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Abstract

Many systems can be described by networks of interacting components. How the system responds to an external shock often depends strongly on how these components interact with each other. In input–output economics, national economies are modelled as a network in which nodes correspond to industrial sectors and the edges to monetary or quantity flow between those. This approach is similar to approach in statistical and computational physics, where a whole (macroscopical) system is also modelled by disecting it into its (microscopical) constituents. The dynamic Leontief model has already been applied to predict how an economy reacts to a shock by assuming that prices stay constant and producers adjust to the shock by changing the quantity of what they produce. The differential equations of this model have the same form as the ones describing the Brownian Motion. Therefore, known results from statistical Physics, like the Green-Kubo equations, can also be applied to this model. However, which role the stochastic Langevin force plays in these models, has been unclear. By showing that three conceptually different expectation values yield the same time evolution, the first part of this work clarifies the extent to which different conceptual approaches to the shock response of networked systems lead to similar behaviour. Until now, these dynamic Leontief models have only allowed shock absorption via quantity adjustments. In the second part, agent based update equations will be derived that allow both, adjustment of quantity and price. Similar to molecular dynamics, here the whole system is simulated by defining and numerically integrating the update equations for each agent/particle. Behavioural parameters determine how much adjustment happens in price or quantity dimension. These are then fitted to input-output data of the Austrian economy for the years 2008-2016. From this, it is concluded that market participants react to shocks in terms of both quantity and price adjustment simultaneously during this timeframe. By considering the behavioural parameters as vectors, not as scalars, future work might use this Agent Based Model to estimate whether there are sector wise differences in adjustment of price and quantity.

Abstract

Viele Systeme können als ein Netzwerk von Komponenten, und wie diese miteinander interagieren, dargestellt werden. Wie ein System auf einen externen Schock reagiert, hängt stark von diesem Netzwerk ab. Input-Output Analysen modellieren nationale Ökonomien als Netzwerk. Die verschiedenen industriellen Sektoren, representieren die Knotenpunkte, die Waren- beziehungsweise Geldflüsse, die Kanten dieses Netzwerks. Die statistische und computergestützte Physik hat ähnliche Herangehensweisen. Diese modellieren (makroskopische) Systeme indem sie sie in ihre (mikroskopischen) Bausteine zerlegen. Unter der Annahme, dass sich Preise nicht verändern, und deshalb Produzenten nur durch Mengenanpassungen auf ökonomische Schocks reagieren, wurde das dynamische Leontief Modell schon erfolgreich verwendet, um vorherzusagen wie die Wirtschaft auf ökonomische Schocks reagiert. Dieses Modell wird durch Differentialgleichungen beschrieben, die die der Brownschen Bewegung gleichen. Deshalb können bekannte Ergebnisse aus der statistischen Physik, wie die Green-Kubo Gleichungen, an dieses Modell angewandt werden. Die Rolle der stochastischen Langevin Kraft, war jedoch bis jetzt unklar. In dem ersten Teil der Arbeit, wird gezeigt das drei konzeptuell verschiedene Erwartungswerte die selbe Zeitentwicklung ergeben. Dadurch wird erklärt, zu welchem Ausmaß die konzeptuell verschiedenen Herangehensweisen zur Schockreaktion des Netzwerksystems zu gleichen Verhalten führt. Bis jetzt konnte das dynamische Leontief Modell Schockanpassungen nur in der Form von Mengenanpassungen abbilden. Im zweiten Teil dieser Arbeit werden Agenten basierte Updategleichungen abgeleitet, die sowohl Mengen- als auch Preisanpassungen erlauben. Ähnlich wie bei der Molekulardynamik wird hier das gesamte System simuliert, indem die Updategleichungen für jeden Agenten/Teilchen definiert und numerisch integriert werden. In dem abgeleiteten Modell bestimmen Verhaltensparameter wie stark die Anpassung in der Preis- oder Mengendimension erfolgt. Diese werden dann an Input-Output-Daten der österreichischen Wirtschaft für die Jahre 2008-2016 gefittet. Es wird festgestellt, dass die Marktteilnehmer in diesem Zeitraum auf Schocks sowohl in Form von Mengen- als auch Preisanpassung gleichzeitig reagieren. Indem die Verhaltensparameter als Vektoren und nicht als Skalare betrachtet werden, könnten zukünftige Arbeiten dieses agentenbasierte Modell verwenden, um abzuschätzen, ob es sektorale Unterschiede in den Preis- und Mengenanpassungen gibt.

Contents

0 Introduction

Applying findings from physics to other research fields has always played an important role in science. Examples range from applying X-rays to medicine [Pla80] and chemistry [Guo19] to the discovery of quantum mechanics, which caused major breakthroughs in chemistry, by explaining how chemical bonds form[Pau31], and in biology [Mar+18], for example by providing a theory that explains the efficiency of photosynthesis [DC66]. In the 1990s, statistical physicists started applying techniques and approaches from their field to explain economic phenomena. [Art21; SM00] Thereby, they founded the field econophysics.

In computational and statistical physics, macroscopic systems are described by depicting them as a collection of their microscopic parts (their particles). Computational and statistical physics often seeks to explain complex macroscopic behaviour (e.g. phase transitions) as an emergent property of the underlying particles and their interactions [LPB15; Gol18]. Very often, such emergent properties can only be studied by computational means.

In the same manner, an economy can be described by depicting it as a collection of sectors. Economic phenomena like the recovery from an economic shock induced by a natural disaster or a pandemic, can be explained by considering how these sectors interact with each other. This analogy can be pushed even further. The state of a physical system is fully defined by the spacial coordinates and momenta of the particles. Likewise, the state of an economy is often defined by the outputs in both monetary and quantity terms of its sectors. Moreover, for physical ensembles, there is a wide variety of conservation laws [Noe18; Lor05]. For closed systems, the total amount of energy, linear momentum and rotational momentum stay constant. Similarly, in economics, the total amount of money exchanged during a financial transaction can be assumed to stay constant. Because of these similarities, approaches already developed in computational and statistical physics to describe physical systems like gases , polymers or proteins [Cha87; Thi07] have been applied to the economy. Not only the propagation of an economic shock, induced by a natural disaster [IT19; Car+20] can be modelled this way, but also the systemic risk of the housing market [Gea+12] and the financial market [TP13; Kli+15].

This work will model the economy as an Input–output (IO) model. As a simplification, IO economics assumes that the economy consists of different sectors, each of which produces one type of homogeneous good. One sector uses goods from other sectors and labor as input to produce an output, namely its sector specific good. These goods are either used by other sectors, or by the end consumer, which makes up the total demand of this sector specific good [MB09].
This demand is seeked to be satisfied by the market participants. This works via the price mechanism: If the demand of a good is higher than its production, buyers will try to outbid each other, which leads to a higher price of this good. Therefore, producers have an incentive to increase their production. Conversely, if the demand of a specific good is lower than its production, producers have to underbid each other, to be able to sell their goods. As this

leads to a lower price, the production of this good will decrease[Var14]. To conclude, the price mechanism regulates the production in such a way that producers will aim to match the demand of the economy: If the demand of a certain good increases, the sector producing this good will increase its accordingly. Likewise, a decrease in demand is followed by a decrease in output.

Changes in economic demand can either develop slowly, for example due to economic growth, or rapidly due to economic shocks, such as a financial crisis or a natural disaster, like a flood or a pandemic. Predicting how an economy reacts to these demand shocks has always been a major question in economics [IT19; Car+20]. This work will try to model how the economy adjusts to and recovers from sudden demand shocks.
When the demand of a sector changes, not only its output has to adjust, but also the supplying sectors. For example, if people buy fewer cars, not only the number of produced cars will decrease, but so will the number of engines and wheels. In turn, less rubber and steel will be produced. These are called upstream effects, which can be described by the Leontief model [Leo86]. In this model, the prices stay constant and each sector tries to match its output to the quantity demanded by the system. For these sudden changes in demand that are considered in the course of this work, it can be assumed that the composition of goods, meaning the proportion of the parts they consist of, stays the same. These proportions are called technical coefficients. For example, the basic composition of a car will not change, it consists of four wheels and one engine.

The first part (§1, §2 and §3) of this work focuses on a dynamic extension [KS00; Hal08] of the classic Leontief model.
The classic Leontief model only describes the time independent static equilibrium case of the economy. As the economy is in an equilibrium, the total demand is assumed to equal the total output. Using the demand of the end consumer in each sector, this model allows us to calculate the total demand and therefore the total output of the economy per sector. Here it should be noted that the total demand of a sector is not only made up of the demand of the end consumer, but also demand of this specific good by the other sectors to produce their outputs. However, the classic Leontief model is unable to describe the time dependent, dynamic adjustment of the output to a change in end consumer demand.
This is made possible by the dynamic Leontief model can account for the fact that a change in demand in a specific sector will not only follow a change in its output, but in all its suppliers and the suppliers of those and so on. Mathematically, changes in output, induced by a demand shock, can be described by three different approaches: An ordinary differential equation (§3.3.1), a stochastic differential equation (§3.3.2) and Linear Response theory (§3.3.3). These approaches originate in statistical mechanics. There, they have also been used to describe how a system, in this case usually a gas, adjusts to a shock, for example compression by an external force [Cha87]. The three approaches yield three different expectation values. The finding of this first part is that nonetheless the time evolution of these three expectation values is the same.

In the second part (§4), approaches from computational physics, to be more precise molec-

ular dynamics, are used to study how an economy reacts to demand shocks. In molecular dynamics, a physical system is simulated by numerically solving the equations of motions of each individual particle. [AW59; Sch96] Agent based modelling simulates an economy by defining update equations which describe how each individual agent, in this case each individual sector, reacts to its evolving environment, in the case of this thesis a demand shock. [HM91] The sectors are equivalent to the particles and the update equations of each sector compare to the equations of motions.
Whilst the dynamic Leontief model only allowed for quantity adjustments, this will allow a formulation of both, quantity and price adjustments in an extended dynamic input-output model. Price and quantity can be seen as two different dimensions, along which the system can adjust to a shock. The behavioural parameters of the update equations of this work determine which portion of the adjustment happens along which dimension. In practical terms, facing reduced demand of a certain produced good can be either met by reducing the produced quantity, by asking for a lower price, or by doing both simultaneously. Finally, fitting these parameters to historical input-output data will allow us to conclude in which years either price or quantity adjustments have been the dominant driver of the economy.

1 Brownian Motion

The aim of this chapter is to introduce concepts from statistical physics used to describe Brownian motion. Later on, these will be applied to economics to develop the dynamic Leontief model.

Brownian motion describes the movements of a particle suspended in a fluid [VPL28]. To model the movement of the suspended particle, we consider the forces that act on it. However, the fluid also consists of particles and due to thermal motion, these move randomly. When they collide with the suspended particle, they transfer their momentum to it. This leads to a random motion of the suspended particle.

For random motions like this, one can either model the realisations of different paths via differential equations with a stochastic term (see 1.1), or via the probability density of all possible impulses and/or locations (see 1.2) of the suspended particle.

1.1 Stochastic differential equations

The fluid surrounding the suspended particle exhibits a dragging force onto the particle. Since the observed velocities v of the particle are very low, we can assume that the drag force F increases proportionally to the velocity [Sto+51]:

$$F = -\alpha v. \tag{1}$$

The proportionality coefficient α is called the friction coefficient or dampening factor. The relation above is known as known as Stoke's Law. With the mass m of the particle, using Newtons second law of motion ($F = m\dot{v}$ [New87]), we get [Ein05]:

$$m\dot{v} + \alpha v = 0.$$

Adding the random force that originates from the fluid's particles bouncing off the suspended one to this equation, we get

$$\dot{v} = -\frac{\alpha}{m} v + \Gamma(t). \tag{2}$$

$\Gamma(t)$ being the random force. Its properties are $\langle \Gamma(t) \rangle = 0$ and $\langle \Gamma(t)\Gamma(t') \rangle = \delta(t - t')$.

Multivariate Ornstein-Uhlenbeck process

If, instead of $v(t) \in \mathbb{R}$ in (2) we model an N dimensional variable $y(t) \in \mathbb{R}^N$, and we take $\gamma \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ as the dampening factor, we get an equation of the following form [UO30]

$$\dot{y}(t) + \gamma y(t) = \Gamma(t). \tag{3}$$

(Since $\Gamma(t)$ is also multidimensional, we now assume the general case with $\langle \Gamma_i(t) \rangle = 0$ and $\langle \Gamma_i(t)\Gamma_j(s) \rangle = \nu_{ij}\delta(t-s)$, ν_{ij} is called the correlation matrix of the Langevin force.) Equations of this form describe a multivariate Ornstein-Uhlenbeck process.

1.2 Fokker-Planck

Differential equations are only one way to model Brownian motion. One can also consider the time evolution of the probability density function of the velocity $f_v(t) \in \mathbb{R}^N$. The equations of motion of the probability density has the form [Fok14; Pla17]:

$$\frac{\partial f_v(t)}{\partial t} = L(t) f_v(t). \tag{4}$$

$L(t)$ is the time dependent time evolution operator. For Brownian motion, this is the linear Fokker-Planck operator. (Here we will not give an explicit form for the brownian equations of motion of the velocity.)

2 Linear Response Theory

This section will provide a short introduction to the Linear Response Theory and the Green-Kubo equations will be derived. This will later on be one of the three equivalent realisations of the dynamic Leontief model (§3.3.3) used to describe the sectorwise ouputs of the economy.

Linear Response Theory describes how a system reacts to external forces. For physical systems this can be either an external force or field.
Using the equilibrium correlation functions, the Green-Kubo [Gre54; Kub66] equations enable us to predict the reaction of the system to an external force. In equilibrium, the different components of the system are still subject to random fluctuations. The equilibrium correlation functions are able to capture, how one component i of the system reacts to a random fluctuation of another component j [Mad07, p. 104]. Linear Response Theory tells us that if j is perturbed, the response of i will be proportional to the response of the random fluctuation. We will later see that the Green-Kubo equations can be applied to all kinds of (non-)physical systems.

2.1 Derivation of the Green-Kubo equations

We consider an N-dimensional stochastic system which is described by $y = \{y_i(t)\}_{i=1,\dots,n}$ variables and the associated vector of probability density functions $f_y(t) = \left(f_{y_1}(t), f_{y_2}(t), \dots, f_{y_N}(t)\right)$. Let us assume, that the system is linear, time invariant and has a stable equilibrium/stationary solution.

Since we assume linearity, the Fokker-Planck equation for this system (similar to equation (4)) is of the form:

$$\frac{\partial f_y(t)}{\partial t} = L_y(t) f_y(t). \tag{5}$$

We assume that the linear operator L can be written as a sum of a time independent operator L^0 and a time-dependent one $L_y^X(t)$ which can be written as the product of the time dependent part $X(t)$ and the time independent one L_y^X as $L_y^X(t) = L_y^X X(t)$

$$L_y(t) = L^0 + L_y^X X(t).$$

Our goal is to describe what happens, when we start applying a force to this system. For $t < 0$ we set $X(t)\big|_{t<0} = 0$ and we then start applying a field for $t > 0$ (and for $t > 0$ we require $X(t)\big|_{t>0} \neq 0$). We consider the system to be in a stationary state til $t = 0$. A system is stationary, when the probability density does not change, i.e. if $\partial_t f_y(t) = 0$. We define this stationary distribution to be f_y^0, therefore

$$\left.\tfrac{\partial f_y(t)}{\partial t}\right|_{t<0} = L^0 f^0 = 0 \tag{6}$$

holds. When applying a small field to this system, the density function will change. Since we only consider small perturbations we try to approximate the change in $f_y(t)$ up to the first order. Choosing

$$f_y(t) \approx f_y^0 + \Delta f_y(t) \tag{7}$$

and plugging this into the Fokker-Planck equations (4) results in

$$\begin{aligned}
\frac{\partial f_y(t)}{\partial t} = \underbrace{\frac{\partial f_y^0}{\partial t}}_{=0} + \frac{\partial \Delta f_y(t)}{\partial t} &= L_y(t) f_y(t) \\
\frac{\partial \Delta f_y(t)}{\partial t} &= \left(L^0 + L_y^X X(t)\right)\left(f_y^0(t) + \Delta f_y(t)\right) \\
\frac{\partial \Delta f_y(t)}{\partial t} &= \underbrace{L^0 f^0}_{=0,\text{ see Eq. (6)}} + L_y^X X(t) + L^0 \Delta f_y(t) + \underbrace{L_y^X X(t)\Delta f_y(t)}_{\substack{\approx 0,\text{ since we only} \\ \text{consider small perturbations}}} \\
\underbrace{\frac{\partial \Delta f_y(t)}{\partial t}}_{\text{l.h.s}} &\approx \underbrace{L^0 \Delta f_y(t) + L_y^X X(t) f^0}_{\text{r.h.s.}} .
\end{aligned} \tag{8}$$

At $(*)$ we use the assumption that the applied field, and the according pertubation is small, therefore $L_y^X X(t)\Delta f(t) \approx 0$ and that f^0 is a stationary solution for L^0, therefore $L^0 f^0 = 0$. This can be solved by applying the Laplace transform. In this case, we will use the following notation: a function $f(t) : t \mapsto f(t)$ transforms to a function $\mathcal{L}[f(t)](s) : s \mapsto \mathcal{L}[t \mapsto f(t)](s)$ or shorter $\mathcal{L}[f(t)](s) : s \mapsto \mathcal{L}[f(t)](s)$, where t will always be the variable in in time/t-domain, and s the one in "Laplace"/s-domain. The transform is given by:

$$\mathcal{L}\left[f(t)\right](s) = \int_0^\infty \exp(-s\tau) f(\tau)\,\mathrm{d}\tau. \tag{9}$$

This is applied to both sides of Eq. (8). (Note, that even though 8 is just an approximation " $\approx$ " for small $L_y^X X(t)$ we will be a bit more generous with the notation from now, and assume equality " $=$ " from now on.)
For the **l.h.s.** we get the following:

$$\begin{aligned}
\mathcal{L}\left[\frac{\partial f_y}{\partial t}\right](s) = \mathcal{L}\left[\frac{\partial \Delta f_y}{\partial t}\right](s) &= \\
&= \int_0^\infty \exp(-s\tau)\frac{\partial \Delta f_y}{\partial t}(\tau)\,\mathrm{d}\tau = \\
&= \exp(-s\tau)\Delta f_y(\tau)\Big|_{\tau=0}^{\infty} - \int_0^\infty \left(-s\exp(-s\tau)\right)\Delta f_y(\tau)\,\mathrm{d}\tau = \\
&= \lim_{\tau\to\infty} \underbrace{\exp(-s\tau)}_{=0}\Delta f_y(t) - \exp(-s0)\underbrace{\Delta f_y(0)}_{=0} + s\int_0^\infty \exp(-s\tau)\Delta f_y(\tau)\,\mathrm{d}\tau = \\
&= 0 + s\mathcal{L}\left[\Delta f_y\right](s) = \\
&= s\,\mathcal{L}\left[\Delta f_y\right](s)
\end{aligned}$$

And for the **r.h.s.**:

$$\mathcal{L}\left[L^0 \Delta f_y(t) + L_y^X X(t) f_y^0\right](s) =$$

$$= L^0 \mathcal{L}\left[\Delta f_y\right](s) + \mathcal{L}\left[L_y^X X(t)\right](s) \cdot f_y^0.$$

Putting these together

$$\begin{aligned}
\textbf{l.h.s.} &= \textbf{r.h.s.} \\
s\mathcal{L}\left[\Delta f_y(t)\right](s) &= L^0 \mathcal{L}\left[\Delta f_y\right](s) + \mathcal{L}\left[L_y^X X(t)\right](s) \cdot f^0 \\
s\mathcal{L}\left[\Delta f_y(t)\right] - L^0 \mathcal{L}\left[\Delta f_y(t)\right] &= \mathcal{L}\left[L_y^X\right](s) \cdot f^0 \\
\mathcal{L}\left[\Delta f_y\right](s) &= \left(s\mathbb{I} - L^0\right)^{-1} \left(\mathcal{L}\left[L_y^X\right](s) \cdot f^0\right) \qquad (10)
\end{aligned}$$

The following properties of Laplace transformations will be used when applying the inverse Laplace transform to this.

(i) The exponential transforms to

$$\mathcal{L}\left[e^{at}\right] = \frac{1}{s-a}$$

(ii) The product of 2 transformed functions equals the Laplace transform of the convolution:

$$\mathcal{L}\left[f(t)\right](s)\mathcal{L}\left[g(t)\right](s) = \mathcal{L}\left[\int_{-\infty}^{t} f(\tau)g(t-\tau)\,\mathrm{d}\tau\right](s) = \mathcal{L}\left[f * g\right](s).$$

(Both can easily be verified by plugging them into the definition of the Laplace transform (9)). These can be used to apply the inverse Laplace transform to (10).

$$\Delta f_y(t) = \int_{-\infty}^{t} e^{L^0\cdot(t-\tau)} L_y^X X(\tau) f^0 \,\mathrm{d}\tau \qquad (11)$$

What we are actually interested in, is the perturbed expectation value of a specific dynamical variable $B(t) = B(y(t))$. To properly write down these expectation values, first we have to introduce the $\langle B(t)\rangle_{f_y}$ notation. The brackets $\langle . \rangle$ indicate that the expectation value of the variable is formed. The subscript $._{f_y}$ highlights over which probability density function it is calculated. In mathematical terms this means:

$$\langle B(t)\rangle_{f_y} := \int B(y) f_y(t)\,\mathrm{d}^N y.$$

As $f_y(t) = f(y,t) \approx f^0_y + \Delta f_y(t)$ (7) is the probability density function of the system at time t:

$$\begin{aligned}\langle B(t)\rangle_{f_y} = \int B(y) f(y,t)\,\mathrm{d}^N y = \\ \approx \langle B(t)\rangle_{f^0} + \int B(y)\Delta f(y,t)\,\mathrm{d}^N y\end{aligned}$$

we can now plug in the obtained solution for $\Delta f_y(t)$ from Eq. (11).

$$\begin{aligned}\langle B(t)\rangle_{f_y} &\approx \langle B(t)\rangle_{f^0_y} + \int B(y)\left(\int_{-\infty}^{t} e^{L^0\cdot(t-\tau)} L^X_y X(\tau) f^0(y)\,\mathrm{d}\tau\right)\mathrm{d}^N y = \\ &= \langle B(t)\rangle_{f^0_y} + \int_{-\infty}^{t} \underbrace{\left(\int B(y) e^{L^0\cdot(t-\tau)} L^X_y f^0\,\mathrm{d}^N y\right)}_{=:R_{B,X}(t-\tau)} X(\tau)\,\mathrm{d}\tau = \qquad (12)\\ &= \langle B(t)\rangle_{f^0_y} + \int_{-\infty}^{t} R_{B,X}(t-\tau) X(\tau)\,\mathrm{d}\tau\end{aligned}$$

Next we will show, how the response function $R_{\cdot,\cdot}$ can be expressed as the equilibrium correlation function

$$c_{A,B}(\tau) = \langle A(y(\tau)) B(y(0))\rangle_0 =$$

With $P(y,\tau \wedge y^0, 0)$ as the probability, that the system is in state y^0 at $t = 0$ and in y at $t = \tau$.

$$= \int A(y) B(y^0) P(y,\tau \wedge y^0, 0)\,\mathrm{d}^N y\,\mathrm{d}^N y^0 =$$

The probability $P(y,\tau \wedge y^0, 0)$ can be rewritten as product of the probability $P(y^0(0))$ and the conditional one $P(y,\tau|y^0,0)$, where the systems goes from $y^0(0)$ to $y(\tau)$.

$$= \int A(y) B(y^0) P(y,\tau|y^0,0) P(y^0,0)\,\mathrm{d}^N y\,\mathrm{d}^N y^0 =$$

The conditional probability can be rewritten as $P(y,\tau|y^0,0) = e^{L^0(y)\cdot\tau}\delta(y-y_0)$ and of course $P(y^0) = f^0(y^0)$ (y^0 is treated here as a variable, as it is one of the integration variables).

$$\begin{aligned}&= \int A(y) B(y) e^{L^0(y)\cdot\tau}\delta(y-y^0) f^0(y^0)\,\mathrm{d}^N y\,\mathrm{d}^N y^0 = \qquad (13)\\ &= \int A(y) e^{L^0(y)\cdot\tau} B(y) f^0(y)\,\mathrm{d}^N y =\end{aligned}$$

Since $f^0(y)$ describes a probability density, and therefore $f^0(y) > 0$ holds for all y, we can introduce the **generalized potential** $\phi(y)$ by

$$f^0(y) = \mathcal{N} e^{-\phi(y)}.$$

Where $\mathcal{N}$ is the normalization constant given by $\mathcal{N}^{-1} := \int e^{-\phi(y)}\,\mathrm{d}y^N$. We can also definine the Observable B as $B(y) := f^0(y)^{-1} L_X(y) f^0(y)$. Plugging this into Eq. (14) the correlation function takes the following form:

$$\begin{aligned} c_{A,B}(\tau) &= \int\ A(y) e^{L^0(y)\cdot\tau} \left(f^0(y)^{-1} L_X(y) f^0(y)\right) f^0(y)\,\mathrm{d}^N y = \\ &= \underbrace{\int\ A(y) e^{L^0(y)\cdot\tau} L_X(y) f^0(y)\,\mathrm{d}^N y}_{=R_{A,X}(\tau)\text{ see Eq. (12)}} \end{aligned} \tag{14}$$

In the last step, one could ask, why its not sufficent to define $B(y) := L_X(y)$. However, $L_X(y)$ is an operator, not an observable (which maps an element from the probability space y to a number). By applying it to a probability distribution, we make sure, that this is actually an observable. Since $f^0(y)^{-1} \cdot \left(L_X(y) f^0(y)\right) \cdot f^0(y)$ is just a product of scalars in each dimension it can be reduced to $\left(L_X(y) f^0(y)\right)$.
This derivation can also be found in [Ris12, pp. 164–165].

Application to stochastic processes I will now apply this result to an Orenstein - Uhlenbeck process. We assume, that an external force $X_i(t)$ is acting on component i of the system. The system can therefore be described by (see (3))

$$\dot{y}_i(t) + \sum_{j=1}^{N} \gamma_{ij} y_j = \Gamma_i(t) + X_i(t)$$

From now on we will use Einsteins index notation, where a summation is implied over those indices that appear twice: $\dot{y}_i(t) + \sum_{j=1}^{N} \gamma_{ij} y_j = \Gamma_i(t) + X_i(t)$ is equivalent to $\dot{y}_i(t) + \gamma_{ij} y_j = \Gamma_i(t) + X_i(t)$. This has the stationary solution (i.e. for $X_i(t) = 0$)

$$f^0(y) = \frac{1}{\sqrt{(2\pi)^N \det(\sigma)}} \exp\left(-\frac{1}{2}(\sigma^{-1})_{ij} y_i y_j\right) \tag{15}$$

(see [Ris12, equation 6.125]). Here $\sigma_{ij} := \left\langle \left(y_i(t) - \langle y_i \rangle_0\right)\ \left(y_j(t) - \langle y_j \rangle_0\right) \right\rangle_{t=0}$ is the correlation matrix before applying X. $L_X(y_i)$ and ϕ can be identified in the following way

$$L_X(y_i) = \tfrac{\partial}{\partial y_i}\ ;\ \tfrac{\partial \phi}{\partial y_i} = \left(\sigma^{-1}\right)_{ij} y_j$$

The response function in the k-th component, to a force acting on the i-th component is

$$R_{y_k, X_i}(t-\tau) = \left(\sigma^{-1}\right)_{ij} \langle y_k(t-\tau) y_j(0) \rangle \cdot X_i(\tau).$$

Plugging this into (12) we get the expectation value for y under the force X (also see [Eva04, p. 99])

$$\langle y(t)\rangle_{X(t);k} = \langle y\rangle_{0;k} + \underbrace{\int_0^t \left(\sigma^{-1}\right)_{ij} \langle y_k(t-\tau)y_j(0)\rangle_0 X_i(\tau)\,\mathrm{d}\tau}_{=:\langle\Delta y(t)\rangle_{X(t);k}} \tag{16}$$

Here $\langle . \rangle_X$ represents the expectation value under the force X. Similarly $\langle . \rangle_0$ is the expectation value in the absence of such a force, here this is the case for $t < 0$.

If the time evolution of the applied force is of the form of a step function, i.e.: $X_i(t) = \Theta(t)X_i$, where $\Theta(t) = \{0 \text{ if } t < 0, 1 \text{ else}\}$. For the expectation value for the new stationary state, we get

$$\begin{aligned}\left\langle \lim_{t\to\infty} \Delta y(t)\right\rangle_{X;k} = \langle\Delta y(\infty)\rangle_{X;k} &= \lim_{t\to\infty}\int_0^t \left(\sigma^{-1}\right)_{ij} \langle y_k(t-\tau)y_j(0)\rangle_0 X_i(\tau)\,\mathrm{d}\tau = \\ &= \underbrace{\left(\lim_{t\to\infty}\int_0^t \left(\sigma^{-1}\right)_{ij} \langle y_k(t-\tau)y_j(0)\rangle_0\,\mathrm{d}\tau\right)}_{=:\rho_{ik}} \cdot X_i = \\ &= \rho_{ik}X_i \overset{\text{in matrix}}{\underset{\text{notation}}{=}} \left(\rho^T X\right)_k\end{aligned} \tag{17}$$

2.2 Applications in different scientific fields

The Green-Kubo relations, i.e. Equation (16) gives us tools to calculate the nonequilibrium expectation value using in equilibrium expectation value and in equilibrium correlation functions. These kind of relations can be found for several physical system.

In §2.1 a linear approximation between the change in expecation value $\langle\Delta\tilde{y}(\infty)\rangle$ and the applied force $\tilde{X}$ was derived. Equation (17), is an expression of the form $\langle\Delta\tilde{y}(\infty)\rangle_{\tilde{X}} = J = \tilde{\rho}\tilde{X}$. In general, $\tilde{X}$ acts as the external perturbation that induces a flux J in the system that is proportional to X, with $\tilde{\rho}$ being a scalar, vector, matrix or tensor that describes this proportionality. The entity $\tilde{\rho}$ is sometimes called susceptibility or the transport coefficient, meaning that the more susceptible a system is, the larger is the flux J that a perturbation $\tilde{X}$ induces. It is particularly useful, that the Green-Kubo relations enable us to determine the factor $\tilde{\rho}$, which describes the non-equilibrium response of a perturbed system, using an equilibrium property of the system, i.e. by the correlation functions.
Due to the langevin force (Γ in Eq. (3)) even in the equilibrium, the system will underly small variations in each component y_j. Since the components of the system are coupled by γ, a small variation in y_j at $t = 0$ might lead to a variation in y_k at $t = \tau$. The equilibrium correlation functions $\langle y_k(\tau)y_j(0)\rangle_0$ in Eq. (16) comprise exactly this information. This is also known as Onsager's regression hypothesis [Ons31].

One can even argue in a handwavy way, why the factor of $(\sigma^{-1})_{ij}$ is necessary. A force X_k is applied in the k-th component to the system at $t=0$. We are assuming, that the the adjustment in reaction to this shock can be expressed by an equation of the form $\langle \Delta y(t)\rangle_{X;j} = \int_0^t \Sigma_{ij}\langle y_j(0)y_k(t-\tau)\rangle_0 X_i(\tau)\,\mathrm{d}\tau$. After an infesitesimal time τ_ϵ, $\langle \Delta y(\tau_\epsilon)\rangle_{X;j} = 0$ (see Eq. (16)) should hold for all $j \neq k$, since the shock has not had time yet, to propagate through the system. Therefore $\Sigma_{ij}\langle y_j(0)y_k(0)\rangle_0 = \delta_{ik}$ should hold. Without loss of generality one can assume that $\langle y_i\rangle_0 = 0$ for all i (otherwise transform $\tilde{y}_i = y_i - \langle y_i\rangle_0$). Therefore $\Sigma_{ij} = \left(\langle y_i(t)y_j(t)\rangle_{t=0}\right)^{-1} \overset{\text{per definition}}{=} \sigma_{ij}^{-1}$.

There is a wide variety of phenomenological laws that are of the form $J = \rho X$ where J is the induced flux, ρ the proportionality factor - usually called the transport coefficient, and X the applied perturbation. In Table 1 some of the most important ones can be found. For

phenomenological law	perturbation X	induced flux J	transport coefficient ρ
$I = \sigma V$	voltage V	current I	resistance σ
$S_{xy} = \sum_{kl}\mu_{ijkl}E_{kl}$	strain rate E	viscuous stress S	viscosity tensor μ
$B_i = \mu H_i$	magnetic field H	magnetic flux density B	magnetic susceptibility μ
$E_i = \epsilon_{ij}D_i$	electric field E	electric displacement D	electric susceptibility ϵ

Table 1: The wide variety of phenomenological laws that are of the form $J = \sigma X$. (see [KPT19, Supplementary Table 1])

all laws of this type, the transport factor σ can be expressed by the equilibrium correlation functions.

Linear Response Theory has also been applied to more complex systems. For example, it was applied to model the effects of geoengineering on the worlds climate[BLL20]. In chemical physics [CM99], Linear Response Theory has been used to model solvation effects. This illustrates that the Linear Response Theory and the Green-Kubo relations have a wide and interdisciplinary range of applications. In the following, we will introduce a new application of the Linear Response Theory to macroeconomics.

3 Application in Economics

Predicting how the economy evolves has always been an important question economics tries to solve. Ordinary, partial and stochastic differential equations have historically played a central role for modelling economic growth [Ach+14].
This section will introduce the sector wise input-output tables and use their basic properties to derive the following three formulations of the dynamic Leontief model: The ordinary differential equation §4.3.6. This is extended to a stochastic differential equation (§3.3.2), to which in turn the Green-Kubo equations §3.3.3 can be applied. All of them can be used to predict the sector wise outputs. This section concludes with the finding that the expectation values of these three different formulations yield the same result.

3.1 Input-Output tables and the flow-consistency equations

We will start by introducing the input-output (IO) tables. An economy can be thought of consisting of different sectors. In the idealized case, each sector produces one type of homogeneous good. To produce those, each sector uses labor and a certain amount of goods from other sectors. In turn, those produced ones can either be used in other sectors or by the end consumer.

This process is described by the IO tables. We assume that the economy consists of n sectors. $(X_0)_{ij}$ is the number of goods flowing from sector i to sector j: Sector $i \xrightarrow{(X_0)_{i,j}}$ Sector j. Therefore, X_0 has to be an $n \times n$ matrix. The end consumer uses $(D_0)_i$ goods in the i-th sector. Each sector also adds value when producing its goods (this can be thought of labor and investments). This is represented by $(v_0)_i$. The total output of a sector is $(Y_0)_i$. The input-output table is then usually written down in the following form:

$$\begin{array}{c|cc} X_0 & D_0 & Y_0 \\ \hline (v_0)^T & - & v_0^T e \\ (Y_0)^T & e^T D_0 & - \end{array} \qquad \text{with } e^T := (1, \dots, 1). \tag{18}$$

The assumption is that all produced goods are either used by other sectors, or by the end consumer, i.e.:

$$\sum_{j=1}^{N} (X_0)_{ij} + (D_0)_i = (Y_0)_i \,. \tag{19}$$

Reversely, the output of a sector is given by its inputs plus its value added:

$$\sum_{i=1}^{N} (X_0)_{ij} + (v_0)_j = (Y_0)_j \,. \tag{20}$$

(19) and (20) are just simple accounting equations for now. They just express that the IO-table is self-consistent, in the sense that the money flows in the network are conserved. These equations will be referred to as flow consistency equations.

3.2 Dynamic IO-tables - The dynamic Leontief model

In the previous section only the equilibrium case was discussed meaning that $\dot{Y}(t) = 0$. However, this framework can also be extended into dynamical update equations. For now let us assume that the composition of the different goods stays constant. Another way to express this is that the technical coefficients, defined by

$$A_{ij} := \frac{(X_0)_{ij}}{(Y_0)_j} \qquad , \tag{21}$$

stay constant. Here, sector j requires A_{ij} units from sector i to produce one unit of itself (e.g.: To produce 1 car, one needs 4 weels and 1 engine). The equilibrium equation (19) can be rewritten to the following

$$(Y_0)_i = \sum_{j=1}^{N} A_{ij}(Y_0)_j + (D_0)_i \qquad \overset{\text{in matrix notation}}{\Leftrightarrow} \qquad Y_0 = AY_0 + D_0 \tag{22}$$

(For now we can neglect (20) since we only concentrate on goods, and therefore do not consider v_0.) Assuming that the A_{ij} stay constant, one can also calculate the total output Y_0, associated to a given demand vector D_0.

$$Y_0 = (\mathbb{I} - A)^{-1} D_0. \tag{23}$$

(Where $\mathbb{I}$ is the suitable identity matrix.) If equation (23) or (22) is satisfied, the system is in an equilibrium, i.e. the output matches the system's demand.

To develop this into a dynamical model, we consider the output $Y(t)_i$ of each sector as a dynamical variable. The amount of goods used at time t in the i-th sector is $\big(AY(t) + D_0\big)_i$. It is assumed that each sector tries to adjust its output such that, $Y(t)_i$ matches $\big(AY(t) + D_0\big)_i$. In contrary to the previously discussed equilibrium case, now $Y(t)_i$ and $\big(AY(t) + D_0\big)_i$ are not necessarily equal at each point in time. This behaviour is expressed by the following differential equation

$$\dot{Y}(t) = AY(t) + D_0 - Y(t) = (A - \mathbb{I})Y(t) + D_0. \tag{24}$$

This is known as the dynamic Leontief model. This model can be extended to include external demand shocks $X(t)$ leading to the following differential equation:

$$\dot{Y}(t) = (A - \mathbb{I})Y(t) + D_0 + X(t). \tag{25}$$

Adding a stochastic force $F(t)$ which models small variations in the demand that act on the system, we end up with the following differential equation (also see [KPT19]) (Here $F(t)$ has the form of white noise, therefore for the mean $\langle F(t)_i \rangle = 0$ and for the covariance $\langle F(t)_i F(s)_j \rangle = \mathbb{I}_{ij}\delta(t-s)$ hold.)

$$\dot{Y}(t) = (A - \mathbb{I})Y(t) + D_0 + X(t) + F(t). \tag{26}$$

3.3 Discussion of different expectation values

In this subsection three conceptually different expectation values will be discussed. All of them are expectation values for the output Y under the application of a demand shock X either in equations (25) or (26). These expectation values are:

- $\langle \Delta Y(t) \rangle_{X;ODE}$ the ordinary differential equation. (Whilst this is not strictly an expectation value, since (25) has no stochastic nature, we will treat it as one for now.)
- $\langle \Delta Y(t) \rangle_{X;SDE}$ of the stochastic differential equation (26).
- $\langle \Delta Y(t) \rangle_{X;LRT}$ the expectation value calculated by applying the formalism of Linear Response Theory (i.e. equation (16)) to (26).

Since both, (25) and (26) have the same equilibrium solution, in absence of a shock X we get $Y_0 = Y_0 = (\mathbb{I} - A)^{-1}D_0$. The changes due to a shock therefore have the following form

$$\langle \Delta Y \rangle_X = Y_X(t) - Y_0 = Y_X(t) - (\mathbb{I} - A)^{-1}D_0 \tag{27}$$

Here $Y_X(t)$ denotes the time evolution under a shock. Performing the transformation $y(t) := Y(t) - Y_0$ equations (25) and (26) can be rewritten as

$$\dot{y}(t) - (A - \mathbb{I})y(t) = X(t), \text{ and } \dot{y}(t) - (A - \mathbb{I})y(t) = X(t) + F(t). \tag{28}$$

It can be seen that these equations are equivalent to the one of the Orenstein-Uhlenbeck process (3) up to very basic substitution of variables.

3.3.1 Derivation of the ODE expectation value, $\langle \Delta Y(t) \rangle_{X;ODE}$

For the differential equation $\dot{y}(t) - (A - \mathbb{I})y(t) = X(t)$ a closed form solution exists. (This can be found in [Ris12, 38f.].) We want $\tilde{y}$ to be a solution. Its initial condition is set by a specific $\tilde{y}(0)$. The homogenous solution of this equation can be assumed to be of the following form

$$\tilde{y}^{\text{hom}}(t)_i = G(t)\tilde{y}(0).$$

(It solves $\dot{\tilde{y}}^{\text{hom}}(t) - (A - \mathbb{I})\tilde{y}^{\text{hom}}(t) = 0$.) Since the initial condition is $y(0) = 0$, G has to satisfy $G(0) = \mathbb{I}$. Plugging G into the homogenous differential equation, results in the following condition

$$\dot{G}(t) - (A - \mathbb{I})G(t) = 0. \tag{29}$$

We therefore get the following solution

$$G(t) = \exp\big(-(A - \mathbb{I})t\big)\,.$$

Using the variation of constants, the following ansatz is chosen for the inhomogenous solution $\tilde{y}^{\text{inhom}}$

$$\tilde{y}^{\text{inhom}}(t) = G(t)c(t). \tag{30}$$

This then has to solve the following

$$\dot{\tilde{y}}^{\text{inhom}}(t) - (A - \mathbb{I})\tilde{y}^{\text{inhom}}(t) = X(t)$$

Plugging $\tilde{y}^{\text{inhom}}(t) = G(t)c(t)$ into this, we get

$$\begin{aligned}
\dot{G}(t)c(t) + G(t)\dot{c}(t) - (A - \mathbb{I})G(t)c(t) &= X(t)\\
c(t)\underbrace{\Big(\dot{G}(t) - (A - \mathbb{I})\Big)}_{\overset{(29)}{=}0} + G(t)\dot{c}(t) &= X(t)\\
G(t)\dot{c}(t) &= X(t).
\end{aligned}$$

Therefore a solution for $\dot{c}$ is $\dot{c}(t) = G^{-1}(t)X(t)$. Integrating and using (30) we get the following solution for $\tilde{y}^{\text{inhom}}$:

$$\begin{aligned}
\tilde{y}^{\text{inhom}}(t) = G(t)c(t) = G(t)\int_0^t G(\tau)^{-1}F(\tau)\,\mathrm{d}\tau &=\\
\overset{(*)}{=} \int_0^t G(t)G(-\tau)X(\tau)\,\mathrm{d}\tau &=\\
\overset{(**)}{=} \int_0^t G(t-\tau)X(\tau)\,\mathrm{d}\tau &=\\
= \int_0^t G(\tau)X(t-\tau)\,\mathrm{d}\tau&
\end{aligned}$$

At $(*)$ we use that $G(\tau)^{-1} = \exp(-(A - \mathbb{I})\tau)^{-1} = \exp(-(A - \mathbb{I})(-\tau)) = G(-\tau)$, for $(**)$: $G(t)G(-\tau) = \exp(-(A-\mathbb{I})t)\exp(-(A-\mathbb{I})(-\tau)) = \exp(-(A-\mathbb{I})(t-\tau)) = G(t-\tau)$. From this we can obtain the general solution $\tilde{y}$ by adding the homogenous and the inhomogenous solutions together

$$\begin{aligned}
\tilde{y}(t) &= \tilde{y}^{\text{hom}}(t) + \tilde{y}^{\text{inhom}}(t) =\\
&= G(t)\tilde{y}(0) + \int_0^t G(\tau)X(t-\tau)\,\mathrm{d}\tau. \qquad (31)
\end{aligned}$$

Since we set $\tilde{y}(t)$ in such a way that its equal to $\langle\Delta Y(t)\rangle_{X;ODE}$, and $y(0) = 0$ we end up with

$$\begin{aligned}\langle\Delta Y(t)\rangle_{X;ODE} &= Y(t) - Y(0) = \\ &= \int_0^t e^{-(A-\mathbb{I})\tau} X(t-\tau)\,\mathrm{d}\tau.\end{aligned} \tag{32}$$

(The differential equation (25) for which this is a solution does not have a stochastic term, therefore the expectation value is equal to the closed form solution.)

3.3.2 Derivation of the SDE expectation value, $\langle\Delta Y\rangle_{X;SDE}$

The ordinary differential equation (25) can be transformed to the stochastic one (26) by transforming $X \to X + F$. Applying this same transformation to the solution (33) we get

$$\Delta Y(t)_{X;SDE} = e^{-(A-\mathbb{I})t}\Delta Y(0) + \int_0^t e^{-(A-\mathbb{I})\tau}\big(X(t-\tau) + F(t-\tau)\big)\,\mathrm{d}\tau. \tag{33}$$

This still includes a stochastic term F. By taking the expecation value this term disappears [Ris12]

$$\begin{aligned}\langle\Delta Y(t)\rangle_{X;SDE} &= \langle Y(t)\rangle_{X;SDE} - Y(0) = \\ &= \int_0^t e^{-(A-\mathbb{I})\tau}\big(X(t-\tau) + \langle F(t-\tau)\rangle\big)\,\mathrm{d}\tau = \\ &= \int_0^t e^{-(A-\mathbb{I})\tau}\left(X(t-\tau) + \underbrace{\langle F(t-\tau)\rangle}_{=0}\right)\mathrm{d}\tau = \\ &= \int_0^t e^{-(A-\mathbb{I})\tau} X(t-\tau)\,\mathrm{d}\tau.\end{aligned} \tag{34}$$

Here we use the linearity of the expectation value and that by definition $\langle F(t)\rangle = 0$.

3.3.3 Derivation of the LRT expectation value, $\langle\Delta Y\rangle_{X;LRT}$

The Green-Kubo formula (16) for the stochastic differential equation in (28) has the following form

$$\langle\Delta Y(t)\rangle_{X;LRT} = \int_0^t \left(\sigma^{-1}\right)_{ij}\langle y_k(\tau)y_j(0)\rangle_0 X_i(\tau)\,\mathrm{d}\tau. \tag{35}$$

To get a closed form solution of this, first we have to calculate the equilibrium expectation value $\langle y_k(\tau)y_j(0)\rangle_0$. Just a short reminder that $\langle.\rangle_0$ is meant to represent the expectation value in absence of an external shock $X = 0$. Therefore y has to solve $\dot{y}(t) - (A - \mathbb{I})y(t) = F(t)$. We can use the closed form solution (33) of this stochastic differential equation (see [Ris12, p. 40] and [Gar04, p. 109])

$$\begin{aligned}
\langle y_k(\tau) y_j(0) \rangle_0 &= \left\langle \left(e^{-(A-\mathbb{I})\tau} y(0) + \int_0^\tau e^{-(A-\mathbb{I})\tau'} F(\tau - \tau') \, \mathrm{d}\tau' \right)_k \Big(y(0) \Big)_j \right\rangle_0 = \\
&= \left\langle \left((e^{-(A-\mathbb{I})\tau})_{hi} y(0)_i \right)_k \Big(y(0) \Big)_j \right\rangle_0 = \\
&= \left\langle \left((e^{-(A-\mathbb{I})\tau})_{ki} y(0)_i \right) \Big(y(0) \Big)_j \right\rangle_0 = \\
&= \left(e^{-(A-\mathbb{I})\tau} \right)_{ki} \underbrace{\langle y(0)_i y(0)_j \rangle_0}_{=:\sigma_{ij}} . \qquad (36)
\end{aligned}$$

$\langle y(0)_i y(0)_j \rangle_0$ is the already known stationary covariance matrix σ_{ij}. Plugging this into (35) we get

$$\begin{aligned}
\langle \Delta Y(t) \rangle_{X;LRT} &= \int_0^t \left(\sigma^{-1} \right)_{ij} \left(e^{-(A-\mathbb{I})(t-\tau)} \right)_{ki} \sigma_{ij} X_i(\tau) \, \mathrm{d}\tau = \\
&\overset{(*)}{=} \int_0^t \underbrace{\left(\sigma^{-1} \right)_{ij} \sigma_{ji}}_{=\mathbb{I}_{ij}} \left(e^{-(A-\mathbb{I})(t-\tau)} \right)_{ki} X_i(\tau) \, \mathrm{d}\tau = \\
&= \int_0^t \left(e^{-(A-\mathbb{I})(t-\tau)} \right) X(\tau) \, \mathrm{d}\tau = \int_0^t \left(e^{-(A-\mathbb{I})\tau} \right) X(t - \tau) \, \mathrm{d}\tau \qquad (37)
\end{aligned}$$

For $(*)$ we used that the covariance matrix is symmetric since $\sigma_{ij} = \langle y(0)_i y(0)_j \rangle_0 = \langle y(0)_j y(0)_i \rangle_0 = \sigma_{ji}$. Again plugging in the solution for the stochastic differential equation (33)

3.4 Equivalence, $\langle \Delta Y \rangle_{X;ODE} = \langle \Delta Y \rangle_{X;SDE} = \langle \Delta Y \rangle_{X;LRT}$

Collecting the results of the previous subsections: Under the assumption that the system is in an equilibrium at $t = 0$ which means that, $\Delta Y(0) = 0$, the equations (31), (34), (37) yield the following result:

$$\langle \Delta Y \rangle_{X;ODE} = \langle \Delta Y \rangle_{X;SDE} = \langle \Delta Y \rangle_{X;LRT}.$$

Therefore, the three different formulations (the ordinary differential equation §3.3.1, the stochastic differential equation §3.3.2 and the Green-Kubo equations §3.3.3) of the dynamic Leontief model are equivalent.

Models based on the dynamic Leontief model (§3.2) have been successfully used to predict economic growth [Joh85], the economic recovery from natural disasters [Hal08; Yas+20] and the recovery from a financial crisis [KPT19] and the effects of a terror attack on the economy [San06]. However, these only allow for quantity adjustments, whilst keeping the prices constant. The next section will discuss agent based modelling which allows both, quantity and price adjustments.

4 Agent Based Model (ABM)

In the following section, first an agent-based extension to the dynamic Leontief model will be motivated. Next the concept of the considered shock to the economy will be discussed. To derive the update equation, price and quantity will be defined for the input-output tables. These are then used, to derive the update equations of the ABM. Using a theoretical example of an economy consisting of 4 sectors, it will be demonstrated how the behavioural parameters influence the trajectories of the ABM. The section will conclude with fitting these parameters to the Austrian economy for the years 2008-2016.

4.1 Motivation

Let us consider a very simple economic market consisting of companies producing goods and consumers buying them. One of the standard assumptions of economics is that if the price P for a good increases, the companies have a higher incentive to produce this good, therefore, the supply S increases proportionally. However, with increasing P the consumers will buy less, in other words the demand D decreases in proportion [Var14, 3f.],[Par10]. Hence, the $S(P)$ and the $D(P)$ curves intersect at a certain P^*, called the equilibrium price (see Figure 1).

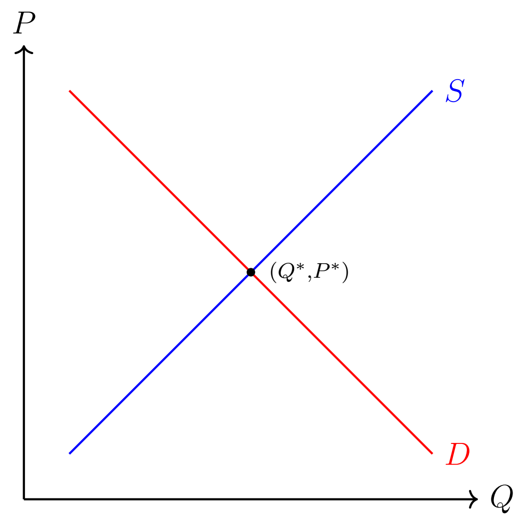

Figure 1: On one hand, the supply S of good increases with increasing price - the producers have an incentive to produce more - on the other hand, the demand D decreases - the consumers have an incentive to consume less. Therefore the $S(P)$ and $D(P)$ intersect at a certain price P^* called the equilibrium price.

However, a priori the form of these functions are unkown and only emerge from the behaviour of the consumers and companies. How this comes into being, is one of the questions

agent based modelling (ABM) tries to solve. Since the 90s [Art94], agent based modelling has been a subdomain of economics. These theories try to model the economy by simulating an ensemble of agents. They have been applied to a wide range of economic problems: Modelling the propagation of an economic shock [IT19; Car+20], the systemic risk of the housing market [Gea+12] and the financial market [TP13; Kli+15]. Usually, the types of agents are companies and end consumers. In each timestep of the simulation, each agent can execute different adjustments, to better comply with its external conditions. Typically these adjustments are expressed as update equations of the variables, describing the state of the different agents.
Since we consider only adjustments that are represented in the already discussed input-output framework, we will assume that the basic adjustments a company can make are changes of price and quantity [Gat+11, p. 55],[Ass+15]. A company wants to sell exactly the amount it produces. If a company sells all of its goods, it will assume that either the price it is charging is too low or it produces too little. On the other hand, if it is not selling all of them, it will either reduce prices or produce less.

This raises the question, how these price and quantity adjustments can be represented in input-output tables, whilst keeping them consistent (i.e.: such that the flow consistency equations (19) and (20) hold). The dynamic Leontief model, discussed in §3.2, already answers this question for quantity changes. Just to reiterate, under the assumption of constant technical coefficients and constant prices,the Leontief model allows us to calculate the output vector in terms of quantity for a certain demand vector [Leo86].
Similarly, a model for price changes exist: the Ghosh model [Gho58]. Parallel to the Leontief model, it assumes constant technical coefficients, but in the case of the Ghosh model the quantities stay constant and only their prices are adjusted. In its usual interpretation, it allows to calculate the price vector for a certain value added vector. However, as we are interested in price adjustments, we will use this model the other way round: to calculate the value added vector for a certain price vector. The adjustments in the Leontief model propagate "upstream", meaning if one sector tries to adjust its output, the sectors delivering goods to it have to adjust their output in quantity terms and so on. On the contrary the adjustments of the Ghosh model propagate "downstream", if one sector adjusts its price, the sectors buying goods from it will have to adjust their prices and so on. Since the price of the goods, which the buying sector uses as inputs to produce its good, increases/decreases, the buying sector will increase/decrease the price of its good in turn.

To sum up, whilst the Leontief model allows us to calculate quantity adjustments under the assumption of constant prices, the Ghosh model allows us to calculate price adjustments, whilst assuming constant quantities. In the following, update equations will be discussed that allow both, quantity and price adjustments whilst keeping the input-ouptut tables consistent. This will be possible, by combining both, the Leontief and the Ghosh model and calculating their adjustments successively.

4.2 Experienced shock $\bar{X}$

Before deriving the update equations of the ABM, we have to discuss what we are actually trying to model. The following example should show, how an agent experiences a change in demand or price as an experienced shock $\bar{X}$.

The flowchart in Figure 2 should illustrate this idea:
In week 1 we start by considering a company in an equilibrium state. By equilibrium state we mean that it sells all goods it produces and would not be able to sell more. Then, in week 2 it experiences a shock $\bar{X}$. As an effect of the shock $\bar{X}$, the company is not able to sell all produced goods anymore. We assume that the company does not know, whether it is facing a quantity or price shock. If the company assumes that it experiences a quantity shock, it will only adjusts its produced quantity. Therefore it will follow the left path for week 3. On the other hand, if it assumes a price shock, it will adjust only its price and follow the right path in the flowchart.
For the middle way, the company will adjust both, its price and its quantity, dependent on the behavioural parameters (h_Q, h_v).

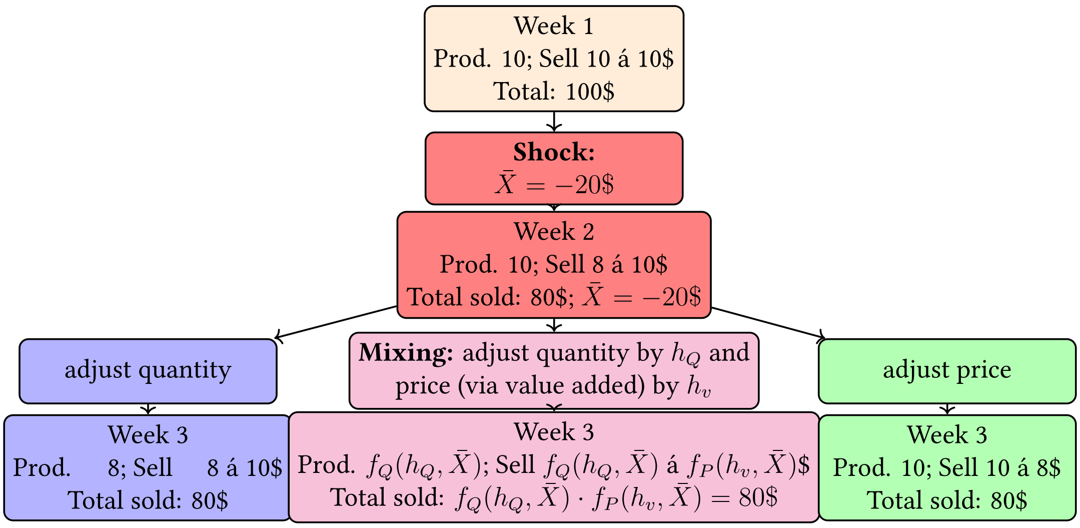

Figure 2: This flowchart illustrates, how a company can react to an experienced shock $\bar{X}$. The function f_Q and f_v are just place holders.

4.3 Unification of the dynamic Leontief and Ghosh model

Given, a whole economy experiences a shock of the type, as discussed in §4.2, we will now derive the update equations of a unified Leontief and Ghosh model.

4.3.1 Relationship of Quantity and Price

Let us again consider the input-output table (as in (18))

$$\begin{array}{c|cc} X_t & D_t & Y_t \\ \hline (v_t)^T & - & v_t^T e \\ (Y_t)^T & e^T D_t & - \end{array} \qquad \text{with } e^T := (1, \ldots, 1), \tag{38}$$

where $\cdot_t$ expresses that these are the variables at time t. Here all values are in monetary terms: $[(X_0)_{ij}] = [(D_t)_i] = [(v_t)_i] = [(Y_t)_i] = \$$. We will define the quantity as a vector $Q_t \in \mathbb{R}^N$ and the price vector as $P_t \in \mathbb{R}^N$. For each sector this leads to the following relation

$$(Y_t)_i = (Q_t)_i \cdot (P_t)_i = (Q_t \circ P_t)_i$$

where $\circ$ is the element-wise multiplication. The dimensions of each variable in this equation are the following

$$[(Y_t)_i] = \$ \quad , \quad [(Q_t)_i] = q_i \quad , \quad [(P_t)_i] = \$/q_i.$$

The goal of the derivation that follows is to get update equations that adjust the input output table according to a demand shock $\bar{X}$ whilst keeping the technical coefficients A_{ij} (21) constant. The update equations should converge such that $D_\infty = D_0 + \bar{X}_0$ holds. The shock $\bar{X}_t$ will be in monetary terms $[(\bar{X}_t)_i] = \$$.

4.3.2 Quantity adjustment

The Leontief model in §3.2 assumes that prices stay constant $(P_t)_i = 1$. Therefore, the flow consistency equation (22) that holds for Y_t in the previous chapter now applies to Q_t:

$$Q_t = \underbrace{\underbrace{AQ_t}_{\text{inputs needed to produce } Q_t} + \underbrace{E_t}_{\text{demand of the end-consumer}}}_{\text{total 'demand'}} . \tag{39}$$

E_t is the demand of the end consumer in quantity terms, $D_t = E_t \circ P_t$ from which $E_t = P_t^{-1} \circ D_t$ follows. This equation can also be read in a way that the total production Q_t matches the inputs needed to produce those AQ_t plus the demand of the end consumer E_t. Adding the experienced shock in quantity terms $\bar{X}_t \circ P_t^{-1}$, the total "experienced" demand at time t is

$$Q_t^{(X)} = AQ_t + E_t + \bar{X}_t \circ P_t^{-1}.$$

Each sector tries to adjust its production $(Q_t)_i$ to match the experienced demand $(Q_t^{(X)})_i$. The logical update equation therefore is of the following form

$$\begin{aligned} Q_{t+1} =& Q_t + h_Q \left(Q_t^{(X)} - Q_t \right) \\ =& Q_t + h_Q \left(AQ_t + E_t + \bar{X}_t \circ P_t^{-1} - Q_t \right) \\ =& Q_t + h_Q \left((A - \mathbb{I}) Q_t + E_t + \bar{X}_t \circ P_t^{-1} \right) \\ \overset{(39)}{=}& Q_t + h_Q \left(\bar{X}_t \circ P_t^{-1} \right). \end{aligned} \tag{40}$$

After each adjustment step, the demand in quantity terms has to be adjusted to keep the IO-tables consistent. (All goods are actually consumed within the same time step.)

$$E_{t+1} = Q_{t+1} - AQ_{t+1} = (\mathbb{I} - A) Q_{1+t} \tag{41}$$

4.3.3 Price/value added adjustment

To understand how each agent can adjust the price in this model, we first need to understand how the price arises from the input-output table (38). By dividing the equilibrium equation (20) by $./(Y_t)_j$ we get

$$\sum_{i=1}^{N} \frac{(X_t)_{ij}}{(Y_t)_j} + \frac{(v_t)_j}{(Y_t)_j} = 1.$$

Defining the diagonal matrix $L := \operatorname{diag}\left((v_t)_1/(Y_t)_1, \ldots, (v_t)_N/(Y_t)_N \right)$, and using the definition of the technical coefficients A_{ij} (see (21)), this can be written as

$$\underbrace{\sum_{j=1}^{N} A_{ij}^T}_{\text{inputs from other sectors}} + \underbrace{L_{ii}}_{\text{value added}} = 1.$$

This expresses, which fraction of the total output in monetary terms consists of the inputs of the other sectors and the value added. If the value added changes to a new vector $v_{\tilde{t}}$, these ratios have to adjust accordingly, satisfying

$$\sum_{j=1}^{N} A_{ij}^T (P_{\tilde{t}})_j + L_{ii} \frac{(v_{\tilde{t}})_i}{(v_t)_i} = (P_{\tilde{t}})_i$$

and in matrix form

$$A^T (P_{\tilde{t}}) + L (v_{\tilde{t}}) \circ (v_t)^{-1} = (P_{\tilde{t}}). \tag{42}$$

These are the new accounting equations also considering the relative price $P_{\tilde{t}}$. Solving this for $v_{\tilde{t}}$ we get

$$v_{\tilde{t}} = \mathrm{diag}(v_t) L^{-1} \left(\mathbb{I} - A^T\right) \left(P_{\tilde{t}}\right).$$

This equation expresses, how much value added the agents can charge for a specific price vector $P_{\tilde{t}}$. (This derivation can also be found in [Die97].) For $P_{\tilde{t}} = (1, \ldots, 1)^T$, it follows that $v_{\tilde{t}} = \mathrm{diag}(v_t) L^{-1} \left(\mathbb{I} - A^T\right)(1, \ldots, 1) = v_t$. So whatever v_t vector we chose in this equations, it will represent the value added vector associated with the unit price $(1, \ldots, 1)^T$. From now on, we will therefore choose $t = 0$ as the reference time, to which we compare the relative price increases/decreases - therefore set $v_{t=0}$ in this equation.
If the agents interpret the experienced shock $\bar{X}_{t+1/2}$ as a price shock, and their current demand in quantity terms is E_{t+1} they assume that they could theoretically charge $P_t + \bar{X}_{t+1/2} \circ E_{t+1}^{-1}$ (Assuming that the ABM has already made an adjustment of $\bar{X} \overset{\text{step}}{\mapsto} \bar{X}_{t+1/2}$ and $E_t \overset{\text{step}}{\mapsto} E_{t+1}$ - this will become clearer in §4.3.5.). Since the experienced shock is in monetary terms, it has to be divided by E_{t+1} the demand in quantity terms. For the value added, an agent could theoretically charge

$$v_t^{(X)} = \mathrm{diag}(v_0) L^{-1} \left(\mathbb{I} - A^T\right) \left(P_t + \bar{X}_{t+1/2} \circ E_{t+1}^{-1}\right). \tag{43}$$

And for the update equation of the value added we get

$$\begin{aligned} v_{t+1} =& v_t + h_v \left(v_t^{(X)} - v_t\right) \\ =& v_t + h_v \left(\mathrm{diag}(v_0) L^{-1} \left(\mathbb{I} - A^T\right) \left(P_t + \bar{X}_{t+1/2} \circ E_{t+1}^{-1}\right) - v_t\right) \\ \overset{(43)}{=}& v_t + h_v \left(\mathrm{diag}(v_0) L^{-1} \left(\mathbb{I} - A^T\right) \left(P_t + \bar{X}_{t+1/2} \circ E_{t+1}^{-1} - P_t\right)\right) \\ =& v_t + h_v \left(\mathrm{diag}(v_0) L^{-1} \left(\mathbb{I} - A^T\right) \left(\bar{X}_{t+1/2} \circ E_{t+1}^{-1}\right)\right). \end{aligned} \tag{44}$$

After an update of the value added, the price has to be adjusted accordingly, again to keep the input-output tables consistent. Solving the accounting equation (42) for P we get

$$P_{t+1} = (\mathbb{I} - A^T)^{-1} L (v_0^{-1} \circ v_{t+1}). \tag{45}$$

4.3.4 Adjustment of the 'experienced' shock

After each iteration the update equations (40), (41), (44) and (45) produce an IO-table. These IO-tables should converge to a table, which has fully adjusted to the shock, by which we mean that $D_\infty = D_0 + \bar{X}_0$. ($._\infty$ expresses the variable after a sufficient ammount of update steps - it has fully converged.) To accomplish, that the system actually converges to this state the shock also has to be updated after each price/quantity update. It has to be adjusted in the following way:

- After a quantity update:

$$\bar{X}_{t+1/2} = \bar{X}_t - (E_{t+1} - E_t) \circ P_t \tag{46}$$

(Since $\bar{X}$ is updated 2 times in one timestep we chose $\bar{X}_{t+1/2}$.)

- After a price/value added update:

$$\bar{X}_{t+1} = \bar{X}_{t+1/2} - E_{t+1} \circ (P_{t+1} - P_t) \tag{47}$$

(Again, assuming that the price adjustment happens after a quantity adjustment $E_t \overset{\text{step}}{\mapsto} E_{t+1}$.)

4.3.5 Pseudo-Code

Putting all these equations together, the pseudo code for one update step is:

$$\begin{aligned}
Q_{t+1} &= Q_t + h_Q \left(\bar{X}_t \circ P_t^{-1}\right) \\
E_{t+1} &= (\mathbb{I} - A) Q_{t+1} \\
\bar{X}_{t+1/2} &= \bar{X}_t - (E_{t+1} - E_t) \circ P_t \\
v_{t+1} &= v_t + h_v \left(\operatorname{diag}(v_0) L^{-1} \left(\mathbb{I} - A^T\right) \left(\bar{X}_{t+1/2} \circ E_{t+1}^{-1}\right) \right) \\
P_{t+1} &= (\mathbb{I} - A^T)^{-1} L (v_0^{-1} \circ v_{t+1}) \\
\bar{X}_{t+1} &= \bar{X}_t - E_t \circ (P_{t+1} - P_t).
\end{aligned} \tag{48}$$

Here, the role of the behavioural parameters h_Q and h_v becomes clearer. h_Q determines how much of the adjustments happens in the quantity dimension, whilst h_v determines the value added/price dimension. The equations were chosen in such a way that $\bar{X}_\infty = 0$ and $\bar{X}_0 = E_\infty \circ P_\infty$ when these update equations converge $t \to \infty$. This relationship is illustrated in Figure 3. If $(h_Q = 1, h_v = 0)$, then $P_\infty = P_0 = (1, \ldots, 1)^T$ and it follows that $E_\infty = \bar{X}_0$. This case represents the Leontief model - only the quantity is adjusted, whilst the value added stays constant. (This is the left side of Figure 3.) On the other hand, if $(h_Q = 0, h_v = 1)$ holds, it follows that $E_t = E_0$ and $P_\infty = \bar{X}_0 / E_0$. This case represents the dynamic Ghosh model - here the quantities are fixed, and only the prices and value added are adjusted. (Right side of Figure 3.)
Assuming that both parameters are sufficently small, the E_∞ and P_∞ only depend upon the fraction h_Q/h_v.

Reflection on the Agent based unification of the Leontief and Ghosh model

Consistently unifying these models was only possible by adjusting both, the shock $\bar{X}$ and (P_t, Q_t) in each timestep. Usually, the dynamic Leontief model is written down as $\dot{Q}(t) =$

$(A - \mathbb{I})Q(t) + E_0 + X(t)$. The natural way to unify this with the Gosh model, would have been to write down an equation that would have the form of $\dot{P}(t) = \mathcal{B} \cdot P(t) + v_0 + X(t)$ (this equation just illustrates the general form). However, keeping the shock $X(t) = X$ constant, the system described by these equations will overshoot the applied shock: $D_\infty - D_0 = E_\infty \circ P_\infty - E_0 \circ P_0 > X$. The IO table of the converged ABM will have increased by more than the initially applied shock X. Both variables Q_t and P_t will try to fully adjust to the shock X without knowing how much adjustment already took place in the other variable. This was only solved by bookkeeping how much adjustment already happened to the shock (see §4.3.4). But this made it necessary to replace $E_0 \mapsto E_t = (\mathbb{I} - A)Q(t)$, otherwise $E_\infty \circ P_\infty = X_0$ would not have been satisfied. (Suppose the system has fully adjusted to the shock, therefore $X_\infty = 0$, since $\dot{Q}_\infty = (A - \mathbb{I})Q_\infty + E_0 + X_\infty = (A - \mathbb{I})Q(t) + E_\infty$ from wich $E_\infty = E_0$ would follow.)

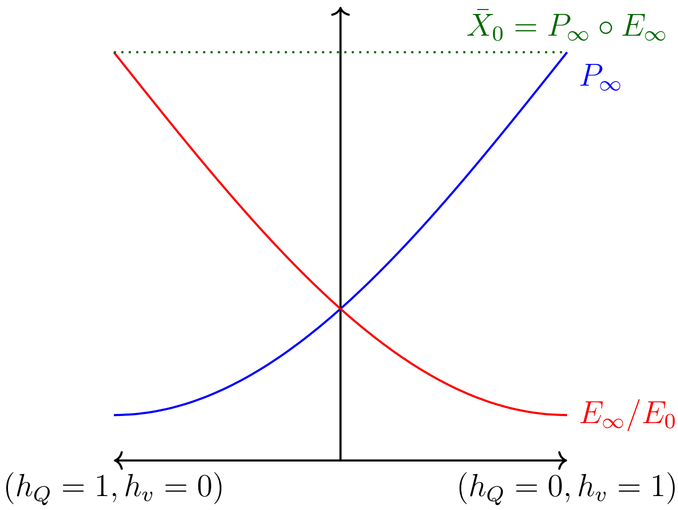

Figure 3: This figure is an illustration of, how the behavioural parameters h_Q, h_v determine the state the sytem ends up in. On the left side of the plot, the adjustment happens only in the quantity dimension - the leontief model dominates, and on the right hand side the adjustmen only happens in the price dimension - the ghosh model dominates. (This is just a schematic plot without actual numeric values.)

4.3.6 From difference to a differential equation formulation

The update equations (48) can be reformulated from a system of difference to a system of differential equations. Note that E_t can be written as $E_t = (\mathbb{I} - A)Q_t = \mathcal{E}Q_t$ and P_t by $P_t = (\mathbb{I} - A^T)^{-1}L(v_0^{-1} \circ v_t) = \mathcal{P}v_t$. (Here the matrizes $\mathcal{E} := (\mathbb{I} - A)$ and $\mathcal{P} := (\mathbb{I} - A^T)^{-1}L \cdot \mathrm{diag}(v_0)^{-1}$ were introduced.) Further, the time domain can be introduced by considering a step $t + \mathrm{d}t$. This gives us the following form of the update equations (48):

$$
\begin{aligned}
Q_{t+\mathrm{d}t} &\overset{(1)}{=} Q_t + \mathrm{d}t \cdot h_Q\left(\bar{X}_t \circ (\mathcal{P}v_t)^{-1}\right) \\
\bar{X}_{t+\mathrm{d}t/2} &\overset{(2)}{=} \bar{X}_t - \mathcal{E}(Q_{t+\mathrm{d}t} - Q_t) \circ (\mathcal{P}v_t) \\
v_{t+\mathrm{d}t} &\overset{(3)}{=} v_t + \mathrm{d}t \cdot h_v\left(\mathcal{P}^{-1}\left(\bar{X}_{t+\mathrm{d}t/2} \circ (\mathcal{E}Q_{t+\mathrm{d}t})^{-1}\right)\right) \\
\bar{X}_{t+\mathrm{d}t} &\overset{(4)}{=} \bar{X}_{t+\mathrm{d}t/2} - \mathcal{E}Q_{t+\mathrm{d}t} \circ \mathcal{P}(v_{t+\mathrm{d}t} - v_t).
\end{aligned}
$$

Considering the limit $\mathrm{d}t \to 0$, (1) and (2) can be rewritten in a differential equation:

$$
\begin{aligned}
\dot{Q}_t &\overset{(1)}{=} \frac{Q_{t+\mathrm{d}t} - Q_t}{\mathrm{d}t} = h_Q\left(\bar{X}_t \circ (\mathcal{P}v_t)^{-1}\right) \\
\dot{v}_t &\overset{(3)}{=} \frac{v_{t+\mathrm{d}t} - v_t}{\mathrm{d}t} = h_v\left(\mathcal{P}\left(\bar{X}_t \circ (\mathcal{E}Q_{t+\mathrm{d}t})^{-1}\right)\right).
\end{aligned}
$$

From plugging in $(2) \hookrightarrow (4)$ it follows that

$$
\begin{aligned}
\bar{X}_{t+\mathrm{d}t} &= \bar{X}_t - \mathcal{E}(Q_{t+\mathrm{d}t} - Q_t) \circ (\mathcal{P}v_t) - \mathcal{E}Q_{t+\mathrm{d}t} \circ \mathcal{P}(v_{t+\mathrm{d}t} - v_t) \\
\bar{X}_{t+\mathrm{d}t} - \bar{X}_t &= -\mathcal{E}(Q_{t+\mathrm{d}t} - Q_t) \circ (\mathcal{P}v_t) - \mathcal{E}Q_{t+\mathrm{d}t} \circ \mathcal{P}(v_{t+\mathrm{d}t} - v_t) \\
\bar{X}_{t+\mathrm{d}t} - \bar{X}_t &= -\mathcal{E}\,\mathrm{d}t\dot{Q}_t \circ (\mathcal{P}v_t) - \mathcal{E}\underbrace{Q_{t+\mathrm{d}t}}_{=Q_t+\mathrm{d}t\cdot(\dots)} \circ \mathcal{P}\,\mathrm{d}t\dot{v}_t \\
\bar{X}_{t+\mathrm{d}t} - \bar{X}_t &= -\,\mathrm{d}t \cdot \mathcal{E}\dot{Q}_t \circ (\mathcal{P}v_t) - \mathrm{d}t \cdot \mathcal{E}Q_t \circ \mathcal{P}\dot{v}_t + \underbrace{(\mathrm{d}t)^2 \cdot (\dots)}_{\approx 0} \\
\dot{\bar{X}}_t = \frac{\bar{X}_{t+\mathrm{d}t} - \bar{X}_t}{\mathrm{d}t} &= -\mathcal{E}\dot{Q}_t \circ (\mathcal{P}v_t) - \mathcal{E}Q_t \circ \mathcal{P}\dot{v}_t.
\end{aligned}
$$

We get the following system of differential equations, that describe the system

$$\dot{Q}_t = h_Q\left(\bar{X}_t \circ (\mathcal{P}v_t)^{-1}\right) \tag{49}$$

$$\dot{v}_t = h_v\left(\mathcal{P}\left(\bar{X}_t \circ (\mathcal{E}Q_{t+\mathrm{d}t})^{-1}\right)\right) \tag{50}$$

$$\dot{\bar{X}} = -\mathcal{E}\dot{Q}_t \circ (\mathcal{P}v_t) - \mathcal{E}Q_t \circ \mathcal{P}\dot{v}_t. \tag{51}$$

These differential equations are equivalent to the difference equations (48). For the numeric applications in the following chapters, the difference equations were used.

4.4 Toy model of an economy of 4 sectors

Let us consider a small sample economy with 4 sectors, which has the following input-output table:

$$\begin{array}{cccc|cc} 0 & 50 & 0 & 0 & 10 & 60 \\ 0 & 0 & 90 & 50 & 20 & 160 \\ 0 & 100 & 0 & 0 & 10 & 110 \\ 0 & 0 & 0 & 0 & 60 & 60 \\ \hline 60 & 10 & 20 & 10 & - & 100 \\ 60 & 160 & 110 & 60 & 110 & - \end{array} . \tag{52}$$

The network of this economy is visualized in Figure 4. This shows the sectors as the different nodes, the flows inbetween the sectors in monetary terms as links. The value added/demand in the sectors is shown as links to/from the nodes without an origin/target node. I chose this form to illustrate the network, because for each node the inflow equals the outflow. This illustrates that the flow consistency equations (19) and (20) ($\sum_{j=0}^{N} X_{ij} + D_i = \sum_{i=0}^{N} X_{ij} + v_j$) hold.

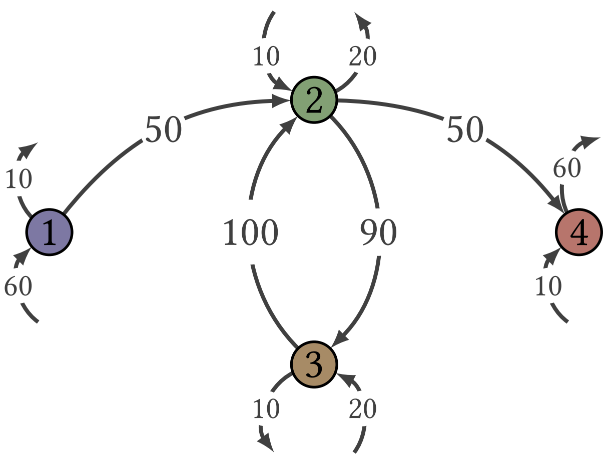

Figure 4: Sketch of the sector-network represented by the input-output table in (52). The demand D_i in each sector is visualized as an arrow to nothing, the value added v_i as an arrow from nothing. This visualiziation highlights that the inputs plus value added equal the outputs plus the other sectors plus the demand. In mathematical terms, it shows that $\sum_{j=0}^{N} X_{ij} + D_i = \sum_{i=0}^{N} X_{ij} + v_j$ holds.

Applying the demand shock $\bar{X}_{250} = (0, (D_0)_2/10, 0, 0)^T$ (the second sector gets shocked by 10% of its initial demand), we will consider how the system converges for some chosen parameter pairs of the step size parameters (h_Q, h_v). In Figure 5 the columns show the trajectories for the pairs $(h_Q = 0.05, h_v = 0)$; $(h_Q = 0.025, h_v = 0.025)$; $(h_Q = 0, h_v = 0.05)$.
Each of the three simulations ran over 1000 timesteps. The shock was applied at the 250th

Variable	$(h_Q = 0.05, h_v = 0)$	$(h_Q = 0.025, h_v = 0.025)$	$(h_Q = 0, h_v = 0.05)$
$(Y_{1000}/Y_0^{-1})_2$	1.026	1.038	1.1
$(E_{1000}/E_0^{-1})_2$	1.1	1.080	1
$(v_{1000}/v_0^{-1})_2$	1	1.007	2.059
$(P_{1000}/P_0)_2$	1	1.019	1.1

Table 2: The state at the end of the simulation for all variables for the second sector.

step. This choice of parameters was mostly chosen for visual reasons, and such that variables converge. This means, that the shock will have disappeared at $t = 1000$:

$$\bar{X}_{1000} = (0, 0, 0, 0)^T \tag{53}$$

and therefore D_t will have increased by 10% in the second sector:

$$D_{1000} \circ D_0^{-1} = (P_{1000} \circ E_{1000})/(E_0 \circ P_0)^{-1} = (1, 1.1, 1, 1)^T. \tag{54}$$

For the left column the parameters are $(h_Q = 0.05, h_v = 0)$. Therefore, the adjustment of D will appear only in the quantity dimension: $(E_{1000}/E_0)_2 = 1.1$ and no adjustment of the price takes place $(P_{1000}/P_0^{-1})_2 = 1$ (see table 2).
The second column represents a mixing between quantity and price adjustments, with $(h_Q = 0.025, h_v = 0.025)$. Since both, the first and third sector are suppliers to the second one, E_t for both decreases at first. In the first column, at the end of the simulation, E_t of the first and third sector returned back to their initial values. However, in the second column, some price adjustment has already taken place. That is why the values of E_t of the first and third sector are actually lower at the end.
In the third column only price adjustments take place $(h_Q = 0, h_v = 0.05)$. Since the seond sector increases its price, the buyers of this sector have to decrease their value added to keep the same price. These are the third and fourth sector. The total output Y_t of the second sector increases the most in this case, i.e.: if only price adjustment takes place.

4.5 Application to the Austrian economy

Until now, the ABM was only discussed in a theoretical context. In this subsection it will be shown, how it can be used to fit the behavioural parameters (h_Q, h_v) to real world data. This will then be realised for publicly available input-output data of the Austrian economy for the years 2008-2016.

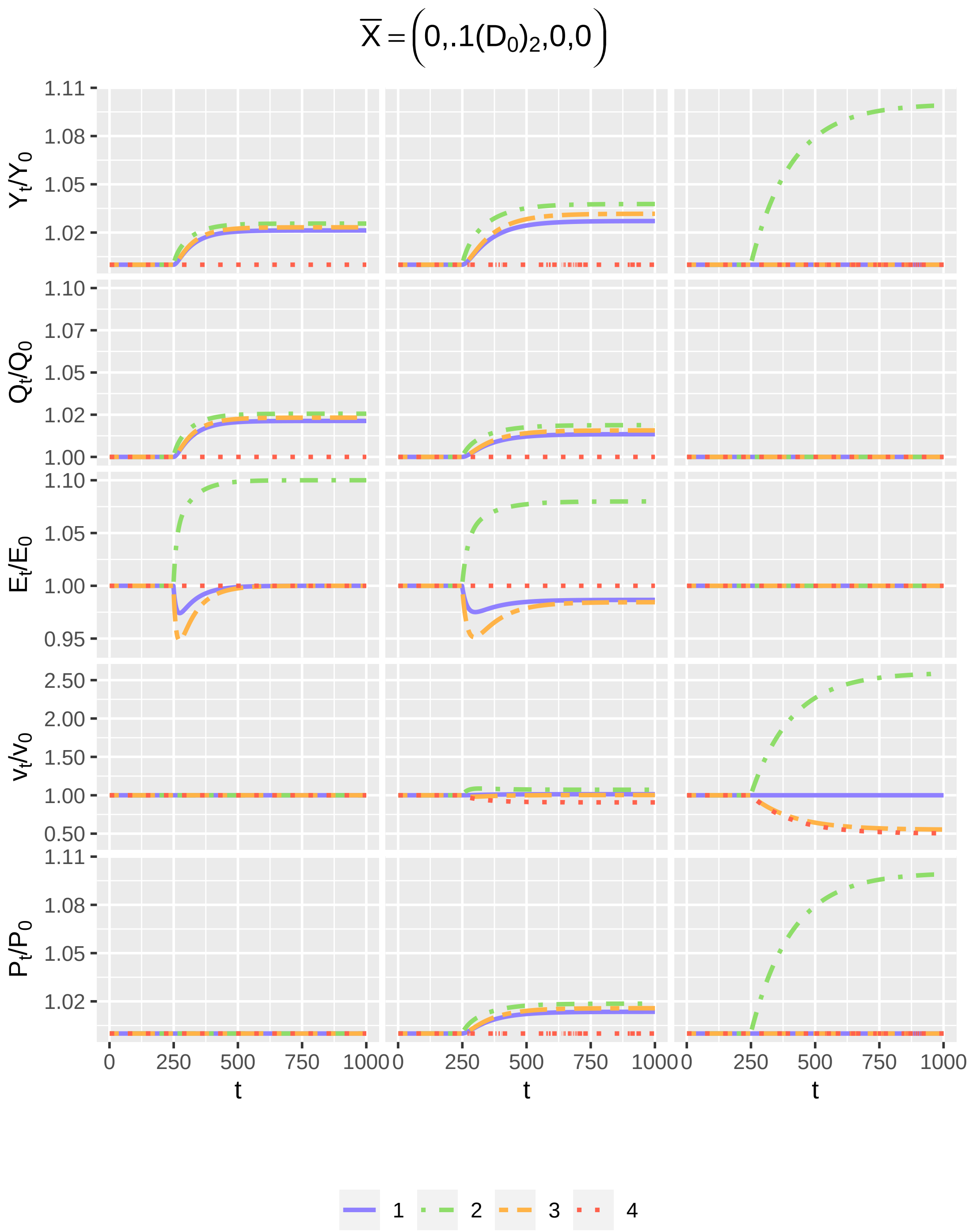

Figure 5: In this graphic one can see the normalized trajectories of the different variables if the system experiences a demand shock $\bar{X}_0 = (0, (D_0)_2/10, 0, 0)$ at $t = 250$. The different colors and linestyles represent the sectors. Each column represents different values for h_Q and h_V. From left to right: $(h_Q = 0.05, h_v = 0)$; $(h_Q = 0.025, h_v = 0.025)$; $(h_Q = 0, h_v = 0.05)$.

4.5.1 Fitting process of the parameters

The parameters (h_Q, h_v) of the model described by (48) are fitted in the following way: We use the input-output tables of two consecutive years

$$\begin{array}{c|cc} X_0 & D_0 & Y_0 \\ \hline (v_0)^T & & \\ (Y_0)^T & & \end{array} \longrightarrow \begin{array}{c|cc} X_1 & D_1 & Y_1 \\ \hline (v_1)^T & & \\ (Y_1)^T & & \end{array}$$

and initialize the ABM discussed in the previous chapter, using the IO-table of the first year. (From now on variables with $\hat{.}$ will denote the variables of the simulated ABM, not the 'real' ones.) The starting values of the ABM therefore are:

$$(\hat{A}_0)_{ij} = \frac{(X_0)_{ij}}{(Y_0)_j}, \quad \hat{D}_0 = D_0, \quad \hat{v}_0 = v_0.$$

We will assume, that the demand shock the system experiences is the difference in demand from the zeroth to the first year $\bar{X} := D_1 - D_0$. Independent from the behavioural parameters h_Q, h_v the simulated ABM will therefore converge to an IO-table with $\hat{D}_\infty = D_0 + \bar{X} = D_0 + D_1 - D_0 = D_1$. Its demand will match the one of the first year.

$$\begin{array}{c|cc} \hat{X}_{0=X_0} & \hat{D}_{0=D_0} & \hat{Y}_{0=Y_0} \\ \hline (\hat{v}_0)^T{}_{=(v_0)^T} & & \\ (\hat{Y}_0)^T{}_{=(Y_0)^T} & & \end{array} \xrightarrow[h_Q,h_v]{\text{Run update steps}} \begin{array}{c|cc} \hat{X}_\infty & \hat{D}_{\infty=D_0+\bar{X}=D_1} & \hat{Y}_\infty \\ \hline (\hat{v}_\infty)^T & & \\ (\hat{Y}_\infty)^T & & \end{array}.$$

However, the form of $\hat{Y}_\infty$ and $\hat{v}_\infty$ is dependent on the h_Q, h_v. We can therefore define the residual R as the difference of the observed IO-table (Y_1, D_1, v_1) and the one the ABM converges to $(\hat{Y}_\infty, \hat{D}_\infty, \hat{v}_\infty)$ as:

$$R := \|Y_1 - \hat{Y}_\infty\| + \|v_1 - \hat{v}_\infty\|.$$

This residual represents the difference between the ABM and the actual observed data. By minimizing the residuals we can estimate the behavioural parameters:

$$R\left((h_Q)_{\min}, (h_v)_{\min}\right) = \min_{h_Q, h_v} R(h_Q, h_v).$$

4.5.2 Fitting the parameters (h_Q, h_v) to the Austrian economy

This fitting process is now applied to the Austrian economy for the years 2008-2016. The IO-tables are publicly available [Aus]. In Figure 6 a heatmap of the residuals $R(h_Q, h_v)$ for the different years can be found. Each square in a heatmap represents the residuals of one run of the ABM with the according values for the behavioural parameters h_Q, h_v.
As already discussed in §4.3.5, in the idealized case $h_v, h_Q << 1$ the form of the steady state

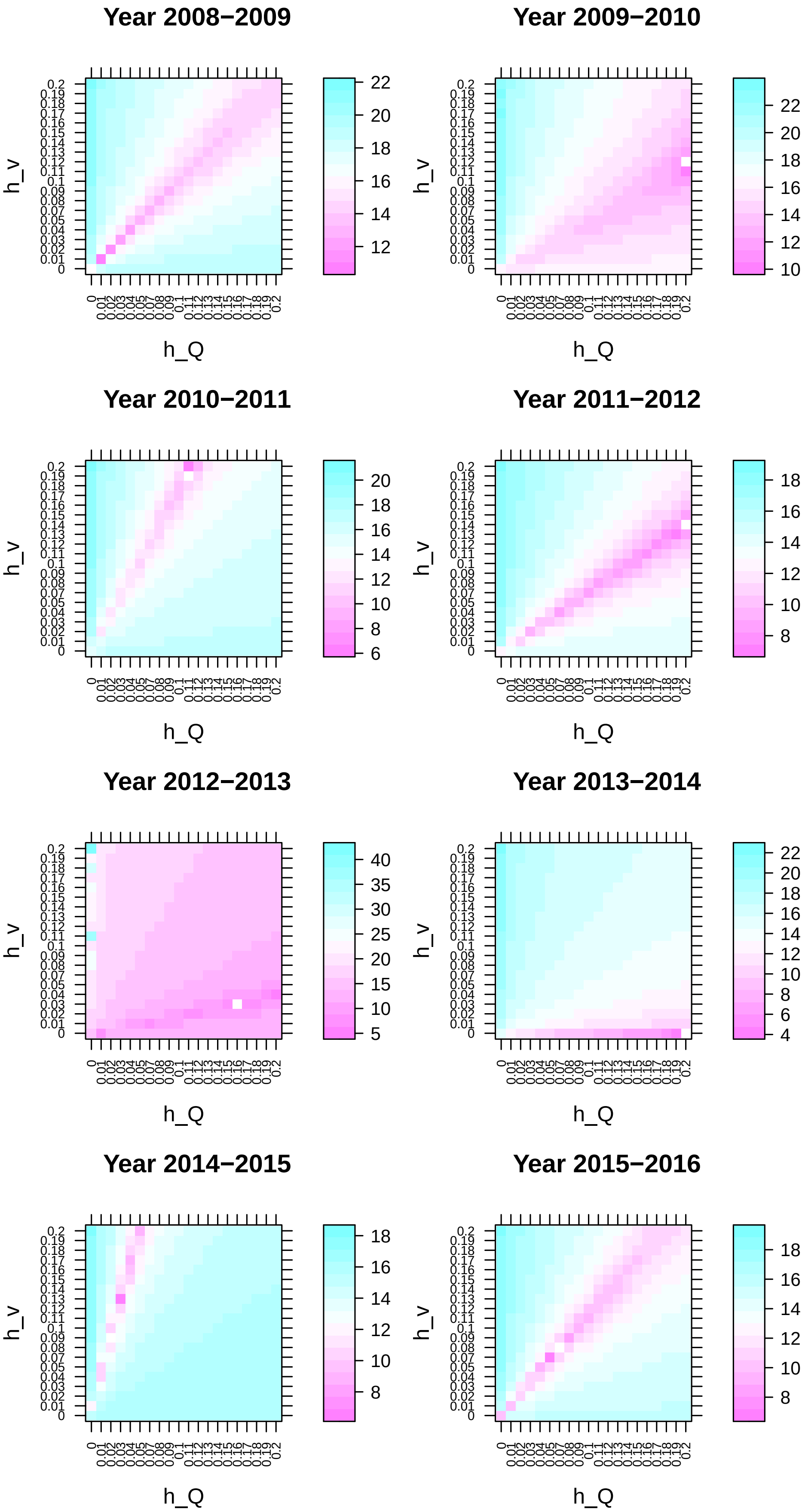

Figure 6: This is a heatmap of $\log\Big(R(h_Q, h_v) - R\left((h_Q)_{\min}, (h_v)_{\min}\right)\Big)$. Note the logarithmic scaling of the difference of the residuals was only chosen for visual purposes.

Year	h_Q/h_v	$\pm$
2008-2009	1.02	0.01
2009-2010	1.77	0.02
2010-2011	0.73	0.03
2011-2012	1.41	0.01
2012-2013	3.86	0.11
2013-2014	10.50	1.32
2014-2015	0.62	0.05
2015-2016	0.91	0.01

Table 3: The values and their standard deviation for h_Q/h_v from a line through the row-wise minima of the heatmap.

(Y_∞, v_∞) only depends on the fraction h_Q/h_v. This is verified by the form of the heatmaps in Figure 6: The minimal values of $R(h_Q, h_v)$ form a straight line without an intercept. (Due to numeric artefacts - h_Q, h_v not being small enough - the values of R along this line are not exactly the same for constant h_Q/h_v.)

The values of the fraction h_Q/h_v were determined by fitting a line through the row-wise minima of these heatmaps. Those values and their standard deviation from the fit can be found in table 3.

Discussion of the results

The fraction h_Q/h_v shows whether price or quantity adjustments are dominant. If $h_Q/h_v > 1$, quantity adjustment has the upper hand, on the other hand if $h_Q/h_v < 1$ price adjustment is dominant. In table 3 it can be seen, that over the years, both take turns. For most years $h_Q/h_v \approx 1$, in this case neither quantity nor price adjustment are dominant. The dynamic Leontief model, which only considers quantity adjustments, would therefore miss half of the effect. This demonstrates that this extension of the typically applied Leontief describes the observed data substantially better.
The years 2012-2013 and especially 2013-2014 are an exception to this general trend. For those years, quantity adjustment is much more dominant. This could be further investigated by considering the concrete sector wise data, however this is beyond the scope this work.

5 Conclusion

This work discusses how the reaction of the economy to a shock can be modelled by considering quantity and price adjustments. The assumption, that the economy consists of different sectors, enables us to use two different approaches: While the differential equation based approach - the dynamic Leontief model - only allows quantity adjustments, the agent based one, allows a more realistic model since both, price and quantity adjustments are incorporated.

The first part of this thesis (§2 and §3) mostly focuses on the application of ordinary differential equations, stochastic differential equations and the Green-Kubo relations to the economy via the dynamic Leontief model. Here, prices stay constant, and each sector tries to match its output to the quantity demanded by the economy. The main result of this discussion is that these three conceptually different expectation values are the same: $\langle \Delta Y \rangle_{X;ODE} = \langle \Delta Y \rangle_{X;SDE} = \langle \Delta Y \rangle_{X;LRT}$ (§3.4). As a result, it can be stated that under certain assumptions, which are met in the case of the dynamic Leontief model, these expectation values are equivalent.
These assumptions are: That the system is of linear and stationary nature. The system is linear, since its state at time $t + \mathrm{d}t$ is linearly dependent on its state at time t. It is stationary since, under the absence of a shock, the system converges to a stationary state. If these expectation values are still equivalent, if one or some of these assumptions are not met, is still to be elucidated.

In the second part (§4) update equations of an Agent Based Model (ABM) that unifies the Ghosh model, in which only price adjustments are possible, and the Leontief model, in which only quantity adjustments are possible, are derived. These update equations allow both, quantity and price adjustments. Depending on the behavioural parameters (h_Q, h_v) either price or quantity adjustment will dominate.
These are then fitted to the Austrian economy for the years 2008-2016, where we conclude that neither price nor quantity clearly dominates.

In this work, the h_Q and h_v were only considered as scalars. The update equations would also allow them to be vectors, which would make it possible to investigate if the sectors have different preferences for the behavioural parameters h_Q, h_v. The fitting process when considering these parameters as vectors is another interesting topic for further investigations. This would allow an estimation, whether there are sector wise differences in adjustment of price and quantity. This gives an outlook for future work, using the update equations developed.

This work discusses different approaches concerning how the economy reacts to and recovers from demand shocks. These enable us to model, how the economy will adjust to demand shocks in certain sectors. In the wake, of the current COVID-19 pandemic, this becomes especially interesting, since it induced drastic changes in demand in different sectors. For example, whilst the demand in tourism and gastronomy decreased substantially[McC], the sales of outdoor sports articles[Bai], for example bikes [Ric], increased. As the demand in

these sectors changed, so will their outputs. For bicycles, price hikes followed[Kha]. This once more underlines the importance of the models developed in this work, as they are able to represent both, price and quantity adjustments at the same time.
Just as well, in the upcoming post-COVID-19 era, models that allow diverse approaches to study economic recovery will play an important role. Understanding the mechanism how economies recover could enable governments to enact focused policies which could support a faster economic recovery.

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