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"The Borel property for non-path-bounded sets in the plane"

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Contents

Abstract

The main objective of this master thesis is to obtain an alternative version of the theorem produced in [8] that characterizes the Borel property for compact path-bounded sets in the complex plane, where path-boundedness as a precondition has been eliminated or substituted by a more general assumption. By Borel property we understand in this context the surjectivity of the 'Borel map' at a point $p \in \hat{K}$, this is a map that associates every function of $A^{\infty}(K)$ to its formal Taylor series at p, where K is a compact set of the complex plane and $\hat{K}$ is the polynomially convex hull of K. We start by contextualizing the work in terms of the setting of CR manifolds and briefly presenting the reasons for the appearance of the algebra $A^{\infty}(K)$. We then explore the structure of the space $A^{\infty}(K)$ and explain the previous work done on [8]. Afterwards we present the two properties that are going to substitute path-boundedness: proper-situatedness (which relates to the behavior of the distance function around the point p) and non-spirality ([1]). We finish exploring the properties of proper-situatedness and providing examples for both properties.

Zusammenfassung

Das Hauptziel dieser Masterarbeit ist eine erweiterte Version des Theorems aus [8] zu beweisen, das die Borel Eigenschaft für kompakte wegbeschränkte (*path-bounded*) Menge auf der komplexen Zahlenebene charakterisiert, in welcher Wegbeschränktheit als Voraussetzung durch allgemeinere Annahmen ersetzt wird. Unter der Borel Eigenschaft verstehen wir in diesem Zusammenhang die Surjektivität der 'Borel Funktion' am Punkt $p \in \hat{K}$. Die Borel Funktion ordnet jedem Element von $A^\infty(K)$ seine formale Taylorreihe mit Entwicklungsstelle p zu, wo K eine kompakte Teilmenge der komplexen Zahlenebene ist und $\hat{K}$ die polynomialkonvexe Hülle von K bezeichnet. Wir beginnen mit einer Kontextualisierung der Arbeit in Hinsicht auf das Setting von CR-Mannigfaltigkeiten und mit einer kurzen Präsentation der Gründe, warum die Algebra $A^\infty(K)$ auftritt. Dann erkunden wir die Struktur des Raumes $A^\infty(K)$ und erklären das vorherige Paper [8]. Nachdem diskutieren wir die zwei Eigenschaften, die Wegbeschränktheit ersetzen werden: *proper-situatedness* (dass verbunden zum Verhalten der Distanzfunktion in der Gegend des Punkts p ist) und *non-spirality* ([1]). Wir schließen mit einem Kapitel ab, wo wir die *proper-situatedness* Eigenschaft erkunden und wo wir Beispiele für beide Eigenschaften vorstellen.

Chapter 1

Introduction

1.1 Preface

Whenever we are working with a system of partial differential equations, if we do not know of any direct method to solve it, it is not an uncommon strategy to try to find what is called a formal solution to the problem, i.e., a solution in the form of a formal power series: possibly infinite series with no concern for convergence. If such a solution exists the next natural step is to relate this formal solution with a solution in the space of functions in which the problem was initially defined. Let us assume we have a partial differential equation where we are looking for solutions in the space of infinitely differentiable functions defined on the real line, $C^\infty(\mathbb{R})$. We can write that mathematically as follows:

$$P(f) = 0, f \in C^\infty(\mathbb{R})$$

where P is a partial differential operator. We are unable at first to find a solution by more direct methods in the space $C^\infty(\mathbb{R})$, but we are able to find a formal solution $f(x) = \sum_{k=0}^{\infty} a_k x^k \in \mathbb{R}[\![x]\!]$, where $\mathbb{R}[\![x]\!]$ is the space of formal power series with coefficients in $\mathbb{R}$. In a sense, we have a solution, but it is not a solution of the original problem, since we do not really know whether the formal solution is actually the Taylor series of a function in $C^\infty(\mathbb{R})$. It is with this conundrum in mind that studying the properties of functions like the following becomes relevant

$$T_0 : C^\infty(\mathbb{R}) \longrightarrow \mathbb{R}[\![x]\!].$$

With T_0 we denote a map associating a function f its Taylor series development at the point 0. If this map T_0 were surjective, then any formal solution of the form $\sum_{k=0}^{\infty} a_k x^k$ would correspond to at least one smooth function. Therefore our formal solution could correspond to an actual solution of the original problem. The question on injectivity also naturally arises in order to study the possible uniqueness of the solution.

In this particular case, for the map T_0, Borel's theorem states that for every formal power series $\sum_{k=0}^{\infty} a_j x^k \in \mathbb{R}[\![x]\!]$ there exists a smooth function f in $C^\infty(\mathbb{R})$ such that $f^{(k)}(0) = k!a_k$, this is, $T_0(f)(x) = \sum_{k=0}^{\infty} a_j x^k$ ([18]). In other words, T_0 is surjective.

This previous results served as inspiration to study, in the setting of CR manifolds, the relationship between CR functions and the *'family of solutions'* of the integrable subbundle. Details about all these concepts will be given in the following section. For now, briefly explained, the so-called subbundle of a CR manifold is integrable whenever for every $p \in M$ there exists a neighborhood U of p and a collection of CR functions $Z_1, \ldots, Z_N$ with linearly independent derivatives on U, satisfying $LZ_j = 0$ for all smooth sections of $\mathscr{V}$ over U and $Z_j(p) = 0, j = 1, \ldots, N$ ([14]). Drawing inspiration from the results of the Borel's Theorem, one can think about this family of solutions as of a family of solutions of a system of linear, first order PDEs. With this in mind one can wonder about the surjectivity of a map like

$$T : \mathfrak{S}_p \longrightarrow \mathbb{C}[\![Z_1, \ldots, Z_{n+d}]\!]$$

where $\mathfrak{S}_p$ is a certain space of CR functions.

Throughout the papers [4], [3], [9], [18], [14] and [15] this previous approach was developed and its relationship to other properties like hypocomplexity or unique continuation was explored.

The problem was eventually translated to the $A^\infty(K)$ Fréchet algebra through the commutative diagram

$$\begin{tikzcd}
\mathfrak{S}_p(K) \arrow[r, "j_p^\infty"] \arrow[d] & J_p^\infty(K) \arrow[d] \\
A^\infty(K) \arrow[r, "\mathfrak{b}_{K,p}"] & \mathbb{C}[\![Z_1, Z_2, \ldots, Z_{n+d}]\!]
\end{tikzcd}$$

And in the setting of this Fréchet algebra the paper [8] offered a solution for the one-dimensional case –since nothing was written in regards to this particular problem–, namely when $K \subset \mathbb{C}$ and $\mathfrak{b}_{K,p} : A^\infty(K) \to \mathbb{C}[\![X]\!]$. The research yielded a comprehensive theorem with the set K under a geometric condition called path-boundedness, which we may quote now:

Definition 1.1 (Path-boundedness)**.** *We say that a compact set $K \subset \mathbb{C}$ is path-bounded if there exists $C > 0$ such that for any neighborhood U of K and any points $p, q \in K$ there exists a rectifiable curve $\gamma \subset U$ of length at most C joining p and q.*

Theorem 1.1. *Let $K \subset \mathbb{C}$ be a path-bounded compact set with $p \in \hat{K}$. Then one and only one of the following properties holds:*

- *$\mathfrak{b}_{K,p}$ is surjective;*

- *$\mathfrak{b}_{K,p}$ is injective and $\hat{K}$ is a neighborhood of p.*

Having achieved this result, the natural follow-up question is whether path-boundedness is necessary to achieve a theorem like the previous one. The work of this master thesis has been to explore the problem in the context of the Fréchet algebra $A^{\infty}(K)$ with the objective of removing path-boundedness as an assumption on the set K, ideally eliminating it or substituting it by a less demanding presupposition or at least an alternative one that allows the theorem to work in sets that are not path-bounded.

I offer two solutions to that question in the form of two properties which I refer to as *'proper-situatedness'* and *'non-spirality'* (the latter coming from [1]).

Definition 1.2. *Let K be a compact set and p a point on the boundary. We will say that p is properly situated if there exists a sequence $\{z_n\} \subset K^c$, $z_n \longrightarrow p$ as $n \longrightarrow \infty$ such that*

$$\lim_{n\to\infty} \frac{d(z_n, K)}{|z_n - p|} > 0.$$

Definition 1.3 (Non-spiraling)**.** *Let $K \subset \mathbb{C}$ compact and $p \in \delta K$, we will say that K is non-spiraling at p if there exists a Jordan curve γ 'connecting p and ∞' such that*

- *$\Gamma \setminus \{p\}$ is fully contained in K^c (the point p either is the endpoint or the beginning of γ).*
- *There exists an analytic branch of the complex logarithm on $\mathbb{C} \setminus \Gamma$ with its imaginary part absolutely bounded.*

Using these two properties I was able to obtain the following results for each property:

Theorem 1.2. *Let K be compact and $0 \in K$. Then the following statements are true:*

- *$\mathfrak{b}_K$ is injective and $\hat{K}$ is a neighborhood of the origin.*
- *$0 \in \delta\hat{K}$ is properly situated and $\mathfrak{b}_K$ is surjective.*

Theorem 1.3. *Let K be compact and $0 \in K$. Then the following statements are true:*

- *$\mathfrak{b}_K$ is injective and $\hat{K}$ is a neighborhood of the origin.*
- *K is non-spiraling at $0 \in \delta\hat{K}$ and $\mathfrak{b}_K$ is surjective.*

In the last section of this work I study the properties of both previous concepts and provide some examples. First, to try to formalize the intuitions about the kind of sets which hold these two properties and secondly, to justify that they offer an alternative or an improvement to path-boundedness.

1.2 Prerequisites

In order to better contextualize the work of this thesis, it will be useful to provide a couple of definitions about the setting in which the work of [4], [18], [14] and [15] was done. We will take a brief detour into differential geometry to then introduce notions of CR geometry and thus build the definitions from the bottom up to offer a more comfortable read, without dwelling excessively into every detail. The definitions and further details on these concepts of smooth manifolds that we will review can be found at [12] and [16].

1.2.1 CR manifolds

Definition 1.4 (Manifold). *A manifold M is a separable Hausdorff space such that for every point p in the manifold there exists a homeomorphism $\varphi : U \to \mathbb{R}^n$, where U is an open neighborhood of p and $\varphi(U) \subset \mathbb{R}^n$ is open. We can call the pair (U, φ) a chart.*

Definition 1.5 (Charts and atlases). *We will call a family of charts $(U_j, \varphi_j)_{j \in J}$ an atlas if $M = \bigcup_{j \in J} U_j$, where $U_j \subset M$ are the domains of the homeomorphisms φ_j.*

An atlas will be called a C^k-atlas if all chart changings $\varphi_{i,j} := \varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \longrightarrow \varphi_i(U_i \cap U_j)$ are differentiable of class C^k for some $k \in \mathbb{N}$.

We can establish an equivalence relationship in the set of atlases of M, calling two atlases (C^k-)equivalent if their union is again an atlas of M. An equivalence class of C^k-atlases of M will be called a C^k-structure on M.

Definition 1.6 (Smooth manifold). *A smooth manifold is a manifold with an infinitely differentiable C^∞-structure.*

In essence, it is a topological space such that every neighborhood of a point is locally homeomorphic to an open set in $\mathbb{R}^n$ such that smoothness properties of the atlases make possible for us to perform calculus in the manifold.

Now let us take a manifold M and $p \in M$ and consider the space T_pM, i.e., the space tangent to M at the point p.

Definition 1.7 (Tangent bundle). *The tangent bundle TM is defined as the disjoint union*

$$TM := \bigsqcup_{p \in M} T_pM.$$

We can characterize the elements of TM as (v, p) or (p, v) where $v \in T_pM$ is the vector and $p \in M$ is the point whose tangent space contains v. This coordinates on TM allow for the local identification with $M \times \mathbb{R}^k$. The tangent bundle comes naturally equipped with a continuous surjective projection map $\pi : TM \to M$, $\pi(p, v) = p$.

In the context of $\mathbb{R}^n$, a vector field on an open subset U is a continuous map from U to $\mathbb{R}^n$. We can extend this notion to smooth manifolds through the tangent bundle:

Definition 1.8 (Vector field)**.** *A vector field on a smooth manifold M is a continuous map $X : M \to TM$, assigning to each point $p \in M$ a tangent vector $X(p)$ in the tangent space T_pM.*

We can generalize the concept of the tangent bundle into what is called a real *vector bundle*:

Definition 1.9 (Vector bundle)**.** *A vector bundle is a topological space V together with a surjective continuous map $\pi : V \to M$ such that*

1. *For each $p \in M$, the so-called fiber $\pi^{-1}(p)$ is isomorphic to a k-dimensional real vector space, $k \in \mathbb{N}$.*

2. *For each $p \in M$ there is a neighborhood U_p of p and a homeomorphism $\varphi : \pi^{-1}(U_p) \to \mathbb{R}^k$ such that $p \circ \varphi = \pi$, ($p : U \times \mathbb{R}^k \to U$ projection) and for each $q \in U_p$, $\varphi(\pi^{-1}(q)) \cong \{q\} \times \mathbb{R}^k \cong \mathbb{R}^k$.*

The idea is to map each point of the manifold M to a vector space –a *fiber*– and to specify how these fibers relate to each other, how to '*sew*' them together, in order words, how these fibers relate to nearby points.

Definition 1.10 (Section)**.** *Let $\mathscr{V}$ be a vector bundle on M and let $\pi : \mathscr{V} \to M$ be the associated surjective continuous map. A (smooth) section L on $\mathscr{V}$ is a (smooth) continuous map $L : M \to \mathscr{V}$ such that $\pi \circ L = Id_M$. Likewise, a local section is a continuous map $L : U \to \mathscr{V}$ for $U \subset M$ open such that $\pi \circ L = Id_U$. We will denote by $\Gamma(U, \mathscr{V})$ the set of all local sections on $\mathscr{V}$ with U as a domain.*

Now suppose that V is a real vector space, we can complexify the space:

$$\mathbb{C} \otimes V := \{\sum_{k=1}^{N} c_k v_k : N \in \mathbb{N}, c_k \in \mathbb{C} \text{ and } v_k \in V\}.$$

We can complexify the tangent bundle to obtain the complexified tangent bundle:

$$\mathbb{C}TM := \mathbb{C} \otimes TM = \bigcup_{p \in M} \mathbb{C} \otimes T_pM.$$

Definition 1.11 (Subbundle)**.** *We will call a subbundle a subset that is a collection of vector spaces which is a bundle of its own. In the case $\mathscr{V} \subset \mathbb{C}TM$, $\mathscr{V}$ is a subcollection of subspaces $V_p \subset T_pM$ with $p \in M$.*

We will call the complex dimension the dimension with respect to $\mathbb{C}$, we can denote it by $\dim_{\mathbb{C}}$, while the dimension as a vector space over the real numbers is going to be denoted by dim. For example, $\dim_{\mathbb{C}} \mathbb{C} = 1$, whilst $\dim \mathbb{C} = 2$.

Definition 1.12 (Smooth function)**.** *Let M is a smooth manifold and $f : M \to R^k$, $k \in \mathbb{N}$ is any function. We say that f is a smooth function if for every $p \in M$ there exists a smooth chart (U, φ) for M whose domain contains p and such that $f \circ \varphi^{-1}$ is smooth on $\varphi(U) \subset \mathbb{R}^n$, $n \in \mathbb{N}$ in the usual sense. We can use $C^\infty(M)$ to denote the set of smooth functions on M or $C^\infty(U)$ if $U \subset M$ open.*

Lastly,

Definition 1.13 (Lie bracket)**.** *Take $f : M \to \mathbb{R}^k$ a smooth function on M and X a vector field also on M. We can define a new function via $(Xf)(p) = X(p)f$. For X and Y vector fields on M we will denote by $[X,Y]$ the Lie bracket, defined as $[X,Y]f = XYf - YXf$, or simply $[X,Y] = XY - YX$. $[X,Y]$ is also a vector field.*

We are now ready to define smooth CR manifolds and CR functions, following [14].

Definition 1.14 (CR manifold)**.** *A CR manifold is a smooth manifold M of dimension $2n+d$, $n,d \in \mathbb{N}$ together with a subbundle $\mathscr{V} \subset \mathbb{C}TM$ such that $\dim_{\mathbb{C}} \mathscr{V} = n$ and $[\mathscr{V},\mathscr{V}] \subset \mathscr{V}$. In order to talk about a CR manifold we would require some notation like $(M,\mathscr{V})$, but we can simplify it to M, with $\mathscr{V}$ being implicit or deduced by context. We will also refer to n as the CR-dimension of M.*

Definition 1.15 (CR function)**.** *Let $U \subset M$ be open. A CR function on U is a smooth function $f \in C^\infty(U)$ such that $Lf = 0$ for every smooth (local) section $L \in \Gamma(U,\mathscr{V})$.*

It is this next definition of integrability the one that can relate CR manifolds to the previous notions of the Borel's theorem and PDEs.

Definition 1.16 (Integrability)**.** *A CR structure $\mathscr{V}$ is integrable if for every $p \in M$ there exists a neighborhood U of p and a basic family of solutions at p, this is, a collection of CR functions $Z_1,...,Z_{n+d}$ with linearly independent derivatives on U, satisfying $LZ_j = 0$ for all smooth sections of $\mathscr{V}$ over U and $Z_j(p) = 0$, $j = 1,...,n+d$.*

The question is whether we can relate any smooth CR function with a formal power series in $\mathbb{C}[\![Z_1,...,Z_{n+d}]\!]$ through some kind of "Borel map"

$$T : C^\infty(M) \longrightarrow \mathbb{C}[\![Z_1,...,Z_{n+d}]\!]$$

and whether this function satisfies a kind of Borel's theorem in this CR setting. We can actually start by proposing a candidate for this "Borel function". Denote $C_p^\infty(U)$ as the ring of smooth germs on p $(p \in U)$; a germ being an equivalence class to which two functions belong if they coincide in some neighborhood of p. We can then define a ring of CR smooth germs:

$$\mathfrak{S}_p := \{f \in C_p^\infty(M) : \text{f is a CR function}\}$$

$$\mathfrak{S}_p := \{f \in C_p^\infty(M) : Lf = 0, \forall L \in \Gamma(U,\mathscr{V})\}$$

We now establish an equivalence relation on $\mathfrak{S}_p$: $f \sim_k g$ whenever $f - g$ vanishes with order k at p. We will denote the sets of equivalence classes as $J_p^k(S)$, these are ring of k-jets of solutions at p. We can establish a quotient map $\pi_p^k : \mathfrak{S}_p \to J_p^k(S)$. Moreover $J_p^k(S) \subset J_p^{k+1}(S)$ and therefore we can establish

$$J_p^\infty(S) = \lim_{k\to\infty} J_p^k(S)$$

as an inductive limit. There also is a natural quotient map

$$j_p^\infty : \mathfrak{S}_p \longrightarrow J_p^\infty(S)$$

whose kernel are CR functions who vanish at p with infinite order, which are call *'flat'*. If a basic family of solutions $Z_1,...,Z_{n+d}$ exists at p, then $J_p^k(S)$ can be identified with the space of polynomials of degree at most k in the variables $Z_1,...,Z_{n+d}$ and hence $J_p^\infty(S)$ with the respective space of formal power series. Taking of all this into account, we reach a setting in the CR context that deeply resembles the classical Borel's theorem setting with a "Borel function" given by

$$j_p^\infty : \mathfrak{S}_p \longrightarrow J_p^\infty(S) \cong \mathbb{C}[\![Z_1,...,Z_{n+d}]\!]$$

We may say that a CR manifold M satisfies the Borel property at p if $j_p^\infty : \mathfrak{S}_p \longrightarrow J_p^\infty(S)$ is surjective.

We will, however, not study this question directly in the spaces $\mathfrak{S}_p$ and $J_p^\infty(S)$, but in spaces that are isomorphic to those two. For a compact set $K \subset \mathbb{C}^n$ such that $p \in K$ we will relate the space $\mathfrak{S}_p(K)$

$$\mathfrak{S}_p(K) := \{f \in C_p^\infty(K) : \text{f is a CR function}\}$$

with $A^\infty(K)$ through the so-called Baouendi-Treves approximation theorem, whose proof and broader context can be found at [2, p. 52].

Theorem 1.4 (Baouendi-Treves approximation theorem)**.** *Let M be a smooth manifold and $\mathcal{V}$ an integrable subbundle. Let $p \in M$ and let $Z = (Z_1,...Z_{n+d})$ be the basic family of solutions near p. Then there exists a compact neighborhood K of p, such that for any continuous solution f, there exists a sequence p_j of polynomials in $n+d$ variables such that*

$$p_j(Z_1,...Z_{n+d}) \longrightarrow f \textit{ uniformly in K.}$$

Through this theorem and its consequences we can establish an isomorphism between $\mathfrak{S}_p(K)$ and $A^\infty(K)$. Also, as mentioned before we can relate $J_p^\infty(K)$ and $\mathbb{C}[\![Z_1,...,Z_{n+d}]\!]$ as well. We can obtain the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{S}_p(K) & \xrightarrow{\quad j_p^\infty \quad} & J_p\infty(K) \\ \downarrow & & \downarrow \\ A^\infty(K) & \xrightarrow{\mathfrak{b}_{K,p}} & \mathbb{C}[\![Z_1,Z_2,...,Z_{n+d}]\!] \end{array}$$

The space $A^\infty(K)$ –a Fréchet algebra– will be dissected itself further on in the next section. However, apart from that, there are some details here that should be addressed as well: perhaps the definition of $\mathfrak{S}_p(K)$ is not clear when K is compact or more information about the isomorphism between the previous space and $A^\infty(K)$ should be given. Although it is known that this works, the exact details are still to be written down more specifically and are the object of a future work of my thesis supervisor, Bernhard Lamel. In any case, the commutativity of the previous diagram makes it equivalent to study the properties of $\mathfrak{b}_{K,p}$ instead of j_p^∞.

As previously mentioned, the definitions and further details on these concepts of smooth manifolds can be found at [12] and [16]. An exposition on the essentials of CR manifolds that I use here and more on the Borel map in the CR setting can be found at [14].

1.2.2 The space $A^\infty(K)$

As previously mentioned, to study the Borel property we will pose the problem in terms of the space $A^\infty(K)$. We can start by talking about its underlying structure as a Fréchet space, starting from the bottom up. We will mostly follow [7].

Definition 1.17 (Topological vector space)**.** *A topological vector space is a vector space X with a topology such that the addition and the scalar product are continuous.*

Definition 1.18 (Locally Convex Space)**.** *A locally convex space is a topological vector space whose topology is induced by a family of seminorms $\mathscr{P}$ such that*

$$\bigcap_{p\in\mathscr{P}} \{x : p(x) = 0\} = \{0\}.$$

The condition on the family of seminorms in the previous definition is chosen to make the topology of the space Hausdorff, i.e., for any two distinct point in the space there exist disjoint neighborhoods that separate them (each neighborhood contains one point and not the other).

The topology induced by the seminorms comes from this characterization of open sets: a subset $U \subset X$ is open if and only if for every $x_0 \in U$ there are $p_1,\ldots,p_n \in \mathscr{P}$ and $\varepsilon_1,\ldots\varepsilon_n > 0$ such that

$$\bigcap_{j=1}^{n} \{x \in X : p_j(x-x_0) < \varepsilon_j\} \subset U.$$

Proposition 1.1. *Let $\{p_1, p_2,\ldots\}$ be a sequence of seminorms on X such that $\cap_{p\in\mathscr{P}}\{x : p(x) = 0\} = \{0\}$. For x and y in X, define*

$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(x-y)}{1+p_n(x-y)}.$$

Then d is a metric on X and the topology defined by d is the same as the topology defined by the seminorms.

Definition 1.19 (Fréchet Space)**.** *A Fréchet space is a (locally convex) space whose topology is defined by a countable family of seminorms which is complete as a metric space, i.e. (X, d) is complete, d being the metric defined by proposition 1.1.*

There are multiple ways of defining a Fréchet space, some of them require including local convexity and some do not, since local convexity is deduced from the axioms. In the previous definition, locally convex is in parenthesis because in the definition of Frechét space I am proposing local convexity is derived from its other properties. In this case, the fact that it is Hausdorff, the fact that the seminorms conform a countable set and completeness are sufficient to prove that the space is locally convex. Given that it is common to refer to Fréchet spaces as locally convex, including 'locally convex' in the definition from above is merely an aesthetic choice.

However, it should also be mentioned that many authors do not actually require local convexity in the definition of a Fréchet space in any cirmcumstance, meaning that they do not require local convexity in definitions of a Fréchet space in which local convexity is not be deduced from the other properties of the space. Be that as it may, our definition of Fréchet spaces does not require to mention local convexity, but since the space is locally convex anyways we are going to mention it in parenthesis.

Lastly we need to define what a Fréchet algebra is

Definition 1.20 (Fréchet Algebra)**.** *A Fréchet algebra is a Frechét space along with an associative continuous multiplication operation, i.e., a bilinear map* $\cdot : X \times X \longrightarrow X$*, such that it makes the space into a ring.*

In essence, Fréchet algebras are generalizations of Banach algebras.

These concepts and more can be found at [7]. Now we can move to study the particular Fréchet algebra we will use, as presented in [8, p. 3-4]. We will denote by $A[\![X]\!]$ the set of formal power series -possibly infinite series with no concern for convergence- with coefficients in A.

Let $K \subset \mathbb{C}$ be a compact subset of the complex plane, then $C(K)[\![X]\!]$ denotes the space of formal power series with coefficients in the space of continuous functions on K. It is a Fréchet algebra with its topology induced by these seminorms:

$$U = \sum_{j=0}^{\infty} u_j X^j \longrightarrow p_n(U) := \sum_{j=0}^{n} \sup_K |u_j|, \qquad n \in \mathbb{N}_0 \left(= \mathbb{N} \bigcup \{0\}\right).$$

Definition 1.21 ($A^\infty(K)$)**.** *Let K be a compact subset of the complex plane. We will define* $A^\infty(K)$ *as the closure in* $C(K)[\![X]\!]$ *of the following subalgebra*

$$H(K) = \left\{ F = \sum_{j=0}^{\infty} \frac{f^{(j)}|_K}{j!} X^j \in C(K)[\![X]\!] : f \in \mathscr{O}(\mathbb{C}) \right\}.$$

Notice that $A^\infty(K)$ *is a closed subalgebra of* $C(K)[\![X]\!]$ *and therefore it is also a Fréchet algebra.*

Remark 1.1. *Given* $X \subseteq \mathbb{C}$ *we will denote by* $\mathscr{O}(X)$

$$\mathscr{O}(X) := \bigcap_{X \subset U \subset \mathbb{C} \text{ open}} \mathscr{H}(U),$$

the set of all functions 'holomorphic on $X \bigcup \delta X$'. In the case $\mathscr{O}(\mathbb{C})$ we are simply referring to entire functions.

1.3 The Borel map

Recall the commutative diagram, now in one dimension

$$\begin{array}{ccc} \mathfrak{S}_p(K) & \xrightarrow{j_p^\infty} & J_p^\infty(K) \\ \downarrow & & \downarrow \\ A^\infty(K) & \xrightarrow{\mathfrak{b}_{K,p}} & \mathbb{C}[\![X]\!] \end{array}$$

We will focus on $p = 0$, $0 \in K$. Still following [8] we can now give a name to the map mapping $A^\infty(K)$ to $\mathbb{C}[\![X]\!]$:

Definition 1.22. *Let $K \subset \mathbb{C}$ compact and $0 \in K$. We will denote as the Borel Map at the origin*

$$\mathfrak{b}_K : A^\infty(K) \longrightarrow \mathbb{C}[\![X]\!]$$

$$\mathfrak{b}_K(F) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} X^n.$$

This is essentially the classical Taylor expansion but with a different domain space.

Remark 1.2. *If $L \subset K \subset \mathbb{C}$ are compact sets, both containing the origin, then there is a natural algebra homomorphism $r_{K,L} : A^\infty(K) \longrightarrow A^\infty(L)$ induced by the restriction of functions and $\mathfrak{b}_K = \mathfrak{b}_L \circ r_{K,L}$.*

Polynomial convexity will be central to the study of this Borel map, as it will appear in almost every result, starting with the next remark 1.3.

Definition 1.23 (Polynomial hull)**.** *Let $K \subset \mathbb{C}^m$. The polynomial hull of K is defined as*

$$\hat{K} = \{z \in \mathbb{C}^m : |P(z)| \leq \sup_{q \in K} |P(q)| \; \textit{for all } P \in \mathbb{C}[z_1, \ldots, z_m]\}.$$

More intuitively, if we think in $\mathbb{C}$ we can see the polynomial hull $\hat{K}$ as 'filling the holes' of K:

$$\hat{K} = K \bigcup_{j \in \mathbb{N}} K_j,$$

where these K_j are the 'holes' of the set K (bounded connected components of K^c).

Definition 1.24 (Polynomial convexity)**.** *If $\hat{K} = K$ one says that K is polynomially convex.*

Remark 1.3. *We know that the Borel map at the origin is never bijective since $\mathbb{C}[\![X]\!]$ is a local algebra and $A^\infty(K)$ is not [8, subsection 2.2].*

We can also claim that whenever the origin lies in the interior of $\hat{K}$, $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} X^n$ is a convergent power series and therefore it is injective but not surjective.

1.4 Previous approaches. Subanalyticity and path-boundedness

The main objective of [8] was to give a complete solution to the question of the Borel property in the previous setting. This was achieved in the form of a comprehensive theorem with the set K under a geometric condition that was called path-boundedness.

Definition 1.25 (Path-boundedness)**.** *We say that a compact set $K \subset \mathbb{C}$ is path-bounded if there exists $C > 0$ such that for any neighborhood U of K and any points $p, q \in K$ there exists a rectifiable curve $\gamma \subset U$ of length at most C joining p and q.*

Remark 1.4. *K is path bounded if and only if $\exists C > 0$ such that $\forall p, q \in K$ $\exists \eta : [0,1] \longrightarrow K$ with $l(\eta) \leq C$. Where l denotes the length.*

Under the path-boundedness condition it was then possible to prove this strong result:

Theorem 1.5. *Let $K \subset \mathbb{C}$ be a path-bounded compact set with $p \in \hat{K}$. Then one and only one of the following properties holds:*

- $\mathfrak{b}_{K,p}$ *is surjective;*
- $\mathfrak{b}_{K,p}$ *is injective and $\hat{K}$ is a neighborhood of p.*

Before the proof of the previous theorem 1.5 it was noted that all subanalytic sets are path-bounded and it was deemed appropriate to offer a proof in the subanalytic case. Both in the case of path-boundedness and subanalyticity the following condition for surjectivity of the Borel map was used:

Theorem 1.6 (Condition for surjectivity of the Borel map)**.** *Let $K \subset \mathbb{C}$ compact and $0 \in K$. $\mathfrak{b}_K$ is not surjective if and only if there are a sequence $\{P_j\} \subset \mathbb{C}[X]$ and $M \in \mathbb{Z}_+$ such that $degP_j \to \infty$ and*

$$|(P_j(\partial)h)(0)| \leq \sum_{k=0}^{M} \sup_{K} |h^{(k)}| = \sum_{k=0}^{M} \sup_{\hat{K}} |h^{(k)}|,$$

for $h \in \mathscr{O}(\hat{K})$ and $j = 1, 2, \ldots$.

Remark 1.5. *Given a polynomial $P \in \mathbb{C}[X]$ of degree $n \in \mathbb{N}_0$ we can write P as $P(X) = \sum_{k=0}^{n} A_k X^k$. Then we can write in operator notation*

$$P(\partial) = \sum_{k=0}^{n} A_k \partial^k.$$

Remark 1.6. *Note that from theorem 1.6 it also follows that the Borel map is not surjective if $0 \in K^\circ$.*

We begin by taking a brief excursus into subanalyticity aided by [5]. In order to define subanalyticity we will first define semianalytic sets. Given R a ring of real-valued functions on a set D and

$$S_0(R) := \{E \subset D : \exists f \in R, \, E = \{x \in X : f(x) > 0\}\}$$

we will denote by $S(R)$ the smallest collection of subsets of X containing $S_0(R)$ which is stable under finite intersections, finite unions and complements.

Let M be a real analytic manifold and $U \subset M$ open, then we will write $\mathscr{H}_{\mathbb{R}}(U)$ the ring of real analytic functions on U.

Definition 1.26 (Semianalytic set)**.** *A subset X of M is semianalytic if each $p \in M$ has a neighbourhood U such that $X \cap U \in S(\mathscr{H}_{\mathbb{R}}(U))$.*

In other words, a subset X is subanalytic if it can be locally 'described' in terms of family of sets generated by real analytic functions.

Definition 1.27 (Subanalytic set)**.** *A subset X of M is subanalytic if for each $p \in M$ there exists a real analytic manifold N and a relatively compact semianalytic subset A of $M \times N$ such that $X \cap U = \pi(A)$, where $\pi : M \times N \to M$ is the projection onto M.*

It can be checked that subanalyticity is preserved under finite unions, finite intersections or differences; that the closure of a subanalytic set is subanalytic, as well as the interior and the complement. Lastly, every connected component will also be subanalytic.

One property of subanalytic sets is relevant for a the proof in the subanalytic case: the existence of a real-analytic function with certain properties [11, p. 482].

Proposition 1.2. *Let A be a subanalytic subset of a real-analytic space M. Let $p \in \overline{A}$, where $\overline{A}$ is the closure of A in $|M|$ (the underlying topological space of M, the elements of the topological space devoid of any further structure). Then there exists a real analytic map $\gamma : (-1,1) \to M$ such that $\gamma(0) = p$ and $\gamma((-1,1) \setminus \{0\}) \subset A$.*

For further reading on subanalytic sets, including alternative definitions or proofs of properties I refer the reader to [5] and [11]. Finally in [8] it was proven that:

Theorem 1.7. *Assume $K \subset \mathbb{C}$ is compact and subanalytic, that $0 \in K$ and $\mathfrak{b}_K$ is not surjective. Then $\hat{K}$ is a neighborhood of the origin.*

The proof is done by contradiction. We assume that $0 \in \delta\hat{K}$, by virtue of theorem 1.6 we know there are a sequence of polynomials and $M \in \mathbb{N}_0$ such that

$$|(P_j(\partial)h)(0)| \leq \sum_{k=0}^{M} \sup_K |h^{(k)}|. \tag{1.1}$$

Proposition 1.2 ensures the existence a certain function $\gamma : (-1,1) \to \mathbb{C}$, with which we can define a family of functions $h_s(z) = (\gamma(s) - z)^{-1}$. When we plug in $h_s(z)$ in equation 1.1 we get a contradiction after some manipulation and the application of the Łojasiewicz inequality. Therefore the point 0 must be in the interior of $\hat{K}$.

Adding to this theorem the knowledge we have about injectivity of the Borel map in the interior 1.3 they concluded with the following result:

Corollary 1.1. *Let $K \subset \mathbb{C}$ be compact and subanalytic with $0 \in K$. Then one and only one of the properties hold:*

- $\mathfrak{b}_K$ *is surjective;*
- $\mathfrak{b}_K$ *is injective and $\hat{K}$ is a neighborhood of* 0.

The two key elements of the proof were the polynomials from theorem 1.6 and the continuous family of functions $h_s(z)$, that 'explodes' at $s = z = 0$. It is natural to imagine that to find a different approach it could suffice to find a sequence of functions f_n in the place of the previous h_s. It is here where path-boundedness comes into play, as a geometrical requirement to find a sequence with as many uniformally bounded derivatives as we desire that explodes at the point 0 after some derivative of sufficiently high order. For that reason the following result was proven:

Proposition 1.3. *Assume that K is a path-bounded polynomially convex compact set and that the origin belongs to the boundary of K. Given $m \in \mathbb{N}$, there exists a sequence $\{f_n\} \in \mathcal{O}(K)$ such that*

$$\sup_{z \in K} |f_n^{(j)}(z)| < 1, \qquad j < m,$$

but $|f_n^{(m)}(0)| \to \infty$ as $n \to \infty$.

Lastly, since the results require the presence of the polynomial convex hull, the following was noted and proved:

Lemma 1.1. *If K is path-bounded then the same is true for $\hat{K}$.*

The proof of the theorem follows the same logical structure of the previous specific case for subanalyticity: one assumes that 0 is in the boundary and the polynomials from theorem 1.6 together with the sequence of functions from proposition 1.3 yield a contradiction that proves the point.

Theorem 1.8. *Assume that K is a path-bounded compact set, that $0 \in K$ and and assume also that $\mathfrak{b}_K$ is not surjective. Then $\hat{K}$ is a neighborhood of the origin.*

Corollary 1.2. *Let $K \subset \mathbb{C}$ be path-bounded compact with $0 \in K$. Then one and only one of the properties hold:*

- $\mathfrak{b}_K$ *is surjective;*
- $\mathfrak{b}_K$ *is injective and $\hat{K}$ is a neighborhood of* 0.

The next natural question is, as already mentioned, whether path-boundedness is strictly necessary to reach the same conclusions. In particular one could ask whether path-boundedness is necessary to find a sequence like in 1.3. It is the purpose of this master thesis to try to obtain a more generalized (or alternative) version of the previous theorem and corollary, where path-boundedness as a precondition has been eliminated or substituted by a weaker, more general assumption; therefore making the result stronger. To deepen into the details of these last theorems and to acquaint oneself with their whole setting, one can of course read the paper that originated this work [8].

Chapter 2

Proposed solutions

In light of the previous work done on the matter –as expounded in the end of the previous chapter–, the strategy is going to be the following: for any $m \in \mathbb{N}$ we will find a sequence of functions $\{f_n\} \subset \mathscr{O}(\mathbb{C})$ that satisfies

$$\sup_{z \in K} |f_n^{(j)}(z)| < 1, \qquad \forall n \in \mathbb{N}, j < m,$$

but $|f_n^{(m)}(0)| \to \infty$ as $n \to \infty$.

Analogous to what has been previously done, we will prove the result by contradiction, assuming that $0 \in \delta\hat{K}$. Then, by virtue of theorem 1.6 we can ensure the existence of a sequence $\{P_j\} \subset \mathbb{C}[X]$ and $M \in \mathbb{Z}_+$ such that $degP_j \to \infty$ and

$$|(P_j(\partial)h)(0)| \leq \sum_{k=0}^{M} \sup_{K} |h^{(k)}| = \sum_{k=0}^{M} \sup_{\hat{K}} |h^{(k)}|.$$

This polynomials, used as differential operator, allow us to bound the value of all derivatives with the value of just the first M derivatives. Then, if we plug in our sequence of functions f_n, which explodes at 0 at the m'th derivative (and we could chose $m = M$ or $m = M+1$ as we find fitting) we reach a contradiction that implies that 0 must necessarily lie on the interior.

The candidate for a sequence of functions with these properties was something in the form

$$f_n(z) := B_n \frac{1}{z - z_n},$$

where $z_n \to 0$ and B_n are constants chosen specifically to control the supreme value of the first m derivatives of the functions. From this protosolution I was able to obtain two kinds of sequences: one in the actual form

$$f_n(z) := B_n \frac{1}{z - z_n},$$

where this B_n will be related to the distance function and also another sequence that makes use of branches of the complex logarithm, in the form

$$f_n(z) := \frac{1}{B_n} \frac{(z - z_n)^m}{m!} \left(\log_n(z - z_n) - \sum_{k=1}^{m} \frac{1}{k} \right),$$

which is essentially derived from integration of $\frac{1}{z-z_n}$. Both require further conditions on the set K or its boundary.

2.1 Solution with distance function condition

Let K be a compact set in the complex plane, we will denote as $d(\cdot,K) : K \to [0,\infty)$ the distance function to K. It will be useful to point out the following:

Property 2.1. *Let K be a compact set in the complex plane and $0 \in \partial K$, then $d(z,K) \leq |z|$. It therefore also holds that*

$$0 \leq \frac{d(z,K)}{|z|} \leq 1.$$

Proof. Since $0 \in \partial K \subset K$, $0 \leq d(z,K) \leq d(z,0) = |z|$, which means $0 \leq \frac{d(z,K)}{|z|} \leq 1$. □

Now we can introduce the geometric condition we will impose on the point 0 in order to be able to get the sequence of functions we want to find.

Definition 2.1. *Let K be a compact set with 0 on the boundary. We will say that 0 is properly situated if there exists a sequence $\{z_n\} \subset K^c$, $z_n \longrightarrow 0$ as $n \longrightarrow \infty$ such that*

$$\lim_{n\to\infty} \frac{d(z_n,K)}{|z_n|} > 0.$$

We can generalize this notion to other points distinct from 0.

Definition 2.2. *Let K be a compact set and p a point on the boundary. We will say that p is properly situated if there exists a sequence $\{z_n\} \subset K^c$, $z_n \longrightarrow p$ as $n \longrightarrow \infty$ such that*

$$\lim_{n\to\infty} \frac{d(z_n,K)}{|z_n - p|} > 0.$$

In this definition we only require the existence of one sequence that satisfies the property, but one could also say

$$\lim_{z\to 0} \frac{d(z,K)}{|z|} > 0.$$

or equivalently

$$d(z,K) = \varepsilon(z)|z|, \text{ where } \lim_{z\to 0} \varepsilon(z) = c \in (0,1]$$

which means that $d(z,K)$ and $c|z|$ are asymptotically equivalent at $z = 0$.

With this property we can now craft a sequence of function that behaves as we wish.

Proposition 2.1. *Assume K polynomially convex, compact and $0 \in \delta K$ properly situated. Given $m \in \mathbb{N}$, the following sequence $\{f_n\} \in \mathscr{O}(K)$ defined by*

$$f_n(z) := B_n \cdot \frac{1}{z - z_n}$$

where

$$B_n := \frac{1}{2}\frac{d(z_n,K)^m}{(m-1)!}$$

satisfies the following properties:

$$\sup_{z\in K}|f_n^{(j)}(z)| < 1, \qquad j < m,$$

but $|f_n^{(m)}(0)| \to \infty$ *as* $n \to \infty$.

Proof. Since 0 is properly situated in K, there exists a sequence $\{z_n\} \subset K^c$, $z_n \longrightarrow 0$ as $n \longrightarrow \infty$ such that

$$\lim_{n\to\infty}\frac{d(z_n,K)}{|z_n|} > 0.$$

In particular we can choose this sequence such that $|z_n| < 1$. The sequence we are working with is defined by

$$f_n(z) := B_n \cdot \frac{1}{z-z_n}$$

with

$$B_n := \frac{1}{2}\frac{d(z_n,K)^m}{(m-1)!}.$$

Notice that since $z_n \notin K$ for every n, $f_n(z) \in \mathscr{O}(K)$. The derivatives take the form:

$$f_n^{(j)}(z) = (-1)^j j! B_n \cdot \frac{1}{(z-z_n)^{j+1}}$$

It is the same case for the derivatives, $f_n^{(j)}(z) \in \mathscr{O}(\mathbb{C})$. Now, for $j < m$:

$$\sup_K ||f_n^{(j)}(z)|| = j!B_n\frac{1}{d(z_n,K)^{j+1}} \leq (m-1)!B_n \cdot \frac{1}{d(z_n,K)^m} = \frac{1}{2}\frac{d(z_n,K)^m}{(m-1)!}\cdot\frac{(m-1)!}{d(z_n,K)^m} = \frac{1}{2} < 1.$$

Where we have used that $d(z_n,K) \leq |z_n| < 1$ in virtue of property 2.1. Therefore $\sup_K ||f_n^{(j)}(z)|| < 1$ for $j < m$.

Now for $j = m$ evaluated at 0:

$$|f_n^{(m)}(0)| = m!B_n \cdot \frac{1}{|z_n|^{m+1}} = \frac{1}{2}\frac{d(z_n,K)^m}{(m-1)!}\cdot\frac{m!}{|z_n|^{m+1}} = \frac{m}{2}\cdot\frac{d(z_n,K)^m}{|z_n|^m}\frac{1}{|z_n|}$$

Since $\lim_{n\to\infty}\frac{d(z_n,K)^m}{|z_n|^m} > 0$ by assumption we have that

$$\lim_{n\to\infty}|f_n^{(m)}(0)| = \infty.$$

So $\{f_n\}$ satisfies the aforementioned properties. □

We can even use generalize this result to any properly situated point p by using the exact same sequence:

Corollary 2.1. *Assume K polynomially convex, compact and $p \in \delta K$ properly situated. Given $m \in \mathbb{N}$, the following sequence $\{f_n\} \in \mathscr{O}(K)$ defined by*

$$f_n(z) := B_n \cdot \frac{1}{z - z_n}$$

where

$$B_n := \frac{1}{2} \frac{d(z_n, K)^m}{(m-1)!}$$

satisfies the following properties:

$$\sup_{z \in K} |f_n^{(j)}(z)| < 1, \qquad j < m$$

but $|f_n^{(m)}(p)| \to \infty$ as $n \to \infty$.

Proof. The calculations do not really change for

$$\sup_{z \in K} |f_n^{(j)}(z)| < 1, \qquad j < m,$$

and notice

$$|f_n^{(m)}(p)| = \frac{m}{2} \cdot \frac{d(z_n, K)^m}{|z_n - p|^m} \frac{1}{|z_n - p|}.$$

□

Before we move on to prove the theorem we still need to define a couple of notions: we would like to impose a condition on K in the theorem, in particular, in the boundary of K. But since in the theorem we will work with $\hat{K}$ and its boundary it is sound to impose it on the boundary that K and $\hat{K}$ share. Since $\hat{K}$ is the result of 'filling the holes' of K, I found it fitting to call it 'exterior boundary'.

Definition 2.3 (Exterior boundary)**.** *Let K be a compact set in the complex plane, then we will call $\delta\hat{K}$ the exterior boundary of K. Equivalently, if U is the unbounded component of $\hat{K}$, δU is the exterior boundary.*

Definition 2.4 (Proper exterior boundary)**.** *Let K be a compact set in the complex plane, if all points of its exterior boundary are properly situated we will say that K has a proper exterior boundary and that $\hat{K}$ has a proper boundary.*

Lastly, it will be useful to note the following lemma:

Lemma 2.1. *Let $\mu \in \mathbb{N}$ and $D_1, D_2, D_3, \ldots, D_\mu \in \mathbb{C}$ with $D_\mu \neq 0$. Then*

$$\lim_{z \to 0} \left| \sum_{k=1}^{\mu} D_k \frac{1}{z^k} \right| = \infty$$

Proof. We can rewrite

$$\sum_{k=1}^{\mu} D_k \frac{1}{z^k} = D_1 \frac{1}{z} + D_2 \frac{1}{z^2} + \ldots + D_\mu \frac{1}{z^\mu} = \frac{D_1 z^{\mu-1} + D_2 z^{\mu-2} + \ldots + D_\mu}{z^\mu}.$$

We now introduce the change of variables $h = 1/z$, therefore we can rewrite the limit as follows,

$$\lim_{z\to 0} \frac{D_1 z^{\mu-1} + D_2 z^{\mu-2} + \ldots + D_\mu}{z^\mu} = \lim_{h\to\infty} \frac{D_1 h^{1-\mu} + D_2 h^{2-\mu} + \ldots + D_\mu}{h^{-\mu}} = \lim_{h\to\infty} \frac{D_\mu}{h^{-\mu}} = \lim_{z\to 0} \frac{D_\mu}{z^\mu}$$

Then we have

$$\lim_{z\to 0} \frac{D_1 z^{\mu-1} + D_2 z^{\mu-2} + \ldots + D_\mu}{z^\mu} = \lim_{z\to 0} \frac{D_1 z^{\mu-1}}{z^\mu} = \lim_{z\to 0} \frac{D_\mu}{z^\mu} = \infty.$$

The continuity of the absolute value function allows us to arrive to the following conclusion:

$$\lim_{z\to 0} \left| \sum_{k=1}^{\mu} D_k \frac{1}{z^k} \right| = \infty.$$

□

Theorem 2.1. *Assume that K is compact with proper exterior boundary, that $0 \in K$ and that $\mathfrak{b}_K$ is not surjective. Then $\hat{K}$ is a neighborhood of the origin.*

Proof. According to Theorem 1.6 there is a sequence of polynomials $\{P_j\} \subset \mathbb{C}[X]$, with degree $P_j = \mu_j \to \infty$ and a non negative integer M such that

$$|P_j(\partial)h(0)| \leq \sum_{l\leq M} \sup_K |h^{(l)}|, \qquad h \in \mathscr{O}(\hat{K}).$$

We can express each $P_j(X)$ as follows:

$$P_j(X) = \sum_{k=0}^{\mu_j} A_{jk} X^k, \qquad A_{j,\mu_j} \neq 0.$$

Assume that $\hat{K}$ is not a neighborhood of the origin. Then $0 \in \delta\hat{K}$. Since the boundary of K (and of $\hat{K}$) is proper, 0 is properly situated and therefore we can find –according to proposition 2.1– a sequence of functions in $\mathscr{O}(\hat{K})$ such that:

$$\sup_{z\in K} |f_n^{(j)}(z)| < 1, \qquad j < M+1,$$

but $|f_n^{(M+1)}(0)| \to \infty$ as $n \to \infty$ (we have taken $m = M+1$). The elements of said sequence $\{f_n\}$ have the form:

$$f_n(z) = B_n \cdot \frac{1}{z - z_n} = \frac{1}{2} \frac{1}{M!} \frac{d(z_n, K)^{M+1}}{z - z_n}.$$

Then it follows that

$$|P_j(\partial) f_n(0)| \leq \sum_{l\leq M} \sup_K |f_n^{(l)}| < M+1.$$

Now in particular, for j sufficiently large $degP_j = \mu_j \geq M+1$ and:

$$|P_j(\partial)f_n(0)| = \left|A_{j,0}f_n(0) + A_{j,1}f_n'(0) + \ldots + A_{j,M+1}f_n^{(M+1)}(0) + \ldots + A_{j,\mu_j}f_n^{(\mu_j)}(0)\right| < M+1$$

We know that $A_{j,0}f_n(0) + A_{j,1}f_n'(0) + \ldots + A_{j,M}f_n^{(M)}(0)$ is bounded, but we will see now that the sum of the rest of the terms becomes unbounded for a sufficiently large n, i.e.,

$$\left|A_{j,M+1}f_n^{(M+1)}(0) + \ldots + A_{j,\mu_j}f_n^{(\mu_j)}(0)\right| \longrightarrow \infty \text{ as } n \to \infty.$$

If we plug in the value of each derivative of f_n at 0 we get

$$\left|A_{j,M+1}(-1)^{M+1}\frac{(M+1)!}{M!}\frac{d(z_n,K)^{M+1}}{z_n^{M+2}} + \ldots + A_{j,\mu_j}(-1)^{\mu_j}\frac{(\mu_j)!}{M!}\frac{d(z_n,K)^{M+1}}{z_n^{\mu_j+1}}\right|.$$

Rearranging things a bit we get

$$\frac{d(z_n,K)^{M+1}}{|z_n|^{M+1}}\left|A_{j,M+1}(-1)^{M+1}\frac{(M+1)!}{M!}\frac{1}{z_n} + \ldots + A_{j,\mu_j}(-1)^{\mu_j}\frac{\mu_j!}{M!}\frac{1}{z_n^{\mu_j-M}}\right|$$

Notice that since $\hat{K}$ has a proper boundary, as $n \longrightarrow \infty$

$$\frac{d(z_n,K)^{M+1}}{|z_n|^{M+1}} = c \in (0,1].$$

We still need to check the behavior of the other half as n goes to infinity,

$$\left|A_{j,M+1}(-1)^{M+1}\frac{(M+1)!}{M!}\frac{1}{z_n} + \ldots + A_{j,\mu_j}(-1)^{\mu_j}\frac{\mu_j!}{M!}\frac{1}{z_n^{\mu_j-M}}\right|.$$

Now we use lemma 2.1 with $D_k = A_{j,M+k}(-1)^{M+k}\frac{(M+k)!}{M!}$, $k = 1,\ldots,\mu - M$ and we conclude that for n big enough, or as $n \longrightarrow \infty$ it can not be bounded, a contradiction. Therefore 0 is a neighborhood of the origin. □

Organizing what we know we can write:

Theorem 2.2. *Let K be compact with proper exterior boundary and $0 \in K$. Then one and only one of the following properties hold:*

- *$\mathfrak{b}_K$ is surjective.*
- *$\mathfrak{b}_K$ is injective and $\hat{K}$ is a neighborhood of the origin.*

The previous formulation is designed to resemble theorem 1.8, but in practicality, we do not require the whole boundary to be proper, we only require information about the point 0 and whether this point is properly situated. Therefore it is only natural to offer a reformulation of the previous theorem:

Theorem 2.3. *Assume K is compact and that $0 \in \delta K$ with 0 properly situated. Then $\mathfrak{b}_K$ is surjective.*

Proof. Assume $\mathfrak{b}_K$ is not surjective, then we can repeat the argument from the previous proof. Since we will reach a contradiction, we conclude that $\mathfrak{b}_K$ is surjective. $\square$

Finally we can say:

Corollary 2.2. *Let K be compact and $0 \in K$. Then the following statements are true:*

- *$\mathfrak{b}_K$ is injective and $\hat{K}$ is a neighborhood of the origin.*
- *$0 \in \delta\hat{K}$ is properly situated and $\mathfrak{b}_K$ is surjective.*

In this case we need to have prior knowledge of the position of the point 0.

2.2 Solution with complex logarithm

In order to rigorously build the logarithm-based sequence of functions we need to be able to define analytic branches of the complex logarithm, $\log_n(z - z_n)$, where $z_n \notin K$ on K polynomially convex, compact and $0 \in \delta K$. We also need said branches to satisfy $(z - z_n)\log_n(z - z_n) \to 0$ as $n \to \infty$. To ensure those conditions we will introduce our own slighty modified version of a property that we may call *non-spiraling* that appears in [1]. We will also include a well-known result from complex analysis that gives information about the existence of logarithm branches related to simply connected sets in order to provide some context.

Definition 2.5 (Curves and Jordan arcs)**.** *A curve on the complex plane is a continuous function $\gamma : I \to \mathbb{C}$, $I \subset \mathbb{R}$. We will call it a Jordan arc if it is one-to-one, or in more geometric words, it never crosses itself. We will denote $\Gamma = \gamma(I)$.*

The following notion is ubiquitous in discussions on logarithm branches and it will be essential to recall it [13, p. 263]:

Definition 2.6 (Simply connected set)**.** *A set U in the complex plane is simply connected if one of the following three equivalent conditions holds: it has no holes, its complement has only one connected component, each closed curve in U is homotopic to a point.*

Remark 2.1. *We say that two curves or functions $\gamma_1, \gamma_2 : X \to Y$ are homotopic whenever there exists a continuous function $H : X \times [0,1] \to Y$ such that*

$$H(x, 0) = \gamma_1(x)$$

$$H(x, 1) = \gamma_2(x).$$

In other words, two curves are homotopic if one can be continuously deformed into the other.

We will now introduce the condition we require this time to make the arguments work, the definition is found in [1, p. 2], a work in which this notion also appeared, although I have changed it slightly to suit our needs.

Definition 2.7 (Non-spiraling)**.** *Let $K \subset \mathbb{C}$ compact and $p \in \delta K$, we will say that K is non-spiraling at p if there exists an one-to-one curve (a Jordan arc) γ 'connecting p and ∞' such that*

- *$\Gamma \setminus \{p\}$ is fully contained in K^c (the point p either is the endpoint or the beginning of γ).*
- *There exists an analytic branch of the complex logarithm on $\mathbb{C} \setminus \Gamma$ with its imaginary part absolutely bounded.*

'Connecting p and ∞' should be understood as precisely that, if one thinks on the Riemann sphere or as the curve starting at the point p and then going to infinity in any direction, if one thinks about it in the complex plane.

There is a difference between the definition given on [1, p. 2] and this one: here we require the curve to go to infinity in contrast to the more local property of the original definition, this is because we want the domain of the branches of the logarithm to include the set K. If it did not include this in the definition we would have to add along the way further demands on the function γ or the set K anyways in order to justify extending the domain of the branch of the logarithm to the whole set K, so we may as well do it now.

With respect to the second point of the definition 2.7: if $\mathbb{C} \setminus \Gamma$ is a simply connected set and $0 \notin \mathbb{C} \setminus \Gamma$, then there exist branches of the logarithm on $\mathbb{C}$. We can quickly see that $\mathbb{C} \setminus \Gamma$ is simply connected:

Proposition 2.2. *Let $\gamma : [0,\infty) \to \mathbb{C}$ a Jordan arc, with $\gamma(0) = \omega$, $\lim_{s\to\infty} |\gamma(s)| = \infty$ ('connecting ω and infinity'). $\Gamma = \gamma([0,\infty))$, then $\mathbb{C} \setminus \Gamma$ is open and simply connected.*

Proof. We will prove that Γ is closed. Let $p_j \in \Gamma$ $j \in \mathbb{N}$ such that $p_j \to p_0 \in \mathbb{C}$ as $j \to \infty$. Since $p_j \in \Gamma$, there exist t_j such that $p_j = f(t_j)$. Given that $\{p_j\}$ is convergent there exists a convergent subsequence t_{j_k} such that $t_{j_k} \to t_0$. From $f(t_{j_k}) = p_{j_k}$ it follows as $k \to \infty$ that $f(t_0) = p \in \Gamma$ through continuity. Therefore, Γ is closed and $\mathbb{C} \setminus \Gamma$ is open.

Since Γ only has one connected component, $\mathbb{C} \setminus \Gamma$ is simply connected (definition 2.6). $\square$

Then the following well-known result from complex analysis suffices to prove the existence of an analytic branch of the logarithm (take $f(z) = z$):

Theorem 2.4 ([17])**.** *Let $\Omega \subset \mathbb{C}$ be a simply connected open set such that $f \in \mathscr{H}(\Omega)$ and $1/f \in \mathscr{H}(\Omega)$, then there exists an analytic branch of* $\log(f)$ *on* Ω .

The same can be said about $\log(z-z_0)$ if $z_0 \notin \mathbb{C}\setminus\Gamma$.

The interesting thing about the second point in the definition 2.7 is the boundedness of the argument of the complex logarithm. Recall that for $z \in \mathbb{C}$ and suitable conditions a certain branch $\log_b$ of the complex logarithm has the form $\log_b(z) = \log|z| + i\arg(z)$, where the imaginary part is the argument of the complex number. We require the argument to be bounded in order to confidently say that $\lim_{z\to 0} z\log(z) = 0$, that the limit exists and that the function $z\log(z)$ is therefore holomorphic in a neighborhood of zero. These observations are essentially what makes the sequence of functions that I will propose further on work.

If we do not require boundedness on the argument we can find problems when dealing with spiral-like sets (hence the name non-spiraling). Imagine a compact set consisting on a self-similar spiral around 0, any Jordan arc γ going to 0 will actually have the shape of a self-similar spiral too. Since γ spirals around zero, the imaginary part of the complex logarithm branch defined on $\mathbb{C}\setminus\Gamma$ will grow (or decrease) indefinitely as we get close to 0, making it perhaps impossible to define the limit at 0 of $z\log(z)$ or to understand $z\log(z)$ as a holomorphic function around 0.

Remark 2.2. *Assume we have suitable conditions for the definition of a branch of the complex logarithm*

$$z\log_b(z) = z\log|z| + izarg(z)$$

Let's first assume $|arg(z)|$ is bounded, then

$$\lim_{z\to 0} z\log|z| + izarg(z) = \lim_{z\to 0} z\log|z|$$

Now let's make the change $z = re^{i\theta}$, were $r = |z|$ and $\theta = arg(z)$, then

$$\begin{aligned}
\lim_{z\to 0} z\log|z| &= \lim_{r\to 0^+} re^{i\theta}\log|re^{i\theta}| \\
&= \lim_{r\to 0^+} r\log(r)e^{i\theta} \\
&= \lim_{r\to 0^+} \frac{\log(r)}{r^{-1}}e^{i\theta} \\
&\overset{\text{L'H}}{=} \lim_{r\to 0^+} \frac{r^{-1}}{-r^{-2}}e^{i\theta} \\
&= \lim_{r\to 0^+} -re^{i\theta} \to 0.
\end{aligned}$$

Independently of θ. Therefore

$$\lim_{z\to 0} z\log_b(z) = 0$$

It is not hard to see that actually $\lim_{z\to 0} z^\alpha \log_b(z) = 0$ for $\alpha > 0$.

If on the other hand, if $|arg(z)| \to \infty$ as $z \to 0$ along any curve going to zero it is no longer clear that $\lim_{z\to 0} z\log|z| + izarg(z)$ exists.

We now have the tools to define branches of the logarithm. We would actually like to define a sequence of logarithm-based functions.

Proposition 2.3. *Let K be compact, $0 \in \delta K$ and K non-spiraling at 0. Let $\{z_n\} \subset \Gamma \setminus \{0\}$ be a sequence such that $z_n \to 0$ as $n \to \infty$. Then, for each n there exists an analytic branch of the logarithm $\log_n(z - z_n)$ such that $\log_n(z - z_n) \in \mathscr{O}(K)$.*

Note again that $\log_n(z - z_n) = \log|z - z_n| + iarg_n(z - z_n)$, what changes is the argument.

Proof. Take $n \in \mathbb{N}$, call γ the Jordan arc that comes from the non-spiraling condition, denote by $\gamma_n : [0,\infty) \to \Gamma_n$ the curve which results from 'truncating' the curve γ to start at z_n ($\gamma(0) = z_n$) and go to infinity. Denote $\Gamma_n := \gamma_n([0,\infty))$, we know from proposition 2.2 that $\mathbb{C} \setminus \Gamma_n$ is a simply connected set.

Since $z_n \notin \mathbb{C} \setminus \Gamma_n$ we know that $z - z_n \in \mathscr{H}(\mathbb{C} \setminus \Gamma_n)$ and $1/(z - z_n) \in \mathscr{H}(\mathbb{C} \setminus \Gamma_n)$, therefore (by Theorem 2.4) there exists an analytic branch of the logarithm $\log_n(z - z_n) \in \mathscr{H}(\mathbb{C} \setminus \Gamma_n)$. Given that $K \subset \mathbb{C} \setminus \Gamma_n$ for each n, for each n we can define $\log_n(z - z_n) \in \mathscr{O}(K)$ as the restriction to K. $\square$

Remark 2.3. *Notice that as explained in remark 2.2*

$$\lim_{z \to z_n} (z - z_n) \log_n(z - z_n) = 0.$$

Remark 2.4. *Call $\log_0(z)$ the branch of the complex logarithm defined on $\mathbb{C} \setminus \Gamma$, recall $\log_n(z - z_n)$ is the branch defined on $\mathbb{C} \setminus \Gamma_n$. We have*

$$\begin{aligned} \log_0(z) - \log_n(z - z_n) &= \log|z| + iarg_0(z) - \log|z - z_n| - iarg_n(z - z_n) \\ &= \log\left|\frac{z}{z - z_n}\right| + iarg_0(z) - iarg_n(z - z_n). \end{aligned}$$

Clearly $\log\left|\frac{z}{z - z_n}\right| \longrightarrow 0$ as $n \to \infty$. On the other hand, we know $arg_0(z)$ is bounded by the non-spiraling assumption; if it isn't the case that $\lim_{n\to\infty} arg_n(z - z_n) = arg_0(z)$, then $arg_0(z)$ and $\lim_{n\to\infty} arg_n(z - z_n)$ must differ by $2\pi k$, $k \in \mathbb{N}$ and simply adding a constant to the argument $arg_n(z - z_n) \pm 2\pi k$ will do. We will assume therefore in the rest of the thesis that $\lim_{n\to\infty} arg_n(z - z_n) = arg_0(z)$ and therefore $\lim_{n\to\infty} \log_n(z - z_n) = \log_0(z)$.

Proposition 2.4. *Let K be compact such that $0 \in \delta K$ and K is non-spiraling at 0. Then the polynomial hull $\hat{K}$ is also non-spiraling at 0 (using in fact the same curve γ).*

Proof. Take K compact and assume that γ is the Jordan arc from the non-spiraling condition. We will prove that $\Gamma \setminus \{0\} \subseteq \hat{K}^c$ so the path also works for $\hat{K}$. W.l.o.g., assume γ is parametrized as $\gamma : [0,\infty)$ with $\gamma(0) = 0$.

We know that $0 \in \delta K$ and that $\delta K = \delta\hat{K} \cup_j B_j$, where B_j is the jth connected component of the boundary of K distinct from $\delta\hat{K}$.

If $0 \in \delta K$ is in $\delta \hat{K}$ (the exterior boundary, since the polynomial hull 'fills the holes of the set') then it follows that actually $\gamma(s) \notin \hat{K}$ for $s > 0$.

If $0 \in B_j$ for some j then we have a contradiction, since we would have that $\gamma(s) \notin K$ for $s > 0$, but γ would necessarily have to cross $\delta \hat{K}$ to go to infinity. $\square$

It will be useful to prove the following lemma to make the incoming proof clearer:

Lemma 2.2. *Let $m \in \mathbb{N}$, $\Omega \subset \mathbb{C}$ and $z_0 \in \Omega$ such that we can define an analytic branch of* $\log(z - z_0)$ *on* $\mathbb{C} \setminus \Omega$. *Then it is true for the following function*

$$f(z) := \frac{(z-z_0)^m}{m!} \left(\log(z-z_0) - \sum_{k=1}^{m} \frac{1}{k} \right)$$

that $f^{(m)}(z) = \log(z-z_0)$ *and moreover*

$$f^{(j)}(z) = \frac{(z-z_0)^{m-j}}{(m-j)!} \left(\log(z-z_0) - \sum_{k=1}^{m-j} \frac{1}{k} \right).$$

Proof. Let's use the following notation

$$f_m(z) := \frac{(z-z_0)^m}{m!} \left(\log(z-z_0) - \sum_{k=1}^{m} \frac{1}{k} \right)$$

We will prove it by induction. Let's start with the simple case $m = 1$. Then f looks like

$$(f(z) =) \, f_1(z) = (z-z_0)\,(\log(z-z_0) - 1)$$

We want to check that $\left(f^{(m)}(z) =\right) f_1'(z) = \log(z-z_0)$. Differentiating yields

$$f_1'(z) = \log(z-z_0) + (z-z_0) \cdot \frac{1}{z-z_0} - 1 = \log(z-z_0) + 1 - 1 = \log(z-z_0)$$

Now assume it is true for a certain m, let's see it also holds for m+1. The function for m+1 has the following form

$$(f(z) =) f_{m+1}(z) = \frac{(z-z_0)^{m+1}}{(m+1!)} \left(\log(z-z_0) - \sum_{k=1}^{m+1} \frac{1}{k} \right)$$

We want to check that $f_{m+1}^{(m+1)}(z) = \log(z-z_0)$. If we differentiate once we obtain

$$\begin{aligned}
f_{m+1}'(z) &= \frac{(z-z_0)^m}{m!} \log(z-z_0) + \frac{(z-z_0)^m}{(m+1)!} - \left(\sum_{k=1}^{m+1} \frac{1}{k} \right) \frac{(z-z_0)^m}{m!} = \\
&= \frac{(z-z_0)^m}{m!} \log(z-z_0) + \frac{(z-z_0)^m}{(m+1)!} - \frac{(z-z_0)^m}{(m+1)!} - \left(\sum_{k=1}^{m} \frac{1}{k} \right) \frac{(z-z_0)^m}{m!} = \\
&= \frac{(z-z_0)^m}{m!} \log(z-z_0) - \left(\sum_{k=1}^{m} \frac{1}{k} \right) \frac{(z-z_0)^m}{m!} = \\
&= \frac{(z-z_0)^m}{m!} \left(\log(z-z_0) - \sum_{k=1}^{m} \frac{1}{k} \right).
\end{aligned}$$

Notice that $f'_{m+1}(z) = f_m(z)$, by the induction assumption, we know that $f_m^{(m)}(z) = \log(z - z_0)$, therefore it follows that

$$f_{m+1}^{(m+1)}(z) = f_m^{(m)}(z) = \log(z - z_0).$$

We conclude that it is therefore true for any $m \in \mathbb{N}$.

Since we have that $f'_{m+1}(z) = f_m(z)$ for every $m \in \mathbb{N}$ further differentiating yields that for $m \geq 1$ and $0 \leq j < m$ we have

$$f_m^{(j)} = f_{m-j}.$$

Therefore $f_m^{(m)}(z) = \log(z - z_0)$ and

$$f_m^{(j)}(z) = \frac{(z - z_0)^{m-j}}{(m-j)!} \left(\log(z - z_0) - \sum_{k=1}^{m-j} \frac{1}{k} \right).$$

□

Proposition 2.5. *Assume that K is polynomially convex, compact, $0 \in \delta K$ and that K is non-spiraling at 0. Given $m \in \mathbb{N}$ and a sequence $\{z_n\}_{n=0} \subset \Gamma \setminus \{0\}$ such that $z_n \longrightarrow 0$, the following sequence of functions*

$$f_n(z) := \frac{1}{B_n} \frac{(z - z_n)^m}{m!} \left(\log_n(z - z_n) - \sum_{k=1}^{m} \frac{1}{k} \right) \in \mathscr{O}(K)$$

where

$$B_n := \max_{0 \leq j < m} B_n^{(j)} + 1$$

and

$$B_n^{(j)} = \max_{0 \leq j < m} \frac{1}{(m-j)!} \sup_K |z - z_n|^{m-j} \left(|\log_n(z - z_n)| + \sum_{k=1}^{m-j} \frac{1}{k} \right),$$

satisfies the following:

$$\sup_{z \in K} |f_n^{(j)}(z)| < 1, \qquad \forall n \in \mathbb{N}, j < m,$$

but $|f_n^{(m)}(0)| \to \infty$ as $n \to \infty$.

Proof. Pick a sequence $\{z_n\}_{n=0} \subset \Gamma \setminus \{0\}$ such that $z_n \longrightarrow 0$. By proposition 2.3 we can define for each n an analytic branch of the logarithm $\log_n(z - z_n)$.

Set

$$f_n(z) := \frac{1}{B_n} \frac{(z - z_n)^m}{m!} \left(\log_n(z - z_n) - \sum_{k=1}^{m} \frac{1}{k} \right) \in \mathscr{O}(K)$$

Where

$$B_n := \max_{0 \leq j < m} B_n^{(j)} + 1$$

and

$$B_n^{(j)} = \max_{0 \leq j < m} \frac{1}{(m-j)!} \sup_K |z - z_n|^{m-j} \left(|\log_n(z - z_n)| + \sum_{k=1}^{m-j} \frac{1}{k} \right).$$

With regards to the properties of the B_n's notice that $B_n < \infty$ since K is compact: the boundedness of K yields $\sup_K |z-z_n|^{m-j} < \infty$ for any n and since $\lim_{z\to z_n}(z-z_n)^{m-j}\log_n(z-z_n) = 0$ (recall remarks 2.3, 2.2), we also have that $\sup_K |z-z_n|^{m-j}|\log_n(z-z_n)| < \infty$ for any n.

It is also relevant to note that:

$$\lim_{n\to\infty} B_n = \frac{1}{(m-j_0)!}\sup_K |z|^{m-j_0}\left(|\log_0(z)| + \sum_{k=1}^{m-j_0}\frac{1}{k}\right) + 1 > 0$$

where we have used that $\log_n(z-z_n) \to \log_0(z)$, (remark 2.4) and j_0 is such that $B_n = B_n^{(j_0)} + 1$.

In virtue of the previous lemma 2.2 we can see that the derivatives for $0 \le j < m$ take the form:

$$f_n^{(j)}(z) = \frac{1}{B_n}\frac{(z-z_n)^{m-j}}{(m-j)!}\left(\log_n(z-z_n) - \sum_{k=1}^{m-j}\frac{1}{k}\right).$$

And of course, at m

$$f_n^{(m)}(z) = \frac{1}{B_n}\log_n(z-z_n).$$

From m on it is easy to see that

$$f^{(m+j)}(z) = \frac{1}{B_n}(-1)^{j-1}(j-1)!\frac{1}{(z-z_n)^j}\ j\in\mathbb{N}.$$

Then

$$\begin{aligned}\sup_K ||f_n^{(j)}(z)|| &\le \frac{1}{B_n}\left[\sup_K \frac{|z-z_n|^{m-j}}{(m-j)!}\left(|\log_n(z-z_n)| + \sum_{k=1}^{m-j}\frac{1}{k}\right)\right] = \\ &= \frac{B_n^{(j)}}{B_n} < 1, \forall 0\le j<m, \forall n\in\mathbb{N}\end{aligned}$$

In a clearer manner:

$$\sup_K ||f_n^{(j)}(z)|| < 1, \forall 0 \le j < m, \forall n\in\mathbb{N}.$$

Now notice that for m

$$\lim_{n\to\infty}|f_n^{(m)}(0)| = \lim_{n\to\infty}\frac{1}{B_n}|log_n(-z_n)| = \infty$$

since $\lim_{n\to\infty} B_n > 0$ as proved above. This is also the case for derivatives of higher order:

$$\lim_{n\to\infty}|f_n^{(m+j)}(0)| = \lim_{n\to\infty}\frac{1}{B_n}(j-1)!\frac{1}{|z_n|^j} = \infty,\ j\in\mathbb{N}.$$

Therefore the sequence of functions $\{f_n\}$ satisfies the aforementioned properties. □

It will be useful to note the following result:

Lemma 2.3. *Let $\mu \in \mathbb{N}$ and $D_0, D_1, D_2, D_3, \ldots, D_\mu \in \mathbb{C}$ with $D_\mu \neq 0$. Then*

$$\lim_{z\to 0}\left|D_0 \log(z) + \sum_{k=1}^{\mu} D_k \frac{1}{z^k}\right| = \infty$$

Proof. We can rewrite

$$\sum_{k=1}^{\mu} D_k \frac{1}{z^k} = D_1\frac{1}{z} + D_2\frac{1}{z^2} + \ldots + D_\mu \frac{1}{z^\mu} = \frac{D_1 z^{\mu-1} + D_2 z^{\mu-2} + \ldots + D_\mu}{z^\mu}.$$

We now introduce the change of variables $h = 1/z$, therefore we can rewrite the limit as follows,

$$\lim_{z\to 0} \frac{D_1 z^{\mu-1} + D_2 z^{\mu-2} + \ldots + D_\mu}{z^\mu} = \lim_{h\to\infty} \frac{D_1 h^{1-\mu} + D_2 h^{2-\mu} + \ldots + D_\mu}{h^{-\mu}} = \lim_{h\to\infty} \frac{D_\mu}{h^{-\mu}} = \lim_{z\to 0} \frac{D_\mu}{z^\mu}$$

Then we have

$$\lim_{z\to 0} D_0 \log(z) + \frac{D_1 z^{\mu-1} + D_2 z^{\mu-2} + \ldots + D_\mu}{z^\mu} = \lim_{z\to 0} D_0 \log(z) + \frac{D_\mu}{z^\mu}.$$

$$\lim_{z\to 0} D_0 \log(z) + \frac{D_\mu}{z^\mu} = \lim_{z\to 0} \frac{D_0 z^\mu \log(z) + D_\mu}{z^\mu}$$

We have already seen that in our setting $z^\mu \log(z) \to 0$ as $z \to 0$ for $\mu > 0$ (which is the case), therefore

$$\lim_{z\to 0} \frac{D_0 z^\mu \log(z) + D_\mu}{z^\mu} = \infty.$$

The continuity of the absolute value/norm function allows us to arrive to the following conclusion:

$$\lim_{z\to 0}\left|D_0 \log(z) + \sum_{k=1}^{\mu} D_k \frac{1}{z^k}\right| = \infty.$$

□

Like in the previous section, we would like to first mimic the structure of theorem 1.8, therefore I will first introduce the following concept:

Definition 2.8 (Non-spiraling set)**.** *A set $K \subset \mathbb{C}$ is non-spiraling if K is non-spiraling at every $p \in \delta\hat{K}$.*

Theorem 2.5. *Assume that K is compact and non-spiraling. Assume that $\mathfrak{b}_K$ is not surjective. Then $\hat{K}$ is a neighborhood of the origin.*

Proof. According to theorem 1.6 there is a sequence of polynomials $\{P_j\} \subset \mathbb{C}[X]$, with $degP_j = \mu_j \to \infty$ and a non negative integer M such that

$$|P_j(\partial)h(0)| \leq \sum_{l\leq M} \sup_K |h^{(l)}|, \qquad h \in \mathscr{O}(\hat{K})$$

We can express each $P_j(X)$ as follows:

$$P_j(X) = \sum_{k=0}^{\mu_j} A_{jk} X^k \qquad A_{j,\mu_j} \neq 0.$$

Assume $0 \in \delta \hat{K}$. We consider the functions f_n in $\mathscr{O}(\hat{K})$ constructed in proposition 2.5 for m=M+1. We then know a bound for the first M derivatives,

$$\sup_K ||f_n^{(j)}(z)|| < 1, \qquad 0 \leq j < M+1.$$

It follows that

$$|P_j(\partial) f_n(0)| \leq \sum_{l \leq M} \sup_K |f_n^{(l)}| < M+1.$$

Now in particular, for j sufficiently large ($degP_j = \mu_j \to \infty$):

$$|P_j(\partial) f_n(0)| = \left| A_{j,0} f_n(0) + A_{j,1} f_n'(0) + \ldots + A_{j,M} f_n^{(M)}(0) + A_{j,M+1} f_n^{(M+1)}(0) + \ldots + A_{j,\mu_j} f_n^{(\mu_j)}(0) \right| < M+1.$$

We know that $A_{j,0} f_n(0) + A_{j,1} f_n'(0) + \ldots + A_{j,M} f_n^{(M)}(0)$ is bounded, but we will see now that the sum of the rest of the terms becomes unbounded for a sufficiently large n:

$$\left| A_{j,M} f_n^{(M+1)}(0) + \ldots + A_{j,\mu_j} f_n^{(\mu_j)}(0) \right| \longrightarrow \infty.$$

We can insert the actual value of the terms and rearrange it to

$$\left| A_{j,M} f_n^{(M+1)}(0) + \ldots + A_{j,\mu_j} f_n^{(\mu_j)}(0) \right| = \left| D_0 \log_{M+1}(-z_n) + D_1 \frac{1}{-z_n} + \ldots + D_{\mu_j - M-1} \frac{1}{(-z_n)^{\mu_j - M-1}} \right|$$

where these constants $D_0, \ldots, D_{\mu_j - M-1}$ are the coefficients that appear alongside with each term after rewriting the previous expression. We know that that according to lemma 2.3

$$\left| D_0 \log_{M+1}(-z_n) + D_1 \frac{1}{-z_n} + \ldots + D_{\mu_j - M-1} \frac{1}{(-z_n)^{\mu_j - M-1}} \right| = \infty \text{ as } n \longrightarrow \infty.$$

Therefore is it impossible for it to be bounded for n big enough, or as $n \longrightarrow \infty$. A contradiction.

$\square$

We can once again take everything we have an write:

Theorem 2.6. *Let K be compact and non-spiraling with $0 \in K$. Then one and only one of the following properties hold:*

- $\mathfrak{b}_K$ *is surjective.*
- $\mathfrak{b}_K$ *is injective and $\hat{K}$ is a neighborhood of the origin.*

As already mentioned, the previous formulation of the theorem is designed to resemble theorem 1.8, but it is possible to offer a reformulation in which the non-spiraling property can be simply required at the point 0:

Theorem 2.7. *Assume K is compact and that $0 \in \delta K$ with K non-spiraling at 0. Then $\mathfrak{b}_K$ is surjective.*

Proof. Assume $\mathfrak{b}_K$ is not surjective, then we can repeat the argument from the previous proof. Since we will reach a contradiction, we conclude that $\mathfrak{b}_K$ is surjective. □

Corollary 2.3. *Let K be compact and $0 \in K$. Then the following statements are true:*

- $\mathfrak{b}_K$ *is injective and $\hat{K}$ is a neighborhood of the origin.*
- *K is non-spiraling at $0 \in \delta\hat{K}$ and $\mathfrak{b}_K$ is surjective.*

Chapter 3

Analyzing proper-stiuatedness and non-spirality

3.1 Proper-situatedness

The first sequence of functions requires that 0 is *properly situated* in K, this is, there exists a sequence $\{z_n\} \subset K^c$, $z_n \to 0$ such that

$$\lim_{n\to\infty} \frac{d(z_n, K)}{|z_n|} > 0.$$

We can start by noting some very intuitive property of properly situated points.

Property 3.1 (Proper-situatedness is preserved under translations, rotations and dilations)**.** *Let $K \subset \mathbb{C}$ and $a, b \in \mathbb{C}$, $a \neq 0$. Then if p is properly situated if and only if $ap + b$ is properly situated in $aK + b$.*

Proof. First recall that $aK + b = \{ak + b : k \in K\}$. Since p is properly situated we have that there exists a sequence $\{z_n\} \subset K^c$, $z_n \longrightarrow p$ as $n \longrightarrow \infty$ such that

$$\lim_{n\to\infty} \frac{d(z_n, K)}{|z_n - p|} > 0.$$

Notice now that $ap + b \in aK + b$ and that $z_n' = az_n + b \to ap + b$ as $n \to \infty$. We have

$$\frac{d(z_n', aK + b)}{|z_n' - (ap + b)|} = \frac{\inf\{|az_n + b - (ak + b)| : k \in K\}}{|az_n + b - (ap + b)|} = \frac{|a| \inf\{|z_n - k| : k \in K\}}{|a||z_n - p|} = \frac{d(z_n, K)}{|z_n - p|}.$$

Therefore

$$\lim_{n\to\infty} \frac{d(z_n', aK + b)}{|z_n' - (ap + b)|} = \lim_{n\to\infty} \frac{d(z_n, K)}{|z_n - p|} > 0.$$

So $ap + b$ is properly situated in $aK + b$. For the other implication we can use the same argument (in that case use $z_n' = \frac{1}{a}(z_n - b)$). ☐

Naturally, the question that follows the previous property is whether proper-situatedness is preserved under maps $f : K \to \mathbb{C}$. For that I offer an answer in the form of conformal mappings, using a Mean-Value-Theorem-like result from [10, p. 859].

Theorem 3.1. *Let f be a holomorphic function defined on an open convex subset D of $\mathbb{C}$. Let a and b be two distinct points of D. Then there exists $z_1, z_2 \in (a,b)$ such that:*

$$\frac{f(b)-f(a)}{b-a} = Re\left(f'(z_1)\right) + iIm\left(f'(z_2)\right).$$

Proposition 3.1 (Proper-situatedness is preserved under conformal mappings)**.** *Let $K \subset \mathbb{C}$, compact and $p \in \delta K$ properly situated through the sequence $\{z_n\}_{n\in\mathbb{N}}$. Let $f : D \to \mathbb{C}$ be a conformal mapping, where $D \subset \mathbb{C}$ is an open convex set containing K, $\{z_n\}_{n\in\mathbb{N}}$ and $[z_n,k]$ for any $n \in \mathbb{N}$ and every $k \in \delta K$. Then if p is properly situated in K, $f(p)$ is properly situated in $f(K)$.*

The domain of the function f must be convex in order to be able to apply theorem 3.1, we could think of the convex hull of K, $\{z_n\}_{n\in\mathbb{N}}$ and $[z_n,k]$, whith p and the z_n's in its interior.

Proof. We have that

$$\frac{d(f(z_n), f(K))}{|f(z_n)-f(p)|} = \frac{\inf\{f(z_n)-f(k) : k \in K\}}{|f(z_n)-f(p)|}.$$

Now, due to theorem 3.1 we know there are $w_1^n, w_2^n, w_3^n, w_4^n \in D$ such that

$$f(z_n)-f(k) = \left(Re(f'(w_1^n)) + iIm(f'(w_2^n))\right)(z_n-k)$$

$$f(z_n)-f(p) = \left(Re(f'(w_3^n)) + iIm(f'(w_4^n))\right)(z_n-p)$$

Now notice that since $w_3^n, w_4^n \in [z_n, p]$, $\lim_{n\to\infty} w_3^n = \lim_{n\to\infty} w_4^n = p$. Notice also that given that $d(z_n,K) \to d(z_n,p)$ as $n \to \infty$ we have that $\lim_{n\to\infty} w_1^n = \lim_{n\to\infty} w_2^n = p$.

Putting it all together we get:

$$\begin{aligned}\frac{d(f(z_n), f(K))}{|f(z_n)-f(p)|} &= \frac{\inf\{f(z_n)-f(k) : k \in K\}}{|f(z_n)-f(p)|} = \\ &= \frac{(Re(f'(w_1^n)) + iIm(f'(w_2^n)))\, d(z_n,K)}{\left(Re(f'(w_3^n)) + iIm(f'(w_4^n))\right)|z_n-p|}\end{aligned}$$

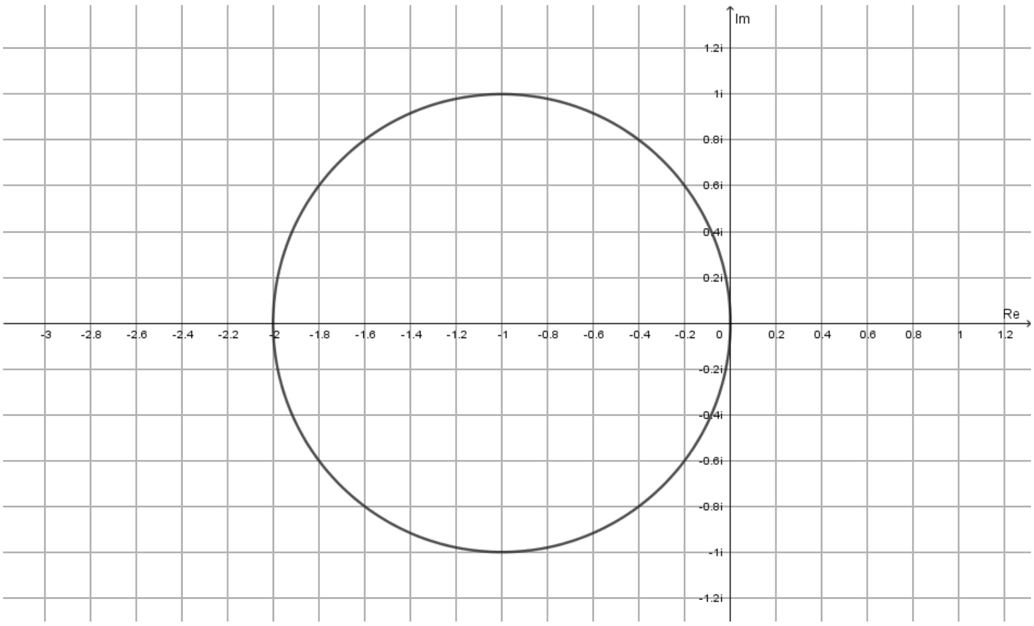
We know that $(Re(f'(p)) + iIm(f'(p))) \neq 0$ since f is a conformal map. We can finally conclude

$$\begin{aligned}\lim_{n\to\infty} \frac{d(f(z_n), f(K))}{|f(z_n)-f(p)|} = \lim_{n\to\infty} &= \frac{(Re(f'(w_1^n)) + iIm(f'(w_2^n)))\, d(z_n,K)}{\left(Re(f'(w_3^n)) + iIm(f'(w_4^n))\right)|z_n-p|} = \\ &= \frac{(Re(f'(p)) + iIm(f'(p)))}{(Re(f'(p)) + iIm(f'(p)))} \lim_{n\to\infty} \frac{d(z_n,K)}{|z_n-p|} = \\ &= \lim_{n\to\infty} \frac{d(z_n,K)}{|z_n-p|} > 0.\end{aligned}$$

$\square$

After this initial snack we can now see some examples. This requirement is related to the behavior of the distance function around 0 and it is therefore naturally related to the shape of the boundary around 0. The simplest case we can see it that of a circumference with 0 on the boundary:

Example 3.1. *Let us take K a circle of radius 1 centered at -1 in $\mathbb{C}$. We will see that the point 0 is properly situated.*



Let us take the sequence $z_n = \frac{1}{n}$, $n \in \mathbb{N}$. Then it is clear that $d(z_n, K) = \frac{1}{n}$, since the closest path to 0 is just a straight line and the boundary of K is convex around 0. It then follows that:

$$\lim_{n\to\infty} \frac{d(z_n, K)}{|z_n|} = \frac{1/n}{1/n} = 1 > 0$$

If in a neighborhood of the origin the shortest distance to the origin is precisely a straight line towards it proper-situatedness follows quite easily. We can formalize this intuitions, for example, using the definition of normal vector to a closed set from [6, p. 280].

Definition 3.1 (Normal vector)**.** *Let $K \subset \mathbb{C}$ be closed. We will say that a vector $v \in \mathbb{C}$ is normal to K at $p \in K$ if there is an open disc contained in K^c centered on a point x_1, such that p is adherent to this disc and $v = \lambda(x_1 - p)$ for some $\lambda > 0$.*

Recall that a point p is adherent to a subset S is $x \in Cl(S)$, where $Cl(\cdot)$ denotes the closure.

We can transform the previous definition 3.1 into a property of a point $p \in K$.

Definition 3.2. *Let $K \subset \mathbb{C}$ be closed. We will say that 'there exists a normal vector to K at $p \in \delta K$' if there exists a normal vector at p like in definition 3.1.*

Along the normal vector the shortest distance is precisely a straight line, i.e., the closest distance to the set along the normal vector is the distance to the point p. We can quickly prove this before moving on:

Property 3.2. *Let $K \subset \mathbb{C}$ be closed such that there exists a normal vector v to K at $p \in \delta K$. Then, along the line $p + tv$ the distance to K is precisely the distance to the point p.*

Proof. Take any sequence $\{t_n\}_{n\in\mathbb{N}} \subset [0,1]$ such that $1 = t_1 > t_2 > t_3 > ... > t_j > t_{j+1} > ... \to 0$, $j \in \mathbb{N}$. Then we can define any sequence along the line generated by the normal vector v like $z_n = p + t_n v$.

By the existence of a normal vector, we have that $D_{d(z_1,p)}(z_1) \subset K^c$ with $p \in Cl(D_{d(z_1,p)}(z_1))$. We will now see that if $D_{d(z_n,p)}(z_n) \subset K^c$ with $p \in Cl(D_{d(z_n,p)}(z_n))$ then necessarily $D_{d(z_{n+1},p)}(z_{n+1}) \subset K^c$ with $p \in Cl(D_{d(z_{n+1},p)}(z_{n+1}))$.

Suppose that $D_{d(z_n,p)}(z_1) \subset K^c$ with $p \in Cl(D_{d(z_n,p)}(z_n))$ but that $D_{d(z_{n+1},p)}(z_{n+1}) \subsetneq K^c$. This means $\exists w \in K$ such that $w \in D_{d(z_{n+1},p)}(z_{n+1})$. It follows:

$$|w - z_n| \geq d(z_n, p) = t_n|v|,$$

$$|w - z_{n+1}| < d(z_{n+1}, p) = t_{n+1}|v|$$

Rewriting we can say

$$t_n|v| \leq |w - z_n|,$$

$$-t_{n+1}|v| < -|w - z_{n+1}|.$$

We have

$$(t_n - t_{n+1})|v| < |w - z_n| - |w - z_{n+1}| \leq |(w - z_n) - (w - z_{n+1})| = |z_{n+1} - z_n| = (t_n - t_{n+1})|v|.$$

Which means

$$(t_n - t_{n+1})|v| < (t_n - t_{n+1})|v|.$$

So the distance along this normal vector is simply the distance to p. □

We can now prove the following:

Proposition 3.2. *Let $K \subset \mathbb{C}$ closed and $p \in \delta K$ such that there exists a normal vector to K at $p \in \delta K$, then, p is properly situated.*

Proof. Recall that $p \in \delta K$ is properly situated in K if there exists a sequence $\{z_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \frac{d(z_n, K)}{|z_n - p|} > 0.$$

Given that *there exists a normal vector v at* $p \in \delta K$ we have a point $x_1 \in K^c$ such that $v = \lambda(x_1 - p)$ for some $\lambda > 0$.

Pick
$$z_n = p + \frac{1}{n}v.$$
Recall that $d(z_n,K) = d(z_n, p)$ (property 3.2). Then:
$$d(z_n,K) = d(z_n, p) = |z_n - p| = \frac{1}{n}|v|.$$
As a consequence
$$\lim_{n\to\infty} \frac{d(z_n,K)}{|z_n - p|} = \frac{1/n|v|}{1/n|v|} = 1 > 0.$$
Therefore p is *properly situated* in K. □

We can expand the intuition we have about the existence of this normal vectors looking at the case in which the boundary is convex around our point of interest.

Definition 3.3 (Convex around p)**.** *Let K be a closed set and let $p \in \delta K$. We will say that the boundary is convex around p if there exists a neighborhood U of p such that $U \cap K$ is convex.*

Proposition 3.3 (Convexity implies the existence of a normal vector)**.** *Let K be a closed set and $p \in \delta K$ such that K is convex around p. Then there exists a normal vector to K at p.*

Proof. Let's see that there exists a tangent line to $U \cap K$ at p. Every line touching p has the expression $p + t\omega$, $\omega \in \mathbb{C}$. In order for it to be tangent it must only touch the point p in $U \cap K$. Assume it touches another point $p' \in K$, then, since $U \cap K$ is convex, every point along the segment $[p, p']$ must also be in $U \cap K$. Since this happens for every line touching p, p is in the interior, which is a contradiction. Therefore there exists a tangent line $p + t\omega_0$ to K at p, $\omega_0 \in \mathbb{C}$. The normal vector will then be $i\omega_0$. □

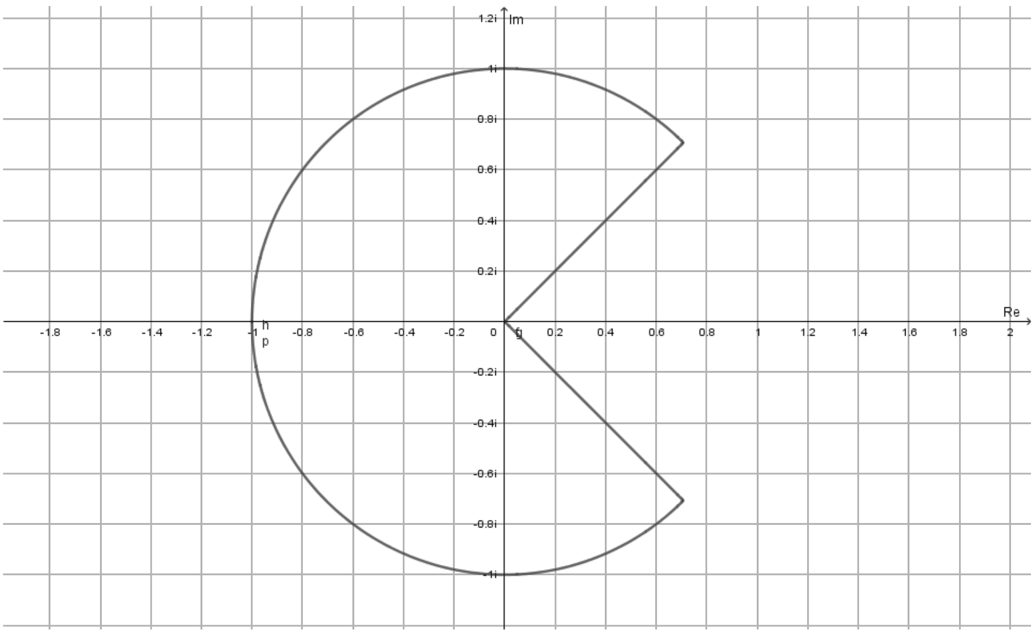
It naturally follows that

Proposition 3.4 (Convexity implies proper-situatedness)**.** *Let K be a closed set and $p \in \delta K$ such that K is convex around p. Then p is properly situated.*

Proof. Convexity around p $\underset{\text{prop. 3.3}}{\Longrightarrow}$ existence of a normal vector at p $\underset{\text{prop. 3.2}}{\Longrightarrow}$ p is properly situated. □

When the boundary is non-convex around p we can still find a normal vector if the boundary is not 'too abrupt' around p. And even if we can not find a normal vector, we might still be able to get proper-situatedness. Therefore, as we will see in the following example, proper-situatedness does not imply the existence of a normal vector:

Example 3.2. *Let's take a circumference in $\mathbb{C}$ in which I 'carved' an entrance towards 0 following the functions x and $-x$. Let's take the sequence $\{\frac{1}{n}\}$, then we have that $d(1/n, K) = \frac{\sqrt{2}}{2n}$.*



It follows

$$\lim_{n\to\infty} \frac{d(z_n, K)}{|z_n|} = \frac{\sqrt{2}}{2} > 0,$$

therefore 0 is properly situated. However, it is impossible to find a normal vector and a disc with the requirements of definition 3.1.

Let's now see an example of a set in which the previous approach does not work. The idea is to find sets where $d(z_n, K)$ goes to zero significantly faster than $|z|$. The boundary around the point 0 is 'too abrupt'.

Example 3.3. *In this case, I took a circumference in $\mathbb{R}^2$ in which I 'carved' an entrance towards 0 following the functions x^2 and $-x^2$.*

Call K the previous set considered to be in $\mathbb{C}$. First, note that $0 \in \delta K$, that K is compact (is closed an bounded) and that it is polynomially convex since its compementary only has one connected component. Take $z \in K^c$ such that $|z| < 1$, then we have that $d(z, K) \leq d(Re(z), K)$, where $Re(\cdot)$ is the real part of the complex number. Now we know

$$d(Re(z), K) \leq Re(z)^2 \leq |z|^2.$$

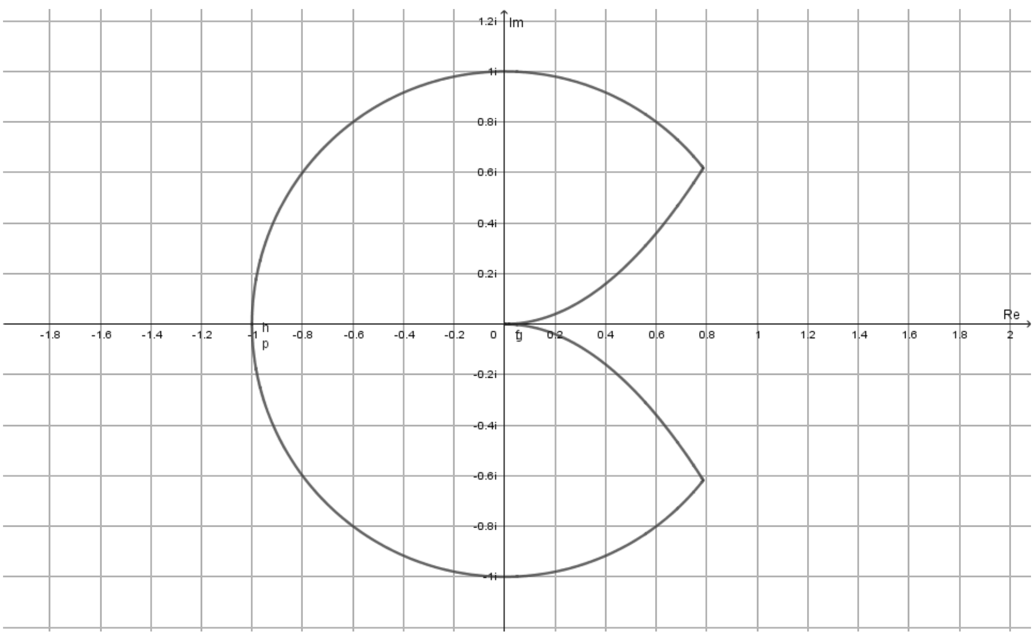
So we get that $d(z, K) \leq |z|^2$. Let $\{z_n\} \subset K^c$ be any sequence going to 0. Then

$$0 \leq \frac{d(z, K)}{|z|} \leq \frac{|z^2|}{|z|} = |z| \longrightarrow 0.$$

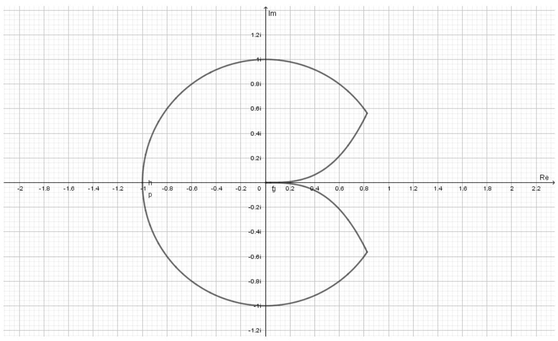
Hence

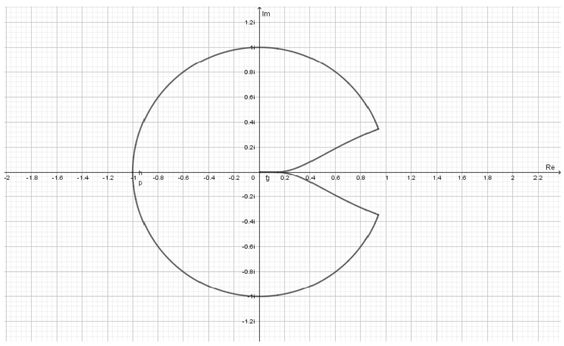
$$\lim_{n\to\infty} \frac{d(z_n, K)}{|z_n|} = 0.$$

Therefore we can not apply the previous approach since we need the limit to be greater than 0.



This is because the distance function 'behaves like' or 'decreases as' the function $|z|^2$ in a neighborhood of zero. Likewise we can craft similar sets in which the approach does not work. For example, if we take the segment of the boundary approaching 0 to be parametrized by $\pm x^3$ or $\pm e^{-\frac{1}{x}}$:





In general, whenever the boundary behaves like a certain function $g(z)$ close to zero we can say

Proposition 3.5. *Let $K \subset \mathbb{C}$ be a compact polynomially convex set such that $0 \in \delta K$. Assume that there is $\varepsilon > 0$ and $g : K^c \to [0, \infty)$ such that $d(z,K) \leq g(z)$ for all $z \in B_\varepsilon(0) \cap K^c$ with*

$$\lim_{z \to 0} \frac{g(z)}{|z|} = 0$$

Then, for any sequence $\{z_n\} \subset K^c$

$$\lim_{n \to \infty} \frac{d(z_n, K)}{|z_n|} = 0.$$

Proof. Easily follows from

$$0 \leq \frac{d(z_n, K)}{|z_n|} \leq \frac{g(z_n)}{|z_n|}$$

□

Remark 3.1. *We can write an analogous proposition using mainly sequences and not necessitating a specific function g.*

The existence of the previous sets, whose boundary is more abrupt around 0 and the own definition of proper-situatedness naturally suggest a new definition

Definition 3.4. *Let K be a compact set and p a point on the boundary. We will say that p is properly situated with order $r > 0$ in K if there exists a sequence $\{z_n\} \subset K^c$, $z_n \longrightarrow p$ as $n \longrightarrow \infty$ such that*

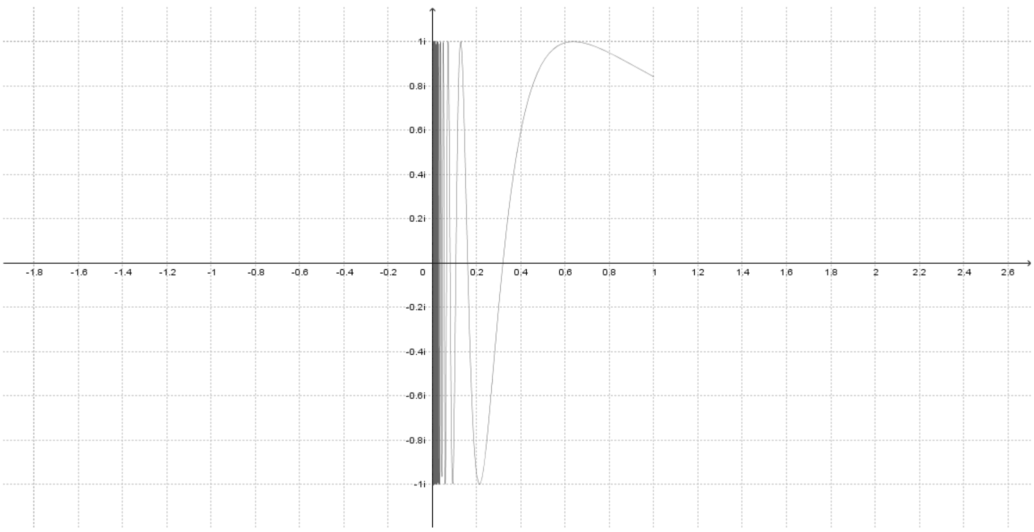
$$\lim_{n\to\infty} \frac{d(z_n, K)}{|z_n - p|^r} > 0.$$

It remains an open question to see whether the same results can be achieved with this more general notion of proper-situatedness.

In order to see whether there is an improvement with respect to the path-boundedness condition we should look for sets which are not path bounded but have 0 properly situated in their boundary. It is the case of the following example:

Example 3.4. *Let $K \subset \mathbb{C}$*

$$K := \{x + i\sin(1/x) : x \in (0,1]\} \bigcup \{iy : -1 \le y \le 1\}$$

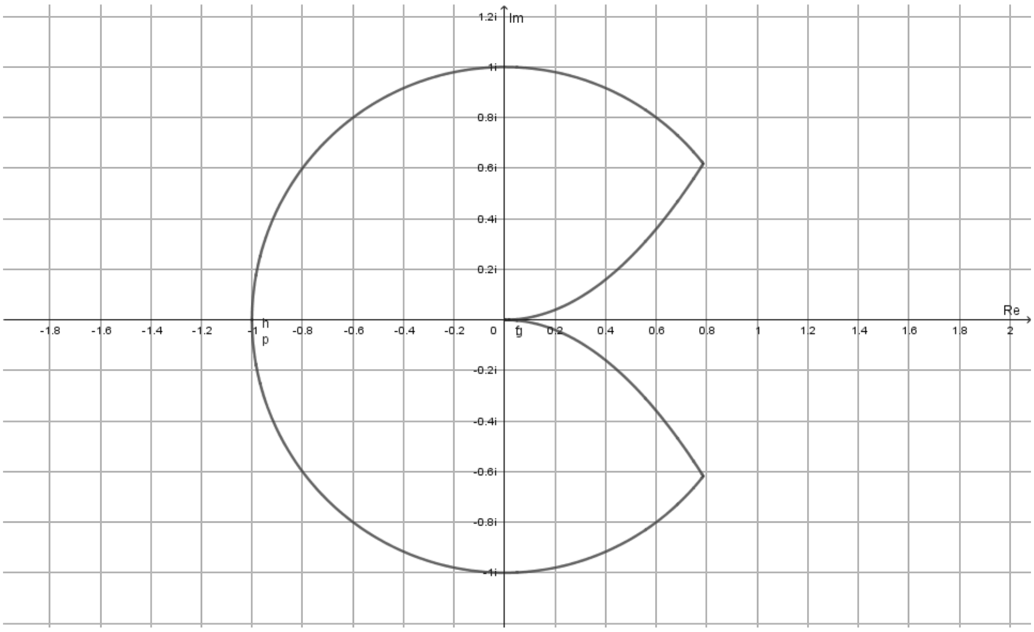


K is compact, since it is closed and bounded, and $0 \in \delta K$. Taking the sequence $\{-1/n\}_{n\in\mathbb{N}}$ we can see that 0 is properly situated. However, K is not path-bounded, take $w \in K \setminus \{iy : -1 \le y \le 1\}$. Then the path connecting 0 and w has infinite length.

3.2 Non-spirality

The logarithmic approach appears to work at the cases in which proper-situatedness fails when facing abrupt or very steep boundaries around 0. Let's check:

Example 3.5. *Recall that in this case the boundary around 0 follows the functions x^2 and $-x^2$ and 0 is not properly situated.*



However, the set is non-spiraling: take the smooth curve $\gamma(t) = t$. γ 'connects 0 and ∞' and $\mathbb{C} \setminus (\Gamma \setminus \{0\})$ is homeomorphic to $\mathbb{C} \setminus \mathbb{R}^-$ in which we can define the main branch of the complex logarithm, whose argument is bounded. Therefore K is non-spiraling at 0.

The same with the other examples we reviewed, where the boundary behaved like $\pm x^3$ and $e^{-\frac{1}{x}}$ close to zero.

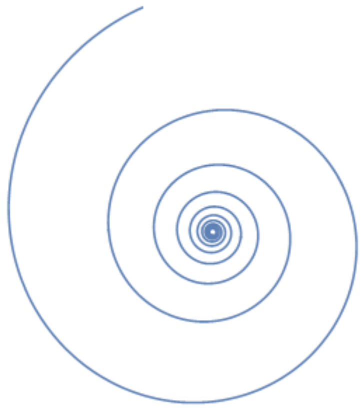
It fails at spiral-like sets, as the name non-spiraling indicates:

Example 3.6. *Take the curve*

$$S := \left\{ \frac{1}{t^4}\cos(t) + i\frac{1}{t^4}\sin(t) : t \in [R, \infty) \right\} \cup \{0\}$$

where R is some fixed positive real number.

The set is compact and also polynomially convex, since its complementary has only one connected component. The set is clearly spiraling. Any curve connecting 0 and infinity will also have the shape of a unending spiral and any branch of the logarithm will therefore have an unbounded argument as one gets closer and closer to the origin.



It is possible that there is a way to circumvent spirality and achieve the same results, and it is possible, given the examples we have seen, that proper-situatedness implies non-spirality. This claims, however, will remain conjectural in this work.

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