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„Future Stability of Relativistic Fluids and Pressureless Dust in Expanding FLRW-type Models“

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Abstract

This thesis investigates the future stability of solutions to the relativistic Euler equations on a fixed FLRW-type background metric that expands with a scale factor $t^{2\alpha}$. To do this we perturb a given solution in a suitable Sobolev space and observe the behaviour of the Sobolev norm of the fluid velocity and energy density over time. First we show local existence of a solution and then global existence by a bootstrap argument. In the case of a pressureless dust and fluid model of the universe we find future stability for constants $\alpha > \frac{1}{2}$ and $\alpha > 1$, respectively. This is done by employing an energy estimate method for partial differential equations. A large portion of the proof revolves around estimating an energy current that is shown to be equivalent to the relevant Sobolev norms. This result generalizes previous work, which was restricted to flat spatial geometries, to arbitrary curved closed smooth spatial geometries.

Zusammenfassung

Diese Masterarbeit untersucht die Zukunftsstabilität von Lösungen der relativistischen Eulergleichungen auf einer festen FLRW-Hintergrundmetrik, die mit einem Skalierungsfaktor $t^{2\alpha}$ expandiert. Dafür stören wir eine als gegeben angenommene Lösung in einem geeigneten Sobolevraum und analysieren das Langzeitverhalten der Sobolevnormen der Fluidgeschwindigkeit und der Energiedichte. Zuerst zeigen wir lokale Existenz einer Lösung und dann globale Existenz mithilfe eines Bootstrap-Arguments. Im Falle eines drucklosen Staubmodells zeigen wir Zukunftsstabilität für $\alpha > \frac{1}{2}$, für ein Fluidmodell für $\alpha > 1$. Dafür nutzen wir die Methode der Energieabschätzungen für partielle Differentialgleichungen. Ein großer Teil des Beweises befasst sich damit, einen Energiestrom abzuschätzen, der äquivalent zu den relevanten Sobolevnormen ist. Dieses Resultat verallgemeinert vorhergehende Arbeiten, welche sich lediglich mit flachen räumlichen Geometrien befassten, auf beliebige gekrümmte abgeschlossene glatte räumliche Geometrien.

Contents

1. Introduction

1.0.1. The Background Metric

In this thesis we investigate the future asymptotic behaviour of solutions to the $3+1$-dimensional relativistic Euler equations. However, we will restrict ourselves to the treatment of a specific class of fixed background spacetimes $([1,\infty)\times M, g)$, where M is a closed Riemannian manifold and g is given by

$$g = -dt^2 + ct^{2\alpha}\gamma, \tag{1.1}$$

where $c \in \mathbb{R}^+$ and $\alpha \in \mathbb{R}^+$ are constants and γ is a closed (i.e. compact without boundary) Riemannian 3-metric. Note that the metric described above is not necessarily one of constant curvature and hence does not have to be an FLRW metric. However, this class of spacetimes is the most explicitly studied among those given by eq. (1.1) and we will thus refer to them as *FLRW-type metrics*. A similar analysis with a focus on the flat expanding case, i.e. where the metric is of the type

$$h = -dt^2 + e^{2\Omega(t)}\sum_{i=1}^{3}(dx^i)^2$$

(where Ω is a non-decreasing function satisfying precise conditions), was performed in [14]. Some of the arguments presented in this treatment resemble the ones in [14]. The present work generalizes [14] to the case of arbitrary closed and smooth spatial geometries.

1.0.2. The Relativistic Euler Equation

The linear fluid equation of state is

$$p = c_s^2\rho, \tag{1.2}$$

where $p \geq 0$ and $\rho \geq 0$ denote the pressure and energy density of the fluid, respectively. c_s denotes the speed of sound and is assumed to be constant. Throughout the following we will make a distinction between the following cases for $\xi := c_s^2$:

- $\xi = 0$, which will be referred to as the *pressureless dust case*,
- $0 < \xi < \frac{1}{3}$, which will be referred to as the *fluid case*.

The *radiation case* ($\xi = \frac{1}{3}$) is not as well-behaved as the ones above and therefore will not be treated in this analysis.
The relativistic Euler equations are given by the system

$$\nabla_\mu T^{\mu\nu} = 0, \tag{1.3}$$

where

$$T^{\mu\nu} = (\rho + p)\tilde{u}^\mu \tilde{u}^\nu + p g^{\mu\nu} \tag{1.4}$$

is the energy momentum tensor of a perfect fluid, ∇ is the Levi-Civita connection and $\tilde{u}$ is the future directed four-velocity of the fluid. It is normalized by

$$g_{\mu\nu}\tilde{u}^\mu \tilde{u}^\nu = -1. \tag{1.5}$$

For an in-depth analysis and derivation of the relativistic model of perfect fluids see e.g. [3, Ch. 5, Sec. 5.1].
The system described in eq. (1.3) and eq. (1.5) can be solved explicitly with solution

$$\tilde{\rho} = \rho_1 t^{-3(1+\xi)\alpha}, \qquad \tilde{u}^\mu = \delta^\mu{}_0, \tag{1.6}$$

where $\mathbb{R} \ni \rho_1 > 0$. To see this, note that, provided ρ is rescaled by $L = \ln\left(t^{3(1+\xi)\alpha}\frac{\rho}{\rho_1}\right)$, by lemma 3 eq. (1.3) and eq. (1.5) can be written as

$$\begin{aligned}
\partial_t L &= -\frac{1}{\tilde{u}^0}\tilde{u}^{a\gamma}\nabla_a L \\
&\quad - (1+\xi)\left(\frac{1}{(\tilde{u}^0)^2}c\alpha t^{2\alpha-1}\gamma_{ab}\tilde{u}^a\tilde{u}^b + \frac{1}{(\tilde{u}^0)^2}ct^{2\alpha}\gamma_{ab}\tilde{u}^a\partial_t\tilde{u}^b + \frac{1}{\tilde{u}^0}{}^\gamma\nabla_a\tilde{u}^a\right), \\
\partial_t\tilde{u}^j &= -\frac{1}{\tilde{u}^0}\tilde{u}^{a\gamma}\nabla_a\tilde{u}^j + \alpha t^{-1}\tilde{u}^j\left(3\xi - 2\right) \\
&\quad - \frac{\xi}{1+\xi}\left(\tilde{u}^j\partial_t L + \frac{1}{\tilde{u}^0}\tilde{u}^j\tilde{u}^a\partial_a L + \frac{1}{c\tilde{u}^0}t^{-2\alpha}\gamma^{ja}\partial_a L\right).
\end{aligned}$$

This system is trivially solved by the energy density and fluid velocity given in eq. (1.6) which models an expanding homogenous isotropic fluid-type universe and is therefore a viable candidate for describing our universe as of our current understanding.

1.0.3. Connection to Previous Work

In the following paragraphs we shall provide a quick recap of the current understanding of relativistic fluids and establish connections to several previous papers regarding the subject.

The Stabilizing Effect of Expanding Spacetimes

Generally speaking, solutions to the relativistic Euler equations are not as well behaved as the ones discussed in this thesis. If one, for example, considers a fairly general fluid equation of state on Minkowski space, problems emerge. As was shown by Christodoulou in [2] the solution forms so called *shock singularities* in finite time. Thus, the resulting solutions are not global in time.
However, this can be remedied though the assumption of an expanding spacetime, as is done in this thesis. The first natural choice for these are FLRW-type spacetimes, which for the case of zero spatial curvature can be written as

$$\left((0,\infty)\times\mathbb{T}^3, -dt^2 + R(t)^2\delta_{ij}dx^i dx^j\right),$$

where R is an increasing function. Interestingly, the expansion rate of the fluid renders some solutions for the fluid globally future stable and thus lead to global regularization of the relativistic Euler equations in specific instances. This is known in the literature as *fluid stabilization*.
This behaviour was discovered and analysed in the Newtonian case for dust (i.e. $\xi = 0$) in [1] by Brauer, Rendall and Reula. A result for irrotational self-gravitating relativistic fluids with $0 < \xi < \frac{1}{3}$ for exponentially expanding spacetimes, i.e.

$$R(t) = e^t,$$

was shown in [10] by Rodnianski and Speck and later generalized to the general case by Speck in [13]. The aforementioned results were built upon and extended in works by Lübbe-Valiente Kroon [8], Hadžić-Speck [6], Oliynyk [9] as well as Friedrich [5].
In this thesis we partly utilize an approach that was used by Speck in [14], where the scale factor is given by $e^{2\Omega}$, e^{Ω} increases without bound and satisfies

$$\int_1^\infty e^{-\Omega(\tau)} \mathscr{B}(\Omega(\tau)) d\tau < \infty$$

for $0 < \xi < \frac{1}{3}$ ($\mathscr{B}$ is a function that increases rather fast, for details, again, see [14]). The equation of state considered by Speck is linear as in eq. (1.2). He finds stability for the fluid case but in the radiation case, i.e. $\xi = \frac{1}{3}$, shock singularities are shown to exist, given that $e^{-\Omega}$ is not integrable.

Power Law Expansion Factors and Non-Accelerated Expansion

The result above appears to imply that in the fluid case $0 < \xi < \frac{1}{3}$ the factor R^{-1} needs to be integrable. If R is a polynomial factor of type $R(t) = t^\alpha$, $\alpha = 1$ is the boundary case between an accelerating (i.e. $\alpha > 1$) and a decelerating (i.e. $\alpha < 1$) case.
However, in [4] by Fajman, Oliynyk and Wyatt, it was shown was shown that regularization even occurs in the non-accelerating case of $\alpha = 1$. This seems to imply that the condition of integrability used in [14] and in this thesis is not sharp in that even in the cases of some non-integrable time factor future stability can be proven.

1.0.4. Motivation

One might inquire why it is worthwhile to analyse the system determined by the metric given in eq. (1.1). As of our current understanding of cosmology our universe is expanding, although the rate at which is unknown. Therefore, a natural approach would be to model the universe by a closed Riemannian manifold (i.e. a compact Riemannian manifold without boundary) that expands with time. If the metric g as given above is a solution to the Einstein equations, one arrives at the question of how predictive a model composed of g and some matter might be. Hence the fundamental question is one of *future stability under a small perturbation of initial data*, which is the subject of this discussion. As this system turns out to be future stable in specific cases, the next step to approximate reality would be to couple the relativistic Euler equations to the Einstein equations and investigate future stability. At the time of completion of this thesis it seems likely that this is the case but has yet to be proven. An analysis of this type should work in a similar fashion to the arguments presented below but might be much more involved due to the coupling and the resulting backreaction and it might also be necessary to sharpen certain requirements on the rate of expansion.

1.1. Outline of the Analysis

In this section we will give a quick rundown of the thesis presented below and summarize important concepts and ideas.

1.1.1. The Continuation Principle

The goal of this thesis is to find out for which expansion rate $t^{2\alpha}$ solutions to the relativistic Euler equations with different matter models are globally future stable. To do this, we first make use of theorem 1 which is a standard PDE result and guarantees that our system as presented in lemma 1 has a set of local solutions launched by small perturbations in a certain Sobolev space of a fixed solution. We will furthermore assume that the solutions have small initial data, i.e.

$$E_N < \epsilon$$

(see definition 3), where $\epsilon > 0$ is small. By continuity of E_N we will conclude that this smallness assumption holds on an extended time interval $[1, T)$. The critical argument is to show that the energy then remains small for all time provided that our initial data is small enough. If that is the case we may then employ proposition 2 and Sobolev embedding to show that since the energy stays finite the solution must exist for all times. Otherwise the norm would have to blow up at some finite time T_{max}.

1.1.2. Energy Estimates and Energy Current

The process, by which we will show that the energy E_N remains finite, will feature the so-called *energy method*, which is a standard technique in treating hyperbolic PDEs. This is done by controlling the time derivative of the energy norms by the norms themselves and concluding their future behaviour from the resulting differential inequality. This is, however, not that straightforward, since the process described above appears flawed in that we may only control the time derivative of the energy of order $N-1$ by its counterpart of order N, but not by the energy of order $N-1$ itself.

To escape the dilemma described above, we will define so-called *energy currents*, which is done precisely in definition 2. The time component of these currents is coercive, i.e.

$$E_N^2 \simeq \sum_{M \leq N} \|j_M^0\|_{\mathscr{L}^1}, \qquad\qquad 0 < \xi < \frac{1}{3},$$

as is shown in proposition 4 by using the divergence theorem for closed Riemannian manifolds. We can, however, show that the $\mathscr{L}^1$-norm of the time component of this current of order N may be controlled by the Nth-order energy and therefore close our energy argument. This is done by means of a key lemma, lemma 8, which gives us the necessary tools to estimate the current.

1.1.3. Estimating the Current

To analyse the problem we introduce a special operator $\tilde{\nabla}$ which acts as an ordinary derivative on the time slot and as a covariant derivative on the spatial slots. The analysis of this operator is crucial as it allows us to exploit the divergence theorem for 3-dimensional Riemannian manifolds. As we find in lemma 7, the $\tilde{\nabla}$-divergence (for the definition, again, see lemma 7) of the current in the fluid case can be expressed as

$$\begin{aligned}
\tilde{\nabla}_\mu j^\mu = & \frac{\xi}{1+\xi}\left(\partial_t u^0 + t^{(3\xi-2)\alpha}\nabla_a u^a\right)\left|\nabla^k L\right|^2 + 2\xi t^{(3\xi-1)2\alpha}\gamma_{ab}\partial_t\left(\frac{u^a}{u^0}\right)\gamma^{IJ}\nabla_I u^b \nabla_J L \\
& + (1+\xi)\left(\partial_t u^0 + t^{(3\xi-2)\alpha}\nabla_a u^a\right)\gamma^{IJ}\left(-\nabla_I u^0 \nabla_J u^0 + c t^{(3\xi-1)2\alpha}\gamma_{ab}\nabla_I u^a \nabla_J u^b\right) \\
& + 4\xi c\alpha(3\xi-1)t^{(3\xi-1)2\alpha-1}\gamma_{ab}\frac{u^a}{u^0}\gamma^{IJ}\nabla_I u^b \nabla_J L \\
& + (1+\xi)(3\xi-1)2\alpha t^{(3\xi-1)2\alpha-1}\gamma_{ab}\gamma^{IJ}\nabla_I u^a \nabla_J u^b \\
& + 2\frac{\xi}{1+\xi}\gamma^{IJ}\nabla_I L F_J - 2(1+\xi)c\gamma^{IJ}\nabla_I u^0 G^0_J + 2(1+\xi)t^{(3\xi-1)2\alpha}\gamma^{IJ}\gamma_{ab}\nabla_I u^a G^b_J.
\end{aligned}$$

Note that all of these terms are shaped in a specific pattern. For example, the first term is

$$\frac{\xi}{1+\xi}\left(\partial_t u^0 + t^{(3\xi-2)\alpha}\nabla_a u^a\right)\left|\nabla^k L\right|^2.$$

The term

$$\left(\partial_t u^0 + t^{(3\xi-2)\alpha}\nabla_a u^a\right)$$

can be estimated in $\mathscr{L}^\infty$ by $t^{(3\xi-2)\alpha}\|u\|_{\mathscr{H}^N}$ by Sobolev embedding, provided the order N is sufficiently large. Employing the Hölder inequality we therefore find

$$\left\|\frac{\xi}{1+\xi}\left(\partial_t u^0 + t^{(3\xi-2)\alpha}\nabla_a u^a\right)\left|\nabla^k L\right|^2\right\|_{\mathscr{L}^1} \lesssim t^{(3\xi-2)\alpha}\|u\|_{\mathscr{H}^N}\left\|\nabla^k L\right\|^2_{\mathscr{L}^2}.$$

If we define the energy to be of the form

$$E_N := \|L\|_{\mathscr{H}^N} + t^{(3\xi-1)\alpha}\|u\|_{\mathscr{H}^N}$$

we find the estimates $\|u\|_{\mathscr{H}^N} \lesssim t^{(1-3\xi)\alpha}E_N$ as well as $\|L\| \lesssim E_N$. This results in an estimate given by

$$\left\|\frac{\xi}{1+\xi}\left(\partial_t u^0 + t^{(3\xi-2)\alpha}\nabla_a u^a\right)\left|\nabla^k L\right|^2\right\|_{\mathscr{L}^1} \lesssim t^{-\alpha}E_N^3.$$

Note, however, that this time factor is integrable,i.e.

$$\int_1^\infty t^{(3\xi-2)\alpha}dt < \infty.$$

The other terms given in the current identity above can be estimated in a similar manner using lemma 8. The details of which can be found in proposition 3. However, there are terms similar to

$$4\xi c\alpha(3\xi-1)t^{(3\xi-1)2\alpha-1}\gamma_{ab}\frac{u^a}{u^0}\gamma^{IJ}\nabla_I u^b\nabla_J L$$

that cannot be estimated in the straightforward fashion described above. Note that if we employed the same argument we would find

$$\left\|4\xi c\alpha(3\xi-1)t^{(3\xi-1)2\alpha-1}\gamma_{ab}\frac{u^a}{u^0}\gamma^{IJ}\nabla_I u^b\nabla_J L\right\|_{\mathscr{L}^1}\lesssim t^{-1}E_N^3.$$

This time factor lacks integrability posing an additional problem. However, we circumvent this by defining a lower order Sobolev norm $\mathscr{E}_{N-1}$ by

$$\mathscr{E}_{N-1}:=t^{(3\xi-1)\alpha+\delta}\|u\|_{\mathscr{H}^{N-1}},$$

where δ is a constant satisfying $0<\delta<\alpha-1$ as well as $\delta\leq(1-3\xi)\alpha$. Given this new norm, by Sobolev embedding and Hölder inequality, we conclude that

$$\begin{aligned}&\left\|4\xi c\alpha(3\xi-1)t^{(3\xi-1)2\alpha-1}\gamma_{ab}\frac{u^a}{u^0}\gamma^{IJ}\nabla_I u^b\nabla_J L\right\|_{\mathscr{L}^1}\\&\lesssim t^{(3\xi-1)2\alpha-1}\|u\|_{\mathscr{H}^{N-1}}\|u\|_{\mathscr{H}^N}\|L\|_{\mathscr{H}^N}\\&\lesssim t^{(3\xi-1)2\alpha-1}t^{(1-3\xi)\alpha-\delta}\mathscr{E}_{N-1}t^{(1-3\xi)\alpha}E_N^2\\&\lesssim t^{-1-\delta}E_N^2.\end{aligned}$$

Now the exponent of the time factor is smaller than -1, making it integrable. It is a priori not clear why the energy $\mathscr{E}_{N-1}$ becomes smaller over time. We will therefore treat this as a separate energy and in our final result, theorem 2, we show that the energy given by the sum of E_N as given above and $\mathscr{E}_{N-1}$ continuously decays.

2. Preliminaries

2.1. Conventions

In this section we introduce the notation needed to treat the concepts of this thesis, i.e. smallness, Lebesgue-norms and Sobolev-norms.

Uniform bounds

To indicate that an object is uniformly bounded by another, we will write $\lesssim$, e.g.

$$f \lesssim g \iff \exists C > 0 : \forall x \in \mathbb{D} : f(x) \leq Cg(x),$$

where f and g are functions on the domain $\mathbb{D}$. We define $\gtrsim$ in an analogous fashion. We say that $f \simeq g$ if $f \lesssim g$ and $f \gtrsim g$.

Lp-norms

Let $p \in (0, \infty)$. We define the Lebesgue p-norm of a scalar function f such that $|f|^P$ is integrable by

$$\|f(t,\cdot)\|_{\mathscr{L}^p} := \left(\int |f(t,\cdot)|^p \right)^{\frac{1}{p}},$$

where integration is always assumed over $\mathbb{R}^3$ with respect to the measure $\mu_\gamma := \sqrt{\det \gamma} d^3x$, if not stated otherwise. Furthermore, we define

$$\|f(t,\cdot)\|_{\mathscr{L}^\infty} := \operatorname*{ess\,sup}_{x \in \mathbb{R}^3} f(x),$$

provided such it exists.

Remark 1. *From now on the dependence of $f(t,\cdot)$ on t will be tacit.*

Remark 2. *To distinguish the Levi-Civita connection of the Lorentzian spacetime $([1,\infty) \times M, g)$ from the one of the closed Riemannian manifold (M, γ) we will refer to the former as ∇ and the latter as ${}^\gamma\nabla$.*

Sobolev-norms of scalars and tensor fields

Let f be a scalar function. We define the N-th order Sobolev norm by

$$\|f\|_{\mathscr{H}^N} := \sqrt{\sum_{k \leq N} \int |{}^\gamma\nabla^k f|^2},$$

where

$$\left|\nabla^k f\right|^2 := \gamma^{i_1 j_1} \cdots \gamma^{i_k j_k \gamma}\nabla_{i_1} \cdots {}^{\gamma}\nabla_{i_k} f {}^{\gamma}\nabla_{j_1} \cdots {}^{\gamma}\nabla_{j_k} f.$$

In an analogous fashion, for a vector field v we define

$$\left|\nabla^k v\right|^2 := \gamma^{i_1 j_1} \cdots \gamma^{i_k j_k}\gamma_{ab} {}^{\gamma}\nabla_{i_1} \cdots {}^{\gamma}\nabla_{i_k} v^a {}^{\gamma}\nabla_{j_1} \cdots {}^{\gamma}\nabla_{j_k} v^b.$$

This definition can be extended to arbitrary tensor fields.

2.2. The Rescaled System

The goal of this section is to rewrite the system given in eq. (1.3) in a convenient form, so that dependence on the time and spatial part are separated. However, we will first derive some useful identities regarding the metric given in eq. (1.1), its Christoffel-symbols and the normalization of the time component $\tilde{u}^0$ of the fluid-4-velocity-vector.

Lemma 1 (Equation satisfied by $\tilde{u}^0$)**.** *The future-directed time component $\tilde{u}^0$ of the fluid velocity vector field satisfies*

$$\tilde{u}^0 = \sqrt{1 + ct^{2\alpha}\gamma_{ab}\tilde{u}^a\tilde{u}^b}.$$

Proof. By eq. (1.5) we have that

$$\begin{aligned} & -1 = g_{\mu\nu}\tilde{u}^\mu\tilde{u}^\nu \\ \iff \quad & -1 = -(\tilde{u}^0)^2 + ct^{2\alpha}\gamma_{ab}\tilde{u}^a\tilde{u}^b \\ \iff \quad & \tilde{u}^0 = \sqrt{1 + ct^{2\alpha}\gamma_{ab}\tilde{u}^a\tilde{u}^b}. \end{aligned}$$

We took the root to be positive since the fluid velocity is future-directed. □

Lemma 2. *The Christoffel symbols of the metric given in eq. (1.1) are*

$$\begin{aligned} \Gamma^0{}_{0\beta} &= 0, \\ \Gamma^0{}_{ab} &= c\alpha t^{2\alpha-1}\gamma_{ab}, \\ \Gamma^j{}_{00} &= 0, \\ \Gamma^j{}_{0b} &= \alpha t^{-1}\delta^j{}_b, \\ \Gamma^j{}_{ab} &= {}^{\gamma}\Gamma^j{}_{ab}. \end{aligned}$$

Proof. See appendix A.1. □

Lemma 3. *Rescaled by $L := \ln\left(t^{3(1+\xi)\alpha}\frac{\rho}{\rho_1}\right)$, where ρ_1 is the constant presented in eq. (1.6), the relativistic Euler equations as given in eq. (1.3) read*

$$\begin{aligned} \partial_t L = & -\frac{1}{\tilde{u}^0}\tilde{u}^{a\gamma}\nabla_a L \\ & - (1+\xi)\left(\frac{1}{(\tilde{u}^0)^2}c\alpha t^{2\alpha-1}\gamma_{ab}\tilde{u}^a\tilde{u}^b + \frac{1}{(\tilde{u}^0)^2}ct^{2\alpha}\gamma_{ab}\tilde{u}^a\partial_t\tilde{u}^b + \frac{1}{\tilde{u}^0}{}^{\gamma}\nabla_a\tilde{u}^a\right), \\ \partial_t\tilde{u}^j = & -\frac{1}{\tilde{u}^0}\tilde{u}^{a\gamma}\nabla_a\tilde{u}^j + \alpha t^{-1}\tilde{u}^j\left(3\xi - 2\right) \\ & - \frac{\xi}{1+\xi}\left(\tilde{u}^j\partial_t L + \frac{1}{\tilde{u}^0}\tilde{u}^j\tilde{u}^a\partial_a L + \frac{1}{c\tilde{u}^0}t^{-2\alpha}\gamma^{ja}\partial_a L\right). \end{aligned} \tag{2.1}$$

Proof. The relativistic Euler equations take the form

$$\begin{aligned}\tilde{u}^\alpha \nabla_\alpha \ln\rho + (1+\xi)\nabla_\alpha \tilde{u}^\alpha &= 0,\\ \tilde{u}^\alpha \nabla_\alpha \tilde{u}^\mu + \frac{\xi}{1+\xi}\left(\tilde{u}^\alpha \tilde{u}^\mu + g^{\alpha\mu}\right)\nabla_\alpha \ln\rho &= 0.\end{aligned}$$

Writing out the covariant derivatives of the scalar equation we get

$$\tilde{u}^0 \partial_t \ln\rho + \tilde{u}^a \nabla_a \ln\rho + (1+\xi)\left(\partial_t \tilde{u}^0 + \partial_a \tilde{u}^a + \Gamma^\sigma{}_{\sigma\beta}\tilde{u}^\beta\right) = 0$$

and using lemma 2, we find

$$\begin{aligned}&\tilde{u}^0 \partial_t \ln\rho + \tilde{u}^a \nabla_a \ln\rho + (1+\xi)\left(\partial_t \tilde{u}^0 + \partial_a \tilde{u}^a + 3\alpha t^{-1}\tilde{u}^0 + {}^\gamma\Gamma^a{}_{ab}\tilde{u}^b\right) = 0\\ \iff\ &\tilde{u}^0 \partial_t \ln\rho + \tilde{u}^a \nabla_a \ln\rho + (1+\xi)\left(\partial_t \tilde{u}^0 + 3\alpha t^{-1}\tilde{u}^0 + {}^\gamma\nabla_a \tilde{u}^a\right) = 0.\end{aligned}$$

Now for the rescaling mentioned above, by straightforward differentiation we find that

$$\partial_t \ln\rho = \partial_t L - 3\alpha(1+\xi)t^{-1}, \quad \partial_a \ln\rho = \partial_a L,$$

and therefore that

$$\tilde{u}^0 \partial_t L + \tilde{u}^{a\gamma}\nabla_a L + (1+\xi)\left(\partial_t \tilde{u}^0 + {}^\gamma\nabla_a \tilde{u}^a\right) = 0.$$

Using $\partial_t \tilde{u}^0 = \frac{1}{\tilde{u}^0}\left(c\alpha t^{2\alpha-1}\gamma_{ab}\tilde{u}^a\tilde{u}^b + ct^{2\alpha}\gamma_{ab}\tilde{u}^a\partial_t\tilde{u}^b\right)$, we find

$$0 = \tilde{u}^0 \partial_t L + \tilde{u}^{a\gamma}\nabla_a L + (1+\xi)\left(\frac{1}{\tilde{u}^0}\left(c\alpha t^{2\alpha-1}\gamma_{ab}\tilde{u}^a\tilde{u}^b + ct^{2\alpha}\gamma_{ab}\tilde{u}^a\partial_t\tilde{u}^b\right) + {}^\gamma\nabla_a \tilde{u}^a\right)$$

yielding the scalar equation of motion. For the 3-velocity of the fluid expanding the covariant derivative gives

$$\begin{aligned}&\tilde{u}^0\partial_t\tilde{u}^j + \tilde{u}^a\partial_a\tilde{u}^j + \Gamma^j{}_{\alpha\beta}\tilde{u}^\alpha\tilde{u}^\beta\\ &\qquad + \frac{\xi}{1+\xi}\left(\tilde{u}^j\left[\tilde{u}^0\partial_t\ln\rho + \tilde{u}^a\partial_a\ln\rho\right] + g^{ja}\partial_a\ln\rho\right) = 0\end{aligned}$$

which again by lemma 2 gives

$$\begin{aligned}&\tilde{u}^0\partial_t\tilde{u}^j + \tilde{u}^{a\gamma}\nabla_a\tilde{u}^j + 2\alpha t^{-1}\tilde{u}^0\tilde{u}^j\\ &\qquad + \frac{\xi}{1+\xi}\left(\tilde{u}^j\tilde{u}^0\partial_t\ln\rho + \tilde{u}^j\tilde{u}^a\partial_a\ln\rho + \frac{1}{c}t^{-2\alpha}\gamma^{ja}\partial_a\ln\rho\right) = 0.\end{aligned}$$

We again use the rescaling to get

$$\begin{aligned}&\tilde{u}^0\partial_t\tilde{u}^j + \tilde{u}^{a\gamma}\nabla_a\tilde{u}^j + 2\alpha t^{-1}\tilde{u}^0\tilde{u}^j\\ &\qquad + \frac{\xi}{1+\xi}\left(\tilde{u}^j\tilde{u}^0\partial_t L - \tilde{u}^j\tilde{u}^0 3\alpha(1+\xi)t^{-1}\right.\\ &\qquad\qquad \left. + \tilde{u}^j\tilde{u}^a\partial_a\ln\rho + \frac{1}{c}t^{-2\alpha}\gamma^{ja}\partial_a\ln\rho\right) = 0\end{aligned}$$

$$
\begin{aligned}
\iff \partial_t \tilde{u}^j &= -\frac{1}{\tilde{u}^0}\tilde{u}^{a\gamma}\nabla_a \tilde{u}^j + \alpha t^{-1}\tilde{u}^j\,(3\xi-2) \\
&-\frac{\xi}{1+\xi}\left(\tilde{u}^j\partial_t L + \frac{1}{\tilde{u}^0}\tilde{u}^j\tilde{u}^a\partial_a L + \frac{1}{c\tilde{u}^0}t^{-2\alpha}\gamma^{ja}\partial_a L\right).
\end{aligned}
$$

□

In the following proposition we proceed by rescaling the spatial fluid velocity and derive the final form of our system of equations.

2.2.1. The Final Form of the Relativistic Euler Equations

Proposition 1 (The Rescaled Euler Equations). *Rescaling the spatial fluid velocity by* $\tilde{u}^j =: t^{(3\xi-2)\alpha}u^j$ *and* $u^0 := \tilde{u}^0$*, we find the equations*

$$
\begin{aligned}
\partial_t L =& -t^{(3\xi-2)\alpha}\frac{1}{u^0}u^{a\gamma}\nabla_a L - (1+\xi)\left(\frac{1}{(u^0)^2}c\alpha t^{(3\xi-1)2\alpha-1}(3\xi-1)\gamma_{ab}u^a u^b\right. \\
&\left. + \frac{1}{(u^0)^2}ct^{(3\xi-1)2\alpha}\gamma_{ab}u^a\partial_t u^b + \frac{1}{u^0}t^{(3\xi-2)\alpha\gamma}\nabla_a u^a\right), \\
\partial_t u^j =& -t^{(3\xi-2)\alpha}\frac{1}{u^0}u^{a\gamma}\nabla_a u^j - \frac{\xi}{1+\xi}\left(u^j\partial_t L\right. \\
&\left. + t^{(3\xi-2)\alpha}\frac{1}{u^0}u^j u^a\partial_a L + \frac{1}{cu^0}t^{-3\xi\alpha}\gamma^{ja}\partial_a L\right), \\
\partial_t u^0 =& -\frac{1}{u^0}t^{(3\xi-2)\alpha\gamma}\nabla_a u^0 + \frac{1}{u^0}(3\xi-1)c\alpha t^{(3\xi-1)2\alpha-1}\gamma_{ab}u^a u^b \\
&-\frac{\xi}{1+\xi}\left(\frac{1}{u^0}ct^{(3\xi-1)2\alpha}\gamma_{ab}u^a u^b\partial_t L + \frac{1}{u^0}t^{(3\xi-2)\alpha}u^{a\gamma}\nabla_a L\right).
\end{aligned}
\tag{2.2}
$$

Proof. For the proof see appendix A.2. □

Remark 3. *Note that the right-hand side of the equations for* $\partial_t u^j$ *and* $\partial_t u^0$ *in proposition 1 significantly simplify in the dust case* $\xi = 0$*. The equation for* $\partial_t u^j$ *even becomes independent of the behaviour of* L*.*

In the arguments presented below it will be useful to have an explicit expression for the time derivative of L. Therefore, we will derive such an expression in the following lemma.

Lemma 4. *Let* $(L, u^1, u^2, u^3)^T$ *be a solution to the system presented in propositions 1. Then* $\partial_t L$ *satisfies the equation*

$$
\begin{aligned}
\partial_t L =& \frac{1}{1-\xi\frac{ct^{(3\xi-1)2\alpha}}{(u^0)^2}\gamma(u,u)}\left(\frac{1}{u^0}t^{(3\xi-2)\alpha}u^{a\gamma}\nabla_a L\,(\xi-1)\right. \\
&- (1+\xi)\alpha ct^{(3\xi-1)2\alpha-1}\gamma(u,u)\frac{1}{(u^0)^2} \\
&\left. - t^{(3\xi-2)\alpha}(1+\xi)\frac{1}{u^0}{}^{\gamma}\nabla_a u^a + (1+\xi)\frac{1}{(u^0)^3}ct^{(9\xi-4)\alpha}\gamma_{ab}u^{c\gamma}\nabla_c u^a u^b\right).
\end{aligned}
\tag{2.3}
$$

Proof. See appendix A.2 □

Remark 4. *From now on we will write the covariant derivative with respect to the metric γ which was previously written as ${}^{\gamma}\nabla$ simply as ∇. Therefore any covariant derivatives appearing in calculations are to be interpreted as connections of the metric γ unless stated otherwise.*

2.3. The Local Existence

2.3.1. The Local Existence Theorem

As previously stated, once we have found a solution to the relativistic Euler equations as stated in proposition 1 with certain initial data, we want to investigate whether slightly perturbed initial data yield a local solution as well. This is guaranteed by the following theorem.

Theorem 1. *Let $N \geq 3$, $j \in \{1, 2, 3\}$ and let*

$$\begin{aligned} L(t=1) =: \bar{L} &\in \begin{cases} \mathscr{H}^{N-1}, & \xi = 0, \\ \mathscr{H}^{N}, & 0 < \xi < \frac{1}{3}, \end{cases} \\ u^j(t=1) =: \bar{u}^j &\in \mathscr{H}^N, \quad 0 \leq \xi < \frac{1}{3}, \end{aligned}$$

be initial data for the system of equations given in (2.2) and furthermore let $\sup_{p\in M} \bar{L} < \infty$. Then there exists a real number $T > 1$ such that the data given above launch a unique classical solution (L, u^1, u^2, u^3) to the system (2.2) on the set $[1, T) \times M$ with the following regularity properties:

$$\begin{aligned} L &\in \mathscr{C}^0\left([1,T) \times M\right), & \xi &= 0, \\ L &\in \mathscr{C}^1\left([1,T) \times M\right), & 0 < \xi &< \frac{1}{3}, \\ u^j &\in \mathscr{C}^1\left([1,T) \times M\right), & 0 \leq \xi &< \frac{1}{3}, \\ L &\in \mathscr{C}^0\left([1,T), \mathscr{H}^{N-1}\right), & \xi &= 0, \\ L &\in \mathscr{C}^0\left([1,T), \mathscr{H}^{N}\right), & 0 < \xi &< \frac{1}{3}, \\ u^0 - 1, u^j &\in \mathscr{C}^0\left([1,T), \mathscr{H}^{N}\right), & 0 \leq \xi &< \frac{1}{3}. \end{aligned}$$

Furthermore, there exists an open neighbourhood $\mathscr{O}$ of the initial data $(\bar{L}, \bar{u}^1, \bar{u}^2, \bar{u}^3)$ in Sobolev space such that all elements of $\mathscr{O}$ launch solutions that exist on $[1, T) \times M$ and have the same regularity properties as the solutions above. The map on $\mathscr{O}$ that maps initial data to solutions is continuous.

Proof. This theorem can be proven by a standard contraction mapping argument using energy estimates similar to those presented in this thesis. For details on how to construct such a proof see e.g. [7, Ch. VI] and [12]. □

2.3.2. The Continuation Principle

Proposition 2 (The Continuation Principle)**.** *Let T be defined as in theorem 1 and let T_{max} be the supremum over all times $T > 1$ for which the solution (L, u^j) exists and satisfies the properties in theorem 1. Then for $0 < \xi < \frac{1}{3}$ we have*

$$\lim_{t \to T_{max}} \sup_{0 \leq \tau \leq t} \left(\|L(\tau, \cdot)\|_{\mathscr{C}_b^1} + \sum_{j=1}^{3} \left\|u^j(\tau, \cdot)\right\|_{\mathscr{C}_b^1} \right) = \infty,$$

as well as for $\xi = 0$

$$\lim_{t \to T_{max}} \sup_{0 \leq \tau \leq t} \left(\|L(\tau, \cdot)\|_{\mathscr{L}^\infty} + \sum_{j=1}^{3} \left\|u^j(s, \cdot)\right\|_{\mathscr{C}_b^1} \right) = \infty.$$

Proof. For ideas of a proof for this proposition see e.g. [11] and [7]. Note that the case $\xi = 0$ is special in the sense that the evolution of u^j decouples from the evolution of L. □

2.4. The Commutation Relations of the Equations of Motion

In this section we establish what form the equations of motion as given in proposition 1 take when commutated with an arbitrary number of covariant derivatives, which will be referred to as equations of variation. To simplify we will only explicitly calculate the first order by which the nature of higher order commutations will become apparent.

Lemma 5. *The equations of variation at first order are given by*

$$\begin{aligned}
u^0\partial_t\nabla_i u^j =& - t^{(3\xi-2)\alpha}u^a\nabla_a\nabla_i u^j\\
&- \frac{\xi}{\xi+1}\left(u^0u^j\partial_t\nabla_i L + t^{(3\xi-2)\alpha}u^j u^a\nabla_a\nabla_i L + \frac{1}{c}t^{-3\xi\alpha}\gamma^{ja}\nabla_a\nabla_i L\right) + G^j{}_i,\\
G^j{}_i :=& - t^{(3\xi-2)\alpha}u^0\nabla_i\frac{1}{u^0}u^a\nabla_a u^j - t^{(3\xi-2)\alpha}\nabla_i u^a\nabla_a u^j - t^{(3\xi-2)\alpha}u^a\,\mathrm{Riem}^j{}_{bia}u^b\\
&- \frac{\xi}{1+\xi}u^0\nabla_i u^j\partial_t L - \frac{\xi}{1+\xi}t^{(3\xi-2)\alpha}\left(u^0\nabla_i u^j u^a\nabla_a L + \nabla_i u^j u^a\nabla_a L\right.\\
&\left.+ u^j\nabla_i u^a\nabla_a L + \frac{1}{c}t^{-3\xi\alpha}u^0\gamma^{ja}\nabla_i\frac{1}{u^0}\nabla_a L\right),\\
u^0\partial_t\nabla_i L =& - t^{(3\xi-2)\alpha}u^a\nabla_a\nabla_i L\\
&- (1+\xi)ct^{(3\xi-1)2\alpha}\frac{1}{u^0}\gamma_{ab}u^a\partial_t\nabla_i u^b - (1+\xi)t^{(3\xi-2)\alpha}\nabla_a\nabla_i u^a + F_i,\\
F_i :=& - t^{(3\xi-2)\alpha}\left(u^0\nabla_i\frac{1}{u^0}u^a\nabla_a L + \nabla_i u^a\nabla_a L\right)\\
&- (1+\xi)c\alpha t^{(3\xi-1)2\alpha-1}(3\xi-1)\left(u^0\nabla_i\frac{1}{(u^0)^2}\gamma_{ab}u^a u^b + 2\frac{1}{u^0}\gamma_{ab}\nabla_i u^a u^b\right)\\
&- (1+\xi)ct^{(3\xi-1)2\alpha}\left(\nabla_i u^0\frac{1}{(u^0)^2}\gamma_{ab}u^a\partial_t u^b + \frac{1}{u^0}\gamma_{ab}\nabla_i u^a\partial_t u^b\right)\\
&- (1+\xi)t^{(3\xi-2)\alpha}\left(u^0\nabla_i\frac{1}{u^0}\nabla_a u^a + \mathrm{Riem}^a{}_{bia}u^b\right).
\end{aligned}$$

Proof. This can be proven by a straightforward differentiation of the equations given in proposition 1. We omit the details. □

Definition 1. *We define the operator ∇_I of order N by*

$$\nabla_I := \nabla_{i_1}\cdots\nabla_{i_N}, \quad \gamma^{IJ} := \gamma^{i_1j_1}\cdots\gamma^{i_Nj_N}.$$

Furthermore, we define the quantity $\nabla_I u^0$ by

$$\nabla_I u^0 := ct^{(3\xi-1)2\alpha}\frac{1}{u^0}\gamma_{ab}u^a\nabla_I u^b.$$

Remark 5. *Note that since*

$$\nabla_I u^0 = ct^{(3\xi-1)2\alpha}\frac{1}{u^0}\gamma_{ab}u^a\nabla_I u^b$$

we have

$$\begin{aligned}
g_{\mu\sigma}\tilde{u}^\mu\nabla_I\tilde{u}^\sigma &= -u^0\nabla_I u^0 + ct^{(3\xi-1)2\alpha}\gamma_{ab}u^a\nabla_I u^b\\
&= -ct^{(3\xi-1)2\alpha}\gamma_{ab}u^a\nabla_I u^b + ct^{(3\xi-1)2\alpha}\gamma_{ab}u^a\nabla_I u^b = 0.
\end{aligned}$$

Lemma 6. *The equation satisfied by $\nabla_i u^0$ is*

$$\begin{aligned}
\left(u^0\partial_t + t^{(3\xi-2)\alpha}u^a\nabla_a\right)\nabla_I u^0 =& - \frac{\xi}{\xi+1}\left(\left((u^0)^2-1\right)\partial_t\nabla_I L\right.\\
&\left.+ u^0t^{(3\xi-2)\alpha}u^m\nabla_m\nabla_I L\right) + G^0{}_I,
\end{aligned}$$

where $G^0{}_I$ is defined by

$$G^0{}_I := \gamma_{ab} c t^{(3\xi-1)2\alpha} \frac{u^a}{u^0} G^b{}_I + u^0 c t^{(3\xi-1)2\alpha} \gamma_{ab} \partial_t \left(\frac{u^a}{u^0} \right) \nabla_I u^b + c t^{(3\xi-1)2\alpha} t^{(3\xi-2)\alpha} u^m \nabla_m \left(\frac{u^a}{u^0} \right) \gamma_{ab} \nabla_I u^b.$$

Proof. Differentiating the expression defined in 1, we find

$$\begin{aligned}
&\left(u^0\partial_t + t^{(3\xi-2)\alpha} u^a \nabla_a\right) \nabla_I u^0 = \left(u^0\partial_t + t^{(3\xi-2)\alpha} u^a \nabla_a\right) \left(c t^{(3\xi-1)2\alpha} \frac{1}{u^0} \gamma_{ab} u^a \nabla_I u^b \right) \\
&= u^0 \left(c t^{(3\xi-1)2\alpha} \gamma_{ab} \partial_t \left(\frac{u^a}{u^0} \right) \nabla_I u^b + \gamma_{ab} \frac{u^a}{u^0} c t^{(3\xi-1)2\alpha} \partial_t \nabla_I u^b + (3\xi-1)2\alpha t^{-1} \nabla_I u^0 \right) \\
&+ c t^{(3\xi-1)2\alpha} t^{(3\xi-2)\alpha} \left(u^m \nabla_m \left(\frac{u^a}{u^0} \right) \gamma_{ab} \nabla_I u^b + \gamma_{ab} \frac{u^a}{u^0} u^m \nabla_m \nabla_I u^b \right).
\end{aligned}$$

Using the equation of variation for u^j as given in lemma 5, we find

$$\begin{aligned}
&\left(u^0\partial_t + t^{(3\xi-2)\alpha} u^a \nabla_a\right) \nabla_I u^0 = \gamma_{ab} c t^{(3\xi-1)2\alpha} \frac{u^a}{u^0} \\
&\times \left(-\frac{\xi}{\xi+1} \left(u^0 u^b \partial_t \nabla_I L + t^{(3\xi-2)\alpha} u^b u^m \nabla_m \nabla_i L + \frac{1}{c} t^{-3\xi\alpha} \gamma^{bm} \nabla_m \nabla_I L \right) + G^b{}_I \right) \\
&+ u^0 c t^{(3\xi-1)2\alpha} \gamma_{ab} \partial_t \left(\frac{u^a}{u^0} \right) \nabla_I u^b + c t^{(3\xi-1)2\alpha} t^{(3\xi-2)\alpha} u^m \nabla_m \left(\frac{u^a}{u^0} \right) \gamma_{ab} \nabla_I u^b \\
&= -\frac{\xi}{\xi+1} \left(c t^{(3\xi-1)2\alpha} \gamma_{ab} u^a u^b \partial_t \nabla_I L + c t^{(3\xi-2)\alpha} t^{(3\xi-1)2\alpha} \frac{1}{u^0} \gamma_{ab} u^a u^b u^m \nabla_m \nabla_i L \right. \\
&\left. + \frac{1}{u^0} t^{(3\xi-2)\alpha} u^m \nabla_m \nabla_I L \right) + G^0{}_I.
\end{aligned}$$

And finally using $c t^{(3\xi-1)2\alpha} \gamma_{ab} u^a u^b = ((u^0)^2 - 1)$ to substitute the second term, we find

$$\begin{aligned}
\left(u^0\partial_t + t^{(3\xi-2)\alpha} u^a \nabla_a\right) \nabla_I u^0 = &-\frac{\xi}{\xi+1} \left(\left((u^0)^2 - 1\right) \partial_t \nabla_I L \right. \\
&\left. + u^0 t^{(3\xi-2)\alpha} u^m \nabla_m \nabla_I L \right) + G^0{}_I.
\end{aligned}$$

□

3. Energies and Norms

In this section we introduce the necessary norms and norm-like quantities used to establish future stability. The energy currents that are defined and treated below do not appear useful immediately, but turn out to be coercive quantities with neat mathematical features.

Definition 2. *In the fluid case $0 < \xi < \frac{1}{3}$ we define the N-th order energy current as*

$$\begin{aligned}
j_N^a :=& t^{(3\xi-2)\alpha}\frac{\xi}{1+\xi}u^a\gamma^{i_1j_1}\cdots\gamma^{i_Nj_N}\nabla_{i_1}\cdots\nabla_{i_N}L\nabla_{j_1}\cdots\nabla_{j_N}L\\
&+2t^{(3\xi-2)\alpha}\xi\gamma^{i_1j_1}\cdots\gamma^{i_Nj_N}\nabla_{i_1}\cdots\nabla_{i_N}L\nabla_{j_1}\cdots\nabla_{j_N}u^a\\
&+(1+\xi)t^{(3\xi-2)\alpha}u^a\gamma^{i_1j_1}\cdots\gamma^{i_Nj_N}\\
&\times\left(-\nabla_{i_1}\cdots\nabla_{i_N}u^0\nabla_{j_1}\cdots\nabla_{j_N}u^0+ct^{(3\xi-1)2\alpha}\gamma_{ab}\nabla_{i_1}\cdots\nabla_{i_N}u^a\nabla_{j_1}\cdots\nabla_{j_N}u^b\right),\\
j_N^0 :=& \frac{\xi}{1+\xi}u^0\gamma^{i_1j_1}\cdots\gamma^{i_Nj_N}\nabla_{i_1}\cdots\nabla_{i_N}L\nabla_{j_1}\cdots\nabla_{j_N}L\\
&+2\xi\gamma^{i_1j_1}\cdots\gamma^{i_Nj_N}\nabla_{i_1}\cdots\nabla_{i_N}L\nabla_{j_1}\cdots\nabla_{j_N}u^0\\
&+(1+\xi)u^0\gamma^{i_1j_1}\cdots\gamma^{i_Nj_N}\\
&\times\left(-\nabla_{i_1}\cdots\nabla_{i_N}u^0\nabla_{j_1}\cdots\nabla_{j_N}u^0+ct^{(3\xi-1)2\alpha}\gamma_{ab}\nabla_{i_1}\cdots\nabla_{i_N}u^a\nabla_{j_1}\cdots\nabla_{j_N}u^b\right).
\end{aligned}$$

In the case of pressureless dust $\xi = 0$, we define

$$\begin{aligned}
j_{u,N}^a :=& u^a\gamma^{i_1j_1}\cdots\gamma^{i_Nj_N}\left(-\nabla_{i_1}\cdots\nabla_{i_N}u^0\nabla_{j_1}\cdots\nabla_{j_N}u^0\right.\\
&\left.+ct^{-2\alpha}\gamma_{mn}\nabla_{i_1}\cdots\nabla_{i_N}u^m\nabla_{j_1}\cdots\nabla_{j_N}u^n\right),\\
j_{L,N}^a :=& t^{-2\alpha}u^a\gamma^{i_1j_1}\cdots\gamma^{i_Nj_N}\nabla_{i_1}\cdots\nabla_{i_N}L\nabla_{j_1}\cdots\nabla_{j_N}L,\\
j_{u,N}^0 :=& t^{2\alpha}u^0\gamma^{i_1j_1}\cdots\gamma^{i_Nj_N}\left(-\nabla_{i_1}\cdots\nabla_{i_N}u^0\nabla_{j_1}\cdots\nabla_{j_N}u^0\right.\\
&\left.+ct^{-2\alpha}\gamma_{ab}\nabla_{i_1}\cdots\nabla_{i_N}u^a\nabla_{j_1}\cdots\nabla_{j_N}u^b\right),\\
j_{L,N}^0 :=& u^0\gamma^{i_1j_1}\cdots\gamma^{i_Nj_N}\nabla_{i_1}\cdots\nabla_{i_N}L\nabla_{j_1}\cdots\nabla_{j_N}L.
\end{aligned}$$

Lemma 7. *Let $\tilde{\nabla}_\mu$ be defined by*

$$\tilde{\nabla}_\mu := \begin{cases}\partial_t, & \mu = 0,\\ {}^{\gamma}\nabla_a, & \mu = a.\end{cases}$$

Then in the case $0 < \xi < \frac{1}{3}$ we have

$$
\begin{aligned}
\tilde{\nabla}_\mu j^\mu =& \frac{\xi}{1+\xi}\left(\partial_t u^0 + t^{(3\xi-2)\alpha}\nabla_a u^a\right)\left|\nabla^k L\right|^2 + 2\xi t^{(3\xi-1)2\alpha}\gamma_{ab}\partial_t\left(\frac{u^a}{u^0}\right)\gamma^{IJ}\nabla_I u^b \nabla_J L \\
&+ (1+\xi)\left(\partial_t u^0 + t^{(3\xi-2)\alpha}\nabla_a u^a\right)\gamma^{IJ}\left(-\nabla_I u^0\nabla_J u^0 + ct^{(3\xi-1)2\alpha}\gamma_{ab}\nabla_I u^a \nabla_J u^b\right) \\
&+ 4\xi c\alpha(3\xi-1)t^{(3\xi-1)2\alpha-1}\gamma_{ab}\frac{u^a}{u^0}\gamma^{IJ}\nabla_I u^b\nabla_J L \\
&+ (1+\xi)(3\xi-1)2\alpha t^{(3\xi-1)2\alpha-1}\gamma_{ab}\gamma^{IJ}\nabla_I u^a \nabla_J u^b \\
&+ 2\frac{\xi}{1+\xi}\gamma^{IJ}\nabla_I L F_J - 2(1+\xi)c\gamma^{IJ}\nabla_I u^0 G_J^0 + 2(1+\xi)t^{(3\xi-1)2\alpha}\gamma^{IJ}\gamma_{ab}\nabla_I u^a G_J^b,
\end{aligned}
$$

and for $\xi = 0$, we find

$$
\begin{aligned}
\tilde{\nabla}_\mu j_u^\mu =& t^{2\alpha}\left(\partial_t u^0 + t^{-2\alpha}\nabla_a u^a\right)\gamma^{IJ}\left(-\nabla_I u^0 \nabla_J u^0 + ct^{-2\alpha}\gamma_{ab}\nabla_I u^a \nabla_b u^b\right) \\
&+ 2\alpha t^{2\alpha-1}u^0\left(-\nabla_I u^0\nabla_J u^0 + ct^{-2\alpha}\gamma_{ab}\nabla_I u^a\nabla_b u^b\right) \\
&+ u^0 t^{2\alpha}c(-2\alpha)t^{-2\alpha-1}\gamma_{ab}\gamma^{IJ}\nabla_I u^a\nabla_J u^b \\
&- 2t^{2\alpha}\gamma^{IJ}\nabla_I u^0 G^0{}_J + 2\gamma_{ab}\gamma^{IJ}\nabla_I u^a G^b{}_J, \\
\tilde{\nabla}_\mu j_L^\mu =& \left(\partial_t u^0 + t^{-2\alpha}\nabla_a u^a\right)\gamma^{IJ}\nabla_I L\nabla_J L \\
&- 2\gamma^{IJ}\nabla_I L t^{-2\alpha}\nabla_a\nabla_J u^a + 2\gamma^{IJ}\nabla_I L F_J \\
&+ 2\gamma^{IJ}\nabla_I L c t^{-4\alpha}\frac{1}{(u^0)^2}\gamma_{ab}u^a u^m\nabla_m\nabla_J u^b - 2\gamma^{IJ}\nabla_I L c t^{-2\alpha}\frac{1}{(u^0)^2}\gamma_{ab}u^a G^b{}_J.
\end{aligned}
$$

Proof. We start with the case $0 < \xi < \frac{1}{3}$. We make repeated use of lemmas 5 as well as 6.

$$
\begin{aligned}
\tilde{\nabla}_\mu j^\mu =& \frac{\xi}{1+\xi}\left(\partial_t u^0 + t^{(3\xi-2)\alpha}\nabla_a u^a\right)\left|\nabla^N L\right|^2 \\
&+ 2\frac{\xi}{1+\xi}\gamma^{IJ}\nabla_I L\left(-(1+\xi)ct^{(3\xi-1)2\alpha}\frac{1}{u^0}\gamma_{ab}u^a\partial_t\nabla_J u^b - \cancel{(1+\xi)t^{(3\xi-2)\alpha}}\nabla_a\nabla_J u^a + F_J\right) \\
&+ \cancel{2\xi\gamma^{IJ}\nabla_I u^0\partial_t\nabla_J L} + \xcancel{2\xi\gamma^{IJ}t^{(3\xi-2)\alpha}\nabla_a\nabla_I L\nabla_J u^a} \\
&+ 2\xi\gamma^{IJ}\partial_t\nabla_I u^0\nabla_J L + \cancel{2\xi\gamma^{IJ}t^{(3\xi-2)\alpha}}\nabla_I L\nabla_a\nabla_J u^a \\
&+ 2\xi\gamma^{IJ}\nabla_I u^0\left(\left((u^0)^2 - \cancel{1}\right)\partial_t\nabla_J L + t^{(3\xi-2)\alpha}u^0 u^a\nabla_a\nabla_J L\right) - 2(1+\xi)\gamma^{IJ}G_I^0\nabla_J u^0 \\
&- 2\xi t^{(3\xi-1)2\alpha}\gamma_{ab}\gamma^{IJ}c\nabla_I u^a\left(u^0 u^b\partial_t\nabla_J L + t^{(3\xi-2)\alpha}u^b u^m\nabla_m\nabla_J L + \xcancel{\frac{1}{c}t^{-3\xi\alpha}\gamma^{bm}\nabla_m\nabla_J L}\right) \\
&+ 2(1+\xi)t^{(3\xi-1)2\alpha}c\gamma^{IJ}\gamma_{ab}\nabla_I u^a G^b{}_J \\
&+ (1+\xi)(3\xi-1)2\alpha c t^{(3\xi-1)2\alpha-1}\gamma^{IJ}\gamma_{ab}\nabla_I u^a\nabla_J u^b \\
&+ (1+\xi)\left(\partial_t u^0 + t^{(3\xi-2)\alpha}\nabla_a u^a\right)\gamma^{IJ}\left(-\nabla_I u^0\nabla_J u^0 + ct^{(3\xi-1)2\alpha}\gamma_{ab}\nabla_I u^a\nabla_J u^b\right)
\end{aligned}
$$

- The terms that are colour-coded in olive already appear in the expression above.
- The terms that are colour-coded in red and blue cancel each other, respectively, due to the identity in remark 5.

We investigate the remaining terms:

$$
\begin{aligned}
& -2\xi c t^{(3\xi-1)2\alpha}\frac{1}{u^0}\gamma^{IJ}\gamma_{ab}u^a\partial_t\nabla_I u^b\nabla_J L+2\xi\gamma^{IJ}\nabla_I L\partial_t\nabla_J u^0 \\
=& \cancel{-2\xi c t^{(3\xi-1)2\alpha}\frac{1}{u^0}\gamma^{IJ}\gamma_{ab}u^a\partial_t\nabla_I u^b\nabla_J L}+2\xi\gamma^{IJ}\nabla_I L \\
& \times\left(\partial_t\left(\frac{u^a}{u^0}\right)t^{(3\xi-1)2\alpha}\gamma_{ab}\nabla_J u^b+\cancel{\frac{u^a}{u^0}c t^{(3\xi-1)2\alpha}}\partial_t\nabla_J u^b+(3\xi-1)2\alpha t^{(3\xi-1)2\alpha-1}c\gamma_{ab}\frac{u^a}{u^0}\nabla_J u^b\right)
\end{aligned}
$$

So these terms yield the last two missing terms of the identity. For the case $\xi=0$ we have

$$
\begin{aligned}
\tilde{\nabla}_\mu j^\mu_u =& t^{2\alpha}\left(\partial_t u^0+t^{-2\alpha}\nabla_a u^a\right)\gamma^{IJ}\left(-\nabla_I u^0\nabla_J u^0+c t^{-2\alpha}\gamma_{ab}\nabla_I u^a\nabla_m u^m\right) \\
&+2\alpha t^{2\alpha-1}u^0\left(-\nabla_I u^0\nabla_J u^0+c t^{-2\alpha}\gamma_{ab}\nabla_I u^a\nabla_m u^m\right) \\
&+u^0 t^{2\alpha}c(-2\alpha)t^{-2\alpha-1}\gamma_{ab}\gamma^{IJ}\nabla_I u^a\nabla_J u^b \\
&-2t^{2\alpha}\gamma^{IJ}\nabla_I u^0 G^0{}_J+2\gamma_{ab}\gamma^{IJ}\nabla_I u^a G^b{}_J, \\
\tilde{\nabla}_\mu j^\mu_L =& \left(\partial_t u^0+t^{-2\alpha}\nabla_a u^a\right)\gamma^{IJ}\nabla_I L\nabla_J L \\
&-2\gamma^{IJ}\nabla_I L c t^{-2\alpha}\frac{1}{u^0}\gamma_{ab}u^a\partial_t\nabla_J u^b-2\gamma^{IJ}\nabla_I L t^{-2\alpha}\nabla_a\nabla_J u^a+F \\
=& \left(\partial_t u^0+t^{-2\alpha}\nabla_a u^a\right)\gamma^{IJ}\nabla_I L\nabla_J L \\
&-2\gamma^{IJ}\nabla_I L t^{-2\alpha}\nabla_a\nabla_J u^a+2\gamma^{IJ}\nabla_I L F_J \\
&-2\gamma^{IJ}\nabla_I L c t^{-2\alpha}\frac{1}{(u^0)^2}\gamma_{ab}u^a\left(-t^{-2\alpha}u^m\nabla_m\nabla_J u^b+G^b{}_J\right) \\
=& \left(\partial_t u^0+t^{-2\alpha}\nabla_a u^a\right)\gamma^{IJ}\nabla_I L\nabla_J L \\
&-2\gamma^{IJ}\nabla_I L t^{-2\alpha}\nabla_a\nabla_J u^a+2\gamma^{IJ}\nabla_I L F_J \\
&+2\gamma^{IJ}\nabla_I L c t^{-4\alpha}\frac{1}{(u^0)^2}\gamma_{ab}u^a u^m\nabla_m\nabla_J u^b-2\gamma^{IJ}\nabla_I L c t^{-2\alpha}\frac{1}{(u^0)^2}\gamma_{ab}u^a G^b{}_J.
\end{aligned}
$$

□

Definition 3 (Energies)**.** *Let ρ be the fluid energy density and u be the fluid spatial velocity. In the fluid case for $N>1$ we define the lower order auxiliary energy*

$$
\mathscr{E}_{N-1}:=t^{(3\xi-1)\alpha+\delta}\|u\|_{\mathscr{H}^{N-1}}, \qquad 0<\xi<\frac{1}{3},
$$

where δ is a constant satisfying $0<\delta<\alpha-1$ as well as $\delta\leq(1-3\xi)\alpha$. For $N\in\mathbb{N}^*$ we define the N-th order fluid energy E_N as

$$
E_N:=\begin{cases}\|L\|_{\mathscr{H}^N}+t^{(3\xi-1)\alpha}\|u\|_{\mathscr{H}^N}+\mathscr{E}_{N-1}, & 0<\xi<\frac{1}{3},\\ \|L\|_{\mathscr{H}^{N-1}}+\|u\|_{\mathscr{H}^N}, & \xi=0.\end{cases} \tag{3.1}
$$

In addition, for pressureless dust we separately define the velocity energy and scalar energy by

$$
\begin{aligned}
E_{u,N}&:=\|u\|_{\mathscr{H}^N},\\
E_{L,N}&:=\|L\|_{\mathscr{H}^N}.
\end{aligned}
$$

3.1. Sobolev Estimates

The following lemmas and propositions form the basis of our proof of future stability in the next chapter. We start by estimating various quantities related to the fluid velocity and energy density. Utilizing these, we will proceed by finding an estimate of the $\mathscr{L}^1$-norm of the divergence of the current that depends on the various energies defined in definition 3.

Lemma 8 (Auxiliary Estimates)**.** *Given the bootstrap assumptions $E_N < \epsilon$ and assuming that N is sufficiently large, we find the following auxiliary estimate for u^0:*

$$\left\|u^0 - 1\right\|_{\mathscr{H}^N} \lesssim \begin{cases} t^{-\delta} E_N, & 0 < \xi < \frac{1}{3}, \\ t^{-2\alpha} E_N, & \xi = 0. \end{cases} \tag{3.2}$$

For the time derivatives of the fluid energy quantity L and the fluid 3-velocity u^j we have

$$\|\partial_t L\|_{\mathscr{H}^{N-1}} \lesssim \begin{cases} t^{-\alpha} E_N + t^{-1-\delta} E_N, & 0 < \xi < \frac{1}{3} \\ t^{-2\alpha} E_{N+1} + t^{-2\alpha-1} E_N + t^{-4\alpha} E_N, & \xi = 0, \end{cases} \tag{3.3}$$

$$\left\|\partial_t u^j\right\|_{\mathscr{H}^{N-1}} \lesssim \begin{cases} t^{-\alpha} E_N + t^{-3\xi\alpha} E_N, & 0 < \xi < \frac{1}{3}, \\ t^{-2\alpha} E_{u,N}, & \xi = 0, \end{cases} \tag{3.4}$$

$$\left\|\partial_t u^0\right\|_{\mathscr{H}^{N-1}} \lesssim \begin{cases} t^{-\alpha-\delta} E_N + t^{-1-\delta} E_N, & 0 < \xi < \frac{1}{3}, \\ t^{-2\alpha} E_N + t^{-2\alpha-1} E_N & \xi = 0. \end{cases} \tag{3.5}$$

For the covariant derivative of u^0 we find

$$\left\|\nabla_I u^0\right\|_{\mathscr{L}^2} \lesssim \begin{cases} t^{-\delta} E_N, & 0 < \xi < \frac{1}{3}, \\ t^{-2\alpha} E_{u,N}, & \xi = 0. \end{cases} \tag{3.6}$$

Furthermore, the inhomogeneous terms can be estimated by

$$\|F_i\|_{\mathscr{L}^2} \lesssim \begin{cases} t^{-\alpha} E_N + t^{-1-\delta} E_N, & 0 < \xi < \frac{1}{3}, \\ t^{-2\alpha} E_N + t^{-2\alpha-1} E_{N+1}, & \xi = 0, \end{cases} \tag{3.7}$$

$$\left\|G^j{}_i\right\|_{\mathscr{L}^2} \lesssim \begin{cases} t^{-3\xi\alpha-\delta} E_N + t^{-2\alpha} E_N, & 0 < \xi < \frac{1}{3}, \\ t^{-2\alpha} E_N, & \xi = 0. \end{cases} \tag{3.8}$$

Remark 6. *Note that all of the time factors in lemma 8 (without the decay property of u^0) decay faster than t^{-1} with the exception of $t^{-3\xi\alpha-\delta}$ in eq. (3.8) and eq.(3.4). However, these terms will be paired with others that are inducing decay in our final estimates.*

Proof. We will treat the more sophisticated case $0 < \xi < \frac{1}{3}$ in detail. The case of pressureless dust follows in a completely analogous fashion. To prove eq. (3.2) we first note that if the order N is sufficiently large, by Sobolev embedding and definition 3 we have

$$\left\|(u^0)^2 - 1\right\|_{\mathscr{H}^N} = \left\|g_{ab}\tilde{u}^a\tilde{u}^b\right\|_{\mathscr{H}^{N-1}} \lesssim t^{(3\xi-1)2\alpha}\gamma_{ab}\|u^a\|_{\mathscr{H}^{N-1}}\left\|u^b\right\|_{\mathscr{H}^N} \lesssim t^{-\delta} E_N.$$

However,

$$u^0 - 1 = \sqrt{1 + g_{ab}\tilde{u}^a\tilde{u}^b} - 1 \leq g_{ab}\tilde{u}^a\tilde{u}^b,$$

and therefore

$$\begin{aligned}\left\|u^0 - 1\right\|_{\mathscr{H}^N} &\simeq \left\|u^0 - 1\right\|_{\mathscr{L}^2} + \left\|\nabla u^0\right\|_{\mathscr{H}^{N-1}} \\ &\lesssim \left\|(u^0)^2 - 1\right\|_{\mathscr{L}^2} + \left\|\nabla (u^0)^2\right\|_{\mathscr{H}^{N-1}} \lesssim \left\|(u^0)^2 - 1\right\|_{\mathscr{H}^N},\end{aligned}$$

proving the desired result. To prove eq. (3.3) we look at the expression derived in lemma 4. The first term is given by

$$\frac{1}{1 - \xi \frac{ct^{(3\xi-1)2\alpha}}{(u^0)^2}\gamma(u,u)} = \frac{1}{1 - \xi \frac{(u^0)^2-1}{(u^0)^2}} = \frac{1}{1 - \xi\left(1 - \frac{1}{(u^0)^2}\right)},$$

which is uniformly bounded. Furthermore,

$$\nabla_i u^0 = \frac{2}{u^0} ct^{(3\xi-1)2\alpha}\gamma_{ab}u^a\nabla_i u^b,$$

which implies that an N-th derivative of the fraction above is at most of order N. Investigating the remaining terms of the expression in lemma 4, e.g.

$$\frac{1}{u^0}t^{(3\xi-2)\alpha}u^a\nabla_a L(\xi - 1),$$

we see that a covariant derivative ∇_I of order N yields a leading order term $\nabla_I\nabla_a L$ and all other terms can be estimated by Sobolev embedding provided N is sufficiently large. We therefore find

$$\begin{aligned}\left\|\frac{1}{u^0}t^{(3\xi-2)\alpha}u^a\nabla_a L(\xi - 1)\right\|_{\mathscr{H}^{N-1}} &\lesssim t^{(3\xi-2)\alpha}\|u\|_{\mathscr{H}^{N-1}}\|L\|_{\mathscr{H}^N} \\ &\lesssim t^{-\alpha-\delta}\mathscr{E}_{N-1}E_N \lesssim t^{-\alpha-\delta}E_N \lesssim t^{-\alpha}E_N.\end{aligned}$$

Taking into consideration the time factors appearing in lemma 4, we find the desired result. To prove eq. (3.4) we employ similar arguments and utilize lemma 1 as well as eq. (3.3). Eq. (3.6) follows immediately from the argument stated above. To prove the estimates regarding the inhomogeneous terms F_i and $G^i{}_j$ we start by investigating the first term of the inhomogeneities of first order as given in lemma 5,

$$t^{(3\xi-2)\alpha}u^0\nabla_i\frac{1}{u^0}u^a\nabla_a L.$$

If this term was derived by a sufficiently high derivative ∇_I of order $N-1$, we would find leading order terms of type

$$\begin{aligned}\left\|t^{(3\xi-2)\alpha}u^0\nabla_i\frac{1}{u^0}u^a\nabla_I\nabla_a L\right\|_{\mathscr{L}^2} &\lesssim t^{(3\xi-2)\alpha}\left\|u^0\right\|_{\mathscr{L}^\infty}\left\|\nabla_i\frac{1}{u^0}\right\|_{\mathscr{L}^\infty}\|u^a\|_{\mathscr{L}^\infty}\|L\|_{\mathscr{H}^N} \\ &\lesssim t^{(3\xi-2)\alpha}\|u\|_{\mathscr{H}^{N-1}}\|L\|_{\mathscr{H}^N} \\ &\lesssim t^{-\alpha-\delta}E_N \lesssim t^{-\alpha}E_N,\end{aligned}$$

using Sobolev embedding and our smallness assumption. For the other terms we employ a similar argument, eq. (3.3), Sobolev embedding and the fact that the Riemann tensor as well as its covariant derivatives are uniformly bounded due to the fact that our manifold M is compact and γ is smooth. Note that the term containing $\partial_t u^j$,

$$t^{(3\xi-1)2\alpha}\frac{1}{u^0}\gamma_{ab}\nabla_I u^a \partial_t u^b,$$

yields a time factor

$$t^{-3\xi\alpha-\delta}t^{(3\xi-1)2\alpha}t^{(1-3\xi)\alpha-\delta} = t^{-\alpha-2\delta} \leq t^{-\alpha}.$$

The estimate for $G^i{}_j$ follows similarly using eq. (3.3) and Sobolev embedding. □

Proposition 3. *The $\tilde{\nabla}$-divergence of the current j^μ defined in 2 and derived in Lemma 7 can be estimated by*

$$\left\|\tilde{\nabla}_\mu j^\mu_N\right\|_{\mathscr{L}^1} \lesssim t^{-\alpha}E_N^2 + t^{-1-\delta}E_N^2, \qquad 0<\xi<\frac{1}{3}, \tag{3.9}$$

$$\left\|\tilde{\nabla}_\mu j^\mu_{u,N}\right\|_{\mathscr{L}^1} \lesssim E_{u,N}^2 + t^{-1}E_{u,N}^2, \qquad \xi = 0, \tag{3.10}$$

$$\left\|\tilde{\nabla}_\mu j^\mu_{L,N-1}\right\|_{\mathscr{L}^1} \lesssim t^{-2\alpha}E_N^2 + t^{-2\alpha-1}E_N^2, \qquad \xi = 0. \tag{3.11}$$

Proof. We will again focus on the more involved case of $0<\xi<\frac{1}{3}$ and omit the details for $\xi=0$. First recall that the $\tilde{\nabla}$-divergence of the current of order N can be expressed as

$$\begin{aligned}
\tilde{\nabla}_\mu j^\mu =& \frac{\xi}{1+\xi}\left(\partial_t u^0 + t^{(3\xi-2)\alpha}\nabla_a u^a\right)\left|\nabla^k L\right|^2 + 2\xi t^{(3\xi-1)2\alpha}\gamma_{ab}\partial_t\left(\frac{u^a}{u^0}\right)\gamma^{IJ}\nabla_I u^b \nabla_J L\\
&+(1+\xi)\left(\partial_t u^0 + t^{(3\xi-2)\alpha}\nabla_a u^a\right)\gamma^{IJ}\left(-\nabla_I u^0\nabla_J u^0 + ct^{(3\xi-1)2\alpha}\gamma_{ab}\nabla_I u^a \nabla_J u^b\right)\\
&+4\xi c\alpha(3\xi-1)t^{(3\xi-1)2\alpha-1}\gamma_{ab}\frac{u^a}{u^0}\gamma^{IJ}\nabla_I u^b\nabla_J L\\
&+(1+\xi)(3\xi-1)2\alpha t^{(3\xi-1)2\alpha-1}\gamma_{ab}\gamma^{IJ}\nabla_I u^a\nabla_J u^b\\
&+2\frac{\xi}{1+\xi}\gamma^{IJ}\nabla_I L F_J - 2(1+\xi)c\gamma^{IJ}\nabla_I u^0 G^0_J + 2(1+\xi)t^{(3\xi-1)2\alpha}\gamma^{IJ}\gamma_{ab}\nabla_I u^a G^b_J.
\end{aligned}$$

First of all, the term

$$(1+\xi)(3\xi-1)2\alpha t^{(3\xi-1)2\alpha-1}\gamma_{ab}\gamma^{IJ}\nabla_I u^a\nabla_J u^b$$

can be dropped entirely due to the fact that it is negative definite. For the term containing $\left|\nabla^k L\right|^2$ we find

$$\left\|\frac{\xi}{1+\xi}\left(\partial_t u^0 + t^{(3\xi-2)\alpha}\nabla_a u^a\right)\left|\nabla^k L\right|^2\right\|_{\mathscr{L}^1} \lesssim t^{(3\xi-2)\alpha}\|\nabla u\|_{\mathscr{L}^\infty}\|L\|^2_{\mathscr{H}^N} \lesssim t^{-\alpha-\delta}E_N^2,$$

where we used lemma 8, Sobolev embedding, the Hölder inequality and the smallness assumption. For the term

$$2\xi t^{(3\xi-1)2\alpha}\gamma_{ab}\partial_t\left(\frac{u^a}{u^0}\right)\gamma^{IJ}\nabla_I u^b\nabla_J L$$

we use an analogous argument and pick up another time factor by eq. (3.4) amounting to a total of

$$t^{(1-3\xi)\alpha} t^{(3\xi-1)2\alpha} t^{-3\xi\alpha} = t^{-\alpha}.$$

To estimate

$$(1+\xi)\left(\partial_t u^0 + t^{(3\xi-2)\alpha}\nabla_a u^a\right)\gamma^{IJ}\left(-\nabla_I u^0 \nabla_J u^0 + ct^{(3\xi-1)2\alpha}\gamma_{ab}\nabla_I u^a \nabla_J u^b\right)$$

we use eq. (3.6) and the aforementioned tools. This nets a time factor of

$$t^{(3\xi-2)\alpha} t^{(3\xi-1)2\alpha} t^{(1-3\xi)2\alpha} t^{(1-3\xi)\alpha-\delta} = t^{-\alpha-\delta}.$$

For

$$4\xi c\alpha(3\xi-1)t^{(3\xi-1)2\alpha-1}\gamma_{ab}\frac{u^a}{u^0}\gamma^{IJ}\nabla_I u^b \nabla_J L$$

we find a net time factor of

$$t^{(3\xi-1)2\alpha-1} t^{(1-3\xi)\alpha} t^{(1-3\xi)\alpha-\delta} = t^{-1-\delta}.$$

All of the remaining terms including those with inhomogeneities F, G and G^0 can be estimated in $\mathscr{L}^1$ using the results of lemma 8, Sobolev embedding and the Hölder inequality. □

Corollary 1. *The $\tilde{\nabla}$-divergence of the current j^μ defined in 2 and derived in Lemma 7 can be estimated by*

$$\left\|\tilde{\nabla}_\mu j^\mu_N\right\|_{\mathscr{L}^1} \lesssim t^{-1-\delta}E_N^2, \qquad 0<\xi<\frac{1}{3}, \tag{3.12}$$

$$\left\|\tilde{\nabla}_\mu j^\mu_{u,N}\right\|_{\mathscr{L}^1} \lesssim E_{u,N}^2, \qquad \xi=0, \tag{3.13}$$

$$\left\|\tilde{\nabla}_\mu j^\mu_{L,N-1}\right\|_{\mathscr{L}^1} \lesssim t^{-2\alpha}E_N^2, \qquad \xi=0. \tag{3.14}$$

Proof. This corollary immediately follows from proposition 3. □

3.2. Relating the Current to the Energy

The goal of this section is to establish coercivity of the energies defined in definition 3 and the time components of the various currents defined in definition 2.

Proposition 4. *Let $E_N \le \epsilon$ on $[1,T)$. Then, if ϵ is small enough and N is sufficiently large, the following equations hold*

$$E_N^2 \simeq \sum_{M\le N}\left\|j^0_M\right\|_{\mathscr{L}^1} + \mathscr{E}^2_{N-1}, \qquad 0<\xi<\frac{1}{3}, \tag{3.15}$$

$$E_{u,N}^2 \simeq \sum_{M\le N} t^{-2\alpha}\left\|j^0_{u,M}\right\|_{\mathscr{L}^1}, \qquad \xi=0, \tag{3.16}$$

$$E_{L,N}^2 \simeq \sum_{M\le N}\left\|j^0_{L,M}\right\|_{\mathscr{L}^1}, \qquad \xi=0, \tag{3.17}$$

$$E_{u,N}^2 + E_{L,N-1}^2 \simeq E_N^2, \qquad \xi=0. \tag{3.18}$$

Proof. We will again focus on the more involved case of $0 < \xi < \frac{1}{3}$ and omit the details for $\xi = 0$ for which the calculation significantly simplifies. First note that by the binomial inequality we have

$$\left|2\xi\gamma^{IJ}\nabla_I u^0 \nabla_J L\right| \leq \frac{\xi}{2(1+\xi)}\gamma^{IJ}\nabla_I L \nabla_J L + 2\xi(1+\xi)\gamma^{IJ}\nabla_I u^0 \nabla_J u^0.$$

Now remember that j^0 was defined by

$$\begin{aligned} j^0_M :=& \frac{\xi}{1+\xi} u^0 \gamma^{IJ}\nabla_I L \nabla_J L + 2\xi\gamma^{IJ}\nabla_I L \nabla_J u^0 \\ &+ (1+\xi)u^0\gamma^{IJ}\left(-\nabla_I u^0 \nabla_J u^0 + ct^{(3\xi-1)2\alpha}\gamma_{ab}\nabla_I u^a \nabla_J u^b\right), \end{aligned}$$

and therefore

$$\begin{aligned} j^0_M \geq& \frac{\xi}{1+\xi}\left(u^0 - \frac{1}{2}\right)\gamma^{IJ}\nabla_I L \nabla_J L + c(1+\xi)t^{(3\xi-1)2\alpha}\gamma_{ab}\gamma^{IJ}\nabla_I u^a \nabla_J u^b \\ &- (u^0 + 2\xi)(1+\xi)\gamma^{IJ}\nabla_I u^0 \nabla_J u^0 \\ \geq& \frac{\xi}{1+\xi}\left(u^0 - \frac{1}{2}\right)\gamma^{IJ}\nabla_I L \nabla_J L + c(1+\xi)t^{(3\xi-1)2\alpha}\gamma_{ab}\gamma^{IJ}\nabla_I u^a \nabla_J u^b \\ &- (u^0 + 2\xi)(1+\xi)\gamma^{IJ}\frac{c^2}{(u^0)^2}t^{(3\xi-1)4\alpha}\gamma_{ab}u^a\nabla_I u^b \gamma_{cd}u^c\nabla_J u^d \\ \geq& \beta\left(\gamma^{IJ}\nabla_I L \nabla_J L + t^{(3\xi-1)2\alpha}\gamma_{ab}\gamma^{IJ}\nabla_I u^a \nabla_J u^b\right) \\ &- \sigma\frac{1}{t^{(3\xi-1)2\alpha}\gamma_{ab}u^a u^b}\gamma_{ab}u^a u^b \gamma^{IJ}\gamma_{mn}\nabla_I u^m \nabla_J u^n \\ \gtrsim& \gamma^{IJ}\nabla_I L \nabla_J L + t^{(3\xi-1)2\alpha}\gamma_{ab}\gamma^{IJ}\nabla_I u^a \nabla_J u^b. \end{aligned}$$

Note that the similarity relation $\gtrsim$ in the last line is independent of u and L. The other direction follows similarly:

$$\begin{aligned} j^0_M \leq& \frac{\xi}{1+\xi}\left(u^0 + \frac{1}{2}\right)\gamma^{IJ}\nabla_I L \nabla_J L + c(1+\xi)t^{(3\xi-1)2\alpha}\gamma_{ab}\gamma^{IJ}\nabla_I u^a \nabla_J u^b \\ &- (u^0 - 2\xi)(1+\xi)\gamma^{IJ}\nabla_I u^0 \nabla_J u^0 \\ \leq& \frac{\xi}{1+\xi}\left(u^0 + \frac{1}{2}\right)\gamma^{IJ}\nabla_I L \nabla_J L + c(1+\xi)t^{(3\xi-1)2\alpha}\gamma_{ab}\gamma^{IJ}\nabla_I u^a \nabla_J u^b \\ &- \underbrace{(u^0 - 2\xi)}_{>0, \quad t\gg 1}(1+\xi)\gamma^{IJ}\frac{c^2}{(u^0)^2}t^{(3\xi-1)4\alpha}\gamma_{ab}u^a\nabla_I u^b \gamma_{cd}u^c\nabla_J u^d \\ \lesssim& \gamma^{IJ}\nabla_I L \nabla_J L + t^{(3\xi-1)2\alpha}\gamma_{ab}\gamma^{IJ}\nabla_I u^a \nabla_J u^b, \end{aligned}$$

where we exploited the fact that the term with a negative sign is positive definite. $\square$

3.3. Integral Inequalities for the Energies

To utilize the results of the estimates above and the coercive property of the current derived in the previous section we will now establish integral inequalities for each of the respective energies for different matter types.

Proposition 5. *Let $E_N \leq \epsilon$ on $[0,T)$ and N be sufficiently large. Then E_N^2 satisfies the inequality*

$$E_N^2(t) \leq E_N^2(t_1) + C\int_{t_1}^t \tau^{-1-\delta}E_N^2(\tau) + \tau^{-\alpha+\delta}E_N^2(\tau)d\tau, \qquad 0<\xi<\frac{1}{3},$$
$$E_{u,N}^2(t) \leq E_{u,N}^2(t_1) + C\int_{t_1}^t \tau^{-2\alpha}E_{u,N}^2(\tau)d\tau, \qquad \xi = 0,$$
$$E_{L,N-1}^2(t) \leq E_{L,N-1}^2(t_1) + C\int_{t_1}^t \tau^{-2\alpha}E_N^2(\tau)d\tau, \qquad \xi = 0.$$

Proof. Again we will focus on the case $0<\xi<\frac{1}{3}$ as the case of pressureless dust is similar. Using definition 2, corollary 1, proposition 4 as well as the divergence theorem, we find

$$\begin{aligned}
\frac{d}{dt}\left(E_N^2 - \mathscr{E}_{N-1}^2\right) &= \frac{d}{dt}\sum_{M\leq N}\left\|j_M^0\right\|_{\mathscr{L}^1} = \sum_{M\leq N}\left\|\frac{d}{dt}j_M^0\right\|_{\mathscr{L}^1} \\
&= \sum_{M\leq N}\left\|\tilde{\nabla}_\mu j_M^\mu - \nabla_a j_M^a\right\|_{\mathscr{L}^1} \leq \sum_{M\leq N}\left\|\tilde{\nabla}_\mu j_M^\mu\right\|_{\mathscr{L}^1} \\
&\lesssim t^{-1-\delta}\sum_{M\leq N}E_M^2 \lesssim t^{-1-\delta}E_N^2.
\end{aligned}$$

Integrating this identity yields the desired expression. For the lower order norm $\mathscr{E}_{N-1}$ we find

$$\begin{aligned}
\frac{d}{dt}\mathscr{E}_{N-1}^2 &= \frac{d}{dt}\left(t^{(3\xi-1)\alpha+\delta}\left(\sum_{k\leq N-1}\int\left|\nabla^k u\right|^2\right)^{\frac{1}{2}}\right)^2 \\
&= 2((3\xi-1)\alpha+\delta)t^{2((3\xi-1)\alpha+\delta)-1}\|u\|_{\mathscr{H}^{N-1}}^2 \\
&\quad + t^{2((3\xi-1)\alpha+\delta)}2\sum_{|I|=|J|\leq N-1}\int\gamma_{ab}\gamma^{IJ}\nabla_I u^a\nabla_J\partial_t u^b \\
&\lesssim 2\underbrace{((3\xi-1)\alpha+\delta)}_{\leq 0}t^{-1}\mathscr{E}_{N-1}^2 + 2t^{(3\xi-1)\alpha+\delta}\mathscr{E}_{N-1}\|\partial_t u\|_{\mathscr{H}^{N-1}} \\
&\lesssim 2t^{(3\xi-1)\alpha+\delta}\mathscr{E}_{N-1}\left(t^{-\alpha}E_N + t^{-3\xi\alpha}E_N\right) \lesssim t^{-\alpha+\delta}E_N^2,
\end{aligned}$$

where we used the Hölder inequality as well as the estimates derived in lemma 8. □

4. The Future Stability Theorem

In this last section we combine the results of the previous chapters to prove future stability of solutions to the system given in proposition 1. We will do so for expansion rates $\alpha > \frac{1}{2}$ and $\alpha > 1$ for the dust and fluid case, respectively.

Theorem 2 (The Future Stability in the Case of Fluids and Pressureless Dust)**.** *Let $\alpha > 1$ (as given in eq. (1.1)) in the case $0 < \xi < \frac{1}{3}$ and $\alpha > \frac{1}{2}$ in the case $\xi = 0$. Furthermore, for integer $N \geq 3$ let*

$$L(t=1) =: \bar{L} \in \begin{cases} \mathscr{H}^{N-1}, & \xi = 0, \\ \mathscr{H}^{N}, & 0 < \xi < \frac{1}{3}, \end{cases}$$

$$u^j(t=1) =: \bar{u}^j \in \mathscr{H}^N, \quad 0 \leq \xi < \frac{1}{3},$$

be initial data for the rescaled Euler equations given in proposition 1. Then there exists a small constant ϵ' such that given $E_N(1) = \epsilon'$ with $\epsilon > \epsilon'$ the classical solution (L, U) provided by theorem 1 exists for all $t \in [1, \infty)$ and furthermore

$$\forall t \geq 1: \quad E_N(t) \leq \epsilon.$$

Proof. We will treat the case $0 < \xi < \frac{1}{3}$ in detail as the case of $\xi = 0$ follows analogously. Now by theorem 1 as well as the continuity of the energy E_N we are given a maximal local solution that exists for the time T^* meaning

$$T^* := \sup\{t \geq 1 | \text{The solution exists on } [1,t) \text{ and } E_N \leq \epsilon \text{ holds on } [1,t)\}.$$

Now by proposition 5 under the assumption that $t_1 \leq t < T^*$ we know that

$$\begin{aligned} E_N^2(t) &\leq E_N^2(t_1) + C\int_{t_1}^t \left(\tau^{-1-\delta}E_N^2(\tau) + \tau^{-\alpha+\delta}E_N^2(\tau)\right) d\tau \\ &\leq DE_N(t_1) + C\epsilon^2 \int_{t_1}^t \left(\tau^{-1-\delta} + \tau^{-\alpha+\delta}\right) d\tau, \end{aligned}$$

for some constants C, D. As we assumed $\alpha > 1$ in the fluid case, we know that $(\tau \mapsto \tau^{-\alpha+\delta})$ as well as $(\tau \mapsto \tau^{-1-\delta})$ are integrable and hence if t_1 is sufficiently large, we have

$$E_N^2(t) \leq DE_N(t_1) + \frac{1}{2}\epsilon^2.$$

Now we just need to find a proper estimate for $E_N(t_1)$ to complete the proof. Applying proposition 5 one more time yields

$$E_N^2(t) \leq E_N^2(1) + C\int_1^t \left(\tau^{-1-\delta}E_N^2(\tau) + \tau^{-\alpha+\delta}E_N^2(\tau)\right) d\tau, \qquad t \in [1, T^*).$$

Furthermore assuming that $t \in [1, t_1]$ and applying Grönwalls lemma we find

$$\begin{aligned} E_N^2(t) &\leq (\epsilon')^2 \exp\left\{ C \int_1^t \left(\tau^{-1-\delta} + \tau^{-\alpha+\delta}\right) d\tau \right\} \\ &\leq (\epsilon')^2 \exp\left\{ C \left(\left\|(\cdot)^{-1-\delta}\right\|_{\mathscr{L}^1([1,\infty))} + \left\|(\cdot)^{-\alpha+\delta}\right\|_{\mathscr{L}^1([1,\infty))} \right) \right\} =: (\epsilon')^2 E, \end{aligned}$$

for $t \in [1, t_1]$ resulting in

$$E_N^2(t) \leq DE(\epsilon')^2 + \frac{1}{2}\epsilon^2, \qquad 1 < t \leq T^*.$$

Now provided that ϵ' is small enough, i.e. $(\epsilon')^2 < \epsilon^2 \frac{1}{2DE}$, we find

$$E_N^2(t) < \epsilon^2, \qquad 1 \leq t < T^*,$$

which is a strict improvement over the assumption $E_N(t) \leq \epsilon$ on $[1, T^*)$. By the continuity of E_N we can therefore extend the smallness condition for times bigger than T^* and by Sobolev embedding and proposition 2, we conclude that our solution exists for all time $t \in [1, \infty)$. $\square$

A. Proofs

A.1. Metric Identities

Proof of Lemma 2. First note that for a metric as given in eq. (1.1) we have $g_{0\beta} = -\delta_{0\beta}$. Thus, we find

$$\begin{aligned}
{\Gamma^0}_{0\beta} &= \frac{1}{2} g^{00} \left(\partial_0 g_{0\beta} + \partial_\beta g_{00} - \partial_0 g_{\beta 0} \right) = 0, \\
{\Gamma^0}_{ab} &= \frac{1}{2} g^{0\sigma} \left(\partial_a g_{\sigma b} + \partial_b g_{\sigma a} - \partial_\sigma g_{ab} \right) \\
&= \frac{1}{2} g^{00} \left(\partial_a g_{0b} + \partial_b g_{0a} - \partial_0 g_{ab} \right) \\
&= \frac{1}{2} 2\alpha c t^{2\alpha-1} \gamma_{ab} = c\alpha t^{2\alpha-1} \gamma_{ab}, \\
{\Gamma^j}_{00} &= \frac{1}{2} g^{j\sigma} \left(2\partial_0 g_{\sigma 0} + \partial_\sigma g_{00} \right) = 0, \\
{\Gamma^j}_{0b} &= \frac{1}{2} g^{j\sigma} \left(\partial_0 g_{g_{b\sigma}} + \partial_b g_{0\sigma} - \partial_\sigma g_{0b} \right) \\
&= \frac{1}{2} g^{js} \partial_0 g_{bs} = \frac{1}{2} \frac{1}{c} t^{-2\alpha} \gamma^{js} c 2\alpha t^{2\alpha-1} \gamma_{bs} = \alpha t^{-1} {\delta^j}_b, \\
{\Gamma^j}_{ab} &= \frac{1}{2} g^{jk} \left(\partial_a g_{kb} + \partial_b g_{ka} - \partial_k g_{ab} \right) \\
&= \frac{1}{2} \frac{1}{c} t^{-2\alpha} \gamma^{jk} c t^{2\alpha} \left(\partial_a \gamma_{kb} + \partial_b \gamma_{ka} - \partial_k \gamma_{ab} \right) = {}^\gamma{\Gamma^j}_{ab}.
\end{aligned}$$

□

A.2. Identities of the Fluid Variables

Proof of Lemma 4. We first derive the scalar equation as given in lemma 3:

$$\begin{aligned}
\partial_t L = &-t^{(3\xi-2)\alpha} \frac{1}{u^0} u^a \, {}^\gamma\nabla_a L \\
&- (1+\xi) \left(\frac{1}{(u^0)^2} c\alpha t^{(3\xi-1)2\alpha-1} \gamma_{ab} u^a u^b + t^{(3\xi-2)\alpha} \frac{1}{u^0} {}^\gamma\nabla_a u^a \right) \\
&- (1+\xi) \frac{1}{(u^0)^2} c t^{(3\xi-1)2\alpha} \gamma_{ab} u_a \Bigg(-t^{(3\xi-2)\alpha} \frac{1}{u^0} u^a \, {}^\gamma\nabla_a u^b \\
&- \frac{\xi}{1+\xi} \left(u^b \partial_t L + t^{(3\xi-2)\alpha} \frac{1}{u^0} u^b u^a \partial_a L + \frac{1}{cu^0} t^{-3\xi\alpha} \gamma^{ba} \partial_a L \right) \Bigg)
\end{aligned}$$

$$
\begin{aligned}
\iff \partial_t L \left(1 - \xi \frac{ct^{(3\xi-1)2\alpha}}{(u^0)^2}\gamma(u,u)\right) = {} & u^{a\gamma}\nabla_a L \left(-t^{(3\xi-2)\alpha}\frac{1}{u^0}\right.\\
& + t^{(3\xi-2)\alpha+(3\xi-1)2\alpha}\xi\frac{1}{(u^0)^3}c\gamma(u,u)\\
& \left. + t^{(3\xi-2)\alpha}\xi\frac{1}{(u^0)^3}\right)\\
& - (1+\xi)\frac{1}{(u^0)^2}c\alpha t^{(3\xi-1)2\alpha-1}\gamma(u,u)\\
& - t^{(3\xi-2)\alpha}(1+\xi)\frac{1}{u^0}{}^{\gamma}\nabla_a u^a\\
& + t^{(9\xi-4)\alpha}(1+\xi)\frac{1}{(u^0)^3}c\gamma_{ab}u^{c\gamma}\nabla_c u^a u^b
\end{aligned}
$$

$$
\begin{aligned}
\iff \partial_t L = {} & \frac{1}{1-\xi\frac{ct^{(3\xi-1)2\alpha}}{(u^0)^2}\gamma(u,u)}\left(\frac{1}{u^0}t^{(3\xi-2)\alpha}u^{a\gamma}\nabla_a L\,(\xi-1)\right.\\
& - (1+\xi)\alpha ct^{(3\xi-1)2\alpha-1}\gamma(u,u)\frac{1}{(u^0)^2}\\
& \left. - t^{(3\xi-2)\alpha}(1+\xi)\frac{1}{u^0}{}^{\gamma}\nabla_a u^a + (1+\xi)\frac{1}{(u^0)^3}ct^{(9\xi-4)\alpha}\gamma_{ab}u^{c\gamma}\nabla_c u^a u^b\right).
\end{aligned}
$$

□

Proof of proposition 1. The equations for $\partial_t L$ and $\partial_t u^j$ immediately follow from lemma 3. We will, however, derive the equation for u^0 in detail. The relativistic Euler equation for $\tilde{u}^0$ reads

$$
\begin{aligned}
\tilde{u}^\alpha\nabla_\alpha\tilde{u}^0 + \frac{\xi}{1+\xi}\left(\tilde{u}^0\tilde{u}^\mu\nabla_\mu\ln\rho + g^{0\mu}\nabla_0\ln\rho\right) = 0\\
\iff \tilde{u}^0\partial_t\tilde{u}^0 + \tilde{u}^a\partial_a\tilde{u}^0 + \Gamma^0{}_{\alpha\beta}\tilde{u}^\alpha\tilde{u}^\beta\\
+\frac{\xi}{1+\xi}\left(((\tilde{u}^0)^2-1)\partial_t\ln\rho + \tilde{u}^a\nabla_a\ln\rho\right) = 0.
\end{aligned}
$$

Using lemma 2, lemma 1 as well as the rescaling of L, we find

$$
\begin{aligned}
\tilde{u}^\mu\partial_\mu\tilde{u}^0 + c\alpha t^{2\alpha-1}\gamma_{ab}\tilde{u}^a\tilde{u}^b\\
+\frac{\xi}{1+\xi}\left(ct^{2\alpha}\gamma_{ab}\tilde{u}^a\tilde{u}^b(\partial_t L - 3\alpha(1+\xi)t^{-1}) + \tilde{u}^a\nabla_a L\right) = 0\\
\iff \tilde{u}^\mu\partial_\mu\tilde{u}^0 + c\alpha t^{2\alpha-1}\gamma_{ab}\tilde{u}^a\tilde{u}^b(1-3\xi)\\
+\frac{\xi}{1+\xi}\left(ct^{2\alpha}\gamma_{ab}\tilde{u}^a\tilde{u}^b\partial_t L + \tilde{u}^a\nabla_a L\right) = 0.
\end{aligned}
$$

Applying the rescaling $\tilde{u} = t^{(3\xi-2)\alpha}u$, we find

$$
\begin{aligned}
u^0\partial_t u^0 + t^{(3\xi-2)\alpha}u^a\partial_a u^0 = {} & c\alpha t^{(3\xi-1)2\alpha-1}\gamma_{ab}u^a u^b(3\xi-1)\\
& - \frac{\xi}{1+\xi}\left(ct^{(3\xi-1)2\alpha}\gamma_{ab}u^a u^b\partial_t L + t^{(3\xi-2)\alpha}u^a\nabla_a L\right).
\end{aligned}
$$

Hence, the final result is given by

$$
\begin{aligned}
\iff \partial_t u^0 = &-\frac{1}{u^0} t^{(3\xi-2)\alpha} u^{a\gamma} \nabla_a u^0 + \frac{1}{u^0} c\alpha t^{(3\xi-1)2\alpha-1} \gamma_{ab} u^a u^b (3\xi - 1) \\
&- \frac{\xi}{1+\xi} \left(\frac{1}{u^0} c t^{(3\xi-1)2\alpha} \gamma_{ab} u^a u^b \partial_t L + \frac{1}{u^0} t^{(3\xi-2)\alpha} u^{a\gamma} \nabla_a L \right).
\end{aligned}
$$

□

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