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Contents

Zusammenfassung

In dieser Arbeit zeigen wir wie dreidimensionale Newton-Cartan Mannigfaltigkeiten für nicht relativistische supersymmetrische Feldtherien konstruiert werden können. Hierfür starten wir mit der neuen minimalen "off shell" Formulierung der Supergravitationstheorie in vier Dimensionen. Die nicht relativistische Feldtheorie in drei Dimensionen kann mit einer sogenannten Null Reduktion konstruiert werden. Wir haben die Lagrangefunktion, die Transformationsregeln und die dazugehörigen Killing Spinor Gleichungen um eine Dimension reduziert. Aus diesen Killing Spinor Gleichungen konnten wir Bedingungen für die Hintergrundfelder gewinnen, sodass global definierte, nicht triviale Lösungen dieser Gleichungen existieren. Aus diesen Bedingungen konnten wir schlussendlich zwei Beispiele für nicht relativistische Manigfaltigkeiten gewinnen die Supersymmtrie erlauben. Das erste Beispiel ist eine integrierbare Blätterung mit räumlichen Blättern die isomorph zur Poincaré Scheibe ist. Das zweite Beispiel entspricht eine nicht integrierbaren Blätterung und führt zu einer Raumzeit ohne Kausalstruktur. Im Gegensatz zu einer vorhergehenden Arbeit mit der alten minimalen Formulierung konnten wir zeigen, dass in der neuen minimalen Formulierung mehr Supersymmetrie erhalten werden kann.

Abstract

In this work we construct curved Newton-Cartan backgrounds for non relativistic supersymmetric field theories in three dimensions. This is done starting from the off shell new minimal supergravity multiplet in four dimensions. The non relativistic field theory in three dimensions can then be obtained via a Null reduction. We reduced the Lagrangian and transformation rules of a chiral multiplet and the corresponding Killing spinor equations. From these Killing spinor equations we obained conditions on the background fields such that global, nowhere vanishing solutions exist. From these conditions we constructed two explicit examples as background manifolds that preserve some supersymmetry. One of these examples is an integrable foliation with spatial slices that are isomorphic to the Poincaré disk. The other example is a non integrable foliation which leads to a non causal spacetime. It is important to note that in contrast to previous work done on the old minimal multiplet, the new minimal multiplet can preserve more supersymmetry on these non relativistic backgrounds.

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Chapter 1

Introduction

Our human intuition often fails when probing the universe on a fundamental level with the first indicator for this phenomenon being the discovery of Quantum mechanics at the beginning of the 20th century. Suddenly particles were found to exhibit wave behaviour leading to the discovery of unintuitive effects. This was a paradigm shift and the beginning of modern physics. It started a lot of new developments as it seemed that fundamental physics works very differently from our experience of physics in daily life.

The next paradigm shift happened when Einstein proposed his theory of Special Relativity in 1905 [1]. It rejected the idea of an absolute time and instead proposed that time is relative and can be different for every observer in the universe. This seems very unnatural from our daily experience because its major implications are only dominant at high speeds near the speed of light c. We can still mostly use Galilean Relativity in our daily lives. Special Relativity led Einstein to reconsider Gravity, ultimately leading to the discovery of General Relativity (GR) in 1915 [2]. General Relativity replaced the theory of Newtonian Gravity and again introduced a new view on space and time. Space and time were combined into a generically curved spacetime. Furthermore this spacetime is not static as the underlying space in Newtonian Gravity, but it is dynamic itself. The curvature of spacetime is under constant change and it dictates how planets and galaxies move in our universe. However, it is very important to emphasize that the prevailing theories should not be considered wrong. They just describe physics in a limited regime, e.g. Galilean Relativity describes the movement of particles as long as they are moving slowly compared to the speed of light. This observation gave rise to the so-called correspondence principle by Niels Bohr [3]. It states that the old theory should be contained in the new theory when taking a certain limit. New physical theories always have to meet this criterion. The principle suggests that a geometric formulation of Newtonian Gravity can be recovered from General Relativity, which is indeed possible.

Following this principle one should be able to recover Newtonian Gravity from General Relativity, which is indeed possible as we will see later in this thesis.

However there are also limitations on General Relativity as it fails to describe processes that happen at length scales of the order of the Planck length $l_P \approx 1.6 \times 10^{-35}$ m, where quantum effects are expected to become important for gravitational interactions. One of the goals in theoretical physics is to extend the regime of validity of physical theories. This is what happened when Quantum mechanics was combined with Special Relativity to give rise to Quantum Field Theory (QFT). QFT describes phenomena in a subatomic regime that deal with relativistic speeds. The next logical step in theoretical physics would be to obtain a consistent theory of Quantum Gravity, which combines General Relativity and Quantum Field Theory and is also valid at the Planck length l_P. This problem turned out to be far more difficult than previously thought. There are many ideas to tackle this problem; one of them is String Theory which describes all particles via different modes of tiny vibrating strings. The great thing about String Theory is that it seems to be able to reproduce General Relativity in a classical low energy limit. But it relies on two important concepts - supersymmetry and extra dimensions - both of which have not yet been observed in experiments. Extra dimensions are quite unintuitive but easy to explain. To define (Super)string theory in a consistent way it needs six extra dimensions. These dimensions are often assumed to be compact and of very small size such that they can not be probed by our experiments. Supersymmetry on the other hand proposes a new symmetry between fermions - the components of matter - and bosons, which mediate the fundamental forces between matter. This extra symmetry reduces a lot of complications that arise* when trying to build a quantum version of General Relativity. We will introduce the concept of supersymmetry in more detail in chapter 4. The correspondence principle then tells us that String theory in its low energy limit reduces to a supersymmetric extension of General Relativity, this is supergravity which we will introduce in chapter 4.

In this thesis we will try to construct supersymmetric non relativistic field theories on curved spacetimes. An important tool that we will employ is Newton-Cartan geometry. Newton-Cartan theory is a diffeomorphism invariant formulation of Newtonian Gravity. This gives a similar framework similar to General Relativity, i.e. a geometric formulation of Newtonian Gravity.

Before we elaborate on what is meant by supersymmetric backgrounds one may ask the question why we go back and study theories like Newtonian Gravity or Newton-Cartan Gravity when we already have General Relativity. There are many reasons why it is interesting and also important, but one of them is that General Relativity is much more complicated than Newtonian Gravity. Many calculations are a lot easier in Newtonian Gravity and still give useful insights.

*These complications are usually infinite values that arise when calculating certain processes, e.g. graviton scattering.

Another reason is the so-called AdS/CFT correspondence [4, 5] which links gravitational theories on Anti-de Sitter (AdS) spacetimes to Conformal Field Theories (CFT). This is a strong/weak duality in the sense that a strongly coupled QFT corresponds to a weakly coupled gravitational theory and vice-versa. This fuelled the hope that strongly coupled theories, in which it is usually hard to do concrete calculations, could be described by weakly coupled ones where those calculations are a lot easier. This correspondence also plays a role in non relativistic physics as there is a non relativistic analog to the AdS/CFT correspondence [9]. There have been attempts to apply it to certain condensed matter systems [6, 7]. In these examples the non relativistic limit was mainly taken on the field theory side, but it is also very interesting to take this limit on the gravitational side which leads to Newton-Cartan geometries [8]. Due to the often non relativistic nature of condensed matter systems there are many applications of Newton-Cartan geometry. Apart from the AdS/CFT correspondence there are more examples like the Quantum Hall Effect [10, 11] or thermal transport [12].

1.1 Supersymmetry on curved spaces

A very important tool in Quantum mechanics and especially in Quantum Field Theory is the concept of a path integral. The path integral formalism was first introduced in the context of Quantum mechanics by Richard Feynman [13]. The path integral generalizes the action principle of classical mechanics. The classical trajectory is replaced with an integral over all possible paths weighted with $\mathrm{e}^{\mathrm{i}\mathcal{S}}$ where $\mathcal{S}$ is the action of the system of consideration. The same concept can be generalized to field theories where the integral is taken over all possible field configurations. It contains all information of a field theory and can in principle be used to obtain correlation functions which in turn allows one to calculate any processes in the corresponding field theory. However, it is hard to define the path integral in a mathematical rigorous way and it is often very hard to compute. The main approach in Quantum Field Theory is to use perturbative methods which work very well for weakly coupled theories, although even there the perturbation series have zero radius of convergence. However, this procedure fails in strongly coupled theories like Quantum chromodynamics, where a perturbative series has not much predictive power. There is therefore an ongoing effort to improve existing, or find new methods, to get better results for strongly coupled theories. One of these approaches is to use the aforementioned AdS/CFT correspondence, another one is to use localisation techniques* which can sometimes be used to solve toy model QFTs exactly. Both applications rely on curved supersymmetric backgrounds, so there is a lot of interest to further study such backgrounds.

*These localisation techniques rely on the stationary phase approximation where one can approximate an integral using fixed points under a certain symmetry group. Under certain circumstances this even yields exact results. The idea of localisation was first discussed in a paper by Duistermaat and Heckman [14]. For a pedagogical introduction to localisation techniques we refer to [15]. Furthermore these two approaches are actually related to each other, respectively localisation techniques are often used in the AdS/CFT context to verify such a correspondence in a certain regime, as localisation allows one to compute explicit path integrals.

We already mentioned supersymmetry as an extra symmetry between fermions and bosons. Supersymmetric field theories are ordinarily considered on Minkowski spacetime, i.e. on flat space. In the context of the AdS/CFT correspondence the question arises if we can replace Minkowski spacetime with a general curved manifold $\mathcal{M}$. The prevailing method until a few years ago was mainly trial and error, until a new approach was developed by Festuccia and Seiberg [16]. This method is easier to apply and also leads to more general results. We will discuss this in detail in chapter 5. The resulting theories are then coupled to a curved background which can be used in supersymmetric localisation techniques mentioned above.

The main goal of this work is to combine the notions of non relativistic Newton-Cartan theory and curved supersymmetric spaces. We will construct a non relativistic supersymmetric field theory in three dimensions using a special kind of dimensional reduction. After that we can determine what manifolds can be used as backgrounds for these theories. As we will see these examples will have some similarities to backgrounds in four dimensional relativistic theories, but there are also some unique features to non relativistic field theories that can not be found in their relativistic counterparts.

1.2 Outline

This thesis is structured as follows. In chapter 2 important concepts of spacetime and how these are implemented in the relativistic and non relativistic case are summarized. This includes a short overview of General Relativity and the Vielbein formalism which is essential when working with fermions in gravitational theories. Chapter 3 will cover the concept of dimensional reduction and show the connection between General Relativity and Newton Cartan Gravity. The fundamentals of Supersymmetry and supergravity will be introduced in chapter 4. Chapter 5 will introduce the already mentioned method by Festuccia and Seiberg to obtain supersymmetric theories on curved manifolds. We then combine all these foundations in chapter 6 where we take this Festuccia-Seiberg limit on a relativistic supergravity theory and perform a special kind of dimensional reduction to obtain the Lagrangian and transformation rules of a non relativistic supersymmetric field theory on a curved background. We will solve the corresponding Killing spinor equations and give examples of possible backgrounds.

Chapter 2

Spacetime geometry

In this chapter we will highlight some differences between relativistic and non relativistic theories and their corresponding geometry of space and time. We will see the transition from non relativistic Newtonian Gravity to the relativistic theory of General Relativity. We then introduce important physical and technical concepts for formulating physical theories on curved spaces.

2.1 Newtonian Gravity

In Newtonian Gravity the magnitude of the force between two particles with respective masses m and M at a distance r in D dimensions is given by

$$F(r) = \frac{GMm}{r^{D-2}}, \tag{2.1}$$

where G is Newton's constant for gravity. Note that this equation is time-independent, i.e. gravity propagates instantaneously in Newtonian physics. Together with Newton's second law we can write down an equation for the radial acceleration of a particle with mass m in a gravitational field of a mass M:

$$\ddot{r}(t) = \frac{GM}{r^{D-2}} \equiv -\frac{\partial \phi}{\partial r}, \tag{2.2}$$

where $\phi(r)$ is the Newtonian potential. Surprisingly the mass m of the particle cancels out*. This is the case because inertial and gravitational mass were found to be equal in several experiments. The Newtonian potential $\phi(r)$ satisfies the Poisson equation, which reads in Cartesian coordinates

$$\Delta\phi(x) = S_{D-2} G \rho(x). \tag{2.3}$$

Here S_D is the volume of the D dimensional unit sphere and $\rho(x)$ is the mass density. Equation (2.3) is invariant under the Galilei group† and the following transformations:

$$x'^i = x^i + \xi^i(t), \quad \phi'(x') = \phi(x) - \delta_{ij}\ddot{\xi}^i x^j. \tag{2.4}$$

*This is best seen in the well known experiment where a feather and a steel sphere take the same time to fall in a vacuum tube, i.e. the fall time is independent of the object's mass.

†For a detailed group theoretical treatment of the Galilei group and its algebra, see [17]

By going to an accelerated frame, gravity appears to vanish. This is known as the equivalence principle which Einstein used to build General Relativity.

2.2 General Relativity

General Relativity as it stands is one of the most successful theories in physics. First described by Einstein in 1915 [2], it has been confirmed by several experimental tests and is an important pillar of modern physics. We will give a short overview of General Relativity here to lay the foundation for coupling fermions to gravity. Of course this does not capture the whole extent of this theory, but it is enough for our purpose here. For a more in depth treatment, we refer to [18, 19, 20].

The foundation of General Relativity is the equivalence principle. It states that locally, a free falling particle in a gravitational field behaves the same as in an uniformly accelerated frame. 'Free falling' in this case means that there are no other forces acting on the particle. It follows that any observer can always accelerate and then use the laws of Special Relativity as if there was no gravitational field. In mathematical terms this boils down to the fact that one can always find a general coordiante transformation which locally gives the Minkowski metric η_{MN}, while globally the curvature is described by a dynamical metric $g_{MN}(x)$.

In General Relativity, the metric transforms under general coordinate transformations as

$$g'_{MN}(x') = \frac{\partial x^P}{\partial x'^M}\frac{\partial x^R}{\partial x'^N} g_{PR}(x) \tag{2.5}$$

which is often described by 'the metric transforms in a covariant way'. (2.5) is just a re-expression of the metric tensor in another set of coordinates. This is a very important property as we want physical laws to be independent of coordinates. For a dynamical theory one also needs the derivative of the metric. The only problem is that ∂_M does not transform covariantly. As usual in gauge theories one defines a covariant derivative ∇_M which also gives a covariantly transforming object:

$$\nabla_M T^{N\ldots}_{P\ldots} = \partial_M T^{N\ldots}_{P\ldots} + \Gamma^N_{ML} T^{L\ldots}_{P\ldots} - \Gamma^L_{MP} T^{N\ldots}_{L\ldots} \pm \ldots \tag{2.6}$$

The connection symbols Γ^R_{MN} themselves do not transform covariantly, they get an extra contribution to cancel the inhomogeneous terms of the partial derivative. In a way, the connection symbols are the analog of a Yang-Mills potential in a non-abelian gauge theory. Indeed there are some general similarities to Yang-Mills theory when interpreting GR as a gauge theory of the Poincaré group [21, 22].

The connection in General Relativity is constrained by metric compatibility $\nabla_R g_{MN} = 0$ and vanishing torsion $\Gamma^R_{[MN]} = 0$. These two constraints give rise to a unique connection which is called the Levi-Civita connection and is given by

$$\Gamma^R_{MN} = \frac{1}{2} g^{RL} \left(\partial_M g_{LN} + \partial_N g_{LM} - \partial_L g_{MN} \right), \tag{2.7}$$

where the Γ^R_{MN} are called Christoffel symbols. The uniqueness of the Levi-Civita connection implies that all the geometric information of the spacetime manifold is contained in the metric. While the partial derivatives commute, this does not hold for covariant derivatives and we get

$$\left[\nabla_R, \nabla_S \right] V^M = R^M{}_{NRS}(\Gamma) V^N \tag{2.8}$$

for any vector V^N and where we defined the Riemann curvature tensor in terms of Christoffel symbols as

$$R^M{}_{NRS}(\Gamma) = \partial_R \Gamma^M_{NS} - \partial_S \Gamma^M_{NR} + \Gamma^L_{NS} \Gamma^M_{LR} - \Gamma^L_{NR} \Gamma^M_{LS}. \tag{2.9}$$

With help of the metric we can take traces and define the Ricci tensor $R_{MN}(\Gamma)$ and the Ricci scalar $R(\Gamma)$

$$R_{MN}(\Gamma) = R^R{}_{MRN}(\Gamma), \quad R(\Gamma) = g^{MN} R_{MN}(\Gamma). \tag{2.10}$$

Now that we have all the necessary ingredients we can go on to formulate Einstein's field equations. We want an equation that is of the form $G = \kappa^2 T$ where G is dependend on the geometry of spacetime, T describes the energy-matter content of the universe and κ is just a constant. This equation then relates the geometry of spacetime (left side) to its energy-matter content (right side). T must be a symmetric tensor with vanishing divergence $\nabla_M T^{MN} = 0$ to satisfy energy-momentum conservation. Of course these constraints must then also be fulfilled by G_{MN}*. When looking for a symmetric second rank tensor with vanishing divergence that is linear in the Riemann tensor, the only possibility left is the Einstein tensor $G_{MN} = R_{MN}(\Gamma) - \frac{1}{2} R(\Gamma) g_{MN}$. The most general equations obeying energy-momentum conservation and are at most second order in derivatives of g_{MN} are the Einstein equations

$$R_{MN}(\Gamma) - \tfrac{1}{2} R(\Gamma) g_{MN} + \Lambda g_{MN} = \kappa^2 T_{MN}, \tag{2.11}$$

where Λ is the cosmological constant. The constant κ is determined by the correspondence principle, i.e. taking the Newtonian limit of (2.11) and comparing to the Poisson equation (2.3). We will now go on to reformulate GR in a different way with the goal of coupling fermions to the theory.

2.3 The Vielbein formalism

In the usual description of General Relativity, the fundamental field is the spacetime metric g_{MN}. In this section we will take a different approach to the geometry of GR which has some very interesting features. Essentially, we will describe spacetime not with the metric, but

*Actually, G_{MN} is automatically conserved and this then implies the conservation of T_{MN}. This is a consequence of the geometrical nature of spacetime. See [18] for more details.

instead use another fundamental field, the Vielbein. This was originally introduced by Élie Cartan in 1923 [23]. We will mostly follow [24] and [25] in this section.

We will now make use of the fact that the metric tensor can be diagonalized. This means that there is an orthogonal transformation that transforms the metric into diagonal form. At a given spacetime point x we can write

$$g_{MN}(x) = E_M{}^A(x)\eta_{AB}E_N{}^B(x) \tag{2.12}$$

where used the fact that the metric tensor can be diagonalized. In this equation we can also see why the Vielbein $E_M{}^A$ is sometimes called the 'square-root of the metric', which is of course not to be taken literally.

The Vielbein (also often called frame field or tetrad) we used in (2.12) can be defined as the set of one forms

$$E^A = E_M{}^A(x)\mathrm{d}x^M, \tag{2.13}$$

that span an orthonormal basis of the cotangent space at each point x of the spacetime manifold. We can then define the dual of (2.13) as

$$E_A = E^M{}_A(x)\partial_M \tag{2.14}$$

which are four independent vector fields spanning an orthonormal basis of the tangent space at each point x in spacetime. These obey the relation $E^M{}_A(x)E_M{}^B(x) = \delta_A^B$. We will also adopt some slight abuse of notation here and call the coefficients $E_M{}^A$ Vielbein (instead of the objects E^A) as it is usually done in the physics literature. $E^M{}_A$ is then often called the inverse Vielbein.

Now we specified a tangent basis in every spacetime point of our manifold. Generally, this can only be done locally. Spacetime is a Lorentzian manifold and by choosing these local frames we can always locally get a Minkowskian structure. This expresses the foundation of General Relativity - the equivalence principle. In a way, this property emerges more naturally in this formalism than in the standard metric formalism of GR.

However, our choice of these orthogonal transformations is not unique, but we can always take another solution like

$$E'_M{}^A(x) = \Lambda^A{}_B(x)E_M{}^B(x), \tag{2.15}$$

where $\Lambda^A{}_B(x)$ is a spacetime-dependent matrix that leaves η_{AB} invariant, i.e. a local Lorentz transformation. This is very important because the unconstrained Vielbein has D^2 components while the metric tensor has just $\frac{1}{2}D(D+1)$ components, as it is a symmetric two-tensor. Thus we get a difference of $D^2 - \frac{1}{2}D(D+1) = \frac{1}{2}D(D-1)$ components. Relation (2.15) explains this difference as the local Lorentz transformations $\Lambda^A{}_B$ are elements of $SO(1, D-1)$

and as such have exactly $\frac{1}{2}D(D-1)$ components. Naively we introduced more variables using Vielbein formulation but due to (2.15) it actually contains the same number as the metric formulation.

Only the flat tangent space indices $A, B, \ldots$ (capital latin letters from the beginning of the alphabet) transform under these local Lorentz transformations while the spacetime indices and the spacetime coordinates do not change. The spacetime indices $M, N, \ldots$ (capital latin letters from the middle of the alphabet) only transform under diffeomorphisms as

$$E'_M{}^A(x') = \frac{\partial x^N}{\partial x'^M} E_N{}^A(x), \tag{2.16}$$

while the frame indices do not transform. Spacetime indices can be raised and lowered by the spacetime metric g_{MN}, while tangent indices can be raised and lowered by the flat Minkoswki metric η_{AB}. We can also define the determinant $|E| \equiv \det E_M{}^A = \sqrt{-\det g}$, since the Vielbein is a non-singular square matrix.

Within this formalism one can express any vector field V as

$$V^M(x) = V^A(x) E^M{}_A(x) \quad \text{with} \quad V^A(x) = V^M(x) E_M{}^A(x). \tag{2.17}$$

The same can of course be done for one-forms as

$$\omega_M(x) = \omega_A(x) E_M{}^A(x) \quad \text{with} \quad \omega_A(x) = \omega_M(x) E^M{}_A(x). \tag{2.18}$$

This can be generalized to any higher rank tensors which can also be of mixed form. Invariants like the inner product can be calculated in either basis and lead to the same result.

A coordinate basis ∂_M is holonomic because the commutator $[\partial_M, \partial_N]$ vanishes, i.e. partial derivatives commute. However this is not the case for the local Lorentz basis. From (2.14) we can see that the local Lorentz basis is indeed not a coordinate basis as the elements do not commute in general and

$$[E_A, E_B] = C_{AB}{}^C E_C, \tag{2.19}$$

where $C_{AB}{}^C$ are the anholonomy coefficients

$$C_{AB}{}^C = E^M{}_A E^N{}_B (\partial_N E_M{}^C - \partial_M E_N{}^C) \quad \text{or} \quad C_{MN}{}^A = 2\partial_{[N} E_{M]}{}^A. \tag{2.20}$$

We can always choose the Vielbein as the fundamental field over the metric. It is not only convenient for some purposes, e.g. calculating curvatures is far more convenient in the Vielbein formalism, but necessary if we want to couple fermions to gravity. Fermions are represented by spinors which are defined by their very specific transformation properties under Lorentz transformations. Thus one needs local Lorentz frames to describe fermions on a curved spacetime.

2.4 Cartan structure equations

Usually the affine connection is written in terms of a coordinate basis with the connection symbols $\Gamma^R_{MN}(x)$ as in (2.7). In supergravity we are required to work with the Vielbein $E_M{}^A$, so we need to introduce a connection again, but this time on the tangent space. This will be the spin connection one form $\Omega^{AB} = \Omega_M{}^{AB}(x)\mathrm{d}x^M$, which we will require to be antisymmetric in its Lorentz indices $\Omega_{AB} = -\Omega_{BA}$. We could also use a more general connection without this imposed constraint but it will turn out that this is exactly what we want for a gravitational theory including fermions. The antisymmetry of the connection ensures that it is an element of the Lie algebra of the Lorentz group $SO(1, D-1)$. The reason Ω_{AB} is called the spin connection comes from the fact that it is needed to describe fermions on curved manifolds.

We introduced the Vielbein as a helpful object to reformulate GR with another fundamental variable. But it is also a one form and we can take a look at the exterior derivative $\mathrm{d}E^A = (\partial_M E_N{}^A)\mathrm{d}x^M \wedge \mathrm{d}x^N$. Note that due to (2.15) the two form $\mathrm{d}E^A$ transforms like

$$\mathrm{d}E'^A = \Lambda^A{}_B \mathrm{d}E^B + \mathrm{d}\Lambda^A{}_B E^B, \tag{2.21}$$

i.e. it does not transform like a local Lorentz tensor due to the second term. To keep Lorentz symmetry in our theory, we want to introduce a Lorentz covariant derivative. To achieve this we add a two form involving the spin connection and consider

$$\mathrm{d}E^A + \Omega^A{}_B \wedge E^B \equiv DE^A \equiv T^A. \tag{2.22}$$

To compensate the unwanted term we get in (2.21) we have to define the Lorentz transformation of $\Omega^A{}_B$ as

$$\Omega'^A{}_B = \Lambda^A{}_C \Omega^C{}_D (\Lambda^{-1})^D{}_B + \Lambda^A{}_C \mathrm{d}(\Lambda^{-1})^C{}_B. \tag{2.23}$$

The resulting two form T^A defined in (2.22) then transforms nicely under Lorentz transformations as $T'^A = \Lambda^A{}_B T^B$. We introduced two new objects in (2.22), the spin connection one form Ω_{AB} and the two form T^A, which is called the torsion two form of the connection Ω_{AB}.

Let us take a look at the transformation properties of the connection components $\Omega_M{}^{AB}(x)$. They transform as a covariant vector under diffeomorphisms and due to (2.23) we have

$$\Omega'_M{}^A{}_B = \Lambda^A{}_C \Omega_M{}^C{}_D (\Lambda^{-1})^D{}_B + \Lambda^A{}_C \partial_M (\Lambda^{-1})^C{}_B. \tag{2.24}$$

This looks very familiar, indeed it corresponds to the transformation behaviour of a Yang-Mills potential in non-abelian Yang-Mills theory. The spin connection $\Omega_M{}^{AB}$ can then be interpreted as the potential of the gauge group $SO(1, D-1)$. It immediately follows that we can also construct a corresponding Yang-Mills field strength in the usual way

$$R^A{}_B = \mathrm{d}\Omega^A{}_B + \Omega^A{}_C \wedge \Omega^C{}_B. \tag{2.25}$$

This equation is called the second Cartan structure equation, while (2.22) is called the first Cartan structure equation. It can be easily seen in a calculation that (2.25) transforms as a local Lorentz tensor, i.e. $R'^A{}_B = \Lambda^A{}_C R^C{}_D (\Lambda^{-1})^D{}_B$.

In (2.22) we defined the object DE^A because we want the derivative of E^A to also behave as a Lorentz tensor. The operator D we introduced is often called the exterior covariant derivative and it acts on Lorentz vector valued p forms as

$$D = d + \Omega \wedge, \tag{2.26}$$

where we suppressed all indices. We can now act with this exterior covariant derivative on both Cartan structure equations (2.22), (2.25) and get (see appendix in [26] for a derivation)

$$DT^A = R^A{}_B \wedge E^B \tag{2.27a}$$

$$DR^A{}_B = 0. \tag{2.27b}$$

These are the first and second Bianchi identities respectively, written in the language of differential forms.

We also want a more general version of (2.26) that is able to act not just on Lorentz vector valued p forms and return a Lorentz vector valued $(p+1)$ form but one that maps Lorentz tensors into Lorentz tensors. To do this we take a look at the transformation behaviour of vector fields

$$V'^A(x) = \Lambda^A{}_B(x) V^B(x) \tag{2.28a}$$

$$U'_A(x) = (\Lambda^{-1})^B{}_A(x) U_B(x). \tag{2.28b}$$

The extension to general (p, q) Lorentz tensors is straightforward. With the same argument as in the previous section, where we introduced the spin connection, we will also use the local Lorentz transformation rules to introduce a Lorentz covariant derivative as

$$D_M V^A = \partial_M V^A + \Omega_M{}^A{}_B V^B \tag{2.29a}$$

$$D_M U_A = \partial_M U_A - \Omega_M{}^B{}_A U_B = \partial_M U_A + \Omega_{MA}{}^B U_B. \tag{2.29b}$$

The extension to general (p, q) Lorentz tensors is once again straightforward. We can now apply this Lorentz covariant derivative to any tensor, for example the metric tensor:

$$D_M \eta_{AB} = \partial_M \eta_{AB} - \Omega_M{}^C{}_A \eta_{CB} - \Omega_M{}^C{}_B \eta_{AC} = -\Omega_{MBA} - \Omega_{MAB} = 0. \tag{2.30}$$

The metric is an invariant tensor of the Lie algebra of the Lorentz group, hence it is not surprising that the derivative vanishes, since Ω_{AB} is an element of the Lorentz group. Now we have a Lorentz covariant derivative but it is still not clear how D_M acts on spacetime tensors. The transition is not too difficult as we can always turn a spacetime tensor into a tangent space tensor by using the Vielbein. Thus it is sufficient to know how a full* covariant derivative acts on the Vielbein. It does so as

$$\nabla_M E_N{}^A = \partial_M E_N{}^A + \Omega_M{}^{AB} E_{NB} - \Gamma^R_{MN} E_R{}^A. \tag{2.31}$$

Similar to metric compatibility we impose the Vielbein postulate

$$\nabla_M E_N{}^A = 0. \tag{2.32}$$

From this we can solve for the Levi-Civita connection Γ and get

$$\Gamma^R_{MN} = E^R{}_A(\partial_M E_N{}^A + \Omega_M{}^{AB} E_{NB}), \tag{2.33}$$

where Γ now only depends on the spin connection and the Vielbein. As we will see the spin connection can also be expressed in terms of the Vielbein, that means we can write $\Gamma = \Gamma(E)$ similar to what we can do in the metric formulation where the Christoffel symbols can be written in terms of the metric and its first order derivative.

The first Cartan structure equation (2.22) is very important for geometry as it gives rise to the Lorentz vector valued torsion two form T^A. The components of the torsion two form is the torsion tensor $T_{MN}{}^A = -T_{NM}{}^A$ which is antisymmetric in its spacetime indices. Usually, when dealing with gravitational theories, one chooses a torsion-free and metric compatible connection - the Levi-Civita connection. If we do so, the first Cartan structure equation reads $\mathrm{d}E^A + \Omega^A{}_B \wedge E^B = 0$. We will see in this thesis that the torsion freeness has to be loosened when dealing with supergravity theories because torsion can arise by coupling gravity to matter. We will see how this works when introducing the Palatini formalism later.

2.4.1 Spin connection

Our goal now is to express the spin connection in terms of just the Vielbein and the torsion. To achieve this let us take a look at the tensor components of (2.22). We get

$$T_{MN}{}^A = C_{MN}{}^A + 2\Omega_{[M}{}^{AB} E_{N]B}, \tag{2.34}$$

where the torsion tensor and the anholonomy coefficients are antisymmetric in the first two indices, while the spin connection is antisymmetric in its last two indices. When taking the permutation $T_{MNP} - T_{NPM} + T_{PMN}$ we can get an expression for the spin connection in terms of the torsion and the anholonomy coefficients (which in turn just contain derivatives of the Vielbein)

$$\Omega_M{}^{AB}(E,T) = \Omega_M{}^{AB}(E) + K_M{}^{AB}(T), \tag{2.35}$$

*By full we mean a derivative that acts in a covariant way on all the indices, i.e. spacetime indices and tangent space indices. We will use ∇ for this operation.

where $\Omega_M{}^{AB}(E)$ is the unique torsion-free spin connection and $K_M{}^{AB}$ is called the contorsion tensor. These are given by

$$\begin{aligned}\Omega_M{}^{AB}(E) &= \tfrac{1}{2}E^{NA}E^{RB}(C_{MNR} - C_{NRM} + C_{RMN}) \\ &= 2E^{N[A}\partial_{[N}E_{M]}{}^{B]} - E^{PA}E^{NB}E_{MC}\partial_{[N}E_{P]}{}^{C},\end{aligned} \tag{2.36}$$

where we just substituted the explicit expression for the anholonomy symbols from (2.19), and

$$K_M{}^{AB} = -\tfrac{1}{2}E^{NA}E^{PB}(T_{MNP} - T_{NPM} + T_{PMN}). \tag{2.37}$$

2.5 Palatini or first order formulation of GR

There are two routes we can take when describing General Relativity in the Vielbein formalism. As we have seen in the previous section, we have two fields $\{E, \Omega\}^*$ to describe the geometric information on a manifold. The more familiar framework is the so-called second order formalism of GR. Here we describe all dynamical properties of gravity by the metric or the Vielbein and the action is

$$S_{EH}(\Omega(E)) = \frac{1}{2\kappa^2}\int |E|E^M{}_A E^N{}_B R_{MN}{}^{AB}(\Omega(E))\,\mathrm{d}^D x, \tag{2.38}$$

where $|E| = \det E_M{}^A$ and the Riemann curvature tensor is expressed as

$$R_{MN}{}^{AB}(\Omega(E)) = 2\partial_{[M}\Omega_{N]}{}^{AB}(E) + 2\Omega_{[M}{}^{AC}(E)\Omega_{N]C}{}^{B}(E), \tag{2.39}$$

which is just (2.25) written in tensor components with the spin connection $\Omega(E)$ expressed in terms of the Vielbein. When varying the action (2.38) with respect to $E_M{}^A$, we get the vacuum Einstein equations

$$R_{MN}{}^{AB}(\Omega(E))E^N{}_B - \frac{1}{2}R(\Omega(E))E_M{}^A = 0 \tag{2.40}$$

in the Vielbein formalism. The name second order formalism stems from the fact that we end up with field equations which are second order in derivatives of $E_M{}^A$. In this formulation we work with a torsion free spin connection.

The other option is the first order or Palatini formalism. In this formalism we treat E and Ω as independent fields and vary the action with respect to both variables. The action is then written as

$$S_{EH}(\Omega, E) = \frac{1}{2\kappa^2}\int |E|E^M{}_A E^N{}_B R_{MN}{}^{AB}(\Omega)\,\mathrm{d}^D x. \tag{2.41}$$

*For a fixed Vielbein one can choose either $\{E, \Omega\}$ or $\{E, T\}$ as independent fields since Ω and T are related by a linear equation. Both formulations are equivalent but we will stick to $\{E, \Omega\}$ in this work as it is usually done in literature.

This leads to two first order field equations for E and Ω. Varying with respect to $E_M{}^A$ again yields the vacuum Einstein equations

$$R_{MN}{}^{AB}(\Omega)E^N{}_B - \frac{1}{2}R(\Omega)E_M{}^A = 0, \tag{2.42}$$

but here the Riemann tensor just depends on an independent spin connection which is not related to $E_M{}^A$. There is a second set of equations of motion obtained by varying with respect to $\Omega_M{}^{AB}$:

$$\partial_{[M}E_{N]}{}^A + \Omega_{[M}{}^{AB}E_{N]B} = 0. \tag{2.43}$$

Note that this is the first Cartan structure equation (2.22) with zero torsion written in components. Hence for GR without matter, we end up with the torsion free spin connection and can conclude that $S_{EH}(\Omega(E)) = S_{EH}(\Omega, E)$. However, when coupled to fermionic matter, equation (2.35) contains a contorsion contribution. This contorsion then gives quartic fermion terms. If one then solves (2.43) for the spin connection and uses the result in the other field equations it turns out that the same result could have also been obtained with quartic fermion terms in the action.

As we have seen, we have two approaches when coupling fermions to GR. Either we deal with quartic fermion terms in the action directly, or we organize them as contorsion contributions in the spin connection. Then why do we even bother to give these two formulations? The answer is that it is often convenient in supergravity calculations, e.g. for proving invariance under local supersymmetry transformations, when the gravitino terms are in the spin connection. For more details see [24].

2.6 Newton-Cartan Gravity

The goal of Newton-Cartan Gravity (NC) is a geometrical reformulation of the classical theory of Newtonian Gravity. To achieve this, we have to abandon the ideas of Special Relativity of an observer dependent time and go back to an absolute time t that is the same for every observer. This leads to separation of space and time. We will see how to geometrically realize these aspects in terms of the metric and the notion of spacetime. Another aspect to take into account is separate conservation of mass and energy. We will see how this is realized when talking about the Bargmann algebra.

In Newtonian Gravity, the equation of motion for a non relativistic test particle in flat Euclidean space $\mathbb{R}^D$ reads

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} + \frac{\partial\phi}{\partial x^i} = 0, \tag{2.44}$$

where ϕ is the gravitational potential and t is the absolute time, which is the same for every observer. The solutions to this equation are usually interpreted as curved trajectories in Euclidean space. Cartan was the first to change this point of view on Newtonian Gravity

[23]. Reflecting the geometrical ideas of General Relativity, he asked if one could also interpret (2.44) as geodesics in a curved spacetime. Indeed, if we parametrize the absolute time with a general parameter $\lambda = at + b$, we can rewrite (2.44) as [18]

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}\lambda^2} + \frac{\partial \phi}{\partial x^i}\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 = 0. \tag{2.45}$$

By comparing this equation to the geodesic equation

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu_{\nu\rho}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda} = 0 \tag{2.46}$$

we can read of the connection coefficients

$$\Gamma^i_{00} = \delta^{ij}\partial_j\phi, \quad \text{all other} \quad \Gamma^\mu_{\nu\rho} = 0, \tag{2.47}$$

where we used the Euclidean metric, $\mu = 0, i$ and $x^0 = t$. From here we can calculate the Riemann tensor arising from this connection. Evaluating (2.9) we get

$$R^i{}_{0j0} = \delta^{ik}\partial_k\partial_j\phi, \quad \text{all other} \quad R^\rho{}_{\sigma\mu\nu} = 0. \tag{2.48}$$

From this, we can impose the equations of motion $R_{00} = 4\pi G\rho$ and get the Poisson equation. Up to now, we did not specify any metric structure, so this is just a reformulation of Newtonian Gravity but not a geometrical one.

One needs to determine the metric that gives rise to the Newtonian connection (2.47). This connection can not be derived from a non-degenerate metric, because it does not obey the usual symmetries of the Riemann tensor known from standard geometry. A better way to see this is to consider the usual Minkowski metric and its inverse

$$\eta_{\mu\nu}/c^2 = \begin{pmatrix} -1 & 0 \\ 0 & \mathbb{1}_{D-1}/c^2 \end{pmatrix}, \qquad \eta^{\mu\nu} = \begin{pmatrix} -1/c^2 & 0 \\ 0 & \mathbb{1}_{D-1} \end{pmatrix}. \tag{2.49}$$

When taking the limit $c \to \infty$, this leads to two degenerate metrics. One covariant temporal metric $\tau_{\mu\nu}$ with three zero eigenvalues and one contravariant spatial metric $h^{\mu\nu}$ with one zero eigenvalue. The Galileo group preserves both of these metrics separately and it seems that the degeneracy comes from the splitting of space and time. The degeneracy then directly implies

$$h^{\mu\nu}\tau_{\mu\rho} = 0. \tag{2.50}$$

From here we could treat Newtonian Gravity in a metric formalism being described by $(\tau_{\mu\nu}, h^{\mu\nu})$. Since we want to describe fermions on a Newton-Cartan background we will instead adopt the Vielbein formalism from this point on. A metric treatment can be found in [18] and [27].

Since $\tau_{\mu\nu}$ is essentially a rank 1 matrix, we can define a temporal Vielbein τ_μ which is given by $\tau_{\mu\nu} = \tau_\mu\tau_\nu$. From the orthogonality relation in (2.50) it then follows that

$$h^{\mu\nu}\tau_\nu = 0 \tag{2.51}$$

for the temporal Vielbein. In NC Gravity one often imposes

$$\tau_\mu = \partial_\mu t(x) \tag{2.52}$$

where $t(x)$ is a time function and is used to foliate spacetime. We will see later in section 3.2.1 how this is implemented and how this can be generalized leading to different notions of NC spacetimes. This is similar to the ADM formalism in General Relativity [28], where there are spatial hypersurfaces of constant time.

In addition to the temporal Vielbein τ_μ defined above, we will also introduce a spatial Vielbein ${e_\mu}^a$ which is defined by

$$h^{\mu\nu}{e_\mu}^a{e_\nu}^b = \delta^{ab}, \tag{2.53}$$

in the same spirit as we did for GR in (2.12). However the similarity is very deceiving here: In the NC case the spatial Vielbeine ${e_\mu}^a$ are defined as cotangent vectors to the spatial slices, while in GR they are defined as the cotangent vectors to the spacetime. We can also introduce inverse temporal τ^μ and spatial ${e^\mu}_a$ Vielbeine, which are defined through the relations

$$\begin{aligned} \tau^\mu\tau_\mu = 1, \quad \tau^\mu{e_\mu}^a = 0, \quad {e^\mu}_a\tau_\mu = 0, \\ {e_\mu}^a{e^\mu}_b = \delta^a_b, \quad {e_\mu}^a{e^\nu}_a = \delta^\nu_\mu - \tau_\mu\tau^\nu. \end{aligned} \tag{2.54}$$

Here we translated the orthogonality relation of (2.50) to the Vielbein fields. Geometrically ${e_\mu}^a{e^\nu}_a$ should be seen as a projector onto the spatial hypersurface, while $\tau^\mu\tau_\nu$ is a projector from spacetime to the temporal direction.

In the next section we will get a better understanding of Newton-Cartan Gravity when performing a Null reduction of General Relativity.

Chapter 3

Dimensional reduction

Dimensional reduction is often used to obtain unified theories in theoretical physics. The approach is to take a higher dimensional theory and compactify all 'unwanted' dimensions, e.g. take a five dimensional theory and compactify one dimension to end up with a four dimensional theory. This technique gained even more importance as Superstring Theories can only be consistently defined in ten dimensions. To make contact with our four dimensional universe one has to, in one way or another, do a dimensional reduction.

Here we will also use this technique extensively. In this chapter we will do a dimensional reduction of relativistic gravity to gain more insight in non relativistic gravity. Later in this thesis we will again use dimensional reduction to reduce relativistic supersymmetric field theories on curved backgrounds.

The idea of dimensional reduction is now almost one hundred years old. Kaluza and Klein were the first to consider a so-called compactification of an additional dimension [29, 30]. We will review this idea here and then go on and do a Null reduction of General Relativity.

3.1 Kaluza Klein reduction

The main idea of Kaluza Klein theories is that our spacetime may have extra dimensions that we can not see. The spacetime symmetries in these dimensions are then seen as internal (gauge) symmetries. This is a unified approach where the gauge symmetries are coming from extra dimensions. In this way, Kaluza and Klein were able to produce a unified four dimensional theory of gravity and a Maxwell and scalar field. It has also been used in a non relativistic setting in [31] where a Kaluza Klein reduction of Newton-Cartan Gravity was done. This gives Galilean electromagnetism and a scalar field in the lower dimensional theory.

To illustrate how this works in an easy example we will take a look at a massless scalar field. We start from a five dimensional theory on $\mathbb{R}^{1,3} \times S^1$ and denote the radius of the circle by R. This is the easiest example, it is however also possible to consider other manifolds for compactification, but we will not do so. We will split the coordinates into $M = (\mu, z)$, where z is the internal dimension on S^1. Usually the radius is taken to be very small (when talking about gravity the Planck scale seems to be a reasonable choice) as we can not observe it. The coordinate set is then $2\pi R$ periodic in the z coordinate

$$z = z + 2\pi R. \tag{3.1}$$

The scalar field $\phi(x, z)$ obeys the massless five dimensional Klein-Gordon equation

$$\Box_5 \phi(x, z) = 0, \tag{3.2}$$

where $\Box_D = \partial_M \partial^M$ is the D'Alembert operator in D dimensions. Since our compact manifold is a circle, we expand our field in Fourier modes $\phi(x, z) = \sum \exp(inz/R)\phi_n(x)$. When using this Ansatz in equation (3.2) each of the ϕ_n obey the following equation

$$(\Box_4 - n^2/R^2)\phi_n(x) = 0. \tag{3.3}$$

As we can see, we now have a 'tower' of n scalar fields ϕ_n with masses $m_n = n/R$, that are dependent on the radius of the compact dimension. Note that the smaller the radius R of the compact dimension becomes, the larger the mass m_n of the scalar ϕ_n gets. This is the reason why we need to choose a small radius R, as we have not yet observed such Kaluza Klein towers in collider experiments. It may however be the case that the first massive state is at such high energies that we are at the moment not able to detect it. Therefore, at low energies it suffices to look at the massless modes. In this example we can see that the dynamics in the extra dimension can have a big impact on physics in the visible dimensions.

Another example which gives a lot of insight is the original procedure of Kaluza and Klein. We start with a $(D + 1)$ dimensional metric* g_{MN} that is independent of the coordinate z, and so are the Vielbeine. This can be equivalently formulated by saying that the Vielbein has to admit a spacelike Killing vector ξ^M, i.e.

$$\mathcal{L}_\xi g_{MN} = 2\nabla_{(M}\xi_{N)} = 0. \tag{3.4}$$

In addition to this local constraint we also have to impose the global condition that the integral curves of the Killing vector ξ^M are closed. The hypersurfaces orthogonal to the Killing vector ξ^M are then the D dimensional spacetime of the lower dimensional theory. In a coordinate system where z is the internal coordinate we will choose $\xi = \xi^z \partial_z$ without loss of generality. In order to do the reduction it is convenient to choose an Ansatz for the Vielbein and its inverse which is given by [35]

$$E_M{}^A = \begin{array}{c c} & \begin{array}{cc} a & * \end{array} \\ \begin{array}{c} \mu \\ z \end{array} & \begin{pmatrix} e_\mu{}^a & \phi A_\mu \\ 0 & \phi \end{pmatrix} \end{array}, \quad E^M{}_A = \begin{array}{c c} & \begin{array}{cc} \mu & z \end{array} \\ \begin{array}{c} a \\ * \end{array} & \begin{pmatrix} e^\mu{}_a & -A_a \\ 0 & \phi^{-1} \end{pmatrix} \end{array}, \tag{3.5}$$

*Note that we differ from our convention defined in appendix A in this section as here capital letters represent five dimensional indices while lower case letters represent four dimensional indices.

where the $(3+1)$ dimensional field content is a Vielbein $e_\mu{}^a$, a massless gauge field A_μ and a massless scalar ϕ. We split the flat coordinates A into $(a,*)$ and we used local Lorentz invariance to set $E_z{}^a = 0$ after which $\lambda^a{}_* = 0$. This is how we end up with this triangular form (see [32] for more details on this). The diffeomorphism parameter also splits under these adapted coordinates into $\zeta^M = (\zeta^\mu, \zeta^z)$, where ζ^μ takes the role of parameter of diffeomeorphisms on the D dimensional hypersurfaces. We can now also state the transformation properties of the lower dimensional fields that transform with the parameters ζ^μ, ζ^z, and $\lambda^a{}_b$ as

$$\delta e_\mu{}^a = \mathcal{L}_\zeta e_\mu{}^a + \lambda^a{}_b e_\mu{}^b \tag{3.6a}$$

$$\delta A_\mu = \mathcal{L}_\zeta A_\mu + \partial_\mu \zeta^z \tag{3.6b}$$

$$\delta\phi = \mathcal{L}_\zeta \phi, \tag{3.6c}$$

where $\mathcal{L}$ denotes the Lie derivative with respect to ζ^μ. One can derive these by plugging the reduction Ansatz (3.5) into the higher dimensional transformation rules of the Vielbein (3.12). As we can see, all fields transform as spacetime tensors under lower dimensional diffeomorphisms and the Vielbein additionally does so as a local Lorentz tensor. The gauge field A_μ has an additional $U(1)$ gauge transformation whose parameter ζ^z corresponds to diffeomeorphisms in the internal dimension.

We could now go on and work out the Spin connection from (2.36) and the Riemann tensor from (2.39) for our Ansatz. However we will omit this here, as we will give the results for the Null reduction in the next section. Details can be found in [33]. All we do is state the reduced action, which can be obtained by plugging Ansatz (3.5) into the Einstein-Hilbert action (2.38):

$$S \sim \int \mathrm{d}^D x \sqrt{|g|}\, \phi[R - \frac{1}{4}\phi^2 F_{ab}F^{ab}]. \tag{3.7}$$

One can clearly see the structure of gravity and electromagnetism with the field strength F_{ab} in this action. Note that the scalar field seems to not have kinetic terms, however when deriving the Einstein equations of motion, one has to use partial integration which then leads to a wave equation for ϕ [33]. An even better way to see this is to do a Weyl rescaling of the metric in action (3.7), i.e. a local rescaling $g_{\mu\nu} = \Omega^2 \tilde{g}_{\mu\nu}$. By doing this rescaling in the action (3.7) one can rewrite the resulting action in the more usual form, consisting of the Einstein-Hilbert term plus matter contributions. In particular, after doing a Weyl rescaling with

$$\Omega = \phi^{-\frac{1}{D-2}}, \quad g_{\mu\nu} = \phi^{-\frac{2}{D-2}} \tilde{g}_{\mu\nu} \tag{3.8}$$

and computing the transformed Ricci scalar using the rescaled metric one gets the action

$$\tilde{S} \sim \int \mathrm{d}^D x \sqrt{|\tilde{g}|} \left[\tilde{R} + \frac{D-1}{D-2}\phi^{-2}(\partial\phi)^2 - \frac{1}{4}\phi^{2\frac{D-1}{D-2}} F_{ab}F^{ab} \right]. \tag{3.9}$$

Finally one can make the following redefinition

$$\phi = e^{\pm\sqrt{2\frac{D-1}{D-2}}\varphi}, \tag{3.10}$$

and the action (3.9) then takes the form

$$\tilde{S} \sim \int \mathrm{d}^D x \sqrt{|\tilde{g}|} \left[\tilde{R} + 2(\partial\varphi)^2 - \frac{1}{4} e^{\pm 2\sqrt{\frac{D-1}{D-2}}\varphi} F_{ab}F^{ab} \right]. \tag{3.11}$$

We thus find the usual Einstein-Hilbert action and the dynamical nature of the scalar field is now clearly reflected in the action.

3.2 Newton-Cartan from Null reduction of General Relativity

In the same spirit as the Kaluza Klein reduction along a spacelike vector was done in the previous section, we now want to do a Null reduction of General Relativity. The idea is again the same, we represent the compact dimension with a Killing vector ξ^M. But instead of a spacelike vector, this time we start from a null, or lightlike, Killing vector, obeying $\xi^2 = 0$. The first ones to use this technique were [34] and [35].

We will take the path of Null reduction to construct Newton-Cartan Gravity. As already mentioned before, General Relativity can be obtained via a gauging procedure of the Poincaré group. The same method can be used to obtain Newton-Cartan Gravity via a gauging procedure of the Bargmann algebra, which is the central extension of the Galilei algebra. For a side to side comparison of these two methods, we refer to [36].

We will start from the Einstein-Hilbert Lagrangian in $(D+1)$ dimensions in the second order formalism (2.38) and we will use the Vielbein formalism throughout. The Vielbein in $(D+1)$ dimensions transforms under infinitesimal diffeomorphisms with parameter ζ^M, and under infinitesimal local Lorentz tranformations with parameter $\lambda^A{}_B$ as

$$\delta E_M{}^A = \zeta^N \partial_N E_M{}^A + \partial_M \zeta^N E_N^A + \lambda^A{}_B E_M{}^B. \tag{3.12}$$

As already mentioned we assume the existence of a null Killing vector $\xi = \xi^M \partial_M$ which then fulfills

$$\mathcal{L}_\xi E_M{}^A = 0, \quad \xi^2 = 0. \tag{3.13}$$

This constraint on the $(D+1)$ dimensional geometry already makes sure that non relativistic geometry will arise in D dimensions. The reason is that when looking at the Poincaré group in lightcone coordinates, there is a subgroup of the Poincaré group that is isomorphic to the Galilei group [37]. In a sense, non relativistic symmetry is embedded in a Lightcone direction of relativistic symmetries. The underlying reason is that the little group of a Null vector is exactly the Bargmann group. Thus we get a Bargmann structure when doing a Null reduction. In contrast, the little group of a spacelike vector gives $SO(1, D-1)$. This is the reason one uses a spacelike vector in the Kaluza-Klein reduction.

We will choose adapted coordinates $x^M = \{x^\mu, z\}$ again, where μ can have D values and will be the coordinate on lower dimensional spacetime. Without loss of generality we will also take the Killing vector to be $\xi = \xi^z \partial_z$. This implies that $\partial_z g_{MN} = 0$ and hence the metric (and the Vielbein) is indeed z independent*.

As we already mentioned before it is convenient to use a Lightcone basis. To do this we write the flat tangent space indices as $A = \{a, -, +\}$, where $(+)$ is the compact Null direction and $(-)$ will be our new time coordinate. Then, in these coordinates, the components of the Minkowski metric will look like $\eta_{ab} = \delta_{ab}$ and $\eta_{+-} = -1$. From $\xi^2 = 0$ we can conclude that we will have two zeros in our Vielbein Ansatz as either ${E_z}^-$ or ${E_z}^+$ can be set to zero. We will choose ${E_z}^- = \xi^2 = 0$ in the following. The other components of the Vielbein can in principle be chosen arbitrarily, however, as we want to reproduce Newton-Cartan theory with our usual fields, we will choose

$$
{E_M}^A = \begin{array}{c} \\ \mu \\ z \end{array} \!\!\begin{array}{c} \begin{array}{ccc} a & - & + \end{array} \\ \begin{pmatrix} {e_\mu}^a & S^{-1}\tau_\mu & -S m_\mu \\ 0 & 0 & S \end{pmatrix} \end{array}, \qquad {E^M}_A = \begin{array}{c} \\ a \\ - \\ + \end{array} \!\!\begin{array}{c} \begin{array}{cc} \mu & z \end{array} \\ \begin{pmatrix} {e^\mu}_a & {e^\mu}_a m_\mu \\ S\tau^\mu & S\tau^\mu m_\mu \\ 0 & S^{-1} \end{pmatrix} \end{array}. \tag{3.14}
$$

Remember that the index a takes one value less than μ, so the matrices in (3.14) are indeed square matrices as they should be. From the $(D+1)$-dimensional relation ${E^M}_A(x){E_M}^B(x) = \delta_A^B$ we immediately find the relations (2.54)

$$
\begin{aligned}
\tau^\mu \tau_\mu = 1, \quad \tau^\mu {e_\mu}^a = 0, \quad {e^\mu}_a \tau_\mu = 0, \\
{e_\mu}^a {e^\mu}_b = \delta_b^a, \quad {e_\mu}^a {e^\nu}_a = \delta_\mu^\nu - \tau_\mu \tau^\nu
\end{aligned} \tag{3.15}
$$

again. The same relations appear when gauging the Bargmann algebra [36] and identifying $\{\tau_\mu, {e_\mu}^a\}$ as the temporal and spatial Vielbein. However, there one defines these relations by hand, while when doing a Null reduction we get them directly.

The fields appearing above are a spatial Vielbein ${e_\mu}^a$ and its projective inverse ${e^\mu}_a$ in D dimensions. Additionally we have a temporal Vielbein τ_μ, a $U(1)$ gauge field m_μ and a scalar S. The gauge field m_μ is subject to a $U(1)$ gauge transformation that corresponds to a central charge

$$
\delta m_\mu = -\partial_\mu \zeta^z, \tag{3.16}
$$

and a central extension of the Galilei algebra. This means that Galilei algebra is extendend with a generator that commutes with all other generators of the Galilei algebra. This extended algebra is then called the Bargmann algebra. Physically this central charge generator

*The Null reduction is a special case due to the Null condition (3.13) on the Killing vector, which implies that the metric component $G_{zz} = 0$. Due to this we will lose an equation of motion, e.g. $R_{zz} = 0$. Hence we can not perform this reduction in the action, but only on the field transformations and the equations of motion. However this argument only applies to dynamical gravity, if we freeze the gravitational fields to fixed background values, as we will do later (see section 5), we can indeed perform this reduction in the action.

is associated with mass conservation in non relativistic theories. As it is shown in [38], this additional generator is convenient when talking about massive representations of the Galilei algebra. The scalar S can be gauge fixed as we will see soon.

As already mentioned, the zero ${E_z}^-$ in the Vielbein Ansatz (3.14) comes from the fact that $\xi^2 = 0$, while for the zero ${E_z}^a$ we used local Lorentz invariance again as in the Kaluza Klein case. This means that we gauge fix the Lorentz parameter ${\lambda^a}_+ = 0$. We are then left with ${\lambda^a}_b$, ${\lambda^a}_-$ and ${\lambda^+}_+ = -{\lambda^-}_-$. We will call ${\lambda^a}_- \equiv \lambda^a$ and ${\lambda^+}_+ \equiv \lambda$, however the latter one can be easily gauge fixed by choosing $S = 1$, because no other field transforms as a gauge field under S. This corresponds to setting $\lambda = 0$. After these gauge fixing steps we can derive the transformation rules for the D dimensional fields directly from the higher dimensional Vielbein transformation rule (3.12) and get

$$\delta\tau_\mu = \mathcal{L}_\zeta \tau_\mu \tag{3.17a}$$

$$\delta {e_\mu}^a = \mathcal{L}_\zeta {e_\mu}^a + {\lambda^a}_b {e_\mu}^b + \lambda^a \tau_\mu \tag{3.17b}$$

$$\delta m_\mu = \mathcal{L}_\zeta m_\mu - \partial_\mu \zeta^z - \lambda_a {e_\mu}^a \tag{3.17c}$$

Similar as in the Kaluza Klein case the gauge field m_μ transforms as a $U(1)$ gauge field with the parameter ζ^z of diffeomorphisms in the compact dimension.

3.2.1 Newton-Cartan geometries

As we already discussed, when talking about Newtonian Gravity there is always a foliation of spacetime as one can introduce a time function $t(x)$. This condition can alternatively be expressed as

$$\tau_{\mu\nu} = 0, \tag{3.18}$$

where $\tau_{\mu\nu} = 2\partial_{[\mu}\tau_{\nu]}$ is called the 'timelike torsion'.* This implies that we can write τ_μ as the gradient of a function $t(x)$ and thus $\tau_\mu = \partial_\mu t(x)$. This 'zero torsion constraint' in (3.18) guarantees a spacetime foliation which ensures that Newtonian causality is fulfilled. Newtonian causality means that there is a distinction between past and future, which is exactly the case when there are constant time slices, i.e. a foliation in the time direction. The time difference T between two events can then be expressed as

$$T = \int_{\mathcal{C}} \mathrm{d}x^\mu \tau_\mu = \int_{\mathcal{C}} \mathrm{d}t, \tag{3.19}$$

where $\mathcal{C}$ is the path connecting the two events. Condition (3.18) ensures that T is independent of the path $\mathcal{C}$ taken, so that T really qualifies as an absolute time difference. This framework is known as torsionless Newton-Cartan geometry and can lead to a geometric formulation of

*In the context of Newton-Cartan theory $\tau_{\mu\nu}$ is often referred to as just the torsion, which is ambiguous as there is also a spacelike torsion τ_{ab}. However, we will also often call $\tau_{\mu\nu}$ torsion, while keeping in mind that there are different kinds of torsion

Newtonian Gravity.

It is however known that the condition (3.18) is sufficient. The Frobenius theorem tells us that a sufficient and necessary condition to obtain a spacetime foliation, is the hypersurface orthogonality condition, $\tau \wedge \mathrm{d}\tau = 0$, or written in components

$$\tau_{[\mu}\partial_\nu \tau_{\rho]} = 0, \tag{3.20}$$

which can also be expressed as

$$\tau_{ab} = e^\mu{}_a e^\nu{}_b \tau_{\mu\nu} = 0. \tag{3.21}$$

In this case there is no notion of absolute time as the time difference T depends on the path taken between two spacelike leaves. However we still have a notion of causality because we can talk about constant time slices. On every spacelike leaf one can distinguish future and past events. Condition (3.21) is called the 'twistless torsion condition' and leads to twistless torsional Newton-Cartan geometries. These types of Newton-Cartan geometries appear when working with Lifshitz holography, i.e. when coupling Newton-Cartan Gravity to a Conformal Field Theory* on the boundary [9]. It also appears in condensed matter applications, for example when studying the Quantum Hall Effect [10, 11].

One can go even further and leave the torsion completely unconstrained, i.e. $\tau \wedge \mathrm{d}\tau \neq 0$. We then get geometries that are well known in mathematics as contact structures, with τ being the contact form, see e.g. [39]. Of course causality is then lost, but there are interesting applications in condensed matter physics, where Newton-Cartan geometries are used as a background to describe non relativistic energy and momentum flux and thermal transport [12].

3.2.2 Making contact with Newtonian Gravity

Now that we showed how to do a Null reduction of General Relativity we will try to make contact with classical Newtonian Gravity. We will use the reduced spin connection (C.1) components together with the Riemann tensor (2.39) and

$$R_{AB}(\Omega(E)) = E^M{}_C E^N{}_A R_{MN}{}^C{}_B(\Omega(E)) \tag{3.22}$$

to obtain the components of the lower dimensional Ricci tensor. We will often turn three dimensional indices μ, $\nu, \ldots$ into flat indices 0, a, $(a = 1, 2)$, using the convention

$$X_0 = \tau^\mu X_\mu, \quad X_a = e^\mu{}_a X_\mu. \tag{3.23}$$

Note that raising or lowering a zero index induces an extra minus sign, i.e. $X_0 = -X^0$, while flat indices $a, b \ldots$ can be freely raised and lowered using the Kronecker delta. We then get

*It does not come as a surprise that torsional NC geometry appears in this context because condition (3.18) is not invariant under dilatations $\delta\tau_\mu = \lambda(x)\tau_\mu$, while (3.21) is indeed invariant under such dilatations.

the components of the Ricci tensor as

$$
\begin{aligned}
R_{++} &= \frac{1}{4}\tau^{ab}\tau_{ab}, & R_{+a} &= \frac{1}{2}D_b\tau^b{}_a - \frac{1}{2}\tau_0{}^a\tau_{ab},\\
R_{+-} &= \frac{1}{2}D_a\tau_0{}^a - \frac{1}{4}\tau_0{}^a\tau_{0a}, & R_{-a} &= R_{0b}(J^b{}_a) - \frac{1}{2}D_0\tau_{0a},\\
R_{--} &= R_{a0}(G^a), & R_{ab} &= R_{ca}(J^c{}_b) - \frac{1}{2}D_a\tau_{0b} + \frac{1}{2}D_0\tau_{ab} + \frac{1}{4}\tau_{0a}\tau_{0b}.
\end{aligned} \tag{3.24}
$$

where the covariant derivatives on $\tau_0{}^a$ and τ^{ab} are given by

$$
\begin{aligned}
D_\mu\tau^a{}_0 &= \partial_\mu\tau^a{}_0 + \omega_\mu{}^{ab}\tau_{b0} + \omega_\mu{}^b\tau_b{}^a,\\
D_\mu\tau^{ab} &= \partial_\mu\tau^{ab} + \omega_\mu{}^a{}_c\tau^{cb} + \omega_\mu{}^b{}_c\tau^{ac},
\end{aligned} \tag{3.25}
$$

and we defined the curvatures for rotations and boosts as

$$
\begin{aligned}
R_{\mu\nu}(J^{ab}) &= 2\partial_{[\mu}\omega_{\nu]}{}^{ab} - 2\omega_{[\mu}{}^a\tau_{\nu]}\tau^{b0} - 2\omega_{[\mu}{}^c e_{\nu]c}\tau^{ab}\\
R_{\mu\nu}(G^a) &= 2\partial_{[\mu}\omega_{\nu]}{}^a + \omega_{[\mu}{}^{ab}\omega_{\nu]b} + \omega_{[\mu}{}^a\tau_{\nu]}{}^0 + \omega_{[\mu}{}^b e_{\nu]b}\tau_0{}^a.
\end{aligned} \tag{3.26}
$$

We also defined the torsion free Newton-Cartan spin and boost connection as

$$
\begin{aligned}
\omega_\mu{}^a &= e^{\nu a}\partial_{[\mu}m_{\nu]} - e_\mu{}^b e^{\nu a}\tau^\rho\partial_{[\nu}e_{\rho]b} - \tau^\nu\partial_{[\mu}e_{\nu]}{}^a - \tau_\mu e^{\nu a}\tau^\rho\partial_{[\nu}m_{\rho]},\\
\omega_\mu{}^{ab} &= 2e^{\nu[a}\partial_{[\mu}e_{\nu]}{}^{b]} - e_\mu{}^c e^{\nu a}e^{\rho b}\partial_{[\nu}e_{\rho]c} - \tau_\mu e^{\nu a}e^{\rho b}\partial_{[\nu}m_{\rho]}.
\end{aligned} \tag{3.27}
$$

We then impose the vacuum equations of motion $R_{MN} = 0$. All components of the Ricci tensor have to be zero separately. From R_{++} we immediately get the twistless torsion condition $\tau_{ab} = 0$. Using this constraint and assuming appropriate boundary conditions R_{+-} implies that $\tau_{a0} = 0$, hence we have zero torsion. Applying the zero torsion condition to the other Ricci tensor components we get the equations of motion for torsion free NC Gravity*

$$
R_{0a}(G^a) = 0, \qquad R_{0a}(J^a{}_b) = 0, \qquad R_{ab}(J^a{}_c) = 0. \tag{3.28}
$$

As we want to make contact with Newtonian Gravity we will perform a partial gauge fixing of Newton-Cartan Gravity. It can be shown that one can consistently impose the gauge fixing conditions [40]

$$
e_\mu{}^a = \delta_\mu{}^a, \quad \tau_\mu = \delta_\mu{}^0, \quad \omega_\mu{}^{ab} = 0, \tag{3.29}
$$

which corresponds to a gauge fixing of local time translations, spatial rotations and spatial translations. Note that due to the gauge fixing of the Vielbein $e_\mu{}^a = \delta_\mu{}^a$, space is flat and there is no distinction between μ and a indices anymore. The only independent fields left at this point are m_μ and m_0, we apply the last gauge fixing condition which fixes the central charge transformations

$$
m_a = \delta^\mu{}_a m_\mu = 0, \quad \text{and} \quad m_0 = \delta_0{}^\mu m_\mu = \phi(x), \tag{3.30}
$$

*Gauge fixing requires equations that are a tiny bit stronger, i.e. $R_{ab}(J) = 0$ and $R_{0a}(J) = 0$, however in three dimensions the last two conditions of (3.28) are equivalent to these and can be obtained via a contraction with the Levi-Civita symbol.

and we identify m_0 with the Newtonian potential. At this point this is quite an obvious choice. All these gauge fixing conditions together give the spin connection

$$\omega_0{}^a = -\delta^{ab}\partial_b\phi(x), \tag{3.31}$$

while all other components give zero. In this gauge the two equations of motion involving the rotation curvature $R_{\mu\nu}(J)$ are identically zero. The only equation left is the one for $R_{0a}(G)$ and we find the vacuum Poisson equation for the Newtonian potential $\phi(x)$

$$\Delta\phi(x) = \delta^{ab}\partial_a\partial_b\phi(x) = 0. \tag{3.32}$$

We showed that it is indeed possible to make contact with Newtonian Gravity if we choose a specific gauge in the Newton-Cartan formulation. Note that this can also be done when gauging the Bargmann algebra [36] and imposing a zero torsion constraint.

Chapter 4

Supersymmetry and supergravity

In the 1960s the question arose what symmetries we can find in particle physics. It was already clear that a Poincaré symmetry was needed for relativistic processes. The Poincaré group $ISO(1, D-1)$ is generated by Lorentz generators J_{mn} and translation generators P_m. Additionally, there are internal or gauge symmetries, like a $U(1)$ symmetry as in electromagnetism, or a $SU(3)$ symmetry in QCD. These internal symmetries form Lie algebras with generators T_m as

$$[T_m, T_n] = f_{mn}{}^l T_l \tag{4.1}$$

with structure constants $f_{mn}{}^l$ depending on the gauge group. The generators of the internal symmetries commute with the Poincaré algebra. One can ask wether it is possible to add symmetry generators to the Poincaré algebra that do not commute with the Poincaré generators. Coleman and Mandula [41] proved that we can not*. They showed that under certain assumptions all such symmetries would lead to a trivial S matrix for all processes. Of course every theorem is dependent on its assumptions and one main assumption of the Coleman-Mandula theorem is that we just consider Lie algebras. But if we loosen this assumption and also allow for graded Lie algebras that include anticimmutation relations along with commutation relations the Coleman-Mandula theorem does not apply anymore. This new algebra is called a graded Lie algebra. A graded Lie algebra generalizes the idea of a Lie algebra by distinguishing in odd and even generators. The generators J_{mn}, P_m and T_m are *even*, while the generators† Q_α are *odd*. The (anti-)commutation relations for a graded Lie algebra are $[even, even] = even$, $\{odd, odd\} = even$ and $[even, odd] = odd$. It is then possible to take the Poncaré generators as even and extend the algebra with additional odd generators obeying commutation relations like

$$[J_{mn}, Q_\alpha] \sim Q_\beta. \tag{4.2}$$

*Note that in the massive case the maximal (maximal as in the sense that otherwise any extension of the group would lead to trivial dyanmics) group is the Poincaré group, while in the massless case we get the conformal group.

†The small greek letters from the beginning of the alphabet $(\alpha, \beta, ..)$ are spinor indices. See appendix B for more details.

It turns out that the generators Q_α have to be realised as a spinor representation of the Lorentz group. A very nice argument why that must be the case can be found in [42]. The Coleman-Mandula theorem is evaded by considering graded Lie algeras.

The relation (4.2) realizes a symmetry between bosons and fermions - this is exactly what supersymmetry is about. The extended Poincaré algebra involving a graded Lie algebra is called a superalgebra. The possible superalgebras with spinor charges are then governed by the Haag-Łopuszański-Sohnius theorem [43], which generalizes the Coleman-Mandula theorem to the supersymmetric case.

An important example where these odd generatos play an important role is a chiral $U(1)$ symmetry called R symmetry which appears in many susy theories. There is an additional generator T_R which acts on Q_α. This R symmetry does not commute but it fulfills the follwing commutation relation.

$$[T_R, Q_\alpha] = -i(\Gamma_*)_\alpha{}^\beta Q_\beta. \tag{4.3}$$

We will see later (section 4.3) that this is one of the major differences between the two different off shell formulations of minimal $D = 4$ supergravity.

4.1 From susy to sugra

Note that supersymmetry (susy) as we discussed it here is a global symmetry. One could now ask what happens if we promote susy to a local symmetry. Similar to gauging the Poincaré algebra we could apply a gauging procedure to the super Poincaré algebra. This will lead to supergravity (sugra). As we have just seen, supersymmetry introduces an extra symmetry between bosonic and fermionic fields. In supersymmetry every bosonic field has a fermionic partner and they transform into each other under a so-called supersymmetry transformations. Gravity is described by the metric (Vielbein), a bosonic field. So we will need a fermionic superpartner for the graviton. As it turns out, this is a spin-$\frac{3}{2}$ field which is usually called gravitino. The gravitino is a vector-spinor ψ_M and we will describe its Lagrangian in the next section. If there appears just one gravitino, we get $\mathcal{N} = 1$ supergravity. If there are more gravitini, then we get extended $\mathcal{N} = 2, .., 8$ supergravity.

The overall structure of the supersymmetry transformation rules can be deduced when thinking about mass dimensions. While bosons are described by kinetic terms with two derivatives (Maxwell, Einstein-Hilbert, Klein-Gordon), fermions are described by kinetic terms with only one derivative (Dirac, Rarita-Schwinger). This leads to a difference in mass dimensions by $\frac{1}{2}$ for bosonic and fermionic fields. Denoting fermions as F and bosons as B the naive approach for such a transformation rule would be $\delta B \sim \epsilon F$ with an anticommuting parameter ϵ. From the above considerations one can deduce that the parameter ϵ must have mass dimension

$[\epsilon] = -\frac{1}{2}$. This rules out a transformation rule for the fermions like $\delta F \sim \epsilon B$ but leads instead to

$$\delta F \sim \partial(\epsilon B). \tag{4.4}$$

As we will see there can be more fields in a supergravity theory than just the graviton-gravitino pair. There can be auxiliary fields with no physical degrees of freedom that are nevertheless important to ensure consistency of the theory. Sugra can also be coupled to matter fields that constitute matter supermultiplets, i.e. a collection of fields that transform into each other under supersymmetry. Note that all possible supermultiplets are a realization of an underlying superalgebra. The respective bosonic and fermionic degrees of freedom in a supermultiplet have to be the same, however there are different ways of counting. These different ways of counting are called on shell and off shell degrees of freedom, depending on using the equations of motion or not. We will explain this in more detail in chapter 4.1.

Note that contrary to bosonic fields, fermions are very sensitive to spacetime dimension. If we want to define supegravity in arbitrary dimensions,* we need to get a better understanding of spinors. Sugra has a lot of applications and important variants in higher dimensions, however we will focus on $D = 4$ here.

As already mentioned we need spinors to describe fermions. Spinors in general are realized as representations of Spin groups which are a double cover of the special orthogonal groups, most relevant for us is the Lorentz group $SO(1, D-1)$. Equivalently the space of spinors is defined as the fundamental representation of the Clifford algebra. The Clifford algebra is spanned by Gamma marices Γ_A which obey the anticommutation relation

$$\{\Gamma_A, \Gamma_B\} = 2\eta_{AB}. \tag{4.5}$$

Note that forming antisymmetric products of these Gamma matrices $\Gamma_{AB} = \frac{1}{2}[\Gamma_A, \Gamma_B]$ leads to elements that are part of the spinor representation of the Lorentz algebra (see discussion around equation (B.5)). Usually one encounters Dirac spinors in the context of particle physics and quantum field theory. Note however that Dirac spinors are usually reducible representations of the spin group. It is often more convenient, and somehow more fundamental to work with irreducible representations, i.e. Weyl or Majorana spinors. We will mostly use Majorana spinors in this work. As already mentioned, spinors are very sensitive to a change in spacetime dimension and not all types of spinors exist in every dimension. Thus we need to be careful when switching from four to three dimensions in a dimensional reduction of fermions, as we will later do. For details on the aforementioned spinor types in different dimensions and the Clifford algebra, see appendix B.

*The maximal dimension where sugra can be defined is actually 11, see [24] for a nice dimensional reduction argument. $D = 11$ sugra is also very important as starting point for dimensional reduction to obtain lowerdimensional sugra theories and a necessary ingredient of M-theory.

In the following section we will take a closer look at the spin-$\frac{3}{2}$ field which we need to construct a supergravity theory.

4.1.1 Rarita-Schwinger field

Supergravity is the gauge theory of local supersymmetry. In the supersymmetry transformations described above there is a global susy parameter ϵ. We want to gauge this symmetry which means that we want to promote our susy parameter to a function $\epsilon(x)$ that depends arbitrarily on spacetime coordinates. The corresponding gauge field happens to be a vector-spinor ψ_M, which is the gravitino field. The gauge transformation of the gravitino field looks like

$$\psi_M(x) \to \psi_M(x) + \partial_M \epsilon(x), \tag{4.6}$$

which we will usually just write as $\delta\psi_M = \partial_M \epsilon(x)$ as in the section above. This gauge transformation has a lot of similarity with electrodynamics. The antisymmetric derivative $\partial_M \psi_N - \partial_N \psi_M$ is also gauge invariant like in electrodynamics. But there is a crucial difference because we are now looking for a first order wave equation as equation of motion for ψ_M. As always when constructing a suitable Lagrangian, we have to consider that it has to be Lorentz invariant, first order in derivatives, invariant under the gauge transformation (4.6) and real. All these requirements are fulfilled by the action [44]

$$S = -\frac{1}{2} \int \mathrm{d}^D x \bar{\psi}_M \Gamma^{MNR} \partial_N \psi_R, \tag{4.7}$$

which is gauge invariant, while the corresponding Lagrangian is invariant up to a total derivative term. From the action (4.7) there follow equations of motions

$$\Gamma^{MNP} \partial_N \psi_P = 0. \tag{4.8}$$

Using some gamma matrix identities these can also be rewritten as

$$2\Gamma^M \partial_{[M} \psi_{N]} = 0. \tag{4.9}$$

4.1.2 Universal part of supergravity

The action we discussed in section 4.1.1 is for a free theory. Introducing self-interactions and interactions with gravity leads to sugra, whose universal part we will discuss here. It consists of the Einstein-Hilbert Lagrangian in second order formalism (2.38) and the Rarita-Schwinger Lagrangian (4.7). We also need local Lorentz and diffeomorphism invariance and thus need to replace the standard derivative on the gravitino by a suitable covariant derivative. This then results in

$$S[E,\psi] = \frac{1}{2} \int \mathrm{d}^D x \, |E| \left(E^{AM} E^{BN} R_{MNAB}(\Omega(E)) - \bar{\psi}_M \Gamma^{MNR} D_N \psi_R \right), \tag{4.10}$$

where $\Gamma^{MNR} \equiv E^{MA} E^{NB} E^{RC} \Gamma_{ABC}$ with the Gamma matrix Γ_{ABC} being the antisymmetric product of three flat space Gamma matrices (see appendix B) and the gravitino covariant curl

$$D_{[N}\psi_{R]} = \partial_{[N}\psi_{R]} + \tfrac{1}{4}\Omega_{[N}{}^{AB}\Gamma_{|AB|}\psi_{R]}, \tag{4.11}$$

with the torsion free spin connection defined in (2.36). Note that there is no Christoffel term in (4.11) because the torsion free Christoffel symbols are symmetric and vanish in the action (4.10) due to the antisymmetric Γ^{MNR}. The fields $E_M{}^A$ and ψ_M form the so-called sugra multiplet, with susy transformation rules

$$\delta E_M{}^A = \frac{1}{2}\bar{\epsilon}\Gamma^A\psi_M, \tag{4.12a}$$

$$\delta\psi_M = D_M\epsilon(x) \equiv \partial_M\epsilon + \frac{1}{4}\Omega_M{}^{AB}\Gamma_{AB}\epsilon. \tag{4.12b}$$

It can be shown that the action (4.10) is invariant under the transformation rules (4.12) to linear order. For more details and a proof of invariance under local susy up to linear order of minimal $D = 4$ on shell supergravity we refer to [24, 45]. Note that in general there will be more terms in the gravitino transformation rule, if additional fields are present are present in the sugra multiplet. This can happen e.g. in off shell formulations of sugra, where extra auxiliary fields are needed (see section 4.2). If we want an off shell formulation of supergravity we will need to add auxiliary fields. These auxiliary fields will also have separate transformation rules and will appear in the gravitino variation. This is similar for additional matter fields. The relation between matter and auxiliary fields in a supersymmetric off shell theory will become crucial when applying the Festuccia-Seiberg method (chapter 5). The equations of motion coming from the action (4.10) are of course the Einstein equations (and the Rarita-Schwinger equations of motion (4.9)) but with gravitino terms ψ_M as sources in the energy-momentum tensor.

Note however, that the action (4.10) is not the full answer. In a complete sugra theory this action will include quartic fermion terms. This was first shown by brute foce calculation by Freedman, van Nieuwenhuizen and Ferrara [46]. This is a very tedious calculation and often very difficult to do analytically. More elegant methods to show susy invariance were developed by Deser and Zumino [47] in 1976.

4.1.3 The algebra of local supersymmetry

In general the commutator of two supersymmetry transformations should again be part of the superalgebra, i.e. the algebra should be closed. For global supersymmetry this commutator on a general field gives a translation, e.g. for a scalar field Φ, one has

$$[\delta_1, \delta_2]\Phi = -\frac{1}{2}\bar{\epsilon}_1\Gamma^M\epsilon_2\partial_M\Phi. \tag{4.13}$$

One would expect that a similar result should also hold for local supersymmetry. In this case, the commutator of two local susy transformations yields a diffeomorphism with parameter $\zeta^M(x) = -\frac{1}{2}\bar{\epsilon}_1(x)\Gamma^M\epsilon_2(x)$ instead of a translation. Since other symmetries such as local Lorentz symmetry and susy are present, these symmetries can and typically do appear in the commutator of two susy transformations (in the form of a so-called *covariant general coordinate transformation*). Note that the structure constants ${f_{AB}}^C$ as they usually appear in such commutators are generalized to structure functions that can depend on various fields of the theory. Such superalgebras are often called soft algebras because they do not close in the conventional sense but with these generalized structure functions.

Furthermore, it can also happen that the local susy algebra only closes off shell. This means that there are terms in the commutator that only vanish when using the equations of motion. In this case, one says that the corresponding theory/multiplet is on shell. Otherwise the theory/multiplet is called off shell. As we will see in the next section the multiplet $\{{E_M}^A, \psi_M\}$ forms only an on shell multiplet. One can sometimes extend this to an off shell multiplet by including auxiliary fields. However an off shell formulation is not available (or has not yet been found) for arbitrary supergravity theories (in arbitrary spacetime dimension).

4.2 Off shell vs. on shell

One can often decide wether a theory/multiplet is on shell/off shell by checking wether the number of bosonic degrees of freedom equals the number of fermionic degrees of freedom. However there are two different ways of counting: On shell and off shell counting. This means that when constructing a sugra theory we need the same number of bosonic and fermionic degrees of freedom. Here the idea of on/off shell plays an important role, as imposing equations of motions always reduces the degrees of freedom because we introduce additional constraints. Especially for our work the off shell formulation will be crucial as we will see later. Let us take a look at the different degrees of freedom on and off shell for our supergravity fields.

Off shell

- Real scalar ϕ, either a dynamical scalar or an auxiliary scalar field S with algebraic equations of motion has always one degree of freedom.

- Gauge field A_M, starting the analysis it seems that there must be D degrees of freedom. But we need to take the gauge condition $\delta A_M = \partial_M \lambda(x)$ into consideration. We have an additional parameter $\lambda(x)$ which reduces the degrees of freedom by one, i.e. we get $D-1$ degrees of freedom.

- Graviton ${E_M}^A$, we can either take the symmetric rank two metric for counting, or the Vielbein while taking local Lorentz transformations into account. The Vielbein has D^2 degrees of freedom while the local Lorentz transformations subtract $\frac{1}{2}D(D-1)$ components. This gives $\frac{1}{2}D(D+1)$, however we have to subtract another D diffeomorphism degrees of freedom to finally get $\frac{1}{2}D(D+1) - D = \frac{1}{2}D(D-1)$ degrees of freedom.

- Spin $\frac{1}{2}$ spinor ψ, for a Majorana spinor we have $n \equiv 2^{\lfloor D/2 \rfloor}$ degrees of freedom (see appendix B).

- Spin $\frac{3}{2}$ gravitino ψ_M, starting the analysis we get nD degrees of freedom, but we need to consider local supersymmetry $\delta\psi_M = D_M\epsilon(x)$, hence we get $n(D-1)$ degrees of freedom.

On shell

- Scalar ϕ, the Klein-Gordon equation does not add any constraints, thus the on shell scalar still has one degree of freedom. However an auxiliary scalar S has no degrees of freedom on shell because its algebraic equation has been solved.

- Gauge field A_M, here the equations of motion are $\partial^M(\partial_M A_N - \partial_N A_M) = 0$. Instead of analyzing this equation we will make use of gauge fixing. We apply the covariant Lorenz gauge $\partial^M A_M = 0$, then the equations of motion reduce to the Klein-Gorden equation $\Box A_M = 0$ which imposes no further constraints. However there is a residual gauge invariance under $A_M \to A_M + \partial_M \Lambda$ with $\Box\Lambda = 0$. This also leave the gauge field invariant and we have to subtract another degree of freedom and end up with $D-2$ degrees of freedom.

	$\phi(x)$	$S(x)$	$A_M(x)$	${E_M}^A(x)$	$\psi(x)$	$\psi_M(x)$	$B_{MN}(x)^*$
off shell	1	1	$D-1$	$\frac{1}{2}D(D-1)$	$2^{\lfloor D/2 \rfloor}$	$2^{\lfloor D/2 \rfloor}(D-1)$	$\frac{1}{2}(D-2)(D-1)$
on shell	1	0	$D-2$	$\frac{1}{2}D(D-3)$	$\frac{1}{2}2^{\lfloor D/2 \rfloor}$	$\frac{1}{2}2^{\lfloor D/2 \rfloor}(D-3)$	$\frac{1}{2}(D-3)(D-2)$

Figure 4.1: On and off shell degrees of freedom of various fields in D dimensional spacetime.

We will stop here at this point. For gauge fields it is always a similar procedure in using a gauge condition, i.e. using de Donder gauge for the graviton to get the Klein-Gorden equation again. The gravitino is a bit more involved but this can be found in detail in [48]. We will summarize all the results in table 4.1.

4.3 Off shell minimal $D = 4$ supergravity

We will now take a look at the two different formulations of minimal off shell supergravity in four dimensions. As we have already seen in the last section, if we want to construct an on shell theory we have $\frac{1}{2}4(4-3) = 2$ degrees of freedom for ${E_M}^A$ and $\frac{1}{2}2^{\lfloor 4/2 \rfloor}(D-3) = 2$ degrees of freedom for ψ_M. They match up and can thus form a multiplet $\{{E_M}^A, \psi_M\}$ for the theory. Now let us turn to the off shell case where we do not use any equations of motion. The off shell case is of particular interest for us because we need it for our procedure here. The method by Seiberg and Festuccia [16] we will describe in section 5 is only possible for a

*The expression for the antisymmetric gauge two tensor comes from the general form for an antisymmetric tensor of rank r with gauge invariance, which has off shell $\binom{D-1}{r}$ and on shell $\binom{D-2}{r}$ degrees of freedom.

theory that is off shell.

To construct this off shell supergravity we need to take a look at table 4.1 again. As we see for $D = 4$ we get $\frac{1}{2}4(4-1) = 6$ degrees of freedom for $E_M{}^A$ but $2^{\lfloor 4/2 \rfloor}(D-1) = 12$ degrees of freedom for ψ_M. To build a sensible susy theory with these fields we need at least 6 more bosonic degrees of freedom. Of course a satisfying result could also be reached if we took 10 bosonic and 4 fermionic additional degrees of freedom. This is indeed possible, and there are such non minimal off shell multiplets, but we will stick to the *minimal* case here. This is done by introducing auxiliary fields which do not propagate, i.e. they have algebraic equations of motion. These equations of motion can be solved and when used in the action and transformation rules we get the on shell formulation of the theory back.

The straightforward option, which was also considered first [49, 50], is the old minimal supergravity multiplet

$$\{E_M{}^A, \psi_M, S, P, V_M\}, \tag{4.14}$$

where we have a real scalar S, a real pseudoscalar P and a real, non-gauge vector V_M, which give together the needed $1 + 1 + 4 = 6$ bosonic degrees of freedom. Sometimes a complex scalar $Z = S + iP$ with its complex conjugate $\bar{Z}$ is used, as it is often more convenient, but has the same number of degrees of freedom as the two real scalar fields.

Another minimal choice of auxiliary fields is the new minimal set [51]

$$\{E_M{}^A, \psi_M, A_M, B_{MN}\}, \tag{4.15}$$

with a real axial vector A_M and an antisymmetric tensor field B_{MN}. The two tensor is a gauge field with the gauge transformation

$$\delta B_{MN} = 2\partial_{[M}\beta_{N]} \tag{4.16}$$

with a gauge parameter β_N. The vector A_M has an additional $U(1)$ gauge invariance and in fact the whole theory has a chiral $U(1)_R$ symmetry

$$\delta_\Lambda A_M = \partial_M \Lambda, \quad \delta_\Lambda \psi_M = -i\Gamma_* \psi_M \Lambda, \tag{4.17}$$

with a symmetry parameter Λ. Note that we have a gauge vector field with $4 - 1 = 3$ off shell degrees of freedom and an antisymmetric gauge tensor field with $\frac{1}{2}(4-2)(4-1) = 3$ degrees of freedom. This gives 6 bosonic degrees of freedom together and is thus a minimal set of auxiliary fields. What is often done is to use the dual V_M of B_{MN} which is a conserved vector field and is given by

$$V^M = \frac{1}{4}\epsilon^{MNRS}\partial_N B_{RS}, \quad \nabla_M V^M = 0. \tag{4.18}$$

With the dual it is even more apparent that we get 3 degrees of freedom for this constrained vector field. Thus the new minimal supergravity multiplet, on which we will perform a Null reduction later on, is

$$\{E_M{}^A, \psi_M, A_M, V_M\}, \tag{4.19}$$

where V_M is defined as in (4.18). The supersymmetry transformation rules for new minimal supergravity are then [51]

$$\delta E_M{}^A = \frac{1}{2}\bar{\epsilon}\Gamma^A\psi_M \tag{4.20a}$$

$$\delta\psi_M = D_M\epsilon + i\Gamma_* A_M\epsilon - i\Gamma_* V_M\epsilon + \frac{i}{2}\Gamma_{MA}\Gamma_* V^A\epsilon \tag{4.20b}$$

$$\delta B_{MN} = \bar{\epsilon}\Gamma_{[N}\psi_{M]} \tag{4.20c}$$

$$\delta A_M = \frac{i}{4}(D_L\bar{\psi}_S + \frac{i}{2}\bar{\psi}_S\Gamma_*\Gamma_K V^K\Gamma_L)\Gamma_*\Gamma^{LS}\Gamma_M\epsilon, \tag{4.20d}$$

with $D_M\epsilon = \partial_M\epsilon + \frac{1}{4}\Omega_M{}^{AB}\Gamma_{AB}\epsilon$. Note that this has the same form as in (4.12), but we get new transformation rules for the auxiliary fields and extra terms in the gravitino transformation.

When solving the algebraic equations of motion, both off shell formulations lead to the same on shell formulation of $N = 1$ supergravity we introduced in the previous section.

Chapter 5

Festuccia-Seiberg method

One of the main motivations to study supersymmetry on curved manifolds come from applications in the AdS/CFT correspondence [4, 5] and in localisation techniques. Supersymmetric localisation techniques are used to obtain exact results for path integrals of field theories. These results can then be used to test holographic correspondences in a certain regime. Without these tools it is usually only possible to evaluate path integrals for free theories. Other non trivial field theories where this is possible were topological field theories. Examples for such theories are [52, 53] where topological field theories were obtained by twisting a supersymmetric field theory. Over the last few years a lot of new exact path integral results could be obtained for susy field theories in curved backgrounds [54]. One of these examples is the work of Pestun [55] who was able to compute the path integral of $4D$ $\mathcal{N} = 2$ and $\mathcal{N} = 4$ supersymmetric gauge theories on the 4-sphere. In the following years a lot of exact results could be obtained in supersymmetric field theories on compact manifolds. All these results used supersymmetric localisation techniques on compact manifolds. For that reason it is very interesting to further investigate which backgrounds can be used to define susy field theories, as this could yield more exactly computable path integrals.

Our goal is to place rigid supersymmetric field theories in curved spacetime, i.e. to consider susy field theories which are not defined on Minkowski spacetime. We want to replace a Minkowski background by some general curved manifold $\mathcal{M}$. One option to do this, is the Noether method where we replace the Minkowski metric by a general curved metric and ordinary derivatives by covariant ones. However, the result will typically not be supersymmetric. We then have to check how it is not invariant and add terms in the Lagrangian and the supersymmetry tranformation rules to cancel the unwanted terms. It becomes clear why this is not the best method, it is not systematic and it is not guaranteed to lead to a supersymmetric theory in the end. For an in depth treatment of the Noether procedure in the context of Newton-Cartan geometries we refer to [56]. A more convenient way of describing susy QFTs on curved backgrounds was developed by Festuccia and Seiberg in [16]. In this method we start from a matter field theory coupled to off shell supergravity and then take a rigid limit to decouple the fluctuations in the gravitational field. In this way we not only obtain the supersymmetric Lagrangian and transformation rules but a better understanding of what backgrounds can be used.

In order to define supersymmetry, one needs susy parameters. In flat space these obey $\partial_M \epsilon = 0$ which is an example of a killing spinor equation. We want to generalize this to suitable Killing spinor equations in curved space and investigate under which conditions on the geometry non trivial solutions for the susy parameters ϵ can be found. In sugra the Killing spinor equations can be found by setting $\delta\psi_M = 0$. This corresponds to the requirement that a classical background preserves some supersymmetry. The fermions have to vanish to keep Lorentz invariance on the background, therefore the boson susy transformations vanish automatically. It then follows that the only restrictions we get on our classical backgrounds come from the gravitino supersymmetry transformations that need to vanish. This constraint gives the Killing spinor equations, depending on the specific off shell supergravity formulation used. Only when these Killing spinor equations admit non trivial, nowhere vanishing solutions on a background, one is able to define supersymmetry on this background.

For simplicity, we restrict ourselves to the old minimal off shell formulation in this chapter. We already introduced the old minimal multiplet in (4.14) and the corresponding Killing spinor equation is obtained by setting the gravitino variation [49, 50] to zero. This yields

$$D_M \epsilon + \frac{i}{2} V_M \Gamma_* \epsilon + \frac{1}{3} \Gamma_M \operatorname{Re}(Z \epsilon_L) - \frac{i}{6} \Gamma_M \not{V} \Gamma_* \epsilon = 0, \tag{5.1}$$

where $D_M = \partial_M \epsilon + \Omega_M{}^{AB} \Gamma_{AB} \epsilon$ and ϵ is the supersymmetry parameter. The resulting Killing spinor equation depends on all off shell supergravity background fields: A complex scalar Z, a real non-gauge vector V_M and the metric which is implicitly contained in the covariant derivative. Once we specify fixed background values for the background fields we can then try to find non trivial solutions for ϵ. Note that if we only allow the metric field to be non trivial, equation (5.1) reduces to

$$D_M \epsilon = 0. \tag{5.2}$$

Under this restriction susy QFTs can only exist on manifolds that admit at least one covariantly constant spinor. However this only leaves Ricci flat manifolds, i.e. Calabi-Yau manifolds. It turns out that replacing the standard derivative by a covariant one is not quite enough to obtain backgrounds that are not strongly restricted. If we want to obtain more general backgrounds we need more fields with non trivial background values. To achieve this we use an off shell supergravity formulation where we can specify background values for the different auxiliary fields in (5.1). When looking for non trivial solutions ϵ of the Killing spinor equation we can then find weaker restriction on possible backgrounds. Once such backgrounds have been obtained, we can take the rigid limit of the corresponding supergravity Lagrangian on this curved space. This results in the Lagrangian and supersymmetry transformation for susy QFTs on non-dynamical curved backgrounds.

Most of the existing literature deals with the old minimal and new minimal multiplet in Lorentzian and Euclidean signature in three or four dimensions [16, 57, 58, 59, 60, 61, 62]. We want to obtain Killing spinor equations to construct non relativistic supersymmetric field theories. This was already done starting from the old minimal mulitplet [63]. We will extend this work to the case of the new minimal multiplet.

In principle it is then a straightforward exercise to apply the method from [16] to non relativistic susy QFTs on Newton-Cartan manifolds. However there arise some difficulties as not a lot is known about non relativistic off shell supergravity. The usual $3D$ Newton-Cartan supergravity described in [40] is on shell. The only off shell formulations known require extra geometric conditions for consistency. Hence it is not clear if the resulting supersymmetry transformations and Killing spinor equations would lead to the most general backgrounds for non relativistic susy QFTs. Furthermore these theories do not have a Lagrangian which makes it more difficult to figure out matter couplings. For these reasons we will take a different approach. We will start from a relativistic supergravity theory in $D = 4$, take the rigid limit in the Lagrangian and the supersymmetry transformations and then go on and perform a Null reduction to obtain a non relativistic susy QFT in $D = 3$. This allows us to analyze the corresponding Killing spinor equations and determine possible backgrounds. It was shown in [57, 58] that relativistic backgrounds for susy QFTs in four dimensions which are obtained from old or new minimal supergravity admit a null Killing vector. We will exploit this fact to obtain non relativistic backgrounds from relativistic ones by performing a Null reduction.

5.1 The rigid limit

The first step is to take the rigid limit in the relativistic Lagrangian. Note again that we will demonstrate this procedure here for old minimal sugra, however this works similarly for new mininal sugra. We start from an off shell supergravity Lagrangian and keep the metric and the auxiliary fields of the sugra multiplet fixed to some arbitrary background values, while setting the background value of the gravitino to zero. To freeze out the fluctuations of the supergravity multiplet, we take the limit in which the Planck mass $M_P \to \infty$. Note that we can omit all gravitino terms in the Lagrangian as the gravitino background value is chosen to be zero. Note that the resulting Lagrangian for a susy QFT on a curved background is not necessarily unique. We can always add terms proportional to some power of the curvature, which will vanish in the flat space limit. However, we will focus on terms that must be added to preserve some supersymmetry. Furthermore, we will not start with the usual ugra Lagrangian with the convential Einstein-Hiblert term. Rather, we will start with a Lagrangian that is related to the usual one via a Weyl rescaling (leading to what is called a Jordan frame Lagrangian). Compare with the discussion around (3.8), where we did such a Weyl rescaling. This gives the advantage that our gravitino supersymmetry variations will not depend on the matter content of the theory but just on the auxiliary fields. In general, the supergravity background fields and the matter fields can be treated completely on their own with this method. Therefore we will get much more general results applicable to a variety of supersymmetric field theories independent of their matter content.

An important feature of this procedure is that the background metric and auxiliary field values are completely arbitrary and not restricted by any equations of motion. We are looking for configurations of the background fields g_{MN} and some auxiliary fields that preserve rigid supersymmetry on the background manifold $\mathcal{M}$, i.e. we want the gravitino variation to vanish for some non trivial ζ that solves equation (5.1). The solutions ζ of equation (5.1) are then the Killing spinors which will be used to construct the supersymmetry parameters. It turns out that it suffices to look for commuting solutions ζ of the Killing spinor equations. However, physical fermions are described by anti commuting spinors because they have to obey the spin-statistics theorem. Hence the supersymmetry parameters ϵ are also anticommuting spinors. Due to the linearity of the Killing spinor equation, we can construct anti commuting solutions ϵ as linear combinations of commuting solutions ζ and constant Grassmann variables θ. Our supersymmetry parameters will therefore be of the form

$$\epsilon = \sum_{i=1}^{n} \theta_i \zeta^i. \tag{5.3}$$

When a commuting solution ζ of the Killing spinor equation exists, one can construct the following vector

$$K^M = i\bar{\zeta}\Gamma^M\zeta, \quad \text{with} \quad K_M K^M = 0. \tag{5.4}$$

In addition, using the Killing spinor equation (5.1), one can show that

$$\nabla_{(M} K_{N)} = 0 \tag{5.5}$$

and thus K^M is a null Killing vector. We see that any background that has at least one solution of the Killing spinor equation admits a null Killing vector K^M.

As mentioned above, we will give some details on how to take the rigid limit by sending $M_P \to \infty$. We will demonstrate this in the case of the old minimal multiplet*, however the procedure works analogously in the new minimal case. We will use n chiral multiplets $\{U^i, \chi^i_L, F^i\}$, where U^i are complex scalars, χ^i_L are left handed Weyl fermions and F^i are complex auxiliary scalars. The complex conjugates of these fields are $\{\bar{U}^{\bar{i}}, \chi^{\bar{i}}_R, \bar{F}^{\bar{i}}\}$, where the $\bar{i}$ is uesd to denote the complex conjugate of the corresponding field with index i. The components of the old minimal multiplet are then given as in (4.14) with a complex scalar Z, a non gauge vector V_M and the graviton-gravitino pair. The Lagrangian can then be written

*Thank you to Jan Rosseel for sharing his private notes for this section.

as [64, 65]

$$\begin{aligned}\frac{1}{M_P^2}|E|^{-1}\mathcal{L} = &-\frac{1}{6}\phi|E|^{-1}\mathcal{L}_{SG} + \phi_{i\bar{j}}\left(-\frac{1}{2}\partial_M U^i\partial^M\bar{U}^{\bar{j}} - \bar{\chi}_L^i \not{D}\chi_R^{\bar{j}} + \frac{1}{2}F^i\bar{F}^{\bar{j}}\right)\\ &-\phi_{ij\bar{k}}\bar{\chi}_L^i\chi_L^j\bar{F}^{\bar{k}} + \phi_{ij\bar{k}}\bar{\chi}_L^i\not{\partial}U^j\chi_R^{\bar{k}} + \frac{1}{2}\phi_{ij\bar{k}\bar{l}}\bar{\chi}_L^i\chi_L^j\bar{\chi}_R^{\bar{k}}\chi_R^{\bar{l}}\\ &+\frac{1}{3}\bar{Z}\left(\phi_i F^i - \phi_{ij}\bar{\chi}_L^i\chi_L^j\right) + \frac{i}{3}V^M\left(\frac{1}{2}\phi_{i\bar{j}}\bar{\chi}_R^{\bar{j}}\Gamma_M\chi_L^i + \phi_i\partial_M U^i\right)\\ &+\frac{1}{M_P^2}\left(-\frac{1}{2}W_{ij}\bar{\chi}_L^i\chi_L^j + \frac{1}{2}W_iF^i + \frac{1}{2}V_MV^M\right) + \text{gravitino terms} + \text{h.c.}. \qquad (5.6)\end{aligned}$$

The covariant derivative of the chiral multiplet fermions χ_L^i is given by

$$D_M\chi_L^i = \left(\partial_M + \frac{1}{4}\Omega_M^{AB}\Gamma_{AB}\right)\chi_L^i, \qquad (5.7)$$

where the connection is the usual torsionless connection of General Relativity. The Lagrangian $\mathcal{L}_{SG}$ is given by:

$$|E|^{-1}\mathcal{L}_{SG} = -\frac{1}{2}R - \frac{1}{3}Z\bar{Z} + \frac{1}{3}V_MV^M + \text{gravitino terms}. \qquad (5.8)$$

The function $W = W(U^i)$ is the superpotential that is holomorphic in the complex chiral multiplet scalars U^i, while $\phi = \phi(U^i, \bar{U}^{\bar{i}})$ is a real function depending on the chiral multiplet scalars U^i and their conjugates $\bar{U}^{\bar{i}}$. Derivatives are denoted with indices i and $\bar{i}$, corresponding to a derivative with respect to U^i or $\bar{U}^{\bar{i}}$. Explicitly this looks like

$$\phi_i = \frac{\partial}{\partial U^i}\phi, \quad \phi_{i\bar{j}} = \frac{\partial^2}{\partial U^i\partial\bar{U}^{\bar{j}}}\phi, \quad \phi_{ij\bar{k}} = \frac{\partial^3}{\partial U^i\partial U^i\partial\bar{U}^{\bar{k}}}, \quad \ldots. \qquad (5.9)$$

The same notation applies to the superpotential and its derivatives. The function ϕ is expressed in terms of a function $K(U^i, \bar{U}^{\bar{i}})$, called the Kähler potential, as follows

$$\phi = -3\exp\left(-\frac{K}{3M_P^2}\right), \qquad (5.10)$$

This brings in an extra dependence on M_P.

We can then take the rigid limit of the Lagrangian (5.6). As already mentioned in the introduction of this chapter, we want to specify arbitrarily fixed but non trivial background values for the metric and the auxiliary fields U and V_M, while keeping the gravitino background value to zero. Then the fluctuations of the supergravity multiplet are frozen out by taking the limit $M_P \to \infty$. As we can already see, the last line in (5.6) does not depend on M_P and hence its rigid limit will be trivial. However we have to pay close attention to the other terms as they do not only depend on M_P through an overall factor but also through ϕ and its derivatives. Thus we need to evaluate the derivatives of (5.10) at least up to the relevant order of M_P. One can find

$$\begin{aligned}
\phi_i =& \frac{K_i}{M_P^2}\exp\left(-\frac{K}{3M_P^2}\right) = \frac{K_i}{M_P^2} + \mathcal{O}(M_P^{-4}),\\
\phi_{ij} =& \frac{K_{ij}}{M_P^2}\exp\left(-\frac{K}{3M_P^2}\right) + \mathcal{O}(M_P^{-4}) = \frac{K_{ij}}{M_P^2} + \mathcal{O}(M_P^{-4}),\\
\phi_{i\bar{j}} =& \frac{K_{i\bar{j}}}{M_P^2}\exp\left(-\frac{K}{3M_P^2}\right) + \mathcal{O}(M_P^{-4}) = \frac{K_{i\bar{j}}}{M_P^2} + \mathcal{O}(M_P^{-4}),\\
\phi_{ij\bar{k}} =& \frac{K_{ij\bar{k}}}{M_P^2}\exp\left(-\frac{K}{3M_P^2}\right) + \mathcal{O}(M_P^{-4}) = \frac{K_{ij\bar{k}}}{M_P^2} + \mathcal{O}(M_P^{-4}),\\
\phi_{ij\bar{k}\bar{l}} =& \frac{K_{ij\bar{k}\bar{l}}}{M_P^2}\exp\left(-\frac{K}{3M_P^2}\right) + \mathcal{O}(M_P^{-4}) = \frac{K_{ij\bar{k}\bar{l}}}{M_P^2} + \mathcal{O}(M_P^{-4}),
\end{aligned} \tag{5.11}$$

where the latter equalities were obtained by a taylor expansion of the exponential terms and omitting all higher order contributions. We can then plug (5.10) into the supergravity Lagrangian to obtain

$$-\frac{M_P^2}{6}\phi\mathcal{L}_{SG} = \frac{M_P^2}{2}\exp\left(-\frac{K}{3M_P^2}\right)\mathcal{L}_{SG} = \frac{M_P^2}{2}\mathcal{L}_{SG} - \frac{K}{6}\mathcal{L}_{SG} + \mathcal{O}(M_P^{-4}). \tag{5.12}$$

We can then use (5.11) and (5.12) in (5.6) to find

$$\begin{aligned}
|E|^{-1}\mathcal{L} =& -\frac{M_P^2}{2}|E|^{-1}\mathcal{L}_{SG} - |E|^{-1}\frac{K}{6}\mathcal{L}_{SG} + K_{i\bar{j}}\left(-\frac{1}{2}\partial_M U^i\partial^M\bar{U}^{\bar{j}} - \bar{\chi}_L^i \not{D}\chi_R^{\bar{j}} + \frac{1}{2}F^i\bar{F}^{\bar{j}}\right)\\
&- K_{ij\bar{k}}\bar{\chi}_L^i\chi_L^j\bar{F}^{\bar{k}} + K_{ij\bar{k}}\bar{\chi}_L^i\not{\partial}U^j\chi_R^{\bar{k}} + \frac{1}{2}K_{ij\bar{k}\bar{l}}\bar{\chi}_L^i\chi_L^j\bar{\chi}_R^{\bar{k}}\chi_R^{\bar{l}} + \frac{1}{3}\bar{Z}\left(K_iF^i - K_{ij}\bar{\chi}_L^i\chi_L^j\right)\\
&+ \frac{i}{3}V^M\left(\frac{1}{2}K_{i\bar{j}}\bar{\chi}_R^{\bar{j}}\Gamma_M\chi_L^i + K_i\partial_M U^i\right) - \frac{1}{2}W_{ij}\bar{\chi}_L^i\chi_L^j + \frac{1}{2}W_iF^i + \frac{1}{2}V_MV^M\\
&+ \text{gravitino terms} + \text{h.c.} + \mathcal{O}(M_P^{-2}).
\end{aligned} \tag{5.13}$$

From here we can take the rigid limit by sending $M_P \to \infty$. Note that the first term diverges, but it only depends on background fields which we fix to arbitrary values. More importantly, it does not depend on any dynamical fields and we can thus disregard it in the rigid limit. We also plug in the background values for the supergravity multiplet and can thus drop the gravitino terms. Then we end up with a Lagrangian coupled to an arbitrary fixed supersymmetric background, valid for n chiral multiplets:

$$\begin{aligned}
|E|^{-1}\mathcal{L} =& -|E|^{-1}\frac{K}{6}\mathcal{L}_{SG} + K_{i\bar{j}}\left(-\frac{1}{2}\partial_M U^i\partial^M\bar{U}^{\bar{j}} - \bar{\chi}_L^i\not{D}\chi_R^{\bar{j}} + \frac{1}{2}F^i\bar{F}^{\bar{j}}\right) - K_{ij\bar{k}}\bar{\chi}_L^i\chi_L^j\bar{F}^{\bar{k}}\\
&+ K_{ij\bar{k}}\bar{\chi}_L^i\not{\partial}U^j\chi_R^{\bar{k}} + \frac{1}{2}K_{ij\bar{k}\bar{l}}\bar{\chi}_L^i\chi_L^j\bar{\chi}_R^{\bar{k}}\chi_R^{\bar{l}} + \frac{1}{3}\bar{Z}\left(K_iF^i - K_{ij}\bar{\chi}_L^i\chi_L^j\right)\\
&+ \frac{i}{3}V^M\left(\frac{1}{2}K_{i\bar{j}}\bar{\chi}_R^{\bar{j}}\Gamma_M\chi_L^i + K_i\partial_M U^i\right) - \frac{1}{2}W_{ij}\bar{\chi}_L^i\chi_L^j + \frac{1}{2}W_iF^i + \frac{1}{2}V_MV^M + \text{h.c.}.
\end{aligned} \tag{5.14}$$

The supersymmetry transformations for the chiral multiplet fields are [64, 65]

$$\begin{aligned}
\delta U^i &= \bar{\epsilon}_L\chi_L^i,\\
\delta\chi_L^i &= \frac{1}{2}\left(\not{\partial}U^i\epsilon_R + F^i\epsilon_L\right),\\
\delta F^i &= \bar{\epsilon}_R\left(\not{D} - \frac{i}{6}\not{V}\right)\chi_L^i - \frac{Z}{3}\bar{\epsilon}_L\chi_L^i.
\end{aligned} \tag{5.15}$$

The Lagrangian (5.14) is invariant under (5.15) up to the use of the Killing spinor equations. This means we first need to know more about the background of Lagrangian (5.14) to check the supersymmetry invariance. This information can be obtained by solving the Killing spinor equations. For old minimal sugra this was done in [60]. The existence of a Killing spinor then implies the existence of a null Killing vector. This can be used to null reduce the set of Killing spinor equations, the Lagrangian ans the susy transformation rule to get those of an non relativistic theory. For old minimal sugra, this, along with an analysis of possible non relativistic susy backgrounds, was done in [63]. In the next chapter we will do this for new minimal sugra.

Chapter 6

Non relativistic backgrounds from relativistic supergravity

In this chapter we will perform a Null reduction of a relativistic supersymmetric theory and then analyze the obtained Killing spinor equations. To achieve this we will couple a chiral multiplet to new minimal supergravity and then take the rigid limit as described in the previous section. We can then perform a Null reduction to obtain a non relativistic supersymmetric theory in three dimensions. In the latter part of this chapter, we will perform the Null reduction on the Killing spinor equations of the new minimal multiplet. This enables us to determine restrictions on the allowed supersymmetric backgrounds. Finally, we will present the main results and give examples for such backgrounds.

However, before we proceed with the Null reduction we need to make sure that there exists a global Null vector. We already discussed this in the context of old minimal supergravity in chapter 5 where we constructed a vector

$$K^M = i\bar{\zeta}\Gamma^M\zeta, \tag{6.1}$$

with ζ being a solution of the corresponding Killing spinor equation. We now need to verify that this vector fulfills

$$K^M K_M, \quad \text{and} \quad \nabla_{(M}K_{N)} \tag{6.2}$$

for new minimal supergravity as well. For convenience we will show this using chiral spinors, i.e. we decompose a solution $\zeta = \zeta_L + \zeta_R$ in a left/right chiral projection. We can use this in (6.1) to get

$$K^M = 2i\bar{\zeta}_L\Gamma^M\zeta_R = 2i\bar{\zeta}_R\Gamma^M\zeta_L. \tag{6.3}$$

Using this property we can rewrite the spinor bilinear $K^M K_M$ as

$$K^M K_M = -4\bar{\zeta}_L\Gamma_M\zeta_R\bar{\zeta}_R\Gamma_M\zeta_L. \tag{6.4}$$

It is now much easier to apply a Fierz identity on $\zeta_R\bar{\zeta}_R$ because in this case all the chiral terms vanish and we end up with

$$K^M K_M = -\bar{\zeta}_L\Gamma_M\bar{\zeta}_R\zeta_R\Gamma^M\zeta_L - \bar{\zeta}_L\Gamma^M\Gamma_{OP}\Gamma_M\zeta_L\bar{\zeta}_R\Gamma^{OP}\zeta_R = 0 \tag{6.5}$$

The first term vanishes due to the commutativity of the spinors, while the second term vanishes because $\Gamma^M\Gamma_{OP}\Gamma_M = 0$ in four dimensions. Thus we can conclude that K^M is a null vector. To show that K^M is indeed a Killing vector, we make use of the Killing spinor equation of new minimal supergravity

$$D_M\epsilon + i\Gamma_* A_M\epsilon - i\Gamma_* V_M\epsilon + \frac{i}{2}\Gamma_{MA}\Gamma_* V^A\epsilon = 0. \tag{6.6}$$

Using that ζ obeys this equation we can obtain

$$\nabla_M K_N = -\epsilon_{MN}{}^{RL}V_R K_L, \tag{6.7}$$

which indeed vanishes when symmetrizing over M and N. Hence K^M is indeed a null Killing vector and can always be constructed if at least one global, non-trivial solution ζ of the Killing spinor equation (6.6) exists.

We will couple a chiral multiplet $\{U, \chi, F\}$ with a scalar U, an auxiliary scalar F and a fermion χ to the new minimal multiplet. Adjusting the Lagrangian given in [16] to our conventions, we obtain

$$\begin{aligned}|E|^{-1}\mathcal{L} = &-\frac{q}{4}(R + 6V_M V^M)U\bar{U} - D_M U D^M\bar{U} + iV^M(\bar{U}D_M U - UD_M\bar{U})\\ &+ F\bar{F} - \bar{\chi}\left(\not{D} - \frac{i}{2}\not{V}\right)\chi_L + FW' + \bar{F}\bar{W}' + \mathrm{Re}\left(2FW' - W''\chi_L\chi_L\right)\end{aligned} \tag{6.8}$$

where q is the $U(1)_R$ charge of the chiral multiplet, W is the superpotential and W' and W'' are the first and second derivative of the superpotential with respect to the scalar U, i.e.

$$W' = \frac{\partial W}{\partial U}, \quad W'' = \frac{\partial^2 W}{\partial U^2}. \tag{6.9}$$

The covariant derivatives appearing in (6.8) are then given by

$$D_M U = (\partial_M - iqA_M)U \tag{6.10a}$$

$$D_M\bar{U} = (\partial_M + iqA_M)\bar{U} \tag{6.10b}$$

$$D_M\chi = (\partial_M + \tfrac{1}{4}\Omega_M{}^{AB}\Gamma_{AB} + i(q-1)A_M\Gamma_*)\chi. \tag{6.10c}$$

The supersymmetry variations for the chiral multiplet are given by

$$\delta U = \bar{\epsilon}\chi_L \tag{6.11a}$$

$$\delta F = \bar{\epsilon}(\not{D} - \tfrac{i}{2}\not{V})\chi_L \tag{6.11b}$$

$$\delta\chi_L = (\not{D}U + F)\epsilon_L. \tag{6.11c}$$

Note that the rigid limit procedure explained in section 5.1 was already performed on Lagrangian (6.8). We can now go on and perform a Null reduction on this Lagrangian to obtain a non relativistic theory.

6.1 Scherk-Schwarz Null reduction

To perform the Null reduction on the fermionic fields, we will assume that the chiral multiplet fields $\{U, \chi, F\}$ only depend on the internal dimension z in a very specific way. Then we will go on and perform a *twisted*, or *Scherk-Schwarz* Null reduction [66]. This is only possible if the theory admits a global symmetry. The higher dimensional fields can then be expressed as symmetry transformations of the lower dimensional fields. The global symmetry is needed to guarantee that the end result is independent of the internal dimension z. In our Lagrangian (6.8) this is the case if the superpotential $W = 0^*$. We then find a global $U(1)$ symmetry for all the multiplet fields with parameter α:

$$\delta U = i\alpha U, \quad \delta\chi_L = i\alpha\chi_L, \quad \delta F = i\alpha F. \tag{6.12}$$

In the spirit of [66] we introduce an extra mass parameter m. The $U(1)$ symmetry then leads to an Ansatz for the fields of the chiral multiplet that includes a phase depending on the internal dimension z:

$$U(x^\mu, z) = \mathrm{e}^{-imz}u(x^\mu), \quad F(x^\mu, z) = \mathrm{e}^{-imz}f(x^\mu). \tag{6.13}$$

Additionally we will use the reduced vector fields V_M and A_M

$$v_\mu = V_\mu + m_\mu V_z = V_\mu + m_\mu v, \quad V_z = v \tag{6.14}$$

$$a_\mu = A_\mu + m_\mu A_z = A_\mu + m_\mu a, \quad A_z = a. \tag{6.15}$$

We use this redefinition to ensure that the fields v_μ and a_μ still transform as a vector under diffeomorphisms. Alternatively one could do the Null reduction in flat coordinates where this redefinition is not needed.

The Null reduction of fermions is a bit more involved as we need to find a suitable decomposition of the Clifford algebra. We already showed how a Null reduction on bosons works in section 3.2. Now we need to find a way to decompose fermions in a consistent way. To achieve this, we have to decompose the four dimensional Clifford algebra. We will try to find a general form $\Gamma_A \sim \gamma_a \otimes \sigma_i$, i.e. the four dimensional Gamma matrices Γ_A get decomposed into three dimensional Gamma matrices γ_a and Pauli matrices σ_i for the internal space. If we use a lightcone basis, we get

$$\Gamma_a = \gamma_a \otimes \mathbb{1}_2, \quad \Gamma_\pm = \gamma_0 \otimes \sigma_\pm, \tag{6.16}$$

with $\sigma_\pm = \frac{1}{\sqrt{2}}(\sigma_1 \pm \ i\sigma_2)$ and $\Gamma_\pm = \frac{1}{\sqrt{2}}(\Gamma_0 \pm \ \Gamma_3)$ The three dimensional Clifford algebra elements γ_0 and γ_a then follow the relations

$$\{\gamma_a, \gamma_b\} = 2\delta_{ab}, \quad \{\gamma_0, \gamma_a\} = 0. \tag{6.17}$$

*This seems to be very restrivctive, however we only consider the simplest case with only one chiral multiplet here. If we consider more chiral multiplets, then one can find $U(1)$ invariant superpotentials.

Using a lightcone basis again, the Lorentz generators take the following convenient form

$$\Gamma_{ab} = \gamma_{ab} \otimes \mathbb{1}_2, \quad \Gamma_{a\pm} = \gamma_{a0} \otimes \sigma_{\pm}, \quad \Gamma_{+-} = -\mathbb{1}_2 \otimes \sigma_3. \tag{6.18}$$

We will also need the decomposition for Γ_* which is given by

$$\Gamma_* = i\gamma_0 \otimes \sigma_3. \tag{6.19}$$

Note that when reducing from four to three dimensions, the Gammma matrices will be 2×2 matrices and the spinors will be two component spinors (see appendix B).

As we want to do a twisted Null reduction we want to include a global factor of $\mathrm{e}^{\pm imz}$ in our chiral fermion reduction Ansatz. This was also done in [63] for the reduction of the old minimal multiplet. Note that the new minimal multiplet admits another global symmetry in form of an $U(1)_R$ symmetry. However this is only a symmetry of the Lagrangian and not the supersymmetry transformation rules. Hence it is not straightforward to use this R symmetry for a twisted Null reduction. It could be possible to let the R symmetry also act on the supersymmetry parameters but it is at this point not clear if the resulting supersymmetry transformation rules would still leave the Lagrangian invariant. This would be an interesting approach to the Null reduction but we did not pursue this further. We instead chose a simple $U(1)$ Ansatz for the left/right chiral fermions

$$\begin{aligned}
\chi_L(x^\mu, z) &= \mathrm{e}^{-izm}\left(\pi\chi_+ \otimes \varphi_- + \bar{\pi}\chi_- \otimes \varphi_+\right) \\
\chi_R(x^\mu, z) &= \mathrm{e}^{+izm}\left(\pi\chi_+ \otimes \varphi_- + \bar{\pi}\chi_- \otimes \varphi_+\right),
\end{aligned} \tag{6.20}$$

with a phase factor that only depends on the internal dimension z and the mass parameter m. In the Gamma matrix decomposition (6.16), the charge conjugation matrix decomposes like in (B.21) and thus we get three dimensional Majorana spinors from four dimensional Majorana spinors. The $\chi_\pm$ are then two component spinors depending on the three dimensional spacetime coordinates, while $\varphi_\pm$ are just constant spinors depending only on the internal space. As they represent the reduced, internal space, we can assign them arbitrary values. For convenience we choose

$$\varphi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \varphi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{6.21}$$

Using this choice for $\varphi_\pm$ we can derive useful relations for the Null reduction:

$$\sigma_\pm \varphi_\mp = \sqrt{2}\varphi_\pm, \quad \sigma_\pm \varphi_\pm = 0 \tag{6.22}$$

We also defined three dimensional operators π and $\bar{\pi}$ as

$$\pi = \frac{1}{2}(\mathbb{1}_2 - i\gamma_0), \quad \bar{\pi} = \frac{1}{2}(\mathbb{1}_2 + i\gamma_0). \tag{6.23}$$

They are projection operators and obey $i\gamma_0\pi = -\pi$ and $i\gamma_0\bar{\pi} = \bar{\pi}$. Note that the resulting three dimensional (pseudo-)chiral fermions are not Majorana anymore, but complex one component spinors. We will refer to $\{\pi\chi_+, \bar{\pi}\chi_-\}$ as (pseudo-)left handed fermions and to $\{\bar{\pi}\chi_+, \pi\chi_-\}$ as (pseudo-)right handed fermions.

As already mentioned in section 3.2, performing a Null reduction will result in a Bargmann symmetry in the lower dimensional theory. The Bargmann algebra includes local spatial rotations, Galilean boosts and a $U(1)$ central charge transformation (3.16) with the corresponding gauge field m_μ. This $U(1)$ symmetry is related to mass/particle number conservation and is a acts on the three dimensional fields in the following way:

$$\delta u(x) = im\beta u(x), \quad \delta f(x) = im\beta f(x), \quad \delta\chi_\pm(x) = \pm im\beta\gamma_0\chi_\pm(x) \tag{6.24}$$

6.2 Chiral multiplet Lagrangian

We will now construct an explicit example of a non relativistic susy QFT on an arbitrary curved Newton-Cartan background. The scalar fields obey a curved space Schrödinger equation, while the fermions obey a Levy-Leblond equation [67] which can be seen as the square root* of the Schrödinger equation. For simplicity we considered only one pseudo-chiral multiplet, but the generalization to more pseudo-chiral multiplets and arbitrary Kähler potentials is straightforward.

Using all Ansätze discussed above, we obtain

$$\begin{aligned}
(\det(e^a,\tau))^{-1}\mathcal{L} = &-\frac{q}{4}\left(\frac{1}{8}R_{ab}(J^{ab}) - 2\tau^{a0}\tau^{a0} + 6h^{\mu\nu}v_\mu v_\nu - \frac{1}{2}v\tau^\mu v_\mu\right)u\bar{u} - h^{\mu\nu}\bar{\nabla}_\mu u\bar{\nabla}_\nu\bar{u} \\
&+ (im + iqa)\tau^\mu(u\bar{\nabla}_\mu\bar{u} - \bar{u}\bar{\nabla}_\mu u) + i(h^{\mu\nu}v_\nu - \tau^\mu v)(\bar{u}\bar{\nabla}_\mu u - u\bar{\nabla}_\mu\bar{u}) \\
&- 2v_0(m + ra)u\bar{u} + f\bar{f} + e^{\mu a}\bar{\chi}_+\gamma_a\left(\bar{\nabla}_\mu - \frac{i}{2}v_\mu\right)\pi\chi_+ \\
&+ e^{\mu a}\bar{\chi}_-\gamma_a\left(\bar{\nabla}_\mu - \frac{i}{2}v_\mu\right)\bar{\pi}\chi_- - \sqrt{2}\tau^\mu\bar{\chi}_-\gamma_0\left(\bar{\nabla}_\mu + \frac{i}{2}v_\mu\gamma_0\right)\pi\chi_+ \\
&+ \sqrt{2}\left(\frac{1}{8}\tau^{ab}\epsilon_{ab} + im\gamma_0 - (q-1) - \frac{1}{2}\gamma_0 v\right)\bar{\chi}_+\bar{\pi}\chi_- \\
&+ \frac{1}{2}\tau^{a0}\left(\bar{\chi}_+\gamma_a\pi\chi_+ - \frac{3}{2}\bar{\chi}_-\gamma_a\bar{\pi}\chi_-\right),
\end{aligned} \tag{6.25}$$

where $\det(e^a,\tau)$ is the determinant of a 3×3 matrix with ${e_\mu}^a$ and τ_μ in its columns, $R_{ab}(J^{ab})$ is the Lorentz rotation curvature defined in (3.26). We also defined the spatial metric of Newton-Cartan geometry $h^{\mu\nu} = e^{\mu a}{e^\nu}_a$ and the covariant derivatives, which are covariantized with respect to local spatial rotations, Galilean boosts, $U(1)$ central charge transformations

*This square root is not meant literally, we have the same situation as in the relativistic case where the Dirac equation can be seen as the square root of the Klein Gordon equation.

as given in (6.24) and $U(1)_R$ transformations, are given by

$$\bar{\nabla}_\mu u = (\partial_\mu - iqa_\mu - imm_\mu)u, \tag{6.26a}$$

$$\bar{\nabla}_\mu \bar{u} = (\partial_\mu + iqa_\mu + imm_\mu)\bar{u}, \tag{6.26b}$$

$$\bar{\nabla}_\mu \pi\chi_+ = \partial_\mu \pi\chi_+ + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\pi\chi_+ - imm_\mu\pi\chi_+ + (q-1)a_\mu\gamma_0\pi\chi_+, \tag{6.26c}$$

$$\bar{\nabla}_\mu \bar{\pi}\chi_- = \partial_\mu \bar{\pi}\chi_- + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\bar{\pi}\chi_- - imm_\mu\bar{\pi}\chi_- - (q-1)a_\mu\gamma_0\bar{\pi}\chi_- - \frac{\sqrt{2}}{2}\omega_\mu{}^a\gamma_{a0}\pi\chi_+. \tag{6.26d}$$

By reducing the four dimensional transformation rules (6.11), one obtains the supersymmetry transformation rules for the three dimensional chiral multiplet $\{u, \pi\chi_+, \bar{\pi}\chi_-, f\}$:

$$\begin{aligned}
\delta u &= \bar{\epsilon}_+\pi\chi_+ + \bar{\epsilon}_-\bar{\pi}\chi_- \\
\delta\pi\chi_+ &= e^{\mu a}\gamma_a\pi\epsilon_+\bar{\nabla}_\mu u + \sqrt{2}i\gamma_0\bar{\pi}\epsilon_-(m+qa)u + f\pi\epsilon_+ \\
\delta\bar{\pi}\chi_- &= e^{\mu a}\gamma_a\bar{\pi}\epsilon_-\bar{\nabla}_\mu u - \sqrt{2}\tau^\mu\gamma_0\pi\epsilon_+\bar{\nabla}_\mu u + f\bar{\pi}\epsilon_+ \\
\delta f &= e^{\mu a}\bar{\epsilon}_+\gamma_a\left(\bar{\nabla}_\mu - \frac{i}{2}v_\mu\right)\pi\chi_+ + e^{\mu a}\bar{\epsilon}_-\gamma_a\left(\bar{\nabla}_\mu - \frac{i}{2}v_\mu\right)\bar{\pi}\chi_- \\
&\quad - \sqrt{2}\tau^\mu\bar{\epsilon}_-\gamma_0\left(\bar{\nabla}_\mu + \frac{i}{2}v_\mu\gamma_0\right)\pi\chi_+ + \frac{1}{2}\tau^{a0}\left(\bar{\epsilon}_+\gamma_a\pi\chi_+ - \frac{3}{2}\bar{\epsilon}_-\gamma_a\bar{\pi}\chi_-\right) \\
&\quad + \sqrt{2}\left(\frac{1}{8}\tau^{ab}\epsilon_{ab} + im\gamma_0 - (q-1) - \frac{1}{2}\gamma_0 v\right)\bar{\epsilon}_+\bar{\pi}\chi_-
\end{aligned} \tag{6.27}$$

Note that the supersymmetry parameters (ϵ_+, ϵ_-) in these transformation rules have to solve the Killing spinor equations which we will derive in the next section.

6.3 Null reduction of Killing spinor equations

The Killing spinor equation is obtained by putting the supersymmetry variation of the gravitino to zero. For the gravitino variation of new minimal supergravity (4.20) we get

$$D_M\epsilon + i\Gamma_* A_M\epsilon - i\Gamma_* V_M\epsilon + \frac{i}{2}\Gamma_{MA}\Gamma_* V^A\epsilon = 0 \tag{6.28}$$

To perform the Null reduction on this equation, we will use the Vielbein Ansatz (3.14) to get the reduced spin connection (C.1). We will also need the three dimensional Clifford algebra elements and an Ansatz for the supersymmetry parameter ϵ. For the susy parameters ϵ we choose an Ansatz that decomposes them into (ϵ_+, ϵ_-) as follows

$$\epsilon(x^\mu, z) = \epsilon_+ \otimes \varphi_- + \epsilon_- \otimes \varphi_+. \tag{6.29}$$

When applying this to equation (6.28) we get four equations. For the case $M = z$, we get two algebraic equations

$$\frac{1}{4}\tau^{ab}\gamma_{ab}\epsilon_+ + 2a\gamma_0\epsilon_+ - 3v\gamma_0\epsilon_+ = 0, \tag{6.30a}$$

$$\frac{1}{8}\tau^{ab}\gamma_{ab}\epsilon_- + \frac{\sqrt{2}}{4}\tau^{a0}\gamma_{a0}\epsilon_+ - a\gamma_0\epsilon_- + \frac{1}{2}v\gamma_0\epsilon_- + \frac{\sqrt{2}}{2}\gamma_a v^a\epsilon_+ = 0, \tag{6.30b}$$

For the case $M = \mu$, we get two differential equations

$$\begin{aligned}
\nabla_\mu \epsilon_+ =& + \frac{1}{4}\tau_{\mu 0}\epsilon_+ - \frac{\sqrt{2}}{4}\tau_\mu{}^a\gamma_{a0}\epsilon_- - a_\mu\gamma_0\epsilon_+ + v_\mu\gamma_0\epsilon_+ + \frac{1}{2}e_\mu{}^a v^b\epsilon_{ab}\epsilon_+ \\
&+ \frac{\sqrt{2}}{2}e_\mu{}^a\gamma_a v\epsilon_- + \frac{\sqrt{2}}{2}\tau_\mu\gamma_a v^a\epsilon_- - \frac{1}{2}\tau_\mu\gamma_0 v_0\epsilon_+, \qquad (6.31a)\\
\nabla_\mu \epsilon_- =& - \frac{1}{4}\tau_{\mu 0}\epsilon_- + a_\mu\gamma_0\epsilon_- - v_\mu\gamma_0\epsilon_- - \frac{1}{2}e_\mu{}^a v^b\epsilon_{ab}\epsilon_- - \frac{\sqrt{2}}{2}e_\mu{}^a\gamma_a v_0\epsilon_+ \\
&- \frac{1}{2}\tau_\mu\gamma_0 v_0\epsilon_-, \qquad (6.31b)
\end{aligned}$$

where the covariant derivatives on the susy parameters are covariantized with respect to local spatial rotations and Galilean boosts:

$$\nabla_\mu\epsilon_+ = \partial_\mu\epsilon_+ + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\epsilon_+ \qquad (6.32)$$

$$\nabla_\mu\epsilon_- = \partial_\mu\epsilon_- + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\epsilon_- - \frac{\sqrt{2}}{2}\omega_\mu{}^a\gamma_{a0}\epsilon_+. \qquad (6.33)$$

Here one can already see one of the major differences to the relativistic case as we get not only differential equations but also algebraic ones. As we will see later, one can not simply disregard the algebraic equations as they are necessary to ensure invariance under Galilean boosts. To check this, we will need the boost transformations of all the fields in the Killing spinor equations. These can be easily obtained from (3.17) by just keeping the λ^a terms. For the inverse fields we demand that the invertibility and orthogonality relations (2.54) are consistent with boosts. The Galilean boost transformations for the Vielbein and central charge gauge fields with boost parameter λ^a are then

$$\begin{aligned}
&\delta\tau_\mu = 0, \quad \delta e_\mu{}^a = \lambda^a\tau_\mu, \quad \delta m_\mu = -\lambda_a e_\mu{}^a \\
&\delta\tau^\mu = -\lambda^a e^\mu{}_a, \quad \delta e^\mu{}_a = 0.
\end{aligned} \qquad (6.34)$$

The fields a and v do not transform under boosts while v_μ and a_μ transform as

$$\delta v_\mu = -\lambda_a e_\mu{}^a v, \quad \delta a_\mu = -\lambda_a e_\mu{}^a a. \qquad (6.35)$$

We will also need the transformation behaviour of the spin connections

$$\begin{aligned}
&\delta\omega_\mu{}^{ab} = -\lambda^{[a}\tau_\mu{}^{b]} - \frac{1}{2}\lambda_c e_\mu{}^c\tau^{ab}, \\
&\delta\omega_\mu{}^a = -\partial_\mu\lambda^a + \lambda_b\omega_\mu{}^{ba} + \frac{1}{2}\lambda^b e_{\mu b}\tau^{a0} - \frac{1}{2}\lambda^a\tau_{\mu 0}
\end{aligned} \qquad (6.36)$$

and of $\epsilon_\pm$

$$\delta\epsilon_+ = 0, \quad \delta\epsilon_- = -\frac{\sqrt{2}}{2}\lambda^a\gamma_{a0}\epsilon_+. \qquad (6.37)$$

Note that the asymmetry appearing in most of the boost transformations is characteristic for the Galilean group.

6.4 Solutions of the reduced Killing spinor equations

We already discussed the construction of commuting and anti commuting solutions of the Killing spinor equation in section 5.1. The only difference to the general discussion is that we now look for a pair (ζ_+, ζ_-) of commuting solutions. This is the case because when performing the Null reduction, we end up with $\mathcal{N} = 2$ supersymmetry in three dimensions. This becomes obvious when looking at the Ansatz (6.29). The overall number of supercharges is four, but in four dimensions they are contained in a single four component Majorana spinor, while in three dimensions we get two Majorana spinors with each two supercharges. Therefore we will end up with a pair of commuting solutions (ζ_+, ζ_-). It is then easy to construct anti commuting solutons (ϵ_+, ϵ_-) again from (5.3).

If we take a closer look at the first algebraic equation (6.30a), we can write it as

$$\left(\frac{1}{4}\tau^{ab}\epsilon_{ab} + 2a - 3v\right)\zeta_+ = 0. \tag{6.38}$$

There are two types of solutions, the first one with $\zeta_+ = 0$ and another one where $(\frac{1}{4}\tau^{ab}\epsilon_{ab} + 2a - 3v)$ is zero. In the following sections we will discuss these two different cases in turn.

6.4.1 The case $\zeta_+ = 0$

We take ζ_+ to be zero, i.e. we are looking for commuting solutions $(0, \zeta_-)$ of our Killing spinor equations. If we apply this condition to equations (6.30b), (6.31a) and (6.31b), we get

$$\left(\frac{1}{8}\tau^{ab}\epsilon_{ab} - a + \frac{1}{2}v\right)\gamma_0\zeta_- = 0 \tag{6.39a}$$

$$\left(\frac{1}{2}\tau_\mu{}^a\gamma_{a0} - e_\mu{}^a v\gamma_a - \tau_\mu v^a\gamma_a\right)\zeta_- = 0 \tag{6.39b}$$

$$D_\mu = \left(-\frac{1}{4}\tau_{\mu 0} + a_\mu\gamma_0 - v_\mu\gamma_0 - \frac{1}{2}e_\mu{}^a v^b\epsilon_{ab} - \frac{1}{2}\tau_\mu\gamma_0 v_0\right)\zeta_- = 0. \tag{6.39c}$$

where

$$D_\mu\zeta_- = \partial_\mu\zeta_- + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\zeta_- \tag{6.40}$$

As already argued in [63], we can assume that any non trivial solution ζ_- of these equations is nowhere vanishing. Using this result we can analyze under which conditions the equations (6.39) have non trivial, nowhere vanishing solutions. We will state these conditions in the following theorem 1 and then reconstruct these results in the course of this subsection.

Theorem 1. *The equations* (6.39a) – (6.39c) *have two non trivial, independent and globally well-defined solutions for* ζ_- *if and only if there exists a globally well-defined unit vector* X_a^-

such that the following conditions hold:

$$\tau^{ab} = 4\epsilon^{ab}(2a - v) \tag{6.41a}$$

$$\tau_{a0} = -2\epsilon_{ab}v^b \tag{6.41b}$$

$$v_\mu = a_\mu - \frac{1}{2}\tau_\mu v_0 - \frac{1}{2}Y^{-a}D_\mu X^-_a \tag{6.41c}$$

$$\nabla_\mu(v^\mu - m^\mu v) = 0, \tag{6.41d}$$

where $Y^{-a} = \epsilon^{ab}X^-_b$ *and* $D_\mu X^-_a = \partial_\mu X^-_a + \omega_{\mu a}{}^b X^-_b$.

Proof. Let us now try to prove this theorem by assuming that there exists one globally well-defined, nowhere vanishing solution ζ_- of equations (6.39a) – (6.39c) and this implies that the conditions (6.41a) – (6.41c) must hold. Note that condition (6.41d) is obtained by reducing the four dimensional constraint $\nabla_M V^M = 0$ on the auxiliary field V^M of the new minimal multiplet to three dimensions. This gives two extra constraints, however one of them is $\partial^z v = 0$, which is trivially fulfilled as we assumed in the beginning that the background fields are independent of the internal coordinate.

Evaluating equation (6.39a) on this solution gives

$$\left(\frac{1}{8}\tau^{ab}\epsilon_{ab} - a + \frac{1}{2}v\right)\zeta_- = 0. \tag{6.42}$$

We assumed that ζ_- is nowhere vanishing, hence the condition (6.41a) must hold. To proceed, we can use this condition in equation (6.39b) and apply τ^μ to both sides* and rewrite it in the form

$$A^a\gamma_a\zeta_- = 0, \quad \text{with} \quad A^a = \frac{1}{2}\tau_{0b}\epsilon^{ba} - v^a \tag{6.43}$$

We assumed that ζ_- is nontrivial hence the matrix $A^a\gamma_a$ is singular and thus its determinant must be zero. Using a gamma matrix identity, we obtain

$$\det(A^a\gamma_a)^2 = (A^aA_a)^2 = 0, \tag{6.44}$$

and thus $A^a = 0$, or equivalently

$$\tau_{a0} = -2\epsilon_{ab}v^b, \tag{6.45}$$

which exactly is condition (6.41b). Before we proceed, we use our well-defined solution ζ_- to introduce the following bilinears, which will be helpful for proving theorem 1:

$$N^- = i\bar{\zeta}_-\gamma_0\zeta_-, \quad X^-_a = \frac{1}{N^-}i\bar{\zeta}_-\gamma_{a0}\zeta_-, \quad Y^-_a = \frac{1}{N^-}i\bar{\zeta}_-\gamma_a\zeta_-. \tag{6.46}$$

Note that they are not independent as they are related by $X^-_a = -\epsilon_{ab}Y^-_b$. They obey the properties

$$X^{-a}\gamma_a\zeta_- = \zeta_- \quad Y^{-a}\gamma_a\zeta_- = \gamma_0\zeta_-, \tag{6.47}$$

*Note that the term where condition (6.41a) is inserted drops immediately out again when applying τ^μ.

and are all unit vectors. We now proceed to equation (6.39c), where we can use condition (6.41b) to rewrite this equation as

$$D_\mu \zeta_- = \left(a_\mu \gamma_0 - v_\mu \gamma_0 - \frac{1}{2} \tau_\mu v_0 \gamma_0 \right) \zeta_- . \tag{6.48}$$

We then rewrite (6.48) as

$$D_\mu \zeta_- = B_\mu^- \gamma_0 \zeta_-, \quad \text{with} \quad B_\mu^- = a_\mu - v_\mu - \frac{1}{2} \tau_\mu v_0 \tag{6.49}$$

From this equation and the definition of the bilinears (6.46), we can see that

$$\partial_\mu N^- = 0, \quad \text{and} \quad D_\mu X_a^- = 2 B_\mu^- Y_a^- \tag{6.50}$$

Using the second relation in the last equation and the expression for B_μ^-, we find condition (6.41c). Hence all conditions hold if there is one well-defined solution ζ_-. To show that there are indeed always two independent solutions, we construct a second solution $\zeta_-^{(2)}$ that is linearly independent from the solution we already found, which we will call $\zeta_-^{(1)}$ for this discussion. We can always write $\zeta_-^{(2)}$ as linear combinations of the eigenvectors $\{\zeta_-^{(1)}, \gamma_0 \zeta_-^{(1)}\}$ of $X^{-a}\gamma_a$. Then linearity of the Killing spinor equations implies that we can take

$$\zeta_-^{(2)} = \gamma_0 \zeta_-^{(1)} \tag{6.51}$$

without loss of generality. If we then do the same analysis for $\zeta_-^{(2)}$, we find that exactly the same conditions from theorem 1 hold. The reason for that is that we find the same global γ_0 factors in all Killing spinor equations. Hence using the new solution just gives an overall minus sign as $\gamma_0^2 = -1$. Thus we automatically get two independent solutions. We observe that the new minimal case behaves somehow different than the old minimal [63]. Starting from the old minimal multiplet there is only one solution guaranteed as a second solution imposes more restrictions.

Let us now reverse the statement of theorem 1 and find a non trivial, globally well-defined unit vector X_a^-, while the conditions (6.41a) – (6.41c) hold. This is clearly true for the first two equations (6.39a) and (6.39b), i.e. they are fulfilled for arbitrary ζ_-. Plugging the conditions (6.41b) and (6.41c) into (6.39c), we get

$$D_\mu \zeta_- = \frac{1}{2} Y^{-a} D_\mu X_a^- \gamma_0 \zeta_- . \tag{6.52}$$

The spin connection terms on both sides turn out to cancel and we end up with

$$\partial_\mu \zeta_- = \frac{1}{2} Y^{-a} \partial_\mu X_a^- \gamma_0 \zeta_- = \frac{1}{2} (X_2^- \partial_\mu X_1^- - X_1^- \partial_\mu X_2^-) \gamma_0 \zeta_-, \tag{6.53}$$

which can be integrated to yield*

$$\zeta_- = \exp\left(\frac{1}{2} \arctan\left(\frac{X_1^-}{X_2^-} \right) \gamma_0 \right) \zeta_0^- . \tag{6.54}$$

Here ζ_0^- is a constant but otherwise unconstrained spinor, hence giving two independent solutions for X_a^-. □

*Here we take the principal value of arctan.

6.4.2 The case $\zeta_+ \neq 0$

As in the case of old minimal supergravity it can be shown that the Killing spinor equations are invariant under boosts. Indeed the second algebraic equation (6.30b) transforms into the first algebraic equation (6.30a), the first differential equation (6.31a) also transforms into the first algebraic equation (6.30a) and the second differential equation transforms into a linear combination of the second algebraic (6.30b) and the first differential equation (6.31a). This covariant transformation behaviour under local Galilean boosts allows us to completely fix the gauge freedom for the case $\zeta_+ \neq 0$. It was already shown in [63] that this can be achieved by choosing $\zeta_- = 0$.

When adopting this choice, the Killing spinor equations (6.30a) – (6.31b) will take the following form

$$\left(\frac{1}{4}\tau^{ab}\epsilon_{ab} + 2a - 3v\right)\gamma_0\zeta_+ = 0 \tag{6.55a}$$

$$\left(\frac{1}{2}\tau^{a0}\gamma_{a0} + \gamma_a v^a\right)\zeta_+ = 0 \tag{6.55b}$$

$$\left(\omega_\mu{}^a\gamma_{a0} - e_\mu{}^a\gamma_a v_0\right)\zeta_+ = 0 \tag{6.55c}$$

$$D_\mu\zeta_+ = \left(\frac{1}{4}\tau_{\mu0} - a_\mu\gamma_0 + v_\mu\gamma_0 + \frac{1}{2}e_\mu{}^a v^b\epsilon_{ab} - \frac{1}{2}\tau_\mu\gamma_0 v_0\right)\zeta_+ \tag{6.55d}$$

where

$$D_\mu\zeta_+ = \partial_\mu\zeta_+ + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\zeta_+. \tag{6.56}$$

We will take the same approach as in the $\zeta_- = 0$ case and formulate the results in the following theorem, and then go on to prove the statement.

Theorem 2. *The equations* (6.55a) – (6.55d) *have two non trivial, independent and globally well-defined solutions for* ζ_+ *if and only if* $\tau_{\mu0}$ *is an exact one form and there exists a globally well-defined unit vector* X_a^+ *such that the following conditions hold:*

$$\tau^{ab} = 4\epsilon^{ab}(3v - 2a), \tag{6.57a}$$

$$\tau_{a0} = 2\epsilon_{ab}v^b, \tag{6.57b}$$

$$\omega_\mu{}^a = \epsilon^{ab}e_{\mu b}v_0, \tag{6.57c}$$

$$v_\mu = a_\mu + \frac{1}{2}\tau_\mu v_0 + \frac{1}{2}Y^{+a}D_\mu X_a^+, \tag{6.57d}$$

$$\nabla_\mu(v^\mu - m^\mu v) = 0, \tag{6.57e}$$

where $Y^{+a} = \epsilon^{ab}X_b^+$ *and* $D_\mu X_a^+ = \partial_\mu X_a^+ + \omega_{\mu a}{}^b X_b^+$.

Proof. We will start with the same assumption for existence of well-defined solutions as before. Then the analysis is very similar and it can be easily seen that the conditions (6.57a) – (6.57c) follow from (6.55a) – (6.55c) for a non trivial ζ_+. We will then again define bilinears, using our well-defined solution:

$$N^+ = -i\bar{\zeta}_+\gamma_0\zeta_+, \quad X_a^+ = -\frac{1}{N^+}i\bar{\zeta}_+\gamma_{a0}\zeta_+, \quad Y_a^+ = -\frac{1}{N^+}i\bar{\zeta}_+\gamma_a\zeta_+. \tag{6.58}$$

Note that they are not independent as they are related by $X_a^+ = -\epsilon_{ab} Y_b^+$. They obey the properties

$$X^{+a}\gamma_a\zeta_+ = \zeta_+ \quad Y^{+a}\gamma_a\zeta_+ = \gamma_0\zeta_+, \tag{6.59}$$

and are all unit vectors. In a similar fashion as before we can use condition (6.57b) in equation (6.55d) to obtain

$$D_\mu\zeta_+ = \left(\frac{1}{2}\tau_{\mu 0} - a_\mu\gamma_0 + v_\mu\gamma_0 - \frac{1}{2}\tau_\mu v_0\right)\zeta_+. \tag{6.60}$$

Simply rearranging this equation to emphasize the Gamma matrix structure yields

$$D_\mu\zeta_+ = \frac{1}{2}\tau_{\mu 0}\zeta_+ + B_\mu^+\gamma_0\zeta_+, \quad \text{with} \quad B_\mu^+ = -a_\mu + v_\mu - \frac{1}{2}\tau_\mu v_0. \tag{6.61}$$

From this equation one can show that

$$\partial_\mu(\ln(N^+)) = \tau_{\mu 0}, \quad \text{and} \quad D_\mu X_a^+ = 2B_\mu^+ Y_a^+, \tag{6.62}$$

i.e. the one form $\tau_{\mu 0}$ is exact. Furthermore the second expression in the last equation finally gives condition (6.57d).

We can now use the same reasoning as in the other case and look for other solutions $\zeta_+^{(2)}$, which we will again, without loss of generality, take to be of the form

$$\zeta_+^{(2)} = \gamma_0\zeta_+^{(1)}. \tag{6.63}$$

Evaluating the Killing spinor equations (6.55a) – (6.55d) on this second solution $\zeta_+^{(2)}$ still gives the same conditions as evaluating on $\zeta_+^{(2)}$. Thus we again find that there are always at least two independent solutions, if the conditions of theorem 2 apply.

We can now again reverse the statement and show that given that there exists an exact $\tau_{\mu 0}$, we can find a X_a^+ such that the conditions (6.57a) – (6.57d) hold. As before the first three Killing spinor equations (6.55a) – (6.55c) are identically satisfied for arbitrary ζ_+. Using the conditions (6.57b) and (6.57d) in equation (6.55d) yields

$$D_\mu\zeta_+ = \frac{1}{2}\tau_{\mu 0} + \frac{1}{2}Y^{+a}D_\mu X_a^+\gamma_0\zeta_+. \tag{6.64}$$

The spin connection terms on both sides again cancel out and we get

$$\partial_\mu\zeta_+ = \frac{1}{2}\tau_{\mu 0} + \frac{1}{2}Y^{+a}\partial_\mu X_a^+\gamma_0\zeta_+ = \frac{1}{2}(X_2^+\partial_\mu X_1^+ - X_1^+\partial_\mu X_2^+)\gamma_0\zeta_+. \tag{6.65}$$

We can now use the exactness of $\tau_{\mu 0}$ to write

$$\frac{1}{2}\tau_{\mu 0} = \partial_\mu\Phi, \tag{6.66}$$

where Φ is some well-defined function. Then equation (6.65) can be integrated and we get

$$\zeta_+ = \mathrm{e}^\Phi \exp\left(\frac{1}{2}\arctan\left(\frac{X_1^+}{X_2^+}\right)\gamma_0\right)\zeta_0^+, \tag{6.67}$$

where we again use the principal value of arctan and ζ_0^+ is again an arbitrary unconstrained spinor, giving two independent solutions for ζ_+. □

6.4.3 Cases with $(0,\zeta_-)$ and $(\zeta_+,0)$

We dealt with the cases where there exist two solutions either of type $(0,\zeta_-)$ or two solutions of type $(\zeta_+,0)$ in the previous sections. Now we take a look at the case where four solutions exist, i.e. all conditions from theorem 1 and theorem 2 have to hold simultaneously. If one applies these conditions, we find

$$\begin{aligned}
\tau^{ab} &= 4\epsilon^{ab}v \\
\tau_{a0} &= 0 \to v^a = 0 \\
\omega_\mu{}^a &= \epsilon^{ab}e_{\mu b}v_0 \\
a_\mu &= -\tau_\mu v_0 - \frac{3}{4}Y^{-a}D_\mu X_a^- - \frac{1}{4}Y^{+a}D_\mu X_a^+ .
\end{aligned} \tag{6.68}$$

In the next section we will give some explicit examples for Newton-Cartan backgrounds.

6.4.4 Examples

We will now give two explicit examples for three dimensional Newton-Cartan backgrounds that admit Killing spinors solving equations (6.30a) – (6.31b). The following examples are inspired from their four dimensional counterparts of relativistic backgrounds [59] of the new minimal multiplet. These Newton-Cartan geometries are also valid as backgrounds for supersymmetric theories stemming from the old minimal multiplet which was analyzed in [63], where the following discussion is done in a very similar fashion. The following two examples are different in their foliation structure. While the first one admits an integrable foliation $\tau \wedge \mathrm{d}\tau = 0$, the opposite is the case for the second example for which $\tau \wedge \mathrm{d}\tau \neq 0$.

Integrable foliation

The class of backgrounds admitting an integrable foliation are characterized by the property $\tau \wedge \mathrm{d}\tau = 0$ or written in components $\tau_{[\mu}\partial_\nu\tau_{\rho]} = 0$. To formulate a suitable Ansatz we will split our coordinates x^μ into $x^\mu = \{x^0 = t, x^i\}$, where $i = 1,2$ corresponds to the spatial coordinates and t represents the time coordinate. We then choose an Ansatz that specifies the Newton-Cartan fields τ_μ, $e_\mu{}^a$ and m_μ and assign them arbitrary functions $\lambda(t,x^i)$, $\kappa(t,x^i)$ and $\phi(t,x^i)$. We choose an Ansatz as

$$\tau_\mu \mathrm{d}x^\mu = \mathrm{e}^\kappa \mathrm{d}t, \quad e_\mu{}^a \mathrm{d}x^\mu = \mathrm{e}^\lambda \delta_i^a \mathrm{d}x^i, \quad \text{and} \quad m_\mu \mathrm{d}x^\mu = \phi\tau_\mu \mathrm{d}x^\mu. \tag{6.69}$$

We then find the projective inverse Vielbeine τ^μ and $e^\mu{}_a$ as

$$\tau^\mu\partial_\mu = \partial_0 = \mathrm{e}^{-\kappa}\frac{\partial}{\partial t}, \quad e^\mu{}_a\partial_\mu = \partial_a = \mathrm{e}^{-\lambda}\delta_a^i\frac{\partial}{\partial x^i}. \tag{6.70}$$

It can be easily checked that this Ansatz obeys $\tau_{[\mu}\partial_\nu\tau_{\rho]} = 0$ but the timelike Vielbein is not closed and we find the torsion

$$\tau_{a0} = \partial_a\kappa, \quad \tau_{ab} = 0. \tag{6.71}$$

The integrability condition $\tau_{[\mu}\partial_\nu\tau_{\rho]} = 0$ implies that the Newton-Cartan manidolf is foliated into two dimensional spacelike slices and a one-dimensional timelike direction. The metric $h^{\mu\nu} = e^\mu{}_a e^{\nu a}$ is then conformally flat on the spatial slices. Furthermore we observe that the central charge gauge field m_μ in our Ansatz only has a non zero timelike component, which is given by ϕ. In the usual context of Newton-Cartan geometries ϕ plays the role of the Newton potential, hence we will also use this identification here. However, this is not fully justified as we do not work with dynamic non relativistic gravity but only on a rigid background.

We will now try to express the background fields $\{v, v_\mu, a, a_\mu\}$ in terms of functions $\{\phi, \kappa, \lambda\}$ that carry the geometric information of our Ansatz. By plugging our Ansatz into (3.27) we obtain the boost and spin connection

$$\omega_\mu{}^{ab} = 2e_\mu{}^a\partial^b\lambda, \quad \omega_\mu{}^a = e_\mu{}^a\partial_0\lambda - \tau_\mu(\partial^a\phi + \phi\partial^a\kappa). \tag{6.72}$$

It is then easy to work out the corresponding curvatures from (3.26). For simplicity we will assume $\partial_0\kappa = \partial_0\lambda = 0$ so that the timelike and the spacelike Vielbeine are time independent. Using this assumption we immediately find

$$R_{\mu\nu}(J^{ab}) = -2(\partial_c\partial_c\lambda + \partial_c\lambda\partial_c\lambda)e_\mu{}^a e_\nu{}^b, \tag{6.73}$$
$$R_{\mu\nu}(G^a) = 2\left(\partial^b\Phi^a + \partial^b\kappa\Phi^a - \partial^a\lambda\Phi^b - \Phi^{(a}\tau^{b)0}\right)\tau_{[\mu}e_{\nu]b} + 2\Phi^b\partial_b\lambda\tau_{[\mu}e_{\nu]}{}^a, \tag{6.74}$$

where $\Phi^a = \partial^a\phi + \phi\partial^a\kappa$. We will demand that there exist two solutions of the Killing spinor equations of the form $(0, \zeta_-)$. This requirement can equivalently be formulated as there exists a well-defined unit vector X_a^-. We will choose the unit vector field X_a^- to be constant $X_1^- = X_2^- = 1/\sqrt{2}$, which corresponds to a local gauge fixing for spatial rotations. It is then straightforward to derive expressions for the background fields from relations (6.41a) – (6.41c) in theorem 1. We find

$$v = 2a, \quad v_0 = \frac{2}{3}a_0, \quad v^a = \frac{1}{2}\epsilon^{ab}\partial_b\kappa,$$
$$a_a = \frac{1}{2}\epsilon_{ab}(\partial^b\kappa + 2X^{-b}\partial^c\lambda X_c^-), \tag{6.75}$$

with a Killing spinor

$$\zeta_- = \mathrm{e}^{\frac{\pi}{8}\gamma_0}\zeta_0^-, \tag{6.76}$$

where ζ_0^- is a constant spinor. We still need to verify that condition (6.57e) on v_μ holds for this example. This is indeed fulfilled if we demand the extra condition

$$\nabla_\mu(m^\mu v) = 0 \tag{6.77}$$

to hold. Plugging our Ansatz into this condition and working out the additional connection term yields

$$\partial_0(\phi a) = 0, \tag{6.78}$$

where we used the fact that the Vielbein fields are time independent. This can be trivially fulfilled if we choose $a = 0$, otherwise we have to use a suitable potential ϕ. Note that the background fields v, a, v_0 and a_0 are in principle not constrained by any geometric information. They only have to obey the conditions $v = 2a$ and $v_0 = 2/3a_0$. Thus we can choose some arbitrary value for these fields and could even set them all to zero.

We will now consider an explicit example for this type of solution. We will consider a manifold with spatial slices that are isomorphic to the Poincaré disk. The Poincaré disk is an example of hyperbolic geometry which is a natural choice because in the relativistic case solutions with AdS backgrounds can be found. We will define

$$\mathrm{e}^{\lambda} = \frac{2}{1 - x^2/r^2}, \quad \text{where} \quad x^2 = x_i x^i < r^2, \tag{6.79}$$

and r denotes the radius of the Poincaré disk. We can then use this Ansatz to evaluate the curvatures above. For the spatial curvature, which is fully determined by Ansatz (6.79), we find $R_{\mu\nu}(J^{ab}) = -2/r^2 e_\mu{}^a e_\nu{}^b$, where we used that $\partial_c \partial_c \lambda = 1/r^2$. However, the boost curvature $R_{\mu\nu}(G^a)$ is not fully determined by this Ansatz because the Newton potential ϕ is not constrained by supersymmetry. This is a consistent choice for our background fields and gives a manifold where the spatial slices admit this hyperbolic structure. Note, however, that this Ansatz does not specify any function for the time direction, i.e. κ is still arbitrary. This is another point where this analysis differs from the one on the old minimal multiplet in [63]. Although there is more supersymmetry preserved in the new minimal case, there is still more freedom in choosing the background fields.

We can do the same procedure for the solutions of type $(\zeta_+, 0)$. We again choose constant unit vectors X_a^+, i.e. $X_1^+ = X_2^+ = 1/\sqrt{2}$. Then we can derive expressions for the background fields from relations (6.57a) – (6.57c) and find

$$\begin{aligned} v &= \frac{2}{3}a, \quad v_0 = a_0 = 0, \quad v^a = -\frac{1}{2}\epsilon^{ab}\partial_b\kappa, \\ a_a &= -\frac{1}{2}\epsilon_{ab}(\partial^b\kappa + 2X^{+b}\partial^c\lambda X_c^+), \end{aligned} \tag{6.80}$$

with Killing spinor

$$\zeta_+ = \mathrm{e}^{\frac{\kappa}{2} + \frac{\pi}{8}\gamma_0}\zeta_0^+, \tag{6.81}$$

where ζ_0^+ is again a constant spinor. However we also have to check condition (6.57c) for the boost connection which has to be consistent with (6.72). We have $v_0 = 0$ and thus $\omega_\mu{}^a = 0$. By comparison we find that $\Phi^a = 0$. This can be solved for ϕ and gives $\phi = \mathrm{e}^{-\kappa}$. When inserting this back into the Ansatz for the central charge gauge field we find that m_μ is a constant one form given by $m_\mu \mathrm{d}x^\mu = dt$. Note that the background fields v and a are again not restricted by geometry and can hence be chosen arbitrarily. In conclusion we find that a twistless torsional Newton-Cartan geometry can be constructed with spatial slices that are isomorphic to the Poincaré disk and a Newton potential of the form $\phi = \mathrm{e}^{-\lambda}$ if the background admits two supercharges of the form $(\zeta_+, 0)$.

Non integrable foliation

We will also discuss an alternative class of backgrounds which are characterized by

$$\tau \wedge \mathrm{d}\tau \neq 0, \tag{6.82}$$

which describes a foliated spacetime which is non integrable. Usually such spacetimes are excluded in physical theories as they do not admit a causal structure as we already discussed in section 3.2.1. However, in this application we consider the resulting geometry not as a spacetime but as a rigid background. As in the previous example we will split our coordinates x^μ into $x^\mu = \{t, x^i\}$ where $i = 1, 2$. We then choose an Ansatz

$$\tau_\mu \mathrm{d}x^\mu = \mathrm{d}t + \alpha_i \mathrm{d}x^i, \quad e_\mu{}^a \mathrm{d}x^\mu = e_i{}^a \mathrm{d}x^i, \quad \text{and} \quad m_\mu \mathrm{d}x^\mu = -\frac{1}{2}\alpha_i \mathrm{d}x^i. \tag{6.83}$$

From this Ansatz the projective inverse Vielbeine τ^μ and $e^\mu{}_a$ follow:

$$\tau^\mu \partial_\mu = \frac{\partial}{\partial t}, \quad e^\mu{}_a \partial_\mu = e^i{}_a \left(\frac{\partial}{\partial x^i} - \alpha_i \frac{\partial}{\partial t} \right). \tag{6.84}$$

For simplicity we will assume time independence of α_i and $e^i{}_a$, i.e. $\partial_t \alpha_i = \partial_t e^i{}_a = 0$. With this assumption we find the torsion

$$\tau_{a0} = 0, \quad \tau_{ab} = 2 e^i{}_a e^j{}_b \partial_{[i} \alpha_{j]}. \tag{6.85}$$

Next we can calculate the spin connection, using (C.1) again, we find

$$\omega_\mu{}^a = \frac{1}{4} e_{\mu b} \tau^{ab}, \quad \omega_\mu{}^{ab} = 2 e^{\nu[a} \partial_{[\mu} e_{\nu]}{}^{b]} - e_\mu{}^c e^{\nu a} e^{\rho b} \partial_{[\nu} e_{\rho]c} + \frac{1}{4} \tau_\mu \tau^{ab}. \tag{6.86}$$

Note the the spin connections are related by $\omega_\mu{}^a = e_{\mu b} \tau^\nu \omega_\nu{}^{ab}$. In contrast to non integrable backgrounds coming from the old minimal multiplet, we can still establish four supercharges here. As shown in [63] supersymmetry gets broken in the context of the old minimal multiplet, i.e. two supercharges get lost. If we demand that all conditions from theorem 1 and 2 have to hold, we still find $\tau_{ab} \neq 0$ and some non trivial background fields.

We will again provide an explicit example. We choose

$$\alpha_i \mathrm{d}x^i = -\frac{r}{2\sin\xi} \mathrm{d}\eta, \quad e_i{}^1 \mathrm{d}x^i = \frac{r}{2} \mathrm{d}\xi, \quad e_i{}^2 \mathrm{d}x^i = \frac{r}{2} \sin\xi \mathrm{d}\eta, \tag{6.87}$$

which is defined on a three manifold with coordinates

$$x^\mu = (t, \xi, \eta), \qquad t \in \mathbb{R}, \quad \xi \in [0, \pi), \quad \eta \in [0, 2\pi). \tag{6.88}$$

Using this example the torsion can be evaluated to be $\tau_{ab} = 2/r \epsilon_{ab}$. We can then derive explicit expressions for the Newton-Cartan connection which yields

$$\omega_\mu{}^{ab} = \frac{1}{r} \epsilon^{ab} (\tau_\mu - 4 \cot\xi e_\mu{}^2), \quad \omega_\mu{}^a = \frac{1}{2r} e_{\mu b} \epsilon^{ab}. \tag{6.89}$$

We can then follow the same procedure as in the example of the integrable foliation and use theorem 1 and 2 to get explicit values for the background fields. This gives

$$a_0 = -v_0, \quad v_a = a_a = 0, \quad v = v_0 = \frac{1}{2r}, \tag{6.90}$$

where we used the relations (6.68) on the background fields. As in the previous example we need to check that condition (6.41d) holds, which again gives $\nabla_\mu(m^\mu v) = 0$. This leads to a differential equation for α_i. By comparison to the case of old minimal sugra in [63] we choose $\alpha_i \mathrm{d}x^i = -\frac{r}{2} f(\xi)\mathrm{d}\eta$ for a general function $f(\xi)$. This then gives the following equation

$$\partial_i f(\xi) + \cot\xi f(\xi) = 0, \tag{6.91}$$

which needs to be obeyed by our Ansatz. Solving this equation gives exactly the form of α_i in (6.87). We can then use Ansatz (6.87) to obtain explicit expressions for the curvatures:

$$R_{\mu\nu}(J^{ab}) = \frac{14}{r^2} e_\mu{}^a e_\nu{}^b, \qquad R_{\mu\nu}(G^a) = -\frac{1}{2r^2}\tau_{[\mu} e_{\nu]}{}^a. \tag{6.92}$$

The Ansatz in (6.87) is inspired by a similar solution with $f(\xi) = \cos(\xi)$ in the null reduction of old minimal supergravity [63]. In that case however, only two supercharges are preserved. Similar torsional Newton-Cartan geometries also appear in the context of non relativistic strings and limits of the AdS/CFT correspondence [69].

To summarise we established a non relativistic supersymmetric theory admitting four supercharges that is coupled to a Newton-Cartan background (6.83) with (6.87).

Chapter 7

Conclusion

I recent years a lot of progress has been made in understanding non perturbative QFTs. This was mainly achieved by studying susy QFTs on curved backgrounds. There are examples of such theories where one is able to compute the path integral exactly. This new knowledge can then often be used to gain more insight and test certain AdS/CFT correspondences. Until now this was mostly done for relativistic theories and only very recently for non relativistic theories [63]. In this thesis we followed the same procedure as in [63] but started from a different supergravity multiplet, i.e. we did not start from the old minimal multiplet but instead used the new minimal multiplet which is often used to couple to field theories that admit an R symmetry.

We were able to construct explicit examples for a non relativistic field theory with rigid supersymmetry on curved Newton-Cartan three manifolds. We were able to derive a Lagrangian and corresponding supersymmetry transformation rules that describe a pseudo chiral multiplet on a non relativistic background in three dimensions. The starting point of this procedure was $\mathcal{N} = 1, D = 4$ new minimal supergravity [51] coupled to a chiral multiplet. After applying a rigid limit with the method of Festuccia and Seiberg [16] we performed a Null reduction on the Lagrangian and transformation rules to end up with this non relativistic theory for a pseudo chiral multiplet. The resulting backgrounds are geometrically restricted by Newton-Cartan background fields and auxiliary fields from the supergravity multiplet. However, the resulting Lagrangian is only consistent if one can find some non trivial supersymmetry parameters. These supersymmetry parameters are constrained by the Killing spinor equations. In contrast to the relativistic case the Killing spinor equations of a non relativistic theory split in a set of differential and algebraic equations which are both needed for consistency. We then showed what conditions need to be satisfied by the background and auxiliary fields, such that global nowhere vanishing solutions exist.

One drawback of this method is that it is only possible to construct non relativistic theories that can be obtained via a Null reduction from a four dimensional relativistic theory. Thus we are limited to such theories and can not survey what backgrounds are possible in a systematic way. A better approach would be to directly apply the Festuccia Seiberg method to three dimensional off shell Newton-Cartan supergravity theories. However, altough there is an on shell Newton-Cartan supergravity theory [40] and quasi off shell realizations that only close upon further further geometric constraints [70], there is no real off shell formulation in three dimensions. For that reason this slightly more cumbersome method has to be used. It would be very interesting to study how the Festuccia Seiberg method could be directly implemented in three dimensional Newton-Cartan supergravity. This would certainly lead to more general non relativistic susy QFTs on curved backgrounds and is still a very interesting open problem. Another natural generalization would be to consider more general matter content or theories with extended supersymmetry. There are many open questions that are worth to be studied in future.

Appendix A

Conventions

We use a metric with mostly plus signature. In four dimensions the metric in Minkowski coordinates $\{x^0, x^1, x^2, x^3\}$ is thus given by diag$(-1,1,1,1)$. The letters $M, N, ..$ are used for four dimensional curved indices, while $A, B, ..$ are used for four dimensional flat indices. For the Null reduction we use adapted coordinates that can be written as $x^M = \{x^\mu, z\}$ with $\mu = 0,1,2$ and z the internal dimension. In three dimensions we use greek letters $\mu, \nu, ..$ from the middle of the alphabet as spacetime indices, while $a, b, ..$ latin letters from the start of the alphabet are reserved for flat indices. For the null reduction we introduce flat Null coordinates

$$x^+ = \frac{1}{\sqrt{2}}(x^0 + x^3), \quad x^- = \frac{1}{\sqrt{2}}(x^0 - x^3). \tag{A.1}$$

The flat coordinates are then split into $x^A = \{a, +, -\}$, with $a = 1,2$. The four dimensional flat metric in Null coordinates $\{x^-, x^+, x^1, x^2\}$ is then given by

$$\eta_{AB} = \begin{array}{c c} & \begin{array}{ccc} - & + & b \end{array} \\ \begin{array}{c} - \\ + \\ a \end{array} & \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \delta_{ab} \end{pmatrix} \end{array}, \tag{A.2}$$

which can also be written more compact as $\eta_{+-} = -1$ and $\eta_{ab} = \delta_{ab}$.

The Levi-Civita symbol in four dimensions ϵ_{ABCD} is defined as

$$\epsilon_{0123} = 1, \quad \epsilon^{0123} = -1. \tag{A.3}$$

The flat Levi-Civita symbol in Null coordinates is then given by

$$\epsilon_{-+ab} = \epsilon_{ab}, \quad \epsilon^{-+ab} = -\epsilon^{ab}, \tag{A.4}$$

where we introduced ϵ^{ab} and ϵ_{ab} with

$$\epsilon_{12} = \epsilon^{12} = 1 \tag{A.5}$$

In three dimensions another often used relation between the Lorentz generators and the Levi-Civita symbol is

$$\gamma_{ab} = \epsilon_{ab}\gamma_0, \quad \gamma_{a0} = \epsilon_{ab}\gamma_b. \tag{A.6}$$

(Anti)symmetrization is defined with weight one, i.e. as

$$A_{[ab]} = \tfrac{1}{2}(A_{ab} - A_{ba}) \quad \text{and} \quad A_{(ab)} = \tfrac{1}{2}(A_{ab} + A_{ba}) \tag{A.7}$$

When we write tensor products and the direct sum we define them as

$$\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac \\ bc \\ ad \\ bd \end{pmatrix}, \qquad \begin{pmatrix} a \\ b \end{pmatrix} \oplus \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}. \tag{A.8}$$

However we will almost never write matrices or tensor products explicitly. We will also often make use of the relation for matrices A, B, C and D

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD), \tag{A.9}$$

if the matrix products AC and BD exist.

Appendix B

Spinors and the Clifford algebra

Spinors are crucial in supersymmetry and for the formulation of supergravity. In this section we will give a short summary of important features of spinors and the Clifford algebra. As is turns out the spin group $Spin(n)$ is the double cover of $SO(n)$. This is most easily seen for $Spin(3)$, as $Spin(3) \cong SU(2)$ and $SU(2)$ is the double cover of $SO(3)$. It turns out that the elements of the Clifford algebra $C\ell_{1,3}(\mathbb{R})$ give a representation of the spin group. In this section, we will give basic properties of representation matrices of the Clifford algebra, i.e. the Γ matrices. We will present everything in D dimensions with one timelike dimension and $D-1$ spacelike dimensions. We will mostly follow [71, 24] here.

B.1 Gamma matrices and their properties

The Clifford algebra in D dimensions is generated by Clifford algebra elements Γ_A, obeying

$$\{\Gamma_A, \Gamma_B\} = \Gamma_A\Gamma_B - \Gamma_B\Gamma_A = 2\eta_{AB} \tag{B.1}$$

To illustrate how an explicit representation of the algebra (B.1) with matrices $(\Gamma_A)_\alpha{}^\beta$ looks like, we will first give examples in two and three dimensions. Here a complete basis of Γ matrices is given by the Pauli matrices

$$\Gamma_0 = i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \Gamma_1 = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \Gamma_2 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{B.2}$$

The Pauli matrices are hermitian and all square to the identity $\sigma_i^2 = \mathbb{1}$. Together with the identity matrix $\mathbb{1}_2$ they form a basis of the real vector space of hermitian 2×2 matrices. Note that in two dimensions, i.e. $(1+1)$ dimensions, we only need $\{\Gamma_0, \Gamma_1\}^*$ to form the Clifford algebra. It is straightforward to check this with the relation $\sigma_i\sigma_j = \delta_{ij}\mathbb{1} + i\epsilon_{ijk}\sigma_k$. For three dimensions $(1+2)$ we can add σ_3 and obtain again a basis of the Clifford algebra. This is a general feature, if we know a representation in $D = 2n$ dimensions, the representation in $2n+1$ dimensions will always have the same size of representation matrices, we just have to

*Note that what is meant here is the set of Γ_0 and Γ_1, not the anticommutator.

add a highest rank element. Thus if we want to obtain representations of higher dimensions we can write

$$\begin{aligned}
\Gamma_0 &= i\sigma_1 \otimes \mathbb{1} \otimes \mathbb{1} \otimes ... \\
\Gamma_1 &= \sigma_2 \otimes \mathbb{1} \otimes \mathbb{1} \otimes ... \\
\Gamma_2 &= \sigma_3 \otimes \sigma_1 \otimes \mathbb{1} \otimes ... \\
\Gamma_3 &= \sigma_3 \otimes \sigma_2 \otimes \mathbb{1} \otimes ... \\
\Gamma_4 &= \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes ... \\
... &= ....
\end{aligned} \tag{B.3}$$

If we take $D = 2n$ above, we need exactly n factors to get a representation for this dimension. Then the dimension of the representation is 2^n or equivalently $2^{D/2}$. If we want to obtain a representation for odd dimensions $D = 2n + 1$ we just need to add one more highest rank element. Thus the dimension of the representation stays the same and we can denote the representation for general dimension D as $2^{\lfloor D/2 \rfloor}$, where $\lfloor \cdot \rfloor$ is the floor function that rounds down to the next integer.

The full Clifford algebra consists of the identity $\mathbb{1}$, the D generators Γ_A and all linear independent combinations of those. Due to relation (B.1), all symmetric products of Γ will reduce again to a lower number of Γ matrices, hence we have to consider antisymmetric products. These are then defined by

$$\Gamma_{AB} = \tfrac{1}{2}(\Gamma_A \Gamma_B - \Gamma_A \Gamma_A) \qquad \Gamma_{A_1..A_r} = \Gamma_{[A_1} ... \Gamma_{A_r]}. \tag{B.4}$$

In fact it can be easily seen at this point that if we define* $\sigma^{MN} = -\frac{i}{4}\Gamma^{MN}$ matrices, which are part of the Clifford algebra $C\ell_{1,3}(\mathbb{C})$, they follow the commutation relation [72]

$$i[\sigma^{MN}, \sigma^{RS}] = \eta^{NR}\sigma^{MS} - \eta^{MR}\sigma^{NS} - \eta^{SM}\sigma^{RN} + \eta^{SN}\sigma^{RM}, \tag{B.5}$$

hence they are also a representation of the Lorentz algebra $\mathfrak{so}(1, 3)$. This is the reason why the Clifford algebra is so important for spinor representations.

In general we have the hermiticity properties like

$$\Gamma_0^\dagger = -\Gamma_0, \quad \Gamma_a^\dagger = \Gamma_a, \tag{B.6}$$

i.e. the timelike Γ matrix is anti hermitian while the spacelike ones are hermitian. We will also adopt the convention that the timelike Γ matrix squares to -1:

$$\Gamma_0^2 = -1. \tag{B.7}$$

*The complex matrices σ^{MN} are often defined when working in a quantum regime, where the Lie algebras are complexified. However we will only work in a classical regime in this work and thus use Γ^{MN} throughout.

Of course we can always find another representation like

$$\Gamma' = U^{-1}\Gamma U, \tag{B.8}$$

where U has to be unitary if (B.6) should still hold. With this in mind, we can also write (B.6) more compact as

$$\Gamma_A^\dagger = \Gamma_0\Gamma_A\Gamma_0. \tag{B.9}$$

In **even dimensions** a complete set $\{\Gamma^{(n)}\}$ with $n = 0, 1, .., D$ is given by

$$\Gamma^{(n)} = \Gamma_{A_1..A_n} \equiv \Gamma_{[A_1}...\Gamma_{A_n]}. \tag{B.10}$$

We can then write the last matrix as*

$$\Gamma_* = (-i)^{D/2+1}\Gamma_0\Gamma_1...\Gamma_{D-1}, \tag{B.11}$$

obeying

$$\{\Gamma_*, \Gamma_a\} = 0, \quad \Gamma_*^2 = 1. \tag{B.12}$$

Γ_* is called the highest rank matrix and provides a link between even and odd dimensions. In even dimensions there is also a very useful relation between Γ matrices, the Levi-Civita symbol and the highest rank matrix:

$$\Gamma_{A_1...A_n} = \frac{i^{D/2+1}}{(D-n)!}\epsilon_{A_1...A_D}\Gamma_*\Gamma^{A_D...A_{n+1}} \tag{B.13}$$

The highest rank element Γ_* anticommutes with Γ_a and can thus be used as Γ_{D+1} in the next higher **odd dimension**. If a basis in even dimensions D is given by $\{\Gamma_0, .., \Gamma_{D-1}\}$, we can easily obtain a basis for $D+1$ like

$$\{\Gamma_0, .., \Gamma_{D-1}, \Gamma_*\}. \tag{B.14}$$

We can just add the highest rank element of an even representation in D dimensions to obtain a representation in $D+1$ dimensions. This is exactly what we did above in the example with the Pauli matrices.

Let us now take a look at what symmetries the Γ matrices have. They can be either symmetric or antisymmetric which will be described by a unitary **charge conjugation matrix** C. The charge conjugation matrix obeys

$$C^T = -\epsilon C, \qquad \Gamma_A^T = -\eta C\Gamma_A C^{-1}, \tag{B.15}$$

where $\epsilon = \pm 1$ and $\eta = \pm 1$. In addition they obey $C = C^\dagger = C^{-1}$. Under a change of representation as in (B.8), the charge conjugation matrix transforms as

$$C' = U^\dagger C U. \tag{B.16}$$

*In physics literature the highest rank matrix is often called Γ_{D+1}, e.g. Γ_5 for $D = 4$.

The exact values for ϵ and η depend on the signature and spacetime dimension. This then leads to a general expression for the transpose of $C\Gamma^{(n)}$ with a Γ matrix of rank n:

$$\left(C\Gamma^{(n)}\right)^T = -\epsilon(-)^{n(n-1)/2}(-\eta)^n C\Gamma^{(n)}. \tag{B.17}$$

We would also like to define the complex conjugation properties of Γ matrices as those play an important role when defining Majorana spinors. For a matrix B we can write

$$\Gamma_A^* = +\eta B\Gamma_A B^{-1}. \tag{B.18}$$

If we use that $\Gamma^* = (\Gamma^\dagger)^T$ and the properties (B.18) and (B.9), we immediately see that

$$B = -C\Gamma_0 \quad \text{and} \quad B^*B = \epsilon\eta. \tag{B.19}$$

In figure B.1 one can see the (ϵ, η) values for different spacetime dimensions, always with one timelike direction. Note that for even dimensions there are always two solutions, i.e. two different charge conjugation matrices, while in odd dimensions C is unique. In table B.1 we give the most convenient choices for supersymmetry in bold.

D	ϵ	η
0	-1	$+1$
	$\mathbf{-1}$	$\mathbf{-1}$
1	-1	-1
2	-1	-1
	$\mathbf{+1}$	$\mathbf{+1}$
3	$+1$	$+1$
4	$\mathbf{+1}$	$\mathbf{+1}$
	$+1$	-1
5	$+1$	-1
6	$+1$	-1
	$\mathbf{-1}$	$\mathbf{+1}$
7	-1	$+1$

Figure B.1: Values for ϵ and η for spacetime dimension $D \mod 8$. Bold values are those which are most convenient for supersymmetry applications.

For our application we only need these properties for $D = 3, 4$. Thus we will give an explicit form here. For $\mathbf{D} = \mathbf{4}$, with the choice $\epsilon = +1$ and $\eta = +1$, this means

$$C_4^T = -C_4, \quad \Gamma_A^T = -C_4\Gamma_A C_4^{-1}. \tag{B.20}$$

If we go from three to four spacetime dimensions we decompose the charge conjugation matrix as

$$C_4 = C_3 \otimes \sigma_1. \tag{B.21}$$

The charge conjugation matrix C_3 for $\mathbf{D} = \mathbf{3}$ then obeys

$$C_3^T = -C_3, \quad \gamma_0^T = -C_3\gamma_0 C_3^{-1}, \quad \gamma_a^T = -C_3\gamma_a C_3^{-1}. \tag{B.22}$$

B.2 Spinors

In particle physics one often encounters Dirac spinors. However Dirac spinors are elements of the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation of the Lorentz group. From this it is clear that there are more fundamental spinors that transform under either the $(\frac{1}{2},0)$ or the $(0,\frac{1}{2})$ representation of the Lorentz group. These are the Weyl spinors. They form the smallest complex representation of the spin group, while Majorana spinors form the smallest real representation of the spin group.

As we just mentioned, we want to find irreducible spinors as Weyl and Majorana spinors. To construct Weyl spinors we use Γ_*, which squares to 1 as introduced in (B.11). Note that Γ_* is only available in even dimensions and hence the existence of **Weyl spinors** is also limited to even dimensions. We can then define left and right chiral (or Weyl) spinors as

$$\lambda_L = \tfrac{1}{2}(\mathbb{1} + \Gamma_*)\lambda, \quad \lambda_R = \tfrac{1}{2}(\mathbb{1} - \Gamma_*)\lambda, \tag{B.23}$$

which are projections of Dirac spinors. For the projection operators the notation $P_{L/R} = \frac{1}{2}(1 \pm \Gamma_*)$ is often used. However in this work we will mostly work with Majorana spinors, which will be introduced now.

Another possible projection is in form of a reality condition. This will lead to **Majorana spinors**. A reality condition on spinors looks like

$$\lambda^* = A\lambda, \tag{B.24}$$

for some matrix A. We then require consistency with Lorentz transformations, i.e. when we take a Lorentz transformation of both sides, we get

$$\left(-\tfrac{1}{4}\Gamma_{AB}\lambda\right)^* = -\tfrac{1}{4}A\Gamma_{AB}\lambda \quad \Rightarrow \quad B\Gamma_{AB}B^{-1}A = A\Gamma_{AB}. \tag{B.25}$$

To fulfill this condition we choose $A = \alpha B$. For consistency we also require $\lambda^{**} = \lambda$ which instantly translates into

$$|\alpha| = 1 \quad \text{and} \quad B^*B = \mathbb{1}. \tag{B.26}$$

It is clear that the second condition can not be fulfilled in every spacetime dimension. If we look at equation (B.19) and table B.1 we immediately see that there can not exist Majorana spinors in $D = 5, 6, 7$*.

Note that we can define even more spinor types. There are also symplectic spinors for extended supersymmetry where $B^*B = -\mathbb{1}$. Or we can combine Weyl and Majorana spinors, i.e. demand that both defining conditions must hold. We will not use them in this work, hence we do not specify them, but details can be found in [24, 71].

*Note that this is only true for our choice of one timelike direction. For cases with more timelike directions there can exist Majorana spinors in these dimensions (see [71]).

There is another approach to Majorana spinors where we demand that the Majorana conjugate equals the Dirac conjugate as defined here

$$\bar{\lambda} = \lambda^T C, \quad \bar{\lambda} = i\lambda^\dagger \Gamma_0. \tag{B.27}$$

It follows that λ must be real and

$$C = i\Gamma_0 \quad \Rightarrow \quad B = i\mathbb{1}, \quad B^* B = \mathbb{1}, \tag{B.28}$$

i.e. we get a concrete realization of the above general result.

B.2.1 Spinor indices

We will mostly avoid using spinor indices in this work, however, sometimes it can be useful to explicitly write them down. To do this in a consistent way, we start by assigning a spinor λ a lower spinor index λ_α. In general we will use greek lower case letters from the start of the alphabet for spinor indices. For an expression like $\Gamma_A \lambda$ we then write $(\Gamma_A)_\alpha{}^\beta \lambda_\beta$, hence the natural index position on Γ matrices is $(\Gamma_A)_\alpha{}^\beta$. To be able to raise and lower indices we need to introduce matrices $\mathcal{C}^{\alpha\beta}$ and $\mathcal{C}_{\alpha\beta}$. We will see soon that they are related to the charge conjugation matrix C we introduced above. The convention we use is the so called 'NorthWest-SouthEast' (NW-SE) convention, where indices are always contracted from upper left to lower right, e.g.

$$\lambda^\alpha = \mathcal{C}^{\alpha\beta} \lambda_\beta, \quad \lambda_\alpha = \lambda^\beta \mathcal{C}_{\beta\alpha}. \tag{B.29}$$

Of course we need to make sure that these two equations in (B.29) are consistent and thus we need to require

$$\mathcal{C}^{\alpha\beta} \mathcal{C}_{\gamma\beta} = \delta^\alpha{}_\gamma, \quad \mathcal{C}_{\beta\alpha} \mathcal{C}^{\beta\gamma} = \delta_\alpha{}^\gamma. \tag{B.30}$$

We then identify a spinor λ^α with the Majorana conjugate $\bar{\lambda}$. If we compare (B.29) and (B.27), we can identify

$$\mathcal{C}^{\alpha\beta} = C^T, \quad \mathcal{C}_{\alpha\beta} = C^{-1} = C. \tag{B.31}$$

Appendix C

Results

C.1 Null reduction results

We can use the Ansatz (3.14) with $S = 1$ to reduce the torsion-free spin connection of General Relativity (2.36). We then get

$$
\begin{aligned}
\hat{\Omega}_\mu{}^{ab} &= \omega_\mu{}^{ab} - \frac{1}{2} m_\mu \tau^{ab}, & \hat{\Omega}_z{}^{ab} &= \frac{1}{2}\tau^{ab}, \\
\hat{\Omega}_\mu{}^{a+} &= -\omega_\mu{}^{a} + \frac{1}{2} m_\mu \tau^{a}{}_0, & \hat{\Omega}_z{}^{a+} &= -\frac{1}{2}\tau^{a}{}_0 \\
\hat{\Omega}_\mu{}^{a-} &= \frac{1}{2}\tau_\mu{}^{a}, & \hat{\Omega}_z{}^{a-} &= 0, \\
\hat{\Omega}_\mu{}^{+-} &= -\frac{1}{2}\tau_{\mu 0}, & \hat{\Omega}_z{}^{+-} &= 0,
\end{aligned} \tag{C.1}
$$

where we defined the torsion-free Newton-Cartan spin connections $\omega_\mu{}^{ab}$, and $\omega_\mu{}^{a}$ and the torsion tensor $\tau_{\mu\nu}$ as

$$
\begin{aligned}
\omega_\mu{}^{a} &= e^{\nu a}\partial_{[\mu} m_{\nu]} - e_\mu{}^{b} e^{\nu a}\tau^\rho \partial_{[\nu} e_{\rho]b} - \tau^\nu \partial_{[\mu} e_{\nu]}{}^{a} - \tau_\mu e^{\nu a}\tau^\rho \partial_{[\nu} m_{\rho]}, \\
\omega_\mu{}^{ab} &= 2e^{\nu[a}\partial_{[\mu} e_{\nu]}{}^{b]} - e_\mu{}^{c} e^{\nu a} e^{\rho b}\partial_{[\nu} e_{\rho]c} - \tau_\mu e^{\nu a} e^{\rho b}\partial_{[\nu} m_{\rho]}, \\
\tau_{\mu\nu} &= 2\partial_{[\mu}\tau_{\nu]}.
\end{aligned} \tag{C.2}
$$

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