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Abstract

Honoring Canjar's work regarding ultrafilters whose Mathias-Prikry forcing does not add a dominating real, this property has been named after Canjar. The broad topic of this thesis is capturing this forcing property by equivalent combinatorial characterizations. In particular, Canjar's original analysis of such filters in [7] is considered to prove that Canjar ultrafilters are necessarily P-points, which do not have rapid Rudin-Keisler predecessors and to prove the existence of such an ultrafilter under the assumption of $\mathfrak{d} = \mathfrak{c}$. Furthermore, the characterization of Canjar filters by Hrušák and Minami in [12] by their notion of P^+-filters is proven to establish the equivalence of Canjar ultrafilters and strong P-points - a notion introduced by Laflamme in [14]. Finally, using the ideas of Blass, Verner and Hrušák in [4] a model which contains a P-point without rapid Rudin-Keisler predecessors, which is not a strong P-point, is constructed, which witnesses that the necessary condition for Canjar ultrafilters given by Canjar is consistently not sufficient.

Zusammenfassung

In Anbetracht von Canjar's Arbeit hinsichtlich Ultrafiltern, deren Mathias-Prikry Forcing keine dominierenden reellen Zahlen hinzufügt, wurde diese Eigenschaft nach Canjar benannt. Das umfassende Thema dieser Arbeit ist das Charakterisieren dieser Forcing Eigenschaft durch equivalente kombinatorische Eigenschaften. Genauer wird zunächst Canjar's ursprüngliche Analyse solcher Filter in [7] untersucht, um zu beweisen, dass Canjar Ultrafilter notwendigerweise P-points ohne rapid Rudin-Keisler Vorgänger sind und dass ein solcher Ultrafilter unter der Annahme von $\mathfrak{d} = \mathfrak{c}$ existiert. Des Weiteren wird die Charakterisierung von Canjar Filtern von Hrušák und Minami in [12] durch P^+-Filter bewiesen um eine Äquivalenz zwischen Canjar Ultrafiltern und strong P-points herzustellen, welche von Laflamme in [14] eingeführt wurden. Abschließend wird nach Blass, Verner und Hrušák in [4] ein Modell konstruiert, welches einen P-point ohne rapid Rudin-Keisler Vorgänger enthält, der aber kein strong P-point ist, was beweist, dass Canjar's notwendige Eigenschaft für Canjar Ultrafilter konsistent nicht hinreichend ist.

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Contents

Introduction

The analysis of the relations between various cardinal characteristics of the continuum has been a rich area of mathematics for a long time. Even though this subfield of set theory has been around for a while, a lot of research is still going on in recent times amounting to various new insights for old problems. One notable example for this is the proof of the consistency of $\omega_1 = \mathfrak{u} < \mathfrak{a} = \omega_2$ by Guzmán and Kalajdzievski in 2019 [10]. There is a long history of consistency proofs of this kind, which we will briefly mention below, but as it turns out the Mathias-Prikry forcing can be used as a useful ingredient for a number of these consistency proofs.

The Mathias-Prikry forcing $\mathbb{M}_{\mathcal{F}}$ is the relativation of the usual Mathias forcing $\mathbb{M}$ to a filter $\mathcal{F}$ on ω. The connection to consistency proofs of cardinal characteristics comes from the following property: It diagonalizes the filter $\mathcal{F}$, i.e. it adds a pseudointersection of $\mathcal{F}$. This means that iterating this forcing with a suitable choice of filters gives us a natural way to increase the splitting number $\mathfrak{s}$ or the almost disjointness number $\mathfrak{a}$. However, many other forcings - for instance the Laver-Prikry forcing - also share this property. What makes the Mathias-Prikry forcing special is the fact that unlike the Laver-Prikry forcing it does not have to add a dominating real. Thus, it is feasible that we do not increase the bounding number $\mathfrak{b}$ in a suitable iteration of the Mathias-Prikry forcing while simultaneously increasing other cardinal characteristics. This means that with the goal of proving consistencies between cardinal invariants in mind it is certainly interesting to study the forcing properties of the Mathias-Prikry forcing and especially characterize when it does not add a dominating real.

In context of this connection of the Mathias-Prikry forcing and the classical cardinal characteristics $\mathfrak{b}, \mathfrak{s}$ and $\mathfrak{a}$ we shall give a brief overview of the consistency results which relate these cardinal characteristics and are of particular interest for this thesis:

The first breakthrough of consistency proofs that established new relations which do not hold in common forcing extensions was due to Shelah. He introduced the notion of creature forcings in [16] and used a countable support iteration of them to obtain models which satisfy the relations $\omega_1 = \mathfrak{b} = \mathfrak{a} < \mathfrak{s} = \omega_2$ and $\omega_1 = \mathfrak{b} < \mathfrak{a} = \mathfrak{s} = \omega_2$, respectively. Later, Brendle and Raghavan noticed in [5] that Shelah's forcing is actually equivalent to a two-step iteration, which first adds a generic ultrafilter and then uses its Mathias-Prikry forcing to diagonalize it. It were then Guzmán, Hrušák and Martínez-Celis in [9], who were able to generalize characterizations for when a Mathias-Prikry forcing adds dominating reals, to prove that this two-step iteration does not add dominating reals, even in the iteration. This means that there is indeed a suitable iteration of the Mathias-Prikry forcing, which gives us new and more straightforward consistency proofs of $\mathfrak{b} < \mathfrak{s}$ and $\mathfrak{b} < \mathfrak{a}$, which allows us to omit the technical notion of creature forcings which Shelah used in his original consistency results.

As already mentioned in the first paragraph, Guzmán and Kalajdzievski used some of these ideas in [10], when they replaced the use of the Mathias forcing with a variant of the Miller forcing to obtain the consistency of $\omega_1 = \mathfrak{u} < \mathfrak{a} = \omega_2$ and to give a new proof of the consistency of $\omega_1 = \mathfrak{u} < \mathfrak{s} = \omega_2$. The latter result had already been proven in [3] by Blass and Shelah before. However, the former result had been an open problem for almost 20 years. Before, in [15] Shelah was able to prove the consistency of $\mathfrak{u} < \mathfrak{a}$ under the assumption of a measurable cardinal, which left open whether the assumption of a measurable cardinal is necessary and whether $\omega_1 = \mathfrak{u} < \mathfrak{a}$ is consistent.

In light of these consistency results we have seen that the following question is of particular interest and will be a focus point of this thesis:

Question 1. When does the Mathias-Prikry forcing $\mathbb{M}_{\mathcal{F}}$ add a dominating real?

Canjar was the first one to study this question, which is why filters, whose Mathias forcing does not add dominating reals, were named Canjar filters. In [7] he was only concerned with ultrafilters, for which he proved that only P-points without rapid Rudin-Keisler predecessors can be Canjar. We will refer to this necessary condition for Canjar ultrafilters as Canjar's condition. Furthermore, he was interested in the question:

Question 2. When is there a Canjar ultrafilter?

As a Canjar ultrafilter is necessarily a P-point, but P-points need not exist in models of ZFC, there are models without Canjar ultrafilters, so we will need an extra assumption to prove the existence of such an ultrafilter. What Canjar proved in [7] is the existence of such an ultrafilter under the extra assumption of $\mathfrak{d} = \mathfrak{c}$. Moreover, he asked if his necessary condition is actually sufficient for Canjar ultrafilters:

Question 3. Is there a P-point without rapid Rudin-Keisler predecessors which is not Canjar?

Shortly after Canjar, Laflamme became interested in these questions and in [14] introduced the notion of a strong P-point for which he proved the implication of Canjar's condition. We will refer to his condition as Laflamme's condition. He noted without proof that his condition is also necessary for Canjar ultrafilters and likewise asked if his condition is actually sufficient:

Question 4. Is there a strong P-point which is not Canjar?

Quite some time later in [12] Hrušák and Minami were able to give a precise combinatorial answer to the first question. As it turns out being Canjar is closely related to the combinatorics of the associated filter $\mathcal{F}^{<\omega}$. They introduced the combinatorial notion of a P^+-filter and proved:

Theorem. $\mathcal{F}$ is Canjar iff $\mathcal{F}^{<\omega}$ is a P^+-filter.

This combinatorial characterization allowed Blass, Verner and Hrušák to finally answer Question 3 and 4. Indeed, they proved in [4] that for ultrafilters this combinatorial characterization is equivalent to Laflamme's condition:

Theorem. $\mathcal{U}^{<\omega}$ is a P^+-filter iff $\mathcal{U}$ is a strong P-point.

This answers Question 4 negatively as together with the other theorem this shows that Laflamme's condition is equivalent to being Canjar. Moreover, under the assumption of MA(countable), i.e. Martin's Axiom for countable partial orders, they proved the existence of a P-point without rapid Rudin-Keisler predecessors which is not a strong P-point. This answers Question 3 positively and shows that Laflamme's condition is consistently stronger than Canjar's condition.

The goal of this thesis will be to thoroughly study and answer each of these questions using the mentioned references. We shall now outline the main results of each section:

In the first section we will fix conventions and introduce basic definitions which are used throughout this thesis. In particular, we will need the usual definitions associated to filters and cardinal invariants like $\mathfrak{b}$ and $\mathfrak{d}$.

In the following section, we will analyse various ways of how we can partially order filters over sets of arbitrary cardinality. Of particular interest will be the Rudin-Keisler order. We will also briefly mention how the Rudin-Keisler order relates to the Katětov and Rudin-Blass order.

In the third section we will restrict our attention from general filters of any cardinality to ultrafilters over ω. We will introduce common notions like P-points, Q-points, Ramsey and rapid ultrafilters and prove various different characterizations of these notions.

In the fourth section we will prove some standard properties of the Mathias-Prikry forcing and partially answer Question 1 by proving the necessity of Canjar's condition. This means that we consider Canjar's original paper on Canjar ultrafilters [7] to prove that a Canjar ultrafilter is necessarily a P-point and cannot have any rapid predecessors in the Rudin-Keisler order.

The fifth section will be devoted to the second result of [7], which is the existence of a Canjar ultrafilter under the assumption of $\mathfrak{d} = \mathfrak{c}$, which gives us a consistent setting for Question 2. Furthermore, we will see why a naive iteration of the Mathias-Prikry forcing of Canjar ultrafilters is not quite enough to get new consistency proofs regarding $\mathfrak{b}$.

Lastly, in the final section we will give a precise answer to Question 1 by proving the combinatorial characterization of Canjar filters given by Hrušák and Minami in [12]. Using this result we will follow the conclusion of Blass, Verner and Hrušák in [4] to answer Question 4 by proving that for ultrafilters this combinatorial characterization is equivalent to Laflamme's condition. Finally, following the ideas of the second part of the same paper we will answer the remaining Question 3 by proving that under the assumption of MA(countable) there exists an ultrafilter which satisfies Laflamme's condition but not Canjar's.

1 Preliminaries

Throughout this thesis we will work with the meta-theory ZFC, the usual axiomatisation of set theory formulated by Zermelo and Fraenkel together with the axiom of choice. All definitions in this section are standard notions commonly used in set theory and in particular in the study of cardinal invariants of the continuum.

1.1 Filters and Ideals

First, we consider the definitions around filters and ideals. We will only be interested in filters and ideals on infinite domains, so we fix an infinite set X. If the underlying set is unambiguous from the context, then we write A^C for the complement $X \setminus A$.

Definition. *A subset $\mathcal{F} \subseteq \mathcal{P}(X)$ is a filter on X iff $\emptyset \notin \mathcal{F}$ and $\mathcal{F}$ is closed under finite intersections and supersets. $\mathcal{F}$ is called free iff it contains the Fréchet filter of X*

$$\operatorname{cofin}(X) := \{A \subseteq X \mid A^C \text{ is finite}\}$$

Furthermore, $\mathcal{F}$ is an ultrafilter iff it is maximal under all filters, or equivalently iff for every $A \subseteq X$ we have that $A \in \mathcal{F}$ or $A^C \in \mathcal{F}$.

We arbitrarily choose to work predominantly with filters throughout this thesis. However, all definitions and proofs can be carried out dually with ideals instead of filters:

Definition. *A subset $\mathcal{I} \subseteq \mathcal{P}(X)$ is an ideal on X iff $X \notin \mathcal{I}$ and $\mathcal{I}$ is closed under finite unions and subsets. $\mathcal{I}$ is called free iff it contains the finite ideal of X*

$$\operatorname{fin}(X) := \{A \subseteq X \mid A \text{ is finite}\}$$

Furthermore, $\mathcal{I}$ is a maximal ideal iff it is maximal under all ideals, or equivalently iff for every $A \subseteq X$ we have $A \in \mathcal{I}$ or $A^C \in \mathcal{I}$.

The duality between filters and ideals is given by the following observation:

Remark. Let $\mathcal{F}$ be a filter on X and $\mathcal{I}$ be an ideal on X. Then

$$\mathcal{F}^* := \{F^C \mid F \in \mathcal{F}\}$$

is an ideal on X, called the dual ideal of $\mathcal{F}$. Furthermore,

$$\mathcal{I}^* := \{I^C \mid I \in \mathcal{I}\}$$

is an filter on X, called the dual filter of $\mathcal{I}$. Note that being free is compatible with taking duals and being an ultrafilter is dual to being a maximal ideal.

Next, we consider when and how we can generate filters from subfamilies of $\mathcal{P}(X)$:

Definition. *Let $\mathcal{G} \subseteq \mathcal{P}(X)$. We say that $\mathcal{G}$ has the finite intersection property (FIP) iff the intersection of finitely many elements of $\mathcal{G}$ is non-empty.*

Definition. *Let $\mathcal{G} \subseteq \mathcal{P}(X)$ satisfy the FIP. We write $\langle \mathcal{G} \rangle$ for the filter generated by $\mathcal{G}$, i.e. the smallest filter on X containing $\mathcal{G}$ or equivalently the intersection of all filters that contain $\mathcal{G}$.*

With generated filters we can define the notion of a basis and the character of a filter:

Definition. *Let $\mathcal{F}$ be a filter on X and let $\mathcal{G} \subseteq \mathcal{P}(X)$ satisfy the FIP and be closed under finite intersections. If $\mathcal{F} = \langle \mathcal{G} \rangle$, then we call $\mathcal{G}$ a basis of $\mathcal{F}$.*
Furthermore, the character of $\mathcal{F}$ is the least cardinality of a basis of $\mathcal{F}$, denoted by $\chi(\mathcal{F})$.

Remark. Given any $\mathcal{G} \subseteq \mathcal{P}(X)$ which satisfies the FIP and $\mathcal{F} = \langle \mathcal{G} \rangle$ we can close $\mathcal{G}$ under finite intersections to obtain a basis of $\mathcal{F}$. Furthermore, if $\mathcal{G}$ is infinite, then this basis will have the same cardinality as $\mathcal{G}$.

Now, we introduce the concept of positive sets relative to a filter or ideal, which will be useful for combinatorial definitions in later chapters:

Definition. *For an ideal $\mathcal{I}$ on X we write*

$$\mathcal{I}^+ := \{A \subseteq X \mid A \notin \mathcal{I}\}$$

for the set of all $\mathcal{I}$-positive sets. For a filter $\mathcal{F}$ on X we define the $\mathcal{F}$-positive sets by

$$\mathcal{F}^+ := (\mathcal{F}^*)^+$$

Remark. Notice that for a filter $\mathcal{F}$ on X the following definitions are equivalent:

$$\mathcal{F}^+ = \{A \subseteq X \mid A^C \notin \mathcal{F}\} = \{A \subseteq X \mid \forall F \in \mathcal{F}\ A \cap F \neq \emptyset\}$$

The middle definition implies that $\mathcal{F} \subseteq \mathcal{F}^+$. Analogously, for an ideal $\mathcal{I}$ on X we have that $\mathcal{I}^* \subseteq \mathcal{I}^+$. Furthermore, the latter definition implies that for any $A \in \mathcal{F}^+$ the set $\mathcal{F} \cup \{A\}$ satisfies the FIP. Together with the ultrafilter lemma we get that

$$\mathcal{F}^+ = \bigcup \{\mathcal{U} \subseteq \mathcal{P}(X) \mid \mathcal{U} \text{ is an ultrafilter extending } \mathcal{F}\}$$

so we have that $\mathcal{F} = \mathcal{F}^+$ iff $\mathcal{F}$ is an ultrafilter.

Next, we want to restrict filters and ideals to smaller domains. This will be particularly important in the analysis of the Rudin-Keisler order in the next section.

Definition. *Let $\mathcal{F}$ be a filter on X and $A \in \mathcal{F}^+$. Define the restriction of $\mathcal{F}$ to A by*

$$\mathcal{F} \restriction A = \{F \cap A \mid F \in \mathcal{F}\}$$

Analogously, let $\mathcal{I}$ be an ideal on X and $A \in \mathcal{I}^+$. Define the restriction of $\mathcal{I}$ to A by

$$\mathcal{I} \restriction A = \{I \cap A \mid A \in \mathcal{I}\}$$

Remark. Notice that the properties free, ultrafilter and maximal transfer to restrictions. For filters if we choose $A, B \in \mathcal{F} \subseteq \mathcal{F}^+$, then also $A \cap B \in \mathcal{F} \subseteq \mathcal{F}^+$ and we have

$$(\mathcal{F} \restriction A) \restriction B = \mathcal{F} \restriction (A \cap B)$$

Analogously, for ideals if we choose $A, B \in \mathcal{I}^* \subseteq \mathcal{I}^+$, then also $A \cap B \in \mathcal{I}^* \subseteq \mathcal{I}^+$ and

$$(\mathcal{I} \restriction A) \restriction B = \mathcal{I} \restriction (A \cap B)$$

In this case we also have that if $A \in \mathcal{F}$, then $\mathcal{F} \restriction A \subseteq \mathcal{F}$ as filters are closed under intersections and if $A, B \in \mathcal{F}$, then $B \subseteq A$ implies that $B \in \mathcal{F} \restriction A$.

1.2 Cardinal invariants of the continuum

We call functions $f : \omega \to \omega$ reals and write ${}^\omega\omega$ for the set of all reals and $[\omega]^{<\omega}$ for the set of all finite subsets of ω. Furthermore, we write $\mathfrak{c} = |{}^\omega\omega|$ for the cardinality of the continuum. Given two reals $f, g \in {}^\omega\omega$ we say that f dominates g and write $g <^* f$ iff

$$\exists N < \omega\ \forall n \geq N\ g(n) < f(n)$$

or equivalently iff the set

$$\{n < \omega \mid g(n) < f(n)\}$$

is cofinite. If $f < g$ pointwise, then we say that f completely dominates g. We define two important cardinal invariants of the continuum using this notion:

Definition. *A family $\mathcal{B} \subseteq {}^\omega\omega$ of reals is called unbounded iff we have that*

$$\forall f \in {}^\omega\omega\ \exists g \in \mathcal{B}\ g \not<^* f$$

We define the bounding number to be the smallest cardinality of an unbounded family.

Similarly, we define the dominating number:

Definition. *A family $\mathcal{D} \subseteq {}^\omega\omega$ of reals is called dominating iff we have that*

$$\forall f \in {}^\omega\omega\ \exists g \in \mathcal{B}\ f <^* g$$

We define the dominating number to be the smallest cardinality of an dominating family.

Clearly, every dominating family is unbounded. Furthermore, it is not hard to prove that $\mathfrak{b}$ and $\mathfrak{d}$ are indeed cardinal invariants of the continuum, i.e. we have that

$$\omega < \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}$$

1.3 Further conventions

Lastly, we will adhere to the following symbolic and forcing conventions:

The choice of symbols will go from small to capital to script letters the higher we go with power sets. For example, when we will later work with a topology on $\mathcal{P}(\omega)$ we will mostly use small letters like m and n for elements of ω, capital letters like A and B for elements of $\mathcal{P}(\omega)$ and script letter like $\mathcal{C}$ and $\mathcal{K}$ for elements of the topology on $\mathcal{P}(\omega)$.

For forcing we use will the convention that $p \leq_{\mathbb{P}} q$ stands for p extends q in the forcing notion $\mathbb{P}$, i.e. p carries more information than q. We will assume that all of our forcings have a weakest element $1_{\mathbb{P}}$ and write $\Vdash_{\mathbb{P}} \varphi$ for $1_{\mathbb{P}} \Vdash_{\mathbb{P}} \varphi$. Also, given a forcing relation $\Vdash_{\mathbb{P}}$ we may often drop the subscript and write $\Vdash$ if the context is unambiguous. As usual, in the forcing language we will identify elements of the ground model $x \in V$ with their canonical check name $\check{x} \in V^{\mathbb{P}}$. Furthermore, we will often switch from the syntactical interpretation of the forcing relation to the semantical interpretation using the forcing theorem. It should be clear that we do not get into trouble, when we assume the existence of generic filters for that, see for example Jech's [13] introduction to forcing on how to interpret this assumption.

2 Ordering of filters

There are many different natural ways to order filters. Here we mainly consider the Rudin-Keisler order and briefly describe its relation to the Katětov and Rudin-Blass order. The Rudin-Keisler section is a generalization of the presentation given in [11] from ultrafilters over ω to ultrafilters over sets of arbitrary infinite cardinality.

2.1 The Rudin-Keisler order

First, we consider the Rudin-Keisler order. We restrict our attention to ultrafilters as in this case the equivalence relation $\equiv_{\mathrm{RK}}$ induced by $\leq_{\mathrm{RK}}$ has a nice characterization. So, for this section, fix ultrafilters $\mathcal{U}, \mathcal{V}$ and $\mathcal{W}$ on X, Y and Z, respectively.

Definition. *Given $f : X \to Y$ we define the image of $\mathcal{U}$ under f by*

$$f(\mathcal{U}) := \{A \subseteq Y \mid f^{-1}[A] \in \mathcal{U}\}$$

Remark. Notice that we have the following alternative definition:

$$f(\mathcal{U}) = \{A \subseteq Y \mid \exists B \in \mathcal{U}\ f[B] \subseteq A\}$$

so $f(\mathcal{U})$ is the filter generated by the basis of all images of sets in $\mathcal{U}$ under f.

The idea of the Rudin-Keisler order will be to order ultrafilters by the ability to express an ultrafilter as the image of another one. We begin our analysis of this image operation by proving that it preserves ultrafilters:

Lemma 2.1.1. *If $f : X \to Y$ and $\mathcal{V} = f(\mathcal{U})$, then $\mathcal{V}$ is also an ultrafilter. Furthermore, $\mathcal{V}$ is free iff f is not constant on any set in $\mathcal{U}$.*

Proof.

- $f^{-1}[Y] = X \in \mathcal{U}$ and $f^{-1}[\emptyset] = \emptyset \notin \mathcal{U}$, thus $Y \in \mathcal{V}$ and $\emptyset \notin \mathcal{V}$.
- Let $A \in \mathcal{V}$ and $B \supseteq A$. Then $f^{-1}[A] \in \mathcal{U}$, so $f^{-1}[B] \supseteq f^{-1}[A] \in \mathcal{U}$ and $B \in \mathcal{V}$.
- Let $A, B \in \mathcal{V}$. Then $f^{-1}[A \cap B] = f^{-1}[A] \cap f^{-1}[B] \in \mathcal{U}$, so $A \cap B \in \mathcal{V}$.
- Let $A \notin \mathcal{V}$. Then $f^{-1}[A] \notin \mathcal{U}$, so $f^{-1}[A^C] = f^{-1}[A]^C \in \mathcal{U}$ and $A^C \in \mathcal{V}$.
- We have that f is not constant on any set in $\mathcal{U}$ iff for every $y \in Y$ we have that $f^{-1}[\{y\}] \notin \mathcal{U}$ iff for every $y \in Y$ $\{y\} \notin \mathcal{V}$ iff $\mathcal{V}$ is free. □

Definition. *We define the Rudin-Keisler order by*

$$\mathcal{V} \leq_{\mathrm{RK}} \mathcal{U} \ :\Longleftrightarrow\ \exists f : X \to Y\ f(\mathcal{U}) = \mathcal{V}$$

Furthermore, we define Rudin-Keisler equivalence by

$$\mathcal{V} \equiv_{\mathrm{RK}} \mathcal{U} \ :\Longleftrightarrow\ \exists U \in \mathcal{U}, V \in \mathcal{V}, f : U \to V\ f(\mathcal{U} \restriction U) = \mathcal{V} \restriction V \text{ and } f \text{ is a bijection}$$

In the definition of $\equiv_{\mathrm{RK}}$ the restriction to sets in $\mathcal{U}$ and $\mathcal{V}$ is the price we have to pay to allow for arbitrary cardinalities of the domain. However, we will later prove that this restriction is not necessary in the case that X and Y have the same cardinality. Before we can prove basic properties for $\leq_{\mathrm{RK}}$ and $\equiv_{\mathrm{RK}}$ to prove that $\equiv_{\mathrm{RK}}$ is the induced equivalence relation of $\leq_{\mathrm{RK}}$ we have to examine the interplay between the image operation of ultrafilters and the restriction of their domains:

Lemma 2.1.2. *If $f : X \to Y$ and $f(\mathcal{U}) = \mathcal{V}$, then for every $U \in \mathcal{U}$ we have that*

$$(f \restriction U)(\mathcal{U} \restriction U) = \mathcal{V} \restriction f[U]$$

Proof. First, let $A \in (f \restriction U)(\mathcal{U} \restriction U)$. Then $(f \restriction U)^{-1}[A] \in \mathcal{U} \restriction U \subseteq \mathcal{U}$. As $f(\mathcal{U}) = \mathcal{V}$ this implies that $f[(f \restriction U)^{-1}[A]] \subseteq A \in \mathcal{V}$. But $A \subseteq f[U]$, so $A \in \mathcal{V} \restriction f[U]$.

Conversely, let $A \in \mathcal{V} \restriction f[U] \subseteq \mathcal{V}$. Then $f^{-1}[A] \in \mathcal{U}$, so we have that

$$(f \restriction U)^{-1}[A] = f^{-1}[A] \cap U \in \mathcal{U} \restriction U$$

which implies that $A \in (f \restriction U)(\mathcal{U} \restriction U)$. □

Corollary 2.1.3. *If $U \in \mathcal{U}, V \in \mathcal{V}$ and $f : U \to V$ is such that $f(\mathcal{U} \restriction U) = \mathcal{V} \restriction V$, then for every $U' \in \mathcal{U}$ with $U' \subseteq U$ we have that*

$$(f \restriction U')(\mathcal{U} \restriction U') = \mathcal{V} \restriction f[U']$$

Proof. We have $U' \in \mathcal{U} \restriction U$, so we can compute $(f \restriction U')(\mathcal{U} \restriction U')$ using the previous lemma:

$$\begin{aligned}(f \restriction U')(\mathcal{U} \restriction U') &= (f \restriction U')(\mathcal{U} \restriction (U \cap U')) = (f \restriction U')((\mathcal{U} \restriction U) \restriction U') \\ &= (\mathcal{V} \restriction V) \restriction f[U'] = \mathcal{V} \restriction (V \cap f[U']) = \mathcal{V} \restriction f[U']\end{aligned}$$ □

Secondly, we need that the image operation commutes with function concatenation:

Lemma 2.1.4. *If $f : X \to Y$, $g : Y \to Z$, then $(g \circ f)(\mathcal{U}) = g(f(\mathcal{U}))$.*

Proof. We have the following equivalences:

$$\begin{aligned}A \in (g \circ f)(\mathcal{U}) &\iff (g \circ f)^{-1}[A] \in \mathcal{U} \iff f^{-1}[g^{-1}[A]] \in \mathcal{U} \\ &\iff g^{-1}[A] \in f(\mathcal{U}) \quad\iff A \in g(f(\mathcal{U}))\end{aligned}$$ □

These simple properties allow us to prove that $\leq_{\mathrm{RK}}$ is a preorder and that $\equiv_{\mathrm{RK}}$ is an equivalence relation:

Lemma 2.1.5.

i) $\leq_{\mathrm{RK}}$ is a preorder.

ii) $\equiv_{\mathrm{RK}}$ is a equivalence relation.

Proof.

i) For the identity id_X we have that $\mathrm{id}_X(\mathcal{U}) = \mathcal{U}$, so $\mathcal{U} \leq_{\mathrm{RK}} \mathcal{U}$.

Now, if $\mathcal{W} \leq_{\mathrm{RK}} \mathcal{V} \leq_{\mathrm{RK}} \mathcal{U}$ choose $f : X \to Y$ and $g : Y \to X$ with $f(\mathcal{U}) = \mathcal{V}$ and $g(\mathcal{V}) = \mathcal{W}$. Then $(g \circ f)(\mathcal{U}) = g(f(\mathcal{U})) = g(\mathcal{V}) = \mathcal{W}$ which implies that $\mathcal{U} \leq_{\mathrm{RK}} \mathcal{W}$.

ii) Again, with $U = V = X$ and id_X we get that $\mathcal{U} \equiv_{\mathrm{RK}} \mathcal{U}$.

Now, if $\mathcal{V} \equiv_{\mathrm{RK}} \mathcal{U}$ choose $U \in U$, $V \in \mathcal{V}$ and a bijection $f : U \to V$ such that $f(\mathcal{U} \restriction U) = \mathcal{V} \restriction V$. Then $f^{-1} : V \to U$ is a bijection and $f^{-1}(\mathcal{V} \restriction V) = \mathcal{U} \restriction U$ as

$$\begin{aligned}A \in f^{-1}(\mathcal{V} \restriction V) &\iff (f^{-1})^{-1}[A] \in \mathcal{V} \restriction V \iff f[A] \in \mathcal{V} \restriction V \\ &\iff f^{-1}[f[A]] \in \mathcal{U} \restriction U \quad \iff A \in \mathcal{U} \restriction U\end{aligned}$$

which implies that $\mathcal{U} \equiv_{\mathrm{RK}} \mathcal{V}$.

Finally, if $\mathcal{W} \equiv_{\mathrm{RK}} \mathcal{V} \equiv_{\mathrm{RK}} \mathcal{U}$, then choose $U \in \mathcal{U}, V_1, V_2 \in \mathcal{V}, W \in \mathcal{W}$ and bijections $f : U \to V_1, g : V_2 \to W$ such that $f(\mathcal{U} \restriction U) = \mathcal{V} \restriction V_1$ and $g(\mathcal{V} \restriction V_2) = \mathcal{W} \restriction W$. Define $V' := V_1 \cap V_2, U' := f^{-1}[V']$ and $W' := g[V']$. Then $U' \in \mathcal{U}, V' \in \mathcal{V}$ and $W' \in \mathcal{W}$, so we get that $(f \restriction U')(\mathcal{U} \restriction U') = \mathcal{V} \restriction V'$ and $(g \restriction V')(\mathcal{V} \restriction V') = \mathcal{W} \restriction W'$ by Corollary 2.1.3. Now, $(g \restriction V') \circ (f \restriction \mathcal{U}') : U' \to W'$ is bijective and

$$((g \restriction V') \circ (f \restriction \mathcal{U}'))(\mathcal{U} \restriction U') = (g \restriction V')(\mathcal{V} \restriction V') = \mathcal{W} \restriction W'$$

which implies that $\mathcal{U} \equiv_{\mathrm{RK}} \mathcal{W}$. □

Next, the compatibility of $\leq_{\mathrm{RK}}$ and $\equiv_{\mathrm{RK}}$ is given by the following theorem:

Theorem 2.1.6. $\equiv_{\mathrm{RK}}$ *is induced by* $\leq_{\mathrm{RK}}$*, i.e.*

$$\mathcal{V} \leq_{\mathrm{RK}} \mathcal{U} \text{ and } \mathcal{U} \leq_{\mathrm{RK}} \mathcal{V} \iff \mathcal{V} \equiv_{\mathrm{RK}} \mathcal{U}$$

For the proof we need the following two lemmata. The first lemma makes use of the De Bruijn–Erdös theorem, which is also known as the compactness theorem for graph colorings, which states that if any finite subgraph of a graph G is k-colorable, then also G is k-colorable. One easy way to prove this theorem is to use the compactness theorem of first-order theories.

Lemma 2.1.7. *If* $f : X \to X$ *is such that* $f(\mathcal{U}) = \mathcal{U}$*, then*

$$\{x \in X \mid f(x) = x\} \in \mathcal{U}$$

Proof. Let $\prec$ be a well-ordering of X. Define the following three sets:

$$\begin{aligned} D &:= \{x \in X \mid f(x) \prec x\} \quad \text{(decreasing)} \\ E &:= \{x \in X \mid f(x) = x\} \quad \text{(equal)} \\ I &:= \{x \in X \mid f(x) \succ x\} \quad \text{(increasing)} \end{aligned}$$

We show that $D, I \notin \mathcal{U}$ so that $E \in \mathcal{U}$ as $\mathcal{U}$ is an ultrafilter.

First, assume that $D \in \mathcal{U}$. Let f^k be the application of f repeated k times. Then for every $x \in D$ there is a least k_x such that $f^{k_x}(x) \notin D$ as there are no infinite decreasing chains in well-founded relations. Define

$$D_{\text{even}} := \{x \in D \mid k_x \text{ is even}\} \qquad \text{and} \qquad D_{\text{odd}} := \{x \in D \mid k_x \text{ is odd}\}$$

Then either $D_{\text{even}} \in \mathcal{U}$ or $D_{\text{odd}} \in \mathcal{U}$, but as $f(\mathcal{U}) = \mathcal{U}$ we have that $D_{\text{even}} \in \mathcal{U}$ implies that $f[D_{\text{even}}] \subseteq D_{\text{odd}} \in \mathcal{U}$ and $D_{\text{odd}} \in \mathcal{U}$ implies that $f[D_{\text{odd}}] \cap D \subseteq D_{\text{even}} \in \mathcal{U}$.

Now, assume that $I \in \mathcal{U}$. Define an undirected graph on I by

$$xEy \;:\iff\; f(x) = y \text{ or } f(y) = x$$

We show that I is a forest, so let $x_1, \ldots, x_n \in I$. We may assume that they are ordered increasingly by $\prec$. Then for every $k > 1$ we have that $x_1 \prec x_k \prec f(x_k)$, so $x_1 E x_k$ is only possible if $f(x_1) = x_k$. But this shows that x_1 has at most one neighbour in $x_2, \ldots, x_n$, so $x_1, \ldots, x_n$ cannot be a cycle.

Thus, every finite subgraph of I is 2-colorable, so by the compactness theorem for graph colorings we can split I into I_0 and I_1 such that they form a 2-coloring of I. Then either $I_0 \in \mathcal{U}$ or $I_1 \in \mathcal{U}$, but as $f(\mathcal{U}) = \mathcal{U}$ we have that $I_0 \in \mathcal{U}$ implies that $f[I_0] \cap I \subseteq I_1 \in \mathcal{U}$ and $I_1 \in \mathcal{U}$ implies that $f[I_1] \cap I \subseteq I_0 \in \mathcal{U}$.

Hence, neither $D \in \mathcal{U}$ nor $I \in \mathcal{U}$, so we get that $E = \{x \in X \mid f(x) = x\} \in \mathcal{U}$. □

The next lemma shows that an ultrafilter is the image of another ultrafilter already if this holds for the restriction to a large set. This allows us to arbitrarily extend functions without disturbing the image of the ultrafilter in this case.

Lemma 2.1.8. *If $U \in \mathcal{U}, V \in \mathcal{V}$ and $f : U \to V$ is such that $f(\mathcal{U} \restriction U) = \mathcal{V} \restriction V$, then for every $\hat{f} : X \to Y$ extending f we have that $\hat{f}(\mathcal{U}) = \mathcal{V}$.*

Proof. First, let $A \in \hat{f}(\mathcal{U})$. Then $\hat{f}^{-1}[A] \in \mathcal{U}$, so $\hat{f}^{-1}[A] \cap U \in \mathcal{U} \restriction U$. But $\hat{f}$ extends f, so we have $\hat{f}^{-1}[A] \cap U = f^{-1}[A \cap V]$. By assumption $f(\mathcal{U} \restriction U) = \mathcal{V} \restriction V$, so we get that $f[f^{-1}[A \cap V]] \subseteq A \cap V \in \mathcal{V} \restriction V$, so $\mathcal{V} \restriction V \subseteq \mathcal{V}$ implies that $A \in \mathcal{V}$.

Conversely, let $A \in \mathcal{V}$. Then $A \cap V \in \mathcal{V} \restriction V$, so $f^{-1}[A \cap V] \in \mathcal{U} \restriction U$. Again, we have that $\hat{f}^{-1}[A] \cap U = f^{-1}[A \cap V]$. Hence, $\hat{f}^{-1}[A] \cap U \in \mathcal{U} \restriction U \subseteq \mathcal{U}$. But then $\hat{f}^{-1}[A] \in \mathcal{U}$ which shows that $A \in \hat{f}(\mathcal{U})$. □

Finally, we can prove the compatibility of $\leq_{\mathrm{RK}}$ and $\equiv_{\mathrm{RK}}$:

Proof of Theorem 2.1.6. First, assume that $\mathcal{V} \leq_{\mathrm{RK}} \mathcal{U}$ and $\mathcal{U} \leq_{\mathrm{RK}} \mathcal{V}$, so choose $f : X \to Y$ and $g : Y \to X$ such that $f(\mathcal{U}) = \mathcal{V}$ and $g(\mathcal{V}) = \mathcal{U}$. Then $(g \circ f)(\mathcal{U}) = g(\mathcal{V}) = \mathcal{U}$, so by the Lemma 2.1.7 there is a $U \in \mathcal{U}$ such that $(g \circ f)(x) = x$ for every $x \in U$. But this means that $(g \circ f) \restriction U$ is a bijection. Thus, $f \restriction U : U \to f[U]$ is a bijection and we have that $U \in \mathcal{U}$ and $f[U] \in \mathcal{V}$. So, by Lemma 2.1.2 we get that $(f \restriction U)(\mathcal{U} \restriction U) = \mathcal{V} \restriction f[U]$ which shows that $\mathcal{V} \equiv_{\mathrm{RK}} \mathcal{U}$.

Conversely, if $\mathcal{V} \equiv_{\mathrm{RK}} \mathcal{U}$, then choose $U \in \mathcal{U}$ and $V \in \mathcal{V}$ and a bijection $f : U \to V$ such that $f(\mathcal{U} \restriction U) = \mathcal{V} \restriction V$. Extend f to a function $\hat{f} : X \to Y$ arbitrarily. Then $\hat{f}(\mathcal{U}) = \mathcal{V}$ by Lemma 2.1.8, so $\mathcal{V} \leq_{\mathrm{RK}} \mathcal{U}$. The other case follows from the symmetry of $\equiv_{\mathrm{RK}}$. □

We finish this subsection by proving that the definition of $\equiv_{\mathrm{RK}}$ can be simplified if X and Y have the same cardinality and by proving that this entire section has the same dual results if we work with maximal ideals instead of ultrafilters. The assumption on the cardinality allows us to extent partial bijections to bijections in the following way:

Lemma 2.1.9. *If $|X| = |Y|$, $U \in \mathcal{U}, V \in \mathcal{V}$ and $f : U \to V$ is a bijection such that $f(\mathcal{U} \restriction U) = \mathcal{V} \restriction V$, then there is a $U' \in \mathcal{U}$ with $U' \subseteq U$ and a bijection $\hat{f} : X \to Y$ extending $f \restriction U'$.*

Proof. Let $\kappa := |X| = |Y|$. If $|X \setminus U| = |Y \setminus V|$, take a bijection $h : (X \setminus U) \to (Y \setminus V)$. Then for $U' := U$ we have the bijection $\hat{f} := f \cup h$ extending f.

Otherwise, $|U| = \kappa$. Split U in two sets $U_0 \in \mathcal{U}$ and U_1 of size κ. As f is a bijection we get that $|Y \setminus f[U_0]| = \kappa$, so take a bijection $h : (X \setminus U_0) \to (Y \setminus f[U_0])$. Then for $U' := U_0$ we have the bijection $\hat{f} := f \restriction U_0 \cup h$ extending $f \restriction U_0$. □

The desired simplification of $\equiv_{\mathrm{RK}}$ now follows from combining this lemma with our previous restriction and extension lemmata:

Corollary 2.1.10. *If $|X| = |Y|$ then*

$$\mathcal{V} \equiv_{\mathrm{RK}} \mathcal{U} \iff \exists f : X \to Y \ f(\mathcal{U}) = \mathcal{V} \text{ and } f \text{ is a bijection}$$

Proof. The non-trivial direction is the implication, so choose $U \in \mathcal{U}$, $V \in \mathcal{V}$ and a bijection $f : U \to V$ such that $f(\mathcal{U} \restriction U) = \mathcal{V} \restriction V$. By the previous lemma there is a $U' \in \mathcal{U}$ with $U' \subseteq U$ and bijection $\hat{f} : X \to Y$ extending $f \restriction U'$. By Corollary 2.1.3 we have $f(\mathcal{U} \restriction U') = \mathcal{V} \restriction f[U']$, so by Lemma 2.1.8 we get $\hat{f}(\mathcal{U}) = \mathcal{V}$. □

Using the duality of maximal ideals and ultrafilters we can define an analogon of the Rudin-Keisler order for maximal ideals. So fix maximal ideals $\mathcal{I}$ and $\mathcal{J}$ on X and Y, respectively, and define:

Definition. *Given $f : X \to Y$ we define the image of $\mathcal{I}$ under f by*

$$f(\mathcal{I}) := \{A \subseteq Y \mid f^{-1}[A] \in \mathcal{I}\}$$

Remark. Notice that we have the following alternative definition:

$$f(\mathcal{I}) = \{A \subseteq Y \mid \exists B \in \mathcal{I}\ A \subseteq f[B]\}$$

so $f(\mathcal{I})$ is the ideal generated by the basis of all images of sets in $\mathcal{I}$ under f.

The key observation is that this definition commutes with taking duals with respect to our previous definition of the image of an ultrafilter:

Lemma 2.1.11. *We have that $f(\mathcal{U})^* = f(\mathcal{U}^*)$ and $f(\mathcal{I}^*) = f(\mathcal{I})^*$.*

Proof. We only have to prove the first equality, because then the second follows from taking double duals. We have that

$$\begin{aligned} A \in f(\mathcal{U})^* &\iff A^C \in f(\mathcal{U}) \iff f^{-1}[A^C] \in \mathcal{U} \iff f^{-1}[A]^C \in \mathcal{U} \\ &\iff f^{-1}[A] \in \mathcal{U}^* \iff A \in f(\mathcal{U}^*) \end{aligned}$$

□

Definition. *We define the Rudin-Keisler order for maximal ideals by*

$$\mathcal{J} \leq_{\mathrm{RK}} \mathcal{I} :\iff \exists f : X \to Y\ f(\mathcal{I}) = \mathcal{J}$$

Now, the previous lemma gives us the following theorem:

Theorem 2.1.12. *We have that*

$$\mathcal{J} \leq_{\mathrm{RK}} \mathcal{I} \iff \mathcal{J}^* \leq_{\mathrm{RK}} \mathcal{I}^*$$

Proof. We have the following equivalences:

$$\begin{aligned} \mathcal{J} \leq_{\mathrm{RK}} \mathcal{I} &\iff \exists f : X \to Y\ f(\mathcal{I}) = \mathcal{J} \\ &\iff \exists f : X \to Y\ f(\mathcal{I}^*)^* = \mathcal{J}^{**} \\ &\iff \exists f : X \to Y\ f(\mathcal{I}^*) = \mathcal{J}^* \iff \mathcal{J}^* \leq_{\mathrm{RK}} \mathcal{I}^* \end{aligned}$$

□

This theorem allows us to transfer all results of the Rudin-Keisler order for ultrafilters to the Rudin-Keisler order for maximal ideal and vice versa. In this section we have restricted ourselves to ultrafilters, but in the following section about the Katětov order we will see that the initial definitions of this section naturally extend to arbitrary filters, which allows us to extend the definition of the Rudin-Keisler order to arbitrary filters. The duality between filters and ideals works in the same way for the extended Rudin-Keisler order.

2.2 The Rudin-Blass order

Next, we will consider an important strengthening of the Rudin-Keisler order by requiring the witnessing function to be finite-to-one. Again, fix ultrafilters $\mathcal{U}, \mathcal{V}$ and $\mathcal{W}$ on X, Y and Z, respectively.

Definition. *We define the Rudin-Blass order by*

$$\mathcal{V} \leq_{\mathrm{RK}} \mathcal{U} \;:\Longleftrightarrow\; \exists f : X \to Y \;\; f(\mathcal{U}) = \mathcal{V} \text{ and } f \text{ is finite-to-one}$$

Clearly, $\leq_{\mathrm{RB}} \subseteq \leq_{\mathrm{RK}}$ and we have that

Lemma 2.2.1. $\leq_{\mathrm{RB}}$ *is a preorder.*

Proof. First, the identity id_X is finite-to-one, so we have that $\mathcal{U} \leq_{\mathrm{RB}} \mathcal{U}$.

Secondly, if we have that $\mathcal{W} \leq_{\mathrm{RB}} \mathcal{V} \leq_{\mathrm{RB}} \mathcal{U}$, then we can choose finite-to-one functions $f : X \to Y$ and $g : Y \to Z$ such that $f(\mathcal{U}) = \mathcal{V}$ and $g(\mathcal{V}) = \mathcal{W}$. But then $g \circ f : X \to Z$ is finite-to-one and Lemma 2.1.4 implies that

$$(g \circ f)(\mathcal{U}) = g(f(\mathcal{U})) = g(\mathcal{V}) = \mathcal{W}$$

which proves that $\mathcal{W} \leq_{\mathrm{RB}} \mathcal{U}$. □

Definition. *We write* $\equiv_{\mathrm{RB}}$ *for the equivalence relation induced by* $\leq_{\mathrm{RB}}$*.*

Remark. Note that Corollary 2.1.10 implies that if $|X| = |Y|$, then we have that

$$\mathcal{V} \equiv_{\mathrm{RB}} \mathcal{U} \iff \mathcal{V} \equiv_{\mathrm{RK}} \mathcal{U}$$

Finally, by combining our previous extension and restriction lemmata we get a useful characterization for when $\leq_{\mathrm{RK}}$ implies $\leq_{\mathrm{RB}}$. Notice that P-points satisfy the condition of the following lemma:

Lemma 2.2.2. *If* $|X| \leq |Y|$, $f : X \to Y$, $f(\mathcal{U}) = \mathcal{V}$ *and there exists* $U \in \mathcal{U}$ *such that* $f \restriction U$ *is finite-to-one, then we have that* $\mathcal{V} \leq_{\mathrm{RB}} \mathcal{U}$*.*

Proof. By Lemma 2.1.2 we know that $(f \restriction U)(\mathcal{U} \restriction U) = \mathcal{V} \restriction f[U]$. As $|X| \leq |Y|$ and $f \restriction U$ is finite-to-one we can pick an extension $\hat{f} : X \to Y$ of $f \restriction U$ which is finite-to-one. By Lemma 2.1.8 we get that $\hat{f}(\mathcal{U}) = \mathcal{V}$ which implies $\mathcal{V} \leq_{\mathrm{RB}} \mathcal{U}$. □

2.3 The Katětov order

Finally, we will extend the Rudin-Keisler order to arbitrary filters and consider the Katětov order. We will see that this order agrees with the Rudin-Keisler order on ultrafilters. Fix filters $\mathcal{F}, G$ and $\mathcal{H}$ on X, Y and Z, respectively.

Definition. *We define the Katětov order by*

$$\mathcal{G} \leq_{\mathrm{K}} \mathcal{F} \;:\Longleftrightarrow\; \exists f : X \to Y \text{ such that } \forall A \in \mathcal{G} \;\; f^{-1}[A] \in \mathcal{F}$$

Remark. Note that Lemma 2.1.1, 2.1.4 and 2.1.5 i) also work if we extend the definition of the image of an ultrafilter to arbitrary filters except for the characterization when the image of a filter is free in Lemma 2.1.1. Using this, we have the following alternative definition for the Katětov order:

$$\mathcal{G} \leq_{\mathrm{K}} \mathcal{F} \iff \exists f : X \to Y \text{ such that } \mathcal{G} \subseteq f(\mathcal{F}) \text{ is a subfilter}$$

This remark allows us to extend the definition of the Rudin-Keisler order to arbitrary filters. Furthermore, the above characterization immediately implies that the Rudin-Keisler order is a subrelation of the Katětov order:

Corollary 2.3.1. *We have that* $\leq_{\mathrm{RK}} \subseteq \leq_{\mathrm{K}}$. □

Lemma 2.3.2. $\leq_{\mathrm{K}}$ *is a preorder.*

Proof. For the identity id_X we have that $\mathcal{F} = \mathrm{id}_X(\mathcal{F})$ such that $\mathcal{F} \leq_{\mathrm{K}} \mathcal{F}$.

Now, if $\mathcal{H} \leq_{\mathrm{K}} \mathcal{G} \leq_{\mathrm{K}} \mathcal{F}$ choose $f : X \to Y$ and $g : Y \to Z$ such that $\mathcal{G} \subseteq f(\mathcal{U})$ and $\mathcal{H} \subseteq g(\mathcal{G})$. Then we have that

$$\mathcal{H} \subseteq g(\mathcal{G}) \subseteq g(f(\mathcal{F})) = (g \circ f)(\mathcal{F})$$

so $g \circ f : X \to Z$ is a witness for $\mathcal{H} \leq_{\mathrm{K}} \mathcal{F}$. □

Definition. *We write* $\equiv_{\mathrm{K}}$ *for the equivalence relation induced by* $\leq_{\mathrm{K}}$.

Using the alternative definition of $\leq_{\mathrm{K}}$ and the maximality of ultrafilters we can also prove that the Katětov order and the Rudin-Keisler order agree if we restrict to ultrafilters:

Corollary 2.3.3. *If* $\mathcal{U}$ *and* $\mathcal{V}$ *are ultrafilters on* X *and* Y*, respectively, then*

$$\mathcal{V} \leq_{\mathrm{RK}} \mathcal{U} \iff \mathcal{V} \leq_{\mathrm{K}} \mathcal{U}$$

Proof. If $\mathcal{V} \leq_{\mathrm{K}} \mathcal{U}$, take $f : X \to Y$ such that $\mathcal{V} \subseteq f(\mathcal{U})$. But $\mathcal{V}$ is an ultrafilter, so by maximality $\mathcal{V} = f(\mathcal{U})$ such that $\mathcal{V} \leq_{\mathrm{RK}} \mathcal{U}$. Corollary 2.3.1 yields the other direction. □

3 Ultrafilters on ω

In this section we take a closer look at special ultrafilters on ω, so every ultrafilter in this section is free and on ω. Fix $\mathcal{U}$ and $\mathcal{V}$ as names for such ultrafilters. Again, the presentation of this section extends the presentation given in [11]. We are interested in ultrafilters which can be characterized by the following combinatorial properties:

Definition. *$\mathcal{U}$ is called …*

i) … Ramsey (or selective) iff for every partition $\langle X_n \mid n < \omega\rangle$ of ω into sets not in $\mathcal{U}$, there exists an $A \in \mathcal{U}$ such that for every $n < \omega$ $|A \cap X_n| \leq 1$.

ii) … a P-point (or weakly selective) iff for every partition $\langle X_n \mid n < \omega\rangle$ of ω into sets not in $\mathcal{U}$, there exists an $A \in \mathcal{U}$ such that for every $n < \omega$ $|A \cap X_n| < \omega$.

iii) … a Q-point iff for every partition $\langle X_n \mid n < \omega\rangle$ of ω into finite pieces, there exists an $A \in \mathcal{U}$ such that for every $n < \omega$ $|A \cap X_n| \leq 1$.

iv) … a semi-Q-point (or rapid) iff for every partition $\langle X_n \mid n < \omega\rangle$ of ω into finite pieces, there exists an $A \in \mathcal{U}$ such that for every $n < \omega$ $|A \cap X_n| \leq n$.

Theorem 3.1. *An ultrafilter is Ramsey iff it is a P-point and a Q-point.*

Proof. Clearly, every Ramsey ultrafilter is a P-point and a Q-point. Conversely, assume that $\mathcal{U}$ is a P-point and a Q-point and let $\langle X_n \mid n < \omega\rangle$ be a partition of ω into sets not in $\mathcal{U}$. $\mathcal{U}$ is a P-point, so choose $A_0 \in \mathcal{U}$ such that for every $n < \omega$ $|A_0 \cap X_n| < \omega$.

To get a partition of ω into to finite pieces enumerate $\omega \setminus \bigcup_{n<\omega}(A_0 \cap X_n) = \langle a_n \mid n < \omega\rangle$ and define $Y_{2n} := A_0 \cap X_n$ and $Y_{2n+1} := \{a_n\}$. $\mathcal{U}$ is a Q-point, so choose $A_1 \in \mathcal{U}$ such that for every $n < \omega$ $|A_1 \cap Y_n| \leq 1$. Then $A := A_0 \cap A_1 \in \mathcal{U}$ and has the desired property. Indeed, let $n < \omega$. Then

$$|A \cap X_n| = |(A_0 \cap X_n) \cap A_1| = |Y_{2n} \cap A_1| \leq 1$$

□

Remark. The existence of Ramsey ultrafilters, P-points, Q-points and rapid ultrafilters are all independent from ZFC. However, CH or more generally $\mathfrak{p} = \mathfrak{c}$ is enough to prove the existence of Ramsey ultrafilters. Furthermore, it is consistent with $\mathfrak{c} = \aleph_2$ that there are no Q-points or P-points, respectively.

We can naturally identify partitions of ω and reals, so every definition above also has an analogous definitions via reals:

Lemma 3.2. *We have the following equivalences:*

i) $\mathcal{U}$ is Ramsey iff for every real f there is an $A \in \mathcal{U}$ such that $f \restriction A$ is constant or one-to-one.

ii) $\mathcal{U}$ is a P-point iff for every real f there is an $A \in \mathcal{U}$ such that $f \restriction A$ is constant or finite-to-one.

iii) $\mathcal{U}$ is a Q-point iff for every finite-to-one real f there is an $A \in \mathcal{U}$ such that $f \restriction A$ is one-to-one.

iv) $\mathcal{U}$ is rapid iff for every finite-to-one real f there is an $A \in \mathcal{U}$ such that $f \restriction A$ is diagonal-to-one (i.e. for every $n < \omega$ $|A \cap f^{-1}[\{n\}]| \leq n$).

Proof. Given a real f define $X_n := f^{-1}[\{n\}]$ to get a partition $\langle X_n \mid n < \omega\rangle \setminus \{\emptyset\}$ of ω. Conversely, given a partition $\langle X_n \mid n < \omega\rangle$ of ω define a real f by $f(k) := n$ if $k \in X_n$. Notice the following equivalences for every $A \in \mathcal{U}$:

a) f is finite-to-one iff each X_n is finite.

b) $f \restriction A$ is constant iff there is a $n < \omega$ such $A \subseteq X_n$.

c) $f \restriction A$ is finite-to-one iff for all $n < \omega$ $|A \cap X_n| < \omega$.

d) $f \restriction A$ is one-to-one iff for all $n < \omega$ $|A \cap X_n| \leq 1$.

e) $f \restriction A$ is diagonal-to-one iff for all $n < \omega$ $|A \cap X_n| \leq n$.

Now, i) follows from b) and d), ii) follows from b) and c), iii) follows from a) and d) and iv) follows from a) and e). □

Our previous study of the Rudin-Keisler order and this characterization of Ramsey ultrafilters gives us an easy proof that these filters are exactly the $\leq_{\mathrm{RK}}$-minimal elements in the Rudin-Keisler order if we restrict to free ultrafilters on ω:

Theorem 3.3. *$\mathcal{U}$ is Ramsey iff $\mathcal{U}$ is minimal in $\leq_{\mathrm{RK}}$ restricted to free ultrafilters on ω.*

Proof. Assume $\mathcal{U}$ is Ramsey and let $\mathcal{V} \leq_{\mathrm{RK}} \mathcal{U}$, so choose a real f such that $f(\mathcal{U}) = \mathcal{V}$. By the previous lemma choose $A \in \mathcal{U}$ such that $f \restriction A$ is constant or one-to-one. By assumption $\mathcal{V}$ is free, so by Lemma 2.1.1 $f \restriction A$ cannot be constant. Hence, the restriction $f \restriction A : A \to f[A]$ is a bijection, $A \in \mathcal{U}$, $f[A] \in \mathcal{V}$ and we have $(f \restriction A)(\mathcal{U} \restriction A) = \mathcal{V} \restriction f[A]$ by Lemma 2.1.2. Thus, $\mathcal{V} \equiv_{\mathrm{RK}} \mathcal{U}$.

Conversely, assume that $\mathcal{U}$ is $\leq_{\mathrm{RK}}$-minimal. Let f be a real and assume that f is not constant on any set in $\mathcal{U}$. By Lemma 2.1.1 $f(\mathcal{U})$ is free and $f(\mathcal{U}) \leq_{\mathrm{RK}} \mathcal{U}$, so by assumption $f(\mathcal{U}) \equiv_{\mathrm{RK}} \mathcal{U}$. By Corollary 2.1.10 we can take a bijection $g \in {}^{\omega}\omega$ such that $g(f(\mathcal{U})) = \mathcal{U}$. By Lemma 2.1.7 there is an $A \in \mathcal{U}$ such that $(g \circ f) \restriction A = \mathrm{id}_A$. But this shows that $f \restriction A$ is one-to-one, so by Lemma 3.2 $\mathcal{U}$ is Ramsey. □

We finish this section by proving three useful lemmata which characterize Q-points and rapid ultrafilters by containing certain sparse sets. The first lemma will give us $\mathfrak{b} \leq \mathfrak{u}$ as a nice byproduct, where $\mathfrak{u}$ is the ultrafilter number:

Lemma 3.4. *The family $\{\mathrm{enum}(A) \mid A \in \mathcal{U}\}$ is unbounded.*

Proof. Let f be a real, w.l.o.g. f is nowhere zero. The idea is to define increasingly big intervals and group them alternatingly into sets X and Y such that their enumerating functions have time to catch up with f. As $\mathcal{U}$ is an ultrafilter either $X \in \mathcal{U}$ or $Y \in \mathcal{U}$, so the result will follow. In detail, define a strictly increasing real g by

$$g(0) := 0 \qquad \text{and for all } n < \omega \qquad g(n+1) := g(n) + f(g(n))$$

Furthermore, define interval partitions

$$X_n := [g(2n), g(2n+1)) \qquad \text{and} \qquad Y_n := [g(2n+1), g(2n+2))$$

Then for $X := \bigcup_{n<\omega} X_n$ and $Y := \bigcup_{n<\omega} Y_n$ we have $X \cup Y = \omega$. Thus, $X \in \mathcal{U}$ or $Y \in \mathcal{U}$, w.l.o.g. $X \in \mathcal{U}$ as in the other case we would get $\mathrm{enum}(Y) \not\leq^* f$ analogously.

We show that $\mathrm{enum}(X) \not\leq^* f$, so let $N < \omega$ and choose $n < \omega$ with $g(2n+1) > N$. By definition of X_k and g we have that $\bigcup_{k\leq n} X_k \subseteq [0, g(2n+1))$, hence

$$|\bigcup_{k\leq n} X_k| \leq g(2n+1)$$

But this gives us that $\mathrm{enum}(X)(g(2n+1)) \in \bigcup_{k>n} X_k$. Thus, $\mathrm{enum}(X) \not\leq^* f$ as

$$\mathrm{enum}(X)(g(2n+1)) \geq \min(X_{n+1}) = g(2n+2) \geq f(g(2n+1)) \qquad \square$$

Corollary 3.5. $\mathfrak{b} \leq \mathfrak{u}$.

Proof. Given a basis $\mathcal{B}$ of $\mathcal{U}$, we can strengthen the previous lemma to

The family $\{\mathrm{enum}(A) \mid A \in \mathcal{B}\}$ is unbounded.

as for any $A \in \mathcal{U}$ we can pick $B \in \mathcal{B}$ such that $B \subseteq A$. But then $\mathrm{enum}(A) \leq \mathrm{enum}(B)$. This strengthened result immediately implies that $\mathfrak{b} \leq \mathfrak{u}$. $\square$

Lemma 3.6. *$\mathcal{U}$ is a Q-point iff for every real f there is an $A \in \mathcal{U}$ such that for all $n < \omega$*

$$f(\mathrm{enum}(A)(n)) < \mathrm{enum}(A)(n+1)$$

Proof. First, assume $\mathcal{U}$ is a Q-point. Let f be a real, w.l.o.g. f is strictly increasing. The idea is again to define increasingly big intervals, where each interval contains the maximum value of the previous interval under f. If we pick one element in every second interval, this sequence will satisfy the equation from above. In detail, define a strictly increasing real g by

$$g(0) := 0, \ g(1) := 1 \qquad \text{and for all } k > 1 \qquad g(k+1) := \max\{f(n) \mid n < g(k)\} + 2$$

Define an interval partition of ω by $I_k := [g(k), g(k+1))$. By assumption on $\mathcal{U}$ choose $A \in \mathcal{U}$ such that for every $k < \omega$ $|A \cap I_k| \leq 1$. Now, let

$$A_0 := \{\mathrm{enum}(A)(2n) \mid n < \omega\} \qquad \text{and} \qquad A_1 := A \setminus A_0$$

Then $A_0 \in \mathcal{U}$ or $A_1 \in \mathcal{U}$, w.l.o.g. $A_0 \in \mathcal{U}$. We claim that A_0 is as desired:

Indeed, let $n < \omega$. Choose $k < \omega$ such that $\mathrm{enum}(A_0)(n) \in I_k$. By definition of I_k we have that $\mathrm{enum}(A_0)(n) < g(k+1)$, so by definition of g we get

$$f(\mathrm{enum}(A_0)(n)) < g(k+2)$$

But A intersects each I_k at most once, so A_0 can only intersect every second I_k at most once. Together with $\mathrm{enum}(A_0)(n) \in I_k$ we get that

$$\mathrm{enum}(A_0)(n+1) \in \bigcup_{l \geq k+2} I_l$$

which gives us $\mathrm{enum}(A_0)(n+1) \geq g(k+2)$ by definition of I_{k+2}. Together, we get

$$f(\mathrm{enum}(A_0)(n)) < g(k+2) \leq \mathrm{enum}(A_0)(n+1)$$

as desired. The case $A_1 \in \mathcal{U}$ follows analogously.

Conversely, we use the characterization of Q-points via reals given by Lemma 3.2, so let f be a finite-to-one real. Define a real g by

$$g(n) := \max\{k \mid f(k) = f(n)\}$$

By assumption choose $A \in \mathcal{U}$ such that for all $n < \omega$ we have that

$$g(\mathrm{enum}(A)(n)) < \mathrm{enum}(A)(n+1)$$

Now, assume $f \upharpoonright A$ was not one-to-one. Then take $n < k < \omega$ such that

$$f(\mathrm{enum}(A)(n)) = f(\mathrm{enum}(A)(k))$$

By choice of A we have that

$$g(\mathrm{enum}(A)(n)) < \mathrm{enum}(A)(n+1) \leq \mathrm{enum}(A)(k)$$

But by definition of g this means that

$$f(\mathrm{enum}(A)(n)) \neq f(\mathrm{enum}(A)(k))$$

which is a contradiction. □

Finally, we will most commonly use the following lemma which characterizes rapid filters as exactly those filters which enumerating functions not only form an unbounded but a dominating family of reals:

Lemma 3.7. *$\mathcal{U}$ is rapid iff the family $\{\mathrm{enum}(A) \mid A \in \mathcal{U}\}$ is dominating.*

Proof. First, assume $\mathcal{U}$ is rapid. Let g be a real, w.l.o.g. g is strictly increasing. Again, the idea is to define increasingly big intervals, but this time we have to account for the fact that our choice set A may intersect the n-th interval n times, which is where the Gaussian sums come from. In detail, define a partition of ω into finite pieces by

$$I_n := [g(\sum_{i=1}^{n} i), g(\sum_{i=1}^{n+1} i))$$

Take $A \in \mathcal{U}$ such that for every $n < \omega$ $|A \cap I_n| \leq n$. We will show that $g < \mathrm{enum}(A)$, so let $n < \omega$. Then we can write $n = (\sum_{i=1}^{k} i) + l$ for some $k < \omega$ and $0 \leq l \leq k$. Now,

$$|A \cap \bigcup_{i=1}^{k} I_i| = \sum_{i=1}^{k} |A \cap I_i| \leq \sum_{i=1}^{k} i \leq n$$

so we get that

$$\mathrm{enum}(A)(n) \in \bigcup_{i \geq k+1} I_i$$

Together, we have

$$g(n) = g((\sum_{i=1}^{k} i) + l) < g(\sum_{i=1}^{k+1} i) = \min(I_{k+1}) \leq \mathrm{enum}(A)(n)$$

which proves that $g < \mathrm{enum}(A)$.

Conversely, assume that $\{\mathrm{enum}(A) \mid A \in \mathcal{U}\}$ is dominating and let f be a finite-to-one real. Define a real g by

$$g(n) := \max(f^{-1}[\{n\}])$$

By assumption choose $A \in \mathcal{U}$ such that $g <^* \mathrm{enum}(A)$. $\mathcal{U}$ is free, so by removing finitely many elements from A we may assume that $g < \mathrm{enum}(A)$. We have to show that $f \restriction A$ is diagonal-to-one:

Indeed, let $n < \omega$, then for every $m \geq n$ we have that

$$\mathrm{enum}(A)(m) \geq \mathrm{enum}(A)(n) > g(n) = \max(f^{-1}[\{n\}])$$

Thus, $f(\mathrm{enum}(A)(m)) = n$ is only possible for $m < n$, so we get that

$$|A \cap f^{-1}[\{n\}]| \leq n$$

By Lemma 3.2 this proves that $\mathcal{U}$ is rapid. □

4 A necessary condition for Canjar ultrafilters

In this section we will take a closer look at the Mathias-Prikry forcing and consider the results in [7] to prove Canjar's necessary condition for ultrafilters which have the property that their associated Mathias-Prikry forcing does not add a dominating real. Furthermore, for an ultrafilter $\mathcal{U}$ we will introduce a strengthening of this property called λ-nondominatingness and consider how this property relates to the cofinality of ${}^{\omega}\omega/\mathcal{U}$.

4.1 The Mathias-Prikry forcing

First, we introduce the Mathias-Prikry forcing and contemplate the properties of the generic added by this forcing:

Definition. *Let $\mathcal{F}$ be a filter on ω. Define the Mathias-Prikry forcing for $\mathcal{F}$ by*

$$\mathbb{M}_{\mathcal{F}} := \{(s, A) \mid s \in [\omega]^{<\omega}, A \in \mathcal{F}\}$$

where $(t, B) \leq (s, A)$ iff $B \subseteq A$, $s \subseteq t$ and $t \setminus s \subseteq A$.

We write $\dot{A}_{\text{gen}}$ for the name of the generic set added by this forcing, i.e. the name for the union of the first coordinates of the generic filter and $\dot{f}_{\text{gen}}$ for the name of the generic real added by this forcing, i.e. the name for the enumeration of $\dot{A}_{\text{gen}}$. Given a $\mathbb{M}_{\mathcal{F}}$-generic filter G in $V[G]$ we write A_G for $\dot{A}^G_{\text{gen}}$ and f_G for $\dot{f}^G_{\text{gen}}$. Notice that $\mathbb{M}_{\mathcal{F}}$ is non-trivial exactly iff $\mathcal{F}$ is free, so from now on we will only consider free filters. With this assumption we have that the set

$$D := \{(s, A) \in \mathbb{M}_{\mathcal{F}} \mid \max(s) < \min(A)\}$$

is dense in $\mathbb{M}_{\mathcal{F}}$. Indeed, for any $(s, A) \in \mathbb{M}_{\mathcal{F}}$ we can intersect A with the cofinite set $\omega \setminus (\max(s) + 1)$ to obtain an extension in D, so we may assume this extra condition when picking extensions in $\mathbb{M}_{\mathcal{F}}$. This is beneficial as for any $(s, A) \in D$ we have that

$$(s, A) \Vdash \operatorname{enum}(s) \subseteq \dot{f}_{\text{gen}}$$

as only elements greater than $\max(s)$ may be added to the generic set. First, we show that a $\mathbb{M}_{\mathcal{F}}$-generic filter G is completely decided by A_G:

Lemma 4.1.1. *Let G be $\mathbb{M}_{\mathcal{F}}$-generic. Then we can reconstruct G from A_G by*

$$G = G_{A_G} := \{(s, A) \mid s \subseteq A_G, A \in \mathcal{F}, A_G \setminus s \subseteq A\}$$

Proof. Clearly, if $(s, A) \in G$, then $s \subseteq A_G$ and $A \in \mathcal{F}$ and if $n \in A_G \setminus s$ choose $(t, B) \in G$ extending (s, A) such that $n \in t$. Then $n \in t \setminus s \subseteq A$, so $(s, A) \in G_{A_G}$.

Conversely, we show that G_{A_G} is a filter, so let $(t, B) \in G_{A_G}$ and $(t, B) \leq (s, A)$. Then $s \subseteq t \subseteq A_G$, $A \in \mathcal{F}$ and $B \subseteq A$, so we have that $(s, A) \in G_{A_G}$ as

$$A_G \setminus s = A_G \setminus t \cup t \setminus s \subseteq B \cup A \subseteq A$$

Now, let $(s, A), (t, B) \in G_{A_G}$. Then $s, t \subseteq A_G$ imply $s \cup t \subseteq A_G$; $A, B \in \mathcal{F}$ imply $A \cap B \in \mathcal{F}$ and $A_G \setminus s \subseteq A, A_G \setminus t \subseteq B$ imply $A_G \setminus s \cup t \subseteq A \cap B$, so $(s \cup t, A \cap B) \in G_{A_G}$. Also, $(s, A), (t, B) \leq (s \cup t, A \cap B)$, because $s \cup t \setminus s \subseteq A_G \setminus s \subseteq A$ and analogously we have that $s \cup t \setminus t \subseteq A_G \setminus t \subseteq B$.

Now, we have that G is $\mathbb{M}_{\mathcal{F}}$-generic, $G \subseteq G_{A_G}$ and G_{A_G} is a filter, so $G = G_{A_G}$. $\square$

Definition. *We say that a set $A \subseteq \omega$ is a $\mathcal{F}$-Mathias set over V if there is a $\mathbb{M}_{\mathcal{F}}$-generic filter G over V such that $A = A_G$. If the context is unambiguous we also just say that A is a Mathias set.*

Proposition 4.1.2. *If A is a Mathias set, then $A = A_{G_A}$.*

Proof. If $n \in A$, then $(\{n\}, \omega) \in G_A$, so $n \in A_{G_A}$. Conversely, if $n \in A_{G_A}$ then take $(s, A) \in G_A$ such that $n \in s$. By definition of G_A we have that $s \subseteq A$, so $n \in A$. □

Thus, together with the previous lemma we have a one-to-one correspondence between $\mathbb{M}_{\mathcal{F}}$-generic filters G over V and Mathias sets A_G over V. This one-to-one correspondence extends to the generic real f_G via range and enum of A_G. Next, we transfer the dense set definition of generic filters to Mathias sets:

Proposition 4.1.3. *A is a $\mathcal{F}$-Mathias set over V iff for every dense set $D \in V$ in $\mathbb{M}_{\mathcal{F}}$ there is a condition $(s, B) \in D$ such that $s \subseteq A$ and $A \setminus s \subseteq B$.*

Proof. Assume that $A = A_G$ for some $\mathbb{M}_{\mathcal{F}}$-generic G and let $D \in V$ be dense in $\mathbb{M}_{\mathcal{F}}$. Take $(s, B) \in G \cap D$, then $s \subseteq A_G$ and $A_G \setminus s \subseteq B$ as for any extension $(t, C) \in G$ of (s, B) we have that $t \setminus s \subseteq B$.

Conversely, let $D \in V$ be dense in $\mathbb{M}_{\mathcal{F}}$. Take $(s, B) \in D$ with $s \subseteq A$ and $A \setminus s \subseteq B$. Then $(s, B) \in G_A$, so G_A is $\mathbb{M}_{\mathcal{F}}$-generic and we have that $A = A_{G_A}$. □

Lastly, we note that Mathias sets are pseudointersections of their filter:

Lemma 4.1.4. *The generic set added by $\mathbb{M}_{\mathcal{F}}$ is a pseudointersection of $\mathcal{F}$.*

Proof. Let $B \in \mathcal{F}$ and $(s, A) \in \mathbb{M}_{\mathcal{F}}$, then $(s, A \cap B) \in \mathbb{M}_{\mathcal{F}}$ extends (s, A) and forces that

$$(s, A \cap B) \Vdash \dot{A}_{\text{gen}} \subseteq^* B$$

as in any $V[G]$ with $(s, A \cap B) \in G$ we have that $A_G \setminus s \subseteq B$. □

4.2 Proof of Canjar's necessary condition

In this section we will consider filters whose associated Mathias-Prikry forcing does not add a dominating real and follow Canjar's work in [7] to prove a necessary condition for them if we restrict to ultrafilters. More specifically, we will prove that such ultrafilters are P-points, which do not have rapid Rudin-Keisler predecessors. As Canjar was the first to study these type of ultrafilters they were named after him:

Definition. *A filter $\mathcal{F}$ is called Canjar if $\mathbb{M}_{\mathcal{F}}$ does not add dominating reals.*

An easy example for a Canjar filter is the cofin-filter. In this case $\mathbb{M}_{\mathcal{F}}$ is forcing equivalent to the Cohen forcing, which does not add dominating reals. Later, we will construct a more interesting example - following Canjar's proof in [7] we will construct a Canjar ultrafilter under the assumption of $\mathfrak{d} = \mathfrak{c}$.

To prove that Canjar's condition is indeed necessary for ultrafilters we will first show that given a Mathias set for $\mathcal{F}$ we can construct Mathias sets for all Rudin-Keisler predecessors of $\mathcal{F}$:

Lemma 4.2.1. *Let A_G be a $\mathcal{F}$-Mathias set over V and $\mathcal{G} = f(\mathcal{F})$. Then we have that $f[A_G]$ is a $\mathcal{G}$-Mathias set over V.*

Proof. Let $D \in V$ be a dense set in $\mathbb{M}_{\mathcal{G}}$. Define

$$D' := \{(s, A) \mid \exists (t, B) \in D \text{ such that } f[s] = t \text{ and } f[A] \subseteq B\} \in V$$

We show that D' is dense in $\mathbb{M}_{\mathcal{F}}$, so let $(s, A) \in \mathbb{M}_{\mathcal{F}}$. Then $(f[s], f[A]) \in \mathbb{M}_{\mathcal{G}}$, so choose an extension $(t, B) \in D$. This means that

$$f[s] \subseteq t, \qquad B \subseteq f[A] \qquad \text{and} \qquad t \setminus f[s] \subseteq f[A]$$

By the last property we can choose some finite $u \subseteq A$ with $t \setminus f[s] = f[u]$. But then

$$f[s \cup u] = f[s] \cup f[u] = f[s] \cup t \setminus f[s] = t$$

Also, $B \in \mathcal{G} = f(\mathcal{F})$, so we may choose $C' \in \mathcal{F}$ such that $f[C'] \subseteq B$, so $C := A \cap C' \in \mathcal{F}$. Then $(s \cup u, C) \in D'$ and $(s \cup u, C) \leq (s, A)$.

Hence, D' is dense in $\mathbb{M}_{\mathcal{F}}$, so as A_G is a $\mathcal{F}$-Mathias set over V take $(s, A) \in D$ such that $s \subseteq A_G$ and $A_G \setminus s \subseteq A$. Take $(t, B) \in D$ such that $f[s] = t$ and $f[A] \subseteq B$. Then $t = f[s] \subseteq f[A_G]$ and

$$f[A_G] \setminus t = f[A_G] \setminus f[s] \subseteq f[A_G \setminus s] \subseteq f[A] \subseteq B$$

so $f[A_G]$ is a $\mathcal{G}$-Mathias set over V. $\square$

Note that by Lemma 4.1.1 we can reconstruct the generic filter from a Mathias set, so $\leq_{\mathrm{RK}}$-smaller filters give us smaller models in the following sense:

Remark 4.2.2. If $\mathcal{G} \leq_{\mathrm{RK}} \mathcal{F}$ and $\mathcal{G}$ is a $\mathbb{M}_{\mathcal{F}}$-generic filter, then we can choose a real f such that $f(\mathcal{F}) = \mathcal{G}$. By the previous corollary we have that $A_H := f[A_G]$ is a $\mathcal{G}$-Mathias set. By Lemma 4.1.1 let H be the associated $\mathbb{M}_{\mathcal{G}}$-generic filter of A_H. Thus, in $V[G]$ the generic H is definable, i.e. $V[H] \subseteq V[G]$.

Moreover, for Rudin-Keisler-equivalent ultrafilters we get the following result:

Corollary 4.2.3. *If $\mathcal{V} \equiv_{\mathrm{RK}} \mathcal{U}$ then $\mathbb{M}_{\mathcal{U}}$ and $\mathbb{M}_{\mathcal{V}}$ are forcing equivalent.*

Proof. If $\mathcal{V} \equiv_{\mathrm{RK}} \mathcal{U}$ we can assume that f is a bijection in the setup of the above remark. Then we can use the same construction using f^{-1} to define $A_G = f^{-1}[f[A_G]] = f^{-1}[A_H]$ in $V[H]$. But then $V[H] = V[G]$. $\square$

Next, we will prove that the generic real $\dot{f}_{\mathrm{gen}}$ added by $\mathbb{M}_{\mathcal{F}}$ is dominating exactly iff $\mathcal{F}$ is rapid. For this we will have to extend our definition of rapid to all filters:

Definition. *A filter $\mathcal{F}$ is called rapid iff $\{\mathrm{enum}(A) \mid A \in \mathcal{F}\}$ is dominating.*

This is a reasonable definition as Lemma 3.7 shows that this definition agrees with our definition of rapid for ultrafilters. It should be noted that as for ultrafilters the existence of a rapid filter is independent of ZFC.

Also note that the following lemma is a condition for every forcing which diagonalizes a rapid filter $\mathcal{F}$ and not only for the Mathias-Prikry forcing. Together with the previous remark this lemma will give us one half of Canjar's condition, namely that a Canjar filter cannot have rapid Rudin-Keisler predecessors:

Lemma 4.2.4. *Let $\mathcal{F} \in V$ be a rapid filter, G be $\mathbb{P}$-generic and $X \in V[G]$ be a pseudointersection of $\mathcal{F}$. Then* $\mathrm{enum}(X)$ *is a dominating real. Conversely for $\mathbb{M}_\mathcal{F}$, if G is $\mathbb{M}_\mathcal{F}$-generic and f_G is a dominating real, then $\mathcal{F}$ is rapid.*

Proof. Assume $\mathcal{F}$ is rapid, let $g \in V$ be a real. W.l.o.g. g is strictly increasing and define a real h by $h(n) := g(2n)$. $\mathcal{F}$ is rapid and free, so take $A \in \mathcal{F}$ such that $h < \mathrm{enum}(A)$. But X is a pseudointersection of $\mathcal{F}$, so $X \subseteq^* A$ and we can choose some $N < \omega$ such that for all $n < \omega$ $\mathrm{enum}(X)(N+n) \in A$. Thus, for all $n < \omega$ we have that

$$\mathrm{enum}(X)(N+n) \geq \mathrm{enum}(A)(n) > h(n) = g(2n)$$

Now, for all $k > 2N$ let $n = \lceil \frac{k}{2} \rceil$. Then $k \leq 2n$ and $N + n \leq k$, so

$$g(k) \leq g(2n) \leq \mathrm{enum}(X)(N+n) \leq \mathrm{enum}(X)(k)$$

But this shows that $\mathrm{enum}(X) <^* g$.

Conversely, assume that f_G is a dominating real after forcing with $\mathbb{M}_\mathcal{F}$. So if $g \in V$, then we have that

$$\Vdash g <^* \dot{f}_{\mathrm{gen}}$$

so we can take $(s, A) \in \mathbb{M}_\mathcal{F}$ and $N < \omega$ such that $\max(s) < \min(A)$ and

$$(s, A) \Vdash \forall n \geq N \; g(n) < \dot{f}_{\mathrm{gen}}(n)$$

Then $s \cup A \in \mathcal{F}$ and we claim that $g <^* \mathrm{enum}(s \cup A)$. Indeed, for $n \geq N$ let t be the set of the $n+1$ smallest elements of A. Then for $(s \cup t, A \setminus t) \leq (s, A)$ we also have that

$$(s \cup t, A \setminus t) \Vdash \forall n \geq N \; g(n) < \dot{f}_{\mathrm{gen}}(n)$$

But $n \in \mathrm{domain}(\mathrm{enum}(s \cup t))$, so we also have that

$$(s \cup t, A \setminus t) \Vdash \dot{f}_{\mathrm{gen}}(n) = \mathrm{enum}(s \cup t)(n)$$

Together, this yields that

$$g(n) < \mathrm{enum}(s \cup t)(n) = \mathrm{enum}(s \cup A)(n)$$

so we get that $g <^* \mathrm{enum}(s \cup A)$ as desired. □

As the Mathias-Prikry forcing diagonalizes its filter we get the following corollary:

Corollary 4.2.5. *When forcing with $\mathbb{M}_\mathcal{F}$ we have that f_G is dominating iff $\mathcal{F}$ is rapid.*

Now, we strengthen the definition of Canjar filters with the following definition:

Definition. *A λ-c.c partial order $\mathbb{P}$ is called λ-nondominating iff for every family of reals in $V[G]$ of size $|D| < \lambda$ there is a $h \in V$ which is not dominated by any real in D.*

In case that $\mathbb{P} = \mathbb{M}_\mathcal{F}$ we also say that $\mathcal{F}$ is λ-nondominating.

Remark. The maximum λ for which there exists a λ-nondominating partial order is $\mathfrak{d}$.

Proof. The trivial forcing is of course $\mathfrak{d}$-nondominating and for $\lambda > \mathfrak{d}$, by definition of $\mathfrak{d}$ the ground model already contains a family of size $\mathfrak{d}$ which dominates all reals, so its canonical name witnesses that the partial order cannot be λ-nondominating. □

In the third subsection we will see that the existence of a $\mathfrak{d}$-nondominating filter is actually consistently stronger than the existence of a Canjar filter in the sense that there are models in which there are Canjar ultrafilters but no $\mathfrak{d}$-nondominating filters.

Lemma 4.2.6. *Assume $\mathbb{P}$ is c.c.c. and does not add dominating reals. Then we have that $\mathbb{P}$ is $\mathfrak{b}$-nondominating.*

Proof. Let $\dot{D}$ be a $V^{\mathbb{P}}$-name for a family of reals of size $\gamma < \mathfrak{b}$. $\mathbb{P}$ is c.c.c., so we can choose $D = \{\dot{f}_\alpha \mid \alpha < \gamma\}$ a family of $V^{\mathbb{P}}$-names for reals which covers $\dot{D}$, i.e.

$$\Vdash \forall f \in \dot{D}\ \exists \alpha < \gamma\ f = \dot{f}_\alpha$$

By assumption $\mathbb{P}$ does not add dominating reals, so for every $\alpha < \gamma$ we have that

$$\Vdash \exists g \in V\ g \not<^* \dot{f}_\alpha \qquad \text{i.e.} \qquad D_\alpha := \{p \mid \exists g\ p \Vdash g \not<^* \dot{f}_\alpha\} \text{ is dense in } \mathbb{P}$$

But $\mathbb{P}$ is c.c.c. so for each $\alpha < \gamma$ we can choose a countable maximal antichain A_α in D_α, so by choosing witnesses g for each $p \in A_\alpha$ there is a countable set of reals G_α such that

$$\Vdash \exists g \in G_\alpha\ g \not<^* \dot{f}_\alpha$$

Thus, as $\omega < \mathfrak{b}$ we know that their union $G := \bigcup_{\alpha<\gamma} G_\alpha$ is in V and has size $|G| \leq \gamma < \mathfrak{b}$. By definition of $\mathfrak{b}$ let h be a real which bounds G. Then h has the desired property. Indeed, as D covers $\dot{D}$ let $\alpha < \gamma$, then

$$\Vdash \exists g \in G_\alpha\ g \not<^* \dot{f}_\alpha \qquad \text{and} \qquad \Vdash \forall g \in G_\alpha\ g <^* h$$

which yields that

$$\Vdash h \not<^* \dot{f}_\alpha$$

□

So as $\mathbb{M}_{\mathcal{F}}$ is clearly c.c.c., in fact σ-centered, we get that being Canjar is actually equivalent to being $\mathfrak{b}$-nondominating. The prior results of this section already give us some structure of λ-nondominating filters in the Rudin-Keisler order:

Corollary 4.2.7. *Each class of λ-nondominating filters is closed downwards in the Rudin-Keisler order.*

Proof. Assume that $\mathcal{G} \leq_{\mathrm{RK}} \mathcal{F}$ and that $\mathcal{G}$ is λ-dominating. As in Remark 4.2.2 let f be a real such that $\mathcal{G} = f(\mathcal{F})$ and let G be $\mathbb{M}_{\mathcal{F}}$-generic and H be the associated $\mathbb{M}_{\mathcal{G}}$-generic filter. Then $V[H] \subseteq V[G]$, so any witness that $\mathcal{G}$ is λ-dominating is also a witness that $\mathcal{F}$ is λ-dominating. □

This result implies the first half of Canjar's necessary condition. Note that we proved that the following condition holds for arbitrary filters and not only for ultrafilters as Canjar noted in [7].

Corollary 4.2.8. *If $\mathcal{F}$ is $\mathfrak{b}$-nondominating, then $\mathcal{F}$ has no rapid predecessors in the Rudin-Keisler order.*

Proof. This follows directly from Corollary 4.2.5, which shows that the Mathias real added by a rapid filter $\mathcal{G}$ is dominating, so in particular $\mathbb{M}_{\mathcal{G}}$ is $\mathfrak{b}$-dominating, and Corollary 4.2.7 by which $\mathcal{G}$ cannot be a $\leq_{\mathrm{RK}}$-predecessor of $\mathcal{F}$. □

The second half of Canjar's condition is concerned with ultrafilters, so for the rest of this section we will restrict our attention to ultrafilters and more closely follow Canjar's proofs in [7]. We will show that every forcing which diagonalizes a non P-point adds a dominating real. As the Mathias-Prikry forcing diagonalizes its filter we get that Canjar filters are necessarily P-points:

Lemma 4.2.9. *Let $\mathcal{U} \in V$ not be a P-point, G be $\mathbb{P}$-generic and $X \in V[G]$ be a pseudointersection of $\mathcal{U}$. Then $\mathbb{P}$ adds a dominating real.*

Proof. By Lemma 3.2 we can choose a real f such that for all $A \in \mathcal{U}$ we have that $f \restriction A$ is not constant or finite-to-one. Thus, for any $k < \omega$ we have that

$$A_k := \{a \mid f(a) > k\} = \omega \setminus \bigcup_{a \leq k} f^{-1}[\{a\}] \in \mathcal{U}$$

But X is a pseudointersection of $\mathcal{U}$, so $X \subseteq^* A_k$, which means that only finitely many elements in X map to a value $\leq k$ under f. Thus, we may define a real h in $V[G]$ as the maximum of these elements:

$$h(k) := \max\{a \in X \mid f(a) \leq k\}$$

Then for $a \in X$ we have that $h(f(a)) \geq a$ and we can define $H(n) := h(\mathrm{Next}(f(X), n))$, where $\mathrm{Next}(A, n)$ is the smallest element of A which is $\geq n$. We claim that H is a dominating real.

Indeed, let $g \in V$, w.l.o.g. g is increasing. Note that for $E := \{n \mid g(f(n)) < n\}$ we have that f is finite-to-one on $\omega \setminus E$. Indeed, let $k < \omega$. Then $n \in f^{-1}[\{k\}] \cap (\omega \setminus E)$ implies that $f(n) = k$, so $g(k) = g(f(n)) \geq n$. Thus,

$$|f^{-1}[\{k\}] \cap (\omega \setminus E)| \leq g(k) + 1$$

By our choice of f we have that $E \in \mathcal{U}$, so again $X \subseteq^* E$. Thus, for almost all elements $a \in X$ we have that $g(f(a)) \leq a$. Now, the proof idea is that using that bound for sufficiently big n we have that $g(n) \leq g(f(a)) < a$, where $f(a)$ is the smallest element in $f(X)$ greater than n. But we have chosen $H(n) = h(f(a)) \geq a$, proving that $g <^* H$.

In detail, choose $N > \max(X \setminus E)$ and let $N' := \max\{f(n) \mid n \leq N\}$. We claim that for all $n > N'$ we have that $g(n) < H(n)$.

Indeed, let $n > N'$ and $\mathrm{Next}(f(X), n) = f(a)$ for some $a \in X$. Then $f(a) \geq n > N'$, so by choice of N' we have that $a > N$. But then by choice of N we have that $a \in E$ such that $g(f(a)) < a$. But g is increasing and $n \leq f(a)$, so we have $g(n) \leq g(f(a)) < a$. Now, $a \in X$, so by definition of H and h we have that

$$g(n) < a \leq h(f(a)) = H(n)$$ □

Finally, if we combine the previous results we get the following necessary condition for Canjar ultrafilters:

Corollary 4.2.10. *If $\mathbb{M}_{\mathcal{U}}$ is $\mathfrak{b}$-nondominating, then $\mathcal{U}$ is a P-point and has no rapid predecessors in the Rudin-Keisler order.* □

Canjar asked in [7] if this condition is also sufficient for Canjar ultrafilters. Following Laflamme [14] we will later introduce the notion of a strong P-point, which will turn out to be an equivalent characterization of Canjar ultrafilters. However, we will also prove that this notion is in fact consistently stronger than the condition we have just considered, which answers Canjar's question negatively.

4.3 λ-nondominatingness and the cofinality of ${}^{\omega}\omega/\mathcal{U}$

We finish this section by studying how the λ-nondominatingness of a forcing which diagonalizes $\mathcal{U}$ relates to the cofinality of $({}^{\omega}\omega/\mathcal{U}, <_{\mathcal{U}})$. In fact, Canjar proved in [6] that there is a rich connection between combinatorial properties of $\mathcal{U}$ and model-theoretic properties of the $\mathcal{U}$-ultrapower of $(\omega, <)$ and more generally the $\mathcal{U}$-ultrapower of the standard model of full arithmetic. So the following result is only a small snapshot of this type of research. We introduce the necessary notions of interest:

Definition. *We write ${}^{\omega}\omega/\mathcal{U}$ for the $\mathcal{U}$-ultrapower of $(\omega, <)$, i.e. the structure which universe is the set of all equivalence classes $[\cdot]_{\mathcal{U}}$ of reals under $\equiv_{\mathcal{U}}$ together with the linear order $<_{\mathcal{U}}$, where for $f, g \in {}^{\omega}\omega$ we define*

$$\begin{aligned} f \equiv_{\mathcal{U}} g \;&:\Longleftrightarrow \{n < \omega \mid f(n) = g(n)\} \in \mathcal{U} \\ [f]_{\mathcal{U}} <_{\mathcal{U}} [g]_{\mathcal{U}} \;&:\Longleftrightarrow \{n < \omega \mid f(n) < g(n)\} \in \mathcal{U} \end{aligned}$$

Remark. If we define for $n < \omega$ that $[n]_{\mathcal{U}}$ is the $\mathcal{U}$-equivalence class of the constant function with value n, then by Łoś Theorem the map $n \mapsto [n]_{\mathcal{U}}$ is an elementary embedding of ω into ${}^{\omega}\omega/\mathcal{U}$, so we can understand ω as an initial segment of ${}^{\omega}\omega/\mathcal{U}$ with respect to the order $<_{\mathcal{U}}$. With this interpretation we say that the constant functions form the standard part in ${}^{\omega}\omega/\mathcal{U}$ and we say that any element in ${}^{\omega}\omega/\mathcal{U}$ which is not in the standard part is non-standard. We write $\mathrm{cf}({}^{\omega}\omega/\mathcal{U})$ for the cofinality of ${}^{\omega}\omega/\mathcal{U}$ with respect to $<_{\mathcal{U}}$.

In the general setting, we can prove the following bounds for $\mathrm{cf}({}^{\omega}\omega/\mathcal{U})$:

Lemma 4.3.1. *We have that $\mathfrak{b} \leq \mathrm{cf}({}^{\omega}\omega/\mathcal{U}) \leq \mathfrak{d}$.*

Proof. Let C' be cofinal in ${}^{\omega}\omega/\mathcal{U}$ and let C be a set which picks a representative from every element of C. We claim that C is unbounded.

Indeed, assume not. Then pick an upper bound f of C. Then for every $g \in C$ we have that $g <^* f$, so $[g]_{\mathcal{U}} <_{\mathcal{U}} [f]_{\mathcal{U}}$ as $\mathcal{U}$ is free. But this contradicts that C' is cofinal in ${}^{\omega}\omega/\mathcal{U}$.

Now, assume D is a dominating family. We claim that

$$C := \{[f]_{\mathcal{U}} \mid f \in D\}$$

is cofinal in ${}^{\omega}\omega/\mathcal{U}$. Indeed, let $[g]_{\mathcal{U}} \in {}^{\omega}\omega/\mathcal{U}$. Take $f \in D$ such that $g <^* f$. But $\mathcal{U}$ is free so this implies that $\{n < \omega \mid g(n) < f(n)\} \in \mathcal{U}$ and $[g]_{\mathcal{U}} <_{\mathcal{U}} [f]_{\mathcal{U}}$. □

If there is a λ-nondominating forcing which diagonalizes $\mathcal{U}$, then we get the following stronger lower bound on $\mathrm{cf}({}^{\omega}\omega/\mathcal{U})$:

Lemma 4.3.2. *Let $\mathbb{P}$ be λ-nondominating, G be $\mathbb{P}$-generic and $X \in V[G]$ be a pseudointersection of $\mathcal{U} \in V$. Then $\lambda \leq \mathrm{cf}({}^{\omega}\omega/\mathcal{U})$.*

Proof. Let C be cofinal in ${}^{\omega}\omega/\mathcal{U}$. In $V[G]$ define for every $f \in C$ a real h_f by

$$h_f(n) := f(\mathrm{Next}(X, n))$$

We claim that this family dominates all ground model reals. Indeed, let $g \in V$ be a real, w.l.o.g. g is increasing. C is cofinal in ${}^{\omega}\omega/\mathcal{U}$, so choose $f \in C$ such that $[g]_{\mathcal{U}} <_{\mathcal{U}} [f]_{\mathcal{U}}$, i.e. $A := \{n < \omega \mid g(n) < f(n)\} \in \mathcal{U}$. By assumption on X we have that $X \subseteq^* A$. Now, let $n > \max(X \setminus A)$. Then $\mathrm{Next}(X, n) \in A$, so as g is increasing we have that

$$g(n) \leq g(\mathrm{Next}(X, n)) < f(\mathrm{Next}(X, n)) = h_f(n)$$

which proves that $g <^* h_f$. □

In case of $\lambda = \mathfrak{d}$ both bounds agree, so we get the following corollaries:

Corollary 4.3.3. *Let $\mathbb{P}$ be $\mathfrak{d}$-nondominating, G be $\mathbb{P}$-generic and $X \in V[G]$ be a pseudointersection of $\mathcal{U} \in V$. Then* $\mathrm{cf}({}^{\omega}\omega/\mathcal{U}) = \mathfrak{d}$*, in particular $\mathfrak{d}$ is regular.* □

Corollary 4.3.4. *If $\mathcal{U}$ is $\mathfrak{d}$-dominating, then* $\mathrm{cf}({}^{\omega}\omega/\mathcal{U}) = \mathfrak{d}$*, in particular $\mathfrak{d}$ is regular.* □

With the proof of the existence of a $\mathfrak{b}$-nondominating filter under the assumption of $\mathfrak{d} = \mathfrak{c}$ of the next section this result implies that the existence of a $\mathfrak{d}$-nondominating ultrafilter is consistently stronger than the existence of a $\mathfrak{b}$-nondominating ultrafilter:

Theorem 4.3.5. *There is a model in which there is a $\mathfrak{b}$-nondominating filter, but not a $\mathfrak{d}$-nondominating filter.*

Proof. Consider any model with $\mathfrak{d} = \mathfrak{c}$ and $\mathfrak{d}$ is singular. By the previous corollary there cannot be a $\mathfrak{d}$-nondominating filter, but the theorem of the next section implies the existence of a $\mathfrak{b}$-nondominating filter. □

Finally, using an easy diagonalization argument we can prove the following sufficient condition for a partial order $\mathbb{P}$ to be $\mathfrak{d}$-nondominating:

Lemma 4.3.6. *If there is a dense subset of $\mathbb{P}$ of size $< \mathfrak{d}$, then $\mathbb{P}$ is $\mathfrak{d}$-nondominating.*

Proof. If D is a dense subset of $\mathbb{P}$, then $\mathbb{P}$ is forcing equivalent to D, so we may as well just assume that $\mathbb{P}$ has size $< \mathfrak{d}$.

Let $\dot{D}$ be a $V^{\mathbb{P}}$-name for a family of reals of size $\gamma < \mathfrak{d}$. $\mathbb{P}$ is $\mathfrak{d}$-c.c., so we can choose a family of $V^{\mathbb{P}}$-names for reals $D := \{\dot{f}_\alpha \mid \alpha < \gamma\}$ which covers $\dot{D}$, i.e.

$$\Vdash \forall f \in \dot{D}\ \exists \alpha < \gamma\ f = \dot{f}_\alpha$$

The idea is to diagonalize against all possible interpretations of the names in D. So, for each $p \in \mathbb{P}$ and $\alpha < \gamma$ define a real $g_{p,\alpha}$ by

$$g_{p,\alpha}(n) := \min\{k \mid \exists q \leq p\ q \Vdash \dot{f}_\alpha(n) = k\}$$

By our size assumptions, we can choose a real g which is not dominated by any of these reals. We claim that g is not dominated by D, i.e.

$$\forall \alpha < \gamma\ \forall N < \omega\ \Vdash\ \exists n > N\ g(n) \geq \dot{f}_\alpha(n)$$

Indeed, let $\alpha < \gamma$ and $N < \omega$. If $p \in \mathbb{P}$, we have that $g \not<^* g_{p,\alpha}$, so we can choose $n > N$ such that $g(n) \geq g_{p,\alpha}(n)$. By definition of $g_{p,\alpha}(n)$ we can choose $q \leq p$ such that

$$q \Vdash \dot{f}_\alpha(n) = g_{p,\alpha}(n)$$

But this proves that $q \Vdash g(n) \geq \dot{f}_\alpha(n)$. □

Note that if $\mathcal{B}$ is a basis of an ultrafilter $\mathcal{U}$, then $D := \{(s, A) \in \mathbb{M}_{\mathcal{U}} \mid A \in \mathcal{B}\}$ is dense in $\mathbb{M}_{\mathcal{U}}$, which immediately implies the following corollary:

Corollary 4.3.7. *If $\mathcal{U}$ has a basis of size $< \mathfrak{d}$, then $\mathcal{U}$ is $\mathfrak{d}$-nondominating.*

Proof. The remark gives us a dense set of size $< \mathfrak{d}$ in $\mathbb{M}_{\mathcal{U}}$, which by the previous lemma implies that $\mathbb{P}$ is $\mathfrak{d}$-nondominating. □

As a final remark, notice that the last corollary implies that $\mathfrak{d}$-nondominating ultrafilters exists in models which satisfy $\mathfrak{u} < \mathfrak{d}$. So under this assumption the existence of a Canjar ultrafilter is trivial. In the next section we will prove the existence of a Canjar ultrafilter in models which satisfy $\mathfrak{d} = \mathfrak{c}$.

5 Existence of a Canjar ultrafilter under $\mathfrak{d} = \mathfrak{c}$

This section will be devoted to the second part of [7] which proves the existence of a Canjar ultrafilter or equivalently of a $\mathfrak{b}$-nondominating ultrafilter under the assumption of $\mathfrak{d} = \mathfrak{c}$. The existence of such ultrafilters is interesting as this is closely connected to proving the consistency of $\mathfrak{b} < \mathfrak{s}$ which we want to examine briefly first.

5.1 Canjar ultrafilters and the consistency of $\mathfrak{b} < \mathfrak{s}$

First, we will consider how the existence of Canjar ultrafilters is a useful step into the direction of proving the consistency of $\mathfrak{b} < \mathfrak{s}$. However, we will see that the notion of a Canjar ultrafilter is not quite enough to work well with iterations, so for the actual consistency proof one has to work with strongly Canjar ultrafilters, which not just preserve the unboundedness of the family of reals, but of every unbounded family - see [9] for more details on the consistency proof of $\mathfrak{b} < \mathfrak{s}$ using this ideas.

We begin by introducing the splitting number $\mathfrak{s}$:

Definition.

- *Let $A, S \in [\omega]^\omega$. We say that S splits A iff $A \cap S$ and $A \cap S^C$ are infinite.*
- *A subset $\mathcal{S} \subseteq [\omega]^\omega$ is called a splitting family iff every $A \in [\omega]^\omega$ is split by an element of $\mathcal{S}$.*
- *$\mathfrak{s}$ denotes the minimal size of a splitting family.*

One natural way to add an unsplit set is to diagonalize an ultrafilter:

Lemma 5.1.1. *If G is $\mathbb{P}$-generic and $X \in V[G]$ is a pseudointersection of an ultrafilter $\mathcal{U} \in V$, then $\mathcal{U}$ is not split by any set in V.*

Proof. If $S \in V$ then either $S \in \mathcal{U}$ or $S^C \in \mathcal{U}$, w.l.o.g. $S \in \mathcal{U}$. But X is a pseudointersection, so $X \subseteq^* S$. This implies that $X \cap S^C$ is finite, so S does not split X. The case $S^C \in \mathcal{U}$ works analogously. $\square$

This means that if we diagonalize an ultrafilter, then we destroy all splitting families, so if we start with a model of CH, then iterating this forcing ω_2 times yields a natural way to increase the splitting number from $\aleph_1$ to $\aleph_2$. To make this more precise we make use of the following facts of countable support iterations of proper forcings:

Fact 5.1.2. *Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \leq \delta \rangle$ be a countable support iteration of proper forcing notions, where $\mathrm{cf}(\delta) > \omega$, then in any extension using this iteration any real is added at a proper initial stage of the iteration.*

One can use this fact to prove a similar result for sets of reals of size $\aleph_1$:

Fact 5.1.3. *Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \leq \omega_2 \rangle$ be a countable support iteration of proper forcing notions, then in any extension using this iteration every set of reals of size $\aleph_1$ is added at a proper initial stage of the iteration.*

Theorem 5.1.4. *There is a model where $\mathfrak{s} = \aleph_2$.*

Proof. Let V be a model of CH and $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \leq \omega_2 \rangle$ be the countable support iteration with $\mathbb{P}_\alpha \Vdash \dot{\mathbb{Q}}_\alpha = \mathbb{M}_{\mathcal{U}}$ for some ultrafilter $\mathcal{U}$. Then in the extension $\mathfrak{s} = \aleph_2$ holds, as by the previous fact any set of reals $\mathcal{S}$ of size $\aleph_1$ is added at a proper initial stage of the iteration. But by the first lemma in the next iterand $\mathcal{S}$ cannot be a splitting family anymore. $\square$

Now, if we had a Canjar ultrafilter available in every step of the iteration, then a naive idea to prove the consistency of $\mathfrak{b} < \mathfrak{s}$ would be to do the above iteration with Canjar ultrafilters at each step. However, this naive approach is doomed to fail due to the following observation:

Fact 5.1.5. *If $\mathbb{P}$ does not add dominating reals and $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ does not add domating reals", then $\mathbb{P} * \dot{\mathbb{Q}}$ might still add dominating reals.*

Thus, the above approach does not guarantee that $\mathfrak{b} = \omega_1$ holds in the extension. In more detail, if G is $\mathbb{P}$-generic over V and H is $\mathbb{Q}$-generic over $V[G]$, then by assumption on $\mathbb{P}$ we have that $({}^\omega\omega)^V$ is unbounded in $V[G]$. However, the assumption on $\mathbb{Q}$ gives us that $({}^\omega\omega)^{V[G]}$ is unbounded in $V[G][H]$, but does not guarantee the same for $({}^\omega\omega)^V$. As described in [1] we can give an example where this situation occurs:

Lemma 5.1.6. *Let $\mathbb{P}$ be the forcing which adds ω_1-Cohen reals and $\dot{\mathbb{Q}}$ be the canonical $V^{\mathbb{P}}$-name for the Hechler forcing over V. Then $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ does not add domating reals".*

Proof. Let G be $\mathbb{P}$-generic over V and $\dot{f}$ be a $V[G]^{\mathbb{Q}}$-name for a real. If we view $\mathbb{P}$ as an iteration of length ω_1 a similar fact to Fact 5.1.2 gives us a $\alpha < \omega_1$ such that $\dot{f} \in V[G \restriction \alpha]$. Let g be the Cohen-real added in $V[G \restriction (\alpha+1)]$. We claim that

$$\Vdash_{\mathbb{Q}} g \not<^* \dot{f}$$

So let $N < \omega$ and $p \in \mathbb{Q}$. Define a subset of the Cohen forcing $\mathbb{C}$ in $V[G \restriction \alpha]$ by

$$D := \{s \in \mathbb{C} \mid \exists n > N \; \exists q \leq p \; q \Vdash_{\mathbb{Q}} s(n) > \dot{f}(n)\} \in V[G \restriction \alpha]$$

We show that D is dense in $\mathbb{C}$, so let $s \in \mathbb{C}$. Choose $n > N$ such that $n \notin \mathrm{domain}(s)$. Choose $q \leq p$ and $k < \omega$ such that $q \Vdash_{\mathbb{Q}} \dot{f}(n) = k$. Then $t := s \cup \{(n, k+1)\} \in D$ as

$$q \Vdash_{\mathbb{Q}} t(n) = k + 1 > \dot{f}(n)$$

Now, in $V[G]$ we have that g is a Cohen real over $V[G \restriction \alpha]$, so choose $s \in D$ with $s \subseteq g$. Then we can choose $n > N$ and $q \leq p$ such that

$$q \Vdash_{\mathbb{Q}} g(n) = s(n) > \dot{f}(n)$$

This proves that $\Vdash_{\mathbb{Q}} g \not<^* \dot{f}$. $\square$

Now, $\mathbb{P}$ does not add dominating reals and the Hechler forcing adds a dominating real, so we have that $\mathbb{P} * \dot{\mathbb{Q}}$ adds a dominating real. Thus, the situation in the above lemma is indeed an example for Fact 5.1.5.

To remedy this problem one not only has to preserve the unboundedness of the family of reals, but of every unbounded family. As shown in [9] this idea can be made precise to give a proof of the consistency of $\mathfrak{b} < \mathfrak{s}$ using strongly Canjar ultrafilters.

5.2 Proof of the existence of a Canjar ultrafilter under $\mathfrak{d} = \mathfrak{c}$

Next, we will follow Canjar's proof in [7] and show that every model of ZFC satisfying $\mathfrak{d} = \mathfrak{c}$ has a Canjar ultrafilter. In particular this result will give us the existence of Canjar ultrafilters in models satisfying CH or more generally Martin's axiom. Note that we proved in the last section that Canjar ultrafilters are necessarily P-points, so in general the existence of a Canjar ultrafilter is independent of ZFC as it is the case for P-points, which is why we need an extra assumption.

5.2.1 Idea of the construction

The general idea of the construction of the Canjar ultrafilter is to construct an increasing chain of filters $\langle \mathcal{F}_\alpha \mid \alpha < \mathfrak{d} \rangle$ and reals $\langle h_\alpha \mid \alpha < \mathfrak{d} \rangle$ such that for any ultrafilter $\mathcal{U}$ extending this chain we have that for any $\mathbb{M}_\mathcal{U}$-name τ for a real there is an $\alpha < \mathfrak{d}$ with

$$\Vdash_\mathcal{U} \; h_\alpha \not<^* \tau$$

Then any $\mathcal{U}$ extending $\langle \mathcal{F}_\alpha \mid \alpha < \mathfrak{d} \rangle$ will be a Canjar ultrafilter. However, in the construction of the chain of filters we do not know how our final ultrafilter will look like, so it does not make sense to talk about $\mathbb{M}_\mathcal{U}$-names for reals, so we have to consider some kind of general Mathias names for reals which are decoupled from any specific filter.

Certainly, we want that for any ultrafilter $\mathcal{U}$ every $\mathbb{M}_\mathcal{U}$-name τ of a real has a representation as such a general name for a real. We will have $\mathfrak{d}$-many of these general names for a real, so the idea for the construction of the above chain of filters will be to handle one of these general names for a real in each step of the iteration and to use the assumption of $\mathfrak{d} = \mathfrak{c}$ to be able to pick a suitable h_α.

With these ideas in mind we call such general names for a real preterms and define the representation of a $\mathbb{M}_\mathcal{U}$-name τ for a real as a preterm as follows:

Definition.

- *We call a partial function from $[\omega]^{<\omega} \times \omega$ to ω a preterm.*
- *If $\mathcal{U}$ is an ultrafilter and τ is a $\mathbb{M}_\mathcal{U}$-name for a real, then we define the associated preterm $\hat{\tau}$ of τ by*

$$\hat{\tau}(s, n) = k \;\; :\Longleftrightarrow \;\; \exists A \in \mathcal{U} \;\, (s, A) \Vdash_\mathcal{U} \tau(n) = k$$

So intuitively, $\hat{\tau}$ captures the information of how an initial segment s of the generic real decides the real τ. The crucial idea is that this information is already enough to construct a real h such that $\mathbb{M}_\mathcal{U}$ forces h not to be dominated by τ. Hence, it is enough to iterate over all preterms and add such a witness h for every preterm.

Remark. Note the associated preterm is indeed a partial function, for if there are k, l and $A, B \in \mathcal{U}$ such that

$$(s, A) \Vdash_\mathcal{U} \tau(n) = k \qquad \text{and} \qquad (s, B) \Vdash_\mathcal{U} \tau(n) = l$$

Then $(s, A \cap B)$ extends both (s, A) and (s, B) and we have that

$$(s, A \cap B) \Vdash_\mathcal{U} k = \tau(n) = l$$

which implies that $k = l$.

Also, notice that in the construction of the chain of filters and reals we only really have to handle preterms which can still appear as the associated preterm of a $\mathbb{M}_{\mathcal{U}}$-name of a real for an ultrafilter $\mathcal{U}$ which extends the current chain of the construction. We show that such preterms satisfy the following completeness condition:

Definition. *We call a preterm $\hat{\tau}$ complete over the set A, iff for every partition of A into finitely many sets $\langle A_i \mid i < l\rangle$ there is some $i < l$ such that*

$$\forall s \in [\omega]^{<\omega}\ \forall n < \omega\ \exists t < (s, A_i)\ \exists k < \omega\ \hat{\tau}(t, n) = k$$

We say $\hat{\tau}$ is complete over a family of sets $\mathcal{A}$ iff τ is complete over every element of $\mathcal{A}$.

Lemma 5.2.1. *If $\mathcal{U}$ is an ultrafilter and τ is a $\mathbb{M}_{\mathcal{U}}$-name for a real, then τ is complete over $\mathcal{U}$ and thus also complete over every subfilter $\mathcal{F}$ of $\mathcal{U}$.*

Proof. Let $A \in \mathcal{U}$ and $\langle A_i \mid i < l\rangle$ be a partition of A into finitely many sets. $\mathcal{U}$ is an ultrafilter, so we can choose $i < l$ such that $A_i \in \mathcal{U}$. We claim that A_i is as desired, so let $s \in [\omega]^{<\omega}$ and $n < \omega$ be given. Deciding the value of τ at n is dense, so we can choose $(t, B) \leq (s, A_i)$ and $k < \omega$ such that

$$(t, B) \Vdash_{\mathcal{U}} \tau(n) = k$$

Then by definition of $\hat{\tau}$ we have that $\hat{\tau}(t, n) = k$ and $t < (s, A_i)$. $\square$

5.2.2 The topology of $\mathcal{P}(\omega)$

Before we further dive into the proof, first we consider how $\mathcal{P}(\omega)$ inherits a topology from ${}^{\omega}2$, how the open sets look like and state a few basic topological properties.

As usual we consider the product topology on ${}^{\omega}2$. By Tychonoff's theorem ${}^{\omega}2$ is compact in this topology, so we do not have to differentiate between compact and closed subsets of ${}^{\omega}2$. For a partial function $s : \omega \to 2$ with finite domain we define its basic open set by

$$\mathcal{O}(s) := \{f \in {}^{\omega}2 \mid s \subseteq f\}$$

By definition of the product topology these $\mathcal{O}(s)$ form a basis of the topology. Now, we can transfer this topology to $\mathcal{P}(\omega)$ with the usual bijection ${}^{\omega}2 \to \mathcal{P}(\omega)$ defined by

$$f \mapsto f^{-1}[\{1\}]$$

Note that if we define for finite disjoint subsets $s, t \subseteq \omega$ the basic open set

$$\mathcal{O}(s, t) := \{A \in \mathcal{P}(\omega) \mid s \subseteq A, t \subseteq A^C\} \qquad \text{and} \qquad \mathcal{O}(s) := \mathcal{O}(s, \emptyset)$$

then these basic open sets are exactly the images of the basic open sets of ${}^{\omega}2$ under the above bijection, so they form a basis of the induced topology on $\mathcal{P}(\omega)$.

Furthermore, by swapping 1's and 0's we can define an involution $\bar{\cdot} : {}^{\omega}2 \to {}^{\omega}2$

$$f \mapsto \overline{f} \text{ where } \overline{f}(n) = 1 - f(n)$$

Note that this is actually an homeomorphism as $\overline{\mathcal{O}(s)} = \mathcal{O}(\overline{s})$ and that the induced homeomorphism $\bar{\cdot}$ on $\mathcal{P}(\omega)$ is the complement function, i.e. for all $A \subseteq \omega$ we have that $\overline{A} = A^C$. In particular we have that $\overline{\mathcal{O}(s,t)} = \mathcal{O}(t, s)$ in $\mathcal{P}(\omega)$.

We can also define a metric on ${}^{\omega}2$ as follows

$$d(f, g) := \sum_{n \geq 0} 2^{-n} \cdot \delta_{f(n), g(n)}$$

It is not hard to prove that the topology induced by this metric agrees with the product definition we chose. Thus, ${}^{\omega}2$ and $\mathcal{P}(\omega)$ are in fact compact, metric spaces.

We need to consider one more topological technicality as the extension lemma only works for filters that are the union of less than $\mathfrak{d}$ many compact sets. The restriction to compact sets will be useful as it allows us to pick finite sets of witnesses for various existential properties. We will see an example of this idea in the proof of the subsequent lemma.

Definition. *For a cardinal $\alpha > \omega$ we call a family of subsets of ω $\mathcal{K}\alpha$ iff it is the union of less than α many compact sets.*

We want to prove that a filter is $\mathcal{K}\alpha$ iff it has a $\mathcal{K}\alpha$ generating set, so we need to consider how we can close compact sets under finite intersections:

Lemma 5.2.2. *Let $\mathcal{K} := \mathcal{K}^{(0)}$ be compact. Then for every $n < \omega$ we have that*

$$\mathcal{K}^{(n)} := \{A \subseteq \omega \mid \exists B_0, \ldots, B_n \in \mathcal{K} \bigcap_{i=0}^{n} B_i \subseteq A\}$$

is compact.

Proof. By induction on n, so assume $K^{(n)}$ is compact. We have that

$$\mathcal{K}^{(n+1)} = \{A \subseteq \omega \mid \exists B \in \mathcal{K}^{(n)} \; \exists C \in \mathcal{K} \; B \cap C \subseteq A\}$$

Let $A \in \mathcal{P}(\omega) \backslash \mathcal{K}^{(n+1)}$. We will use a double compactness argument to prove the existence of an open set $\mathcal{O}$ which satisfies $A \in \mathcal{O} \subseteq \mathcal{P}(\omega) \setminus \mathcal{K}^{(n+1)}$. First, we show that

$$\forall B \in \mathcal{K}^{(n)} \; \exists \text{ finite } t_B \subseteq B \setminus A \; \forall C \in \mathcal{K} \; t_B \cap C \neq \emptyset \qquad (\star)$$

So let $B \in \mathcal{K}^{(n)}$. We claim that

$$\mathcal{K} \subseteq \bigcup_{k \in B \setminus A} \mathcal{O}(\{k\})$$

is an open covering of $\mathcal{K}$. Indeed, let $C \in \mathcal{K}$. By assumption on A we have that $(B \cap C) \nsubseteq A$, so we can choose $k \in (B \cap C) \setminus A$. But then $k \in B \setminus A$ and $C \in \mathcal{O}(\{k\})$.

Now, by compactness of $\mathcal{K}$ we can choose a finite $t_B \subseteq B \setminus A$ such that

$$\mathcal{K} \subseteq \bigcup_{k \in t_B} \mathcal{O}(\{k\})$$

But this shows $(\star)$ as for all $C \in \mathcal{K}$ there is a $k \in t_B$ such that $C \in \mathcal{O}(\{k\})$, i.e. we have that $k \in t_B \cap C$ which implies that $t_B \cap C \neq \emptyset$.

Secondly, we consider the set $T := \{t \subseteq A^C \text{ finite} \mid \forall C \in \mathcal{K} \; t \cap C \neq \emptyset\}$ and claim that

$$\mathcal{K}^{(n)} \subseteq \bigcup_{t \in T} \mathcal{O}(t)$$

is an open covering of $\mathcal{K}^{(n)}$. Indeed, let $B \in \mathcal{K}^{(n)}$. Then by $(\star)$ there is a finite $t_B \subseteq B \setminus A$ such that $t_B \in T$. But then $B \in \mathcal{O}(t_B)$.

Now, by compactness of $\mathcal{K}^{(n)}$ we can choose a finite $T_0 \subseteq T$ such that

$$\mathcal{K}^{(n)} \subseteq \bigcup_{t \in T_0} \mathcal{O}(t)$$

Finally, we can get the finite $s := \bigcup_{t \in T_0} t \subseteq A^C$ which has the desired property that

$$A \in \mathcal{O}(\emptyset, s) \subseteq \mathcal{P}(\omega) \setminus \mathcal{K}^{(n+1)}$$

Indeed, let $D \in \mathcal{O}(\emptyset, s)$. Then $s \subseteq D^C$. Assume that $D \in \mathcal{K}^{(n+1)}$, so choose $B \in \mathcal{K}^{(n)}$ and $C \in \mathcal{K}$ such that $B \cap C \subseteq D$. By definition of T_0 we can choose $t \in T_0$ such that $B \in \mathcal{O}(t)$. By definition of T we can choose a $k \in t \cap C$. But $t \subseteq s$, so we get that $k \in s$. On the other hand, we have that $k \in t \subseteq B$ and $k \in C$. Thus, $k \in B \cap C \subseteq D$. But then by our assumption on D we have that $k \notin s$, which is a contradiction. $\square$

Corollary 5.2.3. *If $\alpha > \omega$, then $\mathcal{F}$ is a $\mathcal{K}\alpha$-filter iff $\mathcal{F}$ has a $\mathcal{K}\alpha$-generating set.*

Proof. The implication is trivial, so let $\mathcal{F} = \langle \mathcal{G} \rangle$ for some $\mathcal{K}\alpha$-generating set $\mathcal{G}$ and choose compact sets $\langle \mathcal{K}_\beta \mid \beta < \gamma < \alpha \rangle$ such that $\mathcal{G} = \bigcup_{\beta < \gamma} \mathcal{K}_\beta$. Note that with the notion of the previous lemma we can write $\mathcal{F}$ as a union of $\gamma < \alpha$ many sets:

$$\mathcal{F} = \langle \mathcal{G} \rangle = \bigcup_{\substack{S \in [\gamma]^{<\omega} \\ n < \omega}} (\bigcup_{\beta \in S} \mathcal{K}_\beta)^{(n)}$$

Note that for each $S \in [\gamma]^{<\omega}$ and $n < \omega$ we have that the finite union of compact sets $\bigcup_{\beta \in S} \mathcal{K}_\beta$ is compact, so by the previous lemma also $(\bigcup_{\beta \in S} \mathcal{K}_\beta)^{(n)}$ is compact. $\square$

Remark 5.2.4. We have that $\mathrm{cofin}(\omega)$ is a $\mathcal{K}\omega_1$-filter. Indeed, we have $|\mathrm{cofin}(\omega)| = \omega$ and every singleton set in $\mathcal{P}(\omega)$ is compact, so we can write

$$\mathrm{cofin}(\omega) = \bigcup_{A \in \mathrm{cofin}(\omega)} \{A\}$$

5.2.3 Proof using the Extension Lemma

Now, in this topological setting we are able to formulate the extension lemma which allows us to prove the existence of a Canjar filter using the construction we described in the first section.

Lemma 5.2.5 (Extension Lemma). *If $\mathcal{F}$ is a $\mathcal{K}\alpha$-filter for $\omega < \alpha \leq \mathfrak{d}$ and $\hat{\tau}$ is a preterm which is complete over $\mathcal{F}$, then there exists a $\mathcal{K}\alpha$-filter $\mathcal{F}'$ extending $\mathcal{F}$ and a real h such that for any ultrafilter $\mathcal{U}$ extending $\mathcal{F}'$ and any $\mathbb{M}_\mathcal{U}$-name τ for a real whose associated preterm is $\hat{\tau}$ we have that*

$$\Vdash_\mathcal{U} \ h \not<^* \tau$$

Theorem 5.2.6. *If $\mathfrak{d} = \mathfrak{c}$, then there exists a Canjar ultrafilter.*

Proof. As $\mathfrak{d} = \mathfrak{c}$ we can enumerate all preterms by $\langle \hat{\tau}_\alpha \mid \alpha < \mathfrak{d} \rangle$. Using the previous lemma we can construct a chain of filters $\langle \mathcal{F}_\alpha \mid \alpha < \mathfrak{d} \rangle$ and a sequence of reals $\langle h_\alpha \mid \alpha < \mathfrak{d} \rangle$ such that we have the following properties:

1. $\mathcal{F}_\alpha$ is $\mathcal{K}[(|\alpha| + \omega)^+]$.
2. If $\beta < \alpha$, then $\mathcal{F}_\beta \subseteq \mathcal{F}_\alpha$.
3. If $\hat{\tau}_\alpha$ is complete over $\mathcal{F}_\alpha$ and $\mathcal{U}$ is any ultrafilter extending $\mathcal{F}_{\alpha+1}$ and τ is any $\mathbb{M}_\mathcal{U}$-name for a real which associated preterm is $\hat{\tau}_\alpha$, then
$$\Vdash_\mathcal{U} \ h_\alpha \not<^* \tau$$

Indeed, by Remark 5.2.4 we can set $\mathcal{F}_0 := \mathrm{cofin}(\omega)$ and for limit ordinals α we can set

$$\mathcal{F}_\alpha := \bigcup \{\mathcal{F}_\beta \mid \beta < \alpha\}$$

At successor stages $\alpha + 1$ we do nothing if $\hat{\tau}_\alpha$ is not complete over $\mathcal{F}_\alpha$ and otherwise we can apply the Extension Lemma to $\mathcal{F}_\alpha$ and $\hat{\tau}_\alpha$ to obtain an extension $\mathcal{F}_{\alpha+1}$ of $\mathcal{F}_\alpha$ and a real h_α satisfying properties 1 to 3.

Now, let $\mathcal{U}$ be any ultrafilter extending the chain we constructed. We claim that $\mathcal{U}$ is a Canjar ultrafilter, so let τ be a $\mathbb{M}_\mathcal{U}$-name for a real. Then we can choose $\alpha < \mathfrak{d}$ such that $\hat{\tau} = \hat{\tau}_\alpha$. By Lemma 5.2.1 $\hat{\tau}_\alpha$ is complete over the subfilter $\mathcal{F}_\alpha$, thus $\mathcal{F}_{\alpha+1}$ and h_α were constructed with the Extension Lemma and we get

$$\Vdash_\mathcal{U} h_\alpha \not<^* \tau$$

which proves that $\mathcal{U}$ is a Canjar ultrafilter. □

5.2.4 Proof of the Extension Lemma

Finally, it remains to prove that the Extension Lemma holds. For the rest of this section we fix some preterm $\hat{\tau}$. First, we will discuss what it means for a real h to satisfy the property of the Extension Lemma, which will motivate the upcoming definitions of good and bad subsets of ω:

Discussion. In the context of the Extension Lemma let τ be a $\mathbb{M}_\mathcal{U}$-name for a real. We want to construct a real h such that $\Vdash_\mathcal{U} h \not<^* \tau$, i.e.

$$\forall m < \omega \ \forall (s, A) \in \mathbb{M}_\mathcal{U} \ \exists n > m \ \exists (t, B) \leq (s, A) \ (t, B) \Vdash_\mathcal{U} h(n) \geq \tau(n)$$

Now, if we try to translate the latter part of this statement for the preterm $\hat{\tau}$ we get the following definition:

Definition. *Let $s \in [\omega]^{<\omega}$ and $n, k < \omega$. Then we say that $A \subseteq \omega$ is (s, n, k)-Good iff*

$$\exists t < (s, A) \ \exists i \leq k \ \hat{\tau}(t, n) = i$$

Otherwise we say that A is (s, n, k)-Bad. We write Good(s, n, k) for all (s, n, k)-Good sets and Bad(s, n, k) for all (s, n, k)-Bad sets.

Discussion. So intuitively, if A is (s, n, k)-Good, then below (s, A) it is possible that $\hat{\tau}$ is decided as a value $\leq k$ at place n. Thus, in the above context we have to construct a real h such that given $m < \omega$ and $(s, A) \in \mathbb{M}_\mathcal{U}$ we can find a $n > m$ such that A is $(s, n, h(n))$-Good. This suggests that we want to extend the given filter $\mathcal{F}$ by the complements of all sets which do not satisfy this property, i.e. the complements of all sets $A \subseteq \omega$ such that

$$\exists s \in [\omega]^{<\omega} \ \exists m < \omega \ \forall n > m \ A \text{ is } (s, n, h(n))\text{-Bad} \qquad (\star)$$

We will see in the next proof, that this choice of an extension $\mathcal{F}'$ works, but the tricky part left to show is that we can choose the real h in such a way that we do not have to add too many sets to $\mathcal{F}$, i.e. that the described generating set of $\mathcal{F}'$ satisfies the FIP, which guarantees that we obtain a filter. Thus, we will use the following lemma to prove the Extension Lemma. Note that $I(h)$ is exactly defined by property $(\star)$:

Lemma 5.2.7. *Let $\mathcal{F}$ be a $\mathcal{K}\alpha$-filter with $\omega < \alpha \leq \mathfrak{d}$. Then there is a real h so that for*

$$I(h) := \{A \subseteq \omega \mid \exists s \in [\omega]^{<\omega} \ \exists m < \omega \ \forall n > m \ A \textit{ is } (s, n, h(n))\textit{-Bad}\}$$

the set $\mathcal{F} \cup \{A^C \mid A \in I(h)\}$ generates a $\mathcal{K}\alpha$ filter.

Proof of the Extension Lemma. Using this lemma we can choose a real h such that the family $\mathcal{F} \cup \{B^C \mid B \in I(h)\}$ generates a $\mathcal{K}\alpha$ filter $\mathcal{F}'$. We claim that these choices of $\mathcal{F}'$ and h satisfy the properties of the Extension Lemma, so let $\mathcal{U}$ be an ultrafilter extending $\mathcal{F}'$ and τ be a $\mathbb{M}_{\mathcal{U}}$-name for a real which associated preterm is $\hat{\tau}$. We have to show that

$$\Vdash_{\mathcal{U}} h \not\leq^* \tau$$

So let $m < \omega$. As $\mathcal{U}$ extends $\mathcal{F}'$ if we are given $(s, A) \in \mathbb{M}_{\mathcal{U}}$, then we have that $A \notin I(h)$. Thus, we can find $n > m$ such that A is $(s, n, h(n))$-good. So choose $t < (s, A)$ and $i \leq h(n)$ such that $\hat{\tau}(t, n) = i$. By definition of $\hat{\tau}$ we can choose $B \in \mathcal{U}$ such that

$$(t, B) \Vdash_{\mathcal{U}} \tau(n) = i$$

But then $(t, A \cap B) \leq (s, A)$ and we have that

$$(t, A \cap B) \Vdash_{\mathcal{U}} \tau(n) = i \leq h(n)$$

□

It is left to show that for some real h we have that $\mathcal{F} \cup \{A^C \mid A \in I(h)\}$ has the FIP, which means that we will have to work with finite unions of bad sets as $\mathcal{F}'$ is defined via complement from elements of $I(h)$, so it will be useful to introduce a notion for that.

From now on fix an enumeration $\langle s_i \mid i < \omega \rangle$ of $[\omega]^{<\omega}$ and define:

Definition. *Given $n, k < \omega$ we say that a set $A \subseteq \omega$ is (n, k)-UBad iff $A = \bigcup_{m<n} B_m$, where for each $m < n$ there is an $i < n$ such that B_m is (s_i, n, k)-Bad. We write UBad(n, k) for the set of all (n, k)-Bad sets.*

First, we consider some topological properties of the families of Good and Bad sets. Clearly, we have that Good(s, n, k) is closed under supersets, so Bad(s, n, k) is closed under subsets. Furthermore, we have that:

Lemma 5.2.8. *Let $s \in [\omega]^{<\omega}$ and $n, k < \omega$. Then the family Good(s, n, k) is open. Thus, the families Bad(s, n, k) and UBad(n, k) are compact.*

Proof. Let A be (s, n, k)-Good. Then choose $t < (s, A)$ and $i \leq k$ such that $\hat{\tau}(t, n) = i$. We have $t \setminus s \subseteq A$, so $A \in \mathcal{O}(t \setminus s)$. We claim that $\mathcal{O}(t \setminus s) \subseteq$ Good(s, n, k), so let $B \in \mathcal{O}(t \setminus s)$. Then $t \setminus s \subseteq B$, so $t < (s, B)$ which shows that B is (s, n, k)-Good. □

The heart of the construction is the following lemma which shows that if $\hat{\tau}$ satisfies the completeness condition, then for every n we can find a k such that A is not (n, k)-UBad:

Lemma 5.2.9. *For each $n < \omega$ and $A \subseteq \omega$ such that $\hat{\tau}$ is complete over A, there is a $k < \omega$ such that A is not (n, k)-UBad.*

Proof. Unraveling the definition of (n, k)-UBad, we have to show that there exists $k < \omega$ such that for all partitions of A into n sets $\mathcal{A} = \langle A_i \mid i < n \rangle$ there is an $i < n$ such that A_i is (s_j, n, k)-Good for all $j < n$. We call a partition of a finite initial segment of A a finite partition of A and all of the following partitions will be into n sets.

The idea will be to assign a $k(\mathcal{B}) < \omega$ to each finite partition $\mathcal{B} = \langle B_i \mid i < n \rangle$ of A such that for every partition $\mathcal{A}$ of A extending $\mathcal{B}$ there is an $i < \omega$ such that A_i is (s_j, n, k)-Good for all $j < n$. In particular, if $\emptyset$ denotes the empty partition, then this $k(\emptyset)$ will be the desired k, because every partition $\mathcal{A}$ of A extends the empty partition. Thus, the $k(\mathcal{B})$ will act like approximations to the final $k := k(\emptyset)$.

Now, in detail consider the tree T of all finite partitions of A ordered by extension. We can find the $k(\mathcal{B}) < \omega$ using the following two rules:

1. For all $\mathcal{B} \in T$ which already have some $B_i \in \mathcal{B}$ such that for every $j < n$ there is an $k_j < \omega$ such that B_i is (s_j, n, k_j)-Good we can define $k(\mathcal{B}) := \max\{k_j \mid j < n\}$. Then for every partition $\mathcal{A}$ of A extending $\mathcal{B}$ we have that A_i extends B_i, so A_i is also $(s_j, n, k(\mathcal{B}))$-Good for every $j < n$.
2. If all immediate successors $\langle \mathcal{C}_j \mid j < n \rangle$ of some $\mathcal{B} \in T$ have been assigned a value $k(\mathcal{C}_j)$ we can set $k(\mathcal{B}) := \max\{k(\mathcal{C}_j) \mid j < n\}$. Clearly, this satisfies the above condition as any partition $\mathcal{A}$ of A extending $\mathcal{B}$ will extend one of the $\mathcal{C}_j$.

We claim that we exhaust T if we assign values to $\mathcal{B} \in T$ using these two rules. For convenience, we say that $\mathcal{B}$ is definite iff $k(\mathcal{B})$ is defined, so it is left to show that $\emptyset \in T$ is definite. We will see that the completeness assumption exactly gives us this property.

Assume $\emptyset$ is not definite, then by rule 2 one of its immediate successors is not definite. Iterating this argument yields a non-definite path $P : \omega \to T$ through T. Then their union $\langle A_i \mid i < n \rangle$ is a partition of A. By completeness of $\hat{\tau}$, choose A_i such that

$$\forall s \in [\omega]^{<\omega}\ \forall n < \omega\ \exists t < (s, A_i)\ \exists k < \omega\ \hat{\tau}(t, n) = k$$

Thus, for all $j < n$ we can choose $t_j < (s_j, A_i)$ and $k_j < \omega$ such that $\hat{\tau}(t_j, n) = k_j$. But now, let $t = \bigcup_{j<n}(t_j \setminus s_j)$. Then t is a finite subset of A_i, so there is some $\mathcal{B} \in T$ on P such that $t \subseteq B_i$. But then B_i is (s_j, n, k_j)-Good for every $j < n$ which contradicts that $\mathcal{B}$ is not definite. □

A simple compactness argument extends this result to compact sets. This will allow us to define our desired real h:

Lemma 5.2.10. *For each $n < \omega$ and compact set $\mathcal{K}$ such that $\hat{\tau}$ is complete over $\mathcal{K}$, there is a $k < \omega$ such that no $A \in \mathcal{K}$ is (n, k)-UBad.*

Proof. For $k < \omega$ define $\mathcal{O}_k := \{A \subseteq \omega \mid A \text{ is not } (n, k)\text{-UBad}\}$. We know that $\mathrm{UBad}(n, k)$ is compact, so each $\mathcal{O}_k$ is open. Thus, by the previous lemma the $\mathcal{O}_k$'s are an open covering of $\mathcal{K}$. Furthermore, the $\mathcal{O}_k$'s are increasing, so by compactness there is a single $k < \omega$ such that $\mathcal{K} = \mathcal{O}_k$. □

Finally, we can prove the remaining Lemma 5.2.7 to conclude this section:

Lemma. *Let $\mathcal{F}$ be a $\mathcal{K}\alpha$-filter with $\omega < \alpha \leq \mathfrak{d}$. Then there is a real h so that for*

$$I(h) := \{A \subseteq \omega \mid \exists s \in [\omega]^{<\omega}\ \exists m < \omega\ \forall n > m\ A \text{ is } (s, n, h(n))\text{-Bad}\}$$

the set $\mathcal{F} \cup \{A^C \mid A \in I(h)\}$ generates a $\mathcal{K}\alpha$ filter.

Proof. Let $\mathcal{F} = \bigcup\{\mathcal{K}_\beta \mid \beta < \lambda\}$ be the union of $\lambda < \alpha$ many compact sets. By the previous lemma for each $\beta < \lambda$ we can choose a real h_β such that

$$\forall A \in \mathcal{K}_\beta\ \forall n < \omega\ A \text{ is not } (n, h_\beta(n))\text{-UBad}$$

As $\lambda < \alpha \leq \mathfrak{d}$ we can choose a real h which is not dominated by any h_β. We have to show that $\mathcal{F} \cup \{A^C \mid A \in I(h)\}$ generates a $\mathcal{K}\alpha$ filter.

As $\mathcal{F}$ is $\mathcal{K}\alpha$ to show that the generated filter is $\mathcal{K}\alpha$ by Corollary 5.2.3 it suffices to show that $\{A^C \mid A \in I(h)\}$ is $\mathcal{K}\alpha$. Note that we can write

$$I(h) = \bigcup_{s \in [\omega]^{<\omega}} \bigcup_{m < \omega} \bigcap_{n > m} \mathrm{Bad}(s, n, h(n))$$

so $I(h)$ is a countable union of compact sets. Under the complement homeomorphism discussed in Section 5.2.2 we get that $\{A^C \mid A \in I(h)\}$ is a countable union of compact sets, i.e. $\{A^C \mid A \in I(h)\}$ is $\mathcal{K}\omega_1$.

It remains to show that $\mathcal{F} \cup \{A^C \mid A \in I(h)\}$ satisfies the FIP. Assume that this was not the case. As $\mathcal{F}$ is closed under finite intersections we can choose a single $A \in \mathcal{F}$ and $\{B_k \mid k < N\} \subseteq I(h)$ such that $A \cap (\bigcap_{k<N} B_k^C) = A \cap (\bigcup_{k<N} B_k)^C = \emptyset$, so by possibly extending A we may assume that $A = \bigcup_{k<N} B_k$.

By definition of $I(h)$ for each $k < N$ choose i_k and m_k such that for all $n > m_k$ we have that B_k is $(s_{i_k}, n, h(n))$-Bad. Let β be such that $A \in \mathcal{K}_\beta$ and pick n such that

1. $h(n) > h_\beta(n)$
2. $n > N$
3. For all $k < N$ we have that $n > m_k$
4. For all $k < N$ we have that $n > i_k$

We can choose such n as $h \not\leq^* h_\beta$, so 1. holds for infinitely many n whereas 2. to 4. hold for cofinitely many n. But by 1. and 3., for all $k < N$ we have B_k is $(s_{i_k}, n, h_\beta(n))$-Bad, thus by 2. and 4. their union A is $(n, h_\beta(n))$-UBad, which contradicts our choice of h_β. $\square$

Unraveling the results of this section we have seen that this lemma allows us to prove the Extension Lemma. Furthermore, we have seen that the Extension Lemma gives as a straightforward iterative construction of a Canjar filter or equivalently a $\mathfrak{b}$-nondominating filter under the assumption of $\mathfrak{d} = \mathfrak{c}$.

As a final note, Canjar also proved in [7] that under the assumption that the set of 'all collections of preterms of size $< \mathfrak{d}$' has size $\mathfrak{d}$ the proof we have described can be modified to give an explicit construction of a $\mathfrak{d}$-nondominating ultrafilter. The main idea is to adapt the extension lemma not to add a suitable real for a single preterm as we did in our construction, but to add a suitable real for every $< \mathfrak{d}$ sized collection of preterms. This real will then witness that the family of reals described by these preterms will not be dominating in the same way the real we added witnessed that a single real is not dominating.

6 Characterizations of Canjar filters

In this final section we want to prove a combinatorial characterization for Canjar filters. As it turns out, for a filter $\mathcal{F}$ to be Canjar is closely related to the combinatorial properties of the associated filter $\mathcal{F}^{<\omega}$. Hrušák and Minami analysed this connection in [12] and the first subsection will be devoted to this characterization. As Blass, Hrušák and Verner concluded in [4] we can use this result to prove that for ultrafilters this combinatorial property is equivalent to the notion of a strong P-point, which Laflamme introduced quite some time earlier in [14].

Remember, that in Section 4 we have seen that Canjar's condition of being a P-point and not having rapid Rudin-Keisler predecessors is necessary for Canjar ultrafilters. So in the last subsection, we will take a look at the construction given in [4] to prove that Laflamme's condition of being a strong P-point is consistently stronger than Canjar's condition, answering Canjar's question whether his condition is sufficient negatively.

6.1 Characterization by P$^+$-filters

In this first subsection we will introduce for a filter $\mathcal{F}$ on ω its associated filter $\mathcal{F}^{<\omega}$, define the notion of a P$^+$-filter and prove that a filter $\mathcal{F}$ is Canjar iff $\mathcal{F}^{<\omega}$ is a P$^+$-filter. Although this result was first proved in [12] we will comply with [4] as the original proof worked predominantly with ideals whereas the latter paper uses filters as we do.

Definition. *If $\mathcal{F}$ is a filter on ω, then we define a filter $\mathcal{F}^{<\omega}$ on $[\omega]^{<\omega} \setminus \{\emptyset\}$ by*

$$\mathcal{F}^{<\omega} := \langle \{[A]^{<\omega} \setminus \{\emptyset\} \mid A \in \mathcal{F}\} \rangle$$

Remark. Note that for $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$ and $[A \cap B]^{<\omega} = [A]^{<\omega} \cap [B]^{<\omega}$, so the generating set of $\mathcal{F}^{<\omega}$ is closed under finite intersections. In particular, the generating set is a basis of the filter $\mathcal{F}^{<\omega}$.

The point of this definition will become more clear with the following proofs, but the general idea will be that $S \subseteq [\omega]^{<\omega} \setminus \{\emptyset\}$ will be families of finite subsets $s \subseteq \omega$ which will contain a witness for a desired property defined by S. This intuition as sets of witnesses justifies the exclusion of the empty set. We have already seen an example of this idea in the proof of Lemma 5.2.2, where we had to find finite subsets such that every element of a compact set had non-trivial intersection with it. As in that proof in the second subsection we will use compactness to prove the existence of such finite witness sets. This interpretation also gives some justification for the definition of $\mathcal{F}^{<\omega}$, because we can interpret $\mathcal{F}^{<\omega}$-positive sets as follows:

Remark. Let $S \subseteq [\omega]^{<\omega} \setminus \{\emptyset\}$ be $\mathcal{F}^{<\omega}$-positive. Then for every $A \in \mathcal{F}$ we have that $[A]^{<\omega} \cap S \neq \emptyset$. In the above interpretation this means that the property defined by S is general enough such that for every $A \in \mathcal{F}$ there is a witness set $s \in S$ which consists entirely of elements of A.

Definition. *We say a filter $\mathcal{F}$ on a countable set X is a P^+-filter iff for every decreasing sequence $\langle S_n \mid n < \omega \rangle$ from $\mathcal{F}^+$ there is a $S \in \mathcal{F}^+$ such that for every $n < \omega$ we have that $S \subseteq^* S_n$.*

In other words a filter is P^+ iff for every decreasing sequence of positive sets, there is a positive pseudointersection. We will use this notion exclusively for $\mathcal{F}^{<\omega}$, i.e. for filters on $[\omega]^{<\omega} \setminus \{\emptyset\}$. In case we have that $\mathcal{F} = \mathcal{U}$ is an ultrafilter the following lemma is a useful characterization of the $\mathcal{U}^{<\omega}$-positive elements:

Lemma 6.1.1. *Let $\mathcal{U}$ be an ultrafilter. Then $S \subseteq [\omega]^{<\omega} \setminus \{\emptyset\}$ is $\mathcal{U}^{<\omega}$-positive iff for every $A \subseteq \omega$ with $\forall s \in S\ s \cap A \neq \emptyset$ we have that $A \in \mathcal{U}$.*

Proof. First, let $S \subseteq [\omega]^{<\omega} \setminus \{\emptyset\}$ be $\mathcal{U}^{<\omega}$-positive and let $A \subseteq \omega$ with $\forall s \in S\ s \cap A \neq \emptyset$. We show that $A \in \mathcal{U}^+ = \mathcal{U}$ as $\mathcal{U}$ is an ultrafilter, so let $B \in \mathcal{U}$, then $[B]^{<\omega} \setminus \{\emptyset\} \in \mathcal{U}^{<\omega}$, so by positivity of S we can choose $s \in S \cap [B]^{<\omega} \setminus \{\emptyset\}$. Now, by assumption on A we have $s \cap A \neq \emptyset$, so $s \subseteq B$ implies $A \cap B \neq \emptyset$, which proves that $S \in \mathcal{U}^+$.

Conversely, assume that for some $S \subseteq [\omega]^{<\omega} \setminus \{\emptyset\}$ we have that for every $A \subseteq \omega$ with $\forall s \in S\ s \cap A \neq \emptyset$ we have that $A \in \mathcal{U}$. We show that S is $\mathcal{U}^{<\omega}$-positive, so let $B \in \mathcal{U}$. Then $B^C \notin \mathcal{U}$, so we can choose $s \in S$ with $s \cap B^C = \emptyset$. Thus, $s \subseteq B$ which shows that $S \cap [B]^{<\omega} \neq \emptyset$, i.e. S is $\mathcal{U}^{<\omega}$-positive. $\square$

Before we can prove our main theorem we need to prove that every dominating family which is split into countably many pieces has a dominating subfamily:

Lemma 6.1.2. *Let $F = \bigcup_{n<\omega} F_n \subseteq {}^\omega\omega$ be a dominating family. Then there exists $n < \omega$ such that F_n is a dominating family.*

Proof. Assume not, then for each $n < \omega$ we can choose $g_n \not<^* F_n$. We claim that the family $\{g_n \mid n < \omega\}$ is unbounded in contradiction to $\mathfrak{b} > \omega$. Let $h \in {}^\omega\omega$, then we can choose $f_n \in F_n$ such that $h <^* f_n$. But $g_n \not<^* f_n$ such that $g_n \not<^* h$. $\square$

Theorem 6.1.3. *The following are equivalent:*

i) $\mathcal{F}$ is a Canjar filter.

ii) $\mathcal{F}^{<\omega}$ is a P^+-filter.

Proof. First, assume that $\mathcal{F}$ is not a P^+-filter, so we can choose a decreasing sequence of $\mathcal{F}^{<\omega}$-positive sets $\langle S_n \mid n < \omega \rangle$ without a $\mathcal{F}^{<\omega}$-positive pseudointersection. Let G be $\mathbb{M}_{\mathcal{F}}$-generic, we will show that $V[G]$ contains a dominating real:

First, notice that for every $n < \omega$ we have that $[A_G \setminus n]^{<\omega} \cap S_n \neq \emptyset$, where A_G is the generic set added by $\mathbb{M}_{\mathcal{F}}$. Indeed, assume that $[A_G \setminus n]^{<\omega} \cap S_n = \emptyset$. Then we can pick $(s, A) \in G$ such that

$$(s, A) \Vdash [\dot{A}_{\text{gen}} \setminus n]^{<\omega} \cap S_n = \emptyset$$

But S_n is $\mathcal{F}^{<\omega}$-positive and $A \setminus n \in \mathcal{F}$, so we can choose $t \in [A \setminus n]^{<\omega} \cap S_n$. Then we have that $(s \cup t, A) \leq (s, A)$, so we get

$$(s \cup t, A) \Vdash t \in [\dot{A}_{\text{gen}} \setminus n]^{<\omega} \cap S_n$$

which contradicts our assumption.

Thus, for every $n < \omega$ we can choose $s_n \in [A_G \setminus n]^{<\omega} \cap S_n$ and define a real f by

$$f(n) := \max(s_n) + 1$$

We claim that f is the desired dominating real, so assume we have a real h which is not dominated by f. Then we can define

$$S := \bigcup_{n<\omega} [h(n)]^{<\omega} \cap S_n$$

As each $[h(n)]^{<\omega}$ is finite and $\langle S_n \mid n < \omega\rangle$ is decreasing we have that S is a pseudointersection of $\langle S_n \mid n < \omega\rangle$. Note that if $h(n) \geq f(n)$, then we have that $h(n) > \max(s_n)$, so we get $s_n \in [h(n)]^{<\omega} \cap S_n \subseteq S$. As $h \not<^* f$ this happens for infinitely many $n < \omega$.

We claim that S is $\mathcal{F}^{<\omega}$-positive. Indeed, if $A \in \mathcal{F}$ we have $A_G \subseteq^* A$, so we can choose $n < \omega$ such that $A_G \setminus n \subseteq A$. Now, choose $m > n$ such that $s_m \in S$, then we have

$$s_m \subseteq A_G \setminus m \subseteq A_G \setminus n \subseteq A$$

which shows that $[A]^{<\omega} \cap S \neq \emptyset$.

But by our choice of $\langle S_n \mid n < \omega\rangle$ we get that S cannot be an element of the ground model. Hence, also h cannot be an element of the ground model, which proves that f is a dominating real.

Conversely, assume that $\mathcal{F}^{<\omega}$ is a P$^+$-filter and assume that $\mathbb{M}_\mathcal{F}$ adds a dominating real. Choose a $\mathbb{M}_\mathcal{F}$-name $\dot{g}$ for a dominating real, then we have for every real $f \in {}^\omega\omega$ that

$$\Vdash \exists N < \omega \; \forall n \geq N \; f(n) \leq \dot{g}(n)$$

Thus, we can choose $(t_f, A_f) \in \mathbb{M}_\mathcal{F}$ and $N_f < \omega$ such that

$$(t_f, A_f) \Vdash \forall n \geq N_f \; f(n) \leq \dot{g}(n)$$

By the previous lemma we can choose t and N such that

$$F := \{f \in {}^\omega\omega \mid t_f = t \text{ and } N_f = N\}$$

is a dominating family. Next, for $n < \omega$ we define

$$S'_n := \{s \in [t^C]^{<\omega} \mid \exists A \in \mathcal{F} \; \exists k \geq n \; \exists i < \omega \; (t \cup s, A) \Vdash \dot{g}(k) = i\}$$

which are all extensions of t which decide a value of $\dot{g}$ above n. The density of deciding $\dot{g}$ yields that every S'_n is $\mathcal{F}^{<\omega}$-positive. Indeed, if $B \in \mathcal{F}$ we have that $B \cap t^C \in \mathcal{F}$, so we can choose $s \subseteq B \cap t^C$, $A \in \mathcal{F}$ and $i < \omega$ such that $(t \cup s, A) \leq (t, B \cap t^C)$ and

$$(t \cup s, A) \Vdash \dot{g}(n) = i$$

But this shows that $s \in S'_n \cap [B]^{<\omega}$.

Thus, $\langle S'_n \mid n < \omega\rangle$ is a decreasing sequence of $\mathcal{F}^{<\omega}$-positive sets. We will need to make this sequence vanishing, i.e. the intersection of the sequence should be empty. We do this by removing the non-vanishing part $T := \bigcap_{n<\omega} S'_n$, which are all extension of t which decide infinitely many values of $\dot{g}$ and define a vanishing decreasing sequence by

$$S_n := S'_n \setminus T$$

We will need to prove that T is not $\mathcal{F}^{<\omega}$-positive to conclude that this is still a decreasing sequence of $\mathcal{F}^{<\omega}$-positive sets. Indeed, assume that T is $\mathcal{F}^{<\omega}$-positive. Then for every $s \in T$ we can choose a maximal function $f_s : N_s \to \omega$ with respect to inclusion of a possible decision of $\dot{g}$, i.e. for all $n \in N_s$ there is a $A^n_s \in \mathcal{F}$ such that

$$(t \cup s, A^n_s) \Vdash \dot{g}(n) = f_s(n)$$

By definition of T each N_s is infinite. Extend f_s to a real by mapping N^C_s to 0. Now, as $\mathfrak{b} < \omega$ the set $\{f_s \mid s \in T\}$ is bounded, so we can choose a real $f \in F$ such that f dominates $\{f_s \mid s \in T\}$. By definition of t and N we can pick $A_f \in \mathcal{F}$ such that

$$(t, A_f) \Vdash \forall n \geq N f(n) \leq \dot{g}(n)$$

But by assumption T is $\mathcal{F}^{<\omega}$-positive, so we can choose $s \in T \cap [A_f]^{<\omega}$. As N_s is infinite and $f_s <^* f$ we can pick a $n \in N_s$ such that $n \geq N$ and $f_s(n) < f(n)$. But then we have that $(t \cup s, A_f \cap A_s^n) \leq (t, A_f), (t \cup s, A_s^n)$ which gives us the contradiction

$$(t \cup s, A_f \cap A_s^n) \Vdash \dot{g}(n) = f_s(n) < f(n) \leq \dot{g}(n)$$

Thus, T is not $\mathcal{F}^{<\omega}$-positive which implies that $\langle S_n \mid n < \omega \rangle$ is a vanishing decreasing sequence of $\mathcal{F}^{<\omega}$-positive sets. By assumption we can choose a $\mathcal{F}^{<\omega}$-positive pseudointersection S of $\langle S_n \mid n < \omega \rangle$. This allows us to define a real f by

$$f(n) := \max(\{i \mid \exists s \in S \setminus S_{n+1}\ \exists A \in \mathcal{F}\ (t \cup s, A) \Vdash \dot{g}(n) = i\})$$

so $f(n)$ is the maximal value $\dot{g}(n)$ can be decided as by an extension of t by an element of $S \setminus S_{n+1}$. Note that this is well-defined, because $S \setminus S_{n+1}$ is finite as S is a pseudointersection and if for some $A, B \in \mathcal{F}$ we have that

$$(t \cup s, A) \Vdash \dot{g}(n) = i \qquad \text{and} \qquad (t \cup s, B) \Vdash \dot{g}(n) = j$$

then $i = j$ as $(t \cup s, A)$ and $(t \cup s, B)$ are compatible. As $S \subseteq^* S_0$ and S is $\mathcal{F}^{<\omega}$-positive, also $S \cap S_0$ is $\mathcal{F}^{<\omega}$-positive, so we may assume that $S \subseteq S_0$. Then the vanishing property allows us to conclude that for every $s \in S$ there is a maximal n_0 such that $s \in S_{n_0}$.

Finally, we get a contradiction as follows. As F is dominating we can choose $h \in F$ such that $f <^* h$. Then we can pick $n \geq N$ such that $f(m) < h(m)$ for all $m \geq n$. As $S \subseteq^* S_n$ and S is $\mathcal{F}^{<\omega}$-positive also $S \cap S_n$ is $\mathcal{F}^{<\omega}$-positive, which allows us to choose a $s \in S \cap S_n \cap [A_h]^{<\omega}$. By the previous discussion we can choose a maximal n_0 such that $s \in S_{n_0}$. As $s \in S_n$ we have $n \leq n_0$ and $s \in S \setminus S_{n_0+1}$, so by definition of f and S_{n_0} we can choose an $A \in \mathcal{F}$ and $i \leq f(n_0)$ such that

$$(t \cup s, A) \Vdash \dot{g}(n_0) = i$$

But then we have that $(t \cup s, A \cap A_h) \leq (t, A_h), (t \cup s, A)$ and $f(n_0) < h(n_0)$ which gives us the desired contradiction

$$(t \cup s, A \cap A_h) \Vdash \dot{g}(n_0) = i \leq f(n_0) < h(n_0) \leq \dot{g}(n_0) \qquad \square$$

Summing up this result, we have seen that the purely forcing theoretic statement of $\mathcal{F}$ being a Canjar filter can be characterized by a purely combinatorial property of $\mathcal{F}^{<\omega}$. Remarkably, one can also prove that there is actually another equivalent characterization called the Menger property, which is a purely topological statement. A proof of this fact is given in [8].

6.2 Characterization by strong P-points

In this section we examine the notion of a strong P-point, which Laflamme introduced in [14]. We will prove that this notion implies Canjar's necessary condition from section 4 and use the result of the previous section to prove that it is actually an equivalent characterization of Canjar ultrafilters, which answers one of the questions Laflamme posed in his work about strong P-points.

6.2.1 Strong P-points

We fix an ultrafilter $\mathcal{U}$ on ω throughout this section. The notion of a strong P-point originates from Laflamme's study of forcing with $\mathcal{F}_\sigma$-filters. In this context Laflamme worked with a different equivalent characterization of P-points, so we begin by proving that his and our definition of P-points coincide. Laflamme worked with the fourth characterization for P-points in the following lemma:

Lemma 6.2.1.1. *The following are equivalent:*

i) $\mathcal{U}$ *is a P-point, i.e. for every partition* $\langle X_n \mid n < \omega \rangle$ *of* ω *into sets not from* $\mathcal{U}$ *there exists an* $A \in \mathcal{U}$ *such that for every* $n < \omega$ *we have that* $|A \cap X_n| < \omega$.

ii) For every sequence $\langle X_n \mid n < \omega \rangle$ *not from* $\mathcal{U}$ *there is an* $A \in \mathcal{U}$ *such that for every* $n < \omega$ *we have that* $|A \cap X_n| < \omega$.

iii) For every sequence $\langle A_n \mid n < \omega \rangle$ *from* $\mathcal{U}$ *there is an* $A \in \mathcal{U}$ *such that for every* $n < \omega$ *we have that* $A \subseteq^* A_n$.

iv) For every sequence $\langle A_n \mid n < \omega \rangle$ *from* $\mathcal{U}$ *there is an interval partition* $\langle I_n \mid n < \omega \rangle$ *of* ω *such that* $\bigcup_{n<\omega}(A_n \cap I_n) \in \mathcal{U}$.

Proof. Clearly, ii) $\Longrightarrow$ i). For i) $\Longrightarrow$ ii) let $\langle X_n \mid n < \omega \rangle$ be a sequence not from $\mathcal{U}$. Disjointify the X_n by recursively defining

$$X'_n := X_n \setminus \bigcup_{m<n} X'_m$$

Enumerate $\omega \setminus \bigcup_{n<\omega} X'_n$ by $\langle a_n \mid n < \omega \rangle$ and define a partition $\langle Y_n \mid n < \omega \rangle$ of ω by

$$Y_{2n} := X'_n \qquad \text{and} \qquad Y_{2n+1} := \{a_n\}$$

Now, by assumption choose $A \in \mathcal{U}$ such that for every $n < \omega$ we have that $|A \cap Y_n| < \omega$. We claim that the same holds for the X_n, so let $n < \omega$. Then we have that

$$|A \cap X_n| \leq |A \cap \bigcup_{m \leq n} X'_m| = |\bigcup_{m \leq n} (A \cap Y_{2m})| \leq \sum_{m \leq n} |A \cap Y_{2m}| < \omega$$

For ii) $\Longrightarrow$ iii) let $\langle A_n \mid n < \omega \rangle$ be a sequence from $\mathcal{U}$. Then $X_n := A_n^C$ is a sequence not from $\mathcal{U}$. By assumption choose $A \in \mathcal{U}$ such that for every $n < \omega$ we have that $|A \cap X_n| = |A \cap A_n^C| < \omega$. But this implies that $A \subseteq^* A_n$ for every $n < \omega$.

For iii) $\Longrightarrow$ ii) let $\langle X_n \mid n < \omega \rangle$ be a sequence not from $\mathcal{U}$. Then $A_n := X_n^C$ is a sequence from $\mathcal{U}$. By assumption choose $A \in \mathcal{U}$ such that for every $n < \omega$ we have that $A \subseteq^* A_n = X_n^C$. But this implies that $|A \cap X_n| < \omega$ for every $n < \omega$.

For iii) $\Longrightarrow$ iv) let $\langle A_n \mid n < \omega \rangle$ be a sequence from $\mathcal{U}$. By assumption choose $A \in \mathcal{U}$ such that for every $n < \omega$ we have that $A \subseteq^* A_n$. Thus, we can choose a strictly increasing real f such that $A \setminus f(n) \subseteq A_n$. Define an interval partition $\langle I_n \mid n < \omega \rangle$ of ω by

$$I_0 := [0, f(1)) \qquad \text{and for } n \geq 1 \qquad I_n := [f(n), f(n+1))$$

We claim that

$$A \setminus f(1) \subseteq \bigcup_{n<\omega} (A_n \cap I_n)$$

so let $k \in A \setminus f(1)$. Choose $n < \omega$ such that $k \in I_n$. We have $k \geq f(1)$, so $n \geq 1$. Then $\min(I_n) = f(n) \leq k$, so we have that $k \in A \setminus f(n) \subseteq A_n$. But this shows that $k \in A_n \cap I_n$. Now, together with $A \in \mathcal{U}$ this implies that $\bigcup_{n<\omega}(A_n \cap I_n) \in \mathcal{U}$ as $\mathcal{U}$ is free.

Finally, for iv) $\Longrightarrow$ iii) let $\langle A_n \mid n < \omega \rangle$ be a sequence from $\mathcal{U}$. Define a decreasing sequence $\langle A'_n \mid n < \omega \rangle$ from $\mathcal{U}$ by

$$A'_n := \bigcap_{m \leq n} A_n$$

By assumption choose an interval partition $\langle I_n \mid n < \omega \rangle$ of ω such that

$$A := \bigcup_{n<\omega} (A'_n \cap I_n) \in \mathcal{U}$$

We claim that A is as desired, so let $n < \omega$. By definition of the A'_m we have for $m \geq n$ that $A'_m \subseteq A_n$, i.e. $A'_m \cap A_n^C = \emptyset$. Thus,

$$|A \cap A_n^C| = |\bigcup_{m<\omega} (A'_m \cap I_m \cap A_n^C)| = |\bigcup_{m<n} (A'_m \cap I_m \cap A_n^C)| < \omega$$

which proves that $A \subseteq^* A_n$. □

Note that with the argument for the last implication we get that ii), iii) and iv) are equivalent to their restrictions to decreasing sequences. Now, using the same topology on $\mathcal{P}(\omega)$ as introduced in section 5.2.2 Laflamme generalized the fourth property of the above lemma to sequences of compact subsets of $\mathcal{U}$ with the following definition:

Definition. *$\mathcal{U}$ is called a strong P-point iff for every sequence of compact subsets of $\mathcal{U}$ $\langle \mathcal{K}_n \mid n < \omega \rangle$ there is an interval partition $\langle I_n \mid n < \omega \rangle$ of ω such that for all choices $A_n \in \mathcal{K}_n$ we have that $\bigcup_{n<\omega}(A_n \cap I_n) \in \mathcal{U}$.*

So the notion of a strong P-point claims that we can choose a common witness for the fourth property of the previous lemma for sequences of compact subsets of $\mathcal{U}$. First, we consider how this new definition relates to our other combinatorial notions of rapid filters and P-points. Of course, a strong P-point is a P-point:

Lemma 6.2.1.2. *If $\mathcal{U}$ is a strong P-point, then $\mathcal{U}$ is a P-point.*

Proof. Let $\langle A_n \mid n < \omega \rangle$ be a sequence of sets from $\mathcal{U}$. In the topology of $\mathcal{P}(\omega)$ the singletons $\mathcal{K}_n := \{A_n\} \subseteq \mathcal{U}$ are compact, so as $\mathcal{U}$ is a strong P-point we can choose an interval partition $\langle I_n \mid n < \omega \rangle$ of ω such that $\bigcup_{n<\omega}(A_n \cap I_n) \in \mathcal{U}$. By the previous lemma $\mathcal{U}$ is a P-point. □

Lemma 6.2.1.3. *If $\mathcal{U}$ is a strong P-point, then $\mathcal{U}$ is not rapid.*

Proof. For $n < \omega$ we define

$$\mathcal{K}_n := \{A \subseteq \omega \mid |A^C| \leq n\}$$

We claim that $\mathcal{K}_n$ is compact, so let $A \in \mathcal{P}(\omega) \setminus \mathcal{K}_n$. Then we can choose $s \subseteq A^C$ such that $|s| = n + 1$. But then $A \in \mathcal{O}(\emptyset, s) \subseteq \mathcal{P}(\omega) \setminus \mathcal{K}_n$.

Furthermore, as $\mathcal{U}$ is free we have that $\text{cofin}(\omega) \subseteq \mathcal{U}$, so $\mathcal{K}_n \subseteq \mathcal{U}$ for every $n < \omega$. Now, by assumption on $\mathcal{U}$ choose an interval partition $\langle I_n \mid n < \omega \rangle$ of ω such that for all choices $A_n \in \mathcal{K}_n$ we have that $\bigcup_{n<\omega}(A_n \cap I_n) \in \mathcal{U}$.

Assume $\mathcal{U}$ is rapid. Then we can choose $A \in \mathcal{U}$ such that for every $n < \omega$ we have that $|A \cap I_n| \leq n$. Thus, we can define $A_n := (A \cap I_n)^C \in \mathcal{K}_n$, so for every $n < \omega$ we get

$$A_n \cap I_n = (A \cap I_n)^C \cap I_n = A^C \cap I_n$$

But then by choice of $\langle I_n \mid n < \omega \rangle$ we get that

$$A^C = \bigcup_{n<\omega} (A^C \cap I_n) = \bigcup_{n<\omega} (A_n \cap I_n) \in \mathcal{U}$$

which contradicts that $A \in \mathcal{U}$. □

Next, we want to prove that being a strong P-point transfers downwards in the Rudin-Keisler order. For this we isolate a restricted characterization of strong P-points similar to the one for P-points remarked above, which simplifies a few of the upcoming proofs:

Lemma 6.2.1.4. *The following are equivalent:*

i) $\mathcal{U}$ is a strong P-point.

ii) For every increasing sequence of compact subsets $\langle \mathcal{K}_n \mid n < \omega \rangle$ of $\mathcal{U}$ there is an interval partition $\langle I_n \mid n < \omega \rangle$ of ω such that for all decreasing choices $A_n \in \mathcal{K}_n$ we have that $\bigcup_{n<\omega}(A_n \cap I_n) \in \mathcal{U}$.

Proof. The non-trivial direction is ii) $\Longrightarrow$ i), so let $\langle \mathcal{K}_n \mid n < \omega \rangle$ be sequence of compact subsets of $\mathcal{U}$. Following the notation of Lemma 5.2.2 we define an increasing sequence of compact subsets of $\mathcal{U}$ by

$$\mathcal{C}_n := (\bigcup_{m \leq n} \mathcal{K}_m)^{(n)}$$

By assumption we can choose an interval partition $\langle I_n \mid n < \omega \rangle$ of ω such that for all decreasing choices $A_n \in \mathcal{C}_n$ we have that $\bigcup_{n<\omega}(A_n \cap I_n) \in \mathcal{U}$.

Now, let $B_n \in \mathcal{K}_n$ for every $n < \omega$. Then we can define decreasing choices $A_n \in \mathcal{C}_n$ by

$$A_n := \bigcap_{m \leq n} B_n \in \mathcal{C}_n$$

By choice of $\langle I_n \mid n < \omega \rangle$ we have that $\bigcup_{n<\omega}(A_n \cap I_n) \in \mathcal{U}$. But for every $n < \omega$ we have that $A_n \subseteq B_n$, so we get $\bigcup_{n<\omega}(B_n \cap I_n) \in \mathcal{U}$ as desired. $\square$

For the next lemma we also need the simple topological observation, which claims that the preimage function of any real f is continuous.

Lemma 6.2.1.5. *Let f be a real. Then $f^{-1} : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ is continuous.*

Proof. Let $A \in (f^{-1})^{-1}[\mathcal{O}(s,t)]$. We show that $A \in \mathcal{O}(f[s], f[t]) \subseteq (f^{-1})^{-1}[\mathcal{O}(s,t)]$:

First, $f^{-1}[A] \in \mathcal{O}(s,t)$ implies $s \subseteq f^{-1}[A]$ and $t \subseteq f^{-1}[A]^C$, so $f[s] \subseteq f[f^{-1}[A]] \subseteq A$ and $f[t] \subseteq f[f^{-1}[A]^C] = f[f^{-1}[A^C]] \subseteq A^C$, so we have that $A \in \mathcal{O}(f[s], f[t])$.

Secondly, for $B \in \mathcal{O}(f[s], f[t])$ we have that $f[s] \subseteq B$ and $f[t] \subseteq B^C$. Thus, we get that $s \subseteq f^{-1}[f[s]] \subseteq f^{-1}[B]$ and $t \subseteq f^{-1}[f[t]] \subseteq f^{-1}[B^C] = f^{-1}[B]^C$, i.e. we have that $f^{-1}[B] \in \mathcal{O}(s,t)$, which implies that $\mathcal{O}(f[s], f[t]) \subseteq (f^{-1})^{-1}[\mathcal{O}(s,t)]$. $\square$

Now, we can combine the previous two results to prove that strong P-points are preserved downwards in the Rudin-Keisler order restricted to free ultrafilters:

Lemma 6.2.1.6. *If $\mathcal{U}$ is a strong P-point and $f \in {}^{\omega}\omega$ is not constant on any set in $\mathcal{U}$, then $f(\mathcal{U})$ is a strong P-point.*

Proof. We use Lemma 6.2.1.4, so let $\langle \mathcal{K}_n \mid n < \omega \rangle$ be an increasing sequence of compact subsets of $f(\mathcal{U})$. By the previous lemma $f^{-1} : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ is continuous, so we get a sequence $\langle \mathcal{C}_n \mid n < \omega \rangle$ of compact subsets of $\mathcal{U}$, where $\mathcal{C}_n := \{f^{-1}[A] \mid A \in \mathcal{K}_n\}$.

Now, choose an interval partition $\langle I_n \mid n < \omega \rangle$ of ω witnessing that $\mathcal{U}$ is a strong P-point. Define a strictly increasing real h by

$$h(n) := \max(f[I_n] \cup \operatorname{range}(h \restriction n)) + 1$$

and an interval partition $\langle J_n \mid n < \omega \rangle$ by

$$J_0 := [0, h(0)) \qquad \text{and for } n \geq 1 \qquad J_n := [h(n-1), h(n))$$

Then by definition of J_n for every $k \in I_n$ we have that $f(k) < h(n)$, so $f[I_n] \subseteq \bigcup_{m \leq n} J_n$. Finally, let $A_n \in \mathcal{K}_n$ be decreasing choices. Then for every $n < \omega$ we have $f^{-1}[A_n] \in \mathcal{C}_n$, so by choice of $\langle I_n \mid n < \omega \rangle$ we have that

$$A := \bigcup_{n<\omega} (f^{-1}[A_n] \cap I_n) \in \mathcal{U}$$

But as $f[A] \in f(\mathcal{U})$, essentially applying f to this equation gives us the desired conclusion:

$$f[A] \subseteq \bigcup_{n<\omega} (A_n \cap J_n) \in f(\mathcal{U})$$

To see that this holds, let $k \in A$. Then we can choose $n < \omega$ such that $k \in f^{-1}[A_n] \cap I_n$. By the previous observation choose $m \leq n$ such that $f(k) \in J_m$. We also have that $f(k) \in f[f^{-1}[A_n]] \subseteq A_n \subseteq A_m$ as the A_n's are decreasing. Hence, $f(k) \in A_m \cap J_m$. □

In particular, Lemma 6.2.1.2, Lemma 6.2.1.3 and Lemma 6.2.1.6 show that Laflamme's condition implies Canjar's condition for Canjar ultrafilters:

Corollary 6.2.1.7. *If $\mathcal{U}$ is a strong P-point, then $\mathcal{U}$ is a P-point and does not have any rapid predecessors in the Rudin-Keisler order.* □

6.2.2 Proof of Laflamme's condition

Finally, using the combinatorial characterization of Canjar filters from the previous section we use Lemma 6.2.1.4 to simplify the proof of Blass, Hrušák and Verner given in [4], which shows that Laflamme's condition is indeed a characterization of Canjar ultrafilters. One should note that Laflamme already mentioned in [14] that his condition is necessary and asked whether it was sufficient, but did not publish a proof of this result.

Theorem 6.2.2.1. *The following are equivalent:*

i) $\mathcal{U}^{<\omega}$ is a P^+-filter.

ii) $\mathcal{U}$ is a strong P-point.

Proof. First, assume that $\mathcal{U}^{<\omega}$ is a P$^+$-filter, but not a strong P-point. Choose an increasing sequence $\langle \mathcal{K}_n \mid n < \omega \rangle$ of compact subsets of $\mathcal{U}$ according to Lemma 6.2.1.4 which witnesses the failure of this property. For $n < \omega$ we define

$$S_n := \{s \in [\omega]^{<\omega} \mid \forall A \in \mathcal{K}_n \; s \cap A \neq \emptyset\}$$

As $\langle \mathcal{K}_n \mid n < \omega \rangle$ is increasing we have that $\langle S_n \mid n < \omega \rangle$ is decreasing. We claim that for every $n < \omega$ we have that S_n is $\mathcal{U}^{<\omega}$-positive, so let $B \in \mathcal{U}$. Then

$$\mathcal{K}_n \subseteq \bigcup_{k \in B} \mathcal{O}(\{k\})$$

is an open covering of $\mathcal{K}_n$. Indeed, let $A \in \mathcal{K}_n \subseteq \mathcal{U}$. $\mathcal{U}$ is a filter, so we can choose $k \in A \cap B$, which shows that $A \in \mathcal{O}(\{k\})$.

Now, $\mathcal{K}_n$ is compact, so we can choose $s \in [B]^{<\omega}$ such that

$$\mathcal{K}_n \subseteq \bigcup_{k \in s} \mathcal{O}(\{k\})$$

But this shows that for every $A \in \mathcal{K}_n$ we have that $s \cap A \neq \emptyset$, so $s \in S_n \cap [B]^{<\omega}$ which yields that S_n is $\mathcal{U}^{<\omega}$-positive.

By assumption on $\mathcal{U}^{<\omega}$ we can choose a $\mathcal{U}^{<\omega}$-positive pseudointersection of $\langle S_n \mid n < \omega \rangle$. Now, define a strictly increasing real h by

$$h(n) := \max(\{\max(s) \mid s \in S \setminus S_n\} \cup \mathrm{range}(h \restriction n)) + 1$$

and an interval partition $\langle I_n \mid n < \omega \rangle$ of ω by

$$I_0 := [0, h(1)) \qquad \text{and for } n \geq 1 \qquad I_n := [h(n), h(n+1))$$

By choice of $\langle \mathcal{K}_n \mid n < \omega \rangle$ we can choose $A_n \in \mathcal{K}_n$ decreasingly such that

$$A := \bigcup_{n<\omega} (A_n \cap I_n) \notin \mathcal{U}$$

To get a contradiction we show that $A \in \mathcal{U}$ using Lemma 6.1.1, so fix $s \in S$, then we have to show that $s \cap A \neq \emptyset$. Define

$$n_0 := \max\{n < \omega \mid s \cap I_n \neq \emptyset\}$$

Assume $s \notin S_{n_0}$. Then by definition of h we have that $\min(I_{n_0}) = h(n_0) > \max(s)$, which contradicts that $s \cap I_{n_0} \neq \emptyset$. Thus, $s \in S_{n_0}$. We have that $A_{n_0} \in \mathcal{K}_{n_0}$, so by definition of S_{n_0} we have that $s \cap A_{n_0} \neq \emptyset$, so choose $k \in s \cap A_{n_0}$. By choice of n_0 we have that $s \subseteq \bigcup_{m \leq n_0} I_m$, so choose $m \leq n_0$ such that $k \in I_m$. As the A_n's are decreasing we get that $k \in A_m \cap I_m$, so $k \in s \cap A$.

Thus, we get $A \in \mathcal{U}$ which is a contradiction.

For the other direction, assume $\mathcal{U}$ is a strong P-point and let $\langle S_n \mid n < \omega \rangle$ be a decreasing sequence of $\mathcal{U}^{<\omega}$-positive sets. Lemma 6.1.1 suggests the following definition of a sequence $\langle \mathcal{K}_n \mid n < \omega \rangle$ of compact subsets of $\mathcal{U}$:

$$\mathcal{K}_n := \{A \mid \forall s \in S_n \; s \cap A \neq \emptyset\}$$

Indeed, $\mathcal{K}_n$ is compact, for if $B \in \mathcal{P}(\omega) \setminus \mathcal{K}_n$ we can choose $s \in S_n$ such that $s \cap B = \emptyset$. But then $s \subseteq B^C$, so $B \in \mathcal{O}(\emptyset, s) \subseteq \mathcal{P}(\omega) \setminus \mathcal{K}_n$.

By assumption on $\mathcal{U}$ we can choose an interval partition $\langle I_n \mid n < \omega \rangle$ of ω such that for all choices $A_n \in \mathcal{K}_n$ we have that $\bigcup_{n<\omega}(A_n \cap I_n) \in \mathcal{U}$. Now, define

$$S := \bigcup_{n<\omega} (S_n \cap \mathcal{P}(I_n))$$

Every $\mathcal{P}(I_n)$ is finite and $\langle S_n \mid n < \omega \rangle$ is decreasing, so S is a pseudointersection, i.e. for every $n < \omega$ we have that $S \subseteq^* S_n$. It remains to show that S is $\mathcal{U}^{<\omega}$-positive, so let $A \in \mathcal{U}$. We have to show that $S \cap [A]^{<\omega} \neq \emptyset$, so assume that for all $n < \omega$ we have that

$$S_n \cap \mathcal{P}(I_n) \cap [A]^{<\omega} = \emptyset$$

We construct choices of $\mathcal{K}_n$ by $A_n := (A \cap I_n)^C$. Then we have that $A^C \cap I_n = A_n \cap I_n$. To see that $A_n \in \mathcal{K}_n$ let $s \in S_n$. If $s \notin \mathcal{P}(I_n)$ we can choose $k \in s \cap I_n^C$ and if $s \in \mathcal{P}(I_n)$ by assumption $s \notin [A]^{<\omega}$, so we can choose $k \in s \cap A^C$. In both cases k witnesses that we have $s \cap A_n \neq \emptyset$.

Finally, note that by choice of the interval partition $\langle I_n \mid n < \omega \rangle$ we get that

$$A^C = \bigcup_{n<\omega} (A^C \cap I_n) = \bigcup_{n<\omega} (A_n \cap I_n) \in \mathcal{U}$$

which contradicts that $A \in \mathcal{U}$. $\square$

6.3 A consistent counterexample to Canjar's condition

In our final section, we complete our study of Canjar ultrafilters by proving that Canjar's condition is consistently not a sufficient condition for Canjar ultrafilters. We will establish this result following the second part of Blass, Verner and Hrušák in [4] by constructing an ultrafilter satisfying Canjar's condition but not Laflamme's condition under the assumption of MA(countable), i.e. Martins Axiom for countable partial orders. With the result from the previous section this proves that such an ultrafilter is not Canjar. As is proven in [2] the assumption MA(countable) is equivalent to $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$.

6.3.1 Summable Ideals and rapid ultrafilters

For the construction in the last subsection we will introduce the notion of summable ideals and use them to prove an alternative characterization of rapid ultrafilters first observed by Vojtáš in [17].

Definition. *Let $g : \omega \to \mathbb{R}^{\geq 0}$ be a function. Then we call*

$$\mathcal{I}_g := \{A \subseteq \omega \mid \sum_{n \in A} g(n) < \infty\}$$

the summable ideal associated to g. If for an ideal $\mathcal{I}$ on ω there is a $g : \omega \to \mathbb{R}^{\geq 0}$ such that $\mathcal{I} = \mathcal{I}_g$, then we call $\mathcal{I}$ a summable ideal.

Remark. Note that $\mathcal{I}_g$ need not be a proper ideal as $\mathcal{I}_g = \mathcal{P}(\omega)$ iff $\sum_{n\in\omega} g(n) < \infty$.

Lemma 6.3.1.1. *Let $g : \omega \to \mathbb{R}^{\geq 0}$ be a function. Then $\mathcal{I}_g$ is an free ideal on ω.*

Proof. First, note that if $A, B \in \mathcal{I}_g$, then

$$\sum_{n \in A\cup B} g(n) \leq \sum_{n \in A} g(n) + \sum_{n \in B} g(n) < \infty$$

so $A \cup B \in \mathcal{I}_g$. Furthermore, if $A \subseteq B \in \mathcal{I}_g$, then

$$\sum_{n \in A} g(n) \leq \sum_{n \in B} g(n) < \infty$$

so $A \in \mathcal{I}_g$. Finally, for every $A \in [\omega]^{<\omega}$ clearly we have that $\sum_{n\in A} g(n) < \infty$. □

Thus, our definition of summable ideals is well-defined. Next, we prove that tall summable ideals are exactly those summable ideals whose generating function g converges to 0:

Definition. *An ideal $\mathcal{I}$ on ω is tall iff for all $B \in [\omega]^\omega$ there is an $A \in \mathcal{I}$ with $|A\cap B| = \omega$.*

Lemma 6.3.1.2. *Let $g : \omega \to \mathbb{R}^{\geq 0}$ be a function. Then $\mathcal{I}_g$ is tall iff $\lim_{n\to\infty} g(n) = 0$.*

Proof. Assume $\mathcal{I}_g$ is tall and let $\epsilon > 0$. We have to show that

$$B := \{n < \omega \mid g(n) \geq \epsilon\}$$

is finite. Assume that B is infinite. As $\mathcal{I}_g$ is tall we can take $A \in \mathcal{I}_g$ such that $A \cap B$ is infinite. But then we get a contradiction as

$$\sum_{n \in A} g(n) \geq \sum_{n \in A\cap B} g(n) \geq \sum_{n \in A\cap B} \epsilon = \infty$$

Conversely, assume that $\lim_{n\to\infty} g(n) = 0$ and let $B \in [\omega]^\omega$. Then we can recursively choose a strictly increasing sequence $a_n \in B$ such that $g(a_n) \leq 2^{-n}$, because B is infinite and for every $n < \omega$ the set

$$\{a < \omega \mid g(a) \leq 2^{-n}\}$$

is cofinite as $\lim_{n\to\infty} g(n) = 0$. Now, set $A := \{a_n \mid n < \omega\}$. Then $A \cap B = A$ is infinite and we have that $A \in \mathcal{I}_g$ as

$$\sum_{n\in A} g(n) = \sum_{n<\omega} g(a_n) \leq \sum_{n<\omega} 2^{-n} = 2 < \infty$$

□

Now, following Vojtáš result in [17] we can prove the following equivalent characterization of rapid filters in terms of summable ideals:

Theorem 6.3.1.3. *An ultrafilter $\mathcal{U}$ is rapid iff it intersects every tall summable ideal.*

Proof. First, assume that $\mathcal{U}$ is rapid. Using the previous lemma we have to consider a function $g : \omega \to \mathbb{R}^{\geq 0}$ with $\lim_{n\to\infty} g(n) = 0$ and show that $\mathcal{U} \cap \mathcal{I}_g$ is non-empty. Define a real $f \in {}^\omega\omega$ by

$$f(n) := \min\{k < \omega \mid \forall l \geq k \; g(l) \leq 2^{-n}\}$$

As g tends to 0 this is well-defined. Now, by assumption on $\mathcal{U}$ choose $A \in \mathcal{U}$ such that $f < \operatorname{enum}(A)$, i.e. by definition of f for all $n < \omega$ we have that $g(\operatorname{enum}(A)(n)) \leq 2^{-n}$. Thus, we get that $A \in \mathcal{U} \cap \mathcal{I}_g$ as

$$\sum_{n\in A} g(n) = \sum_{n<\omega} g(\operatorname{enum}(A)(n)) \leq \sum_{n<\omega} 2^{-n} = 2 < \infty$$

Conversely, assume that $\mathcal{U}$ intersects every tall summable ideal and let $f \in {}^\omega\omega$ be a strictly increasing real. Define a function $g : \omega \to \mathbb{R}^{\geq 0}$ by

$$g(k) := \begin{cases} 1 & k \in [0, f(0)] \\ \frac{\log(n+1)}{n+1} & k \in (f(n-1), f(n)] \text{ for some } n \geq 1 \end{cases}$$

Note that $\lim_{n\to\infty} g(n) = 0$ as $\lim_{n\to\infty} \frac{\log(n+1)}{n+1} = 0$. Thus, by the previous lemma $\mathcal{I}_g$ is a tall summable ideal and by assumption we can choose $A \in \mathcal{U} \cap \mathcal{I}_g$ and claim that $f <^* \operatorname{enum}(A)$, proving that $\mathcal{U}$ is rapid.

Indeed, assume the contrary. Then we have that

$$\sum_{n\in A} g(n) = \sum_{n<\omega} g(\operatorname{enum}(A)(n)) \geq \lim_{n\to\infty} \log(n+1) = \infty$$

contradicting that $A \in \mathcal{I}_g$.

To see that the inequality holds, let $n < \omega$. By assumption we can choose $N \geq n, 3$ such that $f(N) \geq \operatorname{enum}(A)(N)$. But this means that we can pick $\{a_0, \ldots, a_N\} \in [0, f(N)] \cap A$. Note that $N \geq 3$ implies monotonicity of g, i.e. that $g(k) \geq g(f(N))$ for all $k \leq f(N)$, so by definition of g we get for every a_i that $g(a_i) \geq g(f(N)) = \frac{\log(N+1)}{N+1}$ which shows that

$$\sum_{n<\omega} g(\operatorname{enum}(A)(n)) \geq \sum_{i=0}^{N} g(a_i) \geq (N+1)\frac{\log(N+1)}{N+1} = \log(N+1)$$

□

6.3.2 Construction of the counterexample

At last, we consider the construction of an ultrafilter which is not a strong P-point, but a P-point, which does not have rapid Rudin-Keisler predecessors under the assumption of MA(countable) given by Blass, Verner and Hrušák in [4], where MA(countable) is the assumption of Martin's Axiom for all countable partial orders. This will settle the question whether Canjar's condition is consistently weaker as Laflamme's condition.

Before we dive into the construction, we give an overview of the construction, fix some definitions and prove four extension lemmata which are the crucial ingredients needed for our construction.

Theorem 6.3.2.1. *Under* MA(countable) *there is an ultrafilter $\mathcal{U}$ which is not a strong P-point, but a P-point which does not have rapid predecessors in the Rudin-Keisler order.*

Proof sketch. The basic structure of the construction is to iteratively build an increasing chain of filters $\langle F_\alpha \mid \alpha < \mathfrak{c}\rangle$, such that their union $\mathcal{U}$ is our desired ultrafilter. For that in successor steps, we will use MA(countable) to find subsets $A \subseteq \omega$, which when added to our chain, will witness that

1. ...we added enough sets to $\mathcal{U}$, such that $\mathcal{U}$ will be an ultrafilter.
2. ...we killed a potential witness for strong P-pointedness.
3. ...we added a pseudointersection for some instance of P-pointedness.
4. ...we killed a potential witness for a rapid Rudin-Keisler predecessor.

For this, we introduce the following measure on ω by

$$\mu(A) := \sum_{n\in A} \frac{1}{n}$$

Clearly, this is a countably additive measure on ω, which satisfies that $\mu(\omega) = \infty$ as this yields the harmonic series. One of the main ideas is to restrict our construction to μ-infinite filters, i.e. filters $\mathcal{F}$ such that for all $A \in \mathcal{F}$ we have that $\mu(A) = \infty$. In this case we can use the following characterization, for when a generated filter is μ-infinite:

If we have a μ-infinite filter $\mathcal{F}$ and a subset $A \subseteq \omega$, then the filter generated by $\mathcal{F} \cup \{A\}$ is μ-infinite iff for every $B \in \mathcal{F}$ we have that $\mu(A \cap B) = \infty$.

Furthermore, we will construct the witnesses for 2. with respect to the sequence of compact subsets $\langle \mathcal{C}_n \mid n < \omega\rangle$ of $\mathcal{P}(\omega)$ defined by

$$\mathcal{C}_n := \{A \subseteq \omega \mid \mu(A^C) \leq n+1\}$$

i.e. of all subsets of ω whose complement has small measure with respect to μ. Clearly this defines a compact subset of $\mathcal{P}(\omega)$, for if $A \in \mathcal{P}(\omega) \setminus \mathcal{C}_n$, then $\mu(A^C) > n+1$. Thus, we can take a finite $s \subseteq A^C$ such that $\mu(s) > n+1$. But then $A \in \mathcal{O}(\emptyset, s) \subseteq \mathcal{P}(\omega) \setminus \mathcal{C}_n$, because for every $B \in \mathcal{O}(\emptyset, s)$ we have that $\mu(B^C) \geq \mu(s) > n+1$.

Finally, in 4. we only have to consider Rudin-Blass instead of Rudin-Keisler predecessors of $\mathcal{U}$, because by 1. and 3. our final filter $\mathcal{U}$ will be a P-point, for which the following lemma states that all Rudin-Keisler predecessors will be Rudin-Blass predecessors:

Lemma 6.3.2.2. *Let $\mathcal{V} \leq_{\mathrm{RK}} \mathcal{U}$ and $\mathcal{U}$ be a P-point. Then $\mathcal{V} \leq_{\mathrm{RB}} \mathcal{U}$.*

Proof. Let $f \in {}^\omega\omega$ be a real witnessing that $f(\mathcal{U}) = \mathcal{V}$. By Lemma 3.2 choose $A \in \mathcal{U}$ such that $f \restriction A$ is finite-to-one. Then $\mathcal{V} \leq_{\mathrm{RB}} \mathcal{U}$ by Lemma 2.2.2. □

The first extension lemma already follows from the measure properties of μ and does not need the assumption of MA(countable):

Lemma 6.3.2.3. *Let $\mathcal{F}$ be a μ-infinite filter and $A \subseteq \omega$. Then we have that either $\mathcal{F} \cup \{A\}$ or $\mathcal{F} \cup \{A^C\}$ generates a μ-infinite filter.*

Proof. Assume that there is a $B \in \mathcal{F}$ such that $\mu(A \cap B) < \infty$. As $\mu(B) = \infty$ we have that $\mu(A^C \cap B) = \infty$. Now, let $C \in \mathcal{F}$. Then $\mu(A \cap B \cap C) \leq \mu(A \cap B) < \infty$. But $B \cap C \in \mathcal{F}$, so $\mu(B \cap C) = \infty$ implies that $\mu(A^C \cap B \cap C) = \infty$. Thus, we get that $\mu(A^C \cap B \cap C) \leq \mu(A^C \cap C) = \infty$, so $\mathcal{F} \cup \{A^C\}$ generates a μ-infinite filter.

Otherwise, we have that $\mu(A \cap B) = \infty$ for all $B \in \mathcal{F}$, so $\mathcal{F} \cup \{A\}$ generates a μ-infinite filter. $\square$

For the other three extension lemmata we will use MA(countable) to prove the existence of a subset of ω with desirable properties. The second extension lemma proves that we can kill a possible witness to strong P-pointedness:

Lemma 6.3.2.4. *Let $\mathcal{F}$ be a μ-infinite filter with $\chi(\mathcal{F}) < \mathfrak{c}$ and $\langle I_n \mid n < \omega \rangle$ an interval partition of ω. Then there is an $A \subseteq \omega$ and choices $A_n \in \mathcal{C}_n$ such that $\mathcal{F} \cup \{A\}$ generates a μ-infinite filter and $A \cap \bigcup_{n<\omega}(A_n \cap I_n) = \emptyset$.*

Proof. Define a countable forcing notion by

$$\mathbb{P} := \{s \in [\omega]^{<\omega} \mid \forall n < \omega \ \mu(s \cap I_n) \leq n + 1\}$$

ordered by reversed inclusion. As $\chi(\mathcal{F}) < \mathfrak{c}$ we can choose a basis $\mathcal{B}$ of $\mathcal{F}$ with $|\mathcal{B}| < \mathfrak{c}$.

Now, for every $B \in \mathcal{B}$ and $k < \omega$ the set

$$D_B^k := \{s \in \mathbb{P} \mid \mu(s \cap B) \geq k\}$$

is dense. Indeed, let $s \in \mathbb{P}$, and define $N := \max(\{n < \omega \mid s \cap I_n \neq \emptyset\} \cup \{k\})$. Now, $\mu(B) = \infty$ as $\mathcal{F}$ is μ-infinite, so we can choose a finite $t \subseteq B \setminus (\max(I_N) + 1)$ such that $\mu(t) \in [k, k+1]$. But then $s \cup t \leq s$ is as desired as $\mu((s \cup t) \cap B) \geq \mu(t) \geq k$, so $s \cup t \in D_B^k$. Note that $s \cup t \in \mathbb{P}$, because for all $n < \omega$ we have $\mu((s \cup t) \cap I_n) \leq n + 1$ since for $n \leq N$ we get that

$$\mu((s \cup t) \cap I_n) = \mu(s \cap I_n) \leq n + 1$$

whereas for $n > N \geq k$ we have that

$$\mu((s \cup t) \cap I_n) = \mu(t \cap I_n) \leq \mu(t) \leq k + 1 \leq n + 1$$

By MA(countable) there is a generic G for these less than $\mathfrak{c}$ many dense sets. We claim that $A := \bigcup G$ is as desired. Indeed, for every $k < \omega$ and $B \in \mathcal{B}$ there is a $s \in D_B^k \cap G$. Hence, we have that

$$\mu(A \cap B) \geq \mu(s \cap B) \geq k$$

i.e. $\mu(A \cap B) = \infty$, so $\mathcal{F} \cup \{A\}$ generates a μ-infinite filter. Furthermore, if we define

$$A_n := (A \cap I_n)^C$$

we have that $A_n \in \mathcal{C}_n$, because G is directed, so we can use continuity of μ to prove

$$\mu(A \cap I_n) = \mu(\bigcup G \cap I_n) = \mu(\bigcup_{s \in G}(s \cap I_n)) = \sup_{s \in G} \mu(s \cap I_n) \leq n + 1$$

Finally, we have that

$$A \cap \bigcup_{n<\omega}(A_n \cap I_n) = A \cap \bigcup_{n<\omega}(A^C \cap I_n) = \emptyset$$

$\square$

The third extension lemma proves that we can add a pseudointersection for a sequence of sets from our filter, such that P-pointedness is satisfied for this instance:

Lemma 6.3.2.5. *Let $\mathcal{F}$ be a μ-infinite filter with $\chi(\mathcal{F}) < \mathfrak{c}$ and $\langle A_n \mid n < \omega \rangle$ a sequence of sets from $\mathcal{F}$. Then there is an $A \subseteq \omega$ such that $\mathcal{F} \cup \{A\}$ generates a μ-infinite filter and A is a pseudointersection of $\langle A_n \mid n < \omega \rangle$.*

Proof. Define a countable forcing notion by

$$\mathbb{P} := \{(s, I) \mid s, I \in [\omega]^{<\omega}\}$$

and the ordering of $(s, I), (t, J) \in \mathbb{P}$ by

$$(t, J) \leq (s, I) \ :\Longleftrightarrow \ s \subseteq t, I \subseteq J \text{ and } t \setminus s \subseteq \bigcap_{n \in I} A_n$$

Again, as $\chi(\mathcal{F}) < \mathfrak{c}$ we can choose a basis $\mathcal{B}$ of $\mathcal{F}$ with $|\mathcal{B}| < \mathfrak{c}$. Now, for every $B \in \mathcal{B}$ and $k < \omega$ the set

$$D_B^k := \{(s, I) \in \mathbb{P} \mid \mu(s \cap B) \geq k\}$$

is dense. Indeed, let $(s, I) \in \mathbb{P}$, then $B \cap \bigcap_{n \in I} A_n \in \mathcal{F}$, so $\mu(B \cap \bigcap_{n \in I} A_n) = \infty$ as $\mathcal{F}$ is μ-infinite. Hence, we can choose a finite $t \subseteq B \cap \bigcap_{n \in I} A_n$ such that $\mu(t) \geq k$. But then we have that $(s \cup t, I) \leq (s, I)$ and $(s \cup t, I) \in D_B^k$ as

$$\mu((s \cup t) \cap B) \geq \mu(t) \geq k$$

Furthermore, for every $n < \omega$ the set

$$D_n := \{(s, I) \in \mathbb{P} \mid n \in I\}$$

is clearly dense. By MA(countable) there is a generic G for these less than $\mathfrak{c}$ many dense sets. We claim that $A := \bigcup \{s \mid \exists I \in [\omega]^{<\omega} \ (s, I) \in G\}$ is as desired. Indeed, for every $k < \omega$ and $B \in \mathcal{B}$ there is a $(s, I) \in D_B^k \cap G$. Hence,

$$\mu(A \cap B) \geq \mu(s \cap B) \geq k$$

i.e. $\mu(A \cap B) = \infty$, so $\mathcal{F} \cup \{A\}$ generates a μ-infinite filter. It is left to show that A is a pseudointersection of $\langle A_n \mid n < \omega \rangle$, so let $n < \omega$. Then there is a $(s, I) \in D_n \cap G$, i.e. $n \in I$. Thus, we have that $A \setminus A_n \subseteq s$, because for every $k \in A \setminus s$ we can choose some $(t, J) \in G$ with $(t, J) \leq (s, I)$ and $k \in t$. But then by definition of the extension relation and $n \in I$ we get that $k \in A_n$. $\square$

The fourth extension lemma proves that we can kill a possible witness for a rapid Rudin-Blass predecessor:

Lemma 6.3.2.6. *Let $\mathcal{F}$ be a μ-infinite filter with $\chi(\mathcal{F}) < \mathfrak{c}$ and f a finite-to-one real. Then there is an $A \subseteq \omega$ and a function $g : \omega \to \mathbb{R}^{\geq 0}$ with $\lim_{n \to \infty} g(n) = 0$ such that $\mathcal{F} \cup \{A\}$ generates a μ-infinite filter and for all $B \in \mathcal{I}_g$ we have that $\mu(A \cap f^{-1}[B]) < \infty$.*

Proof. For $n < \omega$ we define $b_n := f^{-1}[\{n\}] \in [\omega]^{<\omega}$ and for $s \in [\omega]^{<\omega}$ we define

$$N(s) := \max\{n < \omega \mid s \cap b_n \neq \emptyset\}$$

Furthermore, we define a countable forcing notion by

$$\mathbb{P} := \{(s, p) \mid s \in [\omega]^{<\omega}, p \in \mathbb{Q}\}$$

and the ordering of $(s, p), (t, q) \in \mathbb{P}$ by

$$\begin{aligned}(t, q) \leq (s, p) \ :\Longleftrightarrow \ & s \subseteq t, q \leq p, \forall n \leq N(s) \ t \cap b_n = s \cap b_n \\ & \text{and } \forall n > N(s) \ \mu(t \cap b_n) \leq p\end{aligned}$$

i.e. s and t agree on the first $N(s)$ b_n's and on every other b_n the elements added have small measure. Again, as $\chi(\mathcal{F}) < \mathfrak{c}$ we can choose a basis $\mathcal{B}$ of $\mathcal{F}$ with $|\mathcal{B}| < \mathfrak{c}$. Now, for every $B \in \mathcal{B}$ and $k < \omega$ the set

$$D_B^k := \{(s,p) \in \mathbb{P} \mid \mu(s \cap B) \geq k\}$$

is dense. Indeed, if $(s,p) \in \mathbb{P}$, then let $N > N(s)$ be big enough such that for all $n \geq N$ we have that $\mu(\{\min(b_n)\}) = \frac{1}{\min(b_n)} \leq p$. Note that this implies that if $\mu(b_n \cap B) \geq \frac{p}{2}$, then we can choose $t \subseteq b_n \cap B$ such that $\mu(t) \in [\frac{p}{2}, p]$. Let $I := \{n \geq N \mid \mu(b_n \cap B) \geq \frac{p}{2}\}$, then we have to consider the following two cases:

First, if I is infinite, then let I_0 be the first $\left\lceil \frac{2k}{p} \right\rceil$ elements of I and for each $n \in I_0$ choose some $t_n \subseteq b_n \cap B$ with $\mu(t_n) \in [\frac{p}{2}, p]$. Now, set $t := \bigcup_{n \in I_0} t_n$, then $(s \cup t, p) \leq (s,p)$, because by construction of t we have for every $n > N(s)$ that

$$\mu((s \cup t) \cap b_n) = \mu(t \cap b_n) = \left\{\begin{matrix} \mu(t_n) & \text{if } n \in I_0 \\ 0 & \text{otherwise} \end{matrix}\right\} \leq p$$

Furthermore, we have that $(s \cup t, p) \in D_B^k$ as

$$\mu((s \cup t) \cap B) \geq \mu(t) = \mu(\bigcup_{n \in I_0} t_n) = \sum_{n \in I_0} \mu(t_n) \geq \left\lceil \frac{2k}{p} \right\rceil \cdot \frac{p}{2} \geq k$$

Otherwise, assume that I is finite. Then $\mu(\bigcup_{n \in I \cup N}(b_n \cap B)) < \infty$, but $\mu(B) = \infty$, so we have that $\mu(\bigcup_{n \notin I \cup N}(b_n \cap B)) = \infty$. But this means that there is a finite $J_0 \subseteq (I \cup N)^C$ such that $\mu(\bigcup_{n \in J_0}(b_n \cap B)) \geq k$. Now, set $t := \bigcup_{n \in J_0}(b_n \cap B)$, then $(s \cup t, p) \leq (s,p)$, because by definition of I we have for every $n > N(s)$ that

$$\mu((s \cup t) \cap b_n) = \mu(t \cap b_n) = \left\{\begin{matrix} \mu(b_n \cap B) & \text{if } n \in J_0 \\ 0 & \text{otherwise} \end{matrix}\right\} \leq \frac{p}{2} \leq p$$

Furthermore, we have that $(s \cup t, p) \in D_B^k$ as

$$\mu((s \cup t) \cap B) \geq \mu(t) = \mu(\bigcup_{n \in J_0}(b_n \cap B)) \geq k$$

Also, note that for every $r \in \mathbb{Q}$ the set $D_r := \{(s,p) \in \mathbb{P} \mid p \leq r\}$ is clearly dense. By MA(countable) there is a generic G for these less than $\mathfrak{c}$ many dense sets. We claim that $A := \bigcup\{s \mid \exists p \in \mathbb{Q}\ (s,p) \in G\}$ is as desired. Indeed, for every $k < \omega$ and $B \in \mathcal{B}$ there is a $(s,p) \in D_B^k \cap G$. Hence,

$$\mu(A \cap B) \geq \mu(s \cap B) \geq k$$

i.e. $\mu(A \cap B) = \infty$, so $\mathcal{F} \cup \{A\}$ generates a μ-infinite filter. Furthermore, if we define a function $g : \omega \to \mathbb{R}^{\geq 0}$ by $g(n) := \mu(A \cap b_n)$, then we have that $\lim_{n \to \infty} g(n) = 0$.

Indeed, if $r > 0$, then there is a $(s,p) \in D_r \cap G$, i.e. $p \leq r$. But then for every $n > N(s)$ we can choose some $(t,q) \in G$ with $(t,q) \leq (s,p)$ and $N(t) \geq n$. By definition of the extension relation we get that $A \cap b_n = t \cap b_n$. Hence,

$$g(n) = \mu(A \cap b_n) = \mu(t \cap b_n) \leq p \leq r$$

Finally, let $B \in \mathcal{I}_g$. Then we have that

$$\begin{aligned} \mu(A \cap f^{-1}[B]) &= \mu(A \cap \bigcup_{n \in B} f^{-1}[\{n\}]) = \mu(\bigcup_{n \in B}(A \cap b_n)) \\ &= \sum_{n \in B} \mu(A \cap b_n) = \sum_{n \in B} g(n) < \infty \end{aligned}$$

$\square$

Finally, we can use these extension lemmata to iteratively construct an ultrafilter satisfying our desired properties:

Proof of Theorem 6.3.2.1. We have to enumerate all of the following objects for our construction:

1) Choose an enumeration $\langle A_\alpha \mid \alpha < \mathfrak{c}\rangle$ of $\mathcal{P}(\omega)$.
2) Choose an enumeration $\langle\langle I_n^\alpha \mid n < \omega\rangle \mid \alpha < \mathfrak{c}\rangle$ of all interval partitions of ω.
3) Choose an enumeration $\langle\langle A_n^\alpha \mid n < \omega\rangle \mid \alpha < \mathfrak{c}\rangle$ of all countable sequences of subsets of ω, where each sequence appears cofinally often.
4) Choose an enumeration $\langle f_\alpha \mid \alpha < \mathfrak{c}\rangle$ of all finite-to-one reals.

Now, we iteratively construct an increasing chain of μ-infinite filters $\langle \mathcal{F}_\alpha \mid \alpha < \mathfrak{c}\rangle$ such that the following conditions hold for all $\alpha < \mathfrak{c}$:

i) $\chi(\mathcal{F}_\alpha) \leq |\alpha + \omega|$.
ii) Either $A_\alpha \in \mathcal{F}_{\alpha+1}$ or $A_\alpha^C \in \mathcal{F}_{\alpha+1}$.
iii) There are $A \in F_{\alpha+1}$ and choices $A_n \in \mathcal{C}_n$ such that $A \cap \bigcup_{n<\omega}(A_n \cap I_n^\alpha) = \emptyset$.
iv) If $\langle A_n^\alpha \mid n < \omega\rangle$ is a sequence of sets from $\mathcal{F}_\alpha$ then there is a pseudointersection $A \in \mathcal{F}_{\alpha+1}$ for $\langle A_n^\alpha \mid n < \omega\rangle$.
v) There is a function $g : \omega \to \mathbb{R}^{\geq 0}$ with $\lim_{n\to\infty} g(n) = 0$ and an $A \in \mathcal{F}_{\alpha+1}$ such that for all $B \in \mathcal{I}_g$ we have that $\mu(A \cap f_\alpha^{-1}[B]) < \infty$.

If we succeed with such a construction then the μ-infinite filter $\mathcal{U} := \bigcup_{\alpha<\mathfrak{c}} \mathcal{F}_\alpha$ is as desired. Indeed, ii) implies that $\mathcal{U}$ is an ultrafilter. Together with μ-infiniteness of $\mathcal{U}$ this implies that $\mathcal{U}$ contains all μ-cofinite sets, i.e. all $A \subseteq \omega$ such that $\mu(A^C) < \infty$.

We have that $\mathcal{U}$ is not a strong P-point, because the $\mathcal{C}_n$ are subsets of $\mathcal{U}$ as $\mathcal{U}$ contains all μ-cofinite sets. Now, every interval partition of ω occurs in 2. as some $\langle I_n^\alpha \mid n < \omega\rangle$, so iii) gives us an $A \in U$ and choices $A_n \in \mathcal{C}_n$ such that $A \cap \bigcup_{n<\omega}(A_n \cap I_n) = \emptyset$ which implies that $\bigcup_{n<\omega}(A_n \cap I_n) \notin \mathcal{U}$.

Furthermore, $\mathcal{U}$ is a P-point for if $\langle A_n \mid n < \omega\rangle$ is a sequence of sets from $\mathcal{U}$, then by König's theorem this sequence occurs in 3. as some $\langle A_n^\alpha \mid n < \omega\rangle$, so by iv) there is a pseudointersection of $\langle A_n^\alpha \mid n < \omega\rangle$ in $\mathcal{U}$.

Finally, $\mathcal{U}$ does not have rapid Rudin-Keisler predecessors, for if $\mathcal{V} \leq_{\mathrm{RK}} \mathcal{U}$, then we get from Lemma 6.3.2.2 that $\mathcal{V} \leq_{\mathrm{RB}} \mathcal{U}$. Let f be a finite-to-one real such that $f(\mathcal{U}) = \mathcal{V}$, so f occurs in 3. as some f_α. Now, by v) choose a function $g : \omega \to \mathbb{R}^{\geq 0}$ with $\lim_{n\to\infty} g(n) = 0$ and an $A \in \mathcal{U}$ such that for all $B \in \mathcal{I}_g$ we have that $\mu(A \cap f_\alpha^{-1}[B]) < \infty$.

We show that $\mathcal{I}_g \cap \mathcal{V} = \emptyset$, so let $B \in \mathcal{I}_g$. Then $\mu(A \cap f_\alpha^{-1}[B]) < \infty$, so $(A \cap f_\alpha^{-1}[B])^C \in \mathcal{U}$ as $\mathcal{U}$ contains all μ-cofinite sets. But then $A \cap (A \cap f_\alpha^{-1}[B])^C = A \cap f_\alpha^{-1}[B]^C \in \mathcal{U}$, which implies that $B \notin f_\alpha(\mathcal{U}) = \mathcal{V}$. Thus, by Theorem 6.3.1.3 we get that $\mathcal{V}$ is not rapid.

Now, for the construction we start with $\mathcal{F}_0 := \mathrm{cofin}(\omega)$ which clearly satisfies $\chi(\mathcal{F}_0) = \omega$ and is μ-infinite. At limit α's we take the union of our current chain of μ-infinite filters to obtain a μ-infinite filter which satisfies i).

Finally, for the successor case assume that we have defined a μ-infinite filter $\mathcal{F}_\alpha$. Then we apply Lemma 6.3.2.3, Lemma 6.3.2.4, Lemma 6.3.2.5 and then Lemma 6.3.2.6 after one another to $\mathcal{F}_\alpha$ to obtain a filter $\mathcal{F}_{\alpha+1}$ with $\chi(F_{\alpha+1}) = \chi(F_\alpha)$ and which contains witnesses for properties ii), iii), iv) and v) for the current α, respectively. $\square$

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