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„Evolutionary Dynamics and Rationality"

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Mathias Staudigl

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Education

June 2003	High School Diploma (Matura) with honors
October 2003-October 2004	Alternative civilian service at the "Lebenshilfe Österreich" and the "Behindertenhilfe Bezirk Korneuburg", lower Austria. I was working together with partly disabled persons.
March 2004-October 2007	Studies in Economics at the Department of Economics and Statistics, University of Vienna.
June 2006	First diploma (Erster Studienabschnitt) with honors
October 2006-February 2007	Studies in Economics and Mathematics at the University of Alicante, Spain

Teaching Experience

Tutor for advanced Microeconomics class (emphasis on general equilibrium theory, Prof.Podczeck). Spring term 2007. For this tutorial I have written a short manuscript that should give students some idea about (binary) relations, topology, and correspondences.(Available on request)

Employment

2006 Department of International Finance at the österreichische Kontrollbank in Vienna. My main work covered the estimation of the Fed Funds Rate using the so-called Taylor-Rule.

Various practical trainings during high-school (mainly in the service sector). In particular I was working 3 months at the Front-office of a "Mercure"-Hotel (Accor Group) in Vienna.

Languages

German: Mother tongue
English: Excellent written and spoken proficiency
Spanish: Very good written and spoken proficiency
French: Basic knowledge

Academic Interests and things I would like to do

My field of specialization is the wide range of Microeconomics, especially (evolutionary) Game Theory. But I pay also some interest in rather "classic" topics like General Equilibrium Theory. Due to this microeconomic background I feel myself comfortable in some areas of Macroeconomics, concerning questions of "optimal growth" (I have written a seminar paper about the mathematics of the "Ramsey Model"), and so called "Population games" (Hofbauer and Sigmund) which find their natural application in macroeconomic questions. Recently I become more and more in touch with the theory of ordinary differential equations and their applications to Game Theoretical problems. However, I am not reluctant to applied fields of economics. In particular, questions of experimental economics, Econometrics and environmental economics are areas of my interest.

Honors and Awards

2004 Award at the Austrian prize for Investigations in the field of Tourism (Österreichischer Tourismus-Forschungspreis). Prize granted to the best research works at secondary school level. I wrote the rewarded work for my high-school diploma.

2005 Prize for Academic Excellence (Leistungsstipendium), University of Vienna

2006 Scholarship "ERASMUS". I had the pleasure to visit the University of Alicante, Spain, for the winter term.

EVOLUTIONARY DYNAMICS AND RATIONALITY

MATHIAS STAUDIGL

I dedicate this work to Petra Wührer, not only for years of listening, but more importantly for her moral, emotional and social support. I really admire you. Appraisals also to Ana B. Ania-Martinez, for reading and discussing all the material covered here. I hope the time was not waisted.

Contents

Preface

This thesis elaborates the relations between dynamic evolutionary game theory, the concept of a Nash equilibrium, and the potential use of the theory to serve as an equilibrium selection technique. It emphasizes the importance of different types of evolutionary dynamics, beside the well known replicator dynamic. The vast majority of the literature is concerned with games in normal form. In the last Part I apply evolutionary dynamics to two examples of games in extensive form. One is a perfect information game, where already some general results are known, the other one is a game where one player has imperfect information, where not much is known. In this example we will see how evolutionary dynamics can help to select equilibrium points, when a continuum of Nash equilibria exist. The rest of the thesis is organized as follows:
Part 1 gives a self contained discussion about the well known rationality assumptions that are embodied in the notion of the (nowadays) most common solution method of non-cooperative game theory, the Nash equilibrium. A different interpretation of this solution concept is provided, which was already proposed by John Nash himself in his unpublished Ph.D. Thesis.

Part 2 gives a concise definition of games in normal and extensive form, where we identify players not with single individuals, but much more with large (theoretically infinite) populations. The definition of an extensive form game is influenced by Cressman (2003, in particular chapter 6) and Fudenberg and Tirole (1991, chapter 3). The formalization of decision sets, which establish a correspondence between nodes and players, was introduced by myself. The definition of normal form games follows closely Sandholm (2007a), but also elements of Weibull (1995) and Hofbauer and Sigmund (1998) can be found there.

A short and self-contained introduction to the theory of dynamical systems is given to clarify the meaning of some phrases used in the discussions. The concepts introduced there are of crucial importance to study dynamical systems, and so I found it necessary to add this chapter. Resources therefore have been the books of Hirsch, Smale and Devaney (2004), Königsberger (2000) and Hofbauer and Sigmund (1998).

The core of the thesis deals with three important families of evolutionary dynamics, and fills Part 3 completely. The discussion there is closely related to Sandholm (2007a). Other references can be found in the text. The majority of proofs are adapted versions of Sandholm (2007a), others are made more readable, or are even self-contained.

Immediately after the introduction of the dynamics, an example will be given to demonstrate similarities, differences, and special attributes of the models. Additionally, a self-contained economic derivation of the replicator dynamic is given. Long computations are relegated to the Appendix, which proves the global asymptotic stability of the unique Nash equilibrium in the "Matching Pennies" game under two particular evolutionary dynamics. The proof is formulated by means of Lyapunov functions, which are discussed in Sandholm (2007a) and Hofbauer and Sandholm (2007a, 2007b), and are here generalized to the two-population setting. These proofs are completely formulated by my own.

Part 4 will present three examples of games in extensive form. These examples shall demonstrate the role that evolutionary dynamics might play when we are confronted with games, that have implausible Nash equilibria that pass the test of subgame perfection. Since not much is known about evolutionary dynamics and extensive form games that have not perfect information, these examples are designed as a motivation for further investigations.

Comment of the author: The explicit use of "he" and "she", or "his" and "hers", is not intended to provoke any "battle of the sexes". My intention was much more to make the reading as fluent as possible. In some examples I call one player as "he" while the other as "she". This shall not be misunderstood with any gender role assignment.

Vorwort

Diese Diplomarbeit beschäftigt sich mit den Zusammenhängen zwischen dynamischen Evolutionären Modellen und dem Lösungskonzeptes des Nash Gleichgewichtes, sowie der Möglichkeit eben diese dynamischen Modelle zur Selektion von Gleichgewichten zu verwenden. Besonders hervorgehoben wird die Bedeutung der verschiedenen Modellansätze, die neben der oft verwendeten Replikator Dynamik in der Literatur eingeführt wurden. Die Mehrheit der bekannten Resultate wurde in Spielen in Normalform hergeleitet. Im letzten Teil dieser Arbeit werden Spiele in ihrer extensiven Form unter dynamischen evolutionären Gesichtspunkten analysiert. In diesen Beispielen wird versucht dazustellen wie diese Dynamiken als Selektionsmechanismen zwischen konkurrierenden Gleichgewichten verwendet werden können. Die Diplomarbeit ist wie folgt organisiert:

Der erste Teil ist eine eigenständig verfasste Diskussion der Rationalitätsannahmen, welche implizit in dem Lösungskonzeptes eines Nash Gleichgewichtes stecken. Eine alternative Interpretation eben dieses Gleichgewichtskonzeptes wird dargelegt, welche schon auf die Arbeiten von John Nash zurück gehen.

Der zweite Teil gibt eine formale Definition von Spielen in Normalform, als auch in extensiver Form. Dem üblichen Rahmen der evolutionären Spieltheorie folgend, werden hier Spieler nicht als Individuen betrachtet, sondern durch theoretisch unendlich große Population verkörpert. Dieser Abschnitt beruht auf Cressman (2003, insbesondere Kapitel 6), sowie Fudenberg und Tirole (1991, Kapitel 3). Eine formale Behandlung von "Entscheidungsmengen" (im Text als "decision set" bezeichnet) stammt von mir selbst. Derartige Mengen stellen eine Korrespondenz zwischen Entscheidungsknoten und Spielern dar. Die hier verwendete Definition von Spielen in Normalform ist aufbauend auf Sandholm (2007a), doch auch Elemente von Weibull (1995) und Hofbauer und Sigmund (1998) können gefunden werden.

Eine kurze Einführung in die Theorie dynamischer Systeme wird dargelegt. Dies dient dazu, dem Leser die grundlegenden Begriffe näher zu bringen, die im Laufe der Diplomarbeit verwendet werden. Verwendete Materialen hierfür waren die Bücher von Hirsch, Smale und Devaney (2004), Königsberger (2000) sowie Hofbauer und Sigmund (1998).

Der Hauptteil der Arbeit stellt drei verschiedene Modelle von evolutionären Dynamiken vor. Die Darstellung ist eng verwandt mit der in Sandholm (2007a). Weitere Literaturhinweise sind im Text vermerkt.

Die meisten bewiesenen Sätze können auch in Sandholm (2007a) gefunden werden. Alle sind jedoch von mir selbst überarbeitet, umgeschrieben und teilweise besser lesbar gemacht worden. Manche sind auch von mir selbst durchgeführt worden. Nach jeder besprochenen Dynamik wird ein illustratives Beispiel vorgerechnet, welches die Eigenheiten der jeweiligen Modelle verdeutlichen soll. Längere Rechnungen sind im Appendix anzufinden. Dort wird die globale asymptotische Stabilität des eindeutigen Gleichgewichts im "Matching Pennies" Spiel, in zwei spezifischen evolutionären Dynamiken bewiesen. Der Beweis verwendet geeignete Lyapunov Funktionen, wie sie von Sandholm (2007a) und Hofbauer und Sandholm (2007a, 2007b) definiert wurden. Ich generalisierte diese Ansätze auf Spiele zwischen zwei Population. Der Beweis wurde von mir eigenständig erbracht.

Im vierten Teil sind drei Beispiele von Spielen in extensiver Form angeführt. Hier soll die Rolle von evolutionäre Dynamiken analysiert werden, wenn mehere, unter umständen unplausible, Teilspielperfekte Gleichgewichte exisitieren. Abgesehen von Spielen mit perfekter Information ist noch wenig bekannt auf diesem Gebiet. Dieser Abschnitt soll daher als Motivation zukünftiger Forschungsarbeit dienen.

Part 1. Introduction

1. What to know to play Nash?

Game theory is widely used today in various sciences. The social science, in particular economics, were among the first field were it was applied. But also natural scientists, most of all biologists, discovered this tool quite early. It is often said that the work by Maynard Smith and Price (1973) was among the first which made use of (and then extended) a game-theoretic equilibrium concept to biological systems. Their work is often taken as the birth of evolutionary game theory, which is nowadays one of the most dynamic and interdisciplinary fields in the world of science. Their notion of an evolutionary stable strategy (ESS) opened the way to reinterpret the most common solution concept of non-cooperative game theory, namely the notion of a Nash equilibrium (NE). In particular it opened doors to economists that were not that satisfied with the informational assumptions that have to be made, when applying NE. The fundamental "common knowledge" assumption is something that lies implicitly in the concept of NE. As an example consider the simple 2-player game in extensive form in Figure 1. Player

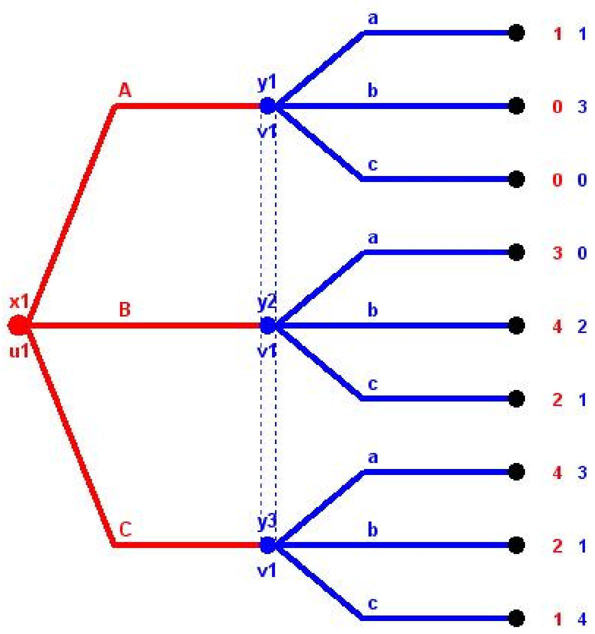

Figure 1. A game with unique Nash equilibrium in pure strategies

1 starts the game by choosing one out of three actions. Player 2 does

not know with certainty which action player 1 has selected. At every decision node, which are indicated by small letters (hence y_1, y_2, y_3, for player 2), player 2 will ask herself what the best action might be, if player 1 has chosen the action that leads to this decision node. For example, if player 1 plays A, player 2 knows that she has to decide at her decision node y_1. Payoffs are received at the terminal nodes of the play. The first numbers are the payoffs to player 1 (red colored), while the last numbers (blue colored) are the payoffs of player 2. Both players know the possible payoffs that might arise in the game. From this information, player 2 can infer that her opponent will never decide to play action A. Independently of what she does, player 1 can perform better by choosing either B or C, so why should he choose A? As a player who wants to win the game (otherwise it needs some justification of why the two meet and play the game), it does not make sense to assume that player 1 will really play action A. But not enough, player 1 also knows that player 2 sees that his action A is never optimal, and moreover he knows that player 2 knows that he is a rational player who wants to get the maximum out of the game. She will eliminate action A mentally, knowing that she plays with a rational guy, and knowing that he knows that she knows that he is rational. In this way she is sure that her node y_1 is never reached during play. After eliminating A, player 1 will know that his opponent's action a is also never an optimal reply, independently of what he actually does. Following the same considerations as player 2, he will eliminate action a mentally out of player 2's action set. But now player 2 sees that player 1 always earns a higher payoff when he selects his second action B (4 is better than 2 and 2 is better than 1). Thus, she knows that player 1 will play his second action, and by maximizing payoffs conditionally on this information, she will play her second action. In this way we have found a unique solution of this game, by making use of the so-called "iterative elimination of strictly dominated strategies" algorithm. If both players are rational in this fairly convincing sense, the game path will be such that player 1 chooses B, player 2 will choose b, and none of them have an incentive to deviate from this "convention", ceteris paribus. We want to emphasize that we could perform this algorithm, just because it is assumed that the players can correctly anticipate the behavior of the other player, or differently, players rely on each other that the opponent plays the game as he would play it.

In the example of Figure 1 the algorithm stopped after four rounds of deletion. In every consecutive round, the assumptions on knowledge became stronger. In the first round player 2 could infer that the first action of player 1, A, is strictly dominated by the other available actions B and C. At the same instant she knows that player 1 also knows that his first action is never optimal, independently of what she does,

so he will never choose this action. Note how important the phrase "she knows that he knows that..." is in this argument. Without it no action can be a priori excluded, and the algorithm breaks down at the first iteration.
In the second round player 1 knows that player 2 will eliminate her first action, because it is strictly dominated. This embodies one more assumption. Now not only player 2 knows that player 1 is rational, in the sense that only optimal actions are chosen, but also player 1 knows that player 2 knows that her first action is never optimal. Then player 2 knows, that player 1 will not choose his third action, and finally player 1 knew that only the second action for player 2 is optimal for her. Player i knows that player j knows that.... That's the way how every new round of the algorithm starts. The apparent problem with this reasoning is that we assume that players know how to play the game a-priori.

Dynamic evolutionary game theory gives a chance to interpret the way how the game is played in a different way. In such a setting we do not look at players in the conventional sense, but rather interpret a game as a situation where different populations interact. The set of all possible populations that could meet to play the game, can be called the society which we are analyzing. The theory starts from the assumption that the individual members of the populations are initially programmed to a certain pure strategy and then gradually adjust their behavior depending on what society does. If we observe that the aggregate behavior converges to Nash equilibrium, we have found a very strong justification for the assumption that people actually play Nash. This is accomplished without any additional assumptions concerning knowledge of players, or any belief that a player might have how his opponent will behave.
There are two other reasons why a dynamic look at a game is something valuable. First, suppose there are two players that meet to play the famous "Matching Pennies" game, which will serve as the "role model" in subsequent sections. It is represented in extensive form in Figure 2. This game possesses one completely mixed NE where each player chooses one of his actions with probability $1/2$. A dynamic analysis of this game offers a straightforward explanation why randomizing is optimal in this game. Particularly in economics it is often very difficult to justify any Nash equilibrium in mixed strategies. However, if we only look at equilibria in pure strategies, we are confronted with games that have no Nash equilibrium (i.e. games of the Matching pennies type). This is very unsatisfactory for a solution concept. However, if we equip "Matching pennies" with a dynamic story, there is no problem with having mixed strategies. To see this point, let us investigate "Matching pennies" a little bit closer. It is a two player game, where

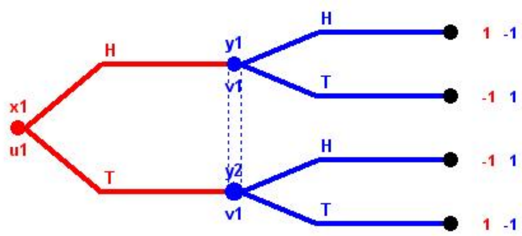

FIGURE 2. The Matching Pennies game in Extensive Form

both have the same set of possible actions. Suppose there are two large populations from which two individuals (from each population one) are randomly drawn to play “Matching Pennies”. Before the game starts, every member of one population has to declare himself being a H-player or a T-player. Both populations are of the same size, which we normalize to unity. When drawn to play the game, individuals do what their pure strategies tell them to do simultaneously, receive the payoffs, and then a new pair of individuals is drawn and told to play “Matching Pennies”. If the populations are sufficiently large we can repeat this random matching process very often, so that the game is played “for a long time” by different members of the population. Since agents initially have to declare themselves being an H or an T strategist, we can partition the player populations into fractions of H-players and fractions of T-players. Thus, a player population can be segregated into two disjoint subpopulations, labeled through the two pure strategies. The sum of these disjoint subpopulations must be equal to the total population size, which is normed to unity. Suppose that at some point of time the player proportions in the populations are unequal. Say, for example, the proportion of H-players in population 1 is $3/4$, so that 75 percent of the population is devoted to pure strategy H, and $1/4$ is a declared member of subpopulation T. Suppose further that the converse is given data in population 2, that is $1/4$ of the individuals in population 2 are declared H-strategists. The expected payoff that an H-strategist of population 1 gains in a game is $1/4$ times the payoff he receives when meeting another H-strategist of population 2, plus $3/4$ times the payoff he will receive when his opponent is a T-player from population 2. Thus, the average payoff to members of subpopulation H equals $-1/2$. Similarly, the expected payoff of T players in population 1 is then $1/2$.[1] On average the more prevalent subpopulation of the first

[1] Note that all what subpopulation H loses on average is absorbed by the subpopulation T. This is a nice characteristic of *zero-sum games*, where all what one player

player population, in our illustration H, loses, while those that appear with a lower frequency gain on average. Now the dynamic story becomes important. Suppose that, as time proceeds, players are allowed to switch strategies. The sizes of the subpopulations are allowed to vary over time. A possible way to model this switching of strategies is by assuming that players imitate the better. At some (rare) moments of time individuals meet an comrade from their own population and get to know the payoff that this agent received. Imitate the better means then, that a player adopts the strategy of his opponent if the comrade receives a higher average payoff with his strategy. In our example this would mean that an H strategist in player population 1 will alter his strategy in favor for strategy T, since the T players receive a higher average payoff. Consequently the fraction of players who decide to play H must decline over time, leading to an increase of the relative size of the T subpopulation in player population 1. A similar adaption process will evolve in population 2, if members of this population behave just as members of population 1 do, where the fraction of T players will decline. The state (or the composition) of the populations change over time, because of players changing their pure strategies. When does this adjustment of subpopulations stop under the proposed *imitative rule*? Obviously only then, when no subpopulation has an higher average payoff than the other, in both player populations. Note that this occurs if and only if the sizes of the subpopulations are equal, thus 50 percent of the individuals are H-strategists and 50 percent are T strategist. This is true from the point of view of population 1 and 2. At not other partition of the populations is such a *stationary state* reached. In terms of evolutionary game theory, one can interpret the Nash equilibrium in Matching Pennies as a partition of the population, where one half of the mass of the population plays H while the other half plays T, and such a partition is mathematically equivalent to a mixed strategy profile.

Second, there are games with more than one Nash equilibrium. A slight change in the payoffs of matching pennis gives us the class of anti-coordination games. One of them is shown in Figure 3. This is a so-called Hawk-Dove game, which possesses three NE, one in mixed and 2 in pure strategies. A solution concept with multiple solutions is clearly something irritating. Which equilibrium shall the theorist predict in such a case? Again, evolutionary dynamic game theory can help as an *equilibrium selection mechanism*. By looking at the dynamic pattern of the strategy adjustment in a "neighborhood" of one of the equilibria, we are able to predict which of the NE are stable in a precise

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wins must be payed by the opponent. The outcome of such a game is therefore a nice model of pure redistributions among players (or, in our framework, of society groups.)

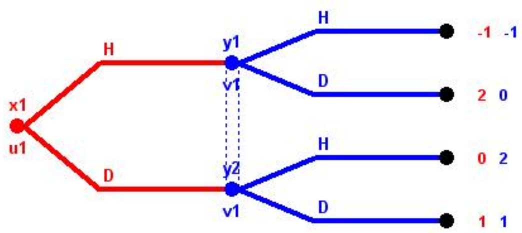

FIGURE 3. An anti-coordination game

way. If some of the found equilibria are stable in this sense, we would have some confident in researchers forecasting such a behavior.

2. Nash equilibrium in large populations

Evolutionary game theory draws a picture of finitely many large populations, from which individuals are randomly drawn and matched to play a certain N-player game. Behind each player, we imagine a large population. What does this have in common with the classical non-cooperative game theory? Suppose that every individual in a population employs a single pure strategy, which is one of possibly many strategies that individuals of this population have. Even if the population is very large (possibly infinite, so that we can speak of a continuum of individuals), we can measure the number of agents in the population that employ some pure strategy. If we set this number in relation to the overall mass of the population, we get the proportion of individuals belonging to the population and devoted to some pure strategy. This proportion can also be interpreted as a *subpopulation* of individuals. We consider only games with finite strategy sets, so, there are only finitely many subpopulations, whose shares of the total population size have to add up to one. The list of sizes of subpopulations is therefore mathematically equivalent to a mixed strategy profile, and a Nash equilibrium can be seen as a distribution of sizes of subpopulations. A central question is now: How do individuals in infinite populations coordinate to arrive at the Nash equilibrium? One suggestion was given by John Nash:

> It is unnecessary to assume that the participants have full knowledge of the total structure of the game, or the ability and inclination to go through any complex reasoning

> process. But the participants are supposed to accumulate empirical information on the relative advantages of the various pure strategies at their disposal.[2]

The relative advantages of which Nash speaks here is the expected payoff of a pure strategy against a mixed strategy profile, which we identify with the composition of the population to which the opponent players belong. What is meant with empirical information is the gathering of data that describes the performance of the various pure strategies, at a given population state. Once the population members found that their currently used strategy performs worse than some other available pure strategy, they are assumed to abandon their strategy and switch to those that promise higher rewards.[3] Through this learning and imitation mechanism, Nash predicts that the state of the large population will converge to his equilibrium concept. In greatest generality, an equilibrium, in the sense of Nash, is then a distribution of sizes of subpopulations, such that these sizes remain constant over time.
We will follow this idea in this work, were especially the link between the equilibrium concept of Nash and the (long-run) outcome of evolutionary dynamics shall be elaborated.

[2]This quotation is taken from Björnerstedt and Weibull (1993, p.156).

[3]One should note that we define player populations as a large mass of human beings that have common preferences in the game to be played.

Part 2. Games in large Populations

3. Definition of a N-population game

In this section we want to introduce a game in a slightly different way as it is done in classical game theory. The standard definition of a game (in normal form) is given by a (finite) set of players, a finite set of pure strategies for each player, and a payoff function for each player associated with the strategies (see Fudenberg and Tirole 1991). Instead we consider a "society" consisting of N, $N < \infty$, different populations. Formally we can describe our model society as a countable set of populations $\mathcal{P} = \{1, \dots, N\}$. Each population is assumed to form a continuum of positive mass, m_k, $1 \leq k \leq N$. If we look at an arbitrary population we simply write k for this population, and k is an index that can vary between 1 and N.
As is usual in evolutionary game theory, we imagine that individuals are drawn randomly from the distinct populations and are matched to play a particular game in question. Each population may engage in a strategic conflict in different positions of the game. A situation of competition between different populations can be described through an N-player game, where behind each player we imagine a large (infinite) population from which individuals are randomly drawn to engage in the conflict. If we want to model competition between individuals belonging to the same population, we can also imagine that N individuals from the same population are drawn and assigned to each player position.[4] As a simple example consider an Entry-Deterrence game.[5] The original story of this game is, that a firm is confronted with the decision whether to enter a market (pure strategy E), that is currently controlled by a monopolist. The monopolist is aware of the possible entrant and has two possible actions: accommodate (A), and share the market, or fight (F), by starting a price war. Its extensive form representation is given in Figure 4. We can view this conflict as a two-population game, where one is the owning population, of a certain asset, who plays the role of the monopolist, and the other population is a (large) group of intruders. For simplicity we assume that both populations are of equal size, so that the members of the populations are drawn with equal probability.
We want to use the extensive form representation of a game as often as possible in this work. Therefore we introduce the notion of a population game for the extensive form, as well as for the normal form of a game. The main difference between the two is the recognition that the player‘s information have huge impacts on the way games are played. The extensive form can incorporate *sequentiality of moves*, while in the

[4]In a 2 player game, we imagine that 2 individuals belonging to the same population are drawn and instructed to act as player 1 or player 2.

[5]In economics this game is better known as the Chain-store game.

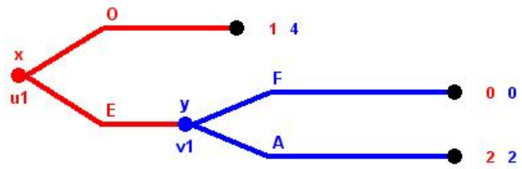

FIGURE 4. An Entry-Deterrence Game

normal form, we have to rely on the assumption that all players make moves "in the same moment".[6] As a motivating example go back to Figures 3 (section 1) and 4. As is rather common, information is visualized through a dotted line that connects the decision nodes of the second player. If two nodes are connected in such a way, we mean that the player in question does not know with certainty which action player 1 has chosen at his decision point. Conversely if the decision nodes are not connected, as in Figure 4, player 2 can observe the action of player 1. This observation of actions is an important attribute of extensive form games. We now introduce these thoughts formally. The beginning will be made with a thorough definition of population games in normal form, as is done by Sandholm (2007a). Afterward we turn to the definition of a population game in extensive form. A discussion of the differences between these two approaches shall conclude the section.

3.1. **Population Games in Normal Form.** Classical game theory defines a game in normal form over the set of players $I = \{1, 2, \ldots, N\}$, the pure strategy space available to each player, $S_k = \left\{e_{k_1}, \ldots, e_{k_{n_k}}\right\}$, and a payoff function for each player

$$\nu^k : S \to \mathbb{R} \tag{3.1}$$

[6]In fact normal form games are always appropriate when players decide on their strategies simultaneously, which does not necessarily imply that choices have to be made at the same point in time. Simultaneity of moves is not necessarily chronologically ordered. For example, to play the matching pennies game it is not necessary that players are in a "face-to-face" situation, where both toss a coin at the same instant of time. It can also be that player 1 decides on his strategy at a completely different point in time as player 2 does, both write their strategies on a sheet of paper, put it in an envelope, and submit it to some independent referee who opens the envelope again at a different point in time, and informs both players about the outcome of the game. This is admittedly a fairly complicated way to play a game, but its nature is the same as if players would decide on their strategy at the same instant of time.

where $S := \times_{k\in I} S_k$, is the pure strategy space, defining all possible strategy combinations in the game. In population games we associate with each player a large population, with continuum of mass m_k, $k \in I$. Instead of N players we look at N populations, building a society $\mathcal{P} = \{1, \dots, N\}$. Each population is endowed with a finite set of pure strategies, which we can identify with the set S_k. For each k, the cardinality of the pure strategy space is $|S_k| := n_k$. This set consist of at least one strategy for all populations, and there exists some population for which $n_k > 1$. Individuals in the player populations are imagined to be programmed to pure strategies. The mass of individuals in population k $(1 \leq k \leq N)$, who are programmed to pure strategy e_{k_h} $(1 \leq h \leq n_k)$ is $0 \leq x_{k_h} \leq m_k$. In this way we can divide each population into finitely many subpopulations, each subpopulation for an available pure strategy. If a certain pure strategy is not used in the population, a zero occurs in the population state. If all individuals in the population are devoted to a single pure strategy, the population state has all its mass concentrated on this strategy, so that this subpopulation's share is 1. The relative size of a subpopulation, measured as the share of total mass m_k that e_{k_h}-players have in population k, is denoted by p_{k_h}, $h \in \{1, 2, \dots, n_k\}$, meanwhile x_{k_h} represents the mass of players in population k choosing pure strategy e_{k_h}. Alternatively we can interpret the number p_{k_h} as the relative frequency with which an e_{k_h} player is observed in population k. To summarize:

Definition 1. *Let S_k be a finite set of pure strategies corresponding to player population $k \in \mathcal{P}$.*

(i) *The set of all possible* ***strategy distributions*** *in this population is*

$$X_k = \left\{ x_k = (x_{k_1}, \dots, x_{k_{n_k}})^T \in \mathbb{R}^{n_k}_+ : \sum_{h=1}^{n_k} x_{i_h} = m_k \right\} \tag{3.2}$$

(ii) *A* ***population state*** *is an element of the $(n_k - 1)$-dimensional simplex* [7]

$$\Delta(S_k) = \left\{ p_k = (p_{k_1}, \dots, p_{k_{n_k}})^T \in \mathbb{R}^{n_k}_+ : \sum_{h=1}^{n_k} p_{k_h} = 1 \right\} \tag{3.3}$$

[7] Consider a finite collection of points in euclidean $\mathbb{R}^n$, given by $\{x_1, x_2, \dots, x_m\}$. Such a set is affinely independent if $\sum_{i=1}^m \lambda_i x_i = 0$ and $\sum_{i=1}^m \lambda_i = 0$ imply that $\lambda_i = 0$, $1 \leq i \leq m$. An $(m-1)$-dimensional simplex is the set of all strictly positive convex combinations of an m element set of affinely independent points.

$$\Delta(x_1 x_2 \cdots x_m) := \left\{ \sum_{i=1}^m \lambda_i x_i | \forall i = 1, \dots, m : \lambda_i \geq 0 \wedge \sum_{i=1}^m \lambda_i = 1 \right\}.$$

See Border (1985).

generated by the pure strategies $\{e_{k_1}, \ldots, e_{k_{n_k}}\}$. *Geometrically each pure strategy* e_{k_h} *is a vertex of an* $n_k - 1$ *dimensional polyhedron, living in* $\mathbb{R}_+^{n_k}$. *For all* $h \in \{1, 2, \ldots, n_k\}$ *the component* $p_{k_h} \equiv \frac{x_{k_h}}{m_k}$ *is the proportion of h-strategists in the player population k, or the frequency with which strategy* e_{k_h} *is observed in population* k.

(iii) *A* ***state of society*** *is the collection of all population states observed at a certain point in time. It describes simultaneously the distribution of frequencies with which the pure strategies in the populations are used, in all* N *populations. The set of possible states of the society is denoted by the set* $\Theta = \Delta(S_1) \times \ldots \times \Delta(S_N)$, *with* $p = (p_1, \ldots, p_N)^T$ *a typical element. It has* $\sum_{\ell=1}^{N}(n_\ell - 1)$ *dimensions.*

In what follows we will assume that the population masses stay constant, so that we do not have to think about possible effects arising from demographic dynamics. Moreover we are going to assume that all populations are of the same size, so every individual in every population is drawn with the same probability. Thus, we can concentrate ourselves on the population states (3.3) and social states.
Strategic interactions might take place between and within the populations. In a given game, individuals from each of the N player populations are randomly drawn and matched to play the game in question. If we fix the strategy sets and the set of populations, we can identify a game with its payoffs. Choose an arbitrary population k. The set of pure strategies available to players from all other populations except k can be written as $S_{-k} = \times_{i \neq k} S_i$. This is the standard notation used in game theory, thus must not be explained further. As a result from the random drawing of agents, we observe the game pattern, induced by the pure strategy profile $e \in S$. If we want to emphasize that a certain player in some population k has chosen a particular pure strategy, we are going to write $e \setminus e_{k_h} \equiv (e_1, \ldots, e_k, \ldots, e_N)$, where $e_{k_h} = e_k \in S_k$ is the pure strategy employed by an agent that belongs to player population k. The probability that an agent from population k is drawn and matched with opponents who jointly play according to the strategy profile e, is given by the product of frequencies with which the agents, in all other populations except k, employ the pure strategy components of e, with exception the k-th component. As is common in game theory, we will denote the strategy profile used by all individuals, except the k-th, as $e_{-k} \in \times_{i \in I \setminus \{k\}} S_i$. Then, we have another writing for $e \setminus e_{k_h}$, namely $(e_{k_h}, e_{-k}) \in S$. Using this notation, the probability

is the product

$$\begin{aligned}\Pr(e_{-k}) : &= \prod_{i \in I \backslash \{k\}} \left\{ p_{i_j} | e_{i_j} \text{ is the pure strategy used by agent } i \text{ in the profile } e \in S \right\} \\ &\equiv \prod_{i \in I \backslash \{k\}} \left\{ p_{i_j} | e_i = e_{i_j} \right\}\end{aligned}$$

Using the pure strategy payoff function (3.1), an h-strategist drawn from population k can expect a payoff

$$\pi_h^k(p) = \sum_{e_{-k} \in S_{-k}} \nu^k(e_{k_h}, e_{-k}) \Pr(e_{-k}) \quad \forall p \in \Theta \tag{3.4}$$

This definition of expected payoffs holds for arbitrary $h \in \{1, 2, \ldots, n_k\}$ and arbitrary population $k \in \mathcal{P}$, so that we are able to write the expected payoff to population k as a vector valued function $\pi^k : \Theta \to \mathbb{R}^{n_k}$, or more explicitly

$$\forall k \in \mathcal{P} : \ \pi^k(p) = \begin{bmatrix} \pi_1^k(p) \\ \vdots \\ \pi_{n_k}^k(p) \end{bmatrix} \quad \forall p \in \Theta \tag{3.5}$$

A component of this vector gives the expected payoff to a randomly drawn h-strategist of population k. The resulting aggregate payoff for a given social state $p \in \Theta$ can then be read off from the list

$$\pi(p) = \begin{bmatrix} \pi^1(p) \\ \vdots \\ \pi^N(p) \end{bmatrix} \tag{3.6}$$

giving a complete description of the earnings that the populations receive. Now we have introduced all elements that we need to define a population game in normal form. To summarize

Definition 2. *A* ***population game in normal form*** *is a N-player game consisting of*

(i) *The society $\mathcal{P} = \{1, \ldots, N\}$ consisting of N large populations, where each population $k \in \mathcal{P}$ forms a continuum of mass $m_k > 0$.*
(ii) *A finite pure strategy space $S_k = \left\{ e_{k_1}, \ldots, e_{k_{n_k}} \right\}$ available to individuals belonging to population k. Behind each pure strategy there is a continuum of individuals playing this pure strategy, when drawn. There might be pure strategies that are unused in a population.*
(iii) *A population state $p_k = (p_{k_1}, \ldots, p_{k_{n_k}})^T$ describing the frequency, measured on the unit interval $[0, 1]$, of individuals appearing in population k, that can be classified to be $h \in \{1, 2, \ldots, n_k\}$ strategists. The set of all potential population states is the simplex $\Delta(S_k) = \left\{ p_k = (p_{k_1}, \ldots, p_{k_{n_k}})^T \in \mathbb{R}_+^{n_k} : \sum_{h=1}^{n_k} p_{k_h} = 1 \right\}$. A*

social state is the collection of all realized population states. The set of all potential social states is the Cartesian product of all population states and denoted by Θ.

(iv) *Payoffs to player populations are described by functions* $\pi^k : \Theta \to \mathbb{R}^{n_k}$, *one for each population* $k \in \mathcal{P}$. *Aggregate payoffs for some social state* $p \in \Theta$ *is then the list* $\pi(p) = [\pi^i(p)]_{i=1,\dots,N}$.

3.2. **Population games in extensive form.** Consider a finite set of players $I = \{1, \dots, N\}$. Each player has a distinct position in a game. These roles are defined through the information structure, which in turn is incorporated in a concise way by the extensive form of a game. Following Cressman (2003, Section 6.1) an N-player extensive form game is defined via a *game tree* and an *ordering of moves*. Distinguished nodes are the origin, or the *root*, of the game tree, and terminal nodes. The set of all decision points is denoted by Υ and the collection of all terminal points is summarized by a set Z. The order of moves is induced by a relation $\curlywedge$ defined on $\Upsilon \times \Upsilon$. A path to an endpoint is called play. A node x is said to *precede* a different node y, if $x \curlywedge y$, and x is on the path to y. Terminal nodes are identified through the equivalence $z \in Z \Leftrightarrow (\nexists x \in \Upsilon) : z \curlywedge x$. Different nodes are connected through actions, chosen by players. The main difference between a terminal and a decision node is consequently, that the latter proceeds some other node, which might possibly be a terminal node. Each node belongs to one of the N players. Formally we can define a mapping

$$\alpha : \Upsilon \to I, \; x \mapsto \alpha(x) \tag{3.7}$$

which assigns to each decision node in Υ a player. Thus the index $\alpha(x) \in \{1, \dots, N\}$ gives the player who has to make a decision at node x. If $x, y \in \Upsilon$ are two decision nodes satisfying $\alpha(x) = \alpha(y) = i$, we say that the nodes x, y belong to player i. The collection of all decision nodes belonging to a certain player is then the set of all nodes, where the function α assumes the same value, hence

$$D_i = \{x \in \Upsilon | \alpha(x) = i\}, \; 1 \leq i \leq N \tag{3.8}$$

shall be called the *decision set* of player $i \in I$. Note that the union of all these decision sets over players must be equal to the set of decision nodes, and further the sets are disjoint. With other words, the collection $\{D_1, D_2, \dots, D_N\}$ is a partition of Υ into N sets, induced by the equivalence relation α.

These decision nodes can be indistinguishable for a player, in the sense that she is not certain that a particular node is really reached during the play of the game. This leads us to the formal definition of information. For each player i there exists a partition $\mathcal{U}_i$ containing the *information sets* of player i. Every node of the tree belongs to one and only one information set. An information set is written as $u \in \mathcal{U}_i$. Fix an arbitrary information set, then $(\forall x, y \in u) : \alpha(x) = \alpha(y) = i$. Every decision node lying in some information set must belong to the same player. Intuitively, an information set is the collection of all nodes that a particular player cannot distinguish a-priori. In a two-player setting, where each player has only one move, player 2 might not be able to observe which decision player 1 selected. She does therefore not know with certainty which of her nodes is reached, so that all nodes coming after the decision of player 1 are "identical" for her. Such a situation is given in Figure 2 for instance. It is as if both players throw the coin at the same instant of time in this game, so player 2 does not know with certainty whether player 1 had head or tail as outcome. If two nodes are contained in the same information set, the player possessing this information set must have the same number of alternatives, or actions, available at these nodes. Suppose this would not be the case. In particular go back to Figure 2, but imagine that at the decision node of player 2 coming after player 1 yielded head, there is an additional action, that is not available at the node coming after player 1 yielded tail. If player 2 is now at the decision node following head of player 1, she can infer that player 1 yielded head with probability 1, since she now has actions available that she could not select if player 1 had yielded tail. Thus, the knowledge of player 2 about the action chosen by player 1 changed dramatically, since now player 2 can distinguish the moves made by player 1 through the actions which are at her disposal. So the nodes cannot belong to the same information set. This is why the set $\mathcal{U}_i$ was introduced as a partition. It partitions the set D_i of decision nodes of player i in such a way that it pools indistinguishable decision nodes together into information sets. Since every node in the tree has to belong to an information set, and every node is assigned to a player via the mapping $\alpha(.)$, we can assemble all nodes and classify them through information sets of player $k \in I$. In a N player setting, the information structure of the game can be represented by a partition $\mathcal{U} = \{\mathcal{U}_1, \dots, \mathcal{U}_N\}$ of the set of decision nodes Υ, where each $\mathcal{U}_k$ is the collection of all information sets that partition the decision set D_k of player $k \in I$.

To every information set $u \in \mathcal{U}_k$ corresponds an *action set* $\mathcal{A}(u)$, specifying all the available actions among which player k can select at information set u. With this notation we make sure, that the possible actions are identical at each node belonging to the same information set. For example in Figure 3 both players have a single information set. The information set of player 2, v_1, contains 2 decision nodes, y_1, y_2, and at both she has the same set of disposable actions, H or D. At each information set a player must make a choice. A *strategy* of a player is the collection of all choices at each information set of a player. Formally we are able to write the space of pure strategies as the Cartesian product of all actions available at each information state

$S_k = \times_{u \in \mathcal{U}_k} \mathcal{A}(u)$. The more segregated the information structure of a game is the more complex can be the pure strategy space of a player. In general if we denote $|\mathcal{A}(u)|$ the number of actions available at information set $u \in \mathcal{U}_k$ of player k, the number of pure strategies will be $|S_k| = \prod_{u \in \mathcal{U}_k} |\mathcal{A}(u)|$. We call this number n_k and write the pure strategy space of player $k \in I$ as a list $S_k = \left\{e_{k_1}, e_{k_2}, \ldots, e_{k_{n_k}}\right\}$. A typical element of this set e_{k_h} $(1 \leq h \leq n_k)$ is a list that specifies an action at each information set of player k. Formally, it is a vector-valued function, mapping information sets to actions,

$$e_{k_h} : \mathcal{U}_k \rightarrow \bigcup_{u \in \mathcal{U}_k} \mathcal{A}(u)$$

$$e_{k_h} = (a(u))_{u \in \mathcal{U}_k} \quad \text{and } \forall u \in \mathcal{U}_k : a(u) \in \mathcal{A}(u)$$

We assume that there is some player whose strategy set consists of more than two elements. If all N players have decided upon a pure strategy, we will write for the profile of pure strategies $e = (e_1, \ldots, e_N) \in S := \times_{i=1}^{N} S_i$, with bearing in mind that behind a component of this profile, say $e_i \in S_i$, lies a list of actions for each information set of player i.

Just like the vast majority of the economic literature, we assume that players have *perfect recall.* Intuitively this means, that no player ever forgets what she once knew. In particular all the chosen actions of player i that have been chosen at some anterior node, are known to her. The definition presented here is from Cressman (2003, p.168).

We say that an action of an arbitrary player k at information set $u \in \mathcal{U}_k$ *comes before* decision point x if the taken action leads to this decision point. Consider two decision nodes of player k at informations set $u' \in \mathcal{U}_k$. Call the decision nodes x, y, and select another information set $u \in \mathcal{U}_k$. The game has perfect recall if for all decision nodes $x, y \in u'$, and all choices $a(u) \in \mathcal{A}(u)$, this action comes before x if and only if it comes before y. This ensures two things. First, if x, y are contained in the same information set, neither $x \curlywedge y$ nor $y \curlywedge x$. Second, the definition ensures that players do not forget their actions. Suppose that player i chooses an action at two decision points x, y that are contained in different information sets, u and u', of the player, and this choice leads to nodes x' and y' that lie in the same information set u''. If the player has to make a decision at these nodes, she does not know which action has taken her to this information set (see Figure 5). At the starting decision nodes, two actions have to be specified, one for each information set. The action chosen at decision node x leads to decision node x' and comes, in our terminology, before x', but not before y' since this node is not reached by the specified action at x. However, x' and y' are in the same information set, but there exists a choice at information set x that comes before x' but not before y'. This violates our definition of perfect recall. Hence, our definition ensures that players never forget a previously chosen action. Terminal nodes

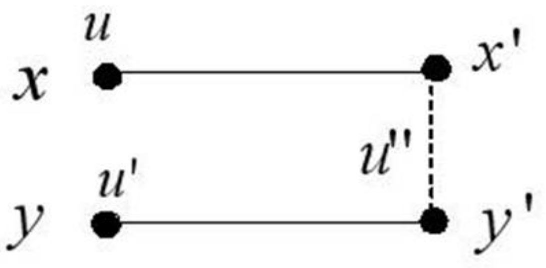

FIGURE 5. A game violating perfect recall

are special nodes of a tree, since these are the only nodes where no node follows. At each terminal node, payoffs to the players are specified. The payoff function to player k at terminal node $z \in Z$ will be written as $\nu^k(z)$.
Once the set of pure strategies is defined for all players, the mixed-strategy extension of the game is easily done. Call $\Delta(S_k)$ the $(n_k - 1)$-dimensional simplex, with vertices generated by the pure strategies in S_k.

A probability distribution $p_k \in \Delta(S_k)$, whose entries represent the relative frequencies with which player k chooses his pure strategies, is called a *mixed strategy*. Thus, $p_k = \left(p_{k_1}, p_{k_2}, \ldots, p_{k_{n_k}}\right)^T$ where $p_{k_j} \in [0,1]$ represents the relative frequency with which pure strategy e_{k_j} is chosen by player k, and by axioms of probability we have $\sum_{j=1}^{n_k} p_{k_j} = 1$. This is why we lose one degree of freedom by generating the mixed strategy space of player k.
The profile $p = (p_1, \ldots, p_N)^T \in \Delta(S_1) \times \ldots \times \Delta(S_N)$ is a list of all mixed strategies used by the players of the game. Call $\Theta := \Delta(S_1) \times \ldots \times \Delta(S_N)$ the $\sum_{k=1}^{N}(n_k - 1)$ dimensional simplex of mixed strategy profiles. For a fixed $p \in \Theta$ there is a certain probability that a terminal node z is reached in the game. Call $\Delta(Z)$ the set of probability distributions over the terminal nodes of the game. Entries in this set are distributions

$$\left\{ (\gamma(z,p))_{z \in Z} : \sum_{z \in Z} \gamma(z,p) = 1, \ \forall p \in \Theta \right\}$$

which can be interpreted as the *outcome* of the profile $p \in \Theta$. This vector assigns to each terminal node a certain probability with which it is reached, when all players act according to p. The payoff function to each player k extends now to expected payoffs, induced by the profile $p \in \Theta$. This expected payoff takes the general form of a function

$$\forall k \in I: \ \pi^k : \Theta \to \mathbb{R}, \ \pi^k(p) = \sum_{z \in Z} \gamma(z,p)\nu^k(z) \tag{3.9}$$

To make all these rather abstract definition more tractable, consider the following example.

Example 1. *Consider the extensive form game in Figure 6. This is a two player game, where player 2 possesses two distinct information sets* $\mathcal{U}_2 = \{\{v_1\}, \{v_2\}\}$. *Player 1's information partition is rather simple,* $\mathcal{U}_1 = \{\{u_1\}\}$. *The game tree consists of 4 decision nodes,* x_1, y_1, y_2, y_3, *where* x_1 *is also the root of the tree. Six terminal nodes are indicated (labeled as* z_1 *until* z_6*), and row vectors represent the payoffs that players receive if the outcome leads to the corresponding terminal node. The first entry in this vector represents the payoff obtained by player 1. The partition of decision nodes is* $D_1 = \{x_1\}$, $D_2 = \{y_1, y_2, y_3\}$. *The pure strategy space of player 1 is* $S_1 \equiv \mathcal{A}(u_1) = \{A, B, C\}$, *and for player 2 we get* $S_2 = \mathcal{A}(v_1) \times \mathcal{A}(v_2) = \{[ac], [ad], [bc], [bd]\}$, *so that the pure strategy* $[ac]$ *has the interpretation: "Play action* a *at information set* v_1 *and action* c *at information set* v_2*."*

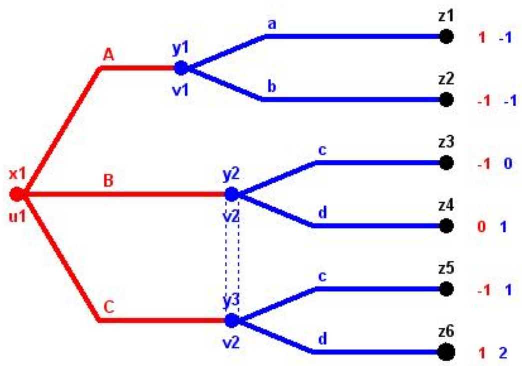

FIGURE 6. A game with one proper subgame

The mixed strategy profile $p = [(4/29, 15/29, 10/29); (0, 0, 1, 0)]$ *has then the interpretation that player 1 chooses* A *with probability* $4/29$, *and player 2 plays according to* $[bc]$. *When the game is played under this profile the following terminal nodes are reachable:*

- *Player 1 plays* A *and player 2 replies with* b *(terminal node* z_2*)* $\Rightarrow$ *Payoffs are* $\nu^1(z_2) = -1, \nu^2(z_2) = -1$
- *Player 1 plays* B *and player 2 replies with* c *(terminal node* z_3*)* $\Rightarrow$ *Payoffs are* $\nu^1(z_3) = -1, \nu^2(z_3) = 0$
- *Player 1 plays* C *and player 2 replies with* c *(terminal node* z_5*)* $\Rightarrow$ *Payoffs are* $\nu^1(z_5) = -1, \nu^2(z_5) = 2$

The expected payoff of the profile p *is then for player 1*

$$\pi^1(p) = 4/29\nu^1(z_2) + 15/29\nu^1(z_3) + 10/29\nu^1(z_5) = -1$$

and for player 2

$$\pi^2(p) = 4/29\nu^2(z_2) + 15/29\nu^2(z_3) + 10/29\nu^2(z_5) = 16/29$$

Now we have a complete description of a game in extensive form, where players may have *imperfect information*. However, there might be situations where a player is not sure of his role in the game. Consider the following interesting example of a (generalized) rock-paper-scissors game, taken from Cressmann (2003, p. 125), and displayed in Figure 7. In this game players may condition their play on whether or not a cer-

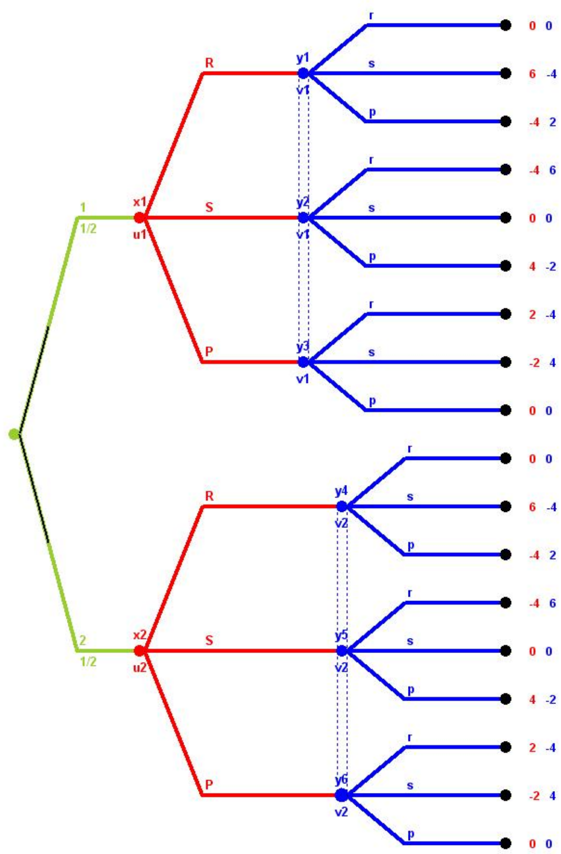

FIGURE 7. A generalized RSP game with incomplete information

tain exogenous event has taken place. For example somebody could be behind this game (an independent referee) who tosses a fair dice. The game is played after an odd and an even number has been observed. These are two equal likely events, on which players can condition their play. Nothing else changes and the base game is a generalized rock-paper-scissors game. For the present discussion, the only thing that is worth noting is that here a third player is introduced, whose only business is to toss the dice, and convey the information to the players whether the resulting number is even or odd. This is actually Harsanyi's transformation used in Bayesian games, where a fictitious player, often called "*nature*", is introduced (See Fudenberg and Tirole, 1991). "Nature" has only one decision node, and her strategies are the distribution of roles in the game. Thus, if necessary, a move by "nature" can be introduced in the definition of an extensive form game. But then "nature" is nothing else than another player in the set I, who also has a decision set, which is singleton, and an information partition, also a singleton set, and payoffs always equal to zero. Therefore, the above description also covers games of *incomplete information* and is consequently an appropriate definition of an extensive form game. To summarize

Definition 3. *A* ***N-player game in extensive form*** *consists of*

(a) *a finite set of players* $I = \{1, 2, \ldots, N\}$,
(b) *a game tree, consisting of:*
  (i) *a set of decision nodes* Υ *and an ordering of moves, described via a precedence relation* $\curlywedge$, *that has the interpretation*

$$\forall x, y \in \Upsilon : \ x \curlywedge y \Leftrightarrow x \text{ comes before } y$$

  (ii) *a rule that states which decision node belongs to which player. This is done via the function (3.7),*
  (iii) *the decision set (3.8), on which the labeling function (3.7) is constant. It is the collection of all decision nodes belonging to a player.*
(c) *the partition of information sets of players,*
(d) *the set of available actions at each information set,*
(e) *payoffs to players received at terminal nodes.*

In the following we call a finite extensive form game Γ *and identify with this all the mentioned characterizing points.*

We can also represent the extensive form as a normal form game. All we need for this is the strategy space S_k of each player k and the payoff functions
nu^k. This leads to the pure strategy space S and a list of payoff functions, $\nu(e) = \left[\nu^1(e), \ldots, \nu^N(e)\right]^T$. The extended normal form game is then the mixed strategy space of all players, together with the expected payoff function (3.9).

Evolutionary game theory has in mind the analysis of *player populations*, instead of single players. We can therefore identify a population game, where N large populations strategically interact, as an N-player extensive form game, where each player has a large population behind him. A game is played when we draw N individuals out of the N different populations. An agent belonging to population k is then assigned to adopt the position of player $1 \leq k \leq N$. The extensive form of a game can already become very complicated in a two-player game, and with more than two players even the visualization of the game tree is fairly difficult. Fortunately, there are some games that can be decomposed in smaller *subgames*, which can be treated as a separate game, be played on Γ. To identify such subgames, we start with the singleton information sets of the game. Suppose $u = \{x\} \in \mathcal{U}_k$ is such an information set, belonging to player $k \in I$. Call Γ_u the game tree with root u. Suppose $x \curlywedge y$, and y has the additional property that its containing information set is disjoint from all information sets that do not belong to Γ_u. Then Γ_u is called a subgame of Γ. If moreover u is not the root of Γ, then Γ_u is a *proper subgame* of Γ. The payoffs arising in such a subgame is the restriction of the payoff function π^k to the terminal nodes of the subgame. The games presented in Figures 7 and 6 are the only games that are decomposable in subgames. In all other games, the only subgame, is the game Γ itself. A proper subgame in Figure 6 starts at information set v_1 of player 2.

4. The extensive versus the Normal Form

The main advantage that the extensive form has over the normal form is the explicit description of the sequential nature of the players' interaction. Games in normal form have the implicit assumption that the agents involved act simultaneously. This might be accurate in the description of gambles like "Matching pennies" or the rock-scissors-paper game. In many situations that are interesting for economists this assumption is rather strict. The most famous example where games that look fairly similar yield possibly different results comes from the Industrial Organization literature. The classical Cournot model assumes that firms act as quantity setters in an oligopolistic market. Technologies of the firms are given and fixed, and each firm knows the technology of its competitors. This is a situation which corresponds to a normal form game. However, assume now that there exists one firm that takes in the leadership in the market. The leader has the possibility to make the first move, hence set its preferred output level. All other firms observe the leaders choice, and adapt their production to the leader's output. This is a different model coming from Industrial Organization, namely the Stackelberg model. It is well known that there are many equilibria in the Stackelberg model.[8] The Cournot

[8]For a concrete example consider an homogeneous market with 2 firms. Inverse demand is given by $P(x) = a - bx$, where $a, b \geq 0, x \equiv x_1 + x_2$. Technology of the firms is given by cost functions $C(x_i) = cx_i, 0 < c \leq a$. For a given output level $\bar{x}_j, j \in \{1, 2\}$ the best response of firm $i \neq j$ is a solution to the problem $\max_{x_i \geq 0} \{P(x)x_i - C(x_i), s.t. : x_j = \bar{x}_j\}$. One can easily compute that output

outcome may also be possible in the Stackelberg model.
We see, due to the sequentiality of the moves of players, the model gains a lot more descriptive power. A clear disadvantage of the extensive form is its complexity. The more possible actions, or states of informations players have, or simply the more players we include in the model, the more complicated it will be. In particular the cardinality of the strategy sets increase very quick, since in a N-player game where each player k has a_k actions at each of his $|\mathcal{U}_k|$ information sets, player k will have a pure strategy space with $a_k^{|\mathcal{U}_k|}$ pure strategies. Thus, a game with 3 actions and 5 information sets has 243 pure strategies, leading to an 242-dimensional mixed strategy space.
Which approach to take depends consequently on the situation of interest. The vast majority in the literature about evolutionary game theory restrict themselves to (symmetric) normal form games, and there are very well-known scientists who reject the use of the extensive form, just because we know that every extensive form game can be brought into a normal form.

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levels on the line $BR^i(\bar{x}_j) = \max[0, \frac{a-c-b\bar{x}_j}{2b}]$ are best responses. Suppose that firm with index 2 is the follower in the market, who uses the constant strategy $BR^2 \equiv \frac{a-c}{2b}$. The Stackelberg leader, firm 1, will anticipate this decision, and choose a profit maximizing output level, given the strategy of the follower. This is obviously $x_1 = 0$. This output combination are best replies to each other, thus form an equilibrium.

Part 3. Foundations of Evolutionary Dynamics

The most exciting feature in an evolutionary dynamic analysis of a game is the description of the way how the population states are formed in the long run. Since a social state is formally identical to a mixed strategy profile of a game, we can apply the concept of Nash equilibrium to population games. The notion of a Nash equilibrium will now be introduced, in the context of a population game in normal form. The theory behind this elegant equilibrium concept is conceptually independent of whether we look at extensive form or normal form games. Intuitively, a Nash equilibrium is a social state in which no population has an incentive to alter its composition, ceteris paribus.

5. Definitions

Let $p \in \Theta$ be a social state describing the joint behavior in a N population game. The components of p are the several population states, as introduced in definition 1. Henceforth, the vector $p_k = (p_{k_1}, \ldots, p_{k_{n_k}})^T$ is a list that contains as entries the share of individuals in population k who are programmed to play the pure strategy e_{k_h}, $1 \leq h \leq n_k$, have. The probability that individuals are matched to play the pure strategy profile $e = (e_1, \ldots, e_N)$ is then the product of the population shares of the individuals contained in the list e. Hence, we can write this probability as $\Pr(e) = \prod_{k=1}^{N} \{p_{k_h} | e_k = e_{k_h}\}$. Let $e \setminus e_{k_h} \equiv (e_{k_h}, e_{-k})$ be the profile where from all populations are agents drawn according to e, with the exception that in population k an agent is drawn, that is programmed to pure strategy e_{k_h}. If this pure strategy was already present in the profile e we can calculate the probability that such a profile will be played as $\Pr(e \setminus e_{k_h}) \equiv \Pr(e) = p_{k_h} \prod_{\ell \in I \setminus \{k\}} \left\{p_{\ell_j} | e_\ell = e_{\ell_j}\right\}$. Using this notation, the average payoff of population k follows directly from equation (3.4)

$$\begin{aligned}
\sum_{h=1}^{n_k} \pi_h^k(p) p_{k_h} &= \sum_{h=1}^{n_k} p_{k_h} \left(\sum_{e_{-k} \in S_{-k}} \nu^k(e_{k_h}, e_{-k}) \Pr(e_{-k}) \right) \\
&= \sum_{e_{-k} \in S_{-k}} \nu^k(e_{k_1}, e_{-k}) \Pr(e_{-k}) p_{k_1} + \ldots + \sum_{e_{-k} \in S_{-k}} \nu^k(e_{k_{n_k}}, e_{-k}) \Pr(e_{-k}) p_{k_{n_k}} \\
&= \sum_{h=1}^{n_k} \sum_{e \in S} \nu^k(e \setminus e_{k_h}) \Pr(e \setminus e_{k_h}) =: \bar{\pi}^k(p) \qquad (5.1)
\end{aligned}$$

Definition 4. *Consider a finite population game with $N \geq 2$ populations. Each population is endowed with a pure strategy set S_k and the space of population states is $\Delta(S_k)$. Payoffs to each population are as in (3.5).*

(i) *The **pure best-response correspondence** of player population k is the correspondence $\beta^k : \Theta \rightarrow\rightarrow S_k$ given through*

$$\beta^k(p) = \left\{e_{k_h} \in S_k | \pi_h^k(p) \geq \pi_j^k(p)\ \forall 1 \leq j \leq n_k, \forall p \in \Theta\right\} \tag{5.2}$$

(ii) *The **mixed best response correspondence** of player population k is $BR^k : \Theta \rightarrow\rightarrow \Delta(S_k)$ and defined through*

$$BR^k(p) = \left\{p_k \in \Delta(S_k) | p_{k_h} > 0 \text{ only if } e_{k_h} \in \beta^k(p)\right\} \tag{5.3}$$

(iii) *A pure strategy $e_{k_i} \in S_k$ is said to be strictly dominated if there is no social state at which it is a best reply,*

$$\forall p \in \Theta : e_{k_i} \notin \beta^k(p) \tag{5.4}$$

(iv) *A social state is a **Nash Equilibrium** of the population game if it belongs to the set*

$$NE = \left\{p \in \Theta | p_k \in BR^k(p)\ \forall 1 \leq k \leq N\right\} \tag{5.5}$$

If we deal with extensive form games we can exploit the sequentiality of moves to select among the possible Nash equilibria those that are optimal in every subgame of the game Γ. Such equilibria are said to be *subgame perfect.* To identify a subgame perfect equilibrium one makes intensive use of the information that the extensive form provides us. The applied solution method is known as *backwards induction.* Loosely speaking, when we solve a game by backwards induction, we start with the smallest subgame of the given extensive form game Γ.[9] At such a subgame we determine the optimal action that a rational agent would choose when this subgame is reached. Working us through to subsequently larger subgames, and determining the optimal actions at all these subgames of the agents, leads us after some time to the root of Γ. Then the algorithm stops, and by combining all the optimal actions of the agents at the subgames gives us a pure strategy profile for every player that ascribes an optimal action at each subgame where this player has a move to make. Such a strategy profile is a Nash equilibrium and is moreover optimal at every subgame of Γ. Thus, we have found (by construction) a subgame perfect equilibrium. For more on backwards induction see Fudenberg and Tirole (1991).

[9]This might be the game as as a whole, when Γ does not possess any proper subgames. Otherwise the game can be decomposed into a finite collection of subgames. On this collection we can write an ordering of subgames labeling a subgame to be the *smallest* if it contains the lowest number of decision nodes that follow it. In terms of the relation $\curlywedge$ we can define the set Υ_{Γ_u}, as the set of decision nodes restricted to the subgame Γ_u, $\Upsilon_{\Gamma_u} = \emptyset$ might be possible, and $\mathcal{F}(\Gamma_u) := \{x \in \Upsilon | \exists y \in \Upsilon_{\Gamma_u} : y \curlywedge x\}$ as the set of decision nodes that follow a decision node in the subgame Γ_u. Since we consider only finite games, for every subgame the sets $\mathcal{F}(\Gamma_u)$ must be finite for every information set u at which a subgame starts. We can then make a comparison of the cardinalities of these sets (which are well defined real numbers by finiteness), and so can classify a subgame to be smallest, if the set $\mathcal{F}(\Gamma_u)$ has the lowest cardinality.

6. Dynamical Systems

This section will introduce those mathematical tools used in the subsequent pages. When we analyze a game through evolutionary dynamics, we actually view the game as a *dynamical system.* The conventional way of solving a game is by calculating its Nash equilibria. The fundamental property of this equilibrium is that nobody can increase his payoff by unilateral deviations from the Nash strategy. Once players coordinated to play the Nash equilibrium, nobody will ever have an incentive to deviate, given the players do not change their strategies simultaneously and in a somehow coordinated way. This makes the Nash equilibrium a very strong solution concept. However, such a static analysis is not able to show how the players actually coordinate to play the Nash equilibrium. Moreover, the argument that no player would have an incentive to deviate uniltateraly from the Nash strategy, relies heavily on the assumption that the opponents will play their Nash strategies. Especially in games played by large populations, it might be fairly hard for the players to anticipate that all opponents really choose their strategy that corresponds to the Nash equilibrium of the game in question. As an alternative one can study a game as a dynamic model, where no assumption concerning any agent's belief is needed. We focus on population games as defined in part 2. In an N player game there are N populations, one population for each player position, from which we randomly draw individuals, who are programmed to some pure strategy available to the player position, and let these individuals play the game. As time proceeds individuals in the populations are allowed to change their pure strategy. This is embodied by so-called behavioral rules, as will be explained in Section 7. Such behavioral rules generate a system of differential equations, which describe the evolution of the relative frequency with which some pure strategy occurs in a population. There is one differential equation for each pure strategy available to a population and every differential equation describes the evolution of the population share that this pure strategy has, that is the number $p_{k_i} \in [0,1]$ for all $1 \leq k \leq N$ and $1 \leq i \leq n_k$. The information how the state of population k behaves over time is then encapsulated by the system of differential equations[10]

$$\dot{p}_k = V^k(p) = \begin{bmatrix} V_1^k(p) \\ \vdots \\ V_{n_k}^k(p) \end{bmatrix}, \quad 1 \leq k \leq N$$

[10]The notation $\dot{p}(t) = \frac{dp(t)}{dt}$ is the time derivative of the continuously differentiable function $p : I \subset \mathbb{R} \to \mathbb{R}^{n_k}$. For ease of notation, we often suppress the time index t.

This system of differential equations defines a *vector field* acting on some open space $M \subset \mathbb{R}^{\sum_{k=1}^{N} n_k}$ that contains the compact set Θ.[11] We will mainly work with vector fields whose Jacobian matrix $DV^k(p) \in \mathbb{R}^{n_k \times n_k}$ is continuously differentiable. We write for such vector fields $V^k \in C^1(M, \mathbb{R}^{n_k})$. All vector fields at which we are going to look at are time independent. Hence, we deal only with *autonomous* systems of ordinary differential equations of order 1.[12]

Typical problems with which we are confronted are so-called *initial value problems*, where we want to solve the system

$$V^k : M \to \mathbb{R}^{n_k} \quad \dot{p}_k = V^k(p),\ p_k(0) = p_k^0 \tag{6.1}$$

where $p_k^0 \in \Delta(S_k)$ is a given initial population state of player population k. Finding a *solution* to (6.1) means finding a continuously differentiable function $\phi^k : I \times \Theta \to \mathbb{R}^{n_k}$, where $I \subset \mathbb{R}$ is a time period on which this function is defined, with the property that for every pair $(t, p) \in I \times \Theta$ we have $\frac{d}{dt}\phi^k(t, p_k^0) = V^k(p)$ and $\phi(0, p_k^0) = p_k^0$. With other words, a solution gives us the state of population k at time t and social state p. If I is the maximal time interval on which solutions are defined, we speak of a maximal solution. The derivative condition implies that we already know the long run behavior of the state of population k, for any initial condition in the state space Θ, by plotting the vector field V^k. If $p_k \in \Delta(S_k)$ is a point on a solution curve of (6.1), then $V^k(p)$ is the tangent vector of the solution curve at this point. Thus, the speed and direction of motion of the population state is encapsulated by the vector field V^k. Since we are mainly interested in the qualitative behavior of population states, the vector field is an important source of information for our studies. Another convenient consequence of the derivative condition of solutions is that we can write a solution in its integral version as

$$\phi^k \text{ is a solution to (6.1)} \Rightarrow \phi^k(t, p_k^0) = \phi^k(0, p_k^0) + \int_0^t V^k(p(s))ds$$

We can also describe the qualitative behavior of all populations at one instance, by summarizing them in the vector

$$V(p) = \begin{bmatrix} V^1(p) \\ V^2(p) \\ \vdots \\ V^N(p) \end{bmatrix}$$

Assuming that the vector field is of the class C^1 allows us to apply the famous *Picard-Lindelöf* Theorem, which ensures that to every initial

[11] A vector field V on a set $M \subset \mathbb{R}^m$ is a mapping assigning to each point $x \in M$ a vector $V(x) \in \mathbb{R}^m$. See Königsberger (2000)

[12] The order of a differential equation is determined by the highest derivative that appears in the equation. Since we deal only with first derivatives, our differential equations are of order 1.

value p_k^0 there exists a unique solution to the problem (6.1) (A nice proof of the Picard-Lindelöf Theorem can be found in Königsberger (2000)). The assumption on the vector field being C^1 is actually more restrictive as we need. To prove the Picard-Lindelöf Theorem it suffices to work with locally Lipschitz continuous vector fields.[13] However it can be shown that C^1 implies Lipschitz continuity, and it allows us to make some statements about the dependence of solutions on initial conditions. For a technical discussion the reader may consult Hirsch, Smale and Devaney (2004).

All the solutions to population specific vector fields that we will encounter, are bounded by the compact subspace $\Delta(S_k)$. This is true since all the dynamics that we will consider have solutions that leave the simplex $\Delta(S_k)$ *forward invariant.* An intuitive understanding of forward invariance under a solution ϕ^k is that once the solution started at some point in the simplex it will always be in this simplex. Two conditions to ensure the forward invariance under solutions are

$$\sum_{i=1}^{n_k} \dot{p}_{k_i}(t) \equiv \sum_{i=1}^{n_k} V_i^k(p(t)) = 0 \quad \forall t \in I, 1 \leq k \leq N, p(t) \in \Theta \tag{6.2}$$

$$(\forall 1 \leq k \leq N)(\forall e_{k_i} \in S_k) : p_{k_i} = 0 \Rightarrow V_i^k(p) \geq 0 \tag{6.3}$$

To understand these conditions suppose that there are only two strategies available to population k. A population state is then a vector with two elements $p_k = (p_{k_1}, p_{k_2})^T$. This vector is defined on the unit square $[0,1] \times [0,1]$ so that at every point in time, the identity $p_{k_2} = 1 - p_{k_1}$ must hold. Taking time derivatives on the left and the right hand side of this identity gives us the above stated invariance condition $\dot{p}_{k_2} = -\dot{p}_{k_1}$. If the proportion of players in population k that choose to play strategy $e_{k_2} \in S_k$ increases, the proportion of players in the same population, that are devoted to pure strategy e_{k_1}, must decrease at the same rate. Henceforth, every motion "forward" has its corresponding "backward" motion, so that at no point in time the dynamics can point outside the unit square. For all positive time population states are constrained to their set of possible population states, which is the simplex $\Delta(S_k)$. The forward invariance of the simplex guarantees that maximal solutions are defined for all future times. We can take $I = [0, +\infty)$ (see Königsberger 2000)

It is clear that solutions are continuously differentiable in time. Moreover, equipped with the assumption that the vector field V^k is C^1, it

[13]A real valued function $f : X \to Y$, where X and Y are some metric spaces with metric d_X, d_Y respectively, is said to be *Lipschitz-continous* if there exists some constant $L \geq 0$ such that for all $x_1, x_2 \in X$ we have $d_Y(f(x_1), f(x_2)) \leq L d_X(x_1, x_2)$. Take X to be an open subset and let $X' \subset X$. The function $f : X \to Y$ is said to be *locally Lipschitz continuous* if each point in X has a neighborhood X' such that that the restriction of f on X' is Lipschitz continuous with Lipschitz constant $L(X')$.

can be shown that solutions also depend continuously on the initial conditions. For a proof see Hirsch, Smale and Devaney (2004, Chapter 17). Now we have all the ingredients to define a dynamic system. A dynamic system is the triplet (I, Θ, ϕ), that is a maximal time interval, a compact state space and a solution indicating the position of a social state for a given initial condition and some point of time in I.

Unfortunately, solving a system of differential equations is a very complicated task, and in high dimensions it is often not possible to write down solutions in closed form. However, for our purposes it is enough to study the qualitative behavior of the population states. With qualitative behavior we mean the description of the path that solution curves trace out when time goes by. This analysis is qualitative in the sense that we actually only look at the vector field, and analyze all possible motions that may arise in the state space Θ. Since we know that solutions must mimic the directions induced by the vector field, a qualitative investigation can be made by studying the phase portrait of differential equations. This is fine in a dynamical system with one variable, but clearly not possible in higher dimensions. In the case of a planar system, we can use the vector field to study the evolution of our dynamical system. In systems with more than two equations one needs however different tools to study the long-run behavior of a dynamical system. One therefore often selects a few interesting points of the dynamical system and tries to investigate the behavior of the system in a neighborhood of these distinguished points. Among these points are so-called *rest points*, or *fixed points* of the dynamical system. Such points have the special property that, once a population state reached such a rest point, there will be no motion away from it, at least if there are no exogenous effects that perturb the system in some way.

Definition 5. *Let $V : M \to \mathbb{R}^n$, $n := \sum_{k=1}^{N} n_k$, be a $C^1(M, \mathbb{R}^n)$ vector field describing the motion of the social state $p \in \Theta$ of a given population game. The point $p^* \in \Theta$ is called a* ***rest point*** *of the vector field if it is a solution to the homogeneous system of equations*

$$(\forall k = 1, 2, \ldots, N) : \; V^k(p^*) = 0 \tag{6.4}$$

Clearly, a rest point must not be unique. For a simple illustration consider the following

Example 2. *Consider the one population, two strategy game, with vector field $\dot{p} = V(p) = p(1-p)$. Here p represents the proportion of individuals in the population that play strategy 1 and $(1-p)$ is the proportion of individuals in the population playing strategy 2. This game has two rest points $p = 0$ and $p = 1$. These two points are constant solutions to the differential equation. Solutions, for initial conditions from the open interval $(0, 1)$ on the real line, can be determined by* *"seperation of variables". For any initial value $p(0)$ in this interval we can rewrite the equation as $\frac{dp}{p(1-p)} = dt$, which can be integrated over some time interval $[0,t]$ on the real line. This gives us*

$$\begin{aligned}
\int_0^t \frac{dp}{p(1-p)} &= \int_0^t dt \\
\int_0^t \frac{dp}{p} + \int_0^t \frac{dp}{1-p} &= t \\
\ln[p(t)] - \ln[p(0)] - \ln[1-p(t)] + \ln[1-p(0)] &= t \\
\ln\left[\frac{p(t)}{1-p(t)}\right] &= \ln\left[\frac{p(0)}{1-p(0)}\right] + t \\
\frac{p(t)}{1-p(t)} &= \frac{p(0)}{1-p(0)}\exp(t) \\
p(t) &= \frac{p(0)\exp(t)}{1-p(0)(1-\exp(t))}
\end{aligned}$$

The long run evolution of this solution for the five initial values $p(0) \in \{0.04, 0.1, 0.5, 1, 1.5\}$ is depicted in Figure 8.[14] *We see that solution*

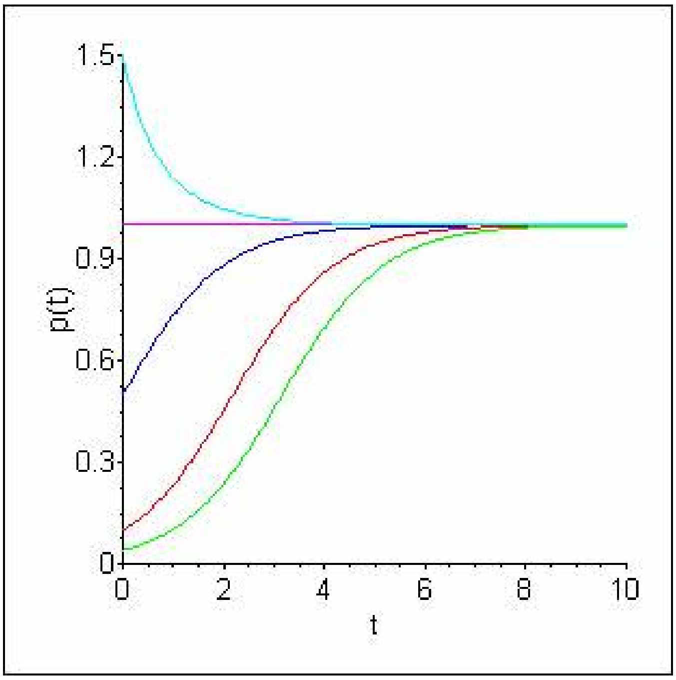

FIGURE 8. Solution curves for 5 different initial values of the differential equation $\dot{p} = p(1-p)$.

tend fairly rapidly to the stationary state $p = 1$, but there is no solution

[14] In a population game there can be certainly no initial value $p(0) = 1.5$. This value is just added to see the general pattern of solution curves, and must be seen independent of the population game context.

starting in the open interval $(0, 1)$ *that tends to the other rest point situated at the origin. To see the motion more clearly there is also one solution plotted whose starting value lies outside the unit interval. We see that this curve decays very fast and approaches the stationary state* $p = 1$ *in future time.*

We have seen in the example that rest points can have fundamentally different properties. While the fixed point $p = 1$ seems to attract all solution curves as time proceeds, the origin does not have this property. More the contrary is the case. A solution may start arbitrary close to the origin, it will never converge to it. It is therefore quite useful to distinguish rest points of a dynamical system according to their *stability properties*.

Definition 6. *Consider the autonomous vector field* $V \in C^1(M, \mathbb{R}^n)$, $\Theta \subset M$, *and suppose the point* $p^* \in \Theta$ *is a rest point of it. Further suppose conditions (6.2) and (6.3) are fulfilled. Then the rest point is said to be*

(i) ***stable*** *if for all open neighborhoods* $\mathcal{U}(p^*)$ *of the rest point there is another open neighborhood* $\mathcal{O}(p^*)$ *such that*

$$(\forall p^0 \in \mathcal{O}(p^*) \cap \Theta)(\forall t \geq 0): \ \phi(t, p^0) \in \mathcal{U}(p^*) \cap \Theta \tag{6.5}$$

(ii) ***locally asymptotically stable*** *if it is stable and moreover*

$$(\forall p^0 \in \mathcal{U}(p^*) \cap \Theta): \ \lim_{t \to \infty} \phi(t, p^0) = p^* \tag{6.6}$$

The rest point is then said to be ***attracting*** *and the set* $\mathcal{U}(p^*) \cap \Theta$ *is its* ***basin of attraction***.

(iii) ***globally asymptotically stable*** *if it is locally asymptotically stable and* $\mathcal{U}(p^*) = \Theta$.

(iv) ***unstable*** *if it is not stable.*

In economics we are mainly interested in rest points that are at least stable. Any forecast that predicts an unstable rest point as a solution of a dynamical system has the very unattractive side effect that only slight perturbation of the initial values are enough to lead the dynamics to a totally different point in the state space. Since in "real-life" we never analyze environments that are protected from exogenous shocks, unstable rest points might even never be observed.[15] The main instruments to detect the stability of the rest points of a vector field, that are used in applications, are techniques of *linearization* of the vector field around the rest point, and the construction of so-called *Lyapunov functions*.

Linearization can be motivated by the familiar method of approximating a known function with a polynomial, hence performing a Taylor expansion around the rest point. Consider the vector field $V \in C^1(M, \mathbb{R}^n)$

[15] Except the probability zero event occurs and the system initially is in the unstable rest point position.

and suppose the point p^* is a rest point of it. A first-order Taylor expansion in a neighborhood $\mathcal{U}(p^*) \cap \Theta$ of the rest point is then given by

$$V(p) = V(p^*) + DV(p^*)(p - p^*) + o(p - p^*)$$

where $o(p-p^*)$ represents the remainder function that converges to zero as p tends to p^* faster then $||p - p^*|| \to 0$ for $p \to p^*$.[16] By definition of a rest point, we have $V(p^*) = 0$, so that we are left with the linear system of differential equations

$$\dot{z} = Az$$

where $z := p - p^*$ and $A := DV(p^*)$ a $n \times n$ matrix. This derivative matrix consists in our case of a lot of components. Since the function V captures all directions of motions of all involved populations, and all the motions of particular populations might in turn depend on the aggregate behavior, there are many derivatives compute. Start with fixing an arbitrary population k $(1 \leq k \leq N)$ whose vector field is described by $V^k(p) = V^k(p_1, \dots, p_N)$, and $p_j = (p_{j_1}, \dots, p_{j_{n_j}})$, $1 \leq j \leq N$. All partial derivatives are then summarized by the matrix

$$DV^k(p) = \left[D_{p_1}V^k(p), \dots, D_{p_N}V^k(p)\right] \quad k = 1, 2, \dots, N$$

where

$$(\forall j, k = 1, 2, \dots, N) : \; D_{p_j}V^k(p) = \begin{bmatrix} \frac{\partial V_1^k}{\partial p_{j_1}}(p) & \frac{\partial V_1^k}{\partial p_{j_2}}(p) & \cdots & \frac{\partial V_1^k}{\partial p_{j_{n_j}}}(p) \\ \frac{\partial V_2^k}{\partial p_{j_1}}(p) & \frac{\partial V_2^k}{\partial p_{j_2}}(p) & \cdots & \frac{\partial V_2^k}{\partial p_{j_{n_j}}}(p) \\ \vdots & \ddots & \ddots & \vdots \\ \frac{\partial V_{n_k}^k}{\partial p_{j_1}}(p) & \frac{\partial V_{n_k}^k}{\partial p_{j_2}}(p) & \cdots & \frac{\partial V_{n_k}^k}{\partial p_{j_{n_j}}}(p) \end{bmatrix}.$$

This matrix does capture all the partial effects that changes in the proportion of individuals playing pure strategy $\ell = 1, 2, \dots, n_j$ in population $j = 1, 2, \dots, N$ have on the motion of the player proportions in population k. It is a matrix of dimension $n_k \times n_j$. The derivative matrix $DV(p)$ is consequently a block matrix of the form

$$DV(p) = \begin{bmatrix} DV^1(p) \\ DV^2(p) \\ \vdots \\ DV^N(p) \end{bmatrix}$$

With our assumptions, in particular the C^1 assumption of the original vector field is here important, this system has a unique solution which approximates solutions of $\dot{p} = V(p)$ in a (possibly small) neighborhood of the rest point p^*. The long run behavior of solutions of the linear system depend on the eigenvalues of the Jacobian matrix $DV(p^*)$, evaluated at the rest point. If all the eigenvalues of this matrix have

[16]The function $||.|| : \mathbb{R}^n \to \mathbb{R}_+$ is here the euclidean norm of $\mathbb{R}^n$.

negative real parts, the rest point is locally asymptotically stable. For a proof see Hirsch, Smale and Devaney (2004, Chapters 3 to 6 and 17). A problem with this linear approximation occurs when there is an eigenvalue whose real part equals to zero. The Jacobian matrix is then said to be *non-hyperbolic*. In such a case the linearized system does not mimic the original vector field accurately, and so different methods have to be found.

Another method to determine the stability of rest points is the construction of a real-valued function that assumes its minimum value at the rest point and is strictly decreasing along solution curves at all other points then the fixed point. Such a function is called a *Lyapunov function*, and will turn out to be very important in later chapters. The inner product of two vectors $x, y \in \mathbb{R}^n$ is denoted by the bilinear map [17] $\langle .,. \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, and computed as $\langle x, y \rangle := \sum_{i=1}^n x_i y_i$.

Lemma 1. *Let $\mathcal{O} \subset \Theta$ be an open subset and consider the continuously differentiable function $L : \mathcal{O} \to \mathbb{R}$. Let $\phi(t, p^0)$ denote a solution to the system of differential equations $\dot{p} = V(p)$, $V \in C^1(M, \mathbb{R}^n)$, $\Theta \subset M$, with initial value $p^0 \in \Theta$. If $\langle \nabla L(\phi(s, p^0)), \phi(s, p^0) \rangle < 0$ for almost all s, then we have $L(\phi(t, p^0)) < L(\phi(s, p^0))$ for all $s < t$.*

Proof:

$$(\forall s < t) : L(\phi(t, p^0)) - L(\phi(s, p^0)) = \int_s^t \left\langle \nabla L(\phi(\tau, p^0)), \phi(\tau, p^0) \right\rangle d\tau < 0$$

q.e.d.

Definition 7. *Consider an autonomous dynamical system $\dot{p} = V(p)$, where $V \in C^1(M, \mathbb{R}^n)$ and M is an open subset of $\mathbb{R}^n$ containing Θ. Denote by $\phi(t, p^0)$ a continuously differentiable solution curve starting from some initial value $p^0 \in \Theta$. Suppose the point p^* is a rest point of the dynamical system and consider the rule $p \mapsto L(p)$ for some $p \in \Theta$, $L(p) \in \mathbb{R}$. If $L(p)$ is continuously differentiable on some open neighborhood $\mathcal{U}(p^*)$ of the rest point, and satisfies*

$$(\forall p \in \mathcal{U}(p^*) \cap \Theta) : \quad L(p) \geq L(p^*)$$

$$(\forall p \in \mathcal{U}(p^*) \cap \Theta) : \quad \tfrac{d}{dt} L(\phi(t, p^0)) = \langle \nabla L(\phi(t, p^0)), \phi(t, p^0) \rangle \leq 0$$

then it is called a ***local Lyapunov function****. If $\langle \nabla L(p), p \rangle < 0$ for all $p \in (\mathcal{U}(p^*) \cap \Theta) \setminus \{p^*\}$ then it is a* ***strict Lyapunov function****.*
p^* is stable if there exists a local Lyapunov function, and asymptotically stable if there exists a strict Lyapunov function.

[17] A map is said to be bilinear if it is linear in both of its arguments, thus in our case

$$(\forall x, y, z \in \mathbb{R}^n)(\forall \lambda \in \mathbb{R}) \quad : \quad \langle x, \lambda(y+z) \rangle = \lambda(\langle x, y \rangle + \langle x, z \rangle)$$
$$\langle \lambda(x+z), y \rangle = \lambda(\langle x, y \rangle + \langle z, y \rangle)$$

A proof of the connection between (asymptotical) stability and the existence of a (strict) Lyapunov function can be found for example in Hirsch, Smale and Devaney (2004, Chapter 9). Once one has found a Lyapunov function, one is able to reconstruct the pattern of solutions to the vector field, in an open neighborhood around the rest point. Note that in the definition of a Lyapunov function we require that the rest point is an isolated minimum and along solution curves we attain only lower level sets of L. Thus, if we encounter a stable rest point of a dynamical system, we can construct a continuously differentiable function that leads us to this rest point, when the solutions start in some neighborhood of it.
There is one more concept we have to introduce. Suppose we find a continuously differentiable function $E : M \to \mathbb{R}$ that is constant along solution curves, thus $\langle \nabla E(\phi(t,p^0)), \phi(t,p^0)\rangle = 0$. We call such a function *first integral*, or a *constant of motion*. By definition, a first integral is a special kind of a Lyapunov function. However, a direct consequence of the constancy along solution curves is that vector fields, that possess a constant of motion, cannot have asymptotically stable fixed points, since there does not exist a strict Lyapunov function. Stability is guaranteed in such systems.

7. From Behavioral Rules to deterministic Dynamics

There are two well established ways to introduce evolutionary dynamics. The "mechanical" approach is by defining growth rate functions that describe the dynamic pattern of strategy frequencies (see Hofbauer and Sigmund, 1998). A drawback of this approach is the flavor of arbitrariness which is accompanied by "simply" writing down functional forms. An alternative takes the behavior of the agents as its primitive. With this framework we are able to model the decision process of individuals in terms of so-called behavioral rules. This term is used by Schlag and Cressman (2003) for decision theoretic problems (extensive form games with a single player). Weibull (1995) and Sandholm (2007a, 2007b) introduce similar concepts for normal form games. As the name suggests, a behavioral rule is a formal description how an agent reviews her strategy, given some information about the state of society. We define such a behavioral rule according to Sandholm (2007a, p.103).

Definition 8. *A* ***behavioral rule*** *is a map from currently aggregate behavior to* ***conditional switch rates****. The map is given by*

$$F^k : \mathbb{R}^{n_k} \times \Delta(S_k) \to \mathbb{R}_+^{n_k \times n_k}$$

The real number $f^k_{ij}(\pi^k(p), p_k)$ is the rate at which a current $e_{k_i} \in S_k$ strategist switches to pure strategy $e_{k_j} \in S_k$. The matrix $F^k(\pi^k(p), p_k) =$

$\left[f_{ij}^k(\pi^k(p), p_k)\right]_{i,j=1,\ldots,n_k}$ *models the decision process of all agents in population k, for a given social state p.*

Note that the conditional switch rates are not probabilities, but agents will switch from strategy e_{k_i} to strategy e_{k_j} with likelihood proportional to the switching rate f_{ij}[18]. With the use of behavioral rules we can define an evolutionary dynamic as an inflow-outflow model. The inflow into pure strategy e_{k_i} captures the mass of individuals that abandon their currently employed pure strategy and play e_{k_i} until the next switching opportunity is granted. The outflow term is the mass of individuals that currently play e_{k_i}, but now decide to play some different pure strategy in S_k. Inflow and outflow affect the number p_{k_i}. Inflows make p_{k_i} growing, while outflows lower this strategy frequency. The netflow is the difference between inflow and outflow and determines whether the frequency with which we observe pure strategy e_{k_i} becomes larger, or declines. Assuming that the populations are sufficiently large, we can describe this inflow-outflow process through a system of differential equations, that assume the form:

$$\dot{p}_{k_i} = \sum_{j=1}^{n_k} p_{k_j} f_{ji}^k - p_{k_i} \sum_{j=1}^{n_k} f_{ij}^k \quad 1 \leq k \leq N, 1 \leq i \leq n_k \tag{7.1}$$

The first term measures the inflow, the second the outflow. This equation induces a continuous law of motion on every population state, which is summarized by the dynamical system

$$\dot{p}_k = \begin{bmatrix} \dot{p}_{k_1} \\ \vdots \\ \dot{p}_{k_{nk}} \end{bmatrix} = V^k(p) \tag{7.2}$$

Hence, every behavioral rule can generate an evolutionary dynamic plus they take individuals behavior as starting point. This explicit microfoundation of the evolution of the populations makes behavioral rules a very attractive tool.

In principle any function, that is in accordance with definition 8, can generate an evolutionary dynamic. Certainly there are some properties that we would like from a "good" behavioral rule to share. One of them can be seen from a technical point, but reflects also a fact coming from large populations. A good behavioral rule shall be Lipschitz-continuous in payoffs and social states. The technical implication of this assumption is that we can apply the *Picard-Lindelöf theorem* (see for example Hirsch, Smale and Devaney, Chapter 17, or the discussion in section 6) on the dynamical system, induced by the equations (7.1).

[18] The following derivation is based on Sandholm (2007a, Chapter 9). It is designed to be more illustrative than mathematically concise. For a thorough mathematical discussion we refer to Sandhalm (2007a, Chapter 9) and Benaim and Weibull (2003)

For every initial social state $p(0) \in \Theta$ there exists a unique and continuous solution $\phi^k : \mathbb{R}_+ \times \Theta \to \mathbb{R}^{n_k}$ for all populations k, satisfying $\frac{d}{dt}\phi^k(t, p(0)) = \dot{p}_k$. On the other hand, it ensures that small changes in initial conditions and the payoffs, alters the way how individuals make their decisions smoothly. If individuals would have perfect information about all payoffs yielded by other pure strategies, and know the state of the population perfectly well, behavioral adjustments would be much faster, possibly leading to discontinuous switches. Perfect and complete information is something rare in large populations, if there does not exist any central authority that distributes information fast and reaches every individual in society. Hence, Lipschitz-continuity reflects an assumption that people have limited knowledge about payoffs and population states, which has some appeal if we look at large populations.

There are certainly other assumptions one may impose on behavioral rules. Indeed we are going to discuss other properties in section 9 and in the following paragraphs. Now we are going to look at some commonly used evolutionary dynamics and the behavioral rules that generate them. In particular we are going to look at three families of evolutionary dynamics. The best-known dynamic, the replicator dynamic, is a member of the class of imitative dynamics. Since it is the best understood dynamic in use in the evolutionary literature, we are going to start with it. Then we consider excess payoff dynamics, where individuals form their decision independently of their current strategy. Agents are more likely to switch to those strategies that promise a higher payoff than the average in the population. The pairwise comparison dynamic is a combination of an imitative dynamic and an excess payoff dynamic.[19]

8. Families of evolutionary dynamics

8.1. **Imitative dynamics.** Behavioral rules driven by imitation have a long tradition in the evolutionary literature. One of the best known evolutionary models, the replicator dynamics, describes an evolutionary process which is driven purely by imitation of other individuals.

[19]We skip one very important dynamic used in the evolutionary game theory literature, namely the so-called best-response dynamic. As its name suggests, it is based on the, from the non-cooperative literature familiar, best-response correspondence of a player. Its treatment has some technical drawbacks, since its vector field is not continuous. Additional theory about dynamical systems is necessary to give a full description of this dynamic. More importantly, the best-response dynamic models a dynamic process in which players are rational in the strict sense, that only current best responses grow, for a given social state. It is clear that the phenomenon of persisting irrational behavior cannot occur in this dynamic, so it is not an interesting dynamic for this work. We do however recognize its importance for economic theory. For a very detailed description of the best-response dynamic see Cressman (2003) or Hofbauer and Sigmund (1998).

All imitative dynamics share a common interpretation. An i-strategist chooses an agent from his own population at random and observes her strategy. The player will imitate the randomly drawn opponent, thus switch to her strategy, according to the rule

$$f_{ij}^{k}(\pi^{k}(p),p) = p_{k_j}\sigma_{ij}^{k}(\pi^{k}(p)) \tag{8.1}$$

where the function σ is Lipschitz continuous, and can be interpreted as the conditional imitation rate of strategy e_{k_j} by an individual that currently plays strategy e_{k_i}.

Example 3. *Suppose σ assumes the form*

$$\sigma_{ij}^{k} = \begin{cases} \pi_j^k(p) - \pi_i^k(p) & \text{if } \pi_j^k(p) > \pi_i^k(p) \\ 0 & \text{otherwise} \end{cases}$$

This revision protocol comes from Schlag (1998) and generates the ***replicator dynamics*** *first introduced by Taylor and Jonker (1978). According to this rule, an agent imitates the randomly drawn opponent if and only if his payoff exceeds the agent's own payoff. It mimics therefore a behavior that imitates via pairwise comparison of payoffs. Combining this rule with equation (7.1) gives us the autonomous differential equation*

$$\begin{aligned} \dot{p}_{k_i} &= \sum_{j=1}^{n_k} p_{k_j} p_{k_i} \left[\pi_i^k(p) - \pi_j^k(p)\right]_+ - p_{k_i} \sum_{j=1}^{n_k} p_{k_j} \left[\pi_j^k(p) - \pi_i^k(p)\right]_+ \\ &= p_{k_i} \sum_{j=1}^{n_k} p_{k_j} (\pi_i^k(p) - \pi_j^k(p)) \\ &= p_{k_i}(\pi_i^k(p) - \bar{\pi}^k(p)) \end{aligned} \tag{8.2}$$

where the expression $[u]_+ = \max[0,u], u \in \mathbb{R}$ and $\bar{\pi}^k(p)$ is the kth population's average payoff as defined in (5.1). The replicator dynamic models a process where only those strategies replicate, which are better than the population's average. One should note the weak information assumptions that are embodied by this proposed behavioral rule. The only thing a revising agent needs to know is the strategy of the randomly met opponent. In an extensive form game this could be modeled through a "message" or signal, coming from the opponent to the revising individual.

A further defining characteristic of imitative behavioral rules is a monotonicity assumption, that links the function σ with the payoffs to strategies. Intuitively we want from a imitative behavioral rule, that agents select those strategies more frequently, that have a higher payoff. This leads us to a general definition as can also be found in Sandholm (2007a).

Definition 9. *A behavioral rule of the form (8.1) generates an* ***imitative dynamic*** *defined on the set of social states* Θ *if for all populations* $1 \leq k \leq N$

(i) *the conditional imitation rate* σ_{ij}^k *is Lipschitz continuous for all* $i, j \in \{1, 2, \ldots, n_k\}$
(ii) $(\forall e_{k_\ell}, e_{k_i}, e_{k_j} \in S_k) : \sigma_{\ell j}^k(\pi^k(p)) - \sigma_{j\ell}^k(\pi^k(p)) \geq \sigma_{\ell i}^k(\pi^k(p)) - \sigma_{i\ell}^k(\pi^k(p))$ *if and only if* $\pi_j^k(p) \geq \pi_i^k(p)$

The dynamic generated by such a behavioral rule takes the general form

$$\dot{p}_{k_i} = p_{k_i} \left\{ \sum_{j=1}^{n_k} p_{k_j} \left[\sigma_{ji}^k(\pi^k(p)) - \sigma_{ij}^k(\pi^k(p)) \right] \right\} \quad k = 1, 2, \ldots, N \quad (8.3)$$

The last condition, that an imitative dynamic has to satisfy, can be interpreted as a monotonicity condition. The net-flow to strategy j, which is the number $\sum_{j=1}^{n_k}[\sigma_{ij}^k - \sigma_{ji}^k]$, must be larger than to strategy i, if the former promises a higher payoff than the latter. The imposition of this restriction appears to be fairly natural if we want to model imitative behavior. We have to emphasize that this condition does not exert that individuals know the payoff functions of all players (strategies) available, nor do individuals need to know the population state exactly. All that we ask for is, that players can observe the strategy of the randomly drawn opponent. For an economic (and self-contained) derivation of the replicator dynamics, see the following

Example 4. *Consider a very large, but finite population of firms, acting in a perfect competitive economy. Take the population sufficiently large, so that all involved variables can be regarded to have a continuum as support. Firms can choose between a finite set of possible investments, denoted by* $S = \{e_1, e_2, \ldots, e_n\}$. *In the initial time period of our observations,* t, *suppose each firm is programmed to a certain investment* e_k, *and firms can only decide upon a single investment type. The number* $x_k(t) \geq 0$ *is the absolute number of firms who choose the investment of type* k *at time* t. *The total number of firms in the economy is then the sum* $\sum_{k=1}^n x_k(t) > 0$. *The list* $p(t) = \frac{1}{\sum_{\ell=1}^n x_\ell(t)}(x_1(t), x_2(t), \ldots, x_n(t))$ *describes the relative share of firms that are devoted to a particular type of investment. Thus, the number* $p_k(t) = \frac{x_k(t)}{\sum_{\ell=1}^n x_\ell(t)}$ *is the relative frequency with which investment of type* k *occurs in the economy, in the sense that* $100 \times p_k(t)$ *percent of all firms choose investment* $e_k \in S$ *at time* t. *Clearly,* $p_k(t) \geq 0$ *and* $\sum_{k=1}^n p_k(t) \equiv 1$ *for all times* t.
By making an investment, firms expect to receive some rent. Suppose this rent depends on the relative occurrence of the particular type of investment in the economy. The rent to a firm, that makes an investment e_k, *is* $\pi_k(p(t)) = \psi(e_k, p(t)) - \delta$. *The function* ψ *is assumed to be continuously differentiable and represents the gains of the investment*

e_k. The constant factor δ is a nonnegative cost term, which is the same for all firms. To model our assumption that the frequency of an investment and its rent is negatively correlated, suppose $\frac{\partial \psi}{\partial p_k} \leq 0$. Average returns to investment in the economy are then given by

$$\bar{\pi}(p(t)) := \pi_1(p(t))p_1(t) + \ldots + \pi_n(p(t))p_n(t) = \sum_{k=1}^{n} \psi(e_k, p(t))p_k(t) - \delta$$

Suppose that only the payoffs out of the investment decides whether a firm survives or not. Hence, investment types that promise a relative higher return, make firms, that have decided on this investment type, grow relatively fast. Since firms are assumed to be programmed to a certain investment type, we can identify the investment with a class of firms. Those firms that have chosen investment of type e_k will then grow according to the differential equation[20]

$$\dot{x}_k(t) = [\psi(e_k, p(t)) - \delta]\, x_k(t)$$

To derive a law of motion of the relative share of firms that are in correspondence with investment e_k, observe that $x_k(t) = p_k(t) \sum_{\ell=1}^{n} x_\ell(t)$. Taking time derivatives, by applying the product rule, we see that $\dot{p}_k(t) \sum_{\ell=1}^{n} x_\ell(t) = \dot{x}_k(t) - p_k(t) \sum_{\ell=1}^{n} \dot{x}_\ell(t)$. From this it follows

$$\begin{aligned} \dot{p}_k(t) \sum_{\ell=1}^{n} x_\ell(t) &= x_k \pi_k(p(t)) - p_k(t) \sum_{r=1}^{n} \pi_r(p(t)) x_r(t) \\ &= p_k(t)\pi_k(p(t)) \sum_{\ell=1}^{n} x_\ell(t) - p_k(t) \sum_{r=1}^{n} \pi_r(p(t)) x_r(t) \end{aligned}$$

Dividing the left and the right side of this equation by $\sum_{\ell=1}^{n} x_\ell(t)$ gives the differential equation

$$\dot{p}_k(t) = p_k \left[\pi_k(p(t)) - \sum_{r=1}^{n} \pi_r(p(t)) p_r(t) \right]$$

what is equivalent to the replicator dynamics, defined for a single population.

8.1.1. ***Properties of Imitative Dynamics***. This section is oriented on works by Sandholm (2007a), Hofbauer and Weibull (1996), Weibull (1995) and Hofbauer and Sigmund (1998). In particular Proposition 1 can be found in Sandholm (2007a), but the proof was done by myself. Proposition 2 is taken from Sandholm (2007a, p.142). Proposition 3 is a well known result and can be found in nearly all textbooks on evolutionary dynamics. The proof given here is however entirely self-contained. Corollary 1 and Lemma 2 are my owns.

We have already mentioned that the net-flows of imitative dynamics

[20]The continuous time assumption can be regarded as a deterministic discrete time approximation. However this is not of importance in this example.

are monotone in payoffs. This implies that informational requirements for modeling imitative behavior are fairly low. However, the weak monotonicity property has some touch of rationality, since we impose that better performing strategies come up with higher net-imitation rates. The notion "better performing" must be understood in a local sense, since only a comparison between two strategies is made. A global monotonicity assumption must clearly involve all strategies available to the population. One can even sharpen the monotonicity of imitative dynamics by the following

Proposition 1. *All imitative dynamics are monotone in the sense that*

$$(\forall k \in \{1,2,\dots,N\})(\forall p_{k_j}, p_{k_i} > 0): \qquad \frac{V_j^k(p)}{p_{k_j}} \geq \frac{V_i^k(p)}{p_{k_i}} \Leftrightarrow \pi_j^k(p) \geq \pi_i^k(p) \tag{8.4}$$

Proof:

($\Leftarrow$) Suppose that $\pi_j^k(p) \geq \pi_i^k(p)$ and $p_{k_i}, p_{k_j} > 0$. Then it follows that

$$\begin{aligned} \frac{V_j^k(p)}{p_{k_j}} &= \sum_{\ell=1}^{n_k} p_{k_\ell}\left[\sigma_{\ell j}^k(\pi^k(p)) - \sigma_{j\ell}^k(\pi^k(p))\right] \\ &\geq \sum_{\ell=1}^{n_k} p_{k_\ell}\left[\sigma_{\ell i}^k(\pi^k(p)) - \sigma_{i\ell}^k(\pi^k(p))\right] \\ &= \frac{V_i^k(p)}{p_{k_i}} \end{aligned}$$

where the inequality follows since the net-flow from all strategies to pure strategy e_{k_j} must be larger than the net-flow to pure strategy e_{k_i} if the latter earns a lower payoff than the first. (This is part (ii) of definition 9.)

($\Rightarrow$) Suppose now that $\frac{V_j^k(p)}{p_{k_j}} \geq \frac{V_i^k(p)}{p_{k_i}}$. We have then that

$$\begin{aligned} \frac{V_j^k(p)}{p_{k_j}} - \frac{V_i^k(p)}{p_{k_i}} &= \sum_{\ell=1}^{n_k} p_{k_\ell}\left\{\left[\sigma_{\ell j}^k(\pi^k(p)) - \sigma_{j\ell}^k(\pi^k(p))\right] - \left[\sigma_{\ell i}^k(\pi^k(p)) - \sigma_{i\ell}^k(\pi^k(p))\right]\right\} \\ &\geq \min_{e_{k_\ell} \in S_k}\left\{\left[\sigma_{\ell j}^k(\pi^k(p)) - \sigma_{j\ell}^k(\pi^k(p))\right] - \left[\sigma_{\ell i}^k(\pi^k(p)) - \sigma_{i\ell}^k(\pi^k(p))\right]\right\} \\ &\geq 0 \end{aligned}$$

only if $\pi_j^k(p) \geq \pi_i^k(p)$ by part (ii) of the definition of an imitative dynamics.

q.e.d.

Proposition 1 tells us that growth rates of strategies are ordered according to their payoffs. This is a very important property of imitative dynamics, and was first investigated by Samuelson and Zhang (1992) (although only for the case of two populations). It is well known that this monotonicity property of the replicator dynamic lead evolutionary dynamics to eliminate strictly dominated strategies, and so offer a dynamic approach, which shares already one very important feature with the "rationalistic" static analysis of games. Elegant proofs of this claim can be found in Samuelson and Zhang (1992), Hofbauer and Weibull (1996) and Sandholm (2007a).

Corollary 1. *For all populations* $1 \leq k \leq N$ *we have for all* p_{k_i}, $p_{k_j} > 0$*:*

$$\pi_j^k(p) = \pi_i^k(p) \Rightarrow \frac{V_j^k(p)}{p_{k_j}} = \frac{V_i^k(p)}{p_{k_i}}$$

.

Proof: $\pi_j^k(p) = \pi_i^k(p)$ is equivalent to $\pi_j^k(p) \leq \pi_i^k(p)$ and $\pi_j^k(p) \geq \pi_i^k(p)$. From the definition of an imitative dynamic, we can conclude

$$\begin{aligned} \pi_j^k(p) \leq \pi_i^k(p) \quad &\Rightarrow \quad (\forall e_{k_\ell} \in S_k) : \sigma_{\ell i}^k - \sigma_{i\ell}^k \geq \sigma_{\ell j}^k - \sigma_{j\ell}^k \\ \pi_j^k(p) \geq \pi_i^k(p) \quad &\Rightarrow \quad (\forall e_{k_\ell} \in S_k) : \sigma_{\ell j}^k - \sigma_{j\ell}^k \geq \sigma_{\ell i}^k - \sigma_{i\ell}^k \end{aligned}$$

If we summarize these two findings, it must be true that

$$(\forall e_{k_\ell} \in S_k) : \sigma_{\ell i}^k - \sigma_{i\ell}^k = \sigma_{\ell j}^k - \sigma_{j\ell}^k$$

And so

$$\sum_{\ell=1}^{n_k} p_{k_\ell} \left[\sigma_{\ell j}^k(\pi^k(p)) - \sigma_{j\ell}^k(\pi^k(p))\right] = \sum_{\ell=1}^{n_k} p_{k_\ell} \left[\sigma_{\ell i}^k(\pi^k(p)) - \sigma_{i\ell}^k(\pi^k(p))\right]$$

or differently put

$$\frac{V_j^k(p)}{p_{k_j}} = \frac{V_i^k(p)}{p_{k_i}}$$

q.e.d.

Another crucial property shared by all imitative dynamics is that there is no *innovation* in a game. Suppose there is an unused strategy, say $e_{k_j} \in S_k$ in population k. This translates itself into $p_{k_j} = 0$, and so, by equation (8.3), $\dot{p}_{k_j} = 0$. A once unexplored strategy will never be discovered by the population. This is a logical consequence from imitative behavior. If individuals imitate strategies that they meet at random, they will never imitate a strategy which they will never meet. Hence, there can be no movement to a strategy, which nobody uses in the population. An immediate consequence from this property is that all "degenerate" population states, states at which all individuals in the population are programmed to a single pure strategy, are necessarily rest points under all imitative dynamics. If everybody in a fixed population only meets individuals that play the same strategy as she already does, then no imitation can take place.

Every imitative dynamic leaves the population masses constantly equal to one. This property makes an imitative dynamics to a *regular dynamic*, following the terminology used by Hofbauer and Weibull (1996), or Weibull (1995, Chapter 5.5). This property states that the sum of all directions of motion in a population has to vanish. For all times, a population state must remain in the simplex $\Delta(S_k)$. Clearly this must also hold at the macro-level of society (which we have defined as the collection of all populations). It is quite easy to establish this property. Fix an arbitrary population k and sum the time derivatives of all strategies contained in S_k:

$$\begin{aligned}\sum_{i=1}^{n_k}\dot{p}_{k_i} &= \sum_{i=1}^{n_k}p_{k_i}\left\{\sum_{j=1}^{n_k}p_{k_j}\left[\sigma^k_{ji}(\pi^k(p))-\sigma^k_{ij}(\pi^k(p))\right]\right\} \\ &= \sum_{i=1}^{n_k}p_{k_i}\left[\sum_{j=1}^{n_k}p_{k_j}\sigma^k_{ji}(\pi^k(p))\right]-\sum_{i=1}^{n_k}p_{k_i}\left[\sum_{j=1}^{n_k}p_{k_j}\sigma^k_{ij}(\pi^k(p))\right] \\ &= \sum_{i=1}^{n_k}p_{k_i}\left[\sum_{j=1}^{n_k}p_{k_j}\sigma^k_{ij}(\pi^k(p))\right]-\sum_{i=1}^{n_k}p_{k_i}\left[\sum_{j=1}^{n_k}p_{k_j}\sigma^k_{ij}(\pi^k(p))\right] \\ &= 0\end{aligned} \tag{8.5}$$

With other words, an imitative process ensures that every population k is in its state space $\Delta(S_k)$ for all times, or in more technical terms, every imitative dynamics leaves the simplex $\Delta(S_k)$ invariant (also its interior and its boundary).[21] Consequently, the space of social states Θ is invariant under imitative dynamics.

In proposition 1 we have established an ordering of growth rates of strategy proportions. This ordering is generated by payoff differences of the several pure strategies available to some player population. Relatively more individuals are going to adopt a certain pure strategy if it offers a higher payoff compared to the strategy that they currently employ. Since imitation does not allow unused strategies to become discovered, we can partition the set of pure strategies into those with positive growth rate, those with negative growth rates and the unused strategies with growth rate zero. Thus, we can define 3 disjoint sets

$$\begin{aligned}\mathcal{G}^k_{\pm}(p) &= \left\{e_{k_i}\in S_k|p_{k_i}>0\wedge\pm\frac{V^k_i(p)}{p_{k_i}}>0\right\} \\ \mathcal{G}^k_0(p) &= \left\{e_{k_i}\in S_k|p_{k_i}=0\vee V^k_i(p)=0\right\}\end{aligned}$$

Lemma 2. *For every imitative dynamics and all populations* $1\leq k\leq N$*, the following is true:*

(i) $\mathcal{G}^k_+\cap\mathcal{G}^k_-=\emptyset$ *and* $\mathcal{G}^k_+\cup\mathcal{G}^k_-\cup\mathcal{G}^k_0(p)=S_k$

(ii) $\min\left\{\pi^k_j(p)|e_{k_j}\in\mathcal{G}^k_+(p)\right\}\geq\max\left\{\pi^k_i(p)|e_{k_i}\in\mathcal{G}^k_-(p)\right\}$

[21]Invariance was also discussed in Section 6.

Proof:

(i) Follows directly from the definitions of these sets.

(ii) Pick pure strategies $e_{k_j} \in \mathcal{G}^k_+(p)$, $e_{k_i} \in \mathcal{G}^k_-(p)$. By definition of these sets, we get $\frac{V^k_j(p)}{p_{k_j}} > 0 > \frac{V^k_i(p)}{p_{k_i}}$. By proposition 1 this holds if and only if $\pi^k_j(p) \geq \pi^k_i(p)$. Since there are only finitely many pure strategies, there must exist a minimal payoff in $\mathcal{G}^k_+(p)$ and a maximal payoff in $\mathcal{G}^k_-(p)$. Henceforth, it must be that $\min\left\{\pi^k_j(p)|e_{k_j} \in \mathcal{G}^k_+(p)\right\} \geq \max\left\{\pi^k_i(p)|e_{k_i} \in \mathcal{G}^k_-(p)\right\}$

q.e.d.

Applying Lemma 2 gives us the possibility to derive a nice, and important, geometric property of payoff vectors and the vector field induced by any imitative dynamic.

Proposition 2. *For every population $1 \leq k \leq N$, the vector of payoffs $\pi^k(p)$ and the vector field $V^k(p)$ of an imitative dynamic must form an acute angle, that is algebraically*

$$\left\langle \pi^k(p), V^k(p) \right\rangle \geq 0$$

Proof: Choose an arbitrary population k and suppose, the vector field vanishes at some social state $p^* \in \Theta$, that is $V^k(p^*) = 0$. In this case we have $\left\langle \pi^k(p), V^k(p) \right\rangle = 0$.
Suppose now the system is not in a resting position. Since imitative dynamics are not innovative, we have

$$\sum_{i=1}^{n_k} V^k_i(p)\pi^k_i(p) = \sum_{e_{k_j} \in \mathcal{G}^k_+(p)} V^k_j(p)\pi^k_j(p) + \sum_{e_{k_i} \in \mathcal{G}^k_-(p)} V^k_i(p)\pi^k_i(p)$$

The directions of motions have to sum to zero, thus

$$\sum_{e_{k_j} \in \mathcal{G}^k_+(p)} V^k_j(p) = - \sum_{e_{k_i} \in \mathcal{G}^k_-(p)} V^k_i(p) \tag{8.6}$$

and on the left and the right side there are positive numbers. Further we can give a lower bound

$$\sum_{e_{k_j} \in \mathcal{G}^k_+(p)} V^k_j(p)\pi^k_j(p) \geq \min\left\{\pi^k_j(p)|e_{k_j} \in \mathcal{G}^k_+(p)\right\} \sum_{e_{k_j} \in \mathcal{G}^k_+(p)} V^k_j(p)$$

Combining this fact together with the equality (8.6) yields

$$\begin{aligned}
\sum_{i=1}^{n_k} V^k_i \pi^k_i(p) &\geq \min\left\{\pi^k_j(p)|e_{k_j} \in \mathcal{G}^k_+(p)\right\} \sum_{e_{k_j} \in \mathcal{G}^k_+(p)} V^k_j(p) + \sum_{e_{k_i} \in \mathcal{G}^k_-(p)} V^k_i(p)\pi^k_i(p) \\
&= \sum_{e_{k_i} \in \mathcal{G}^k_-(p)} V^k_i(p) \left[\pi^k_i(p) - \min\left\{\pi^k_j(p)|e_{k_j} \in \mathcal{G}^k_+(p)\right\}\right] \\
&\geq 0
\end{aligned}$$

since the sum of the growth rates are negative and the term in the brackets must be negative as well by Lemma 2.

q.e.d

The last property we are going to establish is something that all imitative dynamics share, and links the rest points of the dynamics with the Nash equilibria of the underlying game. Consider an N-population game in normal form, where each population is endowed with a finite set of pure strategies. Suppose strategies are fixed, and so is then the space of population states. We can identify the normal form game with the payoff functions of each population. Given this population game, a vector field of an imitative dynamic is induced, according to an behavioral rule as in definition 9. Denote the set of stationary states of this vector field by

$$\mathcal{R} := \left\{\tilde{p} \in \Theta | V^k(\tilde{p}) = 0, \ \forall k = 1, 2 \ldots, N\right\}$$

Proposition 3. *A Nash equilibrium in the N-population normal form game, identified by the payoff functions* $\pi = [\pi^k]_{k=1,2\ldots,N}$*, is always a rest point of an imitative dynamics defined on this game. With other words* $NE \subset \mathcal{R}$

Proof: Suppose $p^* \in \Theta$ is a Nash equilibrium of the underlying population game. Fix an arbitrary population k and note that, if the population state $p^*_k \in \Delta(S_k)$ is a Nash equilibrium in the population, than all the pure strategies that occur with positive mass in this population must have the same average payoff. Thus there exists some constant $c \in \mathbb{R}$, such that:

$$(\forall p^*_{k_i} > 0) : \pi^k_i(p^*) = c := \max_{e_{k_j} \in S_k : p^*_{k_j} > 0} \pi^k_j(p^*)$$

By Corollary 1 we get immediately that all the strategies that occur with positive mass in Nash equilibrium must have the same growth rate. Call this uniform growth rate γ^k. Imitative dynamics are not innovative, what implies that unused strategies in Nash equilibrium have neither inflow nor outflow. This implies

$$(\forall p^*_{k_\ell} = 0) : \ V^k_\ell(p^*) = 0$$

By equation (8.5), the sum of all motions in population k must vanish. In particular this must be true in Nash equilibrium, so that we can conclude

$$\begin{aligned} 0=\sum_{i=1}^{n_k} V_i^k(p^*) &= \sum_{i:p^*_{k_i}>0} V_i^k(p^*) \\ &= \sum_{i:p^*_{k_i}>0} p^*_{k_i}\frac{V_i^k(p^*)}{p^*_{k_i}}, \ \text{since}\frac{V_i^k(p^*)}{p^*_{k_i}} \equiv \gamma^k \\ &= \gamma^k \sum_{i:p^*_{k_i}>0} p^*_{k_i} \end{aligned}$$

Consequently it is true that

$$(\forall p^*_{k_i} > 0) : \ V_i^k(p^*) = 0$$

Since population k was chosen arbitrarily, we get the desired result.

q.e.d.

Unfortunately the reverse implication is not true. *Every imitative dynamics possesses rest points that are not Nash equilibria.* In particular, every vertex of the simplex $\Delta(S_k)$ is a rest point, but will not be a Nash equilibrium in general. This is the major drawback of imitative behavioral rules. However it stands without a doubt that imitation might be one of the most important features of social life. To end up the discussion about imitative dynamics, we are going to present the replicator dynamics applied to a two-population version of the "Matching pennies" game.

Example 5. *"Matching pennies" was already introduced in 1. Its normal form representation is given by Table 1. There are two populations,*

	H	*T*
H	*(1,-1)*	*(-1,1)*
T	*(-1,1)*	*(1,-1)*

TABLE 1. Matching pennies in normal form.

endowed with pure strategy space $S_i = \{H, T\}\,,\ i \in \{1, 2\}$*. These pure strategies can be seen as the vertexes of an closed interval on the real line. A population state is then a partition of this interval into two disjoint parts, and can formally be written as a vector* $p = (p_1, 1 - p_1)$ *satisfying* $p \in [0, 1] \times [0, 1]$*. With this notation we interpret the size of the subpopulation whose individuals play "H" with the number* p_1*, and analogously the size of the subpopulation whose individuals play "T" is the real number* $1 - p_1$*. One population has the role of the row-player and the other population constitutes agents who are assigned to the role of the column player. Let the state of the population in the role of the row player be the vector* p *and the state of the population in the role* *of the column player be $q = (q_1, 1-q_1)$. The space of social states is consequently the cube $\Theta = [0,1]^4$. Payoffs to population 1 are*

$$\pi^1(q) = \begin{bmatrix} \pi_1^1(q) \\ \pi_2^1(q) \end{bmatrix} = \begin{bmatrix} 2q_1 - 1 \\ 1 - 2q_1 \end{bmatrix}$$

and for population 2

$$\pi^2(p) = \begin{bmatrix} \pi_1^2(p) \\ \pi_2^2(p) \end{bmatrix} = \begin{bmatrix} 1 - 2p_1 \\ 2p_1 - 1 \end{bmatrix}$$

Average payoffs in the player populations are then

$$\begin{aligned} \bar{\pi}^1(p,q) &= p_1(2q_1-1) + (1-p_1)(1-2q_1) = (1-2p_1)(1-2q_1) \\ \bar{\pi}^2(p,q) &= q_1(1-2p_1) + (1-q_1)(2p_1-1) = (1-2p_1)(2q_1-1) \end{aligned}$$

It is readily verified that there can be no Nash equilibrium in degenerate population states, and there is a unique Nash equilibrium in the interior of the unit cube given by the profile

$$NE = \{(1/2, 1/2); (1/2, 1/2)\}$$

The replicator dynamics for population 1 can be written in a concise way as

$$\begin{aligned} V^1(p,q) = \begin{bmatrix} p_1 & 0 \\ 0 & 1-p_1 \end{bmatrix} \left\{ \begin{bmatrix} 2q_1 - 1 \\ 1 - 2q_1 \end{bmatrix} - \begin{bmatrix} (1-2p_1)(1-2q_1) \\ (1-2p_1)(1-2q_1) \end{bmatrix} \right\} \\ = \begin{bmatrix} 2p_1(2q_1-1)(1-p_1) \\ 2p_1(1-2q_1)(1-p_1) \end{bmatrix} \end{aligned}$$

The replicator dynamics for population 2 is derived after similar calculation, and equal to

$$V^2(p,q) = \begin{bmatrix} 2q_1(1-q_1)(1-2p_1) \\ 2q_1(2p_1-1)(1-q_1) \end{bmatrix}$$

Obviously we have $V_1^k(p,q) + V_2^k(p,q) = 0$ for $k \in \{1,2\}$. Thus, it suffices to analyze the evolution of one strategy population. The motion of the other subpopulation is then known as the negative of it. Consider the dynamics of the proportion of H-players in population 1. It is clearly visible that the sign of the motion is independent of the frequency of H-players in the own population. Only the proportion of H-players in the opponent population, q_1, affects the direction of the dynamic. If $q_1 > 1/2$ we have $V_1^1(p,q) > 0$ so that the proportion of individuals in population 1 that play H will rise. It will fall only if $q_1 < 1/2$, and the state of population 1, population 2 respectively, is at a rest if $q_1 = 1/2$. Other rest points of the dynamic are when the population is extremely concentrated on one pure strategy, that is if $p_1 \in \{0,1\}$ occurs. No matter what population 2's composition q is, there will be no movement between pure strategies in population 1. This is a direct consequence of the pure imitative behavior of agents under the replicator dynamics. Using the dependence of subpopulation proportions we can identify *a social state with the vector* (p_1, q_1). *From the information that this list conveys, we can easily reconstruct the state of the separate populations. This allows us to restrict ourselves to a phase plane analysis constrained by the unit square* $[0,1] \times [0,1]$. *The set of rest points for both populations can therefore be characterized through a set of points in the unit square, and is given by*

$$\mathcal{R} = \{(0,0), (1,0), (0,1), (1,1), (1/2, 1/2)\}$$

The stability of these rest points can be determined by linearizing the reduced system

$$\tilde{V}(p_1, q_1) = \begin{bmatrix} 2p_1(2q_1 - 1)(1 - p_1) \\ 2q_1(1 - q_1)(1 - 2p_1) \end{bmatrix}$$

The Jacobian matrix of this vector field is for general values (p_1, q_1)

$$D\tilde{V}(p_1, q_1) = \begin{bmatrix} (2 - 4p_1)(2q_1 - 1) & 4p_1(1 - p_1) \\ 4q_1(q_1 - 1) & (2 - 4q_1)(1 - 2p_1) \end{bmatrix}$$

Evaluating the Jacobian at the five points listed in the set $\mathcal{R}$ *gives us*

$$D\tilde{V}(0,0) = D\tilde{V}(1,1) = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} \quad , \quad D\tilde{V}(1,0) = D\tilde{V}(0,1) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$D\tilde{V}(1/2, 1/2) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

The eigenvalues of these matrices can be directly read off form the principle diagonal, except in the case of the Nash equilibrium. Here the eigenvalues are pure imaginary, that is $\lambda = \pm i$, *where* i *is the imaginary unit, corresponding to the complex eigenvectors* $(1,0)^T \pm (0,1)^T i$. *The interior Nash equilibrium is a so-called center of the linearized system. The boundary rest points are saddle points, since eigenvalues always appear in pairs of positive and negative numbers. There are directions of motion that point into their direction, and there are other directions of motion that point away from them. They can therefore not be stable rest points of the replicator dynamics. There is an easy way to find the solution curves of the replicator dynamics in Matching pennies, by means of finding a first integral (see section 6). For sake of convenience we remain in our 2-dimensional restriction. Fix an arbitrary initial social state where* p_1^0 *and* q_1^0 *are not assuming values that make the vector field stationary (i.e. 0,1,1/2). Then, we can compute the ratio*

$$\frac{\dot{p_1}}{\dot{q_1}} = \frac{dp_1}{dq_1} = \frac{p_1(1 - p_1)}{1 - 2p_1} \frac{2q_1 - 1}{q_1(1 - q_1)}$$

This a differential equation which can be solved by separation of variables. Rearranging terms, and integrating the left and the right side of the equation shows

$$\int \frac{1 - 2p_1}{p_1(1 - p_1)} dp_1 = \int \frac{2q_1 - 1}{q_1(1 - q_1)} dq_1$$

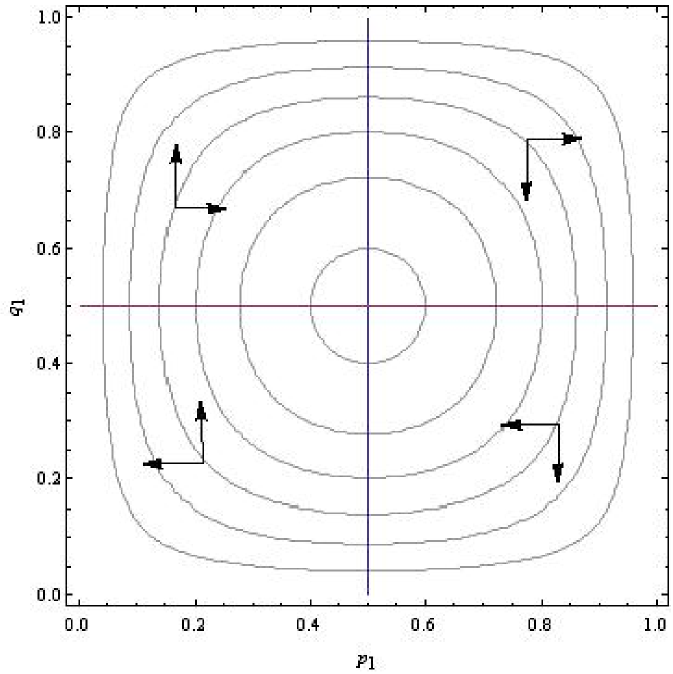

FIGURE 9. Solution curves, given implicitly by equation (8.7), of the Matching pennies game played by two populations under the replicator dynamics. The arrows are added to indicate the directions of motion.

The solution corresponding to the right side is given by $\ln(p_1(1-p_1))+C_1$*, where* C_1 *is a constant of integration. Similarly, the solution to the left side is* $-\ln(q_1(1-q_1))+C_2$*, and so we get*

$$\ln(p_1(1-p_1))+\ln(q_1(1-q_1))=C \tag{8.7}$$

where $C:=-C_1+C_2$ *a constant factor. The expression on the left side is a constant of motion of the replicator dynamics in Matching pennies. It can be shown that all zero-sum games have a first integral for the replicator dynamics (see Sandholm 2007a).*

This example is remarkable, since we see that there is no guarantee that evolutionary dynamics converge to the Nash equilibrium. In fact, population states are captured in a periodic cycle. Another example for non convergence under the replicator dynamic is the "standard" rock-paper-scissors game. As one might have guessed, it is also a zero-sum game.

Example 6. *The standard rock-paper-scissors (standard RPS from now on), is defined through the payoff matrix in Table 2. The best responses of population 1 (again the row player) show cyclic behavior. Let* BR^k *denote the best response of population* k*. Suppose population 2 starts with rock, and continuous then with the best response to the strategy of population 1. A cycle systematically drawn in Table 3 occurs.*

	r	p	s
R	*(0,0)*	*(-1,1)*	*(1,-1)*
P	*(1,-1)*	*(0,0)*	*(-1,1)*
S	*(-1,1)*	*(1-1)*	*(0,0)*

TABLE 2. standard RPS in normal form

After 3 rounds we are back at the initial strategy. It is clear from this

$$\begin{array}{ccccc} r & \to & BR^1(r)=P & \to & BR^2(P)=s \\ & & & & \downarrow \\ BR^1(p)=S & \leftarrow & BR^2(R)=p & \leftarrow & BR^1(s)=R \\ \downarrow & & & & \\ BR^2(S)=r & & & & \end{array}$$

TABLE 3. Cyclic best responses in the standard RPS game.

cyclic behavior that there are no mutually best responses in degenerate population states. Using the same notation as in the previous example, the payoffs in population 1 and 2 are

$$\pi^1(q) = \begin{bmatrix} \pi_1^1(q) \\ \pi_2^1(q) \\ \pi_3^1(q) \end{bmatrix} = \begin{bmatrix} -q_2+q_3 \\ q_1-q_3 \\ -q_1+q_2 \end{bmatrix}$$

and analagously for population 2

$$\pi^2(p) = \begin{bmatrix} -p_2+p_3 \\ p_1-p_3 \\ -p_1+p_2 \end{bmatrix}$$

Average population payoffs are then

$$\begin{aligned} \bar{\pi}^1(p,q) &= p_1(-q_2+q_3)+p_2(q_1-q_3)+p_3(-q_1+q_2) \\ &= -q_1(-p_2+p_3)-q_2(-p_1+p_3)-q_3(-p_1+p_2) \\ &= -\bar{\pi}^2(p,q) \qquad (8.8) \end{aligned}$$

The unique interior NE occurs at that point where every subpopulation earns the same average payoff. This is only then the case, when all three strategies are used with the same probability in the two populations. Thus, the set of NE is a singleton

$$NE = \{(1/3,1/3,1/3),(1/3,1/3,1/3)\}$$

where society neither loses nor gains anything. It is useful to visualize the best-response correspondence for one population. Since the game is ***symmetric****, in the sense that both populations share the same payoff matrix, we get a complete picture of the game by considering a single population. Without loss of generality we can select population 1, and* *calculate the best responses of this population in dependence of the state of its opponent population.*

$$BR^1(q) = \begin{cases} \{R\} & \text{if } q_2 < 2/3 - q_1 \wedge q_2 < 1/3 \\ \{S\} & \text{if } q_1 < 1/3 \wedge q_2 > 1/3 \\ \{P\} & \text{if } q_2 > 2/3 - q_1 \wedge q_1 > 1/3 \\ \{(p_1, p_2, 0)\} & \text{if } q_2 = 2/3 - q_1 \wedge q_1 > 1/3 \\ \{(p_1, 0, p_3)\} & \text{if } q_2 = 1/3 \wedge q_1 < 1/3 \\ \{(0, p_2, p_3)\} & \text{if } q_1 = 1/3 \wedge q_2 > 1/3 \\ \{(p_1, p_2, p_3)\} & \text{if } q_1 = q_2 = 1/3 \end{cases}$$

The replicator dynamics can then be written as

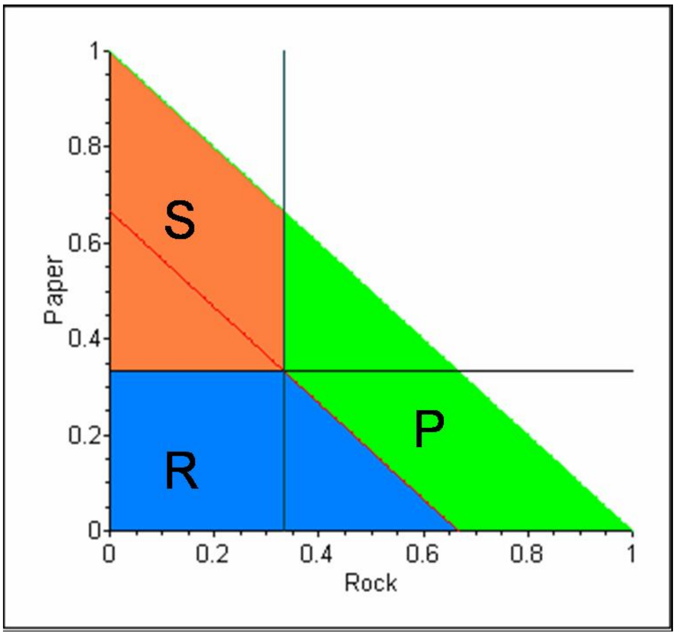

FIGURE 10. Regions of best responses in the standard RPS game.

$$\begin{aligned} V^1(p,q) &= \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix} \left\{ \begin{bmatrix} -q_2 + q_3 \\ q_1 - q_3 \\ -q_1 + q_2 \end{bmatrix} - \begin{bmatrix} \bar{\pi}^1(p,q) \\ \bar{\pi}^1(p,q) \\ \bar{\pi}^1(p,q) \end{bmatrix} \right\} \\ &= \begin{bmatrix} p_1 \left(-q_2 + q_3 - \bar{\pi}^1(p,q)\right) \\ p_2 \left(q_1 - q_3 - \bar{\pi}^1(p,q)\right) \\ p_3 \left(-q_1 + q_2 - \bar{\pi}^1(p,q)\right) \end{bmatrix} \end{aligned}$$

If we want to find a constant of motion, we need to construct a continuously differentiable function, whose time derivative vanishes along solution curves of the replicator dynamics. First of all suppose that population 1 is in its Nash equilibrium state. Per definition of an Nash equilibrium in this game, we have then $\bar{\pi}^1(p^*, q) = 0$ *for all* $q \in \Delta(S_2)$.

Clearly the same holds for population 2, if it is in its Nash equilibrium position. Pick an arbitrary population state $p \in \operatorname{int}[\Delta(S_k)]$*, and consider the sum*

$$\begin{aligned}
\sum_{i:p_i^*>0} p_i^* \frac{V_i^1(p,q)}{p_i} &= \sum_{i:p_i^*>0} p_i^* \left[\pi_i^1(q) - \bar{\pi}^1(p,q)\right] \\
&= \sum_{i:p_i^*>0} p_i^* \pi_i^1(q) - \bar{\pi}^1(p,q) \\
&= \bar{\pi}^1(p^*,q) - \bar{\pi}^1(p,q) \\
&= -\bar{\pi}^1(p,q) \qquad (8.9)
\end{aligned}$$

The same expression holds for population 2, after substituting the letters p *with* q *and changing sub- and superscripts. If we combine these two expressions we get*

$$\sum_{i:p_i^*>0} p_i^* \frac{V_i^1(p,q)}{p_i} + \sum_{i:q_i^*>0} q_i^* \frac{V_i^2(p,q)}{q_i} = 0$$

since the the standard RPS is a zero-sum game (and thus $-\bar{\pi}^1(p,q) = \bar{\pi}^2(p,q)$ *does hold for all social states* $(p,q) \in \Theta \equiv [0,1]^4$*). Our goal is to find a function, whose time derivatives gives us the expression above, for interior solutions of the replicator dynamics. Define the space* $\Theta_{(p^*,q^*)} := \{(p,q) \in \Theta | p_i^* > 0 \Rightarrow p_i > 0, q_i^* > 0 \Rightarrow q_i > 0\}$*. A candidate function for a first integral is then* $H : \Theta_{(p^*,q^*)} \to \mathbb{R}_+$ *given by:*

$$H_{(p^*,q^*)}(p,q) = \sum_{i:p_i^*>0} p_i^* \ln(\frac{p_i^*}{p_i}) + \sum_{i:q_i^*>0} q_i^* \ln(\frac{q_i^*}{q_i})$$

Compute the partial derivatives: $\frac{\partial H_{(p^*,q^*)}(p,q)}{\partial p_i} = -\frac{p_i^*}{p_i}$*, and a similar expression holds for the derivatives with respect to* q_i*. Its time derivative is*

$$\begin{aligned}
\frac{d}{dt} H(p,q) &= \sum_{i:p_i^*>0} \frac{\partial H_{(p^*,q^*)}(p,q)}{\partial p_i} \dot{p}_i + \sum_{i:q_i^*>0} \frac{\partial H_{(p^*,q^*)}(p,q)}{\partial q_i} \dot{q}_i \\
&= -\sum (p_i^* \frac{\dot{p}_i}{p_i} + q_i^* \frac{\dot{q}_i}{q_i}) \\
&= \bar{\pi}^1(p,q) + \bar{\pi}^2(p,q) \quad \text{by (8.9)} \\
&= 0 \quad \text{by (8.8)}
\end{aligned}$$

For the standard RPS game, and in the single population case, it is visualized in Figure 11. Solutions starting in the interior of the space of population states lie on a level set of the function, and remain on these closed curves forever. Again we see that evolutionary dynamics does not guarantee convergence to Nash equilibrium in all games.

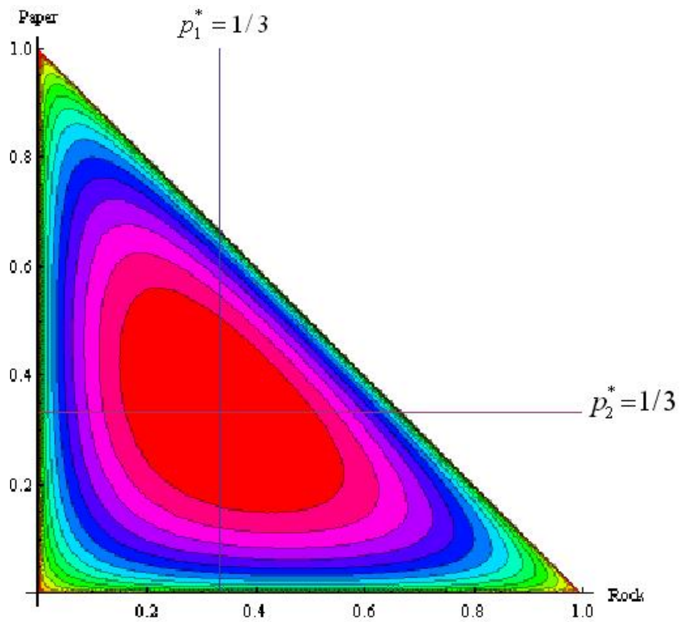

FIGURE 11. Constant of motion for the standard RPS game in the single population case. Here the projection $H(p) = p_1^{1/3} p_2^{1/3} (1 - p_1 - p_2)^{1/3}$ is drawn.

8.2. **Excess Payoff Dynamics.** The name of this family appears in Sandholm (2007a). The discussion presented here is also closely related to this text. Further references are Hofbauer and Berger (2004) and Hofbauer and Sandholm (2007a and 2007b). The strength of behavioral rules driven by imitation is the low amount of information agents need to adopt such a rule. However, we have seen that imitative rules cannot establish a one-to-one correspondence between the rest points of the generated dynamic and the Nash equilibria of the underlying game. Moreover, pure imitation rules do not allow any innovations in society. A once unused strategy remains unused forever. It appears to be necessary to make further assumptions about the knowledge of the individuals. Exactly that is what *excess payoff dynamics* do. Under these dynamics, each agent knows the current population state $p \in \Delta(S_k)$ and evaluates the performance of the various pure strategies directly. She is able to calculate the payoffs arising to each available pure strategy. Then she compares then the payoff with the population's average payoff, and switches to those strategies with some probability, whose payoff is above the population's average. A current i-strategist in population k will therefore switch to pure strategy $e_{k_j} \in S_k$ with probability proportional to the continuous function

$$f_{ij}^k(\pi^k, p_k) = \sigma_j^k \left(\begin{bmatrix} \pi_1^k(p) - \bar{\pi}^k(p) \\ \vdots \\ \pi_{n_k}^k(p) - \bar{\pi}^k(p) \end{bmatrix} \right)$$

Hence the switching rate depends on the difference in payoffs of all pure strategies available to the population and average population payoffs. Note that such a behavioral rule is completely independent of the currently used strategy of an revising agent. Differently put, the switching rates depend solely on the vector of centered payoffs, which measures the deviations of all pure strategies from mean payoff, for given social state $p \in \Theta$. Denote this vector by $\hat{\pi}^k(p)$, and call it the *excess payoff vector*, so that we can write the behavioral rule in a more concise fashion as

$$f_{ij}^k(\pi^k, p_k) = \sigma_j(\hat{\pi}^k(p)) \tag{8.10}$$

Before defining excess payoff dynamics in all generality, we are going to present an important representative of this family.

Example 7. *Suppose that agents in population* $1 \leq k \leq N$ *switch from a pure strategy* $e_{k_i} \in S_k$ *to another pure strategy* $e_{k_j} \in S_k$ *with probability proportional to the rate*

$$f_{ij}^k(\pi^k(p), p) = \begin{cases} \pi_j^k(p) - \bar{\pi}^k(p) & \text{if } \pi_j^k(p) > \bar{\pi}^k(p) \\ 0 & \text{otherwise} \end{cases}$$

The inflow to strategy e_{k_i} *is then*

$$\sum_{j=1}^{n_k} p_{k_j} f_{ji}^k(\pi^k(p), p) = \begin{cases} \pi_i^k(p) - \bar{\pi}^k & \text{if } \pi_i^k(p) > \bar{\pi}^k \\ 0 & \text{otherwise} \end{cases}$$

and the corresponding outflow

$$p_{k_i} \sum_{j=1}^{n_k} f_{ij}^k(\pi^k(p), p) = p_{k_i} \sum_{j:\hat{\pi}_j^k(p)>0} \left[\pi_j^k(p) - \bar{\pi}^k(p)\right].$$

The net-flow, or the law of motion of strategy e_{k_i}*, induced by this behavioral rule, becomes*

$$\dot{p}_{k_i} = \left[\pi_i^k(p) - \bar{\pi}^k(p)\right]_+ - p_{k_i} \sum_{j=1}^{n_k} \left[\pi_j^k(p) - \bar{\pi}^k(p)\right]_+ \tag{8.11}$$

The so obtained system of differential equations, is called the ***Brown-von Neumann-Nash (BNN)*** *dynamics. For a short history of this dynamic see Hofbauer and Berger (2004). The behavioral rule used here is proposed in Sandholm (2007a).*

From the behavioral rule it seems to be natural to assume that strategies with higher excess payoffs influence the switching rates positively. Indeed, Sandholm (2007a, Chapter 4) adds a "positive correlation" condition between excess payoffs and switching rates, as a characteristic feature of excess payoff dynamics. We will follow this approach. Before giving the general definition, note that a vector of excess payoffs cannot be strictly negative, in the sense that every pure strategy earns a strictly lower payoff than the average. Analogously, a vector of excess payoffs cannot be strictly positive, in the sense that every pure strategy earns a strictly higher payoff than the average. It can be that $\hat{\pi}^k(p) = \mathbf{0}$, the null-element of $\mathbb{R}^{n_k}$. In such a case all pure strategies earn the same payoff. Define the set

$$\mathcal{E}^k := \left\{\hat{\pi}^k(p) \in \mathbb{R}^{n_k} | (\exists e_{k_i} \in S_k) : \hat{\pi}^k_i(p) > 0 \text{ for some } p \in \Theta\right\}$$

Hence $\mathcal{E}^k$ is a subspace of $\mathbb{R}^{n_k}$, where there exists some pure strategy that earns a payoff above the population's average. We say that a vector $\mathbf{z}$ is non-positive, $\mathbf{z} \leq \mathbf{0}$, if and only if $\forall i = 1, \ldots, n_k : z_i \leq 0 \wedge \exists i = 1, \ldots, n_k : z_i = 0$.

We have introduced a behavioral rule as a matrix whose elements where the switching rates between two pure strategies. Behavioral rules that generate an excess payoff dynamic, depend only on the excess payoff vector in the population. Hence, a switch from pure strategy, say e_{k_1}, to pure strategy e_{k_3}, will be observed with a probability proportional to the factor $\sigma^k_3(\hat{\pi}^k(p))$. By definition, this factor is independent of the currently used strategy of the revising individual. The behavioral rule of a population k is therefore a matrix with one linear independent row, of the form

$$F^k(\pi^k, p) = \begin{bmatrix} f^k_{11} & f^k_{12} & \cdots & f^k_{1n_k} \\ f^k_{21} & f^k_{22} & \cdots & f^k_{2n_k} \\ \vdots & \ddots & \ddots & \vdots \\ f^k_{n_k1} & f^k_{n_k2} & \cdots & f^k_{n_kn_k} \end{bmatrix} = \begin{bmatrix} \sigma^k_1 & \sigma^k_2 & \cdots & \sigma^k_{n_k} \\ \sigma^k_1 & \sigma^k_2 & \cdots & \sigma^k_{n_k} \\ \vdots & \ddots & \ddots & \vdots \\ \sigma^k_1 & \sigma^k_2 & \cdots & \sigma^k_{n_k} \end{bmatrix}$$

so that actually all the relevant information can be stored in a n_k dimensional column vector, $\left[\sigma^k(\hat{\pi}^k(p))\right]^T = \left[\sigma^k_1(\hat{\pi}^k(p)), \sigma^k_2(\hat{\pi}^k(p)), \ldots, \sigma^k_{n_k}(\hat{\pi}^k(p))\right]$, and the real number $\sigma^k_j(\hat{\pi}^k(p)) \geq 0$ is the conditional switching rate to pure strategy $e_{k_j} \in S_k$.

Definition 10. *(Sandholm, 2007a, pp.144) A behavioral rule of the form (8.10) generates and **excess payoff dynamic** defined on the set of social states Θ if*

(i) *the switching rate $f^k_{ij}(\pi^k(p), p) = \sigma^k_j(\hat{\pi}^k(p))$ is Lipschitz continuous for all populations $1 \leq k \leq N$ and all pure strategies $e_{k_i}, e_{k_j} \in S_k$.*

(ii) *for some $p \in \Theta$ we have $\hat{\pi}^k(p) \in \mathcal{E}^k$, then the vector of excess payoffs and the switching rates point in the same direction. Algebraically, this means*

$$\left\langle \sigma^k(\hat{\pi}^k(p)), \hat{\pi}^k(p) \right\rangle = \sum_{j=1}^{n_k} \sigma^k_j(\hat{\pi}^k(p)) \hat{\pi}^k_j(p) > 0$$

for all populations $1 \leq k \leq N$.

(iii) *it satisfies sign preservation, that is*

$$\operatorname{sgn} \sigma_j^k(\hat{\pi}^k(p)) = \operatorname{sgn}[\hat{\pi}^k(p)]_+ \in \{0, 1\}$$

In particular, this implies that there is no inflow into a pure strategy that earns less than the population's average does.

The dynamic generated by such a behavioral rule assumes the general form

$$\begin{aligned} \dot{p}_{k_i} &= \sum_{j=1}^{n_k} p_{k_j} f_{ji}^k(\pi^k(p), p) - p_{k_i} \sum_{j=1}^{n_k} f_{ij}^k(\pi^k(p), p) \\ &= \sigma_i^k(\hat{\pi}^k(p)) - p_{k_i} \sum_{j=1}^{n_k} \sigma_j^k(\hat{\pi}^k(p)), \quad \forall 1 \leq k \leq N \end{aligned} \tag{8.12}$$

From the evolutionary dynamic (8.12) we see that the inflow term is independent on the frequency with which this pure strategy is played in population k. This clearly distinguishes excess payoff dynamics from imitative dynamics. In particular it makes the class of excess payoff dynamics an *innovative dynamic*, in the sense that there might be currently unused strategies that will be discovered by some individuals in the population and selected if, and only if, they give a positive excess payoff.

Clearly, excess payoff dynamics leave the interior of the state space $\Delta(S_k)$ invariant, and so the interior of the social space Θ. At all points in time, the state of a population k must be contained in this simplex. To see this, sum all differential equations of (8.12).

$$\begin{aligned} \sum_{i=1}^{n_k} \dot{p}_{k_i} &= \sum_{i=1}^{n_k} \sigma_i^k(\hat{\pi}^k(p)) - \sum_{i=1}^{n_k} p_{k_i} \sum_{j=1}^{n_k} \sigma_j^k(\hat{\pi}^k(p)) \\ &= \sum_{i=1}^{n_k} \sigma_i^k(\hat{\pi}^k(p)) - \sum_{j=1}^{n_k} \sigma_j^k(\hat{\pi}^k(p)) \\ &= 0 \end{aligned}$$

since these two sums contain actually the same summands. However, the boundary of the simplex is not invariant under excess payoff dynamics, since currently unused pure strategies may be discovered, and so the population state drifts away from the boundary, into the interior of the simplex.

8.2.1. ***Properties of excess payoff dynamics***. We are now ready to establish the most important properties of excess payoff dynamics. In particular, we are going to investigate the relationship between its rest points, and the Nash equilibria of an underlying population game. Then we are going to establish an analogous monotonicity property as was done for the class of imitative dynamics in proposition (2). All results here are conceptually from Sandholm (2007a, pp.145). Proofs are presented in a self-contained way and all results are written for the multi population case, which is an extension to the reference literature. Corollary 2 is inspired by Sandholm (2007a) and proved individually.

Lemma 3. *Consider an arbitrary N-population game, with fixed strategy sets. Suppose populations play according to a behavioral rule that generates an excess payoff dynamic. Then $\hat{\pi}^k_i(p) \leq 0$ holds for all pure strategies $1 \leq i \leq n_k$ and populations $1 \leq k \leq N$, if and only if p is a Nash equilibrium of the underlying game.*

Proof:

($\Rightarrow$) Suppose $(\forall i = 1, 2, \dots, n_k) : \hat{\pi}^k_i(p) \leq 0$ is true. The definition of average payoffs in population k is $\bar{\pi}^k(p) = \sum_{i=1}^{n_k} p_{k_i}\pi^k_i(p)$, so that our assumption is $\pi^k_i(p) \leq \sum_{i=1}^{n_k} p_{k_i}\pi^k_i(p)$. If $p_{k_i} = 0$ it must therefore be that pure strategy e_{k_i} earns a lower than average payoff, $\pi^k_i \leq \bar{\pi}^k(p)$. It cannot earn a strictly higher payoff, because then it must earn a higher payoff then all the other pure strategies in use, and consequently must earn a higher payoff then every convex combination between these payoffs. Such a convex combination is per definition population k's average payoff, and so the excess payoff of such an unused strategy would be strictly positive. A contradiction.
If on the other hand $p_{k_i} > 0$, then $\pi^k_i(p) = \bar{\pi}^k(p)$ must hold, because $\pi^k_i(p) < \sum_{i=1}^{n_k} p_{k_i}\pi^k_i(p)$ cannot be true, if $\pi^k_i(p)$ appears in the sum on the right hand side of the inequality. Henceforth we get

$$
\begin{aligned}
(\exists c \in \mathbb{R})(\forall p_{k_i} > 0) : \quad & \pi^k_i(p) = c \\
\Leftrightarrow \quad & \pi^k_i(p) = \max_{j: e_{k_j} \in S_k} \pi^k_j(p) \text{ whenever } p_{k_i} > 0
\end{aligned}
$$

But then must the population state $p_k \in \Delta(S_k)$ be a Nash equilibrium in this population. Since k was selected arbitrarily, this must hold for every population.

($\Leftarrow$) If p is a Nash equilibrium, the same steps as in direction ($\Rightarrow$) apply, but in the reversed order.

q.e.d.

The next Lemma is a simple consequence of point (iii) in definition 10 and Lemma 3.

Lemma 4. *Consider an arbitrary N-population game, with fixed strategy sets. Suppose populations play according to a behavioral rule that generates an excess payoff dynamic. If for some social state $p \in \Theta$ we observe that $\hat{\pi}^k(p) \leq \mathbf{0}$ for all populations $1 \leq k \leq N$ then $V^k(p) = \mathbf{0}$ for all populations k.*

Proof: If $p_{k_i} = 0$, then this pure strategy must earn a lower than population average payoff. By definition of a behavioral rule that generates an excess payoff dynamic we get $\sigma_i^k(\hat{\pi}^k(p)) = 0$. If $p_{k_i} > 0$ then $\pi_i^k(p) = \bar{\pi}^k(p)$ and again sign preservation implies that $\sigma_i^k(\hat{\pi}^k(p)) = 0$. The general formula for an excess payoff dynamic (8.12) states that $\dot{p}_{k_i} = 0$, so that $V^k(p) = \mathbf{0}$. Since we have selected population k arbitrarily, we obtain the desired result.

q.e.d.

Equipped with these two Lemmas, it is easy to establish our main result.

Proposition 4. *Consider an arbitrary N-population game, with fixed strategy sets. Suppose populations play according to a behavioral rule that generates an excess payoff dynamic. Independent of the particular specification of the game and the particular form of the dynamics, the following two results are valid:*

(i) $p^* \in \Theta$ *is a NE*$\Leftrightarrow$ $V^k(p^*) = \mathbf{0}\ \forall k \in \{1, 2, \ldots, N\}$
(ii) $(\forall k \in \{1, 2, \ldots, N\}) : \hat{\pi}^k \in \mathcal{E}^k \Rightarrow \left\langle \pi^k(p), V^k(p) \right\rangle > 0$

Proof:

(i) ($\Rightarrow$) If $p^* \in \Theta$ is NE of the population game, with population component $p_k^* \in \Delta(S_k)$, then Lemma 3 implies that the vector of excess payoffs in population k is the zero vector. Lemma 4 states then that the vector field of this population comes to a rest at p^*. This must hold for all populations.
($\Leftarrow$) Can be proved by contraposition. If $p^* \in \Theta$ were not a NE, then by Lemma 3 $\hat{\pi}^k(p^*) \in \mathcal{E}^k$. By definition of an excess payoff dynamics, we get $\left\langle \sigma^k(\pi^k(p^*), p^*), \hat{\pi}^k(p^*) \right\rangle > 0$. But then can the vector field of population k not be in a resting position.
(ii) Call the column vector in $\mathbb{R}^{n_k}$ filled up with ones, as $\mathbf{1}$. We can write the excess payoff vector as $\hat{\pi}^k(p) = \pi^k(p) - \bar{\pi}^k(p)\mathbf{1}$. Exploiting the bilinearity of the inner product, we get

$$
\begin{aligned}
\left\langle \pi^k(p), V^k(p) \right\rangle &= \left\langle \hat{\pi}^k(p) + \bar{\pi}^k(p)\mathbf{1}, V^k(p) \right\rangle \\
&= \left\langle \hat{\pi}^k(p), V^k(p) \right\rangle + \bar{\pi}^k(p) \left\langle \mathbf{1}, V^k(p) \right\rangle \\
&= \sum_{i=1}^{n_k} \hat{\pi}_i^k(p) V_i^k(p) \\
&= \sum_{i=1}^{n_k} \hat{\pi}_i^k(p)\sigma_i^k(\hat{\pi}^k(p)) - \sum_{j=1}^{n_k} \sigma_j^k(\hat{\pi}^k(p)) \sum_{i=1}^{n_k} p_{k_i}\hat{\pi}_i^k(p)
\end{aligned}
$$

where the third equality comes from the already demonstrated fact, that the sum of all directions of motions in a population must sum up to zero. A short calculation beside, reveals that

$$\begin{aligned}\sum_{i=1}^{n_k} p_{k_i}\hat{\pi}_i^k(p) &= \sum_{i=1}^{n_k}\left[\pi_i^k(p)-\bar{\pi}^k(p)\right]p_{k_i}\\ &= \sum_{i=1}^{n_k}\pi_i^k(p)p_{k_i}-\bar{\pi}^k(p)\\ &= 0\end{aligned}$$

Using the result, one sees immediately that

$$\left\langle \pi^k(p), V^k(p)\right\rangle = \left\langle \sigma^k(\pi^k(p)), \hat{\pi}^k(p)\right\rangle > 0$$

by definition of an excess payoff dynamics if $\hat{\pi}^k(p) \in \mathcal{E}^k$.

q.e.d.

Combining the results from Lemma 4 and Proposition 4, we can show that in all excess payoff dynamics, the payoffs and the vector field forms an acute angle. We outline here a different proof, in which point (iii) of definition 10 is not needed as an assumption. However, (iii) simplifies the proof dramatically.

Corollary 2. *In any excess payoff dynamic the vector of payoffs and the induced vector field* $V^k : \Theta \to \Delta(S_k)$ *form an acute angle.*

Proof: Form Proposition 4 we know that we can write

$$\left\langle \pi^k(p), V^k(p)\right\rangle = \left\langle \sigma^k(\pi^k(p)), \hat{\pi}^k(p)\right\rangle$$

This inner product is positive if there is some pure strategy that earns more than the mean. If this is not the case, than $\hat{\pi}^k(p) \leq \mathbf{0}$. We will show that if $\hat{\pi}_i^k(p) < 0$, then $\sigma_i^k(\hat{\pi}^k(p)) = 0$.
Order the pure strategies according to their excess payoffs in an increasing order. The first entry in the vector is then the pure strategy with the lowest excess payoff at p, the second entry is the pure strategy with the second lowest, and so on. By assumption the excess payoff vector is non-positive. Call the number of pure strategies who earn exactly the mean payoff by $n_k^0 \leq n_k$. The rest performs worse than the population's average and we count $n_k^- := n_k - n_k^0$ of such strategies. Define an indicator function as

$$(i = 1, 2\dots, n_k) : \ \mathrm{ind}(\hat{\pi}_i^k(p)) := \begin{cases} 1 & \text{if } \hat{\pi}_i^k(p) \geq 0 \\ 0 & \text{otherwise}\end{cases}$$

Next define the vector $\mathbf{1}_{n_k^0} \in \mathbb{R}_+^{n_k}$ which is filled up with indicator functions. Since we have ordered the payoff entries in an increasing order, the entries 1 to n_k^- are zero, and the entries $n_k^- + 1$ to n_k are 1. Now, if we translate the excess payoff vector of population k by $\epsilon\mathbf{1}_{n_k^0}$, where $\epsilon > 0$ and an arbitrary small parameter, we see that the resulting vector $\hat{\pi}^k(p;\epsilon) := \hat{\pi}^k(p) + \epsilon\mathbf{1}_{n_k^0} \in \mathcal{E}^k$. The linear (and therefore continuous) transformation $g : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ given by $g(\epsilon) := \left\langle \sigma^k(\hat{\pi}^k(p;\epsilon)), \hat{\pi}^k(p;\epsilon)\right\rangle$

assumes, by point (ii) in definition 10, always positive values for $\epsilon > 0$. By continuity, we have $\lim_{\epsilon \longrightarrow 0} g(\epsilon) = g(0) = \langle \sigma^k(\hat{\pi}^k(p)), \hat{\pi}^k(p) \rangle$. This limit cannot be strictly negative, since then, again by continuity, we could find some small $\delta > 0$, and $g(\delta) \leq 0$, what is a contradiction. Henceforth, $g(0) \geq 0$ must be true. By definition 8, conditional switching rates are non-negative, and by our assumption the excess payoff vector is non-positive. To accomplish $\langle \sigma^k(\hat{\pi}^k(p)), \hat{\pi}^k(p) \rangle = \sum_{h=1}^{n_k^-} \sigma_h^k(\hat{\pi}^k)\hat{\pi}^k \geq 0$, we need that $\sigma_h^k = 0$ for $h = 1, 2, \ldots, n_k^-$.

q.e.d.

Excess payoff dynamics have the nice property that they link the fixed points of the induced vector field with the Nash equilibria of the underlying population game. The cost of a model that assumes agents adopt behavioral rules that generate an excess payoff dynamic is that we have to presuppose that agents know a lot about the society in which they are playing the game. In particular, agents need to know the average payoff earned in their populations. This is fairly restrictive when we speak of large populations. Imitative dynamics worked fine without any assumption about the knowledge of individuals, concerning the state of the population they belong to. To end up this section we are going to present examples using the BNN dynamic.

Example 8. *Consider the already familiar Matching pennies game. For sake of completeness, we represent the payoffs to the two populations once again,*

$$\pi^1(q) = \begin{bmatrix} \pi_1^1(q) \\ \pi_2^1(q) \end{bmatrix} = \begin{bmatrix} 2q_1 - 1 \\ 1 - 2q_1 \end{bmatrix}, \; \pi^2(p) = \begin{bmatrix} \pi_1^2(p) \\ \pi_2^2(p) \end{bmatrix} = \begin{bmatrix} 1 - 2p_1 \\ 2p_1 - 1 \end{bmatrix}$$

$$\bar{\pi}^1(p,q) = p_1(2q_1 - 1) + (1 - p_1)(1 - 2q_1) = (1 - 2p_1)(1 - 2q_1) = -\bar{\pi}^2(p,q)$$

Let us start with computing the excess payoff vector for population 1

$$\hat{\pi}^1(p,q) = \begin{bmatrix} \pi^1(q) - \bar{\pi}^1(p,q) \\ \pi^2(q) - \bar{\pi}^1(p,q) \end{bmatrix} = 2 \begin{bmatrix} (2q_1 - 1)(1 - p_1) \\ (1 - 2q_1)p_1 \end{bmatrix}$$

and analogously one calculates the excess payoff vector for population 2

$$\hat{\pi}^2(p,q) = 2 \begin{bmatrix} (1 - 2p_1)(1 - q_1) \\ (2p_1 - 1)q_1 \end{bmatrix}$$

The BNN dynamic for population 1 is then

$$\begin{aligned} \dot{p}_1 &= \left[\hat{\pi}_1^1(p,q)\right]_+ - p_1 \left[\hat{\pi}_1^1(p,q)\right]_+ - p_1 \left[\hat{\pi}_2^1(p,q)\right]_+ \\ \dot{p}_2 &= -\dot{p}_1 \end{aligned}$$

where $[\hat{\pi}_1^1(p,q)]_+ = 2(2q_1 - 1)(1 - p_1)$ *only if* $q_1 > 1/2$*, and 0 otherwise,* $[\hat{\pi}_2^1(p,q)]_+ = 2(1 - 2q_1)p_1$ *only if* $q_1 < 1/2$*. For population 2 we get*

$$\begin{aligned} \dot{q}_1 &= \left[\hat{\pi}_1^2(p,q)\right]_+ - q_1 \left[\hat{\pi}_1^2(p,q)\right]_+ - q_1 \left[\hat{\pi}_2^2(p,q)\right]_+ \\ \dot{q}_2 &= -\dot{q}_1 \end{aligned}$$

where $[\hat{\pi}_1^2(p,q)]_+ = 2(1-2p_1)(1-q_1)$ *only if* $p_1 < 1/2$, *and 0 otherwise,* $[\hat{\pi}_2^2(p,q)]_+ = 2(2p_1-1)q_1$ *only if* $p_1 > 1/2$.
It is obvious from these four equations that it is sufficient to analyze the evolution of the first pure strategy in each population. The path of the second strategy follows then automatically. Our state space is again the unit square in the plane, spanned by the intervals $[0,1]$. *The BNN dynamic divides this unit square into six regions, which have to be studied separately.*

Region 1: $0 \leq p_1, q_1 < 1/2$. *Strategy T has a positive excess payoff in population 1 and strategy H has a positive payoff in population 2. The two relevant differential equations are thus*

$$\begin{bmatrix} \dot{p}_1 \\ \dot{q}_1 \end{bmatrix} = \begin{bmatrix} 2(p_1)^2(2q_1-1) \\ 2(1-q_1)^2(1-2p_1) \end{bmatrix}$$

Wee see that that p_1 *is decreasing, while* q_1 *is increasing. This means that there is an over-proportional outflow in population 1 from pure strategy H, and an over-proportional inflow in population 2 to pure strategy H.*

Region 2: $0 \leq p_1 < 1/2,\ 1/2 < q_1 \leq 1$. *H earns now an payoff above the average in both populations, so that we have to analyze the equations*

$$\begin{bmatrix} \dot{p}_1 \\ \dot{q}_1 \end{bmatrix} = \begin{bmatrix} 2(1-p_1)^2(2q_1-1) \\ 2(1-q_1)^2(1-2p_1) \end{bmatrix}$$

We see that the proportion of H players increases in both populations in this region.

Region 3: $1/2 < p_1 \leq 1,\ 0 \leq q_1 < 1/2$. *T earns an payoff above the average in population 1, and H does so in population 2. The dynamics are in this region*

$$\begin{bmatrix} \dot{p}_1 \\ \dot{q}_1 \end{bmatrix} = \begin{bmatrix} 2(p_1)^2(2q_1-1) \\ 2(q_1)^2(1-2p_1) \end{bmatrix}$$

Both strategy frequencies are decreasing in this region.

Region 4: $1/2 < p_1 \leq 1,\ 1/2 < q_1 \leq 1$. *In this case H earns an above average payoff in population 1, T earns an above average payoff in population 2. This leads us to the consideration of the planar system*

$$\begin{bmatrix} \dot{p}_1 \\ \dot{q}_1 \end{bmatrix} = \begin{bmatrix} 2(1-p_1)^2(2q_1-1) \\ 2(q_1)^2(1-2p_1) \end{bmatrix}$$

We see that the proportion of H strategists in population 1 is increasing in this region, while the proportion of H strategists in population 2 is decreasing.

Region 5: $p_1 = 1/2,\ 0 \leq q_1 \leq 1$. *In this case there is no motion in population 2, since the excess payoff vector is the zero vector.*

The dynamics in population 1 depend on the initial state of population 2.

Region 6: $0 \leq p_1 \leq 1$, $q_1 = 1/2$. *Population 1 is in a steady state position. The motions in population 2 depend on the initial state of population 1.*

One sees that the Nash equilibrium is the unique rest point of the BNN dynamics in Matching pennies, and in particular there can be no rest point at any boundary state. To get a better overview of the dynamics, we can summarize all these cases in two continuous functions

$$\tilde{V}^1(p_1, q_1) = \dot{p}_1 = \begin{cases} 2(p_1)^2(2q_1 - 1) & ;\ 0 \leq q_1 < 1/2 \\ 0 & ;\ q_1 = 1/2 \\ 2(1-p_1)^2(2q_1 - 1) & ;\ 1/2 < q_1 \leq 1 \end{cases}$$

$$\tilde{V}^2(p_1, q_1) = \dot{q}_1 = \begin{cases} 2(1-q_1)^2(1 - 2p_1) & ;\ 0 \leq p_1 < 1/2 \\ 0 & ;\ p_1 = 1/2 \\ 2(q_1)^2(1 - 2p_1) & ;\ 1/2 < p_1 \leq 1 \end{cases}$$

The induced vector field is plotted in Figure 12

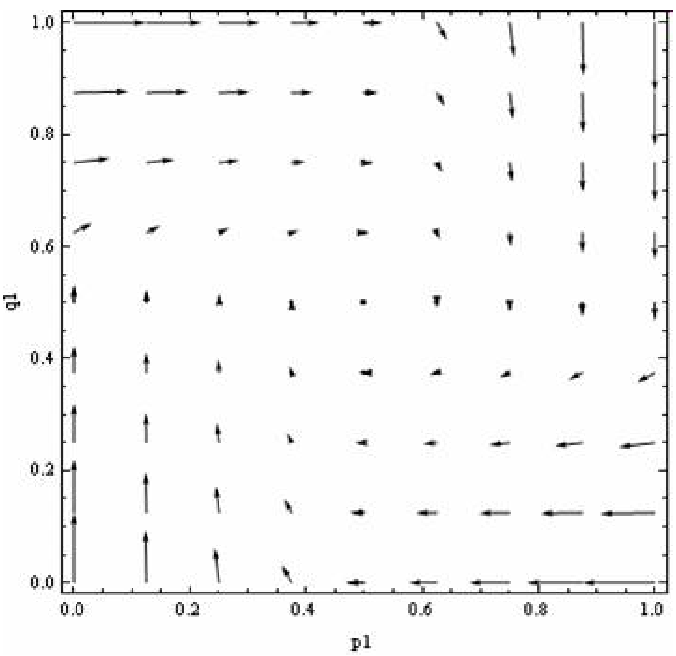

FIGURE 12. Vector field induced by the BNN dynamics in matching pennies

The indicated directions by the vectors mimic the directions of solution curves of the BNN dynamics. It appears that the system approaches its Nash equilibrium in a spiraling motion. The prove that the Nash equilibrium is asymptotically stable under the BNN dynamic needs a long and tedious computation. It is shown how to construct an appropriate Lyapunov function in Appendix A.

8.3. **Pairwise Comparison dynamics.** Maybe the most important property of excess payoff dynamics is their one-to-one correspondence between its rest points and the Nash equilibria of the underlying population game. However, agents that act according to a behavioral rule that generates such dynamics, must be very well informed about the current population state, since they compare every payoff with the population's average. If all individuals in all populations adopt the same behavioral rule that generate an excess payoff dynamic, it is doubtful why there should be a sort of tâttonement process before the Nash equilibrium state is "discovered". In the context of large populations, it is clearly a strong assumption that individuals are perfectly informed about the state of their population, so we seek for a refinement of behavioral rules, that share the connection between rest points and Nash equilibria as excess payoff dynamics, but keep the required level of knowledge of players at a minimum, just like the imitative dynamics do. Such a behavioral rule does exist, and was introduced by Sandholm (2007a, 2007b), which serves also as the general reference for the following lines. Since we search a dynamic that combines all the good facts of the already introduced dynamics, it comes with no surprise that the behavioral rules that generate this dynamic, will be closely related to them. Excess payoff dynamics are derived from agents who pick an individual from the own population at random, and compare the payoff of this agent with the population's average. Strategies are adopted with probability proportional to the difference of expected payoffs of a pure strategy and the population's average. In this case some kind of selection takes place, where only the better than average performing strategies are sorted out. The behavioral rule proposed for the replicator dynamics works similar, whereas agents do not compare the strategy's payoff with the population's average, but with their own payoff. Moreover, switching probabilities are positively influenced by the proportion of individuals that play some pure strategy, since such strategies are more likely to be observed. Pairwise comparison dynamics combine these two properties, as will be clear after looking at the following

Example 9. *Consider a population whose members use the behavioral rule*

$$f^k_{ij}(\pi^k(p),p) = \max\left[0, \pi^k_j(p) - \pi^k_i(p)\right] \equiv \left[\pi^k_j(p) - \pi^k_i(p)\right]_+$$

According to this rule, individuals switch from their current pure strategy $e_{k_i} \in S_k$ to the pure strategy $e_{k_j} \in S_k$ with probability proportional to the absolute difference between the payoffs of these strategies. There will be no flow from e_{k_i} to e_{k_j} if the latter perform worse than the *former. Applying equation (7.1), we get the differential equation*

$$\begin{aligned} \dot{p}_{k_i} &= \sum_{j=1}^{n_k} p_{k_j} f_{ji}^k - p_{k_i} \sum_{j=1}^{n_k} f_{ij}^k \qquad (8.13) \\ &= \sum_{j=1}^{n_k} p_{k_j} \left[\pi_i^k(p) - \pi_j^k(p)\right]_+ - p_{k_i} \sum_{j=1}^{n_k} \left[\pi_j^k(p) - \pi_i^k(p)\right]_+ \end{aligned}$$

The inflow to strategy e_{k_i} depends therefore on the proportion of those strategies, used in population $1 \leq k \leq N$, that perform worse as itself, and the flow out of this pure strategy depends on the proportion of individuals that currently play it, and the payoff difference between it and better performing strategies. Sandholm (2007b) baptized this dynamic with the name ***pairwise difference dynamic.*** *We refer to Sandholm (2007b) for a history of this dynamic.*

It is clear from the just given example that pairwise difference dynamics have the selection characteristic also shared by excess payoff dynamics, since only better performing strategies do spread. Better performing is here understood as a comparison between two available pure strategies to the population. No knowledge about the definite state of the population is assumed, a feature that also imitative dynamics have. In general we introduce pairwise comparison dynamics through a behavioral rule that only depend on the payoffs in the population $k \in \{1, 2, \ldots, N\}$, and so only indirect on the population state.

$$f_{ij}^k(\pi^k, p) = \sigma_{ij}^k(\pi_j^k(p) - \pi_i^k(p)) \qquad (8.14)$$

Equipped with this behavioral rule, we can now state the general

Definition 11. *(Sandholm, 2007a, pp.148) A behavioral rule of the form (8.14) generates a* ***pairwise comparison dynamic*** *defined on the space of social states Θ if*

(i) *the function $\sigma_{ij}^k(\pi_j^k(p) - \pi_i^k(p))$ is Lipschitz continuous in its arguments, for all $1 \leq k \leq N$ and $1 \leq i, j \leq n_k$.*
(ii) *switching rates are positively influenced by the payoff difference between two pure strategies, and worse strategies do not replicate. We can formulate such a condition mathematically as*

$$\operatorname{sgn} \sigma_{ij}^k(\pi_j^k(p) - \pi_i^k(p)) = \operatorname{sgn}\left[\pi_j^k(p) - \pi_i^k(p)\right]_+ \in \{0, 1\}$$

This behavioral rule generates a system of differential equations, one for each pure strategy in population $1 \leq k \leq N$, which is given by

$$\begin{aligned} \dot{p}_{k_i} &= \sum_{j=1}^{n_k} p_{k_j} f_{ji}^k(\pi^k(p), p) - p_{k_i} \sum_{j=1}^{n_k} f_{ij}^k(\pi^k(p), p) \qquad (8.15) \\ &= \sum_{j=1}^{n_k} p_{k_j} \sigma_{ji}^k(\pi_i^k(p) - \pi_j^k(p)) - p_{k_i} \sum_{j=1}^{n_k} \sigma_{ij}^k(\pi_j^k(p) - \pi_i^k(p)) \end{aligned}$$

The last requirement is a monotonicity condition on behalf of the switching rates. If pure strategy $e_{k_i} \in S_k$ performs worse than a randomly drawn opponent strategy, there will be some positive probability that a currently e_{k_i} playing agent changes her strategy in favor for some alternative e_{k_j}. Otherwise there will be no outflow to the randomly drawn strategy, and $\sigma^k_{ij}(\pi^k_j(p) - \pi^k_i(p)) = 0$. Clearly, the sum of all these differential equations vanishes at all population states $p \in \Theta$. To see this, we count

$$\begin{aligned}\sum_{i=1}^{n_k} \dot{p}_{k_i} &= \sum_{i=1}^{n_k}\sum_{j=1}^{n_k} p_{k_j}\sigma^k_{ji}(\pi^k_i(p) - \pi^k_j(p)) - \sum_{i=1}^{n_k} p_{k_i}\sum_{j=1}^{n_k}\sigma^k_{ij}(\pi^k_j(p) - \pi^k_i(p)) \\ &= \sum_{i=1}^{n_k} p_{k_i}\sum_{j=1}^{n_k}\sigma^k_{ij}(\pi^k_j(p) - \pi^k_i(p)) - \sum_{i=1}^{n_k} p_{k_i}\sum_{j=1}^{n_k}\sigma^k_{ij}(\pi^k_j(p) - \pi^k_i(p)) \\ &= 0\end{aligned}$$

8.3.1. ***Properties of pairwise comparison dynamics***. The class of pairwise comparison dynamics establishes a one-to-one correspondence between its rest points and the Nash equilibria of the underlying population game, just as the excess payoff dynamics do. Their remarkable characteristic is that they achieve this without any demanding knowledge restriction on behalf of the individuals. All they need to know is the payoff of their own strategy, and the payoff of the randomly drawn agent, just as in pure imitative behavioral models. We will proof the existence of this correspondence with the help of some Lemmas. Reference literature for the following discussion is Sandholm (2007a,2007b).

Lemma 5. *Consider an arbitrary N population game in normal form. $p^* \in \Theta$ is a Nash equilibrium of this game if and only if for all pure strategies, and all populations $p^*_{k_i} = 0$ or $\sum_{j=1}^{n_k}\left[\pi^k_j(p^*) - \pi^k_i(p^*)\right]_+ = 0$ holds.*

Proof: Fix an arbitrary population $1 \leq k \leq N$. We will proof both directions of this Lemma separately.

($\Rightarrow$) If $p^*_k \in \Delta(S_k)$ is the population k component of the Nash equilibrium p^*, then all the strategies used in this profile must be optimal, given this social state. This means that there exists some scalar $c \in \mathbb{R}$ such that $c = \max_{e_{k_j} \in S_k} \pi^k_j(p^*)$. If some strategy is unused in social state p^*, then this pure strategy cannot earn a higher payoff than c, hence $\pi^k_i(p^*) \leq c$ if $p^*_{k_i} = 0$. On the other hand, a used strategy cannot earn a higher payoff than other used pure strategies, so that we get $\pi^k_i(p^*) = c$ if $p^*_{k_i} > 0$. This implies $\sum_{j=1}^{n_k}\left[\pi^k_j(p^*) - \pi^k_i(p^*)\right]_+ = 0$.

($\Leftarrow$) Suppose first that $p^*_{k_i} = 0$. There can be no outflow from e_{k_i} to some other pure strategy in this population. If there would be no inflow to e_{k_i}, then the pairwise difference dynamic tells us that

$$
\begin{aligned}
\sum_{j=1}^{n_k} p^*_{k_j}\sigma^k_{ji}(\pi^k_i(p^*) - \pi^k_j(p^*)) &= 0 \\
\Leftrightarrow (\forall j : p^*_{k_j} > 0) &: \quad \sigma^k_{ji}(\pi^k_i(p^*) - \pi^k_j(p^*)) = 0 \\
\Leftrightarrow (\forall j : p^*_{k_j} > 0) &: \quad \pi^k_i(p^*) \leq \pi^k_j(p^*)
\end{aligned}
$$

Hence, e_{k_i} cannot be a best-reply to p^*, and therefore receives no weight in Nash equilibrium.
Suppose now that $\sum_{j=1}^{n_k}\left[\pi^k_j(p^*) - \pi^k_i(p^*)\right]_+ = 0$ holds. This translates itself into

$$
\begin{aligned}
(\forall e_{k_j} \in S_k) &: \quad \pi^k_j(p^*) \leq \pi^k_i(p^*) \\
\Leftrightarrow & \quad \pi^k_i(p^*) = \max_{e_{k_j}\in S_k} \pi^k_j(p^*)
\end{aligned}
$$

But this is the definition of a Nash equilibrium.

q.e.d.

The next Lemma establishes a link between the outflows in a player population, and the evolution of the size of the subpopulations, which is the proportion of individuals in the player population, programmed to some pure strategy. It will be useful to write down the dynamical system in a population $k \in \{1, 2, \dots, N\}$ in matrix form. It is readily verified that any pairwise comparison dynamic can be written elegantly as

$$V^k(p) = F^k(\pi^k(p), p)^T p_k - \mathrm{diag}[p_k]F^k(\pi^k(p), p)\mathbf{1} \tag{8.16}$$

where the $n_k \times n_k$ dimensional matrix F^k is the behavioral rule of population k, the n_k dimensional vector $\mathbf{1}$ is a column vector containing only 1 as entries, and finally $\mathrm{diag}[p_k] = \begin{bmatrix} p_{k_1} & 0 & \dots & 0 \\ 0 & p_{k_2} & \dots & 0 \\ \vdots & \dots & \ddots & \vdots \\ 0 & 0 & \dots & p_{k_{n_k}} \end{bmatrix}$.

Lemma 6. *Consider an arbitrary N population game in normal form. The dynamical system of population $1 \leq k \leq N$ has a rest point at the social state $p \in \Theta$ if and only if $\sum_{j=1}^{n_k}\sigma^k_{ij}(\pi^k_j(p) - \pi^k_i(p)) = 0$ for all $e_{k_i} \in S_k$.*

Proof:

($\Leftarrow$) If there is no outflow from any pure strategy $e_{k_i} \in S_k$, the dynamic of all pure strategies reduces to an inflow. This means that $V^k(p) = \left[F^k(\pi^k(p), p)\right]^T p_k$. This is a convex combination of nonnegative numbers. Since the directions of motion have to sum up to zero, the dynamical system has to vanish.

($\Rightarrow$) Suppose $V^k(p) = \mathbf{0}$ holds for some social state $p \in \Theta$. We will construct a finite family of sets that covers S_k and show that all the strategies contained in the sets of the family must have neither inflow nor outflow.

Step 1: Start with the best performing pure strategy in population k, given the social state p. Without loss of generality let this strategy be $e_{k_i} \in S_k$, with payoff $\pi_i^k(p)$. This payoff satisfies the inequalities

$$\begin{aligned} & \pi_i^k(p) \geq \pi_j^k(p) && \forall e_{k_j} \in S_k \\ \Leftrightarrow\quad & \left[\pi_j^k(p) - \pi_i^k(p)\right]_+ = 0 && \forall e_{k_j} \in S_k \\ \Leftrightarrow\quad & \sigma_{ij}^k(\pi_j^k(p) - \pi_i^k(p)) = 0 && \forall e_{k_j} \in S_k \end{aligned}$$

Hence, there is no outflow from a best-performing strategy. Since the dynamical system is in a resting position, there can also be no inflow, thus $\sum_{j=1}^{n_k} p_{k_j}\sigma_{ji}^k = 0$, what occurs if for all $e_{k_j} \in S_k : \ \sigma_{ji}^k = 0$ or $p_{k_j} = 0$. There are two possibilities for this to be observed.

(a) All pure strategies earn the same payoff as the best-performing pure strategy does. In this case all pure strategies have zero outflow and zero inflow, so that we are done.

(b) There exists some worse performing pure strategy than e_{k_i}.

Step 2: Define the set

$$\mathcal{C}_1 := \left\{ e_{k_h} \in S_k : \pi_h^k(p) = \min_{e_{k_j} \in S_k} \left[\pi_i^k(p) - \pi_j^k(p)\right] > 0 \right\}$$

Elements of this set can be interpreted as the "second-best" choices in population k for social state p, since their payoff is closest to the payoff of the best performing strategy. If $\mathcal{C}_1 = \emptyset$, we are in case (a), and therefore done with the proof. We continue with case (b), so that there exists some strategy that lies in $\mathcal{C}_1$. Let e_{k_h} be such a strategy, with associated payoff $\pi_h^k(p)$. Denote the truncated pure strategy space $S_k^1 = S_k \setminus \mathcal{C}_1$. The following inequalities are then satisfied

$$\begin{aligned} & \pi_i^k(p) - \pi_h^k(p) < \pi_i^k(p) - \pi_j^k(p) && \forall e_{k_h} \in \mathcal{C}_1, \forall e_{k_j} \in S_k^1 \\ \Leftrightarrow\quad & \pi_h^k(p) > \pi_j^k(p) && \forall e_{k_h} \in \mathcal{C}_1, \forall e_{k_j} \in S_k^1 \\ \Leftrightarrow\quad & \left[\pi_j^k(p) - \pi_h^k(p)\right]_+ = 0 && \forall e_{k_h} \in \mathcal{C}_1, \forall e_{k_j} \in S_k^1 \\ \Leftrightarrow\quad & \sigma_{hj}^k = 0 && \forall e_{k_h} \in \mathcal{C}_1, \forall e_{k_j} \in S_k^1 \end{aligned}$$

In Step 1 of this proof, we have seen that there is no inflow to the best performing strategies. Since there can be no outflow to other worse performing strategies, who lie in S_k^1, we obtain the result that $\sum_{j=1}^{n_k} \sigma_{hj}^k = 0$. By assumption,

$V_h^k(p) = 0$, so there can also be no inflow to the strategies lying in $\mathcal{C}_1$. Again we have to consider two separate cases, as outlined in Step 1.

Step 3: Define the set

$$\mathcal{C}_2 := \left\{ e_{k_m} \in S_k^1 : \pi_m^k(p) = \min_{e_{k_j} \in S_k^1} \left[\pi_i^k(p) - \pi_j^k(p)\right] > 0 \right\}$$

This is the set of third-best alternatives, given the social state p. Define the truncated pure strategy space $S_k^2 = S_k^1 \setminus \mathcal{C}_2$, and repeat the calculations outlined in Step 2:

$$\begin{array}{rcl} & \pi_i^k(p) - \pi_m^k(p) < \pi_i^k(p) - \pi_j^k(p) & \forall e_{k_m} \in \mathcal{C}_2, \forall e_{k_j} \in S_k^2 \\ \Leftrightarrow & \pi_m^k(p) > \pi_j^k(p) & \forall e_{k_m} \in \mathcal{C}_2, \forall e_{k_j} \in S_k^2 \\ \Leftrightarrow & \left[\pi_j^k(p) - \pi_m^k(p)\right]_+ = 0 & \forall e_{k_m} \in \mathcal{C}_2, \forall e_{k_j} \in S_k^2 \\ \Leftrightarrow & \sigma_{mj}^k = 0 & \forall e_{k_m} \in \mathcal{C}_2, \forall e_{k_j} \in S_k^2 \end{array}$$

Since from the better performing strategies, contained in $S_k \setminus S_k^2$, there is no inflow, there can be no outflow from strategies in $\mathcal{C}_3$, hence $\sum_{j=1}^{n_k} \sigma_{mj}^k = 0$. By our assumption $V_m^k(p) = 0$, forbidding any inflow to this strategy, hence $\sigma_{jm}^k = 0$ for all $e_{k_j} \in S_k$.

Step 4: Since there are only finitely many strategies, this procedure stops after a finite number of steps. By induction we see that $\sum_{j=1}^{n_k} \sigma_{ij}^k(\pi_j^k(p) - \pi_i^k(p)) = 0$ for all $e_{k_i} \in S_k$.

q.e.d.

Using these two Lemmas we are able to proof an important result about pairwise comparison dynamics.

Proposition 5. *Consider an arbitrary N population game in normal form, and let $\dot{p} = V(p)$ be a pairwise comparison dynamic for this game. A social state $p^* \in \Theta$ is a Nash equilibrium of the game if and only if $V(p^*) = \mathbf{0}$.*

Proof:

($\Leftarrow$) Suppose $V(p) = \mathbf{0}$ holds for some $p \in \Theta$, and let p^* denote a Nash equilibrium. From Lemma 6, we know that

$$(\forall e_{k_i} \in S_k) : \sum_{j=1}^{n_k} \sigma_{ij}^k = 0$$

By definition of a pairwise comparison dynamic, the number σ_{ij}^k is nonnegative, and mimics the sign of the payoff difference $\left[\pi_j^k(p) - \pi_i^k(p)\right]_+$. Thus, it must be true that

$$(\forall e_{k_i} \in S_k) : \sum_{j=1}^{n_k} \left[\pi_j^k(p) - \pi_i^k(p)\right]_+ = 0$$

Applying Lemma 5, we see that $p = p^*$ is the case.

(⇒) If p^* is a Nash equilibrium, we know by Lemma 5 that $\sum_{j=1}^{n_k} \left[\pi_j^k(p) - \pi_i^k(p)\right]_+ = 0$ holds for all pure strategies e_{k_i}. By definition of the behavioral rule that generates a pairwise comparison dynamic, it must be true that

$$(\forall e_{k_i} \in S_k) : \sum_{j=1}^{n_k} \sigma_{ij}^k = 0$$

and Lemma 6 establishes the stationarity of the vector field.

q.e.d.

This is a strong result for the class of pairwise comparison dynamics. The assumptions concerning knowledge of players is reduced to a minimum, just like they would obey a strict imitative behavioral rule. As a refinement, the pairwise comparison dynamic eliminate the boundary states as fixed points of the dynamical system, which are always stationary states under imitative behavioral rules.

For a last theoretical result, we are interested in the relation between the evolution of a population state, and the payoffs of the population. An intuitive result would be that the vectors $V^k(p)$ and $\pi^k(p)$ point in the same direction, since higher payoffs of some pure strategy should induce a movement toward the face of the simplex $\Delta(S_k)$, where these especially profitable strategies are used.[22] Indeed, we are able to establish such a positive "correlation", by simply computing the inner product between these two vectors. Pick therefore an arbitrary population $1 \leq k \leq N$ and fix some social state $p \in \Theta$. We calculate

[22] A *face* of a simplex is the convex hull of a finite subset of points that generate the simplex. See Border (1985, Chapter 3).

$$\begin{aligned}
\left\langle V^k(p), \pi^k(p)\right\rangle &= \sum_{i=1}^{n_k} \dot{p}_{k_i} \pi_i^k(p) \\
&= \sum_{i=1}^{n_k} \left[\sum_{j=1}^{n_k} p_{k_j}\sigma_{ji}^k - p_{k_i}\sum_{j=1}^{n_k}\sigma_{ij}^k\right]\pi_i^k(p) = \\
&= \sum_{i=1}^{n_k} \pi_i^k(p)\left(\sum_{j=1}^{n_k} p_{k_j}\sigma_{ji}^k\right) - \sum_{i=1}^{n_k} p_{k_i}\pi_i^k(p)\left(\sum_{j=1}^{n_k}\sigma_{ij}^k\right) = \\
&= \pi_1^k(p)\left(p_{k_1}\sigma_{11}^k + p_{k_2}\sigma_{21}^k + \ldots + p_{k_{n_k}}\sigma_{n_k 1}^k\right) + \\
&+ \pi_2^k(p)\left(p_{k_1}\sigma_{12}^k + p_{k_2}\sigma_{22}^k + \ldots + p_{k_{n_k}}\sigma_{n_k 2}^k\right) + \\
&\vdots \\
&+ \pi_{n_k}^k(p)\left(p_{k_1}\sigma_{1n_k}^k + p_{k_2}\sigma_{2n_k}^k + \ldots + p_{k_{n_k}}\sigma_{n_k n_k}^k\right) - \\
&- p_{k_1}\pi_1^k(p)\left(\sigma_{11}^k + \sigma_{12}^k + \ldots + \sigma_{1n_k}^k\right) - \\
&\vdots \\
&- p_{k_{n_k}}\pi_{n_k}^k(p)\left(\sigma_{n_k 1}^k + \sigma_{n_k 2}^k + \ldots + \sigma_{n_k n_k}^k\right) = \\
&= p_{k_1}\left(\pi_1^k\sigma_{11}^k + \pi_2^k\sigma_{12}^k + \ldots + \pi_{n_k}^k\sigma_{1n_k}^k - \pi_1^k(\sigma_{11}^k + \ldots + \sigma_{1n_k}^k)\right) + \\
&\vdots \\
&+ p_{k_{n_k}}\left(\pi_1^k\sigma_{n_k 1}^k + \ldots + \pi_{n_k}^k\sigma_{n_k n_k}^k - \pi_{n_k}^k(\sigma_{n_k 1}^k + \ldots + \sigma_{n_k n_k}^k)\right) = \\
&= p_{k_1}\left[\sum_{j=1}^{n_k}\sigma_{1j}^k(\pi_j^k(p) - \pi_1^k(p))\right] + \ldots + p_{k_{n_k}}\left[\sum_{j=1}^{n_k}\sigma_{n_k j}^k(\pi_j^k(p) - \pi_{n_k}^k(p))\right] = \\
&= \sum_{i=1}^{n_k}\sum_{j=1}^{n_k} p_{k_i}\sigma_{ij}^k(\pi_j^k(p) - \pi_i^k(p))
\end{aligned}$$

and by definition of the behavioral rule that generates a pairwise comparison dynamic it is $\sigma_{ij}^k = 0$ whenever $\pi_j^k - \pi_i^k \leq 0$, so it follows

$$\left\langle V^k(p), \pi^k(p)\right\rangle \geq 0$$

for all social states $p \in \Theta$ with equality only if, for all $e_{k_i} \in S_k$, we have $\sum_{j=1}^{n_k} p_{k_i}\sigma_{ij}^k(\pi_j^k(p) - \pi_i^k(p)) = 0$. This is a convex combination of nonnegative real numbers, and can therefore only be zero if $p_k = \mathbf{0}$ (what cannot be), or for all e_{k_i}, $\sum_{j=1}^{n_k}\sigma_{ij}^k(\pi_j^k(p) - \pi_i^k(p)) = 0$. By Lemma 6 this occurs only in a Nash equilibrium.

As in all past sections, we are going to investigate the behavior of Pairwise comparison dynamics in an important example, the Matching Pennies game.

Example 10. *Remember the payoffs in the two-populations matching pennies game.*

$$\pi^1(q) = \begin{bmatrix} 2q_1 - 1 \\ 1 - 2q_1 \end{bmatrix}, \ \pi^2(p) = \begin{bmatrix} 1 - 2p_1 \\ 2p_1 - 1 \end{bmatrix}$$

Without losing much, we can focus on the evolution of the relative size of the subpopulations p_1, q_1. The pairwise difference dynamic is in this fairly simple set-up, given by the planar system

$$\begin{aligned} \dot{p}_1 &= p_2 \left[\pi^1_1(q) - \pi^1_2(q)\right]_+ - p_1 \left[\pi^1_2(q) - \pi^1_1(q)\right]_+ \\ \dot{q}_1 &= q_2 \left[\pi^2_1(p) - \pi^2_2(p)\right]_+ - q_1 \left[\pi^2_2(p) - \pi^2_1(p)\right]_+ \end{aligned}$$

A look at the structure of the payoffs in the two populations shows, that $\pi^1_1(q) - \pi^1_2(q) > 0$ only if $q_1 > 1/2$ and $\pi^2_1(p) - \pi^2_2(p) > 0$ only if $p_1 < 1/2$. Thus, there are - just as in the example where the BNN dynamic was discussed - 6 regions we have to analyze:

Region 1 : $0 \leq p_1, q_1 < 1/2$. *The system is given by*

$$\begin{bmatrix} \dot{p}_1 \\ \dot{q}_1 \end{bmatrix} = \begin{bmatrix} -p_1(\pi^1_2(q) - \pi^1_1(q)) \\ (1-q_1)(\pi^2_1(p) - \pi^2_2(p)) \end{bmatrix} = \begin{bmatrix} 2p_1(2q_1 - 1) \\ 2(1-q_1)(1-2p_1) \end{bmatrix}$$

We observe that the fraction of H-strategists in population 1 is decreasing, while the fraction of H-strategists in population 2 is increasing.

Region 2 : $0 \leq p_1 < 1/2, \ 1/2 < q_1 \leq 1$. *Our system becomes*

$$\begin{bmatrix} \dot{p}_1 \\ \dot{q}_1 \end{bmatrix} = \begin{bmatrix} (1-p_1)(\pi^1_1(q) - \pi^1_2(q)) \\ (1-q_1)(\pi^2_1(p) - \pi^2_2(p)) \end{bmatrix} = \begin{bmatrix} 2(1-p_1)(2q_1 - 1) \\ 2(1-q_1)(1-2p_1) \end{bmatrix}$$

The proportion of H-strategists is increasing in population 1 and in population 2.

Region 3: $1/2 < p_1 \leq 1, \ 0 \leq q_1 < 1/2$.

$$\begin{bmatrix} \dot{p}_1 \\ \dot{q}_1 \end{bmatrix} = \begin{bmatrix} -p_1(\pi^1_2(q) - \pi^1_1(q)) \\ -q_1(\pi^2_2(p) - \pi^2_1(p)) \end{bmatrix} = \begin{bmatrix} 2p_1(2q_1 - 1) \\ 2q_1(1-2p_1) \end{bmatrix}$$

The proportion of H-strategists is decreasing in population 1 and in population 2.

Region 4: $1/2 < p_1, \ q_1 \leq 1$.

$$\begin{bmatrix} \dot{p}_1 \\ \dot{q}_1 \end{bmatrix} = \begin{bmatrix} (1-p_1)(\pi^1_1(q) - \pi^1_2(q)) \\ -q_1(\pi^2_2(p) - \pi^2_1(p)) \end{bmatrix} = \begin{bmatrix} 2(1-p_1)(2q_1 - 1 \\ 2q_1(1-2p_1) \end{bmatrix}$$

Region 5: $p_1 = 1/2, \ 0 \leq q_1 \leq 1$. *There is no motion in population 2. The behavior in population 1 depends on the initial position of population 2's state.*

Region 6: $0 \leq p_1 \leq 1 \ q_1 = 1/2$. *There is no motion in population 1. The behavior in population 2 depends on the initial position of the state of population 1.*

A qualitative analysis of this dynamical system, can be performed by plotting the induced vector field in all these regions. Such a vector field is graphically presented in Figure 13. Similar

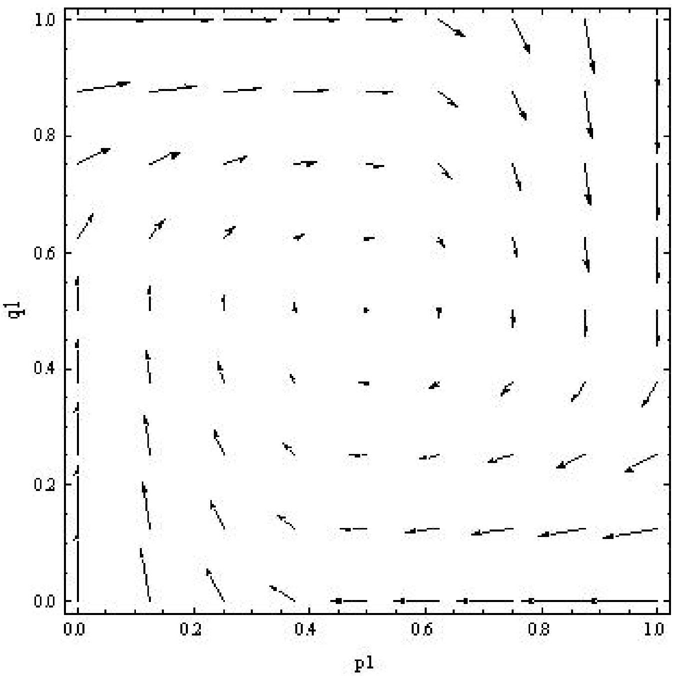

FIGURE 13. Vector field induced by the pairwise difference dynamic in a 2 populaiton matching pennies game.

to the BNN dynamic, we observe some kind of spiraling convergence to the Nash equilibrium. Linearization techniques to determine the stability of the Nash equilibrium under the pairwise difference dynamic gives us no complete answer, since eigenvalues of the Jacobian matrix (Independent from which region one is coming) are pure imaginary, which just reflects the spiraling pattern of solutions. A complete analysis must rely on an appropriate Lyapunov function. Following Hofbauer and Sandholm (2007), we can present such an analysis for the two population case. The computations are relegated to Appendix B. There it is demonstrated that the Nash equilibrium is globally asymptotically stable under the pairwise difference dynamic.

Part 4. Nash equilibrium as an outcome of Evolution?

9. Evolutionary dynamics and strictly dominated strategies

We have discussed the stability of the Nash equilibrium in the "Matching Pennies" game under the main representatives of the introduced evolutionary dynamics. The replicator dynamics, although the most popular dynamic model, did not show convergence to the Nash equilibrium, whereas all other dynamics sent solutions from arbitrary interior initial conditions to the Nash equilibrium of the game.[23] The negative result for the replicator dynamic is a general phenomenon for zero-sum games, played by more than one population.[24] The proof of this fact relies on the existence of first integrals for the replicator dynamics, which was introduced in the examples for the single population, two population case, respectively. Henceforth, stability under the replicator dynamics is given for zero-sum games, but not convergence. An interesting question is then, what made the other dynamics so special to guarantee convergence. The family of excess payoff dynamics model individual behavior where one population gradually adjusts its composition as a reaction to the opponent population, where adjustment is made according to the criterion: "Select the better than average strategies." Moreover, even if there is currently no agent in a population that plays a best-reply to the state of the opponent population, there are some individuals that will discover this strategy and adopt it. Thus, a crucial element of excess payoff dynamics is *innovation*. Technically, in the matching pennies game, the boundary of the unit square in Figure 12 is *repelling*, in the sense that solution trajectories that approach a boundary face will change their direction of motion at some point in time, and move away from the boundary. Even if we start from a social state that lies on the boundary of this unit square, evolutionary forces will push society away from this state.

These two characteristics led to convergence of populations toward their Nash states. In the class of pairwise comparison dynamics, the argument is similar. Here the adjustment criterion is not, "better than the average", but "better than the next one I meet." This criterion is actually imitative behavior, of the sort "imitate the better", and is also a defining characteristic of the replicator dynamic. However, pairwise comparison dynamics are also *innovative*, making an adjustment toward the Nash

[23]For zero-sum games this is generalizable. See Hofbauer and Sandholm (2007b).

[24]See Sandholm (2007a).

equilibrium possible, irrespective of the initial region.

The possibility that some evolutionary dynamics do not lead to a Nash equilibrium is certainly something disappointing. The starting point of the dynamic analysis of games, was the search for a justification of the Nash equilibrium as the result of a game that is played by large populations, where individuals do not necessarily have complete information about the payoff structures and the population states.[25]We have seen through examples that convergence is not guaranteed in all evolutionary models. Even worse, one can construct games with a unique Nash equilibrium, that is repelling under certain dynamics. In fact this is the case for all the main dynamics (replicator, BNN, pairwise difference) applied to the rock-paper-scissors game with payoff matrix $\begin{bmatrix} 0 & -a & b \\ b & 0 & -a \\ -a & b & 0 \end{bmatrix}$, where $a > b > 0$.[26] The first row describes the earnings of "Rock" in all possible constellations, the second "Paper", and the third "Scissors". The loss of one round exceeds the gains that a player can make. Therefore losing the game hurts players, why this version of the game is often called the "bad RSP" game. The unique Nash equilibrium of this game is the center point $p^* = (1/3, 1/3, 1/3)$ for all parameters a, b that are chosen according to the mentioned criteria. With the methods of Appendix A and B, we are able to proof that the unique Nash equilibrium in this family of games is unstable under the BNN and pairwise difference dynamic. For a concrete calculation we refer to Hofbauer and Sandholm (2007a).

Some authors analyzed the relationship between dominance of strategies in normal form games and evolutionary dynamics. A fairly mild postulate of non-cooperative (rational) game theory is that players never play strictly dominated strategies. The reasoning for this postulate was already given in the introduction of this work, where we came to the conclusion that it is indeed not very demanding to assume that players never choose such a strategy. For a reasonable dynamic analysis of a game we would therefore expect that players also do not choose a strictly dominated strategy, or at least the fraction of such players gets extremely small as the game is played for some time.

[25]This is Nash's "mass-action interpretation" of his equilibrium concept. See Björnerstedt and Weibull (1993) or the introductory section 2.

[26]Rock-scissor-paper games are frequently used examples to demonstrate (non)-convergence of solutions. They are extensively analyzed in Hofbauer and Sigmund (1998, in particular Chapter 7), and in Sandholm (2007a).

Fortunately such results are obtained in the class of imitative dynamics, at least if we do not start from a social state where only a strictly dominated strategy is adopted by agents. For a thorough discussion of this we refer to Samuelson and Zhang (1992), Hofbauer and Weibull (1996) and Weibull (1995). Imitative models are certainly important for the social science, but they miss another important ingredient of human societies: innovation. Thus, imitative behavior cannot be the universal (canonical) model of evolutionary game theory and we have to analyze also different dynamics. Here the great disappointments come into light. In two very important papers by Hofbauer and Berger (2004), and Hofbauer and Sandholm (2007a) it is demonstrated that strictly dominated strategies do survive in the long run under the BNN dynamic and the pairwise difference dynamic in some families of games. One family is build on the "bad RSP" game. These are strong results and it is not easy to say what implications they have for an application of evolutionary dynamics in economics.

All the mentioned results were obtained for games in normal form. There is not much literature about dynamics in extensive form games. Instead of presenting a concise theoretical body, we are therefore going to present some interesting examples of games, where our main focus is on establishing some facts about subgame perfect equilibria and evolutionary dynamics.

9.1. **Game 1.** Maybe the easiest example of an extensive form game was already introduced in Figure 4, and is represented in Figure 14 with slightly transformed payoffs. Individuals from

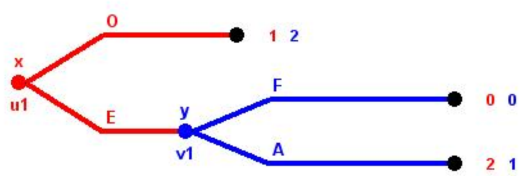

FIGURE 14. Game 1: An entry-deterrence game

population 1 have the first move, and can decide whether to enter the territory of an opponent (action E) or stay out (action O). The owners of the territory can decide whether to accept the entrance of the individual from the different population (action

A), or to fight with the intruder (action F) where both do not make any gains. Denote by $p \in [0,1]$ the share of individuals in population 1 that are programmed to enter the territory, and $q \in [0,1]$ the share of individuals in population 2 that accept the entrant.[27]

9.1.1. ***Static Analysis***. Payoffs in population 1 are given by the vector $\pi^1(q) = \begin{bmatrix} 2q \\ 1 \end{bmatrix}$, where $2q = \pi^1_1(q)$ is the expected payoff of an individual in the first population, entering the opponent's territory when drawn. In population 2 the payoffs are $\pi^2(p) = \begin{bmatrix} 2-p \\ 2(1-p) \end{bmatrix}$, where $2-p = \pi^2_1(p)$ is the expected payoff of an individual in population 2 that accepts an entrant of population 1, when drawn to play the game. Average payoffs in the two populations are then

$$\bar{\pi}^1(p,q) = 2pq + 1 - p, \quad \bar{\pi}^2(p,q) = 2(1-p) + pq$$

The game possesses one proper subgame starting at information set v_1. Suppose we draw an individual belonging to the population of owners, who knows her *population's* payoff function and the state of her population. Such an individual will accept an entrant of population 1, since this will not bring any harm to her and she eventually gains when she really has to make a decision (that is when somebody really entered the territory). Let us now assume that we have drawn an individual from population 1 who knows the same thing as our agent belonging to population 2, and additionally knows his *population's* payoff function and population state.[28] By the assumption of equal information, he knows that an invasion will be accepted by the owner. But then such an individual will enter the territory. Neither of the two drawn individuals can improve their payoff by making an unilateral change from their action. With other words we have already found one Nash equilibrium of this game, in which an intruder enters the territory and the owner accepts the entrant. The acceptance is optimal in the unique proper subgame of the extensive form game, so this equilibrium is subgame perfect. In terms of a population game the social state $(p^S, q^S) = (1,1)$ is a subgame perfect equilibrium, where

[27] In Section 8 the states of the two populations are vectors $p = (p_1, 1-p_1), q = (q_1, 1-q_1)$ for population 1, population 2 respectively. We know that it is enough to look at the evolution of one strategy frequency. In the following analysis we will call the fraction of E players in population 1 by p and the fraction of A players in population 2 by q. The reader may excuse this abuse of notation.

[28] All these considerations are against the spirit of evolutionary analysis.

intruders always enter the territory of the owners, and all owners of the territory accept the new entrants.
This is not the unique Nash equilibrium of this game. Suppose $p = 0$ is given initial data, so that no intruder will enter the territory. In this case individuals from population 2 are never confronted with the intruders, so never have to make a decision whether to accept or fight them. Individuals from population 1 are granted the possibility to change their strategy at some (rare) events. The well informed individual from population 1, mentioned in the paragraph above, will condition his choice on the state of the owning population. He knows that if $q \leq 1/2$, it is better for him to stay out of the territory. Hence the constellation $p = 0$ together with $q \leq 1/2$ forms an equilibrium.
This static analysis demonstrated that the set of Nash equilibria in the game is

$$NE = \Big\{(p,q) \in [0,1]^2 | p = 1, q = 1\Big\} \cup E$$

where $E := \{(p,q) \in [0,1]^2 | p = 0, q \leq 1/2\}$. The logic of backwards induction excludes the possibility that a social state in E will be a reasonable outcome of the game. The argument goes as follows: Viewing the subgame starting at information set v_1 as a game for itself, an intelligent individual should never fight with an intruder, since he never can attain a higher payoff with this action, compared with acceptance of the intruders. Backwards induction makes only a frequency of $q = 1$ a reasonable solution of the game. In an dynamic evolutionary model we will see however, that this logic must not prevail under pure imitative behavior of individuals.

9.1.2. ***Dynamic Analysis.***
The replicator dynamics. The planar system is given by

$$\begin{aligned} \dot{p} &= p(\pi_1^1(q) - \bar{\pi}^1(p,q)) = p(1-p)(2q-1) \\ \dot{q} &= q(\pi_1^2(q) - \bar{\pi}^2(p,q)) = q(1-q)p \end{aligned} \tag{9.1}$$

From this two ordinary differential equations we see that the share of entrants in population 1 is increasing if $0 < p < 1$ and $q > 1/2$. Otherwise the share remains either constant or decreases monotonically. The share of owners that accept an entering individual from population 1 is monotonically increasing if $0 < q < 1$ and $p > 0$. Otherwise it will stay constant. If, by some historical accident, the initial values of the strategy frequencies are in the region, where p is not "too high" and q lies below the threshold $1/2$, there are solutions that hit the boundary of the unit square at some point in the segment of Nash equilibria E. We can also check this by means of a Lyapunov function. Call $\text{int}(\Theta) := \{(p,q) \in [0,1]^2 | 0 < p, q < 1\}$

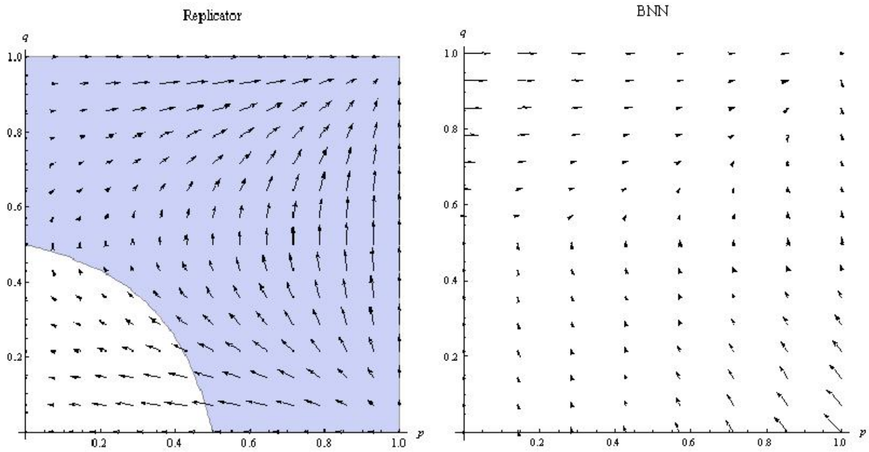

Figure 15. Vector field induced by the replicator and the BNN dynamic in the entry-deterrence game.

the interior of our state space, the unit square. We claim that the continuous differentiable function $H : \mathrm{int}(\Theta) \to \mathbb{R}_+$ given by the rule $H(p,q) = -\log(pq)$ is such a function. Taking the time derivative by applying the chain rule of Calculus gives us,

$$\begin{aligned}\frac{d}{dt}H(p,q) &= -(\frac{\dot{p}}{p} + \frac{\dot{q}}{q}) \\ &= -\left[(\pi_1^1(q) - \bar{\pi}^1(p,q)) + (\pi_1^2(p) - \bar{\pi}^2(p,q))\right] \\ &= -[(2q-1) - p(3q-2)]\end{aligned}$$

This is non-positive iff $(2q-1) - p(3q-2) \geq 0$. We consider several cases.

(1) $0 \leq q < 1/2$: Then we have $3q - 2 < 0$, and therefore a necessary condition is $p \geq \frac{2q-1}{3q-2}$.
(2) $q = 1/2$: We obtain $-(3/2 - 2)p = p/2 \geq 0$ for all $0 \leq p \leq 1$.
(3) $1/2 < q < 2/3$: Then $-(3q-2) > 0$ and $2q - 1 > 0$, and so the time derivative is negative.
(4) $q = 2/3$: $2\frac{2}{3} - 1 = \frac{1}{3}$, and again the time derivative is negative.
(5) $2/3 < q < 1$: Then $2q-1 > 3q-2 > 0$, and so $\frac{2q-1}{3q-2} > 1 > p$ and the time derivative is negative.

A negative time derivative means that H assumes succinctly lower values along solution trajectories of the system (9.1), and must ultimately reach 0. In Figure 15 the flow diagram is provided. The blue region indicates the areas where the time derivative of H is negative. Additionally, at the state $(p,q) = (p^S, q^S)$, the function has a value $H(p^S, q^S) = 0$, what is an isolated minimum on $\mathrm{int}(\Theta)$. Hence, by definition 7 in Section 6, the function H is a Lyapunov function, proving the

local asymptotic stability of the subgame perfect NE.[29]
Already this simple examples shows us that evolutionary dynamics do not have to select the subgame perfect equilibrium as a reasonable outcome. Depending on our initial data, there might be convergence to the segment of Nash equilibria that are not subgame perfect. Even worse, population states in this segment put positive weight on the weakly dominated strategy F. This result is not new. It is a general phenomenon for the replicator dynamics, that it eliminates either the weakly dominated strategy directly (thus, the share of agents using this strategy approaches 0 as $t \to \infty$) or the strategy, that dominates it weakly disappears. For a formal argument we refer to Weibull (1995). In our example this means that either $q \to 1$ for $t \to \infty$ (all individuals in population 2 accept entrants), or $p \to 0$ for $t \to \infty$ (no individual in population 1 enters the territory, so that F fares equally well as A for agents in population 2, and trajectories converge to a unique point in the segment E). The fact that some solution trajectories do not reach the subgame perfect equilibrium is a well known result in the entry-deterrence game. It was already shown in Weibull (1995), or Cressman and Schlag (1998). The latter authors analyze games of perfect information (i.e. all information sets are singletons) and the behavior of trajectories of the replicator dynamic. They come to the conclusion that convergence to the subgame perfect equilibrium is only guaranteed in rather simple games, where not too many decision nodes are involved.

[29]Alternatively one could linearize the system (9.1) in a neighborhood of the point (p^S, q^S) in the relative topology of $[0,1]^2$. The Jacobian matrix of the planar system for general strategy frequencies is given by $DV(p,q) = \begin{bmatrix} (1-2p)(2q-1) & 2p(1-p) \\ (1-q)q & (1-2q)p \end{bmatrix}$. Evaluating this at $(p,q) = (p^S, q^S)$ gives $DV(p^S, q^S) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ with one real eigenvalue $\lambda = -1$ that appears with multiplicity 2. Let $\mathcal{U} \subset \mathbb{R}^2$ denote an open neighborhood of the subgame perfect equilibrium. Its intersection with the unit square gives us then a relative neighborhood of this point. Pick $(x,y)^T \in \mathcal{U} \cap [0,1]^2$, and consider the linear system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x-1 \\ y-1 \end{bmatrix}$$

with rest point $(x,y)^T = (p^S, q^S)$, starting value $(x(0), y(0))^T \in \mathcal{U} \cap [0,1]^2$ and general solution

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} \right\} e^{-t}$$

We see $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for $t \to \infty$, which is an alternative demonstration for the asymptotic stability of the subgame perfect equilibrium.

An interesting topic is how stable this observation is with respect to the evolutionary model we choose. Therefore we analyze Game 1 under the BNN dynamic.
The Brown-von Neumann-Nash dynamic. To be consistent with the notation introduced in Section 8, define the excess payoff vectors of the populations as

$$\begin{aligned}
\hat{\pi}^1(p,q) &:= \pi^1(p) - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \bar{\pi}^1(p,q) \\
&= \begin{bmatrix} (1-p)(2q-1) \\ p(1-2q) \end{bmatrix} \\
\hat{\pi}^2(p,q) &:= \pi^2(p) - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \bar{\pi}^2(p,q) \\
&= \begin{bmatrix} (1-q)p \\ -pq \end{bmatrix}
\end{aligned}$$

The regions where the entries of these vectors are positive are easily determined. Due to the weak dominance of action A over action F in population 2, its excess payoff is never positive. This implies that $[\hat{\pi}^2_2(p,q)]_+ = 0$ for all pairs $(p,q) \in \Delta(S_1) \times \Delta(S_2)$. For population 1 we see $[\hat{\pi}^1_1(p,q)]_+ \neq 0$ if $p < 1$ and $q > 1/2$. The BNN dynamic for Game 1 is the planar system

$$\begin{aligned}
\dot{p} &= \left[\hat{\pi}^1_1(p,q)\right]_+ (1-p) - p\left[\hat{\pi}^1_2(p,q)\right]_+ \\
\dot{q} &= \left[\hat{\pi}^2_1(p,q)\right]_+ (1-q)
\end{aligned} \tag{9.2}$$

whose concrete form depends on the value of q. If $q < 1/2$ the dynamic of population 1 is equal to $\dot{p} = (2q-1)p^2$ and for $q \geq 1/2$ we get the equation $\dot{p} = (1-p)^2(2q-1)$. In the first region, the frequency p is monotonically decreasing, stopping only when $p = 0$. The second region displays an increase of p. Combined with the motion of q, which always tends to increase, we see that, for q initially large enough, the population states converge to the subgame perfect equilibrium. The vector plot in Figure 15 shows that, in strong contrast to the evolutionary pattern under the replicator model, there is no solution trajectory of the system (9.2), that approaches a state in the non-perfect segment of Nash equilibria. By Bernoulli's inequality we know that $(1-q)^2 \geq (1-2q)$. Furthermore, $(p)^2 \leq p$ for all $p \in [0,1]$. Hence, the growht of q exceeds the decay of p, so that even for small values of p trajectories flow rapidly into the region where $q \geq 1/2$ applies. There p is growing and the population states converge to the subgame perfect equilibrium.

9.2. **Game 2.** Consider the two-population extensive form game in Figure 16. Players drawn from population 1 make decisions

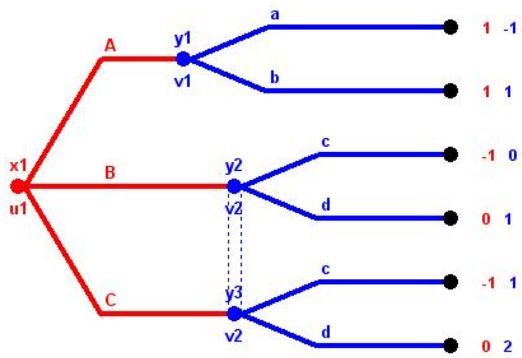

FIGURE 16. Game 2: A game in extensive form where all Nash equilibria are subgame perfect

at node x_1, which happens to be also the root of the game tree. The game is structurally the same as in example 1 given in section 3.2, so we skip the detailed description of the game. There is one proper subgame at population 2's information set v_1, which we will call Γ_{v_1}. Action c is strictly dominated by pure strategy d, and A strictly dominates B and C. Our example is a bimatrix game, with payoff matrices $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 \end{bmatrix}$ for population 1 and $\mathbf{B} = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ for population 2, respectively. Let $\Delta(S_k)$ denote the set of population states in player population $k \in \{1, 2\}$. For population 1 this set is of the form $\Delta(S_1) = \left\{p \in \mathbb{R}^3_+ | p_1 + p_2 + p_3 = 1\right\}$, where for instance p_1 is the frequency with which pure strategy A is observed in population 1. Similarly $\Delta(S_2) = \left\{q \in \mathbb{R}^4_+ | q_1 + q_2 + q_3 + q_4 = 1\right\}$, where q_1 is, for instance, the frequency with which we observe a player from population 2 that plays a at information set v_1 and c at information set v_2, hence follows the pure strategy $[ac]$. The normal form representation of the whole extensive form game is then the 3×4 Table 4. Payoffs are $\pi^1(q) = \begin{bmatrix} 1 \\ -q_1 - q_3 \\ -q_1 - q_3 \end{bmatrix}$

	[ac]	[ad]	[bc]	[bd]
A	(1,-1)	(1,-1)	(1,1)	(1,1)
B	(-1,0)	(0,1)	(-1,0)	(0,1)
C	(-1,1)	(0,2)	(-1,1)	(0,2)

TABLE 4. Normal form representation of Figure 16

in population 1 and $\pi^2(p) = \begin{bmatrix} -p_1 + p_3 \\ -p_1 + p_2 + 2p_3 \\ p_1 + p_3 \\ p_1 + p_2 + 2p_3 \end{bmatrix}$. The equilibrium structure of the game is fairly simple. A is always optimal in population 1, and best replies to A are $[bc]$ and $[bd]$ in population 2. Hence, the set of Nash equilibria is

$$NE = \left\{ (p,q) \in \Delta(S_1) \times \Delta(S_2) | p_1 = 1, q = \begin{pmatrix} 0 \\ 0 \\ \lambda \\ 1-\lambda \end{pmatrix}, 0 \leq \lambda \leq 1 \right\}.$$

Average payoffs in equilibrium are then $\bar{\pi}^1(p^*,q^*) = 1 = \bar{\pi}^2(p^*,q^*)$, for all profiles $(p^*,q^*) \in NE$. For fixed q^* as in NE, strategy A is a strict best reply for agents in population 1, since there is no alternative strategy distribution that achieves the same payoff against q^* as A does. Formally this implies that $\bar{\pi}^1(p,q^*) = p_1(1+\lambda) - \lambda \leq \bar{\pi}^1(p^*,q^*)$ for all $(p^*,q^*) \in NE$ with equality only if $p = p^*$. Similarly, for fixed $p^* = (1,0,0)^T$, there is no strategy combination available to agents in population 2, that performs as good against p^* as q^* does, where q^* is as in NE. Hence, $\bar{\pi}^2(p^*,q) \leq \bar{\pi}^2(p^*,q^*)$ for all $(p^*,q^*) \in NE$ with equality only if $q = q^*$. Cressman (2003, Chapter 3, p.70) defined such a set of Nash equilibria as a *strict equilibrium set (SESet).* Note that this does not mean that the equilibrium points are strict Nash equilibria. In our example no population state q^* that is a component of a social state in NE forms a strict Nash equilibrium in this population, since there is a continuum of population states that get the same payoff against $p^* = (1,0,0)$ as any fixed q^*. Cressman also shows that a set of rest points of the replicator dynamic in any bimatrix game is asymptotically stable iff it is an SESet (Cressman 2003, Theorem 3.2.1, p.72). Thus, we already know that solution trajectories of the replicator dynamic that start in some neighborhood of NE will stay near this set and ultimately converge to a unique point that lies in NE. To see this, we write the replicator dynamic for our example. In population 1 average payoffs are given by $\bar{\pi}^1(p,q) = p_1(1+q_1+q_3) - (q_1+q_3)$, and for population 2 we get

$\bar{\pi}^2(p,q) = (q_1+q_2)(p_3-p_1)+(p_3+p_2)(q_2+q_4)+(q_3+q_4)(p_1+p_3).$ The replicator dynamic in population 1 boils down to

$$\begin{aligned} \dot{p}_1 &= p_1(1-p_1)(1+q_1+q_3) \quad (9.3)\\ \dot{p}_i &= -p_i p_1(1+q_1+q_3) \quad i \in \{2,3\} \quad (9.4) \end{aligned}$$

Well known results are that the replicator dynamic eliminates strictly dominated strategies[30], so that in the long run all individuals in population 1 are going to play the pure strategy A. In population 2 the replicator dynamic takes the form

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = \begin{bmatrix} q_1 & 0 & 0 & 0 \\ 0 & q_2 & 0 & 0 \\ 0 & 0 & q_3 & 0 \\ 0 & 0 & 0 & q_4 \end{bmatrix} \begin{bmatrix} -2p_1(q_3+q_4)-(1-p_1)(q_2+q_4) \\ -2p_1(q_3+q_4)+(1-p_1)(q_1+q_3) \\ 2p_1(q_1+q_2)-(1-p_1)(q_2+q_4) \\ 2p_1(q_1+q_2)+(1-p_1)(q_1+q_3) \end{bmatrix} \quad (9.5)$$

$[ac]$ is strictly dominated, so the replicator dynamic makes q_1 converging to 0 in the long run. On the other hand, we observe that the frequency with which $[bd]$ is played in population 2 increases monotonically. Only the signs of q_2 and q_3 are ambiguous. Choose solution trajectories of (9.5) that are in the interior of the state space $\Delta(S_1) \times \Delta(S_2)$ and are bounded away from zero for some open time interval around $t > 0$. Then compute

$$\frac{d\ln(q_3(t)/q_2(t))}{dt} = \pi_3^2(p) - \pi_2^2(p) = 3p_1 - 1$$

Thus, for $p_1 > 1/3$, the growth rate of the strategy frequency q_3 dominates the growth rate of q_2, implying that the former grows faster than the latter. Since we know that p_1 approaches 1, for every trajectory of (9.3), there must exist a point of time, where it exceeds $1/3$. Hence, evolution will always favor q_3 over q_2 as time goes by. To see that NE is indeed asymptotically stable define an open neighborhood in the relative topology of $\Delta(S_1) \times \Delta(S_2)$, as

$$\mathcal{O}_\epsilon := \{(p,q) \in \Delta(S_1) \times \Delta(S_2) | 1-\epsilon < p_1 \leq 1, 0 \leq q_2 < \epsilon, 1-\epsilon < q_3+q_4 \leq 1\}$$

[31] Trajectories that start in this neighborhood behave according to the following system of ordinary differential equations:

$$\begin{aligned} \dot{p}_1 &= p_1(1-p_1)(1+q_3) \\ \dot{q}_2 &= q_2[q_3(1-3p_1)-2p_1q_4] \\ \dot{q}_3 &= q_3[q_2(3p_1-1)-q_4(1-p_1)] \quad (9.6) \\ \dot{q}_4 &= q_4[2p_1q_2+(1-p_1)q_3] \end{aligned}$$

It can be readily seen that for $p_1 > 1/3$ q_2 decays, and the dynamic leads to some point contained in NE. The linearization

[30] See Hofbauer and Weibull (1996) or Samuelson and Zhang (1992)

[31] We choose $\epsilon > 0$.

of (9.6) has the form ($0 < \lambda < 1$ is assumed, which is the weight of q_3 at some point in NE)

$$\begin{bmatrix} \dot{p}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = \begin{bmatrix} -(1+\lambda) & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ \lambda(1-\lambda) & 2\lambda & 0 & 0 \\ -\lambda(1-\lambda) & 2(1-\lambda) & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 - 1 \\ q_2 \\ q_3 - \lambda \\ q_4 - (1-\lambda) \end{bmatrix} \tag{9.7}$$

The eigenvalues of the Jacobian matrix are -2,$-(1+\lambda)$ and 0 with multiplicity 2. [32] We see that p_1 converges to 1 and q_2 approaches 0, hence the set NE is attracting. Furthermore, small perturbations of the dynamical system do not change the qualitative behavior of solution trajectories. This demonstrates the asymptotic stability of the set NE. What is not obvious is whether the replicator dynamic selects strategy [*bd*] as outcome. Since [*bc*] contains a strictly dominated action (c at information set v_2), and is therefore weakly dominated, it is not plausible that [*bc*] should be chosen by individuals.
Starting from the interior of our state space we see that q_4 always grows faster than q_3 does, since $\frac{d\ln(q_4/q_3)}{dt} = 1 - p_1$. Both have the same growth rate $2(q_1 + q_2)$, iff p_1 reached 1. From the dynamical system (9.5), it can be seen that p_1 influences the motion of q_3 positively, while the growth of q_4 influences it negatively. As p_1 converges to 1, the negative influence of q_4 is however damped, since then $1 - p_1$ converges to 0. Suppose for one moment that $q_1 = q_2 = 0$, which is no real restriction, since both are monotonically falling. The dynamic in population 2 is then completely determined by the planar system $\dot{q}_3 = -q_3q_4(1-p_1) = -\dot{q}_4$. This follows from the fact that the replicator dynamic is a non-innovative dynamic, or in more technical terms, it leaves all the faces of $\Delta(S_2)$ invariant.[33] The differential equation that describes the dynamic for p_1 is given

[32] We can also calculate particular solutions of the linear system (9.7). Choose positive numbers $\delta_1, \delta_2 < \epsilon$ such that $\lambda + \delta_1 + \delta_2 < 1$, and $p_1(0) = 1 - \delta_1$, $q_2(0) = \delta_1$, $q_3(0) = \lambda + \delta_2$, $q_4(0) = 1 - \lambda - \delta_1 - \delta_2$ are given initial conditions, so that trajectories start in $\mathcal{O}_\epsilon$. Particular solutions are then

$$\begin{aligned} p_1(t) &= 1 - \delta_1 \exp\{-(1+\lambda)t\} \\ q_2(t) &= \delta_1 \exp\{-2t\} \\ q_3(t) &= \lambda\frac{1+\lambda+2\lambda\delta_1}{1+\lambda} + \delta_2 + \lambda\delta_1\left(\frac{1-\lambda}{1+\lambda}\exp\{-(1+\lambda)t\} - \exp\{-2t\}\right) \\ q_4(t) &= (1-\lambda-\delta_1-\delta_2) + (1-\lambda)\delta_1\frac{1+2\lambda}{1+\lambda} - (1-\lambda)\delta_1\left(\frac{\lambda}{1+\lambda}\exp\{-(1+\lambda)t\} + \exp\{-2t\}\right) \end{aligned}$$

[33] Integrating this expression over some finite time span $[0,T]$ shows that $q_3(T) + q_4(T) = q_3(0) + q_4(0)$, hence once we are on a face where $q_3 + q_4 = 1$ we will forever stay on this face.

by $\dot{p}_1 = p_1(1-p_1)(1+q_3)$. From this expression we see the following: $\frac{\dot{p}_1}{p_1} = (1-p_1) + q_3(1-p_1) = (1-p_1) + \frac{\dot{q}_4}{q_4}$. Hence, $\frac{\dot{p}_1}{p_1} - \frac{\dot{q}_4}{q_4} = (1-p_1) \geq 0$. This implies that frequency p_1 grows faster than q_4 does, thus must reach unity before the latter can reach this point. But this demonstrates that q_3 can be positive in the long run, even though individuals in population 2 can only decide among the pure strategies $[bc]$ and $[bd]$, and $[bd]$ weakly dominates $[bc]$. This result is not new. In fact we have already encountered this phenomenon for the replicator dynamic in Game 1. In our example, $[bd]$ and $[bc]$ are equally profitable when only A is played by agents from population 1. Hence, an individual that adopts an imitative behavioral rule, will regard them as one and the same strategy, so there is no reason to expect that $[bc]$ gets extinct, if $p_1 = 1$. To summarize we have the result for the replicator dynamic, that evolutionary pressures do not motivate society to select the superior strategy $[bd]$ out of the set of equilibria, although the strategy $[bc]$ imposes the play of a strictly dominated action.

We now take a look at other types of evolutionary dynamics, where we start with the pairwise difference dynamic. In population 2, the pairwise difference dynamic is defined as

$$\dot{q}_i = \sum_{j=1}^{4} q_j[\pi_i^2 - \pi_j^2]_+ - q_i \sum_{j=1}^{4} [\pi_j^2 - \pi_i^2]_+ \tag{9.8}$$

We compute the four differential equations for population 2 step by step.

i=1:

$$\begin{aligned}
\pi_2^2 - \pi_1^2 &= -p_1 + p_2 + 2p_3 + p_1 - p_3 = p_2 + p_3 = 1 - p_1 \\
\pi_3^2 - \pi_1^2 &= p_1 + p_3 + p_1 - p_3 = 2p_1 \\
\pi_4^2 - \pi_1^2 &= p_1 + p_2 + 2p_3 + p_1 - p_3 = 2p_1 + p_2 + p_3 = 1 + p_1
\end{aligned}$$

Substituting these terms into the formula (9.8), we get the differential equation

$$\dot{q}_1 = -q_1[(1-p_1) + 2p_1 + (1+p_1)] = -2q_1(1+p_1) \tag{9.9}$$

The strictly dominated strategy $[ac]$ loses frequency monotonically, hence must ultimately die out.

i=2:

$$\begin{aligned}
\pi_1^2 - \pi_2^2 &= p_1 - 1 \\
\pi_3^2 - \pi_2^2 &= p_1 + p_3 + p_1 - p_2 - 2p_3 = 2p_1 - p_2 - p_3 = 3p_1 - 1 \\
\pi_4^2 - \pi_2^2 &= p_1 + p_2 + 2p_3 + p_1 - p_2 - 2p_3 = 2p_1
\end{aligned}$$

There are two cases we have to distinguish. First, suppose $p_1 \geq 1/3$. Then $\pi_3^2 - \pi_2^2 \geq 0$, and so (9.8) tells us that

$$\dot{q}_2 = q_1(1-p_1) - q_2[(3p_1-1)+2p_1] = q_1(1-p_1) - q_2(5p_1-1)$$

On the other hand (9.8) delegates for $p_1 < 1/3$, that

$$\dot{q}_2 = q_1(1-p_1) + q_3(1-3p_1) - 2q_2p_1 = (q_1+q_3) - p_1(q_1+2q_2+3q_3)$$

To summarize

$$\dot{q}_2 = \begin{cases} (q_1+q_3) - p_1(q_1+2q_2+3q_3) & \text{if } p_1 < 1/3 \\ q_1(1-p_1) - q_2(5p_1-1) & \text{if } p_1 \geq 1/3 \end{cases} \tag{9.10}$$

i=3:

$$\begin{aligned} \pi_1^2 - \pi_3^2 &= -p_1 + p_3 - p_1 - p_3 = -2p_1 \\ \pi_2^2 - \pi_3^2 &= -p_1 + p_2 + 2p_3 - p_1 - p_3 = -2p_1 + p_2 + p_3 = 1 - 3p_1 \\ \pi_4^2 - \pi_3^2 &= p_1 + p_2 + 2p_3 - p_1 - p_3 = p_2 + p_3 = 1 - p_1 \end{aligned}$$

Again there are two cases we have to treat separately, again defined via the threshold value $p_1 = 1/3$. Suppose $p_1 < 1/3$, so that $\pi_2^2 - \pi_3^2 > 0$. (9.8) states then

$$\dot{q}_3 = 2q_1p_1 - q_3[(1-3p_1)+(1-p_1)] = 2q_1p_1 - q_3(2-4p_1) = 2p_1(q_1+2q_3) - 2q_3$$

For $p_1 \geq 1/3$ we have

$$\dot{q}_3 = 2q_1p_1 + q_2(3p_1-1) - q_3(1-p_1) = p_1(2q_1+3q_2+q_3) - (q_2+q_3)$$

Summarizing these two equations gives us

$$\dot{q}_3 = \begin{cases} 2p_1(q_1+2q_3) - 2q_3 & \text{if } p_1 < 1/3 \\ p_1(2q_1+3q_2+q_3) - (q_2+q_3) & \text{if } p_1 \geq 1/3 \end{cases} \tag{9.11}$$

i=4: Since $[bd]$ weakly dominates all the other pure strategies, (9.8) boils down to

$$\dot{q}_4 = q_1(1+p_1) + 2q_2p_1 + q_3(1-p_1) = p_1(q_1+2q_2-q_3) + (q_1+q_3) \tag{9.12}$$

The frequency with which pure strategy $[bd]$ is observed in population 2 will consequently increase monotonically.

In population one, the dynamic is much easier, since A strictly dominates the other two pure strategies, which are furthermore payoff equivalent. Hence $\pi_1^1 - \pi_2^1 \equiv \pi_1^1 - \pi_3^1$ and therefore

$$\begin{aligned} \dot{p}_1 &= (1-p_1)(1+q_1+q_3) \qquad (9.13) \\ \dot{p}_i &= -p_i(1+q_1+q_3) \quad i \in \{2,3\} \end{aligned}$$

It is obvious that solution trajectories of (9.13) converge to the population state where all individuals play the pure strategy A. Now that we have a complete picture of the pairwise difference dynamic, we can observe that evolutionary pressures will eliminate the dominated strategies B and C form population 1 and $[ac]$ from population 2. Since we are mainly interested in the long run pattern of the strategy frequencies, we do not lose much when we restrict ourselves to the dynamics of p_1,q_2, q_3 and q_4. Moreover, since p_1 gains in frequency monotonically we can assume that $p_1 \geq 1/3$ is the case, since all solution trajectories must reach this value at some point in time. On the face $\Delta(S_1)\times\Delta([ad],[bc],[bd])$[34] the dynamical system for population 2 becomes

$$\begin{bmatrix} \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix}_{|p_1\geq 1/3} = \begin{bmatrix} q_2(1-5p_1) \\ q_2(3p_1-1)-q_3(1-p_1) \\ 2p_1q_2+q_3(1-p_1) \end{bmatrix} \tag{9.14}$$

The frequency with which pure strategy $[ad]$ is played in population 2 is decaying on trajectories starting in this region. Furthermore the differential equation that describes the motion of q_2 only depends on p_1, which is a growing positive function. Define the function $\phi_2(t)$ to be a solution of $\dot{q}_2$. Certainly, such a function exists, is continuously differentiable, and further satisfies $\phi_2(0) = q_2(0)$ some known initial value lying in the unit interval. Choose $q_2 > 0$, and $|t| < \delta$. We can approximate the value of $\phi_2(t')$ for $t' \in [0,\delta)$, by a first-order Taylor polynomial

$$\phi_2(t') \approx \phi_2(0) + \dot{\phi}_2(0)t' = q_2(0) + q_2(0)(1-5p_1(0))t'$$

where $p_1(0) \in (1/3, 1)$ is a known initial value. Solving this approximation explicitly for t', we get some feeling how fast q_2 reaches 0. Let $h(p_1(0))$ be the solution of the equation $t' = \frac{1}{5p_1(0)-1}$, which is well defined and has a unique solution. From

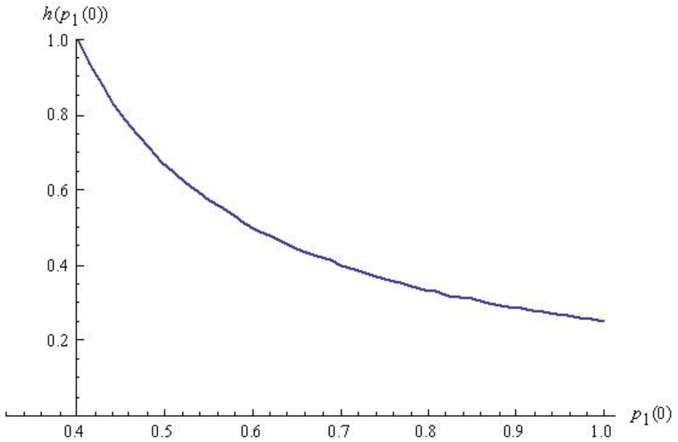

FIGURE 17. Points of time when q_2 vanishes, in dependence of the initial value of p_1.

Figure 17 one can see that even for initial values close to 1/3, we expect q_2 to reach zero approximately after one unit of time

[34]This is the set $\{q \in \Delta(S_2)|q_1 = 0\}$.

has passed. The time span decreases as p_1 has more weight initially. From the law of motion it is clear that q_2 must get eliminated in the long run from population 2, so that only the pure strategies $[bc]$, $[bd]$ remain in the strategy set. We can also say something about the correlation between the growth of p_1 and the decay of q_2. Fix a positive point in time t, which might be close to 0, such that $\phi_2(t') > 0$ holds for all $t' \in (t-\epsilon, t+\epsilon)$, where $\epsilon > 0$ and adequately chosen. The growth rate of q_2 is then $\frac{d\ln(\phi_2(t'))}{dt} = 1 - 5p_1(t')$, so clearly a negative number. The larger p_1 gets, the faster is the decay of q_2, so that evolutionary pressures will even accelerate the extinction of q_2. An infinitesimal small increase of p_1 leads then to a percentage change in the growth rate of

$$\frac{\partial\left(d\ln(\phi_2(t'))/dt\right)}{\partial p_1}\frac{p_1(t')}{d\ln(\phi_2(t'))/dt} = \frac{5p_1(t')}{5p_1(t')-1} = 1 + \frac{1}{5p_1(t')-1}$$

Hence q_2 reacts over proportional on changes of p_1. In more interpretable terms, a one-percent increase in the strategy frequency p_1 leads to a more than one percent fall in the strategy frequency q_2.

The evolution of q_4 is fairly easy to describe. We see that $\dot{q}_4 \geq 0$ for all choices of p_1, q_2, q_3. From previous analysis we know that p_1 converges to 1, while q_2 converges to 0. Henceforth, in the long run q_4 must reach a stationary state. Furthermore, observe that

$$\dot{q}_4 - \dot{q}_3 = (q_2 + 2q_3)(1 - p_1) \geq 0$$

The difference between strategy frequencies q_3 and q_4 is even becoming larger, as long as $p_1 < 1$. Hence it is not clear that q_3 will be played by a fraction of individuals in population 2.

The dynamic of strategy frequency q_3 is positively influenced by p_1 and q_2. We observe that, as soon as $q_2 \leq \frac{q_3(1-p_1)}{3p_1-1}$, q_3 starts decreasing. In the region where $p_1 \geq 1/2$ the ratio $\frac{1-p_1}{3p_1-1}$ lies always below 1. In particular, all agents in population 2 will switch to pure strategy $[bd]$, if $q_2 = 0$. It is therefore crucial for the understanding of the dynamic, to know whether q_2 reaches 0 before p_1 reaches 1, or not. However, we already know that q_2 decays faster than p_1 grows. When we start from a region where p_1 and q_2 were initially played by many agents (this means $q_2 > \frac{q_3(1-p_1)}{3p_1-1}$ and $p_1 > 1/2$), then q_3 will increase. If p_1 is chosen sufficiently large, trajectories will hit the boundary, from which on q_3 keeps on increasing and q_2 decays at exponential rate of 4. Note that in this case q_3 and q_4 grow at the same rate, until q_2 ultimately vanishes.

We make this more precise now. One can identify four principle regions that have to be distinguished and studied separately.

Region I: $0 \leq q_2 < q_3$ and $1/3 \leq p_1 \leq 1/2$.
We get

$$\begin{aligned} q_2(3p_1-1) - q_3(1-p_1) &< q_3(3p_1-1) - q_3(1-p_1) \\ &= 2q_3(2p_1-1) \\ &\leq 0 \end{aligned}$$

Hence q_3 is monotonically decreasing in this region. We know that q_2 falls and p_1 grows over time, where the former decays at a more rapid pace than the latter. Hence, the speed at which q_3 falls must be slowing down as evolution proceeds. Trajectories that do not start from an initial value of q_3 too close to zero, are going to leave this region, as p_1 will exceed $1/2$ at some point of time.

Region II: $0 \leq q_2 < q_3$ and $1/2 < p_1 \leq 1$.
This region has to be divided in two subregions.

(II.a) $q_2 \in [0, \frac{q_3(1-p_1)}{3p_1-1})$. In this case we know that q_3 is decreasing. Note that the given interval will become degenerate (melt to the single point 0), as p_1 converges to 1. This implies that the region shrinks over time, and might be "small" in general. (i.e. Choose $p_1 = 0.9$ then this interval covers a line segment which connects the points 0 and some number below $1/17 \approx 0.05$)

(II.b) $q_2 \in [\frac{q_3(1-p_1)}{3p_1-1}, q_3)$. There we observe $\dot{q}_3 \geq 0$, so that q_3 grows over time.

Region III: $0 \leq q_3 \leq q_2$ and $1/3 \leq p_1 \leq 1/2$.
Again this has to be divided into two subregions:

(III.a) $q_2 \in [q_3, \frac{q_3(1-p_1)}{3p_1-1})$ on which $\dot{q}_3 \leq 0$ is true.

(III.b) $q_2 \in [\frac{q_3(1-p_1)}{3p_1-1}, q_3]$ where q_3 is growing over time. For $p_1 \to 1$ this interval becomes the whole connected set $[0, 1]$, hence the lower bound of it approaches 0.

Region IV: $0 \leq q_3 \leq q_2$ and $1/2 < p_1 \leq 1$.
A trajectory that falls in this region will show an increasing tendency of q_3. This can be seen by $\dot{q}_3 > q_3(3p_1-1) - q_3(1-p_1) = 2q_3(2p_1-1)$

The dynamics on these regions is visualized in Figure 18.[35] Trajectories that start in Region IV will enter Region II, as p_1 grows over time. Starting from Region III we observe a decreasing tendency of q_3. Moreover this negative influence is the

[35]Note that the phase portrait in Figure 18 is not a correct description of the evolutionary pattern in population 2, since q_4 is left out. However, one can gain a lot of information about the dynamical system from it.

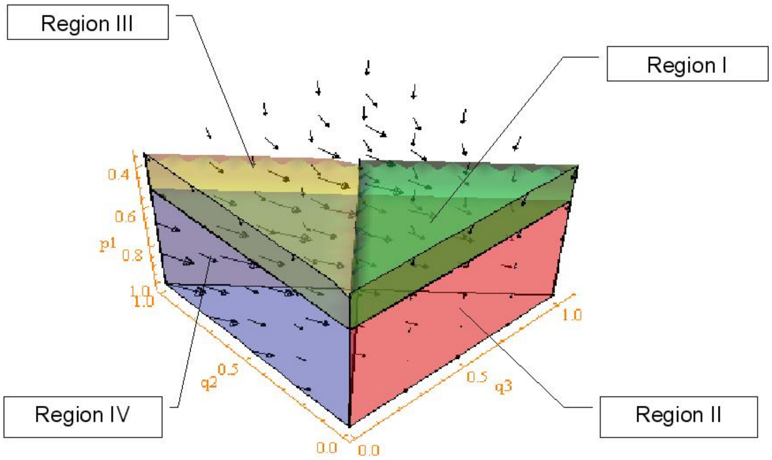

FIGURE 18. Vector Field induced by the pairwise difference dynamic. Note that only the vectors in the colored regions are feasible motions.

larger, the larger q_2 is initially. As time proceeds q_2 falls and the decaying of q_3 slows down. Finally trajectories have to enter Region IV, from which they pass over to Region II.

Trajectories that start in Region I induce also a loss of the strategy frequency q_3. As p_1 grows, they will enter Region II. Once they arrived there, q_3 gains weight again.

It seem like all trajectories have to end up in Region II, thus end up in an area where q_2 becomes arbitrary small, and p_1 converges to 1. We have to be aware that q_4 is also growing in the background. Thus, the growth of q_3 is only due to our ignorance of the superior strategy $[bd]$. However, our discussion was not without any sense, as we can see from Figure 19. There we have set $q_2 = 0$, to see what evolution imposes, when only the two equilibrium strategies are left in population 2. Without loss of generality we can focus ourselves on the evolution of p_1 and q_3, thus we are fine with a two dimensional phase diagram. We see that the vectors are nearly straight lines if q_3 is initially small. One can clearly see that the agents in population 2 favor $[bd]$ over $[bc]$, as the outflow of strategy $[bc]$ dominates its inflow, but this tendency is not strong enough to create a complete specialization in this player population if p_1 is chosen to be large enough. This suggests that population 2 will remain in a polymorphic state, where strategies $[bd]$ and $[bc]$ are both played, provided p_1 is sufficiently large. To see this, we can give a formal statement. The dynamical system is completely

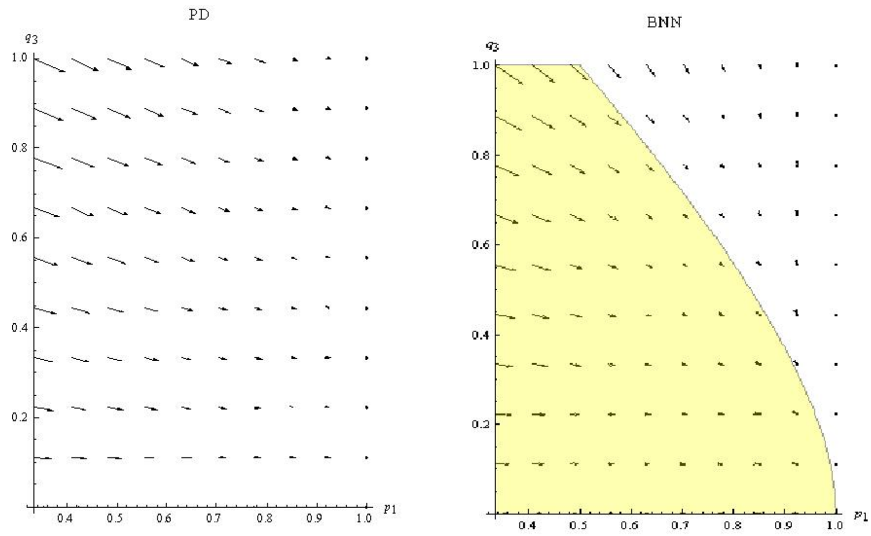

FIGURE 19. Vector Field induced by the pairwise difference dynamic and the BNN dynamic, restricted to the pure strategies A, $[bc]$, $[bd]$

described by three differential equations

$$\begin{aligned}\dot{p}_1 &= (1-p_1)(1+q_3)\\ \dot{q}_3 &= -q_3(1-p_1)\\ \dot{q}_4 &= q_3(1-p_1)\end{aligned}$$

One sees that $\dot{p}_1 - \dot{q}_4 = (1-p_1) \geq 0$ with equality in Nash equilibrium. Thus, p_1 always grows faster than q_4 what in particular implies that it has to reach its equilibrium value before q_4 does. Starting from a point on the plane on which $q_3+q_4 = 1$ holds and where q_3 has sufficiently positive weight, we will observe a positive fraction of individuals in population 2 that play the strictly dominated action c in the limit state of the pairwise difference dynamic. Thus, also the pairwise difference dynamic does not allow us to discard the suboptimal strategy $[bc]$, where some positive fraction of individuals in player population 2 choose a strictly dominated action (admittedly at an information set that will never be reached on a game path). Much more all interior trajectories must end up at some point in the set NE, which was shown to be an SESet under the replicator dynamic. Since we know that the pairwise difference dynamic is closely related to the replicator dynamic,[36] one might conjecture that SESets are also asymptotically stable under pairwise comparison dynamics. Anyway, we have to emphasize that we now have

[36]and therefore to every *monotone selection dynamic*, see Samuelson and Zhang (1992) or Cressman (2003)

encountered two evolutionary dynamics that are not adequate equilibrium selection techniques for more complicated games.

The last missing evolutionary dynamic we have discussed is the BNN dynamic. From its derivation we know that the excess payoff vector is a necessary ingredient to calculate its vector field. In population one this is a fairly simple exercise, since pure strategy A dominates the other two disposable strategies, which additionally are payoff equivalent. Hence we get

$$\begin{aligned}\hat{\pi}^1_1(p,q) &= \pi^1_1(q) - \bar{\pi}^1(p,q) = (1-p_1)(1+q_1+q_3) &(9.15)\\ \hat{\pi}^1_i(p,q) &= -p_1(1+q_1+q_3) \quad i \in \{1,2\} &(9.16)\end{aligned}$$

Under the BNN dynamic only those strategies have a chance to grow that earn a payoff above the population's average. Its differential equations have a convenient form in this example:

$$\begin{aligned}\dot{p}_1 &= (1-p_1)^2(1+q_1+q_3) &(9.17)\\ \dot{p}_i &= -p_i(1-p_1)(1+q_1+q_3) \quad i \in \{1,2\} &(9.18)\end{aligned}$$

It is clear from these expressions that only pure strategy A grows over time, so that population 1 will be completely specialized on this strategy in the long run. This convergence is even monotonically, and independent of the behavior in population 2, so without loss of generality we can concentrate ourselves on the evolution of pure strategy A. In population 2 things are a little bit more complicated. Strategy $[ac]$ is strictly dominated by $[bd]$, and moreover all pure strategies available to this player population weakly dominate it. Hence, we have $\pi^2_1(p) \leq \pi^2_h(p)$ for all $h \in \{1,2,3,4\}$ and $p \in \Delta(S_1)$. It follows directly that also its excess payoff must be non-positive, in all possible constellations (p,q).
Pure strategy $[bd]$ is always a best response in population 2, so that $\pi^2_4(p) \geq \pi^2_h(p)$ for all $h \in \{1,2,3,4\}$ and $p \in \Delta(S_1)$. Henceforth, its excess payoff can never be negative.
For the remaining two pure strategies in population 2 we calculate

$$\begin{aligned}\hat{\pi}^2_2 &= (1-q_2)(-p_1+p_2+2p_3) - q_1(-p_1+p_3) - q_3(p_1+p_3) - q_4(p_1+p_2+2p_3)\\ &= (-p_1+p_3)(1-q_1-q_2) + (1-q_2)(p_2+p_3) - (p_1+p_3)(q_3+q_4) - q_4(p_2+p_3)\\ &= (q_3+q_4)(-p_1+p_3-p_1-p_3) + (p_2+p_3)(1-q_2-q_4) \quad \text{, since } 1-q_1-q_2 = q_3+q_4\\ &= -2p_1(q_3+q_4) + (p_2+p_3)(q_1+q_3)\\ &= -p_1(q_1+3q_3+2q_4) + (q_1+q_3) \quad \text{, since } p_2+p_3 = 1-p_1\end{aligned}$$

This expression is non-negative if $p_1 \leq \frac{q_1+q_3}{q_1+3q_3+2q_4}$, for $q_1+q_3 \neq 0$.
For strategy $[bc]$ we get

$$\begin{aligned}
\hat{\pi}_3^2 &= (1-q_3)(p_1+p_3) - q_1(-p_1+p_3) - q_2(-p_1+p_2+2p_3) - q_4(p_1+p_2+2p_3) \\
&= (-p_1+p_3)(-q_1-q_2) - q_2(p_2+p_3) + (1-q_3-q_4)(p_1+p_3) - q_4(p_2+p_3) \\
&= (q_1+q_2)(p_1+p_3+p_1-p_3) - (p_2+p_3)(q_2+q_4) \quad \text{, since } 1-q_3-q_4 = q_1+q_2 \\
&= p_1(2q_1+3q_2+q_4) - (q_2+q_4) \quad \text{, since } p_2+p_3 = 1-p_1 \\
&\geq 0 \\
&\Rightarrow p_1 \geq \frac{q_2+q_4}{2q_1+3q_2+q_4} \quad \text{, if } q_2+q_4 \neq 0
\end{aligned}$$

To summarize these calculations, we write

$$\hat{\pi}^2(p,q) = \begin{bmatrix} -2p_1(q_3+q_4) - (p_2+p_3)(q_2+q_4) \\ -2p_1(q_3+q_4) + (q_1+q_3)(p_2+p_3) \\ 2p_1(q_1+q_2) - (p_2+p_3)(q_2+q_4) \\ 2p_1(q_1+q_2) + (q_1+q_3)(p_2+p_3) \end{bmatrix}$$

The pure strategy $[ac]$ always earns a payoff below the population's average, which implies a monotone decay of its frequency. Suppose that at some point in time pure strategy $[ac]$ gets extinct from population 2. Since it is a strictly dominated strategy, and there does not exist a second pure strategy that performs equally worse in all possible situations that might arise in the game (i.e. there does not exist a "clone" of $[ac]$), no individual, behaving according to BNN, will rediscover it. Therefore it remains unused forever (in terms of behavioral rules: There is no inflow). By monotonicity, $[ac]$ must get extinct at some point of time, so without loss of generality we can restrict ourselves to an analysis of the faces of $\Delta(S_2)$ on which $q_1 = 0$ holds.
The superior strategy $[bd]$ always earns a payoff above the population's average, which implies that there is a steady inflow of agents applying this strategy. Furthermore $\pi_4^2(p) - \pi_i^2(p) \geq 0$ for all $p \in \Delta(S_1)$ and $i \in \{1,2,3,4\}$, so that its excess payoff dominates the excess payoff of all other available pure strategies in population 2. Consequently, the inflow must exceed the outflow leading to a monotone growth of the frequency with which $[bd]$ is used.
For the remaining two pure strategies, note that $\pi_3^2(p) - \pi_2^2(p) = 2p_1 - p_2 - p_3 = 3p_1 - 1$, which is nonnegative for $p_1 \geq 1/3$. Thus, if p_1 exceeds $1/3$ strategy $[bc]$ becomes a best reply, together with $[bd]$ (which is always an optimal choice).
Setting $q_1 = 0$ in the expression for the excess payoff vector, and rearranging terms, shows that $\hat{\pi}_2^2(p,q) = q_3(1-3p_1) - 2p_1q_4$ and $\hat{\pi}_3^2(p,q) = q_2(3p_1-1) - (1-p_1)q_4$. Since p_1 increases monotonically, every trajectory must exceed the value $1/3$ at some point

in time. The BNN dynamic for strategy frequency q_2 turns then to

$$\dot{q}_2 = -q_2\left([\hat{\pi}_3^2(p,q)]_+ + [\hat{\pi}_4^2(p,q)]_+\right) \leq 0 \tag{9.19}$$

for all social states (p,q), with $p_1 \geq 1/3$. If $p_1 < 1/3$ then frequency q_2 eventually increases some time, until p_1 exceeds this value. Then q_2 monotonically decays according to equation (9.19). For strategy frequency q_3 we observe the law of motion

$$\dot{q}_3 = [\hat{\pi}_3^2(p,q)]_+(1-q_3) - q_3[\hat{\pi}_4^2(p,q)]_+ \tag{9.20}$$

for $p_1 \geq 1/3$. If $p_1 < 1/3$ would apply, then the BNN dynamic boils down to $\dot{q}_3 = -q_3([\hat{\pi}_2^2(p,q)]_+ + [\hat{\pi}_4^2(p,q)]_+)$, so that q_3 decays and might even get extinct, for sufficiently small initial values of q_3. However, as p_1 grows over time, it must pass the barrier of $1/3$, making $[bc]$ a best reply in population 2, so that some agents might discover this strategy again. Therefore a study of the dynamic when $p_1 \geq 1/3$ is needed to understand the evolutionary pattern. Suppose $q_2 = 0$. Then q_3 and q_4 evolve according to the laws of motion $\dot{q}_3 = -q_3(1-p_1)q_3$, $\dot{q}_4 = (1-q_4)(1-p_1)q_3 = -\dot{q}_3$. Thus, q_3 is monotonically falling, and q_4 absorbs its decay. The difference between the frequencies with which these two strategies are applied must become larger over time, as one can also see from $\dot{q}_3 - \dot{q}_4 = 2\dot{q}_3 = -2(q_3)^2(1-p_1)$ which decays at the doubled speed of q_3. If p_1 would increase at a smaller pace than q_4 does, q_4 reaches 1 before the former can. Hence, q_3 must vanish form population 2, what would distinguish the BNN dynamic extremely from all other dynamics we have discussed so far. In population one the dynamic only depends on the frequency q_3. We calculate $\dot{p}_1 - \dot{q}_4 = (1-p_1)[(1-p_1)(1+q_3) - (q_3)^2] \leq 0$ exactly if $p_1 \geq 1 - \frac{(q_3)^2}{1+q_3} = \frac{(q_3 - 1/2(1-\sqrt{5}))(q_3 - 1/2(1+\sqrt{5}))}{1+q_3}$. Thus, for $\frac{(q_3 - 1/2(1-\sqrt{5}))(q_3 - 1/2(1+\sqrt{5}))}{1+q_3} \leq p_1 \leq 1$, we see that $\dot{p}_1 \leq \dot{q}_4$, what means that p_1 grows slower than q_4 does. One should note that the lower bound converges to 1 with the same speed as q_3 converges to 0. Hence, as evolution proceeds it becomes more likely to observe a faster growth of p_1 as q_4. The region where $1/3 \leq p_1 \leq 1 - \frac{(q_3)^2}{1+q_3}$ is the yellow shaded area in Figure 19, where also a phase diagram of the planar system $(\dot{p}_1, \dot{q}_3)$ is provided. Vectors in this area are longer than vectors outside this region, indicating a faster growth of p_1, what is in accordance with our computations. The larger q_3 the stronger is the tendency toward p_1, and the adjustment speed becomes slower the larger p_1 is. Comparing the vector field with the pairwise difference dynamic shows that the force pointing in direction of p_1 is much stronger in the BNN dynamic. On the other hand, if we choose q_3 to be rather small, vectors are nearly straight lines. Thus, as p_1 grows trajectories loose speed, and in particular the decay of strategy frequency q_3 slows down. We can further observe that trajectories leave the yellow shaded region where p_1 grows faster than q_4 fairly late, in the sense that the weight of p_1 must be "large" before curves exit it. Then the increase of q_4 speeds up relative to the increase of p_1 until the latter reaches its equilibrium value 1. Thus, it seems that the BNN dynamic does also not select the superior strategy $[bd]$ out of the Nash component of population 2, although the evolutionary pressure toward this dominant strategy is much stronger than in all other dynamics we have discussed.

To conclude this analysis we formulate the following statement:

Observation 1. *Neither the replicator dynamic, nor the innovative dynamics (BNN and pairwise difference) lead population 2 into a state of complete specialization. One might conjecture that the SESet is an asymptotically stable set under all the discussed evolutionary dynamics.*

One might criticize our example in the direction that optimizing individuals in population 1 actually are not confronted with making a choice in the game. A globally dominates all other pure strategies in this player population, thus there is not really a decision to make. To take this critique into account we re parametrize the payoffs of Game 2.

9.3. **Game 3.** Consider the two-population extensive form game in Figure 20. Our example is the structurally the same bimatrix game as Game 2, but the payoff matrices changed slightly to

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ -1 & 0 & -1 & 0 \\ -1 & 1 & -1 & 1 \end{bmatrix} \text{ for population 1 and } \mathbf{B} = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

for population 2, respectively. The normal form representation of the whole extensive form game is then the 3×4 Table 5.

	[ac]	[ad]	[bc]	[bd]
A	(1,-1)	(1,-1)	(-1,1)	(-1,1)
B	(-1,0)	(0,1)	(-1,0)	(0,1)
C	(-1,1)	(1,2)	(-1,1)	(1,2)

TABLE 5. Normal form representation of Figure 20

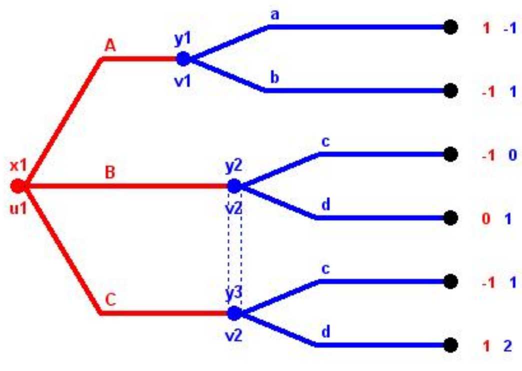

FIGURE 20. Game 3: A game in extensive form with one proper subgame

9.3.1. ***Static Analysis.*** Note that the probability with which we observe action a in subgame Γ_{v_1} is simply $\Pr(a) = q_1 + q_2$, and so is $\Pr(b) = q_3 + q_4$. The mixed strategy (normal form) payoff functions are then for population 1

$$\pi^1(q) = \begin{bmatrix} \pi^1_1(q) \\ \pi^1_2(q) \\ \pi^1_3(q) \end{bmatrix} = \begin{bmatrix} q_1 + q_2 - q_3 - q_4 \\ -q_1 - q_3 \\ -q_1 + q_2 - q_3 + q_4 \end{bmatrix}$$

and for population 2

$$\pi^2(p) = \begin{bmatrix} \pi^2_1(p) \\ \pi^2_2(p) \\ \pi^2_3(p) \\ \pi^2_4(p) \end{bmatrix} = \begin{bmatrix} -p_1 + p_3 \\ -p_1 + p_2 + 2p_3 \\ p_1 + p_3 \\ p_1 + p_2 + 2p_3 \end{bmatrix}$$

We compute the best-response correspondences of the player populations. Start with population 1. The pure strategy A is a best reply if and only if

$$\left.\begin{array}{l} \pi^1_1 > \pi^1_2 \\ \pi^1_1 > \pi^1_3 \end{array}\right\} \Leftrightarrow \begin{array}{c} q_1 + q_2 - q_3 - q_4 > -q_1 - q_3 \\ q_1 + q_2 - q_3 - q_4 > -q_1 + q_2 - q_3 + q_4 \end{array}$$

The second condition is satisfied iff $q_1 > q_4$. Since $-q_1 + q_2 - q_3 + q_4 \geq -q_1 - q_3$, the second condition is more demanding than the first. Hence, if it is satisfied the first one will also be satisfied.

From the payoff matrix $\mathbf{A}$ one sees directly that B can never dominate all other strategies, since it is weakly dominated by C.

C is in turn a pure best reply if and only if it satisfies the system of inequalities

$$\left.\begin{array}{l}\pi_3^1 > \pi_1^1\\ \pi_3^1 > \pi_2^1\end{array}\right\} \Leftrightarrow \begin{array}{c}-q_1+q_2-q_3+q_4 > q_1+q_2-q_3-q_4\\ -q_1+q_2-q_3+q_4 > -q_1-q_3\end{array}$$

The first condition states that $q_4 > q_1$ and the second condition forces $q_2 + q_4 > 0$. If condition 1 is satisfied then in particular $q_4 > 0$, so that condition 2 is satisfied. Hence condition 1 is stronger than condition 2.
Some probability distribution between A and C is a best response exactly if $q_1 = q_4$, and any point in the mixed strategy space $\Delta(S_1)$ is a best response if $q_3 = 1$. To summarize these observations we define the correspondence $BR^1 : \Delta(S_2) \rightarrow\rightarrow \Delta(S_1)$ through[37]

$$BR^1(q) = \begin{cases} \{(1,0,0)\} & \text{if } q_1 > q_4 \\ \{(0,0,1)\} & \text{if } q_4 > q_1 \\ \{(\alpha,\beta,1-\alpha-\beta), 0 \leq \alpha,\beta \leq 1\} & \text{if } q_3 = 1 \\ \{(\alpha,0,1-\alpha), 0 \leq \alpha \leq 1\} & \text{if } q_1 = q_4 > 0 \end{cases} \tag{9.21}$$

Now we turn to player population 2. Strategy $[ac]$ is strictly dominated by $[bd]$, so can never be an optimal choice. Moreover, $[ad]$ and $[bc]$ are weakly dominated by $[bd]$. The best response correspondence for player population 2 is then

$$BR^2(p) = \begin{cases} \{(0,0,0,1)\} & \text{if } 0 < p_1 < 1 \\ \{(0,\delta,0,1-\delta), 0 \leq \delta \leq 1\} & \text{if } p_1 = 0 \\ \{(0,0,\gamma,1-\gamma), 0 \leq \gamma \leq 1\} & \text{if } p_1 = 1 \end{cases} \tag{9.22}$$

A *Nash equilibrium* is a pair of population states (p^*, q^*) such that $p^* \in BR^1(q^*)$ and $q^* \in BR^2(p^*)$. Combining the correspondences (9.21) and (9.22), we see that the set of Nash equilibria in this game is given by

$$NE = \{[(1,0,0),(0,0,1,0)]\} \cup E$$

where $E := \{(p,q) \in \Delta(S_1) \times \Delta(S_2) | p_3 = 1,\ q = (0,\delta,0,1-\delta), \delta \in [0,1]\}$.
The concept of *subgame perfection* leads to a drastic reduction of the set of equilibrium points. Indeed, subgame perfection rules out any Nash component, where the second pure strategy in population 2 has some positive weight. The set of equilibrium population states that pass the subgame perfection test is then

$$SPNE = \{[(1,0,0),(0,0,1,0)]\,, [(0,0,1),(0,0,0,1)]\}$$

[37]With the mixed strategy p=(1,0,0) we identify the pure strategy A of players belonging to population 1.

Even this set contains an equilibrium point that is not plausible in terms of rationality. The social state $[(1,0,0),(0,0,1,0)]$ consists of a population state in the second player population in which all agents choose action b at information set v_1 and action c at information set v_2. However, c is strictly dominated by d at information set v_2! Only the fact that all agents in population 1 choose A makes it possible that this strategy distribution in population 2 can be optimal, since then the information set v_2 of population 2 is de facto never reached when the game is played.

In a dynamic analysis we will want to see, to which equilibrium point evolutionary selection leads. It will be useful to have some expressions by hand. The average payoffs are

$$\begin{aligned}
\bar{\pi}^1(p,q) &= p^T\mathbf{A}q = p_1(q_1+q_2-q_3-q_4) - p_2(q_1+q_3) + p_3(-q_1+q_2-q_3+q_4) \\
\bar{\pi}^2(p,q) &= q^T\mathbf{B}p = q_1(-p_1+p_3) + q_2(-p_1+p_2+2p_3) + q_3(p_1+p_3) + q_4(p_1+p_2+2p_3)
\end{aligned}$$

Excess payoffs are defined as the difference between the payoff that strategy h earns in population k minus the average payoff obtained in this population. Formally,

$$\begin{aligned}
\hat{\pi}^1(p,q) &= \mathbf{A}q - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} p^T\mathbf{A}q = \left(\mathbf{I}_3 - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} p^T\right)\mathbf{A}q \\
\hat{\pi}^2(p,q) &= \left(\mathbf{I}_4 - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} q^T\right)\mathbf{B}p
\end{aligned}$$

where $\mathbf{I}_l$ is the $l \times l$ identity matrix.

We know that strategy B is weakly dominated in population 1 by a combination of A and C, and additionally $\pi^1_2 \leq 0$. There exists some strategy distribution, such that the weakly dominated strategy has non-negative excess payoff. Performing some manipulations we can write

$$\begin{aligned}
\hat{\pi}^1_2(p,q) &= -p_1(q_1+q_2-q_3-q_4) - (1-p_2)(q_1+q_3) - p_3(-q_1+q_2-q_3+q_4) \\
&= (q_1+q_3)(p_3+p_2-1) - p_1(q_1+q_2-q_3-q_4) - p_3(q_2+q_4) \\
&= -p_1(2q_1+q_2-q_4) - p_3(q_2+q_4)
\end{aligned}$$

The second term on the right hand side is always non-positive. To make this expression non-negative, we need that $2q_1+q_2 \leq q_4$ and if the inequality is strict $p_1 \geq -\frac{q_2+q_4}{2q_1+q_2-q_4}p_3$. For a numerical illustration suppose we have the data $q_1 = 0.05$, $q_2 = 0.15$, $q_4 = 0.3$, then $\hat{\pi}^1_2(p,q) = -p_1(0.1+0.15-0.3) - p_3(0.15+0.3) = 0.05p_1 - 0.45p_3$ which is non-negative only if $p_1 \geq 9p_3$. So p_3 is not allowed to exceed $1/9$ to make this possible. However, this numerical example demonstrates that even a weakly dominated

strategies might be "good enough" to be played, even by rational players.
For the two remaining strategies upon which individuals from population 1 can decide, we get

$$\begin{aligned}\hat{\pi}^1_1 &= (1-p_1)(2q_1+q_2-q_4)-(q_2+q_4)p_3\\ \hat{\pi}^1_3 &= 2p_1(q_4-q_1)+p_2(q_2+q_4)\end{aligned}$$

Now we turn to population 2. Payoffs in this population 2 are the same as in Game 2, so we can copy the excess payoff vector, which was given by

$$\hat{\pi}^2(p,q)=\begin{bmatrix}-2p_1(q_3+q_4)-(1-p_1)(q_2+q_4)\\ -2p_1(q_3+q_4)+(1-p_1)(q_1+q_3)\\ 2p_1(q_1+q_2)-(1-p_1)(q_2+q_4)\\ 2p_1(q_1+q_2)+(1-p_1)(q_1+q_3)\end{bmatrix}$$

9.3.2. ***Dynamic Analysis.***
The replicator dynamics. We start with describing the evolution in player population 2. The replicator dynamic in this population is the same as in (9.5). The frequency of pure strategy $[ac]$ develops therefore after the rule

$$\begin{aligned}V^2_1(p,q) &\equiv \dot{q}_1=q_1\hat{\pi}^2_1(p,q)\\ &= q_1[-2p_1(q_3+q_4)-(p_2+p_3)(q_2+q_4)]\end{aligned}$$

This number is never strictly positive, and vanishes for $q_1=0$ or $-2p_1(q_3+q_4)=(p_2+p_3)(q_2+q_4)$. Note that the number q_2+q_4 is exactly the probability (frequency) with which we observe an individual from population 2 choosing action d at information set v_2. Analogously is the number q_3+q_4 the probability that an individual from population 2 chooses to play b at information set v_1. We can therefore write $\Pr(b):=q_3+q_4$ and $\Pr(d):=q_2+q_4$, so that the differential equation boils down to the expression

$$\dot{q}_1=q_1[-2p_1\Pr(b)-(p_2+p_3)\Pr(d)]$$

We can eliminate the frequencies p_2,p_3 by using the identity $p_2+p_3=1-p_1$.

$$\dot{q}_1=q_1[p_1(\Pr(d)-2\Pr(b))-\Pr(d)] \tag{9.23}$$

This vanishes for $q_1=0$ or $p_1=\frac{\Pr(d)}{\Pr(d)-2\Pr(b)}$, whenever this ratio exists.
Doing similar computations for pure strategy $[ad]$ gives us the differential equation

$$\begin{aligned}\dot{q}_2 &= q_2[-2p_1(q_3+q_4)+(q_1+q_3)(p_2+p_3)] \qquad (9.24)\\ &= q_2[-p_1(2\Pr(b)+\Pr(c))+\Pr(c)] \quad \text{where } \Pr(c):=q_1+q_3\end{aligned}$$

which has fixed points at $q_2 = 0$ or $p_1 = \frac{\Pr(c)}{2\Pr(b)+\Pr(c)}$. We see that the share of individuals using the strategy $[ad]$ increases if p_1 is lower than this ratio. This is in accordance with our computations of the excess payoff vector (as it must be) since the ratio $\frac{q_1+q_3}{q_1+3q_3+2q_4} = \frac{\Pr(c)}{2\Pr(b)+\Pr(c)}$.
In the same spirit we can calculate the laws of motion for the pure strategies $[bc], [bd]$, which take the form

$$\begin{aligned}
\dot{q}_3 &= q_3[2(q_1+q_2)p_1 - (p_2+p_3)(q_2+q_4)] \\
&= q_3[p_1(2\Pr(a)+\Pr(d)) - \Pr(d)] \qquad (9.25) \\
\dot{q}_4 &= q_4[2p_1(q_1+q_2) + (q_1+q_3)(p_2+p_3)] \\
&= q_4[p_1(2\Pr(a)-\Pr(c)) + \Pr(c)] \qquad (9.26)
\end{aligned}$$

where $\Pr(a) := q_1 + q_2$.

Observation 2. *The frequencies q_1 and q_4 behave monotonically under the replicator dynamics. q_1 is decreasing, while q_4 increases over time.*

Our goal is to see, whether we can discard the implausible subgame perfect equilibrium $[(1,0,0),(0,0,1,0)]$ using evolutionary arguments. Hence our exercise is to verify the following

Claim 1. *The replicator dynamic is the only model that allows the implausible subgame perfect equilibrium $[(1,0,0),(0,0,1,0)]$ to survive, in a fairly restricitive environment. All other evolutionary dynamics (BNN and pairwise difference) make individuals in population 2 give the superior pure strategy $[bd]$ at least some weight.*

Pick trajectories $q_3(t), q_4(t)$, such that both frequency are always bounded away from 0, and compute

$$\begin{aligned}
\frac{d\ln(q_3(t)/q_4(t))}{dt} &= \frac{\dot{q}_3(t)}{q_3(t)} - \frac{\dot{q}_4(t)}{q_4(t)} \\
&= p_1(2\Pr(a)+\Pr(d)) - \Pr(d) - p_1(2\Pr(a)-\Pr(c)) - \Pr(c) \\
&= p_1 - 1, \quad \text{since } \Pr(c)+\Pr(d) = q_1+q_2+q_3+q_4 = 1
\end{aligned}$$

For $p_1 < 1$, the frequency ratio of pure strategies $[bc]$ and $[bd]$ is strictly decreasing, which means that the growth rate of the frequency with which pure strategy $[bd]$ is used, lies always above the growth rate of the pure strategy $[bc]$. The difference between the two growth rates is a strictly increasing linear function of the frequency p_1, reaching its global minimum at $p_1 = 0$. In this case equation (9.25) shows that q_3 is decreasing, whereas we know from observation 2 that q_4 is always increasing. The gap between the two strategy frequencies is consequently widening up. If $p_1 = 1$, both strategy frequencies grow at the same

rate $2\Pr(a)$. For any other $0 < p_1 < 1$, q_4 is always growing faster than q_3.
We can also compute that (certainly we choose q_2 to be positive)

$$\frac{d\ln(q_2(t)/q_4(t))}{dt} = -2p_1$$

which is also monotonically decreasing, unless $p_1 = 0$. We can make p_1 arbitrary close to 1, and still q_4 will grow faster than q_3. Consider solution trajectories of the replicator dynamics restricted to the face $\Delta([bc],[bd]) \subset \Delta(S_2)$. This face is invariant under the replicator dynamic, since no currently unused strategies are discovered. The laws of motion on this face take the simple form $\dot{q}_3 = q_3q_4(p_1 - 1) = -\dot{q}_4$. Hence, $\dot{q}_3/\dot{q}_4 = -1$, and trajectories approach monotonically the vertex of this face, where only $[bd]$ is played by individuals of population 2. This adaption mechanism can only be disrupted by $p_1 = 1$, or $q_3q_4 = 0$. In this case all trajectories on this face are in a resting position. This illustrates nicely the meaning of $p_1 = 1$. If this degenerate population state occurs, only the subgame Γ_{v_1} is visited on a game path. But on this subgame the pure strategies $[bc]$ and $[bd]$ are identical, since both state that action b is chosen there. As both are identical, an individual following an imitative behavioral rule will not be able to distinguish them "in nature", so both are equally likely to be chosen by agents. However, for any value of $p_1 \in (0,1)$, we cannot observe a tendency of population 2 to tend to the population state $(0,0,1,0)$. More the contrary is the case.
Let us now consider solution trajectories on the face $\Delta([ad],[bd]) \subset \Delta(S_2)$. This face is also an invariant set under the replicator dynamics, and the differential equations boil down to the simple expressions $\dot{q}_2 = -2p_1q_2q_4 = -\dot{q}_4$. Hence, again we can observe a convergence to the vertex where only the pure strategy $[bd]$ is used by agents of population 2. This tendency breaks down only if one of the three involved frequencies vanishes.
Since q_1 is monotonically decreasing, by observation 2, the only remaining interesting case is where trajectories are restricted to the face $\Delta([ad],[bc],[bd]) = \Delta(S_2) \setminus \{[ac]\}$. The differential equation for strategy $[bc]$ takes now the form $\dot{q}_3 = q_3(2p_1q_2 - (q_2+q_4)(1-p_1))$, which can be rearranged as $\dot{q}_3 = q_3(q_2(3p_1 - 1) - q_4(1-p_1))$. If both frequencies q_2 and q_3 are positive we can calculate

$$\frac{\dot{q}_3}{q_3} - \frac{\dot{q}_2}{q_2} = (2p_1 - p_2 - p_3)(q_2 + q_3 + q_4) = 3p_1 - 1$$

We can observe that the strategy frequency q_3 grows faster than q_2 only if $p_1 > 1/3$. For large values of p_1, q_3 dominates q_2 in terms of growth rates, and consequently also in terms of payoffs. However, q_4 is also present, which in turn dominates q_3 and q_2. On this face we get $\dot{q}_4 = q_4[2p_1q_2 + q_3(1-p_1)]$, and so $\frac{d\ln(q_3/q_4)}{dt} = -(1-p_1)$. It follows immediately that q_4 grows faster than q_3 and thus grows also faster than q_2 for $p_1 > 1/3$. We can conclude after all these considerations that a necessary condition for observing a growth of q_3 alone is that q_4 is not present. Therefore we turn our view to the face $\Delta([ad],[bc])$. On this face the dynamic reduces to the equations $\dot{q}_2 = q_2q_3(1-3p_1) = -\dot{q}_3$, where the directions of motions are clearly visible. For large values of p_1 we indeed observe a monotone increase in the strategy frequency q_3, so that a population state $(0,0,1,0)$ seems to be possible.

Now we focus on population 1. Of particular interest is the evolution of the strategy frequency p_1. Its differential equation is given by

$$\dot{p}_1 = p_1\hat{\pi}_1^1(p,q) = p_1[(1-p_1)(2q_1+q_2-q_4) - p_3(q_2+q_4)] \qquad (9.27)$$

On the face $\Delta(S_1)\times\Delta([ad],[bc]) \subset \Delta(S_1)\times\Delta(S_2)$, this reduces to

$$\begin{aligned}\dot{p}_1 &= p_1(q_2(1-p_1) - p_3q_2) = p_1q_2(1-p_1-p_3)\\ &= p_1q_2p_2\end{aligned}$$

Hence, the dynamic does also depend on the frequency with which pure strategy B is chosen. The differential equation is

$$\dot{p}_2 = p_2[-p_1(2q_1+q_2-q_4) - p_3(q_2+q_4)] \qquad (9.28)$$

which gives on the face $\Delta(S_1)\times\Delta([ad],[bc])$ the dynamic $\dot{p}_2 = -p_2q_2(p_1+p_3) = p_2(p_2-1)q_2$. The frequency with which agents employ strategy B in population 1 is decreasing, if only the strategies $[ad]$ and $[bc]$ are observed in population 2. To see the direction of motion of strategy C, we can use the invariance of the state space under the replicator dynamic, so that $\dot{p}_3 = -\dot{p}_1 - \dot{p}_2 = p_3q_2p_2$ on the face $\Delta(S_1)\times\Delta([ad],[bc])$. It is interesting to see that

$$\frac{\dot{p}_3}{p_3} - \frac{\dot{p}_1}{p_1} = 0$$

on $\Delta(S_1)\times\Delta([ad],[bc])$. This means that the frequencies with which C and A are used in population 1 grow at the same rate. The product p_2q_2 describes the speed with which p_1 is growing.

By the product rule, we see

$$\begin{aligned}\frac{d}{dt}(p_2q_2) &= \dot{p}_2q_2 + p_2\dot{q}_2 \\ &= -p_2(q_2)^2(p_1+p_3) + p_2q_2q_3(1-3p_1) \\ &= p_2q_2\left[q_2(p_2-1) + q_3(1-3p_1)\right]\end{aligned}$$

For large initial value of p_1 (in particular for $p_1 > 1/3$) this product is consequently decreasing, implying that the growth rates of p_1 and p_3 slow down over time.

To summarize, the dynamical system defined by the replicator dynamic acting on the face $\Delta(S_1) \times \Delta([ad],[bc])$ is given by

$$\begin{bmatrix}\dot{q}_2\\ \dot{q}_3\\ \dot{p}_1\\ \dot{p}_2\\ \dot{p}_3\end{bmatrix} = \begin{bmatrix} q_2q_3(1-3p_1)\\ q_2q_3(3p_1-1)\\ p_1q_2p_2\\ -p_2q_2(p_1+p_3)\\ p_3p_2q_2\end{bmatrix}$$

The Jacobian matrix of this system is

$$J = \begin{bmatrix} q_3(1-3p_1) & q_2(1-3p_1) & -3q_2q_3 & 0 & 0\\ q_3(3p_1-1) & q_2(3p_1-1) & 3q_2q_3 & 0 & 0\\ p_1p_2 & 0 & q_2p_2 & p_1q_2 & 0\\ -p_2(p_1+p_3) & 0 & -p_2q_2 & -q_2(p_1+p_3) & -p_2q_2\\ p_3p_2 & 0 & 0 & p_3q_2 & p_2q_2\end{bmatrix}$$

The first row gives us all derivatives with respect to q_2, and so on. Evaluation of J at the state $[(1,0,0),(0,0,1,0)]$, we obtain

$$\tilde{J} = \begin{bmatrix} -2 & 0 & 0 & 0 & 0\\ 2 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

The linearized system reduces therefore to the planar system $\dot{q}_2 = -2q_2$ and $\dot{q}_3 = 2q_2$. In particular observe that no motion takes place in population 1. Let $(\tilde{q}_2(0), \tilde{q}_3(0))$ be initial values on the face $\Delta([ad],[bc])$, so that $\tilde{q}_2(0)+\tilde{q}_3(0) = 1$ does hold. The general solution for the first equation is then $\tilde{q}_2(t) = \tilde{q}_2(0)e^{-2t}$. Substituting this into the second equation and integrating gives us

$$\begin{aligned}\tilde{q}_3(t) - q_3(0) &= 2\tilde{q}_2(0)\int_0^t e^{-2u}du \\ &= \tilde{q}_2(0) - \tilde{q}_2(0)e^{-2t}\end{aligned}$$

or equivalently, $\tilde{q}_3(t) = 1 - \tilde{q}_2(0)e^{-2t}$. For $t \to \infty$ we see that $\tilde{q}_3(t) \to 1$ and $\tilde{q}_2(t) \to 0$. This demonstrates that the implausible subgame perfect Nash equilibrium can survive under the replicator dynamics, but only if the pure strategy $[bd]$ is not present in the second player population. Since the replicator dynamic is not innovative, this strategy will never be discovered by any individual in population 2, and so $q_4 = 0$ holds for all times, although it would be in the interest of this population to use $[bd]$.

Observation 3. *A population state $q = (0, 0, 1, 0)$ is only possible when we restrict our attention to the face $\Delta([ad], [bc])$. In this case, convergence to the postulated population state is observed, when $p_1 = 1$ is chosen.*

In the discussion above, we analyzed the game based on its normal form representation. Following Cressman (2003), we can also define dynamics restricted to the information sets of player 2. To do this, we have to find a set of probability distributions on $\Delta(S_2)$ such that the decisions made at information set v_1 are formed independently of the action at information set v_2. Define the conditional probability that action a is chosen at information set v_1, given that c will be played at v_2 as the number $\Pr(a|c) = \frac{q_1}{q_1+q_3}$. To make the probability that a is observed at v_1 independent of the choice at v_2, we need that

$$\begin{aligned} \Pr(a|c) = \Pr(a|d) \quad &\Leftrightarrow \quad \frac{q_1}{q_1 + q_3} = \frac{q_2}{q_2 + q_4} \\ &\Leftrightarrow \quad q_1 q_4 = q_2 q_3 \end{aligned}$$

This condition on population states $q \in \Delta(S_2)$ defines the *Wright manifold* (Cressman, 2003), for player population 2. It is defined via the set

$$\mathcal{W}^2 := \{q \in \text{int}\,[\Delta(S_2)]\,|q_1 q_4 = q_2 q_3\} \tag{9.29}$$

Claim 2. *The Wright manifold of player 2 is an invariant set under the replicator dynamics.*

Proof:

$$\begin{aligned} \frac{d(q_1 q_4)}{dt} &= \dot{q}_1 q_4 + q_1 \dot{q}_4 \\ &= q_1 q_4(\pi_1^2 - \bar{\pi}^2) + q_1 q_4(\pi_4^2 - \bar{\pi}^2) \\ &= q_1 q_4[2p_1(q_1 + q_2 - q_3 - q_4) + (p_2 + p_3)(q_1 + q_3 - q_2 - q_4)] \\ \frac{d(q_2 q_3)}{dt} &= q_2 q_3[2p_1(q_1 + q_2 - q_3 - q_4) + (p_2 + p_3)(q_1 + q_3 - q_2 - q_4)] \end{aligned}$$

On $\mathcal{W}^2$: $q_1 q_4 = q_2 q_3$, so that $\frac{d(q_1 q_4)}{dt} - \frac{d(q_2 q_3)}{dt} = \frac{d}{dt}(q_1 q_4 - q_2 q_3) = 0$.

q.e.d.

Recall that $\Pr(c) := q_1 + q_3$ is the (unconditional) probability that action c is observed at information set v_2. On this manifold we can compute

$$\begin{aligned}\dot{\Pr}(c) &= \dot{q}_1 + \dot{q}_3 \\ &= 2p_1[q_3(q_1+q_2) - q_1(q_3+q_4)] - (q_2+q_4)(p_2+p_3)(q_1+q_3)\end{aligned}$$

On the Wright manifold, we have $q_1(q_3+q_4) = (q_1+q_2)q_3$, so that the first term on the left-hand side vanishes. We see that

$$\dot{\Pr}(c) = -(q_2+q_4)(p_2+p_3)(q_1+q_3) = -(1-p_1)\Pr(d)\Pr(c) \quad (9.30)$$

on $\mathcal{W}^2$. The dynamic of the frequency comes to a rest, if and only if one of the terms $p_2+p_3, \Pr(d), \Pr(c)$ vanishes. Suppose that $p_2+p_3 = 0$. Then information set v_2 is never reached along a game path, so individuals from population 2 never have actually to decide whether to play c or not. Evolutionary selection cannot work for alternatives that do not occur in nature, so the replicator dynamic leaves $\Pr(c)$ unaffected. If $\Pr(d) = 0$, all individuals choose c at v_2, so that $\Pr(c) = 1$. This probability cannot grow further, hence the dynamic must come to a rest. $\Pr(c) = 0$ reflects the imitative and non-innovative behavior modeled by the replicator dynamic.

By equation (9.30) we see that it becomes more likely that action d is chosen if population 2 acts on its Wright manifold. By invariance of this set, we know that, once the population is on this manifold, it stays there forever. So in the long run we can predict a steady increase in the usage frequency of action d at information set v_2.

At information set v_1, the dynamic with which action a evolves is defined as

$$\begin{aligned}\dot{\Pr}(a) &= \dot{q}_1 + \dot{q}_2 \\ &= -2p_1(q_1+q_2)(q_3+q_4) - (p_2+p_3)[q_1(q_2+q_4) + q_2(q_1+q_3)] \\ &= -2p_1(q_1+q_2)(q_3+q_4) = -2p_1\Pr(a)\Pr(b)\end{aligned}$$

where we used the definition of the Wright manifold. Restricted on this set, the frequency with which we observe individuals choosing action b at information set v_1 increases over time. On $\mathcal{W}^2$ the evolutionary pattern is consequently fairly simple to reconstruct. The frequencies of action b and d are rising over time, so that the strategy $[bd]$ becomes prevalent in population 2. In population 1 we can deduce from equation (9.27) that p_1 is negatively influenced by $\Pr(d)$, and increasing in $\Pr(a)$. Since the latter is falling over time, while the former increases, we observe a decaying tendency of p_1. Frequency p_3 depends positively on $\Pr(d)$, thus gains support over time. Further we see that p_2 decays as $\Pr(d)$ rises. Hence, there is a clear tendency toward the subgame perfect equilibrium $[(0,0,1),(0,0,0,1)]$ on the Wright manifold.

Observation 4. *On the Wright manifold of population 2, $\mathcal{W}^2$, populations coordinate upon the subgame perfect Nash equilibrium* $[(0,0,1),(0,0,0,1)]$. *The implausible subgame perfect equilibrium will not be reached on this set.*

BNN dynamics. Again we start with population 2. Under the BNN dynamics a strategy frequency grows only if it earns a payoff above the population's average. We know from our discussion of the excess payoff vector that $\hat{\pi}_1^2(p,q) \leq 0, \forall (p,q) \in \Delta(S_1) \times \Delta(S_2)$. The BNN dynamic is therefore especially simple,

$$\dot{q}_1 = -q_1 \sum_{j=1}^{4} [\hat{\pi}_j^2]_+ \leq 0 \tag{9.31}$$

The strictly dominated strategy loses support monotonically, thus must get eliminated in the long run.
Further, we know about the excess payoff vector that strategy $[bd]$ weakly dominates all other pure strategies, implying that $\hat{\pi}_4^2(p,q) \geq 0, \forall (p,q) \in \Delta(S_1) \times \Delta(S_2)$. This in turn implies that strategy frequency q_4 is monotonically growing over time. The differential equation describing the evolution of frequency q_4 is given by

$$\dot{q}_4 = [\hat{\pi}_4^2]_+ - q_4 \sum_{j=1}^{4} [\hat{\pi}_j^2]_+ \tag{9.32}$$

The following observation is a direct consequence of our discussion in Game 2.

Observation 5. *Once the strictly dominated strategy [ac] disappears from population 2, its usage frequency remains 0 forever.*

This facilitates the analysis a little bit, when we restrict ourselves to a face of $\Delta(S_2)$ on which $q_1 = 0$. [38] Consider trajectories that lie on the face $\Delta([bc],[bd])$ initially, so that $q_3 + q_4 = 1$ holds. The excess payoff vector is on this face given by

$$\hat{\pi}^2(p,q) = \begin{bmatrix} -2p_1 - (1-p_1)q_4 \\ q_3 - p_1(2+q_3) \\ -(1-p_1)q_4 \\ (1-p_1)q_3 \end{bmatrix}$$

[38] Since the BNN dynamics is innovative, this is not a general result. Only in our example this observation does hold, and makes life easier.

so that strategy $[bc]$ never earns a payoff above the average. The dynamics in population 2 are then

$$\begin{aligned}
\dot{q}_2 &= [\hat{\pi}^2_2]_+ - q_2\left\{[\hat{\pi}^2_2]_+ + [\hat{\pi}^2_4]_+\right\} = (1-q_2)[q_3 - p_1(2+q_3)]_+ - q_2[q_3(1-p_1)]_+ \\
\dot{q}_3 &= [\hat{\pi}^2_3]_+ - q_3\left\{[\hat{\pi}^2_2]_+ + [\hat{\pi}^2_4]_+\right\} \\
&= [(p_1-1)q_4]_+ - q_3\{[q_3 - p_1(2+q_3)]_+ + [q_3(1-p_1)]_+\} \\
&= -q_3\{[q_3 - p_1(2+q_3)]_+ + [q_3(1-p_1)]_+\} \\
\dot{q}_4 &= [\hat{\pi}^2_4]_+ - q_4\left\{[\hat{\pi}^2_2]_+ + [\hat{\pi}^2_4]_+\right\} \\
&= [q_3(1-p_1)]_+(1-q_4) - q_4[q_3 - p_1(2+q_3)]_+
\end{aligned}$$

We see that q_3 is decreasing on this face. The evolution of q_2 is ambiguous, since it depends also on the behavior of p_1. Thus, we turn our view on population 1, while we restrict the dynamic to act on $\Delta(S_1)\times\Delta([bc],[bd])$. On this space, the vector of excess payoffs becomes

$$\hat{\pi}^1(p,q) = \begin{bmatrix} -q_4(1-p_1+p_3) \\ q_4(p_1-p_3) \\ q_4(2p_1+p_2) \end{bmatrix}$$

The first entry in this vector, $\hat{\pi}^1_1$, can never be positive. Using these figures, we receive the system of equations

$$\begin{aligned}
\dot{p}_1 &= -p_1\{[q_4(p_1-p_3)]_+ + [q_4(2p_1+p_2)]_+\} \\
\dot{p}_2 &= [q_4(p_1-p_3)]_+(1-p_2) - p_2[q_4(2p_1+p_2)]_+ \\
\dot{p}_3 &= [q_4(2p_1+p_2)]_+(1-p_3) - p_3[q_4(p_1-p_3)]_+
\end{aligned}$$

Hence, p_1 is strictly decreasing. Suppose we start from a region in our restricted state space, where p_1 is close to 1 initially. Then q_2 must be decreasing, since then $-2p_1$ dominates $(1-p_1)q_3$. Only q_4 will gain frequency in population 2, meanwhile q_3 loses support. As p_1 gradually declines, it will reach at some point of time the threshold value $\frac{q_3}{2+q_3}$. Once it falls below this value, q_2 becomes attractive enough for individuals, and there might be some inflow. For population 2 we have found two possible paths. One leads to a full specialization on strategy [bd], while the other leads us away from the face $\Delta([bc],[bd])$. Anyway, there is no chance to reach the implausible subgame perfect equilibrium.

Turn now to the second interesting case where we consider the restricted state space $\Delta(S_1)\times\Delta([ad],[bc])$. On this space, the BNN dynamics turns to

$$\begin{aligned}
\dot{q}_2 &= [q_3(1-3p_1)]_+(1-q_2) - q_2\{[q_2(3p_1-1)]_+ + [2p_1q_2 + (1-p_1)q_3]_+\} \\
\dot{q}_3 &= [q_2(3p_1-1)]_+(1-q_3) - q_3\{[q_3(1-3p_1)]_+ + [2p_1q_2 + (1-p_1)q_3]_+\} \\
\dot{q}_4 &= 2p_1q_2 + (1-p_1)q_3
\end{aligned}$$

The superior strategy [*bd*] will be discovered by agents, as one can see by the positivity of $\dot{q}_4$. If $p_1 > 1/3$, then there is some chance that q_3 is increasing, although there is certainly some outflow where agents settle to [*bd*]. This follows from the law of motion of q_4 which is positively linked with the frequency q_3. Thus, even if we observe q_3 rising, q_4 will also be rising, making perfect concentration on the pure strategy [*bc*] impossible. q_2 will definitely fall in this case. Again it is not possible to observe the implausible subgame perfect equilibrium. Ultimately, we look at the face $\Delta([ad],[bc],[bd])$. Consider the excess payoff vector in population 2,

$$\hat{\pi}^2(p,q) = \begin{bmatrix} -2p_1(q_3+q_4)+(1-p_1)q_3 \\ 2p_1q_2-(1-p_1)(q_2+q_3) \\ 2p_1q_2+q_3(1-p_1) \end{bmatrix}$$

The signs of the entries are not directly visible, with exception of strategy [*bd*], which always earns a payoff above the population's average. From this fact it is clear that there can never occur a population state where all agents decide to play [*bc*], since [*bd*] will always attract some agents. Differently put, even if we would start from the degenerate situation where all agents play [*bc*] initially, some of them will discover [*bd*] and recognize that it is more profitable than their currently used strategy. Thus, some will switch to [*bd*], leading to a growth of q_4, and therefore to a drift away from our initial state. This shows that the implausible subgame perfect Nash equilibrium cannot be a stable population state under the BNN dynamic, as there will always be a tendency that pushes population 2 to another state where strategy [*bd*] has some positive support.

Observation 6. *The BNN dynamic always makes agents in population 2 to coordinate upon a state where the superior strategy* [*bd*] *receives some positive weight in the composition of the population.*

The reason for this difference to the replicator dynamic is obvious the innovative property of excess payoff dynamics. The replicator dynamics allowed strategy [*bc*] to grow in frequency only in the case where [*bd*] is not used. By face invariance, [*bd*] will stay unused forever, and in this case p_1 is equally well performing as p_3. The more support p_1 has the more attractive becomes strategy [*bc*], making it possible that the point $[(1,0,0),(0,0,1,0)]$ is reached. As soon as some agents would use [*bd*], some imitating individuals would learn that this is indeed an exceptionally good choice, and therefore copy it. Only the fact that such [*bd*] players are (and always will be) absent from the player population 2 allows $[bc]$ to grow.

Pairwise difference dynamic. Since the payoffs in population 2 are the same as in Game 2, we can adopt the differential equations that describe the pairwise difference dynamic from this example. We know that the pairwise difference dynamic is an interpolation between the replicator dynamic and the BNN dynamic. In particular it is an innovative dynamic, so we would be surprised if the results differ much from the BNN dynamic. The first similarity between the BNN dynamic and the pairwise difference dynamic is that for both models observation 5 is valid. Hence, we can set $q_1 = 0$. As we have done in the study of the first two dynamics, we look at the evolutionary pattern induced by the pairwise difference dynamic, by analyzing the interesting faces of the tetrahedron $\Delta(S_2)$ separately. Start with the set $\Delta([bc],[bd])$, on which $q_3 + q_4 = 1$. The pairwise difference dynamic describes then the motions in population 2 as

$$\begin{bmatrix} \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix}_{|p_1<1/3} = \begin{bmatrix} q_3(1-3p_1) \\ 2q_3(2p_1-1) \\ q_3(1-p_1) \end{bmatrix}, \quad \begin{bmatrix} \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix}_{|p_1\geq 1/3} = \begin{bmatrix} 0 \\ q_3(p_1-1) \\ q_3(1-p_1) \end{bmatrix}$$

The frequency with which we observe pure strategy $[ad]$ is increasing if we start from a region where $p_1 < 1/3$. If we are in the other region, then there is neither inflow nor outflow of $[ad]$. This is intuitive, since when p_1 is large, $[bd]$ and $[bc]$ become more attractive than $[ad]$. However, if p_1 is a small number, $[ad]$ is nearly as profitable as $[bd]$, so some individuals might get attracted to it.

q_3 is in both cases monotonically decreasing, while q_4 always rises. Thus, we can identify two possible evolutionary patterns, which depend on the frequency p_1. If $p_1 < 1/3$, then the state of population 2 exits the face $\Delta([bc],[bd])$ and turns to an subsimplex on which $[ad]$ has some positive weight. The second possibility arises when $p_1 \geq 1/3$. Then the solution trajectory will remain on the face $\Delta([bc],[bd])$ forever, and approach the vertex where only pure strategy $[bd]$ is played by agents in population 2.

Consider now the face $\Delta([ad],[bc])$, so that $q_2 + q_3 = 1$. The dynamics on this face becomes

$$\begin{bmatrix} \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix}_{|p_1<1/3} = \begin{bmatrix} q_3 - p_1(2+q_3) \\ 2q_3(2p_1-1) \\ 2p_1q_2 + q_3(1-p_1) \end{bmatrix}, \quad \begin{bmatrix} \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix}_{|p_1\geq 1/3} = \begin{bmatrix} -q_2(5p_1-1) \\ p_1(2q_2+1)-1 \\ 2p_1q_2 + q_3(1-p_1) \end{bmatrix}$$

If $p_1 < 1/3$, then $\dot{q}_2 > 0$ if and only if $p_1 < \frac{q_3}{2+q_3} \leq 1/3, \forall q_3 \in [0,1]$. Otherwise, that is for $p_1 \in [\frac{q_3}{2+q_3}, 1/3]$, q_2 is decreasing in

this region. q_3 is decreasing whereas q_4 is increasing.
If $p_1 \geq 1/3$, then q_2 is decaying, since $5p_1 > 1$ in this region. $\dot{q}_3 \geq 0$ only if $p_1 \geq \frac{1}{2q_2+1} \geq 1/3$. For $p_1 \in [1/3, \frac{1}{2q_2+1}]$ q_3 is decreasing. q_4 displays again a tendency to gain support over time.

The remaining interesting case is the behavior of trajectories restricted to the face $\Delta([ad],[bc],[bd])$, so that $q_2 + q_3 + q_4 = 1$. On this set, the pairwise difference dynamic is given by

$$\begin{bmatrix} \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix}_{|p_1<1/3} = \begin{bmatrix} q_3(1-3p_1) - 2p_1q_2 \\ 2q_3(2p_1-1) \\ 2p_1q_2 + q_3(1-p_1) \end{bmatrix}, \begin{bmatrix} \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix}_{|p_1 \geq 1/3} = \begin{bmatrix} -q_2(5p_1-1) \\ p_1(3q_2+q_3) - (q_2+q_3) \\ 2p_1q_2 + q_3(1-p_1) \end{bmatrix}$$

If $p_1 < 1/3$, then $\dot{q}_2 > 0$, if for $q_2, q_3 > 0$, $p_1 < \frac{q_3}{2q_2+3q_3} \leq 1/3$. In case of $q_3 = 0$, then the frequency of pure strategy $[ad]$ is clearly falling. $[bc]$ loses support monotonically, as p_1 is restricted to lie below $1/3$. $[bd]$ will attract individuals from population 2, leading to a growing share of individuals who employ this strategy.
If $p_1 \geq 1/3$, the share of individuals who adopt pure strategy [ad] is monotonically decreasing, and q_4 keeps its increasing tendency. $\dot{q}_3 > 0$ if $p_1 > \frac{q_2+q_3}{3q_2+q_3}$ (certainly we assume that $q_2 + q_3 \neq 0$).

Observation 7. *The pure strategy $[bd]$ is always gaining support under the pairwise difference dynamic. Even though q_3 might increase on some faces of the tetrahedron $\Delta(S_2)$, we cannot observe a tendency such that the state of player population 2 converges to the vertex $(0,0,1,0)$.*

We are now going to look at the pairwise difference dynamic for population 1. Of particular interest is clearly the evolution of the frequency p_1. We compute

$$\begin{aligned} \pi_2^1 - \pi_1^1 &= -q_1 - q_3 - q_1 - q_2 + q_3 + q_4 = -2q_1 - q_2 + q_4 = (q_4 - q_1) - (q_1 + q_2) \\ \pi_3^1 - \pi_1^1 &= -q_1 + q_2 - q_3 + q_4 - q_1 - q_2 + q_3 + q_4 = 2(q_4 - q_1) \end{aligned}$$

If $q_1 \geq q_4$, we see that $\pi_3^1 - \pi_1^1 \leq 0$ and $\pi_2^1 - \pi_1^1 \leq 0$, so that the pairwise difference dynamic boils down to

$$\dot{p}_1 = p_2(2q_1 + q_2 - q_4) + 2p_3(q_1 - q_4)$$

However, we know hat q_1 is strictly decreasing as it represents the frequency with which a strictly dominated strategy is played in population 2. On the other hand q_4 is strictly increasing. Therefore we will not expect that the above law of motion will hold for a long time. Even if $q_4 > q_1$, there might be some inflow to strategy A. This is the case when $q_4 < 2q_1 + q_2$.
On the face $\Delta(S_1) \times \Delta([bc],[bd])$, the dynamic reduces to $\dot{p}_1 =$

$-3p_1q_4 \leq 0$. The rate with which the frequency p_1 decays (hence $2q_4$) is accelerating, as q_4 grows monotonically over time. At some point of time $p_1 < 1/3$ must occur. From our discussion of the dynamics is population 2, we know that in this case q_2 becomes some positive number, and we exit the face $\Delta(S_1) \times \Delta([bc], [bd])$. Therefore we consider trajectories restricted to the space $\Delta(S_1) \times \Delta([ad], [bc])$. On this space $\pi_1^1 = \pi_3^1$, and $\pi_1^1 - \pi_2^1 = q_2$. The pairwise difference dynamic tells us that $\dot{p}_1 = p_2q_2 \geq 0$. Assume we have given data such that $p_1 < 1/3$ holds initially. Then there is some chance that q_2 increases initially, leading to a fast increase of p_1. Even if q_2 is not growing, p_1 rises, but at decreasing pace. Eventually $p_1 \geq 1/3$, where the dynamic in population 2 is fairly simple. q_2, q_3 fall over time, and we move in direction to the vertex where all mass is concentrated on pure strategy $[bd]$.
For strategy C in population 1, we calculate $\pi_3^1 - \pi_2^1 = q_2 + q_4$. On the face $\Delta(S_1) \times \Delta([ac], [bc])$ p_3 stays at a constant level. As soon as q_2 or q_4 have some weight in population 2, there will be some inflow, namely individuals that switch from strategy B to C. If $q_4 > q_1$, what has to be true at some point in time, then there will additionally be some inflow coming from strategy A. Thus the only chance we have to observe coordination on the implausible subgame perfect Nash equilibrium is on the restricted state space $\Delta(S_1) \times \Delta([ad], [bc])$. But we know that q_4 is growing over time, so even if p_1 is rising, there is no tendency in population 2, to specialize on $[bc]$.

Observation 8. *The pairwise difference dynamic excludes the implausible subgame perfect Nash equilibrium* $[(1,0,0),(0,0,1,0)]$. *As soon as the superior strategy* $[bd]$ *is discovered in population 2 (and this will happen with certainty),* p_3 *starts to grow. Hence, there is a clear tendency toward the "intelligent" subgame perfect Nash equilibrium* $[(0,0,1)(0,0,0,1)]$.

Part 5. Appendix

10. Appendix A

The goal of this Appendix is to demonstrate that the Nash equilibrium in the two population "Matching pennies" game is asymptotically stable under the BNN dynamic, as was claimed in Section 8.2.1, Example 8. First, we start writing the system of differential equations in matrix notation. For population 1 this is

$$\dot{p} = \sigma^1(\hat{\pi}^1(p)) - p\mathbf{1}^T\sigma^1(\hat{\pi}^1(p)) \tag{10.1}$$

As a second step we will create an appropriate Lyapunov functions, where the outlines for a single population game is given in Sandholm (2007a). The behavioral rule proposed for the BNN dynamic will play a fundamental role in this exercise. It was described via the Lipschitz continuous function

$$f^k_{ij}(\pi^k, p, q) = \sigma^k_j(\hat{\pi}^k_j(p,q)), \quad k \in \{1,2\}$$

where the monotonicity assumption $\operatorname{sgn} \sigma^k_j(\hat{\pi}^k_j(p,q)) = \operatorname{sgn}[\hat{\pi}^k_j(p,q)]_+$ is a defining characteristic, so that its image is contained in $\mathbb{R}_+$. We will further require that the left-hand derivative does exist, and is positive, hence $\frac{d}{dv^+}\sigma^k_j(v) > 0$. We claim that the continuous function

$$\lambda^k(d) := \sum_{j=1}^{2} \int_0^d \sigma^k_j(v)dv$$

satisfies $\nabla\lambda^k(\hat{\pi}^k) = \sigma^k(\hat{\pi}^k)$ for both populations $k \in \{1,2\}$. To see this, we integrate

$$\begin{aligned}
\int_0^{\pi^k_j} \sigma^k_j(v)dv &= \int_0^{\pi^k_j} \left[v - \bar{\pi}^k\right]_+ dv \\
&= \int_{\bar{\pi}^k}^{\pi^k_j} \left[v - \bar{\pi}^k\right] dv, \quad \text{whenever } \pi^k_j \geq \bar{\pi}^k \\
&= 1/2(\pi^k_j)^2 - \bar{\pi}^k\pi^k_j + 1/2(\bar{\pi}^k)^2 \\
&= 1/2\left(\pi^k - \bar{\pi}^k\right)^2, \quad \text{whenever } \pi^k_j \geq \bar{\pi}^k \\
&= 1/2\left[\pi^k - \bar{\pi}^k\right]^2_+
\end{aligned}$$

Define the additive separable function

$$\begin{aligned}
\Lambda(p,q) : &= \lambda^1(p,q) + \lambda^2(p,q) \qquad (10.2) \\
&= 1/2\sum_{i=1}^{2}\left[\pi^1_i(q) - \bar{\pi}^1(p,q)\right]^2_+ + 1/2\sum_{i=1}^{2}\left[\pi^2_i(p) - \bar{\pi}^2(p,q)\right]^2_+
\end{aligned}$$

Its gradient is the 4 dimensional column vector $\nabla\Lambda(p,q) = \begin{bmatrix} \sigma^1(\hat{\pi}^1(p,q)) \\ \sigma^2(\hat{\pi}^2(p,q)) \end{bmatrix}$. This map will serve as our Lyapunov function. First of all, it is clearly visible, that the function assumes its minimum vale 0 only at the Nash equilibrium, $\Lambda(p,q) = 0 \Leftrightarrow (p,q) = (p^*, q^*)$. We will now see that this function varies monotonically for all solutions of the BNN dynamic.

Suppose that the 4 dimensional column vector $\begin{bmatrix} p(t) \\ q(t) \end{bmatrix}$ is an interior solution trajectory of the BNN dynamic in the two populations Matching pennies game. Taking time derivatives of (10.2), we see

$$\begin{aligned}
\frac{d}{dt}\Lambda(p(t),q(t)) &= \nabla\Lambda(p(t),q(t))^T \begin{bmatrix} \dot{p}(t) \\ \dot{q}(t) \end{bmatrix}, \quad \text{by the chain rule} \\
&= \left(\begin{bmatrix} D_p\hat{\pi}^1(p,q) & D_q\hat{\pi}^1(p,q) \\ D_p\hat{\pi}^2(p,q) & D_q\hat{\pi}^2(p,q) \end{bmatrix}^T \begin{bmatrix} \sigma^1(\hat{\pi}^1(p,q)) \\ \sigma^2(\hat{\pi}^2(p,q)) \end{bmatrix} \right)^T \begin{bmatrix} \dot{p}(t) \\ \dot{q}(t) \end{bmatrix} \\
&= \left[\sigma^1(\hat{\pi}^1(p,q)), \sigma^2(\hat{\pi}^2(p,q))\right]^T \begin{bmatrix} D_p\hat{\pi}^1(p,q) & D_qD\hat{\pi}^1(p,q) \\ D_p\hat{\pi}^2(p,q) & D_qD\hat{\pi}^2(p,q) \end{bmatrix} \begin{bmatrix} \dot{p}(t) \\ \dot{q}(t) \end{bmatrix}
\end{aligned}$$

where

$$D_s\hat{\pi}^k(p,q) = \begin{bmatrix} \frac{\partial\hat{\pi}^k_1}{\partial s_1} & \frac{\partial\hat{\pi}^k_1}{\partial s_2} \\ \frac{\partial\hat{\pi}^k_2}{\partial s_1} & \frac{\partial\hat{\pi}^k_2}{\partial s_2} \end{bmatrix} \qquad s = (s_1, s_2)^T \in \{p,q\}, k \in \{1,2\}$$

is the Jacobian Matrix of population k's excess payoff function. We calculate this matrix in 2 separate steps for population 1. The same computations are then also valid for population 2, where only sub-and superscripts have to be adjusted. We denote the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as $\mathbf{1}$.

(1)

$$\begin{aligned}
D_p\hat{\pi}^1 &= D_p\left[\pi^1 - \mathbf{1}\bar{\pi}^1\right] \\
&= D_p\left[\pi^1 - \mathbf{1}p^T\pi^1\right] \\
&= \begin{bmatrix} \frac{\partial\pi^1_1}{\partial p_1} - \pi^1_1 - p_1\frac{\partial\pi^1}{\partial p_1} - p_2\frac{\partial\pi^1}{\partial p_2}, & \frac{\partial\pi^1_1}{\partial p_2} - \pi^1_2 - p_1\frac{\partial\pi^1}{\partial p_1} - p_2\frac{\partial\pi^1}{\partial p_2} \\ \frac{\partial\pi^1_2}{\partial p_1} - \pi^1_1 - p_1\frac{\partial\pi^1}{\partial p_1} - p_2\frac{\partial\pi^1}{\partial p_2}, & \frac{\partial\pi^1_2}{\partial p_2} - \pi^1_2 - p_1\frac{\partial\pi^1}{\partial p_1} - p_2\frac{\partial\pi^1}{\partial p_2} \end{bmatrix} \\
&= D_p\pi^1 - \mathbf{1}\left(\pi^1\right)^T - \mathbf{1}p^TD_p\pi^1
\end{aligned}$$

(2)

$$
\begin{aligned}
D_q\hat{\pi}^1 &= D_q\left[\pi^1 - \mathbf{1}\bar{\pi}^1\right] \\
&= \begin{bmatrix} \frac{\partial \pi_1^1}{\partial q_1} - p_1\frac{\partial \pi^1}{\partial q_1} - p_2\frac{\partial \pi^1}{\partial q_2}, & \frac{\partial \pi_1^1}{\partial q_2} - p_1\frac{\partial \pi^1}{\partial q_1} - p_2\frac{\partial \pi^1}{\partial q_2} \\ \frac{\partial \pi_2^1}{\partial q_1} - p_1\frac{\partial \pi^1}{\partial q_1} - p_2\frac{\partial \pi^1}{\partial q_2}, & \frac{\partial \pi_2^1}{\partial q_2} - p_1\frac{\partial \pi^1}{\partial q_1} - p_2\frac{\partial \pi^1}{\partial q_2} \end{bmatrix} \\
&= D_q\pi^1 - \mathbf{1}p^T D_q\pi^1
\end{aligned}
$$

We are then able to write the derivative matrix in an elegant way,

$$
\begin{aligned}
D\hat{\pi}(p,q) &= \begin{bmatrix} D_p\hat{\pi}^1(p,q) & D_q D\hat{\pi}^1(p,q) \\ D_p\hat{\pi}^2(p,q) & D_q D\hat{\pi}^2(p,q) \end{bmatrix} \\
&= D\pi(p,q) - \begin{bmatrix} \mathbf{1}\left(\pi^1(q)\right)^T & \mathbf{0} \\ \mathbf{0} & \mathbf{1}\left(\pi^2(p)\right)^T \end{bmatrix} - \begin{bmatrix} \mathbf{1}p^T D_p\pi^1, & \mathbf{1}p^T D_q\pi^1 \\ \mathbf{1}q^T D_p\pi^2, & \mathbf{1}q^T D_q\pi^2 \end{bmatrix} \\
&= D\pi(p,q) - \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \left\{ \begin{bmatrix} \left(\pi^1(q)\right)^T & \mathbf{0}^T \\ \mathbf{0}^T & \left(\pi^2(p)\right)^T \end{bmatrix} + \begin{bmatrix} p^T & \mathbf{0}^T \\ \mathbf{0}^T & q^T \end{bmatrix} D\pi(p,q) \right\}
\end{aligned}
$$

where $\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Substituting this expression into the time derivative of the Lyapunov function (10.2), shows

$$
\begin{aligned}
&\tfrac{d}{dt}\Lambda(p(t),q(t)) = \\
&= \left[\sigma^1(\hat{\pi}^1), \sigma^2(\hat{\pi}^2)\right]^T D\pi(p,q) \begin{bmatrix} \dot{p}(t) \\ \dot{q}(t) \end{bmatrix} \\
&- \left[\sigma^1(\hat{\pi}^1), \sigma^2(\hat{\pi}^2)\right]^T \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p^T & \mathbf{0}^T \\ \mathbf{0}^T & q^T \end{bmatrix} D\pi(p,q) \begin{bmatrix} \dot{p}(t) \\ \dot{q}(t) \end{bmatrix} \\
&- \left[\sigma^1(\hat{\pi}^1), \sigma^2(\hat{\pi}^2)\right]^T \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \left(\pi^1(q)\right)^T & \mathbf{0}^T \\ \mathbf{0}^T & \left(\pi^2(p)\right)^T \end{bmatrix} \begin{bmatrix} \dot{p}(t) \\ \dot{q}(t) \end{bmatrix} \\
&= \left\{ \left[\sigma^1(\hat{\pi}^1), \sigma^2(\hat{\pi}^2)\right]^T - \left[\sigma^1(\hat{\pi}^1), \sigma^2(\hat{\pi}^2)\right]^T \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p^T & \mathbf{0}^T \\ \mathbf{0}^T & q^T \end{bmatrix} \right\} D\pi(p,q) \begin{bmatrix} \dot{p}(t) \\ \dot{q}(t) \end{bmatrix} \\
&- \left[\sigma^1(\hat{\pi}^1), \sigma^2(\hat{\pi}^2)\right]^T \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \left(\pi^1(q)\right)^T & \mathbf{0}^T \\ \mathbf{0}^T & \left(\pi^2(p)\right)^T \end{bmatrix} \begin{bmatrix} \dot{p}(t) \\ \dot{q}(t) \end{bmatrix}
\end{aligned}
$$

The first term is equivalent to the transpose of the BNN dynamic (10.1) in both populations, written in one vector. Hence, the time derivative reduces to the expression

$$\begin{aligned}\frac{d}{dt}\Lambda(p(t),q(t)) &= [\dot{p}(t),\dot{q}(t)]^T D\pi(p,q)\begin{bmatrix}\dot{p}(t)\\ \dot{q}(t)\end{bmatrix}\\ &- \left[\sigma^1(\hat{\pi}^1),\sigma^2(\hat{\pi}^2)\right]^T \begin{bmatrix}1&0\\1&0\\0&1\\0&1\end{bmatrix}\begin{bmatrix}(\pi^1(q))^T & \mathbf{0}^T\\ \mathbf{0}^T & (\pi^2(p))^T\end{bmatrix}\begin{bmatrix}\dot{p}(t)\\ \dot{q}(t)\end{bmatrix}\\ &= h_1(p,q)+h_2(p,q)\end{aligned}$$

First note, that a property of any excess payoff dynamic is that the vector of payoffs and the vector field forms an acute angle, so that the second term $h_2 \leq 0$, with equality only at the Nash equilibrium. This was shown in Corollary 2 of Section 8. To determine the sign of the first term, we exploit the linearity of the payoff function in this game. It is easily seen that the derivative matrix $D\pi(p,q)$ is of the form $\begin{bmatrix}\mathbf{0} & A\\ -A & \mathbf{0}\end{bmatrix}$, where $A = \begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}$ is the payoff matrix of the matching pennies game for player population 1. The quadratic form h_1 is then easily calculated, as

$$[\dot{p}(t),\dot{q}(t)]^T A\begin{bmatrix}\dot{p}(t)\\ \dot{q}(t)\end{bmatrix} - \left[\dot{p(t)},\dot{q}(t)\right]^T A\begin{bmatrix}\dot{p}(t)\\ \dot{q}(t)\end{bmatrix} = 0$$

This establishes that the time derivative is negative along interior solution trajectories, and by definition 7 in Section 6, the Nash equilibrium is globally asymptotically stable under the BNN dynamic.[39]

Our computation demonstrates an important and general tool to determine the stability of rest points of the BNN dynamics (and thus of Nash equilibria). For global aysmptotic stability of the equilibrium point we need that $[\dot{p}(t),\dot{q}(t)]^T D\pi(p,q)\begin{bmatrix}\dot{p}(t)\\ \dot{q}(t)\end{bmatrix} \leq 0$. For linear games the derivative matrix included in this product boils down to the square block matrix, where the blocks are either matrices with zeros as elements, or the payoff matrices of a population engaged in the conflict. Thus, stability classification coincides with determining the definiteness of a square block matrix.

[39]In definition 7 only asymptotic stability is defined. A rest point is *globally asymptotically stable* if the open neighborhood $\mathcal{U}(p^*)$ stated there coincides with the state space Θ.

11. Appendix B

Our task is here to prove the asymptotic stability of the unique Nash equilibrium in the two-population "Matching Pennies" game under the pairwise difference dynamic, as was claimed in Section 8.3.1 example 10. The conditional switching rates that generated the pairwise difference dynamic was proposed as

$$f_{ij}^k(\pi^k(p), p) = \left[\pi_j^k(p) - \pi_i^k(p)\right]_+$$

For an arbitrary population $k \in \{1, 2 \ldots, N\}$, the behavioral rule can be summarized in the matrix

$$F^k(\pi^k(p), p) = \begin{bmatrix} \left[\pi_1^k(p) - \pi_1^k(p)\right]_+, & \left[\pi_2^k(p) - \pi_1^k(p)\right]_+, & \ldots, & \left[\pi_{n_k}^k(p) - \pi_1^k(p)\right]_+ \\ \vdots & \ddots & \ddots & \vdots \\ \left[\pi_1^k(p) - \pi_{n_k}^k(p)\right]_+, & \left[\pi_2^k(p) - \pi_{n_k}^k(p)\right]_+, & \ldots, & \left[\pi_{n_k}^k(p) - \pi_{n_k}^k(p)\right]_+ \end{bmatrix}$$

All entries in the main diagonal of this matrix are certainly equal to zero for all possible social states. The switching rates in population k from a pure strategy $e_{k_i} \in S_k$ to some pure strategy e_{k_j} does therefore only depend on the potential revising strategy and is independent of the incumbent one. Henceforth, we can write the behavioral rule in terms of a continuous function $\sigma_{(i)j}^k : \mathbb{R} \to \mathbb{R}_+$, where $\sigma_{(i)j}^k(\pi_j^k(p) - \pi_i^k(p)) := \left[\pi_j^k(p) - \pi_i^k(p)\right]_+$. Furthermore, the left side derivative of this function exists, so $\frac{d}{dv^+}\sigma_{(i)j}^k > 0$.[40]

The pairwise comparison dynamic in population $k = 1, 2$, was already introduced in matrix notation, in equation (8.16). For two populations simultaneously, we can write

$$V(p, q) = \begin{bmatrix} (F^1)^T & \mathbf{0}^T \\ \mathbf{0} & (F^2)^T \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} - \begin{bmatrix} \text{diag}[p] & \mathbf{0} \\ \mathbf{0}^T & \text{diag}[q] \end{bmatrix} \begin{bmatrix} F^1 & \mathbf{0} \\ \mathbf{0}^T & F^2 \end{bmatrix} \mathbf{1} \tag{11.1}$$

where $\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$,[41] $0_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\text{diag}[p] = \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix}$, and a similar representation is $\text{diag}[q]$.

[40] We keep the index (i) in the description of the function, to remind us in which column of the behavioral rule we are.

[41] The transposition of this matrix in formula 11.1 is redundant here. However we keep it to emphasize that the two distinct populations need not have the same number of pure strategies. In general, were population 1 has n_1, and population n_2 pure strategies, the matrix $\mathbf{0}$ consists of n_1 rows and n_2 columns.

Now, introduce the function

$$x^k_{(i)j}(v) := \int_0^v \sigma^k_{(i)j}(s)ds \qquad i,j,k \in \{1,2\}$$

Per definition it is

$$\frac{d}{dv^+} x^k_{(i)j}(v)\Big|_{v=\pi^k_j-\pi^k_i} = \sigma^k_{(i)j}(\pi^k_j - \pi^k_i) \qquad i,j,k \in \{1,2\}$$

For population $k = 1,2$ we can define the matrix

$$X^k := \begin{bmatrix} x^k_{(1)1}(v) & x^k_{(1)2}(v) \\ x^k_{(2)1}(v) & x^k_{(2)2}(v) \end{bmatrix}$$

By construction, the entries in the main diagonal are always equal to zero. We will show that the continuously differentiable function

$$\begin{aligned} \chi(p,q) &= [p,q]^T \begin{bmatrix} X^1 & \mathbf{0} \\ \mathbf{0} & X^2 \end{bmatrix} \mathbf{1} \\ &= \sum_{i=1}^{2}\sum_{j=1}^{2} p_i x^1_{(i)j}(\pi^1_j(q) - \pi^1_i(q)) + \sum_{i=1}^{2}\sum_{j=1}^{2} q_i x^2_{(i)j}(\pi^2_j(p) - \pi^2_i(p)) \end{aligned}$$

is a Lyapunov function for the pairwise difference dynamic. First of all note, that $\chi : \mathrm{int}(\Theta) \to \mathbb{R}_+$, with minimum value $\chi(p,q) = 0$, which is reached only when $X^1 = X^2 = \mathbf{0}$. By construction of these matrices, this can be achieved only at the Nash equilibrium. We are going to show now that the function is decreasing along interior solution curves of the pairwise difference dynamic, which establishes the asymptotic stability of the Nash equilibrium of Matching pennies. Let $[p(t), q(t)]^T$ be such a solution in the interior of the space of social states Θ, and take the time derivative of our candidate function,

$$\frac{d}{dt}\chi(p(t), q(t)) = \nabla\chi(p(t), q(t))^T \begin{bmatrix} \dot{p}(t) \\ \dot{q}(t) \end{bmatrix}$$

Calculating the gradient vector $\nabla\chi$ is a fairly simple exercise, where only the chain rule of Calculus is used heavily.

$$\begin{aligned}
\frac{\partial\chi(p,q)}{\partial p_1} &= \sum_{j=1}^{2} x^1_{(1)j} + \sum_{i=1}^{2}\sum_{j=1}^{2} q_i\sigma^2_{(i)j}(\pi^2_j(p) - \pi^2_i(p))(\frac{\partial\pi^2_j}{\partial p_1} - \frac{\partial\pi^2_i}{\partial p_1}) \\
&= \sum_{j=1}^{2} x^1_{(1)j} + q_1\left[\pi^2_2(p) - \pi^2_1(p)\right]_+ (\frac{\partial\pi^2_2}{\partial p_1} - \frac{\partial\pi^2_1}{\partial p_1}) \\
&+ \; q_2\left[\pi^2_1(p) - \pi^2_2(p)\right]_+ (\frac{\partial\pi^2_1}{\partial p_1} - \frac{\partial\pi^2_2}{\partial p_1}) \\
&= \sum_{j=1}^{2} x^1_{(1)j} + (q_2\left[\pi^2_1(p) - \pi^2_2(p)\right]_+ - q_1\left[\pi^2_2(p) - \pi^2_1(p)\right]_+)\frac{\partial\pi^2_1}{\partial p_1} \\
&+ \; (q_1\left[\pi^2_2(p) - \pi^2_1(p)\right]_+ - q_2\left[\pi^2_1(p) - \pi^2_2(p)\right]_+)\frac{\partial\pi^2_2}{\partial p_1} \\
&= \sum_{j=1}^{2} x^1_{(1)j}(\pi^1_j(q) - \pi^1_1(q)) + \dot{q}_1\frac{\partial\pi^2_1}{\partial p_1} + \dot{q}_2\frac{\partial\pi^2_2}{\partial p_1}
\end{aligned}$$

and analogously one computes

$$\begin{aligned}
\frac{\partial\chi(p,q)}{\partial p_2} &= \sum_{j=1}^{2} x^1_{(2)j}(\pi^1_j(q) - \pi^1_2(q)) + \sum_{i=1}^{2}\sum_{j=1}^{2} q_i\sigma^2_{(i)j}(\pi^2_j(p) - \pi^2_i(p))(\frac{\partial\pi^2_j}{\partial p_2} - \frac{\partial\pi^2_i}{\partial p_2}) \\
&= \sum_{j=1}^{2} x^1_{(2)j}(\pi^1_j(q) - \pi^1_2(q)) + \dot{q}_1\frac{\partial\pi^2_1}{\partial p_2} + \dot{q}_2\frac{\partial\pi^2_2}{\partial p_2} \\
\frac{\partial\chi(p,q)}{\partial q_1} &= \sum_{j=1}^{2} x^2_{(1)j}(\pi^2_j(p) - \pi^2_1(p)) + \sum_{i=1}^{2}\sum_{j=1}^{2} p_i\sigma^1_{(i)j}(\pi^1_j(q) - \pi^1_i(q))(\frac{\partial\pi^1_j}{\partial q_1} - \frac{\partial\pi^1_i}{\partial q_1}) \\
&= \sum_{j=1}^{2} x^2_{(1)j}(\pi^2_j(p) - \pi^2_1(p)) + \dot{p}_1\frac{\partial\pi^1_1}{\partial q_1} + \dot{p}_2\frac{\partial\pi^1_2}{\partial q_1} \\
\frac{\partial\chi(p,q)}{\partial q_2} &= \sum_{j=1}^{2} x^2_{(2)j}(\pi^2_j(p) - \pi^2_2(p)) + \dot{p}_1\frac{\partial\pi^1_1}{\partial q_2} + \dot{p}_2\frac{\partial\pi^1_2}{\partial q_2}
\end{aligned}$$

In matrix notation, this is summarized as

$$\nabla\chi(p,q) = \begin{bmatrix} X^1 & \mathbf{0} \\ \mathbf{0} & X^2 \end{bmatrix}\mathbf{1} + D\pi(p,q)^T\begin{bmatrix} \dot{p}(t) \\ \dot{q}(t) \end{bmatrix}$$

where

$$D\pi(p,q) = \begin{bmatrix} D_p\pi^1(q) & D_q\pi^1(q) \\ D_p\pi^2(p) & D_q\pi^2(p) \end{bmatrix} = \begin{bmatrix} \mathbf{0} & D_q\pi^1(q) \\ D_p\pi^2(p) & \mathbf{0} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \frac{\partial \pi_1^1}{\partial q_1}(q) & \frac{\partial \pi_1^1}{\partial q_2}(q) \\ 0 & 0 & \frac{\partial \pi_2^1}{\partial q_1}(q) & \frac{\partial \pi_2^1}{\partial q_2}(q) \\ \frac{\partial \pi_1^2}{\partial p_1}(p) & \frac{\partial \pi_1^2}{\partial p_2}(p) & 0 & 0 \\ \frac{\partial \pi_2^2}{\partial p_1}(p) & \frac{\partial \pi_2^2}{\partial p_2}(p) & 0 & 0 \end{bmatrix}$$

so that the time derivative of χ becomes

$$\begin{aligned} \frac{d\chi(p(t),q(t))}{dt} &= \mathbf{1}^T \begin{bmatrix} (X^1)^T & \mathbf{0} \\ \mathbf{0} & (X^2)^T \end{bmatrix} \begin{bmatrix} \dot p(t) \\ \dot q(t) \end{bmatrix} + [\dot p(t), \dot q(t)]^T D\pi(p,q) \begin{bmatrix} \dot p(t) \\ \dot q(t) \end{bmatrix} \\ &= r_1(p,q) + r_2(p,q) \end{aligned}$$

We know that $r_2(p,q) = 0$ for any zero sum game, as was shown in Appendix A.[42] To determine the sign of $r_1(p,q)$, note that this term consists of two parts.

$$r_1(p,q) = [1,1]^T \left(X^1\right)^T \dot p(t) + [1,1]^T \left(X^2\right)^T \dot q(t)$$

The first summand is

$$\begin{aligned} [1,1]^T \left(X^1\right)^T \dot p(t) &= \dot p_1(t)\left(x^1_{(1)1} + x^1_{(1)2}\right) + \dot p_2(t)\left(x^1_{(2)1} + x^2_{(2)2}\right) \\ &= \left(p_2\left[\pi_1^1 - \pi_2^1\right]_+ - p_1\left[\pi_2^1 - \pi_1^1\right]_+\right) x^1_{(1)2} \\ &+ \left(p_1\left[\pi_2^1 - \pi_1^1\right]_+ - p_2\left[\pi_1^1 - \pi_2^1\right]_+\right) x^1_{(2)1} \end{aligned}$$

There are 2 distinct cases we have to consider. Suppose that $\pi_2^1 - \pi_1^1 > 0$. In this case $x^1_{(2)1} = 0$ by definition of this function and so $r_1(p,q) - \mathbf{1}^T (X^2)^T \dot q(t) = -p_1[\pi_2^1 - \pi_1^1]_+ \le 0$. The other case occurs if $\pi_2^1 - \pi_1^1 \le 0$, in which $x^1_{(1)2} = 0$. Hence $r_1(p,q) - \mathbf{1}^T (X^2)^T \dot q(t) = -p_2[\pi_1^1 - \pi_2^1]_+ \le 0$. In both cases we obtain a non-positive number. Clearly the same holds for population 2, so that we can conclude

$$r_1(p,q) \le 0 \quad \forall (p,q) \in \Theta$$

with equality only at the Nash equilibrium. The function χ is indeed a (global) Lyapunov function, and the Nash equilibrium of the two populations "Matching Pennies" game is (globally) asymptotically stable under the pairwise difference dynamic.

[42]Note that $r_2(p,q) \equiv h_1(p,q)$ from Appendix A.

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References

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