int_r^1 dss\left(\phi_1'(t,s)^2 + \dot{\phi}_1(t,s)^2\right)\\ &= \sum_{i,j,k=0}^{\infty}\ddot{c}_k(t)\int_0^1 drr^2 e_l(r)e_k(r)\\ &\int_r^1 dss\left\{c_i(t)c_j(t)e_i'(s)e_j'(s) + \dot{c}_i(t)\dot{c}_j(t)e_i(s)e_j(s)\right\} .\end{aligned} \tag{4.97}$$

The rest of the calculations (4.47)-(4.51) stays the same. We define the integrals

$$-\int_0^1 drr^2 e_k e_l\int_r^1 dsse_i'e_j' = N^*_{ijkl} , \tag{4.98}$$

$$-\int_0^1 drr^2 e_k e_l\int_r^1 dsse_i e_j = N_{ijkl} , \tag{4.99}$$

and after a derivation analogous to the above, we find the resonant system

$$\frac{d\alpha_l}{d\tau} = -\frac{i}{2\omega_l}\left[\sum_{ijk}^{+++}\tilde{R}_{ijkl}\alpha_i\alpha_j\alpha_k + \sum_{ijk}^{-++}\tilde{S}_{ijkl}\bar{\alpha}_i\alpha_j\alpha_k + \sum_{ijk}^{+--}\tilde{S}_{ijkl}\alpha_i\bar{\alpha}_j\bar{\alpha}_k\right] , \tag{4.100}$$

where $\tilde{S}_{ijkl} = 2\tilde{O}_{ijkl} + \tilde{P}_{kjil}$ and $\tilde{R}_{ijkl}$, $\tilde{O}_{ijkl}$, and $\tilde{P}_{ijkl}$ are as in (4.63)-(4.65), with

$$L^{(*)}_{ijkl} \to N^{(*)}_{ijkl} . \tag{4.101}$$

4.4.3 Renormalization flow equations

In order to compare the two approaches, we derive the renormalization flow equations for the weakly non-linear perturbation theory at first non-trivial order derived in Section 4.3. We follow the approach of [11]. We have outlined the renormalization group method as a means to resum secular terms in Section 3.3.2.

To begin, we repeat the weakly non-linear perturbation theory of Section 4.3 formally up to third order. We choose to write down the solution at first order (4.31) using the representation

$$c_j(t) = a_j \cos\theta_j \equiv a_j \cos(\omega_j t + b_j)\,, \tag{4.102}$$

thus

$$\phi_1(t,r) = \sum_{n=0}^{\infty} a_n \cos\theta_n e_n(r)\,, \tag{4.103}$$

where the $e_n(r)$ are as above. In this representation, inserting (4.102) as well as its derivatives and collecting terms, we rewrite the projection S_l in (4.58) as

$$\begin{aligned} S_l = \sum_{ijk} \frac{1}{4}\, a_i a_j a_k \, &\left[\cos(\theta_i+\theta_j+\theta_k) R_{ijkl} + \cos(\theta_i-\theta_j-\theta_k) O_{ijkl}\right. \\ &\left. + \cos(\theta_i-\theta_j+\theta_k) O_{jikl} + \cos(\theta_i+\theta_j-\theta_k) P_{ijkl}\right]\,, \end{aligned} \tag{4.104}$$

where R_{ijkl}, O_{ijkl} and P_{ijkl} are as above and where we have used the relation

$$\cos\theta_1 \cos\theta_2 = \frac{1}{2}\left[\cos(\theta_1-\theta_2) + \cos(\theta_1+\theta_2)\right]\,. \tag{4.105}$$

Using permutations of the indices i and j as well as i and k in the third and fourth term in (4.104), respectively, we write

$$S_l = \sum_{ijk} \frac{1}{4}\, a_i a_j a_k \, \left[\cos(\theta_i+\theta_j+\theta_k) R_{ijkl} + \cos(\theta_i-\theta_j-\theta_k) S_{ijkl}\right]\,, \tag{4.106}$$

where $S_{ijkl} = 2O_{ijkl} + P_{kjil}$.

We proceed in the spirit of Section 3.3.2 and identify the resonant terms in (4.106) as the ones where

$$\theta_i+\theta_j+\theta_k = (\omega_i+\omega_j+\omega_k)\,t + b_i+b_j+b_k = \omega_l t + b_i + b_j + b_k\,, \tag{4.107}$$

$$\theta_i-\theta_j-\theta_k = (\omega_i-\omega_j-\omega_k)\,t + b_i-b_j-b_k = -\left(\omega_l t - b_i + b_j + b_k\right)\,, \tag{4.108}$$

$$\theta_i-\theta_j-\theta_k = (\omega_i-\omega_j-\omega_k)\,t + b_i-b_j-b_k = \omega_l t + b_i - b_j - b_k\,, \tag{4.109}$$

where we note that $\theta_i+\theta_j+\theta_k = -\left(\omega_l t - b_i - b_j - b_k\right)$ cannot be satisfied due to the positivity of the frequencies. We write

$$\begin{aligned} S_l =& \frac{1}{4}\left[\sum_{ijk}^{+++} a_i a_j a_k \cos(\omega_l t + b_i + b_j + b_k) R_{ijkl}\right. \\ &+ \sum_{ijk}^{-++} a_i a_j a_k \cos(\omega_l t - b_i + b_j + b_k) S_{ijkl} \\ &\left.+ \sum_{ijk}^{+--} a_i a_j a_k \cos(\omega_l t + b_i - b_j - b_k) S_{ijkl}\right] + (\ldots)\,, \end{aligned} \tag{4.110}$$

where the ellipsis stands for the non-resonant terms and where we use the same notation for the sums as in (4.89). As we have seen in Section 3.1, resonant terms of the form

(4.110) on the right-hand side of (4.46) will generate secular terms in the solution of (4.46) of the form

$$\begin{aligned} c_l^{(3)}(t) =& \frac{t}{8\omega_l}\left[\sum_{ijk}^{+++} a_i a_j a_k \sin(\omega_l t + b_i + b_j + b_k) R_{ijkl}\right. \\ &+ \sum_{ijk}^{-++} a_i a_j a_k \sin(\omega_l t - b_i + b_j + b_k) S_{ijkl} \\ &\left.+ \sum_{ijk}^{+--} a_i a_j a_k \sin(\omega_l t + b_i - b_j - b_k) S_{ijkl}\right] + (\dots)\,. \end{aligned} \tag{4.111}$$

The naive perturbative expansion for the generalized Fourier coefficients of ϕ expanded in the eigenbasis $\{e_l\}$

$$\langle \phi, e_l \rangle = \epsilon c_l(t) + \epsilon^3 c_l^{(3)}(t) + \mathcal{O}(\epsilon^5)\,, \tag{4.112}$$

in accordance with (4.26) and (4.31), reads

$$\begin{aligned} \epsilon a_l \cos(\omega_l t + b_l) + \epsilon^3 \frac{t}{8\omega_l} & \left[\sum_{ijk}^{+++} a_i a_j a_k \sin(\omega_l t + b_i + b_j + b_k) R_{ijkl}\right. \\ &+ \sum_{ijk}^{-++} a_i a_j a_k \sin(\omega_l t - b_i + b_j + b_k) S_{ijkl} \\ &\left.+ \sum_{ijk}^{+--} a_i a_j a_k \sin(\omega_l t + b_i - b_j - b_k) S_{ijkl}\right] \\ &+ (\dots) + \mathcal{O}(\epsilon^5)\,, \end{aligned} \tag{4.113}$$

where the ellipsis denotes non-secular terms at $\mathcal{O}(\epsilon^3)$. We use the relation

$$\sin(\theta + \varphi) = \sin\theta\cos\varphi + \cos\theta\sin\varphi \tag{4.114}$$

to rewrite the secular terms in (4.113). We rewrite the argument using $\omega_l t + b_i + b_j + b_k = \omega_l t + b_l - b_l + b_i + b_j + b_k$, similarly for the other resonant terms, and get

$$\begin{aligned} \sin(\omega_l t + b_i + b_j + b_k) =& \cos(-b_l + b_i + b_j + b_k)\sin(\omega_l t + b_l) \\ &+ \sin(-b_l + b_i + b_j + b_k)\cos(\omega_l t + b_l)\,, \end{aligned} \tag{4.115}$$

similarly for the other resonant terms. We introduce

$$\begin{aligned} a_l = A_l - \epsilon^2 \frac{\tau}{8\omega_l} & \left[\sum_{ijk}^{+++} A_i A_j A_k \sin(-B_l + B_i + B_j + B_k) R_{ijkl}\right. \\ &+ \sum_{ijk}^{-++} A_i A_j A_k \sin(-B_l - B_i + B_j + B_k) S_{ijkl} \\ &\left.+ \sum_{ijk}^{+--} A_i A_j A_k \sin(-B_l + B_i - B_j - B_k) S_{ijkl}\right]\,, \end{aligned} \tag{4.116}$$

$$
\begin{aligned}
b_l = B_l + \frac{\epsilon^2}{a_l}\frac{\tau}{8\omega_l} \Bigg[& \sum_{ijk}^{+++} A_i A_j A_k \cos(-B_l + B_i + B_j + B_k) R_{ijkl} \\
& + \sum_{ijk}^{-++} A_i A_j A_k \cos(-B_l - B_i + B_j + B_k) S_{ijkl} \\
& + \sum_{ijk}^{+--} A_i A_j A_k \cos(-B_l + B_i - B_j - B_k) S_{ijkl} \Bigg] \, .
\end{aligned} \tag{4.117}
$$

We insert (4.116)-(4.117) into (4.113) and get the renormalized perturbation expansion

$$
\begin{aligned}
\epsilon A_l \cos(\omega_l t + B_l) + \epsilon^3 \frac{1}{8\omega_l} \Bigg[& \sum_{ijk}^{+++} A_i A_j A_k (t-\tau) \cos(-B_l + B_i + B_j + B_k) R_{ijkl} \\
& + \sum_{ijk}^{-++} A_i A_j A_k (t-\tau) \cos(-B_l - B_i + B_j + B_k) S_{ijkl} \\
& + \sum_{ijk}^{+--} A_i A_j A_k \, (t-\tau) \cos\left(-B_l + B_i - B_j - B_k\right) S_{ijkl} \Bigg] \sin(\omega_l t + B_l) \\
+ \epsilon^3 \frac{1}{8\omega_l} \Bigg[& \sum_{ijk}^{+++} A_i A_j A_k (t-\tau) \sin(-B_l + B_i + B_j + B_k) R_{ijkl} \\
& + \sum_{ijk}^{-++} A_i A_j A_k (t-\tau) \sin(-B_l - B_i + B_j + B_k) S_{ijkl} \\
& + \sum_{ijk}^{+--} A_i A_j A_k (t-\tau) \sin\left(-B_l + B_i - B_j - B_k\right) S_{ijkl} \Bigg] \cos(\omega_l t + B_l) \\
& + (\dots) + \mathcal{O}(\epsilon^5) \, .
\end{aligned} \tag{4.118}
$$

We demand the expansion (4.118) be independent of τ. We thus apply the derivative ∂_τ and set it to zero. We evaluate the derivative at the point $\tau = t$. Finally, we use the fact that sine and cosine are linearly independent and find

$$
\begin{aligned}
\frac{dA_l}{d\tau} = \epsilon^2 \frac{1}{8\omega_l} \Bigg[& \sum_{ijk}^{+++} A_i A_j A_k \sin(-B_l + B_i + B_j + B_k) R_{ijkl} \\
& + \sum_{ijk}^{-++} A_i A_j A_k \sin(-B_l - B_i + B_j + B_k) S_{ijkl} \\
& + \sum_{ijk}^{+--} A_i A_j A_k \sin\left(-B_l + B_i - B_j - B_k\right) S_{ijkl} \Bigg] \, ,
\end{aligned} \tag{4.119}
$$

$$
\begin{aligned}
A_l \frac{dB_l}{d\tau} = -\epsilon^2 \frac{1}{8\omega_l} \Bigg[&\sum_{ijk}^{+++} A_i A_j A_k \cos(-B_l + B_i + B_j + B_k) R_{ijkl} \\
+ &\sum_{ijk}^{-++} A_i A_j A_k \cos(-B_l - B_i + B_j + B_k) S_{ijkl} \\
+ &\sum_{ijk}^{+--} A_i A_j A_k \cos\left(-B_l + B_i - B_j - B_k\right) S_{ijkl} \Bigg] ,
\end{aligned} \tag{4.120}
$$

which are the renormalization flow equations for the renormalized amplitude and phase A_l and B_l for flat space enclosed in a cavity at first non-trivial order. Introducing the slow time scale

$$
\tau \to \epsilon^2 \tau \,, \qquad \frac{d}{d\tau} \to \epsilon^2 \frac{d}{d\tau} \,, \tag{4.121}
$$

as well as the transformation

$$
\alpha_l = \frac{1}{2} A_l e^{iB_l} \,, \qquad \frac{d\alpha_l}{d\tau} = \frac{1}{2} \left(\frac{dA_l}{d\tau} + i A_l \frac{dB_l}{d\tau} \right) e^{iB_l} \,, \tag{4.122}
$$

transforms (4.119)-(4.120) into

$$
\frac{d\alpha_l}{d\tau} = -\frac{i}{2\omega_l} \left[\sum_{ijk}^{+++} R_{ijkl} \alpha_i \alpha_j \alpha_k + \sum_{ijk}^{-++} S_{ijkl} \bar{\alpha}_i \alpha_j \alpha_k + \sum_{ijk}^{+--} S_{ijkl} \alpha_i \bar{\alpha}_j \bar{\alpha}_k \right] , \tag{4.123}
$$

which is the same system as in (4.90).

4.4.4 Symmetries

We note that the system (4.90) has the symmetry

$$
\alpha_l \to \alpha_l e^{i(l+1)\theta} \,, \quad \theta \in \mathbb{R} \,. \tag{4.124}
$$

Furthermore, we note that (4.90) is invariant under the rescaling

$$
\alpha_l(\tau) \to \epsilon^{-1} \alpha_l(\tau/\epsilon^2) \tag{4.125}
$$

as well as under the time translation

$$
\alpha_l(\tau) \to \alpha_l(\tau - \tau_0) \,. \tag{4.126}
$$

4.4.5 Comparison with AdS

In contrast to (4.90), the resonant system for AdS [11] lacks the '+++' and '+−−' terms. It can be written in the form [4]

$$
\frac{d\alpha_l}{d\tau} = -\frac{i}{2\omega_l} \sum_{i+j-k=l} C_{ijkl} \alpha_i \alpha_j \bar{\alpha}_k \,, \tag{4.127}
$$

where the C_{ijkl} are the interaction coefficients specific to AdS [11]. The fact that the additional terms are absent is due to selection rules which result from the properties of the AdS mode functions [11, 14]. It follows that in addition to (4.124)-(4.126), the resonant system for AdS also has the symmetry [12]

$$
\alpha_l \to \alpha_l e^{i\theta} \,, \quad \theta \in \mathbb{R} \,. \tag{4.128}
$$

Chapter 5

Numerical and asymptotic analysis

In this chapter, we study the resonant system for flat space enclosed in a cavity, which we derived in Chapter 4, numerically. First, we will discuss how to set up the system for numerical study. We consider the system in a truncated form, prescribe initial data, and use the rescaling symmetry. Second, we discuss the numerical evolution scheme. Third, we give details on the implementation and illustrate the obtained numerical results. Finally, we introduce and apply the analyticity strip method [27] and analyze the asymptotic behavior of the solution to the resonant system.

5.1 Setup

5.1.1 The truncated resonant system

We note that the resonant system in the interior time gauge (4.90) contains on its right-hand side infinite sums. In order to study the system numerically, we note that we need to truncate the sums at some finite number of modes N. The resulting truncated resonant system then approximates the system (4.90) in the limit $N \to \infty$.

Furthermore, we introduce a notation for the coefficients R and S that eliminates one of the indices in accordance with the respective resonance condition. We write

$$R^{+++}_{nij} = R_{i,j,n-i-j-2,n}\,, \quad S^{-++}_{nij} = S_{i,j,n+i-j,n}\,, \quad S^{+--}_{nij} = S_{i,j,i-j-n-2,n}\,, \tag{5.1}$$

where the index k in each of the coefficients has been eliminated in accordance with the resonance condition '$+++$', '$-++$', and '$+--$', respectively. In this notation, we write down the truncated resonant system as

$$\begin{aligned}\frac{d\alpha_n}{d\tau} = -\frac{i}{2\omega_n}\Bigg[&\sum_{\substack{i,j=0\\ n-i-j-2\geq 0}}^{N-1} R^{+++}_{nij}\alpha_i\alpha_j\alpha_{n-i-j-2}\\ &+\sum_{\substack{i,j=0\\ 0\leq n+i-j\leq N-1}}^{N-1} S^{-++}_{nij}\bar{\alpha}_i\alpha_j\alpha_{n+i-j}\\ &+\sum_{\substack{i,j=0\\ i-j-n-2\geq 0}}^{N-1} S^{+--}_{nij}\alpha_i\bar{\alpha}_j\bar{\alpha}_{i-j-n-2}\Bigg]\,, \quad n\in[0,N-1]\,.\end{aligned} \tag{5.2}$$

5.1.2 Initial data

We will study the resonant system using the Gaussian initial data from [21]. Expressed through the auxiliary variables Φ and Π defined in (4.11), we have

$$\Phi(0,r) = 0\,, \quad \Pi(0,r) = \epsilon \exp\left(-64\tan^2\frac{\pi}{2}r\right)\,. \tag{5.3}$$

Using

$$\phi = \epsilon\phi_1 + \mathcal{O}(\epsilon^3)\,, \quad \Phi = \epsilon\phi_1' + \mathcal{O}(\epsilon^3)\,, \quad \Pi = \epsilon\dot{\phi}_1 + \mathcal{O}(\epsilon^3)\,, \tag{5.4}$$

we get

$$\phi_1'(0,r) = 0\,, \tag{5.5}$$
$$\dot{\phi}_1(0,r) = g(r) \equiv \exp\left(-64\tan^2\frac{\pi}{2}r\right)\,. \tag{5.6}$$

We write ϕ_1 in the language of the multiple-scale analysis derivation of Section 4.4 and find

$$\phi_1'(t,\tau,r)\Big|_{\substack{t=0\\ \tau=0}} = \sum_{j=0}^{\infty}\left(\alpha_j(0) + \bar{\alpha}_j(0)\right)e_j'(r) = 2\sum_{j=0}^{\infty}\Re(\alpha_j(0))e_j'(r) = 0\,, \tag{5.7}$$

$$\dot{\phi}_1(t,\tau,r)\Big|_{\substack{t=0\\ \tau=0}} = i\sum_{j=0}^{\infty}\omega_j\left(\alpha_j(0) - \bar{\alpha}_j(0)\right)e_j(r) = -2\sum_{j=0}^{\infty}\omega_j\Im(\alpha_j(0))e_j(r) = g(r)\,, \tag{5.8}$$

where $\Re$ denotes the real part and $\Im$ denotes the imaginary part of α. Projecting (5.7) and (5.8) onto $\{e_n'\}$ and $\{e_n\}$, respectively, we get as initial data for α_n

$$\Re(\alpha_n(0)) = 0\,, \tag{5.9}$$
$$\begin{aligned}\Im(\alpha_n(0)) &= -\frac{1}{2\omega_n}\int_0^1 drr^2\exp\left(-64\tan^2\frac{\pi}{2}r\right)e_n(r)\\ &= -\frac{1}{\omega_n\sqrt{2}}\int_0^1 drr\exp\left(-64\tan^2\frac{\pi}{2}r\right)\sin\omega_j r\,,\end{aligned} \tag{5.10}$$

where we have used $\langle f,g\rangle = \int_0^1 drr^2 f(r)g(r)$ as well as $\langle e_i', e_j'\rangle = \omega_i^2\delta_{ij}$, $\langle e_i, e_j\rangle = \delta_{ij}$.

5.1.3 Rescaling

We will use the fact that the (truncated) resonant system (4.90), (5.2) is invariant under the rescaling

$$\alpha_n(\tau) \to \varepsilon^{-1}\alpha_n(\tau/\varepsilon^2)\,. \tag{5.11}$$

We note that choosing $\varepsilon = 10^{-1}$ and introducing

$$\tilde{\alpha}_n(\tau) = 10\,\alpha_n(100\tau) \tag{5.12}$$

implies

$$\tilde{\alpha}_n(0) = 10\,\alpha_n(0)\,, \tag{5.13}$$

thus

$$\Re(\tilde{\alpha}_n(0)) = 0\,, \tag{5.14}$$
$$\Im(\tilde{\alpha}_n(0)) = -\frac{10}{\omega_n\sqrt{2}}\int_0^1 drr\exp\left(-64\tan^2\frac{\pi}{2}r\right)\sin\omega_j r\,. \tag{5.15}$$

5.2 Numerical evolution scheme

We will use a fourth-order Runge-Kutta method to evolve the system in time. We give a brief introduction into solving ordinary differential equations using a Runge-Kutta method of fourth order based on [26]. We consider a system of N coupled differential equations which can be written as

$$\frac{d\vec{y}(t)}{dt} = \vec{f}(t, \vec{y}) , \quad \vec{y}(t_0) = \vec{y}_0 , \tag{5.16}$$

where $\vec{y} = (y_0, \ldots, y_{N-1})$, $\vec{f} = (f_0, \ldots, f_{N-1})$ and where the functions $f_0, \ldots, f_{N-1}$ are known. We seek an approximate to the solution on the time interval $[t_0, T]$. We define a time-step h and write

$$t_j = t_0 + jh , \quad j = 0, 1, \ldots, \frac{T - t_0}{h} . \tag{5.17}$$

The 'classic' fourth-order Runge-Kutta method [26] is given by

$$\begin{aligned}
\vec{k}_1 &= h\vec{f}(t_j, \vec{y}_j) , \\
\vec{k}_2 &= h\vec{f}(t_j + \frac{1}{2}h, \vec{y}_j + \frac{1}{2}\vec{k}_1) , \\
\vec{k}_3 &= h\vec{f}(t_j + \frac{1}{2}h, \vec{y}_j + \frac{1}{2}\vec{k}_2) , \\
\vec{k}_4 &= h\vec{f}(t_j + h, \vec{y}_j + \vec{k}_3) , \\
\vec{y}_{j+1} &= \vec{y}_j + \frac{1}{6}\vec{k}_1 + \frac{1}{3}\vec{k}_2 + \frac{1}{3}\vec{k}_3 + \frac{1}{6}\vec{k}_4 + \mathcal{O}(h^5) .
\end{aligned} \tag{5.18}$$

where $\vec{y}_j \equiv \vec{y}(t_j)$. The method is called fourth-order since the error term[1] in (5.18) is $\mathcal{O}(h^5)$ [26]. We apply (5.18) to the truncated resonant system (5.2). We write (5.2) in the form (5.16), i.e. the n-th component is

$$\frac{d\alpha_n}{d\tau} = f_n(\alpha_0, \alpha_1, \ldots, \alpha_{N-1}) , \tag{5.19}$$

where $f_n(\alpha_0, \alpha_1, \ldots, \alpha_{N-1})$ is equivalent to the right-hand side of (5.2). We note that the right-hand side of (5.2) does not depend explicitly on τ.

5.3 Implementation and Results

We solved the truncated resonant system for flat space in a cavity (5.2) using the numerical evolution scheme described in Section 5.2. The code was implemented in *Python*[2]. We computed the initial data (5.9)-(5.10) numerically[3] for the first 25 modes, i.e. $n \in [0, 24]$. We used the rescaling (5.12) for the parameter $\varepsilon = 10^{-1}$. We obtained analytic expressions for the interaction coefficients (5.1) consisting of algebraic expressions involving the indices as well as sine integral [30], cosine integral [31] and exponential integral [32] functions in *Mathematica*. As a compromise between accuracy and computational cost, we chose the number of modes $N = 3180$. We used the time-step $h = 10^{-3}$. We evolved the system on the time interval $[0, 57]$.

[1] The order of the error term can be verified by comparing (5.18) to the Taylor expansion of the true solution.

[2] We acknowledge the *Numba* package [19] for *Python*.

[3] The initial data was implemented using the *SciPy* package [29] for *Python*.

5.3.1 Evolution of amplitude and phase

We describe the behavior of the solutions qualitatively. We calculated the quantity $\cos B_n(\tau)$ using the formula

$$\cos B_n(\tau) = \Re\left(\frac{\alpha_n(\tau)}{|\alpha_n(\tau)|}\right), \tag{5.20}$$

which we derived from (4.122). Figures 5.1 and 5.2 show the qualitative behavior of a sample high mode, $n = 1150$. We observe in Figure 5.1 that the solution has an oscillatory character with increasing amplitude. Furthermore, we observe in Figure 5.2 that the frequency of the oscillation begins to grow at later times.

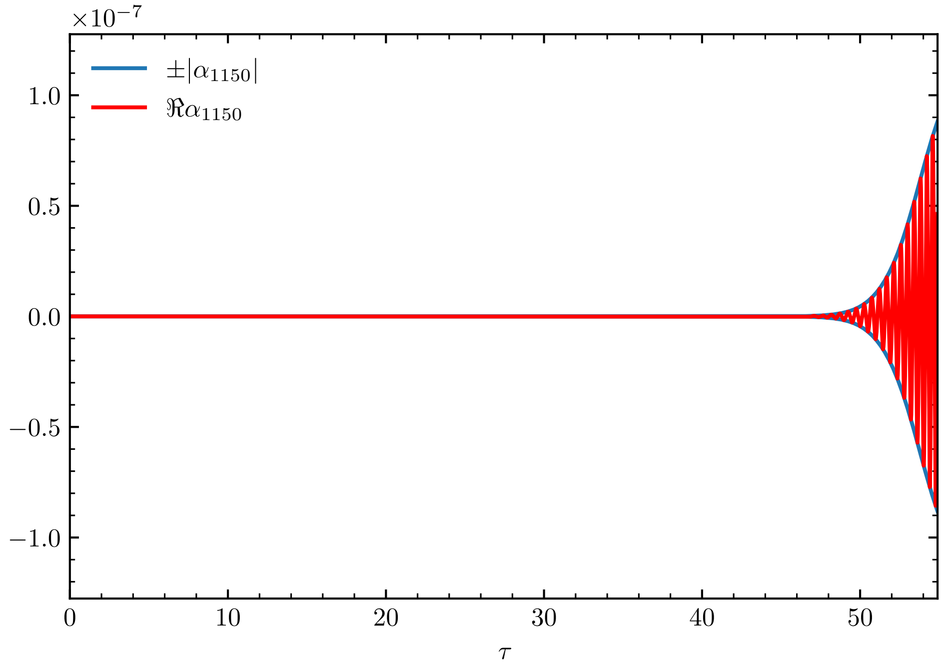

Figure 5.1: Evolution of the real part and amplitude of a sample high mode

5.3.2 Evolution of amplitude spectra

Looking at the evolution of the amplitude spectrum, we find that higher modes are quickly excited. We show the evolution of the amplitude spectrum for several times τ in Figure 5.3. At late times, even the highest modes are excited. We are interested in the question whether the amplitude spectrum evolves to a power law. We will investigate this question in the Section 5.4.

5.3.3 Conserved quantity

We checked whether the quantity

$$E = \sum_{n=0}^{\infty} (n+1)^2 |\alpha_n|^2 , \tag{5.21}$$

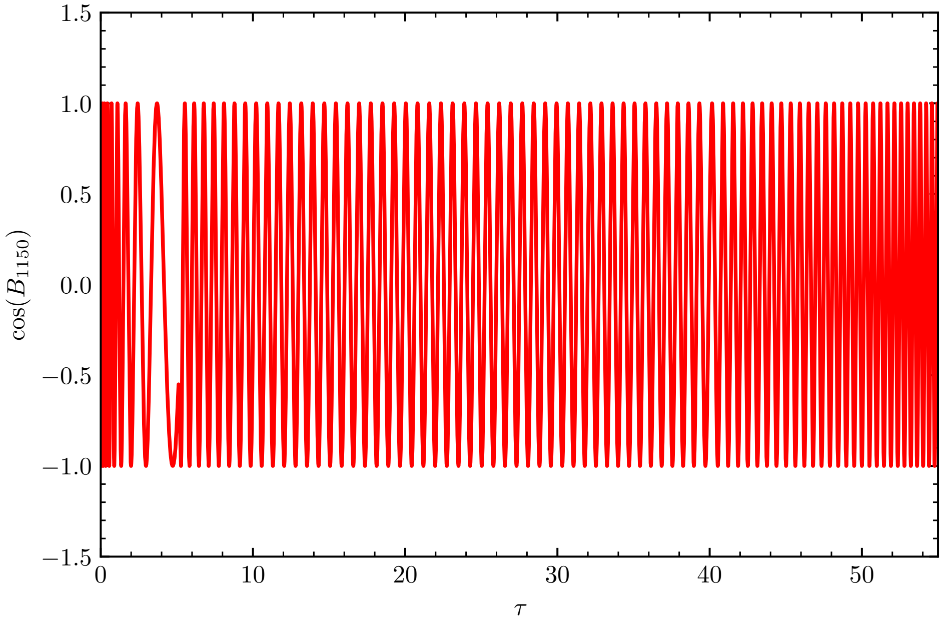

Figure 5.2: Evolution of the phase of a sample high mode

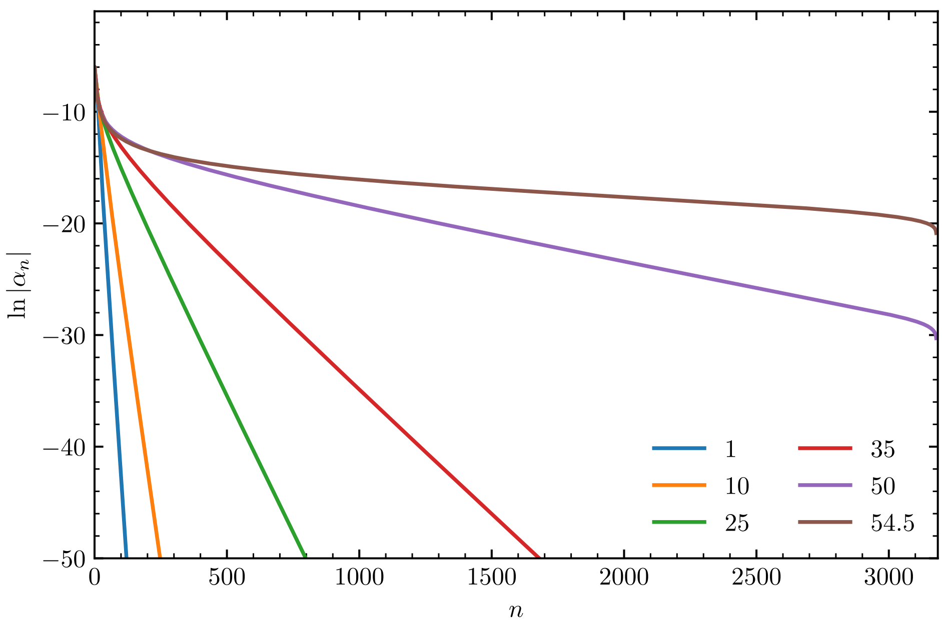

Figure 5.3: Amplitude spectra for several times τ. We find that higher modes are quickly excited.

which can be interpreted as the total energy of the system, is conserved numerically. Given the need for truncation, we used the quantity

$$E = \sum_{n=0}^{N-1} (n+1)^2 |\alpha_n|^2 \tag{5.22}$$

for the numerical analysis. We calculated the quantity at each time step and returned the relative error $\frac{E-E_0}{E_0}$, where E_0 is the value of E at time $\tau = 0$. The results can be seen in Figure 5.4. We note that $\frac{E-E_0}{E_0}$ stays small over the whole time interval of the simulation. We thus verify that the 'total energy' E is indeed (approximately) conserved in the system (5.2).

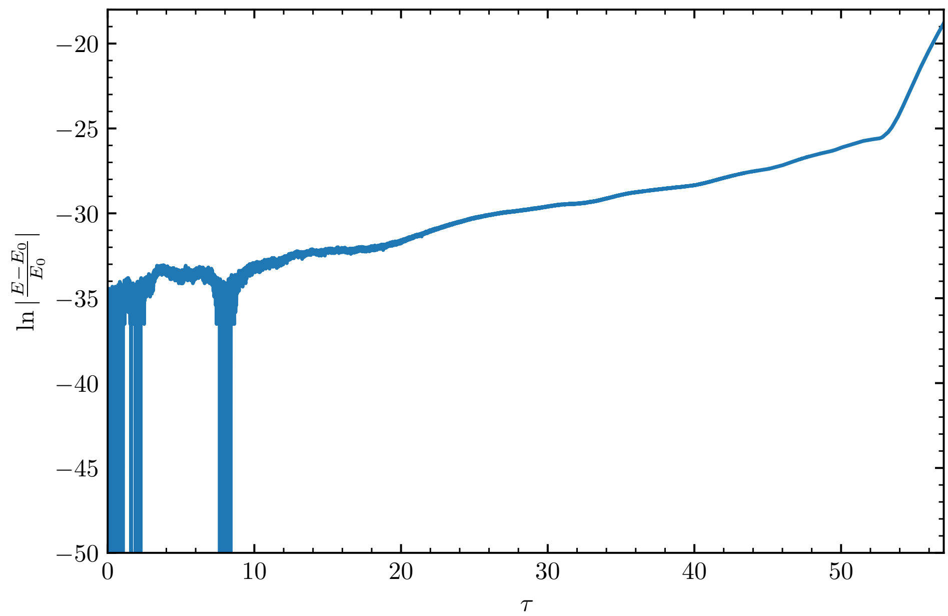

Figure 5.4: Evolution of (the logarithm of) the relative error $\frac{E-E_0}{E_0}$. We verify that E is (approximately) conserved over the course of the simulation.

5.4 Analyticity strip method and asymptotic analysis

In this section, we analyze the numerical data using the analyticity strip method [27]. We introduce the method as a means of establishing that the amplitude spectrum of the system (4.90) becomes singular in finite time.

5.4.1 The method

The analyticity strip method for numerical analysis was introduced in [27]. We give a brief summary based on [27], [20] and [6] here.

The analyticity strip method is based on the asymptotic behavior of the Fourier transform. We consider an analytic function $f(z)$ with singularities at $z = z_j$, $j = 1, \ldots, n$

in the complex plane. We define the Fourier integral

$$\hat{f}_k = \int_\Omega dz f(z) e^{ikz} \,, \tag{5.23}$$

where $\Omega = \mathbb{R}$ or, for periodic functions, $[0, 2\pi]$. In the neighborhood of z_j, $f(z)$ behaves as

$$f(z) \sim (z - z_j)^{\mu_j} \,, \tag{5.24}$$

where μ_j is not a positive integer. The $k \to +\infty$ behavior of the integral (5.23) is dominated by the singularity closest to the real axis, i.e. the singularity with the smallest imaginary part. If the closest singularity is located at $z_j = z_0 \equiv x_0 + i\delta$, $\delta > 0$ with exponent μ, we have [6, 27], at leading order,

$$\hat{f}_k \sim C(\mu)|k|^{-(\mu+1)} e^{ikz_0} = C(\mu)|k|^{-(\mu+1)} e^{-k\delta} e^{ix_0 k} \,, \quad k \to +\infty \,, \tag{5.25}$$

thus

$$|\hat{f}_k| \sim |C(\mu)||k|^{-(\mu+1)} e^{-k\delta} \,. \tag{5.26}$$

The analyticity radius δ determines the width of the analyticity strip, i.e. the distance of the nearest singularity to the real axis. A vanishing δ thus implies that the singularity hits the real axis, which means that the solution loses analyticity and becomes singular. The formula in (5.26) provides a means of establishing that the spectrum of the truncated resonant system (5.2) becomes singular in finite time.

5.4.2 Implementation and Results

We apply the analyticity strip method to the spectrum $|\alpha_n(\tau)| = A_n(\tau)/2$. We make the ansatz

$$|\alpha_n(\tau)| \sim C(\tau) n^{-\gamma(\tau)} e^{-\rho(\tau) n} \,, \quad n \gg 1 \tag{5.27}$$

for the amplitude spectrum, where $\rho(\tau)$ is the analyticity radius. We implement the fit $n \in [50, 1499]$ of this formula to the numerical data using a non-linear least-squares method[4].

The interval where the fit was performed was determined by a sliding fit procedure, which aims at finding an interval such that the parameters returned by the fit do not strongly depend the particular choice of interval. We therefore performed the fit for $n \in [n_{min}, n_{max}]$ first for varying n_{min} and fixed n_{max}. Then, upon fixing n_{min}, we performed the fit for fixed n_{min} and varying n_{max}.

By fitting (5.27) to the amplitude spectrum $|\alpha_n(\tau)|$ as a function of time (i.e. for each time-step on a certain interval), we obtain the time dependence of the analyticity radius $\rho(\tau)$. Figure 5.5 shows the evolution of $\rho(\tau)$. We observe that $\rho(\tau)$ hits zero in finite time τ_*. Figure 5.6 shows the evolution of $\rho(\tau)$ close to $\tau \sim \tau_*$. In Figure 5.7, we plot the exponent $\gamma(\tau)$. We note that Figures 5.5 and 5.6 serve only to illustrate the qualitative behavior of $\rho(\tau)$.

The time evolution of $\rho(\tau)$ can no longer be trusted when $\rho(\tau) \leq 1/(N-1)$ [13, 27]. To get a good estimate of τ_*, we stop the time integration when $\rho(\tau)$ becomes of the order $1/(N-1)$ and make the ansatz

$$\rho(\tau) \sim \rho_0(\tau_* - \tau) \,, \quad \tau \to \tau_* \,. \tag{5.28}$$

[4]The non-linear least-squares method was implemented using *SciPy*.

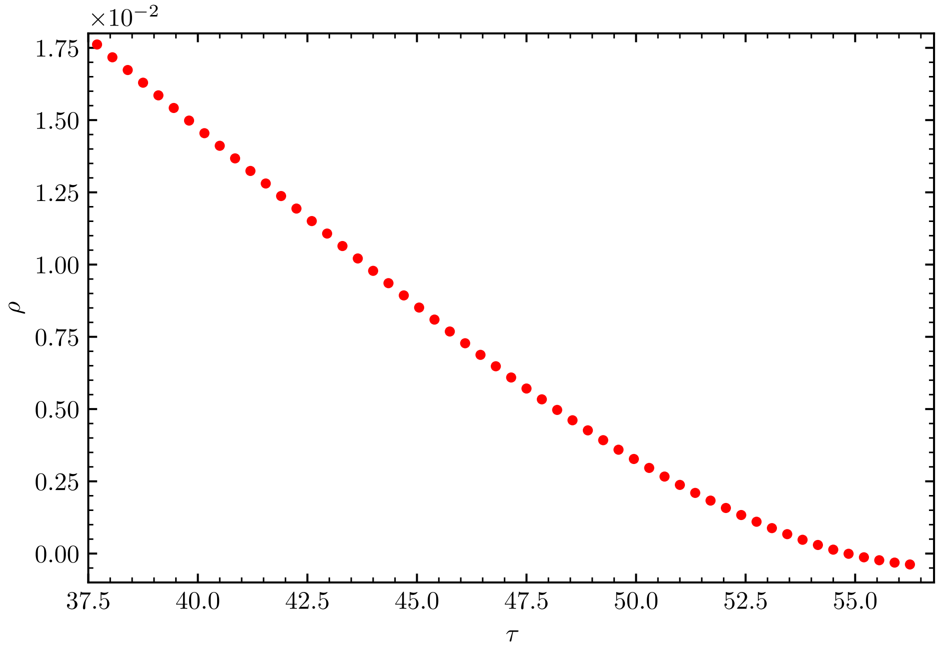

Figure 5.5: Time dependence of the analyticity radius $\rho(\tau)$. The data was obtained by fitting formula (5.27) to the amplitude spectrum $|\alpha_n(\tau)|$ for $n \in [50, 1499]$.

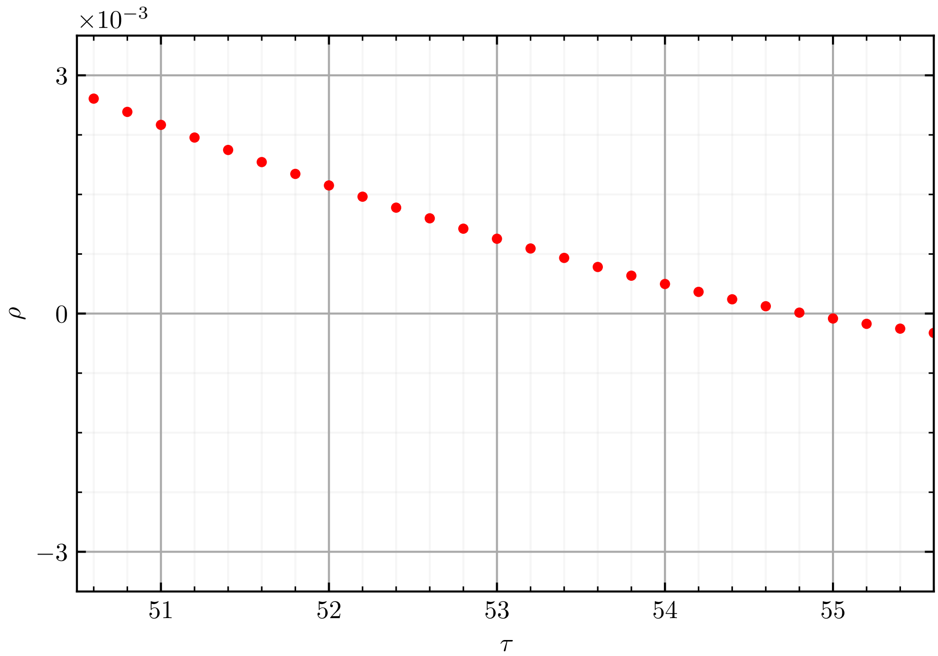

Figure 5.6: Evolution of the analyticity radius $\rho(\tau)$ close to $\tau \sim \tau_*$. We observe that $\rho(\tau)$ hits zero in finite time.

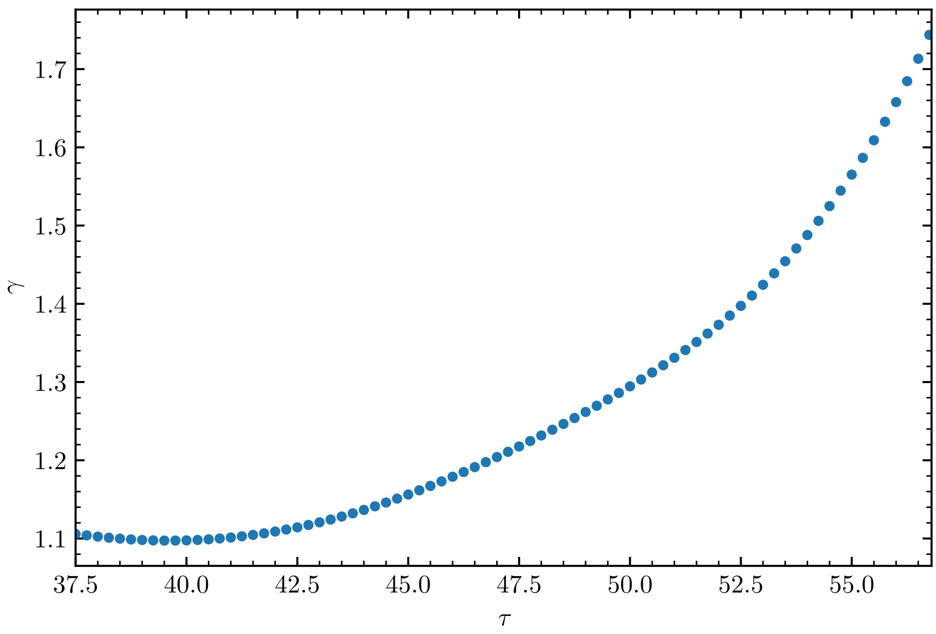

Figure 5.7: Time dependence of the exponent $\gamma(\tau)$.

We extrapolate the value $\tau_* \approx 54.73$ from the fit[5] of (5.28) to the numerical data.

At $\tau \to \tau_*$, the amplitude spectrum (5.27) evolves to a power law with exponent $\gamma_* \equiv \lim_{\tau\to\tau_*} \gamma(\tau)$. Seeking a value for γ_*, we write

$$\lim_{\tau\to\tau_*} |\alpha_n(\tau)| \sim C_* \, n^{-\gamma_*} \,, \quad n \gg 1 \,, \tag{5.29}$$

where $C_* \equiv \lim_{\tau\to\tau_*} C(\tau)$. We fit the formula (5.29) to amplitude spectrum at τ^* for $n \in [50, 1499]$. We give the result in Figure 5.8. We observe the formation of a power law spectrum at τ_* which, by the ansatz (5.27), corresponds to the solution of the resonant system becoming singular in finite time. The divergence for high modes of the spectrum at $\tau \approx \tau_*$ from the power law is due to errors which result from truncating the resonant system for a finite number of modes N. To highlight this fact, we show the amplitude spectrum at a certain time τ for different truncations in Figure 5.9.

In [21], the formation of a power law

$$E_j \sim j^{-\beta} \tag{5.30}$$

was observed, where the exponent was found to be

$$\beta = 1.2 \pm 0.1 \,. \tag{5.31}$$

Equating the energy per mode E_j in (5.30) with the formula

$$E_j = (j+1)^2 |\alpha_j|^2 \sim j^2 j^{-2\gamma_*} = j^{-2(\gamma_* - 1)} \,, \tag{5.32}$$

we find

$$\gamma_* = \frac{\beta + 2}{2} \,. \tag{5.33}$$

[5] The fit was implemented using *SciPy*.

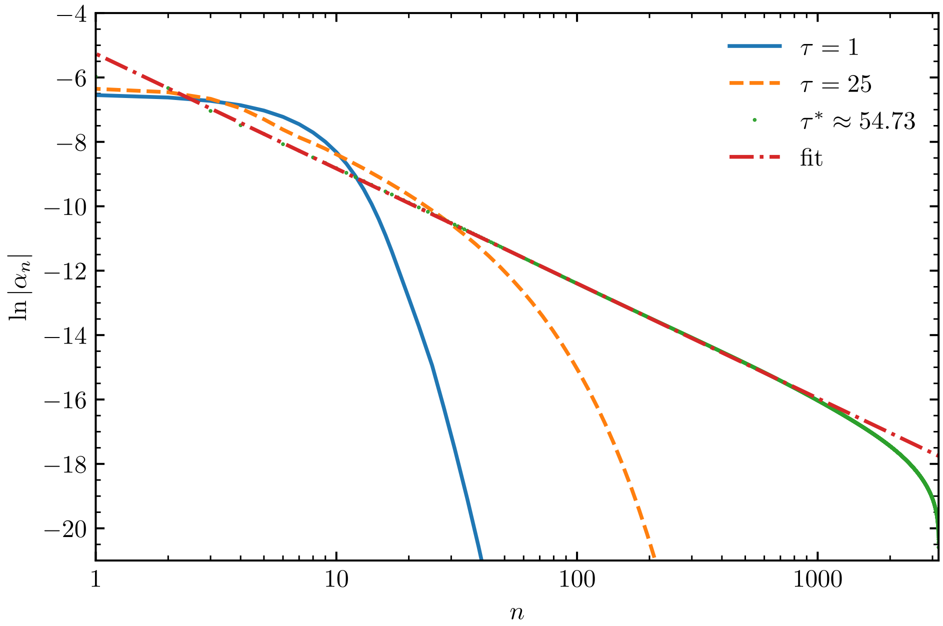

Figure 5.8: Amplitude spectra for several times τ. The fit at time τ_* was performed for $n \in [50, 1499]$ and yields the parameters $C_* \approx 5.2 \cdot 10^{-3}$, $\gamma_* \approx 1.55$. The amplitude spectrum evolves to a power law at τ_*.

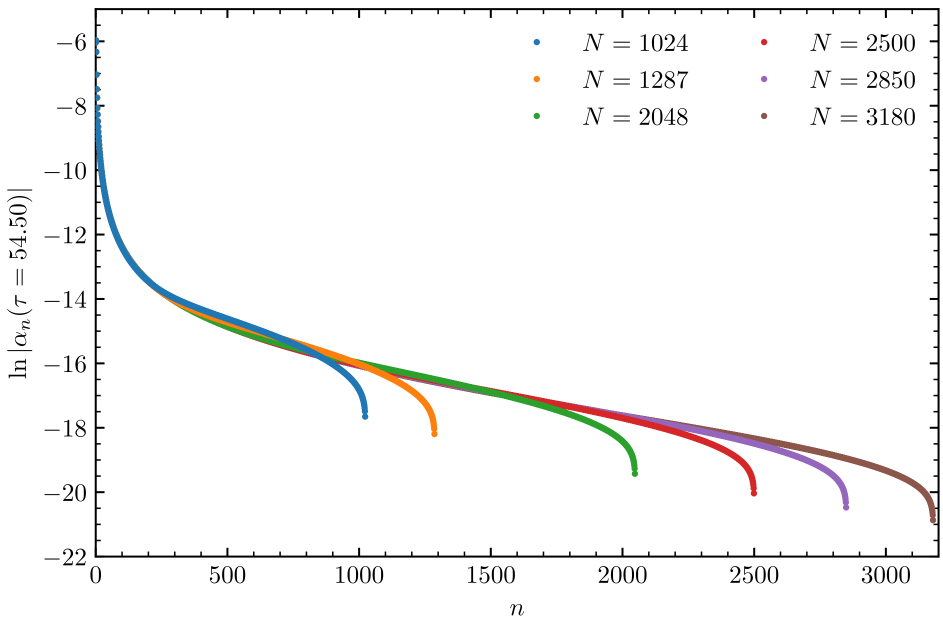

Figure 5.9: Amplitude spectrum $|\alpha_n(\tau)|$ for $\tau = 54.50$ for different truncations N.

Inserting the value (5.31) into (5.33) and comparing (5.33) with the value

$$\gamma_* \approx 1.55 \tag{5.34}$$

obtained from the fit (5.29), we find that the two values (5.31) and (5.34) are in agreement.

To complete our analysis, we study the asymptotic behavior of the phases. From (4.122) we derive

$$\frac{dB_n}{d\tau} = \Im\left(\frac{1}{\alpha_n}\frac{d\alpha_n}{d\tau}\right), \tag{5.35}$$

where $\Im$ denotes the imaginary part. We calculated the quantity (5.35) for a sample mode. We show the result in Figure 5.10 for the mode $n = 150$. We observe that the derivatives $\frac{dB_n}{d\tau}$ do not blow up in finite time. Figure 5.11 shows the evolution of $\frac{dB_n}{d\tau}$ for $n = 150$ for different truncations N.

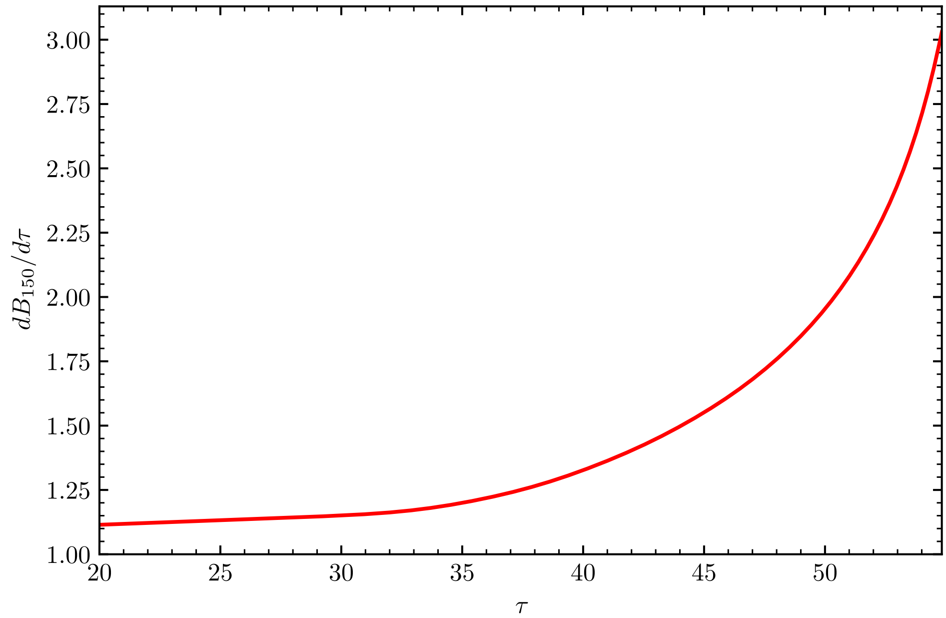

Figure 5.10: Evolution of $\frac{dB_n}{d\tau}$ for $n = 150$. The derivatives of the phases do not blow up in finite time.

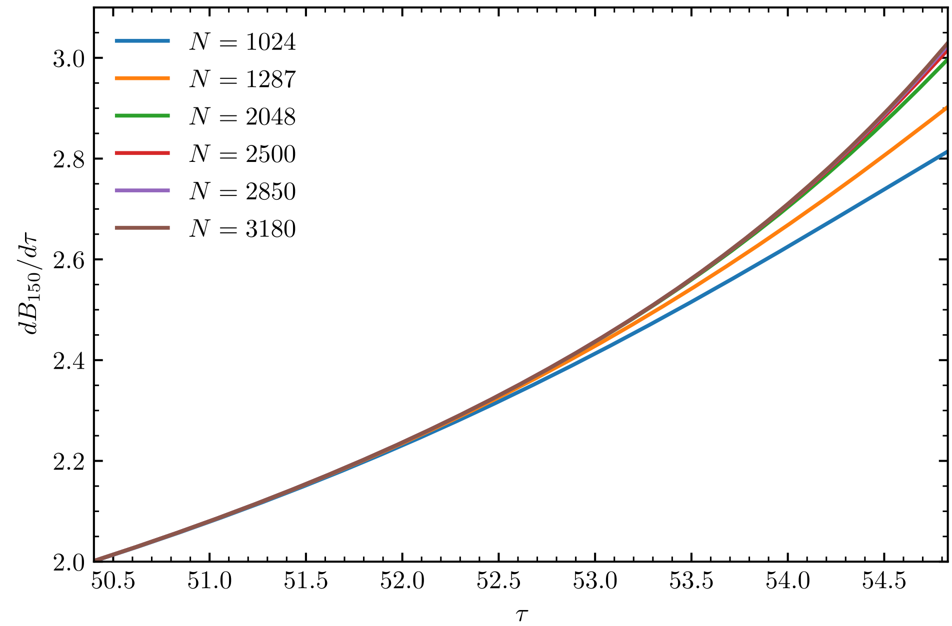

Figure 5.11: Evolution of $\frac{dB_n}{d\tau}$ for $n = 150$ close to τ_* for different values of N.

Chapter 6

Summary and Conclusion

In this thesis, we studied the resonant approximation for the 'flat space enclosed in a cavity' model from [21]. After stating essential preliminaries from gravitational and scalar field theory, we introduced the concept of resonance in oscillatory phenomena as well as of secular terms in perturbative expansions, and reviewed two resummation techniques, the method of multiple scales and the renormalization group method [8], by considering the Duffing equation as an example. Building upon this review, we derived the resonant system for flat space enclosed in a cavity using both multiple-scale analysis and the renormalization group method, discussed its symmetries, and gave a comparison to the resonant system for AdS [4, 11]. Subsequently, we solved the system numerically in the interior time gauge using an explicit Runge-Kutta method, studied its solution and analyzed the asymptotic behavior of the solution using the analyticity strip method [27].

In our derivation of the system, we have found that in comparison to the resonant system for AdS [4, 11], the resonant system for flat space enclosed in a cavity features two extra types of terms, which we denoted by '$+++$' and '$+--$'. The reason lies in the fact that the selection rules for AdS eigenmodes [11] responsible for making these terms vanish are not present for flat space enclosed in a cavity. One consequence of this is that the resonant system for flat space enclosed in a cavity has fewer symmetries than the one for AdS [12]. We also note that in contrast to the ones for AdS, the interaction coefficients we found in our derivation for flat space enclosed in a cavity do not exhibit any symmetries with respect to permutation of the indices.

In our subsequent numerical study, we have observed that solutions to the resonant system are of oscillatory nature and feature an increasing amplitude and phase. We have seen that in the evolution of the amplitude spectrum, higher modes are quickly excited. Furthermore, we have confirmed the (approximate) conservation of the total energy of the system.

By applying the analyticity strip method [27] to study the solution of the resonant system, we have found that its amplitude spectrum evolves to a power law in finite time. By the nature of the asymptotic behavior of the (generalized) Fourier coefficients, this corresponds to the solution of the resonant system becoming singular on that time scale [27]. We note that the value for the power law exponent γ_* determined in this thesis is in agreement with the corresponding value established by Maliborski [21] for the fully non-linear evolution of the system. We interpret this finding as further corroborating the results regarding the instability of flat space enclosed in a cavity reported in [21].

A challenge encountered during the course of this thesis was evolving the system numerically in time with high accuracy. Due to the structure of the equations, the computational cost of evaluating the right-hand side of (5.2) for a certain number of modes N scaled $\sim N^3$. Furthermore, due to the lack of symmetry in the interaction coefficients (5.1), it was not possible to reduce the complexity of the problem by analytical means. Given these limitations, solving the system for $N = 3180$ represented a compromise between accuracy and computational cost. Running on the supercomputer VSC-4, the time evolution of the system using the parameters outlined in Section 5.3 required up to 739 GB of memory (RAM) and took approximately 25000 core hours of computation time.

An open question that remains is the behavior of the phases and their derivatives. In contrast to the resonant system for AdS [4], the derivatives of the phases in the resonant system for flat space enclosed in a cavity do not blow up logarithmically. We suppose more light could be shed on this question by obtaining the asymptotic behavior of the interaction coefficients and conducting an analysis analogous to [4]. Furthermore, we are aware of the studies on AdS in [13], where it is suggested that the resonant system for AdS in $3 + 1$ dimensions is more sensitive to mode truncation than in higher dimensions. Given that flat space enclosed in a cavity also has dimension $3 + 1$, we speculate that more modes may be needed to accurately capture the behavior of the phases.

We have derived and studied the resonant system for flat space enclosed in a cavity. While we established that the solution of the system becomes singular in finite time using the analyticity strip method, more studies may be needed to draw final conclusions on the behavior of the phases.

Appendix A

The eigenvalue problem for L

In this appendix, we consider the eigenvalue problem

$$Ly(r) = \omega^2 y(r) \tag{A.1}$$

with the Dirichlet boundary condition at $r = R$

$$y(R) = 0\,, \tag{A.2}$$

where

$$L = -\frac{1}{r^2}\partial_r\left(r^2\partial_r\right)\,. \tag{A.3}$$

We write (A.1)-(A.3) in the form of a Sturm-Liouville problem [3]

$$\frac{d}{dr}\left[p(r)\frac{dy}{dr}\right] + q(r)y = -\omega_j^2 w(r)y\,, \tag{A.4}$$

where

$$p(r) = r^2\,,\quad q(r) = 0\,,\quad w(r) = r^2\,,\quad 0 \leq r \leq R\,. \tag{A.5}$$

Given (A.5), there are infinitely many eigenvalues ω_j^2, $j \in \mathbb{N}_0$, where ω_j^2 is real (and positive). The eigenfunctions y_j, $j \in \mathbb{N}_0$ form an orthonormal basis on the Hilbert space $L^2([0,R], r^2dr)$ [33] with respect to the inner product

$$\langle y_i, y_j\rangle \coloneqq \int_0^R drr^2 y_i(r)y_j(r)\,, \tag{A.6}$$

i.e.,

$$\langle y_i, y_j\rangle = \delta_{ij}\,, \tag{A.7}$$

where δ_{ij} is the Kronecker delta.

We derive a solution to the Sturm-Liouville problem (A.4)-(A.5). We write (A.4)-(A.5) as

$$r^2y'' + 2ry' + \omega^2r^2y = 0\,, \tag{A.8}$$

where primes refer to derivatives with respect to r. We identify (A.8) as a *spherical Bessel equation* [34]. We introduce [1]

$$y(r) = \frac{z(\omega r)}{\sqrt{\omega r}}\,. \tag{A.9}$$

We calculate y' and y'' in terms of z using the chain rule and insert the expressions into (A.8). Writing $\rho = \omega r$, we get

$$\rho^2 \frac{d^2 z(\rho)}{d\rho^2} + \rho \frac{dz(\rho)}{d\rho} + \left[\rho^2 - \left(\frac{1}{2}\right)^2\right] z(\rho) = 0\,, \tag{A.10}$$

which we identify as Bessel's equation of order $\frac{1}{2}$ [1]. Requiring regularity at zero, the solution to (A.10) is given in terms of the Bessel function $J_{1/2}$. We evaluate [35] $J_{1/2}(\omega r)$ as

$$J_{1/2}(\omega r) = \sqrt{\frac{2}{\pi}} \frac{\sin \omega r}{\sqrt{\omega r}}\,. \tag{A.11}$$

Using (A.9) we find

$$y(r) = C \frac{\sin \omega r}{r}\,. \tag{A.12}$$

We impose the Dirichlet boundary conditions $y(R) = 0$. We get

$$\omega = \omega_j = \frac{(j+1)\pi}{R}\,, \quad j \in \mathbb{N}_0\,. \tag{A.13}$$

Normalizing the eigenfunctions $y_j(r) = C \frac{\sin \omega_j r}{r}$, $j \in \mathbb{N}_0$ using

$$1 = \langle y_i, y_i \rangle = C^2 \int_0^R dr r^2 \frac{\sin \omega_i r}{r} \frac{\sin \omega_i r}{r} = C^2 \frac{R}{2}\,, \tag{A.14}$$

we find the eigenvalues and eigenfunctions for the eigenvalue problem (A.1)-(A.3)

$$\omega_j^2 = \frac{(j+1)^2 \pi^2}{R^2}\,, \quad y_j(r) = \sqrt{\frac{2}{R}} \frac{\sin \omega_j r}{r}\,, \quad j \in \mathbb{N}_0\,. \tag{A.15}$$

Furthermore, we prove that

$$\langle y_i', y_j' \rangle = \omega_j^2 \delta_{ij}\,. \tag{A.16}$$

First, we note that

$$\langle y_i, L y_j \rangle = \langle y_i, \omega_j^2 y_j \rangle = \omega_j^2 \langle y_i, y_j \rangle = \omega_j^2 \delta_{ij}\,, \tag{A.17}$$

where we have used (A.1) and (A.7). Next, we write

$$\langle y_i, L y_j \rangle = \left\langle y_i, -\frac{1}{r^2} \left(r^2 y_j'\right)' \right\rangle \equiv - \int_0^R dr y_i(r) \left(r^2 y_j'(r)\right)'\,. \tag{A.18}$$

We integrate by parts and get

$$\begin{aligned} - \int_0^R dr y_i(r) \left(r^2 y_j'(r)\right)' &= -r^2 y_i(r) y_j'(r) \Big|_0^R + \int_0^R dr r^2 y_i'(r) y_j'(r) \\ &\equiv -r^2 y_i(r) y_j'(r) \Big|_0^R + \langle y_i', y_j' \rangle\,. \end{aligned} \tag{A.19}$$

We complete our proof by noting that the first term evaluates to zero, which means that

$$\langle y_i, L y_j \rangle = \langle y_i', y_j' \rangle\,. \tag{A.20}$$

Appendix B

The cubic radial wave equation

In this appendix, we derive the resonant system for the cubic radial wave equation

$$\Box\phi = -\lambda\phi^3\,, \tag{B.1}$$

for a spherically symmetric scalar field $\phi = \phi(t, r)$ in a fixed Minkowski background. We impose Dirichlet boundary conditions at $r = R = 1$

$$\phi(t, 1) = 0\,. \tag{B.2}$$

The metric in spherical coordinates reads

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_2^2\,, \tag{B.3}$$

where $d\Omega_2^2$ is the round metric on the 2-sphere. Using

$$\Box = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\right)\,, \tag{B.4}$$

Equation (B.1) takes the form

$$\ddot{\phi} + L\phi = \lambda\phi^3\,, \tag{B.5}$$

where dots refer to partial derivatives with respect to t and where

$$L = -\frac{1}{r^2}\partial_r\left(r^2\partial_r\right)\,. \tag{B.6}$$

B.1 Weakly non-linear perturbation theory

We construct the weakly non-linear perturbation theory assuming small initial conditions $\phi(0, r) = \epsilon f(r)$ and $\dot{\phi}(0, r) = \epsilon g(r)$ and using the ansatz

$$\phi(t, r) = \epsilon\phi_1(t, r) + \epsilon^3\phi_3(t, r) + \dots\,, \tag{B.7}$$

where terms involving even powers of ϵ vanish due to the structure of (B.1) and the choice of initial conditions.

At first order of ϵ, we get

$$\ddot{\phi}_1 + L\phi_1 = 0\,, \tag{B.8}$$

with initial conditions $\phi_1(0, r) = f(r)$ and $\dot{\phi}_1(0, r) = g(r)$.

We solve the eigenvalue problem $Le_j(r) = \omega_j^2 e_j(r)$, which is the same as for flat space enclosed in a cavity (see Appendix A), for the Dirichlet boundary conditions $\phi(t,1) = 0$ and get the eigenvalues and eigenfunctions

$$\omega_j^2 = (j+1)^2\pi^2\,, \quad e_j(r) = \sqrt{2}\frac{\sin\omega_j r}{r}, \quad j \in \mathbb{N}_0\,, \tag{B.9}$$

with

$$\langle e_i, e_j\rangle = \int_0^1 drr^2 e_i(r)e_j(r) = \delta_{ij}\,, \tag{B.10}$$

where δ_{ij} is the Kronecker delta.

Expanding the solution to (B.8) in the basis $\{e_j\}$,

$$\phi_1(t,r) = \sum_{j=0}^{\infty} \hat{\phi}_j(t)e_j(r)\,, \tag{B.11}$$

and projecting equation (B.8) onto $\{e_j\}$, we find

$$\ddot{\hat{\phi}}_j(t) + \omega_j^2\hat{\phi}_j(t) = 0\,. \tag{B.12}$$

Equation (B.12) has the solution

$$\hat{\phi}_j(t) = \alpha_j e^{i\omega_j t} + \bar{\alpha}_j e^{-i\omega_j t}\,, \tag{B.13}$$

where α_j is determined by the initial conditions. We thus write down the solution to (B.8) as

$$\phi_1(t,r) = \sum_{j=0}^{\infty}\hat{\phi}_j(t)e_j(r) \equiv \sum_{j=0}^{\infty}\left(\alpha_j e^{i\omega_j t} + \bar{\alpha}_j e^{-i\omega_j t}\right)e_j(r)\,. \tag{B.14}$$

At third order of ϵ, we get the equation

$$\ddot{\phi}_3 + L\phi_3 = \lambda\phi_1^3 \equiv \lambda\sum_{i,j,k=0}^{\infty}\hat{\phi}_i\hat{\phi}_j\hat{\phi}_k e_i e_j e_k\,. \tag{B.15}$$

We expand

$$\phi_3 = \sum_{j=0}^{\infty}\hat{\phi}_{3,j}(t)e_j(r)\,, \tag{B.16}$$

and project (B.15) onto $\{e_l\}$ and find

$$\ddot{\hat{\phi}}_{3,l}(t) + \omega_l^2\hat{\phi}_{3,l}(t) = \lambda\sum_{i,j,k=0}^{\infty}\hat{\phi}_i\hat{\phi}_j\hat{\phi}_k S_{ijkl}\,, \tag{B.17}$$

where

$$S_{ijkl} = \int_0^1 drr^2 e_i(r)e_j(r)e_k(r)e_l(r)\,. \tag{B.18}$$

Evaluating (B.18) yields the interaction coefficients

$$\begin{aligned} S_{ijkl} = \; & \frac{1}{2}\pi\,[\, -(i-j-k+l)\,\mathrm{Si}\,((i-j-k+l)\pi) - \\ & -(i-j+k-l)\,\mathrm{Si}\,((i-j+k-l)\pi) + \\ & +(i-j-k-l-2)\,\mathrm{Si}\,((i-j-k-l-2)\pi) + \\ & +(i-j+k+l+2)\,\mathrm{Si}\,((i-j+k+l+2)\pi) + \\ & +(i+j-k+l+2)\,\mathrm{Si}\,((i+j-k+l+2)\pi) + \\ & +(i+j+k-l+2)\,\mathrm{Si}\,((i+j+k-l+2)\pi) - \\ & -(i+j-k-l)\,\mathrm{Si}\,((i+j-k-l)\pi) - \\ & -(i+j+k+l+4)\,\mathrm{Si}\,((i+j+k+l+4)\pi)]\,, \end{aligned} \tag{B.19}$$

where

$$\mathrm{Si}(x) \equiv \int_0^x \frac{\sin x'}{x'} dx' \tag{B.20}$$

is the sine integral function [30]. We note that S_{ijkl} is symmetric under permutations $i \leftrightarrow j \leftrightarrow k \leftrightarrow l$. Inserting (B.13) into (B.17), we find

$$\begin{aligned} \ddot{\hat{\phi}}_{3,l}(t) + \omega_l^2 \hat{\phi}_{3,l}(t) = \lambda \sum_{ijk} \big[& \alpha_i \alpha_j \alpha_k e^{i(\omega_i+\omega_j+\omega_k)t} S_{ijkl} \\ & +\bar{\alpha}_i \alpha_j \alpha_k e^{i(-\omega_i+\omega_j+\omega_k)t} S_{ijkl} \\ & +\alpha_i \bar{\alpha}_j \alpha_k e^{i(\omega_i-\omega_j+\omega_k)t} S_{ijkl} \\ & +\alpha_i \alpha_j \bar{\alpha}_k e^{i(\omega_i+\omega_j-\omega_k)t} S_{ijkl} \\ & +\alpha_i \bar{\alpha}_j \bar{\alpha}_k e^{i(\omega_i-\omega_j-\omega_k)t} S_{ijkl} \\ & +\bar{\alpha}_i \alpha_j \bar{\alpha}_k e^{i(-\omega_i+\omega_j-\omega_k)t} S_{ijkl} \\ & +\bar{\alpha}_i \bar{\alpha}_j \alpha_k e^{i(-\omega_i-\omega_j+\omega_k)t} S_{ijkl} \\ & +\bar{\alpha}_i \bar{\alpha}_j \bar{\alpha}_k e^{i(-\omega_i-\omega_j-\omega_k)t} R_{ijkl} \big] \end{aligned} \tag{B.21}$$

$$\begin{aligned} = \lambda \sum_{ijk} \big[& \alpha_i \alpha_j \alpha_k e^{i(\omega_i+\omega_j+\omega_k)t} S_{ijkl} \\ & +3\alpha_i \alpha_j \bar{\alpha}_k e^{i(\omega_i+\omega_j-\omega_k)t} S_{ijkl} \\ & +3\alpha_i \bar{\alpha}_j \bar{\alpha}_k e^{i(\omega_i-\omega_j-\omega_k)t} S_{ijkl} \\ & +\bar{\alpha}_i \bar{\alpha}_j \bar{\alpha}_k e^{i(-\omega_i-\omega_j-\omega_k)t} S_{ijkl} \big]\ , \end{aligned} \tag{B.22}$$

where we have used a permutation of the indices i, j, k as well as the symmetry of S_{ijkl} in (B.22). The solution to (B.22) has a secular term $\sim te^{\pm i\omega_l t}$ if the resonance condition $\pm\omega_l = \omega_i \pm \omega_j \pm \omega_k$ is satisfied.

B.2 The resonant system

We derive the resonant system for (B.1)-(B.2) using multiple-scale analysis (which we have introduced in Section 3.3.1).

We introduce the slow time-scale $\tau = \epsilon^2 t$ and redo the perturbation analysis for $\phi = \phi(t, \tau, r)$. In particular, we write down the solution to the equation at first order of ϵ

as

$$\phi_1(t,\tau,r) = \sum_{j=0}^{\infty} \hat{\phi}_j(t,\tau) e_j(r) \equiv \sum_{j=0}^{\infty} \left(\alpha_j(\tau) e^{i\omega_j t} + \bar{\alpha}_j(\tau) e^{-i\omega_j t}\right) e_j(r)\,, \tag{B.23}$$

where we have introduced the slow time dependence in the integration constants $\alpha_j(\tau)$. At third order of ϵ, we find

$$\ddot{\phi}_3 + L\phi_3 = -2\partial_\tau \dot{\phi}_1 + \lambda \sum_{i,j,k=0}^{\infty} \hat{\phi}_i \hat{\phi}_j \hat{\phi}_k e_i e_j e_k\,, \tag{B.24}$$

where dots refer to partial derivatives with respect to t and where we have used

$$\partial_t^2 \phi(t,r) \to \partial_t^2 \phi(t,\tau,r) + 2\epsilon^2 \partial_t \partial_\tau \phi(t,\tau,r) + \mathcal{O}(\epsilon^4)\,. \tag{B.25}$$

We project (B.24) onto $\{e_l\}$ and find

$$\begin{aligned}\ddot{\hat{\phi}}_{3,l}(t) + \omega_l^2 \hat{\phi}_{3,l}(t) = \langle -2\partial_\tau \dot{\phi}_1, e_l\rangle + \lambda \sum_{ijk} \Big[& \alpha_i \alpha_j \alpha_k e^{i(\omega_i+\omega_j+\omega_k)t} S_{ijkl} \\ & + 3\alpha_i \alpha_j \bar{\alpha}_k e^{i(\omega_i+\omega_j-\omega_k)t} S_{ijkl} \\ & + 3\alpha_i \bar{\alpha}_j \bar{\alpha}_k e^{i(\omega_i-\omega_j-\omega_k)t} S_{ijkl} \\ & + \bar{\alpha}_i \bar{\alpha}_j \bar{\alpha}_k e^{i(-\omega_i-\omega_j-\omega_k)t} S_{ijkl} \Big]\,. \end{aligned} \tag{B.26}$$

We derive the condition that removes resonant terms $\sim e^{i\omega_l t} + c.c.$ from the right-hand side of (B.26) by setting the projection of the right-hand side of (B.26) onto $e^{i\omega_l t}$ to zero. We use

$$\left(e^{i\omega_m t}, e^{i\omega_n t}\right) = \delta_{mn}\,, \quad \omega_m = (m+1)\pi, \quad m,n \in \mathbb{N}_0\,, \tag{B.27}$$

where

$$(f,g) = \frac{1}{2}\int_0^2 dt f(t) \bar{g}(t) \tag{B.28}$$

and calculate

$$\begin{aligned}\left(\langle -2\partial_\tau \dot{\phi}_1, e_l\rangle, e^{i\omega_l t}\right) &= -2i\omega_l \left[\frac{d\alpha_l}{d\tau}\left(e^{i\omega_l t}, e^{i\omega_l t}\right) - \frac{d\bar{\alpha}_l}{d\tau}\left(e^{-i\omega_l t}, e^{i\omega_l t}\right)\right] \\ &= -2i\omega_l \frac{d\alpha_l}{d\tau}\,. \end{aligned} \tag{B.29}$$

Projecting the other terms on the right-hand side of (B.26) onto $e^{i\omega_l t}$ we get

$$\begin{aligned}\lambda \sum_{ijk} \Big[& \alpha_i \alpha_j \alpha_k S_{ijkl} \delta_{l+1,i+1+j+1+k+1} \\ & + 3\alpha_i \alpha_j \bar{\alpha}_k S_{ijkl} \delta_{l+1,i+1+j+1-(k+1)} \\ & + 3\alpha_i \bar{\alpha}_j \bar{\alpha}_k S_{ijkl} \delta_{l+1,i+1-(j+1)-(k+1)} \\ & + \bar{\alpha}_i \bar{\alpha}_j \bar{\alpha}_k S_{ijkl} \delta_{l+1,-(i+1)-(j+1)-(k+1)} \Big]\,. \end{aligned} \tag{B.30}$$

We note that $\delta_{l+1,-(i+1)-(j+1)-(k+1)} = 0$ for $i,j,k,l \in \mathbb{N}_0$. Setting the sum of (B.29) and (B.30) to zero, we find the resonant system for the cubic radial wave equation

$$\frac{d\alpha_l}{d\tau} = -\frac{i}{2\omega_l}\lambda \left[\sum_{ijk}^{+++} S_{ijkl} \alpha_i \alpha_j \alpha_k + 3\sum_{ijk}^{++-} S_{ijkl} \alpha_i \alpha_j \bar{\alpha}_k + 3\sum_{ijk}^{+--} S_{ijkl} \alpha_i \bar{\alpha}_j \bar{\alpha}_k\right]\,, \tag{B.31}$$

where

$$\sum_{ijk}^{+++} = \sum_{\substack{i,j,k=0 \\ l+1=i+1+j+1+k+1}}^{\infty} , \qquad \sum_{ijk}^{++-} = \sum_{\substack{i,j,k=0 \\ l+1=i+1+j+1-(k+1)}}^{\infty} , \qquad \sum_{ijk}^{+--} = \sum_{\substack{i,j,k=0 \\ l+1=i+1-(j+1)-(k+1)}}^{\infty} . \tag{B.32}$$

Like the resonant system for flat space enclosed in a cavity (4.90), the resonant system for the cubic radial wave equation (B.31) has '+++' and '+−−' terms, which are absent in the resonant system for AdS [11].

B.3 Symmetries

The system (B.31) is symmetric under the transformation

$$\alpha_l \to \alpha_l e^{i(l+1)\theta} , \quad \theta \in \mathbb{R} . \tag{B.33}$$

Furthermore, (B.31) is invariant under the rescaling

$$\alpha_l(\tau) \to \epsilon^{-1}\alpha_l(\tau/\epsilon^2) \tag{B.34}$$

as well as under the time translation

$$\alpha_l(\tau) \to \alpha_l(\tau - \tau_0) . \tag{B.35}$$

In the following, we check whether the quantity

$$E = \sum_{l=0}^{\infty}(l+1)^2|\alpha_l|^2 \tag{B.36}$$

is conserved. We calculate the time derivative

$$\frac{dE}{dt} = \sum_{l=0}^{\infty}(l+1)^2\frac{d}{dt}|\alpha_l|^2 = \sum_{l=0}^{\infty}(l+1)^2\frac{d}{dt}\left(\alpha_l\bar{\alpha}_l\right) = \sum_{l=0}^{\infty}(l+1)^2\left(\dot{\alpha}_l\bar{\alpha}_l + \alpha_l\dot{\bar{\alpha}}_l\right) \tag{B.37}$$

and insert (B.31), which yields

$$\frac{dE}{dt} = -\frac{i}{2\pi}\lambda\left(A + B + C\right) , \tag{B.38}$$

where

$$A = \sum_{l=0}^{\infty}\sum_{ijk}^{+++} S_{ijkl}\left(\alpha_i\alpha_j\alpha_k\bar{\alpha}_l - \bar{\alpha}_i\bar{\alpha}_j\bar{\alpha}_k\alpha_l\right)(l+1) , \tag{B.39}$$

$$B = 3\sum_{l=0}^{\infty}\sum_{ijk}^{++-} S_{ijkl}\left(\alpha_i\alpha_j\bar{\alpha}_k\bar{\alpha}_l - \bar{\alpha}_i\bar{\alpha}_j\alpha_k\alpha_l\right)(l+1) , \tag{B.40}$$

$$C = 3\sum_{l=0}^{\infty}\sum_{ijk}^{+--} S_{ijkl}\left(\alpha_i\bar{\alpha}_j\bar{\alpha}_k\bar{\alpha}_l - \bar{\alpha}_i\alpha_j\alpha_k\alpha_l\right)(l+1) . \tag{B.41}$$

To show that the right-hand side of (B.38) is zero, we will show that both B and the sum of A and C are zero.

Using the fact that the sum in (B.40), as well as S_{ijkl} and the tensor $\alpha_i\alpha_j\bar{\alpha}_k\bar{\alpha}_l - \bar{\alpha}_i\bar{\alpha}_j\alpha_k\alpha_l$ are symmetric under $k \leftrightarrow l$, we rewrite (B.40) as

$$B = \frac{3}{2}\sum_{l=0}^{\infty}\sum_{ijk}^{++-} S_{ijkl}\left(\alpha_i\alpha_j\bar{\alpha}_k\bar{\alpha}_l - \bar{\alpha}_i\bar{\alpha}_j\alpha_k\alpha_l\right)\left[(l+1)+(k+1)\right] . \tag{B.42}$$

The tensor $S_{ijkl}\left[(l+1)+(k+1)\right]$ is symmetric under $(i,j) \leftrightarrow (k,l)$ whenever the resonance condition $l+1 = i+1+j+1-(k+1)$ is satisfied. The tensor $\alpha_i\alpha_j\bar{\alpha}_k\bar{\alpha}_l - \bar{\alpha}_i\bar{\alpha}_j\alpha_k\alpha_l$ is antisymmetric under $(i,j) \leftrightarrow (k,l)$, thus $B = 0$.

To show that $A + C$ vanishes, we first note that C can be rewritten as

$$C = 3\sum_{l=0}^{\infty}\sum_{ijk}^{+++} S_{ijkl}\left(\bar{\alpha}_i\bar{\alpha}_j\bar{\alpha}_k\alpha_l - \alpha_i\alpha_j\alpha_k\bar{\alpha}_l\right)(i+1)\,, \tag{B.43}$$

by interchanging the indices i and l. We note that the '$+--$' sum becomes a '$+++$' sum under this interchange. Putting $A + C$ together, we find

$$A + C = \sum_{l=0}^{\infty}\sum_{ijk}^{+++} S_{ijkl}\left(\alpha_i\alpha_j\alpha_k\bar{\alpha}_l - \bar{\alpha}_i\bar{\alpha}_j\bar{\alpha}_k\alpha_l\right)\left[(l+1)-3(i+1)\right] . \tag{B.44}$$

Using the fact that the sum in (B.44), as well as S_{ijkl} and the tensor $\alpha_i\alpha_j\alpha_k\bar{\alpha}_l - \bar{\alpha}_i\bar{\alpha}_j\bar{\alpha}_k\alpha_l$ are symmetric under $i \leftrightarrow j \leftrightarrow k$, we rewrite (B.44) as

$$A + C = \sum_{l=0}^{\infty}\sum_{ijk}^{+++} S_{ijkl}\left(\alpha_i\alpha_j\alpha_k\bar{\alpha}_l - \bar{\alpha}_i\bar{\alpha}_j\bar{\alpha}_k\alpha_l\right)\left[(l+1)-(i+1)-(j+1)-(k+1)\right] , \tag{B.45}$$

which is zero since $l+1 = i+1+j+1+k+1$. Thus (B.36) is conserved.

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