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„Non-Relativistic Supergravity in 4D "

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Contents

Zusammenfassung

Eine nichtrelativistische Theorie der reinen Supergravitation, die eine supersymmetrische Erweiterung der Newtonschen Gravitationstheorie darstellt, wurde bisher nur in drei Raumzeitdimensionen konstruiert. Diese Konstruktion wurde mithilfe eines Superalgebra-Eichprinzips durchgeführt, welches nicht offensichtlich auf höhere Dimensionen verallgemeinert werden kann. Eine alternative Methode um vierdimensionale nichtrelativistische Supergravitation zu konstruieren besteht darin, eine Kaluza-Klein Dimensionsreduktion relativistischer fünfdimensionaler Supergravitation entlang einer lichtartigen Isometrierichtung durchzuführen. In der Vergangenheit wurde gezeigt, dass eine solche lichtartige Dimensionsreduktion (0-Reduktion) der Einsteinschen Feldgleichungen zu einer nichtrelativistischen Gravitationstheorie führt. Das Ziel dieser Arbeit ist die Methode der 0-Reduktion auf fünfdimensionale reine on-shell Supergravitation und deren zugehörige Transformationsgleichungen auszuweiten und die daraus abgeleitete nichtrelativistische vierdimensionale on-shell Supergravitation zu studieren.

Stichwörter: Nichtrelativistische 4D $N = 2$ Supergravitation, 5D $N = 2$ Supergravitation, Newton-Cartan Gravitation, Lichtartige Dimensionsreduktion

Abstract

A theory of non-relativistic pure supergravity that leads to a supersymmetric generalization of Newtonian gravity in arbitrary reference frames has so far only been constructed in three space-time dimensions. That construction was based on the use of the method of superalgebra gauging, which can not easily be extended to higher dimensions and one thus has to resort to other methods. One alternative option that leads to four-dimensional non-relativistic supergravity consists of performing a dimensional Kaluza-Klein reduction of relativistic five-dimensional supergravity along a null-like isometry direction. It has been shown in the past that applying such a dimensional null reduction of the Einstein equations leads to non-relativistic gravity. The aim of this thesis is to extend this null reduction to five-dimensional on-shell pure supergravity and its transformation rules and investigate the non-relativistic four-dimensional on-shell supergravity theory that it leads to.

Keywords: Non-Relativistic 4D $N = 2$ Supergravity, 5D $N = 2$ Supergravity, Newton-Cartan Gravity, Null Reduction

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Chapter 1

Introduction

Newtonian physics has been around for more than 300 years and for a long time it was the most successful theory for the description of gravity. Incorporating all basic principles of Newtonian mechanics together with the notions of Euclidean space and absolute time the central equation describing Newtonian gravity is

$$\rho(x) = \frac{1}{4\pi G} \Delta\Phi(x) , \tag{1.1}$$

where a mass distribution $\rho(x)$ gives rise to a scalar Newtonian gravitational potential $\Phi(x)$, with the gravitational constant G. It can be seen from this equation that, at the time, gravity was perceived to be a purely spatial phenomenon, often referred to as action at a distance.

When Einstein formulated his theory of relativity about 100 years ago these previous notions had to make way for new fundamental physical principles. Already with the presentation of the theory of special relativity (SR) the paradigm of absolute time and the independence of time and space were challenged and eventually replaced with the concept of spacetime. Flat Minkowski spacetime of special relativity was still close to the conception of flat Euclidean space. Einstein's general theory of relativity (GR) brought the emancipation of spacetime with the introduction of a general dynamical metric. This theory of gravitation describes gravity as a geometric property of spacetime itself rather than a *force*. The Einsteinian field equations

$$T_{MN} = R_{MN} - \frac{1}{2} R G_{MN} , \tag{1.2}$$

where T_{MN} describes the energy content, and R_{MN} and R describe the curvature of spacetime governed by the metric G_{MN}.

At first glance these laws governing gravity (1.1,1.2) do not share any similarities. When considering the role of mass and energy in Newtonian- and Einsteinian physics respectively, the left hand side of (1.2) can schematically be seen as the relativistic generalisation of the left hand side of (1.1). The curvature tensor terms are first order in derivatives of the connection. How this relates to the Laplacian of the Newton potential is not obvious and will require further inspection (see sec. 2.2.4). Another problem when comparing Newtonian to relativistic gravity is the formalism of differential geometry in which general relativity is formulated. General relativity is formulated in terms of differential geometry and has a pseudo-Riemannian manifold with a metric of spacetime at its core that represents spacetime. This makes the definition of tensors, connections and curvature possible – all concepts (except for tensors) that do not have counterparts in the original formulation of Newtonian gravity. A few years after the publication of general relativity Cartan presented the mathematical framework [15], [16] in which Newtonian gravity is formulated covariantly and geometrically and can thus be compared to relativity. Taking the geometric formulation of relativity as an inspiration, he constructed Newtonian gravity in a time-foliated spacetime in the language of differential geometry. The coordinate independent formulation makes it possible to define a non-relativistic (degenerate) metric structure together with vectors, connections and curvature. This theory is referred to as Newton-Cartan gravity (NC). On the one hand Newton-Cartan gravity is a non-relativistic ($c \to \infty$) limit of general relativity that reproduces the results of Newtonian gravity, and on the other hand a generalisation of Newtonian gravity to general spacetime structures.

A lot of work has been done on the topic of Newton-Cartan gravity ever since. For a good account of the main principles see [47]. While supergravity (SUGRA), a generalisation of general relativity including supersymmetry, i.e. a symmetry that exchanges bosons and fermions, has been studied for approximately 50 years the field of non-relativistic supergravity is relatively young and new terrain in many respects. For an account of recent results in non-relativistic gravity and its matter coupling see [11], [33]. For relativistic supergravity theories including different matter couplings see e.g. [7], [27], [62], [65]. In the field of non-relativistic supergravity some work has been done in three dimensions, while little information exists about higher dimensional theories and theories with higher supersymmetric extensions. A non-relativistic 3D, $\mathcal{N} = 2$ supergravity multiplet was constructed by [4], and in [44] 3D non-relativistic supersymmetric wave equations were developed. There has also been effort to find non-relativistic off-shell supergravity in three dimensions by [14], [42], [67]. This has not yet been generalised to higher dimensions. While part of the interest in non-relativistic supergravity is of academic nature, there have also been proposals for the application of such theories in other fields of physics. For example the quantum Hall effect was described in [61] using a supersymmetric model, building on work [57] in which a theory of the quantum Hall effect was constructed in terms of Newton-Cartan geometry. Another potential field of application of non-relativistic supergravity is localisation of supersymmetric field theories on Newton-Cartan backgrounds. Some research on supersymmetric field theories in non-relativistic backgrounds has been done in three dimensions by [5], [8]. Localisation techniques have previously led to exact results for relativistic field theories on non-trivial curved supergravity backgrounds, see [46], [49]. Extending these results to the non-relativistic regime can, however, benefit from non-relativistic off-shell supergravity theories, which are still not well understood.

In this thesis the null reduction of pure 5-dimensional $\mathcal{N} = 2$ supergravity is performed to obtain a non-relativistic 4-dimensional $\mathcal{N} = 2$ supergravity theory. The first two chapters, 2 and 3, serve as an introduction in which some basic principles about relativistic- and non-relativistic gravity and supergravity are discussed, as conceptually visualised in figure (1.1). In chapters 4-6 the null reduction is performed and the resulting theory is presented.

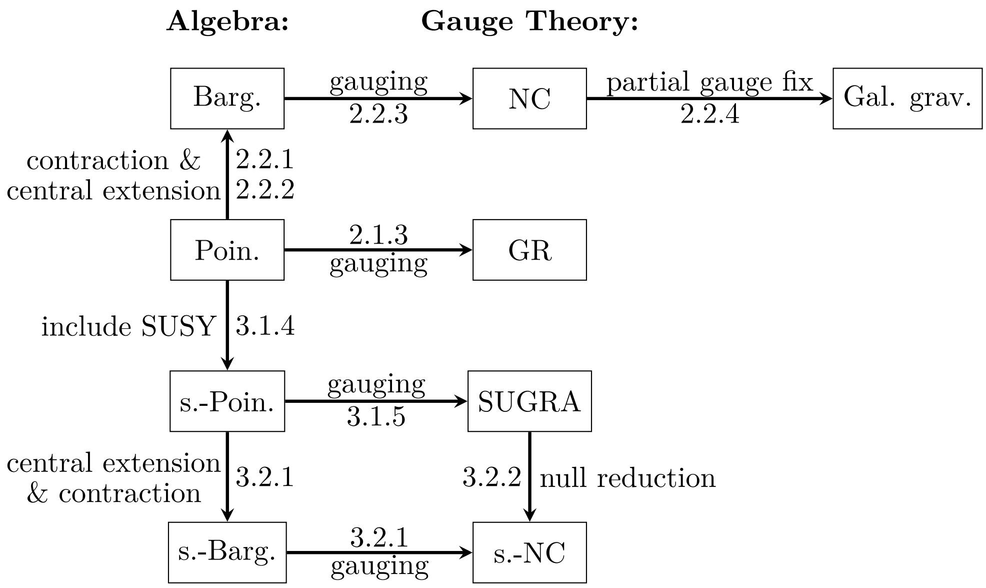

Figure 1.1: This figure visualises schematically the structure of the introductory chapters, 2 about gravity, and 3 about supergravity. The directions taken in the process of gauging algebras are indicated, together with the corresponding sections.

In section 2.1 the Vielbein formalism will be presented as an alternative to (and inspired by) the metric formulation of general relativity. After introducing some basic concepts about gauge theory, they will be put to use to derive general relativity in terms of the Vielbein formalism as the gauge theory of special relativity by gauging the Poincaré algebra. In section 2.2 the Poincaré algebra is contracted to the non-relativistic limit and centrally extended to produce the Bargmann algebra. The latter is gauged to yield Newton-Cartan theory which is then partially gauge fixed to end up with Galilean gravity. To see how the *curvature term* on the right hand side of equation (1.1) relates to the Laplacian of the Newton potential (in the massless case) see section 2.2.4. Finally, the procedure of null reduction is introduced. The following chapter extends the previous discussion to the inclusion of supersymmetry (SUSY). First, in section 3.1, spinors and gamma matrices are introduced in general dimensions as tools to describe fermions. To realise supersymmetry fermionic gauge theory is discussed before repeating the gauging procedure to obtain supergravity from gauging the super-Poincaré algebra. Since with the inclusion of supersymmetry the gauging procedure generally does not suffice anymore to construct multiplets, the construction of a multiplet is presented using an Ansatz. Finally in section 3.2 the super-Bargmann algebra is retrieved as the centrally extended non-relativistic limit of the super-Poincaré algebra and a non-relativistic multiplet is found, first from gauging and then from null reduction.

In chapter 4 the previously constructed relativistic supergravity multiplet is presented together with an Ansatz for null reduction. Following the null reduction the transformation rules of the reduced fields are derived in chapter 5 and the constraint structure of the multiplet is analysed. This 4D, $\mathcal{N} = 2$ super-Newton-Cartan (or equivalently, non-relativistic supergravity) multiplet has not been constructed in previous work and hence is a new result of this thesis. The final chapter 6 treats the linearisation of this multiplet. The linearised constraint structure is discussed and expanded by fermionic constraints and -equations of motion (EOM), which had been omitted in the full theory.

Chapter 2

Gravity

In the first section of this chapter we will discuss relativistic gravity. Standard references are [47], [64]. First the Vielbein formalism will be introduced and discussed, as well as (bosonic) gauge theory. The second section treats non-relativistic gravity constructed as a non-relativistic limit of relativistic gravity as discussed in the publications [3], [6]. The subtleties of mass and energy in non-relativistic theories are discussed before finally establishing a Vielbein formulation of non-relativistic gravity by gauging the Bargmann algebra. The discussions in this chapter will only treat bosonic fields and symmetries.

2.1 About Special- and General Relativity

This section will introduce the Vielbein formulation of GR, motivated by the metric formulation. The symmetry structure of the theory will be analysed to obtain the transformation rules of the Vielbein and the spin connection. Then (bosonic) gauge theory will be introduced in general terms, first giving a short introduction to relevant concepts from group theory before generalising the concepts known from gauge theory and preparing the reader for the following section. There the Vielbein formalism of GR is reproduced by the procedure of gauging the Poincaré algebra, the symmetry algebra of SR. Finally, representations of the Poincaré algebra are discussed to properly define scalars, vectors and tensors.

2.1.1 The Vielbein Formalism

Often general relativity is introduced with the metric tensor as the fundamental field. There is, however, a different formalism in which the fundamental field is an object that produces the metric when contracted with itself. So in some sense one can think of it as the *square root of the metric*. In 4D such a field is referred to as Vierbein or tetrad, while in general dimensions it is called D-bein or, more commonly, Vielbein or frame field. For bosonic general relativity it is not clear at this point what advantages are to be expected from such a formalism. Indeed, for bosonic theories the Vielbein formalism merely represents an alternative formulation to general relativity built from the metric, which can simplify calculations but does not affect results obtained from the theory. The most important reason though, to introduce the Vielbein is to couple the theory to fermions, described by spinors. The problem of the metric formulation of GR is that spinors cannot be expressed in a coordinate basis, but in a local Lorentz basis (which the Vielbeine provide) this becomes possible. While vectors and tensors have their respective transformation behaviour under diffeomorphisms and local Lorentz transformations and one can always simply change the basis from coordinate- to local Lorentz basis (using the Vielbein) this is **not** possible for spinors, which only transform under local Lorentz transformations but not under diffeomorphisms. For now, the formalism will be introduced only in terms of bosonic quantities. The discussion in this chapter will, however, be a preparation for the introduction of spinors in the next chapter and will mostly follow the presentation of [2], [27].

The Vielbein

The metric G is a symmetric non-degenerate covariant 2-tensor. The spectral theorem states that the metric, seen as a $D \times D$-matrix, can be diagonalised using an orthogonal transformation $O^{-1} = O^T$ as

$$G = ODO^T \, , \tag{2.1}$$

where $D = \mathrm{diag}(-\lambda_0, \ldots, \lambda_{D-1})$ and $\lambda_A > 0$ are the eigenvalues. The signs of the eigenvalues compose the signature of the metric. In this thesis the metric of a relativistic spacetime will always have the signature[1] $\mathrm{sig}(G) = (1, D-1)$. For the metric G written with indices relation (2.1) is

$$G_{MN} = O_M{}^A D_{AB} O_N{}^B \, , \tag{2.2}$$

and the diagonal matrix can be written as[2] $D_{AB} = \eta_{AB}\lambda_{\underline{A}}$. Now it is possible to define a new matrix, the frame field

$$E_M{}^A = O_M{}^A \sqrt{\lambda_{\underline{A}}} \, , \tag{2.3}$$

which, contracted with itself using the Minkowski metric yields the curved metric

$$G_{MN} = E_M{}^A \eta_{AB} E_N{}^B \, . \tag{2.4}$$

In words this relation (2.4) states that it is possible to find coordinates for each point on the manifold in which the spacetime metric is locally flat. This is what gave rise to the equivalence principle of relativity in the first place.

[1] In more general cases of pseudo-Riemannian geometry the signature can be different.

[2] The underlined index indicates that the index of this letter is free and not subjected to the Einstein summation convention.

The metric is non-degenerate and thus invertible. Therefore the inverse metric $G^{MN} \equiv (G^{-1})^{MN}$ can be diagonalised in the same fashion as the metric

$$G^{MN} = O^M{}_A \tilde{D}^{AB} O^N{}_B \, , \tag{2.5}$$

where the orthogonal matrices are the same as before $O_M{}^A = O^M{}_A$ and the diagonal matrix contains the inverse eigenvalues $\tilde{D}^{AB} = \eta^{AB}\lambda_{\underline{A}}^{-1}$. The inverse frame field is defined analogously to the frame field; by diagonalising the inverse metric

$$E^M{}_A = O^M{}_A \sqrt{\lambda_{\underline{A}}^{-1}} \, . \tag{2.6}$$

It is easy to check that this is indeed the inverse of the frame field defined above $(E^{-1})^M{}_A = E^M{}_A$. With the Vielbein a new kind of index is introduced. The Latin indices are referred to as tangent space- or flat indices and are contracted with the Minkowski metric, while the Greek indices are referred to as spacetime- or curved indices and are contracted with the spacetime metric. By construction of the Vielbein as an orthogonal matrix (with $D(D-1)/2$ free components) times the square root of the eigenvalues (D free components), its $D(D+1)/2$ degrees of freedom (DOF) agree with the number of free components of a symmetric tensor, i.e. of the metric. Only when introducing an arbitrary Vielbein there seems to be a mismatch in degrees of freedom, bringing new redundancy to the theory.

The one-forms E^A introduce an orthonormal basis of the cotangent space at every point on the manifold and can be expanded using a coordinate basis of one-forms as $E^A = E_M{}^A dx^M$. Equivalently the vector fields E_A introduce an orthonormal basis of the tangent space at every point on the manifold and they can be expanded as $E_A = E^M{}_A \partial_M$. Thereby the Vielbein introduces the ability to switch between flat and spacetime indices at will. For example the vector V can be expanded in flat or spacetime components

$$V^M = E^M{}_A V^A \, , \qquad V^A = E_M{}^A V^M \, . \tag{2.7}$$

In addition to the familiar raising and lowering of curved indices using the metric, one can now raise and lower flat indices with the Minkowski metric and switch between types of indices with the Vielbein. It is, however, important to keep in mind the difference between $\{\partial_M\}$, which form a coordinate basis, and $\{E_A\}$, which form a local Lorentz basis. For more information on local Lorentz bases, see [47]. A coordinate basis, i.e. a basis with commuting elements, is called holonomic. The commutator between two elements of a local Lorentz basis generally does not vanish. If it does vanish, there are no accelerated reference frames, hence there is no gravitational field, hence there is no Riemann curvature as laid out in [1]. In a theory describing gravity the commutator can therefore not vanish globally. As a consequence the frame fields **cannot** be brought to the form $E_A = \frac{\partial}{\partial x^A}$, i.e. they are not a coordinate basis. The non-commutativity of the frame fields is called anholonomy

$$[E_A, E_B] = -C_{AB}{}^C E_C \, , \qquad C_{AB}{}^C = 2E_A{}^M E_B{}^N \partial_{[M} E_{N]}{}^C \, , \tag{2.8}$$

and $C_{AB}{}^{C}$ are referred to as anholonomy coefficients. They will appear again when expressing the spin connection as a field depending on the Vielbein via the anholonomy coefficients (see eq. (2.23)).

Symmetries of the Vielbein

Next, the symmetry structure of the newly introduced Vielbein needs to be studied. The transformation of the Vielbein under local Lorentz- and diffeomorphism symmetry can be obtained from the prior knowledge of vector transformations under said symmetries. We know that a flat vector V_A transforms under local Lorentz transformations as

$$V'_A = \mathbf{\Lambda}_A{}^B V_B \,. \tag{2.9}$$

For vectors the transformation can be expanded in an antisymmetric parameter $\Lambda_{[AB]}$ as $\mathbf{\Lambda}_A{}^B = \delta_A^B - \Lambda_A{}^B + O(\Lambda^2)$ and the infinitesimal transformation then becomes

$$V'_A = V_A - \Lambda_A{}^B V_B \qquad \rightarrow \qquad \delta_\Lambda V_A = -\Lambda_A{}^B V_B \,. \tag{2.10}$$

A curved vector however is invariant under local Lorentz transformations. Since the Vielbein allows us to turn curved- into flat vectors and vice versa, the transformation of the Vielbein is found from the above statements

$$0 = \delta_\Lambda V_M = \delta_\Lambda (V_A E_M{}^A) \qquad \rightarrow \qquad \delta_\Lambda E_M{}^A = -\Lambda^A{}_B E_M{}^B \,. \tag{2.11}$$

The same procedure can be applied for the transformation behaviour under infinitesimal general coordinate transformations. The curved vector field V^M transforms as a spacetime vector, whereas the flat vector field V^A transforms as a collection of spacetime scalars

$$\delta_\Xi V^M = \mathcal{L}_\Xi V^M = \Xi^N \partial_N V^M - V^N \partial_N \Xi^M \,, \tag{2.12a}$$

$$\delta_\Xi V^A = \mathcal{L}_\Xi V^A = \Xi^N \partial_N V^A \,, \tag{2.12b}$$

$$\rightarrow \qquad \delta_\Xi E_M{}^A = \mathcal{L}_\Xi E_M{}^A \,, \tag{2.12c}$$

and the Vielbein transforms as a collection of spacetime one-forms. It is now simple enough to find the transformation rules of the inverse Vielbein by varying the constant contraction of Vielbein and inverse Vielbein. This leads to the set of transformation rules

$$\delta E_M{}^A = \mathcal{L}_\Xi E_M{}^A - \Lambda^A{}_B E_M{}^B \,, \tag{2.13a}$$

$$\delta E^M{}_A = \mathcal{L}_\Xi E^M{}_A - \Lambda_A{}^B E^M{}_B \,. \tag{2.13b}$$

Covariant Derivatives, the Spin Connection and Torsion

From general relativity the derivative covariant under general coordinate transformations[3] ∇_M, and how it acts on vector- and tensor-fields, V and T respectively,

[3] In GR this is what is referred to as *covariant derivative.*

is well known:

$$\nabla_M V_N = \partial_M V_N - \Gamma^P_{MN} V_P \,, \tag{2.14a}$$
$$\nabla_M V^N = \partial_M V^N + \Gamma^N_{MP} V^P \,, \tag{2.14b}$$
$$\begin{aligned}\nabla_M T^{N_1\dots N_r}{}_{P_1\dots P_s} = {} & \partial_M T^{N_1\dots N_r}{}_{P_1\dots P_s} \\ & + \Gamma^{N_1}_{M\Sigma} T^{\Sigma N_2\dots N_r}{}_{P_1\dots P_s} + \dots + \Gamma^{N_r}_{M\Sigma} T^{N_1\dots N_{r-1}\Sigma}{}_{P_1\dots P_s} \\ & - \Gamma^{\Sigma}_{MP_1} T^{N_1\dots N_r}{}_{\Sigma P_2\dots P_s} - \dots - \Gamma^{\Sigma}_{MP_s} T^{N_1\dots N_r}{}_{P_1\dots P_{s-1}\Sigma} \,,\end{aligned} \tag{2.14c}$$

where Γ^P_{MN} is the affine connection. Now the formalism has to be expanded since vectors and tensors transforming under local Lorentz transformations have entered the theory. Explicitly this means that a new kind of derivative has to be defined that transforms covariantly under local Lorentz transformations, which will be denoted as D_M. With it a new gauge field $\Omega_M{}^{AB}$ has to be introduced that transforms with the derivative of the local Lorentz parameter Λ^{AB}. This field will be referred to as spin connection. To produce Lorentz covariant quantities, the Lorentz covariant derivative acts on flat vector- and tensor-fields as

$$D_M V^A = \partial_M V^A + \Omega_M{}^A{}_B V^B \,, \tag{2.15a}$$
$$D_M V_A = \partial_M V_A + \Omega_{MA}{}^B V_B \,, \tag{2.15b}$$
$$\begin{aligned}D_M T^{A_1\dots A_r}{}_{B_1\dots B_s} = {} & \partial_M T^{A_1\dots A_r}{}_{B_1\dots B_s} \\ & + \Omega_M{}^{A_1}{}_C T^{CA_2\dots A_r}{}_{B_1\dots B_s} + \dots + \Omega_M{}^{A_r}{}_C T^{A_1\dots A_{r-1}C}{}_{B_1\dots B_s} \\ & + \Omega_{MB_1}{}^C T^{A_1\dots A_r}{}_{CB_2\dots B_s} + \dots + \Omega_{MB_s}{}^C T^{A_1\dots A_r}{}_{B_1\dots B_{s-1}C} \,.\end{aligned} \tag{2.15c}$$

Taking both these covariant derivatives a completely covariant derivative $\mathcal{D}_M$ can be defined that transforms covariantly under diffeomorphisms and local Lorentz transformations. In relations (2.14) and (2.15) the respective covariant derivative can be replaced by $\mathcal{D}_M$, since these fields only transform under one symmetry. As can be derived from $\mathcal{D}_M V^A = \mathcal{D}_M (V^N E_N{}^A)$, the covariant derivative of the Vielbein takes the expected form

$$\mathcal{D}_M E_N{}^A = \partial_M E_N{}^A + \Omega_M{}^A{}_B E_N{}^B - \Gamma^P_{MN} E_P{}^A \,. \tag{2.16}$$

In general relativity metricity is postulated

$$\nabla_M G_{NP} = \mathcal{D}_M G_{NP} = 0 \,. \tag{2.17}$$

Since $\Omega_M{}^{AB}$ is the gauge field of local Lorentz transformations, of course it is also antisymmetric in its flat indices $\Omega_M{}^{AB} = \Omega_M{}^{[AB]}$ and therefore

$$\mathcal{D}_M \eta_{AB} = 0 \,. \tag{2.18}$$

From these relations metric compatibility can be expressed in terms of the Vielbein

$$\mathcal{D}_M G_{NP} = \mathcal{D}_M \left(\eta_{AB} E_N{}^A E_P{}^B\right) = 2E_{NA}\mathcal{D}_M E_P{}^A \,, \tag{2.19}$$

leading to the equivalent Vielbein postulate

$$\mathcal{D}_M E_N{}^A = 0 \,. \tag{2.20}$$

By inserting this into (2.16) the connection Γ^P_{MN} can be expressed in terms of Vielbein and spin connection

$$\Gamma^P_{MN} = E^P{}_A \mathcal{D}_M E_N{}^A = \left(\partial_M E_N{}^A + \Omega_M{}^{AB} E_{NB}\right) E^P{}_A \,. \tag{2.21}$$

In general relativity torsion is defined as twice the antisymmetric component of the affine connection $T_{MN}{}^P := 2\Gamma^P_{[MN]}$ and set zero to obtain the symmetric Levi-Civita connection that can be expressed using the metric $\Gamma^P_{MN} = \Gamma^P_{MN}(G)$ and is referred to as Christoffel symbol (for the explicit form see [47]). By virtue of (2.21) torsion can also be expressed in terms of Vielbein and spin connection

$$0 = T_{MN}{}^A = 2\partial_{[M} E_{N]}{}^A + 2\Omega_{[M}{}^{AB} E_{N]B} \,. \tag{2.22}$$

This relation makes the spin connection a field depending on the Vielbein. By lowering the frame index of torsion to a spacetime index and adding two cyclic permutations of all indices and subtracting the third, the equation is solved for the spin connection in terms of anholonomy coefficients (see eq. (2.8))

$$\begin{aligned} \Omega_M{}^{AB}(E) &= -\tfrac{1}{2} E^{NA} E^{PB} E_{MC} C_{NP}{}^C + E^{N[A} C_{MN}{}^{B]} \\ &= -\tfrac{1}{2} C^{AB}{}_M + C_M{}^{[AB]} \,. \end{aligned} \tag{2.23}$$

Even though it would, in principle, be possible at this point to eliminate the spin connection completely from the theory, it is convenient to keep it, not only for later generalisations, but also in terms of notation and computation.

Symmetries of the Spin Connection

Now the transformation rule of the spin connection remains to be found. Since the spin connection is a dependent field, its transformation rule could be derived from the transformation rule of the Vielbein. A much simpler way to compute it is, however, to require the covariant derivative of a flat vector to transform appropriately under both symmetry transformations, i.e. as a curved vector under diffeomorphisms and as a flat vector under local Lorentz transformations

$$\delta \mathcal{D}_M V^A \stackrel{!}{=} \mathcal{L}_\Xi \mathcal{D}_M V^A - \Lambda^A{}_B \mathcal{D}_M V^B \,. \tag{2.24}$$

By explicit calculation:

$$
\begin{aligned}
0 &= \delta\mathcal{D}_M V^A - \mathcal{L}_\Xi \mathcal{D}_M V^A + \Lambda^A{}_B \mathcal{D}_M V^B \\
&= \partial_M \left(\Xi^N \partial_N V^A - \Lambda^A{}_B V^B\right) + \Omega_M{}^{AB}\left(\Xi^N \partial_N V_B - \Lambda_B{}^C V_C\right) + \delta\Omega_M{}^{AB} V_B \\
&\quad - \Xi^N \partial_N \left(\partial_M V^A + \Omega_M{}^{AB} V_b\right) - \partial_M \Xi^N \left(\partial_N V^A + \Omega_N{}^{AB} V_B\right) \\
&\quad + \Lambda^A{}_B \left(\partial_M V^B + \Omega_M{}^{BC} V_C\right) \\
&= \delta\Omega_M{}^{AB} V_B - \Xi^N \partial_N \Omega_M{}^{AB} V_B - \partial_M \Xi^N \Omega_N{}^{AB} V_B - \partial_M \Lambda^A{}_B V^B \\
&\quad - \Omega_M{}^{AB} \Lambda_B{}^C V_C + \Omega_M{}^{CB} \Lambda_B{}^A V_C \,.
\end{aligned}
\tag{2.25}
$$

The transformation rule of the spin connection can be read off and it is apparent that it transforms properly: under diffeomorphisms as a vector and under local Lorentz transformations as a non-Abelian gauge field

$$\delta\Omega_M{}^{AB} = \mathcal{L}_\Xi \Omega_M{}^{AB} + \partial_M \Lambda^{AB} + 2\Omega_M{}^{C[A} \Lambda^{B]}{}_C \,. \tag{2.26}$$

Curvature

Similarly to eq. (2.22) an antisymmetrised covariant derivative can be found for the spin connection

$$2\mathcal{D}_{[M}\Omega_{N]}{}^{AB} = 2\partial_{[M}\Omega_{N]}{}^{AB} + 2\Omega_{[M}{}^{AC}\Omega_{N]C}{}^B \,. \tag{2.27}$$

The Riemann tensor, describing the curvature of spacetime, is defined as

$$R_{MN}{}^P{}_\Sigma := 2\partial_{[M}\Gamma^P_{N]\Sigma} + 2\Gamma^P_{T[M}\Gamma^T_{N]\Sigma}. \tag{2.28}$$

By inserting (2.21) into the defining equation of the Riemann tensor it becomes apparent that the antisymmetrised covariant derivative of the spin connection is nothing but the Riemann tensor

$$2\mathcal{D}_{[M}\Omega_{N]}{}^{AB} = R_{MN}{}^P{}_\Sigma E_P{}^A E^{\Sigma B} = R_{MN}{}^{AB} \,. \tag{2.29}$$

2.1.2 General Bosonic Gauge Theory

In this section concepts known from gauge theory will be discussed and generalised, mostly on an operational level. For now, purely bosonic theories will be treated. Everything that is presented will, however, be formulated so as to make the generalisation to fermionic symmetries, i.e. supersymmetries, as straightforward as possible. This generalisation will be made in the next chapter in section 3.1.3. First, different types of symmetries are distinguished, then there is a short summary of concepts relevant for this thesis concerning Lie groups and -algebras, and finally the procedure of gauging an algebra is introduced together with covariant derivatives and curvatures. For a more mathematical motivation of gauge theory concepts see e.g. [34].

Types of Symmetries

Symmetries in physical theories are implemented by transformations of fields. They can be defined on all classes of fields described by the theory by requiring that they map solutions of the equations of motion of the respective fields to other solutions. We will only concern ourselves with symmetry transformations that leave the action invariant. Two symmetry transformations performed consecutively should again yield a symmetry transformation. For every symmetry transformation there should exist a symmetry transformation reversing it. Finally, no transformation at all must of course also be a symmetry transformation. The criteria above lead to the description of physical symmetries as groups. One has to distinguish between the following different kinds of symmetries:

Discrete/Continuous

Continuous symmetries are symmetries, whose composition and inversion are continuous maps, and whose group action is continuous. If these maps are differentiable as well, the symmetry group is a Lie group and infinitesimal transformations are described by the corresponding Lie algebra. Discrete symmetries are non-continuous, meaning that one transformation cannot be smoothly deformed into another. Translations by arbitrary vectors and rotations over arbitrary angles are examples of continuous symmetries, while parity and rotations around $2\pi/n$, with $n \in \mathbb{N}$, are discrete. In this thesis only continuous symmetries will be treated.

Spacetime/Internal

Spacetime symmetries act on Minkowski spacetime, while internal symmetries only act on fields themselves. To be precise, assuming $\phi^i(x)$ are fields of a physical theory and $\boldsymbol{S} = e^S$ and $\boldsymbol{I} = e^I$ are a spacetime- and an internal symmetry transformation respectively, the fields transform schematically as

$$\phi'^i(x) = \boldsymbol{I}\phi^i(x)\,, \tag{2.30a}$$

$$\phi'^i(x) = \boldsymbol{S}^{-1}\phi^i(\boldsymbol{S}x)\,. \tag{2.30b}$$

The transformation rules under infinitesimal symmetry transformations with generators S and I respectively are given by

$$\delta_I\phi^i(x) = I\phi^i(x)\,, \tag{2.31a}$$

$$\delta_S\phi^i(x) = -S\phi^i(x) + Sx^M\partial_M\phi^i(x)\,, \tag{2.31b}$$

where S and I are replaced by the appropriate representations (see eq. (2.40-2.41) and sec. 2.1.4) under which the fields and the coordinates transform. $U(1)$ gauge symmetry of electro-magnetism and $SU(3)$ gauge symmetry of quantum chromodynamics are examples of internal symmetries. Poincaré transformations, diffeomorphisms and supersymmetry transformations are examples of spacetime symmetries.

Both types of symmetries will be encountered in this thesis.

Global/Local

Global, also called rigid symmetries, are characterised by transformations that cannot depend on spacetime coordinates in an arbitrary manner. An isometry is an example of a global symmetry that can be coordinate dependent. Coordinate independent symmetries are automatically global. Local, also called gauge symmetries, on the other hand, are characterised by a symmetry parameter that can have an arbitrary spacetime dependence. Global symmetries can be made local by the procedure that is called *gauging the symmetry.* This will be applied at a later point in this thesis (see sec. 2.1.3, 2.2.3, 3.1.5 and 3.2.1) for Poincaré- and supersymmetry to obtain local Lorentz symmetry, diffeomorphisms from local translations, and supergravity.

Lie Groups and -Algebras

In this section only some of the most fundamental properties and statements about Lie groups and -algebras that will be relevant directly for this thesis are presented. Many details and subtleties will be omitted. For a rigorous discussion of Lie groups and -algebras the reader is referred to [36], [58], [59].

Continuous symmetries usually form Lie groups. Since in many cases it is sufficient to study infinitesimal symmetry transformations, one can work with the Lie algebra instead of dealing with the full group. A Lie group is a group that is also a finite-dimensional smooth manifold. In those terms the Lie algebra is the tangent space at the origin of the manifold, i.e. at the identity element of the Lie group. Since the Lie algebra $\mathfrak{g}$ is a vector space one can chose a basis of Lie algebra elements $\{T_{\boldsymbol{A}}\}$ in which every element θ of the Lie algebra can be expanded as $\theta = \theta^{\boldsymbol{A}} T_{\boldsymbol{A}}$. The basis elements $T_{\boldsymbol{A}}$ are referred to as generators. Lie algebra elements are related to elements of the corresponding simply connected Lie group $\boldsymbol{G}$ by exponentiation

$$T_{\boldsymbol{A}} \in \mathfrak{g} \quad \to \quad \exp\left(\theta^{\boldsymbol{A}} T_{\boldsymbol{A}}\right) \in \boldsymbol{G}\,, \tag{2.32}$$

where $\theta^{\boldsymbol{A}}$ is a parameter. While every Lie group has a uniquely assigned Lie algebra, a Lie algebra cannot uniquely be assigned to a Lie group. This becomes clear when considering that the Lie algebra only encodes the group structure close to the identity. Hence a Lie algebra is the tangent space of the Lie group at the identity element equipped with an antisymmetric bilinear product $[\cdot\,,\cdot]$ called Lie bracket. This is because on the level of the Lie algebra the consecutive action of two symmetry transformations is encoded in the commutator of two general elements of the Lie algebra, $\theta \equiv \theta^{\boldsymbol{A}} T_{\boldsymbol{A}}$ and $\sigma \equiv \sigma^{\boldsymbol{A}} T_{\boldsymbol{A}}$, as can be seen from the Baker-Campbell-Hausdorff formula

$$\exp(\theta)\exp(\sigma) = \exp\left(\theta + \sigma + \tfrac{1}{2}[\theta,\sigma] + O(\theta^2) + O(\sigma^2)\right)\,. \tag{2.33}$$

Let us assume that there is a theory with a set of symmetries generated by the set of generators $\{T_A\}$, with the associated set of parameters $\{\theta^A\}$, and a set of fields $\{\phi^i\}$ that transform under these symmetries as

$$\phi'^{\,i} = \exp\left(\theta^{\boldsymbol{A}} T_{\boldsymbol{A}}\right)\phi^i \qquad \to \qquad \delta\phi^i = \theta^{\boldsymbol{A}} T_{\boldsymbol{A}}\phi^i + O(\theta^2)\,. \tag{2.34}$$

How exactly fields can be acted upon by symmetry transformations will be discussed below. For now the action on the fields should be viewed as schematic. We will only consider infinitesimal transformations, meaning that the $O(\theta^2)$-term is neglected. The consecutive action of two symmetry transformations is encoded in the Lie bracket as

$$[\delta_1, \delta_2]\phi^i = [\theta_1, \theta_2]\phi^i = \theta_2^{\boldsymbol{B}}\theta_1^{\boldsymbol{A}}[T_{\boldsymbol{A}}, T_{\boldsymbol{B}}]\phi^i\,. \tag{2.35}$$

Therefore the commutators of the generators that compose the Lie algebra are the objects of interest. The last statement may seem trivial, but it is in fact only true for bosonic generators and parameters. The inclusion of fermionic quantities will demand the introduction of anti-commutators at this point, as will be discussed in the next chapter (see sec. 3.1.3). This is why the Lie algebra is fully described by its commutation relations

$$[T_{\boldsymbol{A}}, T_{\boldsymbol{B}}] = f_{\boldsymbol{A}\boldsymbol{B}}{}^{\boldsymbol{C}}\, T_{\boldsymbol{C}}\,, \tag{2.36}$$

where $f_{\boldsymbol{A}\boldsymbol{B}}{}^{\boldsymbol{C}}$ are the structure constants. The Lie bracket respects the Jacobi identity

$$[T_{\boldsymbol{A}}, [T_{\boldsymbol{B}}, T_{\boldsymbol{C}}]] + [T_{\boldsymbol{B}}, [T_{\boldsymbol{C}}, T_{\boldsymbol{A}}]] + [T_{\boldsymbol{C}}, [T_{\boldsymbol{A}}, T_{\boldsymbol{B}}]] = 0\,, \tag{2.37}$$

that can, via (2.36), also be expressed in terms of structure constants

$$f_{\boldsymbol{A}\boldsymbol{B}}{}^{\boldsymbol{D}} f_{\boldsymbol{C}\boldsymbol{D}}{}^{\boldsymbol{E}} + f_{\boldsymbol{B}\boldsymbol{C}}{}^{\boldsymbol{D}} f_{\boldsymbol{A}\boldsymbol{D}}{}^{\boldsymbol{E}} + f_{\boldsymbol{C}\boldsymbol{A}}{}^{\boldsymbol{D}} f_{\boldsymbol{B}\boldsymbol{D}}{}^{\boldsymbol{E}} = 0\,. \tag{2.38}$$

Representations

Different fields are typically realised as objects from different vector spaces. To realise the action of a symmetry group on different fields explicitly, one needs to discuss the effect of group elements acting on such vector spaces. The necessary formalism is provided by the theory of representations. A representation P of the group $\boldsymbol{G}$ is a group homomorphism $P : \boldsymbol{G} \to \boldsymbol{GL}(V)$ that assigns a vector space automorphism[4] $P(\boldsymbol{g}) : V \to V$ to every group element $\boldsymbol{g}$. The vector space V is referred to as the representation space. A representation has to fullfill

$$P(\boldsymbol{g}_1)P(\boldsymbol{g}_2) = P(\boldsymbol{g}_1\boldsymbol{g}_2)\,, \qquad \boldsymbol{g}_{1,2} \in \boldsymbol{G}\,. \tag{2.39}$$

In order to implement infinitesimal symmetry transformations acting on fields explicitly one has to work with Lie algebra representations. A Lie algebra representation

[4]An automorphism is a bijective endomorphism. An endomorphism M on the vector space V is a linear map $M : V \to V$.

ρ of a Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ which assigns an endomorphism $\rho(g) : V \to V$ on the vector space V to every Lie algebra element g. In most cases these endomorphisms take the form of matrices. The Lie bracket is transmitted to the space of endomorphisms as

$$\rho([g_1, g_2]) = \rho(g_1)\rho(g_2) - \rho(g_2)\rho(g_1) , \qquad g_{1,2} \in \mathfrak{g} . \tag{2.40}$$

If a representation is injective it is called faithful. This is true for group- and algebra representations

The fields ϕ^i are assumed to transform linearly under the symmetries generated by $T_{\boldsymbol{A}}$ with the matrix representations $t_{\boldsymbol{A}}$ as

$$\delta\phi^i = \theta^{\boldsymbol{A}} T_{\boldsymbol{A}} \phi^i = -\theta^{\boldsymbol{A}} (t_{\boldsymbol{A}})^i{}_j \phi^j . \tag{2.41}$$

We adopt the convention that generators act on the field that they are directly left of. Note that one has to be cautious when acting with multiple generators on a field, since this convention implies that when writing the expression in terms of matrix representations the original order of the abstract generators is reversed, and there is a sign for every generator. Requiring that Lie algebra representation matrices fulfill the same commutation relations as the algebra elements, the representations have to be introduced with a sign (see eq. (2.41)) when replacing a Lie algebra element acting on a field. The following calculation justifies the sign

$$\begin{aligned} [T_{\boldsymbol{A}}, T_{\boldsymbol{B}}]\phi^i =& - T_{\boldsymbol{A}} (t_{\boldsymbol{B}})^i{}_j \phi^j + T_{\boldsymbol{B}} (t_{\boldsymbol{A}})^i{}_j \phi^j \\ =& \, (t_{\boldsymbol{B}})^i{}_j (t_{\boldsymbol{A}})^j{}_k \phi^k - (t_{\boldsymbol{A}})^i{}_j (t_{\boldsymbol{B}})^j{}_k \phi^k \\ =& - [t_{\boldsymbol{A}}, t_{\boldsymbol{B}}]^i{}_j \phi^j , \end{aligned} \tag{2.42a}$$

$$f_{\boldsymbol{AB}}{}^{\boldsymbol{C}} T_{\boldsymbol{C}} \phi^i = - f_{\boldsymbol{AB}}{}^{\boldsymbol{C}} (t_{\boldsymbol{C}})^i{}_j \phi^j . \tag{2.42b}$$

This ensures that

$$[T_{\boldsymbol{A}}, T_{\boldsymbol{B}}] = f_{\boldsymbol{AB}}{}^{\boldsymbol{C}} T_{\boldsymbol{C}} \qquad \leftrightarrow \qquad [t_{\boldsymbol{A}}, t_{\boldsymbol{B}}] = f_{\boldsymbol{AB}}{}^{\boldsymbol{C}} t_{\boldsymbol{C}} . \tag{2.43}$$

For more information on general representation theory and of the representations of the Poincaré algebra see [36], [56] respectively.

Additionally to the above discussion of ordinary representations the more general notion of projective representations can be introduced. A projective representation $\tilde{P}$ of group $\boldsymbol{G}$ fulfills the more general (cf. eq. (2.39)) relation

$$\tilde{P}(\boldsymbol{g}_1)\tilde{P}(\boldsymbol{g}_2) = e^{i\omega(\boldsymbol{g}_1,\boldsymbol{g}_2)} \tilde{P}(\boldsymbol{g}_1\boldsymbol{g}_2) , \qquad \boldsymbol{g}_{1,2} \in \boldsymbol{G} , \tag{2.44}$$

where $\omega : \boldsymbol{G} \times \boldsymbol{G} \to \mathbb{R}$ is a function producing a phase. This leads to an adapted transformation behaviour of fields (cf. eq. (2.41)), assuming that $\boldsymbol{G}$ is a spacetime symmetry group. The transformation is then of the form

$$\phi'^i(x) = e^{if_{\boldsymbol{g}}(\boldsymbol{g}x)} P(\boldsymbol{g}^{-1}) \phi^i(\boldsymbol{g}x) , \qquad \boldsymbol{g} \in \boldsymbol{G} , \tag{2.45}$$

where $f : \boldsymbol{G} \times \mathbb{R}^n \to \mathbb{R}$ is a function and P is an ordinary representation. For more details on the relation between ω and f, the conditions that they obey, and much more on projective representations of groups and algebras the reader is referred to [50], [54], [60].

Graded Lie Algebras

Lie algebras can be equipped with an extra grading structure. Graded Lie algebras will be important to perform İnönü-Wigner contractions to obtain the non-relativistic limit of the Poincaré algebra (see sec. 2.2.1) and it will be necessary to include fermions into the theory (see sec. 3.1.3).

A Lie algebra $\mathfrak{g}$ can be graded with any Abelian group $(\boldsymbol{A}, +)$. As a vector space it decomposes into a direct sum

$$\mathfrak{g} = \bigoplus_{\boldsymbol{a} \in \boldsymbol{A}} \mathfrak{g}_{\boldsymbol{a}} \,, \tag{2.46}$$

and its Lie bracket satisfies the following relation

$$[\mathfrak{g}_{\boldsymbol{a}}, \mathfrak{g}_{\boldsymbol{b}}] = \mathfrak{g}_{\boldsymbol{a}+\boldsymbol{b}} \,, \quad \boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{A} \,. \tag{2.47}$$

Elements T of the graded Lie algebra can now be labelled by their corresponding group element $T^{(a)} \in \mathfrak{g}_{\boldsymbol{a}}$. The grading of an algebra is not unique. If (2.47) is fulfilled, an algebra can be graded by different groups or with different grading structures by the same group. The Poincaré algebra can for example be graded in different ways by the group $\mathbb{Z}_2$, as shown in [41]. The fundamental theorem of finite Abelian groups states that any finite Abelian group is isomorphic to a direct sum of cyclic groups, therefore it suffices to work with the latter. For our purposes, only gradings with $\mathbb{Z}_n$ will be relevant. This simplifies the notation, as the elements of $\mathbb{Z}_n$ can be labelled by $\{0, 1, \dots n-1\}$ and the Lie algebra decomposition becomes

$$\mathfrak{g} = \bigoplus_{a=0}^{n-1} \mathfrak{g}_a \,. \tag{2.48}$$

Saletan- and İnönü-Wigner Contractions

İnönü and Wigner introduced a way to generate an algebra by contraction of a $\mathbb{Z}_2$-graded algebra [35]. Saletan generalised these results to contractions of $\mathbb{Z}_n$-graded algebras [53], as also described by [41]. For this, every element of the Lie algebra is multiplied by a parameter ϵ to the power of the corresponding element of $\mathbb{Z}_n$

$$T^{(a)} \to \epsilon^a T^{(a)} =: H^{(a)} \,, \quad a = 0, \dots n-1 \,. \tag{2.49}$$

All commutators in terms of algebra elements T are now expressed in terms of H and the parameter is eliminated $\epsilon \to 0$. Since some commutators are thereby contracted to zero, this leads to a reduction in commutation relations according to the following relation

$$a+b<n: \quad [T^{(a)}_{\boldsymbol{A}}, T^{(b)}_{\boldsymbol{B}}] = f_{\boldsymbol{AB}}{}^{\boldsymbol{C}} T^{(a+b)}_{\boldsymbol{C}} \quad \to \quad [H^{(a)}_{\boldsymbol{A}}, H^{(b)}_{\boldsymbol{B}}] = f_{\boldsymbol{AB}}{}^{\boldsymbol{C}} H^{(a+b)}_{\boldsymbol{C}} \ , \tag{2.50a}$$

$$a+b\geq n: \quad [T^{(a)}_{\boldsymbol{A}}, T^{(b)}_{\boldsymbol{B}}] = f_{\boldsymbol{AB}}{}^{\boldsymbol{C}} T^{(a+b-n)}_{\boldsymbol{C}} \quad \to \quad [H^{(a)}_{\boldsymbol{A}}, H^{(b)}_{\boldsymbol{B}}] = 0 \ . \tag{2.50b}$$

The Jacobi identity holds for the contracted algebra by construction of the new generators (2.49).

Gauge Fields and Covariant Derivatives

In a field theory one needs to take derivatives to describe kinematics. Derivatives are also required to come up with actions and equations of motion and thereby describe the dynamics of the theory. Ordinary derivatives of fields, however, generally transform with derivatives of parameters. This makes the construction of invariant actions problematic. To cancel these unwanted terms from the derivative, gauge fields are introduced that themselves transform with a derivative of their corresponding parameter. A covariant combination[5] of the derivative and gauge fields, called covariant derivative, can now be defined.

When asking how an ordinary derivative of a field transforms under the symmetries generated by $\{T_{\boldsymbol{A}}\}$, it becomes apparent that there is a non-covariant term in the transformation

$$\delta\partial_M\phi^i = \partial_M(\theta^{\boldsymbol{A}})T_{\boldsymbol{A}}\phi^i + \theta^{\boldsymbol{A}}T_{\boldsymbol{A}}\partial_M\phi^i \ . \tag{2.51}$$

To remove this unwanted term a quantity needs to be introduced, which itself transforms with the derivative of a parameter. Such a quantity h will be a Lie algebra valued one-form $h^{\boldsymbol{A}}_M dx^M T_{\boldsymbol{A}}$, referred to as gauge field. There will not be made any semantic difference between the gauge fields h and its components $h^{\boldsymbol{A}}_M$, which will also be referred to as gauge fields. The object referred to should be understood from the context. The partial derivative together with this new quantity form a gauge-covariant derivative

$$D_M = \partial_M - h_M \ . \tag{2.52}$$

Note that the covariant gauge vector field[6] h_M is an element of the Lie algebra and therefore generally does not commute with other elements of the Lie algebra. When acting on a field transforming linearly under a certain matrix representation, this representation is inserted into (2.52). The gauge field has to transform with the

[5] I.e. its transformation does not include any derivatives of gauge parameters.

[6] The notation $D_M = \partial_M - \delta_{h_M}$ can be found in other texts. This emphasises that h_M acting on a field yields an infinitesimal transformation with $h^{\boldsymbol{A}}_M$ acting as the gauge parameter.

derivative of a parameter since this is why it was introduced in the first place. One can find the complete transformation rules of the gauge fields by simply requiring that the covariant derivative commutes with gauge transformations

$$\delta D_M \phi^i = -\theta^{\boldsymbol{A}} D_M (t_{\boldsymbol{A}})^i{}_j \phi^j \,. \tag{2.53}$$

By explicitly going through the calculation

$$\begin{aligned} \delta D_M \phi^i &= \delta\left(\delta^i_j \partial_M + h^{\boldsymbol{A}}_M (t_{\boldsymbol{A}})^i{}_j\right)\phi^j \\ &= \partial_M \delta\phi^i + \delta(h^{\boldsymbol{A}}_M)(t_{\boldsymbol{A}})^i{}_j \phi^j + h^{\boldsymbol{A}}_M (t_{\boldsymbol{A}})^i{}_j \delta\phi^j \\ &= -\partial_M(\theta^{\boldsymbol{A}})(t_{\boldsymbol{A}})^i{}_j\phi^j - \theta^{\boldsymbol{A}}(t_{\boldsymbol{A}})^i{}_j\partial_M\phi^j \\ &\quad + \delta(h^{\boldsymbol{A}}_M)(t_{\boldsymbol{A}})^i{}_j\phi^j - h^{\boldsymbol{A}}_M\theta^{\boldsymbol{B}}(t_{\boldsymbol{A}})^i{}_k(t_{\boldsymbol{B}})^k{}_j\phi^j \,, \end{aligned} \tag{2.54a}$$

$$-\theta^{\boldsymbol{A}} D_M (t_{\boldsymbol{A}})^i{}_j\phi^j = -\theta^{\boldsymbol{A}}(t_{\boldsymbol{A}})^i{}_j\partial_M\phi^j - \theta^{\boldsymbol{A}} h^{\boldsymbol{B}}_M (t_{\boldsymbol{A}})^i{}_k (t_{\boldsymbol{B}})^k{}_j \phi^j \,, \tag{2.54b}$$

and by comparing the terms on the left and the right side of (2.53) the transformation rule of the gauge fields can be extracted

$$\delta h^{\boldsymbol{A}}_M = \partial_M \theta^{\boldsymbol{A}} + \theta^{\boldsymbol{C}} h^{\boldsymbol{B}}_M f_{\boldsymbol{BC}}{}^{\boldsymbol{A}} \qquad \leftrightarrow \qquad \delta h_M = \partial_M\theta + [h_M, \theta] \,. \tag{2.55}$$

It has not been specified so far what the objects are that covariant derivatives can act on. Covariant derivatives can be defined on all fields that transform covariantly themselves. Not only however can a covariant derivative be defined for covariant fields, but also for any combination of these fields that transforms covariantly. For example a covariant derivative of a covariant derivative can always be defined. How to actually calculate the covariant derivative of a general covariant quantity will be explained in detail in the generalisation of this section in the next chapter (see sec. 3.1.3).

Covariant Curvatures

There is another fundamentally covariant object at hand, called covariant curvature $R_{MN}(T^{\boldsymbol{A}})$, which exists for every generator and basically is the antisymmetrised covariant derivative of the corresponding gauge field. In electro-magnetism and Yang-Mills theory this quantity is called field strength, while in general relativity the curvature of local Lorentz transformations is called Riemann tensor (see eq. (2.29)) and the curvature of local translations is referred to as torsion (see eq. (2.22)). A covariant derivative cannot be defined for a gauge field because the latter already transforms with the derivative of a parameter. Therefore its derivative transforms with a second-derivative on the parameter and there is no field to compensate for such a term. The antisymmetrisation rids the transformation of exactly this second-derivative term, leaving the rest of the expression covariantisable. By examining the transformation of the antisymmetrised partial derivative of a gauge field

$$\delta 2\partial_{[M} h^{\boldsymbol{A}}_{N]} = 2\theta^{\boldsymbol{C}}\partial_{[M} h^{\boldsymbol{B}}_{N]} f_{\boldsymbol{BC}}{}^{\boldsymbol{A}} + 2\partial_{[M}\theta^{\boldsymbol{C}} h^{\boldsymbol{B}}_{N]} f_{\boldsymbol{BC}}{}^{\boldsymbol{A}} \,, \tag{2.56}$$

it is obvious that the second term needs to cancel for the curvature to be covariant. This can easily be achieved by introducing a compensating term that transforms with derivatives on parameters just like the term above. Hence the covariant curvature is defined as

$$R_{MN}(T^{\boldsymbol{A}}) = 2\partial_{[M} h_{N]}^{\boldsymbol{A}} + h_N^{\boldsymbol{C}} h_M^{\boldsymbol{B}} f_{\boldsymbol{BC}}{}^{\boldsymbol{A}} \,. \tag{2.57}$$

Since the curvature is covariant, a covariant derivative acting on it can be defined

$$D_M R_{NP}(T^{\boldsymbol{A}}) = \partial_M R_{NP}(T^{\boldsymbol{A}}) - h_M^{\boldsymbol{C}} R_{NP}(T^{\boldsymbol{B}}) f_{\boldsymbol{BC}}{}^{\boldsymbol{A}} \,. \tag{2.58}$$

It is interesting to take a closer look at the complete antisymmetrisation of this expression

$$D_{[M} R_{NP]}(T^{\boldsymbol{A}}) = \partial_{[M} \left(h_P^{\boldsymbol{C}} h_{N]}^{\boldsymbol{B}} f_{\boldsymbol{BC}}{}^{\boldsymbol{A}} \right) + \left(2\partial_{[N} h_P^{\boldsymbol{C}} + h_{[P}^{\boldsymbol{E}} h_N^{\boldsymbol{D}} f_{\boldsymbol{DE}}{}^{\boldsymbol{C}} \right) h_{M]}^{\boldsymbol{B}} f_{\boldsymbol{BC}}{}^{\boldsymbol{A}} \,. \tag{2.59}$$

It is easy to see that the first and the second term cancel each other. The third term vanishes due to the Jacobi identity for the structure constants (see eq. (2.38)). The above relation

$$D_{[M} R_{NP]}(T^{\boldsymbol{A}}) = 0 \,, \tag{2.60}$$

is valid for all covariant curvatures and is referred to as the Bianchi identity. In electro-magnetism this is known as the homogeneous Maxwell equation.

2.1.3 From Special- to General Relativity by Gauging

While it was possible to construct the Vielbein formalism from the metric formulation of general relativity, it is, however, more systematic to obtain it by so-called gauging of the Poincaré algebra. The gauging procedure, which will be used throughout this thesis was first used by [18] to obtain and interpret supergravity as the gauge theory of supersymmetry. For additional elaboration on gauging, see [27]. To gauge an algebra means that one associates a gauge field, with corresponding gauge transformation rules, to every symmetry (in this case Poincaré) generator. The Vielbein, which was previously introduced without clear motivation, will appear naturally as the gauge field of local translations. To gauge the Poincaré algebra the concepts and methods introduced in section 2.1.2 about gauge theory are put to use. Finally one needs to replace local translations by diffeomorphisms. It will be discussed how this can be achieved.

Transformation Rules and Curvature

The algebra describing the infinitesimal symmetry transformation structure of special relativity is the Poincaré algebra

$$[M_{AB}, M^{CD}] = 4\delta_{[A}^{[C} M^{D]}{}_{B]} \,, \tag{2.61a}$$

$$[P_A, M_{BC}] = 2\eta_{A[B} P_{C]} \,, \tag{2.61b}$$

$$[P_A, P_B] = 0 \,, \tag{2.61c}$$

containing the generators of translations P_A and of Lorentz transformations M_{AB}. The non-zero structure constants of the algebra are

$$f_{[AB][CD]}{}^{[EF]} = 8\delta^{[E}_{[A}\eta_{B][C}\delta^{F]}_{D]} \, , \qquad f_{A[BC]}{}^{D} = 2\eta_{A[B}\delta^{D}_{C]} \, . \tag{2.62}$$

The first structure constant seems to be off by a factor of 2 from the algebra structure. This is because when a contracted generalised index $\boldsymbol{A}$ of generators and structure constants is index-pair-valued $\boldsymbol{A} = [AB]$, then a factor of $\frac{1}{2}$ needs to be introduced to prevent over-counting. This leads to

$$f_{\boldsymbol{AB}}{}^{\boldsymbol{C}} M_{\boldsymbol{C}} = \tfrac{1}{2} f_{\boldsymbol{AB}}{}^{[CD]} M_{[CD]} \, . \tag{2.63}$$

To gauge the algebra means to introduce a gauge field and a spacetime-dependent parameter for every generator.

Generator	Gauge Field	Parameter
P_A	$E_M{}^A$	$\Xi^A(x)$
M_{AB}	$\Omega_M{}^{AB}$	$\Lambda^{AB}(x)$

Table 2.1: The generators of Poincaré symmetry with their associated gauge fields and parameters.

An important attribute is the spacetime-dependence of the parameters which promotes previously global to local symmetries. We are now dealing with local Lorentz and local translational symmetry. The gauge fields have to transform with a derivative of their associated parameter while the remaining part of the transformation rules is determined by inspection of the algebra (see eq. (2.55))

$$\delta E_M{}^A = \partial_M \Xi^A + \Omega_M{}^{AB}\Xi_B - \Lambda^A{}_B E_M{}^B = D_M \Xi^A - \Lambda^A{}_B E_M{}^B \, , \tag{2.64a}$$

$$\delta \Omega_M{}^{AB} = \partial_M \Lambda^{AB} + 2\Omega_M{}^{C[A}\Lambda^{B]}{}_C = D_M \Lambda^{AB} \, . \tag{2.64b}$$

The derivative operator D_M, acting on the parameters in the transformation rules above is in fact the local Lorentz covariant derivative, as generally defined in equation (2.52). The gauge field part acts on the parameter in the appropriate representation (see sec. 2.1.4). In section 2.1.2 the covariant derivative was however defined as a covariant quantity acting on fields.[7] The notation will make no distinction between these different cases for convenience.

The covariant curvatures can be constructed directly from the algebra, by (2.57), to yield

$$R_{MN}(P^A) = 2\partial_{[M} E_{N]}{}^A + 2\Omega_{[M}{}^{AB} E_{N]B} \, , \tag{2.65a}$$

$$R_{MN}(M^{AB}) = 2\partial_{[M} \Omega_{N]}{}^{AB} + 2\Omega_{[M}{}^{CA}\Omega_{N]}{}^B{}_C \, . \tag{2.65b}$$

[7]Since here it is acting on parameters, although it takes the same form, it does not transform covariantly. For this reason it should not be thought of as an actual covariant derivative.

It can be read off that the curvature of local translations is nothing but torsion $R_{MN}(P^A) \equiv T_{MN}{}^A$ as it was defined before in (2.22) while the curvature of local Lorentz transformations is the Riemann tensor $R_{MN}(M^{AB}) \equiv R_{MN}{}^{AB}$ (see eq. (2.29)), as expected. When not written in components but in terms of torsion- and curvature two-forms, these relations can be expressed as

$$R(P^A) = dE^A + \Omega^A{}_C \wedge E^C \,, \tag{2.66a}$$

$$R(M^{AB}) = d\Omega^{AB} + \Omega^A{}_C \wedge \Omega^{CB} \,. \tag{2.66b}$$

They are then referred to as Cartan's first- and second structure equation.

Diffeomorphisms from Local Translations

It is clear that the naïve gauging of the Poincaré algebra produces local translations. This is not yet the desired diffeomorphism symmetry of general relativity. The procedure of relating these two symmetries makes local translations (lt), and therefore general coordinate transformations (gct), special compared to the other symmetries. Along with introducing the Vielbein in section 2.1.1, the no-torsion condition was derived from imposing metricity. Making use of this one can trade local translations for general coordinate transformations as follows

$$\begin{aligned} \delta_{\Xi_{\text{lt}}} E_M{}^A &= \partial_M \Xi^A + \Omega_M{}^{AB}\Xi_B + \Xi^N R_{NM}(P^A) \\ &= \partial_M \left(\Xi^N E_N{}^A\right) + \Xi^N 2\partial_{[N} E_{M]}{}^A + \Xi^N \Omega_N{}^{AB} \\ &= \mathcal{L}_\Xi E_M{}^A + \Xi^N \Omega_N{}^{AB} \\ &= \delta_{\Xi_{\text{gct}}} E_M{}^A + \delta_\Lambda E_M{}^A \,, \end{aligned} \tag{2.67a}$$

$$\text{where:} \qquad \Xi^M_{\text{gct}} = E^M{}_A \Xi^A_{\text{lt}} \,, \qquad \Lambda^{AB} = -\Xi^N \Omega_N{}^{AB} \,. \tag{2.67b}$$

Note that the first two terms in the first line constitute a local translation while the third term does not contribute due to the no-torsion condition (see eq. (2.22)). It follows that for vanishing torsion local translations can be expressed by general coordinate transformations, i.e. diffeomorphisms, additionally to a local Lorentz transformation. To recover the transformation rule from section 2.1.1, the transformations of the Vielbein under diffeomorphisms and local Lorentz transformations are added

$$\delta E_M{}^A = \mathcal{L}_\Xi E_M{}^A - \Lambda^{AB} E_{MB} \,. \tag{2.68}$$

After this the Vielbein is no longer treated as a gauge field and the prefix *covariant* will refer to local Lorentz symmetry.

2.1.4 Representations of the Poincaré Algebra

After having worked with vectors and tensors in the previous sections, this section aims to clarify the proper introduction of scalars, vectors and tensors as irreducible representations of the Poincaré algebra. The Poincaré algebra consists of two types of generators, that is translation- and Lorentz generators, P_A and M_{AB} respectively. Since only coordinates transform under translations as $x^A \to x'^A = x^A - \Xi^A$ and scalars, vectors and tensors do not, the treatment of this symmetry is simpler. All fields transform equally under translations while transformation under Lorentz symmetry depends on the type of field involved. Therefore the types of fields in the theory will be classified by their transformation properties under Lorentz symmetry. For notation see section 2.1.2.

On the group level a general Poincaré transformation has the form

$$\boldsymbol{S}(\Xi, \Lambda) = \exp(\Xi + \Lambda) \,, \tag{2.69}$$

where $\Lambda = \frac{1}{2}\Lambda^{AB}M_{AB}$ and $\Xi = \Xi^A P_A$. By the BCH-formula (see eq. (2.33)) and the fact that a Lorentz transformation and a translation commute to a translation, a redefinition of the translation parameter $\Xi \to \tilde{\Xi}$ can bring every Poincaré transformation to the form[8]

$$\boldsymbol{S}(\tilde{\Xi}, \Lambda) = \exp(-\Xi)\exp(\Lambda) = \boldsymbol{\Xi}^{-1}\boldsymbol{\Lambda} \,. \tag{2.70}$$

Under Poincaré symmetry coordinates transform as

$$x'^A = \boldsymbol{S}^{-1}(\tilde{\Xi}, \Lambda)x^A = \boldsymbol{\Lambda}^{-1}\boldsymbol{\Xi}x^A = \boldsymbol{\Lambda}^{-1}(x^A - \Xi^A) \,. \tag{2.71}$$

On the algebra level the infinitesimal transformation of the coordinate has the form

$$\delta x^A = -\Lambda^A{}_B x^B - \Xi^A \,. \tag{2.72}$$

$\Xi = 0$, *Lorentz Symmetry*

First only the Lorentz algebra is considered to classify objects by their transformation behaviour under $-\Lambda = -\frac{1}{2}\Lambda^{AB}M_{AB}$. Coordinates transform as

$$\delta x^A = -\Lambda x^A = -\Lambda^A{}_B x^B \,. \tag{2.73}$$

A scalar K is an object that does not transform under Lorentz transformations

$$\delta K = -\Lambda K = 0 \,. \tag{2.74}$$

The scalar representation is therefore $m^{\mathrm{s}}_{AB} = 0$. Since the coordinate x^A is a vector, we require the vector V^A to transform in the same fashion, i.e. as

$$\delta V^A = -\Lambda V^A = -\Lambda^A{}_B V^B \,. \tag{2.75}$$

[8] The sign of Ξ is convention.

By requiring that the contraction of a vector V^A and a covector U_A is a scalar, the transformation rule of the covector is computed to be

$$\delta U_A = -\Lambda U_A = -\Lambda_A{}^B U_B \ . \tag{2.76}$$

The vector representation can be read off[9] from these transformation rules $(m^{\mathrm{v}}_{AB})_C{}^D = -2\eta_{C[A}\delta^D_{B]}$. Now that vector- and covector transformations are known, arbitrary tensor representations m^{t}_{AB}, where the tensor indices are suppressed, can be built from them.

The transformation of arbitrary fields $\phi^i(x)$ needs additional attention, since fields are also functions of spacetime. General spacetime transformations of fields have the form (2.31b), where the algebra element Λ in first term, $-\Lambda\phi^i(x)$, is replaced by the appropriate representation, depending on the field. The second term, $\Lambda x^A\partial_A\phi^i(x)$, appears because the field is a function. The coordinate is however a vector, so it transforms under the vector representation

$$\begin{aligned}\Lambda x^A\partial_A\phi^i(x) &= \Lambda^A{}_B x^B\partial_A\phi^i(x)\\ &=: -\tfrac{1}{2}\Lambda^{AB} m^{\mathrm{f}}_{AB}\phi^i(x)\ ,\end{aligned} \tag{2.77}$$

where the function representation is the differential operator $m^{\mathrm{f}}_{AB} = -(m^{\mathrm{v}}_{AB})^{CD}x_C\partial_D$. The complete transformation rule of a field is

$$\delta\phi^i(x) = -\Lambda\phi^i(x) - \Lambda^{AB}x_A\partial_B\phi^i(x)\ . \tag{2.78}$$

$\Lambda = 0$, *Translation Symmetry*

As stated before, only coordinates transform under translations $\Xi = \Xi^A P_A$ as

$$\delta x^A = \Xi x^A = -\Xi^A\ , \tag{2.79}$$

where it has to be noted that the parameter Ξ^A is introduced with a sign[10]. Scalars, vectors and tensors do not transform, making the respective representations: $p^{\mathrm{s}}_A = 0$, $p^{\mathrm{v}}_A = 0$, $p^{\mathrm{t}}_A = 0$. Fields do transform under translations, but only as functions of spacetime coordinates. Like the function representation of the Lorentz algebra, the function representation of translations is a differential operator $p^{\mathrm{f}}_A = \partial_A$. From (2.31b) the infinitesimal transformation rules of arbitrary fields can be derived

$$\delta\phi^i(x) = \Xi^A\partial_A\phi^i(x)\ . \tag{2.80}$$

[9]Remember that whenever replacing a generator with a representation it produces an extra sign, as explained around eq. (2.41).

[10]So actually $-\Xi^A$ is the parameter of translations.

Poincaré Symmetry

Since scalars, vectors and tensors do not transform under translations, their Lorentz transformation rules are the complete Poincaré transformation rules. A general field transforms under Poincaré symmetry as

$$\phi'^i(x) = \mathbf{\Lambda}^{-1}\mathbf{\Xi}\phi^i(\mathbf{\Xi}^{-1}\mathbf{\Lambda}x) = \mathbf{\Lambda}^{-1}\phi^i(\mathbf{\Lambda}x^A + \Xi^A)\,, \tag{2.81}$$

which infinitesimally reduces to the sum of Lorentz transformation rules and translation transformation rules, (2.78) and (2.80) respectively

$$\delta\phi^i(x) = -\Lambda\phi^i(x) + \left(\Xi^B - \Lambda^{AB}x_A\right)\partial_B\phi^i(x)\,. \tag{2.82}$$

2.2 About Galilean- and Newton-Cartan Geometry

After having studied the geometric structure and kinematics of special and general relativity the question arises which concepts can be translated to non-relativistic theories and how this is to be done. In the previous section the procedure of gauging an algebra was introduced as the method to find the corresponding gauge theory; by making the symmetries local.

By gauging the correct algebra the non-relativistic analog of Lorentzian geometry, called Newton-Cartan geometry, can be found. But first, the correct non-relativistic algebra needs to be identified. Taking the non-relativistic limit of the Poincaré algebra yields the Galilei algebra. Note, however, that massive fields can only appear as projective representations of the Galilei algebra. For that reason it is more convenient to work with its central extension, the Bargmann algebra. At this point the Bargmann algebra can be gauged to find the transformation rules of the fields of Newton-Cartan theory. Finally it is shown that Galilean gravity can indeed be recovered from this formulation by partial gauge fixing. In principle one could also recover the non-relativistic Vielbein formalism from a non-relativistic metric formulation of gravity whose fundamental objects are a degenerate temporal- and a degenerate spatial metric that obey certain orthogonality relations as laid out in [2], [33]. This approach is however omitted in favour of gauging the Bargmann algebra.

2.2.1 From Poincaré to Galilei by İ.-W. Contraction

In section 2.1.2 İnönü-Wigner contractions were introduced as a technique to produce a new algebra from a known one. To do this a grading of each algebra element is defined, which is introduced with a small parameter which is then sent towards zero. The motivation for using an İnönü-Wigner contraction is that the inverse speed of light $\frac{1}{c}$ can be used as a small parameter to $\mathbb{Z}_2$-grade the Poincaré algebra which shall then by contraction yield a non-relativistic limit. The Poincaré generators are split up, with the index structure being of the form $A = (0, a)$. The generators corresponding to time translation and rotations are graded with c^0 and the generators of spatial translations and boosts are graded with c^{-1}

Grade 0	$P_0 \to H = P_0$,	$M_{ab} \to J_{ab} = M_{ab}$,
Grade 1	$P_a^{\text{rel}} \to P_a^{\text{n-rel}} = \frac{1}{c} P_a^{\text{rel}}$,	$M_{a0} \to G_a = \frac{1}{c} M_{a0}$.

Table 2.2: Grading of the Poincaré algebra.

Also rotations and boosts are relabelled. It should be noted that this is not the only possible grading of the Poincaré algebra, and therefore the Galilei algebra is not the only possible contraction. It is, however, the correct one in that it reproduces the non-relativistic limit. From this contraction the Galilei algebra emerges with the non-zero commutation relations

$$[J_{ab}, J_{cd}] = 4\delta_{[a[c}J_{d]b]} , \qquad [J_{ab}, G_c] = -2\delta_{c[a}G_{b]} , \tag{2.83a}$$

$$[J_{ab}, P_c] = -2\delta_{c[a}P_{b]} , \qquad [G_a, H] = -P_a . \tag{2.83b}$$

Explicitly, the contraction removes the commutation of two boosts to a rotation and the commutation of a boost and a spatial translation to a time translation.

2.2.2 From Galilei to Bargmann by Central Extension

The Galilei group, and hence the Galilei algebra do not yet describe the full symmetry structure of Newtonian gravity. The problem can be made visible by investigating massive fields. There will be two examples showing that mass in the non-relativistic theory requires special attention. This will motivate the central extension of the Galilei- to the Bargmann algebra. Galilei symmetry can be introduced by the transformation of the coordinates

$$t' = t + \tau , \qquad x'^a = R^a{}_b x^b + v^a t + a^a , \tag{2.84}$$

where $R \in SO(3)$ is a rotation, v and a are constant 3-vectors and τ is a constant.

Covariance of the Schrödinger Equation

First, the transformation of the wave function $\psi(t, x)$ of a massive free particle is derived from the condition of it obeying the Schrödinger equation. A particle that is free for one observer is also free for all observers related to the first observer by a Galilei transformation. Hence

$$\frac{p^2}{2m}\psi(t, x) = i\partial_0\psi(t, x) \qquad \to \qquad \frac{p'^2}{2m}\psi'(t', x') = i\partial'_0\psi'(t', x') , \tag{2.85}$$

where

$$\begin{aligned} &p_a\psi(t, x) = -i\partial_a\psi(t, x) , \\ &\partial'_0 = \partial_0 - v^a R_a{}^b\partial_b , \qquad \partial'_a = R_a{}^b\partial_b , \qquad \partial'_a\partial'^a = \partial_a\partial^a . \end{aligned} \tag{2.86}$$

By inserting the transformed quantities it can be seen that (2.85) is not compatible with the wave function transforming as a scalar, i.e. $\psi'(t', x') \neq \psi(t, x)$. Hence the Ansatz $\psi'(t', x') = e^{if(t,x)}\psi(t, x)$ is presented. The transformed Schrödinger equation can be solved with this Ansatz to find that $f(t, x) = mv^a R_{ab}x^b + \frac{mv^2}{2}t + k$, where k is an arbitrary constant. The transformation-dependent phase in the transformation of the wave function relates to the fact that the wave function of a massive particle transforms with a projective representation (see eq. (2.45)) of the Galilei group. Note that the coordinate-dependent phase vanishes for a massless field, i.e. $m = 0$. It can be shown (see [54]) that any projective representation of a group corresponds to an ordinary representation of a central extension of the group. Since it is more convenient to work with ordinary representations than with projective representations we will work with the central extension of the Galilei algebra which is the Bargmann algebra.

Poisson Algebra of Non-Relativistic Free Particle Action

There are multiple ways (see also sec. 3.2.1) to perform the central extension and find the commutator structure of the Bargmann algebra. One very simple and instructive example, presented in [3], is to calculate the Poisson bracket algebra of the conserved charges of the non-relativistic action of a massive free particle[11]

$$\mathscr{L} = \frac{m\dot{x}^2}{2} \,, \qquad p \equiv \frac{\partial \mathscr{L}}{\partial \dot{x}} \,. \tag{2.87}$$

The conserved[12] charges $Q(T_{\boldsymbol{A}})$ can be calculated with Noether's theorem from the infinitesimal transformations of the coordinates and the Lagrangian. Assuming that the Lagrangian transforms with a total time derivative, Noether's theorem with the included boundary term takes the following form

$$\theta^{\boldsymbol{A}} Q(T_{\boldsymbol{A}}) = p_a \delta_\theta x^a + (\mathscr{L} - p_a \dot{x}^a)\delta_\theta t - \int dt\, \delta_\theta \mathscr{L} \,. \tag{2.88}$$

For each symmetry a corresponding conserved charge is computed

$$\begin{aligned}
&\delta_\tau x^a = 0\,, && \delta_\tau t = \tau\,, && \delta_\tau \mathscr{L} = 0 && \to && Q(H) = -\mathscr{L}\,, && \text{(2.89a)}\\
&\delta_a x^a = a^a\,, && \delta_a t = 0\,, && \delta_a \mathscr{L} = 0 && \to && Q(P_a) = p_a\,, && \text{(2.89b)}\\
&\delta_v x^a = v^a t\,, && \delta_v t = 0\,, && \delta_v \mathscr{L} = m v_a \dot{x}^a && \to && Q(G_a) = p_a t - m x_a\,, && \text{(2.89c)}\\
&\delta_\lambda x^a = \lambda^a{}_b x^b\,, && \delta_\lambda t = 0\,, && \delta_\lambda \mathscr{L} = 0 && \to && Q(J_{ab}) = p_{[a} x_{b]}\,. && \text{(2.89d)}
\end{aligned}$$

As explained in [12], the Lagrangian of a free **relativistic** particle is invariant under Poincaré symmetry. In (2.89c) it can be seen that the Lagrangian of a free **non-relativistic** particle is not invariant under boosts, leading to a change in Noether charges. This ultimately explains the necessity for the central extension of the Galilei algebra that was not encountered when working with the Poincaré algebra. At this point some redefinitions of the charges are performed to later match the convention of the structure constants of the Galilei algebra: $Q(H) \to -Q(H)$ and $Q(J_{ab}) \to 2Q(J_{ab})$. The Poisson bracket of two phase-space functions $f(x,p)$ and $g(x,p)$ is defined as

$$\{f, g\}_{\mathrm{PB}} = \frac{\partial f}{\partial x^a}\frac{\partial g}{\partial p_a} - \frac{\partial g}{\partial x^a}\frac{\partial f}{\partial p_a} \,. \tag{2.90}$$

The Poisson bracket displays the symmetry structure of the action. In this case the Poisson bracket algebra of the conserved charges, additionally to the Galilei algebra,

[11] In [8] Lagrangians for non-relativistic free particles in a curved background and in Newton-Cartan theory, together with their supersymmetric extensions, are presented. These can be used to determine the algebra structure in the respective case, similarly to what is done here.

[12] The expression conserved charge is referring to a quantity conserved in time, i.e. $\partial_0 Q(T_{\boldsymbol{A}}) = 0$.

includes one other non-zero Poisson bracket

$$
\begin{aligned}
\{Q(J_{ab}), Q(J_{cd})\}_{\mathrm{PB}} &= 4\delta_{[a[c}Q(J_{d]b]}) \,, && (2.91a)\\
\{Q(J_{ab}), Q(G_c)\}_{\mathrm{PB}} &= -2\delta_{c[a}Q(G_{b]}) \,, && (2.91b)\\
\{Q(J_{ab}), Q(P_c)\}_{\mathrm{PB}} &= -2\delta_{c[a}Q(P_{b]}) \,, && (2.91c)\\
\{Q(G_a), Q(H)\}_{\mathrm{PB}} &= -Q(P_a) \,, && (2.91d)\\
\{Q(G_a), Q(P_b)\}_{\mathrm{PB}} &= -\delta_{ab}m \,. && (2.91e)
\end{aligned}
$$

The Poisson bracket of two conserved charges is of course also a conserved charge. The new conserved charge $Q(M) := m$ can be introduced that is the generator of the central extension of the Galilei algebra. Since m is a constant it is clear that the Poisson bracket $\{Q(M), \cdot\} = 0$ vanishes. The corresponding set of commutators describes the non-trivial central extension of the Galilei algebra, the Bargmann algebra

$$
\begin{aligned}
[J_{ab}, J_{cd}] &= 4\delta_{[a[c}J_{d]b]} \,, \qquad [J_{ab}, G_c] = -2\delta_{c[a}G_{b]} \,, && (2.92a)\\
[J_{ab}, P_c] &= -2\delta_{c[a}P_{b]} \,, \qquad [G_a, H] = -P_a \,, && (2.92b)\\
[G_a, P_b] &= -\delta_{ab}M \,. && (2.92c)
\end{aligned}
$$

One can also find the Bargmann algebra by modifying the redefinitions of the generators in the contraction procedure (see tab. 3.4). Since the new generator should produce a spacetime symmetry, this symmetry has to emerge from the other generators of spacetime symmetry. The only scalar spacetime symmetry of the Galilei algebra is the generator of time translations H, so it can be guessed that the new scalar generator of the central extension should also emerge from P_0 in the contraction. In fact, the generator of relativistic time translations, when interpreted as energy, can be expanded as the sum of non-relativistic energy and mass in the non-relativistic limit: $P_0 \to H + Mc^2 = P_0$, indicating that in a non-relativistic theory energy and mass (or particle number) are unrelated and are both separately conserved quantities. With the above redefinition the Poincaré algebra can be contracted to the Bargmann algebra. To see how to properly reduce a direct sum of the Poincaré algebra and an Abelian factor with a single generator to the Bargmann algebra, see [14].

2.2.3 From Bargmann to Newton-Cartan by Gauging

Earlier the Poincaré algebra was gauged to find the transformation rules of the Vielbein and the spin connection in general relativity (see sec. 2.1.3). Now the gauging procedure can be applied to the Bargmann algebra to find the transformation rules of the fields of its gauge theory, Newton-Cartan theory. Again every generator is associated with a gauge field and a local parameter

Generator	Gauge Field	Parameter	
P_a	$e_\mu{}^a$	$\zeta^a(x)$	$\xi^\mu(x)$
H	τ_μ	$\zeta(x)$	
M	m_μ	$\sigma(x)$	
J_{ab}	$\omega_\mu{}^{ab}$	$\lambda^{ab}(x)$	
G_a	$\omega_\mu{}^a$	$\lambda^a(x)$	

Table 2.3: The generators of Bargmann symmetry with their associated gauge fields and parameters.

By inspection of the Bargmann algebra the transformation rules of the gauge fields can be read off

$$\delta e_\mu{}^a = \partial_\mu \zeta^a - \omega_\mu{}^{ba}\zeta_b - \omega_\mu{}^a \zeta + e_{\mu b}\lambda^{ba} + \tau_\mu \lambda^a \, , \tag{2.93a}$$

$$\delta \tau_\mu = \partial_\mu \zeta \, , \tag{2.93b}$$

$$\delta m_\mu = \partial_\mu \sigma - \omega_\mu{}^a \zeta_a + e_{\mu a}\lambda^a \, , \tag{2.93c}$$

$$\delta \omega_\mu{}^{ab} = \partial_\mu \lambda^{ab} + 2\omega_\mu{}^{c[a}\lambda^{b]}{}_c \, , \tag{2.93d}$$

$$\delta \omega_\mu{}^a = \partial_\mu \lambda^a - \omega_\mu{}^{ca}\lambda_c + \omega_\mu{}^c \lambda_c{}^a \, . \tag{2.93e}$$

One has to take special care when raising and lowering indices at this point, because simply going from flat to spacetime indices using the Vielbein is no longer possible, as it was in relativity. The spatial Vielbein $e_\mu{}^a$ is not a square matrix, since there is also the temporal Vielbein τ_μ. There is however a way to define the exchange of spacetime and flat indices for covectors using the inverse Vielbeine. The inverse Vielbeine are defined by the following projective invertibility relations

$$e_\mu{}^a e^\mu{}_b = \delta^a_b \, , \qquad e_\mu{}^a e^\nu{}_a = \delta^\mu_\nu - \tau_\mu \tau^\nu \, , \qquad \tau_\mu \tau^\mu = 1 \, , \tag{2.94a}$$

$$e_\mu{}^a \tau^\mu = 0 \, , \qquad \tau_\mu e^\mu{}_a = 0 \, . \tag{2.94b}$$

Take a spacetime covector B_μ, a flat covector B_a and a scalar B_0. These quantities are by definition related by

$$B_a = e^\mu{}_a B_\mu \, , \qquad B_0 = \tau^\mu B_\mu \, , \qquad B_\mu = e_\mu{}^a B_a + \tau_\mu B_0 \, . \tag{2.95}$$

In this notation flat indices are raised (and lowered) using Kronecker deltas as $B^a = \delta^{ab}B_b$. The raising and lowering of the 0-index is not defined.

As was done in relativity, we would like to implement general coordinate transformation symmetry, replacing translation symmetries. In relativity the no-torsion condition was imposed to replace local translations with diffeomorphisms. In the non-relativistic theory three curvature constraints

$$r_{\mu\nu}(P^a) = 2\partial_{[\mu} e_{\nu]}{}^a - 2\omega_{[\mu}{}^{ba} e_{\nu]b} - 2\omega_{[\mu}{}^a \tau_{\nu]} = 0 \, , \tag{2.96a}$$

$$r_{\mu\nu}(M) = 2\partial_{[\mu} m_{\nu]} - 2\omega_{[\mu}{}^a e_{\nu]a} = 0 \, , \tag{2.96b}$$

$$\tau_{\mu\nu} := r_{\mu\nu}(H) = 2\partial_{[\mu}\tau_{\nu]} = 0 \, , \tag{2.96c}$$

are imposed to meet the same goal, to have the gauge fields transform as covectors under diffeomorphisms. The first two constraints are referred to as conventional constraints and the third constraint, the non-relativistic no-torsion condition, defines the foliation[13] of the Newtonian spacetime. The latter condition makes it possible to express the temporal Vielbein as the gradient of a function τ that then allows for the definition of absolute time T between two events, i.e. time that is independent of the path C between these events

$$\tau_{\mu\nu} = 0 \qquad \rightarrow \qquad \tau_\mu = \partial_\mu \tau \qquad \rightarrow \qquad T = \int_C dx^\mu \tau_\mu = \int_C d\tau \, . \tag{2.97}$$

The curvature conditions that were imposed above now allow for replacing local translations with parameters $\zeta = \tau_\mu \xi^\mu$ and $\zeta^a = e_\mu{}^a \xi^\mu$, with general coordinate transformations with parameter ξ^μ and compensating transformations

$$\begin{aligned} & \delta_\xi = \delta_\zeta + \delta_\sigma + \delta_{\lambda_r} + \delta_{\lambda_b} \, , \\ \text{where:} \quad & \xi^\mu = e^\mu{}_a \zeta^a + \tau^\mu \zeta \, , \qquad \sigma = -\xi^\nu m_\nu \, , \\ & \lambda^{ab} = -\xi^\nu \omega_\nu{}^{ab} \, , \qquad \lambda^a = -\xi^\nu \omega_\nu{}^a \, . \end{aligned} \tag{2.98}$$

Note that unlike in the relativistic case different symbols are used for the parameters of local translations and diffeomorphisms. This is because the exchange of curved- for flat coordinates was defined for spacetime covectors only (see eq. (2.95)), and the parameter of diffeomorphisms is a vector. It is this step of replacing local translations with diffeomorphisms in the procedure of gauging the Bargmann algebra that yields the no-torsion condition $\tau_{\mu\nu} = 0$. If one had only gauged the homogeneous Bargmann algebra, i.e. all Bargmann symmetries but translations, and additionally imposed diffeomorphism symmetry, this last no-torsion condition could be dropped, as is done in [6]. The two conventional constraints are, however, retrieved from the closure of the algebra. When obtaining Newton-Cartan geometry from null reduction (see sec. 3.2.2 and ch. 5), $\tau_{\mu\nu} = 0$ is not imposed.

The compensating transformations can be absorbed into the parameters by a redefinition. To match the convention used later in this thesis, the temporal Vielbein is redefined at this point $\tau_\mu \rightarrow -\tau_\mu$. The first two curvature constraints (2.96a,2.96b) can be solved for the spin- and the boost connection (see app. C). The connections written in terms of the remaining independent fields (and their inverse fields) take the form

$$\omega_\mu{}^{ab} = e^{\nu[a} c_{\mu\nu}{}^{b]} - \tfrac{1}{2} e_{c\mu} c^{abc} - \tfrac{1}{2} \tau_\mu c^{ab} \, , \tag{2.99a}$$

$$\omega_\mu{}^a = \tfrac{1}{2} c_{0\mu}{}^a + \tfrac{1}{2} e_{\mu b} c_0{}^{ab} + \tfrac{1}{2} c_\mu{}^a + \tfrac{1}{2} \tau_\mu c_0{}^a \, , \tag{2.99b}$$

$$c_{\mu\nu}{}^a = 2\partial_{[\mu} e_{\nu]}{}^a \, , \qquad c_{\mu\nu} = 2\partial_{[\mu} m_{\nu]} \, , \tag{2.99c}$$

[13] Frobenius' theorem states that the minimal condition for the spacetime to foliate is however not the no-torsion condition but instead the weaker twistless-torsion condition $\tau_{[\mu}\partial_\nu \tau_{\rho]} = 0 \leftrightarrow \tau_{ab} = 0$. In the latter case it is impossible to define absolute time.

where the ${c_{\mu\nu}}^a$ and $c_{\mu\nu}$ are the non-relativistic anholonomy coefficients.

All this leads to the transformation rules of the independent fields

$$\delta {e_\mu}^a = \mathcal{L}_\xi {e_\mu}^a - \lambda^{ab} e_{\mu b} - \lambda^a \tau_\mu \, , \tag{2.100a}$$

$$\delta \tau_\mu = \mathcal{L}_\xi \tau_\mu \, , \tag{2.100b}$$

$$\delta m_\mu = \mathcal{L}_\xi m_\mu + \partial_\mu \sigma + \lambda^a e_{\mu a} \, . \tag{2.100c}$$

An interesting observation can be made from the transformation rules of this NC gravity multiplet. While in relativistic gravity the gravity multiplet (only consisting of the Vielbein) is an irreducible representation of the Poincaré algebra[14], the NC gravity multiplet is an indecomposable but reducible representation of the Bargmann algebra, as is typical for non-relativistic theories (for more details on representations of the Galilei- and Bargmann algebra see e.g. [45]). This can be seen from boost behaviour of the fields – the gauge field of central charge transformations is boosted to the spatial Vielbein, but no field is boosted to the gauge field of central charge transformations.

2.2.4 From Newton-Cartan to Galilean Gravity by Partial Gauge Fixing

Galilean gravity is a generalisation of Newtonian gravity. While the latter only considers freely falling reference frames and includes a time independent Newton potential, Galilean gravity describes freely falling- and accelerated frames with time dependent acceleration. The inclusion of acceleration is what gives rise to a time dependent Newton potential. For extensive elaboration on this see [20], [22]. It is however also possible to impose stronger gauge fixing conditions to only retain Galilei symmetries.

Now that the kinematics of Newton-Cartan gravity has been found, it is still not obvious that this theory in fact reproduces Galilean gravity. To see this a partial gauge fixing is performed in which a lot of fields can consistently be set to zero. The partial gauge fixing will be given rather than derived. The motivation and the derivation that leads to this partial gauge fixing can be found in [4]. Additionally to the conventional constraints and the no-torsion condition (see eq. (2.96)), the curvature of rotations is also set zero, i.e. $r_{\mu\nu}(J^{ab}) = 0$, since Galilean gravity

[14]It may seem obvious for the Vielbein, but this fact remains true also when relativistic supergravity multiplets like (3.68) that include several fields are considered. They are then irreducible representations of the super-Poincaré algebra.

should be recovered in flat space. The gauge fixing conditions are

$$\tau_\mu(x) = \delta_\mu^\emptyset\,, \tag{2.101a}$$
$$\omega_\mu{}^{ab}(x) = 0\,, \tag{2.101b}$$
$$e_i{}^a(x) = \delta_i^a\,, \tag{2.101c}$$
$$t^a(x) := \tau^a(x)\,, \quad \rightarrow \quad e_\emptyset{}^a(x) = -t^a(x)\,, \tag{2.101d}$$
$$\partial_i m(x) := t_i(x) + m_i(x)\,, \tag{2.101e}$$
$$m(x) = 0\,, \tag{2.101f}$$
$$t^a(x) = 0\,, \tag{2.101g}$$
$$\Phi(x) := m_\emptyset(x)\,, \tag{2.101h}$$

where the $\emptyset$-index of a quantity B_μ indicates $B_\emptyset = B_{\mu=0}$ and should not be confused with $B_0 = \tau^\mu B_\mu$. From the Vielbeine one can see that upon using this partial gauge fixing one ends up with Euclidean space with an ordinary Euclidean metric. The only field left in the theory is $m_\emptyset(x)$ which is then identified as the Newton potential $\Phi(x) = m_\emptyset(x)$. Of all connections there is only one non-trivial component left, which is $\omega_\emptyset{}^a(x) = -\partial^a m_\emptyset(x)$ of the boost connection. The only non-zero curvature components reside in the symmetric part of the boost curvature, whose vanishing trace reproduces the massless Poisson equation (i.e. the Laplace equation)

$$r_{\emptyset a}(G^a) = -\partial_a \omega_\emptyset{}^a = \partial_a \partial^a \Phi(x) = \Delta\Phi(x) = 0\,. \tag{2.102}$$

2.2.5 Dimensional Reduction

Dimensional reduction was first proposed by Kaluza and Klein in [39], [40] as a tool to unify electro-magnetism with 4D gravity. By taking one spatial dimension to be compact and small 5D gravity can be viewed as an effective 4D gravitational theory coupled to extra degrees of freedom that emerge from the higher dimension, among which there is a Maxwellian vector potential. Kaluza and Klein originally considered dimensional reduction along a compact spacelike direction, one can, however, consider dimensional reduction along a compact[15] null direction, as described in [23], [37]. Now one may ask if it is also possible to perform a timelike dimensional reduction to obtain a Euclidean theory. This is in fact possible and an example for Euclidean 4D supergravity can be found in [66]. Unlike a spacelike reduction which produces a relativistic effective theory, a null reduction leads to a non-relativistic effective theory. This makes it a useful tool to produce non-trivial lower dimensional non-relativistic theories by null-reducing simpler higher dimensional relativistic theories. The results obtained from contraction of the Poincaré- to the Bargmann algebra and gauging can also be obtained from direct gauging of the Poincaré algebra and null reduction, e.g.:

[15] In fact, there are ways to not impose compactness and still receive effective lower dimensional theories as discussed in [51]. We will, however, assume the dimensions along which the reduction takes place to be compact.

$$
\begin{tikzcd}
\text{Poincaré alg. in } D \arrow[r, "\text{contr. \& centr. ext.}"] & \text{Bargmann alg. in } D \arrow[dr, "\text{gauging}"] & \\
 & & \text{NC in } D \\
\text{Poincaré alg. in } (D+1) \arrow[r, "\text{gauging}"'] & \text{GR in } (D+1) \arrow[ur, "\text{null red.}"'] &
\end{tikzcd}
$$

Figure 2.1: Two paths to D-dimensional Newton-Cartan geometry starting from either the D- or the $(D-1)$-dimensional Poincaré algebra.

When including supersymmetry, which will happen in the next chapter, it is generally no longer possible to follow the steps illustrated in figure 2.1 at will. Especially the naïve gauging procedure will not produce the full supergravity multiplet (e.g. see sec. 3.1.6). With supersymmetry included, finding Newton-Cartan supergravity has to be treated as a separate problem in every dimension.

The dimensional reduction of general fields is non-trivial and will be discussed later (see sec. 3.2.2, 4.3). A D-dimensional vector for example can be reduced to a $(D-1)$-dimensional vector and a scalar. The dimensional reduction of a scalar field as described in [48] is, however, straightforward due to the dimensionally independent number of its components, and shall be presented as an instructive example of dimensional reduction. The reduction of the Vielbein will be discussed subsequently.

Dimensional Reduction of the Scalar Field

The massless[16] scalar field $K(x)$ in D-dimensional flat space obeys the wave equation

$$\Box^{(D)} K(x) = 0 \,, \tag{2.103}$$

where $\Box^{(D)} = \eta^{(D)MN} \partial_M \partial_N$ is the d'Alembertian in D spacetime dimensions.

Spacelike Reduction

In performing a spacelike reduction one assumes that one spacelike dimension is circular. Its coordinate will be denoted as x^* while the coordinates of the remaining $(D-1)$ spacetime directions are denoted by x^A. Note that since the coordinate x^* is circular there is a $U(1)$ symmetry identifying $x^* \sim x^* + 2n\pi R$, $\forall n \in \mathbb{Z}$, where R is the radius of the dimension.

The scalar field $K(x^A, x^*)$ can be expanded in terms of its Fourier modes $K^{(n)}(x^A)$ as

$$K(x^A, x^*) = \sum_{n \in \mathbb{Z}} e^{i \frac{n x^*}{R}} K^{(n)}(x^A) \,. \tag{2.104}$$

[16] The scalar field is chosen to be massless for convenience. If it were massive, the reduced fields would receive a mass contribution from the mass of the original field.

Assuming that the massless field $K(x^A, x^*)$ obeys the D-dimensional wave equation (2.103), its Fourier modes obey an infinite number of $(D-1)$-dimensional Klein-Gordon equations

$$\left(\Box^{(D-1)} - \left(\tfrac{n}{R}\right)^2\right) K^{(n)}(x^A) = 0\,, \tag{2.105}$$

where the mass of each respective Fourier mode is $m_n = |\frac{n}{R}|$.

Null Reduction

If the dimensional reduction is not performed along a spacelike direction but along a lightlike (also referred to as null) direction, the fields in the lower dimensional effective theory exhibit non-relativistic behaviour. In D spacetime dimensions define $(D-2)$ spacelike directions x^a and two null directions as[17] $x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^*)$, where x^* denotes a spacelike coordinate. If the null coordinate x^+ is chosen to be compact (and circular), i.e. $x^+ \sim x^+ + 2n\pi R$, $\forall n \in \mathbb{Z}$, the scalar field $K(x^a, x^-, x^+)$ can be expanded in terms of Fourier modes as

$$K(x^a, x^-, x^+) = \sum_{n\in\mathbb{Z}} e^{i\frac{nx^+}{R}} K^{(n)}(x^a, x^-)\,. \tag{2.106}$$

The wave equation (2.103) of the higher dimensional massless scalar field reduces to the inherently non-relativistic Laplace equation for $n = 0$ and the Schrödinger equation for all other Fourier modes

$$\Delta K^{(n)}(x^a, x^-) = 0 \qquad \text{for } n = 0\,, \tag{2.107a}$$

$$\left(-i\partial_- + \frac{\Delta}{2\frac{n}{R}}\right) K^{(n)}(x^a, x^-) = 0 \qquad \text{for } n \neq 0\,, \tag{2.107b}$$

where $\Delta = \delta^{ab}\partial_a\partial_b$ is the Laplacian, the former null direction x^- serves as the non-relativistic time coordinate and $m_n = \left|\frac{n}{R}\right|$ is the mass of the respective mode.

Massless Modes

The physical justification for choosing higher dimensions to be compact and small is that they are not observable[18]. Since a small compact dimension, i.e. $R \ll 1$, leads to excitations with large masses it is reasonable to truncate the spectrum and only consider the massless modes. This can equivalently be expressed as the following condition on the field

$$\partial_* K(x^A, x^*) = 0 \quad \text{for } x^* \text{ compact and small}\,, \tag{2.108a}$$

$$\partial_+ K(x^a, x^-, x^+) = 0 \quad \text{for } x^+ \text{ compact and small}\,, \tag{2.108b}$$

[17] This defines the components of the Minkowski metric to be $\eta_{+-} = -1$.

[18] As stated before, this is a motivation for this choice, but not a necessary condition according to [51].

where the partial derivatives are in the respective direction of dimensional reduction. In Kaluza and Klein's work this condition is referred to as *cylinder condition*. This condition does not only apply to scalar fields but also to all other fields of a dimensionally reduced multiplet.

Null Reduction of the Vielbein

For a theory describing gravity we can, of course, not limit ourselves to working with flat space, as was done in the previous example. To describe gravity in D spacetime dimensions the Vielbeine $E_M{}^A$ are introduced. When working with Vielbeine there are flat tangent space coordinates $A, B, C, \ldots$ and curved spacetime coordinates $M, N, P, \ldots$. To perform the null reduction along x^+, the set of flat coordinates is expressed in a null basis $x^A = (x^a, x^\mp = \frac{1}{\sqrt{2}}(x^0 \mp x^*))$, where the index a takes $(D-2)$ values. The curved index $x^M = (x^\mu, x^z)$ is split in terms of the direction of reduction, denoted as z, and the remaining spacetime index μ, taking $(D-1)$ values. Since we are assuming the reduced dimension z to be small we are only interested in the massless mode of the Vielbein, i.e.

$$\partial_z E_M{}^A = 0 \,. \tag{2.109}$$

The null reduction Ansatz that will be presented follows closely the discussion of [6]. Formally the condition (2.109) can be viewed as a Killing equation, defining an isometry of the spacetime

$$\partial_z E_M{}^A = \mathcal{L}_\aleph E_M{}^A = 0 \,, \tag{2.110}$$

expressed by a normalised Killing covector $\aleph$ in the direction of reduction, i.e. $\aleph^M = \delta_z^M$. To ensure that the direction z is a null direction, the Killing vector $\aleph$ is taken to be null, i.e. $\aleph^M \aleph_M = 0$. When the (super-)gravity multiplet containing the Vielbein contains other fields as well (e.g. see eq. 4.1), they obey the similar conditions $\mathcal{L}_\aleph\{\text{(su-)gra multiplet}\} = 0$ (e.g. see eq. 4.10), which are again justified by the truncation of the spectra of the respective fields (as in eq. (2.108)). Additionally, the partial gauge fixing condition

$$E_z{}^a = 0 \tag{2.111}$$

is imposed. That, together with the Killing vector being null, requires the Killing vector in flat coordinates to be along a flat null direction $x^\pm$. Since we want to identify the flat null direction x^+ with the reduced direction we choose $\aleph^A = \delta_+^A E_z{}^+$ and equivalently $\aleph^M = E_z{}^+ E^M{}_+$, hence $E_z{}^- = 0$. It also follows that

$$\partial_+ E_M{}^A = 0 \quad \rightarrow \quad E^\mu{}_+ = 0 \,. \tag{2.112}$$

Anticipating the structure of Newton-Cartan geometry found by gauging the Bargmann algebra (see sec. 2.2.3), the Vielbein and the inverse Vielbein can be split in terms of lower dimensional coordinates with the following Ansatz[19]

$$E_M{}^A = \begin{pmatrix} e_\mu{}^a & S^{-1}\,\tau_\mu & -S\,m_\mu \\ 0 & 0 & S \end{pmatrix}\,, \qquad E^M{}_A = \begin{pmatrix} e^\mu{}_a & e^\mu{}_a m_\mu \\ S\,\tau^\mu & S\,\tau^\mu m_\mu \\ 0 & S^{-1} \end{pmatrix}\,, \tag{2.113}$$

where at this point $e_\mu{}^a$, τ_μ and m_μ are general vector- and tensor fields, and S is a scalar field. These fields automatically inherit the orthogonality relations

$$\begin{aligned} e_\mu{}^a e^\mu{}_b = \delta^a_b\,, \qquad \tau_\mu\tau^\mu = 1\,, \\ e_\mu{}^a \tau^\mu = 0\,, \qquad \tau_\mu e^\mu{}_a = 0\,, \\ e_\mu{}^a e^\nu{}_a + \tau_\mu\tau^\nu = \delta^\nu_\mu\,, \end{aligned} \tag{2.114}$$

from the original

$$E_M{}^A E^M{}_B = \delta^A_B\,, \qquad E_M{}^A E^N{}_A = \delta^N_M\,. \tag{2.115}$$

The notation is, however, chosen suggestively because the fields $e_\mu{}^a$ and τ_μ can be identified as the spatial- and the temporal Vielbeine of Newton-Cartan geometry, and m_μ as the gauge field of central charge transformations. The scalar field S does not appear in Newton-Cartan geometry derived by gauging. However, as can be seen later, the scalar S can also be gauge fixed (e.g. see sec. 4.3.4).

[19]Note that the Vielbein $E_M{}^A$ and its inverse are square matrices, owing their seemingly non-square form to the choice of index-splitting $A = (a, -, +)$ and $M = (\mu, z)$.

Chapter 3

Supergravity

In this chapter we will implement supersymmetric extension of the concepts from the previous chapter into the theory. First the Clifford algebra and spinors are introduced as the necessary tools that enable us to talk about supersymmetry. Then supersymmetry is first defined as a global symmetry, as were all symmetries in the last chapter, only to be made local later in the process of gauging. The concepts of bosonic gauge theory are extended to include spinorial symmetries, and with it spinorial algebra generators, gauge fields and gauge parameters. Then an example is presented in which a supergravity multiplet and its transformation rules are found by gauging the super-Poincaré algebra. When supersymmetry is part of the game, it is generally no longer possible to find a supergravity multiplet by gauging the super-Poincaré algebra in an arbitrary spacetime dimension D including $\mathcal{N}$ supersymmetries. Therefore an example is presented of how to construct a supergravity multiplet that cannot be found from gauging in its entirety. Finally, two different derivations of the same non-relativistic supergravity multiplet are stated, to illustrate that the multiplet can be found from gauging the super-Bargmann algebra[1] or from null-reducing a higher dimensional relativistic supergravity multiplet. The latter is the method that will be utilized to find non-relativistic 4D, $\mathcal{N} = 2$ supergravity in chapters 4 and 5.

3.1 About Relativistic Supersymmetry and Supergravity

In this section Clifford algebras and spinors are introduced to be then be included into gauge theory as the main ingredients of fermionic supersymmetry. Then two examples of supergravity multiplets are presented.

[1]In this specific example the non-relativistic supergravity multiplet can be found by gauging. This is generally **not** possible, neither in relativistic nor in non-relativistic settings.

3.1.1 Clifford Algebras and Spinors

Gamma matrices in 4D are known from the Dirac equation, describing the evolution of a massive fermion. In the following section gamma matrices will be introduced more formally to be able to construct and work with them in general dimensions. Gamma matrices act on fermions, which is to say that they act on the space of spinors. Therefore the gamma matrices will be integral to realise Lorentz transformations of spinors. Finally, they will make it possible to define a number of spinor bilinears transforming as tensors under Lorentz symmetry. First, the mathematical structure behind gamma matrices, called Clifford algebra, is discussed. Then, different kinds of spinors will be introduced.

Clifford Algebras

In any dimension D the Clifford algebra can be defined as a unital associative algebra[2] that is generated by objects γ^A that satisfy the anticommutation relation

$$\{\gamma^A, \gamma^B\} = \gamma^A\gamma^B + \gamma^B\gamma^A = 2\eta^{AB}\,\mathbb{1}\,, \tag{3.1}$$

where η^{AB} is the Minkowski metric in D dimensions. We will always work in a matrix representation in which $\mathbb{1}$ is the identity matrix and γ^A are referred to as gamma matrices. There is a simple Ansatz presented in [27] to construct gamma matrices explicitly from tensor products of Pauli matrices for general dimensions. The representation of gamma matrices is, however, not unique and for our purposes any explicit construction is unnecessary since all relevant properties can be stated in general terms. The complete Clifford algebra consist of the unit matrix, the gamma matrices γ^A and all products of gamma matrices (subject to (3.1)). Some important facts on Clifford algebras are:

- We restrict ourselves to irreducible representations that are hermitian, i.e. they obey $\gamma^{A\dagger} = \gamma^0\gamma^A\gamma^0$.
- For every even dimension $D = 2m$ there is a unique equivalence class of 2^m-dimensional irreducible representations $\gamma'^A = U\gamma^A U^{-1}$, related by unitary operators[3] U. For every odd dimension $D = 2m+1$ there are two inequivalent 2^m-dimensional irreducible representations.
- The notation $\gamma^{A_1 \dots A_r}$ is used to denote completely antisymmetrised products of gamma matrices as follows

  $$\gamma^{A_1 \dots A_r} = \gamma^{[A_1} \dots \gamma^{A_r]}\,. \tag{3.2}$$

  The rank r of such a Clifford algebra element refers to its number of indices. The identity matrix has rank $r = 0$.

---

[2] A unital associative algebra is an algebraic structure with addition and associative multiplication, including a multiplicative unit element. It also includes scalar multiplication with elements of a field, e.g. $\mathbb{R}$.

[3] The operators are unitary because the representations are hermitian.

- All Clifford algebra elements $\gamma^{A_1 \dots A_r}$ with rank $r > 0$ are traceless.
- In even[4] dimensions a highest rank element γ_* can be defined as the product of all single gamma matrices

$$\gamma_* = (-i)^{m+1} \gamma_0 \gamma_1 \dots \gamma_{D-1} \, . \tag{3.3}$$

This element is of special interest because it links even and odd dimensions (see eq. (3.7)) and it is related to the chirality of fermions (see eq. (3.18)). It is hermitian and squares to unity. As its name suggests, it is the unique Clifford algebra element with the highest possible rank $r = D$. In 4D this matrix is usually denoted as γ_5.

Basis of Clifford Algebras

Since arbitrary products of gamma matrices reduce to a linear combination of totally antisymmetric products of gamma matrices by the defining equation (3.1), it suffices to only include completely antisymmetric elements (eq. (3.2)) in the basis of a Clifford algebra. In **even** spacetime dimensions the unit matrix, the gamma matrices and all completely antisymmetric products of gamma matrices compose a basis of the algebra

$$\Gamma^{\boldsymbol{A}} = \left\{ \mathbb{1}, \gamma^A, \gamma^{A_1 A_2}, \dots \gamma^{A_1 \dots A_D} \right\} , \tag{3.4}$$

where $\boldsymbol{A}$ is a collective index denoting sets of r indices[5]. Similarly, denoting the inverse basis

$$\Gamma_{\boldsymbol{A}} = \left\{ \mathbb{1}, \gamma_A, \gamma_{A_2 A_1}, \dots \gamma_{A_D \dots A_1} \right\} , \tag{3.5}$$

(note the reverse order of indices) one has the following trace orthogonality relation

$$\mathrm{Tr}(\Gamma^{\boldsymbol{A}} \Gamma_{\boldsymbol{B}}) = 2^m \, \delta^{\boldsymbol{A}}_{\boldsymbol{B}} \, , \tag{3.6}$$

where $\delta^{A_1 \dots A_s}_{B_1 \dots B_s} \equiv \delta^{[A_1}_{B_1} \dots \delta^{A_s]}_{B_s}$. It is shown in [27] that the set $\Gamma^{\boldsymbol{A}}$ form a basis for all $2^m \times 2^m$ matrices.

In **odd** spacetime dimensions $D = 2m + 1$ the structure of the basis is not as obvious. To obtain it one can rearrange the Clifford algebra elements of dimension $D = 2m$. First, the generating gamma matrices are constructed from the gamma matrices of $D = 2m$ by adding the highest rank element as $(2m + 1)^{\mathrm{st}}$ element

$$\gamma^A_\pm = \{ \gamma^0, \gamma^1, \dots \gamma^{2m-1}, \gamma^{2m} = \pm\gamma_* \} \, . \tag{3.7}$$

[4]In odd dimensions the product of all γ^A matrices is proportional to the identity $\gamma_{*(D=2m+1)} = \gamma^2_{*(D=2m)} = \mathbb{1}$, and thus does not lead to a new Clifford algebra element.

[5]Where r denotes the rank of the respective Clifford algebra basis element.

It can easily be verified that the above choice is compatible with equation (3.1). Since in $D = 2m+1$ dimensions every index $A, B, \dots$ takes one more value than in $D = 2m$ dimensions, there would be a lot more basis elements in a basis of the form of (3.4) and (3.5). The $D = 2m$-dimensional basis was, however, already a complete basis for all $2^m \times 2^m$ matrices. To avoid overcompleteness the basis in $D = 2m+1$ spacetime dimensions only contains elements up to rank $r = m$. All elements with rank $r > m$ are related to those with rank $r \leq m$ by duality relations (see app. eq. (B.9)). By counting elements we realise that the basis of dimension $D = 2m+1$ is nothing but a reordering of the basis of dimension $D = 2m$.

Symmetry under Charge Conjugation

For any dimension and any representation it is possible to define a symmetry property for all Clifford algebra elements that only depends on their rank r. This symmetry property can be written out explicitly by using a unitary matrix C, which is referred to as the charge conjugation matrix

$$\left(C\,\Gamma^{(r)}\right)^T = -t_r\, C\,\Gamma^{(r)}\,, \qquad \text{with} \qquad t_r = \pm 1\,. \tag{3.8}$$

There are strong constraints on the coefficients t_r, which can be found by inserting the lowest rank element into (3.8) and continuing with higher rank elements

$$\begin{aligned} C^T &= -t_0\, C\,, \qquad (\gamma^A)^T = t_0 t_1\, C\gamma^A C^{-1}\,, \\ t_2 &= -t_0\,, \qquad t_3 = -t_1\,, \qquad t_{r+4} = t_r\,, \end{aligned} \tag{3.9}$$

thereby determining all symmetries. While for odd dimensions C is unique (see tab. 3.1), there are two options $C_\pm$ for even dimensions. One can switch between the two by replacing C with $C\gamma_*$. In the table below the values of the first two coefficients t_r are listed for arbitrary dimensions.

D mod8	(t_0, t_1) for C_+	(t_0, t_1) for C_-
0	$\mathbf{(-1,-1)}$	$(-1,1)$
1	$(-1,-1)$	—
2	$(-1,-1)$	$\mathbf{(1,-1)}$
3	—	$(1,-1)$
4	$(1,1)$	$\mathbf{(1,-1)}$
5	$(1,1)$	—
6	$(1,1)$	$\mathbf{(-1,1)}$
7	—	$(-1,1)$

Table 3.1: List of symmetry properties of Clifford algebra basis elements that are fully determined by $t_{0,1}$. The bold choices are better suited for supersymmetry in terms of which types of spinors are available to work with. The information in this table is taken from [27].

Since the representation is hermitian and C is used to transpose, another unitary matrix B can be defined from C that determines complex conjugation denoted by a superscript asterisk *

$$B = it_0\, C\gamma^0 \,, \qquad \gamma^{\mu *} = -t_0 t_1\, B\gamma^\mu B^{-1} \,. \tag{3.10}$$

Spinors

In physical theories fermions are described by spinors. A spinor Ψ is an element from the complex vector space that gamma matrices act on, whose components we take to be Grassmann-valued, i.e. anticommuting. It transforms with the spinor representation[6] of the Lorentz algebra, which is defined via the rank $r = 2$-elements of the Clifford algebra, i.e.

$$(m^{\mathrm{Sp}}_{AB})_\alpha{}^\beta = -\tfrac{1}{2}(\gamma_{AB})_\alpha{}^\beta \,. \tag{3.11}$$

The greek indices $\alpha, \beta, \ldots$ are referred to as spinor indices. Spinors Ψ_α have subscript indices. The contraction of spinor indices always affects two indices that are directly next to one another, while the first one is a superscript and the second one is a subscript, e.g. $(\gamma^A\Psi)_\alpha = (\gamma^A)_\alpha{}^\beta\Psi_\beta$. This convention is therefore known as northwest-southeast (NW-SE) convention. Under infinitesimal Lorentz transformations spinors transform as

$$\delta_\Lambda \Psi_\alpha = \tfrac{1}{2}\Lambda^{AB} M_{AB}\Psi_\alpha = -\tfrac{1}{4}\Lambda^{AB}(\gamma_{AB})_\alpha{}^\beta \Psi_\beta \,, \tag{3.12}$$

or, as we will denote more commonly, suppressing spinor indices

$$\delta_\Lambda \Psi = -\tfrac{1}{4}\Lambda^{AB}\gamma_{AB}\Psi \,. \tag{3.13}$$

Although in odd dimensions $D = 2m + 1$ there are two inequivalent representations of the Clifford algebra, this fact will not be of importance to us, since the generating representations of the Lorentz algebra are equivalent

$$\nexists S : \quad S\gamma_+^A S^{-1} = \gamma_-^A \,, \qquad \text{however} \qquad \gamma_*\gamma_+^{AB}\gamma_* = \gamma_-^{AB} \,. \tag{3.14}$$

There is a list of different types of spinors that have different applications. Mostly, however, the type of spinor used in a theory depends on the spacetime dimension.

Dirac Spinors

A general 2^m-complex-component spinor in spacetime dimensions $D = 2m$ or $D = 2m + 1$ is called Dirac spinor. If fermions are to be included in an action, one needs to construct Lorentz scalars from spinors that can contribute to the Lagrangian. To find Lorentz scalars, the Dirac adjoint $\bar{\Psi}^{(D)}$ of a spinor Ψ is defined as

$$\bar{\Psi}^{(D)} = \Psi^\dagger \beta \,, \tag{3.15}$$

[6] It can easily be verified with the commutation relations (2.61a) that this is in fact a representation of the Lorentz algebra.

where[7] $\beta = i\gamma^0$, to obtain the Lorentz invariant spinor bilinear of two Dirac spinors Ψ and Φ

$$\bar{\Psi}^{(D)}\Phi \,. \tag{3.16}$$

This definition of the Dirac conjugate also ensures that the spinor bilinears $\bar{\Psi}^{(D)}\Gamma^{(r)}\Phi$ transform as tensors of rank r under Lorentz transformations. The conjugation superscript (D) will be dropped after the introduction of Majorana conjugation in equation (3.19).

Weyl Spinors

In even dimensions $D = 2m$ one can always choose a Clifford algebra representation that has a 2-block-diagonal highest rank element $\gamma_* = \mathrm{diag}(\mathbb{1}, -\mathbb{1})$ due to $\gamma_*^2 = \mathbb{1}$ and $\mathrm{Tr}(\gamma_*) = 0$. It follows that the generating first rank elements γ^A are 2-block-off-diagonal and the second rank elements γ_{AB} are 2-block-diagonal. This means that the spinor representation of the Lorentz algebra is 2-block-diagonal and therefore reducible to two irreducible subrepresentations. These irreducible representations are called Weyl- or chiral spinors. The highest rank element is used to project the 2^m-complex-component Dirac spinor Ψ onto the two 2^{m-1}-complex-component Weyl spinors λ and χ by

$$\Psi = \begin{pmatrix} \lambda \\ \bar{\chi}^{(D)} \end{pmatrix} , \qquad P_L\Psi = \begin{pmatrix} \lambda \\ 0 \end{pmatrix} , \qquad P_R\Psi = \begin{pmatrix} 0 \\ \bar{\chi}^{(D)} \end{pmatrix} , \tag{3.17}$$

where the two orthogonal chiral projectors are

$$P_{\substack{L \\ R}} = \tfrac{1}{2}(\mathbb{1} \pm \gamma_*) \,. \tag{3.18}$$

In odd dimensions $D = 2m + 1$ the spinor representation of the Lorentz algebra, constituted by the second rank Clifford algebra elements, is irreducible. Hence there are no Weyl spinors.

(Symplectic) Majorana Spinors

The definition of (symplectic) Majorana spinors is highly dependent on the symmetry properties of the gamma matrices under charge conjugation, which, again, are spacetime dimension dependent. In dimensions where[8] $t_1 = -1$ Majorana spinors (MS) can be defined, while in dimensions where[9] $t_1 = 1$ symplectic Majorana spinors (SMS) can be defined. To state the reality conditions that (symplectic) Majorana

[7]The conjugation matrix β is found to be proportional to γ^0 by requiring that $\Psi^\dagger\beta\Psi$ is Lorentz invariant. The imaginary factor is convention.

[8]These are dimensions $D = 0, 1, 2, 3, 4 \mod 8$.

[9]These are dimensions $D = 0, 4, 5, 6, 7 \mod 8$.

spinors have to obey, the operations of charge conjugation ψ^C and Majorana conjugation $\bar{\psi}^{(M)}$ of a spinor ψ are defined as

$$\psi^C = B^{-1}\psi^* \,, \qquad \bar{\psi}^{(M)} = \psi^T C \,. \tag{3.19}$$

The Majorana reality condition that a MS ψ has to obey can be stated equivalently in different versions

$$\psi^C = \psi \quad \leftrightarrow \quad \psi^* = B\psi \quad \leftrightarrow \quad \bar{\psi}^{(\mathrm{D})} = -t_1 \bar{\psi}^{(\mathrm{M})} \,, \tag{3.20}$$

where the superscripts (D) and (M) denote the Dirac- and the Majorana conjugate respectively. These superscripts are now dropped, since from here on the bar will **always** refer to Majorana conjugation. Note that a MS is generally not *real* in the sense of complex conjugation, but only in the sense of charge conjugation.

Symplectic Majorana spinors ψ^i always come in pairs[10], i.e. $i = 1, 2$. They obey the symplectic Majorana reality condition

$$(\psi^i)^C = \psi^j \varepsilon_j{}^i \quad \leftrightarrow \quad (\psi^i)^* = B\psi^j \varepsilon_j{}^i \,, \tag{3.21}$$

where the antisymmetric matrix ε_{ij} and its inverse ε^{ij} are used to raise and lower the pairing index with the NW-SE convention $\psi^i \varepsilon_{ij} = \psi_j$ and $\varepsilon^{ij}\psi_j = \psi^i$, with $\varepsilon^{ki}\varepsilon_{kj} = \delta^i_j$. The above reality conditions reduce the number of components of (symplectic) Majorana spinors from 2^m complex to 2^m real[11] components.

Given two MS ψ and ϕ or two SMS ψ^i and ϕ^i, one can consider the spinor bilinears $\bar{\psi}\phi$ and $\bar{\psi}^i\phi_i$. SMS pairing indices $i, j, \ldots$ that are contracted in the NW-SE convention may also be suppressed for bilinears $\bar{\psi}^i\phi_i \equiv \bar{\psi}\phi$. Both MS- and SMS bilinears are Lorentz invariant since the Majorana conjugate spinors can be replaced by Dirac conjugates. General bilinears of (symplectic) Majorana spinors satisfy the (symplectic) Majorana flip relation

$$\text{MS:} \qquad \bar{\psi}\Gamma^{(r)}\phi = t_r \bar{\phi}\Gamma^{(r)}\psi \,, \tag{3.22a}$$

$$\text{SMS:} \qquad \bar{\psi}^i\Gamma^{(r)}\phi_i = -t_r \bar{\phi}^i\Gamma^{(r)}\psi_i \,. \tag{3.22b}$$

For a generalised version of this relation involving multiple gamma matrices, see equation (B.1). The sign in the SM flip relation comes from the index position in the contraction. (S)MS bilinears are either completely real or completely imaginary according to (B.2, B.3).

[10]The *number* of SMS in a theory can sometimes lead to confusion when it is unclear if it refers to the actual number of SMS or the number of pairs. In this thesis we will always refer to the number of actual spinors.

[11]Or equivalently 2^{m-1} complex components.

Fierz Relations

In index notation a Majorana conjugated spinor $\bar{\psi}$ is denoted with a superscript index ψ^α. A bilinear in index notation is therefore

$$\bar{\psi}\Gamma^{(r)}\phi \equiv \psi^\alpha {\Gamma^{(r)}}_\alpha{}^\beta \phi_\beta \,. \tag{3.23}$$

Since the Clifford algebra basis elements provide a complete basis for the space of $2^m \times 2^m$-matrices, an arbitrary matrix M can be expressed in this basis as

$$M_\alpha{}^\beta = \frac{1}{2^m}\sum_{k=0}^{s(D)} \frac{1}{k!}\left(\Gamma_{A_1\ldots A_k}\right)_\alpha{}^\beta \left(\Gamma^{A_k\ldots A_1}\right)_\gamma{}^\delta M_\delta{}^\gamma \,, \tag{3.24}$$

where[12] $s(D) = \left[D\frac{3+(-1)^D}{4}\right]$. If confronted with a spinor that is multiplied with a spinor bilinear

$$\left(\bar{\phi}\Gamma^{(r)}\lambda\right)\psi_\alpha = \psi_\alpha \phi^\beta {\Gamma^{(r)}}_\beta{}^\gamma \lambda_\gamma \,, \tag{3.25}$$

the above formula can be used to express the matrix $\psi_\alpha\phi^\beta$ in the Clifford basis

$$\psi\bar{\phi} = -\frac{1}{2^m}\sum_{k=0}^{s(D)} \frac{(-1)^{[k/2]}}{k!}\Gamma_{A_1\ldots A_k}\left(\bar{\phi}\Gamma^{A_1\ldots A_k}\psi\right) \,. \tag{3.26}$$

This basis expansion, which is referred to as Fierz relation, moves the spinor ψ of equation (3.25) inside bilinears, while extracting one of the other two spinors. This is an important computational tool to manipulate spinor bilinears, which will be used frequently in chapters 5 and 6.

Minimal Spinors

With the introduction of Weyl spinors it was demonstrated that, depending on the dimension, Dirac spinors are not always irreducible. In even dimensions every Dirac spinor can equivalently be expressed as a sum of two chiral Weyl spinors that are irreducible. Similarly the introduction of (S)MS reduces the components of Dirac spinors by half. Since SMS have to occur in pairs the *reduction* is in fact only a rearrangement of Dirac spinor components. In dimensions $D = 2 \mod 8$, where charge conjugation leaves the highest rank element invariant ($\gamma_*^C = \gamma_*$), the Majorana- and the Weyl conditions

$$\psi^C = \psi \,, \qquad P_L\psi = \psi \quad \text{or} \quad P_R\psi = \psi \,, \tag{3.27}$$

[12]The bracket-function cuts off all decimal places of its argument. The function $s(D)$ is chosen so that the sum runs over all Clifford algebra basis elements. This means that for $D = 2m \to s(D) = 2m$ and for $D = 2m+1 \to s(D) = m$.

are compatible. The most fundamental spinors in these dimensions are called Majorana Weyl spinors with 2^{m-1} real components. Similarly, in dimensions $D = 6$ mod 8 symplectic Majorana Weyl spinors (in pairs), with 2^{m-1} real components each, are the most fundamental spinors. In all other dimensions (symplectic) Majorana spinors are the most fundamental spinors. In this thesis mostly dimensions $D = 5, 4$ will be treated, in which the most fundamental spinors are symplectic Majorana- and Majorana spinors respectively. More detailed background information on minimal spinors can be found in [27], [63].

3.1.2 Global Supersymmetry

In global supersymmetry (SUSY) the set of Poincaré-, and optionally internal symmetries is extended by a symmetry relating bosonic and fermionic degrees of freedom. Such a symmetry is called a supersymmetry. Simple supersymmetry introduces one supersymmetry transformation, while extended supersymmetry includes $\mathcal{N}$ distinct supersymmetry transformations. Due to the dimensional dependence of the formulation of spinors (see sec. 3.1.1) the description of a supersymmetric theory strongly depends on the spacetime dimension D, as well as on $\mathcal{N}$. The number of supersymmetries that is usually discussed is limited to 32 spinor components – in 4D, where Majorana spinors have 4 components, that is $\mathcal{N} = 8$. As explained in [62] the limit for rigid supersymmetries is in fact only 16 spinor components. This is due to the fact that for > 16 spinor components spin-$\frac{3}{2}$ fields, i.e. gravitini that are gauge fields of local supersymmetry, have to be included in the theory. Consequently, gravitons have to be included as well, constituting a supergravity theory. Extensions with > 32 supersymmetry spinor components lead to fields that have spin $\geq \frac{5}{2}$ and there are no known interacting field theories involving a finite number of such higher spin fields.

Under a supersymmetry transformation with the spinorial anticommuting parameter $\mathcal{E}$ a bosonic field B transforms to a fermionic field Ψ schematically as

$$\delta_{\mathcal{E}} B \sim \bar{\mathcal{E}} \Psi \, . \tag{3.28}$$

The boson B can be a scalar, a vector or a tensor and the spinor bilinear can include gamma matrices accordingly. The defining property that is required from supersymmetry is that the consecutive action of two supersymmetry transformations with parameters $\mathcal{E}_{1,2}$ results in a spacetime translation with parameter Ξ, i.e. $[\delta_{\mathcal{E}_1}, \delta_{\mathcal{E}_2}] = \delta_{\Xi}$. This requirement leads to the schematic transformation rule of the spinor Ψ

$$\delta_{\mathcal{E}} \Psi \sim \gamma^A \mathcal{E} \partial_A B \, , \tag{3.29}$$

where γ^A is a Clifford algebra element whose rank depends on whether B is a scalar-, vector- or tensor field.

Another important consequence of this defining feature is the matching number of bosonic- and fermionic degrees of freedom. This calculation follows the reasoning of [27]. To show this, we operate in the Hilbert space of all particle states $|p_A, s, s_z\rangle$ for fixed momentum p_A, arbitrary spin (for massive-) or helicity (for massless particles) s and spin z-component s_z. The spin z-component takes $2s+1$ values for massive states and 2 values for massless states, except for $s=0$, where $s_z=0$ only takes one value. We define the sign operator $(-1)^{2s}$ which produces a positive sign when acting on a bosonic state and a negative sign when acting on a fermionic state. Therefore the trace[13] of the sign operator yields the difference between bosonic- and fermionic degrees of freedom

$$\mathrm{Tr}\left((-1)^{2s}\right) = \begin{cases} \sum_{s>0}(-1)^{2s} n_s 2 + n_0 & \text{for massless states ,} \\ \sum_{s\geq 0}(-1)^{2s} n_s (2s+1) & \text{for massive states .} \end{cases} \tag{3.30}$$

In the sum there is the bosonic/fermionic sign, the number n_s of states of spin/helicity s and the number of values the z-component takes for this spin. The commutator of fermionic transformations implies the anticommutator of fermionic generators[14] $\{Q_\alpha, Q_\beta\} \sim P_A$. The sign operator anticommutes with fermionic generators by definition $\{(-1)^{2s}, Q_\alpha\} = 0$. One can take the trace of both sides of this relation together with the sign operator

$$\mathrm{Tr}\left((-1)^{2s} P_A\right) \sim \mathrm{Tr}\left((-1)^{2s} Q_\alpha Q_\beta + (-1)^{2s} Q_\beta Q_\alpha\right) \tag{3.31}$$

$$= \mathrm{Tr}\left(-Q_\alpha (-1)^{2s} Q_\beta + Q_\alpha (-1)^{2s} Q_\beta\right) = 0 \,, \tag{3.32}$$

where in the last line the first term is due to the anticommutation of the operators and the second term is due to cyclicity of the trace. Since all states in the trace are momentum eigenstates of a fixed momentum, this implies that

$$\mathrm{Tr}\left((-1)^{2s}\right) = 0 \,. \tag{3.33}$$

Hence, in an on-shell supersymmetric theory the number of bosonic and fermionic degrees of freedom is always equal.

For each spacetime dimension D and supersymmetric extension $\mathcal{N}$ there may exist a variety of different multiplets. These multiplets can contain a number of fields that are related in steps of $\pm\frac{1}{2}$ in their spin (as reasoned in e.g. [24]), e.g. there is a multiplet[15] in 4D, $\mathcal{N}=2$ that has a field content with spins $(2, \frac{3}{2}, \frac{3}{2}, 1)$. Since the bosonic and fermionic degrees of freedom have to match in an on-shell theory, i.e. after imposing equations of motion, the field combinations that can form multiplets are limited. By counting and matching bosonic- and fermionic degrees of freedom (DOF) one can make an Ansatz to find a multiplet (see tab. 3.2).

[13] The summation runs over all spins/helicities and spin z-component values for fixed momentum.

[14] For more information on this structure see the next section around equation (3.36).

[15] This example is the supergravity multiplet.

Spin	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
Field	K	ψ	A_M	Ψ_M	${E_M}^A$
on-shell DOF	1	$2^{[D/2]-1}$	$D-2$	$(D-3)2^{[D/2]-1}$	$\frac{D(D-3)}{2}$

Table 3.2: List of fields that can constitute supersymmetry multiplets. K is a scalar field, ψ is a Majorana fermion, A_M is a gauge field, Ψ_M is a Majorana Rarita-Schwinger field (also called gravitino) and ${E_M}^A$ is a graviton. The content of this table is taken from [27].

In off-shell theories bosonic- and fermionic degrees of freedom are counted differently. The equations of motion are not imposed and only redundancies from gauge symmetries are taken into account when counting DOF. Hence, off-shell multiplets can also include auxiliary fields. For details on off-shell DOF-counting see [48].

3.1.3 General Gauge Theory

Most of what is needed to discuss supersymmetric gauge theories has already been introduced in the previous chapter in section 2.1.2 on bosonic gauge theory. It is, however, necessary to discuss some of the relevant concepts that are related to the inclusion of spinors. The concept of an algebra needs to be extended to include fermionic generators. Also, an explicit guide on the practical computation of transformation rules, supercovariant derivatives and -curvatures will be provided. Everything that was introduced in the previous chapter should be considered implied, except for when the differences are made explicit in this section.

Lie Superalgebras

A Lie superalgebra is a generalisation of a Lie algebra that includes fermionic generators. Wherever the prefix “super-” appears from now on, it will refer to the fact that the discussed quantity has been generalised to include fermionic supersymmetry. The respective transformation parameter of a fermionic generator is also fermionic, i.e. an anticommuting Grassmann number. Lie superalgebra elements can also be exponentiated to Lie supergroup elements. For a short introduction to Lie superalgebras and Lie supergroups the reader is referred to [28]. On the level of the Lie algebra the consecutive action of two symmetry transformations is encoded in the commutator of two general elements of the Lie algebra, $\theta \equiv \theta^{\boldsymbol{A}} T_{\boldsymbol{A}}$ and $\sigma \equiv \sigma^{\boldsymbol{A}} T_{\boldsymbol{A}}$, as can be seen from the Baker-Campbell-Hausdorff formula

$$\exp(\theta)\exp(\sigma) = \exp\Big(\theta + \sigma + \tfrac{1}{2}[\theta, \sigma] + O(\theta^2) + O(\sigma^2)\Big) \,. \tag{3.34}$$

Let us assume that there is a theory with a set of **fermionic** symmetries generated by the set of fermionic generators $\{T_A\}$, with the associated set of fermionic parameters $\{\theta^A\}$, and a set of fields $\{\phi^i\}$ that schematically transform under these symmetries as

$$\phi'^{\,i} = \exp(\theta^{\boldsymbol{A}} T_{\boldsymbol{A}})\phi^i \qquad \rightarrow \qquad \delta\phi^i = \theta^{\boldsymbol{A}} T_{\boldsymbol{A}} \phi^i + O(\theta^2) \,. \tag{3.35}$$

We will only consider infinitesimal transformations, meaning that the $O(\theta^2)$-term is neglected. The consecutive action of two fermionic (or super-)symmetry transformations is encoded in the commutator of transformations, or equivalently in the anticommutator of generators as

$$[\delta_1, \delta_2]\phi^i = [\theta_1, \theta_2]\phi^i = \theta_2^{\boldsymbol{B}} \theta_1^{\boldsymbol{A}} \{T_{\boldsymbol{A}}, T_{\boldsymbol{B}}\}\phi^i \, , \tag{3.36}$$

due to the sign from switching the order of Grassmann numbers $\theta_2^{\boldsymbol{B}} \theta_1^{\boldsymbol{A}} = -\theta_1^{\boldsymbol{A}} \theta_2^{\boldsymbol{B}}$. This shows the need to generalise the antisymmetric Lie bracket to the Lie superbracket. Formally this is done by including an inherent $\mathbb{Z}_2$-grading in a superalgebra, where bosonic generators $T_{\boldsymbol{A}}^{(0)}$ are of grade 0 and fermionic generators $T_{\boldsymbol{A}}^{(1)}$ are of grade 1. The Lie superalgebra satisfies the following relation

$$[\mathfrak{g}_a, \mathfrak{g}_b]_S = \mathfrak{g}_{(a+b)\text{mod}2} \, , \quad a, b \in \{0, 1\} \, , \tag{3.37}$$

where $[\cdot\,,\cdot]_S$ is the Lie superbracket which is defined as

$$[T_{\boldsymbol{A}}^{(a)}, T_{\boldsymbol{B}}^{(b)}]_S = (-1)^{a\cdot b+1}[T_{\boldsymbol{B}}^{(b)}, T_{\boldsymbol{A}}^{(a)}]_S \, . \tag{3.38}$$

The Lie superbracket obeys the super-Jacobi identity

$$\begin{aligned} &(-1)^{a\cdot c}[T_{\boldsymbol{A}}^{(a)}, [T_{\boldsymbol{B}}^{(b)}, T_{\boldsymbol{C}}^{(c)}]_S]_S \\ +&(-1)^{b\cdot a}[T_{\boldsymbol{B}}^{(b)}, [T_{\boldsymbol{C}}^{(c)}, T_{\boldsymbol{A}}^{(a)}]_S]_S \\ +&(-1)^{c\cdot b}[T_{\boldsymbol{C}}^{(c)}, [T_{\boldsymbol{A}}^{(a)}, T_{\boldsymbol{B}}^{(b)}]_S]_S = 0 \, . \end{aligned} \tag{3.39}$$

Usually, when denoting the (anti-)commutation relations of a Lie superalgebra, the notation including the Lie superbracket is omitted and replaced by antisymmetric commutators and symmetric anticommutators

$$[T_{\boldsymbol{A}}^{(0)}, T_{\boldsymbol{B}}^{(0)}]_S = [T_{\boldsymbol{A}}^{(0)}, T_{\boldsymbol{B}}^{(0)}] = f_{\boldsymbol{AB}}{}^{\boldsymbol{C}} T_{\boldsymbol{C}}^{(0)} \quad \rightarrow \quad [\text{bos}, \text{bos}] \sim \text{bos} \, , \tag{3.40a}$$

$$[T_{\boldsymbol{A}}^{(0)}, T_{\boldsymbol{B}}^{(1)}]_S = [T_{\boldsymbol{A}}^{(0)}, T_{\boldsymbol{B}}^{(1)}] = f_{\boldsymbol{AB}}{}^{\boldsymbol{C}} T_{\boldsymbol{C}}^{(1)} \quad \rightarrow \quad [\text{bos}, \text{fer}] \sim \text{fer} \, , \tag{3.40b}$$

$$[T_{\boldsymbol{A}}^{(1)}, T_{\boldsymbol{B}}^{(1)}]_S = \{T_{\boldsymbol{A}}^{(1)}, T_{\boldsymbol{B}}^{(1)}\} = f_{\boldsymbol{AB}}{}^{\boldsymbol{C}} T_{\boldsymbol{C}}^{(0)} \quad \rightarrow \quad \{\text{fer}, \text{fer}\} \sim \text{bos} \, . \tag{3.40c}$$

The structure constants also have an intrinsic grading determined by the grading of the two generators in the Lie superbracket. Although it is not written explicitly, this grading is what distinguishes the structure constants in (3.40a,3.40b) and (3.40c), of which the first two are antisymmetric in $\boldsymbol{AB}$ and the third is symmetric in $\boldsymbol{AB}$. For extensive information on Lie superalgebras see [25], [38].

Hard/Soft Lie Algebras

Everything discussed so far holds for *hard* Lie algebras, although the adjective is usually omitted. A hard algebra has structure constants, while a soft algebra has structure functions that can depend on fields. It is important to note that in the mathematical sense a soft Lie algebra is not an algebra. The name is still kept, since a lot of knowledge can be transferred from the theory of Lie algebras. In supersymmetry soft algebras can occur with the structure functions of two supersymmetry transformations containing terms that depend on gauge fields A_M that do not correspond to generators of the algebra. This can have the form

$$[\delta_{\mathcal{E}_1}, \delta_{\mathcal{E}_2}] = \delta_{\Xi} + \delta_\alpha + \dots , \qquad \alpha = -\Xi^M A_M \,. \tag{3.41}$$

For more on such gauge covariant translations see [21]. Soft gauged algebras are usually not written explicitly as a collection of commutation relations, as is the convention for algebras, but instead as the set of transformation parameters resulting from the commutator of two transformations acting on the fields. For example, see the algebras of relativistic 5D, $\mathcal{N} = 2$ supergravity (4.9) and non-relativistic 4D, $\mathcal{N} = 2$ supergravity (5.16-5.20)).

Closed/Open Lie Algebras

Everything discussed so far holds for *closed* Lie algebras, although the adjective is usually omitted. In an open algebra, terms appear in the commutator of two transformations, that cannot be accounted for by the symmetries composing the Lie algebra. These terms are constraints and equations of motion for the fields involved. The equations of motion and constraints produced by the commutator can be thought of as symmetries of the action, if there is one. In this case these symmetries are called zilch symmetries and when including them in the set of symmetries of the theory, or, equivalently, when imposing equations of motion and constraints, the algebra closes. This happens, for example, when taking the commutator of two SUSY transformations of a gravitino, as will be discussed later (see eqs. (6.8-6.9)).

Gauge Fields, Supercovariant Derivatives and Supercovariant Curvatures

There is not much to be added to the bosonic discussion of gauge fields. Analogously to the bosonic generators, the fermionic generators $T_{\boldsymbol{A}}$ have fermionic gauge fields $h_M^{\boldsymbol{A}}$ assigned to them, transforming with the derivative of the respective fermionic parameter $\theta^{\boldsymbol{A}}$. As was derived in the last chapter (see eq. (2.55)), fermionic gauge fields transform as

$$\delta h_M^{\boldsymbol{A}} = \partial_M \theta^{\boldsymbol{A}} + \theta^{\boldsymbol{C}} h_M^{\boldsymbol{B}} f_{\boldsymbol{BC}}{}^{\boldsymbol{A}} \qquad \leftrightarrow \qquad \delta h_M = \partial_M \theta + [h_M, \theta] \,. \tag{3.42}$$

Note that the commutator remains, since the Lie superbracket was only introduced for Lie superalgebra generators. When the covariant derivative includes fermionic gauge fields to transform covariantly under the fermionic symmetries it will be referred to as supercovariant derivative and denoted by a hat as $\hat{D}_M$

$$\hat{D}_M = \partial_M - h_M \,. \tag{3.43}$$

The hat-notation will from now on always be used to denote objects transforming covariantly under the fermionic symmetries, i.e. supercovariant objects. The same goes for supercovariant curvatures

$$\hat{R}_{MN}(T^{\boldsymbol{A}}) = 2\partial_{[M} h_{N]}^{\boldsymbol{A}} + h_N^{\boldsymbol{C}} h_M^{\boldsymbol{B}} f_{\boldsymbol{BC}}{}^{\boldsymbol{A}} \,. \tag{3.44}$$

The super-Bianchi identities for supercovariant curvatures read

$$\hat{D}_{[M} \hat{R}_{NP]}(T^{\boldsymbol{A}}) = 0 \,. \tag{3.45}$$

Note that for all the above equations the order[16] of gauge fields and parameters matters, because the fermionic objects are Grassmann-valued.

Practical Computation of Supercovariant Quantities

In practice, covariant derivatives and curvatures need to be defined, not only for the separate fields of the theory, but for different kinds of covariant combinations. Due to trading local translations for diffeomorphisms, one is always working with soft algebras in supergravity. In higher dimensional and extended ($\mathcal{N} > 1$) supergravity theories one typically needs to include extra fields in the supergravity multiplet that cannot be interpreted as gauge fields of underlying spacetime symmetries. For that reason it is no longer possible to straightforwardly apply the formulae (3.43) and (3.44) from gauge theory. It is, in fact, simpler, as described in [43], [63], to define supercovariant quantities by their transformation property:

> *Supercovariant combinations, supercovariant derivatives acting on supercovariant combinations and supercovariant curvatures of gauge fields transform without derivatives of gauge parameters.*

This technique can be applied like a recipe requiring only the transformation rules of the gauge fields involved. When discussing gauge fields and gauge transformations, from now on Vielbeine and local translations, or equivalently general coordinate transformations, will be excluded. The discussion following illustrates how to find the transformation rules of supercovariant combinations of fields, how to then use the transformation rule to define a supercovariant derivative acting on them, and how to compute supercovariant curvatures of gauge fields in a practical way.

Finding Transformation Rules of Supercovariant Quantities

The gauge transformation of a supercovariant quantity is a supercovariant quantity. This means that the transformation does not contain any derivatives of parameters, all derivatives appearing must be supercovariant derivatives and there cannot be any terms containing gauge fields outside of covariant combinations. With these restrictions in mind the following steps lead to the final transformation rule without

[16]The order of parameters and gauge fields in the equations of section 2.1.2 was chosen to be consistent for the inclusion of fermions. In the purely bosonic case this order does not matter.

without the need to write down all terms from intermediate steps explicitly. The example that is provided to illustrate the procedure is taken from a later point in the thesis, see equation (5.6a), and its meaning and origin will be discussed later as they are not relevant for the following calculations.

- Find the explicit form of the supercovariant quantity in terms of supercovariant quantities and gauge fields.
  Example:
  $$\hat{f}_{\mu\nu} = 2\partial_{[\mu}a_{\nu]} - 2a\hat{\omega}_{[\mu}{}^{a}e_{\nu]a} - 2i\,\bar{\psi}^{+}_{[\mu}\gamma_5\gamma^0\psi^{-}_{\nu]} + a\tfrac{1}{\sqrt{2}}\bar{\psi}^{-}_{\mu}\gamma^0\psi^{-}_{\nu}\,, \tag{3.46}$$
  where a_μ, $\hat{\omega}_\mu{}^a$ and $\psi^{\pm}_{\mu}$ are gauge fields, a is supercovariant and $e_\mu{}^a$ is a supercovariant Vielbein. The first spinor bilinear is symmetric and the second one is antisymmetric.

- To find the transformation rules of completely supercovariant elements of the original supercovariant object, follow the same procedure from the beginning.
  Example: In this example there are no such terms.

- Sort out the terms of the supercovariant quantity that only contain one single gauge field. The transformation rules of these gauge fields will be needed.
  Example:
  $$\begin{aligned}\delta a_\mu &= \partial_\mu\alpha + m_\mu\hat{k} + a\lambda^b e_{\mu b} + i\bar{\varepsilon}^{+}\gamma_5\gamma^0\psi^{-}_{\mu} - i\bar{\varepsilon}^{-}\gamma_5\gamma^0\psi^{+}_{\mu} \\ &\quad - a\tfrac{1}{\sqrt{2}}\bar{\varepsilon}^{-}\gamma^0\psi^{-}_{\mu}\,,\end{aligned} \tag{3.47}$$
  $$\delta\hat{\omega}_\mu{}^a = \partial_\mu\lambda^a + \hat{k}_\mu{}^a\,, \tag{3.48}$$
  where α, λ^a and $\varepsilon^{\pm}$ are gauge parameters, $\hat{k}_\mu{}^a$ consists of supercovariant quantities multiplied with gauge parameters, and $\hat{k}$ is completely supercovariant.

- Insert the transformation rules of the gauge fields into the terms with a single gauge field.
  Example:
  $$\begin{aligned}&\to 2\partial_{[\mu}\Big(\partial_{\nu]}\alpha + m_{\nu]}\hat{k} + a\lambda^b e_{\nu]b} + i\bar{\varepsilon}^{+}\gamma_5\gamma^0\psi^{-}_{\nu]} - i\bar{\varepsilon}^{-}\gamma_5\gamma^0\psi^{+}_{\nu]} - a\tfrac{1}{\sqrt{2}}\bar{\varepsilon}^{-}\gamma^0\psi^{-}_{\nu]}\Big) \\ &\quad - 2a\Big(\partial_{[\mu}\lambda^a + \hat{k}_{[\mu}{}^{a}\Big)e_{\nu]a} - 2i\,\bar{\psi}^{+}_{[\mu}\gamma_5\gamma^0\psi^{-}_{\nu]} + a\tfrac{1}{\sqrt{2}}\bar{\psi}^{-}_{\mu}\gamma^0\psi^{-}_{\nu}\,.\end{aligned} \tag{3.49}$$

- Remove all terms that include derivatives of gauge parameters. All terms with more than one gauge field outside of supercovariant combinations are absorbed in supercovariant combinations automatically, and should therefore be ignored. Also, replace all derivatives with supercovariant derivatives.
  Example:
  $$\begin{aligned}&\to 2\hat{D}_{[\mu}\left(m_{\nu]}\hat{k}\right) + 2\lambda^b\hat{D}_{[\mu}\left(ae_{\nu]b}\right) + 2i\bar{\varepsilon}^{+}\gamma_5\gamma^0\hat{D}_{[\mu}\psi^{-}_{\nu]} - 2i\bar{\varepsilon}^{-}\gamma_5\gamma^0\hat{D}_{[\mu}\psi^{+}_{\nu]} \\ &\quad - 2a\tfrac{1}{\sqrt{2}}\bar{\varepsilon}^{-}\gamma^0\hat{D}_{[\mu}\psi^{-}_{\nu]} - 2a\hat{k}_{[\mu}{}^{a}e_{\nu]a}\,.\end{aligned} \tag{3.50}$$

- Finally, remove all terms that contain gauge fields outside of supercovariant combinations to obtain the correct transformation rule.
  Example:

$$\begin{aligned}\delta\hat{f}_{\mu\nu} &= 2\hat{D}_{[\mu}m_{\nu]}\hat{k} + 2\lambda^b\hat{D}_{[\mu}\left(ae_{\nu]b}\right) + 2i\bar{\varepsilon}^+\gamma_5\gamma^0\hat{D}_{[\mu}\psi^-_{\nu]} \\ &\quad - 2i\bar{\varepsilon}^-\gamma_5\gamma^0\hat{D}_{[\mu}\psi^+_{\nu]} - 2a\tfrac{1}{\sqrt{2}}\bar{\varepsilon}^-\gamma^0\hat{D}_{[\mu}\psi^-_{\nu]} - 2a\hat{k}_{[\mu}{}^a e_{\nu]a}\,.\end{aligned} \tag{3.51}$$

Finding Supercovariant Derivatives of Supercovariant Quantities

A supercovariant derivative of a supercovariant quantity again is a supercovariant quantity. Supercovariant derivatives can be defined for all fields that transform supercovariantly themselves. Not only, however, can a supercovariant derivative be defined for supercovariant fields, but also for any combination of these fields that transforms supercovariantly. For example, a supercovariant derivative of a supercovariant derivative can always be defined. To actually calculate the supercovariant derivative of a supercovariant quantity there is a simple recipe to follow:

- Find the transformation rule of the supercovariant quantity.
  Example:

$$\begin{aligned}\delta\hat{f}_{\mu\nu} &= 2\hat{D}_{[\mu}m_{\nu]}\hat{k} + 2\lambda^b\hat{D}_{[\mu}\left(ae_{\nu]b}\right) + 2i\bar{\varepsilon}^+\gamma_5\gamma^0\hat{D}_{[\mu}\psi^-_{\nu]} \\ &\quad - 2i\bar{\varepsilon}^-\gamma_5\gamma^0\hat{D}_{[\mu}\psi^+_{\nu]} - 2a\tfrac{1}{\sqrt{2}}\bar{\varepsilon}^-\gamma^0\hat{D}_{[\mu}\psi^-_{\nu]} - 2a\hat{k}_{[\mu}{}^a e_{\nu]a}\,.\end{aligned} \tag{3.52}$$

  where all fields and parameters are the ones from the example above. For simplicity we assume that the transformation rule can be simplified by removing $2\hat{D}_{[\mu}m_{\nu]} = 0$, $2\hat{D}_{[\mu}e_{\nu]b} = 0$ and $-2a\frac{1}{\sqrt{2}}\bar{\varepsilon}^-\gamma^0\hat{D}_{[\mu}\psi^-_{\nu]} - 2a\hat{k}_{[\mu}{}^a e_{\nu]a} = 0$ to become

$$\delta\hat{f}_{\mu\nu} = 2\lambda_b e_{[\nu}{}^b\hat{D}_{\mu]}a + 2i\bar{\varepsilon}^+\gamma_5\gamma^0\hat{D}_{[\mu}\psi^-_{\nu]} - 2i\bar{\varepsilon}^-\gamma_5\gamma^0\hat{D}_{[\mu}\psi^+_{\nu]}\,. \tag{3.53}$$

- In the transformation rule replace all transformation parameters with their associated gauge fields to obtain an expression that transforms with the same number of derivatives of parameters as the derivative of the original transformation rule does.
  Example:

$$\begin{aligned}&\lambda^a \;\to\; \hat{\omega}_\rho{}^a\,, \qquad \varepsilon^\pm \;\to\; \psi^\pm_\rho\,, \\ \to\qquad &2\hat{\omega}_{\rho b}e_{[\nu}{}^b\hat{D}_{\mu]}a + 2i\bar{\psi}^+_\rho\gamma_5\gamma^0\hat{D}_{[\mu}\psi^-_{\nu]} - 2i\bar{\psi}^-_\rho\gamma_5\gamma^0\hat{D}_{[\mu}\psi^+_{\nu]}\,.\end{aligned} \tag{3.54}$$

- If there are terms that after replacing the transformation parameters contain multiple gauge fields, one has to be careful not to overcount these terms. A term containing n gauge fields (outside of supercovariant combinations) picks up a factor $1/n$.
  Example: In this example there are no such terms.

- To obtain the supercovariant derivative of the original supercovariant quantity subtract the expression found above from the partial derivative.
  Example:

$$\hat{D}_\rho \hat{f}_{\mu\nu} = \partial_\rho \hat{f}_{\mu\nu} - 2\hat{\omega}_{\rho b} e_{[\nu}{}^b \hat{D}_{\mu]} a - 2i\bar{\psi}^+_\rho \gamma_5 \gamma^0 \hat{D}_{[\mu} \psi^-_{\nu]} + 2i\bar{\psi}^-_\rho \gamma_5 \gamma^0 \hat{D}_{[\mu} \psi^+_{\nu]} \, . \quad (3.55)$$

This is also how the supercovariant derivative of any supercovariant curvature can be computed.

Finding Supercovariant Curvatures of Gauge Fields

Gauge fields are only supercovariant up to the derivative of their associated gauge parameter. Hence, no supercovariant derivative can be defined for them. A fundamental derivative object that can, however, be defined for gauge fields is the supercovariant curvature[17]. The supercovariant curvature of a gauge field is a supercovariant quantity. The computation is very similar to the computation of the supercovariant derivative:

- The first step is to take the transformation rule of a gauge field and remove the derivative of the parameter.
  Example:

$$\begin{aligned} \delta a_\mu = &\cancel{\partial_\mu \alpha} + m_\mu \hat{k} + a\lambda_b e_\mu{}^b + i\bar{\varepsilon}^+ \gamma_5 \gamma^0 \psi^-_\mu - i\bar{\varepsilon}^- \gamma_5 \gamma^0 \psi^+_\mu \\ &- a\frac{1}{\sqrt{2}} \bar{\varepsilon}^- \gamma^0 \psi^-_\mu \, . \end{aligned} \quad (3.56)$$

- Without the derivative of the parameter a supercovariant derivative could be defined as described above. Twice its antisymmetrisation yields the supercovariant curvature.
  Example:

$$\hat{f}_{\mu\nu} := 2\hat{D}_{[\mu} a_{\nu]} = 2\partial_{[\mu} a_{\nu]} - 2a\hat{\omega}_{[\mu}{}^a e_{\nu]a} - 2i\, \bar{\psi}^+_{[\mu} \gamma_5 \gamma^0 \psi^-_{\nu]} + \quad (3.57)$$

$$a\frac{1}{\sqrt{2}} \bar{\psi}^-_\mu \gamma^0 \psi^-_\nu \, . \quad (3.58)$$

3.1.4 The Super-Poincaré Algebra

To extend the Poincaré algebra to include supersymmetry, $\mathcal{N}$ spinorial SUSY generators Q^i_α, also referred to as supercharges, with $1 \leq i \leq \mathcal{N}$, are introduced. The supercharges are chosen to be among the possible minimal irreducible spinors in the dimension under consideration[18]. To state the (anti-)commutation relations describing the super-Poincaré algebra it is convenient to use spinor indices. The

[17]Since the antisymmetrisation of two derivatives vanishes, the term containing the derivative of the parameter drops out.

[18]E.g. in dimensions $D = 3, 4$ the supercharges could be Majorana spinors and in $D = 5$ symplectic Majorana spinors.

bosonic part of the algebra (see (2.61)) is unaltered. The Lie superbrackets that include fermions take a form encoding the expected spinorial transformation behaviour of the fermionic generators under translations and Lorentz transformations, and the commutation of two supersymmetry transformations to a translation

$$
\begin{aligned}
[M_{AB}, M^{CD}] &= 4\delta^{[C}_{[A} M^{D]}{}_{B]} \,, && (3.59a)\\
[P_A, M_{BC}] &= 2\eta_{A[B} P_{C]} \,, && (3.59b)\\
[P_A, P_B] &= 0 \,, && (3.59c)\\
[Q^i_\alpha, M_{AB}] &= \tfrac{1}{2}(\gamma_{AB})_\alpha{}^\beta Q^i_\beta \,, && (3.59d)\\
[Q^i_\alpha, P_A] &= 0 \,, && (3.59e)\\
\{Q^i_\alpha, Q^j_\beta\} &= -\tfrac{1}{2}\delta^{ij}(\gamma^A C^{-1})_{\alpha\beta} P_A \qquad \text{for MS, i.e. } t_1 = -1 \,, && (3.59f)\\
\{Q^i_\alpha, Q^j_\beta\} &= -\tfrac{1}{2}\varepsilon^{ij}(\gamma^A C^{-1})_{\alpha\beta} P_A \qquad \text{for SMS[19], i.e. } t_1 = 1 \,, && (3.59g)
\end{aligned}
$$

where ε^{ij} is an arbitrary completely antisymmetric matrix.

3.1.5 From 4D, $\mathcal{N} = 1$ Supersymmetry to Supergravity by Gauging

In 4D we can work with Majorana spinors. The simplest SUSY theory is $\mathcal{N} = 1$ and it can be found by gauging the super-Poincaré algebra. There also exists an off-shell 4D, $\mathcal{N} = 1$ supergravity multiplet that, like all off-shell multiplets, cannot be obtained from gauging, see [48]. This multiplet will, however, not be discussed here. It is important to note that for arbitrary D and $\mathcal{N}$ a SUGRA theory cannot be obtained by gauging, since the bosonic and fermionic degrees of freedom (see tab. 3.2) of the gauge fields associated with algebra generators do not match. As a consequence, the supergravity multiplet has to contain extra fields that cannot be interpreted as gauge fields of the super-Poincaré algebra. For 4D, $\mathcal{N} = 1$ however the gauging can be performed analogously to the gauging of the Poincaré algebra in section 2.1.3. Additionally to (2.62) the non-vanishing structure constants are

$$
f_{[AB]\alpha}{}^\beta = -\tfrac{1}{2}(\gamma_{AB})_\alpha{}^\beta \,, \qquad f_{\alpha\beta}{}^A = -\tfrac{1}{2}(\gamma^A C^{-1})_{\alpha\beta} \,. \tag{3.60}
$$

The SUSY generator Q is associated with a fermionic gauge field Ψ_M and a fermionic gauge parameter $\mathcal{E}$ (see tab. 3.3).

[19]It can be seen from the anticommutator that in the case of minimal spinors being SMS, the minimal supersymmetric extension will be referred to as $\mathcal{N} = 2$ (some authors refer to it as $\mathcal{N} = 1$). This is compatible with the statement made earlier that SMS always have to come in pairs.

Generator	Gauge Field	Parameter
P_A	${E_M}^A$	$\Xi^A(x)$
M_{AB}	$\hat{\Omega}_M{}^{AB}$	$\Lambda^{AB}(x)$
$(Q^i)_\alpha$	$(\Psi^i_M)^\alpha$	$(\mathcal{E}^i)^\alpha(x)$

Table 3.3: The generators of Poincaré supersymmetry with their associated gauge fields and parameters. The index i indicates possible supersymmetric extensions.

With the structure constants the transformation rules of the gauge fields can be computed

$$\delta {E_M}^A = D_M \Xi^A - {\Lambda^A}_B {E_M}^B + \tfrac{1}{2}\bar{\mathcal{E}}\gamma^A \Psi_M \,, \tag{3.61a}$$

$$\delta \hat{\Omega}_M{}^{AB} = D_M \Lambda^{AB} \,, \tag{3.61b}$$

$$\delta \Psi_M = D_M \mathcal{E} - \tfrac{1}{4}\Lambda^{AB}\gamma_{AB}\Psi_M \,, \tag{3.61c}$$

where the spin connection ${\Omega_M}^{AB}$ from last chapter was replaced with the supercovariant spin connection $\hat{\Omega}_M{}^{AB}$ as the gauge field of local Lorentz transformations. Hence, also the derivative D_M acting on parameters now takes the form[20] $D_M = \partial_M + \hat{\Omega}_M{}^{AB} M_{AB}$. The covariant curvatures can be constructed directly from the algebra, by (3.44), to yield

$$\hat{R}_{MN}(P^A) = 2\partial_{[M}{E_{N]}}^A + 2\hat{\Omega}_{[M}{}^{AB}E_{N]B} - \tfrac{1}{2}\bar{\Psi}_M\gamma^A\Psi_N \,, \tag{3.62a}$$

$$\hat{R}_{MN}(M^{AB}) = 2\partial_{[M}\hat{\Omega}_{N]}{}^{AB} + 2\hat{\Omega}_{[M}{}^{CA}\hat{\Omega}_{N]}{}^{B}{}_{C} \,, \tag{3.62b}$$

$$\hat{\Psi}_{MN} := \hat{R}_{MN}(Q) = 2\partial_{[M}\Psi_{N]} + \tfrac{1}{2}\hat{\Omega}_{[M}{}^{AB}\gamma_{AB}\Psi_{N]} \,. \tag{3.62c}$$

The relativistic *no-torsion condition* now takes a supercovariant form to become the *supercovariant no-torsion condition*

$$\hat{R}_{MN}(P^A) = 0 \,, \tag{3.63}$$

thereby defining the supercovariant spin connection as a dependent field. It can be computed explicitly in the same fashion as was done for the covariant spin connection (see eq. (2.22-2.23))

$$\hat{\Omega}_M{}^{AB}(E,\Psi) = -\tfrac{1}{2}\hat{C}^{AB}{}_M + \hat{C}_M{}^{[AB]} \,, \tag{3.64a}$$

$$\hat{C}_{MN}{}^A = 2\partial_{[M}{E_{N]}}^A - \tfrac{1}{2}\bar{\Psi}_M\gamma^A\Psi_N \,. \tag{3.64b}$$

In a supersymmetric gauge theory local translations amount to diffeomorphisms and other gauge transformations (including SUSY transformations) modulo torsion. These additional gauge transformations can be removed by defining supercovariant general coordinate transformations (see [21]), which include gauge field dependent

[20] This derivative has a positive sign because of the negative sign in the spacetime transformation $\delta\phi = -\Lambda\phi$.

terms cancelling exactly the additional terms in the transformation. This is, however, not done here. We replace local translation symmetry with parameter Ξ^A, with diffeomorphism symmetry with parameter $\Xi^M = \Xi^A E^M{}_A$ for all fields

$$\delta E_M{}^A = \mathcal{L}_\Xi E_M{}^A - \Lambda^A{}_B E_M{}^B + \tfrac{1}{2}\bar{\mathcal{E}}\gamma^A \Psi_M \, , \tag{3.65a}$$

$$\delta \Psi_M = \mathcal{L}_\Xi \Psi_M + D_M \mathcal{E} - \tfrac{1}{4}\Lambda^{AB}\gamma_{AB}\Psi_M \, . \tag{3.65b}$$

Note that as a consequence of imposing the constraint (3.63) the transformation rule of $\hat{\Omega}_M{}^{AB}$ is no longer the same as (3.61b). To be precise, under Lorentz transformations it transforms as in (3.61b), but now it also transforms under supersymmetry. Stating the transformation rule explicitly can, however, be omitted because the supercovariant spin connection is a dependent field.

From this point on the Vielbein is not treated as a gauge field anymore and the prefix *supercovariant* will refer to all symmetries but diffeomorphisms. The 4D, $\mathcal{N} = 1$ supergravity multiplet (3.65) can be null-reduced (see sec. 3.2.2) to find non-relativistic 3D, $\mathcal{N} = 2$ supergravity.

3.1.6 Constructing 5D, $\mathcal{N} = 2$ Supergravity

As mentioned before, not every supergravity multiplet can be fully determined by a gauging procedure. The fields corresponding to algebra generators of spacetime symmetries and part of their transformations under these symmetries can be found by gauging, but the remaining fields of the multiplet as well as their transformation rules need to be found using an Ansatz.

Simply by counting degrees of freedom it becomes apparent that the gauge fields associated with the algebra generators in 5D, i.e. $E_M{}^A$, Ψ^i_M, with $i = 1, 2$, do not have matching bosonic and fermionic degrees of freedom

$$E_M{}^A \;\rightarrow\; 5 \text{ bosonic DOF} \qquad \text{vs.} \qquad \Psi^i_M \;\rightarrow\; 8 \text{ fermionic DOF}. \tag{3.66}$$

This defect can be fixed by including an additional bosonic field in the multiplet. By inspection of table 3.2, a gauge vector field A_M provides the correct number of degrees of freedom (3 bosonic DOF) and is therefore a viable candidate for an Ansatz. The Ansatz for such a multiplet has to obey some minimal conditions. The original transformations coming from the super-Poincaré algebra (3.59) should be preserved and the new vector field should transform as a spacetime vector under diffeomorphisms and as a gauge field under a $U(1)$ gauge transformation[21] with parameter α leading to

$$\delta E_M{}^A = \mathcal{L}_\Xi E_M{}^A - \Lambda^A{}_B E_M{}^B + \tfrac{1}{2}\bar{\mathcal{E}}\gamma^A \Psi_M + \dots \, , \tag{3.67a}$$

$$\delta A_M = \mathcal{L}_\Xi A_M + \partial_M \alpha + \dots \, , \tag{3.67b}$$

$$\delta \Psi^i_M = \mathcal{L}_\Xi \Psi^i_M + D_M \mathcal{E}^i - \tfrac{1}{4}\Lambda^{AB}\gamma_{AB}\Psi^i_M + \dots \, , \tag{3.67c}$$

[21]The vector field A_M is the only field transforming under this $U(1)$ symmetry.

where the i-indices are suppressed[22] in the bilinears of symplectic Majorana spinors. There will be no new terms including Ξ^M, Λ^{AB} and $\hat{\Omega}_M{}^{AB}$ so as not to change the established transformation properties under diffeomorphisms and local Lorentz symmetry. Since it is supersymmetry that demands the introduction of the vector field, it can be assumed that any additional terms in the transformations of $E_M{}^A$ and Ψ^i_M that include the vector field A_M will be supersymmetry transformations with parameter $\mathcal{E}$. We will assume the transformation rule of the Vielbein to not include any additional terms. The transformation rules of the gravitini can, however, include a term of the form $\sim \hat{F}(A)\mathcal{E}$, where $\hat{F}(A)$ is a supercovariant quantity depending on the vector field. The schematic supersymmetry transformation of a fermion to the derivative of a boson (3.29) indicates that $\hat{F}(A)$ is the supercovariant field strength of A_M. The gauge vector field A_M transforms as a vector under diffeomorphisms and as a gauge field under its $U(1)$ symmetry. Under supersymmetry it is assumed to transform with $\sim \bar{\mathcal{E}}\Psi_M$. The following Ansatz, which is compatible with all the above consideration is taken from [7] where a more general matter-coupled theory is considered. The following transformation rules are found by setting all matter fields of the matter-coupled theory to zero

$$\delta E_M{}^A = \mathcal{L}_\Xi E_M{}^A - \Lambda^A{}_B E_M{}^B + \tfrac{1}{2}\bar{\mathcal{E}}\gamma^A\Psi_M \ , \tag{3.68a}$$

$$\delta A_M = \mathcal{L}_\Xi A_M + \partial_M\alpha + i\bar{\mathcal{E}}\Psi_M \ , \tag{3.68b}$$

$$\begin{aligned}\delta\Psi^i_M &= \mathcal{L}_\Xi\Psi^i_M + D_M\mathcal{E}^i - \tfrac{1}{4}\Lambda^{AB}\gamma_{AB}\Psi^i_M \\ &\quad - \tfrac{i}{16}\left[E_{MA}\hat{F}_{BC}\gamma^{ABC} - 4\hat{F}_{MA}\gamma^A\right]\mathcal{E}^i \ ,\end{aligned} \tag{3.68c}$$

where $\hat{F}_{MN}$ is the supercovariant curvature of A_M that will also (in analogy to the terminology of electro-magnetism) be referred to as supercovariant field strength

$$\hat{F}_{NP} = 2\partial_{[N}A_{P]} - i\bar{\Psi}_N\Psi_P \ . \tag{3.69}$$

The gamma matrix structure in the transformation rule of the gravitini is the only non-trivial combination to contract $\hat{F}_{AB}$ and $\Gamma^{(r)}$. The coefficients of the newly added supersymmetry terms can be found from normalisation and the closure of the algebra. They are imaginary, thus ensuring compatibility with the spinor bilinears being imaginary (see app. B.1 for information on reality properties of spinor bilinears). The computation of these coefficients will not be done explicitly but it can be checked that with this choice of coefficients the algebra does in fact close (see sec. 4.2).

[22] I.e. they are contracted with the NW-SE convention.

3.2 About Non-Relativistic Supersymmetry and Newton-Cartan Supergravity

In this section non-relativistic 3D, $\mathcal{N} = 2$ supergravity is derived, first, by gauging the 3D, $\mathcal{N} = 2$ super-Bargmann algebra, and then, by null reduction of relativistic 4D, $\mathcal{N} = 1$ supergravity. An off-shell computation of a non-relativistic 3D, $\mathcal{N} = 2$ supergravity multiplet has been done by [67], which will not be presented here.

3.2.1 From 3D, $\mathcal{N} = 2$ Super-Poincaré to Newton-Cartan Supergravity by Contraction and Gauging

In this section the results of [4] are presented following a similar procedure as described in sections 2.2.1-2.2.3. First, the 3D, $\mathcal{N} = 2$ super-Poincaré algebra is centrally extended and then contracted to the super-Bargmann algebra. The super-Bargmann algebra is gauged to yield the kinematics of 3D, $\mathcal{N} = 2$ Newton-Cartan supergravity. It has to be noted that, contrary to the bosonic theory that was discussed in the previous chapter, both the dimension D and the extension $\mathcal{N}$ of the supersymmetric theory have to be specified due to the dimensional dependence of spinors. By no means, this procedure of gauging the super-Bargmann algebra[23] can be performed for arbitrary D and $\mathcal{N}$. For example, the multiplet of section 5.3 cannot be obtained from gauging. The choice 3D, $\mathcal{N} = 2$ will, however, prove to be adequate for the gauging procedure. The same theory of non-relativistic 3D, $\mathcal{N} = 2$ supergravity can also be found by null-reducing relativistic 4D, $\mathcal{N} = 1$ supergravity, as will be presented in section 3.2.2.

The super-Poincaré algebra is centrally extended[24] by the scalar central charge Z to obtain an additional term in the anticommutator of two Majorana SUSY charges

$$\{Q^i_\alpha, Q^j_\beta\} = -\tfrac{1}{2}\delta^{ij}(\gamma^A C^{-1})_{\alpha\beta} P_A - \tfrac{1}{2}\varepsilon^{ij}(C^{-1})_{\alpha\beta} Z \,. \tag{3.70}$$

The linear combinations

$$Q^\pm = Q^1 \pm \gamma^0 Q^2 \,, \tag{3.71}$$

are defined to then equip the centrally extended super-Poincaré algebra with a grading with the large parameter ω:

[23] The same goes for the super-Poincaré algebra that can generally not be gauged to find a multiplet.

[24] This is the only possible central extension. For information on other non-trivial non-central extensions with spacetime indices see [2], [44], [63].

Grade -2	$\frac{1}{2}(P_0 + Z) \to \omega^{-2} H$,	
Grade -1	$Q^+ \to \omega^{-1} Q^+$,	
Grade 0	$P_a \to P_a$,	$M_{ab} \to J_{ab}$,
Grade 1	$Q^- \to \omega Q^-$,	
Grade 2	$M_{a0} \to \omega^2 G_a$,	$\frac{1}{2}(P_0 - Z) \to \omega^2 M$.

Table 3.4: Grading of the centrally extended super-Poincaré algebra.

The grading stated in table 3.4 is equivalent to a $\mathbb{Z}_5$-grading $(\omega^0, \ldots, \omega^{-5})$. The shift of powers of parameters is, however, necessary to keep the correct (anti-)commutators in the Saletan contraction (see sec. 2.1.2), $\omega \to \infty$, to yield the following non-vanishing (anti-)commutation relations of the super-Bargmann algebra

$$[J_{ab}, J_{cd}] = 4\delta_{[a[c}J_{d]b]} \ , \qquad [J_{ab}, G_c] = -2\delta_{c[a}G_{b]} \ , \tag{3.72a}$$

$$[J_{ab}, P_c] = -2\delta_{c[a}P_{b]} \ , \qquad [G_a, H] = -P_a \ , \tag{3.72b}$$

$$[G_a, P_b] = -\delta_{ab} M \ , \tag{3.72c}$$

$$[J_{ab}, Q^\pm] = -\tfrac{1}{2}\gamma_{ab} Q^\pm \ , \qquad [G_a, Q^+] = -\tfrac{1}{2}\gamma_{a0} Q^- \ , \tag{3.72d}$$

$$\{Q^+_\alpha, Q^+_\beta\} = -(\gamma^0 C^{-1})_{\alpha\beta} H \ , \quad \{Q^-_\alpha, Q^-_\beta\} = -(\gamma^0 C^{-1})_{\alpha\beta} M \ , \tag{3.72e}$$

$$\{Q^+_\alpha, Q^-_\beta\} = -\tfrac{1}{2}(\gamma^a C^{-1})_{\alpha\beta} P_a \ . \tag{3.72f}$$

The bosonic part of the contraction of the centrally extended 3D, $\mathcal{N} = 2$ super-Poincaré algebra produces the same result as the contraction performed in section 2.2.1. Note that the generators of (3.72), excluding Q^+, compose an $\mathcal{N} = 1$ subalgebra. This algebra is, however, not a conventional supersymmetry algebra in which the supercharges anticommute to translations. In fact, this is why it is necessary to include $\mathcal{N} = 2$ supersymmetries in the theory: to obtain spatial- and time translations from SUSY-anticommutators in the non-relativistic algebra, as pointed out in [4]. Now that the algebra is found it can be gauged by introducing the following gauge fields and local parameters:

Generator	Gauge Field	Parameter
P_a	$e_\mu{}^a$	$\zeta^a(x) \searrow \xi^\mu(x)$
H	τ_μ	$\zeta(x) \nearrow$
M	m_μ	$\sigma(x)$
J_{ab}	$\hat{\omega}_\mu{}^{ab}$	$\lambda^{ab}(x)$
G_a	$\hat{\omega}_\mu{}^a$	$\lambda^a(x)$
$(Q^i)_\alpha$	$(\psi^i_\mu)^\alpha$	$(\varepsilon^i)^\alpha(x)$

Table 3.5: The generators of Bargmann supersymmetry with their associated gauge fields and parameters.

The symmetries of spatial- and time translation are replaced by diffeomorphisms, as was already done in the relativistic theory (see eq. (2.67-2.68) and eq. (3.65)) and in the non-supersymmetric non-relativistic theory (see eq. (2.98)). The following conventional supercovariant curvature constraints and the supercovariant no-torsion condition are therefore imposed

$$\hat{r}_{\mu\nu}(P^a) = 0\,, \tag{3.73a}$$
$$\hat{r}_{\mu\nu}(M) = 0\,, \tag{3.73b}$$
$$\hat{\tau}_{\mu\nu} = \hat{r}_{\mu\nu}(H) = 0\,, \tag{3.73c}$$

of which the first two define the supercovariant spin- and boost connections as dependent fields and the last condition ensures the foliation of spacetime. The only novelty in the explicit computation of the supercovariant spin- and boost connections is that they are now built from supercovariantised anholonomy coefficients (see app. C.1). The transformation rules of the gauge fields can be derived from the super-Bargmann algebra. They are of the form

$$\delta e_\mu{}^a = \mathcal{L}_\xi e_\mu{}^a - \lambda^a{}_b e_\mu{}^b - \lambda^a \tau_\mu + \tfrac{1}{2}\bar{\varepsilon}^+\gamma^a\psi_\mu^- + \tfrac{1}{2}\bar{\varepsilon}^-\gamma^a\psi_\mu^+\,, \tag{3.74a}$$
$$\delta\tau_\mu = \mathcal{L}_\xi\tau_\mu + \bar{\varepsilon}^+\gamma^0\psi_\mu^+\,, \tag{3.74b}$$
$$\delta m_\mu = \mathcal{L}_\xi m_\mu + \partial_\mu\sigma + \lambda_b e_\mu{}^b + \bar{\varepsilon}^-\gamma^0\psi_\mu^-\,, \tag{3.74c}$$
$$\begin{aligned}\delta\psi_\mu^- = {}& \mathcal{L}_\xi\psi_\mu^- + \partial_\mu\varepsilon^- - \tfrac{1}{4}\lambda^{ab}\gamma_{ab}\psi_\mu^- + \tfrac{1}{4}\hat{\omega}_\mu{}^{ab}\gamma_{ab}\varepsilon^- \\ & + \tfrac{1}{2}\lambda^a\gamma_a\gamma^0\psi_\mu^+ - \tfrac{1}{2}\hat{\omega}_\mu{}^a\gamma_a\gamma^0\varepsilon^+\,,\end{aligned} \tag{3.74d}$$
$$\delta\psi_\mu^+ = \mathcal{L}_\xi\psi_\mu^+ + \partial_\mu\varepsilon^+ - \tfrac{1}{4}\lambda^{ab}\gamma_{ab}\psi_\mu^+ + \tfrac{1}{4}\hat{\omega}_\mu{}^{ab}\gamma_{ab}\varepsilon^+\,. \tag{3.74e}$$

3.2.2 From 4D, $\mathcal{N} = 1$ Supergravity to 3D, $\mathcal{N} = 2$ Newton-Cartan Supergravity by Null Reduction

The idea of this section is to recreate the results found in the previous section to demonstrate the power of null reduction. A difference with the results obtained in section 3.2.1 is, however, that null reduction does **not** impose the non-relativistic supercovariant no-torsion condition $\hat{\tau}_{\mu\nu} = 0$, hence the reduced theory can generally include torsional contributions. Only after imposing the field equations the twistless-torsion constraint, $\hat{\tau}_{ab} = 0$, is required to hold by Frobenius' theorem for the spacetime to foliate. In this particular example twistless-torsion implies no-torsion[25], so in fact the supercovariant no-torsion condition holds.

When introducing the procedure of null reduction in section 2.2.5, supersymmetry and spinors had not yet been introduced. Since a supergravity multiplet contains gravitini, a null reduction Ansatz for spinors is required. Compared to a spinor in D dimensions, a spinor in $(D-1)$ dimensions has the same number of components if D is odd or half the number of components if D is even. Additionally, one has to make sure that the reduced spinors are minimal spinors in $(D-1)$ and obey the correct dimension-dependent conditions, like the (symplectic) Majorana- or the

[25]This is generally not the case. E.g. see ch. 5.

Weyl condition (see sec. 3.1.1).

Taking the 4D, $\mathcal{N}=1$ supergravity multiplet, which was found in section 3.1.5 as a starting point

$$\delta E_M{}^A = \mathcal{L}_\Xi E_M{}^A - \Lambda^A{}_B E_M{}^B + \tfrac{1}{2}\bar{\mathcal{E}}\gamma^{(4)A}\Psi_M\,, \tag{3.75a}$$

$$\delta\Psi_M = \mathcal{L}_\Xi\Psi_M + D_M\mathcal{E} - \tfrac{1}{4}\Lambda^{AB}\gamma^{(4)}_{AB}\Psi_M\,, \tag{3.75b}$$

where the superscript (4) on the gamma matrices refers to the spacetime dimension, we still need an Ansatz to properly null-reduce the fields. The Vielbein is reduced with the Ansatz described in section 2.2.5. The Ansatz to null-reduce one 4D Majorana gravitino Ψ_M to two 3D Majorana gravitini $\psi^\pm_\mu$ and the results that follow from it were found in [42]. By making a partial basis choice for the Clifford algebra, the gravitini and their parameters are reduced as

$$\Psi_\mu = \begin{pmatrix} \psi^-_\mu - m_\mu\varphi^- \\ \psi^+_\mu - m_\mu\varphi^+ \end{pmatrix}, \qquad \Psi_z = \begin{pmatrix} \varphi^- \\ \varphi^+ \end{pmatrix}, \qquad \mathcal{E} = \begin{pmatrix} \varepsilon^- \\ \varepsilon^+ \end{pmatrix}, \tag{3.76}$$

where the form of Ψ_μ is chosen such that the 3D gravitini $\psi^\pm_\mu$ transform properly under 3D diffeomorphisms. The gamma matrices $\gamma^{(4)}_A$ decompose into lower dimensional gamma matrices $\gamma^{(3)}_0$ and $\gamma^{(3)}_a$, where $a = 1, 2$, as

$$\gamma^{(4)}_a = \begin{pmatrix} \gamma^{(3)}_a & 0 \\ 0 & \gamma^{(3)}_a \end{pmatrix}, \; \gamma^{(4)}_0 = \begin{pmatrix} 0 & \gamma^{(3)}_0 \\ \gamma^{(3)}_0 & 0 \end{pmatrix}, \; \gamma^{(4)}_3 = \begin{pmatrix} 0 & \gamma^{(3)}_0 \\ -\gamma^{(3)}_0 & 0 \end{pmatrix}, \tag{3.77}$$

which when written in null coordinates yields

$$\gamma^{(4)}_\pm = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \gamma^{(3)}_0 \pm \gamma^{(3)}_0 \\ \gamma^{(3)}_0 \mp \gamma^{(3)}_0 & 0 \end{pmatrix}. \tag{3.78}$$

The charge conjugation matrix is reduced according to

$$C^{(4)} = \begin{pmatrix} 0 & C^{(3)} \\ C^{(3)} & 0 \end{pmatrix}. \tag{3.79}$$

With these Ansätze the transformation rules of the fields from the null reduced multiplet can be computed. The reduction being null removes one of the spinors, φ^+, as a constraint. The second spinor, φ^-, is identified as an auxiliary field and set to zero for the on-shell theory. Consequently, torsion vanishes as well

$$\varphi^+ = 0 \quad \xrightarrow{\delta_\varepsilon} \quad \hat{\tau}^{ab}\varepsilon_{ab} = 2\sqrt{2}\bar{\varphi}^-\varphi^-\,, \tag{3.80a}$$

$$\varphi^- = 0 \quad \xrightarrow{\delta_\varepsilon} \quad \hat{\tau}_{\mu\nu} = 0\,. \tag{3.80b}$$

The following on-shell non-relativistic supergravity multiplet is derived

$$
\begin{aligned}
\delta e_\mu{}^a &= \mathcal{L}_\xi e_\mu{}^a - \lambda^a{}_b e_\mu{}^b - \lambda^a \tau_\mu + \tfrac{1}{2}\bar{\varepsilon}^+ \gamma^a \psi_\mu^- + \tfrac{1}{2}\bar{\varepsilon}^- \gamma^a \psi_\mu^+ \,, && \text{(3.81a)}\\
\delta \tau_\mu &= \mathcal{L}_\xi \tau_\mu + \tfrac{1}{\sqrt{2}}\bar{\varepsilon}^+ \gamma^0 \psi_\mu^+ \,, && \text{(3.81b)}\\
\delta m_\mu &= \mathcal{L}_\xi m_\mu + \partial_\mu \sigma + \lambda_b e_\mu{}^b - \tfrac{1}{\sqrt{2}}\bar{\varepsilon}^- \gamma^0 \psi_\mu^- \,, && \text{(3.81c)}\\
\delta \psi_\mu^- &= \mathcal{L}_\xi \psi_\mu^- + \partial_\mu \varepsilon^- - \tfrac{1}{4}\lambda^{ab}\gamma_{ab}\psi_\mu^- + \tfrac{1}{4}\hat{\omega}_\mu{}^{ab}\gamma_{ab}\varepsilon^- \\
&\quad - \tfrac{1}{\sqrt{2}}\lambda^a \gamma_a \gamma^0 \psi_\mu^+ + \tfrac{1}{\sqrt{2}}\hat{\omega}_\mu{}^a \gamma_a \gamma^0 \varepsilon^+ \,, && \text{(3.81d)}\\
\delta \psi_\mu^+ &= \mathcal{L}_\xi \psi_\mu^+ + \partial_\mu \varepsilon^+ - \tfrac{1}{4}\lambda^{ab}\gamma_{ab}\psi_\mu^+ + \tfrac{1}{4}\hat{\omega}_\mu{}^{ab}\gamma_{ab}\varepsilon^+ \,, && \text{(3.81e)}
\end{aligned}
$$

which is equivalent to the multiplet (3.74) that was found by gauging the super-Bargmann algebra. Note that when comparing this multiplet to the results of the upcoming chapters, i.e. the non-relativistic 4D, $\mathcal{N}=2$ supergravity multiplet (5.28) derived by null-reducing the relativistic 5D, $\mathcal{N}=2$ supergravity multiplet (4.1), it can be seen that all transformation rules are the same as (3.81) when ignoring the graviphoton field a_μ.

Chapter 4

Null-Reduction of Relativistic 5D, $\mathcal{N} = 2$ Pure Supergravity

In this chapter relativistic 5D, $\mathcal{N} = 2$ pure supergravity is presented as a starting point for null reduction. Pure 5D supergravity was first developed in 1981 [19]. Since then generalisations of the pure theory were developed by coupling it to matter in the form of vector- (see [29], [30]), tensor- (see [31], [32]) and hyper-multiplets (see [17]). The publication [7] by Bergshoeff et al, in which they present a relativistic 5D, $\mathcal{N} = 2$ theory of supergravity coupled to vector-, hyper- and tensor-multiplets will be used as a starting point for the following calculations. By setting all fields in the vector-, hyper- and tensor-multiplets to zero, the pure theory can be extracted as was shown in section 3.1.6. The only remaining fields $\{E_M{}^A, A_M, \Psi^i_M\}$ compose the Poincaré-supergravity multiplet. The frame field shall be referred to as Vielbein, the vector field as graviphoton, and the two spinor fields with a spacetime index as gravitini.

4.1 Multiplet

The fields of the Poincaré-supergravity multiplet transform under general coordinate transformations with parameter Ξ, local Lorentz transformations with parameter Λ, local SUSY transformations with parameter $\mathcal{E}$, and U(1) gauge transformation with parameter α as

$$\delta E_M{}^A = \mathcal{L}_\Xi E_M{}^A - \Lambda^{AB} E_{MB} + \tfrac{1}{2}\bar{\mathcal{E}}\gamma^A\Psi_M \, , \tag{4.1a}$$

$$\delta A_M = \mathcal{L}_\Xi A_M + \partial_M\alpha + i\bar{\mathcal{E}}\Psi_M \, , \tag{4.1b}$$

$$\begin{aligned}\delta\Psi^i_M &= \mathcal{L}_\Xi\Psi^i_M + D_M\mathcal{E}^i - \tfrac{1}{4}\Lambda^{AB}\gamma_{AB}\Psi^i_M \\ &\quad - \tfrac{i}{16}\hat{F}_{NP}\left[E_{MA}E^N{}_B E^P{}_C\gamma^{ABC} - 4\delta^N_M E^P{}_A\gamma^A\right]\mathcal{E}^i \, .\end{aligned} \tag{4.1c}$$

As is characteristic for the second order formulation[1] of supergravity the supercovariant spin connection $\hat{\Omega}_M{}^{AB}$ is defined as the solution of the supercovariant no-torsion condition

$$\begin{aligned}\hat{R}_{MN}(P^A) &= 2\partial_{[M}E_{N]}{}^A + 2\Omega_{[M}{}^{AB}E_{N]B} \\ &= 2\partial_{[M}E_{N]}{}^A + 2\hat{\Omega}_{[M}{}^{AB}E_{N]B} - \tfrac{1}{2}\bar{\Psi}_M\gamma^A\Psi_N = 0\,,\end{aligned} \tag{4.2}$$

making it a field depending on the Vielbein and the gravitini. Since the transformation rule of the spin connection can be derived from the others, it is not necessary to include it in the multiplet. For the spin connection the hat notation indicates complete covariance with respect to all symmetries *except* for local Lorentz transformations, of which it is the gauge field. The main purpose of this notation is to distinguish the supercovariant spin connection from the covariant spin connection that does not include gravitino terms (see eq. (4.2)). From now on when referring to the spin connection (or later the spin- and boost connections), we will always be the talking about the supercovariantised fields (as will always be indicated by the hat notation). For the explicit form of the spin connection in terms of supercovariant anholonomy coefficients see equation (3.64).

The supercovariant field strength tensor appearing in the gravitino transformation rules is the supercovariant curvature of the graviphoton

$$\hat{F}_{NP} = 2\partial_{[N}A_{P]} - i\bar{\Psi}_N\Psi_P\,. \tag{4.3}$$

The above transformation rules (4.1) are similar to the ones known from pure 4D, $\mathcal{N}=1$ supergravity (see sec. 3.2.2), for which the null reduction was performed by [42] to obtain non-relativistic 3D, $\mathcal{N}=2$ supergravity. The important distinction, however, is the gauge vector, the graviphoton, appearing in the multiplet, due to the fact that we are working with extended $\mathcal{N}=2$ supersymmetry. The null reduction of the vector is straightforward, but a new feature of this theory compared to 4D, $\mathcal{N}=1$. In further comparison the transformation of the Vielbein is unaltered, while the transformation rules of the gravitini are extended by the graviphoton field strength $\hat{F}_{NP}$-terms. Also, it has to be noted that now we are dealing with two[2] symplectic Majorana gravitini, i.e. $i=1,2$, unlike the single Majorana gravitino in 4D $\mathcal{N}=1$. These differences will give rise to a different gravitino reduction Ansatz compared to the one in the null reduction from 4D to 3D. The null reduction of the fields from the supergravity multiplet will be discussed explicitly in section 4.3.

[1]The term *second order formalism* refers to the fact that the Riemann tensor, i.e. the curvature of local Lorentz transformations and hence the field equations are second order in derivatives for a dependent spin connection.

[2]Sometimes these two two 4-real-component symplectic Majorana gravitini are referred to as only *one* 8-real-component SM gravitino, which is because SMS always come in pairs, and therefore the minimal spinor is either one 8-real-component or two 4-real-component SM spinors. The difference is only semantic.

4.2 Algebra Closure

Before engaging in the reduction to 4D it is interesting to check the algebra closure of the above multiplet. If the commutator of two consecutive transformations yields a linear combination of transformations on that same field, the algebra closes off-shell. If, additionally, equations of motion and constraints on some field must be imposed, the algebra is said to close on-shell. One can also view the terms containing the equations of motion as zilch symmetries of the action (see open/closed algebras in sec. 3.1.3), assuming, of course, there is an action. Supersymmetry algebras typically do not close off-shell on the fermionic fields. We will always be working on-shell, so on-shell closure of the algebra will be the relevant concept here. For a certain commutator of transformations, the parameters of the resulting transformations have to be the same regardless of the transformed field. The commutators of bosonic-bosonic- and of bosonic-fermionic transformations only lead to trivial terms and are therefore not made explicit. The calculation of the commutator of two SUSY transformations with parameters $\mathcal{E}_1$, $\mathcal{E}_2$, however, is instructive, since it incorporates the supercovariant no-torsion condition to close the algebra. Requiring that the commutator is a sum of transformations, it can be expressed explicitly as a linear combination of all possible transformations with parameters depending on the original two parameters

$$[\delta_{\mathcal{E}_1}, \delta_{\mathcal{E}_2}] \overset{!}{=} (\delta_\Xi + \delta_\Lambda + \delta_\mathcal{E} + \delta_\alpha) \, , \tag{4.4}$$

where $\Xi^M(\mathcal{E}_{1,2})$ is the parameter of general coordinate transformations, $\Lambda^{AB}(\mathcal{E}_{1,2})$ the parameter of local Lorentz transformations, $\mathcal{E}(\mathcal{E}_{1,2})$ the parameter of a local SUSY transformation and $\alpha(\mathcal{E}_{1,2})$ the parameter of a U(1) gauge transformation. The commutator of two SUSY transformations on the Vielbein yields a general transformation that should take the following form:

$$\begin{aligned}[\delta_{\mathcal{E}_1}, \delta_{\mathcal{E}_2}] \; E_M{}^A &= (\delta_\Xi + \delta_\Lambda + \delta_\mathcal{E}) \; E_M{}^A \\ &= \mathcal{L}_\Xi \; E_M{}^A - \Lambda^{AB} E_{MB} + \tfrac{1}{2}\bar{\mathcal{E}}\gamma^A \Psi_M \, .\end{aligned} \tag{4.5}$$

Applying the transformation rules (4.1), one obtains the following expression

$$\begin{aligned}[\delta_{\mathcal{E}_1}, \delta_{\mathcal{E}_2}] \; E_M{}^A = \tfrac{1}{2}\Big(&\bar{\mathcal{E}}_2\gamma^A\Big(\hat{D}_M\mathcal{E}_1 - \tfrac{i}{16}\hat{F}_{NP}\left[\gamma^{SNP} - 4G^{SN}\gamma^P\right] G_{MS}\mathcal{E}_1\Big) \\ &- \bar{\mathcal{E}}_1\gamma^A\Big(\hat{D}_M\mathcal{E}_2 - \tfrac{i}{16}\hat{F}_{NP}\left[\gamma^{SNP} - 4G^{SN}\gamma^P\right] G_{MS}\mathcal{E}_2\Big)\Big) \\ = \tfrac{1}{2}\Big(&\partial_M(\bar{\mathcal{E}}_2\gamma^A\mathcal{E}_1) + \hat{\Omega}_M{}^{AB}(\bar{\mathcal{E}}_2\gamma_B\mathcal{E}_1) \\ &- \tfrac{i}{8}\hat{F}_{NP}\bar{\mathcal{E}}_2\left[\gamma^{ASNP}G_{MS} - 4\delta^N_M E_S{}^A G^{SP}\right]\mathcal{E}_1\Big) \\ &+ \Xi^N \hat{R}_{NM}(P^A) \, ,\end{aligned} \tag{4.6}$$

which is of the form (4.5), when making use of the fact that the curvature of spatial translations has been set to zero. This relation is the previously imposed supercovariant no-torsion condition (4.2) and it can be seen here that the condition is needed to close the algebra. It allows the theory to be symmetric under general coordinate transformations rather than local translations which is what is required for general relativity.

For consistency the commutator of two SUSY transformations is now applied to the vector field

$$\begin{aligned}[]
[\delta_{\mathcal{E}_1}, \delta_{\mathcal{E}_2}]\, A_M &= (\delta_\Xi + \delta_{\mathcal{E}} + \delta_\alpha)\, A_M \\
&= \mathcal{L}_\Xi\, A_M + i\bar{\mathcal{E}}\Psi_M + \partial_M \alpha\,.
\end{aligned} \tag{4.7}$$

Using the transformation rules (4.1) one finds that

$$\begin{aligned}[]
[\delta_{\mathcal{E}_1}, \delta_{\mathcal{E}_2}]\, A_M = i\Bigg(& \bar{\mathcal{E}}_2\Big(\hat{D}_M\mathcal{E}_1 - \tfrac{i}{16}\hat{F}_{NP}\left[\gamma^{SNP} - 4G^{SN}\gamma^P\right] G_{MS}\mathcal{E}_1\Big) \\
& - \bar{\mathcal{E}}_1\Big(\hat{D}_M\mathcal{E}_2 - \tfrac{i}{16}\hat{F}_{NP}\left[\gamma^{SNP} - 4G^{SN}\gamma^P\right] G_{MS}\mathcal{E}_2\Big)\Bigg) \\
= {} & \partial_M(i\bar{\mathcal{E}}_2\mathcal{E}_1) - \partial_{[M}A_{P]}\,\bar{\mathcal{E}}_2\gamma^P\mathcal{E}_1 - \tfrac{i}{2}\bar{\Psi}^P\Psi_M(\bar{\mathcal{E}}_2\gamma_P\mathcal{E}_1)\,.
\end{aligned} \tag{4.8}$$

By comparing (4.6) and (4.8) it becomes visible that the resulting transformation parameters

$$\Xi^M = \tfrac{1}{2}\bar{\mathcal{E}}_2\gamma^M\mathcal{E}_1\,, \tag{4.9a}$$

$$\Lambda^{AB} = -\Xi^M\hat{\Omega}_M{}^{AB} - \tfrac{i}{16}\left[\hat{F}_{CD}\bar{\mathcal{E}}_2\gamma^{ABCD}\mathcal{E}_1 + 4\hat{F}^{AB}\bar{\mathcal{E}}_2\mathcal{E}_1\right]\,, \tag{4.9b}$$

$$\mathcal{E} = -\Xi^M\Psi_M\,, \tag{4.9c}$$

$$\alpha = i\bar{\mathcal{E}}_2\mathcal{E}_1 - \Xi^M A_M\,, \tag{4.9d}$$

are independent of the transformed field. When applying a commutator of SUSY transformation on the gravitino, the above parameters are recovered in the result. However, additional terms appear in the computation, accounting for a set of equations of motion and constraints on the gravitino. These will appear in the form of supercovariant gravitino curvature constraints. The explicit computation is omitted here, but is performed for the null-reduced linearised theory in section 6.2.3.

4.3 Null Reduction of 5D Fields

For the Vielbein, the graviphoton and the gravitino we need a decomposition Ansatz. Since the reduction is performed along a null Killing vector $\aleph$ it is convenient to work in flat null coordinates with the flat 5D-coordinates split as $A = (a, -, +)$ and the flat metric being $\eta_{ab} = \delta_{ab}$ and $\eta_{+-} = -1$. The Killing vector $\aleph$ is chosen in z-direction (corresponding to the direction of reduction) in adapted spacetime coordinates $M = (\mu, z)$, meaning $\aleph = \partial_z$. By definition, $\mathcal{L}_\aleph G_{MN} = 0$, the Killing vector $\aleph$ describes an isometry. When speaking of a Killing vector here a vector is meant that additionally leaves the complete field content of the supergravity multiplet invariant under

$$\mathcal{L}_\aleph E_M{}^A = 0\,, \qquad \mathcal{L}_\aleph A_M = 0\,, \qquad \mathcal{L}_\aleph \Psi^i_M = 0\,. \tag{4.10}$$

The consequence is complete z-independence of the fields

$$\partial_z E_M{}^A = 0\,, \qquad \partial_z A_M = 0\,, \qquad \partial_z \Psi^i_M = 0\,. \tag{4.11}$$

The z-dependence of the transformation parameters can easily be found by taking the partial z-derivative of a diffeomorphism, a local Lorentz-, a SUSY- or a U(1) gauge transformation (4.1) of the bosonic fields

$$\partial_z \Xi^\mu = 0\,, \qquad \partial_z \Lambda^{AB} = 0\,, \qquad \partial_z \mathcal{E}^i = 0\,, \tag{4.12a}$$

$$\partial_z \Xi^z \equiv -k_\sigma\,, \qquad \partial_z \alpha \equiv k_\alpha\,, \tag{4.12b}$$

where k_σ and k_α are constants. The reduction Ansatz presented above restricts most parameters to be completely z-independent. The two scalar parameters (4.12b) are, however, allowed to have a linear z-dependence and be of the form $\alpha = k_\alpha z + \alpha(x^\mu) + \tilde{k}_\alpha$ (analogously for Ξ^z), where $\tilde{k}_\alpha$ is a constant irrelevant to the theory, since the two parameters containing constants of this kind only appear with derivatives. For now, the two rigid shift and rescaling symmetries with parameters k_α and k_σ are kept in the theory. They will be set to zero at a later point (see sec. 5.2.2, 5.2.5). Equipped with these relations and restrictions the dimensional reduction to 4D can begin.

4.3.1 Vielbein

As discussed in section 2.2.5, the null reduction Ansatz for the Vielbein and the inverse Vielbein has the following form:

$$E_M{}^A = \begin{pmatrix} e_\mu{}^a & S^{-1}\,\tau_\mu & -S\,m_\mu \\ 0 & 0 & S \end{pmatrix}\,, \qquad E^M{}_A = \begin{pmatrix} e^\mu{}_a & m_a \\ S\,\tau^\mu & S\,m_0 \\ 0 & S^{-1} \end{pmatrix}\,. \tag{4.13}$$

The new fields automatically inherit orthogonality relations

$$\begin{aligned} &e_\mu{}^a e^\mu{}_b = \delta^a_b\,, \qquad \tau_\mu \tau^\mu = 1\,, \\ &e_\mu{}^a \tau^\mu = 0\,, \qquad \tau_\mu e^\mu{}_a = 0\,, \\ &e_\mu{}^a e^\nu{}_a + \tau_\mu \tau^\nu = \delta^\mu_\nu\,, \end{aligned} \tag{4.14}$$

from the original

$$E_M{}^A E^M{}_B = \delta^A_B\,, \qquad E_M{}^A E^N{}_A = \delta^N_M\,. \tag{4.15}$$

When applying the 5D transformation rule (4.1) on the z-components of the Vielbein Ansatz (4.13) we find that

$$\delta E_z{}^a = -\Lambda^{-a} S + \tfrac{1}{2}\bar{\mathcal{E}}\gamma^a \Psi_z \,, \tag{4.16a}$$
$$\delta E_z{}^- = \tfrac{1}{2}\bar{\mathcal{E}}\gamma^- \Psi_z \,, \tag{4.16b}$$
$$\delta E_z{}^+ = \delta_z^M \delta_A^+ \mathcal{L}_\Xi E_M{}^A - \Lambda^{-+} S + \tfrac{1}{2}\bar{\mathcal{E}}\gamma^+ \Psi_z \,, \tag{4.16c}$$

where $\gamma^\pm = \frac{1}{\sqrt{2}}(\gamma^0 \pm \gamma_5)$. All but one z-component of the Vielbein vanish. Therefore the above equations give the gauge condition

$$\Lambda^{-a} = \tfrac{1}{2} S^{-1} \bar{\mathcal{E}}\gamma^a \Psi_z \,, \tag{4.17}$$

the constraint coming from the null Killing vector

$$0 = \bar{\mathcal{E}}\gamma^- \Psi_z \tag{4.18}$$

and a transformation rule for the scalar S

$$\delta S = \delta_z^M \delta_A^+ \mathcal{L}_\Xi E_M{}^A - \Lambda^{-+} S + \tfrac{1}{2}\bar{\mathcal{E}}\gamma^+ \Psi_z \,. \tag{4.19}$$

The transformation rules for the other fields coming from the 5D Vielbein $\{e_\mu{}^a, \tau_\mu, m_\mu\}$ will be discussed in chapter 5.

Now the bosonic transformation parameters are reduced by simply relabelling them. Note that the relabelling of the 5D parameter components of local Lorentz- and general coordinate symmetry as

$$\Lambda^{+a} \equiv \lambda^a \,, \qquad \Lambda^{-+} \equiv \lambda \,, \qquad \Xi^\mu \equiv \xi^\mu \,, \qquad \Xi^z \equiv -\sigma \,, \tag{4.20}$$

is suggestive of the new symmetries emerging from the reduction. The reduction of the fermionic parameters is a little more involved and will be discussed together with the reduction of the gravitini in section 4.3.3.

4.3.2 Graviphoton

The graviphoton transformation rule is obtained from (4.1)

$$\delta A_\mu = \delta_\mu^M \mathcal{L}_\Xi A_M + \partial_\mu \alpha + i\bar{\mathcal{E}}\Psi_\mu \,, \tag{4.21a}$$
$$\delta A_z = \delta_z^M \mathcal{L}_\Xi A_M + i\bar{\mathcal{E}}\Psi_z \,. \tag{4.21b}$$

For the reduced vector we have to take care that it transforms appropriately, as a 4D vector and a scalar under general coordinate transformations. The naïve decomposition fails, since the 5-dimensional Lie derivative produces a term which is not accounted for by the 4-dimensional diffeomorphism ($\delta_\mu^M \mathcal{L}_\Xi A_M = \mathcal{L}_\xi A_\mu - A_z \partial_\mu \sigma$). This term is identified with the gauge transformation of m_μ which can be used to cancel it for the 4D vector. Therefore, the 5-dimensional vector is decomposed into a 4D vector and a scalar as

$$a_\mu = A_\mu + m_\mu A_z \,, \qquad a = A_z \,. \tag{4.22}$$

Defining the 4D vector like this trades the privilege of loosing the unwanted derivative term $\partial_\mu \sigma$ for gaining two rigid shift symmetries involving k_σ, k_α and m_μ coming from the transformation of the scalar a. This will be made explicit in the next chapter.

The gauge parameter α of the vector transformation does not have to be null-reduced as it is a scalar.

4.3.3 Gravitini

The reduction Ansatz for the gravitini was inspired by [52]. In the relativistic 5D theory the minimal irreducible spinors we are dealing with are symplectic Majorana spinors. This means that they obey the symplectic Majorana condition

$$(\Psi^i)^* = B^{(5)}\, \Psi^j \varepsilon_j{}^i \,, \tag{4.23}$$

with $i, j = 1, 2$. In total the pair of symplectic Majorana spinors has 8 real components. For the gravitini, which are also spacetime vectors, this results in a total of $8 \times 5 = 40$ real components. In 4D, however, we are dealing with minimal irreducible Majorana spinors, which obey the Majorana condition

$$\psi^* = B^{(4)}\, \psi \,. \tag{4.24}$$

The charge conjugation matrices in 4 and 5 dimensions are related by $C^{(4)} = C^{(5)}\gamma_5$. Majorana spinors in 4D have 4 real components, meaning that a gravitino has $4 \times 4 = 16$ components. Dimensionally the highest possible number of reduced gravitini is therefore 2, accounting for 32 components. The remaining 8 components could be arranged into two additional Majorana spinors. We will keep this reduction structure in mind for the following steps.

Using the chiral projectors $P_{\substack{L\\R}} = \frac{1}{2}(\mathbb{1} \pm \gamma_5)$ on a 5D symplectic Majorana spinor Ψ^i, the following set of chiral spinors can be defined:

$$\Theta^i_L = P_L\, \Psi^i \,, \qquad \Theta^i_R = P_R\, \Psi^j \varepsilon_j{}^i \,. \tag{4.25}$$

These already obey the Majorana-like condition in 4 dimensions

$$(\Theta^i_L)^* = B^{(4)}\, \Theta^i_R \,, \qquad (\Theta^i_R)^* = B^{(4)}\, \Theta^i_L \,, \tag{4.26}$$

and the sum of the two clearly composes a 4D Majorana spinor

$$\Phi^i = \Theta^i_L + \Theta^i_R \,. \tag{4.27}$$

To ensure that the reduced gravitini also transform correctly under general coordinate transformations in 4 dimensions, they take the form

$$\left(\psi_\mu^i,\, \varphi^i\right) := \left(\Phi_\mu^i + \Phi_z^i\, m_\mu,\, \Phi_z^i\right) . \tag{4.28}$$

The previous step in the reduction is reminiscent of the reduction of the graviphoton (4.22), and in fact follows the same reasoning. The same reduction is applied to the 5D SUSY parameters to obtain Majorana spinors

$$\varepsilon_L^i = P_L\, \mathcal{E}^i\, , \qquad \varepsilon_R^i = P_R\, \mathcal{E}^j \varepsilon_j{}^i\, , \tag{4.29a}$$

$$\varepsilon^i = \varepsilon_L^i + \varepsilon_R^i\, . \tag{4.29b}$$

Taking into account Galileian boosts, we can perform yet another decomposition and rearrange the gravitini in terms of their asymmetric boost behaviour. Their parameters are rearranged in the same fashion

$$\psi_\mu^\pm = \tfrac{1}{\sqrt{2}}(\psi_\mu^2 \pm \gamma^0 \psi_\mu^1)\, , \qquad \varphi^\pm = \tfrac{1}{\sqrt{2}}(\varphi^2 \pm \gamma^0 \varphi^1)\, , \tag{4.30a}$$

$$\varepsilon^\pm = \tfrac{1}{\sqrt{2}}(\varepsilon^2 \pm \gamma^0 \varepsilon^1)\, . \tag{4.30b}$$

Already here the null Killing vector constraint (4.18) can be made use of to eliminate one of the spinor fields

$$0 = \bar{\varepsilon}^+ \gamma^0 \varphi^+ \qquad \rightarrow \qquad \varphi^+ = 0\, . \tag{4.31}$$

Since we are left with only one of the two spinors, the superscript will be dropped from here on $\varphi^- \equiv \varphi$. The gauge condition (4.17) now reads

$$\Lambda^{-a} = \tfrac{1}{2}\bar{\varepsilon}^+ \gamma^a \varphi\, , \tag{4.32}$$

For practicality the 5-dimensional gravitino components can be written in terms of projections of the 4-dimensional spinors and 5D gravitino bilinears can be expressed in terms of 4D fields. These relations are necessary to explicitly calculate the reduced transformation rules of all 4D fields and can be looked up in appendix B.

4.3.4 Gauge Fixing of S

One simplification remains to be exploited. After redefining some of the fields with appropriate powers of S,

$$\tau_\mu \rightarrow S^{-1} \tau_\mu\, , \tag{4.33a}$$

$$m_\mu \rightarrow S\, m_\mu \qquad (\text{and } \sigma \rightarrow S\sigma)\, , \tag{4.33b}$$

$$a \rightarrow S^{-1} a\, , \tag{4.33c}$$

$$\varphi \rightarrow S^{-1} \varphi\, , \tag{4.33d}$$

the transformation rules of all emerging 4D fields, i.e. $\{e_\mu{}^a,\, \tau_\mu,\, m_\mu,\, a_\mu,\, a,\, \psi_\mu^\pm,\, \varphi\}$ decouple from S, which leaves us with the freedom to gauge fix $S=1$, thereby also fixing the parameter

$$\lambda = \tfrac{1}{\sqrt{2}}\bar{\varepsilon}^- \gamma^0 \varphi - k_\sigma\, . \tag{4.34}$$

Chapter 5

Non-Relativistic 4D, $\mathcal{N} = 2$ Pure Supergravity

In this chapter the reduction Ansätze of the previous chapter are put into practice to find the transformation rules of the reduced 4D fields. Then, constraints are discussed, and if and how the constraints that have been imposed on the 5D level close the 4D algebra on the 4D multiplet. Finally, the consequences for the multiplet are stated and the multiplet is presented in its final form at the end of this chapter.

5.1 Building the Multiplet

One has to take special care when raising and lowering indices at this point, because simply going from flat to spacetime indices using the Vielbein is no longer possible. The new spatial Vielbein $e_\mu{}^a$ is not even a square matrix anymore. There is, however, still a way to go from curved to flat indices and vice versa for **covectors** and **cotensors**, not for vectors and contravariant tensors, using both the spatial Vielbein $e_\mu{}^a$ and the temporal Vielbein τ_μ. Take a spacetime covector B_μ, a flat vector B_a and a scalar B_0. These quantities are by definition related by

$$B_a = e^\mu{}_a B_\mu \,, \qquad B_0 = \tau^\mu B_\mu \,, \qquad B_\mu = e_\mu{}^a B_a + \tau_\mu B_0 \,. \tag{5.1}$$

In this notation flat indices are raised (and lowered) using Kronecker deltas as $B^a = \delta^{ab} B_b$. The raising and lowering of the 0-index is not defined. This notation is applicable for fields that were defined with spacetime indices in 5D. Gamma matrices in 5D could be defined with spacetime indices by simply raising or lowering the flat indices with the Vielbein. This notation was, however, purposely avoided to prevent misconceptions when dealing with 4D gamma matrices. The object γ_0 appears in the 4D theory and could be confused with something like $\tau^\mu \gamma_\mu$ which it is **not**, because gamma matrices with spacetime indices are not defined in 4D. However, for gamma matrices the 0-index is to be understood as a linear combination of the flat null directions, as is the 5-index, i.e. $\gamma^0 = \frac{1}{\sqrt{2}}(\gamma^+ + \gamma^-)$. This index can, of course, be raised and lowered with the 5D Minkowski metric $\gamma_0 = -\gamma^0$.

Starting with the reduced fields found in the last chapter, it is a computational matter to derive their transformation rules from the 5D multiplet (4.1). Not only are there more fields compared to the 5-dimensional theory, but the symmetry structure of the multiplet has changed from gauged super-Poincaré- to gauged super-Bargmann symmetry.

5.1.1 Vielbein

Taking the remaining transformation relations of the 5D Vielbein

$$\delta E_\mu{}^a = \delta_\mu^M \delta_A^a \mathcal{L}_\Xi E_M{}^A - \Lambda^{ab} e_\mu{}^c \delta_{bc} + \Lambda^{-a} S m_\mu - \Lambda^{+a} S^{-1} \tau_\mu + \tfrac{1}{2}\bar{\mathcal{E}}\gamma^a \Psi_\mu \,, \tag{5.2a}$$

$$\delta E_\mu{}^- = \delta_\mu^M \delta_A^- \mathcal{L}_\Xi E_M{}^A - \Lambda^{-a} e_\mu{}^b \delta_{ab} + \Lambda^{-+} S^{-1} \tau_\mu + \tfrac{1}{2}\bar{\mathcal{E}}\gamma^- \Psi_\mu \,, \tag{5.2b}$$

$$\delta E_\mu{}^+ = \delta_\mu^M \delta_A^+ \mathcal{L}_\Xi E_M{}^A - \Lambda^{+a} e_\mu{}^b \delta_{ab} + \Lambda^{-+} S m_\mu + \tfrac{1}{2}\bar{\mathcal{E}}\gamma^+ \Psi_\mu \,, \tag{5.2c}$$

the 4D transformation rules of $\{e_\mu{}^a, \tau_\mu, m_\mu\}$ are found. This is achieved by inserting the null reduced Vielbein (4.13), the redefined parameters (4.20, 4.32), the reduced spinor bilinears (B.6) and the gauge fixing of S (4.34). The following set of 4D transformation rules is thereby obtained from the 5D Vielbein

$$\delta e_\mu{}^a = \mathcal{L}_\xi e_\mu{}^a - \lambda^a{}_b e_\mu{}^b - \lambda^a \tau_\mu + \tfrac{1}{2}\bar{\varepsilon}^+ \gamma^a \psi_\mu^- + \tfrac{1}{2}\bar{\varepsilon}^- \gamma^a \psi_\mu^+ \,, \tag{5.3a}$$

$$\delta \tau_\mu = \mathcal{L}_\xi \tau_\mu - \tfrac{1}{2} e_{\mu a}\, \bar{\varepsilon}^+ \gamma^a \varphi + \tfrac{1}{\sqrt{2}} \bar{\varepsilon}^+ \gamma^0 \psi_\mu^+ + \tfrac{1}{\sqrt{2}} \tau_\mu \bar{\varepsilon}^- \gamma^0 \varphi \,, \tag{5.3b}$$

$$\delta m_\mu = \mathcal{L}_\xi m_\mu + \partial_\mu \sigma + \lambda_b e_\mu{}^b - \tfrac{1}{\sqrt{2}} \bar{\varepsilon}^- \gamma^0 \psi_\mu^- \,. \tag{5.3c}$$

By looking at the symmetry structure of these fields one can recognise general coordinate transformations with parameter ξ^μ, rotations with parameter λ^{ab}, Galilean boosts with parameter λ^a, a central charge transformation with parameter σ and two local SUSY transformations with parameters $\varepsilon^\pm$. If they were rigid, these symmetries would compose the super-Bargmann algebra.

From these equations it is possible to derive completely supercovariant curvatures (transforming covariantly under rotations, Galilean boosts and SUSY transformations). Replacing the parameters in a transformation rule by their corresponding gauge fields and antisymmetrising, one obtains expressions that under a general variation behave like the variation of the antisymmetrised partial derivative of the gauge field. By subtracting these expressions from one another, covariant quantities called covariant curvatures emerge as was explained in the end of section 3.1.3

$$\hat{r}_{\mu\nu}(P^a) = 2\partial_{[\mu} e_{\nu]}{}^a + 2\hat{\omega}_{[\mu}{}^{ab} e_{\nu]b} + 2\hat{\omega}_{[\mu}{}^a \tau_{\nu]} - \bar{\psi}^+_{[\mu} \gamma^a \psi^-_{\nu]} \,, \tag{5.4a}$$

$$\hat{\tau}_{\mu\nu} \equiv \hat{r}_{\mu\nu}(H) = 2\partial_{[\mu}\tau_{\nu]} - e_{a[\mu}\bar{\psi}^+_{\nu]}\gamma^a\varphi + \sqrt{2}\tau_{[\mu}\bar{\psi}^-_{\nu]}\gamma^0\varphi - \tfrac{1}{\sqrt{2}}\bar{\psi}^+_\mu \gamma^0 \psi^+_\nu \,, \tag{5.4b}$$

$$\hat{r}_{\mu\nu}(M) = 2\partial_{[\mu} m_{\nu]} - 2\hat{\omega}_{[\mu}{}^a e_{\nu]a} + \tfrac{1}{\sqrt{2}}\bar{\psi}^-_\mu \gamma^0 \psi^-_\nu \,. \tag{5.4c}$$

The curvature of time translations $\hat{r}_{\mu\nu}(H)$ will also be referred to as torsion, denoted $\hat{\tau}_{\mu\nu}$, as is usually done in the literature. Its explicit transformation rule can be found in appendix C. At this point the meaning of $\hat{\omega}_\mu{}^{ab}$ and $\hat{\omega}_\mu{}^a$ does not exceed them being the supercovariantised gauge fields of rotations and Galilean boosts, meaning that they transform with derivatives of the parameters λ^a and λ^{ab} respectively. They will be referred to as spin- and boost connection respectively. As is the 5D spin connection, these two fields are actually dependent. Their relation to the 5D spin connection and their dependent structure coming from the null reduction of the 5D supercovariant no-torsion condition will be discussed in section 5.2.1, when analysing constraints.

5.1.2 Graviphoton

The transformation rules of the reduced graviphoton mostly inherit the structure of (4.1b), with some additional terms coming from (5.3c) due to the 4D vector GCT transformation correction. As stated in the previous section 5.1.1 the 5D parameters are relabelled, S is gauge fixed to 1 and the gravitino bilinears are reexpressed in a 4-dimensional fashion to obtain the final form

$$\delta a = \mathcal{L}_\xi a + k_\alpha - a k_\sigma + i\bar{\varepsilon}^+\gamma_5\gamma^0\varphi \,, \tag{5.5a}$$

$$\begin{aligned}\delta a_\mu = {}& \mathcal{L}_\xi a_\mu + \partial_\mu\alpha + m_\mu(k_\alpha - a k_\sigma) + a\lambda_b e_\mu{}^b \\ &+ i\bar{\varepsilon}^+\gamma_5\gamma^0\psi_\mu^- - i\bar{\varepsilon}^-\gamma_5\gamma^0\psi_\mu^+ - a\tfrac{1}{\sqrt{2}}\bar{\varepsilon}^-\gamma^0\psi_\mu^- \,.\end{aligned} \tag{5.5b}$$

The scalar a has picked up two strange symmetries from the null reduction of the 5D diffeomorphism and the U(1) gauge transformation – a constant shift symmetry and a constant scaling respectively. These symmetries also appear in the transformation of the 4D vector coming from the part containing the scalar a.

At this point it is necessary to also have a look at the supercovariant field strength $\hat{f}_{\mu\nu}$ of a_μ and the supercovariant derivative of a. This is because the 5D supercovariant field strength of A_M occurs in the transformation rule of the 5D gravitini (4.1). Its 4D counterpart $\hat{f}_{\mu\nu}$ and the supercovariant derivative $\hat{D}_\mu a$ therefore appear in the reduction process. Following the steps described in the end of section 3.1.3 the supercovariant quantities are found

$$\hat{f}_{\mu\nu} = 2\partial_{[\mu}a_{\nu]} - 2a\hat{\omega}_{[\mu}{}^a e_{\nu]a} - 2i\,\bar{\psi}^+_{[\mu}\gamma_5\gamma^0\psi^-_{\nu]} + a\tfrac{1}{\sqrt{2}}\bar{\psi}^-_\mu\gamma^0\psi^-_\nu \,, \tag{5.6a}$$

$$\hat{D}_\mu a = \partial_\mu a - i\bar{\psi}^+_\mu\gamma_5\gamma^0\varphi \,. \tag{5.6b}$$

Since the transformation rules of the gravitini contain the 5D field strength, it is convenient to write the 5D quantity in terms of covariant 4D quantities. The components of $\hat{F}_{MN}$ can be split into two expressions

$$\hat{F}_{\nu\rho} = \hat{f}_{\nu\rho} - a\hat{r}_{\nu\rho}(M) + 2m_{[\nu}\hat{D}_{\rho]}a \,, \tag{5.7a}$$

$$\hat{F}_{\nu z} = \hat{D}_\nu a \,. \tag{5.7b}$$

The explicit transformation rule of $\hat{f}_{\mu\nu}$ can be found in appendix C.

5.1.3 Gravitini

Using the reduction Ansatz of section 4.3.3 the 5-dimensional gravitino transformation rules imply 4-dimensional transformation rules for the gravitini $\psi_\mu^\pm$ and the spinor in z-direction φ. Since the 5D transformation rules contain the supercovariant spin connection $\hat{\Omega}_M{}^{AB}$ and the supercovariant field strength $\hat{F}_{MN}$, these quantities also have to be null-reduced in order to deal with 4-dimensional quantities in the reduced transformation rules. The contributions of the spin connection will not be added explicitly just yet, but only after discussing the relation between the 5D spin connection and 4D connections together with the 5D supercovariant no-torsion condition in section 5.2.1. The field strength terms, however, can already be made explicit, as shown in the previous section (see eq. (5.7)). The reduced transformation rule of the spinor is

$$\begin{aligned} \delta\varphi &= \mathcal{L}_\xi\varphi - \tfrac{1}{2}k_\sigma\varphi - \tfrac{1}{4}\lambda^{ab}\gamma_{ab}\varphi - \tfrac{1}{2\sqrt{2}}(\bar{\varepsilon}^-\gamma^0\varphi)\varphi + \hat{\Omega}\text{-terms} \\ &\quad - \tfrac{i}{16}\Big[2\hat{D}_b a\gamma^b\gamma_5\gamma^0\varepsilon^- + \sqrt{2}\left((\hat{f}_{bc} - a\hat{r}_{bc}(M))\gamma^{bc} - 4\hat{D}_0 a\right)\gamma_5\varepsilon^+\Big]\,, \end{aligned} \tag{5.8}$$

where $\hat{D}_a \equiv e^\mu{}_a\hat{D}_\mu$ and $\hat{D}_0 \equiv \tau^\mu\hat{D}_\mu$. The complete transformation rule of φ can be found in appendix C. The transformation rules for the gravitini read

$$\begin{aligned} \delta\psi_\mu^- &= \mathcal{L}_\xi\psi_\mu^- + \tfrac{1}{2}k_\sigma\psi_\mu^- - m_\mu k_\sigma\varphi + \partial_\mu\varepsilon^- + e_{\mu a}\lambda^a\varphi - \tfrac{1}{4}\lambda^{ab}\gamma_{ab}\psi_\mu^- \\ &\quad - \tfrac{1}{\sqrt{2}}\lambda^a\gamma_a\gamma^0\psi_\mu^+ - \tfrac{1}{2\sqrt{2}}(\bar{\varepsilon}^-\gamma^0\varphi)\psi_\mu^- - \tfrac{1}{\sqrt{2}}(\bar{\varepsilon}^-\gamma^0\psi_\mu^-)\varphi + \hat{\Omega}\text{-terms} \\ &\quad - \tfrac{i}{16}\Big[\Big((\hat{f}_{\nu\rho} - a\hat{r}_{\nu\rho}(M))(e_{\mu a}e^\nu{}_b e^\rho{}_c\gamma^{abc} - 2\tau_\mu\tau^\nu e^\rho{}_a\gamma^a - 4\delta_\mu^\nu e^\rho{}_a\gamma^a) \\ &\quad - 2e_{\mu a}\hat{D}_0 a\gamma^a\Big)\gamma_5\gamma^0\varepsilon^- \\ &\quad + \sqrt{2}\hat{f}_{\nu\rho}(-2e_{\mu a}e^\nu{}_b\tau^\rho\gamma^{ab} + 4\delta_\mu^\nu\tau^\rho)\gamma_5\varepsilon^+\Big]\,, \end{aligned} \tag{5.9a}$$

$$\begin{aligned} \delta\psi_\mu^+ &= \mathcal{L}_\xi\psi_\mu^+ - \tfrac{1}{2}k_\sigma\psi_\mu^+ + \partial_\mu\varepsilon^+ - \tfrac{1}{4}\lambda^{ab}\gamma_{ab}\psi_\mu^+ \\ &\quad - \tfrac{1}{2\sqrt{2}}(\bar{\varepsilon}^+\gamma^a\varphi)\gamma_a\gamma^0\psi_\mu^- + \tfrac{1}{2\sqrt{2}}(\bar{\varepsilon}^-\gamma^0\varphi)\psi_\mu^+ \\ &\quad + \tfrac{1}{2\sqrt{2}}m_\mu(\bar{\varepsilon}^+\gamma^a\varphi)\gamma_a\gamma^0\varphi - \tfrac{1}{8}m_\mu\hat{\tau}_{ab}\gamma^{ab}\varepsilon^+ + \hat{\Omega}\text{-terms} \\ &\quad - \tfrac{i}{16}\Big[\Big((\hat{f}_{\nu\rho} - a\hat{r}_{\nu\rho}(M))(-e_{\mu a}e^\nu{}_b e^\rho{}_c\gamma^{abc} - 2\tau_\mu\tau^\nu e^\rho{}_a\gamma^a + 4\delta_\mu^\nu e^\rho{}_a\gamma^a) \\ &\quad - 2e_{\mu a}\hat{D}_0 a\gamma^a + 6m_\mu\hat{D}_a a\gamma^a\Big)\gamma_5\gamma^0\varepsilon^+ \\ &\quad + \sqrt{2}\Big(-\tau_\mu(\hat{f}_{ab} - a\hat{r}_{ab}(M))\gamma^{ab} + 2e_{\mu a}\hat{D}_b a\gamma^{ab} - 4\hat{D}_\mu a\Big)\gamma_5\varepsilon^-\Big]\,. \end{aligned} \tag{5.9b}$$

Note that here also the constant scaling symmetry, first seen in the transformation of the graviphoton, takes part in the transformation of all three fields. In the last transformation rule (5.9b), the variation $\delta\varphi^+$ (all terms multiplied by m_μ) appears, preventing the definition of a curvature that transforms covariantly under central charge transformations (one can define a supercovariant curvature transforming covariantly under Galilean boosts, rotations and SUSY transformations). Only upon setting $\delta\varphi^+ = 0$, as will be done when talking about the constraints in section 5.2.4, this becomes possible.

5.2 Constraints

Now we already have a set of transformation rules for the 4D fields, but a few questions remain open:

- How are $\hat{\Omega}$ and the $\hat{\omega}$'s related and what constraints are inherited from the 5D supercovariant no-torsion condition $\hat{R}_{MN}(P^A) = 0$?
- Does the reduced algebra close properly and does the closure produce additional constraints?
- What constraints are produced by Bianchi identities?
- Do further constraint arise from the reduction Killing vector being null?
- How does the field content of the theory change after imposing constraints?

These questions will will be addressed in this section.

5.2.1 Conventional Constraints

It has already been established by imposing the conventional curvature constraint in 5D (the supercovariant no-torsion condition) that the spin connection is a dependent field. In fact, this is how the supercovariant spin connection is defined (see eq. (4.2)). Therefore, it is not necessary to null-reduce the 5D spin connection using its original form in terms of 5D Vielbein and gravitini, but the reduction can be performed using a shortcut. By simply viewing the supercovariant no-torsion condition as a geometric constraint on an independent field and inserting the null reduced Vielbein, a set of equations emerges, relating components of $\hat{\Omega}$. To make the calculation simpler yet, it suffices to consider bosonic terms only, i.e. to neglect gravitini. From the discussion of the parameters (4.12a) we deduce that the components $\hat{\Omega}_z{}^{AB}$ are not gauge fields, but rather completely supercovariant objects. As for $\hat{\Omega}_\mu{}^{-a}$ and $\hat{\Omega}_\mu{}^{-+}$ the compensating transformations from fixing their parameters (4.32, 4.34) must be included. These lead to the only gravitini contributions to the spin connection, which had been ignored in the process before. Then, two 4D supercovariant objects are introduced, transforming as gauge fields with derivatives of λ^{ab} and λ^a respectively, which will be referred to as spin connection $\hat{\omega}_\mu{}^{ab}$ and boost connection $\hat{\omega}_\mu{}^a$. Finally, the 5D quantities have to be adapted to transform as 4D vectors under GCT using the standard method, i.e. $\hat{\Omega}_\mu{}^{AB} + m_\mu \hat{\Omega}_z{}^{AB}$ transforms as a 4D vector, hence

$$\hat{\Omega}_\mu{}^{ab} = \hat{\omega}_\mu{}^{ab} - \tfrac{1}{2} m_\mu \hat{\tau}^{ab} \,, \tag{5.10a}$$

$$\hat{\Omega}_\mu{}^{+a} = \hat{\omega}_\mu{}^{a} - \tfrac{1}{2} m_\mu \hat{\tau}^{a}{}_0 \,. \tag{5.10b}$$

The z-components of the 5D supercovariant no-torsion condition set some components of the spin connection to zero and relates others

$$\hat{\Omega}_z{}^{-a} = 0\,, \tag{5.11a}$$
$$\hat{\Omega}_z{}^{-+} = 0\,, \tag{5.11b}$$
$$\hat{\Omega}_\mu{}^{-a} = \hat{\Omega}_z{}^{ab} e_{\mu b} + \hat{\Omega}_z{}^{+a}\tau_\mu + \psi^\pm\text{-terms}\,, \tag{5.11c}$$
$$\hat{\Omega}_\mu{}^{-+} = \hat{\Omega}_z{}^{+a} e_{\mu a} + \psi^\pm\text{-terms}\,, \tag{5.11d}$$

while the μ-components together with the knowledge about the transformation properties of the spin connection yields

$$\hat{r}_{\mu\nu}(P^a) = 0\,, \tag{5.12a}$$
$$\hat{r}_{\mu\nu}(M) = 0\,, \tag{5.12b}$$
$$\hat{\Omega}_\mu{}^{-a} = -\tfrac{1}{2}\hat{\tau}_\mu{}^a + \tfrac{1}{2}\bar{\psi}^+_\mu\gamma^a\varphi\,, \tag{5.12c}$$
$$\hat{\Omega}_\mu{}^{-+} = \tfrac{1}{2}\hat{\tau}_{\mu 0} + \tfrac{1}{\sqrt{2}}\bar{\psi}^-_\mu\gamma^0\varphi\,. \tag{5.12d}$$

The emerging curvature constraints are reminiscent of the 5D supercovariant no-torsion condition. These are referred to as conventional constraints of Newton-Cartan theory. Similarly to the situation in 5D these constraints define the the connections as dependent fields (depending on $e_\mu{}^a$, τ_μ, m_μ, $\psi^\pm_\mu$). Solving the conventional constraints for the connections is a little more advanced than it was in the relativistic case. For more details on this calculation see appendix B. The equations (5.12a, 5.12b) when solved for the connections yield

$$\hat{\omega}_\mu{}^{ab} = e^{\nu[a}\hat{c}_{\mu\nu}{}^{b]} - \tfrac{1}{2}e_{\mu c}\hat{c}^{abc} - \tfrac{1}{2}\tau_\mu\hat{c}^{ab}\,, \tag{5.13a}$$
$$\hat{\omega}_\mu{}^a = \tfrac{1}{2}\hat{c}_{0\mu}{}^a + \tfrac{1}{2}e_{\mu b}\hat{c}_0{}^{ab} + \tfrac{1}{2}\hat{c}_\mu{}^a + \tfrac{1}{2}\tau_\mu\hat{c}_0{}^a\,, \tag{5.13b}$$
$$\hat{c}_{\mu\nu}{}^a = 2\partial_{[\mu}e_{\nu]}{}^a - \bar{\psi}^+_{[\mu}\gamma^a\psi^-_{\nu]}\,, \tag{5.13c}$$
$$\hat{c}_{\mu\nu} = 2\partial_{[\mu}m_{\nu]} + \tfrac{1}{\sqrt{2}}\bar{\psi}^-_\mu\gamma^0\psi^-_\nu\,, \tag{5.13d}$$

where $\hat{c}_{\mu\nu}{}^a$ and $\hat{c}_{\mu\nu}$ are the non-relativistic supercovariant anholonomy coefficients. Now there are several ways to find the transformation rules of the spin- and boost connection. One relatively simple option is to consider the variation of the conventional constraints (5.12a, 5.12b) and to then solve for the variations of the connections completely analogously to the computation of the connections themselves. The explicit transformation rules of the spin- and boost connections and the explicit supercovariant curvatures $\hat{r}_{\mu\nu}(J^{ab})$ of rotations and $\hat{r}_{\mu\nu}(G^a)$ of Galilean boosts can be found in appendix C.

5.2.2 Algebra Closure

The procedure of checking the closure of the algebra on bosons is analogous to the one performed in 5 dimensions in section 4.2

$$[\delta_1,\delta_2] \overset{!}{=} \delta_\xi + \delta_{\lambda_r} + \delta_{\lambda_b} + \delta_\sigma + \delta_\alpha + \delta_{\varepsilon^+} + \delta_{\varepsilon^-}\,. \tag{5.14}$$

Performing the commutator transformation $[\delta_{\sigma_1}, \delta_{\sigma_2}]$ on the fields a_μ and ψ_μ^- gives a term that is not of the form (5.14). This is an indication that the constant scaling symmetry with parameter k_σ needs to be removed for the algebra to close. Therefore the constant is set zero from now on

$$k_\sigma = 0\,. \tag{5.15}$$

The same argument can, however, not be made to get rid of the constant shift symmetry with parameter k_α. This symmetry is well-behaved in terms of algebra closure.

Since the 5D supercovariant no-torsion condition, which has appeared in the 5D algebra closure, has produced the 4D conventional constraints found in the previous section, no new bosonic constraints are expected. Indeed, the algebra closes on the bosons and produces the following parameters for certain sets of commutators of transformations, at least one of which is a SUSY transformation:

The commutator of a general SUSY transformation ($\varepsilon_1 = \varepsilon_1^+ + \varepsilon_1^-$) and a rotation is trivial

$$\begin{aligned}[][\delta_{\varepsilon_1}, \delta_{\lambda_r}] &= \ \delta_{\varepsilon^\pm}\ , \\ \bar{\varepsilon}^\pm &= \tfrac{1}{4}\lambda^{ab}\bar{\varepsilon}_1^\pm\gamma_{ab}\,. \end{aligned} \tag{5.16}$$

The commutator of a SUSY transformation and a boost is less trivial

$$[\delta_{\varepsilon_1}, \delta_{\lambda_b}] = \ \delta_{\lambda_r} + \delta_{\lambda_b} + \delta_{\varepsilon^-}\ ,$$

$$\lambda^{ab} = -\lambda^{[a}\bar{\varepsilon}_1^+\gamma^{b]}\varphi\ , \tag{5.17a}$$

$$\lambda^a = \tfrac{1}{\sqrt{2}}\lambda^a\bar{\varepsilon}_1^-\gamma^0\varphi\ :, \tag{5.17b}$$

$$\varepsilon^- = -\tfrac{1}{\sqrt{2}}\lambda^a\bar{\varepsilon}_1^+\gamma_a\gamma^0\ . \tag{5.17c}$$

The commutators of SUSY transformations can be treated separately as $(+,+)$, $(+,-)$ and $(-,-)$-parts. The (+,+)-commutator produces the following parameters

$$[\delta_{\varepsilon_1^+}, \delta_{\varepsilon_2^+}] = \delta_\xi + \delta_{\lambda_r} + \delta_{\lambda_b} + \delta_\sigma + \delta_\alpha + \delta_{\varepsilon^\pm}\ ,$$

$$K := \tfrac{1}{\sqrt{2}}\bar{\varepsilon}_2^+\gamma^0\varepsilon_1^+\ , \tag{5.18a}$$

$$\xi^\nu = K\tau^\nu\ , \tag{5.18b}$$

$$\lambda^{ab} = -\,\xi^\nu\hat{\omega}_\nu{}^{ab} - \tfrac{1}{4}\varepsilon^{abc}\hat{f}_{c0}K\ , \tag{5.18c}$$

$$\lambda^a = -\,\xi^\nu\hat{\omega}_\nu{}^a\ , \tag{5.18d}$$

$$\sigma = -\,\xi^\nu m_\nu\ , \tag{5.18e}$$

$$\alpha = -\,\xi^\nu a_\nu\ , \tag{5.18f}$$

$$\bar{\varepsilon}^\pm = -\,\xi^\nu\bar{\psi}_\nu^\pm\ . \tag{5.18g}$$

The $(-,-)$-commutator produces the following parameters

$$
\begin{aligned}
[\delta_{\varepsilon_1^-},\delta_{\varepsilon_2^-}] &= \delta_{\lambda_r}+\delta_{\lambda_b}+\delta_\sigma+\delta_\alpha+\delta_{\varepsilon^-}\,,\\
J &:= \tfrac{1}{\sqrt{2}}\bar{\varepsilon}_2^-\gamma^0\varepsilon_1^-\,, && (5.19a)\\
\lambda^{ab} &= \tfrac{1}{4}J\varepsilon^{abc}\hat{D}_c a-\tfrac{1}{2}J\hat{\tau}^{ab}\,, && (5.19b)\\
\lambda^{a} &= -\tfrac{1}{8}J\varepsilon^{abc}\hat{f}_{bc}-\tfrac{1}{2}J\hat{\tau}^{a}{}_0\,, && (5.19c)\\
\sigma &= -J\,, && (5.19d)\\
\alpha &= -Ja\,, && (5.19e)\\
\bar{\varepsilon}^- &= \tfrac{1}{4\sqrt{2}}\Big[(\bar{\varepsilon}_2^-\gamma_a\varepsilon_1^-)\bar{\varphi}\gamma^{0a}-3(\bar{\varepsilon}_2^-\gamma^0\varepsilon_1^-)\bar{\varphi}\\
&\quad-\tfrac{1}{2}(\bar{\varepsilon}_2^-\gamma_{ab}\varepsilon_1^-)\bar{\varphi}\gamma^{0ab}+(\bar{\varepsilon}_2^-\gamma_{0a}\varepsilon_1^-)\bar{\varphi}\gamma^a\Big] && (5.19f)\\
&= \tfrac{1}{2\sqrt{2}}\Big[(\bar{\varepsilon}_1^-\gamma^0\varphi)\bar{\varepsilon}_2^- -(\bar{\varepsilon}_2^-\gamma^0\varphi)\bar{\varepsilon}_1^- -2(\bar{\varepsilon}_2^-\gamma^0\varepsilon_1^-)\bar{\varphi}\Big]\,.
\end{aligned}
$$

The $(+,-)$-commutator produces the following parameters

$$
\begin{aligned}
[\delta_{\varepsilon_1^+},\delta_{\varepsilon_2^-}] &= \delta_\xi+\delta_{\lambda_r}+\delta_{\lambda_b}+\delta_\sigma+\delta_\alpha+\delta_{\varepsilon^+}+\delta_{\varepsilon^-}\,,\\
K^a &:= \tfrac{1}{2}\bar{\varepsilon}_2^-\gamma^a\varepsilon_1^+\,, && (5.20a)\\
L &:= i\bar{\varepsilon}_2^-\gamma^0\gamma_5\varepsilon_1^+\,, && (5.20b)\\
\xi^\nu &= e^\nu{}_a K^a\,, && (5.20c)\\
\lambda^{ab} &= -\xi^\nu\hat{\omega}_\nu{}^{ab}+\tfrac{1}{4}\hat{D}_0 a\varepsilon^{abc}K_c+\tfrac{1}{4}L\hat{f}^{ab}\,, && (5.20d)\\
\lambda^{a} &= -\xi^\nu\hat{\omega}_\nu{}^{a}+\tfrac{1}{4}\varepsilon^{abc}\hat{f}_{0b}K_c+\tfrac{1}{4}L\hat{f}^a{}_0\,, && (5.20e)\\
\sigma &= -\xi^\nu m_\nu\,, && (5.20f)\\
\alpha &= -\xi^\nu a_\nu+L\,, && (5.20g)\\
\bar{\varepsilon}^+ &= -\xi^\nu\bar{\psi}_\nu^+ +\tfrac{1}{4\sqrt{2}}\Big[(\bar{\varepsilon}_2^-\varepsilon_1^+)\bar{\varphi}\gamma^0+(\bar{\varepsilon}_2^-\gamma_a\varepsilon_1^+)\bar{\varphi}\gamma^{a0}+(\bar{\varepsilon}_2^-\gamma^0\varepsilon_1^+)\bar{\varphi}\\
&\quad-2(\bar{\varepsilon}_2^-\gamma^0\gamma_5\varepsilon_1^+)\bar{\varphi}\gamma_5+(\bar{\varepsilon}_2^-\gamma_5\varepsilon_1^+)\bar{\varphi}\gamma_5\gamma^0\Big] && (5.20h)\\
&= -\xi^\nu\bar{\psi}_\nu^+ +\tfrac{1}{2\sqrt{2}}\Big[(\bar{\varepsilon}_2^-\gamma^0\varphi)\bar{\varepsilon}_1^+ +(\bar{\varepsilon}_1^+\gamma^a\varphi)\bar{\varepsilon}_2^-\gamma_a\gamma^0\Big]\,,\\
\bar{\varepsilon}^- &= -\xi^\nu\bar{\psi}_\nu^-\,. && (5.20i)
\end{aligned}
$$

As expected, the bosonic part of the algebra closes under supersymmetry without requiring additional constraints. When acting on fermions the algebra closes modulo fermionic equations of motion and possibly other fermionic constraints. If there is an action, these *constraints* can in fact be thought of as zilch symmetries of the action (see open/closed algebras in sec. 3.1.3), i.e. symmetries with vanishing transformation rules for fields satisfying the equations of motion. The fermionic equations of motion and other fermionic constraints are not computed explicitly for the full theory, but will be found and discussed for the linearised theory in the next chapter in section 6.2.3 by checking the algebra closure on the fermions.

5.2.3 Bianchi Identities

The following Bianchi identities lead to a set of bosonic equations of motion and constraints for the connections. From

$$\hat{D}_{[\mu}\hat{r}_{\nu\rho]}(M) = 0\,, \tag{5.21}$$

we get the constraints

$$\hat{r}_{[ab}(G_{c]}) = 0\,, \quad \hat{r}_{0[a}(G_{b]}) = 0\,, \tag{5.22}$$

from the Bianchi identity

$$\hat{D}_{[\mu}\hat{r}_{\nu\rho]}(P^a) = 0\,, \tag{5.23}$$

we get

$$\hat{r}_{[ab}(J_{c]d}) = 0\,, \quad 2\hat{r}_{0[a}(J_{b]c}) - \hat{r}_{ab}(G_c) = 0\,. \tag{5.24}$$

Combining the above we additionally get

$$\hat{r}_{0[a}(J_{bc]}) = 0\,, \qquad \hat{r}_{a0}(J_{bc}) - \hat{r}_{bc}(G_a) = 0\,. \tag{5.25}$$

Note that even though the conventional constraints have been imposed before in section 5.2.1, their Bianchi identities give rise to new constraints on the curvatures of rotations and boosts. These constraints could, of course, be varied once more to see if new constraints emerge. The next generation of constraints will be one order higher in derivatives and is expected to be a linear combination of derivatives of constraints. The same argument holds for the Bianchi identities that have not been mentioned. Of course one can write such an identity for every supercovariant curvature, which will produce higher order derivative constraints. This shall not be made explicit here.

5.2.4 Constraints from Null Killing Vector $\aleph$

The first constraint coming from the null Killing vector along which the reduction was performed was already imposed in the end of chapter 4 (see eq. (4.31)). This constraint, however, needs to be varied to possibly yield further constraints. As already mentioned, after deriving the gravitino transformation rule of $\hat{\psi}^+_\mu$ the variation $\delta\varphi^+$ appears and prevents us from successfully defining a covariant curvature. Imposing

$$0 = \delta\varphi^+ = -\tfrac{1}{2\sqrt{2}}(\bar{\varepsilon}^+\gamma^a\varphi)\gamma_a\gamma^0\varphi + \tfrac{1}{8}\hat{\tau}_{ab}\gamma^{ab}\varepsilon^+ + \tfrac{6i}{16}\hat{D}_a a\gamma^a\gamma_5\gamma^0\varepsilon^+ \tag{5.26}$$

rids us of this problem and additionally produces two constraints. One eliminates the spinor φ, while the other one relates torsion and the covariant derivative of the scalar a. This can be seen by analysing the gamma matrix structure of the equation.

To make this structure visible, the first terms is split using Fierz relations (3.26) to pull the parameter out of the spinor bilinear, and the second and third terms need to be rewritten using gamma duality relations (B.12). By varying the constraints, more constraints can be found as is shown in figure 5.1.

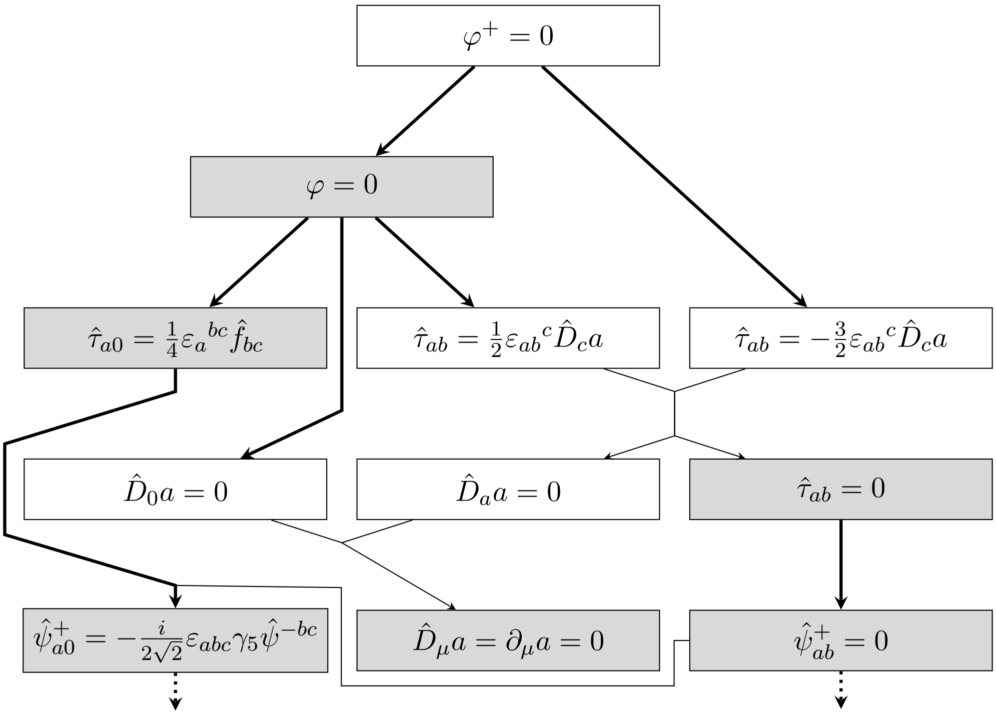

Figure 5.1: The evolution of constraints is made visible here. In this diagram, line n has all constraints of lines $< n$ imposed. Each thick arrow stands for a variation on the constraint it comes from. The thin arrows point to constraints coming from linear combinations of other constraints. The dotted arrows at the end indicate the higher order derivative constraints that could be obtained by further varying the gravitino curvatures. Gray shading highlights the constraints that are useful to simplify the multiplet.

In figure 5.1 the grey constraints simplify the transformation rules of most fields obtained in the previous section and allow us to eliminate $\hat{\tau}_{\mu\nu}$ and $\hat{\psi}^+_{\mu\nu}$ from the theory completely, replacing them by $\hat{f}_{\mu\nu}$ and $\hat{\psi}^-_{\mu\nu}$. An interesting observation can be made here, distinguishing this theory from the results obtained from the reduction 4D $\to$ 3D: Torsion is not entirely set to zero, but only its spatial components. This constraint is referred to as twistless-torsion in the literature and it is the minimal requirement for a spacetime to foliate according to Frobenius' theorem. It is also noteworthy that the covariant derivative of the scalar a is reduced to the simple partial derivative by setting the spinor φ to zero. This makes a a constant.

5.2.5 Removing the Remaining Constant Shift Symmetry

As announced before the constant shift symmetry with parameter k_α is also removed from the theory

$$k_\alpha = 0\,. \tag{5.27}$$

This however is a choice and not a direct consequence of the null reduction or a condition necessary to close the algebra. This choice is convenient and the symmetry is presumed to not have a deeper meaning than being a relic from the null reduction of a 5D U(1) gauge transformation. Since the symmetry is constant it cannot appear in any supercovariant curvatures or derivatives. Therefore, it also does not appear in the constraints presented above. In fact, it does not appear anywhere in the theory except for the transformation rules of a_μ and a, the 5D graviphoton descendent fields. These are some of the motivating reasons to remove this symmetry.

5.3 Final Multiplet

After including the constraints from the last section, the multiplet is now presented in its reduced form

$$\delta e_\mu{}^a = \mathcal{L}_\xi e_\mu{}^a - \lambda^a{}_b e_\mu{}^b - \lambda^a \tau_\mu + \tfrac{1}{2}\bar\varepsilon^+\gamma^a\psi_\mu^- + \tfrac{1}{2}\bar\varepsilon^-\gamma^a\psi_\mu^+\,, \tag{5.28a}$$

$$\delta\tau_\mu = \mathcal{L}_\xi \tau_\mu + \tfrac{1}{\sqrt{2}}\bar\varepsilon^+\gamma^0\psi_\mu^+\,, \tag{5.28b}$$

$$\delta m_\mu = \mathcal{L}_\xi m_\mu + \partial_\mu\sigma + \lambda_b e_\mu{}^b - \tfrac{1}{\sqrt{2}}\bar\varepsilon^-\gamma^0\psi_\mu^-\,, \tag{5.28c}$$

$$\begin{aligned}\delta a_\mu &= \mathcal{L}_\xi a_\mu + \partial_\mu\alpha + a\lambda_b e_\mu{}^b \\ &\quad + i\bar\varepsilon^+\gamma_5\gamma^0\psi_\mu^- - i\bar\varepsilon^-\gamma_5\gamma^0\psi_\mu^+ - a\tfrac{1}{\sqrt{2}}\bar\varepsilon^-\gamma^0\psi_\mu^-\,,\end{aligned} \tag{5.28d}$$

$$\begin{aligned}\delta\psi_\mu^- &= \mathcal{L}_\xi\psi_\mu^- + \partial_\mu\varepsilon^- - \tfrac{1}{4}\lambda^{ab}\gamma_{ab}\psi_\mu^- + \tfrac{1}{4}\hat\omega_\mu{}^{ab}\gamma_{ab}\varepsilon^- \\ &\quad - \tfrac{1}{\sqrt{2}}\lambda^a\gamma_a\gamma^0\psi_\mu^+ + \tfrac{1}{\sqrt{2}}\hat\omega_\mu{}^a\gamma_a\gamma^0\varepsilon^+ \\ &\quad - \tfrac{i}{8}\Big[\hat f_{\nu\rho}\Big(e_{\mu a}e^\nu{}_b e^\rho{}_c\gamma^{abc} - \tau_\mu\tau^\nu e^\rho{}_a\gamma^a - 2\delta_\mu^\nu e^\rho{}_a\gamma^a\Big)\gamma_5\gamma^0\varepsilon^- \\ &\quad + \sqrt{2}\hat f_{\nu\rho}\Big(- e_{\mu a}e^\nu{}_b\tau^\rho\gamma^{ab} + 2\delta_\mu^\nu\tau^\rho\Big)\gamma_5\varepsilon^+\Big]\,,\end{aligned} \tag{5.28e}$$

$$\begin{aligned}\delta\psi_\mu^+ &= \mathcal{L}_\xi\psi_\mu^+ + \partial_\mu\varepsilon^+ - \tfrac{1}{4}\lambda^{ab}\gamma_{ab}\psi_\mu^+ + \tfrac{1}{4}\hat\omega_\mu{}^{ab}\gamma_{ab}\varepsilon^+ \\ &\quad - \tfrac{i}{8}\Big[\hat f_{\nu\rho}\Big(- e_{\mu a}e^\nu{}_b e^\rho{}_c\gamma^{abc} - \tau_\mu\tau^\nu e^\rho{}_a\gamma^a + 2\delta_\mu^\nu e^\rho{}_a\gamma^a\Big)\gamma_5\gamma^0\varepsilon^+ \\ &\quad - \sqrt{2}\tau_\mu\hat f_{ab}\gamma^{ab}\gamma_5\varepsilon^-\Big]\,.\end{aligned} \tag{5.28f}$$

As stated before, the constraints were used to eliminate $\hat\tau_{\mu\nu}$ in favour of $\hat f_{\mu\nu}$. Also, it is interesting to note that the scalar a now only appears as a constant in the transformation rule of the graviphoton. The explicit supercovariant gravitino curvatures $\hat r_{\mu\nu}(Q^\pm) \equiv \hat\psi^\pm_{\mu\nu}$ can be found in appendix C.

Chapter 6

Linearised Non-Relativistic 4D, $\mathcal{N} = 2$ Pure Supergravity

An interesting addition to the multiplet presented at the end of the previous chapter is the set of equations of motion and constraints for the gravitini. In principle these could be found by calculating commutators of symmetry transformations on the gravitini and seeing what has to hold, on top of the regular symmetries, to have algebra closure. These computations are, however, long and cumbersome due to a large amount of terms coming from Fierz relations. The computation becomes more managable in the linearised theory. The linearised theory simply produces non-relativistic rigid 4D, $\mathcal{N} = 2$ supersymmetry in which some of the calculations of last chapter will be repeated, additionally to the expansion of the constraint structure seen before in the full theory in section 5.2.

6.1 Linearised Multiplet

The fields are expanded in powers of a small parameter $\kappa \ll 1$

$$
\begin{aligned}
e_\mu{}^a &= \delta_\mu{}^a + \kappa h_\mu{}^a + O(\kappa^2)\,, &\text{(6.1a)}\\
\tau_\mu &= \delta_{\mu 0} + \kappa\tilde{\tau}_\mu + O(\kappa^2)\,, &\text{(6.1b)}\\
m_\mu &= \kappa\tilde{m}_\mu + O(\kappa^2)\,, &\text{(6.1c)}\\
a_\mu &= \kappa\tilde{a}_\mu + O(\kappa^2)\,, &\text{(6.1d)}\\
\psi_\mu^- &= \kappa\tilde{\psi}_\mu^- + O(\kappa^2)\,, &\text{(6.1e)}\\
\psi_\mu^+ &= \kappa\tilde{\psi}_\mu^+ + O(\kappa^2)\,. &\text{(6.1f)}
\end{aligned}
$$

All transformation parameters p are expanded in a similar fashion as $p = p^{(0)} + \kappa p^{(1)} + O(\kappa^2)$, and the following statements hold when inserting the parameter- and field decomposition into the transformations rules (5.28)

$$\xi^{(0)\nu} = \text{const}\,, \tag{6.2a}$$
$$\lambda^{(0)ab} = 0\,, \tag{6.2b}$$
$$\lambda^{(0)a} = 0\,, \tag{6.2c}$$
$$\sigma^{(0)} = \text{const}\,, \tag{6.2d}$$
$$\alpha^{(0)} = \text{const}\,, \tag{6.2e}$$
$$\varepsilon^{+(0)} = \text{const}\,, \tag{6.2f}$$
$$\varepsilon^{-(0)} = \text{const}\,. \tag{6.2g}$$

The linearisation leads to the above set of rigid symmetries. There are no rigid rotations and boosts because when inserting the linearised quantities into the transformation rules of the full theory, rotations and boosts produce terms of order $O(\kappa^0)$ that are linear in $\lambda^{(0)ab}$ and $\lambda^{(0)a}$. The other rigid symmetries exist because in these cases only the derivatives of the respective parameters appear in the terms of order $O(\kappa^0)$ that need to vanish. The extracted transformation rules of the linearised fields take the form

$$\delta h_\mu{}^a = \xi^{(0)\nu}\partial_\nu h_\mu{}^a + \partial_\mu \xi^{(1)a} - \lambda^{(1)ab}\delta_{\mu b} - \lambda^{(1)a}\delta_{\mu 0} \tag{6.3a}$$
$$+ \tfrac{1}{2}\bar{\varepsilon}^{+(0)}\gamma^a\tilde{\psi}^-_\mu + \tfrac{1}{2}\bar{\varepsilon}^{-(0)}\gamma^a\tilde{\psi}^+_\mu\,, \tag{6.3b}$$
$$\delta\tilde{\tau}_\mu = \xi^{(0)\nu}\partial_\nu\tilde{\tau}_\mu + \partial_\mu\xi^{(1)0} + \tfrac{1}{\sqrt{2}}\bar{\varepsilon}^{+(0)}\gamma^0\tilde{\psi}^+_\mu\,, \tag{6.3c}$$
$$\delta\tilde{m}_\mu = \xi^{(0)\nu}\partial_\nu\tilde{m}_\mu + \partial_\mu\sigma^{(1)} + \lambda^{(1)a}\delta_{\mu a} - \tfrac{1}{\sqrt{2}}\bar{\varepsilon}^{-(0)}\gamma^0\tilde{\psi}^-_\mu\,, \tag{6.3d}$$
$$\delta\tilde{a}_\mu = \xi^{(0)\nu}\partial_\nu\tilde{a}_\mu + \partial_\mu\alpha^{(1)} + i\bar{\varepsilon}^{+(0)}\gamma_5\gamma^0\tilde{\psi}^-_\mu - i\bar{\varepsilon}^{-(0)}\gamma_5\gamma^0\tilde{\psi}^+_\mu\,, \tag{6.3e}$$
$$\begin{aligned}\delta\tilde{\psi}^-_\mu &= \xi^{(0)\nu}\partial_\nu\tilde{\psi}^-_\mu + \partial_\mu\varepsilon^{-(1)} + \tfrac{1}{4}\tilde{\hat{\omega}}_\mu{}^{ab}\gamma_{ab}\varepsilon^{-(0)} + \tfrac{1}{\sqrt{2}}\tilde{\hat{\omega}}_\mu{}^a\gamma_a\gamma^0\varepsilon^{+(0)}\\ &\quad - \tfrac{i}{8}\Big[\Big(-3\delta_{\mu 0}\tilde{\hat{f}}_{a0}\gamma^{a0}\gamma_5 + i\varepsilon^{abc}\delta_{\mu a}\tilde{\hat{f}}_{bc} + 2\delta^a_\mu\tilde{\hat{f}}_{ab}\gamma^{b0}\gamma_5\Big)\varepsilon^{-(0)}\\ &\quad + \sqrt{2}\delta_{\mu a}\Big(-\tilde{\hat{f}}_{b0}\gamma^{ab} + 2\tilde{\hat{f}}^a{}_0\Big)\gamma_5\varepsilon^{+(0)}\Big]\,,\end{aligned} \tag{6.3f}$$
$$\begin{aligned}\delta\tilde{\psi}^+_\mu &= \xi^{(0)\nu}\partial_\nu\tilde{\psi}^+_\mu + \partial_\mu\varepsilon^{+(1)} + \tfrac{1}{4}\tilde{\hat{\omega}}_\mu{}^{ab}\gamma_{ab}\varepsilon^{+(0)}\\ &\quad - \tfrac{i}{8}\Big[\Big(\delta_{\mu 0}\tilde{\hat{f}}_{a0}\gamma^{a0}\gamma_5 - i\varepsilon^{abc}\delta_{\mu a}\tilde{\hat{f}}_{bc} - 2\delta^a_\mu\tilde{\hat{f}}_{ab}\gamma^{b0}\gamma_5\Big)\varepsilon^{+(0)}\\ &\quad - \sqrt{2}\delta_{\mu 0}\tilde{\hat{f}}_{ab}\gamma^{ab}\gamma_5\varepsilon^{-(0)}\Big]\,,\end{aligned} \tag{6.3g}$$

where

$$\tilde{\hat{\omega}}_\mu{}^{ab} = 2\delta^{\nu[a}\partial_{[\mu}h_{\nu]}{}^{b]} - \delta_{\mu c}\partial^{[a}h^{b]c} - \delta_{\mu 0}\partial^{[a}\tilde{m}^{b]}\,, \tag{6.4a}$$
$$\tilde{\hat{\omega}}_\mu{}^a = \partial_{[0}h_{\mu]}{}^a + \delta_{\mu b}\delta^{\nu a}\partial_{[0}h_{\nu]}{}^b + \delta^{\nu a}\partial_{[\mu}\tilde{m}_{\nu]} + \delta_{\mu 0}\delta^{\nu a}\partial_{[0}\tilde{m}_{\nu]}\,, \tag{6.4b}$$
$$\tilde{\hat{f}}_{\mu\nu} = 2\partial_{[\mu}\tilde{a}_{\nu]}\,. \tag{6.4c}$$

The explicit transformation rules of the linearised spin- and boost connections and the linearised supercovariant field strength can be found in appendix C. It can be seen here explicitly that the richness of symmetry transformations of the full theory reduces to simple U(1) gauge symmetries for the first order components of diffeomorphisms and supersymmetry transformations. The symmetries of rotations and boosts also becomes trivial, since the fields do not rotate or boost to themselves anymore. Diffeomorphisms simply become rigid translations and the only interesting transformation structure that remains is supersymmetry, which also becomes rigid.

6.2 Linearised Constraints

In this section linearised constraints will be discussed. The constraints found in the previous chapter in section 5.2 are not reproduced here, but are to be understood as already imposed. It is necessary to repeat the computation of the parameters produced by a commutator of transformations in the linearised version to find the equations of motion and constraints on the gravitini. Then the previously found gravitino curvature constraints are varied to yield new constraints before the newly found gravitino equations of motion and constraints are also varied.

6.2.1 Algebra Closure

First, the parameters produced by acting with the commutators of rigid supersymmetry transformations on the fields are presented. Since the procedure is equivalent to the computation in the full theory, it will not be discussed further here. At first glance it may seem unnecessary to repeat the computation of these parameters, but since the symmetry structure has changed due to the linearisation, the simplest way to obtain the parameters is to repeat the calculation of commutators using the linearised transformation rules. It is not possible to obtain the correct parameters from naïvely *linearising* the parameters (5.18-5.20) of the full theory. For the computation of the commutator of transformations on the gravitini to find equations of motion and constraints it is vital to have the correct linearised parameters at hand.

Calculating the SUSY-transformation commutators, one finds that the $(+,+)$-commutator produces the following non-zero parameters

$$
\begin{aligned}
[\delta_{\varepsilon_1^{+(0)}}, \delta_{\varepsilon_2^{+(0)}}] &= \delta_{\xi^{(0)}} + \delta_{\xi^{(1)}} + \delta_{\lambda_r^{(1)}} + \delta_{\lambda_b^{(1)}} + \delta_{\sigma^{(1)}} + \delta_{\alpha^{(1)}} + \delta_{\varepsilon^{\pm(1)}} \,, \\
K^{(0)} &:= \tfrac{1}{\sqrt{2}}\bar{\varepsilon}_2^{+(0)}\gamma^0\varepsilon_1^{+(0)} \,, && (6.5a)\\
\xi^{(0)\nu} &= K^{(0)}\delta^{\nu 0} \,, && (6.5b)\\
\xi^{(1)a} &= -\,\xi^{(0)\nu} h_\nu{}^a \,, && (6.5c)\\
\xi^{(1)0} &= -\,\xi^{(0)\nu}\tilde{\tau}_\nu \,, && (6.5d)\\
\lambda^{(1)ab} &= -\,\xi^{(0)\nu}\tilde{\hat{\omega}}_\nu{}^{ab} - \tfrac{1}{4}\varepsilon^{abc}\tilde{\hat{f}}_{c0}K^{(0)} \,, && (6.5e)\\
\lambda^{(1)a} &= -\,\xi^{(0)\nu}\tilde{\hat{\omega}}_\nu{}^a \,, && (6.5f)\\
\sigma^{(1)} &= -\,\xi^{(0)\nu}\tilde{m}_\nu \,, && (6.5g)\\
\alpha^{(1)} &= -\,\xi^{(0)\nu}\tilde{a}_\nu \,, && (6.5h)\\
\varepsilon^{\pm(1)} &= -\,\xi^{(0)\nu}\tilde{\psi}_\nu^\pm \,. && (6.5i)
\end{aligned}
$$

The $(-,-)$-commutator only produces a boost

$$
\begin{aligned}
[\delta_{\varepsilon_1^{-(0)}}, \delta_{\varepsilon_2^{-(0)}}] &= \delta_{\lambda_b^{(1)}} \,, \\
J^{(0)} &:= \tfrac{1}{\sqrt{2}}\bar{\varepsilon}_2^{-(0)}\gamma^0\varepsilon_1^{-(0)} \,, && (6.6a)\\
\lambda^{(1)a} &= -\,\tfrac{1}{4}J^{(0)}\varepsilon^{abc}\tilde{\hat{f}}_{bc} \,. && (6.6b)
\end{aligned}
$$

The $(+,-)$-commutator produces the following parameters

$$
\begin{aligned}
[\delta_{\varepsilon_1^{+(0)}}, \delta_{\varepsilon_2^{-(0)}}] &= \delta_{\xi^{(0)}} + \delta_{\xi^{(1)}} + \delta_{\lambda_r^{(1)}} + \delta_{\lambda_b^{(1)}} + \delta_{\sigma^{(1)}} + \delta_{\alpha^{(1)}} + \delta_{\varepsilon^{\pm(1)}} \,, \\
K^{(0)a} &:= \tfrac{1}{2}\bar{\varepsilon}_2^{-(0)}\gamma^a\varepsilon_1^{+(0)} \,, && (6.7a)\\
L^{(0)} &:= i\bar{\varepsilon}_2^{-(0)}\gamma^0\gamma_5\varepsilon_1^{+(0)} \,, && (6.7b)\\
\xi^{(0)\nu} &= K^{(0)a}\delta_a^\nu \,, && (6.7c)\\
\xi^{(1)a} &= -\,\xi^{(0)\nu} h_\nu{}^a \,, && (6.7d)\\
\xi^{(1)0} &= -\,\xi^{(0)\nu}\tilde{\tau}_\nu \,, && (6.7e)\\
\lambda^{(1)ab} &= -\,\xi^{(0)\nu}\tilde{\hat{\omega}}_\nu{}^{ab} + \tfrac{1}{4}L^{(0)}\tilde{\hat{f}}^{ab} \,, && (6.7f)\\
\lambda^{(1)a} &= -\,\xi^{(0)\nu}\tilde{\hat{\omega}}_\nu{}^a + \tfrac{1}{4}\varepsilon^{abc}\tilde{\hat{f}}_{0b}K_c^{(0)} + \tfrac{1}{4}L^{(0)}\tilde{\hat{f}}^a{}_0 \,, && (6.7g)\\
\sigma^{(1)} &= -\,\xi^{(0)\nu}\tilde{m}_\nu \,, && (6.7h)\\
\alpha^{(1)} &= -\,\xi^{(0)\nu}\tilde{a}_\nu \,, && (6.7i)\\
\varepsilon^{\pm(1)} &= -\,\xi^{(0)\nu}\tilde{\psi}_\nu^\pm \,. && (6.7j)
\end{aligned}
$$

The difference at this point, as can be guessed from the inclusion of the parameters $\varepsilon^{\pm(1)}$, is that the commutator was also calculated on the gravitini. The commutator of transformations acting on a fermion does not only produce symmetry transformations of the transformation rule of the respective fermion. It additionally yields terms that vanish when fermionic equations of motion and constraints are imposed

$$\begin{aligned}[\delta_1, \delta_2] \stackrel{!}{=} \delta_{\xi^{(1)}} + \delta_{\lambda_r^{(1)}} + \delta_{\lambda_b^{(1)}} + \delta_{\sigma^{(1)}} + \delta_{\alpha^{(1)}} + \delta_{\varepsilon^{\pm(1)}} \\ + \delta_{\xi^{(0)}} + \delta_{\varepsilon^{\pm(0)}} + \{\text{fermionic EOM \& -constraints}\} \,. \end{aligned} \tag{6.8}$$

If there is an action, these terms can be seen as zilch symmetries of this action, so in this case the statement *"The commutator of two symmetry transformations is equal to a linear combination of symmetry transformations of the theory"*, still holds. It is this calculation that we will get the following fermionic equations of motion and constraints from

$$\gamma^{[ab}\tilde{\hat{\psi}}^{-c]}{}_0 = 0 \quad \rightarrow \quad \gamma^a \tilde{\hat{\psi}}^-_{a0} = 0 \,, \tag{6.9a}$$

$$\gamma^{[a}\tilde{\hat{\psi}}^{-bc]} = 0 \quad \rightarrow \quad \gamma^{ab} \tilde{\hat{\psi}}^-_{ab} = 0 \,. \tag{6.9b}$$

The first equation will be referred to as the fermionic equation of motion because it includes a time derivative on the gravitino, while the second one is a geometric constraint. In each line, the second equations are obtained by contracting the respective first equation with gamma matrices.

6.2.2 Further Constraints from Null Killing Vector ℵ

Now the constraint structure of the previous chapter will be expanded by the variations of the gravitino curvatures that were at the very end of the constraint structure in figure 5.1. In the following process all constraints from the previous chapter, section 5.2, are imposed in their linearised versions. Relations coming from Bianchi identities (see sec. 5.2.3) are used frequently to eliminate or reorganise terms produced by the variations. Additionally, the linearised Bianchi identity

$$\partial_{[\mu}\tilde{\hat{f}}_{\nu\rho]} = 0 \,, \tag{6.10}$$

is needed for simplifications in the following figure 6.1.

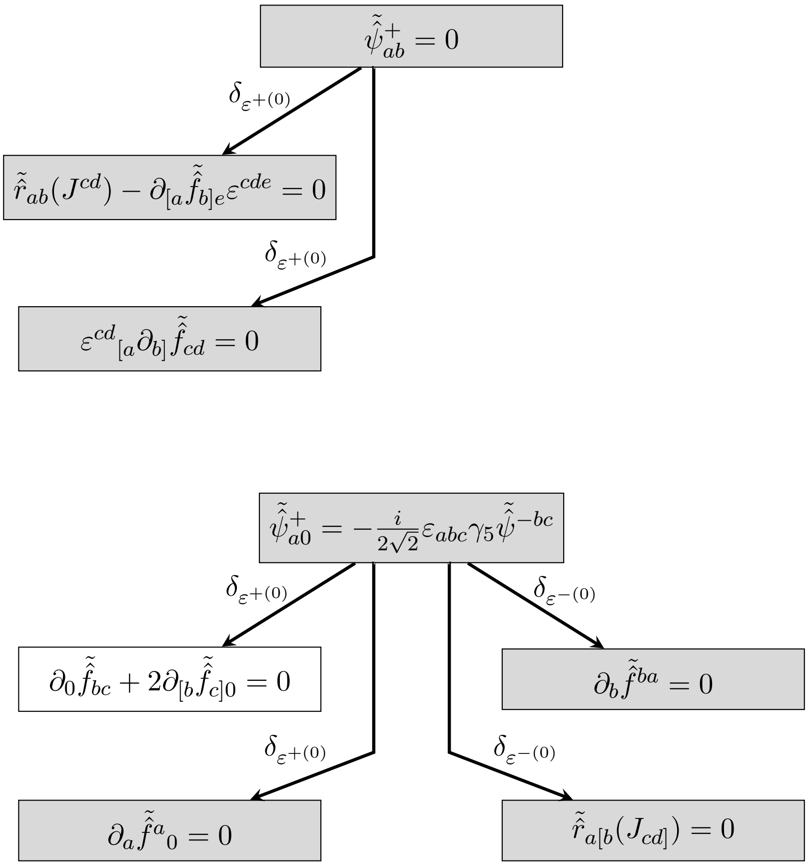

Figure 6.1: The evolution of the vanishing gravitino curvature constraint and the constraint relating gravitino curvatures is made visible here. Each thick arrow stands for a variation on the constraint it comes from. All constraints introduced in the previous chapter and the previous section (sec. 5.2, 6.2.1) are imposed additionally to the ones from previous lines. Bianchi identities are used to simplify expressions. Gray shading highlights final constraints, while a white background marks constraints that become redundant after all constraints have been imposed.

6.2.3 Gravitino Equations of Motion and Constraints

In section 6.2.1 the equations of motion and constraints (6.9) on the gravitini were computed using the closure of the algebra. Now they will be varied to produce new constraints which is illustrated in the following figure 6.2.

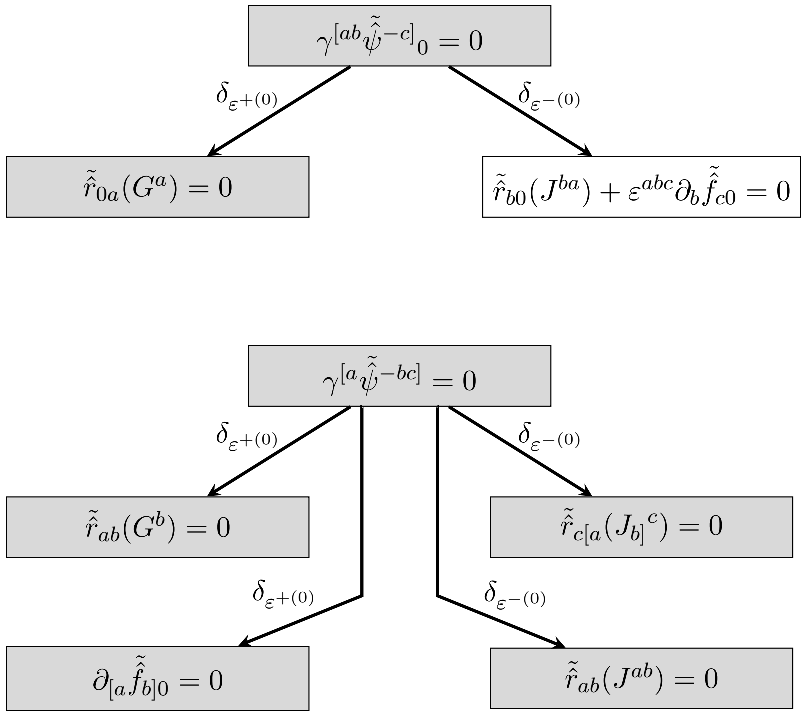

Figure 6.2: The evolution of the gravitino equations of motion and constraints is made visible here. Each thick arrow stands for a variation on the constraint it comes from. All constraints introduced in the previous chapter and the previous sections (sec. 5.2, 6.2.1, 6.2.2) are imposed additionally to the ones from previous lines. Bianchi identities are used to simplify expressions. Gray shading highlights final constraints, while a white background marks constraints that become redundant after all constraints have been imposed.

Combining constraints of figures 6.1 and 6.2 two further constraints

$$\partial_0 \tilde{\tilde{f}}_{ab} = 0\,, \qquad \tilde{\tilde{r}}_{b0}(J^{ba}) = 0\,, \tag{6.11}$$

are obtained by linear combination.

Chapter 7

Conclusion and Outlook

In this thesis the method of gauging an algebra was introduced to motivate the Vielbein formalism of general relativity and Newton-Cartan gravity, as well as their supersymmetric generalisations, supergravity and non-relativistic supergravity respectively. The mathematical tools to obtain non-relativistic physics from the non-relativistic limit of a relativistic theory were discussed. Moreover null reduction, the fundamental ingredient for the results of this thesis, was introduced and then put to use to derive a 4-dimensional non-relativistic $\mathcal{N} = 2$ pure supergravity multiplet and its transformation rules from 5-dimensional relativistic $\mathcal{N} = 2$ pure supergravity. This theory had not been constructed before. It however shares a lot of similarity with the 3-dimensional non-relativistic $\mathcal{N} = 2$ supergravity multiplet, that similarly was derived by null-reducing 4-dimensional relativistic $\mathcal{N} = 1$ supergravity [42], or equivalently by gauging the super-Poincaré algebra [4]. When removing the non-relativistic graviphoton field $a_\mu = 0$, the transformation rules of the 4D multiplet are the same as the ones known from the 3D multiplet. The inclusion of the graviphoton really is the novelty of the results obtained in this thesis. To work with a complete relativistic on-shell multiplet in 5D, $\mathcal{N} = 2$ it is necessary to include the relativistic graviphoton to even out the unequal bosonic (Vielbein) and fermionic (gravitini) degrees of freedom of the gauge fields corresponding to spacetime symmetries. A difference to previous results obtained in 3D is the non-vanishing torsional contribution of the field strength of the graviphoton. In 3D the twistless-torsion constraint implies no-torsion by variation, thereby also removing the gravitino curvature $\hat{\psi}^+_{\mu\nu} = 0$, while in the 4D theory the rich constraint structure discussed in section 5.2 does put restrictions on the gravitino curvatures, on the graviphoton field strength and on torsion but does not set any of them to zero. After imposing twistless-torsion the remaining torsional components can be related to the field strength, as is the case for the remaining components of the gravitino curvature $\hat{\psi}^+_{a0}$ that can be related to components of $\hat{\psi}^-_{\mu\nu}$, thereby eliminating $\tau_{\mu\nu}$ and $\hat{\psi}^+_{\mu\nu}$ in favour of $\hat{f}_{\mu\nu}$ and $\hat{\psi}^-_{\mu\nu}$ respectively from the theory. It can be assumed that beyond the graviphoton the gauge fields, which do not correspond to a spacetime symmetry, of higher dimensional or extended supergravity can also produce torsional contributions.

Since non-relativistic supergravity theories are still in their infancy, there are a lot of directions that possible future research could take. The calculation of partially gauge fixing the non-relativistic 4D, $\mathcal{N}=2$ supergravity multiplet to obtain an analogue of Galilean supergravity, including a graviphoton contribution, was omitted in this thesis. Finding such a partial gauge fix, similarly to what was done in [4], could provide further insight into the gauge fixing structure of higher dimensional supergravity. Another interesting extension of the work in this thesis would be to derive a non-relativistic 4D off-shell multiplet, as has been worked on in three dimension by [14], [42], [67]. Also higher supersymmetric extensions could be considered for null reduction in different dimensions to obtain higher supersymmetric extensions of non-relativistic supergravity. Since in this thesis only pure supergravity was ever discussed an obvious generalisation would be to consider relativistic supersymmetric matter-coupled theories to derive non-relativistic counterparts. Another possible direction for further work is to consider gauge theories of different algebras than the super-Bargmann algebra, like the super-Carroll algebra (describing ultra-relativistic supersymmetry) or the superconformal algebra (extending super-Poincaré symmetry by dilatations and special conformal symmetry) for which one could attempt the reduction to the non-relativistic super-Schrödinger algebra.

Appendix A

Conventions

A.1 General Conventions

- The mostly plus signature $(-,+,+,\dots)$ is used for the metric.
- (Anti-)symmetrisation is defined as

$$A_{[\mu_1\dots\mu_n]} = \frac{1}{n!}\sum_{\sigma\in S_n} \mathrm{sgn}(\sigma) A_{\mu_{\sigma(1)}\cdots\mu_{\sigma(n)}}\ , \tag{A.1}$$

$$A_{(\mu_1\dots\mu_n)} = \frac{1}{n!}\sum_{\sigma\in S_n} A_{\mu_{\sigma(1)}\cdots\mu_{\sigma(n)}}\ , \tag{A.2}$$

  where S_n is the symmetric group of degree n, i.e. containing all permutations of n objects and $\mathrm{sgn}(n)$ gives a positive sign for even- and a negative sign for odd permutations. The complete (anti-)symmetrisation is performed with a total weight of 1.

- The Levi-Civita symbol in $D=(D-1,1)$ spacetime dimensions is defined as:

$$\varepsilon_{01\dots D-1} = 1\ , \qquad \varepsilon^{01\dots D-1} = -1\ , \tag{A.3}$$

$$\text{e.g. in 5D: } \varepsilon^{abc0} = \varepsilon^{abc}\ . \tag{A.4}$$

- The highest rank element in $D=4$ is denoted by γ_5.
- Einstein summation convention: Indices appearing twice in a product, once as sub- and once as superscript, are summed over, e.g. $A^M B_M$. Underlined indices indicate an exception and are not to be summed over, e.g. $A^M B_{\underline{M}}$.
- Hats are used to indicate supercovariant(-ised) objects, e.g. $\hat{R}_{MN}$ denotes supercovariant curvatures.
- Tildes are used to indicate the first order terms in the linearisation of fields, e.g. $\tilde{a}_\mu$ is the linearised gravi-photon.
- The bracket function $[\cdot]$ cuts off all decimal places of its argument.

A.2 Indices

- $M, N, P, \Sigma, \ldots$ denote spacetime (curved) indices on a relativistic spacetime.
- $A, B, C, D, \ldots$ denote local frame (flat) indices on a relativistic spacetime.
- $\mu, \nu, \rho, \sigma, \ldots$ denote spacetime (curved) indices on a non-relativistic spacetime.
- $a, b, c, d, \ldots$ denote local spatial frame (flat) indices on a non-relativistic spacetime.
- 0 denotes a local temporal frame (flat) index on a non-relativistic spacetime, except when on a Clifford algebra element. It is only defined as a subscript index.
- $5, 0$ denotes a fixed index $A = 5, 0$ when on a Clifford algebra element.
- $\alpha, \beta, \gamma, \delta, \ldots$ denote spinor indices.
- $i, j, k, \ldots$ denote the numbering of spinors in supersymmetric extensions $1 \leq i \leq \mathcal{N}$ when on a gravitino, and denote abstract representation indices when on abstract representation matrices and respective abstract fields transforming with them.
- $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}, \ldots$ denote algebra indices, i.e. $\theta = \theta^{\boldsymbol{A}} T_{\boldsymbol{A}}$ is an algebra element, where $T_{\boldsymbol{A}}$ are the generators of the algebra.
- $\emptyset$ denotes the fixed index $\mu = 0$. This notation is used for partial gauge fixing. In the linearised theory the notation becomes redundant $\emptyset \equiv 0$.
- (r, D, n) denotes the rank of a Clifford algebra basis element when on a capital gamma, denotes the spacetime dimension when on the charge conjugation matrix, a gamma matrix, the Minkowski metric or the d'Alembertian, denotes the Fourier mode when on an expanded field, denotes the order of the term of a parameter in the linearised theory, and denotes the grading when on algebra generators. It only appears as superscript.
- R, L denote the chirality of a spinor or the respective chiral projector.

In the process of null reduction (see sec. 4.3) indices are split up as

- $A = (a, -, +)$, where $a = 1, \ldots D-2$ and $\mp = \frac{1}{\sqrt{2}}(0 \mp D-1)$. Hence the Minkowski metric in null coordinates has elements $\eta_{+-} = -1$.
- $M = (\mu, z)$.

A.3 Capitalisation

- Capital letters are used to denote relativistic fields, parameters, curvatures and indices.
- Lower case letters are used to denote non-relativistic fields, parameters, curvatures and indices.
- Capital bold letters are used to denote group elements and algebra indices.
- Capital letters are used to denote algebra generators.
- Lower case letters are used to denote algebra representations.

For the largest part of this thesis the the convention *upper- vs. lower case quantities* $\rightarrow$ *relativistic- vs. non-relativistic theory* coincides with *5D vs. 4D*.

A.4 Abbreviations

- SR and GR: special relativity and general relativity,
- NC: Newton-Cartan,
- (non-)rel.: (non-)relativistic,
- SUSY and SUGRA: supersymmetry and supergravity,
- gct and lt: general coordinate transformations and local translations (**not** Lorentz transformations),
- (S)MS: (symplectic) Majorana spinors,
- EOM: equations of motion,
- DOF: degrees of freedom,
- ch., sec., app., tab. and eq.: chapter, section, appendix, table and equation.

A.5 Parameters of Infinitesimal Transformations

General coordinate transformations act on coordinates x^M, a vector field $A^M(x)$ and a spinor field $\Psi(x)$ as follows

$$\delta_\Xi x^M = -\Xi^M(x) \,, \tag{A.5a}$$
$$\delta_\Xi A^M(x) = \mathcal{L}_\Xi A^M(x) \,, \tag{A.5b}$$
$$\delta_\Xi \Psi(x) = 0 \,. \tag{A.5c}$$

Local Lorentz transformations act on coordinates x^M, a vector field $A^M(x)$ and a spinor field $\Psi(x)$ as follows

$$\delta_\Lambda x^A = -\Lambda^A{}_B(x) x^B \,, \tag{A.6a}$$

$$\delta_\Lambda A^A(x) = -\Lambda^A{}_B(x) A^B(x) \,, \tag{A.6b}$$

$$\delta_\Lambda \Psi(x) = -\tfrac{1}{4}\Lambda^{AB}(x)\gamma_{AB}\Psi(x) \,. \tag{A.6c}$$

Appendix B

Calculating with Spinors and Gamma Matrices

Some additional relations concerning spinors and gamma matrices that aid the explicit computations in chapters 4-6 are presented.

B.1 Calculating with Spinors

In this section some helpful properties of spinor bilinears are denoted. Also, the explicit null reduction of relativistic 5D SMS to non-relativistic 4D MS, and spinor bilinears, are presented.

Symmetry Properties

The following generalisation of the (symplectic) Majorana flip relation allows for a more general gamma matrix structure inside the bilinear

$$\text{MS:} \qquad \bar{\psi}\Gamma^{(r_1)}\dots\Gamma^{(r_n)}\phi = (t_0)^{n-1}t_{r_1}\dots t_{r_n}\bar{\phi}\Gamma^{(r_n)}\dots\Gamma^{(r_1)}\psi\,, \tag{B.1a}$$

$$\text{SMS:} \qquad \bar{\psi}^i\Gamma^{(r_1)}\dots\Gamma^{(r_n)}\phi_i = -(t_0)^{n-1}t_{r_1}\dots t_{r_n}\bar{\phi}^i\Gamma^{(r_n)}\dots\Gamma^{(r_1)}\psi_i\,. \tag{B.1b}$$

Reality Properties

Spinor bilinears of (symplectic) Majorana spinors depending on rank r of the gamma-matrix involved are either purely real or imaginary, according to

$$\left(\bar{\Psi}\Gamma^{(r)}\Phi\right)^* = (-t_0t_1)^{r+1}\,\bar{\Psi}\Gamma^{(r)}\Phi\,. \tag{B.2}$$

This means that, depending on the product t_0t_1, either all spinor bilinears are real or every other one (in terms of rank r) is imaginary, as is the case in $D = 5$. Additionally, in even dimensions the reality property of the (S)MS spinor bilinear, including the highest rank element, can be determined

$$\left(\bar{\Psi}\gamma_*\Phi\right)^* = (-t_0t_1)(-1)^{1+D/2}\,\bar{\Psi}\gamma_*\Phi\,. \tag{B.3}$$

Some Explicit Relations between 5D and 4D Gravitini

In section 4.3 the non-relativistic 4D Majorana gravitini were expressed in terms of relativistic 5D symplectic Majorana gravitini. For actual computations it is practical to also have the expression of 5D gravitini in terms of 4D gravitini. This can be obtained by using the projectors $P_{\pm} = \frac{1}{2}(1 \pm \gamma_5\gamma^0)$

$$\Psi^i_\mu = \tfrac{1}{\sqrt{2}}[P_+(1-(-1)^i\gamma^0)(\psi^-_\mu - m_\mu\varphi) + P_-(1+(-1)^i\gamma^0)\psi^+_\mu]\,, \tag{B.4a}$$
$$\Psi^i_z = \tfrac{1}{\sqrt{2}}P_+(1-(-1)^i\gamma^0)\varphi\,, \tag{B.4b}$$
$$\bar{\Psi}^i_\mu = \tfrac{1}{\sqrt{2}}[(\bar{\psi}^-_\mu - m_\mu\bar{\varphi})(1+(-1)^i\gamma^0)\gamma_5 P_- + \bar{\psi}^+_\mu(1-(-1)^i\gamma^0)\gamma_5 P_+]\,, \tag{B.4c}$$
$$\bar{\Psi}^i_z = \tfrac{1}{\sqrt{2}}\bar{\varphi}(1+(-1)^i\gamma^0)\gamma_5 P_-\,, \tag{B.4d}$$

where the barred spinors are conjugated in the respective dimension, using the appropriate conjugation matrix. The composition of 5D SUSY parameters $\mathcal{E}$ in terms of 4D parameters $\varepsilon^{\pm}$ is analogous. In the above relation the spinor φ^+ does not appear because it has already been set to zero, as the null reduction demands.

The following lists of 5D gravitino bilinear and bilinears of parameters and gravitini in terms of 4D quantities (again $\varphi^+ = 0$) are used for the explicit null reduction:

$$\bar{\Psi}_\mu\Psi_\nu = 2\bar{\psi}^+_{[\mu}\gamma_5\gamma^0\left(\psi^-_{\nu]} - m_{\nu]}\varphi\right)\,, \tag{B.5a}$$
$$\bar{\Psi}_\mu\Psi_z = \bar{\psi}^+_\mu\gamma_5\gamma^0\varphi\,, \tag{B.5b}$$
$$\bar{\Psi}_\mu\gamma^a\Psi_\nu = 2\bar{\psi}^+_{[\mu}\gamma^a\left(\psi^-_{\nu]} - m_{\nu]}\varphi\right)\,, \tag{B.5c}$$
$$\bar{\Psi}_\mu\gamma^a\Psi_z = \bar{\psi}^+_\mu\gamma^a\varphi\,, \tag{B.5d}$$
$$\bar{\Psi}_\mu\gamma^+\Psi_\nu = \sqrt{2}\left(\bar{\psi}^-_\mu\gamma^0\psi^-_\nu + 2m_{[\mu}\bar{\psi}^-_{\nu]}\gamma^0\varphi\right)\,, \tag{B.5e}$$
$$\bar{\Psi}_\mu\gamma^+\Psi_z = \sqrt{2}\bar{\psi}^-_\mu\gamma^0\varphi\,, \tag{B.5f}$$
$$\bar{\Psi}_\mu\gamma^-\Psi_\nu = \sqrt{2}\bar{\psi}^+_\mu\gamma^0\psi^+_\nu\,, \tag{B.5g}$$
$$\bar{\Psi}_\mu\gamma^-\Psi_z = 0\,, \tag{B.5h}$$

$$\bar{\mathcal{E}}\Psi_\mu = \bar{\varepsilon}^+\gamma_5\gamma^0\left(\psi_\mu^- - m_\mu\varphi\right) - \bar{\varepsilon}^-\gamma_5\gamma^0\psi_\mu^+\,, \tag{B.6a}$$
$$\bar{\mathcal{E}}\Psi_z = \bar{\varepsilon}^+\gamma_5\gamma^0\varphi\,, \tag{B.6b}$$
$$\bar{\mathcal{E}}\gamma^a\Psi_\mu = \bar{\varepsilon}^-\gamma^a\psi_\mu^+ + \bar{\varepsilon}^+\gamma^a\left(\psi_\mu^- - m_\mu\varphi\right)\,, \tag{B.6c}$$
$$\bar{\mathcal{E}}\gamma^a\Psi_z = \bar{\varepsilon}^+\gamma^a\varphi\,, \tag{B.6d}$$
$$\bar{\mathcal{E}}\gamma^+\Psi_\mu = \sqrt{2}\bar{\varepsilon}^-\gamma^0\left(\psi_\mu^- - m_\mu\varphi\right)\,, \tag{B.6e}$$
$$\bar{\mathcal{E}}\gamma^+\Psi_z = \sqrt{2}\bar{\varepsilon}^-\gamma^0\varphi\,, \tag{B.6f}$$
$$\bar{\mathcal{E}}\gamma^-\Psi_\mu = \sqrt{2}\bar{\varepsilon}^+\gamma^0\psi_\mu^+\,, \tag{B.6g}$$
$$\bar{\mathcal{E}}\gamma^-\Psi_z = 0\,. \tag{B.6h}$$

B.2 Calculating with Gamma Matrices

This section provides some relations including the highest rank element, duality relations between different rank Clifford algebra elements and finally some explicit non-relativistic 4D gamma matrix relations.

Relations between γ_* and $\Gamma^{(r)}$

The highest rank element γ_* commutes with even rank Clifford algebra elements and and anticommutes with odd rank elements

$$[\gamma_*, \Gamma^{(r)}] = 0 \qquad \text{for} \qquad r = 2n\,, \tag{B.7a}$$
$$\{\gamma_*, \Gamma^{(r)}\} = 0 \qquad \text{for} \qquad r = 2n+1\,, \tag{B.7b}$$

for $n \in \mathbb{N}$.

In even spacetime dimensions $D = 2m$ there is a duality relation between rank r elements and rank $(D-r)$ elements via the highest rank element

$$\gamma^{A_1\ldots A_r}\gamma_* = -\frac{(-i)^{m+1}}{(D-r)!}\,\varepsilon^{A_r\ldots A_1 B_1\ldots B_{D-r}}\,\gamma_{B_1\ldots B_{D-r}}\,. \tag{B.8}$$

In odd spacetime dimension $D = 2m+1$ there is no independent highest rank element, which is why the rank r elements are linked with the duality relation to the rank $(D-r)$ elements

$$\gamma_\pm^{A_1\ldots A_r} = \pm\frac{i^{m+1}}{(D-r)!}\,\varepsilon^{A_1\ldots A_D}\,\gamma^\pm_{A_D\ldots A_{r+1}}\,. \tag{B.9}$$

This is why the Clifford algebra basis in $D = 2m+1$ only consists of elements of rank $r \le m$.

Gamma Matrices in 4D Newton-Cartan Theory

In the following discussion some explicit relations for non-relativistic 4D gamma matrix structures are presented, which are used mainly when working with Fierz relations (see eq. 3.26). The difference between the following non-relativistic gamma matrix relations and working with relativistic 4D gamma matrices is the inherent split of the space- and time-indices, $a, b, c, \dots$ and 0 respectively. The temporal gamma matrix γ_0 anticommutes with γ_a. The spatial γ_a are hermitian, which is why one cannot just take them to be relativistic 3D gamma matrices.

Some non-relativistic 4D gamma matrix combinations:

$$\gamma^a\gamma^b = -\gamma^b\gamma^a + 2\delta^{ab} \qquad = \gamma^{ab} + \delta^{ab} \,, \tag{B.10a}$$

$$\gamma^a\gamma^{bc} = \gamma^{bc}\gamma^a + 4\delta^{a[b}\gamma^{c]} \quad = \gamma^{abc} + 2\delta^{a[b}\gamma^{c]} \,, \tag{B.10b}$$

$$\gamma^{ab}\gamma^c = \gamma^c\gamma^{ab} - 4\delta^{c[a}\gamma^{b]} \quad = \gamma^{abc} - 2\delta^{c[a}\gamma^{b]} \,, \tag{B.10c}$$

$$\gamma^{ab}\gamma_{cd} = \gamma_{cd}\gamma^{ab} - 8\delta^{[a}_{[c}\gamma^{b]}{}_{d]} = 4\gamma^{[a}{}_{[d}\delta^{b]}_{c]} + \delta^{[a}_{d}\delta^{b]}_{c} \,, \tag{B.10d}$$

$$\gamma^{abc}\gamma_{de} = 6\gamma^{[ab}{}_{[e}\delta^{c]}_{d]} + 6\gamma^{[a}\delta^{b}_{[e}\delta^{c]}_{d]} \,. \tag{B.10e}$$

Some non-relativistic 4D gamma matrix contractions:

$$\varepsilon^{abc}\varepsilon_{ade} = 2\delta^{[b}_{d}\delta^{c]}_{e} \,, \tag{B.11a}$$

$$\varepsilon^{abc}\varepsilon_{abd} = 2\delta^{c}_{d} \,, \tag{B.11b}$$

$$\gamma^{ab}\gamma_b = 2\gamma_a \,, \tag{B.11c}$$

$$\gamma^{ac}\gamma_{cb} = \gamma^a{}_b + 2\delta^a_b \,, \tag{B.11d}$$

$$\gamma^{ab}\gamma_{ab} = -6 \,. \tag{B.11e}$$

Some non-relativistic 4D gamma matrix duality relations:

$$\gamma_5 = \tfrac{i}{6}\varepsilon^{abc}\gamma_{abc}\gamma^0 \,, \tag{B.12a}$$

$$\gamma^a\gamma_5 = \tfrac{i}{2}\varepsilon^{abc}\gamma_{bc}\gamma^0 \,, \tag{B.12b}$$

$$\gamma^{ab}\gamma_5 = -i\varepsilon^{abc}\gamma_c\gamma^0 \,, \tag{B.12c}$$

$$\gamma^{abc}\gamma_5 = -i\varepsilon^{abc}\gamma^0 \,. \tag{B.12d}$$

Appendix C

Derived Quantities

In this section some explicit results are presented that were not included in the chapters before. They include derived quantities like connections and curvatures.

C.1 Non-Relativistic 4D, $\mathcal{N}=2$ Quantities

Spinor

The transformation rule of the spinor emerging from the null reduction is

$$\begin{aligned}\delta\varphi &= \mathcal{L}_\xi\varphi - \tfrac{1}{4}\lambda^{ab}\gamma_{ab}\varphi - \tfrac{1}{2\sqrt{2}}(\bar{\varepsilon}^-\gamma^0\varphi)\varphi \\ &+ \tfrac{1}{8}\hat{\tau}^{ab}\gamma_{ab}\varepsilon^- + \tfrac{1}{2\sqrt{2}}\hat{\tau}^a{}_0\gamma_a\gamma^0\varepsilon^+ \\ &- \tfrac{i}{16}\Big[2\hat{D}_b a\gamma^b\gamma_5\gamma^0\varepsilon^- + \sqrt{2}\left(\hat{f}_{bc}\gamma^{bc} - 4\hat{D}_0 a\right)\gamma_5\varepsilon^+\Big]\,.\end{aligned} \tag{C.1a}$$

The spinor is set to zero from here on $\varphi=0$.

(Supercovariant) Spin- and Boost Connections

The computation of the spin- and boost connections and of the supercovariant spin- and boost connections is structurally equivalent. The supercovariant computation will be presented below. For the computation of the non-supercovariant spin- and boost connections simply ignore all gravitino terms.

The supercovariant spin- and boost connections can be computed from the conventional supercovariant curvature constraints

$$\hat{r}_{\mu\nu}(P^a) = 0\,, \qquad \hat{r}_{\mu\nu}(M) = 0\,. \tag{C.2}$$

For the explicit curvatures see equation (2.96). First define the supercovariant anholonomy coefficients

$$\hat{c}_{\mu\nu}{}^a = 2\partial_{[\mu}e_{\nu]}{}^a - \bar{\psi}^+_{[\mu}\gamma^a\psi^-_{\nu]}\,, \qquad \hat{c}_{\mu\nu} = 2\partial_{[\mu}m_{\nu]} + \tfrac{1}{\sqrt{2}}\bar{\psi}^-_\mu\gamma^0\psi^-_\nu\,. \tag{C.3}$$

The supercovariant spin connection $\hat{\omega}_\mu{}^{ab}$ can be found in two steps. The second constraint is contracted with Vielbeine to yield

$$0 = \hat{r}_{\mu\nu}(M) e^{\mu a} e^{\nu b} \quad \rightarrow \quad \hat{\omega}^{[ab]} = \tfrac{1}{2}\hat{c}^{ab} \,, \tag{C.4}$$

an expression that is needed for the second step. By adding and subtracting cyclic permutations of the first conventional constraint

$$\begin{aligned} 0 &= \hat{r}_{\mu\nu}(P^a) e_{\rho a} + \hat{r}_{\rho\mu}(P^a) e_{\nu a} - \hat{r}_{\nu\rho}(P^a) e_{\mu a} && \text{(C.5a)} \\ \rightarrow \quad \hat{\omega}_\mu{}^{ab} &= e^{\nu[a}\hat{c}_{\mu\nu}{}^{b]} - \tfrac{1}{2} e_{\mu c}\hat{c}^{abc} - \tfrac{1}{2}\tau_\mu \hat{c}^{ab} \,, && \text{(C.5b)} \end{aligned}$$

the spin connection is determined.

Similarly the supercovariant boost connection $\hat{\omega}_\mu{}^a$ is computed by contracting the constraints with one spatial- and one temporal Vielbein

$$\begin{aligned} 0 &= \hat{r}_{\mu\nu}(P^a) e^{\mu b}\tau^\nu \quad \rightarrow \quad \hat{\omega}^{ba} = -\hat{c}^b{}_0{}^a + \hat{c}_0{}^{[ab]} - \tfrac{1}{2}\hat{c}^{ab} \,, && \text{(C.6a)} \\ 0 &= \hat{r}_{\mu\nu}(M)\tau^\mu e^{\nu a} \quad \rightarrow \quad \hat{\omega}_0{}^a = \hat{c}_0{}^a \,. && \text{(C.6b)} \end{aligned}$$

The complete boost connection is

$$\begin{aligned} \hat{\omega}_\mu{}^a &= e_{\mu b}\hat{\omega}^{ba} + \tau_\mu \hat{\omega}_0{}^a && \text{(C.7a)} \\ \rightarrow \quad \hat{\omega}_\mu{}^a &= \tfrac{1}{2}\hat{c}_{0\mu}{}^a + \tfrac{1}{2} e_{\mu b}\hat{c}_0{}^{ab} + \tfrac{1}{2}\hat{c}_\mu{}^a + \tfrac{1}{2}\tau_\mu \hat{c}_0{}^a \,. && \text{(C.7b)} \end{aligned}$$

The transformation rules of the spin- and boost connections are

$$\begin{aligned} \delta\hat{\omega}_\mu{}^{ab} &= \mathcal{L}_\xi \hat{\omega}_\mu{}^{ab} + \partial_\mu \lambda^{ab} - \hat{\tau}_\mu{}^{[a}\lambda^{b]} + \tfrac{1}{2}\bar{\varepsilon}^+\gamma^{[b}\hat{\psi}_\mu^{-a]} + \tfrac{1}{2}\bar{\varepsilon}^-\gamma^{[b}\hat{\psi}_\mu^{+a]} \\ &\quad - \tfrac{1}{4} e_{\mu c}\bar{\varepsilon}^+\gamma^c\hat{\psi}^{-ab} + \tfrac{1}{2\sqrt{2}}\tau_\mu\bar{\varepsilon}^-\gamma^0\hat{\psi}^{-ab} \,, && \text{(C.8a)} \\ \delta\hat{\omega}_\mu{}^a &= \mathcal{L}_\xi\hat{\omega}_\mu{}^a + \partial_\mu\lambda^a - e_{\mu b}\hat{\tau}_0{}^{(a}\lambda^{b)} - \tfrac{1}{2\sqrt{2}}\bar{\varepsilon}^-\gamma^0\hat{\psi}_\mu^{-a} - \tfrac{1}{4}\bar{\varepsilon}^+\gamma^a\hat{\psi}_{\mu 0}^- \\ &\quad - \tfrac{1}{4}\bar{\varepsilon}^-\gamma^a\hat{\psi}_{\mu 0}^+ + \tfrac{1}{4} e_{\mu b}\bar{\varepsilon}^+\gamma^b\hat{\psi}_0^{-a} + \tfrac{1}{4} e_{\mu b}\bar{\varepsilon}^-\gamma^b\hat{\psi}_0^{+a} - \tfrac{1}{2\sqrt{2}}\tau_\mu\bar{\varepsilon}^-\gamma^0\hat{\psi}_0^{-a} \,, && \text{(C.8b)} \end{aligned}$$

and the curvatures of the spin- and boost connections are

$$\begin{aligned} \hat{r}_{\mu\nu}(J^{ab}) &= 2\partial_{[\mu}\hat{\omega}_{\nu]}{}^{ab} - 2\hat{\omega}_{[\mu}{}^{[a}\hat{\tau}_{\nu]}{}^{b]} + \bar{\psi}^+_{[\mu}\gamma^{[a}\hat{\psi}^{-b]}_{\nu]} + \bar{\psi}^-_{[\mu}\gamma^{[a}\hat{\psi}^{+b]}_{\nu]} \\ &\quad + \tfrac{1}{2} e_{[\nu}{}^c\bar{\psi}^+_{\mu]}\gamma_c\hat{\psi}^{-ab} + \tfrac{1}{2} e_{[\nu}{}^c\bar{\psi}^-_{\mu]}\gamma_c\hat{\psi}^{+ab} - \tfrac{1}{\sqrt{2}}\tau_{[\nu}\bar{\psi}^-_{\mu]}\gamma^0\hat{\psi}^{-ab} \,, && \text{(C.9a)} \\ \hat{r}_{\mu\nu}(G^a) &= 2\partial_{[\mu}\hat{\omega}_{\nu]}{}^a - 2\hat{\omega}_{[\mu}{}^{(a} e_{\nu]b}\hat{\tau}^{b)}{}_0 + \tfrac{1}{\sqrt{2}}\bar{\psi}^-_{[\mu}\gamma^0\hat{\psi}^{-a}_{\nu]} + \tfrac{1}{2}\bar{\psi}^+_{[\mu}\gamma^a\hat{\psi}^-_{\nu]0} \\ &\quad + \tfrac{1}{2}\bar{\psi}^-_{[\mu}\gamma^a\hat{\psi}^+_{\nu]0} - \tfrac{1}{2} e_{[\nu}{}^b\bar{\psi}^+_{\mu]}\gamma_b\hat{\psi}_0^{-a} - \tfrac{1}{2} e_{[\nu}{}^b\bar{\psi}^-_{\mu]}\gamma_b\hat{\psi}_0^{+a} \\ &\quad + \tfrac{1}{\sqrt{2}}\tau_{[\nu}\bar{\psi}^-_{\mu]}\gamma^0\hat{\psi}_0^{-a} \,. && \text{(C.9b)} \end{aligned}$$

Other Curvatures

The explicit gravitino curvatures are

$$\begin{aligned}
\hat{\psi}^{-}_{\mu\nu} &= 2\partial_{[\mu}\psi^{-}_{\nu]} + \tfrac{1}{2}\hat{\omega}_{[\mu}{}^{ab}\gamma_{ab}\psi^{-}_{\nu]} + \sqrt{2}\hat{\omega}_{[\mu}{}^{a}\gamma_a\gamma^0\psi^{+}_{\nu]} \\
&\quad - \tfrac{i}{4}\Big[\hat{f}_{\sigma\rho}\Big(e^{\sigma}{}_b e^{\rho}{}_c\gamma_a{}^{bc}e_{[\mu}{}^{a} - \tau^{\sigma}e^{\rho}{}_a\tau_{[\mu}\gamma^a - 2e^{\rho}{}_a\delta^{\sigma}_{[\mu}\gamma^a\Big)\gamma_5\gamma^0\psi^{-}_{\nu]} \qquad \text{(C.10a)} \\
&\quad + \sqrt{2}\hat{f}_{\sigma\rho}\left(-e^{\sigma}{}_b\tau^{\rho}\gamma_a{}^{b}e_{[\mu}{}^{a} + 2\tau^{\rho}\delta^{\sigma}_{[\mu}\right)\gamma_5\psi^{+}_{\nu]}\Big]\,, \\
\hat{\psi}^{+}_{\mu\nu} &= 2\partial_{[\mu}\psi^{+}_{\nu]} + \tfrac{1}{2}\hat{\omega}_{[\mu}{}^{ab}\gamma_{ab}\psi^{+}_{\nu]} \\
&\quad - \tfrac{i}{4}\Big[\hat{f}_{\sigma\rho}\Big(-e^{\sigma}{}_b e^{\rho}{}_c\gamma_a{}^{bc}e_{[\mu}{}^{a} - \tau^{\sigma}e^{\rho}{}_a\tau_{[\mu}\gamma^a + 2e^{\rho}{}_a\delta^{\sigma}_{[\mu}\gamma^a\Big)\gamma_5\gamma^0\psi^{+}_{\nu]} \qquad \text{(C.10b)} \\
&\quad - \sqrt{2}\hat{f}_{ab}\tau_{[\mu}\gamma^{ab}\gamma_5\psi^{-}_{\nu]}\Big]\,.
\end{aligned}$$

The transformation rules of torsion and field strength are

$$\delta\hat{\tau}_{\mu\nu} = \mathcal{L}_{\xi}\hat{\tau}_{\mu\nu} + \tfrac{1}{\sqrt{2}}\bar{\varepsilon}^{+}\gamma^0\hat{\psi}^{+}_{\mu\nu}\,, \qquad \text{(C.11a)}$$

$$\delta\hat{f}_{\mu\nu} = \mathcal{L}_{\xi}\hat{f}_{\mu\nu} + 2\tau_{[\mu}e_{\nu]}{}^{b}\lambda_b\hat{D}_0 a + i\bar{\varepsilon}^{+}\gamma_5\gamma^0\hat{\psi}^{-}_{\mu\nu} - i\bar{\varepsilon}^{-}\gamma_5\gamma^0\hat{\psi}^{+}_{\mu\nu}\,. \qquad \text{(C.11b)}$$

C.2 Non-Relativistic 4D, $\mathcal{N}=2$ Linearised Quantities

Linearised Transformation Rules

The linearised transformation rules of the connections and the field strength are

$$\begin{aligned}
\delta\tilde{\hat{\omega}}_{\mu}{}^{ab} &= \xi^{(0)\nu}\partial_\nu\tilde{\hat{\omega}}_{\mu}{}^{ab} + \partial_\mu\lambda^{(1)ab} - \tfrac{1}{2}\delta_{\mu c}\bar{\varepsilon}^{+(0)}\gamma^{c}\tilde{\hat{\psi}}^{-ab} + \tfrac{1}{2}\delta_{\mu 0}\bar{\varepsilon}^{+(0)}\gamma^{[a}\tilde{\hat{\psi}}^{-b]}{}_{0} \\
&\quad + \tfrac{3}{4\sqrt{2}}\delta_{\mu 0}\bar{\varepsilon}^{-(0)}\gamma^0\tilde{\hat{\psi}}^{-ab}\,, \qquad \text{(C.12a)} \\
\delta\tilde{\hat{\omega}}_{\mu}{}^{a} &= \xi^{(0)\nu}\partial_\nu\tilde{\hat{\omega}}_{\mu}{}^{a} + \partial_\mu\lambda^{(1)a} + \tfrac{i}{4\sqrt{2}}\delta_{\mu b}\bar{\varepsilon}^{-(0)}\varepsilon^{cd(a}\gamma^{b)}\gamma_5\tilde{\hat{\psi}}^{-}_{cd} \\
&\quad - \tfrac{1}{2}\delta_{\mu b}\bar{\varepsilon}^{+(0)}\gamma^{(a}\tilde{\hat{\psi}}^{-b)}{}_{0} + \tfrac{1}{\sqrt{2}}\delta_{\mu 0}\bar{\varepsilon}^{-(0)}\gamma^0\tilde{\hat{\psi}}^{-a}{}_{0} + \tfrac{1}{2\sqrt{2}}\delta_{\mu b}\bar{\varepsilon}^{-(0)}\gamma^0\tilde{\hat{\psi}}^{-ab}\,, \qquad \text{(C.12b)} \\
\delta\tilde{\hat{f}}_{\mu\nu} &= \xi^{(0)\rho}\partial_\rho\tilde{\hat{f}}_{\mu\nu} + i\delta^{a}_{\mu}\delta^{b}_{\nu}\bar{\varepsilon}^{+(0)}\gamma_5\gamma^0\tilde{\hat{\psi}}^{-}_{ab} + 2i\delta^{a}_{[\mu}\delta_{\nu]0}\bar{\varepsilon}^{+(0)}\gamma_5\gamma^0\tilde{\hat{\psi}}^{-}_{a0} \\
&\quad + \tfrac{1}{\sqrt{2}}\delta^{a}_{[\mu}\delta_{\nu]0}\varepsilon_{abc}\bar{\varepsilon}^{-(0)}\gamma^0\tilde{\hat{\psi}}^{-bc}\,. \qquad \text{(C.12c)}
\end{aligned}$$

Linearised Curvatures

All linearised curvatures are reduced to antisymmetrised partial derivatives of the corresponding gauge fields, e.g.

$$\tilde{\hat{r}}_{\mu\nu}(J^{ab}) = 2\partial_{[\mu}\tilde{\hat{\omega}}_{\nu]}{}^{ab}\,, \qquad \tilde{\hat{r}}_{\mu\nu}(G^{a}) = 2\partial_{[\mu}\tilde{\hat{\omega}}_{\nu]}{}^{a}\,, \qquad \tilde{\hat{\psi}}^{\pm}_{\mu\nu} = 2\partial_{[\mu}\tilde{\psi}^{\pm}_{\nu]}\,. \qquad \text{(C.13)}$$

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