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„The Cauchy Problem in General Relativity"

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Abstract

Einstein's field equations can be, under suitable boundary conditions, formulated as an initial value problem. In this way one can see the strong relation between Einstein's general theory of relativity and the theory of hyperbolic PDE's. By the formulation of Einstein's equation as an IVP it became clear that the initial data cannot be chosen freely, they have to fulfill some constraints. This also leads to non-linear PDE's on manifolds.

It turns out that it is not trivial to show the existence of a maximal development due to the fact that it is not unique. To handle this problem one has to restrict to a specific class of developments in order to get a maximal and unique development. With given initial data, it is possible to show that there is a unique maximal globally hyperbolic development (MGHD).

In this Master's thesis we show how one can reduce the Cauchy problem in General Relativity to that of solving a system of hyperbolic differential equations. Next we prove the existence and uniqueness of a solution for symmetric hyperbolic systems and make the connection to linear wave equations (Chapter 1). Afterwards the existence and uniqueness of a solution to non-linear wave equations is shown (Chapter 2), which then is extended to manifolds of Lorentzian signature. At the end there is a proof for the existence of a maximal globally hyperbolic development (Chapter 3).

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Zusammenfassung

Die Einstein'schen Feldgleichungen der allgemeinen Relativitätstheorie können unter bestimmten Umständen als ein Anfangswertproblem formuliert werden. Auf diese Weise wird die enge Verbindung zwischen der Allgemeinen Relativitätstheorie und der Theorie der hyperbolischen Differentialgleichungen sichtbar. Durch die Formulierung der Feldgleichungen als ein Cauchy Problem wird klar, dass die Anfangswerte nicht beliebig gewählt werden können, sondern gewissen Einschränkungen unterworfen sind. Am Ende stehen nichtlineare partielle Differentialgleichungen auf Mannigfaltigkeiten.

Es zeigt sich weiters, dass die Existenz einer maximalen Entwickung eines Cauchy-Problems in der allgemeinen Relativitätstheorie aufgrund von Nichteindeutigkeiten der Lösungen schwierig zu beweisen ist. So muss man sich letzten Endes auf eine bestimmte Klasse von Entwicklungen beschränken, um zu einer maximalen und eindeutigen Entwicklung zu kommen. Es ist also möglich, bei gegebenen Anfangswerten, die Existenz einer maximalen gobalen hyperbolischen Entwicklung zu zeigen.

In dieser Masterarbeit wird gezeigt wie man das Cauchy Problem der allgemeinen Relativitätstheorie auf das Problem der Lösung von hyperbolischen Differentialgleichungen reduzieren kann (Kapitel 1). Danach wird die Eindeutigkeit/Existenz der Lösung von symmetrischen hyperbolischen Systemen bewiesen und anschließend deren Verbindung zu linearen Wellengleichungen. Danach wird die Eindeutigkeit/Existenz der Lösung von nichtlinearen Wellengleichungen gezeigt (Kapitel 2) und dies auf Mannigfaltigkeiten mit Lorentz-Signaltur erweitert. An Ende wird der Beweis zur Existenz einer maximalen globalen hyperbolischen Entwicklung erläutert (Kapitel 3).

Contents

Introduction

After Einstein published the field equations of general relativity, some explicit solution (like the one of Karl Schwarzschild) where found. Due to the fact that some of these spacetimes have singularities, the question arose whether these singularities are of physical relevance or only generated by a high degree of symmetry (of the solutions). Even Einstein was of the opinion that these singularities disappear in case of less symmetry.

If one tries to understand the Schwarzschild solution an additional problem arises due to the appearance of the $r = 2M$ hypersurface in standard coordinates, namely whether this hypersurface is a singularity or not. From a cosmological viewpoint this model, called the Einstein static universe, is not a natural model, expanding models are much more natural in general relativity (for stability reasons). Indeed the solution which Einstein suggested was unstable and not a fensible model of the universe.

The work of Hawking and Penrose in the 50's and 60's gave an important contribution to the issue of singularities. They proved, simply speaking, that spacetimes generally exhibit singularities (the so-called 'singularity theorems'). Nevertheless, it is important to remark that the notion of singulatities which they used in their paper is that of the incompleteness of causal geodesics. In fact they don't state that there are singularities where curvature blow up happens, although there are examples of solutions which have singularities in the sense of causal geodesic incompleteness, but do not have curvature singularities. Let's have a look on a simple example of such a solution, the so-called flat Kasner solution

$$g_F = -dt^2 + t^2 dx^2 + dy^2 + dz^2$$

on $(0, \infty) \times \mathbb{R}^3$. This solution is past causally geodesically incomplete, and Hawking's theorem applies to it. Nevertheless, the Riemannian curvature tensor of g_F is identically zero and so the $t = 0$ hypersurface cannot be considered to be a singularity. Indeed one can embed the above solution into

Minkowski space. Due to this embedding one can extend the solution (without any singularities appearing) beyond the hypersurface for $t = 0$. One can say that the singularity theorems are without any doubt important, but leave some questions of interest unanswered. Within Lorentzian geometry, the study of singularity theorems is an important area of current research. From this perspective one has the chance to understand some of the fundamental solutions (such as the Schwarzschild solution) much better and in particular the $r = 2M$ hypersurface in Schwarzschild's solution gets satisfactory interpretation. Parallel to the Lorentz geometric perspective, the idea of formulating Einstein's equations as an initial value problem was developed.

In a seminal work Yvonne Choquet-Bruhat showed in 1952 that that Einstein's field equations can be formulated as an initial value problem. She proved that, if the the initial data are given, there is a corresponding solution to the equation. The problem here was that there are infinitely many (inequivalent) developments corresponding to a given initial data set. From this point of view it was important to show whether there is a (initial data associated) unique development or not. To achieve such a development, it was necessary to introduce some form of maximality.

Despite of this maximality condition it turns out that one cannot get a unique object in the class of all developments. But if this condition is required in the class of globally hyperbolic developments it will work, which was proven by Choquet-Bruhat and Robert Geroch in 1962. It should be noted that there exist also non-extendable globally hyperbolic developments and sometimes there are inequivalent maximal extensions of the maximal globally hyperbolic development, which means that general relativity is not a deterministic theory. There is a lack of predictability in Einstein's general theory of relativity.

But the existing examples are very special, an observation which leads to the strong cosmic censorship conjecture. One formulation of the strong cosmic censorship conjecture is that for generic initial data (in the asymptotically flat or spatially compact setting), the associated maximal globally hyperbolic development is inextendible.

This has led to a strong focus on the initial value point of view in the study of Einstein's equations. The same is true for numerical studies, which are often based on this point of view. Finally we enumerate some of the mathematical problems associated with the Cauchy problem:

- **Stability:** The models which are used in physics for the universe and isolated systems have a high degree of symmetry. The question which arise is whether one gets a similar solution if the corresponding initial data vary slightly, i.e. if the solutions are stable under slightly perturbed initial conditions. One prominent example which is unstable is Einstein's static universe. From the viewpoint of isolated systems the proof of the stability of the Kerr spacetime is also of fundamental importance. One further topic of central significance in cosmology is the proof of the future stability of the standard cosmological model.

- **Predictability:** If it turned out that the strong cosmic censorship is not valid, it would be a drawback for the consideration of Einstein's equations as an initial value problem. Indeed it would no longer be possible to describe the behavior of a system out of its initial data. So the proof of the strong cosmic censorship is of fundamental importance.

- **Singularities:** As we described before, if we have a look at standard solutions of models of isolated systems and the universe, singularities can occur due to the high degree of symmetry. Actually singularities are described in the sense of causal geodesic incompleteness, but it is not clear that (if we have generic initial data) we get a blowup in curvature in the directions of incomplete causal geodesics on the corresponding maximal Cauchy development.

  Based on the considerations the aim is to get a proof of the strong cosmic censorship conjecture.

- **Homogenization/isotropization:** If one starts working in cosmology, one of the first assumptions which is stressed is the cosmological principle; i.e., the assumption that the universe is spatially homogeneous and isotropic. For sure it would be better if one can deduce this as a consequence of the evolution from Einstein's field equations, than imposing it as an assumption.

  Indeed there is a conjecture (by using a positive cosmological constant), the so-called cosmic no-hair conjecture, which states that the universe can approach de Sitter space asymptotically (in systems with no black holes etc.). Additionally, conjectures without cosmological constant exist, even though they are harder to prove.

  One interesting and helpful way is to compare the theoretical results with numerical simulations. They have the benefit of delivering real numbers, which then can be compared with observations, something which is not possible if one takes exclusively the analytical perspective. It also doesn't make sense to try to calculate mathematically the gravitational wave signal resulting from inspiralling black holes. For sure there are a lot of further examples.

The drawback of this approach is that only a finite amount of solutions can be considered and it can happen that something may be overlooked, which would be no surprise if one has in mind that the space of initial data is infinite dimensional. In contrast, this is no problem from the analytic viewpoint where one can consider an infinite number of initial data simultaneously. Also the analysis of asymptotics is (at least in some special situations) better taken from the analytic perspective. An additional benefit of this approach is that one gets a deeper understanding how the entire theory works, if exact results can be proven.

Chapter 1

General Relativity

The main focus is on the development of Einstein's field equations of general relativity for a special matter model, the so called non-linear scalar field. The question is whether it is possible to find out how a system/universe look like at a given time if there are some given initial conditions, so to say the evolution of the space-time. It would be desirable that one can formulate the problem as an initial value problem (ivp) which is solvable in the context of PDE-theory. To get an idea about the problem let us start with Einsteins field equations.

$$G = \mathrm{Ric} - \frac{1}{2}Sg = T \tag{1.1}$$

where S is the scalar curvature of (M, g) with the energy-stress tensor

$$T = d\phi \otimes d\phi - \left[\frac{1}{2}\langle \nabla\phi, \nabla\phi\rangle + V(\phi)\right] g \tag{1.2}$$

where $\langle\cdot,\cdot\rangle := g$, or in abstract index notation

$$T_{ab} = \nabla_a\phi\nabla_b\phi - \left(\frac{1}{2}\nabla^c\phi\nabla_c\phi + V(\phi)\right) g_{ab} \tag{1.3}$$

This matter model is known as a non-linear scalar field , with a smooth function $V : \mathbb{R} \to \mathbb{R}$ which is the *potential* V and a smooth function $\phi : M \to \mathbb{R}$, referred to as the *scalar field*. So if we want $T = G$ we have

$$d\phi \otimes d\phi - \left[\frac{1}{2}\langle \nabla\phi, \nabla\phi\rangle + V(\phi)\right] g = \mathrm{Ric} - \frac{1}{2}Sg \tag{1.4}$$

This means (we use the signature $(-,+,...,+)$ for g and $dim\ M = (n+1)$):

$$\nabla_a\phi\nabla_b\phi - \left(\frac{1}{2}\nabla^c\phi\nabla_c\phi + V(\phi)\right)g_{ab} = R_{ab} - \frac{1}{2}R^c{}_c g_{ab}$$

$$\Rightarrow \nabla^a\phi\nabla_b\phi - \left(\frac{1}{2}\nabla^c\phi\nabla_c\phi + V(\phi)\right)g^a{}_b = R^a{}_b - \frac{1}{2}R^c{}_c g^a{}_b$$

$$\overset{trace}{\Rightarrow} \nabla^a\phi\nabla_a\phi - \left(\frac{1}{2}\nabla^c\phi\nabla_c\phi + V(\phi)\right)\underbrace{g^a{}_a}_{(n+1)} = \underbrace{R^a{}_a}_{S} - \frac{n+1}{2}R^a{}_a = \left(1 - \frac{n+1}{2}\right)S$$

$$\Rightarrow \left(1 - \frac{n+1}{2}\right)\nabla^a\phi\nabla_a\phi - (n+1)V(\phi) = \left(1 - \frac{n+1}{2}\right)S$$

$$\Rightarrow \frac{n-1}{2}\left(S - \langle\nabla\phi,\nabla\phi\rangle\right) = (n+1)V(\phi) \tag{1.5}$$

So (1.4) holds only if

$$\begin{aligned}\mathrm{Ric} &\overset{(1.4)}{=} d\phi\otimes d\phi + \frac{1}{2}\left(S - \langle\nabla\phi,\nabla\phi\rangle\right)g - V(\phi)g \overset{(1.5)}{=} \\ &= d\phi\otimes d\phi + \left(\frac{n+1}{n-1} - 1\right)V(\phi)g = d\phi\otimes d\phi + \frac{2}{n-1}V(\phi)g\end{aligned} \tag{1.6}$$

Conversely, we suppose that

$$\mathrm{Ric} = d\phi\otimes d\phi + \frac{2}{n-1}V(\phi)g \tag{1.7}$$

$$\Rightarrow\ R_{ab} = \nabla_a\phi\nabla_b\phi + \frac{2}{n-1}V(\phi)g_{ab}\ \Rightarrow R^a{}_b = \nabla^a\phi\nabla_b\phi + \frac{2}{n-1}V(\phi)g^a{}_b$$

$$\overset{trace}{\Rightarrow} S = \langle\nabla\phi,\nabla\phi\rangle + 2\frac{n+1}{n-1}V(\phi) \tag{1.8}$$

$$\begin{aligned}\overset{(1.7,8)}{\Rightarrow} \mathrm{Ric} - \frac{1}{2}Sg &= d\phi\otimes d\phi + \frac{2}{n-1}V(\phi)g - \frac{1}{2}\langle\nabla\phi,\nabla\phi\rangle g - \frac{n+1}{n-1}V(\phi)g = \\ &= d\phi\otimes d\phi - \left[\frac{1}{2}\langle\nabla\phi,\nabla\phi\rangle + V(\phi)\right]g + \frac{2}{n-1}V(\phi)g - \left(\frac{n+1}{n-1}+1\right)V(\phi)g\end{aligned} \tag{1.9}$$

$$\Rightarrow (1.4).$$

Therefore we can write for $G = T$ equivalently

$$\mathrm{Ric} = d\phi\otimes d\phi + \frac{2}{n-1}V(\phi)g$$

This equation together with a matter equation for ϕ constitutes our system of interest, which we will call the:

Einstein non-linear scalar field system.

$$\mathrm{Ric} - d\phi \otimes d\phi - \frac{2}{n-1} V(\phi) g = 0 \tag{1.10}$$

$$\Box_g \phi - V'(\phi) = 0 \tag{1.11}$$

For these equations our task is to develope an initial value problem. We will give a definition for the ivp, although we don't know now how the initial data should look like,i.e. what the constraint equations are. It should serve as a place to start (for the following chapters) to avoid to lose sight of the final goal.

Definition 1.0.1. *Initial data to the Einstein non-linear scalar field system*

$$\mathrm{Ric} - d\phi \otimes d\phi - \frac{2}{n-1} V(\phi) g = 0 \tag{1.12}$$

$$\Box_g \phi - V'(\phi) = 0 \tag{1.13}$$

consist of:

An n-dimensional manifold Σ with a Riemannian metric g_0, a covariant 2-tensor k and two functions ϕ_0 and ϕ_1 on Σ, all assumed to be smooth and to satisfy (the constraint equations)

$$r - k_{ij}k^{ij} + (tr\, k)^2 = \phi_1^2 + D^i\phi_0 D_i\phi_0 + 2V(\phi_0)$$

$$D^j k_{ij} - D_i(tr\, k) = \phi_1 D_i \phi_0$$

whit D as the Levi-Civita connection of g_0, r is the associated scalar curvature and indices are raised and lowered by g_0.

*Once the initial data are given, the **initial value problem** (ivp) is that of finding*

- *an $n+1$ dimensional manifold M with a Lorentz metric g, a function $\phi \in \mathcal{C}^\infty(M)$ such that our Einstein non-linear scalar field system is satisfied,*
- *and an embedding $i : \Sigma \hookrightarrow M$ where $i^*g = g_0$, $\phi \circ i = \phi_0$ and if N regarded as the future directed unit normal, K the second fundamental form of $i(\Sigma)$, then $i^*K = k$ and $(N\phi) \circ i = \phi_1$.*

*Such a triple (M, g, ϕ) is called a **development** of the initial data (while we assume that an embedding i exists). If additionally $i(\Sigma)$ is a Cauchy hypersurface in (M, g), we say that (M, g, ϕ) is a **globally hyperbolic development**.*

Now we want to state the main problems that we have to face.

- we have to handle the constraint equations, due to the foliation of the space-time (M, g) into (Σ, g_0). So we need the compatibility to the curvature in (M, g).

- we have to derive the proper initial data, with respect to the constraint equations.

- we have to formulate the system of field equations as a hyperbolic differential equation (due to the fact, that the Ricci tensor in Einstein's field equations seen as a differential operator is not hyperbolic).

- handle the problem of proof of existence and uniqueness for linear and nonlinear hyperbolic differential equations and transfer the whole setup to Lorentzian manifolds.

1.1 Differentiation and Curvature on Lorentz manifolds

An orthonormal basis for T_pM is called a **frame** on M at p. If $n = \dim M$ then a set $E_1, ..., E_n$ of n mutually orthogonal unit vector fields is called a frame field, since it assigns a frame to each point. On $\mathbb{R}^n$ the natural vector fields $\partial_i := \frac{\partial}{\partial x^i}$ for $i = 1, ...n$ form a frame field.

We often use the notation of O'Neill [2], but in the end we want take the usual notations as used in General Relativity. Unfortunately this differs in parts. Therefore we define some operators, to be used in this work. We start with the **covariant derivative**. If (M, g) is a Lorentz manifold, there is a unique metric and torsion free connection, the so-called Levi-Civita connection (see [2], Theorem 11,p.61). For two vector fields $X = \sum_i X^i\partial_i,\ Y = \sum_j Y^j\partial_j$ on M we get for the covariant derivative of Y along X (by use of Einstein's summation convention)

$$\begin{aligned}\nabla_X Y &= \nabla_{X^i\partial_i} Y^j\partial_j = X^i\left(\nabla_{\partial_i} Y^j\partial_j\right) = X^i\left(\partial_i Y^j\partial_j + Y^j\nabla_{\partial_i}\partial_j\right)\\ &= X^i\left(\partial_i Y^j\partial_j + Y^j\Gamma^k_{ij}\partial_k\right) = X^i\partial_i Y^j\partial_j + Y^j X^i\Gamma^k_{ij}\partial_k \\ &= \left(X(Y^k) + Y^j X^i\Gamma^k_{ij}\right)\partial_k\end{aligned} \tag{1.14}$$

Let now be $\mathcal{T}^r_s(M)$ be the space of smooth tensorfield on M, covariant of order s and contravariant of order r. If $T \in \mathcal{T}^r_s(M)$, then the covariant derivative is:

$$\left(\nabla_\alpha T^{\beta_1...\beta_r}_{\alpha_1...\alpha_s}\right) X^\alpha Y_1^{\alpha_1}...Y_s^{\alpha_s}\theta_{1\beta_1}...\theta_{r\beta_r} = \left(\nabla_X T\right)\left(Y_1, ..., Y_s, \theta_1, ..., \theta_r\right) \tag{1.15}$$

The right hand side can be computed as follows,

$$\begin{aligned}&(\nabla_X T)(Y_1, ..., Y_s, \theta_1, ..., \theta_r) = X\left[T(Y_1, ..., Y_s, \theta_1, ..., \theta_r)\right] \\ &-T(\nabla_X Y_1, ..., Y_s, \theta_1, ..., \theta_r) - ... - T(Y_1, ..., Y_s, \theta_1, ..., \nabla_X \theta_r),\end{aligned} \tag{1.16}$$

where we used that ∇ commutes with contractions. With respect to coordinates, we get

$$\begin{aligned}\nabla_\alpha T^{\beta_1 \ldots \beta_r}_{\alpha_1 \ldots \alpha_s} &= (\nabla_{\partial_\alpha} T)\left(dx^{\beta_1}, ..., dx^{\beta_r}, \partial_{\alpha_1}, ..., \partial_{\alpha_s}\right) = \\ &= \partial_\alpha T^{\beta_1 \ldots \beta_r}_{\alpha_1 \ldots \alpha_s} - \Gamma^\mu_{\alpha\alpha_1} T^{\beta_1 \ldots \beta_r}_{\mu\alpha_2 \ldots \alpha_s} - ... - \Gamma^\mu_{\alpha\alpha_s} T^{\beta_1 \ldots \beta_r}_{\alpha_1\alpha_2 \ldots \alpha_s - 1\mu} + \\ &\quad + \Gamma^{\beta_1}_{\alpha\mu} T^{\mu\beta_2 \ldots \beta_r}_{\alpha_1 \ldots \alpha_s} + ... + \Gamma^{\beta_r}_{\alpha\mu} T^{\beta_1\beta_2 \ldots \beta_r - 1\mu}_{\alpha_1 \ldots \alpha_s}\end{aligned} \tag{1.17}$$

Let $T = \nabla X$ and let us compute

$$\begin{aligned}(\nabla_\alpha \nabla_\beta X^\gamma) Y^\alpha Z^\beta \theta_\gamma = Y\left[T(Z, \theta)\right] - T(\nabla_Y Z, \theta) - T(Z, \nabla_Y \theta) = \\ Y\left[\theta(\nabla_Z X)\right] - \theta(\nabla_{\nabla_Y Z} X) - \nabla_Y \theta(\nabla_Z X) = \theta(\nabla_Y \nabla_Z X - \nabla_{\nabla_Y Z} X).\end{aligned} \tag{1.18}$$

Because the Levi-Civita connection is torsion free we get after applying the last calculation with twisted indices, a (1,3) tensor field on M the so called **Riemann tensor** $R : \mathfrak{X}(M)^3 \longrightarrow \mathfrak{X}(M)$. It measures the path dependence of parallel transport (where we use the Levi-Civita connetion) in that it can be seen as a measure for the change of a vector to return to its start value when parallel transported around a small loop. It is a function

$$\begin{aligned}(\nabla_\alpha \nabla_\beta X^\gamma - \nabla_\beta \nabla_\alpha X^\gamma) Y^\alpha Z^\beta \theta_\gamma = \\ \theta\underbrace{\left(\nabla_Y \nabla_Z X - \nabla_Z \nabla_Y X + \nabla_{[Y,Z]} X\right)}_{=:R_{YZ}X}.\end{aligned} \tag{1.19}$$

Here we have the opposite sign to that of [2] p.74. To go align with the conventions as in [3], cf.(3.2.11), p.38,

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) X^c = -R_{abd}{}^c X^d, \tag{1.20}$$

From the Riemann tensor we get the **Ricci tensor** Ric of M by the following contraction

$$R_{ab} = R_{acb}{}^c, \tag{1.21}$$

It is symmetric and relative to a frame field it is given by

$$\mathrm{Ric}(X, Y) = \sum_m \epsilon_m \langle R_{XE_m} E_m, Y\rangle \tag{1.22}$$

where $\epsilon_m = \langle E_m, E_m\rangle$.

1.2 Constraint equations

The idea for an ivp is that given some initial data one can give a solution to this problem at a later time, or at least say something about existence and uniqueness of a solution. It turns out that these initial data cannot be arbitrarily chosen (in most field theories) but have to fulfill some constraints. To illustrate this lets begin with a simpler problem. If we start with the fundamental equations of electrodynamics, the Maxwell equations and use the so called scalar- and vector potentials V and A, it is possible to write them in the following form:

$$\Box V = 0$$
$$\Box A = 0$$
$$\partial_t V + \mathrm{div} A = 0$$

The first two are just wave equations, which we can solve for given initial data. But we have to solve the whole system and the third equation, which is the so called Lorenz gauge condition (it is used to decouple the original system of PDE's, where the electric and magnetic field are coupled), imposes some conditions. As long as the initial data satisfies the constraint equation, one can find a solution as mentioned above.

The same problem arises in General Relativity. For our ivp we construct a foliation of the Lorentzian spacetime (M, g) into a set of $3-$dim. Riemannian manifolds (Σ, g_0) which are parameterized by the time coordinate. In this way we can rewrite Einsteins equations in terms of the induced metric $g_{ij}(t)$ and the second fundamental form $k_{ij}(t)$ of the $3-$dim. manifolds. But as in electrodynamics one cannot just choose an arbitrary set of initial data. They have to fulfill also some constraint equations.

To derive the constraint equations for our system we use the Gauß ([2] thm.5, p.100) and Codazzi equations ([2] prop. 33, p.115). For this we use temporarily the notation as in O'Neil [2], later on we change again to the original setting. Let (M, g) a spacelike hyperplane in a time oriented Lorentzian manifold $(\overline{M}, \overline{g})$. If we have vector fields $V, W, X \in \mathfrak{X}(M)$ which are tangent to $M \subset \overline{M}$, then the Gauß equation describes tan $\bar{R}_{VW}X$ in terms of the so-called shape tensor $II(V, W) =$ nor $\overline{D}_V W$(which is symmetric and bilinear over smooth functions, [2] Lemma 4, p.100), whereas the Codazzi equation analogously describes nor $\overline{R}_{VW}X$.

Definition 1.2.1. *Let $(\overline{M}, \overline{g})$ be a time oriented Lorentz manifold and M a spacelike hypersurface in $(\overline{M}, \overline{g})$, furthermore let $i : M \hookrightarrow \overline{M}$ be the em-*

bedding and N a future directed unit timelike vector field such that for every $p \in M$, $\overline{g}(N_p, v) = 0$ for every $v \in T_pM$.
The second fundamental form of M is the covariant 2-tensor field k on M defined by $k(v, w) = \overline{g}(\overline{D}_v N, w)$, for $v, w \in T_pM$, where $\overline{D}$ is the Levi-Civita connection of $(\overline{M}, \overline{g})$.

Proposition 1.2.1. *Let $(\overline{M}, \overline{g})$ be a time oriented Lorentz manifold and M a spacelike hypersurface with induced metric, let $i : M \to \overline{M}$ be the embedding and N be a future directed unit timelike vector field such that for every $p \in M$, $\overline{g}(N_p, w) = 0$ for every $w \in T_pM$ and let k be the second fundamental form of M. Then*

$$\overline{G}(N_p, N_p) = \frac{1}{2}\left[S - k_{ij}k^{ij} + (tr_g\, k)^2\right](p), \tag{1.23}$$

$$\overline{G}(N_p, v) = \left[D^j k_{ij} - D_i(tr_g\, k)\right] v^i, \tag{1.24}$$

where D is the Levi-Civita connection on M, S the scalar curvature of (M, g) and $\overline{\mathrm{G}} := \overline{\mathrm{Ric}} - \frac{1}{2}\overline{S}\overline{g}$ the Einstein tensor of $(\overline{M}, \overline{g})$, $p \in M$, $v \in T_pM$.

Proof. Let $\langle ., . \rangle = \overline{g}$ and $X : M \to T\overline{M}$ be an $\overline{M}$ vector field on M. Let us denote the separation of the vector field X into its normal and tangential components by $\operatorname{nor} X$ and $\tan X$ respectively.

$$\begin{tikzcd} M \arrow[r, "X"] \arrow[dr, "i"'] & T\overline{M} \arrow[d, "\pi"] \\ & \overline{M} \end{tikzcd}$$

Let $e_0 = N$ and let V and W be vector fields on M. Extending W to a vector field $\overline{W}$ on $\overline{M}$ which is always orthogonal to e_0, we get at $p \in M$,

$$\begin{aligned} \langle II(V, W), e_0 \rangle &= \left\langle \overline{D}_V W, e_0 \right\rangle \\ &= \overline{D}_V \left\langle \overline{W}, e_0 \right\rangle - \left\langle \overline{D}_V e_0, W \right\rangle = -k(V, W), \end{aligned} \tag{1.25}$$

the first step holds by definition, the second uses the product rule and the last uses that $\left\langle \overline{W}, e_0 \right\rangle$ is zero as proposed. So we get $II(V, W) = k(V, W)e_0$.

We wish to compute $\overline{G}(e_0, e_0)$ for $\overline{G}$, the Einstein tensor of $(\overline{M}, \overline{g})$. For this we assume that in a neighborhood of $p \in M$, e_i, $i = 1, ..., n$, is an orthonormal basis for T_pM. By taking the trace of the curvature tensor we get

$$\overline{\mathrm{Ric}}(e_j, e_j) = -\left\langle \overline{R}_{e_0 e_j} e_j, e_0 \right\rangle + \sum_{i=1}^{n} \left\langle \overline{R}_{e_i e_j} e_j, e_i \right\rangle$$

for the Ricci tensor, and by summation over j we obtain

$$\sum_{j=1}^{n} \overline{\mathrm{Ric}}(e_j, e_j) = \underbrace{-\sum_{j=1}^{n} \left\langle \overline{R}_{e_0 e_j} e_j, e_0 \right\rangle}_{\overline{\mathrm{Ric}}(e_0,e_0)} + \sum_{i,j=1}^{n} \left\langle \overline{R}_{e_i e_j} e_j, e_i \right\rangle, \tag{1.26}$$

where we used the symmetry of the curvature tensor $\left\langle \overline{R}_{e_0 e_j} e_j, e_0 \right\rangle = \left\langle \overline{R}_{e_j e_0} e_0, e_j \right\rangle$, (see [2] Prop.36(4) p.75). Before the next calculation let us state the Gauss-equation (for a proof see [2],p 100 Thm.5), which we will use. For vector fields V, W, X, Y tangent to M, we have the so-called Gauss-equation:

$$\begin{aligned} \langle \mathrm{R}_{VW} X, Y \rangle = \left\langle \overline{\mathrm{R}}_{VW} X, Y \right\rangle + \langle I\!I(V, X), I\!I(W, Y) \rangle \\ - \langle I\!I(V, Y), I\!I(W, X) \rangle \end{aligned} \tag{1.27}$$

So we get the first equation.

$$\begin{aligned} 2\overline{G}(e_0, e_0) &= 2\overline{\mathrm{Ric}}(e_0, e_0) + S(\overline{g}) \\ &= 2\overline{\mathrm{Ric}}(e_0, e_0) + \left(-\overline{\mathrm{Ric}}(e_0, e_0) + \sum_{j=1}^{n} \overline{\mathrm{Ric}}(e_j, e_j) \right) \\ &= \overline{\mathrm{Ric}}(e_0, e_0) + \sum_{j=1}^{n} \overline{\mathrm{Ric}}(e_j, e_j) \overset{(1.26)}{=} \sum_{i,j=1}^{n} \left\langle \overline{R}_{e_i e_j} e_j, e_i \right\rangle \\ &\overset{(1.27)}{=} \sum_{i,j=1}^{n} \left\langle R_{e_i e_j} e_j, e_i \right\rangle - \sum_{i,j=1}^{n} \langle I\!I(e_i, e_j), I\!I(e_j, e_i) \rangle \\ &\quad + \sum_{i,j=1}^{n} \langle I\!I(e_i, e_i), I\!I(e_j, e_j) \rangle = S(g) - k_{ij} k^{ij} + (tr_g k)^2 \end{aligned} \tag{1.28}$$

For the second equation we have to calculate $\overline{G}(e_0, e_i) = \overline{\mathrm{Ric}}(e_0, e_i)$, because of $\overline{g}(e_0, e_i) = 0$. By (1.22) we get

$$\begin{aligned} \overline{\mathrm{Ric}}(e_i, e_0) &= -\underbrace{\left\langle \overline{R}_{e_0 e_i} e_0, e_0 \right\rangle}_{=0} + \sum_{j=1}^{n} \left\langle \overline{R}_{e_i e_j} e_j, e_0 \right\rangle \\ &= \sum_{j=1}^{n} \left\langle \overline{R}_{e_i e_j} e_j, e_0 \right\rangle \end{aligned} \tag{1.29}$$

Let us furthermore start with the covariant derivative of the second fundamental form $(\nabla_V II)(X,Y)$. For a vector field V on M and an $\overline{M}$ vector field Z on M such that $\operatorname{nor} Z = Z$ we have $D_V^{\perp} Z = \operatorname{nor} \overline{D}_V Z$ and $(\nabla_V II)(X,Y) = D_V^{\perp}[II(X,Y)] - II(D_V X, Y) - II(X, D_V Y)$. With this and V, W and X vector fields on M we can state the Codazzi equation ([2] prop. 33, p.115, note the difference in sign) as

$$\operatorname{nor} \overline{R}_{VW} X = (\nabla_V II)(W,X) - (\nabla_W II)(V,X) \tag{1.30}$$

With (1.25) and $\left\langle D_V^{\perp} e_0, e_0 \right\rangle = \left\langle \bar{D}_V e_0, e_0 \right\rangle = \frac{1}{2} \bar{D}_V \left\langle e_0, e_0 \right\rangle = 0$ we can write $\nabla_V II$ in terms of k:

$$\begin{aligned} D_V^{\perp}[II(X,Y)] &= D_V^{\perp}[k(X,Y)e_0] = V[k(X,Y)]\, e_0 + \underbrace{k(X,Y) D_V^{\perp} e_0}_{=0} \\ &= (D_V k)(X,Y)e_0 + k(D_V X, Y)e_0 + k(X, D_V Y)e_0 \\ &\overset{(1.25)}{=} (D_V k)(X,Y)e_0 - II(D_V X, Y) - II(X, D_V Y) \end{aligned} \tag{1.31}$$

So we have $\nabla_V II(X,Y) = (D_V k)(X,Y)e_0$ and from (1.29) and the Codazzi equation (1.30), we get the second equation

$$\overline{\mathrm{Ric}}(e_0, e_i) = \sum_{j=1}^{n} \left\langle (D_{e_i} k)(e_j, e_j)e_0 - (D_{e_j} k)(e_i, e_j)e_0, e_0 \right\rangle$$

Now we have to couple the above with the non-linear scalar field. The first equation, the so-called Hamilton constraint is the coupling with the energy density, which follows from $\bar{G}(e_0, e_0) = T(e_0, e_0) = \rho$. Let now ϕ be a smooth function on $\bar{M}$. Assume that $\overline{g}$ and ϕ satisfy $G = T$. From (1.23) we get

$$\overline{G}(N,N) = \frac{1}{2}\left[S - k_{ij}k^{ij} + (\operatorname{tr}_g k)^2\right] = \rho.$$

To get ρ we start with $\overline{G}(N,N) = \overline{T}(N,N) = \rho$, where $N = e_0$. We have to calculate $T(N,N)$. From (1.2) we get

$$\begin{aligned} T(N,N) &= d\phi(N)^2 - \left[\frac{1}{2}\left\langle \mathrm{grad}\phi, \mathrm{grad}\phi \right\rangle + V(\phi)\right] \underbrace{g(N,N)}_{=-1} \\ &= N(\phi)^2 + \frac{1}{2}\left\langle \mathrm{grad}\phi, \mathrm{grad}\phi \right\rangle + V(\phi) \end{aligned}$$

For the middle term we compute

$$\langle \mathrm{grad}\phi, \mathrm{grad}\phi \rangle = \sum_i \epsilon_i \langle \mathrm{grad}\phi, e_i \rangle^2 = \sum_{i=0}^{n} D_i\phi$$

$$= -D_0\phi^2 + \sum_{i=1}^{n} D_i\phi = N(\phi)^2 + D_i\phi D^i\phi$$

due to $\langle \mathrm{grad}\phi, e_i \rangle = e_i(\phi) = D_{e_i}\phi = D_i\phi$. So we obtain

$$\rho = \frac{1}{2}\left[(N\phi)^2 + D^i\phi D_i\phi\right] + V(\phi)$$

The second constraint equation, the so-called momentum constraint for any $v \in TM$ can be derived by a computation of $\overline{G}(N, v)$:

$$\overline{G}(N, v) \overset{(1.24)}{=} \left[D^j k_{ij} - D_i(\mathrm{tr}_g k)\right] v_i \overset{(1.1)}{=} T(N, v)$$

from this we use (1.2) and obtain

$$T(N, v) \overset{(1.2)}{=} N(\phi) \cdot v(\phi) - \left[\frac{1}{2}\langle \mathrm{grad}\phi, \mathrm{grad}\phi \rangle + V(\phi)\right] \underbrace{g(N, v)}_{=0}$$

$$= N(\phi) \cdot v(\phi)$$

with $v(\phi) = \sum_i v^i e_i(\phi) = \sum_i v^i D_i(\phi)$ we get finally

$$\left[D^j k_{ij} - D_i(\mathrm{tr}_g k)\right] = N(\phi) D_i(\phi).$$

□

1.3 Modified system of equations

If we express the system (1.12), (1.13) in local coordinates we don't obtain a system of non-linear wave equations. By a modification, we can under certain circumstances get what we want. Let us start with the Ricci tensor

$$R_{\mu\nu} = -\frac{1}{2} g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} + \nabla_{(\mu}\Gamma_{\nu)} + g^{\alpha\beta}g^{\gamma\delta}\left[\Gamma_{\alpha\gamma\mu}\Gamma_{\beta\delta\nu} + \Gamma_{\alpha\gamma\mu}\Gamma_{\beta\nu\delta} + \Gamma_{\alpha\gamma\nu}\Gamma_{\beta\mu\delta}\right]$$

with

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2}\left[\partial_\alpha g_{\gamma\beta} + \partial_\gamma g_{\alpha\beta} - \partial_\beta g_{\alpha\gamma}\right]$$

$$\Gamma^\mu_{\alpha\gamma} = g^{\mu\beta}\Gamma_{\alpha\beta\gamma}$$

with the notation

$$\nabla_\mu \Gamma_\nu = \partial_\mu \Gamma_\nu - \Gamma^\alpha_{\mu\nu}\Gamma_\alpha$$

$$\nabla_{(\mu}\Gamma_{\nu)} = \frac{1}{2}\left(\nabla_\mu \Gamma_\nu + \nabla_\nu \Gamma_\mu\right)$$

$$\Gamma_\alpha = g^{\mu\nu}\Gamma_{\mu\alpha\nu}$$

Now the problem is that the Ricci-tensor seen as a differential operator is not hyperbolic, which indeed we could not expect because of the diffeomorphism invariance. The problem lies in the term $\nabla_{(\mu}\Gamma_{\nu)}$ of the latter formulation of the Ricci tensor.

The first term itself is a hyperbolic differential operator which acts on the metric also the last term would not make problems because it only contains first derivatives of the metric. To overcome the problem we want to define a new modified Ricci tensor. To get what we want we have to act on the second term, which causes the problems. Therefore the idea is to define a modified system of equations, where we replace the Ricci tensor by

$$\hat{R}_{\mu\nu} = -\frac{1}{2}g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} + \nabla_{(\mu}F_{\nu)} + g^{\alpha\beta}g^{\gamma\delta}\left[\Gamma_{\alpha\gamma\mu}\Gamma_{\beta\delta\nu} + \Gamma_{\alpha\gamma\mu}\Gamma_{\beta\nu\delta} + \Gamma_{\alpha\gamma\nu}\Gamma_{\beta\mu\delta}\right]$$

which is, if we relate it to $R_{\mu\nu}$

$$\hat{R}_{\mu\nu} = R_{\mu\nu} + \nabla_{(\mu}\mathcal{D}_{\nu)}$$

where

$$\mathcal{D}_\nu = F_\nu - \Gamma_\nu$$

Then the modified system of equation is

$$\hat{R}_{\mu\nu} - \nabla_\mu\phi\nabla_\nu\phi - \frac{2}{n-1}V(\phi)g = 0 \tag{1.32}$$

$$\nabla^\mu\nabla_\mu\phi - V'(\phi) = 0 \tag{1.33}$$

If h is a fixed Lorentz metric on M inducing a linear connection $\overline{\nabla}$, we can define

$$A(X,Y,\eta) = \eta\left(\nabla_X Y - \overline{\nabla}_X Y\right), \tag{1.34}$$

where X, Y are vector fields and η is a 1-form. As A is multilinear over the functions, it is a tensor field. In index notation we get for the above expression (1.34)

$$A^\mu_{\alpha\beta} = A(\partial_\alpha, \partial_\beta, dx^\mu) = \Gamma^\mu_{\alpha\beta} - \overline{\Gamma}^\mu_{\alpha\beta},$$

with the Christoffel symbol on $\overline{M}$ and the fixed metric h

$$\overline{\Gamma}^{\mu}_{\alpha\beta} = \frac{1}{2} h^{\mu\nu} \left(\partial_\alpha h_{\beta\nu} + \partial_\beta h_{\alpha\nu} - \partial_\nu h_{\alpha\beta}\right)$$

If we apply the metric onto A we get the contraction Γ_μ for the Christoffel symbol on M, but not for the Christoffel symbol on $\overline{M}$ because there we have the metric h

$$g_{\mu\nu} g^{\alpha\beta} A^{\mu}_{\alpha\beta} = g_{\mu\nu} g^{\alpha\beta} \Gamma^{\mu}_{\alpha\beta} - g_{\mu\nu} g^{\alpha\beta} \overline{\Gamma}^{\mu}_{\alpha\beta} = \Gamma_\nu - \underbrace{g_{\mu\nu} g^{\alpha\beta} \overline{\Gamma}^{\mu}_{\alpha\beta}}_{=:F_\nu}$$

Now we get

$$\mathcal{D}_\nu = -g_{\mu\nu} g^{\alpha\beta} A^{\mu}_{\alpha\beta}$$

The benefit now is that F only depends on the metric g but not on its derivatives, which in the end gives us the opportunity to handle the modified system (where $\hat{R}$ acts as a differential operator on g) by methods of PDE theory, which will be developed in the following chapters. Now we want show that solving the problem (1.12), (1.13) is reduced to solving (1.32), (1.33) as we are able find initial data for these equations in a way that $\mathcal{D}_\nu = 0$ and $\nabla_\mu \mathcal{D}_\nu = 0$ initially.

Assume (M, g) is a globally hyperbolic Lorentz manifold and Σ a smooth spacelike Cauchy hypersurface. Additionally, assume that ϕ is a smooth function on M such that both ϕ and g satisfy the modified system (1.32), (1.33). Let h be the Lorentz metric associated to $\overline{\Gamma}^{\mu}_{\alpha\beta}$. First we want to show that, if $\mathcal{D}_\nu$ and $\nabla_\mu \mathcal{D}_\nu$ vanish on some subset Ω of Σ, then $\mathcal{D}$ vanishes also on $D(\Omega)$, which is the union of the future and past Cauchy development $D^+(\Omega)$, resp. $D^-(\Omega)$ of Ω [see [1] p.109, 10.2.7 Cauchy surfaces]. To show this we need an expression for $G_{\mu\nu} - T_{\mu\nu} = 0$ in terms of our modified system. We start with

$$\hat{R}_{\mu\nu} = R_{\mu\nu} + \nabla_{(\mu} \mathcal{D}_{\nu)} = R_{\mu\nu} + \frac{1}{2} \nabla_\mu \mathcal{D}_\nu + \frac{1}{2} \nabla_\nu \mathcal{D}_\mu \;\; \text{... apply } g^{\mu\nu}$$

$$g^{\mu\nu} \hat{R}_{\mu\nu} = g^{\mu\nu} R_{\mu\nu} + \frac{1}{2} g^{\mu\nu} \nabla_\mu \mathcal{D}_\nu + \frac{1}{2} g^{\mu\nu} \nabla_\nu \mathcal{D}_\mu = R + \frac{1}{2} \nabla_\mu \mathcal{D}^\nu + \frac{1}{2} \underbrace{\nabla_\nu \mathcal{D}^\nu}_{\nu \to \gamma}$$

$$\Longrightarrow \hat{R} = R + \nabla_\gamma \mathcal{D}^\gamma \tag{1.35}$$

Now we use this to change from the original to the modified system

$$\underbrace{R_{\mu\nu} + \nabla_{(\mu} \mathcal{D}_{\nu)}}_{\hat{R}_{\mu\nu}} - \frac{1}{2} \underbrace{\left(R + \nabla_\gamma \mathcal{D}^\gamma\right)}_{\hat{R}} g_{\mu\nu} - T_{\mu\nu} = 0.$$

So we have

$$G_{\mu\nu} - T_{\mu\nu} = -\nabla_{(\mu}\mathcal{D}_{\nu)} + \frac{1}{2}\nabla_\gamma \mathcal{D}^\gamma g_{\mu\nu} = -\nabla_{(\mu}\mathcal{D}_{\nu)} + \frac{1}{2}\nabla^\delta \mathcal{D}_\delta g_{\mu\nu} \tag{1.36}$$

by using $g^{\gamma\delta}g_{\gamma\delta}$. The Einstein tensor is divergence free and so is $T_{\mu\nu}$ due to (1.1), so we can calculate

$$\begin{aligned} 0 &= \operatorname{div}(G_{\mu\nu} - T_{\mu\nu}) = \operatorname{div}\left(-\nabla_{(\mu}\mathcal{D}_{\nu)} + \frac{1}{2}\nabla^\delta \mathcal{D}_\delta g_{\mu\nu}\right) \\ &= -\frac{1}{2}\nabla^\mu\nabla_\mu \mathcal{D}_\nu - \frac{1}{2}\nabla^\mu\nabla_\nu \mathcal{D}_\mu \\ &\quad + \underbrace{\frac{1}{2}\nabla^\mu\nabla^\delta \mathcal{D}_\delta g_{\mu\nu}}_{\nabla_\nu\nabla^\delta\mathcal{D}_\delta = \nabla_\nu\nabla^\mu\mathcal{D}_\mu} + \frac{1}{2}\left(\nabla^\delta\mathcal{D}_\delta\right)\underbrace{\nabla^\mu g_{\mu\nu}}_{=0} \end{aligned} \tag{1.37}$$

$$\begin{aligned} \Rightarrow 0 &= \nabla^\mu\nabla_\mu\mathcal{D}_\nu + \nabla^\mu\nabla_\nu\mathcal{D}_\mu - \nabla_\nu\nabla^\mu\mathcal{D}_\mu \\ &= \nabla^\mu\nabla_\mu\mathcal{D}_\nu + (\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)\mathcal{D}^\mu \end{aligned}$$

By use of the Bianchi identity ([see [1] p.103, (10.9),(10.11)]) we get

$$(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)\mathcal{D}^\mu = -R_{\mu\nu\delta}{}^{\mu}\mathcal{D}^\delta = R_{\nu\delta}\mathcal{D}^\delta = R_{\nu\mu}\mathcal{D}^\mu \tag{1.38}$$

and finally

$$\nabla^\mu\nabla_\mu\mathcal{D}_\nu + R_{\nu\mu}\mathcal{D}^\mu = \nabla^\mu\nabla_\mu\mathcal{D}_\nu + R_\nu{}^\mu\mathcal{D}_\mu = 0 \tag{1.39}$$

By a uniqueness result for tensor wave equations ([1], p.140 Cor.12.12) which we postpone to the next chapters, we conclude that $\mathcal{D}_\nu = 0$ in $D(\Omega)$. So, g and ϕ solve our original system (1.12), (1.13) in $D(\Omega)$. If $\Omega = \Sigma$, we get a solution for all of M.

1.4 Initial data

Assume now we have initial data $(\Sigma, g_0, k, \phi_0, \phi_1)$ to (1.12), (1.13). If we take the initial data for the modified system (1.32), (1.33), we get more freedom, what can be used to obtain $\mathcal{D}_\nu = 0$ at $t = 0$. As mentioned in the definition on the initial value problem, we want a globally hyperbolic development in M that is diffeomorphic to $\mathbb{R}\times\Sigma$ (see [1], p.112, Prop. 11.3). The initial data are defined on the hypersurface $0\times\Sigma$. Let

$$h = -dt^2 + g_0$$

be our fixed Lorentz metric on M. Let $(x^1, ..., x^n)$, $(x^0, x^1, ..., x^n)$ be fixed coordinate systems on an open subset U of Σ resp. on $\mathbb{R} \times U$, where $x^0 = t$. We start with initial data for $g_{\mu\nu}$, where the spatial part g_{ij} is noted by g_0

$$g_{ij}\big|_{t=0} = g_0(\partial_i, \partial_j) \tag{1.40}$$

Due to the splitting of M we can choose g_{00} and g_{0i} in such a way as to get ∂_t as the future directed unit normal to the hypersurface at $t = 0$.

$$g_{00}\big|_{t=0} = -1, \quad g_{0i}\big|_{t=0} = 0 \tag{1.41}$$

Then we can formulate the second fundamental form k to g as follows

$$k(e_i, e_j) = \overline{g}(\overline{D}_{e_i} e_0, e_j) \overset{\text{symmetry of k}}{=}$$

$$= \frac{1}{2}\left(\overline{g}(e_i, \overline{D}_{e_j} e_0) + \overline{g}(e_j, \overline{D}_{e_i} e_0)\right) = \frac{1}{2}\left(\overline{g}(e_i, \overline{D}_{e_0} e_j) + \overline{g}(e_j, \overline{D}_{e_0} e_i)\right) =$$

$$= \frac{1}{2}\overline{D}_{e_0} g_{ij}$$

where we used : $\nabla_\alpha e_\beta - \nabla_\beta e_\alpha = [\partial_\alpha, \partial_\beta] = \partial_\alpha\partial_\beta - \partial_\beta\partial_\alpha = 0$ for basis vectors under the Levi-Civita connection. From this we make the definition

$$\partial_0 g_{ij}\big|_{t=0} = 2k_{ij} = 2k(\partial_i, \partial_j) \tag{1.42}$$

What remains are $\partial_0 g_{00}$ and $\partial_0 g_{0i}$ and we determine these objects under the condition that $\mathcal{D}_\mu\big|_{t=0} = 0$. If we have a metric g then we calculate for $t = 0$

$$\begin{aligned}
\Gamma_{\mu l \nu} &= \frac{1}{2}\left(\partial_\mu g_{\nu l} + \partial_\nu g_{\mu l} - \partial_l g_{\mu\nu}\right) \\
&= \frac{1}{2}\left(\partial_0 g_{0l} + \partial_0 g_{0l} - \partial_l g_{00}\right) + \frac{1}{2}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right) \\
&= \frac{1}{2}\left(2\partial_0 g_{0l} - \partial_l g_{00}\right) + \frac{1}{2}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right) \\
&= \partial_0 g_{0l} - \frac{1}{2}\partial_l g_{00} + \frac{1}{2}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right)
\end{aligned} \tag{1.43}$$

Now we calculate Γ_l at $(t = 0)$

$$\Gamma_l = g^{\mu\nu}\Gamma_{\mu l \nu} \overset{(1.41)}{=} -\partial_0 g_{0l} + \frac{1}{2} g^{ij}\left(2\partial_i g_{jl} - \partial_l g_{ij}\right)$$

and also Γ_0 at $(t=0)$

$$\begin{aligned}\Gamma_0 = g^{\mu\nu}\Gamma_{\mu 0\nu} &= g^{\mu\nu}\left(\frac{1}{2}\left[\partial_\mu g_{\nu 0} + \partial_\nu g_{\mu 0} - \partial_0 g_{\mu\nu}\right]\right) \\ &= g^{00}\left(\frac{1}{2}\left[\partial_0 g_{00} + \partial_0 g_{00} - \partial_0 g_{00}\right]\right) + g^{0i}\left(\frac{1}{2}\left[\partial_0 g_{i0} + \partial_i g_{00} - \partial_0 g_{0i}\right]\right) \\ &\quad + g^{i0}\left(\frac{1}{2}\left[\partial_i g_{00} + \partial_0 g_{i0} - \partial_0 g_{i0}\right]\right) + g^{ij}\left(\frac{1}{2}\left[\partial_i g_{j0} + \partial_j g_{i0} - \partial_0 g_{ij}\right]\right) \\ &= g^{00}\left(\frac{1}{2}\partial_0 g_{00}\right) + \underbrace{g^{0i}\left(\frac{1}{2}\partial_i g_{00}\right) + g^{i0}\left(\frac{1}{2}\partial_i g_{00}\right)}_{=0 \text{ due to (1.41)}} - g^{ij}\left(\frac{1}{2}\partial_0 g_{ij}\right) \\ &\overset{(1.42)}{=} g^{00}\left(\frac{1}{2}\partial_0 g_{00}\right) - g^{ij}k_{ij} \overset{(1.41)}{=} -\frac{1}{2}\partial_0 g_{00} - \text{tr } k\end{aligned}$$

To achieve the above requirement (concerning Γ_0), we need

$$\partial_0 g_{00}|_{t=0} = -2F_0|_{t=0} = -2\text{tr } k, \tag{1.44}$$

where $F_\mu|_{t=0}$ is well defined since the metric was specified for $(t=0)$ and we need (concerning Γ_l), again for $(t=0)$

$$\partial_0 g_{0l}|_{t=0} = \left[-F_l + \frac{1}{2}g^{ij}(2\partial_i g_{jl} - \partial_l g_{ij})\right]\Bigg|_{t=0} \tag{1.45}$$

For our ϕ, we set

$$\phi|_{t=0} = \phi_0, \quad (\partial_t\phi)|_{t=0} = \phi_1 \tag{1.46}$$

From (1.44), (1.45) we know that $\mathcal{D}_\mu = 0$ for $t = 0$. What is left to know is that $\partial_0\mathcal{D}_\mu = 0$ for $t = 0$, to conclude that $\mathcal{D}_\mu = 0$ in our development. For the last step, we assume that we already have a solution for the modified system of the field equations with given initial data, at least in some coordinate patch. What we should have in mind is first that the original initial data must in turn be fulfill the constraint equations and second that the solution fulfills (1.36). By contraction of (1.36) with $n^\mu X^\nu$ at $t = 0$, where $X \perp n$, we get zero for the left side, due to the fact that the constraint equations are fulfilled. Furthermore the partial derivative is all what remains from the covariant derivative, because the parts involving the Christoffel symbols vanish on account of $\mathcal{D}_\mu = 0$ originally. What is left now is the right-hand side

$$-\frac{1}{2}n^\mu X^\nu(\partial_\mu\mathcal{D}_\nu + \partial_\nu\mathcal{D}_\mu)$$

The term $n^\mu X^\nu(\partial_\mu \mathcal{D}_\mu) g_{\mu\nu} = n^\mu X_\mu(\partial_\mu \mathcal{D}_\mu)$ is zero due to $X \perp n$. We get now $\partial_0 \mathcal{D}_i = 0$ for $t = 0, i = 1, ..., n$ due to the fact that $X^\nu \partial_\nu \mathcal{D}_\mu = 0$ for $t = 0$. In the same way we get $\partial_0 \mathcal{D}_0 = 0$ by contraction of (1.36) with $n^\mu n^\nu$.

So we have the modified system (which we can handle), the constraint equations (due to the geometry of the spacetime) and the initial conditions $(1.40) - (1.46)$ for the modified system. In the next chapter we start with the necessary PDE theory to prove one of the main theorems, which we will state here without proof for further motivation.

Theorem 1.4.1. *Let* $(\Sigma, g_0, k, \phi_0, \phi_1)$ *be initial data to* $(1.13), (1.13)$*. then there is a globally hyperbolic development of the data.*

Chapter 2

PDE theory

In the next chapters we will develop the necessary PDE theory to handle the modified system of the field equations. A strategy which is often used is to separate the existence and smoothness of a problem. So we work with weak solutions, where existence and uniqueness can be handled more easily and afterwards have a look about the regularity behavior of the weak solution. Maybe it turns out that the weak solution is smooth enough to qualify as a classical solution. Our way is as follows

$$\begin{gathered}\text{Symmetric hyperbolic systems}\\ \downarrow \\ \text{Linear wave equation}\\ \downarrow \\ \text{Local existence and uniqueness for non-linear wave equations}\\ \downarrow \\ \text{Linear wave equations on globally hyperbolic Lorentz manifolds}\\ \downarrow \\ \text{Einstein non-linear scalar field setting}\end{gathered}$$

2.1 Symmetric hyperbolic systems

We start now with the existence and uniqueness of solutions to linear symmetric hyperbolic systems. For this we use energy estimates, which have often the problem that the function we want to estimate is indeed bounded in terms of itself. In our case Gronwall's lemma can help, which we we will state but not prove.

Lemma 2.1.1. *(Gronwall's lemma). Let $T_0 \in \mathbb{R}$, $T > T_0$ and f, k and G be non-negative functions on $[T_0, T]$ such that $f \in L^\infty([T_0, T])$, $k \in L^1([T_0, T])$*

and G is non-decreasing. Assuming

$$f(t) \leq G(t) + \int_{T_0}^{t} k(s)f(s)ds,$$

$\forall t \in [T_0, T]$, *we obtain*

$$f(t) \leq G(t)\mathrm{exp}\left(\int_{T_0}^{t} k(s)ds\right),$$

$\forall t \in [T_0, T]$.

Proof. For a proof see standard literature for Differential Equations.

2.2 The basic energy inequality

We begin with the definition of the symmetric hyperbolic system where we use Einstein's summation convention.

Definition 2.2.1. *A symmetric hyperbolic system is of the form*

$$Lu = A^{\mu}\partial_{\mu}u + Bu = f \tag{2.1}$$

$$u(0,.) = u_0 \tag{2.2}$$

where $A^{\mu}, \mu = 0, ..., n$ *and* B *are smooth* $N \times N$ *real matrix-valued functions on* $\Omega \subseteq \mathbb{R}^{n+1}$ *(i.e.* $C^{\infty}(\Omega, M_{\mathbb{R}}(N, N))$*) all of whose derivatives are bounded. Additionally* $A^{\mu}, \mu = 0, ..., n$ *is symmetric and we have a positive definite* A^0 *which has an uniform positive lower bound (which means that there exists a constant* $c_0 > 0$ *where* $A \geq c_0$ *holds). Finally,* u *is an* $\mathbb{R}^N$*-valued function on* Ω*,* $f \in C^{\infty}\left(\Omega, \mathbb{R}^N\right)$ *and* $u_0 \in C^{\infty}(\mathbb{R}^n)$*.*

Assume there is a solution u of $(2.1), (2.2)$ on $S_T = [0, T] \times \mathbb{R}^n$ (a so-called slab). We also assume that u and $\partial_t u$ satisfy uniform Schwartz bounds, which means that for every pair of multiindices α and β, there exists a constant $C_{\alpha,\beta}$ such that

$$|x^{\alpha}|\left[\left|\partial^{\beta}u\right| + \left|\partial^{\beta}\partial_0 u\right|\right](t, x) \leq C_{\alpha,\beta}$$

on the slab. From this it follows that also f satisfies uniform Schwartz bounds. To show existence and uniqueness of solutions and that they depend continuously on initial data we use energy methods, where we use a certain type of energy for the PDE. Now we want to derive the basic energy inequality. Let

$$E = \frac{1}{2}\int_{\mathbb{R}^n} u^t A^0 u \, dx$$

If we differentiate we get

$$\partial_t E = \int_{\mathbb{R}^n} \frac{1}{2}\partial_t \left[u^t A^0 u\right] dx = \int_{\mathbb{R}^n} \frac{1}{2}\left(\partial_t u^t\right) A^0 u + \frac{1}{2} u^t \left(\partial_t A^0\right) u + \frac{1}{2} u^t A^0 \partial_t u \, dx$$

$$= \int_{\mathbb{R}^n} \frac{1}{2} u^t \left(\partial_t A^0\right) u + u^t A^0 \partial_t u \, dx,$$

where we used the symmetry of A^0 for the last term ($u^t A^0 \partial_t u = u^t(\partial_t u^t A^0)^t = (\partial_t u^t A^0 u)^t$). To handle the second term in the last expression, we use (2.1) and split up the time and spatial components to obtain

$$\int_{\mathbb{R}^n} u^t A^0 \partial_t u dx = \int_{\mathbb{R}^n} \left[-u^t A^i \partial_i u - u^t B u + u^t f\right] dx.$$

For the first term on the right hand side of the last equation we calculate, by use of partial integration, the properties of u due to the Schwartz bounds and also the symmetry of A^i

$$\begin{aligned}\int_{\mathbb{R}^n} u^t A^i \partial_i u \, dx &= -\int_{\mathbb{R}^n} \partial_i \left(u^t A^i\right) u \, dx \\ &= -\int_{\mathbb{R}^n} \left(\partial_i u^t\right) A^i u dx - \int_{\mathbb{R}^n} u^t \left(\partial_i A^i\right) u dx\end{aligned}$$

which simply is

$$\int_{\mathbb{R}^n} u^t A^i \partial_i u dx + \int_{\mathbb{R}^n} \left(\partial_i u^t\right) A^i u dx = -\int_{\mathbb{R}^n} u^t \left(\partial_i A^i\right) u dx$$

If we use again the symmetry of A^i, we can rewrite the first term in the last equation as

$$-\int_{\mathbb{R}^n} u^t A^i \partial_i u \, dx = \frac{1}{2}\int_{\mathbb{R}^n} u^t \left(\partial_i A^i\right) u \, dx$$

If we sum up the above terms we get

$$\begin{aligned}\partial_t E &= \frac{1}{2}\int_{\mathbb{R}^n} u^t \left(\partial_t A^0\right) u\, dx + \frac{1}{2}\int_{\mathbb{R}^n} u^t \left(\partial_i A^i\right) u\, dx - \int_{\mathbb{R}^n} u^t B u\, dx + \int_{\mathbb{R}^n} u^t f\, dx \\ &= \underbrace{\int_{\mathbb{R}^n} u^t \left(\frac{1}{2}\partial_t A^0 + \frac{1}{2}\partial_i A^i - B\right) u\, dx}_{\leq CE} + \int_{\mathbb{R}^n} u^t f\, dx\end{aligned}$$

The first term on the right-hand side is $\leq CE$ for some constant C, because $\partial_\mu A^\mu$ and B have an upper bound. A^0 is positive definite and has a positive lower bound. Due to the positive lower bound c_0 of A^0 we have: $c_0 \left|u\right|^2 \leq u^t A^0 u$. So we have an estimate for

$$\begin{aligned}\int_{\mathbb{R}^n} \left|u^t f\right| dx &\overset{(CSI)}{\leq} \left(\int_{\mathbb{R}^n} \left|u\right|^2 dx\right)^{1/2} \left\|f(t,.)\right\|_2 \\ &\leq C\left(\int_{\mathbb{R}^n} \left(u^t A^0 u\right) dx\right)^{1/2} \left\|f(t,.)\right\|_2 \\ &\leq C E^{1/2} \left\|f(t,.)\right\|_2\end{aligned}$$

where we used the Cauchy-Schwartz inequality (CSI). So we get

$$\partial_t E \leq CE + CE^{1/2} \left\|f(t,.)\right\|_2 . \tag{2.3}$$

C denotes a constant which can change from line to line, but should be clear from the context. To apply Grönwall's lemma to get a helpful estimation for the energy, we have to take a small detour. To be able to divide the last expression (2.3) by $E^{1/2}$ we define $E_\varepsilon = E + \epsilon$ where $\varepsilon > 0$. The above expression holds also for E_ε and we can divide by $E_\varepsilon^{1/2}$ to obtain

$$\partial_t E_\varepsilon^{1/2} \leq C E_\varepsilon^{1/2} + C\left\|f(t,.)\right\|_2 .$$

After integration we get (for $t \geq 0$)

$$E_\varepsilon^{1/2}(t) \leq E_\varepsilon^{1/2}(0) + C\int_0^t \left\|f(s,.)\right\|_2 ds + C\int_0^t E_\varepsilon^{1/2}(s) ds.$$

In this setup we can now apply Grönwall's lemma, where $G := E_\varepsilon^{1/2}(0) + C\int_0^t \left\|f(s,.)\right\|_2 ds$ and $k(t) = C$. So we obtain

$$E_\varepsilon^{1/2}(t) \leq \left(E_\varepsilon^{1/2}(0) + C\int_0^t \left\|f(s,.)\right\|_2 ds\right) e^{Ct}.$$

Letting $\varepsilon \to 0+$, we obtain

$$E(t) \leq \left(E^{1/2}(0) + C \int_0^t \|f(s,.)\|_2 \, ds \right) e^{Ct}. \tag{2.4}$$

This can be used for showing the uniqueness of the solution u of (2.1), (2.2) if one consider the difference of the energies of the two solutions.
It will turn out that we need estimations for a negative number of derivatives of the solution to handle the nonlinear problem of our modified equations. To achieve this we need:

Lemma 2.2.1. *Consider a solution to* (2.1), (2.2) *under the assumptions we mentioned above. We define*

$$E_k[u] = \frac{1}{2} \sum_{|\alpha| \leq k} \int_{\mathbb{R}^n} (\partial^\alpha u)^t A^0 \partial^\alpha u \, dx. \tag{2.5}$$

Then we have the following inequality

$$\partial_t E_k \leq C E_k + C E_k^{1/2} \|f\|_{H^k}, \tag{2.6}$$

where the constants depend on the bounds on A^μ, B and especially on c_0 (which is the uniform lower bound of A^0).

Proof. First we make an useful estimation, which we will also use for later proofs. A^0 has a positive lower bound c_0 which led to $x^t A^0 x \geq c_0 |x|^2 \quad \forall x \in \mathbb{R}^n$. Furthermore we can use this also for $\partial^\alpha u$, i.e from (2.5) we get due to the positive lower bound c_0 of A^0:

$$\frac{c_0}{2} \sum_{|\alpha| \leq k} \int_{\mathbb{R}^n} |\partial^\alpha u|^2 \, dx \leq E_k[\partial^\alpha u]$$

and from this, we obtain for some $C > 0$:

$$\|\partial^\alpha u\|_2 \leq C E_k^{1/2}[\partial^\alpha u].$$

Because of (2.3) which is $\partial_t E \leq CE + CE^{1/2} \|f(t,.)\|_2$ we get (2.6) for $E = E_0$. To derive the inequality as desired, we note that due to (2.1) we have

$$L\partial^\alpha u = \partial^\alpha f + [L, \partial^\alpha] u.$$

If we use the (desired) inequality (2.6) for $k = 0$ and the new expression for $L\partial^\alpha u$, we get

$$\partial_t E[\partial^\alpha u] \leq CE[\partial^\alpha u] + CE^{1/2}[\partial^\alpha u] \|\partial^\alpha f + [L, \partial^\alpha] u\|_2 \tag{2.7}$$

Let us make an estimation for the last term $\|[L,\partial^\alpha]\,u\|_2$ on the right side of (2.7) . For $|\alpha|\leq k$, we can calculate

$$\begin{aligned}[L,\partial^\alpha]\,u &= A^\mu\partial_\mu\partial^\alpha u + B\partial^\alpha u - \partial^\alpha\left(A^\mu\partial_\mu u + Bu\right)\\ &= A^\mu\partial_\mu\partial^\alpha u + B\partial^\alpha u - A^\mu\partial^\alpha\partial_\mu u - \left(\partial^\alpha A^\mu\right)\partial_\mu u - \left(\partial^\alpha B\right)u - B\partial^\alpha u\\ &= -\left(\partial^\alpha A^\mu\right)\partial_\mu u - \left(\partial^\alpha B\right)u\end{aligned}$$

Due to the fact that all the derivatives of A^μ and B are bounded, we can write

$$|[L,\partial^\alpha]\,u| \leq C\left(|u| + \sum_{|\alpha|=1}|\partial^\alpha u|\right)$$

From this by using the estimations for $|u|^2$ and $|\partial^\alpha u|^2$ at the beginning of the proof, we estimate

$$|[L,\partial^\alpha]\,u|^2 \leq C\left(|u|^2 + \sum_{\mu=0}^{n}|\partial_\mu u|^2\right) \leq C\left(u^tA^0u + \sum_{\mu=0}^{n}(\partial_\mu u)^tA^0\partial_\mu u\right)$$

If we integrate and take the square root we get on the left hand side the $\|.\|_2$-norm. On the right hand side, if we keep the definition of E_k in mind, we get $E_1[u]$:

$$\|[L,\partial^\alpha]\,u\|_2 \leq CE_1^{1/2}[u]$$

and for the higher order of E_k, i.e. E_k for $k\geq 1$ we obtain due to the positivity of the additional terms

$$CE_1^{1/2}[u] \leq CE_k^{1/2}[u]$$

Now we put the last results into (2.7) to get:

$$\begin{aligned}\partial_tE_k[u] &\leq C\sum_{|\alpha|\leq k}\partial_t\left(E[\partial^\alpha u]\right)\\ &\leq CE_k[u] + CE_k^{1/2}[u]\,\|f\|_{H^k} + CE_k^{1/2}[u]\underbrace{\|[L,\partial^\alpha]\,u\|_2}_{\leq CE_k^{1/2}[u]}\\ &\leq CE_k[u] + CE_k^{1/2}[u]\,\|f\|_{H^k}\end{aligned}$$

which yields the desired inequality. □

Corollary 2.2.1. . *Consider a solution to* (2.1), (2.2) *again under the assumptions mentioned above. For* $t \in [0, T]$ *we obtain*

$$E_k^{1/2}(t) \leq C \left[E_k^{1/2}(0) + \int_0^t \|f(s, .)\|_{H^k} \, ds \right]. \tag{2.8}$$

Where the constants C *depend on* k*, the bounds on* A^μ*,* B *and on* T*.*

Proof. Let $\mathcal{E}_k = e^{-Ct} E_k + \varepsilon$, where the constant C appears at first on the right-hand side of (2.6) and $\varepsilon > 0$. Then by differentiating $\mathcal{E}_k$ we get also one term containing $\partial_t E_k$. Here we can use (2.6) for a further estimation. Then we get through a direct calculation

$$\partial_t \mathcal{E}_k \leq C e^{Ct/2} \mathcal{E}_k^{1/2} \|f\|_{H^k}.$$

Since we allow that C depend on T on the right hand side of (2.8), we can bound $e^{Ct/2}$ by a constant. As we have done before, we divide by $\mathcal{E}_k^{1/2}$ (which is allowed due to $\mathcal{E}_k \geq \varepsilon > 0$ and integrate, we get

$$\mathcal{E}_k^{1/2}(t) \leq \mathcal{E}_k^{1/2}(0) + C \int_0^t \|f(s, .)\|_{H^k} \, ds.$$

For $\varepsilon \to 0$, we obtain the appropriate outcome, because there is only a constant factor between E_k and $\mathcal{E}_k$ due to the fact that the constants allowed to depend on T. □

Lemma 2.2.2. *Assume* u *is a solution to* (2.1)*, where we use the same setting as before, and* $k \in \mathbb{Z}$*. For* $t \in [0, T]$ *we then have*

$$\|u(t, .)\|_{(k)} \leq C \left[\|u(0, .)\|_{(k)} + \int_0^t \|f(s, .)\|_{(k)} \, ds \right]. \tag{2.9}$$

Proof. We know that (2.9) holds if $k \geq 0$. Therefore we assume now that k is a negative integer. Let $U(t, .) = (1 - \Delta)^k u(t, .) \ \ \forall t \in [0, T]$. Here we use U because this function is in the dual of $H^{(k)}$ (for more information see [1], p.39, Def.5.13). This leads to the fact that although we use negative integers for k, we can use for U our formulas which we have derived for positive integers. Because of our setup we know that U fulfills the uniform Schwartz bounds. So we get, by using (2.8):

$$\begin{aligned} \|u(t, .)\|_{(k)} = \|U(t, .)\|_{(-k)} &\leq C E_{-k}^{1/2}[U](t) \\ &\leq C \left[C E_{-k}^{1/2}[U](0) + \int_0^t \|LU(s, .)\|_{(-k)} \, ds \right] \\ &\leq C \left[\|u(0, .)\|_{(k)} + \int_0^t \|LU(s, .)\|_{(-k)} \, ds \right] \end{aligned}$$

Finally we have to estimate the last term on the right side. Therefore we formulate the equation in the following form

$$\begin{aligned} f = Lu = (1-\Delta)^{-k} LU + \left[L, (1-\Delta)^{-k}\right] U \\ \Rightarrow (1-\Delta)^{-k} LU = f - \left[L, (1-\Delta)^{-k}\right] U \end{aligned} \tag{2.10}$$

If we take the $\|.\|_{(k)}$-norm we get

$$\begin{aligned} \left\|(1-\Delta)^{-k} LU(t,.)\right\|_{(k)} &= \left\|f - \left[L, (1-\Delta)^{-k}\right] U(t,.)\right\|_{(k)} \\ &\leq \|f\|_{(k)} + \left\|\left[L, (1-\Delta)^{-k}\right] U(t,.)\right\|_{(k)} . \end{aligned}$$

Now we use the fact that $\left\|(1-\Delta)^{t/2} u\right\|_{(s-t)} = \|u\|_{(s)}$ and for $s = k$ and $t = 2k$ we get $\left\|(1-\Delta)^{k} u\right\|_{(-k)} = \|u\|_{(k)}$, also a correspondence between $H_{(k)}$ and its dual space $H_{(-k)}$. So we have

$$\|LU(t,.)\|_{(-k)} \leq \|f\|_{(k)} + \left\|\left[L, (1-\Delta)^{-k}\right] U(t,.)\right\|_{(k)}$$

To estimate the last term $\left[L, (1-\Delta)^{-k}\right]$, we split up the operator $L = A^{\mu}\partial_{\mu} + B$ into its time $(A^0\partial_0)$ and spatial $(A^j\partial_j)$, $j = 1, ..., n$ components.

$$\begin{aligned} &\left\|\left[A^j\partial_j, (1-\Delta)^{-k}\right] U(t,.)\right\|_{(k)} \\ &= \left\|(1-\Delta)^{k} \left[A^j\partial_j, (1-\Delta)^{-k}\right] U(t,.)\right\|_{(-k)} \\ &= \left\|(1-\Delta)^{k} \left(A^j\partial_j (1-\Delta)^{-k} U(t,.)\right) - A^j\partial_j U(t,.)\right\|_{(-k)} \\ &= \left\|\cancel{A^j\partial_j U(t,.)} - \cancel{A^j\partial_j U(t,.)} + P(\partial)(\partial_j (1-\Delta)^{-k} U(t,.))\right\|_{(-k)} \\ &\leq C \left\|(1-\Delta)^{-k} U(t,.)\right\|_{(k)} = C \left\|U(t,.)\right\|_{(-k)} \end{aligned} \tag{2.11}$$

for the last estimate we used ([1], Corollary 5.19, p42) about estimations of partial derivatives in Sobolev spaces. To use this corollary we used in the last calculations that $(1-\Delta)^k (f \cdot g) = \left((1-\Delta)^k f\right) \cdot g + P(\partial)(f)$, (for f and g proper differentiable) where the order of $P(\partial)$ is $2k-1$. To get this we can use the Leibniz rule where we see that there is only one term on which the highest order derivative acts on f, in all other cases at least one derivative acts on g. So $\mathrm{ord}(P(\partial)) \leq 2k-1$ together with ∂_{μ} we have the final order of $\leq 2k$. In the same way we proceed with the time derivative.

$$\begin{aligned} \left\|\left[A^0\partial_t, (1-\Delta)^{-k}\right] U(t,.)\right\|_{(k)} &= \left\|P(\partial)(\partial_t (1-\Delta)^{-k} U(t,.))\right\|_{(-k)} \\ &\leq C \left\|(1-\Delta)^{-k} \partial_t U(t,.)\right\|_{(k-1)} = C \left\|\partial_t U(t,.)\right\|_{(-k-1)} \end{aligned}$$

The term $\left[B,(1-\Delta)^{-k}\right]U$ can be estimated by a constant due to the boundedness of B and its derivatives. So we will take account of this and put this into the final constant of the estimation

$$\left\|\left[L,(1-\Delta)^{-k}\right]U(t,.)\right\|_{(k)} \leq C\left[\|U(t,.)\|_{(-k)} + \|\partial_t U(t,.)\|_{(-k-1)}\right].$$

To get an estimate for the last term $\|\partial_t U(t,.)\|_{(-k-1)}$, we define $L_0 u = (A^0)^{-1}Lu$ which means $L_0 u = \partial_t u + (A^0)^{-1}A^i\partial_i u + (A_0)^{-1}Bu$ and obtain (similar to (2.10))

$$(A^0)^{-1}f = L_0 u = (1-\Delta)^{-k}L_0 U(t,.) + \left[L_0,(1-\Delta)^{-k}\right]U(t,.). \tag{2.12}$$

Due to $(L_0-\partial_t) = (A^0)^{-1}A^i\partial_t + B$, we can apply ([1], p.40 Lemma 5.14) (and our duality property) to get

$$\left\|(1-\Delta)^{-k}\left(L_0-\partial_t\right)U(t,.)\right\|_{(k-1)} \leq C\|U(t,.)\|_{(-k)}. \tag{2.13}$$

Additionally we get for our new L_0 out of (2.11) the following useful estimate.

$$\begin{aligned} &\left[L_0,(1-\Delta)^{-k}U(t,.)\right] \\ &\quad= \underbrace{\left[\partial_t,(1-\Delta)^{-k}\right]U(t,.)}_{=0} - \left[(A^0)^{-1}A^i\partial_i,(1-\Delta)^{-k}\right]U(t,.) \\ &\quad\Rightarrow \left\|\left[L_0,(1-\Delta)^{-k}U(t,.)\right]\right\|_{k-1} \leq C\|U(t,.)\|_{k-1} \end{aligned} \tag{2.14}$$

Due to $(1-\Delta)^{-k}\partial_t U(t,.) = (A^0)^{-1}f-(1-\Delta)^{-k}\overbrace{((A^0)^{-1}A^i\partial_i U(t,.) + BU(t,.))}^{=(L_0-\partial_t)U(t,.)}-\left[L_0,(1-\Delta)^{-k}\right]$U,which we get directly from (2.12) we can make the following estimation

$$\begin{aligned} \|\partial_t U(t,.)\|_{(-k-1)} &= \left\|(1-\Delta)^{-k}\partial_t U(t,.)\right\|_{(k-1)} \\ &\leq \left\|(A^0)^{-1}f\right\|_{(k-1)} + \left\|(1-\Delta)^{-k}(L_0-\partial_t)U(t,.)\right\|_{(k-1)} \\ &\quad + \left\|[L_0,(1-\Delta)^{-k}]U(t,.)\right\|_{(k-1)} \\ &\leq \underbrace{C\|f(t,.)\|_{(k-1)}}_{\text{([1], p.41, Lemma 5.17)}} + \underbrace{C\|U(t,.)\|_{(-k)}}_{(2.13)} + \underbrace{C\|U(t,.)\|_{(-k+1)}}_{(2.14)} \\ &\leq C\|f(t,.)\|_{(k-1)} + C\|U(t,.)\|_{(-k)} \end{aligned}$$

If we subsume the last calculation we get, again by use of our duality property the following inequality

$$\|u(t,.)\|_{(k)} \leq C\left[\|u(0,.)\|_{(k)} + \int_0^t (\|u(s,.)\|_{(k)} + \|f(s,.)\|_{(k)})ds\right].$$

This gives the final result, by using Grönwall's lemma. $\square$

Corollary 2.2.2. *Assume that u is a solution of* (2.1) *with the same assumptions as in definition 2.1, furthermore let $k \in \mathbb{N}$. Then, for $t \in [0,T]$,*

$$\|u(t,.)\|_{(k)} \leq C\left[\|u(T,.)\|_{(k)} + \int_t^T \|f(s,.)\|_{(k)}\, ds\right]. \tag{2.15}$$

Proof. see [1], p.63

Proposition 2.2.1. ***(Uniqueness)*** *For $x_0 \in \mathbb{R}^n, r > 0, s_0 > 0$ and $T_1 < 0 < T_2$, we define*

$$C_{x_0,r,s_0,T_1,T_2} = \left\{(t,x) \in [T_1,T_2] \times \mathbb{R}^n : |t| < \frac{r}{s_0}, x \in B_{r-s_0|t|}(x_0)\right\}. \tag{2.16}$$

Assume A^μ, B to be maps from $\mathbb{R}^{n+1}$ to the set of real values $N \times N$ matrices $M_\mathbb{R}(N,N)$, with A^μ symmetric and C^1, B in C^0. Furthermore A^0 is positive definite on $[T_1,T_2] \times \mathbb{R}^n$ (where $[T_1,T_2]$ is compact) with constant positive lower bound and the matrices A^μ are all bounded on the same set and $f \in C(\mathbb{R}^{n+1}, \mathbb{R}^N)$.
Assume we have two C^1-solutions u_1, u_2 to (2.1), (2.2)*, defined on $(a,b)\times\mathbb{R}^n$ with $a < 0 < b$ and corresponding to the initial data u_{01} and u_{02}.*
Let $[T_1,T_2]$ be a compact subinterval of (a,b) with $T_1 \leq 0 \leq T_2$, then there is an $s_0 > 0$, which depends on the lower bound of A^0 and the upper bound of A^i in $[T_1,T_2]$ such that if $u_{01}(x) = u_{02}(x)$ for $x \in B_r(x_0)$, then $u_1(t,x) = u_2(t,x)$ for $(t,x) \in C_{x_0,r,s_0,T_1,T_2}$. Similarly, assume u to be a C^1-solution to (2.1), (2.2) *on $[T_1,T_2] \times \mathbb{R}^n$, $u_0(x) = 0$ for $x \in B_r(x_0)$ and $f(t,x) = 0$ for $(t,x) \in C_{x_0,r,s_0,T_1,T_2}$, then $u(t,x) = 0$ for $(t,x) \in C_{x_0,r,s_0,T_1,T_2}$.*

Remark. If the data coincide, then it follows from the proposition that also the solutions coincide where they are defined. Also if u_0 has compact support and for any $[T_1,T_2]$, there is a compact set K where $f(t,x) = 0$ for $t \in [T_1,T_2]$ and $x \notin K$, then there is a compact set K_1 such that $u(t,x) = 0$ for $t \in [T_1,T_2]$ and $x \notin K_1$.

Proof. We get the first statement from the second if we consider $u_1 - u_2$. Let us start with the following equation

$$\begin{aligned}
\partial_\alpha[e^{-kt}u^tA^\alpha u] &= \partial_\alpha(e^{-kt}u^t)A^\alpha u + e^{-kt}u^t\partial_\alpha A^\alpha u + e^{-kt}u^tA^\alpha\partial_\alpha u\\
&= -ke^{-kt}u^tA^0u + e^{-kt}\underbrace{\partial_0 u^tA^0u}_{u^tA^0\partial_0 u} + e^{-kt}\underbrace{\partial_j u^tA^ju}_{u^tA^j\partial_j u}\\
&\quad + e^{-kt}u^t\partial_\alpha A^\alpha u + e^{-kt}u^t(-Bu+f)\\
&= -ke^{-kt}u^tA^0u + e^{-kt}u^t\underbrace{A^\alpha\partial_\alpha u}_{=-Bu+f}\\
&\quad + e^{-kt}u^t\partial_\alpha A^\alpha u + e^{-kt}u^t(-Bu+f)\\
&= e^{-kt}u^t[-kA^0 + (\partial_\alpha A^\alpha) - 2B]u + 2e^{-kt}u^tf,
\end{aligned}$$

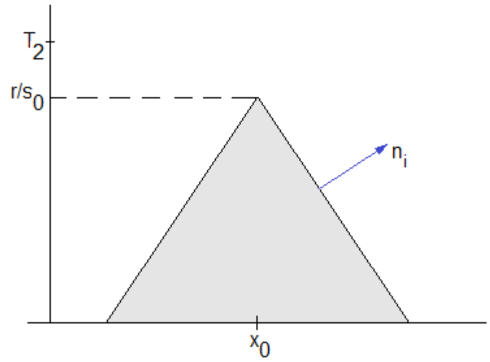
with k a constant. If we integrate this expression over $\mathcal{D} := C_{x_0,r,s_0,0,T_2}$, and use Stokes'theorem (see [1], p. 100, Lemma 10.8)

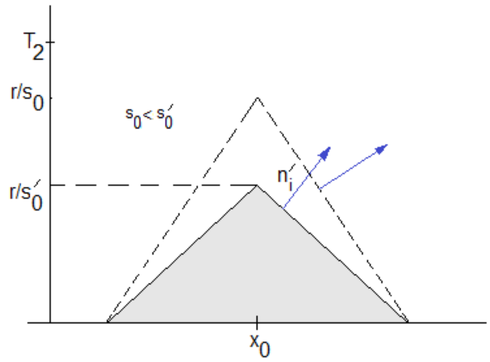
$$\int_M \mathrm{div}\xi\varepsilon_M = \int_{\partial M}\frac{\langle\xi,N\rangle}{\langle N,N\rangle}\varepsilon_{\partial M}$$

(with the Euclidean metric on $\mathbb{R}^{n+1}$), we get (by using for N the outwards pointing normal vector n_i to the respective plane)

$$\begin{aligned}
\int_{\mathcal{D}}\partial_\alpha[e^{-kt}u^tA^\alpha u]\,\varepsilon_{\mathcal{D}} &= \int_{\partial\mathcal{D}}\left\langle e^{-kt}u^tA^\alpha u, n_i\right\rangle\,\varepsilon_{\partial\mathcal{D}}\\
&= \int_{\partial\mathcal{D}} n_0\cdot e^{-kt}u^tA^\alpha u\,\varepsilon_{\partial\mathcal{D}} + \sum_{i=1}^n\int_{\partial\mathcal{D}} n_i\cdot e^{-kt}u^tA^\alpha u\,\varepsilon_{\partial\mathcal{D}}\\
&= e^{-kt}\left(\int_{\partial\mathcal{D}} n_0\cdot u^tA^\alpha u\,\varepsilon_{\partial\mathcal{D}} + \sum_{i=1}^n\int_{\partial\mathcal{D}} n_i\cdot u^tA^\alpha u\,\varepsilon_{\partial\mathcal{D}}\right)
\end{aligned}$$

Depending on the dimension we get at the end two or three terms. The part of the integrals over $\partial\mathcal{D}$ at $t = 0$ vanishes. Now we can adjust the shape of $\mathcal{D}$ by choosing s_0 to be sufficiently large, such that the outward normal n_α of the remaining surfaces guarantees that $n_\alpha A^\alpha$ is positive definite. Note that we can choose s_0 in this way, because it depends only on a lower bound of A^0 and an upper bound of A^i on $[T_1,T_2]\times\mathbb{R}^n$. Let us now have a look on $\mathcal{D}$ to visualize this. We start with an s_0 and increase afterwards s_0 to an s_0' with $s_0 < s_0'$. Now we fix this new s_0, then the boundary integrals are all non-negative. Assuming that, $f = 0$ is in $\mathcal{D}$. On the other hand, the integral





on the right hand side of the above equation can be estimated by

$$\int_{\mathcal{D}} e^{-kt} u^t [-kA^0 + (\partial_\alpha A^\alpha) - 2B]u + 2e^{-kt} u^t f \, dx < -c_0 \int_{\mathcal{D}} e^{-kt} u^t u \, dx$$

where $c_0 > 0$ is a constant and choosing k large enough which can be done due to the fact that the positive definite matrix A^0 has a positive uniform lower bound, whereas the other matrices have a uniform upper bound, because the region is bounded and the involved terms are continuous. So we have in our equation one non-negative and one non-positive term, which as a result gives that in fact both sides have to be zero. So we obtain that u is zero in $\mathcal{D}$. If we reverse the time, we get the argument for negative times. $\square$

Theorem 2.2.1. ***(Existence)*** *Let $S_T = [0, T] \times \mathbb{R}^n$, with $T > 0$. Consider the initial value problem* (2.1), (2.2)*, where $u_0 \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^N)$, $f \in C_0^\infty(\mathbb{R}^{n+1}, \mathbb{R}^N)$, A^μ, $B \in C^\infty(\mathbb{R}^{n+1}, M_{\mathbb{R}}(N, N))$, all of whose derivatives are bounded, with A^μ symmetric and A^0 positive definite with a uniform positive lower bound. Then there is a unique $u \in C^\infty([0, T) \times \mathbb{R}^n, \mathbb{R}^N)$ solving* (2.1), (2.2) *and a compact set $K \subset \mathbb{R}^n$ such that $u(t, x) = 0$ for $t \in [0, T]$ and $x \notin K$.*

Proof. The adjoint operator for L is

$$
\begin{aligned}
L^* &= -A^0\partial_t u - A^j\partial_j u + (-\partial_t A^0 - \partial_j A^j + B^t)u \\
&= -\partial_t(A^0 u) - \partial_j(A^j u) + B^t u.
\end{aligned}
$$

which we get by integration by part as follows:

$$
\begin{aligned}
\iint Lu \cdot \phi \, dtdx &= \iint \left(A^0\partial_t u + A^j\partial_j u + Bu\right) \cdot \phi \, dtdx \\
&= \iint \underbrace{(-\partial_t(A^0\phi) - \partial_j(A^j\phi) + B^t\phi)}_{L^*\phi} \cdot u \, dtdx
\end{aligned}
$$

We want to use the estimate (2.15) for an arbitrary $\phi \in C_0^\infty(\mathbb{R}^{n+1}, \mathbb{C}^N)$ where $\phi(t,x) = 0 \ \ \forall t \geq T$. Because $-L^*$ is of the same type as L we can write by using (2.15)

$$
\|\phi(t,.)\|_{(-k)} \leq C \int_t^T \|(L^*\phi)(s,.)\|_{(-k)} \, ds \tag{2.17}
$$

$\forall t \in [0,T]$. The first term of the right side of (2.15) vanishes due to the requirements made for ϕ. As a consequence it follows that $\phi(t,.) = \psi(t,.) \ \ \forall t \in [0,T]$ if $\phi, \psi \in C_0^\infty(\mathbb{R}^{n+1}, \mathbb{C}^N)$ both vanish for $t \geq T$ and $L^*\phi = L^*\psi$ holds. This is important for the next definition, where it guarantees that the functional only depends on $L^*\phi$ but not on ϕ itself. Given a ϕ as before and an $f \in \left\{L^1[0,T], H_{(k)}(\mathbb{R}^n, \mathbb{C}^N)\right\}$, we define

$$
F(L^*\phi) = \langle f, \phi \rangle := \int_0^T (\phi(t), f(t))_{L^2} dt.
$$

For more information see ([1], p 44, Proposition 5.21). Now we use (2.17):

$$
\begin{aligned}
|F(L^*\phi)| &\leq \int_0^T |(\phi(t), f(t))_{L^2}|\, dt \overset{[1],p42}{\leq} \int_0^T \|\phi(t)\|_{(-k)} \|f(t)\|_{(k)}\, dt \\
&\leq C \int_0^T \int_t^T \|L^*\phi(s,.)\|_{(-k)}\, ds \, \|f(t)\|_{(k)}\, dt \\
&\leq C \int_0^T \int_0^T \|L^*\phi(s,.)\|_{(-k)}\, ds \, \|f(t)\|_{(k)}\, dt \\
&\leq C \int_0^T \|L^*\phi(s,.)\|_{(-k)}\, ds \cdot \underbrace{\int_0^T \|f(t)\|_{(k)}\, dt}_{\leq const} \\
&\leq C \int_0^T \|L^*\phi(t,.)\|_{(-k)}\, dt
\end{aligned}
$$

which we can do because we change the lower integral boundary from t to 0, which isn't a problem (maybe the integral is bigger, but this doesn't matter). We can consider $L^*\phi \in X := L^1\left\{[0,T], H_{(-k)}(\mathbb{R}^n, \mathbb{C}^N)\right\}$. Let M be the subspace of X which is spanned by $L^*\phi$ for ϕ as before. Then we can consider F as a linear functional on M which is bounded.

What we have done so far was to prepare a setting which allows us now to use the Hahn-Banach theorem, which gives an extension of the bounded linear functional F to the whole space X (where $\|L^*\phi\||_X = \|L^*\phi\||_M$) and guarantees that the dual space is not empty. By the above duality statement (see ([1], p 44, Proposition 5.21)) we get a $u \in L^\infty\left\{[0,T], H_{(k)}(\mathbb{R}^n, \mathbb{C}^N)\right\}$ such that

$$
\int_0^T (\phi(t), f(t))_{L^2} dt = \int_0^T (L^*\phi(t,.), u(t))_{L^2} dt
$$

holds for all ϕ as above. This equation supplies the idea for the existence of a weak solution and so we get a hint for the needed next steps. As a first step let us take an $f \in C_0^\infty(\mathbb{R}^{n+1}, \mathbb{R}^N)$, such that $f(t,.) = 0 \ \ \forall t \leq 0$. Then u can be considered as an element of $L^\infty\left\{(-\infty, T], H_{(k)}(\mathbb{R}^n, \mathbb{C}^N)\right\}$ if it is zero for $t < 0$. So we get

$$
\int_{-\infty}^T (\phi(t), f(t))_{L^2} dt = \int_{-\infty}^T (L^*\phi(t,.), u(t))_{L^2} dt
$$

$\forall \phi \in C_0^\infty(\mathbb{R}^{n+1}, \mathbb{C}^N)$ with $\phi(t,.) = 0 \ \ \forall t \geq T$. Now we can apply ([1], p.271 Lemma A.5), which gives us for our $u \in L^\infty$ a k-times weakly differentiable

$U \in L^2_{loc}(I \times \mathbb{R}^n, \mathbb{C}^N)$ where all the derivatives are also in L^2_{loc} such that

$$\int_{-\infty}^{T} \int_{\mathbb{R}^n} \phi \cdot \overline{f} \, dxdt = \int_{-\infty}^{T} \int_{\mathbb{R}^n} L^* \phi \cdot \overline{U} \, dxdt \tag{2.18}$$

$\forall \phi \in C_0^\infty[(-\infty, T) \times \mathbb{R}^n, \mathbb{C}^N]$. So the lemma shifts the focus of the time variable t only of $u \in L^\infty[(-\infty, T], H_{(k)}(\mathbb{R}^n, \mathbb{C}^N)]$ to $U \in L^2_{loc}(I \times \mathbb{R}^n, \mathbb{C}^N)$ which evens out so to say, the significance of t and x.

What we want in the end is a smooth u, but up to now we only know there are weak spatial derivatives, we don't know whether u has also weak time derivatives. Therefore we make an assumption by induction that for $j + |\alpha| \leq k$ and $j \leq l \leq k-1$, there is a $U_{j,\alpha} \in L^2_{loc}((-\infty, T) \times \mathbb{R}^n, \mathbb{C}^N)$ such that

$$\int_{-\infty}^{T} \int_{\mathbb{R}^n} \phi \cdot \overline{U}_{j,\alpha} \, dxdt = (-1)^{j+|\alpha|} \int_{-\infty}^{T} \int_{\mathbb{R}^n} \partial_t^j \partial^\alpha \phi \cdot \overline{U} \, dxdt$$

$\forall \phi \in C_0^\infty[(-\infty, T) \times \mathbb{R}^n, \mathbb{C}^N]$ and $\partial_t^j \partial^\alpha U = U_{j,\alpha}$. For $l = 0$, the inductive statement is true (as we know from before). Because we want an additional time derivative (from l to $l+1$), we reformulate (2.18) into a convenient expression. For this we define $\psi = A^0 \phi$ and $g = (A^0)^{-1}[f - A^j \partial_j U - BU]$. Every $\psi \in C_0^\infty[(-\infty, T) \times \mathbb{R}^n, \mathbb{C}^N]$ can be written in this way, where ϕ and ψ are in the same class. So we get:

$$\int_{-\infty}^{T} \int_{\mathbb{R}^n} \psi \cdot \overline{g} \, dxdt = - \int_{-\infty}^{T} \int_{\mathbb{R}^n} \partial_t \psi \cdot \overline{U} \, dxdt. \tag{2.19}$$

This is indeed the same as (2.18). First for the expression under the left integral we have

$$\psi \cdot \overline{g} = \psi^T \overline{g} = (A^0 \phi)^T \overline{g} = \phi^T (A^0)^T \overline{g} = \phi^T A^0 \overline{g}.$$

If we use now our definition for g, we get

$$\phi^T A^0 \overline{g} = \phi^T A^0 \cdot (A^0)^{-1} [\overline{f} - A^j \partial_j \overline{U} - B\overline{U}] = \phi^T \overline{f} - \phi^T A^j \partial_j \overline{U} - \phi^T B \overline{U}$$

Now we put this into the following integral:

$$\int_{-\infty}^{T} \int_{\mathbb{R}^n} \phi \cdot \overline{f} - \phi \cdot A^j \partial_j \overline{U} - \phi \cdot B\overline{U} \, dxdt$$

$$\overset{(2.18)}{=} \int_{-\infty}^{T} \int_{\mathbb{R}^n} L^* \overline{U} - \phi A^j \partial_j \cdot \overline{U} - \phi B \cdot \overline{U} \, dxdt$$

By using the term for $L^*\overline{U}$, the symmetry of the A^j and partial integration we can cancel out the terms up to $\partial_t\psi$ which gives us at the end (2.19).

Let α be a multiindex and j be a non-negative integer with $|\alpha| + j \leq k - 1$ and $j \leq l$. Then there exists the weak derivative $\partial^\alpha \partial_t^j g$ and is contained in $L^2_{loc}[(-\infty, T) \times \mathbb{R}^n, \mathbb{C}^N]$. Therefore we can replace ψ (for any α with $|\alpha| \leq k - l - 1$) with $\partial_t^j \partial^\alpha \psi$. The integral on the left side can be replaced by an integral of the scalar product of ψ with a function, say $h \in L^2_{loc}[(-\infty, T) \times \mathbb{R}^n, \mathbb{C}^N]$.

$$\int_{-\infty}^{T} (\psi, h)_{L^2}\, dt = -\int_{-\infty}^{T} \int_{\mathbb{R}^n} \partial_t^{j+1} \partial^\alpha \psi \cdot \overline{U}\, dxdt.$$

So the induction from $l \to l+1$ holds and we obtain that U is k times weakly differentiable with respect to x and t in $(-\infty, T) \times \mathbb{R}^n$. From ([1], p.50, Lemma 6.7), which says for a $u \in L^2(\Omega)$ (where Ω is open in $\mathbb{R}^n$) which is l times weakly differentiable and the weak derivatives are in $L^2_{loc}(\Omega)$, that $u \in C^k(\Omega)$ if $l > k + n/2$, we see that U is also continuously differentiable. So (2.18) implies that $LU = f$. In addition, $U = 0$ for $t \leq 0$ holds. Up to now we only know that for each k there is a U but not that these $U's$ coincide. That this is indeed the case is a result of the uniqueness result (Proposition 2.1), as long the $U's$ are in C^1 and so for large enough k (which we can choose freely) we have a unique U. So we get a smooth solution.

Let now relax the conditions to $f \in C_0^\infty(\mathbb{R}^{n+1}, \mathbb{R}^N)$ in so far as we no longer require f to vanish for $t \leq 0$. For this we take a smooth function $\eta \in C^\infty(\mathbb{R}, \mathbb{R})$ whith $0 \leq \eta \leq 1$, $\eta(t) = 0$ for all $t \leq 0$ and $\eta(t) = 1$ for all $t \geq 0$. We define $f_\varepsilon(t, x) = \eta(t/\varepsilon) f(t, x)$. So we get for every $\varepsilon > 0$ a smooth solution u_ε for $Lu_\varepsilon = f_\varepsilon$ with $u_\varepsilon = 0\ \forall t \leq 0$. Again from our uniqueness result we get a compact set K such that $u_\varepsilon(t, x) = 0$ for $x \notin K$ and $t \leq T$ and $\forall \varepsilon > 0$. From the inequality (2.9) we get

$$\|(u_{\varepsilon_1} - u_{\varepsilon_2})(t, .)\|_{(k)} \leq C \int_0^t |\eta(s/\varepsilon_1) - \eta(s/\varepsilon_2)|\, \|f(s, .)\|_{(k)}\, ds.$$

So for any $t \in [0, T]$ the solution $u_\varepsilon(t, .)$ converges in any H^k-norm and for $\varepsilon \to 0+$ also in any C^k-norm (by the Sobolev embedding theorem). By using the equation in a recursive way, we obtain also convergence of any number of time derivatives (as long we are away from $t = 0$) and became a solution u on $(0, T) \times \mathbb{R}^n$ which is smooth. As stated in the theorem we have to extend now to $t = 0$. We define $u(0, .) = 0$ and again out of this, the higher derivatives are given recursively by the equation. If $\partial_t u$ converges, what should it converge to for $\varepsilon \to 0+$? If we use again the inequality (2.9), we get

$$\|(u_\varepsilon)(t,.)\|_{(k)} \le C \int_0^t \|f_\varepsilon(s,.)\|_{(k)}\; ds \le C \int_0^t \|f(s,.)\|_{(k)}\; ds.$$

This shows that due to the independence from ε on the right side, the inequality holds also for u. If we now choose k large enough (which we can do freely) and the H^k-norm dominates (by the Sobolev embedding theorem) the C^0-norm, we obtain that $u(t,.) \overset{\varepsilon\to 0+}{\to} 0$ in C^0 and also in every C^k-norm. So $\partial_t u$ converges properly in any H^k-norm and also for all higher time derivatives(which we get again recursively).
At this point we have proven the theorem for $u_0 = 0$. To get it for $u_0 \neq 0$ we only have to do a small modification. By considering the equation for $u - u_0\psi$, with a $\psi \in C_0^\infty(\mathbb{R})$, such that $\psi(t) = 1$ for $t \in [-1, T+1]$, we obtain a smooth solution, as far we additionally (to the assumptions for f) demand that $u_0 \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^N)$. □

Corollary 2.2.3. . *Let* $u_0 \in C^\infty(\mathbb{R}^n, \mathbb{R}^N)$, $f \in C^\infty(\mathbb{R}^{n+1}, \mathbb{R}^N)$ *and* A^μ, $B \in C^\infty(\mathbb{R}^{n+1}, M_{\mathbb{R}}(N,N))$. *Assume that* A^μ *are symmetric and that for any compact interval* $[T_1, T_2]$, *there are constants* a_0, $b_0 > 0$ *with* $A^0 \ge a_0$ *and* $\|A^\mu\| \le b_0$ *on* $[T_1,T_2] \times \mathbb{R}^n$. *Then there exists a unique solution* $u \in C^\infty(\mathbb{R}^{n+1}, \mathbb{R}^N)$, *to the initial value problem* (2.1),(2.2).

Proof. see [1], p.67

2.3 Linear wave equations

In the actual chapter we pay attention to the following equation

$$g^{\mu\nu}\partial_\mu\partial_\nu u + a^\mu \partial_\mu u + bu = f \tag{2.20}$$

where $g^{\mu\nu} \in C^\infty(\mathbb{R}^{n+1}, Symm_{\mathbb{R}}(n+1, n+1))$ such that $g^{00} < 0$ and g^{ij} are the components of a positive definite matrix. Additionally let be $a^\mu, b \in C^\infty(\mathbb{R}^{n+1}, M_{\mathbb{R}}(N,N))$ and $f \in C^\infty(\mathbb{R}^{n+1}, \mathbb{R}^N)$. Assume that the solution to the equation is $u \in C^\infty(\mathbb{R}^{n+1}, \mathbb{R}^N)$ with the property that for a compact interval $I \subseteq \mathbb{R}$, there exists a compact set $K_I \subset \mathbb{R}^n$ such that $u(t,x) = 0$ if $t \in I$ and $x \notin K_I$. With these conditions we can define the energy by

$$\mathcal{E} = \frac{1}{2}\int_{\mathbb{R}^n} [-g^{00}\,|\partial_t u|^2 + g^{ij}\partial_i u \cdot \partial_j u + |u|^2] dx. \tag{2.21}$$

Lemma 2.3.1. *By the conditions fixed above, let $g^{\mu\nu}$, its first derivatives, a^μ and b all have uniform bounds. Let g^{00} have a uniform negative upper bound and g^{ij} a uniform positive lower bound. Then there exists a constant C, which depends on the bounds, such that for $t \geq 0$, the energy from* (2.21) *satisfies*

$$\mathcal{E}^{1/2}(t) \leq \left(\mathcal{E}^{1/2}(0) + C \int_0^t \|f(s,.)\|_2 \, ds \right) e^{Ct}. \tag{2.22}$$

Remark. By the lemma we obtain that two solutions of (2.20) satisfying the above conditions coincide for all $t \geq 0$ as far the initial data of these solutions also coincide at $t = 0$, by considering the inequality obtained for the difference of the two solutions.

Proof. Let us first differentiate (2.21) with respect to time:

$$\begin{aligned} \partial_t \mathcal{E}^{1/2}(t) = \int_{\mathbb{R}^n} \Big[& -\frac{1}{2}(\partial_t g^{00}) \left|\partial_t u\right|^2 - g^{00} \partial_t \cdot \partial_t^2 u \\ & + \frac{1}{2}(\partial_t g^{ij}) \partial_i u \cdot \partial_j u + g^{ij} \partial_i u \cdot (\partial_j \partial_t u) + u \cdot \partial_t u \Big] \, dx \end{aligned} \tag{2.23}$$

The first, third and fifth terms can be bounded in terms of the energy. Due to the respective bounds for g^{00} and g^{ij} we get a uniform constant C (which can for convenience change from line to line) such that the following inequality

$$\int_{\mathbb{R}^n} \left[-\frac{1}{2}(\partial_t g^{00}) \left|\partial_t u\right|^2 + \frac{1}{2}(\partial_t g^{ij}) \partial_i u \cdot \partial_j u + u \cdot \partial_t u \right] dx \leq C\mathcal{E} \tag{2.24}$$

holds. For the remaining terms, we have

$$\begin{aligned} & \int_{\mathbb{R}^n} g^{ij} \partial_i u \cdot (\partial_j \partial_t u) dx \\ & = \int_{\mathbb{R}^n} [\partial_j (g^{ij} \partial_i u \cdot \partial_t u) - (\partial_j g^{ij}) \partial_i u \cdot \partial_t u - g^{ij} \partial_i \partial_j u \cdot \partial_t u] dx \\ & = - \int_{\mathbb{R}^n} [(\partial_j g^{ij}) \partial_i u \cdot \partial_t u + g^{ij} \partial_i \partial_j u \cdot \partial_t u] dx \end{aligned}$$

Now in this form we can estimate the integrand of the last line as in (2.24). If we combine these estimates and subtract the remaining terms, resp. what is left of these, then we get

$$\partial_t \mathcal{E} \leq C\mathcal{E} - \int_{\mathbb{R}^n} (g^{00} \partial_t^2 u + g^{ij} \partial_i \partial_j u) \cdot \partial_t u \, dx.$$

If we have all the entries of $g^{\mu\nu}$ in mind, we see that there is a missing part, namely the entries for g^{0i}. So we have also to take the term $-2g^{0i}\partial_i\partial_t u \cdot \partial_t u$ (the factor 2 comes from the symmetry of $g^{\mu\nu}$) into account.

$$\int_{\mathbb{R}^n} 2g^{0i}\partial_i\partial_t u \cdot \partial_t u \, dx = \int_{\mathbb{R}^n} g^{0i}\partial_i(|\partial_t u|^2) \, dx = -\int_{\mathbb{R}^n} (\partial_i g^{0i}) \left|\partial_t u\right|^2 \, dx$$

Here we used integration by parts and due to the boundedness of the first derivatives of the metric we can again estimate the right-side by $C\mathcal{E}$. Now we are able to merge the different parts of the metric and get

$$\partial_t \mathcal{E} \leq C\mathcal{E} - \int_{\mathbb{R}^n} g^{\mu\nu}\partial_\mu\partial_\nu u \cdot \partial_t u \, dx.$$

From this point we can use the equation and reformulate it to $g^{\mu\nu}\partial_\mu\partial_\nu u = -a^\mu\partial_\mu u - bu + f$. We assume that a^μ and b are bounded and so we can insert the estimate for $-(a^\mu\partial_\mu u + bu)\cdot\partial_t u$ into the term $C\mathcal{E}$ and the remaining part $f \cdot \partial_t u$ can also be estimated due to the negative upper bound of g^{00}. So we have: $g^{00}\left|\partial_t u\right|^2 \leq (\partial_t u)^t g^{00}\partial_t u$ and (take care of the minus) get

$$\begin{aligned}
\int_{\mathbb{R}^n} |\partial_t u \cdot f| \, dx &\overset{(CSI)}{\leq} \left(\int_{\mathbb{R}^n} |\partial_t u|^2 \, dx\right)^{1/2} \|f(t,.)\|_2 \\
&\leq C\left(\int_{\mathbb{R}^n} \left(\partial_t u)^t g^{00}\partial_t u\right) dx\right)^{1/2} \|f(t,.)\|_2 \\
&\leq C\mathcal{E}^{1/2} \|f(t,.)\|_2 \, .
\end{aligned}$$

Therefore we get

$$\partial_t \mathcal{E} \leq C\mathcal{E} + C\mathcal{E}^{1/2} \|f(t,.)\|_2 \, .$$

To get a more convenient form we introduce $\mathcal{E}_\varepsilon = \mathcal{E} + \varepsilon$ (for $\varepsilon > 0$) because we want to divide by $\sqrt{\mathcal{E}_\varepsilon}$. Indeed we get the same estimate for $\mathcal{E}_\varepsilon$ and can now divide by $\sqrt{\mathcal{E}_\varepsilon}$ to arrive at

$$\partial_t \mathcal{E}_\varepsilon^{1/2} \leq C\mathcal{E}_\varepsilon^{1/2} + C\|f(t,.)\|_2 \, .$$

By integration (fundamental theorem) we obtain, for $t \geq 0$

$$\mathcal{E}_\varepsilon^{1/2}(t) \leq \mathcal{E}_\varepsilon^{1/2}(0) + C\int_0^t \|f(s,.)\|_2 \, ds + C\int_0^t \mathcal{E}_\varepsilon^{1/2}(s) \, ds$$

By using Grönwall's lemma, we get the form

$$\mathcal{E}_\varepsilon^{1/2}(t) \leq \left(\mathcal{E}_\varepsilon^{1/2}(0) + C\int_0^t \|f(s,.)\|_2 \, ds\right) e^{Ct}.$$

We obtain the final statement by letting $\varepsilon \to 0+$. $\square$

In our case it is natural to use a Lorentz metric, which in turn entails certain restriction on the different entries $g^{\mu\nu}$. In the end we will consider components of the inverse of a Lorentz metric. In the following we are concerned with this topic, respective the properties of such an inverse.

2.4 Some Linear Algebra

Let $g \in Symm_{\mathbb{R}}(n+1)$ with components $g_{\mu\nu},\ \mu,\ \nu = 0, ..., n$. Let the submatrix of g be $g_\flat \in M_{\mathbb{R}}(n)$ with components $g_{ij},\ i,\ j = 1, ..., n$. If g is invertible, the components of the inverse are denoted by $g^{\mu\nu}$ and the components for the inverse of $g_\flat$ (which we call $g^\sharp$) are g^{ij}. The vector with components g_{0i} shall be denoted by $v[g]$.

For a symmetric, positive definite $n \times n$-matrix ξ and an n-dim. vector v, we write

$$|v|_\xi = \left(\sum_{i,j=1}^{n} \xi_{ij} v^i v^j \right)^{1/2}.$$

In this sense it follows that $|v| = |v|_\delta$ with δ_{ij} the Kronecker delta.
If $A \in M_{\mathbb{R}}(n+1, n+1)$ (not required to be symmetric), we denote the $(n+1)\times(n+1)-$dimensional matrix as M_A with the components $(M_A)_{00} = 1$ and $(M_A)_{0i}, (M_A)_{i0} = 0$ where the remaining ij components are given by A.

$$M_A = \begin{pmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & A_{ij} \\ 0 & \end{pmatrix}$$

If now $\rho \in Symm_{\mathbb{R}}(n+1)$ has one negative eigenvalue and n positive ones, we say that it is a Lorentz matrix.

Lemma 2.4.1. *Let $\rho \in Symm_{\mathbb{R}}(n+1)$. Assume that $\rho_{00} < 0$ and $\rho_\flat$ is positive definite. Then ρ is a Lorentz matrix.*

Proof. Pick $A \in O(n)$ which diagonalizes the submatrix $\rho_\flat$. Furthermore let $h = M_A^t \rho M_A$. Then the components $h_{00} = \rho_{00} < 0$, which can easily be calculated and $h_\flat = A^t \rho_\flat A$ is a diagonal submatrix which have positive entries λ_i. Furthermore the eigenvalues of ρ and h are the same. Now we want to calculate the determinant of $h - \lambda\mathbb{I}$ with the diagonal submatrix $h_\flat = A^t \rho_\flat A$. To do this we use the determinant expansion theorem of Laplace, where we in turn can use the nice properties of the diagonal matrices. For

$$h = \begin{pmatrix} h_{00} & \cdots & h_{0n} \\ \vdots & & h_\flat \\ h_{n0} & & \end{pmatrix}$$

we get

$$p(\lambda) = \underbrace{\left(\rho_{00} - \lambda - \frac{h_{01}^2}{\lambda_1 - \lambda} - \ldots - \frac{h_{0n}^2}{\lambda_n - \lambda}\right)}_{=:f(\lambda)}(\lambda_1 - \lambda)\ldots(\lambda_n - \lambda).$$

For $f(\lambda)$ we obtain

$$f'(\lambda) = -1 - \frac{h_{01}^2}{(\lambda_1 - \lambda)^2} - \ldots - \frac{h_{0n}^2}{(\lambda_n - \lambda)^2}.$$

Note that due to the positive definiteness all $\lambda_1, \ldots, \lambda_n$ are positive. Let us define the smallest $\lambda_i := \lambda_{min}$. Now we look at the interval $(-\infty, \lambda_{min})$ (λ_{min} is not an argument of $f(\lambda)$). In this interval $f'(\lambda) < 0$ holds. Due to the continuity of $f(\lambda)$, because of $f(-\infty) = \infty$ and $f(0) < 0$ and the fact that $f(\lambda)$ is strongly decreasing, we see that there is a unique negative value of λ, let us call it λ_0, for which $f(\lambda) = 0$. This is also an eigenvalue of ρ. By differentiating $p(\lambda)$ (product rule to $p(\lambda) = f(\lambda)(\lambda_1 - \lambda)\ldots(\lambda_n - \lambda)$ and set $\lambda = \lambda_0$) we get $p'(\lambda_0) \neq 0$ (remember that $\rho_{00} < 0$).
Therefore λ_0 is a root with multiplicity one of the polynomial $p(\lambda) = 0$ and so we have only one eigenvalue within $(-\infty, \lambda_{min})$.The remaining n eigenvalues are positive, because of ρ is a symmetric matrix and therefore has only real eigenvalues. □

Definition 2.4.1. *A canonical Lorentz matrix is a symmetric* $(n+1)\times(n+1)$*-dimensional real-valued matrix* g *with components* $g_{\mu\nu}$ *such that* $g_{00} < 0$ *and* $g_\flat > 0$. $\mathcal{C}_n$ *denotes the set of* $(n+1)\times(n+1)$*-dimensional canonical Lorentz matrices. Let* $a_1, a_2, a_3 \in \mathbb{R}$ *and* $a = (a_1, a_2, a_3)$*, then we define* $\mathcal{C}_{n,a}$ *(which is a subset of* $\mathcal{C}_n$*) to consist of all matrices* g *with*

$$g_{00} \leq -a_1, \qquad g_\flat \geq a_2, \qquad \sum_{\mu,\nu=0}^{n} |g_{\mu\nu}| \leq a_3$$

By $a > 0$*, we mean that* a *contains only strictly positive components.*

Lemma 2.4.2. *Let $g \in \mathcal{C}_n$. Then g is a Lorentz matrix,*

$$g^{00} = \frac{1}{g_{00} - d^2} \tag{2.25}$$

where $d = |v[g]|_{g_\flat^{-1}}$, $g^\sharp$ is positive definite, with

$$\frac{g_{00}}{g_{00} - d^2} |w|^2_{g_\flat^{-1}} \leq |w|^2_{g^\sharp} \leq |w|^2_{g_\flat^{-1}} \tag{2.26}$$

for any $w \in \mathbb{R}^n$ and

$$v[g^{-1}] = \frac{1}{d^2 - g_{00}} g_\flat^{-1} v[g]. \tag{2.27}$$

Note that $g^{00} < 0$ since $g_{00} < 0$ and that there is an upper bound which depends only on g_{00} and d. Especially $g^{-1} \in \mathcal{C}_n$.

Proof. That g is a Lorentz matrix follow from the last lemma. Assume $A = \sqrt{g_\flat^{-1}}$, which is positive definite and symmetric. Then we have

$$A^t g_\flat A = \left(\sqrt{g_\flat^{-1}}\right)^t \cdot g_\flat \cdot \sqrt{g_\flat^{-1}} = \left(\sqrt{g_\flat^{-1}} \cdot g_\flat\right)^t \cdot \sqrt{g_\flat^{-1}} = g_\flat \cdot g_\flat^{-1} = Id.$$

Therefore we get for the h which we defined in the last proof $h_{00} = g_{00}$, $h_\flat = Id$ and $v[h] = A^t v[g]$. Now we choose a $B \in O(n)$ in such a way that the following holds: $B^t A^t v[g] = |A^t v[g]| \, e_1$, with $e_1 = (1, 0, \ldots, 0)^t$. For d we get

$$d = |v[g]|_{g_\flat^{-1}} = \left|A^t v[g]\right|.$$

If we now define $\rho = M_B^t M_A^t g M_A M_B$, we get again $\rho_{00} = g_{00}$, $\rho_\flat = Id$ and $v[h] = de_1$, due to the special structure of our matrices (M_A, M_B and $A^t g_\flat A = Id$) which ensures the work with proper submatrices. Note that the matrices preserve especially the 00-component.

The inverse of the 2×2-matrix with components $\rho_{\mu\nu}$ ($\mu, \nu = 0, 1$) is

$$\frac{1}{g_{00} - d^2} \begin{pmatrix} 1 & -d \\ -d & g_{00} \end{pmatrix},$$

and we can directly read off the g^{00} component due to

$$g^{-1} = M_A M_B \rho^{-1} M_B^t M_A^t,$$

and the fact that M_A and M_B preserve the 00-component, which was our claim for (2.25). From the form of the M_A, M_B matrices we get for the inverse components of the $n \times n$-matrix g^{ij} the expression $g^\sharp = AB\rho^\sharp B^t A^t$. To get an estimate as we have in (2.26), we look for $w \neq 0$ at

$$\frac{|w|^2_{g^\sharp}}{|w|^2_{g_\flat^{-1}}} = \frac{(g^{\sharp w}, w)}{(Aw, Aw)} = \frac{(\rho^\sharp B^t A^t w, B^t A^t w)}{(B^t A^t w, B^t A^t w)},$$

where we have first used the definition for $|w|_g$ and the definition for $A^2 = g_\flat^{-1}$ and secondly that B is orthogonal and A is symmetric. Now, after this we have to examine $\rho^\sharp$. $\rho^\sharp$ is diagonal and as we can see from above for the inverse the 11-component is $g_{00}/(g_{00} - d^2) \leq 1$, whereas the ii-components are one for $i > 1$. With this we directly get (2.26).

Again from the inverse and the explanation from the orthogonal matrix B we get the expression for

$$v[\rho^{-1}] = -\frac{d}{g_{00} - d^2} e_1 \quad \text{and} \quad dBe_1 = A^t v[g].$$

So we get (2.27) by

$$v[\rho^{-1}] = ABv[\rho^{-1}] = -\frac{1}{g_{00} - d^2} A^2 v[g] = -\frac{1}{g_{00} - d^2} g_\flat^{-1} v[g].$$

$\square$

2.5 Existence of solutions to linear wave equations

The idea here is to formulate linear wave equation in a very general form and sort the respective parts (which turns out to be possible due to our previous linear algebra considerations) in such a manner that at the end we get a symmetric hyperbolic system, which as we know have a unique solution. To begin with we have to do some sorting work to bring our objects into a convenient shape.

Let $g_I \in C^\infty(\mathbb{R}^{n+1}, \mathcal{C}_n)$, $I = 1, ..., N$ (also a Lorentz matrix-valued function), with components $g_{I\mu\nu}$ (usefully the components of the inverse are denoted by $g_I^{\mu\nu}$). Let $a = (a_1, a_2, a_3)$, with $a_i > 0$, $i = 1, 2, 3$ constants for any compact interval $[T_1, T_2]$, in such a way that $g_I(t, x) \in \mathcal{C}_{n,a}$ for all (t, x) in the slab $[T_1, T_2] \times \mathbb{R}^n$. Assume that $b_I^{J\alpha}, c_J^I, f^I \in C^\infty(\mathbb{R}^{n+1})$ with $J = 1, ..., N$ and $\alpha = 0, ..., n$. Furthermore $u_0^I, u_1^I \in C^\infty(\mathbb{R}^n)$. Now we define the initial value problem which we are interested in:

$$g_I^{\mu\nu}\partial_\mu\partial_\nu u^I + b_J^{I\alpha}\partial_\alpha u^J + c_J^I u^J = f^I \tag{2.28}$$

$$u^I(0, .) = u_0^I \tag{2.29}$$

$$\partial_t u^I(0, .) = u_1^I, \tag{2.30}$$

where we sum over all indices except I. From the linear algebra consideration we know now that $g_I^\sharp$ is positive definite and $g_I^{00} < 0$. Additionally we have a constant vector $b = (b_1, b_2, b_3)$, where the components $b_i > 0, i = 1, 2, 3$ for any compact interval $[T_1, T_2]$, such that $g_I^{-1} \in \mathcal{C}_{n,b}$ for all (t, x) in the slab $[T_1, T_2] \times \mathbb{R}^n$. When convenient, we divide (2.28) by $-g_I^{00}$ to get $g_I^{00} = -1$. Assume

$$v^I = (\partial_1 u^I, ..., \partial_n u^I, \partial_t u^I, u^I) = (v_1^I, ..., v_{n+2}^I). \tag{2.31}$$

Let us now define the following $(n+2)\times(n+2)$ symmetric matrices A^{I0}, A^{Ik}. For A^{I0} with

$$A_{ij}^{I0} = g_I^{ij} \quad \text{and} \quad A_{n+1,n+1}^{I0} = A_{n+2,n+2}^{I0} = 1$$

and the remaining components are zero we get

$$
\begin{aligned}
A^{I0} &= \begin{pmatrix} A^{I0}_{11} & \dots & A^{I0}_{1,n+1} & A^{I0}_{1,n+2} \\ \vdots & \ddots & \vdots & \vdots \\ A^{I0}_{n+1,1} & \dots & A^{I0}_{n+1,n+1} & A^{I0}_{n+1,n+2} \\ A^{I0}_{n+2,1} & \dots & A^{I0}_{n+2,n+1} & A^{I0}_{n+2,n+2} \end{pmatrix} \\
&= \begin{pmatrix} & & & 0 & 0 \\ & g_I^{ij} & & \vdots & \vdots \\ & & & 1 & 0 \\ 0 & \dots & 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

And for A^{Ik} with

$$A^{Ik}_{n+1,i} = A^{Ik}_{i,n+1} = g_I^{ik} \quad \text{and} \quad A^{Ik}_{n+1,n+1} = 2g_I^{0k}$$

and the remaining components are zero we get

$$
A^{Ik} = \begin{array}{cc} & \begin{array}{cccc} \phantom{A^{Ik}_{n+1,1}} & & {\scriptstyle A^{Ik}_{i,n+1}} \atop \downarrow & \phantom{A^{Ik}_{n+1,n+2}} \end{array} \\ \begin{array}{c} \\ \\ {\scriptstyle A^{Ik}_{n+1,i}} \rightarrow \\ \\ \end{array} & \begin{pmatrix} A^{Ik}_{11} & \dots & A^{Ik}_{1,n+1} & A^{Ik}_{1,n+2} \\ \vdots & \ddots & \vdots & \vdots \\ A^{Ik}_{n+1,1} & \dots & A^{Ik}_{n+1,n+1} & A^{Ik}_{n+1,n+2} \\ A^{Ik}_{n+2,1} & \dots & A^{Ik}_{n+2,n+1} & A^{Ik}_{n+2,n+2} \end{pmatrix} \end{array}
$$

$$
= \begin{array}{cc} & \begin{array}{cccc} & & {\scriptstyle A^{Ik}_{i,n+1}} \atop \downarrow & \end{array} \\ \begin{array}{c} \\ \\ {\scriptstyle A^{Ik}_{n+1,i}} \rightarrow \\ \\ \end{array} & \begin{pmatrix} 0 & \dots & g_I^{1k} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ g_I^{1k} & \dots & 2g_I^{0k} & 0 \\ 0 & \dots & 0 & 0 \end{pmatrix} \end{array}
$$

Furthermore let

$$
\begin{gathered}
d^I_{Jn+1,i} = -b^{Ii}_J, \;\; d^I_{Jn+1,n+1} = -b^{I0}_J, \\
d^I_{Jn+1,n+2} = -c^I_J, \;\; d^I_{Jn+2,n+1} = -\delta^I_J, \;\; h^I_{n+1} = -f^I
\end{gathered}
$$

and the remaining components are zero. So we get

$$d^I_J = \begin{array}{r} \\ \\ d^I_{Jn+1,i}\rightarrow \\ \\ \end{array} \begin{pmatrix} d^I_{J1,1} & \dots & d^I_{J1,n+1} & d^I_{J1,n+2} \\ \vdots & \ddots & \vdots & \vdots \\ d^I_{Jn+1,1} & \dots & d^I_{Jn+1,n+1} & d^I_{Jn+1,n+2} \\ d^I_{Jn+2,1} & \dots & d^I_{Jn+2,n+1} & d^I_{Jn+2,n+2} \end{pmatrix}$$

$$= \begin{array}{r} \\ \\ d^I_{Jn+1,i}\rightarrow \\ \\ \end{array} \begin{pmatrix} 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ -b^{I1}_J & \dots & -b^{I0}_J & -c^I_J \\ 0 & \dots & -\delta^I_J & 0 \end{pmatrix}$$

So d^I_J is a smooth function into the set of $(n+2)\times(n+2)$-matrices and h^I is also a smooth function into $\mathbb{R}^{n+2}$. Assume u satisfies (2.28) and v is given by (2.31), then we have

$$A^{I0}\partial_t v^I - A^{Ik}\partial_k v^I + \sum_J d^I_J v^J = h^I. \tag{2.32}$$

If all indexed v^I are put into one vector v and if we have a look at the structure of the matrices A^{Ik} and A^{I0} (symmetric, positive definite), we see that the equation is symmetric hyperbolic. Additionally if we put all A^{I0} into one matrix A^0, we see that for every compact interval $[T_1, T_2]$ there is a lower constant bound, i.e. $A^0 \geq a_0$ on $[T_1, T_2]\times\mathbb{R}^n$. This also holds for the A^k on the same slab.

Let u^I be a solutions to (2.28) – (2.30). If v^I is defined as in (2.31), we get a solution for the symmetric hyperbolic system (2.32) in such a way that the initial data are

$$\partial_i v^I_{n+2}(0,.) = v^I_i(0,.) \tag{2.33}$$

for $i = 1, ..., n$, which we get by differentiating the relevant component in (2.31). If we have a solution to (2.32) which is smooth and where the initial data (2.33) are satisfied we get from (2.31)

$$\begin{aligned} \partial_t v^I_j &= \partial_t\partial_j u^I = \partial_j\partial_t u^I = \partial_j v^I_{n+1}, \\ \partial_t v^I_{n+2} &= \partial_t u^I = v^I_{n+1}, \end{aligned} \tag{2.34}$$

and from (2.32) we get $\partial_i u^I(0,.) = v_i^I(0,.)$. By using (2.34), we get

$$\begin{aligned}\partial_i u^I(t,x) &= v_i^I(0,.) + \int_0^t \partial_i\partial_t v_{n+2}^I(\tau,x)\,d\tau \\ &= v_i^I(0,.) + \int_0^t \partial_t v_i^I(\tau,x)\,d\tau = v_i^I(t,x).\end{aligned}$$

and from this it follows that v^I is of the form of (2.32). So u^I constitutes a solution for (2.28) – (2.30) with $u^I(0,x) = v_{n+2}^I(0,x)$ and $\partial_t u^I(0,x) = v_{n+1}^I(0,x)$. □

From Proposition 2.2.1 and the Remark following it and from Corollary 2.2.3, we get now (under consideration of the above) the next theorem.

Theorem 2.5.1. *Assume $J, I = 1, ..., N$, $\alpha = 0, ..., n$ and $i = 1, 2, 3$. Let $g_I \in C^\infty(\mathbb{R}^{n+1}, \mathcal{C}_n)$, with components $g_{I\mu\nu}$. Assume that for every compact interval $[T_1, T_2]$, there are positive constants a_i, such that $g_I(t,x) \in \mathcal{C}_{n,a}$, with $a = (a_1, a_2, a_3)$, $\forall (t,x) \in [T_1, T_2] \times \mathbb{R}^n$. The components of the inverse are denoted by $g_I^{\mu\nu}$. Assume that $b_I^{J\alpha}, c_J^I, f^I \in C^\infty(\mathbb{R}^{n+1})$ and that $u_0^I, u_1^I \in C^\infty(\mathbb{R}^n)$. Then we have an unique solution $u \in C^\infty(\mathbb{R}^{n+1}, \mathbb{R}^N)$ for (2.28) – (2.30). If the initial data have compact support and $f^I(t,x) = 0$ for $t \in [T_1, T_2]$ (where $T_1 < 0 < T_2$) and $x \notin K_1$ (where K_1 is compact), then there is a compact set K such that $u(t,x) = 0$ for $t \in [T_1, T_2]$ if $x \notin K$.*

We also have an uniqueness statement of greater generality, which follows from Proposition 2.2.1.

Proposition 2.5.1. *Assume $J, I = 1, ..., N$, $\alpha = 0, ..., n$ and $i = 1, 2, 3$. Let $g_I \in C^1(\mathbb{R}^{n+1}, \mathcal{C}_n)$, with components $g_{I\mu\nu}$. Assume that for every compact interval $[T_1, T_2]$, there are constants $a_i > 0$, such that $g_I(t,x) \in \mathcal{C}_{n,a}$, with $a = (a_1, a_2, a_3)$, $\forall (t,x) \in [T_1, T_2] \times \mathbb{R}^n$. The components of the inverse are denoted by $g_I^{\mu\nu}$. Assume that $b_I^{J\alpha}, c_J^I, f^I \in C(\mathbb{R}^{n+1})$ and that $u_0^I, u_1^I \in C^\infty(\mathbb{R}^n)$. Then, given a compact interval $[T_1, T_2]$ (with $T_1 \le 0 \le T_2$), there exists an $s_0 > 0$ such that the following holds: if $u \in C^2$ is a solution of (2.28), the initial data vanish inside $B_r(x_0)$ and $f^I(t,x) = 0$ for $(t,x) \in C_{x_0,r,s_0,T_1,T_2}$, then $u(t,x) = 0$ in C_{x_0,r,s_0,T_1,T_2}.*

2.6 Local existence, non-linear wave equations

We start with some definitions, which we will later frequently use.

Definition 2.6.1. *Let $1 \leq m, n \in \mathbb{Z}$. A function $h : \mathbb{R}^{n+1} \to \mathbb{R}^m$ is said to be of locally x-compact support if for any compact interval $[T_1, T_2]$ there is a compact set $K \subset \mathbb{R}^n$, such that $h(t,x) = 0$ if $t \in [T_1, T_2]$ and $x \notin K$.*

This notion is important because a smooth function of locally x-compact support $u : \mathbb{R}^{n+1} \to \mathbb{R}$ can be viewed as an element of $C^l(\mathbb{R}, H^k(\mathbb{R}^n, \mathbb{R}^m))$ for any k, l and that is why we will use this concept often later on. Note that this is not the case if we consider a function $u \in C^\infty$ where $u(t,.)$ has compact support for every fixed t.

Definition 2.6.2. *Let $1 \leq n, N \in \mathbb{Z}$ and $0 \leq k \in \mathbb{Z}$ or $k = \infty$. Let $g \in C^k(\mathbb{R}^{nN+2N+n+1}, \mathcal{C}_n)$ and $f \in C^k(\mathbb{R}^{nN+2N+n+1}, \mathbb{R}^N)$. If*

- *for every multiindex $\alpha = (\alpha_1, ..., \alpha_{nN+2N+n+1})$ such that $|\alpha| < k+1$, and compact interval $I = [T_1, T_2]$, there are increasing functions $h^1_{I,\alpha}, h^2_{I,\alpha} \in C(\mathbb{R}, \mathbb{R})$ such that*

$$|(\partial^\alpha g_{\mu\nu})(t,x,\xi)| \leq h^1_{I,\alpha}(|\xi|) \tag{2.35}$$

$$|(\partial^\alpha f)(t,x,\xi)| \leq h^2_{I,\alpha}(|\xi|) \tag{2.36}$$

  $\forall \mu, \nu = 0, ..., n$, $t \in I$, $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^{nN+2N}$, and

- *for every I, there is an $a = (a_1, a_2, a_3)$, $a_i > 0, i = 1, 2, 3$ such that $g(t,x,\xi) \in \mathcal{C}_{n,a}$ for every $(t,x,\xi) \in I \times \mathbb{R}^{nN+2N+n}$,*

- *the function f_b defined by $f_b(t,x) = f(t,x,0)$, has locally x-compact support,*

we call g a $C^k - N, n$-admissible metric (third point not needed) respectively f a $C^k - N, n$-admissible non-linearity (second point not needed).

If g is a $C^0 - N, n$-admissible metric, then by applying Lemma 2.4.1 we know that g is a Lorentz matrix. From the assumptions made before and Lemma 2.4.2, there are for every compact interval I, constants $b_i, i = 1, 2, 3$ with $b = (b_1, b_2, b_3)$, such that $g^{-1}(t,x,\xi) \in \mathcal{C}_{n,b}$ $\forall (t,x,\xi) \in I \times \mathbb{R}^{nN+2N+n}$. Let $u : \Omega \to \mathbb{R}^N$ for some $\Omega \subseteq \mathbb{R}^{n+1}$ be a differentiable function, then we define $g[u]$ to be a function on Ω as

$$g[u](t,x) = g\left\{t, x, u(t,x), \partial_0 u(t,x), ..., \partial_n u(t,x)\right\},$$

and analogously for $f[u]$. We additionally need some further terminology for the proof of the local existence of solutions, concerning constants and functions which depend on the metric and non-linearity.

Definition 2.6.3. *Let $1 \leq N, n \in \mathbb{Z}$. Let be g a $C^\infty - N, n$-admissible metric and f a $C^\infty - N, n$-admissible non-linearity and $I = [T_1, T_2]$ a compact interval.*

- *A map κ which associates to every g, f a function $\kappa_I[g,f] \in C(\mathbb{R}^m, \mathbb{R}_{\geq 0})$ such that $\kappa_{I_1}[g,f] \leq \kappa_{I_2}[g,f]$ for $I_1 \subseteq I_2$ is called a $C^\infty - N, n$-admissible majorizer .*
- *A map C which associates to every g, f a real number $C_I[g,f] > 0$ such that $C_{I_1}[g,f] \leq C_{I_2}[g,f]$ for $I_1 \subseteq I_2$ is called a $C^\infty - N, n$-admissible constant .*

Remark. We often will omit the argument $[g,f]$ and write κ_I, C_I when it is clear from the context. We also use, for brevity the same notation κ_I, C_I for all N, n-admissible majorizer and constants, which can change their values from line to line.

Now we want to pay attention to the initial value problem

$$g^{\mu\nu}\partial_\mu\partial_\nu u = f, \tag{2.37}$$

$$u(T_o, .) = U_0, \tag{2.38}$$

$$\partial_t u(T_0, .) = U_1, \tag{2.39}$$

where $T_0 \in \mathbb{R}$, $g = g[u]$ and $f = f[u]$. All the assumptions hold if we divide, for convenience (to get $g^{00} = -1$), the equation by $-g^{00}$.

Lemma 2.6.1. *Let $1 \leq N, n \in \mathbb{Z}$, let g be a $C^1 - N, n$-admissible metric and f be a $C^1 - N, n$-admissible non-linearity. Examine two C^2 solutions u and v to (2.37) on $[T_0, T) \times \mathbb{R}^n$, where we do not require the initial data to have compact support. If the initial data coincide at $t = T_0$, then $u = v$ on $[T_0, T) \times \mathbb{R}^n$. The statement in the opposite time direction is similar.*

Proof. For what follows we write $g_v = g[v]$ and analogously for the other related objects. If there are two solutions u, v to (2.37) we get from the difference

$$g_v^{\mu\nu}\partial_\mu\partial_\nu v - g_u^{\mu\nu}\partial_\mu\partial_\nu u = f_v - f_u$$

by adding $g_u^{\mu\nu}\partial_\mu\partial_\nu v$ on both sides the following

$$g_v^{\mu\nu}\partial_\mu\partial_\nu v + f_u - f_v - g_u^{\mu\nu}\partial_\mu\partial_\nu v = g_u^{\mu\nu}\partial_\mu\partial_\nu u - g_u^{\mu\nu}\partial_\mu\partial_\nu v$$

which is

$$g_u^{\mu\nu}\partial_\mu\partial_\nu(u-v) = (g_v^{\mu\nu} - g_u^{\mu\nu})\partial_\mu\partial_\nu v + f_u - f_v. \tag{2.40}$$

Let us now introduce the following concept.

Let $h_u = h[u] = h(t, x, u(t, x), \partial_0 u(t, x), ..., \partial_n u(t, x))$ then

$h_u(t, x) - h_v(t, x) =$

$$= \int_0^1 \frac{d}{d\lambda} h \underbrace{(t, x, \lambda u(t, x) + (1-\lambda)v(t, x), \lambda\partial_0 u(t, x) + (1-\lambda)\partial_0 v(t, x), ...)}_{*} d\lambda$$

$$= \int_0^1 \partial_u h(*)(u-v) + \sum_{j=0}^n \partial_{\partial_j u} h(*)(\partial_j u - \partial_j v) d\lambda$$

$$=: h_1(u-v) + \sum_{j=0}^n h_2^j \partial_j(u-v)$$

where we used first the fundamental theorem of calculus and afterwards the chain rule. This can be written in short as

$$h(u) - h(v) = \int_0^1 h'(\lambda u + (1-\lambda)v) d\lambda \cdot (u-v) \tag{2.41}$$

So we have continuous functions f_1 and f_2^μ such that

$$f_u - f_v = f_1 \cdot (u-v) + f_2^\mu \cdot \partial_\mu(u-v). \tag{2.42}$$

We get an similar expression for $g_v^{\mu\nu} - g_u^{\mu\nu}$ and by inserting this into (2.40), we have (together with our assumptions) the conditions of Proposition 2.5.1 applied to (2.37) which concludes the proof. □

Before we start with the next lemma, we introduce some helpful notation, which will be afterwards frequently used. Let ,

$$M_k[v](t) = \|v(t, .)\|_{H^{k+1}} + \|\partial_t v(t, .)\|_{H^k}, \tag{2.43}$$

$$m[v](t) = \sum_{|\alpha|+j\leq 2} \sup_{x\in\mathbb{R}^n} \left|\partial^\alpha \partial_t^j v(t, x)\right|, \tag{2.44}$$

supposing that the right-hand sides are defined. Additionally note that $M[v] = M_0[v]$.

The main use of the next lemma is for the proof that the elements of our sequence (which we use for proving the existence of local solutions) are uniformly bounded on a suitable time interval.

Lemma 2.6.2. *Let $1 \leq N, n \in \mathbb{Z}$. Then there are N, n-admissible majorizers κ_1, κ_2 and N, n-admissible constants C_1, C_2, C_3 such that the following holds. Assume g is a C^∞-N, n-admissible metric, f a C^∞-N, n-admissible non-linearity, $U_0, U_1 \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^N)$ and assume $v \in C^\infty(\mathbb{R}^{n+1}, \mathbb{R}^N)$ is of locally x-compact support. Let $I = [T_0, T_1]$ with $T_0 \leq T_1$. Additionally we write $g_v = g[v], f_v = f[v]$. Let furthermore u be a solution of*

$$g_v^{\mu\nu} \partial_\mu \partial_\nu u = f, \tag{2.45}$$

$$u(T_0, .) = U_0, \tag{2.46}$$

$$\partial_t u(T_0, .) = U_1. \tag{2.47}$$

If $t \in I$, then

$$\begin{aligned} M_k[u](t) \leq\ & C_{1,I} M_k[u](T_0) \\ & + \int_{T_0}^t C_{2,I} + \kappa_{1,I} m[v] \left\{(1 + m[u]) M_k[v] + M_k[u]\right\} ds. \end{aligned} \tag{2.48}$$

If $f_b = 0$ (with $f_b(t, x) = f(t, x, 0)$), we can in the integral omit the constant term. If

$$E_k[v, u] = \frac{1}{2} \sum_{|\alpha| \leq k} \int_{\mathbb{R}^n} [-g_v^{00} |\partial^\alpha \partial_t u|^2 + g_v^{ij} \partial^\alpha \partial_i u \cdot \partial^\alpha \partial_j u + |\partial^\alpha u|^2] dx. \tag{2.49}$$

we have the inequality

$$\partial_t E_k[v, u] \leq C_{3,I} + \kappa_{2,I}(m[u], m[v])(M_k^2[v] + E_k[v, u]). \tag{2.50}$$

Remark: For (2.45)-(2.47) there is a unique solution $u \in C^\infty(\mathbb{R}^{n+1}, \mathbb{R}^N)$ by Theorem 2.5.1. The solution is also of locally x-compact support. In the expression for the energy E_k the term $g_v^{00} = -1$. Nevertheless we write it as above to assure that we don't run into a problem if this condition is not satisfied.

Proof: Due to the assumptions regarding g, there exists an N, n-admissible constant C for $t \in I = [T_1, T_2]$ such that,

$$\frac{1}{C_I} E_k^{1/2}[v, w](t) \le M_k[w](t) \le C_I E_k^{1/2}[v, w](t), \tag{2.51}$$

where w is such that all involved objects are defined. For convenience we write for $E_k[v, u]$ now E_k and for E_0 we write E. By differentiating we get

$$\begin{aligned}\partial_t E = \partial_t &\left[\frac{1}{2} \int_{\mathbb{R}^n} -g_v^{00} \left|\partial_t u\right|^2 + g_v^{ij} \partial_i u \partial_j u + |u|^2 \, dx \right] = \\ = &\frac{1}{2} \int_{\mathbb{R}^n} (-\partial_t g_v^{00}) \left|\partial_t u\right|^2 - 2 g_v^{00} \partial_t u \partial_t^2 u \, dx \\ &+ \int_{\mathbb{R}^n} \frac{1}{2} (\partial_t g_v^{ij}) \partial_i u \partial_j u + g_v^{ij} (\partial_t \partial_j u) \partial_i u \, dx + \\ &+ \int_{\mathbb{R}^n} |u| \, \partial_t u \, dx.\end{aligned}$$

Now we have (by additionally using μ, ν indices)

$$g_v^{\mu\nu} \partial_\mu \partial_\nu u \cdot \partial_t u - 2 g_v^{0i} \partial_t \partial_i u \cdot \partial_t u = g_v^{ij} \partial_i \partial_j u \cdot \partial_t u + g_v^{00} \partial_t^2 u \cdot \partial_t u$$

and also by using partial integration

$$\begin{aligned}\int_{\mathbb{R}^n} g_v^{ij} \partial_i u \cdot \partial_t \partial_j u \, dx &= -\int_{\mathbb{R}^n} \partial_j (g_v^{ij} \partial_i u) \cdot \partial_t u \, dx \\ \int_{\mathbb{R}^n} 2 g_v^{0i} \partial_t \partial_i u \cdot \partial_t u \, dx &= -\int_{\mathbb{R}^n} \partial_i (g_v^{0i}) \left|\partial_t u\right|^2 dx,\end{aligned}$$

we get the following expression

$$\begin{aligned}\partial_t E = \int_{\mathbb{R}^n} &\left[-g_v^{\mu\nu} \partial_\mu \partial_\nu u \cdot \partial_t u - \partial_i (g_v^{0i}) \left|\partial_t u\right|^2 - \frac{1}{2} (\partial_t g_v^{00}) \left|\partial_t u\right|^2 \right. \\ &\left. - (\partial_i g_v^{ij}) \partial_j u \cdot \partial_t u + \frac{1}{2} (\partial_t g_v^{ij}) \partial_j u \cdot \partial_i u + u \cdot \partial_t u \right] dx.\end{aligned} \tag{2.52}$$

The first term of (2.52) can be estimated by the use of (2.37), the Cauchy-Schwarz inequality and the expression of E_k for $k = 0$. Because $\partial_t g_v^{00} = 0$ and all $\partial_i g_v^{0i}, \partial_i g_v^{ij}$ and $\partial_t g_v^{ij}$ are bounded due to the fact that we used an admissible metric (by this also its entries $g_v^{\mu\nu}$ are bounded on the interval I), we can estimate the residual terms by κ_I, $m[v](t)$ and the energy term E. So we get

$$\partial_t E \leq \kappa_I(m[v])E + C \left\| f_v(t,.) \right\|_2 E^{1/2}, \tag{2.53}$$

where the constant C is a number and κ is an N, n-admissible majorizer. Now to obtain such an estimate for E_k, we use that $\partial^\alpha u$ fulfill the following equation (by using (2.45))

$$g_v^{\mu\nu}\partial_\mu\partial_\nu\partial^\alpha u = \partial^\alpha f_v + \left[g_v^{\mu\nu}\partial_\mu\partial_\nu, \partial^\alpha\right] u.$$

If we now increase from E_0 to E_k we get from (2.53) and (2.49)

$$\partial_t E_k \leq \kappa_I(m[v])E_k + C \left\| f_v \right\|_{H^k} E_k^{1/2} + C \sum_{|\alpha| \leq k} \left\| \left[g_v^{\mu\nu}\partial_\mu\partial_\nu, \partial^\alpha\right] u \right\|_2 E_k^{1/2},$$

where we have taken the higher order of derivatives into account, which leads to an additional term (the last summation) and we get the $\|.\|_{H^k}$-norm (where we had for $k = 0$ the $\|.\|_2$-norm). Note that the H^k-norm covers only the x-variables.

If we consider the term $\left[g_v^{\mu\nu}\partial_\mu\partial_\nu, \partial^\alpha\right] u$, we can write this up to constant factors (and keeping in mind that we use Einstein's summation convention) as

$$\left(\partial^\beta\partial_i g_v^{\mu\nu}\right)\partial^\gamma\partial_\mu\partial_\nu u, \tag{2.54}$$

with $|\beta| + |\gamma| = |\alpha| - 1$. We can assume that there is at most one of μ, ν equal zero, due to the fact that $g_v^{00} = -1$ and we differentiate with respect to x^i. Now we can write, by defining $g_0 = g[0]$:

$$\left(\partial^\beta\partial_i g_v^{\mu\nu}\right)\partial^\gamma\partial_\mu\partial_\nu u = \left[\partial^\beta\partial_i(g_v^{\mu\nu} - g_0^{\mu\nu})\right]\partial^\gamma\partial_\mu\partial_\nu u + (\partial^\beta\partial_i g_0^{\mu\nu})\partial^\gamma\partial_\mu\partial_\nu u.$$

If we look at the sup norm of the first factor of the second term over $[T_0, T] \times \mathbb{R}^n$ in the last equation, we see that it depends only on the interval I. If we estimate now this second term in L^2, we can put this first factor into the sup norm and get

$$\left\|(\partial^\beta\partial_i g_0^{\mu\nu})\partial^\gamma\partial_\mu\partial_\nu u\right\|_2 \leq \kappa_I(m[v])M_k[u]$$

For the first term, we use ([1] Lemma 6.16, p.54) and get for order $|\alpha| = (k-1)$ by using $\phi_1 = \partial_i(g_v^{\mu\nu} - g_0^{\mu\nu})$ and $\phi_2 = \partial_\mu\partial_\nu u$:

$$\left\|\partial^\beta\phi_1 \cdot \partial^\gamma\phi_2\right\|_2 \leq C(\left\|D^{k-1}\phi_1\right\|_2 \underbrace{\|\phi_2\|_\infty}_{\leq m[u]} + \underbrace{\|\phi_1\|_\infty}_{\leq \kappa_I(m[v])} \underbrace{\left\|D^{k-1}\phi_2\right\|_2}_{\leq M_k[u]})$$

and for the first term in the last estimation we get

$$\left\|D^{k-1}\phi_1\right\|_2 \cong \|g_v^{\mu\nu} - g_0^{\mu\nu}\|_{H^k} \le C\,\|v\|_\infty \cdot \|v\|_{H^k} = \kappa_I(m[v])M_k[v]$$

where we used ([1] Lemma 6.17, p.55) with $\|g_v^{\mu\nu} - g_0^{\mu\nu}\|_{H^k} \cong F(t,x,.)$ and (2.35) from the definition of the C^k N,n-admissible metric for the estimation.

If we sum up the last estimation we get finally

$$\underbrace{\left\|\left(\partial^\beta\partial_i g_v^{\mu\nu}\right)\partial^\gamma\partial_\mu\partial_\nu u\right\|_2}_{\left\|\left[g_v^{\mu\nu}\partial_\mu\partial_\nu,\partial^\alpha\right]u\right\|_2} \le \kappa_I(m[v])(M_k[u] + m[u]M_k[v])$$

The last estimation is for f_v in H^k, where we use again ([1] Lemma 6.17, p.55) to obtain

$$\|f_v\|_{H^k} \le C_I + \kappa_I(m[v])M_k[v]$$

Here we get the constant term from f_b (which is $f_b(t,x) = f(t,x,0)$). A final summation yields

$$\begin{aligned}\partial_t E_k \le& \\ &\kappa_I(m[v])E_k + (C_I + \kappa_I(m[v])(M_k[v] + M_k[u] + m[u]M_k[v]))\,E_k^{1/2}.\end{aligned} \tag{2.55}$$

If we apply Young's inequality and use (2.51) we get (2.50).

To be able also to differentiate at zero, we use instead of E_k the term $\hat{E}_k = E_k + \varepsilon$ and obtain now

$$\partial_t \hat{E}_k^{1/2} = C_I + \kappa_I(m[v])(M_k[u] + m[u]M_k[v] + M_k[u]).$$

By integration and letting $\varepsilon \to 0$ we get, by using (2.51) finally (2.48). □

Lemma 2.6.3. *Let $1 \le N, n \in \mathbb{Z}$. Then there are N,n-admissible majorizers κ_3, κ_4 and an N,n-admissible constant C_4 such that the following holds. Let g, f be a $C^\infty -$ N,n-admissible metric resp. non-linearity. Let $T_0 \le T_1 \in \mathbb{R}$, $U_{0,i}, U_{1,i} \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^N)$ and assume $v_i \in C^\infty(\mathbb{R}^{n+1}, \mathbb{R}^N)$ be functions with locally x-compact support, where $i = 1,2$. Let $g_i = g[v_i]$, $f_i = f[v_i]$ and u_i be solutions to*

$$\begin{aligned} g_i^{\mu\nu}\partial_\mu\partial_\nu u_i &= f_i, \\ u_i(T_0,.) &= U_{0,i}, \\ \partial_t u_i(T_0,.) &= U_{1,i}. \end{aligned}$$

For $u = u_2 - u_1$, $v = v_2 - v_1$ and $t \in I = [T_0, T_1]$ we get

$$M[u](t) \leq C_{4,I} \exp\left(\int_{T_0}^{t} \kappa_{4,I}(m[v_2])ds\right) \cdot \left(M[u](T_0) + \int_{T_0}^{t} \kappa_{3,I}(m[u_1], m[v_1], m[v_2])M[v]ds\right)$$

where $M[u] = M_0[u]$.

Proof: We define the energy by

$$\mathcal{E} = \frac{1}{2}\int_{\mathbb{R}^n} [-g_2^{00}\,|\partial_t u|^2 + g_2^{ij}\partial_i u \cdot \partial_j u + |u|^2]dx$$

and note that (as can be easily calculated)

$$g_2^{\mu\nu}\partial_\mu\partial_\nu u = (g_1^{\mu\nu} - g_2^{\mu\nu})\partial_\mu\partial_\nu u_1 + f_2 - f_1.$$

By differentiating $\mathcal{E}$ with respect to t (which is (2.52) due to $\mathcal{E} = E_0$ from (2.49)) we get

$$\partial_t\mathcal{E} = \int_{\mathbb{R}^n} (-g_2^{\mu\nu})\partial_\mu\partial_\nu u_1 \partial_t u - \underbrace{(\partial_i g_2^{i0})}_{\leq \kappa_I(m[v_2])}\,|\partial_t u|^2 - \frac{1}{2}\underbrace{(\partial_t g_2^{00})}_{\leq \kappa_I(m[v_2])}\,|\partial_t u|^2 \tag{2.56}$$

$$- \underbrace{(\partial_i g_2^{ij})}_{\leq \kappa_I(m[v_2])}\,\partial_j u \partial_t u + \frac{1}{2}\underbrace{(\partial_t g_2^{ij})}_{\leq \kappa_I(m[v_2])}\,\partial_i u \partial_j u + u\partial_t u\,dx \tag{2.57}$$

$$\leq \int_{\mathbb{R}^n} (-g_2^{\mu\nu})\partial_\mu\partial_\nu u_1 \partial_t u - \kappa_I(m[v_2])\underbrace{M[u]^2}_{\leq C_I \mathcal{E}}\,dx \tag{2.58}$$

The remaining part is the first term in the integral, which we estimate by:

$$\int_{\mathbb{R}^n} (g_1^{\mu\nu} - g_2^{\mu\nu})\partial_\mu\partial_\nu u_1 \partial_t u\,dx + \int_{\mathbb{R}^n} (f_2 - f_1)\partial_t u\,dx$$

$$\overset{p.I.}{=} \int_{\mathbb{R}^n} -\partial_\mu(g_1^{\mu\nu} - g_2^{\mu\nu})\partial_\nu u_1 \partial_t u - (g_1^{\mu\nu} - g_2^{\mu\nu})\partial_\nu u_1 \partial_t \partial_\mu u + (f_2 - f_1)\partial_t u\,dx$$

$$\overset{(2.41)}{=} \int_{\mathbb{R}^n} a^{\mu\nu}\overbrace{(v_1 - v_2)}^{v}\,\partial_\nu u_1 \partial_t u + b^{\mu\nu\delta}\partial_\delta(v_1 - v_2)\partial_\nu u_1 \partial_t u$$
$$+ c^{\mu\nu}(v_1 - v_2)\partial_\nu u_1 \partial_t \partial_\mu u + d^{\mu\nu\delta}\partial_\delta(v_1 - v_2)\partial_\nu u_1 \partial_t u$$
$$+ e(v_1 - v_2)\partial_t u + h^\delta \partial_\delta(v_1 - v_2)\partial_t u\,dx$$

where $a^{\mu\nu}, b^{\mu\nu\delta}, c^{\mu\nu}, d^{\mu\nu\delta}, e, h^{\delta}$ are suitable functions (see proof of Lemma 2.6.1). Now let us do a first estimation of one of the above terms in the last integral. This should be done exemplary in a way that can be applied in the same way to all other terms.

$$\begin{aligned}\int_{\mathbb{R}^n} \left| b^{\mu\nu\delta}\partial_\delta \left|v\right| \partial_\nu u_1 \partial_t u \right| dx &\leq \kappa_I(m[u_1]) \left\|b^{\mu\nu}\right\|_\infty \int_{\mathbb{R}^n} \left|\nabla v\right| \cdot \left|\partial_t u\right| dx \\ &\leq \kappa_I(m[u_1]) \quad \underbrace{\left\|b^{\mu\nu}\right\|_\infty}_{\leq \kappa_I(m[v_1], m[v_2])} \quad M[v] \cdot \underbrace{M[u]}_{\leq C\mathcal{E}^{1/2}}\end{aligned}$$

where the estimation $M[u] \leq C\mathcal{E}^{1/2}$ comes from (2.51) and the estimation for $\left\|b^{\mu\nu}\right\|_\infty$ is due to the property of the N, n-admissible metric and the dependency on the functions v_1 and v_2 from Lemma 2.6.1. If we summarize this and include also the estimates (2.58) from the beginning, we get finally

$$\partial_t \mathcal{E} \leq \kappa_I(m[v_2])\mathcal{E} + \kappa_I(m[u_1], m[v_1], m[v_2]) M[v] \mathcal{E}^{1/2})$$

In short, for the main point we applied (2.41) to get this result. This leads by using the fundamental theorem of calculus to

$$\mathcal{E}^{1/2}(t) \leq \mathcal{E}^{1/2}(T_0) + \int_{T_0}^{t} \left\{ \kappa_I(m[v_2])\mathcal{E}^{1/2} + \kappa_I(m[u_1], m[v_1], m[v_2]) M[v] \right\} ds.$$

By applying Grönwall's lemma, we obtain

$$\begin{aligned}&\mathcal{E}^{1/2}(t) \\ &\quad \leq \left(\mathcal{E}^{1/2}(T_0) + \int_{T_0}^{t} \kappa_I(m[u_1], m[v_1], m[v_2]) M[v] ds \right) \exp\left(\int_{T_0}^{t} \kappa_I(m[v_2]) ds \right).\end{aligned}$$

From this we get the lemma. □

2.6.1 Local existence

Proposition 2.6.1. *Let $1 \leq N, n \in \mathbb{Z}$. Then there is an N, n-admissible majorizer κ and an N, n-admissible constant C such that the following holds. Assume g, f is a $C^\infty - N, n$-admissible metric resp. non-linearity. Furthermore let $k > n/2+1$, $U_0 \in H^{k+1}(\mathbb{R}^n, \mathbb{R}^N)$, $U_1 \in H^k(\mathbb{R}^n, \mathbb{R}^N)$ and $I = [T_1, T_2]$ a compact interval. Then there is a $T > 0$, depending continuously on the H^{k+1}-norm of U_0, on the H^k-norm of U_1 and on I, such that if $T_0 \in [T_1, T_2]$, then there exists a unique solution $u \in C_b^2([T_0, T_0+T] \times \mathbb{R}^n, \mathbb{R}^N)$ to* (2.37)-(2.39). *Furthermore,*

$$\begin{aligned} u &\in C\left\{[T_0, T_0+T], H^{k+1}(\mathbb{R}^n, \mathbb{R}^N)\right\} \\ \partial_t u &\in C\left\{[T_0, T_0+T], H^{k}(\mathbb{R}^n, \mathbb{R}^N)\right\} \end{aligned} \tag{2.59}$$

and for $t \in [T_0, T_0+T]$

$$\mathcal{E}_k(t) \leq [\mathcal{E}_k(T_0) + C_I \cdot (t - T_0)] \exp\left(\int_{T_0}^t \kappa_I(m[u]) d\tau\right), \tag{2.60}$$

where $\mathcal{E}_k(t) = E_k[u, u](t)$ and $E_k[v, u](t)$ is as in (2.49). *One can also assume that the bound on $\mathcal{E}_k(t)$ for $t \in [T_0, T_0+T]$ depends only on I, an upper bound on the H^{k+1}-norm of U_0 and an upper bound on the H^k-norm of U_1.*

Remark. This result does not show immediately the existence of local smooth solutions if one has smooth initial data because the time where the solution exists could depend on the degree of regularity of the initial data.

Proof: Let $U_{0,l}, U_{1,l} \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^N)$ be sequences with $U_{0,l} \xrightarrow{H^{k+1}} U_0$ and $U_{1,l} \xrightarrow{H^k} U_1$. Furthermore let $C_0 = \|U_0\|_{H^{k+1}} + \|U_1\|_{H^k}$, and $\|U_{0,l}\|_{H^{k+1}} + \|U_{1,l}\|_{H^k} \leq C_0 + 1$. We get the inequality if the sequence is chosen suitable.

Next we define w_0 by $w_0(t, x) = U_{0,0}(x)$ and if w_l has been defined (with x-compact support), that $g_{l+1} = g[w_l]$ and analogously $f_{l+1} = f[w_l]$. The element w_l of the sequence should be a solution to the following system, which is also part of an ongoing iteration

$$g_{l+1}^{\mu\nu} \partial_\mu \partial_\nu w_{l+1} = f_{l+1}, \tag{2.61}$$

$$w_{l+1}(T_0) = U_{0,l+1}, \tag{2.62}$$

$$\partial_t w_{l+1}(T_0) = U_{1,l+1}, \tag{2.63}$$

From Theorem 2.5.1 we know that there is a solution $w_{l+1} \in C^\infty(\mathbb{R}^{n+1}, \mathbb{R}^N)$ which is unique with x-compact support solving (2.61) − (2.63).

Boundedness. First we prove the boundedness of the sequence in

$$C^0\left\{[T_0, T_0+T], H^{k+1}(\mathbb{R}^n, \mathbb{R}^N)\right\} \cap C^1\left\{[T_0, T_0+T], H^k(\mathbb{R}^n, \mathbb{R}^N)\right\}$$

We do this by induction and start with the assumption

$$M_k[w_{l-1}](t) \leq \mathcal{M} \tag{2.64}$$

for $T_0 \leq t \leq T_0 + T$ and a constant $\mathcal{M}$. By our assumtion at the beginning it is clear that the case $l = 1$ is valid for any T and $\mathcal{M}$ depending only on C_0. If we now assume the validity for l and $l+1$ then we get

$$m[w_l](t) \leq \kappa_I(\mathcal{M})\{1 + M_k[w_l](t)\} \leq \kappa_I(\mathcal{M})$$

by the equation and Sobolev embedding on the interval $[T_0, T_0 + T]$ with $T \leq 1$. By splitting up the terms of $m[w_l](t)$ we see that we need for the first estimate for l only (2.64) (the 1 in the middle comes from the estimate for the (splitted-)part $\partial_t^2 w_l$) and we also use the equation. Because the function w_l and its derivatives are bounded (which we know from the equation resp. the properties of f and g) the sup norm can be controlled by the $\|.\|_{H^{k+1}}$ norm and a constant, resp. a majorizer. And here (2.64) acts in the desired way.

By using the equation we only need to control the sup norm of the derivatives of order 0 and 1 for w_{l-1} (which we get by (2.64)). For the next step, which is w_l we have to control the sup norm of the second derivatives of w_l, which again is given by (2.64) if we replace l by $l+1$ and assuming that we have at least one spatial derivative. Due to the boundedness of $m[w_0]$ we don't need to know something about w_{-1} for the induction from $l = 1$ to $l = 2$, which one can see below. If now (2.64) holds for $l = 1$ or for l and $l - 1$ then we have $m[w_{l-1}] \leq \kappa_I(\mathcal{M})$ and

$$m[w_l](t) \leq \kappa_I(\mathcal{M})\{1 + M_k[w_l](t)\}$$

By applying this to the estimation (2.64) which we get from Lemma 2.6.2

$$M_k[w_l](t) \leq C_I\left(M_k[w_l](T_0) + \kappa_I(\mathcal{M})\int_{T_0}^t (1 + M_k[w_l])ds\right).$$

If we use Grönwall's lemma, we obtain

$$M_k[w_l](t) \leq C_I\left[M_k[w_l](T_0) + \kappa_I(\mathcal{M})(t - T_0)\right]\exp[(t - T_0)\kappa_I(\mathcal{M})].$$

For $\mathcal{M} = 4C_l(C_0 + 1)$ we get $\mathcal{M} \geq 4C_l M_k[w_l](T_0)$ for all l. If now T is chosen small enough, depending only on $\mathcal{M}$ (resp. C_0) and I, the induction is completed and (2.64) holds for all l.

Convergence in the low norm. In this section we have to prove that the sequence converges in

$$C^0 \left\{[T_0, T_0 + T], H^1(\mathbb{R}^n, \mathbb{R}^N)\right\} \cap C^1 \left\{[T_0, T_0 + T], L^2(\mathbb{R}^n, \mathbb{R}^N)\right\}.$$

First we verify whether the iterations contained in the right space. To do this we use Lemma 2.6.3 and do the following replacements: $v_2 = u_1 = w_l, v_1 = w_{l-1}, u_2 = w_{l+1}$. Then we define

$$a_l = \sup_{T_0 \leq t \leq T_0+T} M[w_{l+1} - w_l].$$

Letting T small enough, such that it depends only on C_0 and I. Furthermore we choose the initial data sequence in such a way that the following holds

$$2C_{4,l} M[w_{l+1} - w_l](T_0) \leq 2^{-l},$$

then we get from the following estimation

$$
\begin{aligned}
M[w_{l+1} - w_l] &\overset{Lemma\, 2.6.3}{\leq} \\
&\leq C_{4,I} \exp\left(\int_{T_0}^{T_0+T} \kappa_{4,I}(m[w_l]) ds\right) \cdot \\
&\quad \cdot \left[M[w_{l+1} - w_l](T_0) + \int_{T_0}^{T_0+T} \kappa_{3,I}(m[w_l], m[w_{l-1}]) M[w_l - w_{l-1}] ds\right] \\
&\leq \underbrace{\exp\left(\int_{T_0}^{T_0+T} \kappa_{4,I}(m[w_l]) ds\right)}_{\to 1 \text{ for } T\to 0, \text{so} <2 \text{ for T small}} \cdot \\
&\quad \cdot [2^{-l-1} + \underbrace{\int_{T_0}^{T_0+T} \kappa_{3,I}(m[w_l], m[w_{l-1}]) M[w_l - w_{l-1}] ds}_{\to 1 \text{ for } T \to T_0} \cdot C_{4,I} \cdot a_{l-1}] \\
&\leq 2 \cdot 2^{-l-1} + \underbrace{2 \cdot C_{4,I} \cdot \int_{T_0}^{T_0+T} \kappa_{3,I}(m[w_l], m[w_{l-1}]) M[w_l - w_{l-1}] ds}_{<1/2 \text{ for T small}} \cdot a_{l-1}
\end{aligned}
$$

$$\leq 2^{-l} + \frac{1}{2}a_{l-1}$$

the inequality for A_l as

$$a_l \leq 2^{-l} + \frac{1}{2}a_{l-1}.$$

and from this we get

$$a_l \leq \frac{l-1}{2^l} + \frac{1}{2^{l-1}}a_1.$$

This already works because of the uniform boundedness of $\kappa_{4,I}(m[w_l])$ and $\kappa_{3,I}(m[w_l], m[w_{l-1}])$, which follows from the first part of the proof. Because a_l is summable, we have the convergence in the right space.

Convergence in higher norms. Now we prove the boundedness of the sequence and that it is also a Cauchy sequence in

$$C^0\left\{[T_0, T_0+T], H_{(s+1)}(\mathbb{R}^n, \mathbb{R}^N)\right\} \cap C^1\left\{[T_0, T_0+T], H_{(s)}(\mathbb{R}^n, \mathbb{R}^N)\right\}$$

Let $0 < s < k$ and apply ([1], Lemma 5.20, p.42) with $a_s, b_s \in (0,1)$ and $a_s + b_s = 1$ such that

$$\|w_l(t,.) - w_m(t,.)\|_{(s+1)} \leq \underbrace{\|w_l(t,.) - w_m(t,.)\|_{(k+1)}^{b_s}}_{\text{bounded for } T_0 \leq t \leq T_0+T} \cdot \underbrace{\|w_l(t,.) - w_m(t,.)\|_{(2)}^{a_s}}_{\to 0}$$

So we see that w_l is a Cauchy sequence in $C^0\left\{[T_0, T_0+T], H_{s+1}(\mathbb{R}^n, \mathbb{R}^N)\right\}$ and by a similar conclusion also in $C^1\left\{[T_0, T_0+T], H_s(\mathbb{R}^n, \mathbb{R}^N)\right\}$.

Let us assume that $k > n/2 + 1$. By use of the equation, where we always get a relation between the solution respectively the second derivative of the solutions and the admissible non-linearity (also keep in mind the admissible metric and the properties which are connected to these entities), we can use Sobolev embedding to show that $\|f_l\|_{C_b^1} \leq C\|f_l\|_{(k)}$. From this and the above we get that w_l converges in $C_b^2([T_0, T_0+T] \times \mathbb{R}^n, \mathbb{R}^N)$ and so we have a C^2-solution of the equation, which we will hence forth call u.

Now we have to show that $u(t,.)$ is in H^{k+1} and that $\partial_t u(t,.)$ is in H^k. If $0 < s < k$ we have $u(t,.) \in H_{(s+1)}$ and $\partial_t u(t,.) \in H_{(s)}$ and also

$$\|u(t,.)\|_{(s+1)} + \|\partial_t u(t,.)\|_{(s)} = \lim_{l\to\infty}[\|w_l(t,.)\|_{(s+1)} + \|\partial_t w_l(t,.)\|_{(s)}] \leq C\mathcal{M}.$$

The right hand side does not depend on s and we have monotone convergence ($\|u\|_{(s_1)} \leq \|u\|_{(s_2)}$ for $s_1 \leq s_2$). By this the Lebesgue's monotone convergence theorem is applicable to deduce that $u(t,.) \in H^{k+1}$ and $\partial_t u(t,.) \in H^k$. In this way one sees that the defining integrals for the (s)-norms converge to the integral which defines the k-norm. Additionally we have

$$\|u(t,.)\|_{H^{k+1}} + \|\partial_t u(t,.)\|_{H^k} \leq C\mathcal{M}.$$

Weak continuity. Now we want to show that (2.59) holds. Let us begin to show that the solution is weakly continuous. Assume that f is in the dual space of $H_{(k+1)}(\mathbb{R}^n, \mathbb{R}^N)$. So we know that there is a $\phi \in H_{(-k-1)}(\mathbb{R}^n, \mathbb{R}^N)$ such that

$$f(w) = \int_{\mathbb{R}^n} \hat{w}(\xi) \cdot \hat{\phi}(\xi) d\xi$$

$\forall w \in H_{(k+1)}(\mathbb{R}^n, \mathbb{R}^N)$. If now ϕ_j is a sequence of Schwartz functions which converges to ϕ in $H_{(-k-1)}$, we get (see also [1], p.43)

$$|f[u(t,.)] - f[w_l(t,.)]| \leq C\mathcal{M}\,\|\phi - \phi_j\|_{(-k-1)} + \left| \int_{\mathbb{R}^n} [\hat{u}(t,\xi) - \hat{w}_l(t,\xi)] \cdot \hat{\phi}_j(\xi) d\xi \right|,$$

where we write for simplicity $\hat{u}(t,\xi)$ for the Fourier transform of $u(t,.)$ at ξ. If j is large enough, the first term on the right is less or equal $\varepsilon/2$, due to the convergence in $H_{(-k-1)}$. Additionally by choosing l large enough (which depends on j) also the second term is less or equal $\varepsilon/2$. So the left-hand side is less than ε and therefore $f[w_l(t,.)]$ converges uniformly to $f[u(t,.)]$ which means that the past function is continuous and so u is weakly continuous. Analogously one can show this for $\partial_t u$.

Deriving a Bound. If we substitute $v \to w_l, U_0 \to U_{0,l+1}$ and $U_1 \to U_{1,l+1}$ then we can apply Lemma 2.6.2, with w_{l+1} as the corresponding solution. If we further use the inequality (2.50) with the fundamental theorem of calculus we get for $[T_0, T_0 + T]$:

$$E_k[w_l, w_{l+1}](t) \leq E_k[w_l, w_{l+1}](T_0) + \int_{T_0}^{t} [C_I + \kappa_I(m[w_l], m[w_{l+1}])(M_k^2[w_l] + E_k[w_l, w_{l+1}])] ds. \tag{2.65}$$

Now let us analyse the various terms inside the integral. Note that for $t \in [T_0, T_0 + T]$,

$$\lim_{l\to\infty} [E_k[w_l, w_{l+1}](t) - E_k[u, w_{l+1}](t)] = 0,$$

again since w_l converges to u with respect to the C^2-norm and the boundedness of $M_k[w_l]$ due to (2.64). Furthermore we have $M_k^2[w_l](t) \le C_I E_k[u, w_l](t)$. If we now insert this into (2.65), we obtain

$$E_k[u, w_l](t) \le E_k[u, w_l](T_0) + C_I(t - T_0) + \int_{T_0}^t C_I \kappa_I(m[u], m[w_l]) E_k[u, w_l] ds.$$

Because of $m[w_l] \to m[u]$, due to the fact that w_l converges to u with respect to the C^2-norm and that $E_k[w_l, w_{l+1}](T_0) \to E_k[u, u](T_0)$, we get from the above and the assumptions about the initial data $U_{0,l}$ and $U_{1,l}$ we get by applying Fatou's lemma

$$\mathfrak{E}_k(t) \le \mathcal{E}_k(T_0) + C_I(t - T_0) + \int_{T_0}^t \kappa_I(m[u]) \mathfrak{E}_k(\tau) d\tau,$$

where $\mathcal{E}_k(t) = E_k[u, u](t)$ and $\mathfrak{E}_k(t) = \limsup_{l\to\infty} E_k[u, w_l](t)$. If we use Grönwall's lemma, we get

$$\mathfrak{E}_k(t) \le [\mathcal{E}_k(T_0) + C_I(t - T_0)] \exp\left(\int_{T_0}^t \kappa_I(m[u]) d\tau\right).$$

Because of the weak convergence $w_l \overset{H^{k+1}}{\rightharpoonup} u$ and $\partial_t w_l \overset{H^k}{\rightharpoonup} \partial_t u$, we obtain

$$\begin{aligned} \mathcal{E}_k &= \lim_{l\to\infty} \frac{1}{2} \sum_{|\alpha|\le k} \int_{\mathbb{R}^n} [(\partial^\alpha \partial_t u) \cdot (\partial^\alpha \partial_t w_l) + g_u^{ij} (\partial^\alpha \partial_i u) \cdot (\partial^\alpha \partial_j w_l) + \partial^\alpha u \cdot \partial^\alpha w_l] dx \\ &\le \mathcal{E}_k^{1/2} \limsup_{l\to\infty} E_k^{1/2}[u, w_l], \end{aligned}$$

and so we have $\mathcal{E}_k(t) \le \mathfrak{E}_k(t)$ and (2.60) holds.

Strong continuity. Now we want to show that u and $\partial_t u$ are right continuous at T_0 so we have to prove that

$$\lim_{t\to T_{0+}} (\|u(t, .) - u(T_0, .)\|_{H^{k+1}} + \|\partial_t u(t, .) - \partial_t u(T_0, .)\|_{H^k}) = 0, \qquad (2.66)$$

First we define, by using $h^{ij}(x) = g^{ij}[u](T_0, x)$ an inner product on $H^{k+1}(\mathbb{R}^n, \mathbb{R}^N)$ by

$$\begin{aligned}&\langle (v_1, v_2), (w_1, w_2) \rangle = \\ &= \frac{1}{2} \sum_{|\alpha| \le k} \int_{\mathbb{R}^n} \left[(\partial^\alpha v_2) \cdot (\partial^\alpha w_2) + h^{ij} (\partial^\alpha \partial_t v_1) \cdot (\partial^\alpha \partial_t w_1) + (\partial^\alpha v_1)(\partial^\alpha w_1) \right] dx.\end{aligned}$$

By direct computation, we get

$$\begin{aligned}&\langle (u(t,.) - U_0, \partial_t u(t,.) - U_1), (u(t,.) - U_0, \partial_t u(t,.) - U_1) \rangle = \\ &\quad = \langle (u(t,.), \partial_t u(t,.)), (u(t,.), \partial_t u(t,.)) \rangle \\ &\qquad - \underbrace{2 \langle (u(t,.), \partial_t u(t,.)), (U_0, U_1) \rangle}_{\to -2\mathcal{E}_k(T_0), \text{ by weak continuity of } u} + \underbrace{\langle (U_0, U_1), (U_0, U_1) \rangle}_{= \mathcal{E}_k(T_0)}.\end{aligned} \tag{2.67}$$

This can easily verified by inserting both sides into the formula of the inner product, split the term up and shorten both sides. So we see that the detour over the inner product gives us a split into terms which can in turn be estimated by known relations which we have derived earlier. Because of (2.60) we have

$$\limsup_{t \to T_0+} \mathcal{E}_k(t) \le \mathcal{E}_k(T_0),$$

and from the continuity and boundedness of u we get

$$\lim_{t \to T_0+} \langle (u(t,.), \partial_t u(t,.)), (u(t,.), \partial_t u(t,.)) \rangle - \mathcal{E}_k(T_0) = 0,$$

With these observations we see that all terms are cancelled, so that we get (2.66). Furthermore one can set up an iteration which converges in a neighborhood of $t \in [T_0, T_0 + T]$. For this one has to use the above arguments together with the uniqueness property to show that u and $\partial_t u$ as functions from $[T_0, T_0 + t]$ to H^{k+1} resp. H^k are right continuous at t. One gets in addition left continuity by time reversal and both together imply continuity. □

Summary of the proof: We showed that we have 1.) locally a unique solution $u \in C^2$, which is also 2.) continuous from a time interval into some Sobolev spaces and which 3.) fulfills an energy inequality. Therefore we split the proof into several parts. Mainly we showed that an iteration of solutions w_l for the local linear problem converges to the solution for the local non-linear problem. For 1.) we showed the boundedness of the sequence, the convergence in the low norm and the convergence in the higher norms. For 2.) we showed the weak continuity and the strong continuity and for 3.) we found the relevant part in the section for the bound.

Lemma 2.6.4. *Let $1 \leq N, n \in \mathbb{Z}$, g be a $C^\infty - N, n$-admissible metric and f be a $C^\infty - N, n$-admissible non-linearity and $T_0 \in \mathbb{R}$. Assume that $u \in C^2([T_0, T_0 + T] \times \mathbb{R}^n, \mathbb{R}^N)$ (for some $T \in (0, \infty)$) is a solution to the equations* (2.37) $-$ (2.39) *with initial conditions $U_0 \in H^{k+1}$ and $U_1 \in H^k$ for some $k > n/2 + 1$. Then u satisfies* (2.59) *and* (2.60) *(here the κ, C are an N,n-admissible majorizer, and constant, respectively). We write $I = [T_0, T_0 + T]$ and let T_k (independent from the k above) be the supremum of times T such that there is a C^2 solution on $[T_0, T_0+T] \times \mathbb{R}^n$ satisfying* (2.59)*. Then either $T_k = \infty$ or*

$$\lim_{t \to T_k -} \sup_{T_0 \leq \tau \leq t} m[u](t) = \infty. \tag{2.68}$$

Proof: Let u be a C^2 solution with the initial data in $H^{k+1} \times H^k$. Although we know that the estimate (2.60) holds on the time interval stated in Proposition 2.6.1, we don't know whether it is valid on the interval claimed in this lemma. Let $T_* \in [T_0, T_0 + \varepsilon]$ and $\mathcal{A}$ the set of all T_* for which (2.59) and the estimate (2.60) is valid on the interval $[T_0, T_*]$. From Proposition 2.6.1 we know that $\mathcal{A}$ contains an open neighborhood of T_0 and therefore it is non-empty. To prove the statement in the lemma we have to show that $\mathcal{A}$ is open and closed.

Openness: Let $T_* \in \mathcal{A}$. Proposition 2.6.1 guarantees a solution on $[T_*, T_* + \varepsilon]$ for some $\varepsilon > 0$ and we get directly (2.59) on $[T_0, T_* + \varepsilon]$. For the estimate (2.60) on $[T_0, T_* + \varepsilon]$ we split up the interval. For $[T_0, T_*]$ it holds by assumption and for the interval $[T_*, T_* + \varepsilon]$ we use again Proposition 2.6.1 to get

$$\mathcal{E}_k(t) \leq [\mathcal{E}_k(T_*) + C_I \cdot (t - T_*)] \cdot \exp\left(\int_{T_*}^{t} \kappa_I(m[u]) d\tau\right)$$

and if we replace in the estimate (2.60) t with T_*, we get

$$\mathcal{E}_k(T_*) \leq [\mathcal{E}_k(T_0) + C_I \cdot (T_* - T_0)] \cdot \exp\left(\int_{T_0}^{T_*} \kappa_I(m[u]) d\tau\right)$$

and by combining this we obtain

$$\begin{aligned}\mathcal{E}_k(t) \leq & \left[[\mathcal{E}_k(T_0) + C_I \cdot (T_* - T_0)] \cdot exp\left(\int_{T_0}^{T_*} \kappa_I(m[u])d\tau\right) + C_I \cdot (t - T_*)\right] \\ & \cdot \exp\left(\int_{T_*}^{t} \kappa_I(m[u])d\tau\right) \\ & \leq [\mathcal{E}_k(T_0) + C_I \cdot (t - T_0)] \cdot \exp\left(\int_{T_0}^{t} \kappa_I(m[u])d\tau\right)\end{aligned}$$

Closedness: Let T_* be in the closure of $\mathcal{A}$. If we now take a $T_{**} < T_*$ the estimate (2.60) holds on $[T_0, T_{**}]$. From this we can deduce a uniform bound on $u(t,.)$ in H_{k+1} and $\partial_t u(t,.)$ in H_k for $t \in [T_0, T_*)$. The time T in Proposition 2.6.1 for which the solution exists, only depends on the compact interval in which the initial data are defined and the magnitude of the initial data. From this we get a solution which is beyond T_*, and also (2.59) (by the proposition itself) and (2.60) (due to continuity) holds.
Let $T_k < \infty$ and assume that (2.68) does not hold. Then $m[u]$ is bounded on $[T_0, T_k)$ and the inequality (2.60) provides a universal bound on this interval for $\mathcal{E}_k$. The local existence result would then give a solution beyond T_k, which contradicts the definition of T_k. □

Corollary 2.6.1. *Let $1 \leq N, n \in \mathbb{Z}$, g be a $C^\infty - N, n$-admissible metric and f be a $C^\infty - N, n$-admissible non-linearity. Assume $U_0, U_1 \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^N)$ and $T_0 \in \mathbb{R}$. Then there are $T_1 < T_0 < T_2$ and an unique solution $u \in C^\infty[(T_1, T_2) \times \mathbb{R}^n, \mathbb{R}^N]$ to the equations (2.37) – (2.39). The solution is of locally x-compact support and either $T_2 = \infty$ or $\lim_{\tau \to T_2 -} \sup_{T_0 \leq t \leq \tau} m[u](t) = \infty$. An analogous statement holds for T_1.*

2.6.1.1 Stability

Assume u is a background solution for Einstein's equations on (T_-, T_+). If we have two points $T_0, T_1 \in (T_-, T_+)$ and a sequence of initial data (indexed by l) which converges to the initial data of u at T_0, then

- the associated solution u_l exists until T_1 (for l large enough)
- the initial data of u_l at T_1 converges to the initial data of u at T_1.

Proposition 2.6.2. *Let $1 \leq N, n \in \mathbb{Z}$, g be a $C^\infty - N, n$-admissible metric and f be a $C^\infty - N, n$-admissible non-linearity. The initial value problem is* *as follows:*

$$g^{\mu\nu}\partial_\mu\partial_\nu u = f, \tag{2.69}$$

$$u(T_0,.) = U_0, \tag{2.70}$$

$$\partial_t u(T_0,.) = U_1, \tag{2.71}$$

with the (assumed) solution $u \in C^\infty[(T_-,T_+)\times\mathbb{R}^n,\mathbb{R}^N]$ *corresponding to the initial data* $U_0, U_1 \in C_0^\infty(\mathbb{R}^n,\mathbb{R}^N)$. *Assume that there exists a sequence of functions* $U_{0,l}, U_{1,l} \in C_0^\infty(\mathbb{R}^n,\mathbb{R}^N)$ *such that*

$$\lim_{l\to\infty}[\|U_{0,l}-U_0\|_{H^{k+1}} + \|U_{1,l}-U_1\|_{H^k}] = 0$$

for some $k > n/2+1$. *Let* $u_l \in C^\infty[(T_{l,-},T_{l,+})\times\mathbb{R}^n,\mathbb{R}^N]$ *be the solution to* (2.69)-(2.71) *(where* U_0, U_1 *are replaced by* $U_{0,l}, U_{1,l}$*). Let* $T_1 \in (T_-,T_+)$*, then there is an* l_0 *such that for* $l \geq l_0$ *we get* $T_1 \in (T_{l,-},T_{l,+})$. *Additionally,*

$$\lim_{l\to\infty}[\|u_l(T_1,.)-u(T_1,.)\|_{H^{k+1}} + \|\partial_t u_l(T_1,.)-\partial_t u(T_1,.)\|_{H^k}] = 0$$

We assume that for the solution the existence intervals $(T_-,T_+)$and $(T_{l,-},T_{l,+})$ are maximal.

Proof: We can assume that $T_0 < T_1$ and that T_1 is in the interior of a fixed compact subset of (T_-,T_+). Let $g = g[u], f = f[u], g_l = g_l[u_l]$ and $f_l = f_l[u_l]$. We write the difference of the respective equations for u and u_l as

$$g_l^{\mu\nu}\partial_\mu\partial_\nu v_l = F_l,$$

where we set $v_l = u_l - u$ and $F_l = (g^{\mu\nu} - g_l^{\mu\nu})\partial_\mu\partial_\nu u + f_l - f$. There exists a function $G \in C^\infty[(T_-,T_+)\times\mathbb{R}^{nN+2N+n},\mathbb{R}^N]$ for which the first condition of a $C^\infty - N,n$-admissible non-linearity is valid if we restrict to compact subintervals of (T_-,T_+), such that $G(t,x,0)=0$ for $(t,x)\in(T_-,T_+)\times\mathbb{R}^n$ and that $F_l(t,x) = G\{t,x,v_l(t,x),...,\partial_n v_l(t,x)\}$.
A similar statement which can found in ([1] lemma. 6.17, p.55) states for a compact subinterval I of T_-,T_+ that

$$\|F_l(t,.)\|_{H^k} \leq \sigma_I(m[v])[\|v_l(t,.)\|_{H^{k+1}} + \|\partial_t v_l(t,.)\|_{H^k}]$$

$\forall t \in I$. Here σ has the role of an N,n-admissible majorizer, which depends on f,g and (additional) on u and the intervals I where $\sigma_I[g,f,u]$ is defined on compact subintervals of (T_-,T_+) (for convenience we use the notation σ_I

also for different objects with the same properties, but this should be clear from the respective context). Define

$$\mathcal{E}_{l,j} = \frac{1}{2} \sum_{|\alpha| \le j} \int_{\mathbb{R}^n} [-g_l^{00} \left|\partial^\alpha \partial_0 v_l\right|^2 + g_l^{ij} \partial^\alpha \partial_i v_l \cdot \partial^\alpha \partial_j v_l + \left|\partial^\alpha v_l\right|^2] dx.$$

Due to our assumptions on g, $\mathcal{E}_{l,j}$ is equivalent to $\|v_l(t,.)\|_{H^{j+1}} + \|\partial_t v_l(t,.)\|_{H^j}$. We also write $\mathcal{E}_l = \mathcal{E}_{l,0}$. Now if we differentiate and carry out estimations analogous to Lemma 2.6.2 we get

$$\begin{aligned}\partial_t \mathcal{E}_l = \int_{\mathbb{R}^n} \Big[& -g_l^{\mu\nu} \partial_\mu \partial_\nu v_l \cdot \partial_t v_l - \partial_i (g_l^{0i}) \left|\partial_t v_l\right|^2 - \frac{1}{2} (\partial_t g_l^{00}) \left|\partial_t v_l\right|^2 \\ & - (\partial_i g_l^{ij}) \partial_j v_l \cdot \partial_t v_l + \frac{1}{2} (\partial_t g_l^{ij}) \partial_j v_l \cdot \partial_i v_l + v_l \cdot \partial_t v_l \Big] dx \\ \le & \; C \left\|F_l(t,.)\right\|_2 \mathcal{E}_l^{1/2} + \sigma_I(m[v_l]) \mathcal{E}_l \le \sigma_I(m[v_l]) \mathcal{E}_l,\end{aligned}$$

where $t \in I$, I is a compact subset of (T_-, T_+) and all derivatives of u are bounded on compact subsets of (T_-, T_+). Due to the fact that $\partial^\alpha v_l$ satisfies

$$g_l^{\mu\nu} \partial_\mu \partial_\nu \partial_t v_l = [g_l^{\mu\nu} \partial_\mu \partial_\nu, \partial^\alpha] v_l + \partial^\alpha F_l$$

which can be verified by a direct calculation, we obtain the same estimate for $\partial^\alpha v_l$ if we take $\partial^\alpha F_l + [\partial^\alpha, g_l^{\mu\nu} \partial_\mu \partial_\nu] v_l$ instead of F_l. By summing the above we get

$$\partial_t \mathcal{E}_{l,j} \le \sigma_I(m[v_l]) \mathcal{E}_{l,j} + C \sum_{|\alpha| \le j} \left\|[\partial^\alpha, g_l^{\mu\nu} \partial_\mu \partial_\nu] v_l\right\|_2 \mathcal{E}_{l,j}^{1/2}.$$

We observe that, again analogously as in the proof of Lemma 2.6.2, $[\partial^\alpha, g_l^{\mu\nu} \partial_\mu \partial_\nu] v_l$ is up to a factor, with $|\beta| + |\gamma| = |\alpha| - 1$, a sum of terms of the form

$$(\partial^\beta \partial_i g_l^{\mu\nu}) \partial^\gamma \partial_\mu \partial_\nu v_l = [\partial^\beta \partial_i (g_l^{\mu\nu} - g^{\mu\nu})] \partial^\gamma \partial_\mu \partial_\nu v_l + (\partial^\beta \partial_i g^{\mu\nu}) \partial^\gamma \partial_\mu \partial_\nu v_l.$$

Due to $g_l^{00} = g^{00} = -1$ we get from ([1],Lemma.6.16, p.54)

$$\left\|(\partial^\beta \partial_i g_l^{\mu\nu}) \partial^\gamma \partial_\mu \partial_\nu v_l\right\|_2 \le \sigma_I(m[v_l]) \mathcal{E}_{l,j}^{1/2}.$$

So we get

$$\partial_t \mathcal{E}_{l,j} \le \underbrace{\sigma_I(m[v_l]) \mathcal{E}_{l,j} + C \underbrace{\sum_{|\alpha| \le j} \left\|[\partial^\alpha, g_l^{\mu\nu} \partial_\mu \partial_\nu] v_l\right\|_2}_{\le \sigma_I(m[v_l]) \mathcal{E}_{l,j}^{1/2}} \mathcal{E}_{l,j}^{1/2}}_{\le \sigma_I(m[v_l]) \mathcal{E}_{l,j}}. \tag{2.72}$$

Now we define a set $\mathcal{A}$ as follows

$$\mathcal{A} = \{t \in [T_0, T_1] : \exists l_1 \text{ such that } \forall s \in [T_0, t], l \geq l_1 \text{ it holds} \\ t < T_{l,+} \text{ and } M_k[u_l](s) \leq M_k[u](s) + 1\}.$$

The set $\mathcal{A}$ is non-empty because $T_0 \in \mathcal{A}$. Let $t \in \mathcal{A}$. Then there is an l_1 such that $[T_0, t] \subset (T_{l,-}, T_{l,+})$ and

$$M_k[u_l](s) \leq M_k[u](s) + 1 \tag{2.73}$$

where $s \in [T_0, t]$ and $l \geq l_1$. Due to Proposition 2.6.1 the existence time depends only on the norm of the data and therefore there exists an $\varepsilon > 0$ such that u_l can be extended to an interval of length ε beyond t for all $l \geq l_1$. Additionally we get a bound on $M_k[u_l](s)$ which is uniform in l for $l \geq l_1$ and $s \in [T_0, t + \varepsilon]$ and from the formula we get also a uniform bound on $m[v_l](s)$ for $s \in [T_0, t+\varepsilon]$ and $l \geq l_1$. From (2.72) we see that (2.73) holds on $[T_0, t+\varepsilon]$ if l is large enough. Thus $\mathcal{A}$ is open.

Assume that $t_i \in \mathcal{A}$ where $t_i \to T$. Now we want to show that $T \in \mathcal{A}$. Because $\mathcal{A}$ is connected and $T_0 \in \mathcal{A}$, $T > t_i$ holds. Suppose also that for a suitable l_i (2.73) holds on $[T_0, t_i]$ for $l \geq l_i$. From Proposition 2.6.1 we get an $\varepsilon > 0$ such that by specifying the initial data $u_l(t_i, .), \partial_t u_l(t_i, .)$ at t_i (where $l \geq l_i$), we obtain existence up to $t_i + \varepsilon$ and for $s \in [T_0, t_i + \varepsilon]$ a bound on $M_k[u_l](s)$ independent of $l \geq l_i$. Now we fix an i in such a way that $T - t_i < \varepsilon/2$. Then we get for $l \geq l_i$, $[T_0, T + \varepsilon/2] \subset (T_{l,-}, T_{l,+})$ and a uniform bound on $M_k[u_l](s)$ and also on $m[u_l](s)$ for $l \geq l_i$ and $s \in [T_0, T+\varepsilon/2]$. From (2.72), we get for large enough l, that (2.73) on $[T_0, T + \varepsilon/2]$ holds. Therefore $\mathcal{A}$ is a non-empty, open and closed subset of $[T_0, T_1]$ and so $T_1 \in \mathcal{A}$.

Again by using the equation, we see that there exists an l_1 such that $m[v_l](s)$ is uniformly bounded for $s \in [T_0, T_1]$ and $l \geq l_1$. By inserting this into (2.72), integration and letting $l \to \infty$ the result follows. $\square$

Chapter 3

Developments

One problem with general Lorentz manifolds (M, g) is the occurence of pathologies, such as closed timelike curves on compact Lorentz manifolds. Therefore we want to restrict the manifolds in such a way that we can exclude such pathologies. At first we say a Lorentz manifold (M, g) which does not allows closed timelike curves satisfies the **chronology condition** . Furthermore, if there is for any neighborhood U of $p \in M$ a neighborhood of p with $V \subseteq U$ such that every causal curve with endpoints in V lies completely in U, then we say that the **strong causality condition** holds at p.

Let be $p, q \in M$, then the relation $p \ll q$ means that there exists a future pointing timelike curve from p to q. The pair $p < q$ means the existence of a future pointing causal curve in M from p to q whereas $p \leq q$ means $p < q$ or $p = q$.

A Lorentz manifold (M, g) is called **globally hyperbolic** if it fulfills for all $p \in M$ the strong causality condition and that for every pair $p < q$ the set $J(p, q) = J^+(p) \cap J^-(q)$ is compact, with

$$J^+(A) = \{p \in M : q \leq p \text{ for some } q \in A\},$$

where A is a subset of M and analogously we get $J^-(A)$.There are also sets $I^+(A), I^-(A)$ which we get by taking future/ past pointing timelike curves instead of causal curves.

If we have a subset A of a Lorentz manifold (M, g) such that there is no pair of points $p, q \in A$ which can be connected by a timelike/causal curve it is called **achronal/acausal** . Suppose we have an achronal subset A of M, then we define the **future Cauchy development** of A as

$$D^+(A) = \{p \in M : \text{every past inextendible causal curve through } p \text{ meets } A\}$$

Analogously we define the past Cauchy development and define the Cauchy development as $D(A) = D^+(A) \cup D^-(A)$. Let $\gamma : [0, a) \to M$ be a piecewise smooth curve. If it has a continuous extension $\tilde{\gamma} : [0, a] \to M$ it is called **extendible**, otherwise **inextendible** (casually speaking inextendible means that the curve goes on forever and does not end at a finite point). If a subset $S \subseteq M$ is met exactly once by every inextendible timelike curve in M we call S a **Cauchy (hyper)surface** . Note that a Cauchy hypersurface is a closed achronal topological hypersurface which is met exactly once by every inextendible causal curve. Additionally if M has a Cauchy hypersurface then it is globally hyperbolic see ([2], Lemma 29, p.415).

3.1 Characterization of global hyperbolicity

(see [1], Ch.11) Here we briefly note some facts without proving. It should serve as a collection of statements which we will use later.

$\to$ **Existence of a Cauchy hypersurface.** Here we note some facts from ([1], Ch.11.1). Main statement: A globally hyperbolic Lorentz manifold (M, g) has a smooth spacelike Cauchy hypersurface S and (M, g) is diffeomorphic to $\mathbb{R} \times S$.

Proposition 3.1.1. *Suppose there is a connected, time oriented Lorentz manifold (M, g) and let S be a Cauchy hypersurface. Then it follows that S is a topological manifold and M is homeomorphic to $\mathbb{R} \times S$.*
If additionally S is a smooth, spacelike Cauchy hypersurface and S' another smooth, spacelike Cauchy hypersurface, then M is diffeomorphic to $\mathbb{R} \times S$ and S and S' are diffeomorphic.

Proof. see ([1], Proposition 11.3, p.112)

Theorem 3.1.1. *Assume (M, g) is an oriented and time oriented Lorentz manifold which is globally hyperbolic. Then there is a surjective continuous function $\tau : M \to \mathbb{R}$ which along any causal curve is strictly increasing. If we have an inextendible causal curve $\gamma : (t_-, t_+) \to M$, then we get $\tau[\gamma(t)] \to \pm\infty$ as $t_\pm \to \pm\infty$. Especially, $\tau^{-1}(a)$ is an acausal Cauchy hypersurface for every $a \in \mathbb{R}$. (Similar statements are true for past and future inextendible causal curves.)*

Sketch of proof. see ([1], Theorem 11.6, p.114). Here one constructs first a Lebesgue measure μ and a Borel-σ algebra on subsets of the manifold

(M, g). This is done by the use of a partition of the unitiy on M (see [1], Lemma 10.3, p.96) which gives smooth functions ϕ to define

$$m_i = \int \phi_i \underbrace{\sqrt{\det h}\, dx^1 \wedge ... \wedge dx^n}_{\mu_h}$$

where h is a Riemannian metric on M (see [2], Lemma 25, p.140).This can now used to define a smooth n-form ω in M by

$$\omega = \sum_{i=1}^{\infty} \frac{1}{m_i 2^i} \phi_i \mu_h.$$

If we now use this to define a linear functional

$$\Lambda(f) = \int_M f\omega$$

on the $\mathbb{C}$-valued functions with compact support, we can use the Riesz representation theorem, which gives us the upper positive measure μ and the Borel-σ algebra. So we get the the correlation

$$\int f d\mu = \int_M f\omega$$

for all $f \in C_c(M)$. One shows now that $\mu(U) > 0$ for a non-empty open set U and that $\mu(C_p^+) = 0$ where $C_p^+ = J^+(p) - I^+(p)$ with $p \in M$. This can be done by the help of the exponential map, where one maps a causal geodesic curve from N (the future directed null vectors in T_pM) to M. If all the image points of $\exp_p\big| N$ are critical points, the conditions of Sard's theorem are fulfilled and we get measure zero. The properties of the function $\tau : M \to \mathbb{R}$,

$$\tau(p) = \ln \frac{f_-(p)}{f_+(p)}$$

which consists of two continuous functions $f_\pm(p) = \mu[J^\pm(p)]$ are shown by measure theoretic arguments. It should be noted that the function f_- is strictly increasing and f_+ strictly decreasing along causal curves and if $\gamma : [0, a) \to M$ is a future inextendible causal curve, we get $f_+ \circ \gamma(t) \to 0$ as $t \to a-$. Analogously $f_- \circ \gamma(t) \to 0$ as $t \to b+$ for past inextendible causal curves. Now τ is along causal curves stricly increasing, which gives together with an inextendible causal curve the convergence to ∞ in the case of $t \to a-$ and $-\infty$ in the case of $t \to b+$.

→ **Smooth time functions.** Here we briefly note some facts from ([1], Ch.11.2/3). Main statement: Every globally hyperbolic Lorentz manifold has a temporal function whose level sets are Cauchy hypersurfaces.

Assume that (M, g) is a Lorentz manifold. A **time function** on (M, g) is a function which is strictly increasing along any causal curve and a **temporal function** τ is a smooth function where $\nabla\tau$ is timelike and past directed.

Definition 3.1.1. *Assume $t_- < t_a < t < t_b < t_+$ and define $S_\pm = S_{t_\pm}$ and $S = S_t = \tau^{-1}(t)$. A **temporal step function** around t, compatible with the outer extremes t_-, t_+ and inner extremes t_a, t_b is a function $\sigma : M \to \mathbb{R}$ which fulfills*

- *$\nabla\sigma$ is timelike and past pointing in $V = \{p \in M : \nabla\sigma(p) \neq 0\}$*
- *$\sigma(p) \in [-1, 1]\, \forall p \in M$*
- *$\sigma(p) = \begin{cases} -1 \ldots p \in J^-(S_-) \\ +1 \ldots p \in J^+(S_+) \end{cases}$*
- *$S_{t'} \subset V$, $\forall t' \in (t_a, t_b)$.*

Theorem 3.1.2. *Assume (M, g) is an oriented, time oriented and connected globally hyperbolic Lorentz manifold. Then there exists a temporal function $\mathcal{T} : M \to \mathbb{R}$ and for each inextendible causal curve $\gamma : (t_-, t_+) \to M$ we have $\mathcal{T}[\gamma(t)] \to \pm\infty$ as $t \to t_\pm\mp$. Especially each hypersurface $\mathcal{T}^{-1}(t)$ is a smooth spacelike Cauchy hypersurface.*

Sketch of proof. (see [1], Theorem 11.18, p.123). This proof contains the construction of a convenient function which fulfills the necessary properties of a temporal function. One starts by designing a map h ([1], Proposition 11.10, p.119) which has the properties w.r.t support and gradient (which is timelike and past pointing). This function is then modified into two functions h^+, h^- ([1], Lemma 11.12/13, p.121,122) in which also the sets I^-, I^+, J^- and J^+ are involved while preserving the old properties of h. The next step is the construction of a function σ ([1], Proposition 11.14, p.121) where also the Cauchy hypersurface S_t is contained in the set of points of M where $\nabla\sigma(p) \neq 0$. In ([1], Corollary 11.15, p.122) one finds σ in such a way that there are compact subsets K in the Cauchy hypersufaces from an interval of τ^{-1}. Now we have smooth functions σ_j on compact sets K_j. If one now sum up all these functions and shows convergence one can additionally build out of these functions the desired function $\mathcal{T}$.

$\rightarrow$ **Smooth temporal functions comply to Cauchy hypersurfaces.** Here we briefly note some facts from ([1], Ch.11.4). Main statement: For a spacelike Cauchy hypersurface S, there exists a smooth temporal function $\mathcal{T}$ whose properties are noted in the last subsection. The main theorem in this subsection is

Theorem 3.1.3. *Assume (M, g) is an oriented, time oriented and connected globally hyperbolic Lorentz manifold and S is a smooth spacelike Cauchy hypersurface. Then there exists a smooth function $\mathcal{T} : M \rightarrow \mathbb{R}$ which*

- *has a past directed timelike gradient for all $p \in M$*
- *$\mathcal{T}^{-1}(t)$ is a Cauchy hypersurface for every $t \in \mathbb{R}$*
- *$\mathcal{T}^{-1}(0) = S$*
- *for every inextendible causal curve $\gamma : (t_-, t_+) \rightarrow M$ the following holds $\mathcal{T}[\gamma(t)] \rightarrow \pm\infty$ as $t \rightarrow t_\pm \mp$.*

Sketch of proof. see ([1], Theorem 11.27, p.127). The design of the function $\mathcal{T}$ again precedes in several step by using the concept of a normal neighborhood of submanifolds ([2], p.197-200) and also by using the exponential map.

3.2 Uniqueness/Existence for linear wave equations

(See [1], Ch.12) Here we briefly note some facts without proving them. It should serve as a collection of statements which we will use later. This section deals with the uniqueness of linear wave equations on a globally hyperbolic Lorentz manifold from a geometric viewpoint. The point is, if we have two solutions u_1 and u_2 for the linear wave equation and these solutions and also their normal derivatives coincide on the initial hypersurface Ω they can be extended to the future (past) domain $D(\Omega)$ and we get $u_1 = u_2$ on $D(\Omega)$. It should be noted that the equations only required to be satisfied on the respective sets $(J^-(p) \cap J^+(S),\ D^+(\Omega))$ and the statements hold also for negative time by similar arguments.

Lemma 3.2.1. *Let (M, g) be an $(n+1)$-dimensional Lorentz manifold and S a smooth spacelike Cauchy hypersurface. Assume that p is a point to the future of S and (V, ϕ) are geodesic normal coordinates centered at p such that the set $J^-(p) \cap J^+(S)$ is in V and compact.*

1.) Let $u : V \to \mathbb{R}^l$ be a solution to

$$\Box_g u + Xu + \kappa u = 0$$

with X an $l \times l$-matrix of smooth vector fields on V and κ an $l \times l$-matrix matrix-valued function on V. If now u, ∇u vanish on $J^-(p) \cap S$ it follows that u, ∇u vanish in $J^-(p) \cap J^+(S)$.

2.) If $A \in \mathcal{T}^r_s(M)$, $B \in \mathcal{T}^{r+s+1}_{r+s}(M)$ and $C \in \mathcal{T}^{r+s}_{r+s}(M)$ (smooth tensor fields on M) solves

$$(\Box_g A)^{\alpha_1 \ldots \alpha_r}_{\beta_1 \ldots \beta_s} + B^{\alpha_1 \ldots \alpha_r\, \gamma_1 \ldots \gamma_{s+1}}_{\beta_1 \ldots \beta_s\, \delta_1 \ldots \delta_r} \nabla_{\gamma_1} A^{\delta_1 \ldots \delta_r}_{\gamma_2 \ldots \gamma_{s+1}} + C^{\alpha_1 \ldots \alpha_r\, \gamma_1 \ldots \gamma_s}_{\beta_1 \ldots \beta_s\, \delta_1 \ldots \delta_r} A^{\delta_1 \ldots \delta_r}_{\gamma_1 \ldots \gamma_s} = 0$$

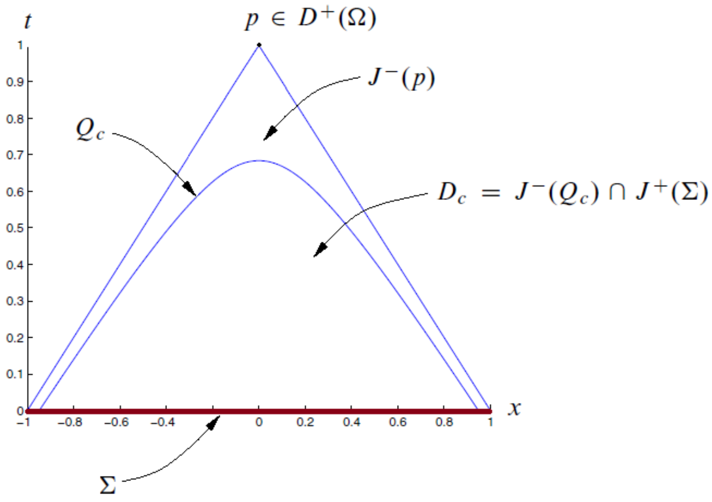
and A, ∇A vanish on $J^-(p) \cap S$ then u, ∇u vanish in $J^-(p) \cap J^+(S)$.

Sketch of proof. see ([1], Lemma 12.8/Corollary 12.10, p.135-140). The geometrical formulation of uniqueness is to show that for an open subset Ω of a spacelike Cauchy hypersurface Σ and a solution u of $\Box_g u + Xu + \kappa u = 0$ which vanishes on Ω and also its normal derivative vanishes on Ω we get that $u = 0$ on $D(\Omega)$.

Let us start with a 2-dim. Minkowski space on Σ (base line, red) the initial data are zero and we want to show that in the triangle the solution also vanishes. Now we take the auxiliary set Q_c which we get by considering the points q in the past of p and note all the points for which the timelike vector $q - p$ has squared length $c < 0$. Then we have $D_c = J^-(Q_c) \cap J^+(\Sigma)$. The point is now that $\overline{\bigcup_{c<0} D_c} \equiv D$. So one proves the vanishing of the solution any D_c. This can be done by constructing a smooth vector field ξ over D_c and integrating the divergence of the vector field by using

$$\int_M \operatorname{div} \xi \, \varepsilon_M = \int_{\partial M} \frac{\langle \xi, N \rangle}{\langle N, N \rangle} \varepsilon_{\partial M}$$

where $\varepsilon_M, \varepsilon_{\partial M}$ are volume forms. Now one concludes that the boundary terms are vanish and so we have $u = 0$ on D_c. If we have a general Lorentz manifold this equation does not apply due to the fact that the region is not a manifold with boundary. To make this working one can subtract where the



hyperbola with the $t = 0$ hypersurface intersects from the region of interest to get a convenient region. An additional problem is that the vector field cannot be assumed to have compact support and therefore one has to apply a proper cut-off function (this can be done by using a tubular neighborhood of the above intersection ([1], Section 12.1, p.131-135)). So in the setting of a general Lorentz manifold it is more challenging, but nevertheless the idea is the same.

Corollary 3.2.1. *Assume (M, g) is a oriented, time-oriented and connected globally hyperbolic $(n+1)$-dimensional Lorentz manifold, S a smooth space-like Cauchy hypersurface and $\Omega \subseteq S$.*

1.) If $A \in \mathcal{T}^r_s(M)$, $B \in \mathcal{T}^{r+s+1}_{r+s}(M)$ and $C \in \mathcal{T}^{r+s}_{r+s}(M)$ (smooth tensor fields on M) solves

$$(\Box_g A)^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s} + B^{\alpha_1\ldots\alpha_r\,\gamma_1\ldots\gamma_{s+1}}_{\beta_1\ldots\beta_s\,\delta_1\ldots\delta_r} \nabla_{\gamma_1} A^{\delta_1\ldots\delta_r}_{\gamma_2\ldots\gamma_{s+1}} + C^{\alpha_1\ldots\alpha_r\,\gamma_1\ldots\gamma_s}_{\beta_1\ldots\beta_s\,\delta_1\ldots\delta_r} A^{\delta_1\ldots\delta_r}_{\gamma_1\ldots\gamma_s} = 0$$

and A, ∇A vanish on Ω then u, ∇u vanish in $D^+(\Omega)$.

2.) Assume that $D^+(\Omega) \subseteq U$ with U open and $u : V \to \mathbb{R}^l$ satisfy

$$\Box_g u + Xu + \kappa u = 0$$

with X an $l \times l$ matrix of smooth vector fields on U and κ an $l \times l$ matrix-valued function on U. If u, ∇u vanish on Ω then u, ∇u vanish in $D^+(\Omega)$.

Proof. see ([1], Corollary 12.12/12.14, p.140-141).

Theorem 3.2.1. *Assume (M, g) is a oriented, time-oriented and connected globally hyperbolic $(n+1)$-dimensional Lorentz manifold, S a smooth spacelike Cauchy hypersurface and t a temporal function as in Theorem* 3.1.3 *where $S = t^{-1}(0)$ and we write $S_{t_0} = t^{-1}(t_0)$ for $t_0 \in \mathbb{R}$. Let $B \in \mathcal{T}^{r+s+1}_{r+s}(M)$, $C \in \mathcal{T}^{r+s}_{r+s}(M)$ and $E \in \mathcal{T}^{r}_{s}(M)$. Given smooth tensor fields $A_0, A_1 \in \mathcal{T}^{r}_{s}(M)$, there exists a smooth tensor field $A \in \mathcal{T}^{r}_{s}(M)$ which solves the initial value problem*

$$\begin{aligned}(\Box_g A)^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s} + B^{\alpha_1\ldots\alpha_r\,\gamma_1\ldots\gamma_{s+1}}_{\beta_1\ldots\beta_s\,\delta_1\ldots\delta_r}\nabla_{\gamma_1}A^{\delta_1\ldots\delta_r}_{\gamma_2\ldots\gamma_{s+1}}&\\ +C^{\alpha_1\ldots\alpha_r\,\gamma_1\ldots\gamma_s}_{\beta_1\ldots\beta_s\,\delta_1\ldots\delta_r}A^{\delta_1\ldots\delta_r}_{\gamma_1\ldots\gamma_s} &= E^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s},\\ A(p) &= A_0(p)\\ \nabla_N A(p) &= A_1(p)\end{aligned} \tag{3.1}$$

$\forall p \in S$ and N is the future directed unit normal vector to S_τ which is $N = \frac{-\nabla t}{|\nabla t|}$.

Sketch of proof. See ([1], Theorem 12.17/12.19, p.142-144). The idea is that one apply Theorem 2.5.1 from which one gets the existence for a global solution. To use this theorem it is important to take a viewpoint on this equation as an equation on $\mathbb{R}^{n+1}$. Therefore one have to find the right set of charts (here it is important that the necessary causal structures are compact and in the respective charts). After applying Theorem 2.5.1 one have to go the way back to the manifold level and show the validity of the solution.

3.3 Existence of a globally hyperbolic development

(See [1], Ch.14.3). Here we continue the work from the first chapter (General Relativity) and show the existence of a globally hyperbolic development (ghd). The functions ϕ, metrics $g_{\mu\nu}$, etc. are defined there. Since we have now collected the necessary know-how in the field of non-linear PDE's we can use them for the next theorem.

Theorem 3.3.1. *Let $(\Sigma,\, g_0,\, k,\, \phi_0,\, \phi_1)$ be initial data to*

$$\begin{aligned}\mathrm{Ric} - d\phi\otimes d\phi - \frac{2}{n-1}V(\phi)g = 0\\ \Box_g\phi - V'(\phi) = 0.\end{aligned} \tag{3.2}$$

Then there is a globally hyperbolic development (ghd) of the data.

Sketch of proof. (See [1], Theorem 14.2, p.156-158). First a convenient matrix $A_{\mu\nu}$ is constructed by suitable choosing (w.r.t. bounds) $A_{00} = g_{00}$, $A_{0i} = g_{0i}$ and $A_{ij} = g_{ij}$ the latter one to be everywhere positive definite also with appropriate bounds. The bounds are chosen w.r.t the necessary conditions which we get from the PDE theory for our problem. After this the ivp would be modified in that the $g^{\mu\nu}$ are substituted by $A^{\mu\nu}$ and the modified Ricci tensor is multiplied by a mollifier and so are the initial data. The resulting equation can now be considered as an equation on $\mathbb{R}^{n+1}$ and Corollary 2.6.1 is applicable, from which we obtain a local smooth solution. From the smoothness one get neighborhoods W_p of $p \in \mathbb{R} \times \Sigma$ (or take two points $p, q \in \mathbb{R} \times \Sigma$ with $W_p \cap W_q \neq \emptyset$), which can at the end be used to build up the manifold M by union of all the W_p.

Now one has to design a suitable metric on M. This works due to the closures of the $W_p's$ being contained in open sets (e.g. W_1 and W_2 for the sets W_p and W_q) which we can, when endowed with with coordinates, consider as charts for the manifold. Assume u to be the vector which assembles ϕ and $g_{\mu\nu}$ for $\mu, \nu = 0, ...n$. From the structure of the initial value problem and the constructed (but permissible) modification we made, one can show that u_i, or more exactly the respective ϕ_i and $g_{i\mu\nu}$ (with i as an index for the different points) are solutions of the ivp and if the initial data coincide the resulting solutions also coincide (on $W_p \cap W_q$ for two points $p, q \in \mathbb{R} \times \Sigma$). In the end one obtains the manifold as the union of all the W_p and the embedding $i : \Sigma \to M$ as the inclusion $i(p) = (0, p)$. Furthermore $i^*g = g_0, i^*K = k, i^*\phi = \phi_0$ and $i^*(N\phi) = \phi_1$. Finally it turns out that Σ is a Cauchy hypersurface.

Theorem 3.3.2. *(Two developments are extensions of a common development) Let* $(\Sigma, g_0, k, \phi_0, \phi_1)$ *be initial data to* (3.2)*. If we have two developments* (M_a, g_a, ϕ_a) *and* (M_b, g_b, ϕ_b) *with the respective embeddings, then we have a globally hyperbolic development* (M, g, ϕ) *with corresponding embedding* $i : \Sigma \to M$ *and smooth maps* $\psi_a : M \to M_a$ *and* $\psi_b : M \to M_b$ *which are time orientation preserving. These maps are diffeomorphisms onto their image s.t.* $\psi_a^* g_a = g$, $\psi_a^* \phi_a = \phi$, $\psi_b^* g_b = g$ *and* $\psi_b^* \phi_b = \phi$*. Furthermore we have* $\psi_a \circ i = i_a$ *and* $\psi_b \circ i = i_b$*.*

Proof. see ([1], Theorem 14.3, p.158-163).

3.4 Cauchy stability

(See [1], Ch.15). Here we briefly note some facts without proving them. It should serve as a collection of statements which we will use later.

Definition 3.4.1. ***(Sobolev spaces on manifolds)*** *Let M be a compact n-dimensional manifold and $\phi_i, i = 1, ..., l$ a finite partition of unity such that $\phi_i \subset U_i$ where the U_i are open sets. Assume that (x_i, U_i) are coordinates. Let be $T \in \mathcal{T}^r_s(M)$ then we define the norm as follows:*

$$\|T\|_{H^k} = \left(\sum_{i=1}^{n} \sum_{j_1,...,j_r=1}^{n} \sum_{i_1,...,i_s=1}^{n} \sum_{|\alpha|\leq k} \int_{x_i(U_i)} (\phi_i \left|\partial^\alpha T^{j_1,...,j_r}_{i_1,...,i_s}\right|^2) \circ x_i^{-1} dx_i^1 ... dx_i^n \right)^{1/2}$$

Here $T^{j_1,...,j_r}_{i_1,...,i_s}$ are the components of T relative to the coordinates x_i.

If we use different partitions of unity we get for sure different norms, but the next lemma shows that they are equivalent and so, if we have a sequence $T_j \to T$ with respect to one norm (say $|.|_{H^k}$), then we get also $T_j \to T$ with respect to the other norm (say $\|.\|_{H^k}$). Therefore one can say $T_j \to T$ in H^k without referring to the partition of unity.

Lemma 3.4.1. ***(Equivalent norms)****. Assume that (ϕ_i, U_i), $i = 1, ..., l$ and also (ψ_i, U_i), $i = 1, ..., m$ are partitions of unity which satisfying the properties as fixed in the previous definition. Furthermore the corresponding norms are $\|.\|_{H^k}$ and $|.|_{H^k}$ as in the previous definition. Then the norms are equivalent, i.e. there are $C_{i,k} > 0$, $i = 1, 2$, depending on k, r, s and on the partitions of unity, such that*

$$C_{1,k} |T|_{H^k} \leq \|T\|_{H^k} \leq C_{2,k} |T|_{H^k}$$

$\forall T \in \mathcal{T}^r_s(M)$.

Proof. see ([1], Theorem 15.3, p.165).

Definition 3.4.2. ***(Canonical foliation)*** *Let $(\overline{M}, g)$ be an $(n+1)$-dimensional Lorentz manifold. A canonical foliation of $(\overline{M}, g)$ is a diffeomorphism*

$$\chi : \mathbb{R} \times M \to \overline{M}$$

for a smooth n-dimensional manifold M, where ∂_t is timelike and the hypersurfaces $\{t\} \times M$ are spacelike with respect to $\chi^* g$.

Note. A Lorentz manifold does not necessary admit canonical foliations.

Proposition 3.4.1. *Assume $(\overline{M}, g)$ is an oriented, time oriented, connected and globally hyperbolic Lorentz manifold. Then $(\overline{M}, g)$ has a canonical foliation. If M_0 is a spacelike Cauchy hypersurface, then the n-dimensional manifold M from Definition* 3.4.2 *can be chosen to be M_0 and the diffeomorphism $\chi(0, p) = p$. If in contrast $(\overline{M}, g)$ has a canonical foliation $\chi : \mathbb{R} \times M \to \overline{M}$ with M compact, then $(\overline{M}, g)$ is globally hyperbolic.*

To give a precise statement about stability, we have to introduce the notion of a background solution.

Definition 3.4.3. ***(Background solutions)*** *Assume M to be a compact n-dim. manifold and g a smooth Lorentz metric on $I \times M$, with I an open interval. Assume t to be the function $t(\tau, p) = \tau$, $g(\partial_t, \partial_t) = g_{00} < 0$ and the hypersurfaces , $\{\tau\} \times M$ (with $\tau \in I$) are spacelike with respect to g. If $\phi : I \times M \to \mathbb{R}$ is a smooth function, which together with g, fulfills*

$$\begin{aligned} \mathrm{Ric} - d\phi \otimes d\phi - \frac{2}{n-1} V(\phi) g &= 0 \\ \Box_g \phi - V'(\phi) &= 0. \end{aligned} \tag{3.3}$$

then we call $(I \times M, g, \phi)$ a background solution .

Definition 3.4.4. ***(Induced initial data)*** *Assume g is a Lorentz metric and ϕ a smooth function on $(T_-, T_+) \times M$ with $\tau \in (T_-, T_+)$. Furthermore, $\{\tau\} \times M$ should be spacelike with respect to g and $i : M \to \mathbb{R} \times M$ be defined by $i(p) = (\tau, p)$. Assume g_0 to be the Riemannian metric on M which we get by pulling back the Riemannian metric induced on $\{\tau\} \times M$ by g through i and also let k_0 be the covariant 2-tensor which we get by pulling back the second fundamental form induced on $\{\tau\} \times M$ by g through i. Finally, let $\phi_0 = \phi \circ i$ and $\phi_1 = (N\phi) \circ i$, where N is the future directed unit normal to $\{\tau\} \times M$.*
Then we call $(g_0, k_0, \phi_0, \phi_1)$ the initial data induced on $\{\tau\} \times M$ by (g, ϕ), or simply the initial data induced on $\{\tau\} \times M$ if it is clear from the context.

Theorem 3.4.1. ***(Cauchy stability in general relativity)*** *Assume that $(I \times M, g, \phi)$ is a background solution and $(g_0, k_0, \phi_0, \phi_1)$ the initial data induced on $\{0\} \times M$ by (g, ϕ). Let ρ_j be a sequence of Riemannian metrics on M, κ_j a sequence of covariant 2-tensors and let $\psi_{0,j}, \psi_{1,j}$ be sequences of functions which are smooth such that*

$$\left.\begin{aligned} \rho_j &\to g_0 \\ \psi_{0,j} &\to \phi_0 \end{aligned}\right\} \text{ in } H^{l+1} \qquad \text{and} \qquad \left.\begin{aligned} \kappa_j &\to k_0 \\ \psi_{1,j} &\to \phi_1 \end{aligned}\right\} \text{ in } H^l,$$

where $l > n/2+1$. Assume that $(\rho_j, \kappa_j, \psi_{0,j}, \psi_{1,j})$ satisfy the constraint equations from Definition 1.0.1

$$r - k_{ij}k^{ij} + (tr\, k)^2 = \phi_1^2 + D^i\phi_0 D_i\phi_0 + 2V(\phi_0)$$
$$D^j k_{ij} - D_i(tr\, k) = \phi_1 D_i \phi_0$$

with $(g_0, k_0, \phi_0, \phi_1)$ replaced by $(\rho_j, \kappa_j, \psi_{0,j}, \psi_{1,j})$. Then

1.) *there are $T_{j,+}, T_{j,-} \in \mathbb{R}$, a Lorentz metric h_j on $\overline{M}_j = (T_{j,-}, T_{j,+}) \times M$ and a smooth function ϕ_j on $\overline{M}_j$ such that (h_j, ϕ_j) satisfy* (3.3) *on $\overline{M}_j$.*

2.) *the initial data induced on $\{0\} \times M$ by (h_j, ψ_j) are $(\rho_j, \kappa_j, \psi_{0,j}, \psi_{1,j})$, ∂_t is timelike with respect to h_j and $\{\tau\} \times M$ is a spacelike Cauchy hypersurface with respect to h_j for all $\tau \in (T_{j,-}, T_{j,+})$.*

3.) *if $T \in I$, then $T \in (T_{j,-}, T_{j,+})$ for j large enough and the initial data induced on $\{T\} \times M$ by (h_j, ψ_j) converge to the corresponding initial data of (g, ϕ).*

Sketch of proof. see ([1], Theorem 15.10, p.167-175). We search for a set $(\rho_j, \kappa_j, \psi_{0,j}, \psi_{1,j})$ a Lorentz metric h and a smooth function ψ which satisfy

$$\hat{R}_{\mu\nu} - \nabla_\mu\phi\nabla_\nu\phi - \frac{2}{n-1}V(\phi)g = 0 \tag{3.4}$$
$$\nabla^\mu\nabla_\mu\phi - V'(\phi) = 0 \tag{3.5}$$

First we have to construct the initial data for all the components of h and also their time derivatives. this can be done by using the initial data for the spatial components ($h_{im}|_{t=0} = \rho_j(\partial_i, \partial_m)$) and for the residual components one uses the connection to the unit normal N on $\{0\} \times M$. Although $h_{\mu\nu} \stackrel{j\to\infty}{\longrightarrow} g_{\mu\nu}$ in H^{k+1} some care is required due to the fact that neither $g_{\mu\nu}$ nor $h_{\mu\nu}$ are intrinsic to $\{0\} \times M$. If we conversely regard $g_{\mu\nu}$ as

- a Riemannian metric g_{ij},
- a one form g_{0i} and
- a function g_{00}

then we can interpret $g_{\mu\nu}|_{t=0}$ as consisting of three objects which are intrinsic to $\{0\} \times M$. To obtain $\partial_t h_{\mu\nu}|_{t=0}$ one uses the properties of the second fundamental form. From this (by additionally checking existence and tensor properties) one gets the expression:

$$g_{\mu\nu}|_{t=0} = \partial_m h_{i0} - \partial_i h_{m0} + \frac{2}{N^0}\left[\kappa_j(\partial_i, \partial_m) - \partial_i(N^0)h_{0m} - \langle \nabla_{\partial_i} X, \partial_m \rangle\right]$$

with X defined by $N = N^0\partial_t + X$. For the remaining time derivatives $\partial_t h_{00}|_{t=0}$ and $\partial_t h_{0i}|_{t=0}$ we demand that $\mathcal{D}_\mu|_{t=0} = 0$. Here some work on the Christoffel symbol, respectively on its contraction is necessary, to see the right behavior of the respective terms (and to extract $\partial_t h_{00}$) at $t = 0$ and to the independence of the choice of coordinates on M. From this (mainly technically) work one gets the right behavior and the convergence from h_j to g for $j \to \infty$. Finally one gets $\psi|_{t=0}$ from the initial data and the derivative again with the help of the unit normal N. From this we also get the convergence of the ψ and $\partial_t\psi$ to the ϕ and $\partial_t\phi$ in H^{l+1} and H^l for $j \to \infty$.

Geometric setup. Finally we want prove that

- If $T \in I$ then it is in the existence interval of (h_j, ψ_j) as long as j is large enough.
- The initial data induced on $\{T\} \times M$ converges to the initial data of (g, ϕ).

Due to the fact that the manifold we have is compact, we may argue in coordinate charts. If we now have a look at the Riemannian metrices which are induced by g on $\{\tau\} \times M$ and we split them again up the into the corresponding metric on M, which we will call $g_\flat(\tau)$ and a 1-form on each hypersurface $\{\tau\} \times M$ we find arguments for constant boundaries (C_1, C_2 and C_3) for the respective parts of the metric (with respect to a fixed Riemannian metric ρ on M).

If there is a causal curve γ with respect to g and the projection onto M is denoted by $\gamma_\flat$, such that $\gamma = (\gamma^0, \gamma_\flat)$ we get a constant $C > 0$ for a causal curve $\gamma : [a, b] \to I \times M$, with respect to g such that

$$l_\rho(\gamma_\flat) \leq C\left(t[\gamma(b)] - t[\gamma(a)]\right),$$

where $l_\rho(\gamma_\flat)$ is the length of $\gamma_\flat$ with respect to a fixed Riemannian metric on M and $t \circ \gamma$ takes values in a proper interval. We obtain this after some calculations, where the expressions in local coordinates are used and the constant restriction for the different parts is taken into account. The constant C depends only on the constants C_1, C_2 and C_3 for the metric parts. So we get for all families of metrics for which such bounds hold, such an estimate with a constant that is regardless of the choice of the Lorentz metric in the considered family.

Local existence. Let B_r be a covering of M and consider equations (3.4), (3.5) on $V_i = I \times B_r$. Let $x^1, ..., x^n$ be normal coordinates on the elements of the covering on V_i. In the following we focus our attention to the set V_i on which the above coordinates are defined. All objects which are used are defined in terms of the specified coordinates. Let A be a smooth function which maps Lorentz matrices to Lorentz matrices, which is also in the range of positive definite $n \times n$.matrices. The different parts of the matrix have to be specified in suitable compact sets. Now we replace $h^{\mu\nu}$ in (3.4), (3.5) by $A^{\mu\nu}$ and modify also F_μ in an admissible way. Building on this setup we can use the local existence result. Finally there are some additional technical tasks (modifying the initial data by a mollifier) to apply Corollary 2.5.1, from which we get a local smooth solution u.

Patching together. Due to the fact that it is not enough to start always at $\tau = 0$, but to start at any $\tau \in [0, T]$ there is still some work left. Now let $h_{\mu\nu}|_{t=\tau}$, $\partial_t h_{\mu\nu}|_{t=\tau}$, and $\psi|_{t=\tau}$, $\partial_t \psi|_{t=\tau}$ be the initial data (now at τ). As in the last paragraph we interpret $\partial_t h_{\mu\nu}|_{t=\tau}$ as a metric consisting of different parts. Additionally, as before, we assume that the initial data are such that they satisfy the constraint equations and also $\mathcal{D}_\mu|_{t=\tau} = 0$.

We start at $\{\tau\} \times B_r(p)$, where $B_r(p)$ is the set of points in M, which have a distance strictly less than r from p, where the distance is measured w.r.t. ρ. We use again the matrix A and demand that χ is a cutoff function. To patch the solutions together we have to cut the initial data in a proper way and pull them back to $\mathbb{R}^n$ (same for the initial data of $g_{\mu\nu}$ and ϕ) and then solve the modified equations. In this way we get solutions u_j and u. It should be mentioned that all this happens in the proper Sobolev spaces. In this way we get for the solutionalso an existence time. After this one has to show that, on a suitable set, the solution u coincides with (g, ϕ). To this end, global hyperbolicity and the use of suitable Cauchy hypersurfaces are necessary to formulate an equation regarding the difference of the two solutions (u_j, u). After this is done one has to compare $h_{j,i,\tau}, \psi_{j,i,\tau}$ (which is now a solution on a specific set for the original equations) for two different $i's$, say i_1, i_2. This also turns out to be working properly and can be iterated for $i = 1$ to N. So we get convergence of the perturbed initial data on, say, $\tau' \in (\tau - \varepsilon_0, \tau + \varepsilon_0)$ to the initial data of the background solution.

3.5 Existence of a MGHD

In this chapter the main result is proven. An important point in the proof is the axiom of choice, so below we provide a short reminder on this topic.

3.5.1 Set theory reminder

Definition 3.5.1. ***(partial ordering, partially ordered set)*** *A partial ordering on a set X is a relation $\leq$ on X which is*

- *Reflexive: $a \leq a$, $\forall a \in X$*
- *Antisymmetric: $a \leq b$ and $b \leq a \Rightarrow a = b$, $\forall a, b \in X$*
- *Transitive: $a \leq b$ and $b \leq c \Rightarrow a \leq c$, $\forall a, b, c \in X$*

We call set together with a partial ordering a partially ordered set .

Definition 3.5.2. ***(totally ordered)*** *We call a partially ordered set totally ordered if $a, b \in X$ implies $a \leq b$ or $b \leq a$.*

Definition 3.5.3. ***(upper bound, maximal element)*** *If $(X, \leq)$ is a partially ordered set and $A \subseteq X$, then $x \in X$ is an upper bound for A if $a \in A$ implies $a \leq x$. A maximal element of X is an $x \in X$ such that $x' \in X$ implies $x' \leq x$.*

Theorem 3.5.1. *We have an equivalence between the next statements*

- ***(Maximality principle)*** *Assume that X has a partially ordering in that way that every totally ordered subset has an upper bound, then there exists a maximal element in X.*
- ***(Axiom of choice)*** *If $\{S_\alpha : \alpha \in A\}$ is an indexed family of nonempty sets S_α, then there is a function $f : A \to \bigcup S_\alpha$ such that $f(\alpha) \in S_\alpha$ for all $\alpha \in A$.*

3.5.2 Existence of a maximal ghd

As remarked before, we call a development (M, g, ϕ) of the initial data $(\Sigma, g_0, k, \phi_0, \phi_1)$ to (3.3) such that $i(\Sigma)$ is a Cauchy hypersurface in M a globally hyperbolic development of the initial data.

Definition 3.5.4. ***(maximal globally hyperbolic development)*** *Let $(\Sigma, g_0, k, \phi_0, \phi_1)$ be initial data to* (3.3)*. We call a globally hyperbolic development (M, g, ϕ), with an associated embedding $i : \Sigma \to M$, a maximal globally hyperbolic development of the initial data if there is for every other globally hyperbolic development (M', g', ϕ') with an associated embedding $i' : \Sigma \to M'$ a function $\psi : M' \to M$ which is a time orientation preserving diffeomorphism onto its image in that way that the following holds $\psi^* g = g'$, $\psi^* \phi = \phi'$ and $\psi \circ i' = i$.*

If we have two maximal globally hyperbolic developments, (M, g, ϕ) and (M', g', ϕ'), of the same initial data and the respective maps $\psi : M \to M'$ and $\psi' : M' \to M$, then we have $(\psi' \circ \psi)^* g = g$ and $\psi' \circ \psi$ is the identity map on $i(\Sigma)$. Assume that $p \in J^-[i(\Sigma)]$ and γ is a future directed inextendible timelike geodesic with $\gamma(0) = p$, then there is an $s_0 > 0$ such that we have

$$\gamma(s_0) \in i(\Sigma).$$

and so $\psi' \circ \psi \circ \gamma(s_0) = \gamma(s_0)$. Additionally, $\psi' \circ \psi$ maps a tangent vector for $q \in i(\Sigma)$ to itself and due to the fact that it is a time orientation preserving isometry, it maps also the future directed unit normal on itself, that is $(\psi' \circ \psi)_*$ is the identity on the tangent space $T_q M$ for $q \in i(\Sigma)$. From this we get $(\psi' \circ \psi \circ \gamma)'(s_0) = \gamma'(s_0)$ so that $\psi' \circ \psi \circ \gamma' = \gamma$ and $\psi' \circ \psi(p) = p$. An analogous argument holds for the inverse time direction (i.e. $p \in J^+[i(\Sigma)]$) and so we finally conclude that

$$\psi' \circ \psi = Id \text{ and similarly } \psi \circ \psi' = Id.$$

From this we see that M and M' are diffeomorphic with a diffeomorphism $\psi : M \to M'$ such that $\psi^* g' = g$, $\psi^* \phi' = \phi$ and $\psi \circ i = i'$. So we can say the above developments are **isometric**. This means that the diffeomorphism is indeed an isometry and it preserves the scalar fields (which is the same thing as $\psi^* \phi = \phi'$ in Definition 3.5.4) and respects also the embedding. So we can say that:

"Up to isometry we can say that a maximal globally hyperbolic development is unique."

If we are given initial data $(\Sigma, g_0, k, \phi_0, \phi_1)$ to the system (3.3), we denote the family of all globally hyperbolic developments by $\mathcal{M}$. Due to the fact that a globally hyperbolic development (M, g, ϕ) has the property that $M \cong \mathbb{R} \times i(\Sigma)$ (where $\cong$ signifies the existence of a diffeomorphism) and because i is a diffeomorphism from Σ to $i(\Sigma)$ we get $M \cong \mathbb{R} \times \Sigma$. From this viewpoint we can look at $\mathcal{M}$ as a subset of pairs (g, ϕ) on $\mathbb{R} \times \Sigma$, where g is the Lorentz metric and ϕ a smooth function. Next we are interested in the collection of equivalence classes of globally hyperbolic developments in which the isometry is the inherent equivalence relation.

Theorem 3.5.2. *Let $(\Sigma, g_0, k, \phi_0, \phi_1)$ be initial data to*

$$\mathrm{Ric} - d\phi \otimes d\phi - \frac{2}{n-1} V(\phi) g = 0$$
$$\Box_g \phi - V'(\phi) = 0.$$

Then there exists a maximal globally hyperbolic development of the initial data which is up to isometry unique.

Proof. From Theorem 3.3.1 we know that $\mathcal{M} \neq \emptyset$. Let $N, N' \in \mathcal{M}$ (we write briefly N instead of (N, g, ϕ).) Now we look at N and get from Theorem 3.3.2 that there is an open subset U of N and we have a time orientation preserving isometry ψ from U onto an open subset of N'. Additionally U is a globally hyperbolic development of the initial data and ψ respects the embedding.

Let $C(N, N')$ be the collection of the pairs (U, ψ), where

- U is an open subset which contains $i(\Sigma)$, with $i(\Sigma)$ a Cauchy hypersurface in U.
- ψ is an isometry from U to an open subset of N' such that $\psi \circ i' = i$.

Note. We remark that there is an error in [1]. The idea, which came from [7] was wrongly continued and leads to the error about the statement of upper limits under the given conditions. Afterwards we will describe the idea and do some modifications (as described in [6]) to handle this difficulty. Indeed there is a more elegant way to perform this proof (see [6]). The idea from [7] was to find a point $p' \in H^+$ (where $p \in N, p' \in N'$ are points, that can't be openly separated and H^+ is the set of this points and is also restricted to $J^+(i(\Sigma)))$ and a spacelike hypersurface S, in N such that $S \setminus \{p'\} \subset U$. Given such a point with such a hypersurface one can argue that $\psi(S \setminus \{p'\}) \cup p$ is a spacelike hypersurface in N, which we will call S'. Additionally ψ can be extended to a smooth map $(S \to S')$ which maps the initial data induced on S (by (g, φ)) to the initial data induced on S' (by (g', φ')). It can then be shown (see [6]) that ψ can be extended. But this would be a contradiction to the statement that (U, ψ) is maximal. So if one can prove that H^+ is non-empty, there exists such a point p and the above spacelike hypersurface with the respective properties. We will refer later (above the definition of φ) to this.

Additionally (for later use, definition of φ) we will introduce further conditions on U. We say that U has a *spacelike boundary at an appropriate point, say* p', if for $p' \in \partial U$, there is a spacelike hypersurface $S \subseteq N$, such that $p' \in S$ and $S \setminus \{p'\} \subset U$. If U, U' are two globally hyperbolic developments in N, where both have spacelike boundaries at an appropriate point, then $U \cup U'$ is also a globally hyperbolic development with spacelike boundary at an appropriate point.

Now we will show that there is a globally hyperbolic development $U \subseteq N$ with spacelike boundary. Let $U \subseteq N$ be a globally hyperbolic development whose boundary need not to be spacelike. Suppose that $i : \Sigma \to N$ is the corresponding embedding to N. If now $q \in i(\Sigma)$, we can assume that there is a neighbourhood $O_q \subseteq U$ of q such that O_q is globally hyperbolic with Cauchy hypersurface $O_q \cap i(\Sigma)$. By the help of Lemma 12.5 (see [1], p.135, where local coordinates were constructed) we can design O_q in that way way that the boundary is spacelike. By taking the union of all the O_q we get finally the wanted development, which we will call U.

Let us have a look at two of them, namely (U, ψ) and $(\tilde{U}, \tilde{\psi})$ and suppose furthermore that $p \in U \cap \tilde{U}$. Assume that γ is an inextendible timelike geodesic in $U \cap \tilde{U}$. Due to $U \cap \tilde{U}$ being globally hyperbolic we get $\gamma(s_0) \in i(\Sigma)$ for some s_0. Similarly to the introductory arguments before the Theorem, we have $\psi \circ \gamma(s_0) = \tilde{\psi} \circ \gamma(s_0)$ and analogously for the first derivative and so $\psi \circ \gamma = \tilde{\psi} \circ \gamma$ so that we can conclude $\psi(p) = \tilde{\psi}(p)$. As a consequence,there is an isometry from $\tilde{U} \cup U$ into N' and (see above) $\tilde{U} \cup U$ is globally hyperbolic with spacelike boundary. Now we apply Theorem 3.5.1 to deduce that $C(N, N')$ has a maximal element (U, ψ), which by the last argument is also unique.

If $U = N$ we write $N \leq N'$. Because the maximal elements of $C(N, N')$ are unique, the relation $\leq$ induces a partial ordering on $\mathcal{M}$, where two elements are identical if they are isometric. Assume that $\{N_\alpha\}$, $\alpha \in A$ is a totally ordered subset of $\mathcal{M}$. In case of $N_\alpha \leq N_\beta$, there exists a map $\psi_{\beta\alpha} : N_\alpha \to N_\beta$ with the properties as above. Let

$$K = \coprod_{\alpha \in A} N_\alpha$$

be the disjoint union of the N_α and let us establish the following equivalence relation on K by stating that the two elements $p_\alpha \in N_\alpha$ and $p_\beta \in N_\beta$ are equivalent ($p_\alpha \sim p_\beta$), if one of the following statements are true:

$$N_\alpha \leq N_\beta, \;\; \psi_{\beta\alpha}(p_\alpha) = p_\beta \;\; \text{or} \;\; N_\beta \leq N_\alpha, \;\; \psi_{\alpha\beta}(p_\beta) = p_\alpha$$

We denote by Q the quotient $(K/\sim)$ of K with respect to this equivalence relation.

Q is a topological Hausdorff space: Let be $p_\alpha \in N_\alpha$, then we define

$$\pi : K \to Q$$
$$\pi(p_\alpha) = [p_\alpha]$$

and write for the resulting equivalence class $q = [p_\alpha]$. The open sets for the topology on Q are defined to be the union of finite intersections of sets of the form $\pi(U_\alpha)$, where $U_\alpha \subseteq N_\alpha$, with U_α open. Thus Q is a topological space. For the Hausdorff property assume $q_1, q_2 \in Q$ such that $q_1 \neq q_2$ and $p_\alpha \in q_1$ and $r_\beta \in q_2$. Without loss of generality we assume $N_\alpha \leq N_\beta$. Then it follows that $[\psi_{\beta\alpha}(p_\alpha)] = q_1$ and $[\psi_{\beta\alpha}(p_\alpha)] \neq r_\beta$ and we have disjoint open neighborhoods U_β and V_β of $\psi_{\beta\alpha}(p_\alpha)$ respective r_β. Finally $\pi(U_\beta)$ and $\pi(V_\beta)$ are disjoint open neighborhoods containing q_1 and q_2, respectively, and so Q is Hausdorff.

Q has a differentiable structure: Assume $q \in Q$, then there is a $p_\alpha \in q$. Let U_α be an open neighborhood of p_α with coordinates x_α. We are able to define an injective map y_α on $\pi(U_\alpha)$ by

$$y_\alpha([q_\alpha]) = x_\alpha(q_\alpha)$$

due to the fact that given any $q \in \pi(U_\alpha)$, there exists a unique $q_\alpha \in U_\alpha$ with $\pi(q_\alpha) = q$. Because we want the y_α to establish coordinate charts, we have first to show that the y_α are homeomorphisms. Because $y_\alpha^{-1}(W) = \pi[x_\alpha^{-1}(W)]$ for an open set W, we conclude the continuity of y_α. Because of

$$\pi(U_\alpha) \cap \pi(U_\beta) = \pi[U_\alpha \cap \psi_{\alpha\beta}(U_\beta)] \;\; \text{or} \;\; \pi(U_\alpha) \cap \pi(U_\beta) = \pi[U_\alpha \cap \psi_{\alpha\beta}^{-1}(U_\beta)],$$

any open subset of $\pi(U_\alpha)$ can be written as $\pi(V_\alpha)$ for some open subset $V_\alpha \subseteq U_\alpha$. Out of this it follows that y_α is a homeomorphism, because it takes open sets to open sets. Let y_β be similarly defined on $\pi(U_\beta)$ and suppose that $\pi(U_\alpha) \cap \pi(U_\beta) \neq \emptyset$. Again let $N_\alpha \leq N_\beta$, without loss of generality. Then we get $y_\beta \circ y_\alpha^{-1} = x_\beta \circ \psi_{\beta\alpha} \circ x_\alpha^{-1}$ where $y_\beta \circ y_\alpha^{-1}$ and $y_\alpha \circ y_\beta^{-1}$ are both smooth. Finally $[y_\alpha, \pi(U_\alpha)]$ constitutes an atlas. This atlas can be extended to a maximal atlas.

Existence of a Lorentz metric on Q: First we define a map

$$\pi_\alpha : N_\alpha \to Q$$
$$\pi_\alpha(p_\alpha) = [p_\alpha]$$

which is injective and because of $y_\alpha \circ \pi_\alpha \circ x_\alpha^{-1} = id$, if defined, we see that π_α is a diffeomorphism onto its image.

Assume $p \in Q$, $X, Y \in T_pQ$ and let p_α be such that $p = [p_\alpha]$ and let $X_\alpha, Y_\alpha \in T_{p_\alpha}N_\alpha$ such that $\pi_{\alpha *}X_\alpha = X$ and analogously for Y. Due to π_α being a diffeomorphism, X_α and Y_α are unique. Now we define

$$g(X, Y) = g_\alpha(X_\alpha, Y_\alpha).$$

In the following we have to show that this definition makes sense. Let $[p_\beta] = p$ and define X_β, Y_β as X_α, Y_α before. Furthermore assume (without loss of generality) that $N_\alpha \leq N_\beta$. Because of uniqueness we have $\psi_{\beta\alpha *}X_\alpha = X_\beta$ (and similarly for Y_β) and get

$$g_\beta(X_\beta, Y_\beta) = g_\beta(\psi_{\beta\alpha *}X_\alpha, \psi_{\beta\alpha *}Y_\alpha) = \psi^*_{\beta\alpha}g_\beta(X_\alpha, Y_\alpha) = g_\alpha(X_\alpha, Y_\alpha).$$

Due to

$$g(\,\partial_{y^\mu_\alpha}\big|_p\,, \partial_{y^\nu_\alpha}\big|_p) = g_\alpha(\,\partial_{x^\mu_\alpha}\big|_{\pi_\alpha^{-1}(p)}\,, \partial_{x^\nu_\alpha}\big|_{\pi_\alpha^{-1}(p)}),$$

g is smooth. The proof that we can define a smooth function ϕ on Q is simple. Furthermore we see that all relevant relations will be valid by defining an embedding $i : \Sigma \to Q$. If we assume that all maps $\psi_{\alpha\beta}$ preserve the time orientation and also the orientation, we can suppose that Q is also time oriented and oriented. Additionally the embedding $i(\Sigma)$ is a spacelike Cauchy hypersurface in Q. If we demand that Σ is connected, we get that Q is also connected.

Q is second countable: By using the present geometric structures we prove the second countability of Q. For this we define on TQ a vector field G in a way that the projection $\pi_1 : TQ \to Q$ results in a correspondence which is one-to-one between the integral curves of the vector field G and geodesics in Q, see ([2], Proposition 28, p.70). Let be γ_v the geodesic with $\gamma_v(0) = \pi_1(v)$ and $\gamma_v^{'}(0) = v$. For $v \in TQ$ we get G_v then by the initial velocity of $\gamma_v^{'}(s)$. By ([2], Proposition 28, p.70) there exists an open subset $\mathcal{D} \subseteq \mathbb{R} \times TQ$ containing $\{0\} \times TQ$ and a smooth flow $\alpha : \mathcal{D} \to TQ$ (i.e. $\alpha^{'}(t, v) = G_{\alpha(t,v)}$). So we have

with the latter notation $\pi_1 \circ \alpha(t, v)$ and $\alpha(t, v) = \gamma_v^{'}(t)$. Let C_pQ denote the set of all timelike vectors in T_pQ and define

$$E = \bigcup_{p \in i(\Sigma)} \mathbb{R} \times C_pQ \cap \mathcal{D}.$$

Because of the second countability of Σ we see that E is also second countable. If Σ has dimension n, the manifold E is $2n+2$ dimensional.

Assume now that W_j is a countable basis for the topology of the manifold E and let be $O_j = \pi_1 \circ \alpha(W_j)$. Then we want to show that the O_j constitute a basis for the topology of Q. Let $q \in Q$ be contained in an open neighborhood U. Because $i(\Sigma)$ is a Cauchy hypersurface, we have a $(t, v) \in E$ with $\pi_1 \circ \alpha(t, v) = q$. Due to the continuity of $\pi_1 \circ \alpha$ there exists a W_j in such a way that $(t, v) \in W_j$ and $\pi_1 \circ \alpha(W_j) \subseteq U$. The remaining question is whether the O_j are open or not. Before treating this problem let us note the following. If $v \in TQ$ is timelike, assume γ_v to be the geodesic with initial velocity v. Due to the fact that $i(\Sigma)$ is a Cauchy hypersurface, there exists a unique $s_v \in \mathbb{R}$ such that $\gamma_v(s_v) \in i(\Sigma)$. As a result we get a map $\sigma : \mathcal{T} \to \mathbb{R}$ defined by $\sigma(v) = s_v$. Here $\mathcal{T} \subset TQ$ is the set of timelike vectors.We have to show that the map σ is continuous. Assume v is future directed and $v_i \to v$, furthermore suppose $\sigma(v_i) \leq \sigma(v) - \varepsilon$ for some $\varepsilon > 0$. Then we have

$$\gamma_{v_i}[\sigma(v_i)] \leq \gamma_{v_i}[\sigma(v) - \varepsilon] \ll \gamma_v[\sigma(v)]$$

if i is large enough and we can choose ε so small that $\gamma_{v_i}[\sigma(v) - \varepsilon]$ is well defined. The strict inequality on the right side of the estimation is due to the fact that $\gamma_{v_i}[\sigma(v) - \varepsilon]$ converges to $\gamma_v[\sigma(v) - \varepsilon]$. Because $\gamma_{v_i}[\sigma(v_i)]$ and $\gamma_v[\sigma(v)]$ belong to $i(\Sigma)$ we obtain a timelike curve with endpoints in $i(\Sigma)$. Therefore $i(\Sigma)$ cannot be a Cauchy hypersurface, which is a contradiction to the initial statements. Analogously we cannot have $\sigma(v_i) \geq \sigma(v) + \varepsilon$. From this we conclude that σ is continuous.

Assume now that $q \in O_j$, then there exists a $(t, v) \in E$ with $\pi_1 \circ \alpha(t, v) = q$. Let $O \subseteq TQ$ be an open neighborhood of $w = \alpha(t, v)$, which is small enough that it only contains timelike vectors. Note that $\pi_1(O)$ is an open neighborhood of q and $\alpha(\sigma(u), u)$ converges to $\alpha(\sigma(w), w) = v$ and $\sigma(u)$ converges to $-t$ as O shrinks, with $u \in O$. From this we get the convergence

$$[-\sigma(u), \alpha(\sigma(u)), u] \to (t, v). \tag{3.6}$$

If O_j is small enough (3.6) is contained in W_j and $\pi_1(O)$ contained in O_j which is open. Consequently Q is second countable.

Existence of a maximal element: Due to the fact that Q is a development and we have $N_\alpha \leq Q$, $\forall \alpha \in A$, Q is an upper bound for $\{N_\alpha\}$, $\alpha \in A$. So, each totally ordered subset of $\mathcal{M}$ has an upper bound. out of this we draw the conclusion that $\mathcal{M}$ has a maximal element M. The only extension of M is M itself (up to isometry). That M it is an extension of every development is the final step.

The maximal element is a globally hyperbolic development. Let M, M' be two different developments of the same initial data. Let (U, ψ) be the maximal element of the partially ordered set $C(M', M)$. Let us design an equivalence relation on $M \coprod M'$ by $q \sim q\ \forall q$

$$q \sim \psi(q) \text{ for } q \in U \quad \text{and} \quad \psi(q) \sim q \text{ for } q \in U$$

Setting $\tilde{M} = M \coprod M' / \sim$, we have a topological space and we call π_2 the projection from $M \coprod M'$ to $\tilde{M}$. In this space there is a differentiable structure and it is also second countable. $\tilde{M}$ would be a manifold if it turned out that it is additionally Hausdorff and then it follows immediately that it is a globally hyperbolic development of the initial data. Since $\tilde{M}$ is an extension of M we deduce that $U = M'$, due to the fact that M is the only extension of M itself.

Note. If we can show that $\tilde{M}$ is Hausdorff, one can argue that $(\tilde{M}, \tilde{g}, \tilde{\varphi})$ is a globally hyperbolic development of the data and also an extension of (M, g, φ) and (M', g', φ'). Because of the fact that the isometry class of (M, g, φ) is a maximal element of $\mathcal{M}$, it can be shown that one can embed (M', g', φ') isometrically into (M, g, φ). From this it is clear that the central problem is to show that $\tilde{M}$ is Hausdorff.

Fundamental consequences of non-Hausdorffness. We prove that $\tilde{M}$ is Hausdorff, by contradiction. Assume $\tilde{M}$ is not Hausdorff, then we have elements $p, p' \in \tilde{M}$ such that for every pair V, V' (open neighborhoods of p, p' respectively) we have $V \cap V' \neq \emptyset$. Without loss of generality we can assume that

$$p' \in \partial U \cap J^+(i'(\Sigma), M') \quad \text{and} \quad p \in \partial\psi(U) \cap J^+(i(\Sigma), M)$$

Note that

$$I^-(p, M) \cap J^+(i(\Sigma), M) \subseteq \psi(U)$$

$$I^-(p', M') \cap J^+(i'(\Sigma), M') \subseteq U$$

The reason for the latter statement is that if

$$p \in I^-(p', M') \cap J^+(i'(\Sigma), M')$$

for instance and $q' \notin U$, then the set $I^+(q', M')$ and its projection to $\tilde{M}$ are open neighborhoods of p' which belong to the complement of U. Otherwise we could construct an inextendible timelike geodesic in U where the intersection with $i'(\Sigma)$ is empty. Therefore we get the required contradiction to $p' \in \partial U$. In the same way one proves the first inclusion above.

Let us write H for the set of all points of ∂U which are "non-separable" (as described above). If we assume that $H \neq \emptyset$ we get a contradiction and so as H is empty, we get the Hausdorffness of $\tilde{M}$. Due to the fact that H is certainly open in ∂U and given any null geodesic in H, its endpoint in ∂U must also be in H. Then the corresponding null geodesic in $\partial(\psi(U))$ must have an endpoint in M. Using these two properties of H one can find a point $p' \in H$ and a spacelike surface T' through p' such that $T' \setminus p' \in U$. This motivates the term *spacelike boundary at an appropriate point.*

Assume that $q' \ll p'$, then we have that $I^+(q', M')$ and its projection to $\tilde{M}$ are open neighborhoods of p' in M' and $\tilde{M}$ respectively. If now $q \in M$ is such that $p \ll q$, then $I^-(q, M)$ and its projection to $\tilde{M}$ are open neighborhoods of p. Due to the fact that the intersection of these neighborhoods has to be non-empty, we obtain that $\psi(q') \ll q$. Because M is globally hyperbolic, we have $\psi(q') \leq p$ since the relation $\leq$ is closed on M. But in the case $q' \ll p'$ there exists a q'' such that $q' \ll q'' \ll p'$ such that $\psi(q'') \leq p$ and so $\psi(q') \ll p$. Similarly, if $q \ll p$ and $q \in \psi(U)$, then it follows that $\psi^{-1}(q) \ll p'$. Let now γ be a future directed timelike curve in M' such that $\gamma(0) = p'$ holds and assume $r \ll p$. Then $\psi^{-1}(r) \ll p'$ so that $\psi^{-1}(r) \ll \gamma(t)$ for t close enough to 0. Consequently there is an $\varepsilon > 0$ such that $r \ll \psi \circ \gamma(t) \ll p$ for $t \in (-\varepsilon, 0)$. Because, for every neighborhood V of p there exists an $r \ll p$ such that $(J^+(r, M) \cap J^-(p, M)) \subseteq V$ (because M is globally hyperbolic), it follows that $\psi \circ \gamma(t) \to p$ for $t \to 0-$.

Definition of φ. Because the boundary of U is spacelike at p', there exists a smooth spacelike hypersurface S such that $p' \in S$ and $S \setminus \{p'\} \subseteq U$. We define $\varphi : S \to M$ by

$$\varphi(q) = \psi(q) \text{ for } q \in S \setminus \{p'\} \text{ and } \varphi(p') = p.$$

φ is smooth. This is clear in every point except p'. Assume that n is the future directed unit normal vector to S. There exists a neighborhood V of p' in S and an $\varepsilon > 0$ such that geodesics with starting point $q \in V$ and initial velocity n_q are defined on $(-\varepsilon, \varepsilon)$. The map from V to TM taking $q \to \gamma'_{n_q}(-\varepsilon/2)$ is smooth due to reasons mentioned above. If ε is small enough $\gamma'_{n_q}(-\varepsilon/2)$ belongs to $J^+[i(\Sigma)]$ and therefore also to U. Otherwise one could construct an inextendible timelike geodesic in U which can not intersect $i'(\Sigma)$. In this way, the map $\eta(q) = \psi_* \gamma'_{n_q}(-\varepsilon/2)$ from V to TM is smooth.

Now we want to evaluate γ'_{n_q} at $(-\varepsilon/2)$, however is is not clear whether this is defined. Because of the existence of the isometry it is defined for $q \neq p'$, since

$$\gamma_{\eta(q)}(s) = \psi \circ \gamma_{n_q}(s - \frac{\varepsilon}{2})$$

with $q \neq p'$. The statement at the end of the paragraph on the consequences of not being Hausdorff ensures that it is defined in the case of $q = p'$. Due to the fact that $\gamma_v(\varepsilon/2)$ is a smooth map of v on the open set where it is defined, we see that $\gamma_{\eta(q)}(\varepsilon/2)$ is a smooth function of $q \in V$. Consequently, φ is smooth because $\gamma_{\eta(q)}(\varepsilon/2)$ coincides with φ on V.

φ is an embedding. Assume that $e_i, i = 1, ..., n$ is an orthonormal basis of TS in a neighborhood of p'. Then we have in this neighborhood for $q \neq p'$,

$$\delta_{ij} = g'(e_i|_q\,, e_j|_q) = \psi^* g(e_i|_q\,, e_j|_q) = g(\varphi_* e_i|_q\,, \varphi_* e_j|_q).$$

The same equality is valid for $q = p'$ by continuity. Therefore we get that φ_* restricted to $T_{p'}S$ is injective and so we can say that φ is an immersion. By the help of ([2], p.19) we can ensure that φ (by restriction to S if necessary) is an embedding and the pullback of the initial data which is induced on $\varphi(S)$ by (g, ϕ) under ϕ correspond with the initial data which are induced on S by (g', ϕ') since ψ is an isometry and φ is smooth.

M and M' are both developments of S, which is a consequence of the existence of the embedding φ. By the help of Theorem 3.3.2 we see that there is an open neighborhood D of S in such a way that D is globally hyperbolic with respect to the metric g' and S is a Cauchy hypersurface. Additionally there is an isometry $\chi : D \to M$ such that $\chi = \varphi$ when restricted to S. We suppose that D has a spacelike boundary. We also have $\chi = \psi$ on $D \cap U$ by a uniqueness argument (we have done this several times). Additionally we have that $D \cup U$ is a globally hyperbolic development of Σ with spacelike boundary.

From this we are able to extend ψ to a larger domain, but this contradicts the maximality of (U, ψ), so that we can deduce that $\tilde{M}$ is indeed Hausdorff, such that $\tilde{M}$ is an extension of M. Consequently $U = M'$ by the maximality of M, so we finally arrive at $M' \leq M$. $\square$

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