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"The Linearization Principle in Infinite Dimensional Algebraic Geometry"

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Overview

Consider maps $f : \mathbb{C}[[x]]^p \to \mathbb{C}[[x]]^q$ between Cartesian products of formal or convergent power series rings in $x = (x_1, \dots, x_n)$. Such an f is called *tactile* if it is given by substitution of power series $a(x) = (a_1(x), \dots, a_p(x)) \in \mathbb{C}[[x]]^p$ for the variables $y = (y_1, \dots, y_p)$ in a power series vector $g(x, y) \in \mathbb{C}[[x, y]]^q$,

$$f\colon a(x) \mapsto g(x, a(x)).$$

Maps of this type – and their associated zerosets $\{a \in \mathbb{C}[[x]]^p; g(x, a) = 0\}$ which will be called *felts* – are ubiquitous in local analytic geometry: arc spaces, local automorphism groups of varieties, K-equivalence and approximation theorems. They are also the main object of study in chapters 1 to 3 of the present thesis. Chapter 4 is joint work with D. Wagner and offers a proof for the fact that the set of normal crossings points on a variety (over an algebraically closed field of arbitrary characteristic) is open. We continue with an overview of the chapters:

Chapter 1

In most of the above settings one is interested in finding zeros of tactile maps in spaces of (formal or convergent) power series. This means solving infinitely many equations in infinitely many variables. In the finite dimensional situation, i.e., for analytic maps between affine spaces $\mathbb{C}^p$ and $\mathbb{C}^q$, the first instance for solving is given by the Implicit Function Theorem, or its more general companion, the Rank Theorem. Both results are used locally at points where the zeroset is smooth. In the infinite dimensional situation which we encounter with spaces of power series, the corresponding theorems are much more subtle. At their best, they allow to solve tactile equations *from a certain degree on*, meaning that if one knows an approximate power series solution up to sufficiently high degree, then an exact solution exists and its expansion can be determined.

In chapter 1 we utilize and extend in a purely algebraic setting a Rank Theorem for (convergent) tactile maps as it has been established in [HM94]: By local automorphisms of source and target, the tactile map can be linearized in the neighbourhood of a chosen power series. The assumptions of the Theorem involve a *Rank Condition* and an *Order Condition*, which both together are sufficient to yield local linearization. The Rank Condition is also necessary, as it is trivially satisfied by any linear map, and preserved by composition with automorphisms. The Order Condition, formulated in terms of standard bases of ideals, ensures that the higher order terms of f are suitably dominated by its linear terms. It turns out that the linearizing local automorphisms lie beyond the class of tactile maps. They are of a more general type, so-called *textile* maps. These can be characterized by saying that the coefficients of the output series are polynomials in the coefficients of the input series. Actually it turns out that by allowing textile maps as automorphisms, the Rank Theorem can be extended to this more general class of maps between power series spaces (see Theorem 2 below).

Let k be a field of characteristic 0. A *cord* is a sequence $c = (c_\alpha)_{\alpha\in\mathbb{N}^n}$ of constants c_α in k. The local k-algebra of formal power series $k[[x]] = k[[x_1,\dots,x_n]]$ in n variables and coefficients in k naturally identifies with the space $\mathcal{C} = \mathcal{C}_n(k)$ of cords over k. The maximal ideal of $k[[x]]$ of power series without constant term is denoted by $\mathfrak{m}$. We define $\operatorname{ord} c$ as the order of c as a power series. The space $\mathcal{C}$ comes equipped with the $\mathfrak{m}$-adic topology induced by the 0-neighborhoods $\mathfrak{m}^k$ of series of order $\geq k$. Open sets in the $\mathfrak{m}$-adic topology will be referred to as $\mathfrak{m}$-*open*, though we will drop the "$\mathfrak{m}$" if the topology is clear out of context. In the case $n = 1$ we shall speak of *arcs* and write $\mathcal{A}$ for $\mathcal{C}$, respectively $k[[t]]$ for $k[[x_1]]$.

Elements of $\mathcal{C}$ are sequences of elements in k indexed by $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n$. We will consider elements of a Cartesian product $\mathcal{C}^p$, $p \in \mathbb{N}$, as sequences in k which are indexed by $(\alpha,\alpha_{n+1}) \in \mathbb{N}^n \times \mathbb{N}$ with $0 \leq \alpha_{n+1} \leq p-1$. A map $f\colon \mathcal{C}^m \to \mathcal{C}^p; c \mapsto (f_\alpha(c))_{\alpha\in\mathbb{N}^{n+1}}$ is called *textile* if for all α the component $f_\alpha(c)$ is a polynomial in the coefficients of c. Let U be a subset of $\mathcal{C}^m$. A map $f\colon U \to \mathcal{C}^p$ is called textile if it is the restriction of a textile map $\mathcal{C}^m \to \mathcal{C}^p$ to U. Denote by $k[\underline{\mathbb{N}}^m_n]$ the polynomial ring $k[x^j_\alpha; 1 \leq j \leq m, \alpha \in \mathbb{N}^n]$. The *felt defined by a textile map* $f\colon \mathcal{C}^m \to \mathcal{C}^p$ is the closed subscheme $\operatorname{Spec} k[\underline{\mathbb{N}}^m_n]/(f_\alpha; \alpha \in \mathbb{N}^{n+1})$ of $\operatorname{Spec} k[\underline{\mathbb{N}}^m_n]$. For most applications in section 2 the underlying set of k-points $\{c \in \mathcal{C}^m; f(c) = 0\}$ is of primary interest. It is an "algebraic" subset of a countable Cartesian product of copies of k, where the "equations" are infinitely many polynomials in a polynomial ring of countably many variables. Sometimes we will use the term "felt" also for Zariski-open subsets of some felt defined by a textile map.

Any vector of formal power series $g \in k[[y_1,\dots,y_m]]^p$ induces a textile map $f : \mathfrak{m}\cdot\mathcal{C}^m \to \mathcal{C}^p$ via substitution $f(c) = g(c)$. As we mentioned before, such maps are called *tactile*. Their zerosets in case $n = 1$ are called *arc spaces*. The tangent map of f at 0 is given by the linear terms of g.

Textile maps define by restriction morphisms between felts, thus allowing to speak of the *category of felts*. A felt X is *smooth at a cord* c if there is an $\mathfrak{m}$-adic neighbourhood U of c in $\mathcal{C}^m$ such that $X \cap U$ is isomorphic to an $\mathfrak{m}$-open subset of a felt defined by a linear textile map. Tangent spaces and maps can be defined in a natural way.

Examples. 1. As a special case of tactiles, let X be an affine algebraic variety defined in $\mathbb{A}^m_k$ by polynomials $f_1,\dots,f_p \in k[x_1,\dots,x_m]$. Then

$$X_\infty = \{a \in \mathcal{A}^m,\ f_1(a) = \ldots = f_p(a) = 0\}$$

is the classical arc space associated to X (see [Nas95]). The point $a(0)$ lies in X; we say that the arc *goes through* or *is centered at* $a(0)$. The image $\pi_q(X)$

of X in $(\mathcal{A}/\mathfrak{m}^{q+1})^m$ under the canonical projection is the space of q-jets which can be lifted to arcs on X, whereas

$$X_q = \{a \in (\mathcal{A}/\mathfrak{m}^{q+1})^m; f_1(a) = \ldots = f_p(a) = 0 \bmod \mathfrak{m}^{q+1}\}$$

is the *space of q-jets on X*.

2. Let $f \in k[x_1, \ldots, x_m]$ and identify $\mathcal{A}$ with $k[[t]]$ via $a = (a_i)_{i\in\mathbb{N}} \mapsto \sum a_i t^i$. Consider the felt Z defined by f. For an integer $d > 0$, decompose $a \in \mathcal{A}^m$ into $a = \bar{a}+\hat{a}$, where $\bar{a}$ denotes the expansion of a up to degree $d-1$ (its truncation) and $\hat{a}$ has order $\geq d$. We identify $\bar{a}$ as an element in $\pi_{d-1}(Z)$ (see example 1) and consider $f(\bar{a}+\hat{a}) = 0$ as an equation in $\hat{a}$, say $g(\hat{a}) = 0$. Of course, g induces a tactile map $G : \mathfrak{m}^d \cdot \mathcal{A}^m \to \mathcal{A}$. One of the main aspects of this paper will be to show that for d sufficiently large, the map G can be linearized locally (w.r.t. the $\mathfrak{m}$-adic topology) in a well defined way (see Theorem 6). Intuitively, this will signify that the recursion equations resulting from $f(a) = 0$ for the coefficients of a will become eventually as good as linear equations. In particular, if $\bar{a}$ is an approximate solution for some sufficiently high d, then the existence of the "remainder" $\hat{a}$ such that $\bar{a} + \hat{a}$ is an exact solution is automatically ensured. Geometrically speaking, this says that the fibre over $\bar{a}$ under the truncation map $b \mapsto \bar{b}$ is trivial. For a more general triviality result see Theorem 10.

3. Consider a polynomial recursion of order d indexed by the naturals:

$$a_i = f_i(a_{i-1}, \ldots, a_{i-d}), \qquad i \geq d,$$

with $f_i \in k[y_1, \ldots, y_d]$. It induces a textile map

$$F : \mathcal{A} \to \mathcal{A} : a \to (a_0, \ldots, a_{d-1}, a_d - f_d(a), a_{d+1} - f_{d+1}(a), \ldots).$$

Prescribing any initial conditions $a_0, \ldots, a_{d-1}$ gives rise to a unique solution $(a_i)_{i\in\mathbb{N}}$ of the recursion. As a variation, let the polynomial recursion be of infinite order having now the form $a_i = f_i(a_{i-1}, \ldots, a_0)$ with polynomials $f_i \in k[y_1, \ldots, y_i]$. Again we get a textile map $F : \mathcal{A} \to \mathcal{A}$, given by $a \to (a_i - f_i(a))_{i\in\mathbb{N}}$.

For several constructions on the way to linearization it is necessary to have a textile version of the Inverse Mapping Theorem. In contrast to the corresponding theorem in analysis, the result and its proof are mostly algebraic. The basic assumption is that the non-linear terms of the map increase the order of cords stronger than the linear terms do: Write $f = \ell+h$ with ℓ the (invertible) tangent map of f at the cord c in question, and h the higher order terms. The hypothesis is then that the composition $\tilde{\ell}h$ (with $\tilde{\ell}$ linear s.t. $\tilde{\ell}\ell = \mathrm{id}$) is *contractive*, i.e., increases the order of cords (see section 1.1):

The following theorem is a generalization of the inverse function theorem in [HM94] to the setting of textile maps (see Theorem 3 in 1.1).

Theorem 1 (Inverse Function Theorem)**.** *Let $U \in \mathcal{C}^m$ be an $\mathfrak{m}$-adic neighbourhood of 0 and $f\colon U \subseteq \mathcal{C}^m \to \mathcal{C}^m$ be textile with $f(0) = 0$. Assume that $f = \ell + h$, $f(U) \subseteq W$ (neighbourhood of 0) , where $\ell\colon U \to W$ is linear, textile and invertible, i.e., there exists a linear map $\tilde{\ell} : W \to U$ with $\ell \circ \tilde{\ell} = \mathrm{id}_W$ and $\tilde{\ell} \circ \ell = \mathrm{id}_U$. If $\tilde{\ell}h$ is contractive on U, then f admits on U an inverse.*

The above result will be further generalized to a parametric version (see 1.3.2) and a version for cords with coefficients in a ring with nilpotent elements (see section 1.1).

Example. Consider $f\colon \mathfrak{m} \to \mathfrak{m}^2$ given by $a \to t \cdot a + a^2$. Here we set $\ell(a) = t \cdot a$ and $\tilde{\ell}(a) = t^{-1} \cdot a$. For $a \in \mathfrak{m}$, $\tilde{\ell}h(a) = t^{-1} \cdot a^2$ is not contractive (the order remains constant for $a \in \mathfrak{m} \setminus \mathfrak{m}^2$), whereas it is contractive if we restrict to $a \in \mathfrak{m}^2$. It follows from the theorem that the restriction $f|_{\mathfrak{m}^2}\colon \mathfrak{m}^2 \to \mathfrak{m}^3$ is locally invertible at 0.

In finite dimensional analysis, the Rank Theorem is a trivial consequence of the Inverse Mapping Theorem. In the infinite dimensional context, say Banach spaces or locally convex spaces, and also in the present situation of cord spaces, this is no longer true. In many applications it is the Rank Theorem which is actually needed (despite the numerous important applications of theorems like the Nash-Moser Inverse Function Theorem, [Ham82]) to show the manifold structure of fibres of smooth maps or to prove local triviality.
A first step in this respect is the correct definition of the hypothesis of constant rank. We follow the concept proposed by Hauser and Müller in [HM94] for analytic maps between convergent power series spaces. Working with formal instead of convergent series, the definition slightly simplifies, but is nevertheless somewhat unhandy to grasp. It goes as follows:

Let V be an $\mathfrak{m}$-adic neighbourhood of 0 in $\mathcal{C}$. A textile map $\gamma : V \subseteq \mathcal{C} \to \mathcal{C}^m$ is called a *curve* (over V) in $\mathcal{C}^m$. If $\mathrm{im}(\gamma) \subseteq U \subseteq \mathcal{C}^m$ we say that γ defines a curve *in* U. Let $f : U \subseteq \mathcal{C}^m \to \mathcal{C}^p$ be a textile map and let J be a closed linear direct complement in $\mathcal{C}^p$ of the image $\mathrm{im}(T_{a_0} f)$ of the tangent map of f at a cord a_0. Then f has *constant rank at* a_0 *with respect to* J if for all curves γ in U with $\gamma(0) = a_0$ and all curves η in $\mathcal{C}^p$ there exist unique curves ρ in $\mathcal{C}^m$ and τ in J such that

$$\eta = T_\gamma f \cdot \rho + \tau.$$

If f is tactile, say induced by a polynomial $g \in k[y_1, \ldots, y_m]$, then $T_\gamma f \cdot \rho$ is given by

$$\partial_1 g(\gamma) \cdot \rho^1 + \cdots + \partial_m g(\gamma) \cdot \rho^m.$$

Here, $\partial_i g$ is the ith partial derivative of g.

In the applications the occurring textile map f is often *quasi-submersive*, i.e., if $f = \ell + h$ with $\ell = T_{a_0} f$, then $\mathrm{im}(h) \subseteq \mathrm{im}(\ell)$. If ℓ is surjective, f is a *submersion.*

Let $\ell\colon \mathcal{C}^m \to \mathcal{C}^p$ be textile linear; a textile linear map $\sigma\colon \mathcal{C}^p \to \mathcal{C}^m$ satisfying $\ell\sigma\ell = \ell$ is called a *scission of* ℓ. Note that a scission provides a left inverse on the image of ℓ and the kernel of $\ell\sigma$ defines a closed direct complement of this image. In the case of tactile maps we follow [HM94] and construct scissions using the Grauert-Hironaka-Galligo Division Theorem for formal power series (see section 1.2.1). After that we give the construction for arbitrary textile linear maps. This will be achieved by a suitable generalization of the Gauss Algorithm to this context.

Here is one extension of the Rank Theorem in [HM94] which is used in chapter 2:

Theorem 2 (Rank Theorem)**.** *Let $f : U \subseteq \mathcal{C}^m \to \mathcal{C}^p$ be a textile map on a neighbourhood $U = \mathfrak{m}^l \cdot \mathcal{C}^m$ of 0 with $f(0) = 0$. Write f in the form $f = \ell + h$ with $\ell = \mathrm{T}_0 f$ the tangent map of f at 0. Assume that ℓ admits a scission $\sigma : \mathcal{C}^p \to \mathcal{C}^m$ for which σh is contractive on U, and that f has constant rank at 0 with respect to* $\ker(\ell\sigma)$. *Then there exist locally invertible textile maps $u : U \to \mathcal{C}^m$ and $v : f(U) \to \mathcal{C}^p$ at 0 such that*

$$v \circ f \circ u^{-1} = \ell.$$

This theorem will be extended in sections 1.3.2 and 1.3.3 to the relative case with parameters and a version for cords with coefficients in a ring with nilpotent elements. In the case of quasi-submersions the order condition gives a sufficient criterion for linearization (see section 1.3.1; and sections 2.2 and 2.4 for applications). We continue with our list of examples:

Examples. 5. Consider the tactile map induced by an element $f \in k[x_1, \dots, x_m]$. Let a be an arc of X with $a(0)$ a smooth point of $X = V(f) \subseteq \mathbb{A}^m_k$. Using the classical Implicit Function Theorem for power series or Theorem 2 it's easy to see that a is a smooth cord of X_∞. In section 2.1.1 we treat the more interesting case when a is an arc through a point in the singular locus $\mathrm{Sing}(X)$, but does not lie entirely in it, i.e., $a(0) \in \mathrm{Sing}(X)$ but $a \notin \mathrm{Sing}(X)_\infty$.

6. Consider example 2 above in the special case $f = x + xy \in k[x, y]$. The induced textile map $F : \mathfrak{m} \cdot \mathcal{A}^2 \to \mathcal{A}; a \mapsto f(a)$ has constant rank at 0 with respect to the complement of $(t) \cdot \mathcal{A}$. The composition σh is contractive, where $h(a^1, a^2) = a^1 a^2$ and σ is a scission for the linear map $\ell : (a^1, a^2) \mapsto a^1$. Thus the Rank Theorem applies and F can be linearized at 0. The linearizing maps u, v of the Rank Theorem are given by $v = \mathrm{id}$ and $u(a) = (a^1(1 + a^2), a^2)$. Indeed, this u would be the natural candidate for linearization when viewing f in the form $f = x(1 + y)$. Note: It is due to the very simple form of the linear part of f that we can write down u and v so explicitly. Taking $f = tx + xy \in k[[t]][x, y]$ it is still possible to linearize the induced map on $\mathfrak{m}^2$, but it is not possible to give closed formulae for u, v. The reason for this lies in the fact that the scission one has to construct is a textile map which is not tactile.

7. In several variables it is in general difficult to prove that a textile map has constant rank. The following example gives a simple tactile map which does not have constant rank. By necessity of the rank condition this map is not locally linearizable: Let $F = x^2 - y^2 \in k[x, y]$. Set $\eta := x_1 x_2 + \ldots + (x_1 x_2)^l$, $l \in \mathbb{N}$, and try to lift the exact solution $(\eta, \eta) \in k[[x_1, x_2]]^2$ of $F(x, y) = 0$. Consider the tactile map

$$g : (x_1, x_2)^{l+1} \cdot k[[x_1, x_2]]^2 \to k[[x_1, x_2]]; g(x, y) = F(\eta + x, \eta + y).$$

To check whether g has constant rank we have to lift any relation (a, b) between $\partial_1 g(0) = \eta$ and $\partial_2 g(0) = -\eta$ to a relation (A, B) between $\partial_1 g(\gamma)$ and $\partial_2 g(\gamma)$ for any $\gamma = (\gamma_1, \gamma_2) \in k[[s, x_1, x_2]]$ of sufficiently high order (in the x_i) such that $\gamma \equiv 0 \mod (s)$ and $A \equiv a \mod (s)$, $B \equiv b \mod (s)$. Specify $(a, b) = (1, 1)$; a short computation shows that then one has to find $A', B' \in (s) \subseteq k[[s, x_1, x_2]]$ such that

$$(\eta + \gamma_1)A' - (\eta + \gamma_2)B' = \gamma_2 - \gamma_1.$$

But this will be impossible (independently of the order of γ) if $\mathrm{supp}(\gamma_i) \cap \mathrm{supp}(x_1 x_2) = \emptyset$. Therefore g does not have constant rank at 0. Note that the situation does not improve by providing an approximative solution of higher degree, since the above calculation is valid for arbitrary $l \in \mathbb{N}$. However, considering g as a tactile map on $k[[t]]^2$, it fulfills the rank condition on some $\mathfrak{m}$-adic neighbourhood of 0.

Chapter 2

This is a collection of six important results in singularity theory being direct corollaries or special instances of the Rank Theorem. We shall briefly describe these applications.

(a) Denef and Loeser use the following local triviality result as a main step to set up motivic integration in the context of singular varieties [DL99] (see also Lemma 9.1 in [Loo02]). Let $X \subseteq \mathbb{A}_k^N$ be an affine variety over a field k of characteristic 0. Assume X is given by $f_1, \ldots, f_p \in k[x_1, \ldots, x_N]$. An arc (or more precisely a k-arc) of X is a $k[[t]]$-point of X, i.e., a solution to $f_1 = \cdots = f_p = 0$ in $k[[t]]^N$. The set of arcs of X is denoted by X_∞. Similarly, an m-jet is a solution to these equations in $\left(k[t]/(t)^{m+1}\right)^N$. Write X_m for the set of m-jets of X. There is a natural projection $\pi_m : X_\infty \to X_m$. Define $\mathcal{A}_e = X_\infty \setminus \pi_e^{-1}((\mathrm{Sing}X)_e)$ as the space of arcs of X which do not lie in the singular locus up to order e. Moreover assume that X is of pure dimension d. Denef and Loeser then show that, for sufficiently large $n \in \mathbb{N}$, the map

$$\theta_n : \pi_{n+1}(X_\infty) \to \pi_n(X_\infty)$$

is a piecewise trivial fibration over $\pi_n(\mathcal{A}_e)$ with fibre $\mathbb{A}_k^d$, [DL99] (2.5). Specializing to the case of a hypersurface this means the following: Let γ be an n-jet of X such that not all partial derivatives vanish on γ modulo $(t)^{e+1}$. Then finding an $(n+1)$-jet η of X with $\eta = \gamma \mod (t)^{n+1}$ for $n \geq e$ sufficiently large can be done by solving a linear system of rank 1. An extension of this result can be found in [Reg06].
The triviality of the above fibration θ_n is one of the key technical results of the paper [DL99]. Extending the result of Denef and Loeser we show that trivialization can already be obtained on the level of arcs, i.e., the map

$$X_\infty \to \pi_n(X_\infty)$$

is a piecewise affine bundle over $\pi_n(\mathcal{A}_e)$ (therefore all the θ_n are simultaneously trivialized). The trivializing map will be textile and can be explicitly described (see section 2.1).

(b) Grinberg, Kazhdan and Drinfeld show in [GK00] (for $k = \mathbb{C}$) and [Dri02] (for arbitrary k) a similar factorization result: Let X be a scheme over k and $\gamma_0 \in X_\infty \setminus (\mathrm{Sing}X)_\infty$. Then the formal neighbourhood $X_\infty[\gamma_0]$ of X_∞ in γ_0 is a product of the form $Y[y] \times D^\infty$. Here $Y[y]$ is the formal neighbourhood of a scheme of finite type over k in a point $y \in Y$ and D^∞ is the product of countably many formal schemes Spf $k[[t]]$. Section 2.2 shows that this local factorization theorem follows from the Rank Theorem (for $\mathrm{char}\, k = 0$).

(c) Denef and Lipshitz in [DL84] and Winkel in [Win] study power series solutions in one variable x with coefficients in k, char $k = 0$, of a system of algebraic differential equations. Here an algebraic differential equation in variables x and $y_1, \dots, y_n$ is a differential equation which is polynomial in $x, y_1, \dots, y_n$ and the derivatives of the y_i's. Algebraic differential operators in this sense naturally define a textile map $k[[x]]^n \to k[[x]]^n$. Thus it might prove useful to study the techniques of this paper in the framework of algebraic differential equations. A first instance can be found in section 2.3. There systems of n explicit differential equations, that is systems of the form $y^{(q)} = P(x, y, y^{(1)}, \dots, y^{(q-1)})$, with a vector of polynomials $P = (P_1, \dots, P_n)$ and $q \in \mathbb{N}$, are considered. Solving such a system in $k[[x]]^n$ for given initial conditions, i.e., the coefficients of the solutions are given up to order $q-1$, is equivalent to solving an equation of the form $\ell(y) = b$, where $b \in k[[x]]^n$ and ℓ is a linear textile map.

(d) Tougeron's Implicit Function Theorem can be seen as a special case of the Rank Theorem. In fact, the proof of the theorem in [Tou68] is based on the following assertion: Let $F \in k[[x, y]]$, k a complete valued field of characteristic 0, and $F(0) = 0$. Denote by $\delta = (\delta_1, \dots, \delta_p)$ the Jacobian of F w.r.t. $y = (y_1, \dots, y_p)$ evaluated at $(x, 0)$, and for $1 \le i \le r \le p$ set $y^i = (y_1^i, \dots, y_p^i)$. Then there are $Y^i \in k[[x, y_j^l; 1 \le l \le r, 1 \le j \le p]]^p$, $1 \le i \le r$, such that

$$F(x, \sum_{i=1}^r \delta_i Y^i) = F(x, 0) + \delta \left(\sum_{i=1}^r \delta_i y^i \right).$$

In section 2.4 this will be proven by linearization of the map $(y^1, \dots, y^r) \mapsto F(x, \sum_{i=1}^r \delta_i y^i) - F(x, 0)$ using the Rank Theorem.

(e) Wavrik's Approximation Theorem, a variation of Artin's Approximation Theorem, is based on Tougeron's Implicit Function Theorem, Thm. I$_n$ in [Wav75], and [Art68]. Let k be a complete valued field of characteristic 0 and let $F \in k[[x, y]][z]$ be irreducible, $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_p)$ and $z = (z_1, \dots, z_r)$. Wavrik proves that for any integer $q > 0$ there exists an N such that if $(\bar{y}, \bar{z}) \in k[[x]]^{p+r}$ satisfies $F(x, \bar{y}, \bar{z}) = 0 \mod (x)^N$, then there exist series $(z(x), y(x)) \in k[[x]]^{p+r}$ with

$$F(x, y(x), z(x)) = 0$$

and

$$(y(x), z(x)) = (\bar{y}, \bar{z}) \mod (x)^q.$$

A solution to $F = 0$ can be obtained as follows: Set $y_i(x) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha^i x^\alpha$, $z_i(x) = \sum_{\alpha \in \mathbb{N}^n} b_\alpha^i x^\alpha$ and substitute these expressions in $F(x, y, z)$ for y respectively z. This gives for each power of x a polynomial equation in the a_α^i, b_α^i. The claim then is that if these equations can be solved up to a sufficiently high degree, then the remaining equations also have a solution. Indeed, it is shown in section 2.4 that solving the remaining equations is equivalent to solving a system of *linear* equations. In one single variable x, the equivalence follows directly from our Rank Theorem. In several variables, one can use Tougeron's Implicit Function Theorem.

(f) Lamel and Mir investigate the following question [LM07]: Let $f\colon (\mathbb{C}^n, 0) \to \mathbb{C}^n$ be a germ of a holomorphic map of generic rank n, i.e., its Jacobian determinant is a nonzero power series. For germs u of biholomorphic maps (preserving the origin) we consider the map defined by $u \mapsto f \circ u$. In local coordinates this map is given by substitution of power series, hence is a tactile map. Here naturally the question for a left inverse to this map arises. Lamel and Mir prove that it is possible to find one if the derivative $u'(0)$ is known, see [LM07] Theorem 2.4 and section 2.6 for the precise statement. This result provides information on the biholomorphic solutions of the equation $f(u) = b(x)$, where b is a holomorphic germ. The formal version of this result is in fact a special instance of the Rank Theorem (for the convergent case, apply the same proof to the result in [HM94]).

It should by now have become clear that the proofs of all the above results are based on one common principle, *linearization*: The respective problems are expressed through certain maps between spaces of power series, and the solution of the problem corresponds to locally linearizing the map at a given point. This process is governed by the Rank Theorem.

Chapter 3

This chapter contains various partial results and open questions concerning the combinatorial, algebraic and geometric structure of jet and arc algebras. These are – in the case of hypersurfaces – defined as follows. Let $f \in k[x_1, \dots, x_m]$ and denote by $k[[t]]$ the formal power series ring in one variable t over the field k. Substitute $\sum_{i=0}^{\infty} x_i^j t^i$ for x_j in f and expand the result as a power series in t. It will be of the form

$$f\left(\sum_{i\geq 0} x_i^j t^i\right) = F_0 + F_1 t + F_2 t^2 + \dots,$$

where $F_e \in k[x_i^j; 1 \leq i \leq e, 1 \leq j \leq m]$, a polynomial ring in countable many variables. Especially we have $F_0 = f(x_0^1, \dots, x_0^m)$. For $q \in \mathbb{N}$ the qth jet scheme X_q of $X = \operatorname{Spec} k[x]/(f)$ is the affine scheme

$$X_q = \operatorname{Spec} k[x_i^j; 1 \leq j \leq m, 0 \leq i \leq q]/(F_0, \dots, F_q).$$

The coordinate ring of X_q will be called the *qth jet algebra*. Moreover, the arc space X_∞ of X is the (in general non-Noetherian) scheme defined by the *arc algebra*

$$k[x_i^j; 1 \leq j \leq m, i \in \mathbb{N}_0]/(F_i; i \in \mathbb{N}_0).$$

Basic Properties of these algebras are studied in section 3.1. The defining equations for jet and arc algebras can be computed from the defining equation f of the base variety using a specific k-derivation D on the polynomial ring $k[\underline{\mathbb{N}}^m] = k[x_i^j; 1 \leq j \leq m, i \in \mathbb{N}_0]$. We show that this operator D has a canonical form in terms of partial derivatives of f and the multivariate Bell-Polynomials (see section 3.2): Denote by D^i the i-fold composition of D with itself, by $\partial_{0,j}$ the partial derivative $\partial/\partial x_0^j$, and write $D^i|_{k[0;m]}$ for the restriction of D^i to the subalgebra $k[0; m]$, where

$$k[q; m] = k[x_i^j; 1 \leq j \leq m, 0 \leq i \leq q]$$

(note that $k[q;m]$ is graded by the weight $\mathrm{wt}(x_i^j) = i$). Then the following holds (cf. Proposition 22 and Corollary 23 in Chapter 3):

Proposition. *The operator $D^i|_{k[0;m]} = D_{i-1}\dots D_0$ has a representation of the form*

$$D^i|_{k[0;m]} = \sum_{|\alpha|\le i} d^i_\alpha \cdot \partial_0^\alpha$$

with $d^i_\alpha \in k[i;m]$, $\mathrm{wt}(d^i_\alpha) = i$ and $\partial_0^\alpha = \prod_{j=1}^m \partial_{0,j}^{\alpha_j}, \alpha = (\alpha_1,\dots,\alpha_m) \in \mathbb{N}_0^m$. Moreover, the coefficients d^i_α fulfill the following recursion:

$$d^i_\alpha = \sum_{j=1}^m \left(\sum_{e=0}^{i-1} x^j_{e+1} \frac{\partial}{\partial x^j_e}(d^{i-1}_\alpha) + x^j_1 d^{i-1}_{\alpha - E_j} \right) \tag{1}$$

with E_j the jth standard basis vector of $\mathbb{R}^m$.

It turns out that the d^i_α are natural generalizations of the Bell-polynomials (see [Com74] or [Rio80]) to the multivariate case, Proposition 27 in section 3.2. This gives a natural and combinatorial way to describe the defining ideal of the jet respectively arc space of X (see Proposition 28):

Proposition. *Let $f \in k[x]$ with qth jet algebra defined by $F_0,\dots,F_q \in k[q;m]$, $q \in \mathbb{N}\cup\{\infty\}$. Then for all i*

$$F_i = \sum_{|\alpha|\le i} d^i_\alpha \cdot \partial_0^\alpha f(x_0^1,\dots,x_0^m),$$

with $d^i_\alpha \in k[i;m]$, $\mathrm{wt}(d^i_\alpha) = i$ the multivariate Bell polynomials, and $\partial_0^\alpha = \prod_{j=1}^m \partial_{0,j}^{\alpha_j}, \alpha = (\alpha_1,\dots,\alpha_m) \in \mathbb{N}_0^m$.

In Question 4.11 of [Mus01] Mustaţa conjectures that the singular locus of the jet scheme of a local complete intersection is precisely the preimage of the singular locus of the base variety under the canonical projection map, i.e.:

Let X be a l.c.i. variety with qth jet space X_q and canonical projection $\pi_q : X_q \to X$. Is the following assertion true?

$$(X_q)_{\mathrm{reg}} = \pi_q^{-1}(X_{\mathrm{reg}}).$$

The canonical morphism $X_q \to X$ is the morphism induced by the inclusion

$$k[0;m] \to k[q;m].$$

In section 3.3 we discuss some examples and approaches towards the problem. We give a prove for a simple special case, namely for hypersurfaces X with pure dimensional jet schemes.

After this we rise the question for the Hilbert-Poincare (in short HP) series of arc algebras with the grading given by $\mathrm{wt}(x_i^j) = i$. For smooth base varieties it turns out to be some power of the generating series of the partition function (see section 3.4):

Proposition. *Let X be an irreducible hypersurface with arc space given by $k[\mathbb{N}^m]/I$, where $I = (F_0, F_1, \ldots)$ is the ideal induced by a defining equation $f \in k[x]$ as before and $I_q = I \cap k[q; m]$. Then with $L = \mathrm{Quot}(k[x_0^j]/(F_0))$ we get*

$$\mathrm{HP}_{L \otimes_{k[x_0^j]} k[q;m]/I_q}(t) = \mathbb{H}_q^{m-1},$$

where $\mathbb{H}_q = \mathrm{HP}_{k[1;m]}$.

The chapter closes with the computation of some completions of polynomial rings in countable many variables. The ultimate goal would be to be able to describe the generators for the defining ideal of the completion of the local ring of an arc algebra in a given arc. This would be quite useful for a constructive version of the Grinberg-Kazhdan-Drinfeld formal arc theorem (see section 2.2). In the Noetherian case of say $k[x]/(g_1, \ldots, g_n)$ this can easily be answered using flatness of $k[[x]]$ over $k[x]$: $\widehat{k[x]_{(0)}} = k[[x]]/(g_1, \ldots, g_n)$ – the generators of the ideal in $k[[x]]$ are the same generators as for the ideal in $k[x]$, considered as elements of $k[[x]]$. In the non-Noetherian case one loses flatness and thus in general this nice result.

Chapter 4

This is joint work with D. Wagner. It deals with an assertion known to experts in resolution of singularities. Nevertheless, up to the knowledge of the author a reference for the presented situation is lacking.
Let X be an affine subvariety of $\mathbb{A}_k^n$, k algebraically closed (but of arbitrary characteristic). We say that X *is algebraic normal crossings* (in short *anc*) *at a point* $p \in X$ if there are local coordinates $y_1, \ldots, y_n$ at p such that X is locally at p given by $y_1 \cdots y_e = 0$ with $e \leq n$ (in the literature this property is also referred to as *simple* or *strict normal crossings*, see [Kol]). By *"local coordinates"* we mean a regular system of parameters for the local ring $\mathcal{O}_{\mathbb{A}^n,p}$. We say that X *is normal crossings* (in short *nc*) *at* p if p is an algebraic normal crossings point for $\widehat{X}_p$, i.e., if the formal germ of X at p is defined by $y_1 \cdots y_e = 0$, where $y_1, \ldots, y_n$ is a formal coordinate system at p. A *formal coordinate system* is a regular system of parameters for $\widehat{\mathcal{O}}_{\mathbb{A}^n,p}$. The locus of points in which X is algebraic normal crossings (resp. normal crossings) is called the *algebraic normal crossings locus of X (resp. normal crossings locus of X)* and will be denoted by X_{anc} (resp. X_{nc}). We prove (see Theorem 18):

Theorem. *The normal crossings locus X_{nc} of a hypersurface $X \subseteq \mathbb{A}^n$ is Zariski-open in X.*

This is done by using the concept of étale neighbourhoods and the Artin Approximation Theorem (see [Art69], Thm. 1.10). All used techniques are applicable to the non-hypersurface case.

Chapter 1

The Rank Theorem for Textile Maps

1.1 Felts and textile maps

In this section we will distinguish subsets of $\mathcal{C}^m$, which are given as the vanishing set of *textile* maps. For these maps we will provide an inverse mapping theorem and the notion of tangent maps.

1.1.1 Space of Cords and Textile Maps

Let k be a field of characteristic 0. A *cord* is a sequence $c = (c_\alpha)_{\alpha \in \mathbb{N}^n}$ of constants c_α in k. Sometimes we will speak more precisely of an *n-cord over k*. The entry c_α is the *coefficient* of c at index α or *α-coefficient* of c. The space $\mathcal{C} = \mathcal{C}_n = \mathcal{C}(k)$ of cords over k naturally identifies with the local k-algebra of formal power series $k[[x_1, \ldots, x_n]]$ in n variables and coefficients in k. The maximal ideal of $k[[x]]$ of power series without constant term is denoted by $\mathfrak{m}$. Substitution of the variables by power series in $\mathfrak{m}$ provides an additional algebraic structure on $\mathcal{C}$. We define $\operatorname{ord} c$ as the order of c as a power series. The space $\mathcal{C}$ comes equipped with the $\mathfrak{m}$-adic topology induced by the 0-neighborhoods $\mathfrak{m}^l$ of series of order $\geq l$. Vectors of cords $c = (c^1, \ldots, c^m) \in \mathcal{C}^m$ will be encoded in the following way: For the coefficients c^i_α, $\alpha \in \mathbb{N}^n$, $i = 1, \ldots, m$, we write $c_{(\alpha, i-1)}$ and understand these coefficients as a family indexed by $\mathbb{N}^{n+1}$. Thus a vector of n-cords $c \in \mathcal{C}^m$ corresponds to an $(n+1)$-cord $(c_\beta)_{\beta \in \mathbb{N}^{n+1}}$ where $c_\beta = 0$ for $\beta_{n+1} \geq m$ and $c_\beta = c^i_{\bar\beta}$ for $\beta_{n+1} = i - 1$, $\bar\beta = (\beta_1, \ldots, \beta_n)$.

A *textile map* $f \colon \mathcal{C}^m \to \mathcal{C}^p$, $m, p \in \mathbb{N}$, maps cords $c = (c_\alpha)_{\alpha \in \mathbb{N}^{n+1}} \in \mathcal{C}^m$ to cords with coefficients consisting of polynomials in the c_α. If the coefficient polynomials are linear, such mappings are called linear. Precisely speaking: f is a textile map if the image of $c \in \mathcal{C}^m$ is of the form

$$f(c) = (f_\alpha(c))_{\alpha \in \mathbb{N}^{n+1}}$$

with $f_\alpha \in k[c_\beta, \beta \in \mathbb{N}^{n+1}]$. The f_α is referred to as the α-component of f. Note: Although the definition of textile maps involves infinitely many variables, in many interesting examples they are given by finite data. See the examples in the introduction. Let U be a subset of $\mathcal{C}^m$. A map $f\colon U \to \mathcal{C}^p$ is called textile if it is the restriction of a textile map $\mathcal{C}^m \to \mathcal{C}^p$ to U. Denote by $k[\underline{\mathbb{N}}_n^m]$ the polynomial ring $k[x_\alpha^j; 1 \leq j \leq m, \alpha \in \mathbb{N}^n]$. The *felt defined by a textile map* $f\colon \mathcal{C}^m \to \mathcal{C}^p$ is the closed subscheme $\operatorname{Spec} k[\underline{\mathbb{N}}_n^m]/(f_\alpha; \alpha \in \mathbb{N}^{n+1})$ of $\operatorname{Spec} k[\underline{\mathbb{N}}_n^m]$. For most applications in section 2 the underlying set of k-points $\{c \in \mathcal{C}^m; f(c) = 0\}$ is of primary interest. It is an "algebraic" subset of a countable Cartesian product of copies of k, where the "equations" are infinitely many polynomials in a polynomial ring of countably many variables. Sometimes we will use the term "felt" also for Zariski-open subsets of some felt defined by a textile map.

Let $Y \subseteq \mathcal{C}^m$ be a felt. A map $f : Y \to k$ is called *regular at* $p \in Y$ if there are a Zariski-open neighbourhood U of p and polynomials $g, h \in k[\underline{\mathbb{N}}_n^m]$ with h non-zero on U such that

$$f = \frac{g}{h}$$

on U. If f is regular at all points of Y we call it *regular (on Y)*. We write $\mathcal{O}_Y$ for the k-algebra of *regular functions on Y*. A textile map $f : Y \to \mathcal{C}$ is called regular if its components f_α are regular functions on Y,i.e., $f_\alpha \in \mathcal{O}_Y$ for all $\alpha \in \mathbb{N}^n$.

The sum of textile maps $f, g : Y \to \mathcal{C}$ is defined naturally by coefficientwise addition. Moreover we introduce the product of these textile maps by Cauchy-multiplication:

$$f \cdot g := (\sum_{\nu+\mu=\alpha} f_\nu g_\mu)_\alpha.$$

The *composition* of textile maps $f : \mathcal{C}^m \to \mathcal{C}^p$ and $g : \mathcal{C}^q \to \mathcal{C}^m$ is defined as follows: $f_\alpha \in k[x_\beta; \beta \in B_\alpha]$, then

$$(f \circ g)_\alpha(c) := f_\alpha(g(c)) = f_\alpha(g_\beta(c); \beta \in B_\alpha)$$

for $c \in \mathcal{C}$. In each coefficient one has a substitution of polynomials. Call f *(right) invertible*, if there exists a textile map g such that $f \circ g = \mathrm{id}$.

Any vector of formal power series $g \in k[[y_1, \ldots, y_m]]^p$ induces a textile map $f : \mathfrak{m} \cdot \mathcal{C}^m \to \mathcal{C}^p$ via substitution $f(c) = g(c)$. Such maps are called *tactile*. The felts defined by tactile maps in the case $n = 1$ are called *arc spaces*. Note that the composition of tactile maps is the same as the tactile map defined by substitution of the corresponding power series.

Sometimes it will be convenient to represent textile linear maps by $\mathbb{N}^2$-matrices, i.e., elements in $k^{\mathbb{N}^2}$: Let $\ell : \mathcal{A} \to \mathcal{A}$ be given by $\ell_i(a) = \sum_{j=0}^\infty \ell_{ij} a_j$. Then we identify ℓ with the $\mathbb{N}^2$-matrix $\underline{\ell} := (\ell_{ij})_{i,j\in\mathbb{N}}$. In analogy to linear algebra the composition of two textile linear maps ℓ^1, ℓ^2 corresponds to the product of the matrices

$$\underline{\ell}^1 \cdot \underline{\ell}^2 = (\sum_{l=0}^\infty \ell_{il}^1 \ell_{lj}^2)_{i,j}.$$

Note that the last sum is well-defined, since $\ell_{il}^j = 0$, $j \in \{1,2\}$, except for a finite number of $l \in \mathbb{N}$.

Let $L : \mathbb{R}^n_{>0} \to \mathbb{R}_{>0}$ be a positive linear form. Moreover, let $<$ be a monomial ordering on $k[[x_1,\ldots,x_n]]$ induced by L, e.g., $x^\alpha < x^\beta$ if $L\cdot\alpha < L\cdot\beta$ or $L\cdot\alpha = L\cdot\beta$ and $\alpha < \beta$ lexicographically. Embed $k[[x_1,\ldots,x_n]]^m$ in $k[[x_1,\ldots,x_{n+1}]]$ by $x^\alpha \cdot e_i \mapsto x^{(\alpha,i-1)}$. For $\beta \in \mathbb{N}^{n+1}$ define $L\cdot\beta$ as $L\cdot\bar{\beta}$. Then we get an ordering on $k[[x_1,\ldots,x_{n+1}]]$ by $x^\alpha < x^\beta$ if: (i) $L\cdot\alpha < L\cdot\beta$ or (ii) $L\cdot\alpha = L\cdot\beta$ and $\alpha < \beta$ lexicographically. This ordering defines a module ordering on $k[[x_1,\ldots,x_n]]^m$ and thus on $\mathcal{C}^m$. Given a module ordering induced by L as above and an element $b \in k[[x]]^m$ the *order of b with respect to L* is defined as

$$\operatorname{ord}_L b = \min\{L\cdot\alpha; b_\alpha \neq 0\}.$$

Analogously we define the *order of a textile map $f : \mathcal{C}^m \to \mathcal{C}^p$ with respect to L* as

$$\operatorname{ord}_L f = \min_{\alpha\in\mathbb{N}^{n+1}}\{L\cdot\alpha; f_\alpha \neq 0\}.$$

Usually, if L has been specified, we simply write $\operatorname{ord} b$ respectively $\operatorname{ord} f$. If no linear form has been defined, we will assume L to be $L : (a_1,\ldots,a_n) \mapsto 1\cdot a_1 + \ldots + 1\cdot a_n$ (as in the beginning of this section). The ideals of textile maps $f : \mathcal{C}^m \to \mathcal{C}$ of order greater equal l, $l \in \mathbb{N}$, define a natural topology on the ring of textile maps $\mathcal{C}^m \to \mathcal{C}$. For obvious reasons we will denote them by $\mathfrak{m}^l$. It should be clear from context what is meant by $\mathfrak{m}^l$, an ideal of power series or an ideal of textile maps.

Consider a textile map $h : U \subseteq \mathcal{C}^m \longrightarrow \mathcal{C}^p$, U an $\mathfrak{m}$-adic neighbourhood of 0. Without loss of generality one may assume $h(0) = 0$. For $\Gamma \in L(\mathbb{Z}^n)$ the textile map h is called *Γ-shifting on U* if

$$\operatorname{ord}(h(a) - h(b)) \geq \operatorname{ord}(a-b) + \Gamma \quad \text{for all } a,b \in U. \tag{1.1}$$

Lemma 1. *A textile map $h : U \subseteq \mathcal{C}^m \to \mathcal{C}$ is Γ-shifting on U if and only if for all $\alpha \in \mathbb{N}^n$ the coefficient $h_\alpha(c)$ is a polynomial in those c_ν for which $L\cdot\nu \leq L\cdot\alpha - \Gamma$.*

For $f \neq 0$ on U the *degree of contraction of f on U (with respect to L)*, denoted by $\kappa(f)$, is defined as the maximal Γ such that (1.1) is fulfilled. If $f = 0$ on U, then we set $\kappa(f) = \infty$. We also say f is *contractive of degree* Γ *(on U with respect to L)*. In case $\Gamma > 0$ the map h is simply called *contractive*. By the Lemma above it is easy to see that the contraction degree is well-defined. Let f, g be textile maps of contraction degree Γ resp. Θ on their domains of definition U respectively V. Then $\kappa(f+g) \geq \min\{\Gamma,\Theta\}$ on $U \cap V$. Next assume that the composition $f\circ g$ is well-defined and that $\kappa(f|_{g(V)}) = \Gamma$. Then $\kappa(f\circ g) = \Gamma + \Theta$ on V.

It turns out that tactile maps have nice contraction properties. This can be seen in the following example:

Example 1. Consider the “monomial” map $f : U \subseteq \mathcal{C}^m \to \mathcal{C}; c \mapsto c^\alpha$, $\alpha \in \mathbb{N}^m$, with $U = \mathfrak{m}^{p_1} \times \cdots \times \mathfrak{m}^{p_m}$. Denote by $p_{\min}$ the minimum of the p_i's. It is immediate that $\kappa(f) = \sum_{i=1}^m \alpha_i p_i - p_{\min}$.

This example shows that one has some control over the degree of contraction of tactiles. By shrinking the domain of definition U the contraction degree increases!

The notion of contractivity will be used in the following more general context (see section 2.2). Let A be a local, commutative and unital k-algebra with nilpotent maximal ideal $\mathfrak{n}$, i.e., there exists an integer $N \in \mathbb{N}$ such that $\mathfrak{n}^N = 0$. The set of cords with coefficients in A will be denoted by $\mathcal{C}_A$. For $a \in A$ set

$$\mathrm{pow}(a) = \begin{cases} \max\{i; a \in \mathfrak{n}^i\} & a \neq 0 \\ N & a = 0 \end{cases}$$

where $\mathfrak{n}^0 = A$. For an element $c \in \mathcal{C}_A^m$ we define the *refined order of* a as the pair

$$\textsc{ord}(c) = (\mathrm{ord}(c), \mathrm{pow}(c_{\mathrm{ord}\, c})) \in \mathbb{N}^2,$$

where $\mathbb{N}^2$ is considered with the lexicographic ordering. If $(c_i)_{i\in\mathbb{N}}$ is a family of cords in $\mathcal{C}_A^m$ with increasing sequence of modified orders $(\textsc{ord}(c_i))_i$, then the family is summable. A map $h\colon U \subseteq \mathcal{C}_A^m \to \mathcal{C}_A^p$, U an $\mathfrak{m}$-adic neighbourhood of 0, is called contractive if for all cords a and b in U

$$\textsc{ord}(h(a) - h(b)) > \textsc{ord}(a - b)$$

holds. If $A = k$ the only nilpotent ideal in A is the zero-ideal and both notions of contractivity coincide.

Remark. For other applications it might be reasonable to interchange the order of ord and pow in the definition of the modified order.

Theorem 3 (Inverse Mapping Theorem)**.** *Let $U \in \mathcal{C}_A^m$ be an $\mathfrak{m}$-adic neighbourhood of 0 and $f\colon U \subseteq \mathcal{C}_A^m \to \mathcal{C}_A^m$ be textile with $f(0) = 0$. Assume that $f = \ell + h$, $f(U) \subseteq W$ (neighbourhood of 0), where $\ell\colon U \to W$ is textile, linear and invertible, i.e., there exists a linear map $\tilde{\ell} : W \to U$ with $\ell \circ \tilde{\ell} = \mathrm{id}_W$ and $\tilde{\ell} \circ \ell = \mathrm{id}_U$. If $\tilde{\ell}h$ is contractive on U, then f admits on U an inverse, i.e., there is a textile map $g : f(U) \to U$ with $f \circ g = \mathrm{id}_{f(U)}$ and $g \circ f = \mathrm{id}_U$.*

Proof. First simplify to the case $f = \mathrm{id} + h$ with h contractive on U by considering $\tilde{\ell}f = \mathrm{id} + \tilde{\ell}h$. Then recursively define a sequence $(g_j)_{j\in\mathbb{N}}$ of textile maps $U \to \mathcal{C}_A^m$ by $g_0 = 0$ and $g_{j+1} = \mathrm{id} - h \circ g_j$ for $j \geq 1$. The sequence $(g_j)_j$ converges pointwise on U. Indeed, g_{j+1} can be written as

$$g_{j+1} = g_0 + \sum_{i=1}^{j+1} D_i,$$

with $D_j = g_j - g_{j-1}$. For any $a \in U$ the family $(D_j(a))_{j\in\mathbb{N}}$ is summable: For $j \geq 2$ we can write D_j as

$$D_j = h \circ g_{j-2} - h \circ g_{j-1},$$

and thus by contractivity of h

$$\textsc{ord}\, D_j(a) > \textsc{ord}\, D_{j-1}(a)$$

for any $a \in U$. Therefore, $g = \lim g_j$ is a well-defined textile map $U \to \mathcal{C}_A^m$. It remains to show that g is a (right-) inverse for f. But for any $j \in \mathbb{N}$

$$f \circ g - \mathrm{id} = (g - g_j) + (h \circ g - h \circ g_j) + (h \circ g_j - h \circ g_{j-1}),$$

and each of the three summands on the right hand side tends to 0 for $j \to \infty$. Therefore, g is a right-inverse for f.
It's not hard to see that g is in fact also a left-inverse for f. It suffices to show that

$$(\mathrm{ORD}(g_m(f(a)) - a))_{m \in \mathbb{N}}$$

is an increasing sequence (for all $a \in U$) which follows immediately from contractiveness of h. $\square$

Remarks. (a) Note that there are textile invertible maps which are not of the special form demanded in the theorem. An example would be

$$f\colon \mathcal{A} \to \mathcal{A}; (a_i)_i \mapsto (a_0 + a_1^2, a_1, a_2 + a_3^2, a_3, + \ldots).$$

Nevertheless this textile map may be transformed into a form which allows the use of Theorem 3.
(b) Theorem 3 implies as a special case the classical Inverse Mapping Theorem for formal power series: Let $f = (f_1, \ldots, f_n) \in k[[x_1, \ldots, x_n]]^n$ with $f(0) = 0$. Denote by ∂f its Jacobian matrix $(\partial f_i / \partial x_j)_{i,j}$. If $\det \partial f(0) \neq 0$ then there exist $g_1, \ldots, g_n \in \mathfrak{m} \cdot k[[x_1, \ldots, x_n]]$ with

$$f(g_1(x), \ldots, g_n(x)) = (x_1, \ldots, x_n).$$

Lemma 2. *Assume that $h\colon U \subseteq \mathcal{C}_A^m \to \mathcal{C}_A$, U an $\mathfrak{m}$-adic neighbourhood of 0, is textile with $h(0) = 0$. Moreover, let its α-coefficient h_α be of the form $h_\alpha = h'_\alpha + h''_\alpha$ with $h'_\alpha \in A[x_l^j; 1 \leq j \leq m, 0 \leq l < e]$ and $h''_\alpha \in \mathfrak{n}[x_e^j, 1 \leq j \leq m,]$, where $e = |\alpha|$. Then h is contractive.*

Proof. For simplicity of notation we restrict to the case $h\colon \mathcal{A}_A^m \to \mathcal{A}_A$. Let $a, b \in U$ with $\mathrm{ord}(a - b) = i$. Write $a = b + \zeta$ for some $\zeta \in U$, $\zeta_i \neq 0$. By assumption on h we have $h_j(a) - h_j(b) = 0$ for $0 \leq j \leq i - 1$. Using Taylor expansion we see for the ith coefficient

$$h_i(b + \zeta) - h_i(b) = \partial h''_i(b) \cdot \zeta_i + \partial^2 h''_i(b) \cdot \zeta_i^2 + \ldots \tag{1.2}$$

where $\partial^l h''_i(b) \cdot \zeta_i^l$ abbreviates all order l terms in the Taylor expansion. By assumption on h all partial derivatives in (1.2) lie in $\mathfrak{n}$, thus

$$\mathrm{pow}(h_i(a) - h_i(b)) \geq \mathrm{pow}(\zeta_i) + 1 > \mathrm{pow}((a - b)_i),$$

so h is contractive. $\square$

Denote by $\mathcal{C}(q)$ the *set of q-cords*, that is

$$\mathcal{C}(q) = \{(c_\alpha)_{\alpha \in \mathbb{N}^n} \in k^{n(q+1)}; |\alpha| < q + 1\} \cong \mathcal{C} \mod \mathfrak{m}^{q+1}.$$

Analogously we introduce $\mathcal{C}^m(q)$, and especially if X is the felt defined by a textile map $f\colon \mathcal{C}^m \to \mathcal{C}$ the *set of q-cords of X*

$$X(q) = \{(c_\alpha)_{\alpha \in \mathbb{N}^{n+1}} \in (k^{n(q+1)})^m; |\alpha| < q + 1, f(c) = 0 \mod \mathfrak{m}^{q+1}\}.$$

Clearly, $X(q)$ has a natural structure as a subscheme of $\mathbb{A}^{n(q+1)m}$. The canonical projections $X \to X(q)$, resp. $X(q) \to X(p)$ for $q \geq p$, induced by truncation of cords, resp. q-cords, will be denoted by π_q, resp. π_p^q. We will identify $\mathcal{C}(q)$ with those cords $c \in \mathcal{C}$ such that $c_\alpha = 0$ for all $\alpha \in \mathbb{N}^n$ with $|\alpha| > q$. Then a textile map $f\colon \mathcal{C}^m \to \mathcal{C}^r$ induces a map $f_q\colon \mathcal{C}(q)^m \to \mathcal{C}(q)^r$ with components $(f_q)_\alpha$. It's easy to see that if f is 0-shifting, then f is compatible with the truncation maps π_q in the sense that the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{C}^m \arrow[r, "f"] \arrow[d, "\pi_q"'] & \mathcal{C}^r \arrow[d, "\pi_q"] \\
\mathcal{C}^m(q) \arrow[r, "f_q"] \arrow[d, "\pi_p^q"'] & \mathcal{C}^r(q) \arrow[d, "\pi_p^q"'] \\
\mathcal{C}^m(p) \arrow[r, "f_p"] & \mathcal{C}^r(p)
\end{tikzcd}$$

Let $f : \mathcal{C}^m \to \mathcal{C}^p$ be a textile map. The felt $X = f^{-1}(0)$ is called *smooth at a cord* $c \in X$ if there is an $\mathfrak{m}$-open neighbourhood U of c such that $X \cap U$ is isomorphic to the kernel of a linear textile map. Smoothness of a (k-)point in classical algebraic geometry can be embedded into the theory of felts in the following way: Let X be a subvariety of $\mathbb{A}_k^n$ given by the ideal $\langle f_1, \ldots, f_p \rangle \subseteq k[x_1, \ldots, x_n]$. Consider the tactile map

$$F_X : \mathcal{A}^n \to \mathcal{A}^p; a \mapsto (f_1(a), \ldots, f_p(a)).$$

The k-points of X, i.e., elements $b = (b^1, \ldots, b^n) \in k^n$ with $f_i(b) = 0$ for all i, correspond to arcs $a = (a^1, \ldots, a^n) \in X_\infty$ with $a^i = (b^i, 0, \ldots)$. Using the classical Implicit Function Theorem or the Rank Theorem from section 1.3 it is obvious that X_∞ is smooth at a if b is a smooth point of X. With the same methods one shows that all arcs $a \in X_\infty$ with $a(0)$ a smooth point of X are smooth cords of X_∞.

1.1.2 Differential Calculus

This section provides the basic definitions for the differential apparatus in the context of textile maps. Consider a textile map $f : \mathcal{C}^m \to \mathcal{C}; c \mapsto (f_\alpha(c))_\alpha$. For $\nu \in \mathbb{N}^{n+1}$ the *partial derivative of f with respect to ν* is defined by coefficientwise differentiation:

$$\partial_\nu f(c) := \left(\frac{\partial f_\alpha(c)}{\partial c_\nu} \right)_{\alpha \in \mathbb{N}^n}.$$

Obviously ∂_ν is a k-derivation on $\mathcal{C}$. We write $\partial f(a)$ for the vector

$$(\partial_\nu f(a))_{\nu \in \mathbb{N}^{n+1}}.$$

The *directional derivative of f in $a \in \mathcal{C}^m$ in direction of $v \in \mathcal{C}^m$*, is given by

$$T_a f \cdot v := \frac{1}{s} \left(f(a + sv) - f(a) \right) \Big|_{s=0}.$$

Using the notation $(\partial f)(a) \bullet v := \sum_{\nu \in \mathbb{N}^{n+1}} \partial_\nu f(a) v_\nu$ this yields for a textile map f:

$$T_a f \cdot v = (\partial f)(a) \bullet v.$$

In the case of tactile maps the directional derivative can be calculated as usually. Let $F : \mathcal{C}^m \to \mathcal{C}$ be given by $c \mapsto f(c)$, with $f \in k[x_1, \ldots, x_m]$, then

$$T_a F \cdot v = \partial_1 f(a) \cdot v_1 + \cdots + \partial_m f(a) \cdot v_m.$$

It's easy to see that each f has a Taylor expansion around a given by

$$f(y) = \sum_{\nu \in \mathbb{N}^{n+1}} \frac{1}{\nu!} (\partial_\nu f)(a)(y-a)^\nu.$$

Consider textile maps $f : \mathcal{C}^m \to \mathcal{C}^p$ and $g : \mathcal{C}^l \to \mathcal{C}^m$ with well-defined composition $f \circ g$. The chain-rule for partial derivatives is given by: $\partial_\nu (f \circ g)(a) = \partial f(g(a)) \bullet \partial_\nu g(a)$. More generally one deduces the following properties of the derivative:

Proposition 3. *With the above assumptions the following holds:*
1. $\partial(f \circ g)(a) = (\partial f(g(a)) \bullet \partial_\nu g(a))_\nu$
2. Let ϕ be a textile linear map, then

$$T_a(\phi \circ f) \cdot v = \phi(T_a f \cdot v).$$

3. Let ϕ be of the form $\phi = \mathrm{id} + g$, and $f \circ g$ well-defined, then

$$\begin{aligned} T_a(f \circ \phi) \cdot v &= \partial f(\phi(a)) \bullet v + \partial f(\phi(a)) \bullet (\partial g(a) \bullet v) \\ &= T_{\phi(a)} f \cdot v + T_{\phi(a)} f \cdot (T_a g(a) \cdot v) \ . \end{aligned}$$

Proof. For the easy calculations see [Bru05]. □

1.2 Scissions of linear maps

We recall the notion of scissions of linear maps (cf. [HM94], p. 99). In Theorem 3 the inverse of a map is constructed using the inverse of its tangent map. For the proof of the Rank Theorem (see section 1.3) a similar construction is needed. There the linearizing automorphisms rely on inverting the tangent map on its image (and going back to a direct complement of its kernel).
Let $\ell : V \to W$ be a linear map of k-vector spaces V and W. A *scission* of ℓ is a linear map $\sigma : W \to V$ with $\ell\sigma\ell = \ell$. Scissions provide projections $\pi_{\ker} := \mathrm{id} - \sigma\ell$ onto $\ker(\ell)$ and $\pi_{\mathrm{im}} := \ell\sigma$ onto $\mathrm{im}(\ell)$. Thus, σ induces direct sum decompositions

$$V = \ker(\ell) \oplus \mathrm{im}(\sigma\ell) \quad \text{and} \quad W = \mathrm{im}(\ell) \oplus \ker(\ell\sigma).$$

On the other hand each direct sum decomposition of the form

$$V = \ker(\ell) \oplus L, \quad W = \mathrm{im}(\ell) \oplus J$$

induces a scission σ of ℓ by

$$\sigma := (\ell|_L)^{-1} \circ \pi,$$

where π is the projection from W onto $\operatorname{im}(\ell)$ with kernel J. In the following sections we will explicitly construct scissions for textile linear maps. This is necessary to obtain precise information on the degree of contraction of the scission used to linearize a map of constant rank. For the case of $k[[x]]$-linear maps ℓ, the degree of contraction will be given by the maximal order of a standard basis of the module $\operatorname{im}(\ell)$.

1.2.1 Scissions of $k[[x]]$-linear maps

We start with a short reminder on standard bases. Most of the notation has already been introduced in section 1.1.1. Write $x^{\alpha,j}$, $\alpha \in \mathbb{N}^n$, $j \in \{1,\dots,m\}$, for the vector $(0,\dots,x^\alpha,0,\dots,0)^\top$ in $k[[x]]^m$, where x^α is at the jth component. For $\alpha \in \mathbb{N}^n$ we use the usual multiindex notation, i.e., $x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n}$. Thus, each element $a \in k[[x]]^m$ can be uniquely written as

$$a = \sum_{\alpha,j} a_{\alpha,j} x^{\alpha,j}.$$

Fix a monomial order $<$ on $\mathbb{N}^n$. We assume that it is induced by a positive linear form L on $\mathbb{R}^n_{>0}$. Extend this order to a module ordering on $k[[x]]^m$, again denoted by "$<$". The *support* of an element $a \in k[[x]]^m$ is defined as

$$\operatorname{supp}(a) := \{(\alpha,j) \in \mathbb{N}^{n+1}; a_{\alpha,j} \neq 0\}.$$

By $\operatorname{in}(a)$ we denote the initial monomial vector $x^{\alpha,j}$ of a, i.e.,

$$(\alpha,j) = \min_{<} \operatorname{supp}(a).$$

An element $a \in k[[x]]^m$ is called *L-monic* or just *monic*, if the coefficient of $\operatorname{in}(a)$ is 1. Let M be a submodule of $k[[x]]^p$. Then $\operatorname{in}(M)$ denotes the initial module of M: $\operatorname{in}(M) := \langle \operatorname{in}(a); a \in M\rangle$. A subset $\mathcal{F} = \{f_1,\dots,f_l\}$ of M is called a *standard basis* of M (w.r.t. $<$) if

$$\operatorname{in}(\langle f_1,\dots,f_l\rangle) = \langle \operatorname{in}(f_1),\dots,\operatorname{in}(f_l)\rangle.$$

Consider a $k[[x]]$-linear map $\ell : k[[x]]^m \to k[[x]]^p; a = (a^1,\dots,a^m) \mapsto \sum_{i=1}^m f_i a^i$. Assume that $f_1,\dots,f_m$ form a standard basis for $\operatorname{im}(\ell)$. Choose a partition of $S = \operatorname{supp}(\operatorname{in}(\operatorname{im}(\ell))) \subseteq \mathbb{N}^n \times \{1,\dots,p\}$ by disjoint sets $S_1,\dots,S_m$, such that $S_i \subseteq \operatorname{supp}(k[[x]]\cdot \operatorname{in}(f_i))$. For a subset A of $\mathbb{N}^n \times \{1,\dots,p\}$ we will consider the projection $p_1(A)$ onto the first factor $p_1(A) := \{\alpha \in \mathbb{N}^n; \exists j \in \{1,\dots,p\} : (\alpha,j) \in A\}$. Moreover, set

$$\Delta(\ell) := \{b \in k[[x]]^p; \operatorname{supp}(b) \cap \operatorname{supp}(\operatorname{in}(\operatorname{im}(\ell))) = \emptyset\}$$

and

$$\nabla(\ell) := \{a \in k[[x]]^m; \operatorname{supp}(a_i f_i) \subseteq S_i, \quad \text{for all} \quad i\}.$$

With $\ell^0 : k[[x]]^m \to k[[x]]^p; a \mapsto \sum_i \operatorname{in}(f_i)a^i$ this means

$$k[[x]]^m = \ker(\ell^0) \oplus \nabla(\ell) \quad\text{and}\quad k[[x]]^p = \operatorname{im}(\ell^0) \oplus \Delta(\ell).$$

As the next theorem shows even more is true: This decomposition still holds with ℓ^0 replaced by ℓ.

Theorem 4. *Let $f_1, \ldots, f_m \in k[[x]]^p$ be a monic standard basis. Denote* $\mathrm{in}(f_i)$ *by x^{α^i, j_i}. Let $S = \uplus_{i=1}^m S_i$ be a partition of the support of the initial module generated by $f_1, \ldots, f_m$. Then for all $f \in k[[x]]^p$ there exist unique quotients $g_j \in k[[x]]$, $1 \leq j \leq m$, and a unique remainder $h \in k[[x]]^p$ such that*

1. $f = g_1 f_1 + \ldots + g_m f_m + h$ with $\mathrm{supp}(g_i) \subseteq p_1(S_i) - \alpha^i$ *and* $\mathrm{supp}(h) \subseteq \mathbb{N}^n \times \{1, \ldots, p\} \setminus S$.

2. The linear map $\ell : k[[x]]^m \to k[[x]]^p; a = (a^1, \ldots, a^m) \mapsto \sum_i f_i a^i$ induces a direct sum decomposition

$$k[[x]]^m = \ker(\ell) \oplus \nabla(\ell) \quad \text{and} \quad k[[x]]^p = \mathrm{im}(\ell) \oplus \Delta(\ell).$$

For a proof of this theorem we refer to [Gra72], [Hir77], [Gal74], [Gal79], [Sch80] or [HM94]. Using the notation of the theorem, a scission σ of ℓ can be constructed as follows: Denote by $D_{\mathcal{F}}$ the map

$$D_{\mathcal{F}} : \mathrm{im}(\ell) \to \nabla(\ell); f \mapsto (g_1, \ldots, g_m).$$

A projection onto the image of ℓ is given by

$$\pi : k[[x]]^p \to \mathrm{im}(\ell); f \mapsto f - h.$$

Then $\sigma := D_{\mathcal{F}} \circ \pi$ is a scission of ℓ.

If $\mathcal{F}$ doesn't form a standard basis, we may choose a (monic) standard basis $\hat{\mathcal{F}} = \{F_1, \ldots, F_l\}$, inducing a $k[[x]]$-linear map $\rho\colon k[[x]]^l \to k[[x]]^m$ with

$$\rho_j(a) = \sum_{i=1}^{l} \beta_j^i a^i$$

where $F_i = \sum_j \beta_j^i f_j$. Thus $\ell \circ \rho = \hat{\ell}$, where

$$\hat{\ell}\colon k[[x]]^l \to k[[x]]^p; a = (a^1, \ldots, a^l) \mapsto \sum_i F_i a^i.$$

By the preceding remarks $\tau := D_{\hat{\mathcal{F}}} \circ \pi_{\mathrm{im}(\hat{\ell})}$ is a scission of $\hat{\ell}$ and thus

$$\sigma := \rho \tau \pi_{\mathrm{im}(\ell)} \tag{1.3}$$

is a scission of ℓ. We will refer to this scission of ℓ as the *scission of ℓ corresponding to the standard basis $\hat{\mathcal{F}}$.*

Proposition 4. *Let ℓ be a $k[[x]]$-linear map. Then the scission σ of ℓ corresponding to a standard basis $\mathcal{F}$ of* $\mathrm{im}(\ell)$ *has degree of contraction $\kappa(\sigma)$ bounded by*

$$\kappa(\sigma) \geq -\max_{f \in \mathcal{F}} \{\mathrm{ord}(f)\}.$$

Proof. Consider a linear map $\ell : k[[x]]^m \to k[[x]]^p; a \mapsto \sum_{i=1}^m f_i a^i$ with $\mathcal{F} = \{f_1, \ldots, f_m\}$ forming a standard basis of $\mathrm{im}(\ell)$. Define σ as the scission $D_{\mathcal{F}} \circ \pi$ of ℓ corresponding to $\mathcal{F}$. Further, set for $1 \leq i \leq m$

$$\Delta_i = \{a \in k[[x]]; \mathrm{supp}(a) \subseteq p_1(S_i) - \alpha^i\}$$

and

$$\bar{\Delta} = \{a \in k[[x]]^p; \mathrm{supp}(a) \subseteq \mathbb{N}^n \times \{1, \ldots, p\} \setminus S\}.$$

The division map $D_{\mathcal{F}}$ is given by (see [Sch80])

$$D_{\mathcal{F}} = \sum_{k=0}^{\infty} \Phi_1^{-1} \circ \left(\Phi_2 \circ \Phi_1^{-1}\right)^k .$$

Here, Φ_1 and Φ_2 are defined as

$$\Phi_1 : (\oplus_{i=1}^m \Delta_i) \oplus \bar{\Delta} \longrightarrow k[[x]]^p; (g_1, \ldots, g_m, h) \mapsto g_1 x^{\alpha^1, j_1} + \ldots + g_m x^{\alpha^m, j_m}$$

and

$$\Phi_2 : (\oplus_{i=1}^m \Delta_i) \oplus \bar{\Delta} \longrightarrow k[[x]]^p; (g_1, \ldots, g_m, h) \mapsto g_1 u^1 + \ldots + g_p u^p,$$

with u^i given by $\ell^i = x^{\alpha^i, j_i} - u^i$. After a change of the linear form L (see [Sch80]) we may assume that $\mathrm{ord}\, x^{\alpha^i, j_i} < \mathrm{ord}\, u^i$. Thus, $\kappa(\left(\Phi_2 \circ \Phi_1^{-1}\right)^k) \geq 0$: Φ_1^{-1} operates on a power series $a \in k[[x]]^p$ by singling out the x^{α^i, j_i}. Hence, $\mathrm{supp}(\Phi_1^{-1}(a))_i \cap p_1(S_i) \subseteq p_1(S_i) - \alpha^i$, $i = 1, \ldots, m$. The maximal shift is given by the maximal order of the standard basis $\mathcal{F}$. The second map Φ_2 multiplies each component of $\Phi_1^{-1}(a)$ by a u^i, which has higher order than x^{α^i, j_i} – the order of a has increased after applying $\Phi_2 \circ \Phi_1^{-1}$, which means positive contraction degree! By section 1.1.1

$$\kappa(D_{\mathcal{F}}) \geq \min_k \{\kappa(\Phi_1^{-1} \circ \left(\Phi_2 \circ \Phi_1^{-1}\right)^k)\}.$$

So $\kappa(D_{f_i}) = \kappa(\Phi_1^{-1}) = -\max_{f \in \mathcal{F}}\{\mathrm{ord}\, f\}$. If $\mathcal{F}$ is not a standard basis, a scission σ is constructed as in equation (1.3). Then $\kappa(\sigma) \geq \kappa(\rho) + \kappa(\tau) \geq \kappa(\tau)$. For τ the claim was just proven. □

We finish this section with a special $k[[x]]$-linear mapping:

Construction 1. Let $\ell : k[[x]]^n \to k[[x]]^n$ be given by a matrix $A \in k[[x]]^{n \times n}$ with $\det A \neq 0 \in k[[x]]$. In this case it is sufficient to use the Weierstrass division theorem instead of Theorem 4. A scission σ for ℓ will be defined by multiplication with A^{adj}, the adjoint matrix of A, and then taking the quotient by Weierstrass division. For the last step, recall the Weierstrass division theorem: Let $h \in k[[x]]$, then there exists a linear change of coordinates φ so that $h\varphi$ is x_n-regular of order $\mathrm{ord}\, h$. The theorem asserts that for any $g \in k[[x]]$ there exist *unique* power series $B, R \in k[[x]]$ such that

$$h\varphi = B \cdot g + R,$$

with $\deg_{x_n} R < \operatorname{ord} h$. The series $B\varphi^{-1}$ is called the quotient of h by g using Weierstrass division, and will henceforth be denoted by $Q(h;\varphi,g)$. Note that Q is textile.

Let $y = (y_1,\dots,y_n)^\top$ be defined by $y = A^{adj}\cdot z$ for given $z = (z_1,\dots,z_n)^\top \in k[[x]]^n$. In addition assume that φ_i are linear coordinate changes such that $y_i\varphi_i$ is x_n-regular of order $\operatorname{ord} y_i$. Then we define

$$\sigma(z) = (Q(y_1,\varphi_1,\det A),\dots,Q(y_n,\varphi_n,\det A)).$$

Proposition 5. *The map σ as defined above is textile and a scission for ℓ. Moreover, its degree of contraction is*

$$\kappa(\sigma) \geq -\operatorname{ord}\det A + \kappa(A^{adj}\cdot(-)).$$

Proof. That σ is textile follows from a proof of the Weierstrass Division theorem. An easy calculation shows that it is indeed a scission and the contraction degree follows from the results in section 1.1. □

Note that the last construction differs from the one obtained earlier by means of standard basis. It's advantage lies in the relative easy computation of its degree of contraction – no estimates on the order of a standard basis of its column module are necessary!

1.2.2 Scissions of arbitrary linear maps

For simplicity of notation we restrict our considerations in this section to textile linear maps on $\mathcal{A}$. All results have generalizations to the multivariate case. An arc $a \in \mathcal{A}$ is a vector of infinite (countable) length. In 1.1.1 textile linear maps $\mathcal{A} \to \mathcal{A}$ were identified with matrices of countable many rows and columns, called $\mathbb{N}^2$-matrices. By definition of textile maps, each row contains just finitely many nonzero entries. We say a matrix S is a *scission* of a matrix A, if $ASA = A$ holds. First consider the case of linear maps between finite dimensional k-vectorspaces. There, linear algebra provides a simple way of constructing scissions: For example let $\ell : k^3 \to k^2$ be the k-linear map given by the matrix

$$A := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

According to the Gauss Algorithm there are matrices $P \in \mathrm{Gl}_2(k)$ and $Q \in \mathrm{Gl}_3(k)$ such that A is equivalent to

$$\tilde{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

i.e., $PAQ = \tilde{A}$. P operates on A by elementary row operations, Q by elementary column operations. A scission of $\tilde{A}$ is given by the matrix

$$\tilde{S} := \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Indeed, $\tilde{A}\tilde{S}\tilde{A} = \tilde{A}$. Thus, the matrix $S := Q\tilde{S}P$ is a scission of ℓ. The same reasoning applies for textile linear maps $\ell : \mathcal{A} \to \mathcal{A}$. Two textile linear maps

$\ell_1, \ell_2 : \mathcal{A} \to \mathcal{A}$ are called *equivalent* if there exist linear automorphisms P, Q of $\mathcal{A}$ such that

$$P\ell_1 Q = \ell_2.$$

Let $\ell : \mathcal{A} \to \mathcal{A}$ be a textile linear map. We represent ℓ by an $\mathbb{N}^2$-matrix $(\ell_{ij})_{ij}$ denoted by $\underline{\ell}$. Since the maps are textile for all $i \in \mathbb{N}$ there exists an $N_i \in \mathbb{N}$ such that $\ell_{ij} = 0$ for $j > N_i$. The submatrix of $\underline{\ell}$ obtained by fixing the first index $i \in \mathbb{N}$ is called the ith row of $\underline{\ell}$. Analogously, the jth column of $\underline{\ell}$ is defined as the submatrix where the second index is fixed to j. The $\mathbb{N}^2$-matrix $\underline{\ell}$ is said to be in *canonical form* if all rows and columns contain at most one nonzero entry. Denote by $\underline{\epsilon}^{ij}$ the matrix $(\delta_{ik}\delta_{lj})_{kl}$ with δ_{kl} the Kronecker symbol.
Let $(\underline{s_l})_{l\in\mathbb{N}}$ be a sequence of $\mathbb{N}^2$-matrices. An $\mathbb{N}^2$-matrix $\underline{s}$ is the *limit* of $(\underline{s_l})_l$ if s is the limit of $(s_l)_l$ as textile linear maps (see section 1.1.1). This means that for all $N \in \mathbb{N}$ there exists an N_0 such that for all $l \geq N_0$ the jth row of $(\underline{s} - \underline{s_l})$ is zero for $0 \leq j \leq N$. In analogy to the notation for power series, we write for the last condition $\operatorname{ord}(\underline{s} - \underline{s_l}) > N$.

Theorem 5. *Every textile linear map $\ell : \mathcal{A} \to \mathcal{A}$ is equivalent to a textile linear map in canonical form. The transformation maps P, Q and the canonical form $\tilde{\ell}$ are obtained by the algorithm described below.*

Algorithm.

(i) $i := 0$, $\underline{\tilde{\ell}} := \underline{\ell}$, $\underline{P} := \underline{\mathrm{id}}$, $\underline{Q} := \underline{\mathrm{id}}$;

(ii) Set $N_i := \max\{j; \ell_{ij} \neq 0\}$, w.l.o.g. the *pivot element* ℓ_{iN_i} equals 1;

(iii) Eliminate the (l, N_i) entries of $\underline{\ell}$ for $l > i$ by applying

$$\underline{P}_i = \underline{\mathrm{id}} - \sum_{l=1}^{\infty} \ell_{i+l,N_i} \cdot \underline{\epsilon}^{i+l,i}$$

from the left;

(iv) Eliminate the (i, k) entries of $\underline{\ell}$ for $k < N_i$ by applying

$$\underline{Q}_i := \underline{\mathrm{id}} - \sum_{j=0}^{N_i-1} \ell_{i,j} \cdot \underline{\epsilon}^{N_i,j}$$

from the right;

(v) Set $\underline{P} := \underline{P}_i \circ \underline{P}$, $\underline{Q} := \underline{Q} \circ \underline{Q}_i$, $\underline{\tilde{\ell}} := \underline{P} \circ \ell \circ \underline{Q}$ and $i = i + 1$. Return to (ii).

Remark. Note that the constructed $\mathbb{N}^2$-matrices $\underline{P}$ (resp. $\underline{Q}_i$) are lower diagonal $\mathbb{N}^2$-matrices. Hence the corresponding linear maps P (resp. Q) are invertible.

Example 2. Consider the difference operator $\Delta = s^d + h_{d-1}s^{d-1} + \ldots + h_0 \in k[s]$, $s \circ (a_0, a_1, \ldots) = (a_1, \ldots, a_2, \ldots)$, with induced map $\ell : \mathcal{A} \to \mathcal{A}$,

$$a \mapsto \ell(a) = (h_0 a_0 + \cdots + h_{d-1}a_{d-1} + a_d, h_0 a_1 + \cdots + h_{d-1}a_d + a_{d+1}, \ldots),$$

for $a \in \mathcal{A}$. The algorithm yields as the canonical form of ℓ

$$a \mapsto \tilde{\ell}(a) = (a_d, a_{d+1}, a_{d+2}, \ldots, a_{d+i}, \ldots).$$

This implies the well known fact that the set of solutions to the equation $\Delta \circ a = b$ is isomorphic to k^d.

Proof. The finite composition $\underline{P}_j \circ \cdots \circ \underline{P}_0$ is denoted by $\underline{P}^j$ and analogously $\underline{Q}^j := \underline{Q}_0 \circ \cdots \circ \underline{Q}_j$. The sequence $(P^j)_{j \in \mathbb{N}}$ converges to an $\mathbb{N}^2$-matrix $\underline{P}$, since

$$\mathrm{ord}(\underline{P_j} - \underline{\mathrm{id}}) \geq j.$$

Thus, $\mathrm{ord}(\underline{P}^{j+1} - \underline{P}^j) \geq j+1$. All $\underline{P_j}$ are lower diagonal, hence, the same is true for $\underline{P}$. For $(\underline{Q}^j)_j$ it is sufficient to show that for all $l \in \mathbb{N}$ there exists an n_l such that for all $m, n > n_l$

$$\mathrm{ord}(\underline{Q}^m - \underline{Q}^n) > l.$$

Fix an $l \in \mathbb{N}$ and define n_l as a natural number such that all $j \leq l$ appear as the column element of a pivot element $\ell_{rj}, r < n_l$. This gives convergence of $(\underline{Q}^j)_j$. The algorithm obviously transforms $\underline{\ell}$ into its canonical form. $\square$

Construction 2. A scission σ of a k-linear map $\ell : \mathcal{A} \to \mathcal{A}$ can be constructed as follows: First one reduces to the case of linear maps in canonical form. Assume a scission $\sigma_{\tilde{\ell}}$ is constructed for the canonical form $\tilde{\ell}$ of ℓ. A scission of ℓ is obtained by $\sigma_\ell := Q \sigma_{\tilde{\ell}} P$, where P, Q are automorphisms of $\mathcal{A}$ with $P \ell Q = \tilde{\ell}$. Thus, without loss of generality ℓ has matrix representation $\underline{\ell}$ in canonical form. Denote by I_1 the set of indices $i \in \mathbb{N}$ s.t. the ith row of $\underline{\ell}$ is nonzero and by I_2 the set of indices $j \in \mathbb{N}$ such that the jth column is nonzero. Then

$$\mathrm{im}(\ell) = \{\gamma \in k^{\mathbb{N}}; \gamma_k = 0 \ \text{ for all } \ k \in \mathbb{N} \setminus I_1\}$$

and

$$\ker(\ell) = \{\gamma \in k^{\mathbb{N}}; \gamma_k = 0 \ \text{ for all } \ k \in I_2\}.$$

Then $(\ell|_{\mathcal{L}})^{-1} \circ \pi$ is a scission of ℓ, where $\pi : k^{\mathbb{N}} \to \mathrm{im}(\ell)$ is the natural projection and $\mathcal{L} := \{\gamma \in \mathcal{A}; \gamma_k = 0 \ \text{ for } \ k \in \mathbb{N} \setminus I_2\}$ is a direct complement of $\ker(\ell)$ as a k-vector space.

1.3 The Rank Theorem

Consider a textile map $f \colon \mathcal{C}^m \to \mathcal{C}^p$ with tangent map $\ell = T_{a_0} f$ at a_0. Our purpose is to linearize f locally at a_0 by applying local automorphisms of source and target. This is obtained by generalizing the Rank Theorem in [HM94] (proven there for tactile maps) in three directions: We allow textile maps, we give a version with parameters, and also a statement with nilpotent coefficients. The main argument of the proof from [HM94] is recalled and extended to textile maps in section 1.3.1. We introduce textile maps depending on parameters in 1.3.2 and prove the corresponding theorem. The nilpotent case is treated in section 1.3.3.

1.3.1 The Rank Theorem for textile maps

We start with some technical preliminaries. The notation used here is abutted to the analytic case treated in [HM94]. A textile map $\gamma : U \subseteq \mathcal{C} \to \mathcal{C}^m$ is called a *curve* (over U) in $\mathcal{C}^m$. The argument of γ is called the parameter of the curve and is mostly denoted by s. For an arbitrary subvector space $J \subseteq \mathcal{C}^m$ a textile map $\gamma\colon U \subseteq \mathcal{C} \to \mathcal{C}^m$ is called a *curve in* J if $\mathrm{im}(\gamma) \subseteq J$. The set of all curves in J is denoted by $\Gamma(J)$. The tangent space $T_a J$ of J at $a \in J$ is defined as the k-vector space with elements

$$\partial\gamma(0),$$

for a curve γ in J with $\gamma(0) = a$. Obviously $\mathrm{T}_a\mathcal{C}^m$ can be identified with $\mathcal{C}^m$ for any $a \in \mathcal{C}^m$. Thus, a curve in the tangent bundle of $\mathcal{C}^m$ is given by a pair of curves $(\gamma, b) \in \Gamma(\mathcal{C}^m)^2$.

A textile map $f : \mathcal{C}^m \to \mathcal{C}^p$ induces an appropriate tangent map $Tf : \mathrm{T}\mathcal{C}^m \to \mathrm{T}\mathcal{C}^p$ and further by composition a map

$$\Gamma(\mathrm{T}\mathcal{C}^m) \to \Gamma(\mathrm{T}\mathcal{C}^p); (\gamma, b) \mapsto (f\gamma, T_\gamma f \cdot b).$$

In the case of tactiles $T_\gamma f \cdot b$ is given by

$$T_\gamma f \cdot b = \sum_{i=1}^{n} (\partial_i f)\,(\gamma(s)) \cdot b^i(s).$$

Let $U \subseteq \mathcal{C}^m$ be an $\mathfrak{m}$-adic neighbourhood of $a_0 \in \mathcal{C}^m$. Assume that the image $\mathrm{im}(T_{a_0} f)$ has (as a topological vector space) a direct complement J, i.e.

$$\mathrm{im}(T_{a_0} f) \oplus J = \mathcal{C}^p.$$

Moreover let γ be a curve in U with $\gamma(0) = a_0$; then we say f has *constant rank at* a_0 *with respect to* J if for all curves of the form $(f\gamma, \eta)$ in $\mathrm{T}\mathcal{C}^p$ there exist unique curves ρ and τ such that (γ, ρ) defines a curve in $\mathrm{T}U$ and $\tau \in \Gamma(\mathrm{T}J)$ with the property

$$\eta = T_\gamma f \cdot \rho + \tau. \tag{1.4}$$

A textile map $f : U \subseteq \mathcal{C}^m \to \mathcal{C}^p$ is called *flat at* a_0 if for all curves γ in U with $\gamma(0) = a_0$ the evaluation at $s = 0$ defines a surjective map

$$\ker(T_\gamma f : \mathrm{T}_\gamma U \to \mathrm{T}_{f\gamma}\mathcal{C}^p) \longrightarrow \ker(T_{a_0} f : \mathrm{T}_{a_0} U \to \mathrm{T}_{f(a_0)}\mathcal{C}^p).$$

Similar to [HM94] one proves the following proposition (see [Bru05]):

Proposition 6. *Let $f : U \subseteq \mathcal{C}^m \to \mathcal{C}^p$ be a textile map of type $f = \ell + h$. Assume there exists a scission of ℓ, such that σh is contractive. Then f has constant rank at* 0 *if and only if f is flat at* 0.

Remark. The contractivity condition on σ and h is necessary to conclude that flatness means constant rank. The other direction doesn't need the construction of a scission of ℓ. A typical class of flat maps is given by maps with injective tangent map.

Theorem 6 (Rank Theorem)**.** *Let $f : U \subseteq \mathcal{C}^m \to \mathcal{C}^p$ be a textile map on an $\mathfrak{m}$-adic neighbourhood U of 0 with $f(0) = 0$. Write f in the form $f = \ell + h$ with $\ell = T_0 f$ the tangent map of f at 0. Assume that ℓ admits a scission σ for which σh is contractive on U, and that f has constant rank at 0 with respect to $\ker(\ell\sigma)$. Then there exist locally invertible textile maps $u : U \to \mathcal{C}^m$ and $v : f(U) \to \mathcal{C}^p$ at 0 such that*

$$vfu^{-1} = \ell.$$

Proof. The proof naturally falls into two parts. First construct two invertible textile maps u, v; second show that these maps indeed linearize f. This will be ensured by the rank condition. Set $u = \mathrm{id}_{\mathcal{C}^m} + \sigma h$. Obviously $u(0) = 0$ and $T_0 u = \mathrm{id}$. By contractiveness of σh on U this u is invertible on U. Note:

$$\ell\sigma f = \ell\sigma(\ell + h) = \ell + \ell\sigma h = \ell(\mathrm{id} + \sigma h) = \ell u.$$

Thus, fu^{-1} fulfills $\ell\sigma f u^{-1} = \ell$. Without loss of generality one can assume for the rest of the proof that $\ell\sigma f = \ell$. The second map v is defined by $v = \mathrm{id}_{\mathcal{C}^p} - (\mathrm{id}_{\mathcal{C}^p} - \ell\sigma) f\sigma\ell\sigma$. Again: $v(0) = 0$ and v is invertible with inverse $v^{-1} = \mathrm{id}_{\mathcal{C}^p} + (\mathrm{id}_{\mathcal{C}^p} - \ell\sigma) f\sigma\ell\sigma$. For later use we write

$$\begin{aligned} v &= \mathrm{id} - f\sigma\ell\sigma + \ell\sigma f\sigma\ell\sigma \\ &= \mathrm{id} - f\sigma\ell\sigma + \ell\sigma \\ &= \mathrm{id} - h\sigma, \end{aligned}$$

with $\kappa(h\sigma) > 0$. Note: The diffeomorphisms u and v are constructed without using the rank condition at all. Assume f fulfills the equation $(*)$: $f = f\sigma\ell$. This would imply that

$$\begin{aligned} vf &= f - (\mathrm{id} - \ell\sigma) f\sigma\ell\sigma f \\ &= f - (\mathrm{id} - \ell\sigma) f\sigma\ell \\ &= f - (\mathrm{id} - \ell\sigma) f \\ &= \ell\sigma f \\ &= \ell. \end{aligned}$$

Equation $(*)$ is a consequence of the Rank Condition. The composition $\sigma\ell$ is a projection onto the complement of $\ker(\ell)$ in $\mathcal{C}^m$. It is sufficient to show that $T_a f|_{\ker(\ell)} = 0$ for all $a \in \mathcal{C}^m$ near 0. From $\ell\sigma f = \ell$ follows $f = \ell + (\mathrm{id}_{\mathcal{C}^p} - \ell\sigma) f$ and subsequently

$$T_a f = \ell + (\mathrm{id}_{\mathcal{C}^p} - \ell\sigma) T_a f.$$

Let a be fixed in U. Define a curve $\gamma(s) := s \cdot a$. For $b \in \ker(\ell)$

$$T_{\gamma(s)} f \cdot b = (\mathrm{id}_{\mathcal{C}^p} - \ell\sigma) T_{\gamma(s)} f \cdot b$$

holds. By uniqueness of the decomposition in equation (1.4)

$$T_{\gamma(s)} f \cdot b = 0$$

and thus $T_a f|_{\ker(\ell)} = 0$. The assertion follows. □

A textile map $f = \ell + h$ is called a *quasi-submersion* if

$$\mathrm{im}(h) \subseteq \mathrm{im}(\ell).$$

Especially, if ℓ is surjective, f is a *submersion*. Quasi-submersions play an important role in the applications in section 2. A quasi-submersive map f is linearizable if the *order condition* (i.e., there exists a scission σ for ℓ such that $\kappa(\sigma h) > 0$) is fulfilled. Indeed, by construction of v in the proof of Theorem 6 we see that in the case of quasi-submersive f this map v equals the identity. Therefore, checking the Rank Condition becomes obsolete.

Some simplification: The Rank Condition is technical and may be hard to check. In some applications the following simpler condition is sufficient. Let $f : U \subseteq \mathcal{C}^m \to \mathcal{C}^p$ be a textile map. Denote by J a direct complement of $\mathrm{im}(T_0 f)$. Then f is said to have *pointwise constant rank at* 0 *with respect to* J if for all $a \in U$

$$\mathrm{im}(T_a f) \oplus J = \mathcal{C}^p.$$

Proposition 7. *Let $f : U \subseteq \mathcal{C}^m \to \mathcal{C}^p$ be a tactile, U an $\mathfrak{m}$-adic neighbourhood of 0. If for all $a \in U$*

$$\mathrm{in}(\mathrm{im}(T_a f)) = \mathrm{in}(\mathrm{im}(T_0 f))$$

then f has pointwise constant rank at 0 with respect to

$$\Delta(T_0 f) = \{b \in \mathcal{C}^p; \mathrm{supp}(b) \cap \mathrm{in}(\mathrm{im}(T_0 f)) = \emptyset\}.$$

1.3.2 The Parametric Rank Theorem

The Rank Theorem ensures linearization of textile maps $f : U \subseteq \mathcal{C}^m \to \mathcal{C}^p$ of constant rank near a fixed cord $c \in U$. The Parametric Rank Theorem will deal with families of textile maps, thus providing a tool to trivialize vector bundles (see section 2.1).

Let $\mathcal{S} \subseteq \mathbb{A}_k^q$ be constructible and $Y \subseteq \mathcal{C}^m$ a subfelt. A map

$$f_{\mathcal{S}} : \mathcal{S} \times Y \to k; (s, c) \mapsto f_{\mathcal{S}}(s, c)$$

is called a *regular k-valued family of functions over $\mathcal{S}$* if there is a function $g : U \times Y \to k$, where $U \subseteq \mathbb{A}_k^q$ containing $\mathcal{S}$ is a subvariety and g locally looks like a quotient $\frac{h}{l}$ of $h, l \in k[s_1, \ldots, s_q] \otimes_k k[\mathbb{N}_m^n]$, l locally non-zero, such that $g|_{\mathcal{S} \times Y} = f_{\mathcal{S}}$. More generally we define a *regular family of textile maps over $\mathcal{S}$ on Y* as a map

$$f_{\mathcal{S}} : \mathcal{S} \times Y \to \mathcal{C}^p; (s, c) \mapsto (f_{\mathcal{S},\alpha}(s, c))_{\alpha \in \mathbb{N}^{n+1}}$$

where each $f_{\mathcal{S},\alpha}$ is a regular k-valued family over $\mathcal{S}$. Thus for fixed $s = (s_1, \ldots, s_q) \in \mathcal{S}$ the restricted map

$$f_s := f_{\mathcal{S}}(s, -) : Y \to \mathcal{C}^p$$

is a (regular) textile map. The s_i will be referred to as *parameters* and $\mathcal{S}$ as the *parameter space*. If $f_{\mathcal{S}}$ is a family such that all f_s are linear maps, we call it a *linear family over* $\mathcal{S}$. As a special case the family $\mathrm{id}_{\mathcal{S}} : \mathcal{S} \times Y \to Y$, where $\mathrm{id}_s = \mathrm{id}$ for all $s \in \mathcal{S}$, appears.

If $f_{\mathcal{S}} : \mathcal{S} \times \mathcal{C}^m \to \mathcal{C}^p$ and $g_{\mathcal{S}} : \mathcal{S} \times \mathcal{C}^q \to \mathcal{C}^m$ are two regular families we define their composition as the family $\mathcal{S} \times \mathcal{C}^q \to \mathcal{C}^p$ given by

$$f_{\mathcal{S}} \circ g_{\mathcal{S}} := f_{\mathcal{S}} \circ (proj_1, g_{\mathcal{S}}),$$

where $proj_1 : \mathcal{S} \times \mathcal{C}^q \to \mathcal{S}$ is the projection onto the first factor. Thus $(f_{\mathcal{S}} \circ g_{\mathcal{S}})_s = f_s \circ g_s$. A regular family $f_{\mathcal{S}} : \mathcal{S} \times \mathcal{C}^m \to \mathcal{C}^p$ is called *invertible* if there exists a *regular* family $g_{\mathcal{S}} : \mathcal{S} \times \mathcal{C}^p \to \mathcal{C}^m$ over $\mathcal{S}$ with $f_{\mathcal{S}} \circ g_{\mathcal{S}} = \mathrm{id}$.

Example 3. Consider

$$\ell_{\mathbb{A}_k} : \mathbb{A}_k \times \mathcal{A} \to \mathcal{A}; (s, c) \mapsto (0, sc_0, sc_1, \ldots).$$

Obviously $\ell_{\mathbb{A}_k}$ is a regular family. The family $\sigma_{\mathcal{S}}$, $\mathcal{S} = \mathbb{A}_k \setminus \{0\}$, defined by

$$\sigma_{\mathcal{S}} : \mathcal{S} \times \mathfrak{m} \cdot \mathcal{A} \to \mathcal{A}; c \mapsto \frac{1}{s} \cdot (c_1, c_2, \ldots)$$

is regular with (right) inverse $\ell|_{\mathcal{S}}$. Thus $\sigma_{\mathcal{S}}$ is an invertible family.

The defining property for contractive textile maps naturally generalizes to families of textile maps over $\mathcal{S}$. Let $U = \mathfrak{m}^l \cdot \mathcal{C}^m$ be an $\mathfrak{m}$-adic neighbourhood of 0. A family $f_{\mathcal{S}} : \mathcal{S} \times U \to \mathcal{C}^p$ is called *contractive on* U if it fulfills condition (1.1) independently of $\mathcal{S}$, or equivalently, it fulfills (1.1) pointwise: f_s is contractive on U for all $s \in \mathcal{S}$.

Theorem 7. *Let $f_{\mathcal{S}} : \mathcal{S} \times U \to \mathcal{C}^m$, $U = \mathfrak{m}^l \cdot \mathcal{C}^m$, an $\mathfrak{m}$-adic neighbourhood of 0, be a regular family over $\mathcal{S}$ with $f_{\mathcal{S}}(-, 0) = 0$. Assume $f_{\mathcal{S}} = \ell_{\mathcal{S}} + h_{\mathcal{S}}$ with $\ell_{\mathcal{S}} : \mathcal{S} \times U \to W$ linear with inverse $\tilde{\ell}_{\mathcal{S}} : \mathcal{S} \times W \to U$. If $\tilde{\ell}_{\mathcal{S}} \circ h_{\mathcal{S}}$ is contractive on U, then $f_{\mathcal{S}}$ admits on U an (right) inverse, i.e., there is a regular family $g_{\mathcal{S}} : \mathcal{S} \times f(U) \to \mathcal{C}^m$ with $f_{\mathcal{S}} \circ g_{\mathcal{S}} = \mathrm{id}_{\mathcal{S}}$.*

Proof. The same reasoning as in the proof of Theorem 3 can be applied in the present situation. The inverse for $f_{\mathcal{S}}$ is constructed as the limit of the sequence defined by the recursion $g_{\mathcal{S}}^0 = 0$,

$$g_{\mathcal{S}}^{i+1} = \mathrm{id}_{\mathcal{S}} - h_{\mathcal{S}} \circ g_{\mathcal{S}}^i.$$

By induction one shows that $g_{\mathcal{S}}^i$ is a regular family, the case $i = 0$ being trivial. For $i > 0$: The map $h_{\mathcal{S}}$ is a regular family on $U = \mathfrak{m}^l \cdot \mathcal{C}^m$. Therefore – using the induction hypotheses – we see that $h_{\mathcal{S}} \circ g_{\mathcal{S}}^{i-1}$ is regular and so is $g_{\mathcal{S}}^i$. Moreover, $(\mathrm{ord}(g_{\mathcal{S}}^{i+1} - g_{\mathcal{S}}^i))_i$ is an increasing sequence. Thus the limit of $(g_{\mathcal{S}}^i)_i$ exists and the α-coefficient of $g_{\mathcal{S}}$, $\alpha \in \mathbb{N}^{n+1}$, is the same as the α-coefficient of $g_{\mathcal{S}}^{i_\alpha}$ for some $i_\alpha \in \mathbb{N}$, for which we have just shown that it is a regular family. □

Consider a regular family $f_{\mathcal{S}}$ on $U \subseteq \mathcal{C}^m$. The *tangent map* with respect to $\mathcal{C}^m$ of the family $f_{\mathcal{S}}$ (see section 1.1.2) is the regular family $Tf_{\mathcal{S}}$ given by linear maps Tf_s which are the tangent maps to the f_s, $s \in \mathcal{S}$.
Let $\ell_{\mathcal{S}} : \mathcal{S} \times U \to \mathcal{C}^p$, $U \subseteq \mathcal{C}^m$, be a regular linear family. A *scission* for $\ell_{\mathcal{S}}$ is a regular linear family $\sigma_{\mathcal{S}} : \mathcal{S} \times \mathcal{C}^p \to \mathcal{C}^m$ such that $\ell_{\mathcal{S}}\sigma_{\mathcal{S}}\ell_{\mathcal{S}} = \ell_{\mathcal{S}}$.
Assume that J is a direct complement of $\mathrm{im}(T_{a_0} f_s)$ in $\mathcal{C}^p$, for all $s \in \mathcal{S}$, where

$a_0 \in U$. The family $f_{\mathcal{S}}$ is *of constant rank at* a_0 *with respect to* J (or *fulfills the parametric rank condition at* a_0 *w.r.t.* J) if for all $s \in \mathcal{S}$ the restricted maps f_s are of constant rank at a_0 with respect to J (see section 1.3).

Theorem 8 (Parametric Rank Theorem)**.** *Let $f_{\mathcal{S}} : \mathcal{S} \times U \to \mathcal{C}^p$ be a regular family, $U = \mathfrak{m}^l \cdot \mathcal{C}^m$; write $f_{\mathcal{S}} = \ell_{\mathcal{S}} + h_{\mathcal{S}}$ with $\ell_{\mathcal{S}} := T_0 f_{\mathcal{S}}$. Assume $f_{\mathcal{S}}(-,0) = 0$ and that the following conditions hold: (i) the linear family $\ell_{\mathcal{S}}$ admits a scission $\sigma_{\mathcal{S}}$ s.t. $\sigma_{\mathcal{S}} h_{\mathcal{S}}$ is contractive; (ii) the family $f_{\mathcal{S}}$ fulfills the parametric rank condition w.r.t. a direct complement of* $\mathrm{im}(T_0 f_{\mathcal{S}})$. *Then there exist regular (locally) invertible families $u_{\mathcal{S}}$ and $v_{\mathcal{S}}$ with*

$$v_{\mathcal{S}} f_{\mathcal{S}} u_{\mathcal{S}}^{-1} = \ell_{\mathcal{S}}.$$

Proof. The arguments used in the proof of Theorem 6 may be applied here. By the assumptions and Theorem 7 the families $u_{\mathcal{S}} = \mathrm{id}_{\mathcal{S}} + \sigma_{\mathcal{S}} h_{\mathcal{S}}$ respectively $v_{\mathcal{S}} = \mathrm{id}_{\mathcal{S}} - (\mathrm{id}_{\mathcal{S}} - \ell_{\mathcal{S}} \sigma_{\mathcal{S}}) f_{\mathcal{S}} \sigma_{\mathcal{S}} \ell_{\mathcal{S}} \sigma_{\mathcal{S}}$ are regular and invertible. By condition (ii) and Theorem 6 these families linearize $f_{\mathcal{S}}$ pointwise, thus

$$v_{\mathcal{S}} f_{\mathcal{S}} u_{\mathcal{S}}^{-1} = \ell_{\mathcal{S}}.$$

□

1.3.3 The Rank Theorem for test-rings

Let A be *test-ring*, i.e., a local, commutative, and unital k-algebra with nilpotent maximal ideal $\mathfrak{n}$. Denote by $\mathcal{C}_A$ the space of n-cords with coefficients in A. By Theorem 3 it follows that for a textile map $h \colon U \subseteq \mathcal{C}_A^m \to \mathcal{C}_A^m$ which is contractive with respect to the refined order ORD, the map $\mathrm{id} + h$ is invertible. Again U denotes an $\mathfrak{m}$-adic neighbourhood of 0. The proof of Theorem 6 can be adopted to this setting, thus giving the analogous assertion:

Theorem 9 (Rank Theorem for test-rings)**.** *Let A be a test-ring and $f : U \subseteq \mathcal{C}_A^m \to \mathcal{C}_A^p$ be a textile map on an $\mathfrak{m}$-adic neighbourhood U of 0 with $f(0) = 0$. Write f in the form $f = \ell + h$ with $\ell = T_0 f$ the tangent map of f at 0. Assume that ℓ admits a scission σ for which σh is contractive on U, and that f has constant rank at 0 with respect to* $\ker(\ell\sigma)$. *Then there exist locally invertible textile maps $u : U \to \mathcal{C}_A^m$ and $v : f(U) \to \mathcal{C}_A^p$ at 0 such that*

$$v f u^{-1} = \ell.$$

1.4 The Rank Theorem and Polynomial Rings in Countable Many Variables

The study of textile maps and felts inevitably leads to the investigation of polynomial rings in countably many variables. A felt is the spectrum of the quotient of such a polynomial ring by an ideal, and textile maps are algebra homomorphisms between such (in general) non-Noetherian algebras. We restrict our attention mostly to felts defined by tactile maps, especially including the situation as it appears in Denef-Loeser's local triviality result, see Lemma 10. Section 1.4.1 starts with the definition of polynomial and – in preparation for chapter 3 – formal power series rings in infinitely many variables as it can be found in [Bou98]. Afterwards, in section 1.4.2 an inverse mapping theorem and a rank theorem are proven. These results are extended to felts defined by the solution set of an algebraic system of equations in the ring of generalized formal power series, see section 1.4.2:

1.4.1 Polynomial rings in countable many variables

Let S be a monoid with composition written multiplicatively). For a commutative ring A we consider $A^{(S)}$, the A-module of formal linear combinations of elements of S with coefficients in A. A basis for $A^{(S)}$ is given by elements $e_s : t \mapsto \delta_{st}$, where δ is the Kronecker-δ. So every element of $A^{(S)}$ can be written uniquely as

$$\sum_{s\in S} \alpha_s e_s$$

for $\alpha_s \in A$, just finitely many non-zero. The A-module $A^{(S)}$ obviously becomes an associative A-algebra with multiplication defined by

$$e_s \cdot e_t := e_{st}.$$

Moreover, if A is commutative so is $A^{(S)}$. This algebra is called the *A-algebra of the monoid S*, see [Bou89], III.2.6.

Example 4. Consider $S = \mathbb{N}^{(I)}$ where I is an arbitrary index set. In this case $A^{(S)}$ is also called the *free commutative associative algebra of I over A*. It will be denoted by $A[x_i; i \in I]$ and is also known as the *A-algebra of polynomials in variables $x_i, i \in I$.*

For arbitrary elements $x = \sum_s x_s e_s$, $y = \sum_s y_s e_s \in A^{(S)}$ their product can be written as

$$xy = \sum_{s\in S} \left(\sum_{ut=s} x_u y_t \right) e_s. \tag{1.5}$$

Formula (1.5) even makes sense for elements of A^S if for arbitrary $s \in S$ the set

$$\{(u,t) \in S \times S; ut = s\}$$

is finite. Call this property (*). If S has property (*), then A^S becomes an A-algebra, too. It is called the *total algebra of S over A*, see [Bou89], III.2.10. The algebra $A^{(S)}$ is a natural subalgebra of A^S, and for $T \subseteq S$ a submonoid we can naturally identify A^T as a subalgebra of A^S. For an element $x = \sum_s x_s e_s \in A^S$ we define its *support* $\operatorname{supp}(x)$ as

$$\operatorname{supp}(x) = \{s \in S; x_s \neq 0\}.$$

Example 5. For any set I the monoid $\mathbb{N}^{(I)}$ has property (*), since $\mathbb{N}$ has it. We may thus consider the total algebra of $\mathbb{N}^{(I)}$ over A. It is denoted by $A[[x_i; i \in I]]$ or $A[[\underline{I}]]$ and called the *A-algebra of formal power series in variables $x_i, i \in I$.*

For the rest of this section we will restrict to rings which are unital and commutative.

1.4.2 A Rank Theorem for Polynomial Rings

Consider $k[\underline{\mathbb{N}}^m]$, the polynomial ring in variables x_i^j, $1 \leq j \leq m$, $i \in \mathbb{N}_0$, with coefficients in k. In the case $m = 1$ we will simply write $k[\underline{\mathbb{N}}]$ and denote its variables by $x_0, x_1, \ldots$. Obviously $k[\underline{\mathbb{N}}^m]$ is not Noetherian but every ideal $J \subseteq k[\underline{\mathbb{N}}^m]$ can be generated by countably many elements. This follows immediately from the fact that each $J_n = J \cap k[x_0, \ldots, x_n]$, $n \in \mathbb{N}_0$, is finitely generated by the Hilbert Basissatz. Thus we represent – after a choice of generators – each ideal $J \subseteq k[\underline{\mathbb{N}}^m]$ by a sequence of elements in $k[\underline{\mathbb{N}}^m]$. In the sequel $k[q; m]$ will denote the polynomial ring $k[x_i^j; 1 \leq j \leq m, 0 \leq i \leq q]aux :$.

An *affine felt* is a subscheme of $\mathbb{A}_\infty^m = \operatorname{Spec} k[\underline{\mathbb{N}}^m]$. Closed subfelts of $\mathbb{A}_\infty^m$ are given by ideals $J \subseteq k[\underline{\mathbb{N}}^m]$. A morphism of schemes $X \to Y$ between two felts X and Y is called a *textile* map. These notions are equivalent to those in 1.1.

Any k-algebra homomorphism $\varphi \colon k[\underline{\mathbb{N}}^m] \to k[\underline{\mathbb{N}}^q]$, $m, q \in \mathbb{N}$, is given by $\varphi(x_i^j) = \varphi_i^j \in k[\underline{\mathbb{N}}^q]$. In this way φ will be identified with an element in $k[\underline{\mathbb{N}}^q]^{\mathbb{N}_0 \times \{1,\ldots,m\}}$. Let $(\varphi^l)_{l \in \mathbb{N}}$ be a sequence of k-algebra homomorphisms $k[\underline{\mathbb{N}}^m] \to k[\underline{\mathbb{N}}^q]$. We say that this sequence *converges to a k-algebra homomorphism* $\varphi \colon k[\underline{\mathbb{N}}^m] \to k[\underline{\mathbb{N}}^q]$ if it converges in $k[\underline{\mathbb{N}}^q]^{\mathbb{N}_0 \times \{1,\ldots,m\}}$ endowed with the topology of coefficientwise convergence. The limit of $(\varphi^l)_l$ w.r.t. this topology defines – in case of existence – a new k-algebra homomorphism, called the *limit* of $(\varphi^l)_l$.

A textile map $f \colon \mathbb{A}_\infty^q \to \mathbb{A}_\infty^m$ induced by a k-algebra homomorphism

$$f^\sharp \colon k[\underline{\mathbb{N}}^m] \to k[\underline{\mathbb{N}}^q]$$

is called *contractive of degree* $\Gamma \in \mathbb{Z}$ if $f^\sharp(x_i^j)$ is polynomial in x_ν^l with $\nu \leq i - \Gamma$. We simply say f is *contractive* if $\Gamma > 0$.

If $f \colon X \to Y$ is a morphism of affine felts we will usually use the symbol f for both, the morphism of affine schemes and its corresponding homomorphism of rings. If no confusion can arise we will write x_i^j for the whole family $x_i^j, 1 \leq j \leq m$, $i \in \mathbb{N}_0$, and skip any specification of indices. The set of units of a ring L is denoted by $L^\times$.

Proposition 8 (Inverse Mapping Theorem). *Let $f \colon \mathbb{A}_\infty^m \to \mathbb{A}_\infty^m$ be a textile map of the form $\ell + h$ with ℓ some linear isomorphism, $h(0) = 0$, and $\ell^{-1}h$ contractive. Then f is an isomorphism.*

Proof. Without loss of generality we can assume that $f(x_i^j) = x_i^j + h_i^j$ with

$$h_i^j \in (x_n^l; 1 \leq l \leq m, 0 \leq n \leq i - 1) \subseteq k[i-1; m],$$

i.e., $\ell = \mathrm{id}$ and h is contractive. We inductively construct an inverse for f by the following sequence of automorphisms: Set $g_0 = \mathrm{id}$ and assume that automorphisms $g_1, \ldots, g_{n-1}$ have already been constructed such that $(g_l \circ f)(x_i^j) = x_i^j$

for $i \le l \le n-1$. Define $g_n\colon k[\underline{\mathbb{N}}^m] \to k[\underline{\mathbb{N}}^m]$ by

$$g_n(x_i^j) := \begin{cases} x_n^j - g_{n-1}(h_n^j) & \text{for } i = n \\ g_{n-1}(x_i^j) & \text{otherwise} \end{cases}$$

The sequence $(g_n)_{n\in\mathbb{N}}$ converges since $g_n(x_i^j) = g_{n+1}(x_i^j)$ for $i \le n$, thus its limit defines a textile map $g\colon k[\underline{\mathbb{N}}^m] \to k[\underline{\mathbb{N}}^m]$ with $g(x_i^j) = g_i(x_i^j)$. This g is the inverse of f. Indeed, for all i, j we have

$$(g \circ f)(x_i^j) = g_i(x_i^j) + g_i(h_i^j) = x_i^j - g_{i-1}(h_i^j) + g_{i-1}(h_i^j) = x_i^j,$$

where we used the fact that h_i^j does not depend on x_n^l with $n \ge i$. $\square$

Example 6. Consider $f\colon k[\underline{\mathbb{N}}] \to k[\underline{\mathbb{N}}]$ defined by $x_i \mapsto x_i + \sum_{j<i} x_j^2$. Then g_2 is given by $g_2(x_0) = x_0$, $g_2(x_1) = x_1 - x_0^2$, $g_2(x_2) = x_2 - x_0^2 - (x_1 - x_0^2)^2$ and $g_2(x_i) = x_i$ for $i \ge 3$ and indeed $g_2(f(x_i)) = x_i$ for $i \le 2$.

Every $F \in k[\underline{\mathbb{N}}^m]$ can be written in the form

$$F = \sum_\alpha F_\alpha x^\alpha$$

where the sum is taken over all $\alpha = (\alpha_{ij})_{ij} \in \mathbb{N}^{(\mathbb{N}_0 \times \{1,\dots,m\})}$, such that all but finitely many F_α vanish. The *support* $\operatorname{supp}(F)$ is defined as the set of indices α such that $F_\alpha \ne 0$. We say a variable x_i^j *appears in* F, and write shortly $(i,j) \in \operatorname{supp}(F)$, if there exists an $\alpha \in \operatorname{supp}(F)$ with $\alpha_{ij} \ne 0$. If $F \in k[\underline{\mathbb{N}}^m]$, then we will denote its homogenous part of degree 1 by F' and the remaining terms $F - F'$ by F''.

Proposition 9 (Rank Theorem)**.** *Let L be a commutative ring with one and $J = (F_0, F_1, \dots)$ an ideal contained in $(x_i^j; i \in \mathbb{N}_0, 1 \le j \le m) \subseteq L[\underline{\mathbb{N}}^m]$. Assume that*

1. *each F_r' contains a term $A_{i_r}^{j_r} x_{i_r}^{j_r}$ with $(i_r, j_r) \notin \cup_{l<r}\operatorname{supp}(F_l')$, $A_{i_r}^{j_r} \in L^\times$ and $i_{r-1} < i_r$;*
2. *$F_r'' \in L[x_i^j; i < i_r]$.*

Then there exists an L-algebra automorphism $u\colon L[\underline{\mathbb{N}}^m] \to L[\underline{\mathbb{N}}^m]$ such that $u(J)$ is generated by F_r', $r \in \mathbb{N}_0$.

Proof. We construct the automorphism u inductively as follows. Set

$$u_0\colon k[\underline{\mathbb{N}}^m] \to k[\underline{\mathbb{N}}^m]; x_i^j \mapsto \begin{cases} x_{i_0}^{j_0} - (A_{i_0}^{j_0})^{-1} F_0'' & \text{for } (i,j) = (i_0, j_0) \\ x_i^j & \text{otherwise.} \end{cases}$$

Clearly $u_0(F_0) = F_0'$ and u_0 is an automorphism by Proposition 8 since assumption (2) implies contractiveness of $u_0 - \mathrm{id}$. Assume that an automorphism u_{n-1} has been constructed such that $u_{n-1}(F_r) = F_r'$ for all $r < n$, and define

$$u_n\colon k[\underline{\mathbb{N}}^m] \to k[\underline{\mathbb{N}}^m]; x_i^j \mapsto \begin{cases} x_{i_n}^{j_n} - (A_{i_n}^{j_n})^{-1} \left(u_{n-1}(F_n)\right)'' & \text{for } (i,j) = (i_n, j_n) \\ u_{n-1}(x_i^j) & \text{otherwise.} \end{cases}$$

By the induction hypothesis and construction of u_n clearly $u_n(F_l) = u_{n-1}(F_l) = F_l'$ for $l < n$. Moreover

$$\begin{aligned} u_n(F_n) &= u_{n-1}(F_n' - A_{i_n}^{j_n} x_{i_n}^{j_n}) + A_{i_n}^{j_n}\left(x_{i_n}^{j_n} - (A_{i_n}^{j_n})^{-1}(u_{n-1}(F_n))''\right) + u_n(F_n'') \\ &= u_{n-1}(F_n) - (u_{n-1}(F_n))'' \\ &= F_n'. \end{aligned}$$

The last equality follows from the fact that the u_i have linear part equal to the identity. Using Proposition 8 we see that u_n is invertible: if x_i^j appears in $(u_{n-1}(F_n))''$ then $i < i_n$. The $(u_i)_i$ converge towards an automorphism since

$$u_n(x_i^j) = u_{n'}(x_i^j)$$

if $i_n, i_{n'} \geq i$ and the sequence $(i_r)_r$ is increasing. This completes the proof. $\square$

Example 7. Let $J \subseteq k[\underline{\mathbb{N}}^2]$ be the ideal with generators $F_r = x_{r+1}^1 + x_r^1 x_r^2$, $r \in \mathbb{N}_0$. So for $r = 0, 1$ this gives

$$F_0 = x_1^1 + x_0^1 x_0^2,\ F_1 = x_2^1 + x_1^1 x_1^2.$$

We will follow the algorithm provided in the proof of Proposition 9 to linearize the F_0, F_1. Set $(i_0, j_0) = (1, 1)$ and define u_0 as

$$u_0 : x_i^j \mapsto \begin{cases} x_1^1 - x_0^1 x_0^2 & \text{for } (i,j) = (1,1) \\ x_i^j & \text{otherwise.} \end{cases}$$

Then $u_0(F_0) = x_1^1$ and $u_0(F_1) = x_2^1 + (x_1^1 - x_0^1 x_0^2)x_1^2$. Continue with $(i_1, j_1) = (2, 1)$. This gives with

$$u_1 : x_i^j \mapsto \begin{cases} x_2^1 - (x_1^1 - x_0^1 x_0^2)x_1^2 & \text{for } (i,j) = (2,1) \\ x_1^1 - x_0^1 x_0^2 & \text{for } (i,j) = (1,1) \\ x_i^j & \text{otherwise.} \end{cases}$$

Then an easy computation shows that

$$u_1(J) = (x_1^1, x_2^1, x_3^1 + (x_2^1 - (x_1^1 - x_0^1 x_0^2)x_1^2)x_2^2, \ldots).$$

If $x(t)$ is a k-arc of a variety $X \subseteq \mathbb{A}^n$ (cf. section 2.1) and $d \in \mathbb{N}$, we will naturally identify $a = x(t) \mod (t)^d$, i.e., $a_i = x_i(t) \mod (t)^d$, with a vector of polynomials of degree less than d. Moreover, a corresponds to a maximal ideal in $k[d-1; m]$ (induced by the coefficients of the powers of t), which will be denoted by $\mathfrak{m}_a$. We will use the same notation when considering the ideal generated by $\mathfrak{m}_a$ in $k[\underline{\mathbb{N}}^m]$. The following corollary provides another geometric consequence of Proposition 9. It is treated in a different context in section 2.1 (see also Lemma 4.1 in [DL99], and [Loo02]).

Corollary 10. *Let $X \subseteq \mathbb{A}^n$ be a (reduced) hypersurface given by a polynomial $f \in k[x_1, \ldots, x_m]$ and let $x(t)$ be a k-arc of X which does not lie in* $\mathrm{Sing}X$ *up to order $d \in \mathbb{N}$, i.e., $e = \min_j\{\mathrm{ord}\, \partial_j f(x(t))\} \leq d - 1$. Set $a = x(t) \mod (t)^d$. Then there exists a textile isomorphism $u\colon k[\underline{\mathbb{N}}^m]/\mathfrak{m}_a \to k[\underline{\mathbb{N}}^m]/\mathfrak{m}_a$ such that $u(I_\infty)$ is a linear ideal.*

Remark. By the corollary the truncated arc a can be lifted to an arc $x(t)$ of X by solving linear equations. Put differently: The preimage of a under the truncation map $x \mapsto x \mod (t)^d$ has the structure of a manifold.

Proof. Taylor expansion yields

$$f(a+y) = f(a) + \sum_{j=1}^{n} \partial_j f(a) \cdot y^j + h(a;y),$$

with h at least of order two in y. Write the jth component of a as $\sum_{i=0}^{d-1} a_i^j t^i$, $a_i^j \in k$, and the jth component of y as $\sum_{i \geq d} x_i^j t^i$. By the assumptions the F_r have non-vanishing linear part ($N = \max_j \{\deg \partial_j f(a)\}$)

$$F'_r = \sum_{j=1}^{m} \sum_{l+i=r} A_l^j x_i^j, \; e \leq l \leq N.$$

We will consider the class of I_∞ in the factor ring

$$R = k[x_l^j; l \leq d-1]/\mathfrak{m}_a[x_l^j; l \geq d].$$

Note that the first non-vanishing $\bar{F}_r \in R$ has index $r = e+d$. Indeed since $x(t)$ is by assumption an arc of X its coefficients provide a solution to the equations F_r, $r \in \mathbb{N}_0$. Thus all F_r in which just x_i^j with $i \leq d-1$ appear will vanish modulo $\mathfrak{m}_a$. For arbitrary $r < e+d$ the F'_r depend on x_i^j with $e+i < e+d$, thus on variables with index $i \leq d-1$. Let M denote the order of h as a power series in y, by definition of h we have $M \geq 2$. Therefore h is contractive of degree $(M-1)d$ and so h_r depends on x_i^j with $i \leq r-(M-1)d$. For $M = 2$ this bound attains its maximum $r-d$. If $r < e+d$ this gives $i < e$, i.e., in F_r with $r < e+d$ appear just x_i^j with $i \leq d-1$, thus these polynomials vanish in R.
We will apply Proposition 9 to $\bar{I}_\infty \subseteq R$ where $\bar{F}_r = 0$ for $r < e+d$. By assumption there exists a $p \in \{1, \ldots, m\}$ with $A_e^p \neq 0$. Assume $r \geq e+d$. In that case the term $A_e^p x_{r-e}^p$ appears in $\bar{F}_r$. By the previous considerations

$$(r-e, p) \notin \cup_{l<r} \mathrm{supp}(\bar{F}'_l),$$

and $\bar{F}''_r$, which equals $\bar{h}_r$ above, lies in $k[x_i^j; i < r-e]$. All assumptions of Proposition 9 are fulfilled and this gives the desired linearizing isomorphism u. □

Generalized Power Series and their Felts

Let G be a group with identity element 0. Usually we will write the group operation additively using "+". A group is called *ordered* if there is an ordering "$\leq$" on G which is compatible with the group structure, i.e., if $x \leq y$ then $x+z \leq y+z$ for all $z \in G$. The ordering on G is called *total*, if it is total as an ordering on a set. From now on we denote by G a totally ordered group. Any such G has a natural decomposition into subsets

$$G_+ = \{g \in G; g \geq 0\}$$

$$G_- = \{g \in G; g \leq 0\}$$

with $G_+ \cap G_- = \{0\}$. The set G_+ is called the set of *positive elements of* G, and the elements of G_- are accordingly called *negative.* Note that G_+ is a monoid with "+", which has the induced total ordering.

In what follows we will restrict to Abelian groups. Further let k be a field of characteristic 0. We consider formal expressions of the form

$$\xi = \sum_{g \in G} \xi_g t^g$$

where $\xi_g \in k$. The set of all such elements is a k-vectorspace with coefficientwise addition and k-multiplication. For an element ξ of this vectorspace we define the *support of* ξ as

$$\operatorname{supp}(\xi) = \{g \in G; \xi_g \neq 0\}.$$

The coefficient ξ_0 will be referred to as the *constant term of* ξ.

Proposition 11. *Let* $\xi = \sum_{g \in G} \xi_g t^g$ *and* $\eta = \sum_{g \in G} \eta_g t^g$ *have well-ordered support. Then the multiplication*

$$\left(\sum_{g \in G} \xi_g t^g \right) \cdot \left(\sum_{g \in G} \eta_g t^g \right) = \sum_{g \in G} \left(\sum_{g_1 + g_2 = g} \xi_{g_1} \eta_{g_2} \right) t^g$$

is well-defined.

For a proof we refer to [vdH01], Prop. 3.1. We define the *ring of (formal) G-series (in one variable t with coefficients in k)* as the k-vectorspace

$$k[[t^G]] = \{\xi = \sum_{g \in G} \xi_g t^g;\ \operatorname{supp}(\xi) \text{ is well ordered}\}$$

together with the ring structure defined in Proposition 11. In the literature these series are also called *generalized power series* or *Noetherian series.* Clearly $k[[t^G]]$ is commutative with unit element t^0. We will distinguish the subring $k[[t^+]]$ of elements of $k[[t^G]]$ with support contained in G_+:

$$k[[t^+]] = \{\xi \in k[[t^G]]; \operatorname{supp}(\xi) \subseteq G_+\}.$$

Example 8. For $n \in \mathbb{N}$ let G be $(\mathbb{Z}, +, <)$ the additive group of integers with the usual order. Then $G_+ = \mathbb{N}$ and $k[[t^+]] \cong k[[t]]$ the ring of formal power series in t.

The following Proposition collects some simple properties of these rings:

Proposition 12. *(1) Every* $\xi \in k[[t^G]]$ *has a unique representation of the form* $\xi = ct^w(1 + \eta)$ *with* $c \in k$, $w \in G$ *and* $\eta \in k[[t^+]]$.
(2) The ring $k[[t^+]]$ *is a local domain with maximal ideal* $\mathfrak{m} = (t^g; g > 0)$ *and* $k[[t^G]]$ *is a field.*

The ring $k[[t^+]]$ is endowed with a topology given by the G-valuation

$$v\colon k[[t^G]] \to G; v(\xi) = \min \operatorname{supp}(\xi).$$

The neighbourhoods of 0 are given by the ideals $U_g = \{\xi \in k[[t^+]]; v(\xi) \geq g\}$, $g \in G_+$.

We briefly recall the concept of *transfinite induction*. Let $\mathcal{N}$ be a well-ordered set and $A \subseteq \mathcal{N}$. If for every $g \in \mathcal{N}$ the fact that $o \in A$ for all $o < g$ implies $g \in A$, then $A = \mathcal{N}$. In other words: If some property P holds for $\min \mathcal{N}$ and if from the fact that P holds for all $o < g$ follows that g has P, then P holds for all $g \in \mathcal{N}$.

A map $\varphi\colon k[[t^+]]^m \to k[[t^+]]$ is called *textile* if for any $\xi = (\xi^1, \dots, \xi^m) \in k[[t^+]]^m$, $\xi^j = \sum_{g \in G} \xi^j_g t^g$, the image $\varphi(\xi) = \sum_{g \in G} \varphi_g(\xi) t^g$ is a G-series with $\varphi_g(\xi)$ polynomial in a finite number of ξ^j_g, $1 \leq j \leq m$, and $g \in G$. We call φ *tactile* if it is induced by some polynomial expression, i.e., if there exists a polynomial $f \in k[x_1, \dots, x_m]$ such that $\varphi(\xi) = f(\xi^1, \dots, \xi^m)$. Tactile maps are clearly textile. Note that though we demand that $\varphi_g(\xi)$ is polynomial for all ξ, we cannot in general identify φ_g with a polynomial. If we restrict to elements with support contained in a fixed well-ordered submonoid of G_+, then we may identify φ_g with a polynomial.

Let $\mathcal{N} \subseteq G_+$ be a submonoid, which is well-ordered in the induced ordering. Set $w = \min \mathcal{N}$. Then we consider the subalgebra $k[[t^{\mathcal{N}}]] \subseteq k[[t^+]]$ of $\mathcal{N}$*-series* given by elements ξ with $\operatorname{supp}(\xi) \subseteq \mathcal{N}$. A polynomial $f \in k[x_1, \dots, x_m]$ defines a hypersurface $X \subseteq \mathbb{A}^m$ and a tactile map

$$\varphi\colon k[[t^{\mathcal{N}}]]^m \to k[[t^{\mathcal{N}}]]; \xi \mapsto f(\xi).$$

The image of a vector ξ of $\mathcal{N}$-series can be written as $\varphi(\xi) = \sum_{g \in \mathcal{N}} \varphi_g(\xi) t^g$ with $\varphi_g(x)$ being a polynomial in the indeterminates x^j_g, $1 \leq j \leq m$, $g \in \mathcal{N}$. The coefficients $F_g = \varphi_g(x)$ define an ideal I in

$$k[\underline{\mathcal{N}}^m] = k[x^j_g; 1 \leq j \leq m, g \in \mathcal{N}].$$

This gives a subscheme $X_{\mathcal{N}} = \operatorname{Spec} k[\underline{\mathcal{N}}^m]/I$ of $\mathbb{A}^m_{\mathcal{N}} = \operatorname{Spec} k[\underline{\mathcal{N}}^m]$. In this framework a *textile map* $\mathbb{A}^m_{\mathcal{N}} \to \mathbb{A}^m_{\mathcal{N}}$ is just a k-algebra homomorphism

$$\varphi\colon k[\underline{\mathcal{N}}^m] \to k[\underline{\mathcal{N}}^m].$$

Especially φ is called *contractive*, if for all $g \in \mathcal{N}$ the $\varphi_g(x)$ is polynomial in x^j_o with $o < g$.

Remark. In the case $\mathcal{N} = \mathbb{N}$ the arc space $X_\infty = X_{\mathbb{N}}$ is defined as the projective limit of the jet spaces of X. Recall that the pth jet space X_p of X is defined as the set of truncated polynomial solutions (of degree p in one variable, say t) to the defining equations of X. That is, for $X = V(f) \subseteq \mathbb{A}^m$ we have

$$X_p(k) = \{x(t) \in \left(k[t]/(t^{p+1})\right)^m; f(x(t)) = 0 \mod t^{p+1})\}.$$

It is easy to see that X_p is a k-scheme of finite type! In the situation of arbitrary $\mathcal{N}$ or $k[[t^+]]$ it is a priori not clear how to define an analog for the X_p having nice properties like being of finite type over k. A generalization of the jet scheme construction could be to set for $g \in G_+$

$$X_g = \operatorname{Spec} k[x^j_o; 1 \leq j \leq m, o < g]/(F_o; o < g),$$

which respects the topology induced by the valuation v. But there is one major drawback - the X_g need not be of finite type. Take $\mathcal{N}$ to be the monoid generated by $\left(\frac{n-1}{n}; n \in \mathbb{N}\right) \subseteq \mathbb{Q}$ and 1. For $g = 1$

$$X_1 = \operatorname{Spec} k[x_o^j; 1 \leq j \leq m; o < 1]/(F_o; o < 1)$$

and this is the spectrum of a polynomial ring in infinitely many variables.

Proposition 13. *Let $f\colon \mathbb{A}^m_{\mathcal{N}} \to \mathbb{A}^m_{\mathcal{N}}$ be a textile map of the form $f = \ell + h$ with ℓ a linear isomorphism and $\ell^{-1}h$ contractive. Moreover assume that $f(0) = 0$. Then f is an isomorphism.*

Proof. Without loss of generality $\ell = \mathrm{id}$ and h is contractive, i.e.,

$$h_g^j \in (x_o^l; 1 \leq l \leq m, w \leq o < g) \subseteq k[x_o^l; 1 \leq l \leq m, w \leq o < g],$$

for all $(g, j) \in \mathcal{N} \times \{1, \ldots, m\}$. We inductively construct an inverse for f by the following sequence of ringautomorphisms: Set $u_w = \mathrm{id}$ and assume for $o < g$ automorphisms u_o have already been constructed such that $(u_o \circ f)(x_\omega^j) = x_\omega^j$ for $\omega \leq o$. Define $u_g\colon k[\underline{\mathcal{N}}^m] \to k[\underline{\mathcal{N}}^m]$ by

$$u_g(x_o^j) := \begin{cases} x_o^j - u_p(h_g^j) & \text{for } \ o = g \\ u_o(x_o^j) & o < g, \\ x_o^j & \text{otherwise} \end{cases}$$

where $p \in \mathcal{N}$ is the maximal index ω of an x_ω^j appearing in h_g^j. Since h is contractive $p < g$. The sequence $(u_g)_{g \in \mathcal{N}}$ converges since $u_g(x_o^j) = u_s(x_o^j)$ for $o \leq g$ and $s > g$, thus defining a textile map $u\colon k[\underline{\mathcal{N}}^m] \to k[\underline{\mathcal{N}}^m]$ with $u(x_g^j) = u_g(x_g^j)$. This u is the inverse of f. Indeed for all g, j we have

$$(u \circ f)(x_g^j) = u_g(x_g^j) + u_g(h_g^j) = x_g^j - u_p(h_g^j) + u_p(h_g^j) = x_g^j.$$

□

As in the previous section we will write F' for the homogenous part of degree 1 of an element $F \in L[\underline{\mathcal{N}}^m]$, set $F'' = F - F'$ and define the support of a polynomial $F \in k[\underline{\mathcal{N}}^m]$ analogously to the definition in the classical setting

Proposition 14. *Let L be a commutative ring with one and $J = (F_g; g \in \mathcal{N})$ in $L[\underline{\mathcal{N}}^m]$ an ideal in $(x_o^j, o \in \mathcal{N}, 1 \leq j \leq m) \subseteq k[\underline{\mathcal{N}}^m]$. Assume that*

1. *every F_g' contains a term $A_{\omega_g}^{j_g} x_{\omega_g}^{j_g}$ such that $(\omega_g, j_g) \notin \cup_{o<g} \mathrm{supp}(F_o')$, $A_{\omega_g}^{j_g} \in L^\times$ and $\omega_o < \omega_g$ for $o < g$;*
2. *$F_g'' \in L[x_o^j; o < \omega_g]$.*

Then there exists an L-algebra automorphism $u\colon L[\underline{\mathcal{N}}^m] \to L[\underline{\mathcal{N}}^m]$ such that $u(J)$ is generated by F_g', $g \in \mathcal{N}$.

Proof. We proceed analogously to the proof of Proposition 9 and construct the automorphism u inductively as follows. Set

$$u_w\colon L[\underline{\mathcal{N}}^m] \to L[\underline{\mathcal{N}}^m]; x_o^j \mapsto \begin{cases} x_{\omega_w}^{j_w} - (A_{\omega_w}^{j_w})^{-1} F_w'' & \text{for } \ (o, j) = (\omega_w, j_w) \\ x_o^j & \text{otherwise.} \end{cases}$$

Clearly $u_w(F_w) = F'_w$ and u_w is an automorphism by Lemma 3 since $\widetilde{\mathrm{supp}}(F''_w) \subseteq \{(o,j); o < \omega_w\}$. Assume that for all $o < g$ the automorphisms u_o have been constructed such that $u_o(F_\eta) = F'_\eta$ for all $\eta \leq o$. Then we define

$$u_g : x^j_\eta \mapsto \begin{cases} x^{j_g}_{\omega_g} - (A^{j_g}_{\omega_g})^{-1}\left(u_{p_g}(F_g)\right)'' & (\eta, j) = (\omega_g, j_g) \\ u_\eta(x^j_\eta) & (\eta, j) = (\omega_o, j_o) \;\text{ for some }\; o < g \\ x^j_\eta & \text{otherwise.} \end{cases}$$

Here p_g is the maximal index o of elements x^j_o appearing in F_g. By the induction hypothesis for $o < g$ we have $u_g(F_o) = u_{p_o}(F_o) = F'_o$. Moreover

$$\begin{aligned} u_g(F_g) &= u_{p_g}(F_g) - \left(u_{p_g}(F_g)\right)'' \\ &= F'_g. \end{aligned}$$

The last equality follows from the fact that the u_o have linear part equal to the identity. Using Lemma 13 we see u_g is invertible: if $(o,j) \in \widetilde{\mathrm{supp}}(u_{p_g}(F_g))''$ then $o < \omega_g$. The $(u_o)_o$ converge towards an automorphism of $L[\underline{\mathcal{N}}^m]$ since

$$u_g(x^j_o) = u_{g'}(x^j_o)$$

if $\omega_n, \omega_{n'} \geq o$ and the sequence $(\omega_r)_r$ is increasing. This completes the proof. $\square$

Chapter 2

Six instances of the linearization principle

With the preparations presented in Chapter 1 we are now ready to prove and partially extend the following six results in singularity theory by one argument (namely the Linearization Principle): Denef-Loeser's local-triviality result for arc spaces [DL99], the Grinberg-Kazhdan-Drinfeld formal arc theorem [GK00, Dri02], triviality of the solution space of certain polynomial differential operators, Tougeron's Implicit Function Theorem [Tou68], Artin's respectively Wavrik's Approximation Theorem [Art68, Wav75], and an inversion theorem by Lamel-Mir [LM07].
Recall that $\mathcal{C}$ and $\mathcal{A}$ denote the k-algebras of n respectively 1 cords. In the latter case we speak of arcs. They naturally identify with elements of $k[[x_1, \ldots, x_n]]$ respectively $k[[t]]$.

2.1 A fibration theorem by Denef and Loeser

Let $X \subseteq \mathbb{A}_k^n$ be an affine variety over a field k of characteristic 0 defined by elements $f_1, \ldots, f_p$ of $k[x_1, \ldots, x_n]$. These polynomials define by substitution of power series $a^i \in k[[t]]$ for the x_i a tactile map $F : \mathcal{A}^n \to \mathcal{A}^p$ with coefficients $(F_i)_{i \in \mathbb{N}}$. The felt defined by F is called the *arc space of* X and denoted by X_∞. In terms of power series

$$X_\infty = \{a = (a^1, \ldots, a^n) \in k[[t]]^n; f_1(a) = \ldots = f_p(a) = 0\}$$

For $q \in \mathbb{N}$ we write π_q for the natural projection $X_\infty \to \left(\mathcal{A}/\mathfrak{m}^{q+1}\right)^n \cong (k^{q+1})^n$ sending an arc a to the vector of its first $q + 1$ coefficient-vectors $(a_0, \ldots, a_q)$. The image of X_∞ under π_q is contained in

$$X_q = \{a = (a_0, \ldots, a_q) \in (k^n)^{q+1}; F_0(a) = \ldots = F_q(a) = 0\},$$

the set of “approximate solutions” called the *qth jet-space of* X. Note: $\pi_q(X_\infty)$ consists of those approximate solutions which can be lifted to (exact) solutions in $\mathcal{A}^n$. For an element $g \in k[x_1, \ldots, x_n]$ and an arc $a \in \mathcal{A}^n$ we define the *order of a with respect to g* as

$$\mathrm{ord}_g(a) := \mathrm{ord}(g(a)).$$

Here $g(a)$ is obtained by substituting a^i for x_i in g. Let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal. The *order of a with respect to I* is defined as

$$\operatorname{ord}_I(a) := \min\{\operatorname{ord}_g a; g \in I\}.$$

Especially if I denotes the ideal of $\operatorname{Sing}(X)$, we will write $\mathcal{A}_e(X)$, $e \in \mathbb{N}$, for the set of arcs on X lying in $\operatorname{Sing}(X)$ with order at most e, i.e.,

$$\mathcal{A}_e(X) = \{a \in X_\infty; \operatorname{ord}_I(a) \leq e\}.$$

Theorem 10. *([DL99], Lemma 4.1) Let X be an affine algebraic variety over k of pure dimension d; let $e \in \mathbb{N}$. The map*

$$\pi_e : X_\infty \to \pi_e(X_\infty) \subseteq (k^n)^{e+1}$$

is a piecewise trivial fibration over $\pi_e(\mathcal{A}_e(X))$ with fibre $\mathfrak{m}^{e+1} \cdot \mathcal{A}^d$. This means: $\pi_e(\mathcal{A}_e(X))$ can be covered by finitely many locally (Zariski-) closed subsets $\mathcal{S}$ such that there exist textile invertible maps φ with commutative diagram

$$\begin{tikzcd}
\pi_e^{-1}(\mathcal{S}) \arrow[rr, "\varphi"] \arrow[dr, "\pi_e"'] & & \mathcal{S} \times (\mathfrak{m}^{e+1} \cdot \mathcal{A}^d) \arrow[dl, "\mathrm{proj}_1"] \\
& \mathcal{S} &
\end{tikzcd}$$

This result implies Lemma 4.1 in [DL99]: the trivializing textile map φ is by construction (see below) compatible with the truncation maps π_q, respectively π_p^q, $q \geq p$, as described in section 1.1.1, thus inducing a trivial fibration $\pi_{q+1}(X_\infty) \to \pi_q(X_\infty)$, $q \geq e$, over $\pi_q(\mathcal{A}_e(X))$ (with respective affine bundle structure) in the sense of [DL99]. The proof of Theorem 10 will be given in the next two sections.

The proof of the trivialization of π_e in [DL99] and [Loo02] makes use of Hensel's Lemma (see e.g. Corollary 1 in III, §4.5 of [Bou98]), which in turn follows from Thm. 2, III, §4.5 there. Notice that the statement of this Thm. 2 is analogous to the Rank Theorem in the respective setting. Our proof of Thm. 10 gives in addition an explicit description of the trivializing map φ.

2.1.1 The Hypersurface Case

Let $X = V(f) \subseteq \mathbb{A}_k^n$ with $f \in k[x_1, \ldots, x_n]$ be a (reduced) hypersurface. The singular locus $\operatorname{Sing}(X)$ is given by the Jacobian ideal

$$J_f = \langle f, \partial_1 f, \ldots, \partial_n f \rangle.$$

Each $a \in \mathcal{A}_e(X)$ can be decomposed as $a = \bar{a} + \hat{a}$ where $\bar{a}$ denotes the arc $(a_0, \ldots, a_e, 0, \ldots)$ and $\hat{a} = a - \bar{a}$. Thus, $\operatorname{ord}(\hat{a}) \geq e + 1$. We identify $\bar{a}$ with the element $\pi_e(a)$. The polynomial f induces a tactile $\mathcal{A}^n \to \mathcal{A}$. We view f as a regular family over the parameter set $\mathcal{S} = \pi_e(\mathcal{A}_e(X))$, which is constructible, see e.g. [Reg95], §1, or section (4.4) in [DL99]. The parameter set admits a partition into locally closed subsets $\mathcal{S}_{i,e'}$ (w.r.t. the Zariski-topology on $\mathbb{A}_k^{n(e+1)}$), $(i, e') \in \Lambda := \{1, \dots, n\} \times \{0, \dots, e\}$, defined as

$$\mathcal{S}_{i,e'} := \{a \in \pi_e(\mathcal{A}_e(X)); \min_j\{\mathrm{ord}(\partial_j f(a))\} = \mathrm{ord}(\partial_i f(a)) = e'\}.$$

Consider for $\alpha \in \Lambda$ the family

$$f_{\mathcal{S}_\alpha} : \mathcal{S}_\alpha \times \mathfrak{m}^{e+1} \cdot \mathcal{A}^n \to \mathcal{A}; (\bar{a}, \eta) \mapsto f(\bar{a} + \eta).$$

For simplicity of notation we will denote this family again by f. Taylor expansion yields

$$f(\bar{a} + \eta) = f(\bar{a}) + \widehat{f}(\bar{a}, \eta),$$

where $\widehat{f}(\bar{a}, \eta) = \sum \partial_j f(\bar{a}) \cdot \eta_j + h(\bar{a}, \eta)$ defines a regular family on $\mathcal{S}$. The map h is of order at least 2 in η.

The map $\widehat{f}$ fulfills all assumptions of the Parametric Rank Theorem. Its linear part $\ell := T_0 \widehat{f}$ has image

$$\mathrm{im}(\ell(\bar{a}, -)) = \langle \partial_i f(\bar{a}) \rangle = \mathfrak{m}^{e'+e+1}$$

for all $\bar{a} \in \mathcal{S}$. The image has direct complement $J = \oplus_{i=0}^{e'+e} k t^i$.

Moreover,

$$T_\varrho \widehat{f} \cdot \eta = \sum_j (\partial_j f(\bar{a}) + \partial_j h(\bar{a}, \varrho)) \cdot \eta_j$$

and $\mathrm{ord}(\partial_i f(\bar{a}) + \partial_i h(\bar{a}, \varrho)) = \mathrm{ord}(\partial_i f(\bar{a}))$ for all $\varrho \in \mathfrak{m}^{e+1} \cdot \mathcal{A}^n$. Hence, by Proposition 7, $\widehat{f}$ fulfills the Rank Condition for all $\bar{a} \in \mathcal{S}$ on $\mathfrak{m}^{e+1} \cdot \mathcal{A}^n$ (with respect to J). By Construction 2, the linear part ℓ admits a scission σ with contraction degree $\kappa(\sigma) = -e'$ (and complement $\ker \ell\sigma = J$). Indeed, σ is constructed by taking the quotient when dividing through $\partial_i f(\bar{a})$ (cf. section 1.2). Since the division theorem (theorem 4) requires monic divisors, the coefficients of the family $\sigma = \sigma_{\mathcal{S}}$ will be rational in the coefficients of $\bar{a}$ but regular on the chosen $\mathcal{S}$. It is easy to see that

$$\kappa(h) \geq 2(e+1) - (e+1) = e + 1 > e'.$$

Hence, σh is contractive. The Parametric Rank Theorem gives regular invertible families $u_{\mathcal{S}}$ and $v_{\mathcal{S}}$ such that

$$v_{\mathcal{S}} \widehat{f}_{\mathcal{S}} u_{\mathcal{S}}^{-1} = \ell_{\mathcal{S}}.$$

The map $\mathfrak{m}^{e+1} \cdot \mathcal{A}^n \to \mathfrak{m}^{e+1} \cdot \mathcal{A}^n; \eta \mapsto u_{\bar{a}}^{-1}(\eta)$ defines an isomorphism between solutions x of

$$f(\bar{a} + x) = 0$$

and solutions y of

$$\ell(y) = v(-f(\bar{a})).$$

By linear algebra the set of solutions to the last equation is isomorphic to $\mathfrak{m}^{e+1} \cdot \mathcal{A}^{n-1}$ via a linear map $\psi : \mathfrak{m}^{e+1} \cdot \mathcal{A}^n \to \mathfrak{m}^{e+1} \cdot \mathcal{A}^{n-1}$. This results in the commutative diagram

$$\begin{tikzcd}
\pi_e^{-1}(\mathcal{S}) \arrow[rr, "\varphi"] \arrow[dr, "\pi_e"'] & & \mathcal{S} \times \mathfrak{m}^{e+1} \cdot \mathcal{A}^{n-1} \arrow[dl, "\mathrm{proj}_1"] \\
& \mathcal{S} &
\end{tikzcd}$$

where φ is the invertible textile map defined by $\varphi(\bar{a} + \hat{a}) = (\bar{a}, \psi u_{\bar{a}}^{-1}(\hat{a}))$.

Consider now the general situation $\mathcal{S} = \cup_{\alpha \in \Lambda} \mathcal{S}_\alpha$. Then for each $\alpha \in \Lambda$ there exists a trivializing map φ_α. It is a simple matter to check that the transition maps $\varphi_\alpha \varphi_\beta^{-1}$ are linear isomorphisms for $\alpha, \beta \in \Lambda$. This completes the proof in the hypersurface case.

2.1.2 The General Case

Let X be an algebraic variety over k of pure dimension d defined by the ideal $\langle f_1, \dots, f_N \rangle$ in $k[x_1, \dots, x_n]$. Its singular locus is given by the ideal $\langle f_1, \dots, f_N, \theta \rangle$ where θ is running through all $(N-d)$-minors of $\partial f = \left(\frac{\partial f_i}{\partial x_j}\right)_{ij}$. In this case $\mathcal{A}_e(X)$ is the set of arcs $a \in X_\infty$ for which $\operatorname{ord} \theta(a) \leq e$ for at least one $(N-d)$-minor θ.

We proceed analogously to section 2.1.1 and first show that pointwise linearization is possible. Let $\bar{a}$ again be the truncation of an arc $a \in \mathcal{A}_e(X)$ to order e and consider $\partial f(\bar{a}) \in k[[t]]^{N \times n}$. Since $k[[t]]$ is a principal ideal domain, we can find linear automorphisms P, Q of $k[[t]]^N$ respectively $k[[t]]^n$ such that $P \partial f(\bar{a}) Q$ is in Smith-form, i.e.,

$$\begin{pmatrix}
t^{\varepsilon_1} E_1 & 0 & \cdots & 0 & \cdots & 0 \\
0 & \ddots & & \vdots & & \vdots \\
\vdots & & t^{\varepsilon_{N-d}} E_{N-d} & \vdots & & \vdots \\
0 & \cdots & \cdots & 0 & \cdots & 0 \\
\vdots & & & & & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & 0
\end{pmatrix}$$

with $E_i \in k[[t]]^\times$ and $\varepsilon_1 \leq \ldots \leq \varepsilon_{N-d}$. Endow $k[[t]]^N$ with the module ordering defined in section 1.2. Then it is clear that the first $N-d$ columns form a standard basis. Its maximal order is given by ε_{N-d}. The greatest common divisor of all $(N-d)$-minors of $\partial f(\bar{a})$, which is invariant under equivalence of matrices, is equal to $\prod_i E_i t^{\varepsilon_i}$. Therefore $\varepsilon_{N-d} \leq e$. Constructing a scission σ for $\ell(z) = \partial f(\bar{a}) z$ as in section 1.2 thus shows that σh is contractive. By Proposition 7 we see that the Rank Condition is fulfilled. This gives pointwise linearization of $\hat{a} \mapsto f(\bar{a} + \hat{a})$ where $f = (f_1, \dots, f_N)$.

We leave it to the reader to verify that one can carry out the same procedure as in section 2.1.1 to prove the local triviality of $\pi_e : X_\infty \to \pi_e(X_\infty) \subseteq (k^n)^{e+1}$.

2.2 The Grinberg-Kazhdan-Drinfeld formal arc theorem

Let X be a scheme of finite type over k and denote by $X_\infty[\gamma_0]$ the formal neighbourhood[1] of the arc space X_∞ in a k-arc γ_0 which does not lie in $(\mathrm{Sing}X)_\infty$. The following theorem is proved for $k = \mathbb{C}$ by Grinberg-Kazhdan in [GK00] and for arbitrary fields k in [Dri02]:

Theorem 11. *Under the above assumptions we have*

$$X_\infty[\gamma_0] \cong Y[y] \times D^\infty,$$

where $Y[y]$ is the formal neighbourhood of a (possibly singular) scheme Y of finite type over k in a suitable point $y \in Y$ and D^∞ is a countable product of formal discs, i.e., the formal spectrum[2] of $k[[t]]$.

Remark. Denote the structure sheaf of X_∞ by $\mathcal{O}_\infty$. The assertion can be reformulated (see [Reg06]) as

$$\widehat{\mathcal{O}}_{\infty,\gamma_0} \cong B[z_i; i \in \mathbb{N}]\widehat{\ }.$$

Here B denotes a complete local Noetherian ring (corresponding to $Y[y]$) and $B[z_i; i \in \mathbb{N}]\widehat{\ }$ is the completion of $B[z_i; i \in \mathbb{N}]$ w.r.t. the ideal $(z_i; i \in \mathbb{N})$.

We will show that in the case of hypersurfaces this local factorization theorem can be seen as a consequence of the Rank Theorem (over a field of characteristic 0). The restriction to hypersurfaces is not essential but keeps notation simple. The relation between these two results is as follows: As in [Dri02] we use the characterization of the formal neighbourhood by its functor of points from test-rings to sets. A *test-ring* A is a local, commutative, and unital k-algebra with nilpotent maximal ideal $\mathfrak{n}$ and residue field k. The A-points of $X_\infty[\gamma_0]$ are $A[[t]]$-points of X whose reduction modulo $\mathfrak{n}$ coincides with γ_0.

Consider a hypersurface X given by $f \in k[x_1, \dots, x_n, y]$ where k is a field of characteristic 0. Let $\gamma_0 = (x_0, y_0) \in k[[t]]^{n+1}$ be a k-arc of X, i.e., $f(\gamma_0) = 0$. Moreover, we assume that $\gamma_0 \notin (\mathrm{Sing}X)_\infty$, thus, w.l.o.g. $\mathrm{ord}_t\, \partial_y f(x_0, y_0) = d > 0$ (the case $d = 0$ is trivial and will henceforth be excluded). In this setting finding A-points for $X_\infty[\gamma_0]$ translates into finding $\gamma = (x, y) \in A[[t]]^n \times A[[t]]$ with $\gamma = \gamma_0 \mod \mathfrak{n}$ and

$$f(x, y) = 0.$$

[1] If Z is a scheme with structure sheaf $\mathcal{O}_Z$ and I is an ideal sheaf of a point $p \in Z$, then the formal neighbourhood of Z in p, denoted by $Z[p]$, is the topological space $\{p\}$ equipped with the structure sheaf $\lim_\leftarrow \mathcal{O}_Z/I^n$ (given by $\widehat{\mathcal{O}}_{Z,p}$); see for example [Har77], Chap. II.9. For the characterization of a formal neighbourhood by its functor of points see the main text.

[2] see [Har77], Chap. II.9.

By the Weierstrass Preparation Theorem there are a unit $u \in A[[t]]^\times$ and a distinguished polynomial $q \in A[t]$, both unique, such that

$$\partial_y f(x, y) = u \cdot q.$$

Taking this modulo $\mathfrak{n}$ shows that $\deg_t q = d$ (d depends on γ_0 but not on γ). For any integer $r > 1$ we define the textile isomorphism

$$\psi_q \colon A[[t]]^{n+1} \to A[t]^n_{<(r+1)d} \times A[t]_{<rd} \times A[[t]]^n \times A[[t]]; (x, y) \mapsto (\bar{x}, \bar{y}, \xi, \eta)$$

induced by Weierstrass division $x = \bar{x} + q^{r+1}\xi$, $y = \bar{y} + q^r\eta$. Now work with $f \circ \psi_q^{-1}$. This will have the advantage that after fixing $\bar{x}$ and $\bar{y}$ the induced map is a quasi-submersion. Note: ψ_q is not textile for general A – we are strongly using the nilpotency of $\mathfrak{n}$ in A. By Taylor expansion of $\partial_y f(\bar{x} + q^{r+1}\xi, \bar{y} + q^r\eta)$ it is easy to see that

$$\partial_y f(\bar{x}, \bar{y}) = \tilde{u} \cdot q \tag{2.1}$$

for some $\tilde{u} \in A[[t]]^\times$. Fixing $\bar{x}$ and $\bar{y}$ consider

$$\varphi \colon A[[t]]^n \times A[[t]] \to A[[t]]; (\xi, \eta) \mapsto f(\bar{x} + q^{r+1}\xi, \bar{y} + q^r\eta) - f(\bar{x}, \bar{y}),$$

which has Taylor expansion

$$\varphi(\xi, \eta) = \partial_x f(\bar{x}, \bar{y})q^{r+1} \cdot \xi + \partial_y f(\bar{x}, \bar{y})q^r \cdot \eta + H(q^{r+1}\xi, q^r\eta),$$

where H is at least of order two in its entries. Denote the linear part of φ by ℓ. From equation (2.1) we deduce that $\mathrm{im}(\ell) = (q)^{r+1}$ and by construction for $h(\xi, \eta) = H(q^{r+1}\xi, q^r\eta)$ the relation $\mathrm{im}(h) \subseteq \mathrm{im}(\ell)$ holds, i.e., φ is quasi-submersive. Therefore we don't have to check the rank condition. Set $e = \mathrm{ord}_t(q)$ and define σ to be the scission for ℓ constructed via division by $\tilde{u}q^{r+1}$. It is obvious that the contraction degrees satisfy $\kappa(\ell) = -(r+1)e$ and $\kappa(h) \geq 2re$. In the case that $e > 0$ the order condition is fulfilled: $\kappa(h) = 2re > (r+1)e = \kappa(\sigma)$. If $e = 0$ the situation is more complicated. The composition σh is still contractive with respect to the refined order ORD by Lemma 2. Therefore, the linearizing automorphism u in Thm. 6 can be constructed. This allows linearization of φ. Solvability of the linearized equation

$$\ell(\xi, \eta) = f(\bar{x}, \bar{y}) \tag{2.2}$$

is given, since by assumption $f(\bar{x}, \bar{y}) \in (q)^{r+1}$. In fact, the solution space to (2.2) is a free $A[[t]]$-module of rank n.

Conversely, given $(q, \bar{x}, \bar{y}) \in A[t] \times A[t]_{<d(r+1)} \times A[t]_{<rd}$ fulfilling conditions (C) and equations (E) below, the induced map φ is linearizable and the corresponding linear equation (2.2) has a solution. The conditions and equations are as follows:

$$(\mathrm{C}) \qquad \begin{cases} \mathrm{C}_q: & q = t^d \mod \mathfrak{n}, q \text{ monic} \\ \mathrm{C}_{\bar{x}}: & \bar{x} \mod \mathfrak{n} = x_0 \mod (t)^{d(r+1)} \\ \mathrm{C}_{\bar{y}}: & \bar{y} \mod \mathfrak{n} = y_0 \mod (t)^{dr} \end{cases}$$

and

$$(\mathrm{E}) \quad \begin{cases} \mathrm{E}_1: & \partial_y f(\bar{x},\bar{y}) = 0 \mod (q) \\ \mathrm{E}_2: & f(\bar{x},\bar{y}) = 0 \mod (q^{r+1}). \end{cases}$$

Indeed, by (E_1) we see that $\partial_y f(\bar{x},\bar{y}) = B \cdot q$ for some $B \in A[[t]]$ (here we naturally identify $\bar{x},\bar{y}$ with elements in $A[t]$). From (C) it follows that $B \in A[[t]]^\times$. Therefore, $\mathrm{im}(\ell) = (q)^{r+1}$; φ fulfills rank and order condition as follows analogously to the presentation above, and the equation

$$\varphi(\xi,\eta) = f(\bar{x},\bar{y}) \tag{2.3}$$

has a solution by (E_2). In fact the set of solutions to equation (2.3) is isomorphic to some $\mathcal{A}_A^n$.

To sum up, any A-deformation of the arc γ_0 is determined by $(q,\bar{x},\bar{y})$ fulfilling conditions (C) and (E) and by (ξ,η), which have to fulfill a linear equation. The first data defines a scheme Y of finite type over k. As in [Dri02] for any k-algebra R the set $Y(R)$ consists of triples $(q,\bar{x},\bar{y})$ fulfilling (C) and (E) (with A replaced by R); the distinguished element $y \in Y$ appearing in the Grinberg-Kazhdan-Drinfeld formal arc theorem corresponds then to $(t^d, x_0 \mod (t)^{(r+1)d}, y_0 \mod (t)^{rd})$.

2.3 On solutions of polynomial ODE's

Textile maps appear naturally in the context of differential and difference equations. For $q,n \in \mathbb{N}$ and polynomials P_i in qn indeterminates we consider the following system of ordinary differential equations:

$$(*) \begin{cases} x_1^{(q)} &= P_1(x_1,\ldots,x_n,x_1^{(1)},\ldots,x_n^{(1)},\ldots,x_1^{(q-1)},\ldots,x_n^{(q-1)}) \\ \vdots & \\ x_n^{(q)} &= P_n(x_1,\ldots,x_n,x_1^{(1)},\ldots,x_n^{(1)},\ldots,x_1^{(q-1)},\ldots,x_n^{(q-1)}) \end{cases}$$

We are searching for solutions $x = (x_1,\ldots,x_n) \in k[[t]]^n$ where $x_i^{(l)}$ denotes the lth derivative (with respect to t) of a power series $x_i \in k[[t]]$. The problem of finding solutions to this system can be formulated via cords and textile maps as follows: Identify again $k[[t]]$ with $\mathcal{A}$ and denote by D the *differential operator*

$$D : \mathcal{A} \to \mathcal{A}; a = (a_i)_{i\in\mathbb{N}} \mapsto ((i+1)a_{i+1})_{i\in\mathbb{N}}.$$

For $j \in \mathbb{N}$ its jth power is written as D^j, $D^0 := \mathrm{id}$. We denote by p_i the tactile map induced by P_i. Solutions to the system $(*)$ are zeros of the textile map $f : \mathcal{A}^n \to \mathcal{A}^n$;

$$f(x_1,\ldots,x_n) = \begin{pmatrix} D^q(x_1) & - & p_1(x_1,\ldots,x_n,\ldots,D^{q-1}x_1,\ldots,D^{q-1}x_n) \\ & \vdots & \\ D^q(x_n) & - & p_n(x_1,\ldots,x_n,\ldots,D^{q-1}x_1,\ldots,D^{q-1}x_n) \end{pmatrix}$$

Taking into account initial conditions $D^j x_i(0) = x_{0,i}^j$, $i = 1,\ldots,n$, $j = 0,\ldots,q-1$ yields a regular family

$$f_{\mathfrak{B}} : \mathfrak{B} \times (\mathfrak{m}^q \cdot \mathcal{A}^n) \to \mathcal{A}^n,$$

where $\mathfrak{B} = (k^q)^n$. Denote by f_{x_0} the restricted map $f_{\mathfrak{B}}(x_0, -)$. Zeros of f_{x_0} are solutions of the differential equation with initial condition $x_0 \in \mathfrak{B}$.

Theorem 12. *Solutions of the polynomial differential equation $(*)$ with initial vector x_0 are isomorphic to solutions $y \in \mathfrak{m}^q \cdot \mathcal{A}^n$ of a linear system of the form*

$$\ell(y) = b_0,$$

where $\ell : \mathcal{A}^n \to \mathcal{A}^n$ is textile linear and $b_0 \in \mathcal{A}^n$.

The proof is mostly a consequence of the following lemma:

Lemma 15. *(a) Let $\nu_1, \ldots, \nu_p \in \mathbb{N} \cup \{0\}$ and $\mu_1, \ldots, \mu_p \in \mathbb{N}$; the map*

$$g : \mathcal{A} \to \mathcal{A}; x \mapsto (D^{\nu_1} x)^{\mu_1} \cdots (D^{\nu_p} x)^{\mu_p}$$

has contraction degree $\kappa(g) \geq -\max\{\nu_1, \ldots, \nu_p\}$.
(b) Let U be a neighbourhood of 0 and let $h : U \subseteq \mathcal{A} \to \mathcal{A}$ be a textile map. Then

$$\kappa(T_0 h) \geq \kappa(h).$$

Proof. (Lemma) (a) is obvious. For (b): write h_i for the ith coefficient of h. Then

$$\begin{aligned} T_0 h \cdot y = \partial h \bullet y &= \sum_{\gamma \in \mathbb{N}} (\partial_\gamma h)(0) \cdot y_\gamma \\ &= \left(\sum_{\gamma \leq i + \kappa(h)} (\partial_\gamma h_i)(0) y_\gamma \right)_{i \in \mathbb{N}} . \end{aligned}$$

Hence, $\kappa(T_0 h) \geq \kappa(h)$.

□

To prove the theorem we proceed as follows: Using Taylor expansion gives f_{x_0} as $f_{x_0}(y) = f(x_0) + \widehat{f}(x_0; y)$ with $\widehat{f}(x_0; 0) = 0$. This $\widehat{f}(x_0, -) : \mathfrak{m}^q \cdot \mathcal{A}^n \to \mathcal{A}^n$ fulfills the assumptions of the Rank Theorem: According to (a) in Lemma 15, $\kappa(p) \geq -(q-1)$, and further by (b) $\kappa(T_0 p) \geq -(q-1)$ follows. Since $T_0 \widehat{f} = D^q - T_0 p$, we deduce $\kappa(T_0 \widehat{f}) = -q$. By Construction 2 there is a scission σ for $T_0 \widehat{f}$ with $\kappa(\sigma) = +q$. Hence, σ composed with $h := f - T_0 \widehat{f}$ is contractive. The Rank Condition is fulfilled since $T_0 \widehat{f}$ is injective and, thus, flat (see Proposition 6).
By the Rank Theorem there exist linearizing automorphisms u, v such that $v \widehat{f} u^{-1} = \ell$ with $\ell := T_0 \widehat{f}$. Similar to section 2.1.1 we conclude that solutions of $\ell(x_0; \eta) = v(-f(x_0))$ are isomorphic to solutions of $f(x_0 + u^{-1}(\eta)) = 0$.

2.4 Tougeron's Implicit Function Theorem

Let k be a complete valued field of characteristic 0. We denote by $k\{x, y\}$ and $k[[x, y]]$ the ring of convergent and formal power series respectively in the variables $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_p)$ with coefficients in k. The respective maximal ideals will be denoted by $\mathfrak{m}$ and $\bar{\mathfrak{m}}$.
Using Theorem 6 we prove the following result, known as Tougeron's Implicit Function Theorem (see Proposition 1, p. 206 ff. in [Tou68]):

Theorem 13. *Let $F \in k[[x,y]]^q$, $q \le p$, with $F(0,0) = 0$. Denote by A the ideal generated by all q-minors of the Jacobian $\delta = \partial_y F(x,0)$, and by B any proper ideal in $k[[x]]$. If $F(x,0) \in BA^2$, then there exist $y_1(x), \ldots, y_p(x) \in BA$ such that*

$$F(x, y_1(x), \ldots, y_p(x)) = 0.$$

An analogous assertion holds for $F \in k\{x,y\}^q$. In that case the $y_i(x)$ will be convergent, too. For simplicity of notation we will restrict our considerations to the hypersurface case, i.e., we assume $q = 1$. We denote the ith component of the Jacobian δ by δ_i. So A is generated by $\delta_1, \ldots, \delta_p$. The proof of the theorem reduces to a proof of the following lemma (see Lemma 3.1 in [Tou68], Lemma 1.2 in Chap. V of [Rui93] or Lemma 2.8 in [Art68]):

Lemma 16. *With the notation of the theorem and new indeterminates $y^i = (y_1^i, \ldots, y_p^i)$, $i = 1, \ldots, r$, the following assertion holds: for $r \le p$ there are power series $Y^i \in k[[x, y_j^l; 1 \le l \le r, 1 \le j \le p]]^p$, $1 \le i \le r$, so that*

$$F(x, \sum_{i=1}^{r} \delta_i Y^i) = F(x,0) + \delta(\sum_{i=1}^{r} \delta_i y^i)$$

and $Y^i(x,0) = 0$ for $1 \le i \le r$.

The lemma obviously asserts that the tactile map

$$g\colon (y^1, \ldots, y^r) \mapsto F(x, \sum_{i=1}^{r} \delta_i y^i) - F(x,0) \tag{2.4}$$

can be linearized. As we know from the introduction, linearizing tactile maps in several variables is not always possible. In the situation of the lemma, however, we work with a quasi-submersion. Indeed, the textile map f at hand is induced by a rather arbitrary $F \in k[[x,y]]$, in which for y a linear combination of δ_i is substituted. Thus, the image of f is contained in the image of the tangent map given by δ: f is a quasi-submersion, the Rank Condition is obsolete and linearization is possible as soon as the order condition is fulfilled. Note: the Rank Theorem ensures linearization by textile maps. Thus we will have to check that the linearizing automorphisms obtained from the Rank Theorem in this case are indeed tactile!

Remark. A similar situation appears in the proof of the Grinberg-Kazhdan-Drinfeld formal arc theorem. There a given map is modified in order to become a quasi-submersion by composing it with a linear combination of appropriate power series (related to partial derivatives of the original map), see section 2.2. To prove the theorem it suffices again to work with the modified map.

Remark. More generally Lemma 16 holds for $\delta_1, \ldots, \delta_r \in \mathrm{ann}(M)$ where $M = k[[x]]^q/\mathrm{im}(\lambda)$ and λ is the map induced by the Jacobian δ.

Note that the assumption $F(x,0) \in BA^2$ in Theorem 13 is irrelevant for the linearization described by Lemma 16. However, we will see that it ensures that the linearization of g has a zero: In order to prove Theorem 13 choose generators $\delta_1, \ldots, \delta_r$ of A. By Lemma 16 there exist series $Y^i \in k[[x, y_j^l; 1 \le l \le r, 1 \le j \le p]]^p$ such that

$$F(x, \sum_{i=1}^{r} \delta_i Y^i) = F(x,0) + \delta(\sum_{i=1}^{r} \delta_i y^i).$$

Thus any solution $y^i \in k[[x]]^p$, $1 \le i \le r$, to

$$F(x,0) + \delta(\sum_{i=1}^{r} \delta_i y^i) = 0 \tag{2.5}$$

yields a solution $y_1(x), \ldots, y_p(x)$ to $F(x,y) = 0$ by setting

$$y_j(x) = \sum_{i=1}^{p} \delta_i Y_j^i(x, y^1, \ldots, y^p).$$

Equation (2.5) is linear. Since $F(x,0) \in BA^2$ there exist $\beta^i \in B^p$ with

$$F(x,0) = \delta \left(\sum_{i=1}^{r} \delta_i \beta^i \right),$$

and setting $y^i = -\beta^i$, $1 \le i \le r$, gives a solution to (2.5). This proves Theorem 13 (we were following the proof in Chap. V of [Rui93]).

Proof of Lemma 16: Write $F(x,y) = F(x,0) + \delta y + H(x,y)$ where H is at least quadratic in y; set $z = (z_1^1, \ldots, z_1^r, \ldots, z_p^1, \ldots, z_p^r)$. Without loss of generality $F(x,0) = 0$; otherwise consider $F(x,y) - F(x,0)$. Define the matrix D as the $p \times p$ block matrix $\operatorname{diag}(\delta_-, \ldots, \delta_-)$, where $\delta_- = (\delta_1, \ldots, \delta_r)$. Then consider the tactile map

$$g : k[[x]]^{rp} \to k[[x]]; z \mapsto F(x, D \cdot z).$$

The tangent map of g with respect to z at 0 is given by ∂g(0)$= \partial F(x,0) \cdot D$ and thus $I = \operatorname{im}(\partial g(0)) = (\delta_i \delta_j; 1 \le i \le p, 1 \le j \le r)$. We first check the Rank Condition. The non-linear part of g is $H(x, D \cdot z)$. Its image is obviously contained in I. So $\operatorname{im}(g) \subseteq I$. The automorphism v of Theorem 6 is in this case defined as $v = \mathrm{id} - (\mathrm{id} - \partial g(0)\sigma) g \sigma \partial g(0) \sigma$ for a scission σ of $\partial g(0)$. But since $\mathrm{id} - \partial g(0)\sigma$ is a projection onto the complement of I we see that $v = \mathrm{id}$ on $\operatorname{im}(g)$, i.e., g has constant rank. In particular v is a tactile map.
It remains to prove that σh is contractive. Here we use a scission σ as constructed in section 1.2. For this write $h(x,z) = \sum_{1 \le i,j \le r} \delta_i \delta_j h_{ij}(x,z)$ for appropriate $h_{ij} \in k[[x,z]]$. Note that in the definition of h_{ij} we keep track of the order of i, j; this is just a technical convention. The scission σ is given by division through the $\delta_i \delta_j$, $1 \le i, j \le r$. Since $h(x,z)$ has this special form there is a natural way how to divide: $\sigma h(x,z)$ has on position $(i-1) \cdot r + j$ the term h_{ij} and 0 else. It's easy to see that $\kappa(\sigma h) > 0$, since the $h_{ij}(x,z)$ are at least of order 2 in $z \in (x_1, \ldots, x_n) \subseteq k[[x]]$. Moreover, $\mathrm{id} + \sigma h$ is tactile, since the $h_{i,j}(x,z)$ are power series in x, z. Thus the lemma follows from Theorem 6. □

2.5 Artin's and Wavrik's Approximation Theorem

Let $F \in k[[x,y]][z]$ with indeterminates $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_p)$ and $z = (z_1, \ldots, z_r)$. We are now interested in solutions $(y(x), z(x)) \in k[[x]]^{p+r}$ to $F(x,y,z) = 0$. The solution is called an *N-order solution* if

$$\operatorname{ord} F(x, y(x), z(x)) \ge N + 1.$$

The following assertion has been proved for F polynomial in [Art69] and for F a power series in [Wav75]: Given a positive integer q, there is an $N_0 = N_0(F, q)$ such that any N-order solution, $N \geq N_0$, can be approximated to order q by an exact solution. Let us review this theorem in the special case $n = p = m = 1$, F an irreducible polynomial as is proven in Theorem I of [Wav78]:

Theorem 14. *Given $q > 0$ and $F(x, y) \in k[x, y]$ irreducible there exists an N such that if $\bar{y} \in k[[x]]$ satisfies $F(x, \bar{y}) \equiv 0 \mod (x)^N$, then there exists a series $y(x) \in k[[x]]$ with $F(x, y(x)) = 0$ and $y(x) \equiv \bar{y} \mod (x)^q$.*

Proof. Using the following argument (see [Wav78], proof of Theorem I) we can further assume that $\operatorname{ord}_x \partial_y F(x, \bar{y}) < r \in \mathbb{N}$: Consider the discriminant $\Delta \in k[x]$ of F and $\partial_y F$ and polynomials $A(x, y)$ and $B(x, y)$ such that $\Delta = AF + B\partial_y F$. Denote by d the order of Δ. If $\operatorname{ord} F(x, \bar{y}) > d$ then $\operatorname{ord} \partial_y F(x, \bar{y}) < d$.
Take an element $y_0 \in k[[x]]$, $\deg y_0 \leq d$, such that $\operatorname{ord} F(x, y_0) \geq d + 1$ and $\operatorname{ord} \partial_y F(x, y_0) = e < d$. The textile map $g : (x)^{d+1} \subseteq k[[x]] \to k[[x]]$, $g(z) = F(x, y_0 + z)$ has constant rank, since $\operatorname{im}(F) \subseteq \operatorname{im}(\partial_y F(x, y_0)) = (x)^e$. It is easy to see that $\kappa(\sigma h) > 0$, where σ is a scission of $\partial_y F(x, y_0)$ as in section 1.2 and $h(z) = g(z) - \partial_y g(y_0) z$. Thus we may linearize g at 0 and lift any such y_0 to a solution $y(x)$ of $f(x, y) = 0$. This linearization is possible as soon as y_0 is known up to order d. □

Remark. One should note that the index N obtained in the proof by using the Rank Theorem is much lower than the one provided in [Wav78], p. 411. The proof provided there needed $N = 2d + q$. This follows from the fact that Tougeron's Implicit Function Theorem is used, which is not as efficient as the Rank Theorem. In a subsequent section of [Wav78] a minimal N is computed by means of resolution of singularities. It is not clear whether this optimal N is the same as the one obtained in the proof above.

2.5.1 The Analytic Artin Approximation Theorem

The results of this and the last section give a new view on the proof of Artin's Approximation Theorem as it was published in [Art68]. In contrast to Wavrik's Approximation Theorem by assumption a solution $\bar{y} \in k[[x]]$, and not an approximate solution, to analytic equations in $k\{x, y\}$ is given, and one obtains an analytic solution y approximating $\bar{y}$ in the $\bar{\mathfrak{m}}$-adic topology. The precise statement is (Theorem 1.2 in [Art68]) is as follows:

Theorem 15 (Artin's Analytic Approximation Theorem)**.** *Consider*

$$f = (f_1, \ldots, f_q) \in k\{x, y\}^q$$

with $f(0, 0) = 0$. Let $\bar{y} \in k[[x]]^p$ be a formal solution (without constant term) to $f(x, y) = 0$, i.e., $f(x, \bar{y}) = 0$. Then for any $\alpha \geq 1$ there exists a convergent solution $y \in k\{x\}^p$ of $f(x, y) = 0$ such that

$$y - \bar{y} = 0 \mod \bar{\mathfrak{m}}^\alpha.$$

Recall the strategy of proof for Theorem 15 (see for example V.3 in [Rui93] or III.4 in [Tou68] for well-arranged accounts): (i) reduce the general question to a special case with more structure (this involves induction on the height of

$I = (f_1, \ldots, f_q) \subseteq k\{x, y\}$), (ii) solve the special case using Tougeron's implicit function theorem (using induction on $\dim k\{x\}$).

Denote by I the ideal generated by $f_1, \ldots, f_q$ in $k\{x, y\}$. Then the main reduction steps are:

(A) I is a prime ideal

(B) $I = (h_1, \ldots, h_s)$ with $\delta(x, \bar{y}) \neq 0$ for $\delta = \det \left(\frac{\partial h_i}{\partial y_j} \right)_{1 \le i,j \le s}$.

Consider the reduced case, i.e., assume that (A) and (B) hold. So $I = (h_1, \ldots, h_s)$ is a prime ideal in $k\{x, y\}$ and $\delta(x, \bar{y}) \neq 0$. The series $h_1, \ldots, h_s$ induce a tactile map $h\colon k\{x\}^p \to k\{x\}^s$. If $\delta_1, \ldots, \delta_r$ are s-minors of $\partial_y h(x, 0)$ the modified tactile map

$$(y^1, \ldots, y^r) \mapsto h(x, \sum_{i=1}^{r} \delta_i y^i)$$

is a quasi-submersion and hence linearizable (compare section 2.4). But the linearized map

$$(y^1, \ldots, y^r) \mapsto h(x, 0) + \partial_y h(x, 0) \left(\sum_{i=1}^{r} \delta_i y^i \right)$$

does not necessarily have a zero. After an additional modification – this is the assertion of Lemma 17 below – existence of solutions is ensured, and it is sufficient to find an approximating analytic solution to

$$h(x, 0) + \partial_y h(x, 0) \left(\sum_{i=1}^{r} \delta_i y^i \right) = 0.$$

The last steps are precisely where Tougeron's implicit function theorem comes into play in the original proof.

Lemma 17. *Let $f_1, \ldots, f_p \in k\{x, y\}$, let $\Delta(x, y) \in k\{x, y\}$, $\bar{y} \in k[[x]]^p$ such that $\Delta(0) = 0$, $\bar{y}(0) = 0$, and $\Delta(x, \bar{y}) \neq 0$. Suppose that $f_i(x, \bar{y}) = 0$, $1 \le i \le p$. Then there exist for all $j \in \mathbb{N}$ a $z^j \in k\{x\}^p$ with $y - z = 0 \mod (x)^j$ such that*

$$f_i(x, z^j(x)) \in \Delta(x, z^j(x)) \cdot k\{x\}, 1 \le i \le p.$$

For a proof of this Lemma, which is the crucial part in the proof of Artin's Approximation Theorem, we refer to [Art68], [Rui93] and [Tou68].

Remarks. (a) The proof of Lemma 17 has two parts: (i) it is shown that such a $z(x)$ exists formally; (ii) there exists a convergent $z(x)$. For (ii) induction on the number of variables (via Weierstrass division theorem) is used. The assertion of (i) is rather trivial: Write $\bar{y} = \bar{q} \cdot \Delta(x, \bar{y})^{\nu+1} + \bar{z}$. By assumption $\operatorname{ord} \Delta(x, \bar{y}) \ge 1$, so $\bar{y} - \bar{z} = 0 \mod (x)^{\nu+1}$. Moreover,

$$0 = f(\bar{y}) = f(\bar{z}) + \partial f(\bar{z}) \cdot \bar{q} \Delta^{\nu+1}(x, \bar{y}) + \text{hot},$$

thus $f(\bar{z}) \in \Delta(x, \bar{y})$. Since $\Delta(x, \bar{y})$ and $\Delta(x, \bar{z})$ generate the same ideal in $k[[x]]$ the formal statement follows (note that for this conclusion the weaker assertion $(\Delta(x, \bar{y})) \subseteq (\Delta(x, \bar{z}))$ would be sufficient).

(b) The convergent $z(x)$ is obtained by the above mentioned induction on the number of variables, see for example [Tou68] for the details.

2.6 An inversion theorem by Lamel and Mir

In their paper [LM07], Lamel and Mir investigate the group of local holomorphic automorphisms of $\mathbb{C}^n$ preserving a real analytic submanifold M (Cauchy-Riemann automorphisms). Their key result to describe this group is as follows. Denote by $\mathcal{O}_n$ the space of holomorphic germs $(\mathbb{C}^n, 0) \to \mathbb{C}$, and by $\mathrm{Aut}(\mathbb{C}^n, 0) \subset \mathcal{O}_n^n$ the group of biholomorphic germs of $(\mathbb{C}^n, 0)$. An element $f \in \mathcal{O}_n^n$ is said to have *generic rank* n if its Jacobian determinant $\det(\partial f)$ is nonzero as an element in $\mathcal{O}_n$. Any $f \in \mathcal{O}_n^n$ induces a map $F : \mathrm{Aut}(\mathbb{C}^n, 0) \to \mathcal{O}_n^n$ given by the composition $u \mapsto f \circ u$.

Theorem 16. *(Thm. 2.4 in [LM07]) Let $f \in \mathcal{O}_n^n$ be a germ of a holomorphic map of generic rank n. There exists a holomorphic map $\Psi : \mathcal{O}_n^n \times \mathrm{GL}_n(\mathbb{C}) \to \mathrm{Aut}(\mathbb{C}^n, 0)$ inverting F in the sense that $\Psi(F(u), \partial u(0)) = u$ for all $u \in \mathrm{Aut}(\mathbb{C}^n, 0)$. Furthermore Ψ can be chosen such that $\partial(\Psi(b, \lambda))(0) = \lambda$ for all $b \in \mathcal{O}_n^n$ and $\lambda \in \mathrm{GL}_n(\mathbb{C})$.*

Proof. We shall show how the Rank Theorem allows to construct a suitable Ψ in the formal context taking the space $\mathcal{C}_n = \mathbb{C}[[x_1, \ldots, x_n]]$ of formal power series (cords) instead of $\mathcal{O}_n$. In order to treat convergent power series one has to refer to the Rank Theorem from [HM94], with the same reasoning as in the formal case. The argument in [LM07] is much more involved, their computations invoking tacitly various instances of the proof of the Rank Theorem.
The map $F : \mathcal{C}_n^n \to \mathcal{C}_n^n$ is tactile with Jacobian determinant $\det(\partial f) \neq 0 \in \mathcal{C}_n$. Our goal is to linearize it locally at the linear part λx of a given u. Decompose $u \in \mathrm{Aut}(\mathbb{C}^n, 0)$ as $u(x) = \lambda \cdot x + v(x)$ with $\lambda \in \mathrm{GL}_n(\mathbb{C})$ and $v \in \mathcal{C}_n^n$ of order 2. Consider the Taylor expansion (with the obvious abbreviation)

$$G(x) = f(u(x)) - f(\lambda x) = \partial f(\lambda x) \cdot v + \partial^2 f(\lambda x) \cdot v^2 + \ldots,$$

with linear part $\ell(v) = \partial f(\lambda x) \cdot v$. Since f is of generic rank n and $\lambda \in \mathrm{GL}_n(\mathbb{C})$, the map ℓ is injective. Thus G has constant rank at 0, by the remark after Proposition 6. Next we show that G fulfills the order condition of the Rank Theorem (Thm. 6). In order to keep notation short we assume that λ is the identity matrix. Denote the initial form of lowest degree of f^i by f^i_* and write e_i for the order of f^i. By the same triangularization argument as in Lemma 4.5 of [LM07] we may assume that $f_* = (f^1_*, \ldots, f^n_*)$ has generic rank n. This gives

$$\operatorname{ord}\det(\partial f_*) = e_1 + \ldots + e_n - n.$$

Moreover for $\beta \in \mathbb{N}^n$, $|\beta| = m$, and $l \in \{1, \ldots, n\}$ we have

$$\operatorname{ord}(\partial_\beta f^l_*) \geq e_l - m$$

for $e_l \geq m$ and $\operatorname{ord}(\partial_\beta f^l_*) = \infty$ else. As the map f has generic rank n a scission σ for ℓ can be obtained as in Proposition 5 by the adjoint matrix of ∂f. The nonlinear part $h = G - \ell$ consists of all terms of G which are of order at least 2 in $v_1, \ldots, v_n$. We write

$$h(v) = \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| \geq 2}} h_\alpha \cdot v^\alpha,$$

with $h_\alpha \in \mathcal{C}_n^n$. For $|\alpha| = m \geq 2$, the coefficient vector h_α has ith component h_α^i involving the mth order partial derivatives of f^i. Multiplication with the adjoint matrix of ∂f yields in the ith component

$$\begin{aligned}\big((\partial f)^{adj} \cdot h_\alpha\big)_i &= \sum_{l=1}^n (\partial f)_{il}^{adj} \cdot h_\alpha^l \\ &= \sum_{j=1}^n (-1)^{i+l} \det(\partial f)^{(l,i)} \cdot h_\alpha^l.\end{aligned}$$

Here, $A^{(i,j)}$ denotes the $(n-1) \times (n-1)$ matrix obtained from an $n \times n$ matrix A by deleting the ith row and the jth column. From the last equation it follows that each term in the sum is of type

$$T = \Big(\prod_{\substack{1 \leq j \leq n \\ j \neq l}} \partial_{i_j} f^j\Big) \cdot (\partial_\beta f^l),$$

where $i_j \in \{1, \ldots, n\}$ and $\beta \in \mathbb{N}^n$, $|\beta| = m$. Obviously

$$\operatorname{ord} T \geq \sum_{i=1}^n e_i - n - m + 1.$$

This allows to compute the contraction degree κ of the map

$$\varphi_\alpha : v \mapsto \frac{1}{\det(\partial f)} \cdot (\partial f)^{adj} \cdot h_\alpha \cdot v^\alpha$$

if v varies in $(x_1, \ldots, x_n)^2 \cdot \mathcal{C}_n^n$. We get

$$\begin{aligned}\kappa(\varphi_\alpha) &\geq -\Big(\sum_{i=1}^n e_i - n\Big) + \Big(\sum_{i=1}^n e_i - n - m + 1\Big) + (2m - 2) \\ &= m - 1.[\end{aligned}$$

So $\kappa(\varphi_\alpha) > 0$ for all α with $|\alpha| = m \geq 2$. Now

$$(\sigma h)(z) = \sum_{\alpha \in \mathbb{N}^n, |\alpha| \geq 2} \varphi_\alpha(z)$$

yields

$$\kappa(\sigma h) \geq \min_\alpha \kappa(\varphi_\alpha) = 1 > 0.$$

The order condition is fulfilled. Therefore G can be linearized to ℓ by local automorphisms of source and target. Once this is done, it suffices to solve the linear equations in order to construct the required map Ψ of the theorem. $\square$

Chapter 3

Jet and Arc Algebras

This chapter contains a collection of partial results and open questions concerning the combinatorial, algebraic and geometric structure of jet and arc algebras (the coordinate rings of affine jet and arc spaces). Section 3.1 introduces these algebras and gives some simple properties. The defining equations for jet and arc algebras can be computed from the defining equations f_i of the base variety using a specific k-derivation D on $k[\underline{\mathbb{N}}^m]$, the polynomial ring in variables x_i^j, $1 \leq j \leq m$ and $i \in \mathbb{N}_0$, with coefficients in k. We show that this operator D has a canonical form in terms of partial derivatives of the f_i's and the multivariate Bell-Polynomials, see section 3.2. Mustaţa conjectures in [Mus01] that the singular locus of the jet scheme of a local complete intersection is precisely the preimage of the singular locus of the base variety under the canonical projection map. In section 3.3 we discuss some examples and approaches towards this problem. We give a prove for a special case. After this we rise the question for the Poincare series of arc algebras with the grading given by $\mathrm{wt}(x_i^j) = i$. For smooth base varieties it turns out to be some power of the generating series of the partition function, section 3.4. We conclude this chapter with some completions of polynomial rings in countably many variables. The ultimate goal would be to describe the generators for the defining ideal of the completion of the local ring of an arc algebra in a given arc. This would be useful for a constructive version of the Grinberg-Kazhdan-Drinfeld formal arc theorem (see section 2.2).

3.1 Definition and Basic Properties

Let $f \in k[x_1, \ldots, x_m]$. For $1 \leq j \leq m$ substitute $\sum_{i=0}^{\infty} x_i^j t^i$ for x_j in f and expand the result as a power series in t. It will be of the form

$$f(\sum x_i^j t^i) = \tilde{F}_0 + \tilde{F}_1 t + \tilde{F}_2 t^2 + \ldots,$$

with polynomials $\tilde{F}_i \in k[i;m]$. Recall that $k[i;m]$ was set to be $k[x_e^j; 1 \leq j \leq m, 0 \leq e \leq i]$. Especially, for $i = 0$ we obtain

$$\tilde{F}_0 = f(x_0^1, \ldots, x_0^m).$$

Alternatively we may consider the substitution

$$\varphi\colon x_j \mapsto \sum_{i=0}^{\infty} \frac{x_i^j}{i!} t^i.$$

Again $\varphi(f)$ is a power series in t with coefficients $F_i \in k[i;m]$. The two families of polynomials are related by

$$\tilde{F}_e(x_i^j) = F_e(i! x_i^j).$$

The F_i can also be constructed in the following way: Introduce on $k[\underline{\mathbb{N}}^m]$ a k-derivation D by

$$D : k[\underline{\mathbb{N}}^m] \to k[\underline{\mathbb{N}}^m]; x_i^j \mapsto D(x_i^j) = x_{i+1}^j.$$

The i-fold composition of D with itself will be denoted by D^i, and D^0 is set to be the identity. Using D the series $\varphi(f)$ can be rewritten as

$$\varphi(f) = \sum_{i=0}^{\infty} \frac{D^i F_0}{i!} t^i. \tag{3.1}$$

This is checked on the variables x_j, since both sides are additive and multiplicative in their arguments. By definition, φ is additive and multiplicative. For the right-hand side the generalized Leibniz-rule, i.e.,

$$D^i(F_0 G_0) = \sum_{l=0}^{i} \binom{i}{l} D^l F_0 \cdot D^{i-l} G_0,$$

implies that

$$\sum_i \frac{1}{i!} D^i(F_0 G_0) t^i = \sum_i \sum_{l=0}^{i} \frac{D^l F_0}{l!} \frac{D^{i-l} G_0}{(i-l)!} t^i = \left(\sum_i \frac{D^i F_0}{i!} t^i \right) \cdot \left(\sum_i \frac{D^i G_0}{i!} t^i \right).$$

From (3.1) we immediately see that

$$F_i = \frac{1}{i!} D^i F_0.$$

3.1.1 The global situation

In what follows we will be mainly interested in the ideal (Df) generated by the $F_i, i \in \mathbb{N}_0$. Therefore we will leave out the constant factor $\frac{1}{i!}$ and use the following notation: $F_i = D^i f(x_0^1, \ldots, x_0^m)$.

Remark. In section 3.2 we will give an even more explicit way to construct the polynomials F_i by means of multivariate Bell polynomials.

Example 9. Consider $f = y^2 - x^3$ and write a_i for x_i^1 and b_i for x_i^2. Then $(F_0, F_1, \ldots)$ is given by

$$\begin{aligned}(F_0, F_1, \ldots) &= (b_0^2 - a_0^3, 2b_0 b_1 - 3a_0^2 a_1, 2b_0 b_2 + 2b_1^2 - 3a_0^2 a_2 - 6a_0 a_1^2, \\ &\quad 2b_0 b_3 + 4b_1 b_2 - 3a_0^2 a_3 - 12 a_0 a_1 a_2 - 6a_1^3, \ldots)\end{aligned}$$

There are two canonical gradings on $k[\underline{\mathbb{N}}^m]$:

(A) "deg": each variable hast the same degree $\deg(x_i^j) = 1$; the k-vector space of elements of degree n will be denoted by $k[\underline{\mathbb{N}}^m]^{(n)}$.

(B) "wt": the *weight* is defined by $\mathrm{wt}(x_i^j) = i$; thus $k[\underline{\mathbb{N}}^m]$ becomes a graded $k[0;m]$-algebra. The $k[0;m]$-submodule of homogenous elements of weight n will be denoted by $k[\underline{\mathbb{N}}^m]_{(n)}$.

For any $F_0 \in k[0;m]$ the derivatives $D^i F_0$, $i \in \mathbb{N}$, are wt-homogenous of weight i. Especially, the ideal (Df) introduced above is a weighted homogenous ideal in $k[\underline{\mathbb{N}}^m]$.

Remark. The derivation D is a derivation of degree 1 with respect to wt, i.e.,

$$Dk[\underline{\mathbb{N}}^m]_{(i)} \subseteq k[\underline{\mathbb{N}}^m]_{(i+1)}.$$

It is of degree -1 with respect to deg, i.e., $Dk[\underline{\mathbb{N}}^m]^{(i)} \subseteq k[\underline{\mathbb{N}}^m]^{(i-1)}$.

Let $X \subseteq \mathbb{A}_k^n$ be an affine scheme defined by an ideal $I = (f_1, \ldots, f_n) \subseteq k[x]$. Using the notation introduced above we write F_i^j for $D^i f_j(x_0^1, \ldots, x_0^m)$, $1 \le j \le n$. For $q \in \mathbb{N}$ the algebras

$$J_q(X) = k[q;m]/(F_i^j; 1 \le j \le n, 0 \le i \le q)$$

respectively

$$J_\infty(X) = k[\underline{\mathbb{N}}^m]/(F_i^j; 1 \le j \le n, i \in \mathbb{N}_0)$$

are called the *qth jet and arc algebra of* X respectively. Their associated schemes are called *qth jet space of* X and *arc space of* X. In the sequel we will write (DI) for the ideal $(F_i^j; 1 \le j \le n, 0 \le i \le q)$.

Proposition 18. *Consider two closed subschemes X, Y of $\mathbb{A}^m$ defined by ideals $I_X = (f_1, \ldots, f_p)$ and $I_Y = (g_1, \ldots, g_q)$. Then $I_X = I_Y$ if and only if $(DI_X) = (DI_Y)$.*

Proof. The direction left to right is trivial. So let us consider the other direction, and assume that $(DI_X) = (DI_Y)$. For every $j \in \{1, \ldots, p\}$ we can write

$$F_0^j = \sum_{e=1}^{q} c_e G_0^e.$$

For weight reasons no other G_i^j may appear in this expression. By this $F_0^j \in I_Y$ for all j, i.e., $I_X \subseteq I_Y$. Interchanging F's and G's proves the reverse inclusion and thus the assertion. ☐

Remark. There are examples of schemes X and Y which have at all levels isomorphic jet schemes $X_m \cong Y_m$, $m \in \mathbb{N}$, but are not isomorphic: $X \ncong Y$. In fact, X and Y can be chosen to be non-singular, cf. [IW].

Proposition 19. *The following properties for elements of $k[\underline{\mathbb{N}}^m]$ hold:*

1. *If $F \in k[\underline{\mathbb{N}}^m]$ is weighted homogenous and $q_1, \dots, q_l \in k[\underline{\mathbb{N}}^m]$ are such that $F = q_1 \cdots q_l$, then the q_i are weighted homogenous.*

2. *Let $F_0 \in k[0; m]$; for all $i \in \mathbb{N}$ the ith derivative F_i of F_0 is of the form*

$$F_i = c_{i,0} \cdot q_i$$

with $\mathrm{wt}(c_{i,0}) = 0$, i.e., it is an element of $k[0; m]$, and q_i irreducible and wt-homogenous of weight i. If o and d denote order and degree of F_0, then for $i \geq o$ it holds that $c_{i,0}(0) \neq 0$ and for $i \geq d$ that $c_{i,0} \in k$, thus F_i is irreducible.

3. *Let $F_0 \in k[0; m]$; then for all $i \in \mathbb{N}_0$: $D^i F_0 \nmid D^{i+1} F_0$ and $D^{i+1} F_0 \nmid D^i F_0$.*

Proof. 1.It is sufficient to prove the assertion for $l = 2$. The general assertion then follows by induction. So assume that $f = q_1 q_2$. Let $\mathrm{wt}(f) = n$ and let q_1 and q_2 have order e_1 respectively e_2 and degree d_1 respectively d_2 with respect to wt. Then $\mathrm{ord}_{\mathrm{wt}}\, q_1 q_2 = e_1 + e_2$ and $\deg_{\mathrm{wt}}\, q_1 q_2 = d_1 + d_2$. Since $e_i \leq d_i$ we conclude that $e_i = d_i$: if $e_1 < d_1$ then $e_2 > d_2$ which gives a contradiction (analogously we treat the case $e_2 < d_2$). Thus q_1 and $q_2 h$ are weighted homogeneous, too.
2. Assume that $F_i = p_1 \cdots p_l$ with p_i irreducible. By 1. we see that the p_i are wt-homogenous. Since F_i is an ith derivative there exists a $j \in \{1, \dots, m\}$ such that x_i^j appears in F_i. But then this variable has to appear in one of the p_e, say in p_l. Thus $\mathrm{wt}(p_l) = i$, $\mathrm{wt}(p_1 \cdots p_{l-1}) = 0$ and we may set $c_{i,0} = p_1 \cdots p_{l-1}$ and $q_i = p_l$.
Next decompose F_0 into its deg-homogenous parts $F_0^o + \cdots + F_0^d$: $\deg(F_0^i) = i$. Then F_0^o contains a term of the form $c x_0^{j_1} \cdots x_0^{j_o}$, $j_i \in \{1, \dots, m\}$. Applying D^i, $i \geq o$ we see that $D^i F_0^o$ is of the form

$$D^i F_0^o = c x_1^{j_1} \cdots x_1^{j_{o-1}} x_{i-(o-1)}^{j_o} + \cdots$$

The first term has degree o and weight i, hence appears in q_i. Moreover, since $\mathrm{ord}\, F_i = \mathrm{ord}\, F_0$ we get

$$o = \mathrm{ord}\, F_i = \mathrm{ord}\, c_{i,0} + \mathrm{ord}\, q_i = \mathrm{ord}\, c_{i,0} + o$$

which forces $\mathrm{ord}\, c_{i,0} = 0$. The same argumentation with deg instead of ord and d instead of o shows that $\deg c_{i,0} = 0$ for $i \geq d$.
4. The second statement is trivial for weight reasons. For the first assume that the converse holds, i.e., there exists a $q \in k[\underline{\mathbb{N}}^m]$ such that $qF_i = F_{i+1}$. Then necessarily $\mathrm{wt}(q) = 1$, thus $\deg(q) \geq 1$. Since $\deg(F_i) = \deg(F_{i+1})$ we get

$$\deg(F_{i+1}) = 1 + \deg(F_i)$$

a contradiction. □

Remark. Note, that 2. holds even if f is reducible.

Example 10. For simplicity of notation we will drop the upper indices and simply write x_i instead of x_i^j. Accordingly, the expression x_i^j will denote here the jth power of the variable x_i.

$$\begin{aligned}
F_0 &= x_0^2(1+x_0)\\
F_1 &= x_0x_1(2+3x_0)\\
F_2 &= 2x_1^2+2x_0x_2+6x_0x_1^2+3x_0^2x_2\\
F_3 &= 6x_1x_2+2x_0x_3+\underbrace{6x_1^3}_{\text{wt } 3,\ \deg\ 3}+18x_0x_1x_2+3x_0^2x_3
\end{aligned}$$

The polynomials F_2, F_3 are irreducible. This shows, that irreducibility might already occur earlier than for $i \geq \deg(F_0)$.

Remark. Many properties of a scheme X are reflected in properties of its jet schemes X_m, $m \in \mathbb{N}$, and vice versa. Let $\mathcal{P}$ be a property of a scheme. For example if $\mathcal{P}$ is one of the properties "irreducible", "connected", "normal" or "$\mathbb{Q}$-factorial", then if X_m has $\mathcal{P}$ for some $m \in \mathbb{N}$, so has X (cf. [Ish]). Also a scheme X of finite type over an algebraically closed field (of characteristic 0) is non-singular if and only if there are integers $m \leq m'$ with flat truncation morphism $\pi_m^{m'} : X_{m'} \to X_m$, see [Ish].

3.1.2 The local situation

Next we will consider the *focussed jet and arc space of a variety X in a closed point $p \in X$*. For simplicity of notation we restrict our considerations to hypersurfaces through the origin, i.e., $X = \operatorname{Spec} k[x]/(f)$ and p the origin. Instead of considering all power series (respectively all truncated power series) solutions to the equation $f = 0$ we restrict to those with vanishing constant terms: $\sum_{q \geq i \geq 1} x_i^j t^i$, $1 \leq j \leq m$ and $q \in \mathbb{N} \cup \{\infty\}$. This corresponds to the following substitution:

$$(-)^\bullet : k[\underline{\mathbb{N}}^m] \to k[x_i^j; 1 \leq j \leq m, i \in \mathbb{N}]; x_i^j \mapsto \begin{cases} 0 & i = 0 \\ x_i^j & i \neq 0 \end{cases}.$$

We will write $(Df)^\bullet$ for the ideal generated by $F_i^\bullet, i \in \mathbb{N}$. The k-algebras

$$J_q(X;0) = \operatorname{Spec} k[x_i^j; 1 \leq i \leq q, 1 \leq j \leq m]/(F_1^\bullet, \ldots, F_q^\bullet)$$

respectively

$$J_\infty(X;0) = \operatorname{Spec} k[x_i^j; i \in \mathbb{N}, 1 \leq j \leq m]/(Df)^\bullet$$

will be called the *focussed qth jet algebra of X at* 0 respectively *the focussed arc algebra of X at* 0. The corresponding (affine) schemes will be called focussed qth jet space and focussed arc space of X at 0.

Note that the above substitution is also well-defined for $f \in k[[x_1, \ldots, x_m]]$, i.e., it makes sense to speak of the *focussed jet space of a formal germ.*

Example 11. Consider $f = y^2 - x^3$ and write a_i for x_i^1 and b_i for x_i^2. Then $(Df)^\bullet$ is given by

$$(Df)^\bullet = (b_1^2, b_1b_2 - a_1^3, \ldots)$$

For the generators F_i of (Df) we had the simple and useful formula $F_i = DF_{i-1}$. In the local situation we lose such an expression, but one can recover the defining polynomial (resp. power series) f from the ideal $(Df)^\bullet$. If $g \in k[\underline{\mathbb{N}}^m]$ is weighted homogeneous denote by $g[l]$ all terms of g which are built up by variables of weight l.

Proposition 20. *Let $f \in k[[x_1, \ldots, x_m]]$ and $(Df)^\bullet = (F_0^\bullet, F_1^\bullet, \ldots)$. Then:*

$$f = \sum_{l=0}^{\infty} \frac{1}{l!} F_l^\bullet[1]$$

Proof. Write $f = \sum f_\alpha x^\alpha$. It is clear that $\tilde{F}_i[1] = \sum_{|\alpha|=i} f_\alpha x_1^\alpha$. Indeed, substitution of $\sum_{l=1}^{\infty} x_l^j t^i$ for x_j in the expression of f gives a power series in t. The terms of the coefficient $\tilde{F}_i^\bullet$ of t^i which are built up just by $x_1^j, 1 \leq j \leq m$, are precisely $\sum_{|\alpha|=i} f_\alpha x_1^\alpha$, where $x_1 = (x_1^1, \ldots, x_1^m)$. Moreover, $F_i^\bullet = i! \tilde{F}_i^\bullet(\frac{x_l^j}{l!})$ and thus $f = \sum_l \frac{1}{l!} F_l^\bullet[1]$. □

Example 12. Consider the wt-homogenous ideal

$$I = (2b_1^2, 6b_1b_2, 8b_1b_3 + 6b_2^2 - 24a_1^4, \ldots).$$

From this we can deduce that if I is the D-ideal of some polynomial f, then it will be – up to degree 5 – of the form $f = y^2 - x^4$.

Proposition 21. *As before we consider $k[q;m]$, $q \in \mathbb{N}$, in a canonical way as a k-subalgebra of $k[\underline{\mathbb{N}}^m]$.*

1. *For any $F_0 \in k[0;m]$ its* deg*-order equals* $\min_l\{l; F_l^\bullet \neq 0\}$.
2. *If $(g) \subseteq (f)$, then $(Dg)^\bullet \subseteq (Df)^\bullet$.*
3. *$F_l^\bullet$ is* wt*-homogenous of weight l. $DF_l^\bullet$ is* wt*-homogenous of weight $l+1$.*

Proof. This is follows easily from Proposition 19. □

Remark. Analogously to Proposition 18 one might ask the following question (*Local Isomorphism Problem*): Let $f\colon (X,x) \to (Y,y)$ be a morphism of formal germs. Assume that $f_m\colon (\pi_m^X)^{-1}(x) \cong (\pi_m^Y)^{-1}(y)$ for $m \gg 0$. Thus this imply that f is already an isomorphism of formal germs? A positive answer was given by S. Ishii in the case that (Y,y) is reduced, cf. [Ish]. Note that the coordinate ring of $(\pi_m^X)^{-1}(0)$ is precisely $J_m(X;0)$.

3.2 The canonical form of the Derivation D

The objective in this section is to find a combinatorial description for the defining equations of the jet-schemes. For simplicity of notation we will restrict our considerations to hypersurfaces in $\mathbb{A}^m$. If a hypersurface is given by a polynomial $f \in k[x_1, \ldots, x_m]$, then defining equations for its qth jet scheme are given by $D^i f \in k[q;m]$, $0 \leq i \leq q$, where D is the k-derivation defined at the beginning of section 3.1, acting on the variables by $Dx_i^j = x_{i+1}^j$. We will write $\partial_{i,j}$

to denote the partial derivative $\frac{\partial}{\partial x_i^j}$ respectively $\partial_{i,x}$ for $\frac{\partial}{\partial x_i}$ (if the variable is clear out of context we simply write ∂_i). Moreover

$$k[q;m] = k[x_i^j; 1 \leq j \leq m, 0 \leq i \leq q],$$

is considered in the canonical way as a k-subalgebra of $k[\underline{\mathbb{N}}^m]$. We want to obtain a canonical form for $D|_{k[0;m]}$ (and its powers) in terms of the partial derivatives $\partial_{0,j}$.

Example 13. Write x and y instead of x^1 resp. x^2. Consider $f = x_0y_1^2 + x_0x_1y_0$ and its D-derivative

$$Df = x_1y_1^2 + 2x_0y_1y_2 + x_1^2y_0 + x_0y_0x_2 + x_0x_1y_1.$$

The same result can be obtained by applying the operator

$$T = x_1\partial_{0,x} + x_2\partial_{1,x} + y_1\partial_{0,y} + y_2\partial_{1,y}$$

to f.

For a set of variables $y_1, \ldots, y_N$ we denote by $W(y_1, \ldots, y_N)$ the Weyl-Algebra in indeterminates $y_1, \ldots, y_N$. It's easy to see that for all $f \in k[q-1;m]$ the relation $Df = Tf$ holds, where T is the operator

$$T = \sum_{j=1}^{m}\sum_{i=0}^{q-1} x_{i+1}^j \partial_{i,j} \in W(q;m) = W(x_i^j; 1 \leq j \leq m, 0 \leq i \leq q). \tag{3.2}$$

Indeed, both D and T define k-derivations on $k[\underline{\mathbb{N}}^m]$ which agree on the x_i^j. The generators of the qth jet ideal of the hypersurface X defined by $f \in k[x_1, \ldots, x_m]$ are

$$F_i = D^i f, 0 \leq i \leq q,$$

where we naturally identify f with $F_0 = f(x_0^1, \ldots, x_0^m) \in k[0;m]$. Recall that $D^2 f$ is iteratively defined as $D(Df)$. Using (3.2) the element $D^2 f$ can alternatively be obtained as follows: compute the composition of T with itself in the Weyl-Algebra $W(q;m)$, and apply the resulting differential operator to f:

$$\begin{aligned} T^2 &= \sum_{i,j} x_{i+1}^j \partial_{i,j} \circ \sum_{e,l} x_{e+1}^l \partial_{e,l} \\ &= \sum_{i,j,e,l} \left(\delta_{i,e+1}\delta_{j,l} x_{i+1}^j \partial_{e,l} + x_{i+1}^j x_{e+1}^l \partial_{i,j}\partial_{e,l}\right). \end{aligned}$$

Applied to $f \in k[\underline{\mathbb{N}}^m]$ this gives exactly $D(D(f))$. For simplicity of notation we define k-derivations $D_q : k[q;m] \to k[q+1;m]$ by restriction of D to $k[q;m]$:

$$D_q = D|_{k[q;m]} = \sum_{j=1}^{m}\sum_{i=0}^{q} x_{i+1}^j \partial_{i,j}.$$

Then X_q is given by

$$\begin{aligned} F_0 &= f(x_0^1, \ldots, x_0^m) \\ F_i &= D^i F_0, 1 \leq i \leq q. \end{aligned}$$

Example 14. We compute $D^2|_{k[0;m]} = D_1 D_0$:

$$\begin{aligned} D_1 D_0 &= (x_1\partial_0 + x_2\partial_1)(x_1\partial_0) = x_1\partial_0 x_1\partial_0 + x_2\partial_1 x_1\partial_0 \\ &= x_1^2\partial_0^2 + x_2\partial_0 + x_2 x_1\partial_1\partial_0 \\ &= x_1^2\partial_0^2 + x_2\partial_0. \end{aligned}$$

The term $\partial_1\partial_0$ is identically zero on $k[0;m]$ since $\partial_0 f \in k[0;m]$ for any $f \in k[0;m]$. This example suggests that there exists a canonical form of $D^i|_{k[0;m]}$ involving only partial derivatives ∂_0 and its powers.

Proposition 22. *The operator $D^i|_{k[0;m]} = D_{i-1}\dots D_0$ has a representation of the form:*

$$D^i|_{k[0;m]} = \sum_{\substack{\alpha \\ |\alpha|\leq i}} d^i_\alpha \cdot \partial_0^\alpha,$$

with $d^i_\alpha \in k[i;m]$, $wt(d^i_\alpha) = i$ and $\partial_0^\alpha = \prod_{j=1}^m \partial_{0,j}^{\alpha_j}, \alpha = (\alpha_1,\dots,\alpha_m) \in \mathbb{N}_0^m$.

Proof. We prove this by induction on i, the case $i = 1$ being trivial. Assume that $D^{i-1}|_{k[0;m]} = \sum_{|\beta|\leq i-1} b_\beta \partial_0^\beta$ with coefficients b_β of weight $i-1$. Composing $D^{i-1}|_{k[0;m]}$ with $D_{i-1} = \sum_l \sum_{e=0}^{i-1} x^l_{e+1}\partial_{e,l}$ gives (E_l denotes the lth standard basis vector in $\mathbb{R}^m$)

$$\begin{aligned} D_{i-1}D^{i-1}|_{k[0;m]} &= \sum_{e=0}^{i-1}\sum_{l=1}^{m} x^l_{e+1}\partial_{e,l} \circ \sum_{\substack{\beta \\ |\beta|\leq i-1}} b_\beta \partial_0^\beta \\ &= \sum_{e,l}\sum_{\beta} \left(x^l_{e+1}\partial_{e,l}(b_\beta)\partial_0^\beta + x^l_{e+1} b_\beta \partial_{e,l}\partial_0^\beta \right) \\ &= \sum_\beta \left(\sum_{e,l} x^l_{e+1}\partial_{e,l}(b_\beta) \right) \partial_0^\beta + \sum_\beta \left(\sum_l x_1^l b_\beta \partial_0^{\beta+E_l} \right). \end{aligned}$$

If $x^l_{e+1}\partial_{e,l}(b_\beta) \neq 0$ then it has weight $(e+1)+(i-1)-e = i$. Similar, if $x_1^l b_\beta \neq 0$ it has weight $1 + (i-1) = i$. Thus $D^i|_{k[0;m]}$ has the desired form. $\square$

Corollary 23. *The coefficients $d^i_\alpha \in k[i;m]$ of the operator*

$$D^i|_{k[0;m]} = \sum_{|\alpha|\leq i} d^i_\alpha \cdot \partial_0^\alpha,$$

$i \geq 1$, in Proposition 22 fulfill the following recursion:

$$d^i_\alpha = \sum_{j=1}^{m} \left(\sum_{e=0}^{i-1} x^j_{e+1} \frac{\partial}{\partial x^j_e}(d^{i-1}_\alpha) + x^j_1 d^{i-1}_{\alpha-E_j} \right) \tag{3.3}$$

with E_j denoting the jth standard basis vector of $\mathbb{R}^m$.

Proof. This follows immediately from the proof of Proposition 22. $\square$

Specialized to the case $m = 1$ the recurrence (3.3) becomes:

$$d^i_\alpha = \sum_{l=0}^{i-1} x_{l+1}\partial_l(d^{i-1}_\alpha) + x_1 d^{i-1}_{\alpha-1}. \tag{3.4}$$

Clearly, the d^i_α fulfill the following initial conditions:

$$\begin{aligned} d^i_\alpha &= 0, \text{ for } \alpha \geq i+1 \\ d^i_\alpha &= \delta_{i,\alpha}, \text{ for } i = 0 \text{ or } \alpha = 0. \end{aligned} \tag{3.5}$$

Using this we compute the first powers of D:

$$\begin{aligned} D^1|_{k[0;1]} &= x_1\partial_0 \\ D^2|_{k[0;1]} &= x_2\partial_0 + x_1^2\partial_0^2 \\ D^3|_{k[0;1]} &= x_3\partial_0 + 3x_1x_2\partial_0^2 + x_1^3\partial_0^3 \\ D^4|_{k[0;1]} &= x_4\partial_0 + (3x_2^2 + 4x_1x_3)\partial_0^2 + 6x_1^2x_2\partial_0^3 + x_1^4\partial_0^4. \end{aligned}$$

Corollary 24. *In case $m = 1$ we have for $i \geq 1$:*

1. $d^i_1 = x_i$
2. $d^i_i = x_1^i$
3. $d^{i+1}_i = \frac{i(i+1)}{2} x_1^{i-1} x_2$

Proof. The first and second assertion are obvious from (3.4) and (3.5). The claim in 3. follows from the fact that $d^{i+1}_i = i x_1^{i-1} x_2 + x_1 d^i_{i-1}$. By this the coefficient a_i of $x_1^{i-1}x_2$ fulfills the recursion $a_i = a_{i-1} + i, a_1 = 1$. □

We now give an explicit description of the coefficient polynomials d^i_α. This was investigated in cooperation with Ch. Koutschan [Kou]. Especially the assertion of Proposition 25 was experimentally found by Ch. Koutschan.
Let $P(i, \alpha)$ denote the set of partitions of the integer i into α parts $p_1, \ldots, p_\alpha$. For simplicity of language we will henceforth assume that $p_1 \leq p_2 \ldots \leq p_\alpha$. Moreover we define a function $s = s(p)$ returning for each partition p the number of ways to split a set of $p_1 + \cdots + p_\alpha$ elements into subsets of size given by the summands of the partition p. By x_p we denote the monomial $x_{p_1} \cdots x_{p_\alpha}$. Then:

Proposition 25. *With the above notation a solution to recurrence (3.4) is given by:*

$$d^i_\alpha = \sum_{p \in P(i,\alpha)} s(p) \cdot x_p.$$

Proof. Set $g^i_\alpha = \sum_{p\in P(i,\alpha)} s(p) \cdot x_p$. We prove that g^i_α fulfills recurrence (3.4) and initial conditions (3.5). It's easy to check that the initial conditions are satisfied. For example if $i = 0$, then for $\alpha \neq 0$ the set $P(0, \alpha) = \emptyset$. If $\alpha = 0$, then $P(0,\alpha)$ has one element,i.e., $s(p)=1$. Substituting g^i_α for d^i_α in the right hand side of (3.4) yields

$$\sum_{l=1}^{i-1}\sum_{p\in P(i-1,\alpha)} s(p)x_{l+1}\frac{\partial}{\partial_{x_l}}(x_p)+\sum_{p\in P(i-1,\alpha-1)} s(p)x_1\cdot x_p. \tag{3.6}$$

Let us denote the first sum expression in (3.6) by RHS_1 and the second one by RHS_2. For $q\in P(i,\alpha)$ the expression of g^i_α contains the term $s(q)x_q$. We next determine the coefficient of x_q in (3.6).
Obviously, x_q appears in RHS_2 if and only if $q_1=1$. Then x_q has coefficient $s(p^0)$, where $p^0\in P(i-1,\alpha-1)$ is the partition obtained from q by cancelling the first 1.
For the appearance of x_q in RHS_1 we write

$$q=(\underbrace{1,\dots,1}_{j_1},\underbrace{2,\dots,2}_{j_2},\dots,\underbrace{i,\dots,i}_{j_i}),$$

i.e., j_e is the number of e appearing in q. Clearly, q is coming from those elements of $P(i-1,\alpha)$ which have the form

$$p^e=(\underbrace{1,\dots,1}_{j_1},\dots,\underbrace{e,\dots,e}_{j_e+1},\underbrace{e+1,\dots,e+1}_{j_{e+1}-1},\dots,\underbrace{i,\dots,i}_{j_i})\in P(i-1,\alpha)$$

with coefficient $(j_e+1)s(p^e)$. In total the coefficient of x_q in (3.6) is

$$s(p^0)+(j_1+1)s(p^1)+\ldots+(j_{i-1}+1)s(p^{i-1}).$$

The assertion follows with the next Lemma. □

Lemma 26. *For $q\in P(i,\alpha)$ let j_e, p^e, $e\in\{0,\dots,i-1\}$ be defined as in the proof of Proposition 25. Then:*

$$s(q)=\sum_{e=0}^{i-1}(j_e+1)s(p^e).$$

Proof. For given $q\in P(i,\alpha)$ the integer $s(q)$ is the number of possibilities to decompose a set of i elements into subsets of size $q_1,\dots,q_\alpha$. Each such decomposition may be obtained from a decomposition of $i-1$ into α sets of size equal to $q_1,\dots,q_{e-1},q_e-1,q_{e+1},\dots,q_\alpha$, $1\le e\le\alpha$. If the exceptional set is one of the first j_1 many, then one of these sets is empty, and there are $s(p^0)$ many possibilities for this (note that for a partition the entries are ordered by increasing value!). If the exceptional set has $e\neq 1$ many elements, then we have $(j_e+1)s(p^e)$ many possibilities: there are j_e+1 many possibilities where to put the "new" set of size e. This gives in total $\sum_{e=0}^{i-1}(j_e+1)s(p^e)$, proving the lemma.
Alternatively one may proceed as follows: basic combinatorics shows that for a partition q as above the following holds:

$$\begin{aligned} s(q) &= \frac{1}{1!^{j_1}\dots(i-\alpha+1)!^{j_{i-\alpha+1}}}\cdot\begin{pmatrix} & i & \\ j_1 & \dots & j_{i-\alpha+1}\end{pmatrix} \\ &= \frac{1}{j_1!\dots j_{i-\alpha+1}!}\cdot\frac{i!}{1!^{j_1}\dots i!^{j_{i-\alpha+1}}}.\end{aligned}$$

Next it's easy to see that $s(p^e) = \frac{e+1}{i} \cdot \frac{j_{e+1}}{j_e+1} \cdot s(q)$ and thus

$$\begin{aligned}
\sum_{e=0}^{i-1} (j_e+1) s(p^e) &= \sum_{e=0}^{i-1} \frac{j_{e+1}(e+1)}{i} s(q) \\
&= \frac{1}{i} \left(\sum_{e=1}^{i} j_e \cdot e \right) s(q) \\
&= s(q),
\end{aligned}$$

which proves the lemma. □

Remark. The polynomials d_α^i as in the last Proposition are known in the literature as the *(exponential) partial Bell polynomials*, see e.g. [Com74] or [Rio80]. This was pointed out to the author by G. Regensburger.

In literature the partial Bell polynomials are introduced by the relation

$$\frac{1}{\alpha!} \left(\sum_{e \geq 1} x_e \frac{t^e}{e!} \right)^\alpha = \sum_{i \geq \alpha} d_\alpha^i \frac{t^i}{i!},$$

$\alpha \in \mathbb{N}_0$. We will refer to the partial Bell polynomials as the *univariate Bell polynomials*. In analogy to the univariate case we define for $i \in \mathbb{N}$ and $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$ the *multivariate partial Bell polynomial* d_α^i by the relation

$$\frac{1}{\alpha_1!} \left(\sum_{e \geq 1} x_e^1 \frac{t^e}{e!} \right)^{\alpha_1} \cdots \frac{1}{\alpha_m!} \left(\sum_{e \geq 1} x_e^m \frac{t^e}{e!} \right)^{\alpha_m} = \sum_{i \geq 0} d_\alpha^i \frac{t^i}{i!}.$$

Thus we may express the multivariate Bell polynomials in terms of univariate ones:

$$d_\alpha^i = \sum_{j_1 + \cdots + j_m = i} \begin{pmatrix} & i & \\ j_1 & \ldots & j_m \end{pmatrix} d_{\alpha_1}^{j_1}(x^1) \cdots d_{\alpha_m}^{j_m}(x^m). \tag{3.7}$$

Here we indicate the dependence of the polynomials on a particular subset of variables by adding an argument to the d_α^i.

Proposition 27. *The multivariate Bell polynomials fulfill recurrence (3.3).*

Proof. During this proof we will abbreviate the multinomial coefficient

$$\begin{pmatrix} & i & \\ j_1 & \ldots & j_m \end{pmatrix}$$

by $c_{i,j}$, $j = (j_1, \ldots, j_m)$. Moreover, in this context d_α^j will denote the product $d_{\alpha_1}^{j_1} \cdots d_{\alpha_m}^{j_m}$. Then substituting (3.7) into the RHS of equation (3.3) yields

$$\sum_{\gamma=1}^{m} \left(\sum_{e=0}^{i} x_{e+1}^\gamma \frac{\partial}{\partial x_e^\gamma} \left(\sum_{|j|=i-1} c_{i-1,j} \cdot d_\alpha^j \right) + x_1^\gamma \sum_{|j|=i-1} c_{i-1,j} \cdot d_{\alpha - E_\gamma}^j \right)$$

which simplifies to

$$\sum_{\gamma=1}^{m}\sum_{|j|=i-1} c_{i-1,j} d_{\alpha_1}^{j_1}\cdots\left(\sum_{e=0}^{j_\gamma} x_{e+1}^{\gamma}\frac{\partial}{\partial x_e^{\gamma}}(d_{\alpha_\gamma}^{j_\gamma}) + x_1^{\gamma} d_{\alpha_\gamma-1}^{j_\gamma}\right)\cdots d_{\alpha_m}^{j_m}. \tag{3.8}$$

Using recurrence (3.4) for the univariate Bell polynomials (3.8) then equals

$$\begin{aligned} & \sum_{\gamma=1}^{m}\sum_{|j|=i-1} c_{i-1,j}\cdot d_{\alpha_1}^{j_1}\cdots d_{\alpha_\gamma}^{j_\gamma+1}\cdots d_{\alpha_m}^{j_m} \\ = & \sum_{|j|=i} c_{i,j}\cdot d_{\alpha}^{j} \end{aligned}$$

which proves the Proposition. □

As an immediate consequence of Proposition 22 we obtain the next proposition. It shows that the defining equations of the jet spaces are completely determined by the partial derivatives of generators of the ideal corresponding to the base scheme. This gives heuristic reasons why the local and global isomorphism problem "should" have a positive answer.

Proposition 28. *Let $f \in k[x]$ with qth jet algebra defined by $F_0, \ldots, F_q \in k[q;m]$, $q \in \mathbb{N}\cup\{\infty\}$. Then for all i:*

$$F_i = \sum_{\substack{\alpha \\ |\alpha|\le i}} d_\alpha^i \cdot \partial_0^\alpha f(x_0^1, \ldots, x_0^m),$$

with $d_\alpha^i \in k[i;m]$, $wt(d_\alpha^i) = i$ the multivariate Bell polynomials, and $\partial_0^\alpha = \prod_{j=1}^m \partial_{0,j}^{\alpha_j}, \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}_0^m$. Especially for the focussed jet algebras (at a closed point p):

$$F_i^\bullet = \sum_{|\alpha|\le i} d_\alpha^i \cdot \partial_0^\alpha f(p).$$

3.3 The regular locus of Jet Spaces

This section is devoted to the following question raised by M. Mustaţă in [Mus01] on local complete intersection (in short "l.c.i.") varieties:

Question 1 (Question 4.11 in [Mus01])**.** *Let X be a l.c.i. variety with qth jet space X_q, $q \geq 1$, and canonical projection $\pi_q : X_q \to X$. Is the following assertion true?*

$$(X_q)_{\mathrm{reg}} = \pi_q^{-1}(X_{\mathrm{reg}}).$$

In Proposition 4.12 of the same paper Mustaţă gives a positive answer for the case $q = 1$:

Proposition (Prop. 4.12 in [Mus01])**.** *If X is a l.c.i. variety, then*

$$(X_1)_{\mathrm{reg}} = \pi_1^{-1}(X_{\mathrm{reg}}).$$

We investigate some examples and give a positive answer for hypersurfaces X with pure-dimensional jet schemes.

Example 15. The assumption that X is a variety – this implies in particular that X is reduced – is vital for the Proposition. Consider $X = \operatorname{Spec} k[x,y]/(x^2y)$ with first jet-space given by

$$\begin{cases} x_0^2 y_0 = 0 \\ 2x_0y_0x_1 + x_0^2 y_1 = 0 \end{cases}$$

It has two components; $Y = \{x_0 = 0\}$ of codimension 1 and $Z = \{y_0 = y_1 = 0\}$ of codimension 2. The first component Y is exactly $\pi_1^{-1}(X_{\text{sing}})$. Using the Jacobian criterion (see for example section 16.6 in [Eis95]) we see that

$$(X_1)_{\text{sing}} : \{x_0 = 0, x_1y_0 = 0\}.$$

Indeed for all points $p \in Y$ with $x_1y_0 \neq 0$, thus $p \notin Z$, the Jacobian matrix has rank 1, so p is a regular point of X_1. Hence we conclude that $(X_1)_{\text{sing}} \subsetneq \pi_1^{-1}(X_{\text{sing}})$. This example essentially boils down to the next example:

Example 16. Consider for $n \geq 2$ the singular scheme $X(n) = \operatorname{Spec} k[x]/(x)^n$ with first jet-scheme

$$X_1(n) = \operatorname{Spec} k[x_0, x_1]/(x_0^n, x_0^{n-1}x_1).$$

It has one codimension 1 component, namely $\{x_0 = 0\}$. The Jacobian matrix J of the defining equations for $X_1(n)$ is

$$J = \begin{pmatrix} nx_0^{n-1} & 0 \\ (n-1)x_0^{n-2}x_1 & x_0^{n-1} \end{pmatrix},$$

from which we see

$$(X_1(n))_{\text{sing}} = \begin{cases} V(x_0, x_1) & \text{if } n = 2 \\ V(x_0) & \text{if } n > 3 \end{cases}.$$

So for $n = 2$ clearly $(X_1(n))_{\text{sing}} \subsetneq \pi_1^{-1}(X_{\text{reg}})$, whereas the conjecture is true for $n > 3$.

Remark. In [Mus01] it is proven for any l.c.i. variety X that if X_q is irreducible for some $q \geq 1$, then X_q is reduced. Is the converse true? This is Question 4.10 of [Mus01]. A solution for this problem would be given by a positive answer to Question 1. Indeed, assume that X_q is reduced and reducible. Then any irreducible component in $\pi_q^{-1}(X_{\text{sing}})$ is generically smooth. But if $(X_q)_{\text{reg}} = \pi_q^{-1}(X_{\text{reg}})$ this gives a contradiction. The last proposition answers the problem for the case $q = 1$.

3.3.1 Hypersurfaces with pure-dimensional jet scheme

Let X be a hypersurface in $\mathbb{A}^m$ given by $f \in k[x_1, \ldots, x_m]$, k a field of characteristic 0. Recall that the q-jet scheme X_q of X is given by the ideal $(F_0, \ldots, F_q) \subseteq k[q;m]$, where $F_0 = f(x_0^1, \ldots, x_0^m)$ and $F_i = DF_{i-1}$.

By Prop. 1.4 in [Mus01] it is known that the qth jet scheme of a l.c.i. variety Y is pure dimensional if and only if $\dim Y_q \leq \dim Y \cdot (q+1)$. In that case Y_q is again a l.c.i. variety. In the case of the hypersurface X this condition is:

$\dim X_q \leq (m-1)\cdot(q+1)$. If we assume that X_q is pure dimensional we may use the Jacobian criterion for regularity (see section 16.6 in [Eis95]) to determine the singular locus of X_q, see Proposition 29.

Remark. Note that if Y is a l.c.i. variety and Y_{q+1} is pure-dimensional, so is Y_q, $q \geq 1$ (Proposition 1.6, [Mus01]). Especially, Y_q is pure-dimensional if it is irreducible, which is the case if and only if X has just canonical singularities. For more information on the relation between pure-dimensionality of Y_q and the question for log canonical singularities of Y see [Mus01], Thm. 3.4 and Prop. 3.6.

Proposition 29. *Let $X \subseteq \mathbb{A}^m$ be a hypersurface with pure dimensional qth jet scheme, $q \geq 1$, then*

$$(X_q)_{\mathrm{reg}} = \pi_q^{-1}(X_{\mathrm{reg}}).$$

Proof. Since $\pi_q : X_q \to X$ is a bundle over the regular part of X_q it's clear that $\dim X_q = (m-1)\cdot(q+1)$. Denote by F the column vector $(F_0, \dots, F_q)^\top$ and by $\partial_{i,j}$ the partial derivative $\partial_{x_i^j}$. Moreover set ∂F_0 for the row vector $(\partial_{0,1} \dots \partial_{0,m})F_0$. Applying the Jacobian Criterion for regularity we consider the $(q+1)$-minors of ∂F. Taking into account the special form of the F_i we see that ∂F is a $(q+1) \times m(q+1)$ matrix with entries in $k[q;m]$ which has the shape

$$\partial F = \begin{pmatrix} \partial F_0 & 0 & \dots & \dots & 0 \\ * & \partial F_0 & 0 & \dots & 0 \\ * & * & & \ddots & 0 \\ * & * & \dots & \dots & \partial F_0 \end{pmatrix}$$

Let $A = (A_{ij})_{i,j}$ be a $(q+1) \times (q+1)$ submatrix of ∂F. The determinant of A is given by

$$\det A = \sum_{\sigma \in S_{q+1}} \operatorname{sgn}(\sigma) \prod_{i=0}^{q} A_{i\sigma(i)}.$$

It's easy to see that each product of A_{ij} in the above sum contains exactly one factor stemming from the first row, which is either zero or $\partial_{0,j}F_0$ for some $j \in \{1, \dots, m\}$. Thus the ideal J of $(q+1)$-minors of ∂F is contained in the ideal $(\partial_{0,1}F_0, \dots, \partial_{0,m}F_0)$ of F_0. Hence

$$\pi_q^{-1}(X_{\mathrm{sing}}) \subseteq (X_q)_{\mathrm{sing}}.$$

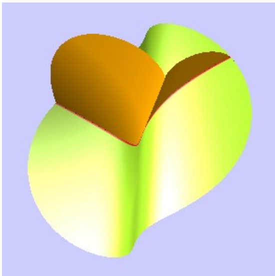
The reverse inclusion is trivial and we conclude that $(X_q)_{\mathrm{reg}} = \pi_q^{-1}(X_{\mathrm{reg}})$. $\square$

Example 17. Let $f \in k[x_1, \dots, x_m]$ define a hypersurface X in $\mathbb{A}^m$, k algebraically closed. Assume that X has a singularity at $p \in X$. Is it possible to find a second hypersurface Y, given by $g \in k[x_1, \dots, x_m]$ such that $p \in X \cap Y$ is a non-singular point? For generic Y we will have a complete intersection and its singular locus is given by (f, g) and the 2-minors of

$$\begin{pmatrix} \partial_1 f & \dots & \partial_m f \\ \partial_1 g & \dots & \partial_m g \end{pmatrix}$$

which are of the form $\partial_i f \partial_j g - \partial_j f \partial_i g \subseteq (\partial_1 f, \dots, \partial_m f)$. Thus if $\mathfrak{p}$ is the prime ideal corresponding to $p \in X \cap Y$ we get – since $I(X_{\mathrm{sing}}) \subseteq \mathfrak{p}$ – that

$I((X \cap Y)_{\mathrm{sing}}) \subseteq \mathfrak{p}$.
Look at the example $f = y^2 - x^3 \in k[x, y, z]$, the cylinder over the Neil parabola. We choose as Y the cylinder over $g = y + z^3$ which gives in the real picture below at the origin a smooth intersection:



The previous computation shows that this is not the case over $\mathbb{C}$. Indeed, the minimal associated primes of $X \cap Y$ are given by $I_1 = (y + z^3, x - z^2)$ and $I_2 = (y + z^3, z^4 + xz^2 + x^2)$. The intersection is not (locally) irreducible at the origin, thus singular.

Remark. The last example gives an easy proof for $(X_1)_{\mathrm{reg}} = (\pi_1)^{-1}(X_{\mathrm{reg}})$ in the case of arbitrary hypersurfaces X for which X_1 is a l.c.i. (this is a slightly stronger assumption than pure dimensionality). Indeed X_1 is given by F_0 and F_1, both defining hypersurfaces in $\mathbb{A}^{2m}$. The singular locus of $V(F_0)$ is a cylinder over X_{sing}. The intersection of $V(F_0)$ and $V(F_1)$ cannot be smooth at any point of $V(F_0)_{\mathrm{sing}}$ as we saw above. Thus $\pi_1^{-1}(X_{\mathrm{sing}}) = (X_1)_{\mathrm{sing}}$.

3.3.2 Iterated Jet Schemes

Let X be a variety and X_1 its space of 1-jets. We study the *iterated* $(1,1)$*-jet space* $X_{1,1}$ defined as

$$X_{1,1} = (X_1)_1 .$$

Clearly, $X_{1,1}$ is again an affine scheme. If $X = \operatorname{Spec} k[x_1, \ldots, x_m]/(f_1, \ldots, f_p)$ then we obtain the defining equations of $X_{1,1}$ as follows: every f_j gives two defining equations $F_0^j, (f_j)_1 = F_1^j \in k[1; m]$, each of which produces another two. For later calculations it is useful to see the precise structure: Substitute

$$(x_{00}^e + x_{01}^e \cdot s) + (x_{10}^e + x_{11}^e \cdot s) \cdot t$$

for x_e in the expression of f_j. The coefficients of $1, s, t, st$ give the defining equations for $X_{1,1}$. These are (if some index in the argument of a polynomial is not specified, we assume that it depends on all variables with possible value for the index)

$$(*) \begin{cases} F_{00}^j &= f_j(x_{00}^e) \\ F_{01}^j &= \sum_{e=1}^m \partial_{x_e} f_j(x_{00}^1, \ldots, x_{00}^m) \cdot x_{01}^e = (f_j)_1(x_{00}^e, x_{01}^e) \\ F_{10}^j &= \sum_{e=1}^m \partial_{x_e} f_j(x_{00}^1, \ldots, x_{00}^m) \cdot x_{10}^e = (f_j)_1(x_{00}^e, x_{10}^e) \\ F_{11}^j &= DF_{01}^j, \text{ with } Dx_{il}^j = x_{i(l+1)}^j \quad \text{for all } i, j, l \end{cases} .$$

Remark. Note that if $X \subset \mathbb{A}^m$ then $X_1 \subseteq \mathbb{A}^{2m}$, $X_2 \subseteq \mathbb{A}^{3m}$ and $X_{1,1} \subseteq \mathbb{A}^{4m}$.

Analogously to the natural map $\pi_m^X : X_m \to X$ we introduce the map $\pi_{1,1}^X : X_{1,1} \to X$ given by the k-algebra homomorphism ($1 \le e \le m$)

$$k[x_{00}^e]/(F_{00}^j, 1 \le j \le p) \to k[x_{il}^e; 0 \le i,l \le 1]/(F_{il}^j, 1 \le j \le p, 0 \le i,l \le 1).$$

Obviously, we have $\pi_{1,1}^X = \pi_1^{X_1} \circ \pi_1^X$.

Proposition 30. *Let $X \subseteq \mathbb{A}^m$ be an affine scheme. Then X_2 is a closed subscheme of $X_{1,1}$ and $\pi_2 = \pi_{1,1} \circ \theta$ where $\theta : X_2 \to X_{1,1}$ is induced by*

$$\theta : k[x_{il}^e; 0 \le i,l \le 1]/(F_{il}^j, 1 \le j \le p, 0 \le i,l \le 1) \to k[x_0^e, x_1^e, x_2^e]/(F_0^j, F_1^j, F_2^j),$$

with

$$x_{il}^j \mapsto \begin{cases} x_0^j & (i,l) = (0,0) \\ x_1^j & (i,l) \in \{(0,1),(1,0)\} \\ x_2^j & (i,l) = (1,1) \end{cases}.$$

Proof. By $(*)$ this θ is well defined. Obviously it defines an isomorphism between X_2 and

$$Y = \operatorname{Spec} k[x_{il}^e; e,i,l]/(F_{00}^j, \dots, F_{11}^j, x_{01}^e - x_{10}^e; 1 \le e \le m).$$

The second claim is straight forward. □

Remark. The bijection between k-points of X_2 and those of Y is given by

$$\begin{array}{lll} X_2 \to Y & ; & (x_0^j, x_1^j, x_2^j) \mapsto (y_{00}^j, y_{01}^j, y_{10}^j, y_{11}^j) = (x_0^j, x_1^j, x_1^j, x_2^j) \\ Y \to X_2 & ; & (y_{00}^j, y_{01}^j, y_{10}^j, y_{11}^j) \mapsto (y_{00}^j, y_{01}^j, y_{11}^j). \end{array}$$

Remark. It is tempting to try to relate $(X_2)_{\text{sing}}$ to $(X_{1,1})_{\text{sing}}$, but be careful: If X is a variety, then X_1 need not be a variety! Thus the question arises when

$$(X_{1,1})_{\text{sing}} \ne \left(\pi_1^{X_1}\right)^{-1} \left((\pi_1^X)^{-1}(X_{\text{sing}})\right).$$

Example 18. Let $X : xy = 0$, then X_2 is given by $x_0y_0, x_0y_1 + x_1y_0, 2x_1y_1 + x_0y_2 + x_2y_0$ with singular locus $(X_2)_{\text{sing}} : x_0, y_0, x_1y_1$. X_2 has four components

$$\begin{array}{ll} C_1 : & y_0 = y_1 = y_2 = 0 \\ C_2 : & x_0 = y_0 = y_1 = 0 \\ C_3 : & x_0 = y_0 = x_1 = 0 \\ C_4 : & x_0 = x_1 = x_2 = 0. \end{array}$$

The $(1,1)$-jet scheme is given by $x_{00}y_{00}, x_{00}y_{01} + x_{01}y_{00}, x_{00}y_{10} + x_{10}y_{00}, x_{01}y_{10} + x_{10}y_{01} + x_{00}y_{11} + x_{11}y_{00}$. Its singular locus is determined by

$$(X_{1,1})_{\text{sing}} : x_{00}, y_{00}, x_{01}y_{10}, x_{10}y_{01}.$$

Note that in this example $(X_{1,1})_{\text{sing}} \cap X_2 = (X_2)_{\text{sing}}$ (where X_2 is identified with a closed subscheme of $X_{1,1}$ as in Proposition 30).

Question 2. *Let X be a variety. When does $(X_{1,1})_{\text{sing}} \cap X_2 = (X_2)_{\text{sing}}$ hold?*

3.4 Hilbert-Poincare Series of Jet Algebras

The k-algebras $J_q(\mathbb{A}^m_k)$, $q \in \mathbb{N} \cup \{\infty\}$, are graded with respect to the weight $\mathrm{wt}(x_i^j) = i$. Moreover, the defining equations of the jet space of a variety are weighted-homogenous. Thus it is natural to ask for the Hilbert-Poincare series $\mathrm{HP}_R(t)$ of a jet algebra R with respect to this weight. We restrict our considerations to hypersurfaces, in fact to irreducible hypersurfaces (especially reduced). Let $f \in k[x]$ define a hypersurface X in $\mathbb{A}^m$ and let $F_0, \ldots, F_q$ be the induced homogenous defining equations of the qth jet space. The homogenous part of weight 0 is $J_0(X) = k[0; m]/(F_0)$, which is not Artinian in general. Therefore we will consider the Hilbert-Poincare series, in short HP-series, of the focussed jet algebra

$$J_q(X; \mathfrak{p}) = J_q(X) \otimes_{J_0(X)} \kappa(\mathfrak{p})$$

of X in $\mathfrak{p}$. Clearly, $J_q(X; \mathfrak{p})$ is the coordinate ring of the fibre of $\pi_q^X : X_q \to X$ over $\mathfrak{p}$. By Proposition 21 the image $F_i^\bullet$ of F_i under the canonical map $J_q(X) \to J_q(X; \mathfrak{p})$ is still weighted homogenous. Since not all generators of the (focussed) jet algebras have in general weight 1 the Hilbert-function is not polynomial but quasi-polynomial. It is well known that in this situation the Hilbert-Poincare series is rational for each $q < \infty$. If $I \subseteq k[\underline{\mathbb{N}}^m]$ is an ideal we will write I_q for its intersection with the subalgebra $J_q(\mathbb{A}^m)$. Note:

Lemma 31. *Let $I \subseteq k[\underline{\mathbb{N}}^m]$ be a (weighted) homogenous ideal. Then*

$$\mathrm{HP}_{J_\infty(\mathbb{A}^m)/I}(t) = \lim_{q \to \infty} \mathrm{HP}_{J_q(\mathbb{A}^m)/I_q}(t).$$

where the limit is taken with respect to the (t)-adic topology.

Proof. This is clear since $[J_\infty(\mathbb{A}^m)/I]_i = [J_q(\mathbb{A}^m)/I_q]_i$ for $i \leq q$ and the coefficient of t^i in $\mathrm{HP}_{J_\infty(\mathbb{A}^m)/I}$ is exactly $\dim_k [J_\infty(\mathbb{A}^m)/I]_i$. $\square$

Example 19. Consider $R = J_\infty(\mathbb{A}^1; 0) = k[x_i; i \in \mathbb{N}]$. We compute its HP-series. The nth weighted piece R_n of R is generated by monomials of weight n. A monomial $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ has weight

$$\mathrm{wt}(x^\alpha) = 1 \cdot \alpha_1 + \ldots + n \cdot \alpha_n.$$

The number of monomials of weight n is thus equal to the number of solutions in $\mathbb{N}^n$ to the Diophantine equation $\sum_{i=1}^n i \cdot \alpha_i = n$, which is the number $p(n)$ of partitions of n. Therefore, $\mathrm{HP}_R(t) = \sum p(n) t^n$ is the generating series of the partition function $p(n)$ which is:

$$\sum_{n=0}^{\infty} p(n) t^n = \frac{1}{\prod_{i \geq 1}(1 - t^i)}.$$

Thus we see that for $q \in \mathbb{N} \cup \{\infty\}$ the HP-series of the qth jet algebra of $\mathbb{A}^1$ focussed at 0 is

$$\mathrm{HP}_{J_q(\mathbb{A}^1;0)} = \mathbb{H}_q$$

with

$$\mathbb{H}_q = \frac{1}{\prod_{1 \leq i \leq q}(1 - t^i)}.$$

This especially shows that for $q = \infty$ the HP-series is not rational.

Example 20. We will continue the last example and compute the HP-series for $R = J_\infty(\mathbb{A}^2; 0) = k[x_i, y_i; i \in \mathbb{N}]$. Again we have to compute the number of monomials $x^\alpha y^\beta$ of given weight n, which is the number of solutions in $\mathbb{N}^n \times \mathbb{N}^n$ to the equation

$$1 \cdot (\alpha_1 + \beta_1) + \ldots + n \cdot (\alpha_n + \beta_n) = n.$$

For this we compute the number of monomials such that $\mathrm{wt}(x^\alpha) = j$ and $\mathrm{wt}(x^\beta) = n - j$ which we have already computed as $p(j)$ resp. $p(n-j)$, and vary j. Thus

$$\dim_k R_n = \sum_{j=0}^{n} p(j)p(n-j).$$

This is nothing but the nth coefficient of $\mathbb{H}^2$ (resp. of $\mathbb{H}_q^2$ for $n \leq q$), i.e., $\mathrm{HP}_{J_q(\mathbb{A}^2;0)} = \mathbb{H}_q^2$. More generally we can conclude that $\mathrm{HP}_{J_q(\mathbb{A}^m;0)} = \mathbb{H}_q^m$.

The result of the last example can also be deduced from the well-known fact (see e.g. [GP02], p. 276):

Lemma 32. *Let $A = \oplus_{i \geq 0} A_i$ be a graded k-algebra, M be a graded A-module and $f \in A_d$, $d \in \mathbb{N}$. Then* $\ker(\cdot f)$ *and* $\mathrm{coker}\,(\cdot f)$ *are graded $A/(f)$-modules with induced grading and*

$$\mathrm{HP}_M(t) \cdot (1 - t^d) = \mathrm{HP}_{\mathrm{coker}\,(\cdot f)}(t) - t^d \cdot \mathrm{HP}_{\ker(\cdot f)}(t).$$

Indeed, the elements $x_1^1, \ldots, x_1^m, x_2^1, \ldots, x_q^m$ build a regular sequence in $k[q; m]$. Let $S_{e,l}$ denote the algebra

$$k[x_0^1, \ldots, x_0^m, \ldots x_e^1, \ldots, x_e^l]/(x_0^1, \ldots, x_e^{l-1}).$$

Then the kernel of $(\cdot x_e^l)\colon S_{e,l}(-e) \to S_{e,l}$ is trivial and one can conclude by induction that $\mathrm{HP}_{J_q(\mathbb{A}^m;0)}(t) = \mathbb{H}_q^m$.

Proposition 33. *Let $X \subseteq \mathbb{A}^m$ be an irreducible hypersurface with jet algebra $J_q(X) = J_q(\mathbb{A}^m)/I_q$, where $I = (F_0, F_1, \ldots)$ as introduced before. Then if $\mathfrak{p}$ is the generic point of X:*

$$\mathrm{HP}_{J_q(\mathbb{A}^m;\mathfrak{p})}(t) = \mathbb{H}_q^{m-1}.$$

Proof. Set $L = \mathrm{Quot}(k[0; m]/(F_0))$. We will show that $F_1^\bullet, \ldots, F_q^\bullet$ define a regular sequence in $J_q(\mathbb{A}^m; \mathfrak{p})$. Indeed, $(F_1^\bullet, \ldots, F_m^\bullet) \neq (1)$. It remains to show that for all $i \in \{2, \ldots, q\}$ the element $F_i^\bullet$ is a non-zerodivisor in

$$J_q(\mathbb{A}^m; \mathfrak{p})/(F_1^\bullet, \ldots, F_{i-1}^\bullet).$$

For this reason we write $F_i^\bullet = c_1 x_i^1 + \cdots + c_m x_i^m + d$ with $c_l \in L$ and

$$d \in L[x_e^j; 1 \leq j \leq m, 1 \leq e \leq i-1].$$

At least one c_l does not vanish, w.l.o.g. this is c_1. Consider the following homogenous k-algebra homomorphism:

$$\varphi : x_e^j \mapsto \begin{cases} x_e^j & (e,j) \neq (i,1) \\ \frac{1}{c_1}\left(x_i^1 - \sum_{l=2}^m c_l x_i^l - d\right) & (e,j) = (i,1). \end{cases}$$

This coordinate change transforms $F_i^\bullet$ into x_i^1, which is clearly a non-zero divisor in $J_q(\mathbb{A}^m;\mathfrak{p})/(F_1^\bullet,\dots,F_{i-1}^\bullet)$. Now using the above Lemma finishes the proof of the Proposition. $\square$

Corollary 34. *With the notation of the previous Proposition let $p \in X$ be an arbitrary smooth point. Then:*

$$\mathrm{HP}_{J_q(\mathbb{A}^m;p)}(t) = \mathbb{H}_q^{m-1}.$$

3.5 Completions of Polynomial Rings

This section starts the study of completions of polynomial rings in countably many indeterminates over a field k of characteristic 0. In the Noetherian situation (i.e., for polynomial rings in finitely many variables) completions are a powerful tool to study local properties of a variety. Its usefulness is based on flatness of the completion over the given local ring: If $I = (f_1,\dots,f_n)$ is an ideal in a Noetherian local ring R, then its closure in the completion $\widehat{R}$ is generated again by $f_1,\dots,f_n$. Thus the completion of a quotient can be described easily. In the non-Noetherian setting flatness does not hold in general. Completions of such kind have been investigated by Reguera in [Reg06] and [Reg09]. There it is shown that if z is the generic point of an irreducible generically stable subset of X_∞, X some variety, then $\widehat{\mathcal{O}}_{\infty,z}$ is Noetherian (Cor. 4.6, [Reg06]). More explicit results can be found in [Reg09]. In the following we compute the completion of $k[\underline{\mathbb{N}}^m]_\mathfrak{p}$, $\mathfrak{p} \in \mathbb{A}^m_\infty$, in some simple cases.

Example 21. Consider the polynomial ring $k^{(\mathbb{N}^n)} = k[x] = k[x_1,\dots,x_n]$, $n \in \mathbb{N}$. Denote the (maximal) ideal $(x_1,\dots,x_n)$ by $\mathfrak{m}$ and endow $k[x]$ with the $\mathfrak{m}$-adic topology, i.e., the topology induced by the filtration

$$k[x] \supseteq \mathfrak{m} \supseteq \mathfrak{m}^2 \supseteq \dots.$$

Completion with respect to this topology is another way to obtain the ring of formal power series in n variables (compare section 1.4.1):

$$\varprojlim_i k[x_1,\dots,x_n]/\mathfrak{m}^i = k[[x_1,\dots,x_n]].$$

Indeed, the following map defines an isomorphism of k-algebras:

$$\theta\colon k[[x]] \to \prod_{i\geq 1} k[x]/\mathfrak{m}^i \to \varprojlim_i k[x]/\mathfrak{m}^i; \xi \mapsto (\xi + \hat{\mathfrak{m}}^i)_i,$$

with $\hat{\mathfrak{m}}$ denoting the maximal ideal in $k[[x]]$. Its inverse is given by

$$\psi\colon (\bar{\xi}_i)_i \mapsto \sum_{i\geq 0}(\xi_{i+1} - \xi_i),$$

where the ξ_i are representatives of $\bar{\xi}_i$ and ξ_0 is set to 0. Note that $\psi(\xi)$ is independent of the chosen representatives (since defined by a telescoping sum). Moreover, by coherence of the sequence ξ we have $\xi_{i+i} - \xi_i \in \mathfrak{m}^i$, thus the sum in the definition of ψ converges in the $\mathfrak{m}$-adic topology. Therefore ψ is well-defined. A short computation shows that both θ and ψ are k-algebra homomorphisms:

It's clear that ψ is k-linear. For the ringhomomorphism property note that for $i \leq j$ the ith homogenous part of $\psi(\xi)$ is given by the degree i part ξ_{j+1}^i of a representative ξ_{j+1}. But then

$$(\psi(\xi\eta))^i = \psi((\xi_l\eta_l)_l)^i = (\xi_{j+1}\eta_{j+1})^i$$

The last term being

$$\sum_{a+b=i} \xi_{j+1}^a \eta_{j+1}^b = (\psi((\xi_l))\psi((\eta_l)))^i .$$

For every $f \in k[\underline{\mathbb{N}}]$ we define its *degree* $\deg f$ as

$$\deg f = \max\{|\alpha|; \alpha \in \operatorname{supp}(f)\}.$$

If $f = 0$, then we set $\deg f = \infty$. Analogously, we define the *order* of an element $f \in k[[\underline{\mathbb{N}}]]$ as the minimum of all $|\alpha|$, $\alpha \in \operatorname{supp}(f)$, respectively ∞ in case of $f = 0$. Note: for simplicity of notation we write $k[\underline{\mathbb{N}}]$ instead of $k[\underline{\mathbb{N}_0}]$. Consider for every $n \in \mathbb{N}$ the set of exponents of monomials in $k[\underline{\mathbb{N}}]$ of the same degree n, i.e.,

$$M_n = \{\alpha \in \mathbb{N}; |\alpha| = n\}.$$

In contrast to the last example the completion of $k[\underline{\mathbb{N}}]$ with respect to $\mathfrak{m} = (x_0, x_1, \ldots)$ is *not* the power series ring $k^{\mathbb{N}^{\mathbb{N}}}$ in variables $x_i, i \in \mathbb{N}_0$, which would contain elements like $\sum_{i=0}^{\infty} x_i$. We define the *degree-finite power series*

$$k[[\underline{\mathbb{N}}]]_{\text{fin}} = \{f \in k[[\underline{\mathbb{N}}]]; |\operatorname{supp}(f) \cap M_n| < \infty, \quad \text{for all } n \in \mathbb{N}\}.$$

We denote by $\mathfrak{m}_1$ the closure of $\mathfrak{m} = (x_0, x_1, \ldots)$ in $k[[\underline{\mathbb{N}}]]_{\text{fin}}$. It is given by the ideal of degree-finite power series in variables $x_i, i \in \mathbb{N}$ without constant term. More generally we consider

$$\mathfrak{m}_i = \{f \in k[[\underline{\mathbb{N}}]]_{\text{fin}}; \operatorname{ord} f \geq i\}.$$

Proposition 35. *Let $\{k[\underline{\mathbb{N}}]/\mathfrak{m}^i, \pi_i^j\}$ be the projective system given by the canonical maps*

$$\pi_i^j \colon k[\underline{\mathbb{N}}]/\mathfrak{m}^i \leftarrow k[\underline{\mathbb{N}}]/\mathfrak{m}^j$$

for $j \geq i$. Then:

$$\varprojlim k[\underline{\mathbb{N}}]/\mathfrak{m}^i \cong k[[\underline{\mathbb{N}}]]_{fin}.$$

Proof. Analogously to Example 21 we consider the map

$$\psi \colon \varprojlim_i k[\underline{\mathbb{N}}]/\mathfrak{m}^i \to k[[\underline{\mathbb{N}}]]_{\text{fin}}; \xi = (\xi_i)_i \mapsto \sum_{i \geq 0} (\xi_{i+1} - \xi_i)$$

with $\xi_0 = 0$. By the same reasoning as in the example ψ is well-defined. Moreover, $\operatorname{im}(\psi) \subseteq k[[\underline{\mathbb{N}}]]_{\text{fin}}$, since $\operatorname{ord} \xi_{i+1} - \xi_i \geq i$, i.e., just finitely many terms of given order appear in $\psi(\xi)$.
Obviously

$$k[[\underline{\mathbb{N}}]]_{\text{fin}}/\mathfrak{m}_i \cong k[\underline{\mathbb{N}}]/\mathfrak{m}^i.$$

Therefore we get a surjective map

$$\theta\colon k[[\mathbb{N}]]_{\text{fin}} \to \varprojlim_i k[\mathbb{N}]/\mathfrak{m}^i; f \mapsto (f + \mathfrak{m}_i)_i$$

with $\psi\theta = \mathrm{id}$ and $\theta\psi = \mathrm{id}$. Both maps are k-algebra homomorphisms. □

Remark. Consider $k[\mathbb{N}]$ with the linear topology defined by $\mathfrak{m}^i$, $i \in \mathbb{N}$. This topology is Hausdorff, i.e.,

$$\bigcap_{i \in \mathbb{N}} \mathfrak{m}^i = \{0\}.$$

Indeed, by definition $f \in \mathfrak{m}^i$ if and only if $\operatorname{ord} f \geq i$. Thus, $f \in \cap_{i\in\mathbb{N}}\mathfrak{m}^i$ implies $\operatorname{ord} f \geq i$ for all $i \in \mathbb{N}$, i.e., $f = 0$, and the assertion follows. From this we conclude that the canonical map $k[\mathbb{N}] \to k[[\mathbb{N}]]_{\text{fin}}$ is injective (see [Mat89]).

Remark. Let $f \in k[[\mathbb{N}]]_{\text{fin}}$ and $\{x_j; j \in J\}$ be a set of variables. Then we may expand f as a power series in the x_j, $j \in J$ with coefficients which are power series in the remaining indeterminates x_j, $j \in J' = \mathbb{N}_0 \setminus J$. In that case we will use the following notation:

$$f = \sum_{\alpha \in \mathbb{N}^{(J)}} f_\alpha x^\alpha.$$

If $J = \{0, \ldots, d\}$ we simply write $f = \sum_{\alpha\in\mathbb{N}^{d+1}} f_\alpha x^\alpha$, $f_\alpha \in k[[x_j; j \in J']]$.

Proposition 36. *For $d \in \mathbb{N}_0$ consider the ideal $\mathfrak{n} = (x_0, \ldots, x_d) \subseteq k[\mathbb{N}]$, $J = \{0, \ldots, d\}$, and set $\mathfrak{p} = (x_j; j \in J')$. Let $\{k[\mathbb{N}]/\mathfrak{n}^i, \pi_i^j\}$ be the canonical projective system, then:*

1. $\varprojlim k[\mathbb{N}]/\mathfrak{n}^i = k[x_{d+1}, \ldots][[x_0, \ldots, x_d]]$.
2. *For any $i \in \mathbb{N}$ the ideal*

$$\mathfrak{p}_i = \{f = \sum_{\alpha\in\mathbb{N}^{d+1}} f_\alpha x^\alpha \in k[x_{d+1}, \ldots][[x_0, \ldots, x_d]]; f_\alpha \in (x_j, j \in J')^i\}$$

is the closure of the ideal $\mathfrak{p}^i$ in $k[x_j; j \in J'][[x_0, \ldots, x_d]]$ with respect to the topology given by $\mathfrak{n}_j = \ker(p_j)$, where

$$p_e : K[x_j; j \in J'][[x_0, \ldots, x_d]] \to k[\mathbb{N}]/\mathfrak{n}^e$$

is the natural projection.

Remark. More explicitly the ideals $\mathfrak{n}_e$ are of the form

$$\mathfrak{n}_e = \{f \in k[x_j; j \in J'][[x_0, \ldots, x_d]]; \operatorname{ord}_{(x_0,\ldots,x_d)} f \geq e\}.$$

Proof. Ad 1. The first result can be proved in exactly the same way as the last Proposition. Nevertheless we will vary the proof a bit: Write every element $f \in R = k[x_j; j \in J'][[x_0, \ldots, x_d]]$ as

$$f = \sum_{\alpha\in\mathbb{N}^{d+1}} f_\alpha x^\alpha$$

with $f_\alpha \in k[x_j; j \in J']$ and $x^\alpha = x_0^{\alpha_0} \cdots x_d^{\alpha_d}$. The homogenous part $f^{(i)}$ of degree i of f is then

$$\left(\sum_{|\alpha| \leq i} f_\alpha x^\alpha \right)^{(i)},$$

hence $R \subseteq k[[\mathbb{N}]]_{\mathrm{fin}}$. Consider

$$\theta : \varprojlim k[\mathbb{N}]/\mathfrak{n}^i \to k[[\mathbb{N}]]_{\mathrm{fin}}; (\xi_i) \mapsto \sum (\xi_{i+1} - \xi_i)$$

with $\xi_0 = 0$. As before one can see that θ is a well-defined homomorphism of k-algebras. Moreover, $R \subseteq \mathrm{im}(\theta)$ since for any $f \in R$ we have

$$f = \theta((f \mod \mathfrak{n}_i)_i)$$

using the natural identification

$$R/\mathfrak{n}_i \cong k[\mathbb{N}]/\mathfrak{n}^i.$$

Clearly, θ is injective: $\theta(\xi) = 0$ if and only if all homogenous parts of degree $j \in \mathbb{N}$ are zero. But

$$\theta(\xi)^{(j)} = (\xi_i)^{(j)}$$

for $j < i$. Thus $(\xi_i)_i = (0)$. It remains to show that $\mathrm{im}(\theta) \subseteq R$, but this is clear from the above statements.

Ad 2. The closure $\overline{\mathfrak{p}^i}$ of $\mathfrak{p}^i$ with respect to the topology defined by the filtration $\mathfrak{n}_1 \supseteq \mathfrak{n}_2 \supseteq \cdots$ is given by

$$\overline{\mathfrak{p}^i} = \cap_{j \in \mathbb{N}} \left(\mathfrak{p}^i + \mathfrak{n}_j \right).$$

We first show that $\mathfrak{p}_i \subseteq \overline{\mathfrak{p}^i}$: Let $f = \sum f_\alpha x^\alpha$ be an element of $\mathfrak{p}_i$. Then for all $\alpha \in \mathbb{N}^{d+1}$ the corresponding coefficient f_α lies in $\mathfrak{p}^i$, and for all $j \in \mathbb{N}$ we can write f as

$$f = f'_j + j''_j,$$

with $f'_j = \sum_{|\alpha|<j} f_\alpha x^\alpha$ and $f''_j = \sum_{|\alpha| \geq j} f_\alpha x^\alpha$. Since all $f_\alpha \in \mathfrak{p}^i$ the polynomial f'_j lies in $\mathfrak{p}^i$. By construction $f''_j \in \mathfrak{n}_j$. Therefore $f \in \overline{\mathfrak{p}^i}$.
For the reverse inclusion $\mathfrak{p}_i \supseteq \overline{\mathfrak{p}^i}$ assume that there exists an $f \in \overline{\mathfrak{p}^i}$ which does not lie in $\mathfrak{p}_i$. As we have already shown that $\mathfrak{p}_i \subseteq \overline{\mathfrak{p}^i}$, we may assume without loss of generality that $f = f_\alpha x^\alpha$ with $\deg_{(x_j; j \in J')} f_\alpha < i$. Moreover, since $f \in \overline{\mathfrak{p}^i}$ for every integer $j \geq 1$ we have $f \in \mathfrak{p}^i + \mathfrak{n}_j$. So we see that no term of f belongs to $\mathfrak{p}^i$, thus $f \in \cap_j \mathfrak{n}_j$ and $f = 0$.

□

Remark. With the notation from above we conjecture that $\mathfrak{p}_i \neq \mathfrak{p}_1^i$. A possible counterexample is given in $R = k[x_1, x_2, \ldots][[y]]$ by

$$f = \sum_{i \geq 1} x_i^2 \cdot x_0^{i-1} = x_1^2 + x_2^2 y + x_3^2 y^2 + \ldots \in \mathfrak{m}_2.$$

Proposition 37. *With the notation introduced above there exists an isomorphism*
$$k[x_{d+1},\dots][[x_0,\dots,x_d]]/\mathfrak{p}_i \cong k[x_{d+1},\dots]/\mathfrak{p}^i[[x_0,\dots,x_d]],$$
inducing an isomorphism of the corresponding inverse systems. Moreover

$$\varprojlim_i \left((\varprojlim_j k[\underline{\mathbb{N}}]/\mathfrak{n}^j)/\mathfrak{m}_i \right) \cong k[[\underline{\mathbb{N}}]]_{\mathrm{fin}}.$$

Proof. For the first assertion consider the map

$$\varphi\colon k[x_j; j\in J'][[x_0,\dots,x_d]] \to k[x_j; j\in J']/\mathfrak{p}^i[[x_0,\dots,x_d]]$$
given by $f=\sum_{\alpha\in\mathbb{N}^{d+1}} f_\alpha x^\alpha \mapsto \sum_\alpha \bar f_\alpha x^\alpha$. It is obviously a well-defined surjective k-algebra homomorphism. Moreover,

$$\varphi(f)=0 \Leftrightarrow \bar f_\alpha = 0 \text{ for all } \alpha \Leftrightarrow f_\alpha\in\mathfrak{p}^i \text{ for all } \alpha \Leftrightarrow f\in\mathfrak{p}_i.$$

The last claim is proven as follows. By the first assertion and Proposition 36, 1, it is enough to show that there exists an isomorphism

$$\varprojlim \left(k[x_j; j\in J']/(x_j; j\in J')^i[[x_0,\dots,x_d]]\right) \to k[[\underline{\mathbb{N}}]]_{\mathrm{fin}}.$$
We define such a θ by

$$\theta(\xi)=\sum_{i\geq 0}\left(\xi_{i+1}-\xi_i\right),$$

where again $\xi_0=0$. Clearly, θ is well-defined: it is independent of the chosen representatives ξ_i and $\mathrm{im}(\theta)\subseteq k[[\underline{\mathbb{N}}]]_{\mathrm{fin}}$. Moreover, θ is surjective: let $f=\sum_{\alpha\in\mathbb{N}^{d+1}} f_\alpha x^\alpha$, $f_\alpha\in(x_j; j\in J')$, be an element of $k[[\underline{\mathbb{N}}]]_{\mathrm{fin}}$. Consider $\xi_i=\sum f'_\alpha x^\alpha$ where $f_\alpha=f'_\alpha+f''_\alpha$ with $\deg_{x_j, j\in J'} f'_\alpha < i$ and $\mathrm{ord}_{x_j, j\in J'} f''_\alpha\geq i$. The sequence $\xi=(\xi)_i$ is coherent, and clearly $\theta(\xi)=f$. Injectivity of θ follows from the fact that if $\theta(\xi)=\theta(\eta)$, then $\xi_i^{(j)}=\eta_i^{(j)}$ for all $j<i$, since $\theta(\xi)^{(j)}=\xi_i^{(j)}$. □

Let I be an ideal in $k[\underline{\mathbb{N}}]$. We are interested in the completion of the quotient $k[\underline{\mathbb{N}}]/I$ with respect to the filtration induced by $(\mathfrak{m}^i)_{i\in\mathbb{N}}$. Since the inverse system $\{k[\underline{\mathbb{N}}]/\mathfrak{m}^i, \pi_i^j\}$ is surjective, i.e., all the π_i^j are surjective, the exact sequence

$$0\to I\to k[\underline{\mathbb{N}}]\to k[\underline{\mathbb{N}}]/I\to 0$$

induces an exact sequence

$$0\to \varprojlim I/(I\cap\mathfrak{m}^i)\to k[[\underline{\mathbb{N}}]]_{\mathrm{fin}}\to \varprojlim (k[\underline{\mathbb{N}}]/I)/\bar{\mathfrak{m}}^i\to 0,$$

see for example [Har77], II.9. In what follows we will as usual abbreviate the projective limit by "$\hat{\ }$". For example we will write $\hat I$ instead of $\varprojlim I/(I\cap\mathfrak{m}^i)$. Thus, the above exact sequence reads as

$$\widehat{k[\underline{\mathbb{N}}]/I} = k[[\underline{\mathbb{N}}]]_{\mathrm{fin}}/\widehat{I}.$$

In order to work with this object it would be advantageous to know an explicit description of $\hat I$. Recall the finite dimensional situation, i.e, the case of an ideal I in a Noetherian ring R. Completion w.r.t. some ideal $J\subseteq R$ induces

$\widehat{R/I} \cong \hat{R}/\hat{I}$. Since $\hat{R}$ is a flat R-module we know that if $f_1, \ldots, f_n$ are generators for I in R then they are generators for $\hat{I}$ in $\hat{R}$.
There are two immediate possibilities to compute $\hat{I}$. Either $\hat{I}$ can be seen as $\varprojlim I/(I \cap \mathfrak{m}^i)$ or as the closure of $\mathrm{inj}(I)$, the image of I under the canonical map $k[\underline{\mathbb{N}}] \to \widehat{k[\underline{\mathbb{N}}]} = k[[\underline{\mathbb{N}}]]_{\mathrm{fin}}$. The closure can be computed as the intersection

$$\overline{\mathrm{inj}(I)} = \cap_{j \in \mathbb{N}} \left(\mathrm{inj}(I) + \mathfrak{m}_i\right).$$

Here $\mathfrak{m}_i$ is the kernel of the canonical map $p_i : k[[\underline{\mathbb{N}}]]_{\mathrm{fin}} \to k[\underline{\mathbb{N}}]/\mathfrak{m}^i$.
Example 22. Consider $I = (x_0) \subseteq k[\underline{\mathbb{N}}]$. Then $I \cap \mathfrak{m}^i = (x_0^i, x_0^{i-1}x_1, \ldots)$, the generators being monomials of degree i with x_0 appearing in non-zero power. So $I/(I \cap \mathfrak{m}^i)$ consists of multiples of x_0 which have degree less than i. Hence we conclude that

$$\hat{I} = \{f \in k[[\underline{\mathbb{N}}]]_{\mathrm{fin}}; x_0 | f\} = {}_{k[[\underline{\mathbb{N}}]]_{\mathrm{fin}}}(x_0). \tag{3.9}$$

Indeed, denoting the right-hand side of equation (3.9) by A, it is obvious that $A \subseteq \overline{\mathrm{inj}(I)}$. To see the reverse inclusion assume that there exists an $f \in \overline{\mathrm{inj}(I)}$ such that $x_0 \nmid x^\alpha$ for some $\alpha \in \mathrm{supp}(f)$. In fact we may assume that this holds for all $\alpha \in \mathrm{supp}(f)$. But then no term in f lies in $\mathrm{inj}(I)$, i.e., $f \in \cap_{i=0}^\infty \mathfrak{m}_i$ which is 0.

Example 23. Let $f = x - y \in k[x, y]$. The induced felt (see chapter 1 and section 1.4.2) is given by the ideal $I = (x_0 - y_0, x_1 - y_1, \ldots)$. Since the $x_i - y_i$ are homogenous we see that $I \cap \mathfrak{m}^i$ can be identified with (see Proposition 38 below)

$$\{g \in k[\underline{\mathbb{N}}]; g = \sum_{j \in J} \alpha_j (x_j - y_j),\ |J| < \infty,\ \mathrm{ord}\ \ \alpha_j \geq i - 1\}.$$

More generally we have:

Proposition 38. *Let $f \in k[x_1, \ldots, x_m]$ be homogenous of degree d. Then for $I = (F_0, F_1, \ldots)$:*

$$I \cap \mathfrak{m}^i = \mathfrak{m}^{i-d} \cdot I.$$

Proof. Since f is homogenous of degree d so are all F_i. Any element $g \in I \cap \mathfrak{m}^i$ is of the form

$$g = \sum_{j \in J} \alpha_j F_j,$$

with $|J| < \infty$ and $\alpha_j \in k[\underline{\mathbb{N}}^m]$. Set $e = \min\{\mathrm{ord}\, \alpha_j\}$; we may assume that

$$e = \min\{l \in \mathbb{N}; \sum \alpha_j^{(l)} f_j \neq 0\}.$$

Under these assumptions: $g \in I \cap \mathfrak{m}^i \Leftrightarrow e \geq i - d$. Hence $I \cap \mathfrak{m}^i = \mathfrak{m}^{i-d} \cdot I$. □

Remark. We conjecture for the general case: For an arbitrary ideal

$$I = (f_0, f_1, \ldots)$$

the closure $\hat{I}$ equals the ideal I_{fin} of $\sum_{i=0}^\infty \alpha_i f_i \in k[[\underline{\mathbb{N}}]]_{\mathrm{fin}}$ with $\alpha_i \in k[[\underline{\mathbb{N}}]]_{\mathrm{fin}}$. Up to now we cannot even answer the question: *Is every ideal in* $k[[\underline{\mathbb{N}}]]_{\mathrm{fin}}$ *generated by countably many elements?*

Remark. In this remark we collect further interesting questions concerning the algebraic structure of $k[[\underline{\mathbb{N}}]]_{\text{fin}}$. First: *Is* $k[[\underline{\mathbb{N}}]]_{\text{fin}}$ *a unique factorization domain?* It is well-known that every polynomial ring in finitely many variables over a field is UFD. Using the Weierstrass Preparation Theorem one concludes that also formal (resp. convergent) power series rings in finitely many variables are UFD's. We ask wether the analog holds for $k[\underline{\mathbb{N}}]$ respectively $k[[\underline{\mathbb{N}}]]_{\text{fin}}$? Clearly we have:

Proposition. $k[\underline{\mathbb{N}}]$*is a unique factorization domain.*

Proof. Let $f \in k[\underline{\mathbb{N}}]$, then $f \in R_N = k[x_i; 0 \leq i \leq N]$ for some $N \in \mathbb{N}$. Since R_N is a UFD f has in R_N a unique factorization into irreducible elements $f = f_1 \cdots f_p$. If there would be another set of irreducible polynomials $f_1', \dots, f_q' \in k[\underline{\mathbb{N}}]$, $q \in \mathbb{N}$, with $f = f_1' \cdots f_q'$, then all f_i' would be contained in an $R_{\tilde{N}}$, w.l.o.g. $\tilde{N} \geq N$. But then since $R_N \subseteq R_{\tilde{N}}$, and $R_{\tilde{N}}$ is UFD, we see that $q = p$ and $\{f_i; 1 \leq i \leq p\} = \{f_i'; 1 \leq i \leq q\}$. We conclude that $k[\underline{\mathbb{N}}]$ is a UFD. $\square$

It is not clear how to relate the UFD property of $k[\underline{\mathbb{N}}]$ with $k[[\underline{\mathbb{N}}]]_{\text{fin}}$. The way in the classical, i.e., finite dimensional, setting is via the Weierstrass Preparation Theorem. More precisely $f \in k[[x_1, \dots, x_n]]$ is written (after a linear change of coordinates) as an element in $k[[x_1, \dots, x_{n-1}]][x_n]$. Using induction and the Gauss Lemma one concludes that $k[[x_1, \dots, x_n]]$ is a UFD.
However, in the setting of power series rings in infinitely many variables it is not clear how to generalize x_n-regularity and the Weierstrass Theorem. Even if that is done the induction argument will be more involved than in the finite dimensional case. Since the Weierstrass Theorem is vital to a large number of results in singularity theory on might wonder for an appropriate substitute in the present setting: *What is a substitute for the Weierstrass Preparation Theorem in* $k[[\underline{\mathbb{N}}]]_{\text{fin}}$*?*.

Chapter 4

Constructions with Étale Neighbourhoods

A basic but powerful tool in analysis and differential geometry is the inverse function theorem, which asserts that if the tangent map at a point is an isomorphism, so is the original map, at least in a sufficiently small neighbourhood of the point. Exactly the invertibility in a *"small neighbourhood"* causes troubles in algebraic geometry. If one works over the real or complex numbers the Euclidean topology is fine enough to offer sufficiently small neighbourhoods. This is not any longer the case for arbitrary fields, where the notion of "Euclidean topology" is not defined. The Zariski topology on the other hand is much to coarse to allow an inverse function theorem, see example 25 below. The notion of *étale neighbourhood* is a profitable tool to exploit morphisms which "would fulfill the assumptions of the inverse mapping theorem if considered over $\mathbb{C}$".
A short introduction to étale morphisms is given in section 4.1. For a detailed introduction we refer to the literature, e.g. [Har77], [KPR75] or [Mil80]. The most used properties of étale morphisms in the present chapter are openness and the fact that they induce isomorphisms of the completions of the respective local rings.

The key problem is the following: Let X be a variety over an algebraically closed field k and $x \in X$ a point. Denote the completion of the local ring of X at x by $\widehat{\mathcal{O}}_{X,x}$. Let $\mathcal{P}$ be a property of $\widehat{\mathcal{O}}_{X,x}$. What information can be gained on the set $X_{\mathcal{P}}$ of all $x \in X$ such that $\widehat{\mathcal{O}}_{X,x}$ has property $\mathcal{P}$? Such properties of the completion are for instance: being an integral domain, reduced, a principal ideal domain or being normal crossings. We especially prove that the normal crossings locus X_{nc} is open in X, a hypersurface in some $\mathbb{A}^n$ (a generalization to arbitrary varieties is straight forward). This result is well-known to experts in resolution of singularities, but up to the knowledge of the author there is no reference for this question in the above general setting. Note, that it is quite easy to prove this fact if a Euclidean topology (and thus a notion of analytic maps) is available. In fact, one could see étale neighbourhoods as a substitute for the respective method of proof.

The structure of the proof for the normal crossings case is as follows: First prove

a "global" version of the assertion, i.e., show that for a union of hypersurfaces the set of points where it is *algebraic normal crossings* (also known as *simple normal crossings* in the literature) is open, see Proposition 41. Next translate the formal property of being normal crossings into the global property of being algebraic normal crossings. This uses étale neighbourhoods and the powerful Artin Approximation Theorem [Art69]. In a last step openness of the algebraic normal crossings locus in the étale neighbourhood is transferred to the original variety by openness of étale morphisms.
This chapter was developed in joint work with Dominique Wagner and will be part of a forthcoming survey paper on basic constructions using étale neighbourhoods.

4.1 Étale and Formal Neighbourhoods

Let k be an algebraically closed field of arbitrary characteristic. Denote by $k[x]$ the polynomial ring in the variables $x = (x_1, \dots, x_n)$ over k. Its completion w.r.t. the maximal ideal (x) is the ring of formal power series $k[[x]]$. For an ideal $I \subseteq k[x]$ we will denote by $V(I)$ the scheme defined by I.

Let X be a Noetherian scheme and Y a closed subscheme of X given by a sheaf of ideals $\mathcal{I}$. The *formal neighbourhood* $(\widehat{X}, \mathcal{O}_{\widehat{X}})$ *of* X *in* Y is the ringed space defined by the topological space Y and the sheaf of rings

$$\mathcal{O}_{\widehat{X}} = \varprojlim_n \mathcal{O}_X/\mathcal{I}^n$$

(see [Har77], II.9). If $X = \operatorname{Spec} A$ is affine and $p \in X$ is a closed point, then the formal neighbourhood of X in p is the one point space $\{p\}$ together with the sheaf of rings given by $\widehat{A}_p$, the completion of the local ring A_p w.r.t. its maximal ideal.

Example 24. Let $X = \operatorname{Spec} k[x, y]/(y^2 - x^2 - x^3)$ and $p = 0$. The structure sheaf of the formal neighbourhood of X in 0 is given by

$$k[[x, y]]/(y^2 - x^2 - x^3),$$

which is not an integral domain. Indeed, $y^2 - x^2 - x^3$ factors in $k[[x, y]]$ as

$$y^2 - x^2 - x^3 = (y + x\sqrt{1 + x})(y - x\sqrt{1 + x}).$$

Therefore the formal neighbourhood of X in 0 is reducible.

Let X and Y be varieties over k. For a point $p \in X$ we denote by $C_p(X)$ the *tangent cone of* X *at* p. It is given by the associated graded algebra of the local ring of X at p:

$$\operatorname{gr}(\mathcal{O}_{X,p}) = \bigoplus_{i=0}^{\infty} \mathfrak{m}_p^i/\mathfrak{m}_p^{i+1},$$

where $\mathfrak{m}_p^0 = \mathcal{O}_{X,p}$.

More explicit, we may define $C_0(X)$ for an affine variety $X \subseteq \mathbb{A}^n$ given by an ideal $I \subseteq k[x_1,\dots,x_n]$ as follows: Denote by I_* the ideal of initial forms of elements of I, where the initial form of an element $f \in k[x]$ is its homogenous part of lowest degree. Then $C_0(X) \cong k[x]/I_*$.

For a point x of a variety (or scheme) X we denote by $\kappa(x)$ its residue field, i.e., $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$. Let $\varphi\colon X \to Y$ be a morphism of varieties such that the induced map $\varphi_p : \kappa(\varphi(p)) \to \kappa(p)$ is an isomorphism. We call φ *étale at* $p \in X$ if the induced map on the tangent cones $d_p\varphi : C_p(X) \to C_{\varphi(p)}(Y)$ is an isomorphism. If $p \in X$ is a regular closed point then the tangent cone agrees with the tangent space, and an étale morphism is a morphism whose tangent map is an isomorphism. Especially, if φ is étale we know that $p \in X_{\text{reg}}$ if and only if $\varphi(p) \in Y_{\text{reg}}$.

Example 25. Let $X = \operatorname{Spec} k[x,y]/(x-y^2)$ and $Y = \operatorname{Spec} k[x]$ with $\varphi\colon X \to Y$ induced by the inclusion $k[x] \hookrightarrow k[x,y]$. Clearly for any $p = (p_x,p_y) \in X \setminus \{0\}$ the map $d_p\varphi$ gives an isomorphism between $C_{p_x}(Y)$ and $C_p(X)$, thus defines an étale map. Note that this is not the case for the point $p = 0$. Note further that in the case of $k = \mathbb{C}$ we may consider X and Y as manifolds over $\mathbb{C}$. Then the implicit function theorem is applicable, and states that X can (Euclidean-) locally at $p \in X \setminus \{0\}$ be parametrized by a neighbourhood of $p_x \in Y$.

For arbitrary schemes X and Y a morphism $\varphi\colon X \to Y$ is called *étale* if it is flat and unramified. It is called *étale at p* if the induced morphism of local schemes $\varphi\colon X_p \to Y_{\varphi(p)}$ is étale. For the convenience of the reader we summarize some properties of étale (resp. flat and unramified) morphisms in the next Proposition. Details, especially proofs, can be found for example in the excellent sources [Har77], [KPR75] or [Mil80].

Proposition 39. *Let X, Y be schemes. For a point $x \in X$ we denote by $\kappa(x)$ its residue field, i.e., $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$.*

1. *A flat morphism of finite type between Noetherian schemes is open.*
2. *If $f\colon X \to Y$ is locally of finite type, then f is unramified if and only if the sheaf of relative differentials vanishes, i.e., $\Omega^1_{X/Y} = 0$.*
3. *Open immersions, compositions of étale morphisms and any base changes of étale morphisms are étale.*
4. *Let $f : X \to Y$ be of finite type, Y locally Noetherian, $x \in X$ and $y = f(x)$ so that $\kappa(x) = \kappa(y)$. Moreover, let $\theta : \widehat{\mathcal{O}}_{Y,y} \to \widehat{\mathcal{O}}_{X,x}$ be the canonical morphism. Then f is étale (in some neighbourhood of x) if and only if θ is an isomorphism.*

The last observation especially implies that $\dim_x X = \dim_y Y$. Étale morphisms of varieties over $\mathbb{C}$ are morphisms which are local isomorphisms in the analytic sense, see example 25. Locally all étale morphisms are of the form

$$F : \operatorname{Spec} R[T_1,\dots,T_n]/(f_1,\dots,f_n) \to \operatorname{Spec} R$$

with $\det \frac{\partial F}{\partial T}$ a unit in $R[T]/(f_1,\dots,f_n)$ for some ring R. This is equivalent to F being flat and unramified (see [Mil80], Cor. 3.16, and [Mum99], III §5).

An *étale neighbourhood of a point* $x \in X$ is a pair (U, u) consisting of a scheme U and a point $u \in U$ with an étale morphism $\varphi : U \to X$ such that $\varphi(u) = x$.

Example 26. Let X be the node in the plane with coordinate ring

$$A = k[x, y]/(y^2 - x^2 - x^3).$$

Clearly X is irreducible and so is $\operatorname{Spec} A_{(x,y)}$. In the formal neighbourhood of the origin the germ of X is reducible (see example 24), since

$$y^2 - x^2 - x^3 = (y + x\sqrt{1+x})(y - x\sqrt{1+x}).$$

In the case of $k = \mathbb{C}$ this factorization holds in an Euclidean neighbourhood of the origin, since the factors are algebraic power series, thus convergent (see [Rui93], p. 106). Though we cannot obtain this decomposition in a Zariski-open neighbourhood of 0, it will be possible in an étale neighbourhood. Consider

$$U = \operatorname{Spec} A_a[T]/(T^2 - a)$$

where $a = 1 + x$ and the canonical map $\varphi\colon U \to X$ induced by

$$A_a \to A_a[T]/(T^2 - a).$$

Let us denote the coordinate ring of U by B, and set $f = T^2 - a$. By the remark above φ is étale if and only if $\partial f/\partial T = 2T$ is a unit in B. But $2a$ is a unit in B, hence $2T(2a)^{-1}T = 1$. Thus φ is étale. On U the polynomial $y^2 - x^2 - x^3$ factors into $y - Tx$ and $y + Tx$. Therefore U is an étale neighbourhood which is reducible with two smooth branches intersecting transversally.

The connected étale neighbourhoods of $x \in X$ form a filtered system. The *local ring* $\mathcal{O}_{X,\bar{x}}$ *of* X *at* x *w.r.t. the étale topology* is defined as

$$\mathcal{O}_{X,\bar{x}} = \varinjlim_{(U,u)} \Gamma(U, \mathcal{O}_U),$$

where the limit is taken over the system of connected étale neighbourhoods (U, u) of x. Let $(A, \mathfrak{m})$ be a local ring. It is called *Henselian* w.r.t. $\mathfrak{m}$ if it has the following property: If $F \in A[T]$ with $F(0) \in \mathfrak{m}$ and $F'(0) \in (A/\mathfrak{m})^\times$, then there exists an $a \in \mathfrak{m}$ with $F(a) = 0$. As usual, F' denotes the derivative of F w.r.t. T. The *Henselization* A^h *of* A is defined to be the smallest Henselian ring containing A. More precisely this means: The ring A^h is Henselian, there is a local homomorphism $i : A \to A^h$, and any other local homomorphism $\theta : A \to B$, with B Henselian, factors through i. Important examples of Henselian rings are complete local rings. Especially $k[[x]]$, the completion of $k[x]$ w.r.t. (x), is Henselian. But it is not the smallest Henselian ring containing $k[x]_{(x)}$. In fact, $k[x]^h_{(x)}$ equals $k\langle x\rangle$ the ring of *algebraic power series*, see [Art71]. Recall that a power series $f \in k[[x]]$ is called algebraic if there exists a non-zero polynomial $P(x, t) \in k[x, t]$ with $P(x, f) = 0$. The local ring of a scheme X at x w.r.t. the étale topology is the Henselization (w.r.t. $\mathfrak{m}_x$) of the local ring w.r.t. the Zariski-topology:

$$\mathcal{O}_{X,\bar{x}} = \mathcal{O}^h_{X,x}.$$

For further details, including proofs, see [Mil80].

Remark. Note that the étale neighbourhoods are not the open sets of a topology, but take their part in a *Grothendieck topology* (see for example [FGI$^+$05], I.2). Although it is not a "true" topology on X it is still enough to allow analogous constructions (like cohomology theories).

Properties of formal and étale neighbourhoods are strongly related via the powerful Artin Approximation Theorem, see Thm. 1.10 in [Art69]:

Theorem 17 (Artin Approximation Theorem)**.** *Let k be a field or an excellent discrete valuation ring, and let A^h be the Henselization of a k-algebra of finite type at a prime ideal. Let I be a proper ideal of A^h. Given an arbitrary system of polynomial equations in $Y = (Y_1, \ldots, Y_N)$,*

$$f(Y) = 0,$$

with coefficients in A^h, a solution $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_N)$ in the I-adic completion $\widehat{A}^h$ of A, and an integer c, there exists a solution $y = (y_1, \ldots, y_N) \in A^h$ with

$$y_i = \bar{y}_i \mod \mathfrak{m}^c.$$

4.2 Global Constructions

Let X be a finite union of algebraic varieties over k (i.e., integral separated schemes of finite type over k). All properties which will be studied here are local, hence we may assume that all varieties are affine. In fact, we will consider mainly subvarieties of some $\mathbb{A}^n_k$, $n \in \mathbb{N}$. We say that X *is algebraic normal crossings* (in short *anc*) *at a point* $p \in X$ if there are local coordinates $y_1, \ldots, y_n$ at p such that X is locally at p given by $y_1 \cdots y_e = 0$ with $e \leq n$ (in the literature this property is also referred to as *simple* or *strict normal crossings*, e.g. [Kol]). By *"local coordinates"* we mean a regular system of parameters for the local ring $\mathcal{O}_{\mathbb{A}^n,p}$. We say that X *is normal crossings* (in short *nc*) *at* p if p is an algebraic normal crossings point for $\widehat{X}_p$, i.e., if the formal germ of X at p is defined by $y_1 \cdots y_e = 0$, $e \leq n$, where $y_1, \ldots, y_n$ is a formal coordinate system at p. A *formal coordinate system* is a regular system of parameters for $\widehat{\mathcal{O}}_{\mathbb{A}^n,p}$. The locus of points in which X is algebraic normal crossings (resp. normal crossings) is called the *algebraic normal crossings locus of X (resp. normal crossings locus of X)* and will be denoted by X_{anc} (resp. X_{nc}).

Example 27. The hypersurface $X = \operatorname{Spec} k[x, y, z]/(x^2 - y^2 z^2) \subseteq \mathbb{A}^3$ is algebraic normal crossings, thus also normal crossings, at all points except the origin. The origin is not a normal crossings point. In contrast, the hypersurface $Y = \operatorname{Spec} k[x, y, z]/(x^2 - y^2 z)$ is irreducible, thus has no algebraic normal crossings points. Its normal crossings locus is $Y_{\mathrm{nc}} = Y \setminus \{0\}$ (see Figure 4.1), which is open in Y.

Let $f \in \mathcal{O}_{\mathbb{A}^n,p}$ be vanishing at $p \in \mathbb{A}^n$, i.e., $f \in \mathfrak{m}_p$ (where we write $\mathfrak{m}_p$ for the maximal ideal of $\mathcal{O}_{\mathbb{A}^n,p}$). Denote *the differential of f at* p, i.e., the class $\bar{f} \in \mathfrak{m}_p/\mathfrak{m}_p^2$, by $df(p)$.

Lemma 40. *Let X be a union of hypersurfaces $X_i = \operatorname{Spec} k[x]/(f_i)$, $1 \leq i \leq s$, $s \in \mathbb{N}$. Then X is anc at $p \in \mathbb{A}^n$ if and only if $p \notin (X_i)_{\mathrm{sing}}$ for all i and the $\{df_i(p); f_i(p) = 0\}$ are k-linearly independent.*

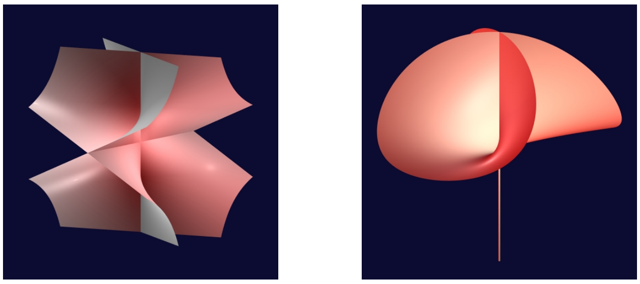


Figure 4.1: $X = V(x^2 - y^2z^2)$ and $Y = V(x^2 - y^2z)$.

Proof. The "if" part is obvious. For the other direction let X and p fulfill the conditions above. Since the $df_i(p)$ are k-linearly independent, the f_i vanishing at p are part of a regular system of parameters of $\mathcal{O}_{\mathbb{A}^n,p}$ (see [Mat89], Thm. 14.2) from which the assertion follows immediately (see also [Bod04]). $\square$

Proposition 41. *The algebraic normal crossings locus of a finite union of hypersurfaces is open.*

Proof. Let $X = \cup_{1\leq i\leq s} X_i$ where X_i is the hypersurface defined by $f_i \in k[x]$. If p lies in the intersection $X_i \cap X_j$ of two hypersurfaces and is not an anc point of $X_i \cup X_j$, then $p \in X_i \cap X_j \cap X_l$ is not an anc point of $X_i \cup X_j \cup X_l$ for any l. Define the following (Zariski-) closed subsets of X. For $q \geq 1$:

$$K_q = \cup_{(i_1,\ldots,i_q)} \left(X_{i_1} \cap \cdots \cap X_{i_q} \cap V(M_q(df_{i_1}, \ldots, df_{i_q}))\right),$$

where $M_q(df_{i_1}, \ldots, df_{i_q})$ denotes the ideal generated by all q-minors of the matrix $(df_{i_1}, \ldots, df_{i_q})$ and the union is taken over all q-tuples $(i_1, \ldots, i_q)$ with distinct entries $i_j \in \{1, \ldots, s\}$. Finally, set $K = \cup_{q=1}^{s} K_q$. Clearly K is closed and contains by Lemma 40 all non-anc points. Conversely, no point of K is an anc point. Thus $X_{\text{anc}} = X \setminus K$ is open. $\square$

Example 28. Figure 4.2 illustrates Proposition 41. Here X is the union of three plane curves.

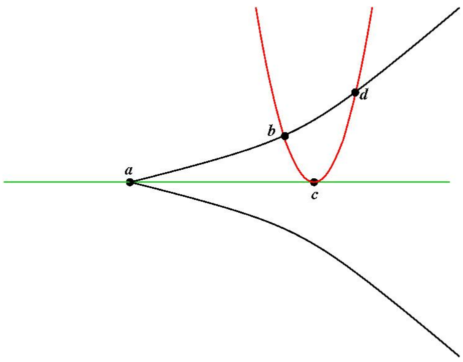


Figure 4.2: The points a and c are not anc, but b and d are.

4.3 Applications

In this section we give some applications of étale neighbourhoods. The prototype of questions we study will be the following: Let X be a variety, $p \in X$ a closed point. Moreover, let $\mathcal{P}$ be a property of the formal neighbourhood $\widehat{\mathcal{O}}_{X,p}$, e.g., normal crossings, reducible, ... Is the set of points $q \in X$ for which $\widehat{\mathcal{O}}_{X,q}$ has property $\mathcal{P}$ open (resp. closed or locally closed) in the Zariski-topology?

4.3.1 Normal Crossings Locus

As a first example we study the property of being normal crossings, see 4.2 and Theorem 18 below.

Theorem 18. *The normal crossings locus X_{nc} of a hypersurface $X \subseteq \mathbb{A}^n$ is open in X.*

Proof. (i) We first show that if $X = \operatorname{Spec} k[x]/(f)$ is normal crossings at p, then there exists an étale neighbourhood $\varphi\colon (U, u) \to \mathbb{A}^n$ of p such that u is an algebraic normal crossings point of $\varphi^{-1}(X)$. Without loss of generality we may assume $p = 0$. Since $p \in X_{\mathrm{nc}}$ there exist $\bar{g}_1, \ldots, \bar{g}_m \in k[[x]]$, $m \leq n$, building part of a regular system of parameters of $\widehat{\mathcal{O}}_{\mathbb{A}^n,0}$ such that

$$f = \bar{g}_1 \cdots \bar{g}_m.$$

By Theorem 17 there exists an étale neighbourhood $\varphi\colon (U, u) \to \mathbb{A}^n$ of $p \in \mathbb{A}^n$ with $\varphi(u) = p$,

$$\varphi^*(f) = g_1 \cdots g_m,$$

on U and $g_i = \bar{g}_i \mod (x)^c$. Note that $g_1, \ldots, g_m$ are regular on U. By choosing the constant c of Theorem 17 equal to 2 we can assure that the g_i are part of a regular system of parameters of $\mathcal{O}_{U,u}$. Thus $\varphi^{-1}(X)$ is algebraic normal crossings at u.

(ii) If $w \in U$ is an algebraic normal crossings or normal crossings point, then $\varphi(w)$ is a normal crossings point of X. This follows immediately from Proposition 39, (4).

(iii) By (i) every point $p \in X_{\mathrm{nc}}$ has an étale neighbourhood

$$\varphi_x\colon (U_x, u_x) \to X$$

so that $\varphi_x^{-1}(X)$ is anc at u_x. By Proposition 41 we see that $(U_x)_{\mathrm{anc}} \subseteq U_x$ is open. Openness of étale maps (see Proposition 39, (1)) implies that $\varphi_x((U_x)_{\mathrm{anc}}) \subseteq X$ is open, and so is

$$\bigcup_{x \in X} \varphi_x((U_x)_{\mathrm{anc}}) \subseteq X.$$

$\square$

Let $X = \operatorname{Spec} k[x]/(f)$ be a scheme defined by a not necessarily reduced polynomial $f \in k[x]$. Analogous to the normal crossings locus of X we ask for the *monomial locus X_{mon} of X*. This is the locus of points $p \in X$ so that there exist formal coordinates $y_1, \ldots, y_n$ with $f = y^\alpha$ for some element $\alpha \in \mathbb{N}^n$. Denote by

X_{red} the reduced scheme associated to X. Then $X_{mon} = (X_{red})_{\mathrm{nc}}$. Indeed, let $f_1, \ldots, f_s$ be the distinct irreducible factors of f in $\mathcal{O}_{\mathbb{A}^n,p}$ defining hypersurfaces X_i. By definition $\mathcal{O}_{X_{red},p}$ is reduced. Since it is essentially of finite type (see [Mat89], p. 232, 260), we conclude that $\widehat{\mathcal{O}}_{X_{red},p}$ is reduced. The same is true for $\widehat{\mathcal{O}}_{X_i,p}$. If $p \in X_{mon}$ then each irreducible component of $f_i \in \widehat{\mathcal{O}}_{\mathbb{A}^n,p}$ corresponds to one of the $y_1, \ldots, y_n$ after a change of coordinates. Therefore:

$$p \in X_{mon} \Leftrightarrow p \in \cap_i \left(X_i\right)_{\mathrm{nc}} \Leftrightarrow p \in \left(X_{red}\right)_{\mathrm{nc}}.$$

The last theorem thus implies:

Corollary 42. *Let $X = \operatorname{Spec} k[x]/(f)$ be a hypersurface in $\mathbb{A}^n$ defined by a not necessarily reduced polynomial $f \in k[x]$. Then the monomial locus X_{mon} of X is open in X.*

4.3.2 Mikado Schemes

Let X be an excellent scheme. Denote by $X_p^1, \ldots, X_p^N$, $N = N(p) \in \mathbb{N}$ the components of X passing through p. Then X is said to be *mikado at p* if $p \in (X_p^i)_{\mathrm{reg}}$ for all $1 \leq i \leq N$ and $p \in (Z_p)_{\mathrm{reg}}$, where $Z_p = X_p^1 \cap \cdots \cap X_p^N$. The locus of all mikado points of X will be denoted by X_{mik}.

Example 29. Let $X = \operatorname{Spec} k[x,y]/(y(y-x^2))$ and $Y = \operatorname{Spec} k[x,y]/(xy(y-x^2))$. Clearly, X is not mikado at 0, but Y is; neither of them is normal crossings at the origin.

In order to successfully apply the étale construction to consider the locus of "formally mikado points" in an irreducible scheme it will be necessary that X_{mik} is open in X. The next example gives a counterexample: $X \setminus X_{\mathrm{mik}}$ is locally closed but not closed. In this way we can construct examples of schemes/varieties which have only constructible X_{mik}, which is – in the algebraic category – the worst possible behaviour.

Example 30. Consider $X = \operatorname{Spec} k[x,y,z]/(yz(x-z)(y-x^2))$, which is a union of four hypersurfaces in $\mathbb{A}^3$. It's easy to see that $X \setminus X_{\mathrm{mik}} = V(x,y) \setminus V(x,y,z)$, which is locally closed in X but not open, see figure 4.3.

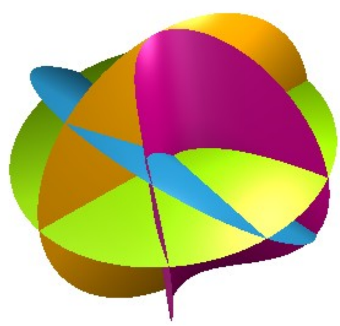


Figure 4.3: The set $X \setminus X_{\mathrm{mik}}$ is locally closed but not open.

4.3.3 Formally Irreducible

Let $X = \cup_{1\leq i\leq s} X_i$ be a union of algebraic varieties (not necessarily hypersurfaces). We say that X is *irreducible (resp. formally irreducible) at* $p \in X$ if $\mathcal{O}_{X,p}$ (resp. $\widehat{\mathcal{O}}_{X,p}$) is an integral domain. Otherwise X is called *reducible (resp. formally reducible) at* p. We denote the locus of irreducible resp. formally irreducible $p \in X$ by X_{ire} resp. X_{fire}. Analogously we denote by X_{re} the complement of X_{ire} in X.

It is natural to try to prove an analogous result for X_{fire} as for X_{nc} with the same method of proof. Let X be a finite union of algebraic varieties $X_1, \ldots, X_s$. Then $X_{\text{re}} \subseteq X$ is closed. This follows from the simple fact that

$$X_{\text{re}} = \cup_{i\neq j} \left(X_i \cap X_j\right).$$

Thus X_{ire} is open. Note that X_{ire} will in general contain points where one of the X_i is formally reducible. This causes troubles in step (iii) of the proof of Theorem 18. Indeed, using the notation there, the sets $\varphi_x((U_x)_{\text{ire}})$ are useless for our purpose (since they might contain formally reducible points).

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Appendix

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Abstract English

The main objective of this PhD thesis is the study of the linearization principle in algebraic and local analytic geometry.

The Rank Theorem for analytic maps between convergent power series spaces developed by Hauser and Müller is extended to a purely algebraic setting: A Rank Theorem for a new class of maps on power series spaces (over a field of characteristic zero), so-called textile maps, is proven. This class includes the classic case of substitutions of power series. Moreover, we consider a Rank Theorem with parameters and give a corresponding result for power series with coefficients in a test-ring (basically a local ring with nilpotent maximal ideal).

The developed theory is used to show that several important results from singularity theory are consequences or special instances of the linearization principle, i.e., the corresponding Rank Theorem. These include a trivialization theorem by Denef-Loeser, the Grinberg-Kazhdan-Drinfeld formal arc theorem, an inversion theorem by Lamel-Mir, Tougeron's implicit function theorem as well as the approximation theorems by Artin and Wavrik.

In the second part of the thesis approaches and ideas for the study of zero sets of textile maps, so-called felts, especially in the case of arc spaces, are given. This leads to the study of quotients of polynomial rings in countably many indeterminates.

Finally, some constructions using étale neighbourhoods are used to proof that the normal crossings locus of a variety over an algebraically closed field of arbitrary characteristic is open. The same question is treated for other geometric properties which have applications in resolution of singularities.

Abstract Deutsch

Die vorliegende Dissertation beschäftigt sich mit dem Linearisierungsprinzip in der algebraischen und lokal analytischen Geometrie.

Zunächst wird der von Hauser und Müller bewiesene Rangsatz für analytische Abbildungen zwischen Räumen konvergenter Potenzreihen auf eine rein algebraische Situation erweitert: Es wird der entsprechende Rangsatz für eine neue Klasse von Abbildungen zwischen Potenzreihenringen (über Körpern der Charakteristik Null), sogenannten textilen Abbildungen, bewiesen. Ein Spezialfall sind Abbildungen, die durch Substitution von Potenzreihen induziert werden. Weitere Varianten des Rangsatzes beinhalten eine parametrische Version sowie einen Rangsatz für Potenzreihen mit Koeffizienten in einem Testring (im Wesentlichen ein lokaler Ring mit nilpotentem maximalem Ideal).

Die entwickelte Theorie für textile Abbildungen wird verwendet, um zu zeigen, dass zahlreiche wichtige Resultate der Singularitätentheorie direkte Folgerungen oder Spezialfälle des Linearisierungsprinzips, d.h. des entsprechenden Rangsatzes, sind. Darunter fallen etwa ein Satz über die Trivialisierung von Jet Spaces, das Grinberg-Kazhdan-Drinfeld Formal Arc Theorem, ein Umkehrsatz von Lamel-Mir, Tougeron's Satz über Implizite Funktionen und die Approximationssätze von Artin und Wavrik.

Im zweiten Teil der Arbeit werden Ansätze und Ideen zum Studium der Nullstellenmengen von textilen Abbildungen, sogenannten Filzen, erörtert. Dies führt zu Untersuchungen von Polynomringen in abzählbar vielen Variablen und deren Quotienten. Wichtigstes Beispiel stellen hier die Arc Spaces dar.

Das letzte Kapitel beschäftigt sich mit der Menge aller Punkte einer Varietät, in denen sie normale Kreuzung ist. Durch Konstruktionen mit étalen Umgebungen wird gezeigt, dass diese Menge offen ist. Die gleiche Frage wird für andere geometrische Problemstellungen behandelt, die zum Teil Anwendungen in der Auflösung von Singularitäten haben.

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Curriculum Vitae

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Research Interests

Algebraic and Analytic Geometry, Interrelations with Commutative Algebra and Complex Analysis, especially CR-Geometry.

Personal Information

Date of Birth	May 7, 1981
Place of Birth	Salzburg, Austria
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Education

2008 – Present **PhD Student in Pure Mathematics**, *University of Vienna*, Department of Mathematics, Vienna, Austria.

2005 – 2008 **PhD Student in Pure Mathematics**, *University of Innsbruck*, Department of Mathematics, Innsbruck, Austria.

2000 – 2005 **MS in Technical Mathematics**, *University of Innsbruck*, Department of Mathematics, Innsbruck, Austria.
Graduated with high honours.

1999 – 2000 **Military Service**

1991 – 1999 **High school** education at the BRG Akademiestrasse Salzburg.
Matura with excellence in June 2000.

1987 – 1991 **Elementary school** in Salzburg

Employment History

2006 – Present **Research Assistant**, FWF-project P18992, Innsbruck and Vienna.

Fall 06/07 **Teaching Assistant**, University of Innsbruck, Excs. in "Math. for Physicists 1".

Fall 05/06 **Teaching Assistant**, University of Innsbruck, Excs. in "Algebra".

Oct. '04 – Dec '05 **Research grant** ("Forschungsbeihilfe") in the framework of the FWF project "Resolution of Singularities", *University of Innsbruck*.

Scientific Activities

2009 Participant in the conference FPSAC09 in Hagenberg, *Austria*
Research stay at the University of Versailles, *France*

2008 Research stay at the Tokyo University of Technology, *Japan*; Talk (at Nihon University, Tokyo): "Linearization Problems in Geometry".
Participation in the conference "On the resolution of singularities" in Kyoto, *Japan*.
Participant in the conference "Deformations of Surfaces" at the Alfréd Rényi Institute of Mathematics, Budapest, *Hungary*
Research stay at RISC Linz, *Austria*; Talk: "Basic Properties of Jet and Arc Algebras".
Coorganizer of the "Vienna Algebra Day" in Vienna, *Austria*.
Coorganizer of the "Vienna Geometry Day" in Vienna, *Austria*.
Participant in the "YMIS winter school – Arc spaces, Integration and Combinatorial Algebra" in Sedano, *Spain*

2007 Organizer of the "First α–ω–Conference in Algebraic Geometry" in Obergurgl, *Austria*.
Participant in the workshop "Algebraic Geometry" in Nové Hrady, *Czech Republic*; Talk: "Primary Decomposition of ideals of Jet Spaces".
Participant in the conference "Singularities, Computing and Visualization" in Segovia, *Spain*.
Participant in the "YMIS winter school – Algebra and Topology of Singularities" in Sedano, *Spain*.
Research stay at Tokyo Institute of Technology, *Japan*; Talk: "A Rank Theorem for Formal Power Series."

2006 Organizer of the workshop "Singularities" in Obergurgl, *Austria*.
Participant in the "International Congress of Mathematicians" in Madrid, *Spain*.
Participant in the "CIMPA summer school – new trends in singularity theory" in Madrid, *Spain*.
Participant in the summer school "Resolution of singularities" in Trieste, *Italy*.
Research stay in Madrid, *Spain*.
Participant in the "YMIS winter school – Combinatorial Convexity and Algebraic Geometry. Applications" in Sedano, *Spain*.
Research stay in Valladolid, *Spain*.

2005 Participant in the workshop "Algebraic geometry & Singularities" in Aschau, *Austria*; Talk: "The Rank Theorem for Analytic Power Series".
Participant at the winter school "Singularities" in Marseille, *France*.

Academic Honors and Awards

2006 – 2007 Research fellowship for writing my Ph.D. Thesis (**“Doktoratsstipendium”**) from the University of Innsbruck.

2006 and 2007 Scholarship for writing scientific papers (“**Förderungsstipendium”**) from the University of Innsbruck in the years 2006 and 2007.

2005 Prize Winner at the Week for **Industrial Modelling** in Innsbruck; Topic: Bindemitteloptimierung.

2002 – 2004 Scholarships for extraordinary achievements (“**Leistungsstipendium**”) from the University of Innsbruck in the years 2002, 2003 and 2004.

2002 – 2004 Scholarships of the **Julius-Raab-Stiftung** (Linz) in the years 2002, 2003 and 2004.

Publications and Preprints

Publications

[1] **C. Bruschek and C. Niederegger, M. Koppi, H.-P. Schroecker, D. Wagner**, *Verbesserung von Frisch- und Festbetoneigenschaften durch Minimierung der Haufwerksporosität von Bindemitteln mittels Approximation der Fuller-Kurve durch Mischen von Kornfraktionen*, Beton, January 2007.

Preprints

[2] **C. Bruschek and H. Hauser**, *Arcs, Cords and Felts – Six Instances of the Linearization Principle*, 2009, submitted.

[3] **C. Bruschek and D. Wagner**, *Basic Constructions in the Étale Topology*, 2009.

[4] **C. Bruschek and D. Wagner**, *Ansichtssache Algebra*, to appear in “Bildwelten des Wissens”, 2009.

[5] **C. Bruschek, S. Gann, H. Hauser, D. Wagner, D. Zeillinger**, *UFOs – Unidentified figurative objects, a geometric challenge*, arXiv:math/0512160v1, 2005, 1–14.