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The History of Partitions - from Euler's Beginnings to the Rogers-Ramanujan Identities

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Introduction

It was Gottfried Leibniz who first considered partitions of integers in a letter of 1674 to Jacob Bernoulli asking about the number of partitions of integers. Although he was the first one who mentioned partitions, the deeper theory of partitions began with another famous name in mathematics: Leonhard Euler. He became interested in partitions after receiving a letter from Philip Naudé in 1740. Euler was asked how many partitions there are of 50 into 7 distinct parts. He solved this question using generating functions and became interested in a more fundamental question: What is the total number of partitions of n? At this point the theory of partitions began. After Euler's work with generating functions, the next milestone was set by James Joseph Sylvester by representing partitions graphically. This was more than 100 years after Euler's starting point. Without a doubt the Rogers-Ramanujan identities are one of the most beautiful results in the history of partitions. They are called after the American mathematician Leonard James Rogers, who was hardly recognized from other mathematicians at this time, and the Indian genius Srinivasa Ramanujan. The story behind them is quite curious and without Godfrey Harold Hardy we perhaps would not know them today.

This thesis will be a journey through the history of partitions. Starting with the work of Euler and Sylvester, we go on with hypergeometric series and Jacobi's triple product identity and end up with the Rogers-Ramanujan identities and a generalization of them.

I would like to thank my supervisor Prof. Markus Fulmek, for encouraging me to discover the world of partitions, giving me the freedom to write about topics I am interested in and for answering my questions. Furthermore, I would like to thank my friends and colleagues, we spent a lot of time together, helped each other and had so much fun together. Without them it would have been much more difficult. Last, but not least, I would like to thank my family, who made it possible for me to study, although it took a while.

Contents

1 Partitions

Definition 1.1. A partition λ of a positive integer n is a finite non-increasing sequence of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_l)$, such that $\sum_{i=1}^{l} \lambda_i = n$. The λ_i are called the parts of λ and l is the number of parts or the length of the partition.
One can also use the notation $(1^{f_1}2^{f_2}3^{f_3}\ldots)$ for a partition λ, where f_j is the multiplicity of j in λ and only finitely many of them are non-zero. Therefore,
$\sum_{j\geq 1} f_j \cdot j = n$.
Moreover, we will denote the number of partitions of n by the function $p(n)$ and call it partition function. We define $p(0) := 1$ and the only partition that is either of a non-positive integer or has a non-positive number of parts is the empty partition of 0.

Example.

$p(1) = 1$	1	(1)
$p(2) = 2$	$2, 1+1$	$(2), (1,1)$
$p(3) = 3$	$3, 2+1, 1+1+1$	$(3), (2,1), (1,1,1)$
$p(4) = 5$	$4, 3+1, 2+2, 2+1+1, 1+1+1+1$	$(4), (3,1), (2,2), (2,1,1), (1,1,1,1)$

The partition function increases quite rapidly. For example $p(50) = 204226$ *and* $p(100) = 190569292$.

In the following let λ always be a partition of a positive integer n. We will also restrict partitions to some specific properties. For example, we might consider all parts of λ to be odd or distinct from each other. For these cases we want to adapt our partition function.

Definition 1.2. Let $\mathcal{P}$ be the set of all partitions. For a subset $P \subseteq \mathcal{P}$ we denote by $p(P, n)$ the number of partitions of n which belong to P.

Example. *Let* $\mathcal{O}$ *be the set of all partitions with odd parts and* $\mathcal{D}$ *the set of all partitions with distinct parts and take a look at the corresponding partition functions:*

$p(\mathcal{O},1)=1$	1	(1)
$p(\mathcal{O},2)=1$	$1+1$	$(1,1)$
$p(\mathcal{O},3)=2$	$3, 1+1+1$	$(3),(1,1,1)$
$p(\mathcal{O},4)=2$	$3+1, 1+1+1+1$	$(3,1),(1,1,1,1)$
$p(\mathcal{O},5)=3$	$5, 3+1+1, 1+1+1+1+1$	$(5),(3,1,1),(1,1,1,1,1)$

$p(\mathcal{D},1)=1$	1	(1)
$p(\mathcal{D},2)=1$	2	(2)
$p(\mathcal{D},3)=2$	$3, 2+1$	$(3),(2,1)$
$p(\mathcal{D},4)=2$	$4, 3+1$	$(4),(3,1)$
$p(\mathcal{D},5)=3$	$5, 4+1, 3+2$	$(5),(4,1),(3,2)$

We see that $p(\mathcal{O},n) = p(\mathcal{D},n)$ for $n \leq 5$. This is not by accident, we will proof the corresponding theorem, which shows that this is true for all positive integer n, soon.

1.1 Generating functions

We are now going into the world of generating functions for the different types of partitions. We begin with the generating function for $p(n)$. Therefore we consider the product

$$\frac{1}{1-q} \cdot \frac{1}{1-q^2}.$$

If we expand each factor of the product as a geometric series we get

$$(1 + q^1 + q^{1+1} + q^{1+1+1} + \dots) \cdot (1 + q^2 + q^{2+2} + q^{2+2+2} + \dots).$$

This is the same as

$$1 + q^1 + (q^2 + q^{1+1}) + (q^{2+1} + q^{1+1+1}) + (q^{2+2} + q^{2+1+1} + q^{1+1+1+1}) + \dots$$

What we see is, that all possible partitions employing only 1s and 2s as parts are generated in the exponent of the indeterminate q. This implies, that the coefficient of q^n is the number of partitions of n where only parts equal to one or two are allowed. For the same reason we get

$$\sum_{n=0}^{\infty} p_m(n) q^n = \prod_{n=1}^{m} \frac{1}{1-q^n}, \tag{1.1}$$

where $p_m(n)$ denotes the number of partitions of n into parts no greater than m. If we let $m \to \infty$ we obtain the generating function for the partition function $p(n)$:

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n} \tag{1.2}$$

As mentioned before, we now want to prove that the number of partitions of n into odd parts is the same as the number of partitions of n into distinct parts. This is a theorem by Euler, he proved it in 1748 in the same we as we do now.

Theorem 1.3 (Euler)**.** *$p(\mathcal{O}, n) = p(\mathcal{D}, n)$ for all positive integer n.*

Proof. First we want to know the generating functions of these objects.
If we take a look at (1.2), we can write the infinite product on the right hand side as follows:

$$\begin{aligned}&\left(1+q^1+q^{1+1}+q^{1+1+1}+\dots\right)\cdot\left(1+q^2+q^{2+2}+q^{2+2+2}+\dots\right)\\ &\cdot\left(1+q^3+q^{3+3}+q^{3+3+3}+\dots\right)\cdot\left(1+q^4+q^{4+4}+q^{4+4+4}+\dots\right)\cdots\end{aligned} \tag{1.3}$$

If we restrict the infinite sums to their first two parts, we get

$$\left(1+q^1\right)\cdot\left(1+q^2\right)\cdot\left(1+q^3\right)\cdot\left(1+q^4\right)\cdots = \prod_{n=1}^{\infty}\left(1+q^n\right), \tag{1.4}$$

where we see, that each positive integer only appears once and therefore this is the generating function for partitions of n into distinct parts.
We now do the same, but we cancel every other infinite sum of (1.3) and get

$$\begin{aligned}&\left(1+q^1+q^{1+1}+q^{1+1+1}+\dots\right)\cdot\left(1+q^3+q^{3+3}+q^{3+3+3}+\dots\right)\cdot\\ &\qquad\cdot\left(1+q^5+q^{5+5}+q^{5+5+5}+\dots\right)\cdots = \prod_{n=1}^{\infty}\frac{1}{1-q^{2n-1}},\end{aligned}$$

where we see, that only those parts with odd positive integers remain and therefore this is the generating function for partitions of n into odd parts.

Now we want to show that $\sum\limits_{n\geq 0} p(\mathcal{D}, n)q^n = \sum\limits_{n\geq 0} p(\mathcal{O}, n)q^n$:

$$\begin{aligned}\sum_{n\geq 0} p(\mathcal{D}, n)q^n &= \prod_{n\geq 1}\left(1+q^n\right)\\ &= \prod_{n\geq 1}\left(1+q^n\right)\frac{(1-q^n)}{(1-q^n)}\\ &= \prod_{n\geq 1}\frac{1-q^{2n}}{1-q^n}\\ &= \prod_{n\geq 1}\frac{1-q^{2n}}{\left(1-q^{2n-1}\right)\left(1-q^{2n}\right)}\\ &= \prod_{n\geq 1}\frac{1}{1-q^{2n-1}}\\ &= \sum_{n\geq 0} p(\mathcal{O}, n)q^n\end{aligned}$$

□

This proof shows perfectly an elegant method of proving partition identities. Every time we can find an equality of infinite products or sums and interpret them as a generating function for certain partitions, we have a partition identity. For example:

$$\begin{aligned}
\prod_{n=0}^{\infty}\left(1+q^{3n+1}\right)\left(1+q^{3n+2}\right) &= \prod_{n=0}^{\infty}\frac{\left(1-q^{6n+2}\right)\left(1-q^{6n+4}\right)}{\left(1-q^{3n+1}\right)\left(1-q^{3n+2}\right)} \\
&= \prod_{n=0}^{\infty}\frac{\left(1-q^{6n+2}\right)\left(1-q^{6n+4}\right)}{\left(1-q^{6n+1}\right)\left(1-q^{6n+2}\right)\left(1-q^{6n+4}\right)\left(1-q^{6n+5}\right)} \\
&= \prod_{n=0}^{\infty}\frac{1}{\left(1-q^{6n+1}\right)\left(1-q^{6n+5}\right)}
\end{aligned}$$

Next we observe that

$$\begin{aligned}
\prod_{n=0}^{\infty}\left(1+q^{2n+1}+q^{(2n+1)+(2n+1)}\right) &= \prod_{n=0}^{\infty}\left(1+q^{2n+1}+q^{4n+2}\right) \\
&= \prod_{n=0}^{\infty}\left(1+q^{2n+1}+q^{4n+2}\right)\frac{1-q^{2n+1}}{1-q^{2n+1}} \\
&= \prod_{n=0}^{\infty}\frac{1-q^{6n+3}}{1-q^{2n+1}} \\
&= \prod_{n=0}^{\infty}\frac{1-q^{6n+3}}{\left(1-q^{6n+1}\right)\left(1-q^{6n+3}\right)\left(1-q^{6n+5}\right)} \\
&= \prod_{n=0}^{\infty}\frac{1}{\left(1-q^{6n+1}\right)\left(1-q^{6n+5}\right)}
\end{aligned}$$

So we get

$$\begin{aligned}
\prod_{n=0}^{\infty}\left(1+q^{3n+1}\right)\left(1+q^{3n+2}\right) &= \prod_{n=0}^{\infty}\left(1+q^{2n+1}+q^{(2n+1)+(2n+1)}\right) \\
&= \prod_{n=0}^{\infty}\frac{1}{\left(1-q^{6n+1}\right)\left(1-q^{6n+5}\right)}
\end{aligned}$$

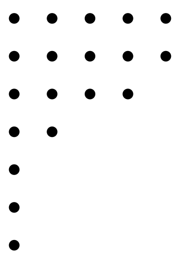
Using the method of Euler's proof, we have proven the following theorem:

Theorem 1.4. *The number of partitions of n into distinct non-multiples of 3 equals the number of partitions of n into odd parts, where no part appears more than twice and this is the same as the number of partitions of n into parts congruent to 1 or 5 modulo 6.*

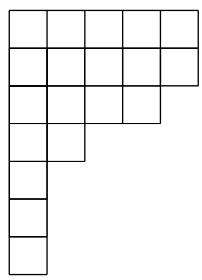
1.2 Graphical representation of partitions

The next big achievement in the theory of partitions was reached more than 100 years after Euler's beginnings. It was in the 1880s, when James Joseph Sylvester worked with some of his students on a new approach to visualize partitions. The result was the so-called "Ferrers diagram", named after Norman Macleod Ferrers.
The idea is to represent a partition λ of n of length l in a Ferrers diagram with a total of n dots in l left-justified rows where the number of dots in row i is λ_i.

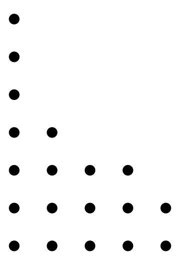
Example. *Consider the partition (5,5,4,2,1,1,1). The corresponding Ferrers diagram looks as follows:*



There are several ways to represent a partition graphically. For example one can draw unit squares instead of dots:



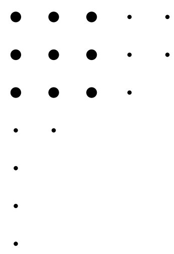
Another way is to turn the diagram upside down:



In this thesis we will only use the first notation.

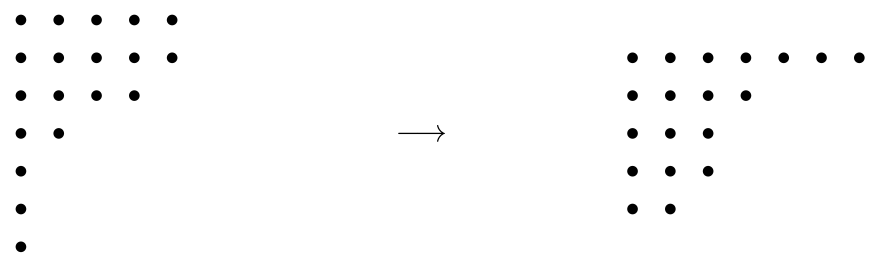
Definition 1.5. The Durfee square of a partition λ is the largest square of dots in the upper left corner of the Ferrers diagram of λ. We denote the side length of the Durfee square by $d(\lambda)$, which is given by the number of parts λ_i such that $\lambda_i \geq i$.

Example. *For λ=(5,5,4,2,1,1,1), $d(\lambda)$=3.*



Definition 1.6. The conjugate of a partition λ is denoted by $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_m)$, where $m = \lambda_1$ and λ'_i is the number of parts of λ that are $\geq i$.
In terms of Ferrers diagrams, the conjugate is the partition obtained from λ by reflecting its Ferrers diagram along its main diagonal. (This is the same operation as the transposition of a matrix.)
Note, that λ and λ' are two partitions of the same integer.

Example. *To understand the formal definition we will give an example for the conjugate of (5,5,4,2,1,1,1) which is (7,4,3,3,2).*



Definition 1.7. A partition λ is called self-conjugate if $\lambda' = \lambda$.

We see that the number of rows of λ is the number of its parts and this is the same as the number of columns of λ' and therefore this is the largest part of λ'. The number of columns of λ is the largest part of λ and this is the same as the number of rows of λ' and therefore this is the number of parts of λ'. Moreover, the conjugate of the conjugate of λ is again λ.

1.3 Some partition identities

Theorem 1.8. *The number of partitions of n with m parts equals the number of partitions of n with largest part m.*

Proof. Mapping a partition to its conjugate is obviously a bijection between the two classes of partitions. □

Remark 1.9. For the same reasons we can rewrite the theorem by writing 'at most' in front of m. So the number of partitions of n with at most m parts is the same as the number of partitions of n with largest part at most m, which we denoted in (1.1) by $p_m(n)$.

Definition 1.10. For a shorter notation of a few equations we define the so-called q-Pochhammer symbol or q-shifted factorial:

$$(a)_n = (a;q)_n \coloneqq (1-a)(1-aq)\cdots(1-aq^{n-1})$$
$$(a)_\infty = (a;q)_\infty \coloneqq \lim_{n\to\infty}(a;q)_n$$
$$(a)_0 \coloneqq 1.$$

Using Theorem 1.8 and Remark 1.9 we can prove the following identity:

Theorem 1.11.

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty}\frac{1}{(1-q^n)} = \sum_{m=0}^{\infty}\frac{q^m}{(q;q)_m} \tag{1.5}$$

Proof. As already mentioned in Remark 1.9, we know that $p_m(n)$ also denotes the number of partitions of n into at most m parts. The corresponding generating function is

$$\sum_{n=0}^{\infty} p_m(n)q^n = \frac{1}{(1-q)\cdots(1-q^m)}.$$

So $p_m(n) - p_{m-1}(n)$ denotes the number of partitions of n into exactly m parts, which we denote by $p(m,n)$. Let us take a look at the corresponding generating functions:

$$\frac{1}{(1-q)\cdots(1-q^m)} - \frac{1}{(1-q)\cdots(1-q^{m-1})} = \frac{1-(1-q^m)}{(1-q)\cdots(1-q^m)} = \frac{q^m}{(1-q)\cdots(1-q^m)}$$

So we proved

$$\sum_{n=0}^{\infty} p(m,n)q^n = \frac{q^m}{(1-q)\cdots(1-q^m)} = \frac{q^m}{(q;q)_m}.$$

If we now sum over the length, we get

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty}\frac{1}{(1-q^n)} = \sum_{m=0}^{\infty}\frac{q^m}{(q;q)_m}.$$

□

The next theorem shows another identity including the partition generating function.

Theorem 1.12.

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty}\frac{1}{(1-q^n)} = \sum_{m=0}^{\infty}\frac{q^{m^2}}{(q;q)_m^2} \tag{1.6}$$

Proof. We take an arbitrary partition of n with a Durfee square of size m. The square dissects the Ferrers diagram into three parts, the square, a partition into at most m parts to the right of it and a partition with largest part at most m below it. For these two parts we know that the generating function is

$$\frac{1}{(1-q)\cdots(1-q^m)}.$$

So the number of partitions of n with Durfee square of size m is the coefficient of q^{n-m^2} in

$$\left(\frac{1}{(1-q)\cdots(1-q^m)}\right)^2,$$

which is the same as the coefficient of q^n in

$$\frac{q^{m^2}}{(1-q)^2\cdots(1-q^m)^2} = \frac{q^{m^2}}{(q;q)_m^2}.$$

If we sum over all sizes of the Durfee square we get the generating function for all partitions and so

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)} = \sum_{m=0}^{\infty} \frac{q^{m^2}}{(q;q)_m^2}.$$

□

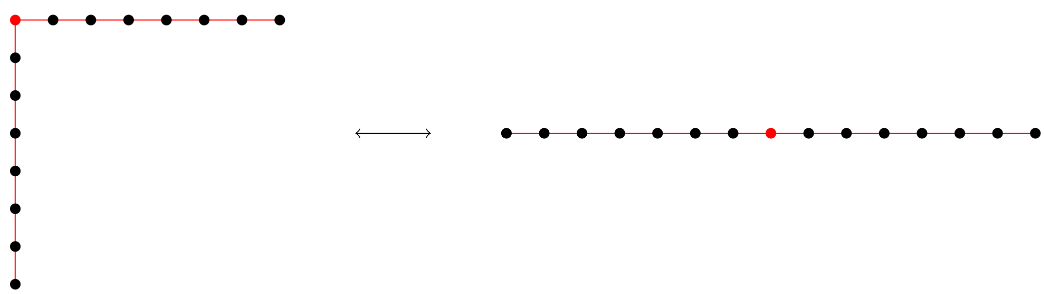
Theorem 1.13. *The number of partitions of n into odd and distinct parts is equal to the number of self-conjugate partitions of n. If we take a look at the generating functions of these subjects, this means that*

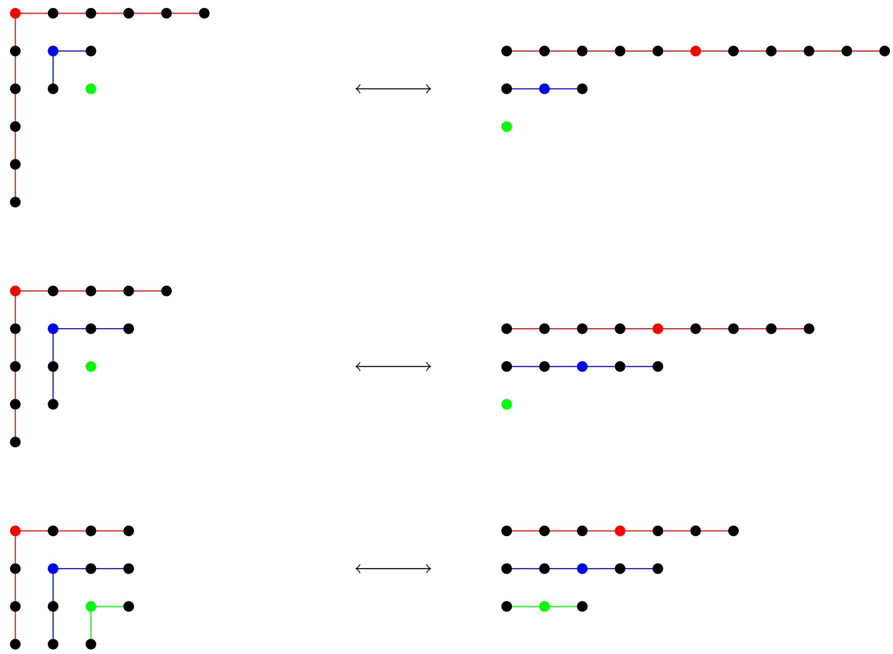
$$\prod_{n=1}^{\infty}(1+q^{2n-1}) = \sum_{m=0}^{\infty} \frac{q^{m^2}}{(q^2;q^2)_m}.$$

Proof. We see from (1.4), that the generating function for partitions into odd and distinct parts is

$$\prod_{n=1}^{\infty}(1+q^{2n-1}).$$

Moreover, there exists a bijection between partitions into odd and distinct parts and self-conjugate partitions. Consider this in an example for $n = 15$. $15 = 11+3+1 = 9+5+1 = 7+5+3$ are the 4 partitions of 15 into odd and distinct parts and $8+1+1+1+1+1+1+1 = 6+3+3+1+1+1 = 5+4+3+2+1 = 4+4+4+3$ are the 4 self-conjugate partitions. A self-conjugate partition, can be transformed into a partition into odd and distinct parts as follows: take the corner of the first line of its Ferrers diagram and consider it as the middle point by adding everything below this point to the left and continue this procedure for all corner points. In the other direction, we take the dot in the middle of each part and see it as the corner of the bent line. It is easy to see what is going on graphically.





We are done, if we can show that

$$\sum_{m=0}^{\infty} \frac{q^{m^2}}{(q^2;q^2)_m}$$

is the generating function for self-conjugate partitions.
A self-conjugate partition of n can be dissected into its Durfee square of say m^2 dots and two partitions of $\frac{1}{2}(n-m^2)$, each of them is the conjugate of the other. The partition to the right of the Durfee square is one with at most m parts, where the one below the Durfee square is a partition with largest part at most m.
By Remark 1.9, we know that the numbers of those types of partitions are equal. So what we want is the coefficient of $q^{\frac{1}{2}(n-m^2)}$ in the generating function for partitions with at most m parts, which is

$$\frac{1}{(1-q)(1-q^2)\cdots(1-q^m)},$$

but this is the same as the coefficient of q^n in the expression

$$\frac{q^{m^2}}{(q^2;q^2)_m}.$$

Summing over all m gives us the generating function for the self-conjugate partitions of n

$$\sum_{m=0}^{\infty} \frac{q^{m^2}}{(q^2;q^2)_m}.$$

$\square$

2 A famous theorem of Euler

2.1 Euler's pentagonal number theorem

Let us proceed with a very famous and important theorem of Euler. He was interested in the reciprocal of the generating function for partitions, which is

$$\prod_{n=1}^{\infty}(1-q^n).$$

Expanding this product by direct multiplication, one gets:

$$\prod_{n=1}^{\infty}(1-q^n) = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + q^{26} - ..., \tag{2.1}$$

which led Euler to the conjecture of the famous pentagonal number theorem, which he proved a few years later.

Theorem 2.1.

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{1}{2}n(3n-1)}.$$

For the original proof of Euler the interested reader is referenced to [4]. We will give the combinatorial proof of Fabian Franklin, who earned his Ph.D. under J.J. Sylvester in 1880.
The pentagonal numbers are given by $\frac{n(3n-1)}{2}$ for $n \geq 1$. The n-th pentagonal number counts the dots of a pentagon, where each side consists of n dots. If we let $n = 0, +1, -1, +2, -2, +3, -3, ...$ we get the series 0,1,2,5,7,12,15,..., which are the generalized pentagonal numbers. We see that these are the exponents of q in (2.1). To connect equation (2.1) with partitions, let us again multiply the terms of the left hand side of (2.1) step by step.

$(1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^5)\cdots =$
$1 - q - q^2 - q^3 + q^{1+2} - q^4 + q^{1+3} - q^5 + q^{1+4} + q^{2+3} - q^6 - q^{1+2+3} + q^{2+4} + q^{1+5} + \cdots$

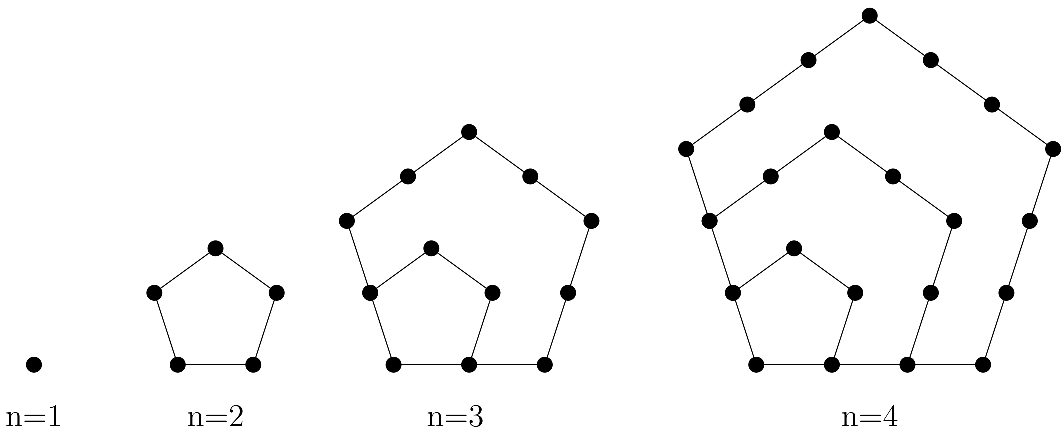


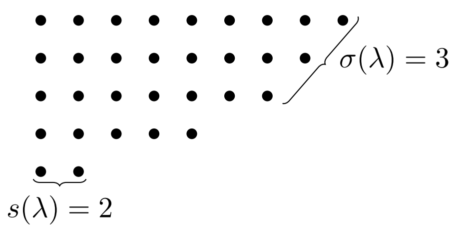
Figure 2.1: Pentagonal numbers for n=1,...,4

The following observation is due to Legendre. We see that in the exponent of q we have partitions into distinct parts, the coefficient of q^n is +1 if the number of parts of the partition of n is even and -1 if the number is odd. So if the number of partitions of n into distinct parts of even number is the same as the number of partitions of n into distinct parts of odd number, there is no q^n left and as we see in (2.1) the only exponents of q which remains are generalized pentagonal numbers. So we want to prove the following theorem.

Theorem 2.2. *Denote by $p_e(\mathcal{D}, n)$, respectively $p_o(\mathcal{D}, n)$, the number of partitions of n with an even, respectively odd, number of distinct parts. Then*

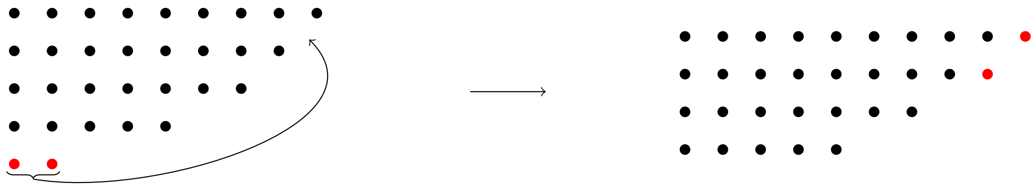
$$p_e(\mathcal{D}, n) - p_o(\mathcal{D}, n) = \begin{cases} (-1)^m & \textit{if } n = \frac{1}{2}m(3m \pm 1) \textit{ for some } m \in \mathbb{N} \\ 0 & \textit{otherwise.} \end{cases}$$

Proof (Franklin 1881). Let $\lambda = (\lambda_1, ..., \lambda_l)$ be a partition with l distinct parts. We define $s(\lambda) := \lambda_l$ and $\sigma(\lambda)$ to be the length of the largest sequence of consecutive integers appearing in λ beginning with λ_1, or the maximal j such that $\lambda_j = \lambda_1 - j + 1$. Let us describe this definitions graphically for $\lambda = (9, 8, 7, 5, 2)$



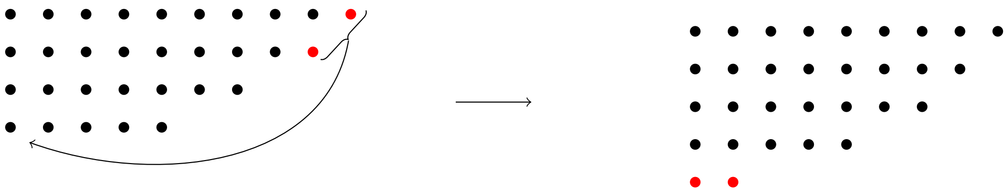
Our goal is finding a one-to-one correspondence between the number of partitions into distinct parts with odd and even number, but as stated in the theorem, this will not be possible for all partitions.
With these notations we distinguish four cases and transform the partitions as follows:

Case 1: $s(\lambda) \leq \sigma(\lambda)$
In this case we add 1 to the first $s(\lambda)$ parts and delete the smallest part. Graphically we move the bottom row in the Ferrers diagram to its rightmost diagonal. For example, if λ is like before we do the following:



(9,8,7,5,2) (10,9,7,5)

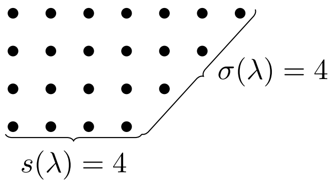
Case 2: $s(\lambda) > \sigma(\lambda)$
In this case we subtract 1 of the first $\sigma(\lambda)$ parts and add a new smallest part of the size $\sigma(\lambda)$. Graphically we move the first $\sigma(\lambda)$ diagonal dots to the bottom of the Ferrers diagram.



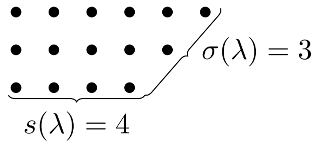
(10,9,7,5) (9,8,7,5,2)

We see that in both cases this procedure changes the parity of the number of parts. If we do the transformation twice, we come back to the original partition, so it looks like we found a bijection and $p_e(\mathcal{D}, n) - p_o(\mathcal{D}, n)$ is almost always zero, but there are two cases left where neither case 1 nor case 2 is applicable, though the conditions are satisfied.

Exceptional case 1: $s(\lambda) = \sigma(\lambda) = l$
In this case we cannot do the transformation required in case 1, because this would not give the Ferrers diagram of a partition. The number being partitioned is $l + (l+1) + ... + (2l-1) = \frac{1}{2}l(3l-1)$.



Exceptional case 1



Exceptional case 2

Exceptional case 2: $s(\lambda) = l + 1, \sigma(\lambda) = l$
In this case the procedure of case 2 would not give a partition into distinct parts. The number being partitioned is $(l + 1) + (l + 2) + ... + 2l = \frac{1}{2}l(3l + 1)$.
So we see, if n is not a generalized pentagonal number, $p_e(\mathcal{D}, n) = p_o(\mathcal{D}, n)$, otherwise $p_e(\mathcal{D}, n) - p_o(\mathcal{D}, n) = (-1)^l$ for $n = \frac{1}{2}l(3l \pm 1)$.

□

Using Theorem 2.2, we can now easily prove Euler's pentagonal number theorem.

Proof of Theorem 2.1. Recall that we want to prove:

$$\prod_{n=1}^{\infty}(1 - q^n) = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{1}{2}n(3n-1)}.$$

We start with the right-hand side of the equation:

$$\begin{aligned}
\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{1}{2}n(3n-1)} &= 1 + \sum_{n=1}^{\infty}(-1)^n q^{\frac{1}{2}n(3n-1)} + \sum_{n=-1}^{-\infty}(-1)^n q^{\frac{1}{2}n(3n-1)} \\
&= 1 + \sum_{n=1}^{\infty}(-1)^n q^{\frac{1}{2}n(3n-1)} + \sum_{n=1}^{\infty}(-1)^n q^{\frac{1}{2}n(3n+1)} \\
&= 1 + \sum_{n=1}^{\infty}(-1)^n q^{\frac{1}{2}n(3n-1)}(1 + q^n) \\
&= 1 + \sum_{n=1}^{\infty}(p_e(\mathcal{D}, n) - p_o(\mathcal{D}, n))q^n.
\end{aligned}$$

We see that the exponents of q are only generalized pentagonal numbers and therefore the last equation holds because of Theorem 2.2. It remains to show that

$$\prod_{n=1}^{\infty}(1 - q^n) = 1 + \sum_{n=1}^{\infty}(p_e(\mathcal{D}, n) - p_o(\mathcal{D}, n))q^n.$$

Since in the product the exponents of q are all partitions with distinct parts and the coefficients are +1 whenever the partition has an even number of parts and -1 whenever the partition has an odd number of parts, we conclude:

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{a_1=0}^{1}\sum_{a_2=0}^{1}\sum_{a_3=0}^{1}\cdots(-1)^{a_1+a_2+a_3+\cdots}q^{a_1\cdot 1+a_2\cdot 2+a_3\cdot 3+\cdots}$$
$$= 1+\sum_{n=1}^{\infty}(p_e(\mathcal{D},n)-p_o(\mathcal{D},n))q^n$$

$\square$

After the proof of Euler's pentagonal number theorem, I want to present a theorem of N.J. Fine, which he stated without a proof in [10], in a slightly different notation as Fine did. It was found more than 118 years after Legendre's observation.

Theorem 2.3. *Let $p_E(\mathcal{D},n)$, respectively $p_O(\mathcal{D},n)$, denote the number of partitions of n with distinct parts and the largest part is even, respectively odd. Then*

$$p_E(\mathcal{D},n)-p_O(\mathcal{D},n) = \begin{cases} 1 & \text{if } n=\frac{1}{2}m(3m+1) \text{ for some } m\in\mathbb{N} \\ -1 & \text{if } n=\frac{1}{2}m(3m-1) \text{ for some } m\in\mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

In [10, p. 617] Fine remarked:

"[The theorem] bears some resemblance to the famous pentagonal number theorem of Euler, but we have not been able to establish any real connection between the two theorems."

Let us take a look at an example for Legendre's observation and Fine's theorem:

Example. *The partitions of $n=12$ with distinct parts are:*
12, 11+1, 10+2, 9+3, 9+2+1, 8+4, 8+3+1, 7+5, 7+4+1, 7+3+2, 6+5+1, 6+4+2, 6+3+2+1, 5+4+3, 5+4+2+1. Let us write the different numbers in a table:

$p_e(\mathcal{D},12)$
11+1
10+2
9+3
8+4
7+5
6+3+2+1
5+4+2+1

$p_o(\mathcal{D},12)$
12
9+2+1
8+3+2
7+4+1
7+3+2
6+5+1
6+4+2
5+4+3

$p_E(\mathcal{D},12)$
12
10+2
8+4
8+3+1
6+5+1
6+4+2
6+3+2+1

$p_O(\mathcal{D},12)$
11+1
9+3
9+2+1
7+5
7+4+1
7+3+2
5+4+3
5+4+2+1

So we get: $p_e(\mathcal{D},12)-p_o(\mathcal{D},12)=7-8=-1$ *and* $p_E(\mathcal{D},12)-p_O(\mathcal{D},12)=7-8=1$ *as it should be by our theorems. To point out the difference between the two theorems we take a look at $n=15$:*
$p_e(\mathcal{D},15)-p_o(\mathcal{D},15)=13-14=-1$ *and* $p_E(\mathcal{D},15)-p_O(\mathcal{D},15)=14-13=1.$

2.2 Euler's recurrence formula

Using the pentagonal number theorem and the partition generating function, it is possible to get an efficient algorithm for computing values of $p(n)$.

Theorem 2.4. *For $n > 0$,*

$$\begin{aligned} p(n) =& p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) + ... \\ &+ (-1)^{m-1} p\left(n - \frac{1}{2}m(3m-1)\right) + (-1)^{m-1} p\left(n - \frac{1}{2}m(3m+1)\right) + ... \end{aligned}$$

Proof. From the fact that

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}$$

and

$$\sum_{m=-\infty}^{\infty} (-1)^m q^{\frac{1}{2}m(3m-1)} = \prod_{n=1}^{\infty} (1-q^n),$$

we get

$$\left(\sum_{n=0}^{\infty} p(n)q^n\right)\left(\sum_{m=-\infty}^{\infty} (-1)^m q^{\frac{1}{2}m(3m-1)}\right) = 1.$$

Thus,

$$\left(\sum_{n=0}^{\infty} p(n)q^n\right)(1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + ...) = 1,$$

or by multiplying,

$$\sum_{n=0}^{\infty} p(n)q^n - \sum_{n=0}^{\infty} p(n)q^{n+1} - \sum_{n=0}^{\infty} p(n)q^{n+2} + \sum_{n=0}^{\infty} p(n)q^{n+5} + \sum_{n=0}^{\infty} p(n)q^{n+7} - ... = 1.$$

If we now shift the indices and recall that $p(n) = 0$ for $n < 0$, we get:

$$\sum_{n=0}^{\infty} (p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) - ...)\, q^n = 1$$

Comparing the coefficients of q^n gives the result. □

3 Generating functions in two variables

After the first results of some generating functions, we extend this concept by another variable. Let us consider $p(P, m, n)$, the number of partitions of n into exactly m parts that lie in a subset $P \subseteq \mathcal{P}$. This leads us to the two-variable generating function

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} p(P, m, n) z^m q^n = \sum_{\lambda \in P} z^{l(\lambda)} q^{s(\lambda)},$$

where $\lambda = (\lambda_1, ..., \lambda_r)$, $l(\lambda) = r$ denotes the length of the partition and $s(\lambda) = \lambda_1 + ... + \lambda_r$. Moreover, we prove many corollaries to prepare the proof of the famous triple product identity of Jacobi. Finally, we will present two proofs of a theorem from Sylvester. Let us start with the generating function for partitions of n into m parts.

Theorem 3.1.

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} p(\mathcal{P}, m, n) z^m q^n = \prod_{n=1}^{\infty} \frac{1}{(1 - zq^n)} \tag{3.1}$$

Proof. We can write the product as follows:

$$\begin{aligned}
&\frac{1}{1-zq} \cdot \frac{1}{1-zq^2} \cdot \frac{1}{1-zq^3} \cdot \frac{1}{1-zq^4} \cdots = \\
&(1 + zq + z^2q^2 + z^3q^3 + ...) \cdot (1 + zq^2 + z^2q^4 + z^2q^6 + ...) \cdot \\
&(1 + zq^3 + z^2q^6 + z^3q^9 + ...) \cdot (1 + zq^4 + z^2q^8 + z^3q^{12} + ...) \cdots = \\
=&(1 + zq + z^2q^{1+1} + z^3q^{1+1+1} + ...)(1 + zq^2 + z^2q^{2+2} + z^3q^{2+2+2} + ...) \cdot \\
&(1 + zq^3 + z^2q^{3+3} + z^3q^{3+3+3} + ...)(1 + zq^4 + z^2q^{4+4} + z^3q^{4+4+4} + ...) \cdots
\end{aligned}$$

We see that this is the required generating function. □

3.1 Basic hypergeometric series

In this section there will be many identities including basic hypergeometric series and many identities related to partitions, more or less one corollary after the other. We start with the q-binomial theorem, which is an example for a basic hypergeometric series.

Theorem 3.2 (q-binomial series)**.** *For* $|q| < 1$, $|z| < 1$,

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \prod_{n=0}^{\infty} \frac{(1-azq^n)}{(1-zq^n)} = \frac{(az;q)_\infty}{(z;q)_\infty}. \tag{3.2}$$

Proof. Let

$$F(z) := \frac{(az;q)_\infty}{(z;q)_\infty} = \sum_{n=0}^{\infty} A_n z^n.$$

We see that

$$\begin{aligned}(1-z)F(z) &= (1-az) \cdot \prod_{n=1}^{\infty} \frac{(1-azq^n)}{(1-zq^n)} \\ &= (1-az) \cdot \prod_{n=0}^{\infty} \frac{(1-azq^{n+1})}{(1-zq^{n+1})} \\ &= (1-az)F(zq)\end{aligned}$$

Therefore we get:

$$(1-z)\sum_{n=0}^{\infty} A_n z^n = (1-az)\sum_{n=0}^{\infty} A_n q^n z^n$$
$$\sum_{n=0}^{\infty} A_n z^n - \sum_{n=1}^{\infty} A_{n-1} z^n = \sum_{n=0}^{\infty} A_n q^n z^n - \sum_{n=1}^{\infty} A_{n-1} a q^{n-1} z^n$$

By comparing coefficients of z^n, we get:

$$A_n - A_{n-1} = q^n A_n - aq^{n-1} A_{n-1},$$

being equivalent to

$$A_n = \frac{(1-aq^{n-1})}{(1-q^n)} A_{n-1}.$$

Iterating this gives us

$$A_n = \frac{(1-aq^{n-1})(1-aq^{n-2})\cdots(1-a)}{(1-q^n)(1-q^{n-1})\cdots(1-q)} A_0.$$

Starting with $A_0 = 1$, we get

$$A_n = \frac{(a;q)_n}{(q;q)_n},$$

which proves the theorem. □

Let us consider two special cases of this theorem, which will be useful later.

Corollary 3.3. *For $|q| < 1$, $|z| < 1$,*

$$\sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = \prod_{n=0}^{\infty} \frac{1}{(1-zq^n)}, \tag{3.3}$$

$$\sum_{n=0}^{\infty} \frac{z^n q^{\frac{n(n-1)}{2}}}{(q;q)_n} = \prod_{n=0}^{\infty} (1+zq^n). \tag{3.4}$$

Proof. We obtain (3.3) by setting $a = 0$ in equation (3.2).
For equation (3.4) we set $z = -\frac{z}{a}$ in (3.2) to get

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} \cdot \left(-\frac{z}{a}\right)^n = \frac{(-z;q)_\infty}{\left(-\frac{z}{a};q\right)_\infty}$$

$$\sum_{n=0}^{\infty} \frac{\left(-\frac{z}{a}+z\right)\left(-\frac{z}{a}+zq\right)\cdots\left(-\frac{z}{a}+zq^{n-1}\right)}{(q;q)_n} = \frac{(-z;q)_\infty}{\left(-\frac{z}{a};q\right)_\infty}.$$

For $a \to \infty$ we get

$$\sum_{n=0}^{\infty} \frac{z \cdot zq \cdots zq^{n-1}}{(q;q)_n} = \sum_{n=0}^{\infty} \frac{z^n q^{\frac{n(n-1)}{2}}}{(q;q)_n} = (-z;q)_\infty = \prod_{n=0}^{\infty} (1+zq^n).$$

□

Remark 3.4. If we set $z = q$ in equation (3.3) we get

$$\sum_{n=0}^{\infty} \frac{q^n}{(q;q)_n} = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)},$$

which is exactly the same as equation (1.5).

Now we take a look at these two identities, when we set $z = zq$. Let us start with equation (3.3):

$$\sum_{n=0}^{\infty} \frac{z^n q^n}{(q;q)_n} = \prod_{n=0}^{\infty} \frac{1}{(1-zq^{n+1})} = \prod_{n=1}^{\infty} \frac{1}{(1-zq^n)},$$

where the last product is the same as in Theorem 3.1. So what we proved is

$$\sum_{n=0}^{\infty} \frac{z^n q^n}{(q;q)_n} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} p(\mathcal{P}, m, n) z^m q^n = \prod_{n=1}^{\infty} \frac{1}{(1-zq^n)}$$

Now we do the same for equation (3.4) and get

$$\sum_{n=0}^{\infty} \frac{z^n q^{\frac{n(n+1)}{2}}}{(q;q)_n} = \prod_{n=0}^{\infty} (1+zq^{n+1}) = \prod_{n=1}^{\infty} (1+zq^n)$$

The last product can be written as $(1+zq^1)(1+zq^2)(1+zq^3)\cdots$, where we see, that a typical term looks like $(zq^{i_1})(zq^{i_2})\cdots(zq^{i_j}) = z^j q^{i_1+i_2+\cdots+i_j}$, which arises from the partition $i_1 + \cdots + i_j$ with j distinct parts. So what we proved is, that the generating function for partitions of n into m distinct parts is given by:

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} p(\mathcal{D}, m, n) z^m q^n = \prod_{n=1}^{\infty}(1+zq^n) = \sum_{n=0}^{\infty} \frac{z^n q^{\frac{n(n+1)}{2}}}{(q;q)_n}$$

At this point we come back to the very beginning, to the question of Naudé stated to Euler: How many partitions are there of 50 into 7 distinct parts. We are now able to answer the question in the same way as Euler did, compute the coefficient of $z^7 q^{50}$ in the above equation and we see that there are 522 such partitions.

For our next interesting result, we have to do some preparations.

Corollary 3.5.

$$\sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n z^n}{(q;q)_n (c;q)_n} = \frac{(b;q)_\infty (az;q)_\infty}{(c;q)_\infty (z;q)_\infty} \sum_{n=0}^{\infty} \frac{\left(\frac{c}{b};q\right)_n (z;q)_n b^n}{(q;q)_n (az;q)_n}$$

Proof.

$$\sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n z^n}{(q;q)_n (c;q)_n} = \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{n=0}^{\infty} \frac{(a;q)_n z^n}{(q;q)_n} \frac{(cq^n;q)_\infty}{(bq^n;q)_\infty}, \tag{3.5}$$

since

$$(b;q)_n = \frac{(b;q)_\infty}{(bq^n;q)_\infty}.$$

We now apply Theorem 3.2 twice. First, we set $z = bq^n$ and $a = \frac{c}{b}$, to see that

$$\frac{(cq^n;q)_\infty}{(bq^n;q)_\infty} = \sum_{m=0}^{\infty} \frac{\left(\frac{c}{b};q\right)_m b^m q^{nm}}{(q;q)_m}$$

and equation (3.5) becomes

$$\sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n z^n}{(q;q)_n (c;q)_n} = \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(a;q)_n z^n}{(q;q)_n} \frac{\left(\frac{c}{b};q\right)_m b^m q^{nm}}{(q;q)_m} \tag{3.6}$$

Then we apply Theorem 3.2, with $z = zq^m$, to see that

$$\sum_{n=0}^{\infty} \frac{(a;q)_n (zq^m)^n}{(q;q)_n} = \frac{(azq^m;q)_\infty}{(zq^m;q)_\infty}$$

and equation (3.6) becomes

$$\sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n z^n}{(q;q)_n (c;q)_n} = \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{m=0}^{\infty} \frac{\left(\frac{c}{b};q\right)_m b^m}{(q;q)_m} \frac{(azq^m;q)_\infty}{(zq^m;q)_\infty}. \tag{3.7}$$

For the last step, we notice that $(azq^m;q)_\infty = \frac{(az;q)_\infty}{(az;q)_m}$ and the same holds for $(zq^m;q)_\infty$, so equation (3.7) becomes

$$\sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_n z^n}{(q;q)_n(c;q)_n} = \frac{(b;q)_\infty(az;q)_\infty}{(c;q)_\infty(z;q)_\infty}\sum_{m=0}^{\infty}\frac{\left(\frac{c}{b};q\right)_m(z;q)_m b^m}{(q;q)_m(az;q)_m}$$

□

This result allows us to prove the following two corollaries.

Corollary 3.6.

$$\sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_n\left(-\frac{q}{b}\right)^n}{(q;q)_n\left(\frac{aq}{b};q\right)_n} = \frac{(aq;q^2)_\infty(-q;q)_\infty\left(\frac{aq^2}{b^2};q^2\right)_\infty}{\left(\frac{aq}{b};q\right)_\infty\left(-\frac{q}{b};q\right)_\infty}$$

Proof. Interchanging a and b in Corollary 3.5, set $z=-\frac{q}{b}$ and $c=\frac{aq}{b}$ gives

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{(b;q)_n(a;q)_n\left(-\frac{q}{b}\right)^n}{(q;q)_n\left(\frac{aq}{b};q\right)_n} &= \frac{(a;q)_\infty(-q;q)_\infty}{\left(\frac{aq}{b};q\right)_\infty\left(-\frac{q}{b};q\right)_\infty}\cdot\sum_{n=0}^{\infty}\frac{\left(\frac{q}{b};q\right)_n\left(-\frac{q}{b};q\right)_n\cdot a^n}{(q;q)_n(-q;q)_n}\\
&= \frac{(a;q)_\infty(-q;q)_\infty}{\left(\frac{aq}{b};q\right)_\infty\left(-\frac{q}{b};q\right)_\infty}\cdot\sum_{n=0}^{\infty}\frac{\left(\frac{q^2}{b^2};q^2\right)_n\cdot a^n}{(q^2;q^2)_n}\\
&\overset{\substack{Thm\\3.2}}{=} \frac{(a;q)_\infty(-q;q)_\infty\left(\frac{aq^2}{b^2};q^2\right)_\infty}{\left(\frac{aq}{b};q\right)_\infty\left(-\frac{q}{b};q\right)_\infty(a;q^2)_\infty}\\
&= \frac{(aq;q^2)_\infty(-q;q)_\infty\left(\frac{aq^2}{b^2};q^2\right)_\infty}{\left(\frac{aq}{b};q\right)_\infty\left(-\frac{q}{b};q\right)_\infty},
\end{aligned}$$

where the last equality holds because

$$\frac{(a;q)_\infty}{(a;q^2)_\infty} = \left(aq;q^2\right)_\infty.$$

□

Corollary 3.7.

$$\sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_n\left(\frac{c}{ab}\right)^n}{(q;q)_n(c;q)_n} = \frac{\left(\frac{c}{a};q\right)_\infty\left(\frac{c}{b};q\right)_\infty}{(c;q)_\infty\left(\frac{c}{ab};q\right)_\infty}$$

Proof. Let us set $z=\frac{c}{ab}$ in Corollary 3.5 to get

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_n\left(\frac{c}{ab}\right)^n}{(q;q)_n(c;q)_n} &= \frac{(b;q)_\infty\left(\frac{c}{b};q\right)_\infty}{(c;q)_\infty\left(\frac{c}{ab};q\right)_\infty}\sum_{n=0}^{\infty}\frac{\left(\frac{c}{b};q\right)_n\left(\frac{c}{ab};q\right)_n b^n}{(q;q)_n\left(\frac{c}{b};q\right)_n}\\
&= \frac{(b;q)_\infty\left(\frac{c}{b};q\right)_\infty}{(c;q)_\infty\left(\frac{c}{ab};q\right)_\infty}\sum_{n=0}^{\infty}\frac{\left(\frac{c}{ab};q\right)_n b^n}{(q;q)_n}.
\end{aligned}$$

Then we use Theorem 3.2 with $z = b$ and $a = \frac{c}{ab}$ to get

$$\sum_{n=0}^{\infty} \frac{\left(\frac{c}{ab}; q\right)_n b^n}{(q;q)_n} = \frac{\left(\frac{c}{a}; q\right)_\infty}{(c;q)_\infty}$$

All together we get

$$\sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n \left(\frac{c}{ab}\right)^n}{(q;q)_n (c;q)_n} = \frac{(b;q)_\infty \left(\frac{c}{b}; q\right)_\infty}{(c;q)_\infty \left(\frac{c}{ab}; q\right)_\infty} \frac{\left(\frac{c}{a}; q\right)_\infty}{(b;q)_\infty} = \frac{\left(\frac{c}{a}; q\right)_\infty \left(\frac{c}{b}; q\right)_\infty}{(c;q)_\infty \left(\frac{c}{ab}; q\right)_\infty}$$

□

Now we can easily prove the next corollary.

Corollary 3.8.

$$\sum_{n=0}^{\infty} \frac{q^{n^2-n} z^n}{(q;q)_n (z;q)_n} = \prod_{n=0}^{\infty} \frac{1}{1 - zq^n} \tag{3.8}$$

Proof. If we set $a = \alpha^{-1}$, $b = \beta^{-1}$ and $c = z$ in Corollary 3.7, we get

$$\begin{aligned} \frac{(z\alpha;q)_\infty (z\beta;q)_\infty}{(z;q)_\infty (z\alpha\beta;q)_\infty} &= \sum_{n=0}^{\infty} \frac{(\alpha^{-1};q)_n (\beta^{-1};q)_n (\alpha\beta z)^n}{(q;q)_n (z;q)_n} \\ &= \sum_{n=0}^{\infty} \frac{(1-\alpha^{-1}) \cdots (1-\alpha^{-1}q^{n-1}) (1-\beta^{-1}) \cdots (1-\beta^{-1}q^{n-1}) (\alpha\beta z)^n}{(q;q)_n (z;q)_n} \\ &= \sum_{n=0}^{\infty} \frac{(\alpha-1)(\alpha-q) \cdots (\alpha-q^{n-1})(\beta-1)(\beta-q) \cdots (\beta-q^{n-1}) z^n}{(q;q)_n (z;q)_n}. \end{aligned}$$

If we set $\alpha = \beta = 0$, this identity becomes

$$\frac{1}{(z;q)_\infty} = \prod_{n=0}^{\infty} \frac{1}{1 - zq^n} = \sum_{n=0}^{\infty} \frac{q^{n^2-n} z^n}{(q;q)_n (z;q)_n}$$

□

Corollary 3.9.

$$\sum_{n=0}^{\infty} \frac{(a;q)_n \cdot q^{\frac{n(n+1)}{2}}}{(q;q)_n} = \left(aq; q^2\right)_\infty (-q;q)_\infty$$

Proof. Set $b = \beta^{-1}$ in Corollary 3.6:

$$\sum_{n=0}^{\infty} \frac{(a;q)_n \cdot (\beta-1)(\beta-q) \cdots (\beta-q^{n-1}) \cdot (-q)^n}{(q;q)_n (a\beta q;q)_n} = \frac{(aq;q^2)_\infty (-q;q)_\infty (a\beta^2 q^2; q^2)_\infty}{(-q\beta;q)_\infty (a\beta q;q)_\infty}$$

Now we set $\beta = 0$ and we get the equation. □

Remark 3.10. 1. If we take $z = q$ in (3.8) we get (1.6).

2. If we take a look at (3.3) and (3.8) we see that we have shown

$$\sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = \prod_{n=0}^{\infty} \frac{1}{(1 - zq^n)} = \sum_{n=0}^{\infty} \frac{q^{n^2-n} z^n}{(q;q)_n (z;q)_n}.$$

3.2 Jacobi's triple product identity

Now we are able to prove the famous triple product identity of Jacobi, which is a consequence of Corollary 3.3, in a more or less simple way. The identity was proven and published by Jacobi in 1829 in his book *Fundamenta nova theoriae functionum ellipticarum*. It is the key to the proofs of many other identities, e.g. the Rogers-Ramanujan identities.

Theorem 3.11.

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = \prod_{n=0}^{\infty} \left(1 - q^{2n+2}\right) \left(1 + zq^{2n+1}\right) \left(1 + z^{-1} q^{2n+1}\right) \tag{3.9}$$

Proof. First we set $z = zq$ and $q = q^2$ in (3.4) to get

$$\begin{aligned} \prod_{n=0}^{\infty} \left(1 + zq^{2n+1}\right) &= \sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q^2;q^2)_n} \\ &= \frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} z^n q^{n^2} \left(q^{2n+2};q^2\right)_\infty \\ &= \frac{1}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} z^n q^{n^2} \left(q^{2n+2};q^2\right)_\infty , \end{aligned} \tag{3.10}$$

where the second equation holds because

$$\frac{1}{(q^2;q^2)_n} = \frac{(q^{2n+2};q^2)_\infty}{(q^2;q^2)_\infty}$$

and the last equation holds because $(q^{2n+2};q^2)_\infty = 0$ for negative n.
Again we take a look at Corollary 3.3 and set $z = -q^{2n+2}$ and $q = q^2$ in (3.4) to get

$$\sum_{j=0}^{\infty} \frac{\left(-q^{2n+2}\right)^j q^{j(j-1)}}{(q^2;q^2)_j} = \prod_{j=0}^{\infty} \left(1 - q^{2n+2}\left(q^2\right)^j\right).$$

We can rewrite this equation as follows

$$\sum_{j=0}^{\infty} \frac{(-1)^j q^{j^2+2nj+j}}{(q^2;q^2)_j} = \left(q^{2n+2};q^2\right)_\infty .$$

If we substitute this result into equation (3.10) we get

$$\begin{aligned} \frac{1}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} z^n q^{n^2} \left(q^{2n+2};q^2\right)_\infty &= \frac{1}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} z^n q^{n^2} \sum_{j=0}^{\infty} \frac{(-1)^j q^{j^2+2nj+j}}{(q^2;q^2)_j} \\ &= \frac{1}{(q^2;q^2)_\infty} \sum_{j=0}^{\infty} \frac{(-1)^j z^{-j} q^j}{(q^2;q^2)_j} \sum_{n=-\infty}^{\infty} q^{(n+j)^2} z^{n+j} \\ &\overset{\text{n+j=n}}{=} \frac{1}{(q^2;q^2)_\infty} \sum_{j=0}^{\infty} \frac{(-\frac{q}{z})^j}{(q^2;q^2)_j} \sum_{n=-\infty}^{\infty} q^{n^2} z^n. \end{aligned} \tag{3.11}$$

For the next step we take a look at equation (3.3) and set $z = -\frac{q}{z}$ and $q = q^2$. We get

$$\sum_{j=0}^{\infty} \frac{(-\frac{q}{z})^j}{(q^2;q^2)_j} = \prod_{n=0}^{\infty} \frac{1}{1+\frac{q}{z}(q^2)^n} = \prod_{n=0}^{\infty} \frac{1}{1+\frac{q^{2n+1}}{z}} = \frac{1}{(-\frac{q}{z};q^2)_\infty}. \tag{3.12}$$

Combining (3.10), (3.11) and (3.12) we finally get

$$\begin{aligned}\prod_{n=0}^{\infty}\left(1+zq^{2n+1}\right) &\overset{(3.10)}{=} \frac{1}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty} z^n q^{n^2}\left(q^{2n+2};q^2\right)_\infty \\ &\overset{(3.11)}{=} \frac{1}{(q^2;q^2)_\infty}\sum_{j=0}^{\infty}\frac{(-\frac{q}{z})^j}{(q^2;q^2)_j}\sum_{n=-\infty}^{\infty} q^{n^2}z^n \\ &\overset{(3.12)}{=} \frac{1}{(q^2;q^2)_\infty\left(-\frac{q}{z};q^2\right)_\infty}\sum_{n=-\infty}^{\infty} q^{n^2}z^n.\end{aligned}$$

Hence,

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = \prod_{n=0}^{\infty}\left(1-q^{2n+2}\right)\left(1+zq^{2n+1}\right)\left(1+z^{-1}q^{2n+1}\right),$$

which completes the proof.

□

Remark 3.12. Euler's pentagonal number theorem is a special case of Jacobi's triple product identity. If we replace q by $q^{\frac{3}{2}}$ and z by $-q^{\frac{1}{2}}$, we get

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(3n+1)}{2}} = \prod_{n=0}^{\infty}\left(1-q^{3n+3}\right)\left(1-q^{3n+2}\right)\left(1-q^{3n+1}\right) = \prod_{n=1}^{\infty}(1-q^n).$$

Corollary 3.13.

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n q^{(2k+1)n(n+1)/2-in}\left(1-q^{(2n+1)i}\right) &= \sum_{n=-\infty}^{\infty}(-1)^n q^{(2k+1)n(n+1)/2-in} \\ &= \prod_{n=0}^{\infty}\left(1-q^{(2k+1)(n+1)}\right)\left(1-q^{(2k+1)n+i}\right)\left(1-q^{(2k+1)(n+1)-i}\right).\end{aligned}$$

Proof. The second equation follows immediately from Jacobi's triple product identity (3.9) by replacing q by $q^{k+\frac{1}{2}}$ and z by $-q^{k+\frac{1}{2}-i}$. It remains to show the first equation:

$$\begin{aligned}&\sum_{n=0}^{\infty}(-1)^n q^{(2k+1)n(n+1)/2-in}\left(1-q^{(2n+1)i}\right) \\ &= \sum_{n=0}^{\infty}(-1)^n q^{(2k+1)n(n+1)/2-in} + \sum_{n=1}^{\infty}(-1)^n q^{(2k+1)n(n-1)/2+in} \\ &= \sum_{n=0}^{\infty}(-1)^n q^{(2k+1)n(n+1)/2-in} + \sum_{n=-1}^{-\infty}(-1)^n q^{(2k+1)n(n+1)/2-in} \\ &= \sum_{n=-\infty}^{\infty}(-1)^n q^{(2k+1)n(n+1)/2-in}.\end{aligned}$$

□

Corollary 3.14.

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} = \prod_{n=1}^{\infty}\frac{(1-q^n)}{(1+q^n)},$$
$$\sum_{n=0}^{\infty} q^{n(n+1)/2} = \prod_{n=1}^{\infty}\frac{(1-q^{2n})}{(1-q^{2n-1})}.$$

Proof. We set $z=-1$ in (3.9) and get

$$\begin{aligned}\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} &= \prod_{n=0}^{\infty}\left(1-q^{2n+2}\right)\left(1-q^{2n+1}\right)\left(1-q^{2n+1}\right)\\ &= \prod_{n=1}^{\infty}(1-q^n)\left(1-q^{2n+1}\right)\\ &= \prod_{n=1}^{\infty}\frac{(1-q^n)}{(1+q^n)},\end{aligned}$$

where the last equation holds, because we have seen in Theorem 1.3, that

$$\prod_{n=1}^{\infty}(1+q^n) = \prod_{n=1}^{\infty}\frac{1}{(1-q^{2n-1})},$$

which is equivalent to

$$\prod_{n=1}^{\infty}\frac{1}{(1+q^n)} = \prod_{n=1}^{\infty}\left(1-q^{2n-1}\right). \tag{$\bigstar$}$$

For the second equation, we observe that

$$\sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{1}{2}\sum_{n=-\infty}^{\infty} q^{n(n+1)/2}.$$

Now we set z and q equal to $q^{\frac{1}{2}}$ in (3.9) and get

$$\frac{1}{2}\sum_{n=-\infty}^{\infty} q^{n(n+1)/2} = \frac{1}{2}\prod_{n=0}^{\infty}\left(1-q^{n+1}\right)\left(1+q^{n+1}\right)(1+q^n).$$

Since

$$\frac{1}{2}\prod_{n=0}^{\infty}(1+q^n) = \prod_{n=1}^{\infty}(1+q^n)$$

and

$$\prod_{n=0}^{\infty}\left(1-q^{n+1}\right)\left(1+q^{n+1}\right) = \prod_{n=1}^{\infty}\left(1-q^{2n}\right)$$

we get

$$\frac{1}{2}\prod_{n=0}^{\infty}\left(1-q^{n+1}\right)\left(1+q^{n+1}\right)(1+q^n) = \prod_{n=1}^{\infty}(1+q^n)\left(1-q^{2n}\right).$$

Using the same argument ($\bigstar$) as for the first equation we get

$$\prod_{n=1}^{\infty}(1+q^n)(1-q^{2n}) = \prod_{n=1}^{\infty}\frac{(1-q^{2n})}{(1-q^{2n-1})}.$$

$\square$

3.3 A partition theorem by Sylvester

Next, we will present two proofs for a theorem by J.J. Sylvester. The first proof was given by Sylvester in [18], here we will give two other proofs, the first being due to Hansraj Gupta in [12] and the second using generating functions.

Theorem 3.15. *Let $A_k(n)$ denote the number of partitions of n into odd parts with exactly k different parts, where repetitions are allowed. Let $B_k(n)$ denote the number of partitions of n into distinct parts such that exactly k maximal subsequences of consecutive integers appear. Then for each k and n, $A_k(n) = B_k(n)$.*

Proof 1. We will give an algorithm where we transform bijectively a partition of one type into a partition of the other type. This algorithm will be illustrated by an example.
We start with a partition of 77, which belongs to $A_5(77)$:

$$19, 19, 11, 5, 5, 5, 5, 3, 3, 1, 1$$

The algorithm starts with the biggest part of the given partition by splitting it into two consecutive integers. This is possible, because we deal with odd numbers. Our biggest part is 19, so we split it into

$$10, 9.$$

Then we take the next part, from this we allot 1 to each of the parts of the new partition and split the rest into two consecutive integers and write them to the right:

$$11, 10, 9, 8.$$

We do the same with the next part:

$$12, 11, 10, 9, 4, 3$$

In general, one does this up to the point, where the next part of the given partition is smaller than the number of parts of the new one. In our example, we are at this point, because the next part we have to deal with is 5 and we already have six parts. So we just add 1 to the first five parts:

$$13, 12, 11, 10, 5, 3.$$

We do this also for the remaining 5's:

$$16, 15, 14, 13, 8, 3.$$

We do the same for the two 3's:

$$18, 17, 16, 13, 8, 3.$$

Finally, the two 1's get added to the first part:

$$20, 17, 16, 13, 8, 3.$$

We should obtain 5 sequences of consecutive integers and indeed we find the 5 subsequences (20), (17,16), (13), (8),(3). So our new partition belongs to $B_5(77)$.

It remains to show the inverse of the algorithm. For this we take the partition obtained above: 20,17,16,13,8,3.
If the first two parts do not form a sequence, subtract a suitable number from the first part, such that afterwards it will be equal to the second part plus 1. Then subtract a suitable number from the first three parts, such that afterwards the third part will be one more than the fourth. Then from the first five parts such that the fifth part will be one more than the sixth and so on till the process cannot be applied any more. If the number of parts is odd, then at the last step we subtract the last element itself, such that we are left with an even number of parts. The subtracted numbers give us parts of the new partition. Let us take a look what this means by executing it on our partition.

	20	17	16	13	8	3	
-	2						(two 1's)
	18	17	16	13	8	3	
-	2	2	2				(two 3's)
	16	15	14	13	8	3	
-	4	4	4	4	4		(four 5's)
	12	11	10	9	4	3	

Now the first part cannot be applied any more. In the second part we subtract 1 from each of the parts except the last two. Adding these 1's to the sum of the last two parts, we get a part of our required partition into odd parts. We do this until the last parts disappeared.

	12	11	10	9	4	3	
-	1	1	1	1	4	3	(11)
	11	10	9	8			
-	1	1	9	8			(19)
	10	9					
-	10	9					(19)

So as we expected our generated partition is 19,19,11,5,5,5,5,3,3,1,1. It is easy to see, that this gives a bijection between the two considered types of partitions.

□

Proof 2. The generating function for $A_k(n)$ is given by

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} A_k(n) z^k q^n = \prod_{j=1}^{\infty}\left(1 + zq^{2j-1} + zq^{2(2j-1)} + zq^{3(2j-1)} + ...\right).$$

Since

$$\begin{aligned} zq^{2j-1} + zq^{2(2j-1)} + zq^{3(2j-1)} + ... &= zq^{2(j-1)}(1 + q^{2j-1} + (q^{2j-1})^2 + (q^{2j-1})^3 + ... \\ &= zq^{2j-1} \cdot \frac{1}{1-q^{2j-1}}, \end{aligned}$$

we can write

$$\begin{aligned} \prod_{j=1}^{\infty}\left(1 + zq^{2j-1} + zq^{2(2j-1)} + zq^{3(2j-1)} + ...\right) &= \prod_{j=1}^{\infty}\left(1 + \frac{zq^{2j-1}}{1-q^{2j-1}}\right) \\ &= \prod_{j=1}^{\infty} \frac{1-(1-z)q^{2j-1}}{1-q^{2j-1}} \\ &= \frac{((1-z)q;q^2)_\infty}{(q;q^2)_\infty} \\ &\overset{Cor\ 1.3}{=} \left((1-z)q;q^2\right)_\infty (-q;q)_\infty. \end{aligned}$$

In the next step we want to show that the generating function for $B_k(n)$ is given by:

$$\begin{aligned} \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} B_k(n) z^k q^n =& \sum_{j=1}^{\infty} z\left(q + zq^2 + zq^3 + ...\right)\left(q^2 + zq^4 + zq^6 + ...\right)\cdots \\ & \cdots\left(q^{j-1} + zq^{2(j-1)} + zq^{3(j-1)} + ...\right)\left(q^j + q^{2j} + q^{3j} + ...\right). \end{aligned} \tag{3.13}$$

Let us therefore consider the m-th term of this, for some $m \geq k$,

$$\begin{aligned} & z\left(q + zq^2 + zq^3 + ...\right)\left(q^2 + zq^4 + zq^6 + ...\right)\cdots \\ & \cdots\left(q^{m-1} + zq^{2(m-1)} + zq^{3(m-1)} + ...\right)\left(q^m + q^{2m} + q^{3m} + ...\right) \end{aligned}$$

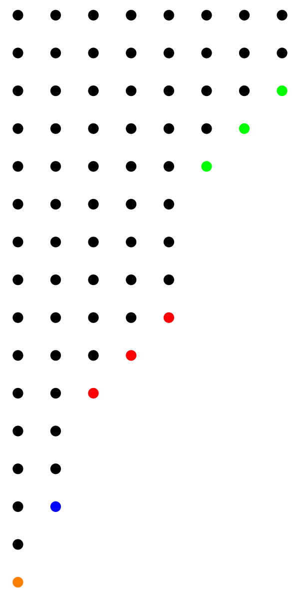
and expand it in terms of z and q. A term $z^k q^n$ occurs in this expansion when

$$n = 1 \cdot p_1 + 2 \cdot p_2 + ... + m \cdot p_m,$$

where the p_i are positive integers and exactly $k-1$ are greater than 1. Of course, we interpret this expression of n as a partition, where every integer from 1 to m occurs at least once.

Let $p_{i_1}, ..., p_{i_{k-1}}$ be the p_i's greater than 1. We see that a maximal sequence of consecutive integers ends, when it reaches one i_j $(1 \leq j \leq k-1)$ and the last sequence ends

at m, which gives us in total k sequences.
Therefore the parts of the conjugate partition are distinct and the number of sequences stays the same. So the coefficient of $z^k q^n$ enumerates all partitions of n into distinct parts with k sequences of consecutive integers and this is $B_k(n)$.
Let us take a look at an example, where $k = 4$, $m = 8$ and $n = 72$. For the partition (8,8,8,7,6,5,5,5,5,5,4,3,2,2,2,1,1) we get: $p_1 = 2$, $p_2 = 3$, $p_5 = 4$, $p_8 = 3$, $p_3 = p_4 = p_6 = p_7 = 1$. We also take a look at its Ferrers diagram and mark the sequences there, which are 4 sequences (1), (2), (3,4,5), (6,7,8).



We see that the first sequence of the partition ends at $i_1 = 1$, the second at $i_2 = 2$, the third at $i_3 = 5$ and the last at $m = 8$. The conjugate partition is

$$(16, 14, 11, 10, 9, 5, 4, 3),$$

which also has 4 sequences and therefore belongs to $B_4(72)$.
It remains to show that

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} A_k(n) z^k q^n = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} B_k(n) z^k q^n.$$

Therefore we take a look at (3.13):

$$\begin{aligned}
\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} B_k(n) z^k q^n &= \sum_{j=1}^{\infty} z(q + zq^2 + zq^3 + ...)(q^2 + zq^4 + zq^6 + ...)\cdots \\
&\cdots (q^{j-1} + zq^{2(j-1)} + zq^{3(j-1)} + ...)(q^j + q^{2j} + q^{3j} + ...) \\
&= \sum_{j=1}^{\infty} q^{\frac{j(j+1)}{2}} \cdot z \cdot (1 + zq + zq^2 + ...)\cdots \\
&\cdots (1 + zq^{j-1} + zq^{2(j-1)} + ...)(1 + q^j + q^{2j} + ...).
\end{aligned} \tag{3.14}$$

We can write

$$1 + zq^s + zq^{2s} + ... = 1 + \frac{zq^s}{1-q^s} = \frac{1+(z-1)q^s}{1-q^s}.$$

Thus (3.14) becomes

$$\begin{aligned}
&1 + \sum_{j=1}^{\infty} q^{\frac{j(j+1)}{2}} \cdot z \cdot \frac{((1-z)q;q)_{j-1}}{(q;q)_j} \\
&= \sum_{j=0}^{\infty} \frac{((1-z);q)_j q^{\frac{j(j+1)}{2}}}{(q;q)_j} \\
&\overset{\substack{Cor\\3.9}}{=} \left((1-z)q;q^2\right)_\infty (-q;q)_\infty \\
&= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} A_k(n) z^k q^n.
\end{aligned}$$

□

4 Gaussian polynomial

Next we are going to take a look at the q-binomial coefficient, often called the Gaussian polynomial. Not only the coefficient itself, but also its connection to partitions will be part of this chapter.

Definition 4.1. The Gaussian polynomial, denoted by $\begin{bmatrix} n \\ m \end{bmatrix}$, is defined by

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{cases} \frac{(q;q)_n}{(q;q)_m (q;q)_{n-m}} & \text{if } 0 \leq m \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Let us prove some properties of this polynomial.

Theorem 4.2. *Let $0 \leq m \leq n$ be integers. The Gaussian polynomial $\begin{bmatrix} n \\ m \end{bmatrix}$ is a polynomial of degree $m(n-m)$ in q and satisfies the following relations:*

$$\begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} n \\ n \end{bmatrix} = 1;$$

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n \\ n-m \end{bmatrix};$$

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n-1 \\ m \end{bmatrix} + q^{n-m} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}; \tag{4.1}$$

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} + q^{m} \begin{bmatrix} n-1 \\ m \end{bmatrix}; \tag{4.2}$$

$$\lim_{q \to 1} \begin{bmatrix} n \\ m \end{bmatrix} = \frac{n!}{m!(n-m)!} = \binom{n}{m}. \tag{4.3}$$

Proof. The first two equations follow directly from the definition, so let us start with (4.1).

$$
\begin{aligned}
\begin{bmatrix} n \\ m \end{bmatrix} - \begin{bmatrix} n-1 \\ m \end{bmatrix} &= \frac{(q;q)_n}{(q;q)_m (q;q)_{n-m}} - \frac{(q;q)_{n-1}}{(q;q)_m (q;q)_{n-m-1}} \\
&= \frac{(q;q)_{n-1}}{(q;q)_m (q;q)_{n-m}} \Big(1 - q^n - (1 - q^{n-m})\Big) \\
&= q^{n-m} \frac{(q;q)_{n-1}}{(q;q)_{m-1} (q;q)_{n-m}} \\
&= q^{n-m} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}.
\end{aligned}
$$

Equation (4.2) follows by replacing m by $n-m$ in equation (4.1) and using the symmetry property.

$$
\begin{aligned}
\begin{bmatrix} n \\ n-m \end{bmatrix} &= \begin{bmatrix} n-1 \\ n-m \end{bmatrix} + q^{n-(n-m)} \begin{bmatrix} n-1 \\ n-m-1 \end{bmatrix} \\
\begin{bmatrix} n \\ m \end{bmatrix} &= \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} + q^{m} \begin{bmatrix} n-1 \\ m \end{bmatrix}
\end{aligned}
$$

Now we take a look a the last property:

$$
\begin{aligned}
\lim_{q\to 1} \begin{bmatrix} n \\ m \end{bmatrix} &= \lim_{q\to 1} \frac{1-q^n}{1-q^m} \cdot \frac{1-q^{n-1}}{1-q^{m-1}} \cdots \frac{1-q^{n-m+1}}{1-q} \\
&= \lim_{q\to 1} \frac{\frac{1-q^n}{1-q}}{\frac{1-q^m}{1-q}} \cdot \frac{\frac{1-q^{n-1}}{1-q}}{\frac{1-q^{m-1}}{1-q}} \cdots \frac{\frac{1-q^{n-m+1}}{1-q}}{\frac{1-q}{1-q}} \\
&= \lim_{q\to 1} \frac{1+q+...+q^{n-1}}{1+q+...+q^{m-1}} \cdot \frac{1+q+...+q^{n-2}}{1+q+...+q^{m-2}} \cdots \frac{1+q+...+q^{n-m}}{1} \\
&= \frac{n}{m} \cdot \frac{n-1}{m-1} \cdots \frac{n-m+1}{1} \\
&= \frac{n!}{m!(n-m)!} = \binom{n}{m}.
\end{aligned}
$$

The fact that $\begin{bmatrix} n \\ m \end{bmatrix}$ is a polynomial of degree $m(n-m)$ follows by induction on n and using (4.2).

$\square$

4.1 The q-binomial theorem

Theorem 4.3 (q-binomial theorem)**.**

$$(z;q)_n = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} (-1)^j z^j q^{\frac{j(j-1)}{2}}; \tag{4.4}$$
$$\frac{1}{(z;q)_n} = \sum_{j=0}^{\infty} \begin{bmatrix} n+j-1 \\ j \end{bmatrix} z^j.$$

Proof. This is an immediate consequence of Theorem 3.2. For the first equation we set $z = zq^n$ and $a = q^{-n}$ in (3.2) to get

$$\begin{aligned}
(z;q)_n &= \frac{(z;q)_\infty}{(zq^n;q)_\infty} \overset{\substack{Thm\\3.2}}{=} \sum_{j=0}^{\infty} \frac{(q^{-n};q)_j z^j q^{nj}}{(q;q)_j} \\
&= \sum_{j=0}^{\infty} \frac{(1-q^{-n})(1-q^{-n+1})\cdots(1-q^{-n+j-1})\, z^j q^{nj}}{(q;q)_j} \\
&= \sum_{j=0}^{n} \frac{(1-q^{-n})(1-q^{-n+1})\cdots(1-q^{-n+j-1})\, z^j q^{nj}}{(q;q)_j} \\
&= \sum_{j=0}^{n} \frac{(1-q^{-n})\cdot q\,(q^{-1}-q^{-n})\cdots q^{j-1}\,(q^{-j+1}-q^{-n})\, z^j q^{nj}}{(q;q)_j} \\
&= \sum_{j=0}^{n} \frac{q^{\frac{j(j-1)}{2}}\,(1-q^{-n})\,(q^{-1}-q^{-n})\cdots(q^{-j+1}-q^{-n})\, z^j q^{nj}}{(q;q)_j} \\
&= \sum_{j=0}^{n} \frac{q^{\frac{j(j-1)}{2}} q^{-n}\,(q^{n}-1)\cdot q^{-n}\,(q^{n-1}-1)\cdots q^{-n}\,(q^{n-j+1}-1)\, z^j q^{nj}}{(q;q)_j} \\
&= \sum_{j=0}^{n} \frac{q^{\frac{j(j-1)}{2}-nj}\,(q^{n}-1)\,(q^{n-1}-1)\cdots(q^{n-j+1}-1)\, z^j q^{nj}}{(q;q)_j} \\
&= \sum_{j=0}^{n} \frac{(-1)^j q^{\frac{j(j-1)}{2}-nj}\,(1-q^{n})\,(1-q^{n-1})\cdots(1-q^{n-j+1})\, z^j q^{nj}}{(q;q)_j} \\
&= \sum_{j=0}^{n} \frac{(q;q)_n}{(q;q)_j (q;q)_{n-j}} (-1)^j z^j q^{\frac{j(j-1)}{2}} \\
&= \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} (-1)^j z^j q^{\frac{j(j-1)}{2}}.
\end{aligned}$$

For the second equation we set $a = q^n$ in Theorem 3.2 to get

$$
\begin{aligned}
\frac{1}{(z;q)_n} = \frac{(zq^n;q)_\infty}{(z;q)_\infty} &\overset{Thm\ 3.2}{=} \sum_{j=0}^{\infty} \frac{(q^n;q)_j\, z^j}{(q;q)_j} \\
&= \sum_{j=0}^{\infty} \frac{(q;q)_{n+j-1}}{(q;q)_j (q;q)_{n-1}} z^j \\
&= \sum_{j=0}^{\infty} \begin{bmatrix} n+j-1 \\ j \end{bmatrix} z^j .
\end{aligned}
$$

□

Remark 4.4. A very interesting fact about the q-binomial theorem is, that Jacobi's triple product identity is in fact a corollary of it. To see this, we first rewrite our version (3.9) of the triple product identity,

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = \prod_{n=0}^{\infty} \left(1 - q^{2n+2}\right)\left(1 + zq^{2n+1}\right)\left(1 + z^{-1}q^{2n+1}\right).$$

The right hand side can be rewritten as

$$\left(q^2;q^2\right)_\infty \left(-zq;q^2\right)_\infty \left(-\frac{q}{z};q^2\right)_\infty .$$

Now we set $z = -\frac{z}{q}$,

$$\sum_{n=-\infty}^{\infty} (-1)^n z^n q^{n(n-1)} = \left(q^2;q^2\right)_\infty \left(z;q^2\right)_\infty \left(\frac{q^2}{z};q^2\right)_\infty .$$

In the last step we set $q = q^{\frac{1}{2}}$ and get another version of Jacobi's triple product identity:

$$\sum_{n=-\infty}^{\infty} (-1)^n z^n q^{\frac{n(n-1)}{2}} = (q;q)_\infty (z;q)_\infty \left(\frac{q}{z};q\right)_\infty . \tag{4.5}$$

We now can easily show, that Jacobi's triple product identity is a consequence of the q-binomial theorem by setting $n = 2m$ and $k = j - m$ in (4.4) to get

$$(z;q)_{2m} = \sum_{k=-m}^{m} \begin{bmatrix} 2m \\ k+m \end{bmatrix} (-1)^{k+m} z^{k+m} q^{\frac{(k+m)(k+m-1)}{2}} . \tag{4.6}$$

Substituting z by zq^{-m} and rewriting $(zq^{-m};q)_{2m}$ as $(zq^{-m};q)_m\,(z;q)_m$ we can make the following observation:

$$
\begin{aligned}
\left(zq^{-m};q\right)_m &= \left(1-zq^{-m}\right)\left(1-zq^{-m+1}\right)\cdots\left(1-zq^{-1}\right)\\
&= (-1)^m\left(zq^{-m}-1\right)\left(zq^{m-1}-1\right)\cdots\left(zq^{-1}-1\right)\\
&= (-1)^m z^m\left(q^{-m}-\frac{1}{z}\right)\left(q^{-m+1}-\frac{1}{z}\right)\cdots\left(q^{-1}-\frac{1}{z}\right)\\
&= (-1)^m z^m q^{-m^2}\left(1-\frac{q^m}{z}\right)\left(q-\frac{q^m}{z}\right)\cdots\left(q^{m-1}-\frac{q^m}{z}\right)\\
&= (-1)^m z^m q^{-m^2} q^{\frac{m(m-1)}{2}}\left(1-\frac{q^m}{z}\right)\left(1-\frac{q^{m-1}}{z}\right)\cdots\left(1-\frac{q}{z}\right)\\
&= (-1)^m z^m q^{-m^2+\frac{m(m-1)}{2}}\left(\frac{q}{z};q\right)_m.
\end{aligned}
$$

Hence, equation (4.6) becomes

$$
\begin{aligned}
(zq^{-m};q)_m(z;q)_m &= (-1)^m z^m q^{-m^2+\frac{m(m-1)}{2}}\left(\frac{q}{z};q\right)_m (z;q)_m\\
&= \sum_{k=-m}^{m}\left[\begin{array}{c}2m\\k+m\end{array}\right](-1)^{k+m}z^{k+m}q^{\frac{(k+m)(k+m-1)}{2}-m(k+m)}.
\end{aligned}
$$

Therefore we get

$$
\left(\frac{q}{z};q\right)_m (z;q)_m = \sum_{k=-m}^{m}\frac{(q;q)_{2m}(-1)^k z^k q^{\frac{k(k-1)}{2}}}{(q;q)_{m+k}(q;q)_{m-k}}.
$$

Jacobi's triple product identity (4.5) follows when we let $m\to\infty$.

Remark 4.5. Clearly, there is a connection between the q-binomial theorem and the ordinary binomial theorem,

$$
\sum_{j=0}^{n}\binom{n}{j}x^j y^{n-j} = (x+y)^n.
$$

If we set $z=-\frac{x}{y}$ in (4.4), we get

$$
\left(-\frac{x}{y};q\right)_n = \sum_{j=0}^{n}\left[\begin{array}{c}n\\j\end{array}\right](-1)^j\left(-\frac{x}{y}\right)^j q^{\frac{j(j-1)}{2}}.
$$

We then multiply both sides by y^n to get

$$
(y+x)(y+xq)\cdots\left(y+xq^{n-1}\right) = \sum_{j=0}^{n}\left[\begin{array}{c}n\\j\end{array}\right]x^j y^{n-j}q^{\frac{j(j-1)}{2}}.
$$

For $q\to 1$ we obtain the ordinary binomial theorem (see (4.3)).

4.2 Connection with partitions

One may now ask how this polynomial is related to partitions and their generating functions. Although we mentioned many different types of partitions, there is one we did not deal with yet, namely partitions of n into at most l parts, each at most k and denote the number of these with $p_{k,l}(n)$. Two conditions are quite obvious, namely

$$p_{k,l}(n) = 0 \text{ if } n > kl$$
$$p_{k,l}(kl) = 1.$$

Hence its generating function, denoted by

$$G(k,l;q) = \sum_{n\geq 0} p_{k,l}(n)q^n,$$

is a polynomial of degree kl.

Theorem 4.6. *Let $k, l \geq 0$ be fixed, then*

$$G(k,l;q) = \frac{\left(1-q^{k+l}\right)\left(1-q^{k+l-1}\right)\cdots\left(1-q^{l+1}\right)}{(1-q^k)(1-q^{k-1})\cdots(1-q)} = \frac{(q;q)_{k+l}}{(q;q)_k(q;q)_l} = \begin{bmatrix} k+l \\ k \end{bmatrix}$$

Proof. Let

$$g(k,l;q) = \frac{(q;q)_{k+l}}{(q;q)_k(q;q)_l},$$

then

$$g(k,0;q) = g(0,l;q) = 1.$$

Moreover,

$$\begin{aligned}
g(k,l;q) - g(k,l-1;q) &= \frac{(q;q)_{k+l}}{(q;q)_k(q;q)_l} - \frac{(q;q)_{k+l-1}}{(q;q)_k(q;q)_{l-1}} \\
&= \frac{(q;q)_{k+l}}{(q;q)_k(q;q)_l} - \frac{(q;q)_{k+l-1}(1-q^l)}{(q;q)_k(q;q)_l} \\
&= \frac{(q;q)_{k+l-1}}{(q;q)_k(q;q)_l}\left(1-q^{k+l}-\left(1-q^l\right)\right) \\
&= \frac{(q;q)_{k+l-1}}{(q;q)_k(q;q)_l}\left(q^l-q^{k+l}\right) \\
&= \frac{(q;q)_{k+l-1}}{(q;q)_k(q;q)_l}q^l\left(1-q^k\right) \\
&= q^l\frac{(q;q)_{k+l-1}}{(q;q)_{k-1}(q;q)_l} \\
&= q^l g(k-1,l;q).
\end{aligned}$$

These two conditions uniquely define $g(k, l; q)$ for all non-negative integers k and l. Notice that the following holds:

$$p_{k,0}(n) = p_{0,l}(n) = \begin{cases} 1 & \text{if } k = l = n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore we get

$$G(k, 0; q) = G(0, l; q) = 1.$$

Furthermore, $p_{k,l}(n) - p_{k,l-1}(n)$ denotes the number of partitions of n into exactly l parts, each less or equal than k.
We subtract 1 from each part to get a partition of $n - l$ into at most l parts, each less or equal than $k - 1$. This procedure is reversible for given n, k and l and therefore establishes a bijection between partitions enumerated by $p_{k,l}(n) - p_{k,l-1}(n)$ and those enumerated by $p_{k-1,l}(n - l)$. Therefore

$$p_{k,l}(n) - p_{k,l-1}(n) = p_{k-1,l}(n - l).$$

Translating this into a generating function identity, we obtain

$$G(k, l; q) - G(k, l - 1; q) = q^l G(k - 1, l; q)$$

On the one hand, $g(k, l; q)$ and $G(k, l; q)$ satisfy the same initial conditions, on the other hand, they satisfy the same defining recurrence, so they must be identical, i.e.

$$G(k, l; q) = g(k, l; q) = \frac{(q; q)_{k+l}}{(q; q)_k (q; q)_l} = \begin{bmatrix} k + l \\ k \end{bmatrix}.$$

$\square$

5 The Rogers-Ramanujan identities

After the study of partitions, generating functions, q-series and some famous identities, like Euler's pentagonal number theorem and Jacobi's triple product identity, we now come to the next very famous and interesting identities, the Rogers-Ramanujan identities, given by

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}, \tag{5.1}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{5.2}$$

Before we go to the history of the Rogers-Ramanujan identities, let us take a look at equation (1.5), to recall, this was

$$\sum_{n=0}^{\infty} \frac{q^n}{(q;q)_n} = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)}. \tag{1.5}$$

This identity is due to Euler and although it is similar to the Rogers-Ramanujan identities, it was discovered about 150 years before them.
We are going to prove the identities analytically first, then we will show their combinatorial interpretation and at the end we will prove a generalization by Gordon.

5.1 The indian genius Srinivasa Ramanujan

The following summary of the life of Srinivasa Ramanujan is due to G.H. Hardy and can be found in detail in [13, Ch. 1].
He was born in 1887, near Kumbakonam, into a very poor family. He was recognized as a quite abnormal boy at the age of twelve or thirteen. A curious story about him is for example, soon after he began to study trigonometry, he discovered "Euler's theorems for the sine and cosine" by himself and was disappointed when he found out that they where known already. In 1904 he won the Subrahmanyam scholarship and joined the Government College of Kumbakonam, but due to his permanently study of mathematics, even during other courses, he lost his scholarship. In 1913 he wrote a

letter to G.H. Hardy, which contained about 120 theorems, mostly formal identities extracted from his notebooks and stated without proofs. About three of them Hardy wrote in [13, p. 9]:
"*[These formulas] defeated me completely; I had never seen anything in the least like them before. A single look at them is enough to show that they could only be written down by a mathematician of the highest class. They must be true because, if they were not true, no one would have had the imagination to invent them.*"
Hardy managed it to bring him to England one year later. He was not the first person who recognized Ramanujan's genius, but the first who gave him the chance to work in an good environment. Unfortunately Ramanujan became ill in 1917 and never really recovered, though he continued to work until his death in 1920.

5.2 The curious story behind the identities

Before we come to the proof of the Rogers-Ramanujan identities, a few words about the discovery of them. The interested reader can look the following up in detail in [13, p. 91]. The Rogers-Ramanujan identities were first discovered and proven by Rogers in 1894. He was a mathematician who was hardly known and so no one realized his remarkable discovery. In 1913 Ramanujan stated the identities without proof, he knew that he had none and none of the mathematicians who tried to find one were able to prove them. A few years later Ramanujan managed to find one, Hardy writes about this discovery in [13, p. 91]:
"*The mystery was solved, trebly, in 1917. In that year Ramanujan, looking through old volumes of the Proceedings of the London Mathematical Society, came accidentally across Rogers's paper. I can remember very well his surprise, and the admiration which he expressed for Rogers's work.*"
About the same time Issai Schur, who was cut off from England by the war, rediscovered them again. One of the two proofs he published, is a combinatorial one and quite different than any other one known at this time.

5.3 Proof of the Rogers-Ramanujan identities

Let us prove the Rogers-Ramanujan identities. We prepare the proof by two Lemmas, from which both identities follow immediately. The proof follows the one in [9, 299 ff.]. We start with the definition of Ramanujan's theta function.

Definition 5.1. Ramanujan's theta function is given by

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}.$$

Remark 5.2. Using this definition we can rewrite Jacobi's triple product identity as follows:

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{5.3}$$

Notice that we get this identity by setting $q = ab$ and $z = -a$ in (4.5).

Remark 5.3. Replacing a and b by $-q$ respectively $-q^2$ in (5.3) we easily see that

$$f\left(-q, -q^2\right) = \left(q; q^3\right)_\infty \left(q^2; q^3\right)_\infty \left(q^3; q^3\right)_\infty = (q; q)_\infty.$$

Moreover, by the definition we immediately get

$$f\left(-q, -q^2\right) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}}.$$

Combining those two results we got another proof of Euler's pentagonal number theorem

$$(q; q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}}.$$

Another consequence of (5.3) are the following two identities, where we first replace a and b by $-q^2$ respectively $-q^3$ and second replace them by $-q$ respectively $-q^4$.

$$f\left(-q^2, -q^3\right) = \left(q^2; q^5\right)_\infty \left(q^3; q^5\right)_\infty \left(q^5; q^5\right)_\infty, \tag{5.4}$$

$$f\left(-q, -q^4\right) = \left(q; q^5\right)_\infty \left(q^4; q^5\right)_\infty \left(q^5; q^5\right)_\infty. \tag{5.5}$$

If we divide these two equations by $(q; q)_\infty$ we get

$$\frac{f\left(-q^2, -q^3\right)}{(q; q)_\infty} = \frac{\left(q^2; q^5\right)_\infty \left(q^3; q^5\right)_\infty \left(q^5; q^5\right)_\infty}{(q; q)_\infty} = \frac{1}{\left(q; q^5\right)_\infty \left(q^4; q^5\right)_\infty},$$

$$\frac{f\left(-q, -q^4\right)}{(q; q)_\infty} = \frac{\left(q; q^5\right)_\infty \left(q^4; q^5\right)_\infty \left(q^5; q^5\right)_\infty}{(q; q)_\infty} = \frac{1}{\left(q^2; q^5\right)_\infty \left(q^3; q^5\right)_\infty}.$$

Therefore we can write (5.1) and (5.2) as

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \frac{f\left(-q^2, -q^3\right)}{(q; q)_\infty}, \tag{5.6}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q; q)_n} = \frac{f\left(-q, -q^4\right)}{(q; q)_\infty}. \tag{5.7}$$

Moreover, if we set $z = q^2$ (respectively $z = q$) and $q = q^5$ in (4.5), we get

$$f\left(-q^2, -q^3\right) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{5n^2-n}{2}}$$

respectively

$$f\left(-q, -q^4\right) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{5n^2-3n}{2}}.$$

Lemma 5.4. *Let*

$$G(z) = 1 + \sum_{n=1}^{\infty} (-1)^n q^{\frac{5n^2-n}{2}} z^{2n} \left(1 - zq^{2n}\right) \frac{(zq;q)_{n-1}}{(q;q)_n}. \tag{5.8}$$

Then

$$G(1) = f\left(-q^2, -q^3\right) \overset{(5.4)}{=} \left(q^2;q^5\right)_\infty \left(q^3;q^5\right)_\infty \left(q^5;q^5\right)_\infty$$

and

$$(1-q)G(q) = f\left(-q, -q^4\right) \overset{(5.5)}{=} \left(q;q^5\right)_\infty \left(q^4;q^5\right)_\infty \left(q^5;q^5\right)_\infty .$$

Moreover, the following three-term functional equation holds:

$$\frac{G(z)}{1-zq} = G(zq) + zq\left(1 - zq^2\right) G\left(zq^2\right). \tag{5.9}$$

Proof. The identity of $G(1)$ and $f(-q^2, -q^3)$ respectively $(1-q)G(q)$ and $f(-q, -q^4)$ is a straight-forward calculation:

$$\begin{aligned} G(1) &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n q^{\frac{5n^2-n}{2}} \left(1-q^{2n}\right)}{1-q^n} = 1 + \sum_{n=1}^{\infty} (-1)^n q^{\frac{5n^2-n}{2}} \left(1+q^n\right) \\ &= \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{5n^2-n}{2}} = f\left(-q^2, -q^3\right), \end{aligned}$$

$$\begin{aligned} (1-q)G(q) &= 1 - q + \sum_{n=1}^{\infty} \frac{(-1)^n q^{\frac{5n^2-n}{2}} q^{2n} \left(1-q^{2n+1}\right) \left(q^2;q\right)_{n-1}}{\left(q^2;q\right)_{n-1}} \\ &= \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{5n^2-3n}{2}} = f\left(-q, -q^4\right). \end{aligned}$$

For the second equality recall (5.4) and (5.5).
It remains to show the recurrence relation. Therefore we split the series by writing

$$1 - zq^{2n} = 1 - q^n + q^n - zq^{2n} = q^n\left(1 - zq^n\right) + \left(1 - q^n\right)$$

to obtain

$$\begin{aligned} G(z) = 1 &+ \sum_{n=1}^{\infty} (-1)^n q^{\frac{5n^2-n}{2}} z^{2n} q^n \frac{(zq;q)_n}{(q;q)_n} \\ &+ \sum_{n=1}^{\infty} (-1)^n q^{\frac{5n^2-n}{2}} z^{2n} \frac{(zq;q)_{n-1}}{(q;q)_{n-1}}. \end{aligned}$$

By replacing n by $n+1$ in the second sum, we get

$$\begin{aligned} G(z) &= 1 + \sum_{n=1}^{\infty} (-1)^n q^{\frac{5n^2-n}{2}} z^{2n} q^n \frac{(zq;q)_n}{(q;q)_n} \\ &\quad - \sum_{n=0}^{\infty} (-1)^n q^{\frac{5n^2+9n+4}{2}} z^{2n+2} \frac{(zq;q)_n}{(q;q)_n} \\ &= 1 - z^2q^2 + \sum_{n=1}^{\infty} (-1)^n q^{\frac{5n^2+n}{2}} z^{2n} \left(1 - z^2 q^{4n+2}\right) \frac{(zq;q)_n}{(q;q)_n}. \end{aligned} \tag{5.10}$$

Now consider $\frac{G(z)}{1-zq} - G(zq)$. Using (5.10) for $\frac{G(z)}{1-zq}$ and (5.8) for $G(zq)$, we obtain

$$\begin{aligned}
\frac{G(z)}{1-zq} - G(zq) &= 1 + zq + \sum_{n=1}^{\infty}(-1)^n q^{\frac{5n^2+n}{2}} z^{2n}\left(1 - z^2q^{4n+2}\right)\frac{(zq^2;q)_{n-1}}{(q;q)_n} \\
&\quad - 1 - \sum_{n=1}^{\infty}(-1)^n q^{\frac{5n^2-n}{2}} q^{2n} z^{2n}\left(1 - zq^{2n+1}\right)\frac{(zq^2;q)_{n-1}}{(q;q)_n} \\
&= zq + \sum_{n=1}^{\infty}(-1)^n q^{\frac{5n^2+n}{2}} z^{2n}\frac{(zq^2;q)_{n-1}}{(q;q)_n}\left(zq^{3n+1}\left(1 - zq^{n+1}\right) + (1-q^n)\right).
\end{aligned}$$

When we split the series again, according to the two terms within the braces, we get

$$\begin{aligned}
\frac{G(z)}{1-zq} - G(zq) = zq &+ \sum_{n=1}^{\infty}(-1)^n q^{\frac{5n^2+7n+2}{2}} z^{2n+1}\frac{(zq^2;q)_n}{(q;q)_n} \\
&+ \sum_{n=1}^{\infty}(-1)^n q^{\frac{5n^2+n}{2}} z^{2n}\frac{(zq^2;q)_{n-1}}{(q;q)_{n-1}}.
\end{aligned}$$

We separate out the first term in the last sum and then replace n by $n+1$ for the remaining terms to obtain

$$\begin{aligned}
\frac{G(z)}{1-zq} - G(zq) &= zq + \sum_{n=1}^{\infty}(-1)^n q^{\frac{5n^2+7n+2}{2}} z^{2n+1}\frac{(zq^2;q)_n}{(q;q)_n} - z^2q^3 \\
&\quad - \sum_{n=1}^{\infty}(-1)^n q^{\frac{5n^2+11n+6}{2}} z^{2n+2}\frac{(zq^2;q)_n}{(q;q)_n} \\
&= zq\left(1 - zq^2\right) + zq\sum_{n=1}^{\infty}(-1)^n q^{\frac{5n^2+7n}{2}} z^{2n}\left(1 - zq^{2n+2}\right)\frac{(zq^2;q)_n}{(q;q)_n} \\
&= zq\left(1 - zq^2\right)\left(1 + \sum_{n=1}^{\infty}(-1)^n q^{\frac{5n^2-n}{2}} q^{4n} z^{2n}\left(1 - zq^{2n+2}\right)\frac{(zq^3;q)_{n-1}}{(q;q)_n}\right) \\
&= zq\left(1 - zq^2\right)G(zq^2).
\end{aligned}$$

$\square$

Lemma 5.5. *For $G(z)$ as defined in (5.8), we have*

$$G(z) = \left(1 + \sum_{n=1}^{\infty}\frac{z^n q^{n^2}}{(q;q)_n}\right)(zq;q)_\infty.$$

Proof. Let

$$H(z) = \frac{G(z)}{(zq;q)_\infty},$$

then the three-term functional equation in Lemma 5.4 implies

$$H(z) = H(zq) + zqH\left(zq^2\right),$$

since

$$
\begin{aligned}
H(zq) + zqH(zq^2) &= \frac{G(zq)}{(zq^2;q)_\infty} + \frac{zqG\left(zq^2\right)}{(zq^3;q)_\infty} \\
&= \frac{(1-zq)G(zq)}{(zq;q)_\infty} + \frac{zq(1-zq)\left(1-zq^2\right)G\left(zq^2\right)}{(zq;q)_\infty} \\
&= \frac{(1-zq)\left(G(zq) + zq\left(1-zq^2\right)G\left(zq^2\right)\right)}{(zq;q)_\infty} \overset{(5.9)}{=} \frac{G(z)}{(zq;q)_\infty} = H(z)
\end{aligned}
$$

Consider the following expansion

$$
H(z) = \sum_{n=0}^{\infty} h_n z^n,
$$

then the coefficients satisfy the following recurrence relation

$$
\begin{aligned}
h_n &= q^n h_n + h_{n-1} q^{2n-1} \\
&= \frac{q^{2n-1}}{1-q^n} h_{n-1}.
\end{aligned}
$$

Solving this recurrence relation with $h_0 = 1$ we get

$$
h_n = \frac{q^{n^2}}{(q;q)_n}.
$$

Since $H(0) = 1$, we get

$$
H(z) = 1 + \sum_{n=1}^{\infty} \frac{z^n q^{n^2}}{(q;q)_n}.
$$

□

With these lemmas it is easy to prove the identities (5.6) and (5.7).

Proof of the Rogers-Ramanujan identities. For the first identity we set $z = 1$ in Lemma 5.5 to get

$$
G(1) = \left(1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q;q)_n}\right)(q;q)_\infty.
$$

Recall that we already know from Lemma 5.4 that

$$
G(1) = f(-q^2, -q^3).
$$

Therefore we get (5.6)

$$
\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{f\left(-q^2,-q^3\right)}{(q;q)_\infty} = \frac{1}{\left(q;q^5\right)_\infty \left(q^4;q^5\right)_\infty}.
$$

For the second identity we set $z = q$ in Lemma 5.5 to get

$$G(q) = \left(1 + \sum_{n=1}^{\infty} \frac{q^{n^2+n}}{(q;q)_n}\right) \left(q^2;q\right)_\infty .$$

Recall, that by Lemma 5.4

$$(1-q)G(q) = f(-q,-q^4),$$

hence

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{f\left(-q,-q^4\right)}{(q;q)_\infty} = \frac{1}{\left(q^2;q^5\right)_\infty \left(q^3;q^5\right)_\infty},$$

which proves (5.7). □

5.4 Combinatorial interpretation

Although the identities are analytic equations, they also have a combinatorial interpretation as generating functions of special partitions. Neither Rogers nor Ramanujan considered their combinatorial meaning, it was 1916 when Percy A. MacMahon published the first partition theoretical interpretation of them in [15, p. 33]. At this time the identities were not proven, MacMahon writes in [15, p. 33]:

"*This most remarkable theorem has been verified as far as the coefficient of x^{89} by actual expansion so that there is practically no reason to doubt its truth; but it has not been established.*"

Taking a look at the right hand side of both identities we see, in the same way as we got the generating function of n into distinct parts, that

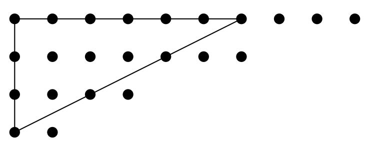
$$\frac{1}{\left(q;q^5\right)_\infty \left(q^4;q^5\right)_\infty} = \prod_{n=0}^{\infty} \frac{1}{\left(1-q^{5n+1}\right)\left(1-q^{5n+4}\right)}$$

is the generating function for partitions of n into parts congruent to 1 or 4 modulo 5 and

$$\frac{1}{\left(q^2;q^5\right)_\infty \left(q^3;q^5\right)_\infty} = \prod_{n=0}^{\infty} \frac{1}{\left(1-q^{5n+2}\right)\left(1-q^{5n+3}\right)}$$

is the generating function for partitions of n into parts congruent to 2 or 3 modulo 5.

For the left hand side of (5.1) we consider a partition of n into parts differing by at least 2. If the partition has exactly m parts, the number being partitioned is at least $1 + 3 + ... + 2m - 1 = m^2$, these are the dots in the triangle in the following Ferrers diagram for the partition $(10, 7, 4, 2)$, where $n = 23$ and $m = 4$.



Therefore a partition of n into m parts differing by at least 2 can be represented by a triangle of m^2 dots and a partition of $n - m^2$ into at most m parts. As we know already, the number of such partitions is the coefficient of q^{n-m^2} in

$$\frac{1}{(q;q)_m},$$

which is the same as the coefficient of q^n in

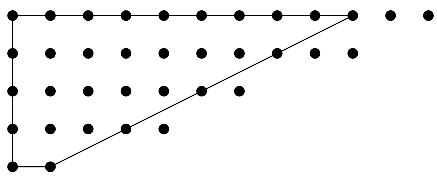
$$\frac{q^{m^2}}{(q;q)_m}.$$

If we sum over all m we get

$$\sum_{m=0}^{\infty} \frac{q^{m^2}}{(q;q)_m},$$

which is therefore the generating function for partitions of n into parts differing by at least 2.

For the left hand side of (5.2) we consider a partition of n into parts differing by at least 2 and 1 is excluded as a part. Such a partition with exactly m parts, must have at least $2+4+...+2m = m^2+m$ dots, which now build a trapezoid instead of a triangle in its Ferrers diagram. For the partition (12,10,7,5,2), where $n = 36$ and $m = 5$, it looks like the following



Therefore, a partition of n into exactly m parts differing by at least 2 where 1 is excluded as a part can be represented as a trapezoid of $m^2 + m$ dots and a partition of $n - (m^2 + m)$ into at most m parts. Like before, the number of such partitions is the coefficient of $q^{n-\left(m^2+m\right)}$ in

$$\frac{1}{(q;q)_m},$$

which is the same as the coefficient of q^n in

$$\frac{q^{m^2+m}}{(q;q)_m}.$$

If we again sum over all m we get

$$\sum_{m=0}^{\infty} \frac{q^{m^2+m}}{(q;q)_m},$$

which is therefore the generating function for partitions of n into parts differing by at least 2 and 1 is excluded as a part.

We now can state the Rogers-Ramanujan identities using their combinatorial interpretation.

Theorem 5.6. *(1) The number of partitions of n into parts differing by at least 2 is equal to the number of partitions of n into parts congruent to 1 or 4 modulo 5.*

(2) The number of partitions of n into parts differing by at least 2 and 1 is excluded as a part is equal to the number of partitions of n into parts congruent to 2 or 3 modulo 5.

5.5 Gordon's generalization

A few years after the rediscovery of the Rogers-Ramanujan identities, many mathematicians searched for generalizations of them. The first analytic generalization, which was proved in 1954, is due to H. L. Alder [1], though he was not able to interpret this identities partition theoretically. He proved the existence of polynomials $G_{k,n}(q)$, such that

$$\prod_{\substack{n\geq 1;\\ n\not\equiv 0,\pm k (\bmod 2k+1)}} \frac{1}{1-q^n} = \sum_{n=0}^{\infty} \frac{G_{k,n}(q)}{(q;q)_n}$$

and

$$\prod_{\substack{n\geq 1;\\ n\not\equiv 0,\pm 1 (\bmod 2k+1)}} \frac{1}{1-q^n} = \sum_{n=0}^{\infty} \frac{G_{k,n}(q)q^n}{(q;q)_n},$$

where $G_{2,n}(q) = q^{n^2}$ and so these identities reduce to the Rogers-Ramanujan identities for $k = 2$.
The first combinatorial generalization of the Rogers-Ramanujan identities was given by B. Gordon [11] in 1961.

Theorem 5.7 (Gordon 1961)**.** *Let $A_{k,i}(n)$ denote the number of partitions of n into parts not congruent to* 0 *or $\pm i$ (mod* $2k+1$*). Let $B_{k,i}(n)$ denote the number of partitions of n of the form $(b_1, b_2, ..., b_l)$, where $b_j - b_{j+k-1} \geq 2$ and at most $i-1$ parts equal* 1*. Then $A_{k,i}(n) = B_{k,i}(n)$.*

Remark 5.8. If we set $k = i = 2$ in Theorem 5.7, we get (1) in Theorem 5.6.
If we set $k = 2$ and $i = 1$ in Theorem 5.7, we get (2) in Theorem 5.6.

We are going to prove the theorem, by the help of a few Lemmas, in a similar way as we proved the Rogers-Ramanujan identities.

Definition 5.9. We define

$$H_{k,i}(a,z,q) = \sum_{n=0}^{\infty} \frac{z^{kn}q^{kn^2+n-in}a^n\left(1-z^iq^{2ni}\right)\left(azq^{n+1};q\right)_\infty\left(a^{-1};q\right)_n}{(q;q)_n\left(zq^n;q\right)_\infty}$$
$$J_{k,i}(a,z,q) = H_{k,i}(a,zq,q) - zqaH_{k,i-1}(a,zq,q),$$

where any value for a is admissible, even $a = 0$, since
$a^n\left(a^{-1};q\right)_\infty = (a-1)(a-q)\cdots(a-q^{n-1})$ is a polynomial in a whose value at 0 is $(-1)^nq^{\frac{n(n-1)}{2}}$.

Lemma 5.10.

$$H_{k,i}(a,z,q) - H_{k,i-1}(a,z,q) = z^{i-1} J_{k,k-i+1}(a,z,q)$$

Proof. We first note that

$$q^{-in}\left(1 - z^i q^{2ni}\right) - q^{-(i-1)n}\left(1 - z^{i-1}q^{2n(i-1)}\right) = q^{-in}(1-q^n) + z^{i-1}q^{n(i-1)}(1-zq^n).$$

Using this observation, we see that

$$\begin{aligned}
H_{k,i}(a,z,q) - H_{k,i-1}(a,z,q) &= \sum_{n=0}^{\infty} \frac{z^{kn}q^{kn^2+n}a^n\left(azq^{n+1};q\right)_\infty\left(a^{-1};q\right)_n}{(q;q)_n\left(zq^n;q\right)_\infty} q^{-in}(1-q^n) \\
&+ \sum_{n=0}^{\infty} \frac{z^{kn}q^{kn^2+n}a^n\left(azq^{n+1};q\right)_\infty\left(a^{-1};q\right)_n}{(q;q)_n\left(zq^n;q\right)_\infty} z^{i-1}q^{n(i-1)}(1-zq^n) \\
&= \sum_{n=1}^{\infty} \frac{z^{kn}q^{kn^2+n}a^n\left(azq^{n+1};q\right)_\infty\left(a^{-1};q\right)_n q^{-in}}{(q;q)_{n-1}\left(zq^n;q\right)_\infty} \\
&+ \sum_{n=0}^{\infty} \frac{z^{kn}q^{kn^2+n}a^n\left(azq^{n+1};q\right)_\infty\left(a^{-1};q\right)_n z^{i-1}q^{n(i-1)}}{(q;q)_n\left(zq^{n+1};q\right)_\infty} \\
&= \sum_{n=0}^{\infty} \frac{z^{kn+k}q^{kn^2+n+2kn+k+1}a^{n+1}\left(azq^{n+2};q\right)_\infty\left(a^{-1};q\right)_{n+1} q^{-in-i}}{(q;q)_n\left(zq^{n+1};q\right)_\infty} \\
&+ \sum_{n=0}^{\infty} \frac{z^{kn}q^{kn^2+n}a^n\left(azq^{n+1};q\right)_\infty\left(a^{-1};q\right)_n z^{i-1}q^{n(i-1)}}{(q;q)_n\left(zq^{n+1};q\right)_\infty} \\
&= z^{i-1}\sum_{n=0}^{\infty} \frac{z^{kn}q^{kn^2+in}a^n\left(azq^{n+2};q\right)_\infty\left(a^{-1};q\right)_n}{(q;q)_n(zq^{n+1};q)_\infty} \\
&\cdot\left(az^{k-i+1}q^{2n(k-i)+n+k-i+1}\left(1-\frac{q^n}{a}\right) + (1-azq^{n+1})\right) \\
&= z^{i-1}\sum_{n=0}^{\infty} \frac{z^{kn}q^{kn^2+in}a^n\left(azq^{n+2};q\right)_\infty\left(a^{-1};q\right)_n}{(q;q)_n(zq^{n+1};q)_\infty} \\
&\quad\cdot\left(1-(zq)^{k-i+1}q^{2n(k-i+1)}\right) \\
&- z^{i-1}\sum_{n=0}^{\infty} \frac{z^{kn}q^{kn^2+in}a^n\left(azq^{n+2};q\right)_\infty\left(a^{-1};q\right)_n}{(q;q)_n(zq^{n+1};q)_\infty}\cdot \\
&\quad\cdot\left(azq^{n+1}\left(1-(zq)^{k-i}q^{2n(k-i)}\right)\right) \\
&= z^{i-1}\left(H_{k,k-i+1}(a,zq,q) - azqH_{k,k-i}(a,zq,q)\right) \\
&= z^{i-1}J_{k,k-i+1}(a,z,q).
\end{aligned}$$

□

Lemma 5.11.

$$J_{k,i}(a,z,q) - J_{k,i-1}(a,z,q) = (zq)^{i-1}\left(J_{k,k-i+1}(a,zq,q) - aJ_{k,k-i+2}(a,zq,q)\right) \tag{5.11}$$

Proof.

$$\begin{aligned} J_{k,i}(a,z,q) - J_{k,i-1}(a,z,q) &= (H_{k,i}(a,zq,q) - H_{k,i-1}(a,zq,q) \\ &\quad - azq(H_{k,i-1}(a,zq,q) - H_{k,i-2}(a,zq,q)) \\ &= (zq)^{i-1}J_{k,k-i+1}(a,zq,q) - a(zq)^{i-1}J_{k,k-i+2}(a,zq,q), \end{aligned}$$

where the last equation follows from the previous Lemma. □

Lemma 5.12.

$$J_{k,i}(0,1,q) = \prod_{\substack{n\geq 1;\\ n\not\equiv 0,\pm i (\text{mod } 2k+1)}} \frac{1}{1-q^n}. \tag{5.12}$$

Proof. Notice that

$$\frac{1}{(q;q)_\infty} = \frac{1}{(q;q)_n\,(q^{n+1};q)_\infty}. \tag{$\bigstar$}$$

By the definition of $J_{k,i}(a,z,q)$ we get

$$\begin{aligned} J_{k,i}(0,1,q) &= H_{k,i}(0,q,q) \\ &\overset{(\bigstar)}{=} \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty} q^{kn^2+n(k-i+1)}(-1)^n q^{\frac{n(n-1)}{2}}\left(1-q^{(2n+1)i}\right) \\ &= \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{\frac{(2k+1)n(n+1)}{2}-in}\left(1-q^{(2n+1)i}\right) \\ &\overset{\substack{Cor\\3.13}}{=} \frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{(2k+1)n(n+1)/2-in} \\ &\overset{\substack{Cor\\3.13}}{=} \frac{1}{(q;q)_\infty}\prod_{n=0}^{\infty}\left(1-q^{(2k+1)(n+1)}\right)\left(1-q^{(2k+1)n+i}\right)\left(1-q^{(2k+1)(n+1)-i}\right) \\ &= \prod_{\substack{n\geq 1;\\ n\not\equiv 0,\pm i (\text{mod } 2k+1)}} \frac{1}{1-q^n} \end{aligned}$$

□

We now proceed with Gordon's theorem and follow the proof in [7, 109 ff.].

Proof of Theorem 5.7. Let $b_{k,i}(m,n)$ denote the number of partitions of n of the form $(\lambda_1,\ldots,\lambda_m)$ with exactly m parts, such that $\lambda_j \geq \lambda_{j+1}$, $\lambda_j - \lambda_{j+k-1} \geq 2$ and at most $i-1$ of the λ_j equal 1. Then for $1 \leq i \leq k$

$$b_{k,i}(m,n) = \begin{cases} 1 \text{ if } m = n = 0 \\ 0 \text{ if } m \leq 0 \text{ or } n \leq 0 \text{ but } (m,n) \neq (0,0), \end{cases} \tag{5.13}$$

$$b_{k,0}(m,n) = 0. \tag{5.14}$$

These conditions are obvious, since the only partition that is either of a non-positive number or has a non-positive number of parts is the empty partition of 0.
Now we consider $b_{k,i}(m,n) - b_{k,i-1}(m,n)$: This enumerates the same kind of partitions of n as $b_{k,i}(m,n)$ does, except the condition, that exactly $i-1$ parts are equal to 1. We now transform this set of partitions. At first we delete the $i-1$ ones and then subtract 1 from each of the remaining parts. The resulting partitions $(\lambda'_1, ..., \lambda'_{m-i+1})$ have $m-i+1$ parts, they partition $n-m$ and the parts satisfy $\lambda'_j - \lambda'_{j+k-1} \geq 2$. Since originally 1 appeared $i-1$ times and, because of the difference condition, the total number of ones and twos can not exceed $k-1$, we see that originally 2 appeared at most $k-1-(i-1)$ times. Therefore after the transformation 1 appears at most $k-i$ times. This transformation establishes a one-to-one correspondence between partitions enumerated by $b_{k,i}(m,n) - b_{k,i-1}(m,n)$ and those enumerated by $b_{k,k-i+1}(m-i+1, n-m)$, so the following holds

$$b_{k,i}(m,n) - b_{k,i-1}(m,n) = b_{k,k-i+1}(m-i+1, n-m). \tag{5.15}$$

We have to convince ourselves, that the $b_{k,i}(m,n)$ are uniquely determined for $0 \leq i \leq k$ by (5.13), (5.14) and (5.15). Equations (5.13) and (5.14) handle the case where $n, m \leq 0$, $i > 0$ and $n > 0$, $i = 0$. These are the starting points of our recursion (5.15), which represents $b_{k,i}(m,n)$ as a sum of two terms, where the first term has a lower i index and the second a lower n index, since we can assume $m > 0$, so the $b_{k,i}(m,n)$ are uniquely determined. Now let us consider

$$J_{k,i}(0,z,q) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{k,i}(m,n) z^m q^n.$$

From the fact that for $1 \leq i \leq k$

$$J_{k,i}(0,0,q) = J_{k,i}(0,z,0) = 1,$$

we see that for $1 \leq i \leq k$

$$c_{k,i}(m,n) = \begin{cases} 1 \text{ if } m = n = 0 \\ 0 \text{ if } m \leq 0 \text{ or } n \leq 0 \text{ but } (m,n) \neq (0,0). \end{cases}$$

From the fact that

$$J_{k,0}(0,z,q) = H_{k,0}(0,zq,q) = 0,$$

we see that

$$c_{k,0}(m,n) = 0.$$

With this preparation we take a look at equation (5.11) for $a = 0$ we get

$$c_{k,i}(m,n) - c_{k,i-1}(m,n) = c_{k,k-i+1}(m-i+1, n-m).$$

Hence $c_{k,i}(m,n)$ also satisfies equations (5.13)-(5.15) that uniquely define $b_{k,i}(m,n)$. Therefore $b_{k,i}(m,n) = c_{k,i}(m,n)$ for all m and n with $0 \leq i \leq k$. Since

$$\sum_{m=0}^{\infty} b_{k,i}(m,n) = B_{k,i}(n),$$

we see that

$$\begin{aligned}
\sum_{n=0}^{\infty} B_{k,i}(n)q^n &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} b_{k,i}(m,n)q^n \\
&= J_{k,i}(0,1,q) \\
&\overset{\substack{Lemma\\ 5.12}}{=} \prod_{\substack{n\geq 1;\\ n\not\equiv 0,\pm i \pmod{2k+1}}} \frac{1}{1-q^n} \\
&= \sum_{n=0}^{\infty} A_{k,i}(n)q^n.
\end{aligned}$$

So we get $A_{k,i}(n) = B_{k,i}(n)$.

□

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Appendix

Abstract

The theory of partitions started with the famous mathematician Leonhard Euler, who was asked how many partitions there are of 50 into 7 distinct parts. He solved this problem using generating functions and was then interested in the generating functions for partitions of n, whose number of partitions is given by $p(n)$. After that, he studied the reciprocal of the generating function of $p(n)$, the result was Euler's pentagonal number theorem. The next important mathematician in this topic was J.J. Sylvester, his approach to consider partitions graphically was a huge innovation and gave a new point of view. Using the theory of generating functions and the graphical representation of partitions we prove several partition identities, for example that the number of partitions of n into odd and distinct parts is equal to the number of self-conjugate partitions of n. Considering partitions with some restrictions and properties lead us to two variable generating functions, where we have to handle hypergeometric functions and come across Gaussian polynomials and Jacobi's triple product identity. This identity will be the key for proving the major result of the last part; The Rogers-Ramanujan identities. Finally we prove a generalization of them, which was the work of Gordon.

Zusammenfassung

Die Theorie der Partitionen begann mit dem berühmten Mathematiker Leonhard Euler, der die Frage gestellt bekam, wie viele Partitionen es von 50 in 7 unterschiedliche Teile gibt. Er löste dieses Problem mit erzeugenden Funktionen und war danach an der erzeugenden Funktion für Partitionen von n, deren Anzahl mit $p(n)$ angegeben wird, interessiert. Danach untersuchte er die reziproke Funktion der erzeugenden Funktion von $p(n)$ und formulierte als Ergebnis den Pentagonalzahlensatz bekannt ist. Ein weiterer wichtiger Mathematiker in diesem Gebiet war J.J. Sylvester, dessen Ansatz Partitionen graphisch darzustellen eine große Innovation war und einen neuen Zugang ermöglichte.

In dieser Arbeit beweisen wir mit erzeugenden Funktionen und der graphischen Darstellung von Partitionen einige Identitäten von Partitionen, wie zum Beispiel, dass die Anzahl der Partitionen von n in ungerade und verschiedene Teile gleich der An-

zahl von selbstkonjugierten Partitionen von n ist. Beim Betrachten von Partitionen mit unterschiedlichsten Einschränkungen und Eigenschaften nutzen wir erzeugenden Funktionen mit zwei Variablen, arbeiten mit hypergeometrischen Funktionen und treffen auf Gauß'sche Polynome und Jacobi's Tripleprodukt Identität. Diese Identität ist der Schlüssel um das Hauptthema des letzten Abschnittes zu beweisen; die Rogers-Ramanujan Identitäten. Zum Schluss beweisen wir eine ihrer Verallgemeinerungen, die auf Gordon zurückgeht.