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Describing an Algorithm that relates Kirby- and Trisection diagrams

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Abstract

The aim of this thesis is to describe and compare two natural ways of decomposing smooth closed oriented 4-manifolds, namely handle decompositions and trisections. Moreover we compare the associated combinatorial descriptions of smooth closed oriented 4-manifolds derived from these decompositions, called Kirby diagrams and trisection diagrams, respectively. The thesis culminates in an algorithm that takes a Kirby diagram and produces a trisection diagram describing the same manifold.
The first part serves to familiarize the reader with basic notions of handle decompositions and to introduce important concepts in the study of low dimensional topology such as Heegaard-decompositions and Kirby diagrams.
In the second part we acquaint the reader with the notion of a trisection of a smooth closed oriented 4-manifold and prove some of their fundamental properties as well as their existence. We then move on to a discussion of the aforementioned algorithm and apply it to a simple example. Finally, we consider special cases for which the algorithm can be simplified.

Das Ziel der vorliegenden Arbeit ist die Beschreibung sowie der Vergleich zweier natürlicher Arten glatte geschlossene orientierte 4-Mannigfaltigkeiten zu zerlegen, nämlich die der Henkelzerlegung und die der Trisection. Darüber hinaus vergleichen wir die von diesen Zerlegungen abgeleiteten kombinatorischen Beschreibungen von glatten geschlossenen orientierten 4-Mannigfaltigkeiten die als Kirby Diagramme beziehungsweise Trisection Diagramme bekannt sind. Die Arbeit mündet in der Erläuterung eines Algorithmus welcher ein Kirby Diagramm nimmt und ein Trisection Diagramm, welches dieselbe Mannigfaltigkeit beschreibt wie das ursprüngliche Kirby Diagramm, produziert.
Der erste Teil dient dazu die Leser*in mit den grundlegenen Begriffen der Henkelzerlegung vertraut zu machen und wichtige Konzepte niedrigdimensionalen Topologie, wie etwa Heegaard-Zerlegungen und Kirby Diagramme, einzuführen.
Im zweiten Teil stellen wir der Leser*in das Konzept der Trisection einer glatten abgeschlossenen orientierten Mannigfaltigkeit vor und beweisen einige ihrer fundamentalen Eigenschaften einschließlich ihrer Existenz. Wir wenden uns dann einer Diskussion des vorhin genannten Algorithmus und wenden diesen auf ein einfaches Beispiel an. Schließlich erwähnen wir Spezialfälle in denen der Algorithmus vereinfacht werden kann.

Acknowledgements

This thesis as well as me getting to this point in my life would not have been possible without the support of many people. First and foremost I would like to thank my parents for all the loving care they blessed me with, for teaching me to value education and competence. I want to thank my siblings for always being there for me and for being great role models and for encouraging me in everything I have done so far. I also want to thank my girlfriend for her extraordinary warmth and my newborn nephew for his lovely first smile.
There are many important people in my life and it is certainly not (reasonably) possible to name all of them here but I would like to say that I am deeply grateful to my friends for making my life as enjoyable and bright as it is, for being pillars of support if need be and for being great conversation partners.
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It is important to me to thank Ass.-Prof. Anton Mellit for giving a great course on algebraic topology and sparking my interest in differential topology as well as for many inspiring conversations that have helped me to develop a better sense of what "the right" or "interesting" questions are.
I am indebted to my friend Anne-Sophie Ritter for being of invaluable help in digitizing hand-drawn sketches for this thesis. Without her my thesis would not look nearly as pretty as it does.
The person without whose close mentoring this thesis definitely would not have turned out nearly as well is my advisor, Ass.-Prof. Vera Vértesi. I want to thank her for investing an incredible amount of energy into my Masters thesis and for her patience in explaining new concepts to me. It is a common occurrence that advisors only glance at the Masters theses of their students but this was not at all the case here and I am much better of for it. There are several instances in which she noticed that I had only partially understood a concept which she then helped me to properly grasp. For all this, her friendly encouragement, and more I want to genuinely thank her.

Contents

1 Introduction

In this section we will give a rough overview of handle decompositions and trisections, the two different ways of dividing (4-)manifolds into standard parts and we will briefly explain how the combinatorial descriptions arise from these decompositions. To do this we will have to use some terms which will only be properly introduced in the later sections. The following short introduction will culminate in a result which provides the link between these two combinatorial descriptions and proves their equivalence. The proof of this is what the present thesis builds up to. We begin with the basic definitions, keeping in mind that we work in dimension 4 only. From this point on all manifolds in this thesis will be smooth with or without boundary and sometimes with corners, and they will always be compact. In the following short introduction we will specialize handle decomposition theory to dimension 4.

We will beginn with a description of handle decompositions (in dimension 4) and an outline of how one derives Kirby diagrams from them.

Definition 1.0.1. *[handles and handle decompositions] A 4-dimensional k-handle h_k $(0 \le k \le 4)$ is the set $B^k \times B^{4-k}$ together with a diffeomorphism $\phi : \partial B^k \times B^{4-k} = \mathbb{S}^{k-1} \times B^{4-k} \longrightarrow \partial X$ for an 4-manifold X. We say that ϕ attaches the handle to ∂X and denote the resulting object by $X \cup_\phi h_k = \big(X \sqcup h_k\big)/x \sim \phi(x)$. Attaching a 0-handle to X means taking the disjoint union of X with a copy of B^4.*
A handle decomposition of a closed n-manifold X is a finite sequence of handles that are, starting with one or more 0-handles, successively attached to the handles that were attached before and whose union (after smoothing out corners) is diffeomorphic to X. Note that this means that any handle decomposition has at least one 0-handle since this is the only type of handle that attaches to the empty set.

Handles have various parts which we will frequently make reference to.

Definition 1.0.2. *Given a 4-dimensional k-handle $h_k = B^k \times B^{4-k}$, $B^k \times \{0\}$ is called its core, $\{0\} \times B^{4-k}$ its cocore, $\mathbb{S}^{k-1} \times B^{4-k}$ its attaching region, $\mathbb{S}^{k-1} \times \{0\}$ its attaching sphere, and $\{0\} \times \mathbb{S}^{4-k-1}$ its belt sphere. All of these are sketched in Figure 1.*

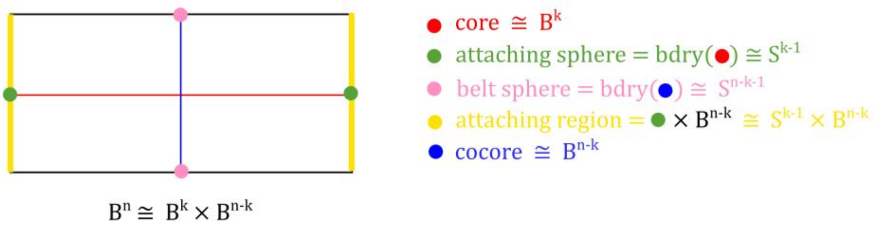

Figure 1: The various parts of a handle

One way of thinking about handle decompositions is that they are the manifold version of cell decompositions from algebraic topology in that handle attachment is just cell attachment only with the cell made thicker by crossing it with as many factors of I as needed in order for the resulting object to have the dimension of the manifold. (Also the attaching maps are diffeomorphisms as opposed to continuous maps) This thickening introduces an additional complication as it is not canonically clear how to thicken the attaching spheres, in fact different choices will in general lead to non-diffeomorphic (even non-homeomorphic) manifolds so the specific way in which this thickening happens will have to be accounted for. We will now try to illustrate how this extra data can be specified.

It is not obvious but nevertheless true (see Theorem 2.1.1) that every n-manifold can be decomposed into handles. So let X be a connected closed oriented 4-manifold with a handle decomposition. One can always reduce any handle decomposition to one that has only one 0- and one 4-handle (see Lemma 2.1.3). Since we build up the manifold X starting with the 0-handle, a 4-dimensional ball, and since this building process only really involves its boundary, we can visualize the handle decomposition by thinking about $\partial B^4 = \mathbb{S}^3$ and the attaching regions of all subsequent handles. These attaching regions can always be chosen to miss a point so we might as well think about $\mathbb{R}^3$ and actually draw the attaching regions. Let us first think about what these regions look like: for a 1-handle we get two disjoint B^3's, for a 2-handle we get an $\mathbb{S}^1 \times B^2$. It turns out that the collection of the 0-handle, the 1-handles, and the 2-handles, together with the number of 3-handles, already uniquely determines the rest of the decomposition (provided that there is only a single 4 handle in this decomposition) and therefore X, see [LP72, Theorem A]. Another fact that might seem surprising at first is that we do not need to know the gluing diffeomorphism for the entire attaching regions of the handles: for 1-handles it is enough to know it for the core, so just the position of the pair of points, and for 2-handles we need to know both where the core of the handle is embedded as well as the so called framing, a concept which we will explain in detail in section 2.1. The framings, i.e. different ways of thickening up the attaching sphere of 4-dimensional 2-handles, turn out to be parametrized by $\mathbb{Z}$, though only affinely. By this we mean that we can only assign the framing of the knot a numerical value once a certain reference has been fixed. We will revisit the issue shortly.
Taking all of this together means that the 1-handles, just a collection of pairs of thickened up points, and embedded $\mathbb{S}^1$'s with (affine) numbers in $\mathbb{Z}$ associated to them already fully determine a closed manifold, provided this collection of 1- and 2-handles can be closed up using some 3- and a 4-handle. It should be emphasized at this point that not any arbitrary combination of pairs of thickened-up points and knots defines a <u>closed</u> 4-manifold, this subtlety will be discussed in section 2.3.
One final simplification we can do is that we project the entire attaching region, which lives in $\mathbb{R}^3$, onto $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$ in such a way that the images under the projection of the attaching spheres of a 1- and of a 2-handle if the 2-handle goes over the 1-handle in the original diagram in $\mathbb{R}^3$. We furthermore stipulate that the attaching link of the 2-handles is embedded into $\mathbb{R}^2$ except for double-point intersections, where we remember which curve ran over which in the original diagram by breaking up the one that larger z-coordinate at the intersection point. One may interpret the breaking up of the curves as them running over bridges so as to avoid intersecting the curve below them. Thus we can think about the attaching regions being projected onto a genus g surface Σ_g where g is the number of intersections of the attaching link in this projection. See Figure 33 for clarification.
One advantage of projecting the entire diagram to $\mathbb{R}^2 \times \{0\}$ is that the plane provides a reference which allows us to encode the framing in a number, this number is then called the framing with respect to the "blackboard framing".
The projection can always be done after isotoping the attaching regions and choosing an appropriate plane to project onto. See Figure 2 for an example. The resulting diagram will be referred to as a Kirby diagram. We have not lost any information in passing from the 3-dimensional picture to the 2-dimensional one. Some examples of Kirby diagrams can be seen in Figure 3.

We will now describe trisections and outline how to represent them diagrammatically. To make the following discussion easier we introduce the notion of an n-dimensional genus k handlebody, $H^n_k = \natural_k \mathbb{S}^1 \times B^{n-1}$. Here $\natural$ denotes the so called **boundary sum**, though the reader does not have to be familiar with this concept as H^n_k is diffeomorphic to the union of an n-dimensional 0-handle with k n-dimensional 1-handles. We will henceforth use the symbol $" \cong "$ for equality up to diffeomorphism.

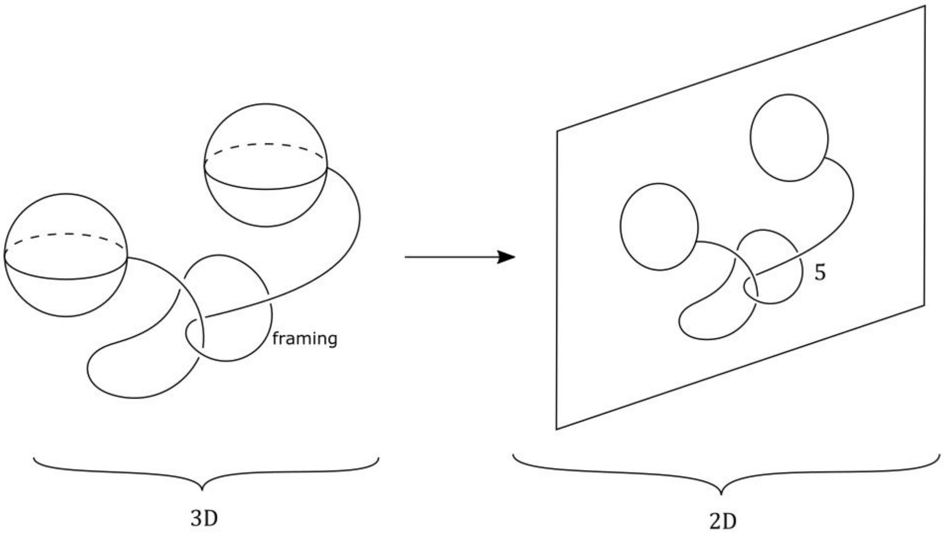

Figure 2: Getting a Kirby diagram from the attaching regions of 4-dimensional 1- and 2-handles. The framing can, after projecting, be compared to the projection surface, thus assigning it a number. The diagrams describe $\mathbb{S}^4$ though in a slightly complicated way.

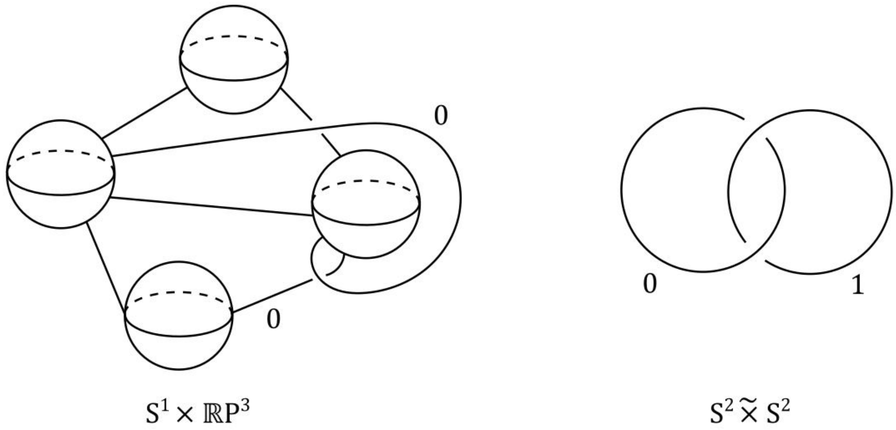

Figure 3: Two (very) simple examples of Kirby diagrams

Definition 1.0.3. *[Trisections] Let X be a closed oriented 4-manifold. A $(g; k_1, k_2, k_3)$-trisection of X is a decomposition*

$$X = X_1 \cup X_2 \cup X_3$$

where $X_i \cong H^4_{k_i}$, $H_{i,i+1} = X_i \cap X_{i+1} \cong H^3_g$, and $\Sigma = X_1 \cap X_2 \cap X_3 \cong \Sigma_g$. We call the tuple (X, X_1, X_2, X_3) a trisected manifold.

We claim (and prove in section 3.2) that one can equivalently think about trisections of 4-manifolds as starting out with a genus g surface Σ_g (sometimes called the core of the trisection) that we glue three 3-dimensional genus g handlebodies to, resulting in a structure called the spine of the trisection. After thickening the spine up we get a 4-manifold with three boundary components that we can cap off with three 4-dimensional genus k_i handlebodies provided the original gluing satisfies a certain technical condition. A Theorem due to Laudenbach and Poenaru [LP72, Theorem A] guarantees that this process of capping off the boundary components is unique up to diffeomorphism. We already hinted at the fact that not every gluing of three 3-dimensional genus g handlebodies to Σ_g can be capped off in this way: the

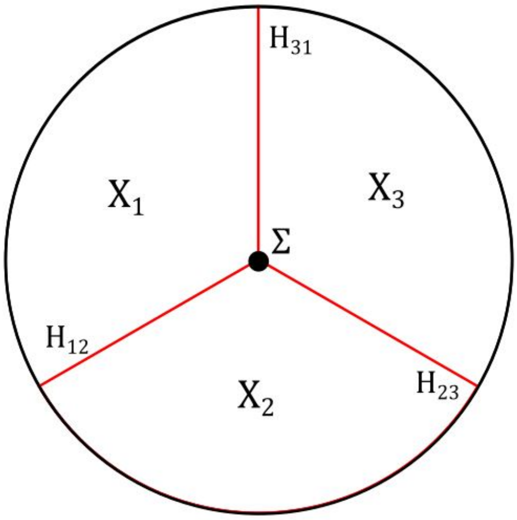

Figure 4: Cartoon of a trisection

condition is that each pair of the three handlebodies, when glued together along Σ_g, is diffeomorphic to $\sharp_{k_i} \mathbb{S}^1 \times \mathbb{S}^2 \cong \partial X_i$, which is awkward to check in practice. The two interesting ideas that trisections bring to the table are first that they reveal an inherent symmetry of closed oriented 4-manifolds and second, and more importantly, that they allow us to play the understanding of closed oriented 4-manifolds back to genus g surfaces. This opens up using 3-manifold techniques for the analysis of 4-manifolds.
To encode how to attach the three 3-dimensional handlebodies $H_{i,i+1}$ to Σ we use three sets of g curves. This works because a 3-dimensional genus g handlebody really is just a 3-dimensional 0-handle with g 3-dimensional 1-handles attached to it and if one wants to glue such a handlebody to a genus g surface Σ_g one only needs to know where all the belt spheres of the 1-handles go, meaning that this information is specified by a collection of g simple closed curves on Σ_g. The gluing of three 3-dimensional genus g handlebodies to Σ_g is therefore determined by 3 sets of g simple closed curves which, together with the genus g surface is precisely the combinatorial data we need to uniquely (up to diffeomorphism) describe a trisection of the 4-manifold. Since any two of the handlebodies combine to give $\sharp_k \mathbb{S}^1 \times \mathbb{S}^2$ the systems of curves can be manipulated to have a particularly nice form, which we shall call X_1-standard form. In this form two of the systems of curves (the red and the blue ones in the pictures) are in the simple arrangement depicted in 5.
As in the case of handle decompositions there is a "flat" version of trisection diagrams, i.e. one that lives in $\mathbb{R}^2$. Note that a surface of genus g can be seen as the boundary of a 3-dimensional handlebody of genus g. We can represent this by thinking about the the boundary of the 0-handle, which is an $\mathbb{S}^2$, and, after forgetting about a point which the attaching regions of the 1-handles can always avoid, this can be thought of as $\mathbb{R}^2$, and to this we attach the 1-handles. The only thing that matters about these attachments though is the attaching regions and this is just g pairs of disks. Similar to the 1-handles, all of the curves in a trisection diagram can also avoid a single point in the boundary of the 0-handle, and so we can instead depict all of the attaching information in a plane. We call the resulting diagram a flat trisection diagram and note that we have not lost any information in passing from the trisection diagram to a flat trisection diagram. See Figure 6 for examples of this.

To summarize, we have two combinatorial ways of describing closed oriented 4-manifolds: the first one uses pairs of balls and knots in 3-dimensional space (respectively in a plane after projecting), the other uses 3 sets of g simple closed curves on a genus g surface.

It is important to realize that neither topological decomposition is unique, i.e. there is never just

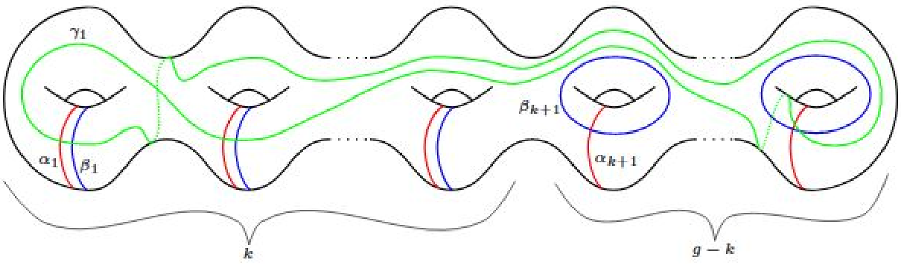

Figure 5: Trisection diagram in X_1-standard form. Note that only one of the g green γ curves is drawn.

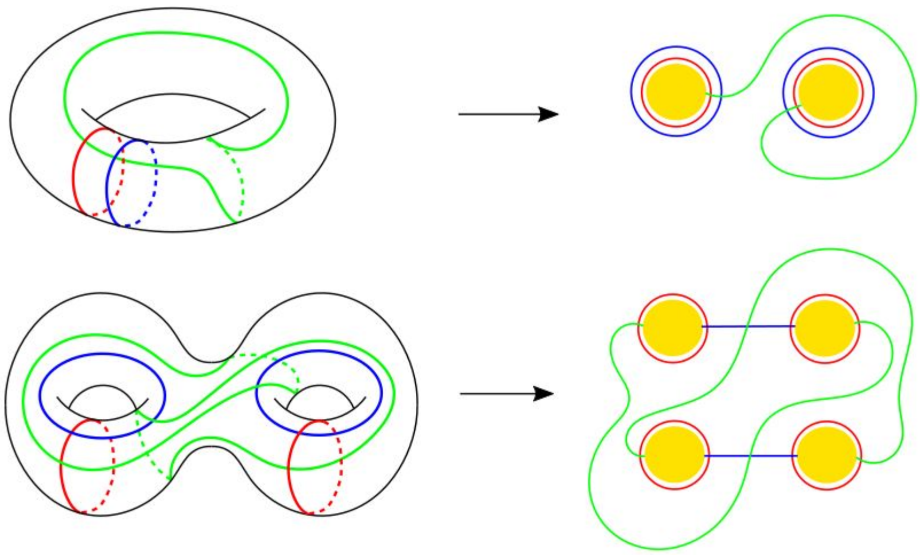

Figure 6: Two (very) simple examples of trisection diagrams, both in regular and in flat form. The upper diagram describes $\mathbb{C}P^2$ and the lower one describes $\mathbb{S}^2 \times \mathbb{S}^2$. The diagrams on both sides are in X_1-standard form.

one handle decomposition or one trisection of a given manifold X, so the combinatorial description isn't either unless one mods out some moves which we will talk about in later sections.

The main result of this thesis (Theorem 3.4.1) is that we describe an algorithm that takes a Kirby diagram and produces a (flat) trisection diagram. The details of the constructions and the proof of Theorem 3.4.1 will be presented in section 3.4. We apply Theorem 3.4.1 to a concrete example and simplify the algorithm for special cases in section 3.5.
The opposite direction, so going from a (flat) trisection diagram to a Kirby diagram, is much simpler and is essentially a byproduct of developing all the language we need.

2 Handle decompositions and Kirby diagrams

Handle decompositions in general and Kirby diagrams in particular provide convenient and combinatorial means to describe all possible closed 4-manifolds. We will first introduce handle decompositions of manifolds and handle moves before we specialize to the 3- and 4-dimensional cases.

2.1 Handle decompositions of closed manifolds

As already mentioned in the introduction, handles are a particularly convenient way to decompose manifolds into simpler parts. We define them here for general dimension n.

Definition 2.1.1. *An n-dimensional k-handle h_k $(0 \leq k \leq n)$ is the set $B^k \times B^{n-k}$ together with a diffeomorphism onto its image $\phi : \partial B^k \times B^{n-k} = \mathbb{S}^{k-1} \times B^{n-k} \longrightarrow \partial X$ for an n-manifold X. We say that ϕ attaches the handle to ∂X and denote the resulting object by $X \cup_\phi h_k = \big(X \sqcup h_k\big)/x \sim \phi(x)$. Attaching a 0-handle to X means taking the disjoint union of X with a copy of B^n.*

It is worth pointing out that handle attachments lead to manifolds with corners which we have to smooth out in order to obtain a manifold with boundary, allowing us to proceed with further handle attachments. Fortunately there is a canonical way of doing this, see [Wal16, Proposition 2.6.2].
The idea of using only handles to build up a manifold gives rise to

Definition 2.1.2. *A handle decomposition of a smooth closed n-manifold X is a finite sequence of handles, starting with one or more 0-handles, that are successively attached to the handles that were attached before and whose union is diffeomorphic to X.*

The point of this notion is that we want to understand manifolds by understanding easier substructures. To this end we need to understand handles a little better, a little terminology is useful at this point.

Definition 2.1.3. *Given an n-dimensional k-handle $h_k = B^k \times B^{n-k}$, $B^k \times \{0\}$ is called its core, $\{0\} \times B^{n-k}$ its cocore, $\mathbb{S}^{k-1} \times B^{n-k}$ its attaching region, $\mathbb{S}^{k-1} \times \{0\}$ its attaching sphere, and $\{0\} \times \mathbb{S}^{n-k-1}$ its belt sphere. All of these are sketched in Figure 1.*

For some examples of both handles and handle attachments, see Figures 7, 11, and 24.

The issue of how exactly one attaches a handle is slightly subtle. It is a standard result that the diffeomorphism type of $X \cup_\phi h_k$ only depends on the isotopy type of ϕ, i.e. isotopic gluing diffeomorphisms lead to diffeomorphic results. This can be found for example in [GS99, Chapter 4.1].

One can use this fact to show that $X \cup_\phi h_k$ is specified by a diffeomorphism $\phi_0 : \mathbb{S}^{k-1} \longrightarrow \partial X$ and a trivialization of the normal bundle $\mathcal{N}(K)$ of $K = Im(\phi_0)$, i.e. a bundle isomorphism $\mathcal{N}(K) \longrightarrow \mathbb{S}^{k-1} \times \mathbb{R}^{n-k}$. Such a trivialization of the normal bundle of K is called a framing of the knot K. Any attaching map ϕ clearly determines both an embedding $\phi_0 := \phi|_{\mathbb{S}^{k-1} \times \{0\}}$ and a trivialization of the normal bundle via the tubular neighbourhood theorem. Conversely, any embedded knot K together with a framing, a so-called **framed knot**, determines an embedding of $\mathbb{S}^{k-1} \times \mathbb{R}^{n-k}$ that induces both the embedding of K when restricted to $\mathbb{S}^{k-1} \times \{0\}$ and the original framing. Furthermore any two such embeddings obtained from a framed knot can only differ by a rotation in every $s \times B^{n-k}$-factor for $s \in \mathbb{S}^{k-1}$, so they are isotopic. This means that a framed knot determines the isotopy class of an embedding of $\mathbb{S}^{k-1} \times B^{n-k}$, so the isotopy class of a handle attachment. Therefore framed knots are enough to specify the diffeomorphism type of $X \cup_\phi h_k$.
Now fix a framing for a knot $K \cong \mathbb{S}^{k-1}$ in ∂X. Any other framing of K can now be compared to this

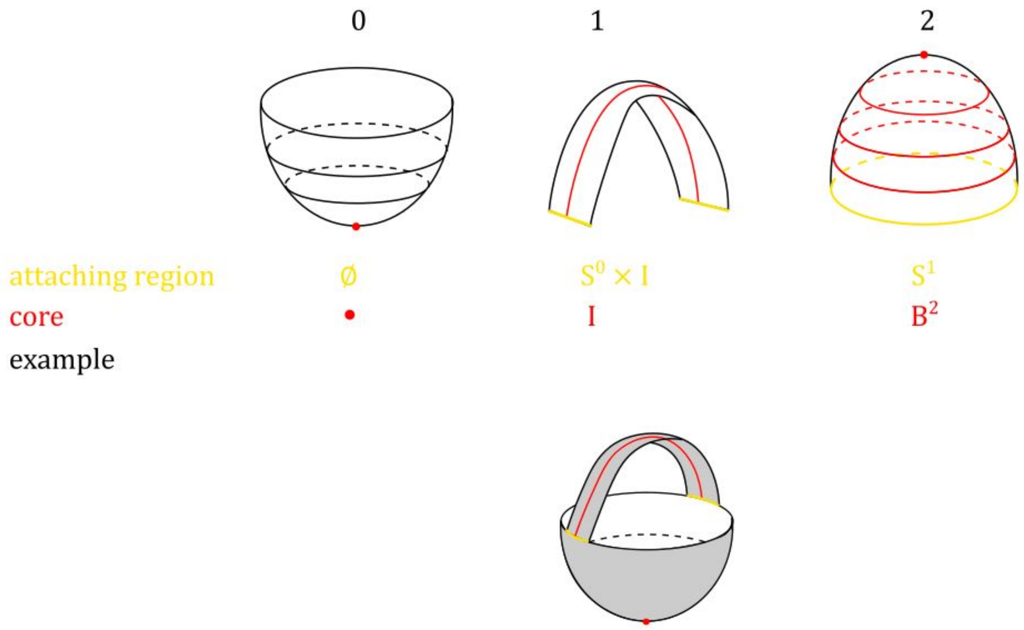

Figure 7: 2-dimensional handles

framing. This comparison gives an element in $GL(n-k)$ for every point in K, meaning that we define an element in $\pi_{k-1}(GL(n-k)) \cong \pi_{k-1}(O(n-k))$ (the basepoint does not matter since the connected components of these groups are diffeomorphic to one another and the connected components are path connected, so any choice of basepoint will lead to an isomomorphic group. We therefore suppress the reference to the basepoint). We used that $GL(n-k)$ deformation retracts to $O(n-k)$ using the Gram-Schmidt procedure. We may conclude that there is a bijection between framings of K and elements in $\pi_{k-1}(O(n-k))$, though this bijection relies on our initial choice of framing. Framings of K can thus be thought of an affine $\pi_{k-1}(O(n-k))$.

The case that will be relevant for the purposes of this thesis is the one where $n = 4$ and $k = 2$. Then the framings will be parametrized by $\pi_1(O(2)) \cong \mathbb{Z}$ once a reference framing is chosen. This reference framing can be specified in a variety of ways, for example by fixing a normal vectorfield along K. Our choice will be to induce it by means of a surface, see the discussion in section 2.3.

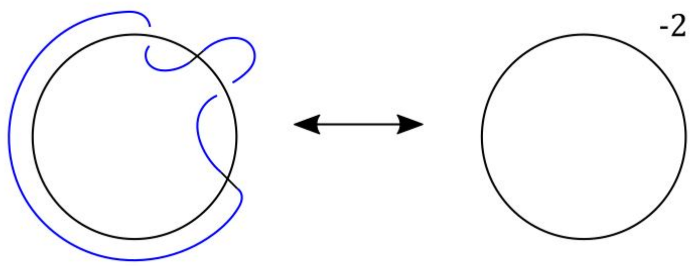

Figure 8: A −2-framed unknot (with respect to the blackboard framing). The blue knot represents the framing of the black knot with respect to the blackboard framing by winding around as many times as the framing requires.

We will now carry on with a more general discussion of handles and handle decompositions.

As handle decomposition is a manifold-version of cell decomposition one might expect to it to be a very useful tool to understand the topology of manifolds. This is true indeed and we are pleased to state the following theorem:

Theorem 2.1.1. *Every manifold X admits a handle decomposition. If X is compact, the associated sequence of handles in the decomposition is finite.*

This is proven in [Wal16, Lemma 5.1.8].

Handle decompositions of manifolds are far from unqiue. For example, given a handle decomposition we can make the observation that every n-dimensional k-handle can be seen as an n-dimensional $n-k$-handle simply by switching the roles of the two factors in $H_k = B^k \times B^{n-k}$. A handle can always be thought of being attached along B^{n-k} to whatever is "above" it. If we flip all the handles in our handle decomposition this way, making k-handles into $n-k$-handles, we have in effect "turned the handlebody upside-down". The resulting handle decomposition of X is called the handle decomposition dual to our original one.

Closer inspection tells us that there are far more handle decompositions corresponding to any given manifold. There is a basic set of two (families of) moves on handle decompositions that do not change the associated manifold and that are complete in the sense that any two handle decompositions can be transformed into one another using these moves. The first family of moves is isotopies of the attaching maps.

The other family of moves is called birth or cancellation of pairs of handles of adjacent dimensions.

Proposition 2.1.1. *A $(k-1)$-handle h_{k-1} and a k-handle h_k $(1 \leq k \leq n)$ can be cancelled, provided that the attaching sphere of h_k intersects the belt sphere of h_{k-1} transversely in a single point.*

Let us see why this should be true in the case of a 3-dimensional 1- and 2-handle attached to the boundary of some 3-manifold Y. The situation which we will describe is depicted in Figure 9. The point is that the union of the two handles is a 3-dimensional ball, and the union of the attaching region of the 1-handle and of the part of the attaching region of the 2-handle that lies in ∂Y is a 2-dimensional disk. This means that the union of the two handles is a ball that is glued to ∂Y along a disk and intuitively speaking one can press the ball flat without changing the diffeomorphism type of the resulting manifold. We thus end up with Y without any handles attached. A proof of Proposition 2.1.1 is presented in [GS99, Proposition 4.2.9].

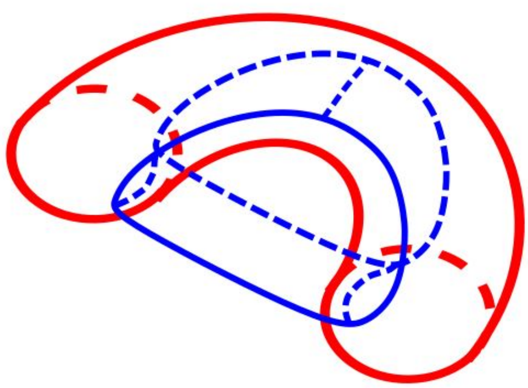

Figure 9: An example of how a 3-dimensional 1- and 2-handles fit together to give B^3.

One consequence of this is that one can always add or remove redundant pairs of handles that cancel one another to get a new handle decomposition from an old one.
The following theorem guarantees that these two families of moves really are enough to get from one handle decomposition of a manifold to any other.

Theorem 2.1.2. *Given two n-manifolds X and X' as well as two handle decompositions of X and X' respectively, X and X' are diffeomorphic iff there exists a sequence of handle creations and cancellations, as well as isotopies of the attaching maps, after which the handle decompositions coincide.*

For a reference see [GS99, Theorem 4.2.12].

Right now we have to think about handle decompositions as gluing all the handles successively onto each other, but it would be much nicer if we could just start with one handle and glue all the other ones onto it one after another in order of increasing index. This can be done but to prove it we first remind the reader of the intuitive statement that a (proper) submanifold can always avoid other submanifolds of suffiently small dimension simply by moving it around using an isotopy.

Lemma 2.1.1. *Given a finite collection of submanifolds A_α of an m-dimensional manifold M, each of dimension $< (m - \nu)$, any embedding of a ν-dimensional manifold V is isotopic to an embedding that avoids all of the A_α.*

A proof of this can be found in [Wal16, Lemma 4.5.9].

When two submanifolds A and B of an n-dimensional manifold X have $dim(A) + dim(B) < n \iff codim(A) + codim(B) > n$ do not intersect we say that A and B are in general position. The previous lemma tells us that, when such A and B are not in general position, we can always apply an isotopy to one of the them so that afterwards they are.

Going back to handle attachments and the order of their attachment, we can now prove

Lemma 2.1.2. *Given a handle decomposition of a smooth manifold X we can always assume that the handles are attached with increasing index. Handles that have the same index can be attached in arbitrary order or simultaneously.*

Proof. Let h_k and h_l be handles of order k and l, respectively, with $k \le l$, and let us assume that h_l is attached before h_k is. Denote the attaching sphere of h_k, a manifold of dimension $k-1$, by A_k and the belt sphere of h_l, a manifold of dimension $n-l-1$, by B_l. Then $codim(A_k \cap B_l) = codim(A_k) + codim(B_l) = (n-1-(k-1)) + (n-1-(n-l-1)) = n-1+1+l-k \ge n > n-1$ (remember that $n-1$ is the dimension of the ambient manifold in this case), so by general position we can always assume this intersection to be empty. Now we can just isotope A_k off of h_l using a flow, for example the one that is orthogonal to the cocore with respect to the euclidean metric (and extend that flow past the attaching region of the handle). □

This was the first step in thinking about handle decompositions more systematically. For example we can use Lemma 2.1.2 to make the

Definition 2.1.4. *Given a handle decomposition of an n-dimensional manifold X, we define $X^k = 0-handles \cup \cdots \cup k-handles$ (and smooth out corners after every handle attachment), so we have a nested sequence of manifolds with boundary $X^0 \subset X^1 \subset \cdots \subset X^n = X$.*

The next step is to show that it is always possible to build up a connected manifold from a single 0-handle.

Lemma 2.1.3. *Let X be a connected n-manifold with a handle decomposition that has k_i i-handles, in particular the number of* 0- *and n-handles might be larger than one. We can always use the moves on handle decompositions to reduce this to a handle decomposition with only a single* 0- *and n-handle.*

Proof. Recall that every handle decomposition has always at least one 0-handle and, dually, at least one n-handle. Suppose that there are two 0-handles. Using Lemma 2.1.2 we build X up starting with the 0-handles. The two 0-handles are disjoint from one another, now when attaching 1-handles the attaching regions are either both on the boundary of the same 0 handle or they connect the two. Since the manifold is connected and all higher handles have connected attaching regions and therefore have nothing to do with the manifold being connected, at least one of the 1-handles connects the two 0-handles. The part of the attaching sphere that is on each 0-handle is only a point and the entire boundary of the 0-handle is its belt sphere, so we can automatically cancel one of the 0-handles with said 1-handle.
To get statement for n-handles we just pass to the dual handle decomposition and note that the process we just described does not introduce any new handles, so we do not destroy the fact that the manifold has only one 0-handle by reducing to a single n-handle. $\square$

Lemma 2.1.3 tells us that we can build up a manifold simply by starting out with a 0-handle and attaching handles to its boundary in order of increasing index. We will use this fact in the next two sections to argue that everything of interest in handle decompositions of 3- and 4-manifolds really takes place in $\mathbb{S}^2$ and $\mathbb{S}^3$, the boundaries of 3- and 4-dimensional 0-handles.

We want to conclude this section by giving some examples of handle decompositions of manifolds. The first class of examples is provided by triangulations. Every triangulation of a manifolds induces a handle decomposition since, in particular, every triangulation is a cell decomposition. Thickening up all the cells gives the desired handle decomposition.
A class of examples that is somewhat related to this are manifolds that are defined by taking a polygon with $2k$ sides that are identified in pairs. We then put a disk, the 0-handle, inside this polygon. The 1-handles are attached by letting a loop go through a pair of faces and then thickening it up, the 2-handles by letting them run along the boundary of the 1-handles. We can do something similar in dimension 3, replacing polygons by polyhedra, though one has to be more careful for the resulting gluing to be a manifold. For some examples, see Figure 10.

The most efficient way of dealing with handle decompositions is using Morse functions, that is smooth functions whose (finitely many) critical points are nondegenerate. Using these functions and the associated machinery most of the statements in this section can be proven rather conveniently.

2.2 Dimension 3 and Heegaard splittings

The aim of this section is to specialize the concept of handle decompositions to closed oriented 3-manifolds, giving rise to a particularly nice decomposition of 3-manifolds that can be seen as the 3-dimensional precursor of trisections, so called Heegaard-decompositions.
Let us start by examining the types of handles that can occur. We start with a 3-dimensional 0-handle, in other words with B^3, and attach handles to its boundary $\partial B^3 = \mathbb{S}^2$. Next up are 3-dimensional 1-handles, these are intervalls thickened up to be 3-dimensional, so full cylinders $I \times B^2$, that are glued to $\mathbb{S}^2$ along $\{0\} \times B^2 \sqcup \{1\} \times B^2 = \mathbb{S}^0 \times B^2$. In view of framing, we determine the handle fully by picking out two points in $\mathbb{S}^2$ and making a choice of orthonormal bases in the tangent spaces for both of those. Now either the orientation of both agree or disagree with the orientation of $\mathbb{S}^2$, leading to an orientation preserving or reversing handle, respectively. Since we want to analyze oriented manifolds, we are only

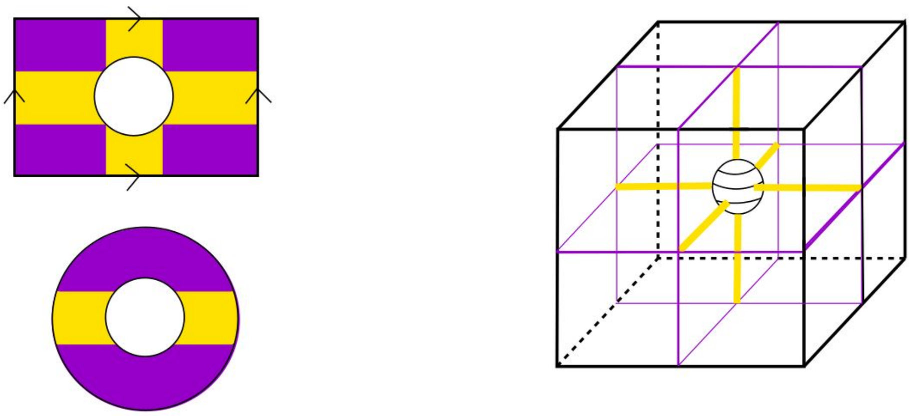

Figure 10: Some examples of handle decompositions of 2- and 3-manifolds. On the lefthand side we see a 2-dimensional torus and $\mathbb{R}P^2$ (a disk with a boundary whose points are identified antipodally) and on the righthand side we see a 3-dimensional torus. Note that the 2-handles have not been drawn thick for the sake of clarity.

left with one choice and thus there is only one way of attaching 3-dimensional 1-handles. The boundary of the union of the 0-handle and the 1-handles is a genus g surface and, while the attaching sphere of 2-handles already uniquely determines the attaching of the 2-handle, there are many different ways of embedding an $\mathbb{S}^1$ into such a surface. This really is where the complicated topology of 3-manifolds comes from since otherwise we do not have a lot of choices in building the manifold. Lastly, up to isotopy, the identity is the only one orientation preserving diffeomorphism from $\mathbb{S}^2$ to itself, so there is a unique way of attaching a 3-dimensional 3-handle. See Figure 11 for some drawings of 3-dimensional handles and Figure 10 for an example of a handle decomposition of a 3-dimensional manifold.

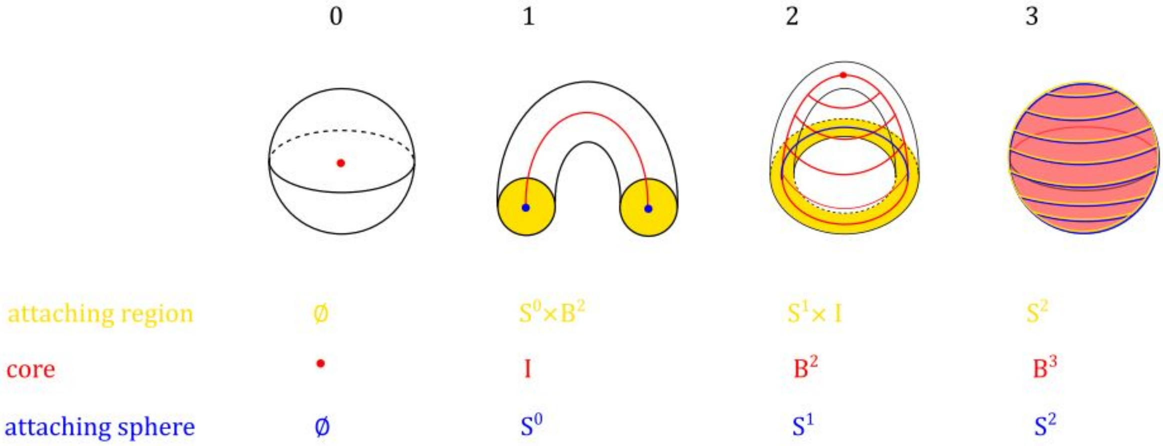

Figure 11: 3-dimensional handles

Recall that isotopies within levels played an important role in the uniqueness of handle decompositions. We now want to introduce a specialized notion which describes situations in which the attaching region of a k-handles is isotoped over another k-handle.

Let Y be a 3-dimensional manifold with boundary and let h_1 and h_1' be two 3-dimensional 1-handles attached to ∂Y. A **handle slide** of h_1 over h_1' is an isotopy of the attaching region of h_1 over h_1' as depicted in Figure 12. Given two 3-dimensional 2-handles h_2 and h_2' we perform a handle slide of h_2 over h_2' by sliding the attaching region of h_2 over h_2' as depicted in Figure 13.

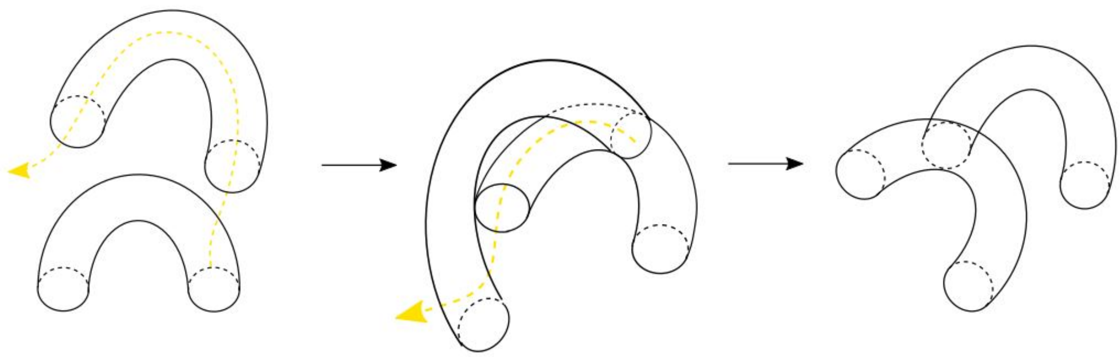

Figure 12: A handle slide among 3-dimensional 1-handles

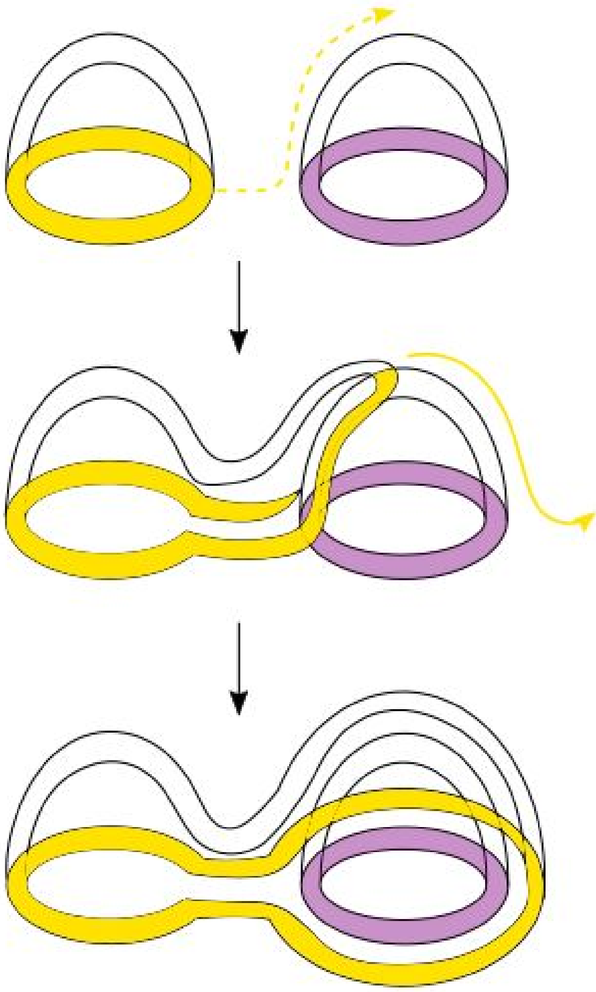

Figure 13: A handle slide among 3-dimensional 2-handles

A more careful description of 2-handle slides goes as follows: Recall that given a 3-manifold Y, Y^1 is the union of the (unique) 0-handle and the 1-handles, ∂Y^1 is therefore diffeomorphic to a genus g

surface where g is the number of 1-handles. Now given two disjoint simple closed curves α_1 and α_2 on a genus g surface Σ_g we connect them by a properly embedded arc that intersects both α_1 and α_2 in just one point. Sliding α_1 over α_2 along c then means that we replace α_1 by a curve α_1' that snugly envelopes $\alpha_1 \cup c \cup \alpha_2$, see Figure 14. Another way of saying this is that α_1' is the band-connect sum of α_1 and α_2 along c. In total the handleslide transforms (α_1, α_2) into (α_1', α_2). When performing a handleslide among two members of a larger collection of such disjoint simple closed curves living on Σ_g we take care that c is disjoint from every curve not involved in the handleslide. Two sets of g simple closed curves $\alpha = (\alpha_1, \dots, \alpha_g)$ and $\alpha' = (\alpha_1', \dots, \alpha_g')$ on the same surface are called **slide equivalent** if α can be made into α' using a sequence of handle slides and isotopies. Naturally we also want to compare systems of curves when they live on different but diffeomorphic genus g surfaces (for example when we attach the 1-handles slightly differently), so we call the pair (Σ_g, α) of a genus g surface and a system of g simple closed curves **slide-diffeomorphic** to another such pair (Σ_g', α') if there exists a diffeomorphism $\phi : \Sigma \longrightarrow \Sigma'$ such that $\phi(\alpha) = (\phi(\alpha_1), \dots, \phi(\alpha_g))$ is slide equivalent to α'. Similarly we say that $(\Sigma_g, \alpha, \beta)$ is slide-diffeomorphic to $(\Sigma_g', \alpha', \beta')$ if there exists a diffeomorphism $\phi : \Sigma \longrightarrow \Sigma'$ such that $\phi(\alpha)$ and $\phi(\beta)$ are slide equivalent to α' and β' respectively (where we only slide amongst α's and β's but don't allow mixed slides e.g. an α over a β curve).

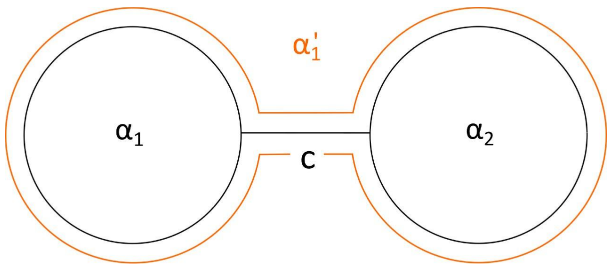

Figure 14: Sliding α_1 over α_2 along c to obtain α_1'

With the general notions specialized to three dimensions we introduce a notion that has proven extremely useful in the study of 3-manifolds and that can be seen as the 3-dimensional precursor to trisections. The construction captures the idea that we really only need to know how 1- and 2-handles are attached to one another in order to know a 3-manifold.

Definition 2.2.1. *Let Y be a closed oriented 3-manifold. A genus g Heegaard splitting $Y = H_g^+ \cup_{\Sigma_g} H_g^-$ is a decomposition of Y into two genus g handlebodies H_g^+ and H_g^- whose boundaries, Σ_g and $-\Sigma_g$ respectively, are glued to one another using an orientation reversing diffeomorphism. Σ_g is then called a Heegaard-splitting surface for Y.*
A diffeomorphism $\psi : Y = H_g^+ \cup_{\Sigma_g} H_g^- \longrightarrow N = G_{g'}^+ \cup_{\Sigma_{g'}} G_{g'}^-$ between two Heegaard-split manifolds Y and N is called a split-diffeomorphism if $\psi(H_g^\pm) = G_{g'}^\pm$. In particular this implies that $\psi(\Sigma_g) = \Sigma_{g'}$ and by the classification of closed oriented 2-manifolds $g = g'$.

It is easy to see that, using the technology provided by the theory of handle decompositions, Heegaard-splittings occur quite naturally: Given a closed oriented 3-manifold Y with a handle decomposition that has k_0 0-handles, k_1 1-handles, k_2 2-handles, and k_3 3-handles we have already established that we can assume $k_0 = k_3 = 1$. Combining the fact that $H_{k_1}^+ =$ 0-handle $\cup\, k_1$ 1-handles is a genus k_1 handlebody with boundary Σ_{k_1} and $H_{k_2}^- = k_2$ 2-handles $\cup$ 3-handle is an upside-down genus k_2 handlebody with boundary $-\Sigma_{k_2}$ with the observation that, X being a closed oriented manifold, Σ_{k_1} has to be orientation-reversing diffeomorphic to $-\Sigma_{k_2}$, we conclude that $k_1 = k_2 =: g$ by the classification of closed oriented surfaces, and so $Y = H_g^+ \cup_{\Sigma_g} H_g^-$. In conclusion we have:

Theorem 2.2.1. *Every closed oriented* 3*-manifold admits a Heegaard splitting.*

One easily checks that, conversely, every Heegaard-split manifold admits a handle decomposition that induces the splitting.
As one might expect, Heegaard splittings are far from unique. In fact, given any genus g Heegaard splitting for a manifold there exists a genus $g + k$ Heegaard splitting for all $k \in \mathbb{N}$ that is derived from the original Heegaard splitting.

Definition 2.2.2. *Let H^3_g be a* 3*-dimensional genus g handlebody. A properly embedded arc c with endpoints in $\partial H^3_g = \Sigma_g$ is called boundary parallel if there exists an embedded disk that is bounded by c and some properly embedded line $s \subset \Sigma_g$ that connects the endpoints of c.*

Definition 2.2.3. *Let $Y = H^+_g \cup_{\Sigma_g} H^-_g$ be a genus g Heegaard-split manifold, let furthermore c be a boundary parallel arc. Denote by $N(c)$ a tubular neighbourhood of c and set $\tilde{H}^+_g = H^+_g \cup N(c)$ and $\tilde{H}^-_g = H^-_g \smallsetminus \mathring{N}(c)$. Then $Y = \tilde{H}^+_g \cup_{\tilde{H}^+_g \cap \tilde{H}^-_g} \tilde{H}^-_g$ is called a stabilization of the original Heegaard splitting.*

Seeing that this is in fact a Heegaard splitting is not hard and could be done directly from this definition. Since an analogous argument to this will be made in section 3.1 for trisections, we will instead prove it in another way using the tight connection to handle decompositions.

Proposition 2.2.1. *Let $Y = H^+_g \cup_{\Sigma_g} H^-_g$ be a genus g Heegaard-split manifold, pick a handle decomposition of Y that induces this Heegaard-splitting. Stabilizing the Heegaard-splitting is equivalent to introducing a canceling $1-2$ handle pair in the handle decomposition and collecting the* 1*-handles into one and the* 2*-handles into the other handlebody. Consequently a stabilization produces a genus $g+1$ Heegaard-splitting.*

Proof. $'' \Longrightarrow ''$ First notice that the tubular neighbourhood $N(c)$ of the curve c along which we stabilize the Heegaard-splitting intersects H^+_g in a pair of disks, so we add a 1-handle with core c to H^+_g, call it H^+_{g+1}. Remembering that c being boundary parallel means that there exists an embedded disk D as in Figure 15, attached to H^+_{g+1} along a circle, we take a tubular neighbourhood of this disk D and interpret it as a 2-handle with core D and attaching sphere ∂D. This attaching sphere intersects the cocore of the 1-handle once and transversely, so we have introduced a canceling $1-2$ handle pair.
$'' \Longleftarrow ''$ Attaching a 1-handle to H^+_g can be interpreted as attaching the core c with its endpoints to Σ_g, with its interior lying in H^-_g, and then taking a tubular neighbourhood of it. The fact that we attach a 2-handle that cancels the 1-handle at the same time implies that there exists an embedded disk, the core of this 2-handle, over which one can slide c onto the surface Σ_g. The curve c is thus boundary parallel. We see that $Y = \big(H^+_g \cup \text{1-handle}\big) \cup_{\Sigma_{g+1}} \big(H^-_g \smallsetminus \text{interior of 1-handle}\big)$ and, using that the $1-2$ handle pair combines to give a 3-ball, taking away a 1-handle from H^-_g is the same as attaching the 2-handle to it. □

A key observation in the context of Heegaard-decompositions is that the attachment of the attaching spheres of 2-handles, both ordinary and dual, to the surface uniquely determines the associated manifold. To understand this remember that in dimension 3, 1-handles are dual 2-handles and we can attach them "above", i.e. to the genus g surface, by tracking where the belt sphere of the 1-handle, which is the attaching sphere of the same handle interpreted as a 2-handle, goes. Call the curves corresponding to the belt spheres of the 1-handles $\alpha = (\alpha_1, \dots, \alpha_g)$ and the attaching spheres of the 2-handles $\beta = (\beta_1, \dots, \beta_g)$. Notice that if a 2-handle is attached along a contractible curve then we need to have an additional 3-handle as indicated in Figure 16, so contractible curves really correspond to cancelling $2-3$ pairs. Thus by the uniqueness of handle decompositions there is a second 3-handle iff there is some combination of handle slides among the β's such that some β_j is made contractible, see for example Figure 16.

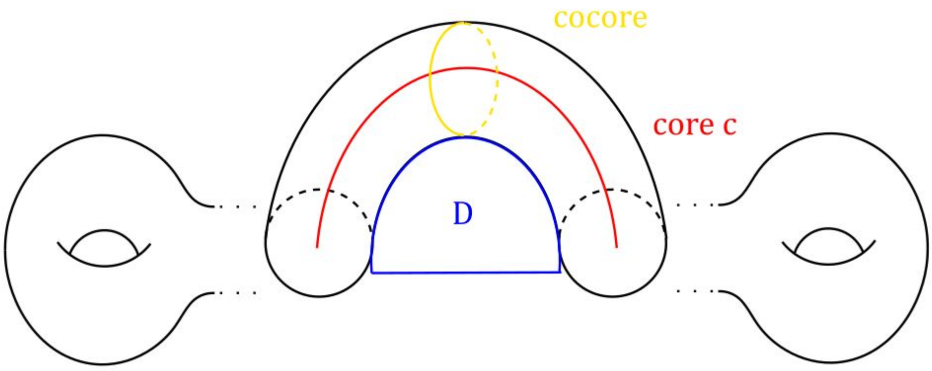

Figure 15: Viewing a stabilization as a $1-2$ pair creation and vice versa

Everything we discussed here for the β-curves is dually true for the α-curves. A system of curves that satisfies that no combination of handle slides among them makes any of them contractible will be called **independent**, much in the sense of linear independent sets of vectors in a vectorspace (they actually are a basis in $H_1(\Sigma_g, \mathbb{Z}) = \mathbb{Z}^g$). This property is equivalent to the property that each set of g 2-handles can be completed to a genus g handlebody using a single 3-handle, which is precisely what we wanted.

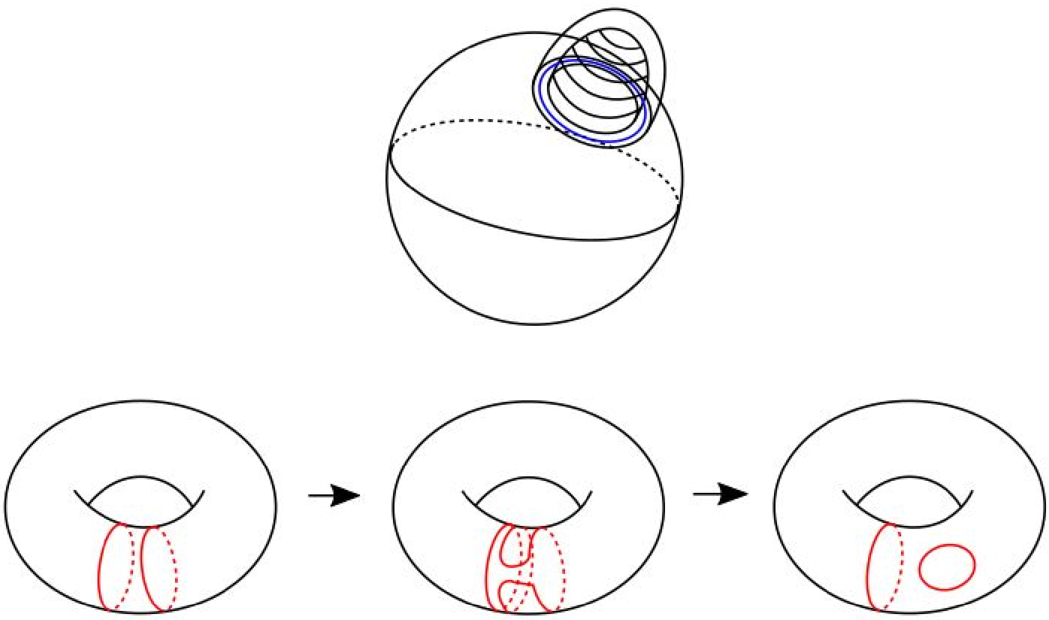

Figure 16: An explanation of how homotopically trivial attaching spheres of 2-spheres lead to additional 3-handles and an example of a dependent system of curves

A genus g surface together with two independent systems of g disjoint noncontractible simple closed curves therefore determines a 3-manifold.

One can now lift these considerations to a more abstract level, independent of concrete surfaces embedded in concrete 3-manifolds:

Definition 2.2.4. *[Heegaard-diagrams] A Heegaard diagram $(\Sigma_g, \alpha, \beta)$ is a genus g surfaces Σ_g together with two sets of g independent closed oriented curves $\alpha = (\alpha_1, \ldots, \alpha_g)$ and $\beta = (\beta_1, \ldots, \beta_g)$.*
A stabilization of a Heegaard diagram is the connect sum of a Heegaard diagram with the one of the 3-sphere presented in Figure 18.

When drawing a concrete Heegaard diagram one typically chooses the α-curves to look like in Figure 17.

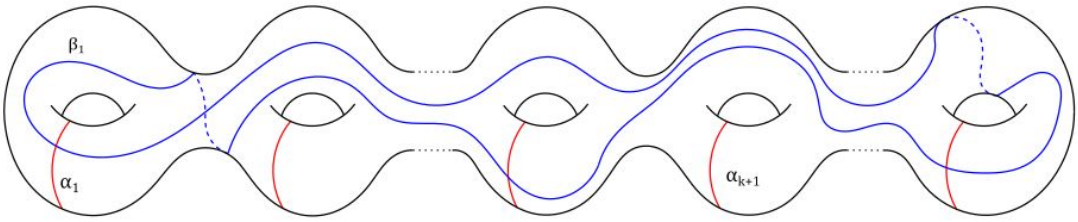

Figure 17: Heegaard diagram in standard position.

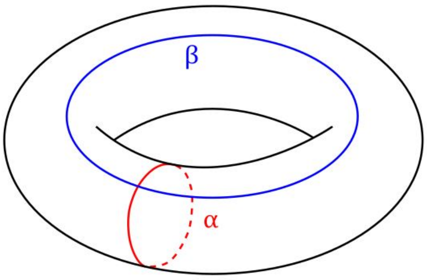

Figure 18: A Heegaard diagram of $\mathbb{S}^3$

By the previous considerations, a Heegaard decomposition determines a Heegaard diagram and vice versa. As one might expect, stabilizations of Heegaard diagrams precisely correspond to stabilizations of Heegaard decompositions. It is clear that both structures are tightly connected, we conclude this discussion by quoting

Theorem 2.2.2. *Let M be a closed connected 3-manifold.*

1. *Any two Heegaard splittings become isotopic after suitable stabilizations*
2. *More precisely, let D_0 and D_1 be two Heegaard diagrams. Then there are stabilizations D_0', D_1' by adding pairs of cancelling handles of index 1 and 2, such that one can pass from D_0' to D_1' by an ambient isotopy and a finite sequence of handle slides.*

a proof of which can be found in [Lau14, Theorem 1.1]. Here the author of [Lau14] is thinking of the embedded version of Heegaard diagrams.

Another way to represent a Heegaard diagram is that we take away a point from $\partial(0 - handle) \cong \mathbb{S}^2$ since we can always avoid this with the attaching regions of 1- and 2-handles, and then we simply draw the attaching regions of those handles and not the handles themselves. The attaching regions of the 1-handles can be thought of as "teleporters" that are identified by reflecting across a plane between the two B^2's. Clearly this construction contains equivalent information to Heegaard diagrams, we call it the flat Heegaard diagram. Flat Heegaard diagrams admit the same moves (and uniqueness statement) as proper Heegaard diagrams, these can easily be worked out by the reader. For some examples of flat Heegaard diagrams and an idea of what stabilizations and handle slides look like in this setting, see Figure 19.

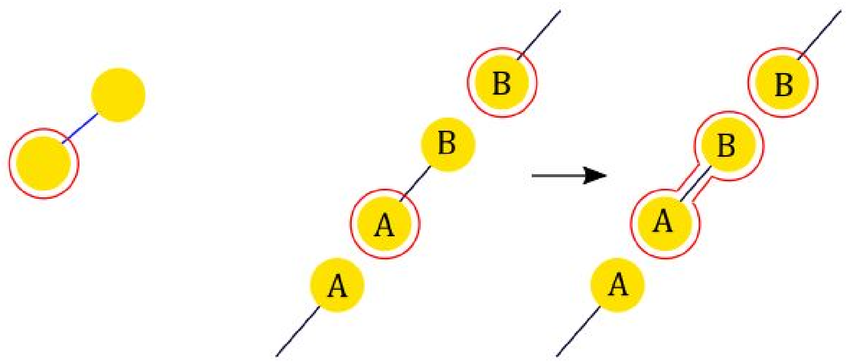

Figure 19: An illustration of what stabilizations and handle slide look like in flat Heegaard diagrams

For the purpose of this thesis it is not necessary to know any examples of Heegaard diagrams with the exception of the so called standard genus g Heegaard-splitting of $\sharp_k \mathbb{S}^1 \times \mathbb{S}^2$. By this one means the diagram depicted in Figure 20 and we want to argue here why this describes $\sharp_k \mathbb{S}^1 \times \mathbb{S}^2$. First note that this diagram has been stabilized $g-k$ times and so we just have to understand why the first k factors correspond to $\sharp_k \mathbb{S}^1 \times \mathbb{S}^2$. The other observation is that

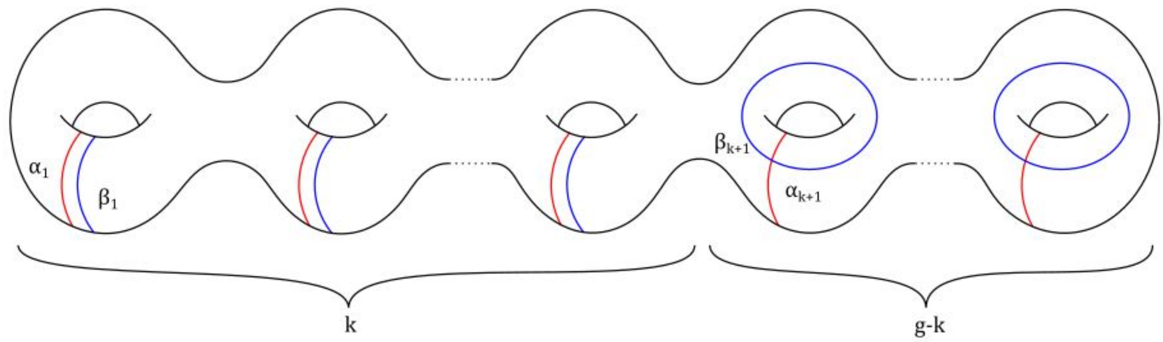

Figure 20: The standard genus g Heegaard diagram of $\sharp_k \mathbb{S}^1 \times \mathbb{S}^2$.

Lemma 2.2.1. *Let Y and Y' be two closed oriented 3-manifolds described by the Heegaard diagrams $(\Sigma_g, \alpha, \beta)$ and $(\Sigma_{g'}, \alpha', \beta')$ respectively. Then $Y \sharp Y'$ is described by $(\Sigma_g \sharp \Sigma_{g'}, \alpha \sqcup \alpha', \beta \sqcup \beta')$.*

Proof. Since it does not matter which B^3 we delete in Y and Y', simply take one that deletes a disk in a Heegaard surface corresponding to Σ_g and similiarly for a Heegaard surface corresponding to $\Sigma_{g'}$ and do this in such a way that the connected sum precisely glues the two Heegaard surfaces together. The resulting surface is a Heegaard surface of $Y \sharp Y'$ because the connected sum summed together the outside and inside handlebodies of Y and Y'. The result follows by passing to the level of Heegaard diagrams. □

We now want to argue that the Heegaard diagram depicted in Figure 21 describes $\mathbb{S}^1 \times \mathbb{S}^2$, and then use Lemma 2.2.1 to conclude that the Heegaard diagram seen in Figure 20 describes $\sharp_k \mathbb{S}^1 \times \mathbb{S}^2$. The diagram in Figure 21 tells us that we need to attach a disk inside the torus and outside the torus, both touching at their boundary, so we have attached an $\mathbb{S}^2$. Since the attaching spheres can be shifted anywhere on the torus, we really attach an $\mathbb{S}^2$ along every meridinal circle, so we obtain $\mathbb{S}^1 \times \mathbb{S}^2$.

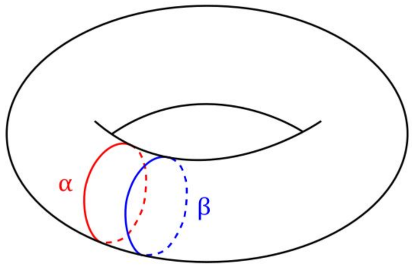

Figure 21: The standard genus 1 Heegaard diagram of $\mathbb{S}^1 \times \mathbb{S}^2$.

Finally, there is a lemma that we are going to need in our investigation of trisections. It is the fact that any 3-dimensional manifold admits a smooth regular flow that is defined everywhere except on codimension 2 submanifolds.

Lemma 2.2.2. *Let $Y = H_k^+ \cup_\Sigma H_k^-$ be a closed oriented 3-manifold. Let furthermore $W_k^\pm \subset H_k^\pm$ be a wedge of k circles, where each circle is obtained by connecting the core of a 1-handle (2-handle) to the core of the 0-handle (3-handle) by radial lines in the 0-handle (3-handles). Then there exists a smooth flow $\phi : (-\infty, \infty) \times \left(H_k^+ \setminus W_k^+\right) \cup_\Sigma \left(H_k^- \setminus W_k^-\right) \longrightarrow \left(H_k^+ \setminus W_k^+\right) \cup_\Sigma \left(H_k^- \setminus W_k^-\right)$ that does not have any critical points.*

Proof. The key idea here is that we can get a flow without critical points on each handle minus the core simply by pushing everything away from the core. When putting together the handles, one needs to avoid flowlines that lead to the core of the other handles. We appeal to the intuition of the reader that this flow can be smoothed out close to the attaching regions. See Figure 22 for the idea of this proof. □

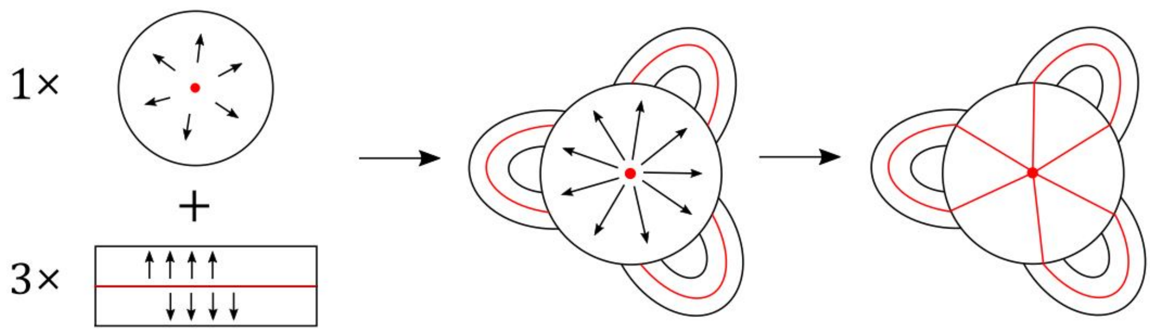

Figure 22: Using handles to define a non-singular smooth flow outside of the cores of the handles and the flowlines connecting them

2.3 Dimension 4 and Kirby diagrams

Remarkably, just as in dimension 3, keeping track of the way 1- and 2-handles are attached is enough to uniquely specify a closed oriented 4-manifold with a single 0- and 4-handle. The way in which 1-handles are attached in dimension 4 is combinatorially as complicated as in dimension 3 as we just need to know

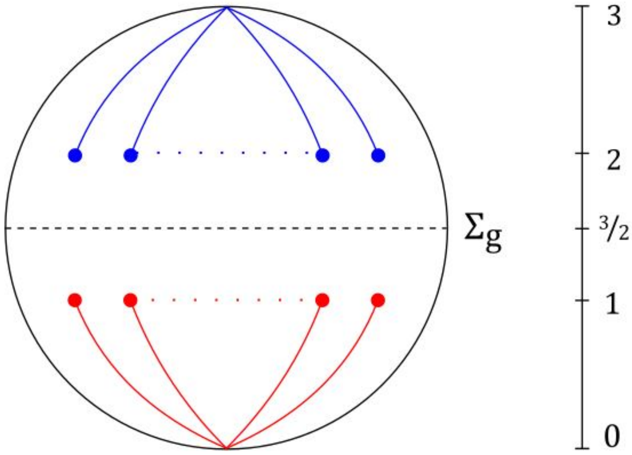

Figure 23: The flow we constructed in Lemma 2.2.2 defines a deformation retract of the entire manifold minus the red and blue lines to the Heegaard surface.

where a pair of points goes. 2-handles however are substantially more complicated as we will see.
The boundary of a 4-dimensional 0-handle is an $\mathbb{S}^3$, so up to a point which we can avoid it is $\mathbb{R}^3$. To this we attach 1-handles along $\mathbb{S}^0 \times B^3$, two balls in space that are identified with one another by reflecting along a plane in between the two B^3's.
We attach 2-handles by embedding an $\mathbb{S}^1$ into $\partial\big(0 - handle \cup k\, 1 - handles\big) = \sharp_k \mathbb{S}^1 \times \mathbb{S}^2$, we call this embedded circle K, and by prescribing a framing for K. Throughout this chapter we will use the so called blackboard framing for Kirby diagrams. This means that the reference framing is given by the surface on which the diagram lies (which is $\mathbb{R}^2$ together with some bridges to account for both the 1-handles and the "breaks" in the undercrossings, so Σ_g minus a point, where g is the number of 1-handles plus the number of intersections in the projected link), or more precisely by the normal vectorfield defined by the surface. Using this reference framing, the framing of the knot takes values in $\mathbb{Z}$ and determines the attaching diffeomorphism of the 2-handle up to isotopy. We want to emphasize that there is a huge number of non-isotopic embeddings of $\mathbb{S}^1$ into $\mathbb{R}^3$ as a quick look into any textbook on knot theory will corroborate, 4-manifolds really are much more complicated than 3-manifolds.
In lower dimensions the question of whether a collection of handles even can be completed to a closed manifold has not been so important because it is usually quite easy to tell if this can be done or not, but this is not true in dimension 4. We will return to this question shortly, but let us assume for a moment that we have a collection of a single 0-handle, some 1- and 2-handles that can be completed to a closed 4-manifold using some 3-handles and a single 4-handle. Then a theorem by Laudenbach and Poenaru [LP72, Theorem A] guarantees that there is only one way to "cap off" said collection of 0-, 1-, and 2-handles with a union of 3-handles and a single 4-handle. This means that we only have to specify 1- and 2-handles in order to uniquely determine a closed 4-manifold (that has a single 0- and 4-handle), provided that we know that it is possible to complete this collection of 1- and 2-handles to such a closed 4-manifold, which is equivalent to $\partial\big(0{-}handle{\cup}1{-}handles{\cup}2{-}handles\big) = \sharp_m \mathbb{S}^1 {\times} \mathbb{S}^2 = -\partial\big(3{-}handles{\cup}4{-}handle\big)$ for some m. As discussed in the introduction, we may project the attaching regions of the 1- and 2-handles to $\mathbb{R}^2 \times \{0\}$ where we can measure the framing of the 2-handles with respect to $\mathbb{R}^2 \times \{0\}$, so we can assign a number to specify the framing of the 2-handles. Closable or not, a collection of 1-handles together with a framed link, once projected, is called a Kirby diagram. For some examples of 4-dimensional handles, see Figure 24.

Like in dimension 3 we also want to introduce the notion of handle slides in dimension 4. Given

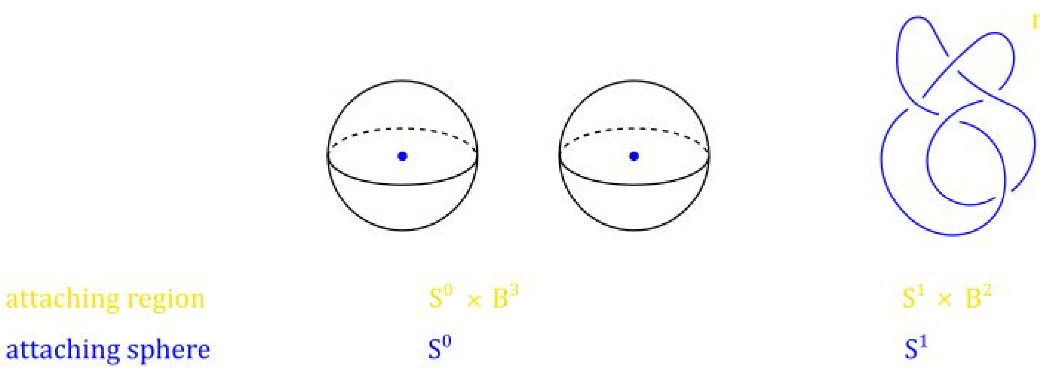

Figure 24: 4-dimensional handles

what we just discussed, we only need to analyze handle decompositions of 4-manifolds up to the level of 2-handles, so the only handle slides that concern us are slides of 1- and 2-handles. As in dimension 3, the idea of sliding a 4-dimension k-handle h_k over another such handle h'_k (both attached to the boundary ∂X of a 4-manifold X) is that one isotopes the attaching region of h_k over h'_k in $\partial\big(X \cup h'_k\big)$. Since this is quite exhausting to make precise we define handle slides by referring to what they do on the diagrammatic level.

Slides of 1-handles work as in dimension 3 so they are not particularly interesting (see Figure 25 for an example), but slides among 2-handles are not as easy as they have been so far. To make things easier for us we remember that we can keep track of the framing of a knot by drawing a parallel knot that encodes the framing (with respect to the blackboard framing) by twisting around the original knot that number of times, see Figure 8. Let K_1 and K_2 be two attaching spheres of 2-handles having blackboard framing r_1 and r_2 respectively. Let furthermore K_1^{fr} and K_2^{fr} be the knots encoding the framing of K_1 and K_2 respectively. Let furthermore c be a properly embedded arc in $\mathbb{R}^3$ connecting K_1 and K_2, with c interesecting either of the knots at its endpoints. A handle slide of K_1 over K_2 along c is performed by band-summing K_1 to a copy of K_2^{fr} along c, we call the result K'_1, and the associated parallel knot that encodes the framing is obtained by band-summing K_1^{fr} to another copy of K_2^{fr} along c, the result of which we call K'^{fr}_1. It follows that, if K_1 had framing r_1 and K_2 had framing r_2, then K' will have framing $r_1 + r_2$, where all framings are measured with respect to the blackboard framing. See Figure 26 for an Illustration of this process. Of course, knowing how to compute the framing of the new knot after the handle slide, we can also do handle slides without the help of these parallel knots, see Figure 27 for an example.

Returning to the issue of closability of Kirby diagrams we have

Theorem 2.3.1. *[Kirby] A Kirby diagram can be closed iff there exists a sequence of blow-ups or blow-downs, handle slides, and handle cancellations such that afterwards the diagram consists of unlinked unknots with framing in* $\{-1, 0, 1\}$.

Where a blow-up/down means adding a ± 1-framed unknot to the framed link which turns out to be equivalent to connect summing with a $\mathbb{C}P^2$ or $\overline{\mathbb{C}P^2}$ respectively. A proof of 2.3.1 can be found in Kirby's book [Kir80, Theorem 1.5.1].

With this out of the way, we want to know when exactly the closures of closable Kirby diagrams coincide. For this we need to understand what canceling $1-2$ and $2-3$ pairs look like. It is fairly clear that a cancelling $1-2$ pair looks like the one depicted in Figure 28 where the pair cancels no matter the

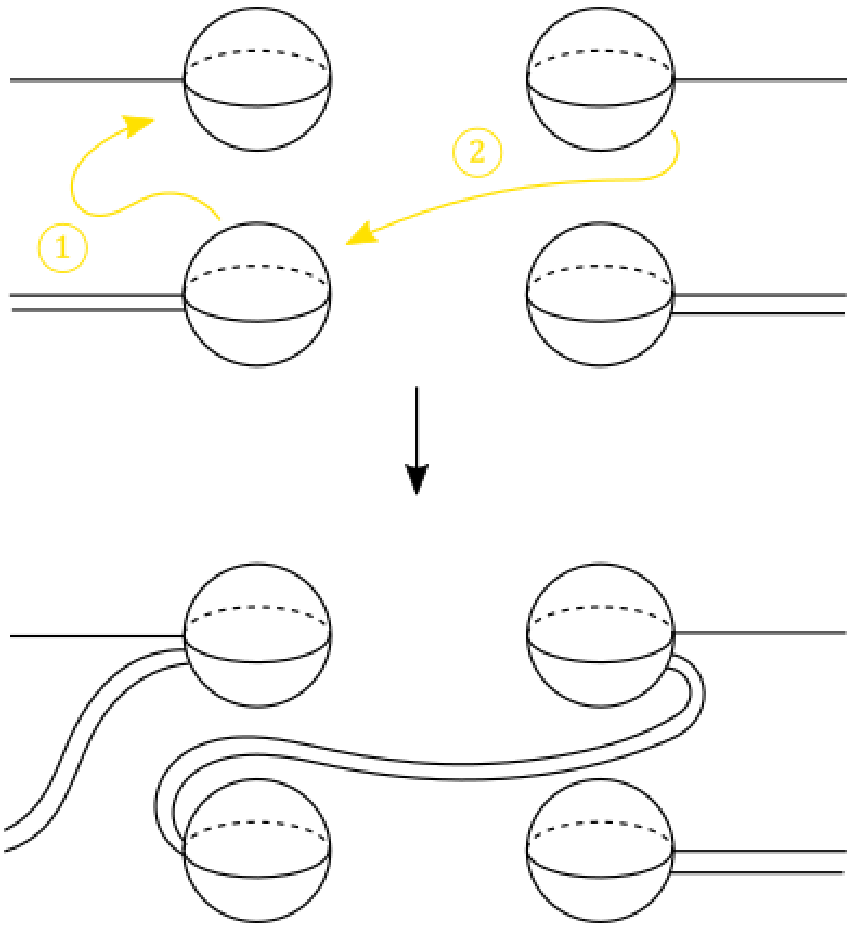

Figure 25: An example of a slide among 4-dimensional 1-handles.

framing of the 2-handle. Figure 28 also shows a canceling $2-3$ pair though it is not as obvious how such a pair looks, for a detailed argument of this see [GS99, Chapter 5.1].

There is a different way of representing 4-dimensional 1-handles which is sometimes more convenient. Think back to Proposition 2.1.1 and the way cancelling pairs of handles combine to a ball. One way to describe the situation is that we first attach a k-handle and then we add a $k+1$-handle which we think of as a tubular neighbourhood of its cocore. Applying this image to the case of a cancelling pair of a 4-dimensional 1- and 2-handle we might say that adding the 1-handle is the same as taking away the 2-handle that cancels it. We take away a 2-handle by immersing its cocore, a disk B^2, and pushing its interior from the boundary of the 0-handle into the 0-handle. We then consider a tubular neighbourhood of the cocore, which is a 4-dimensional submanifold of the 0-handle, and delete it. The boundary of the cocore of the 2-handle is an unknot (since it bounds a disk), we put a dot on it to distinguish it from proper 2-handles. Thus one can equivalently denote 1-handles as dotted circles, knots passing through this circle are being thought of as going over the 1-handle. This is called the dotted circle notation for Kirby diagrams and an example of this can be seen in Figure 28.

Finally there is a family of examples of Kirby diagrams which is particularly nice to work with called plumbings. They take the building blocks depicted in Figure 29 all of which are associated to surfaces though we will not go into this here. Every building block contains an f framed unknot and we put them together by linking them together like the Hopf link. See Figure 30 for an example of this. Since we have specified the way in which they are put together we can actually encode the combination of plumbings diagrams in a so called plumbing graph in which every vertex represents one of the elementary

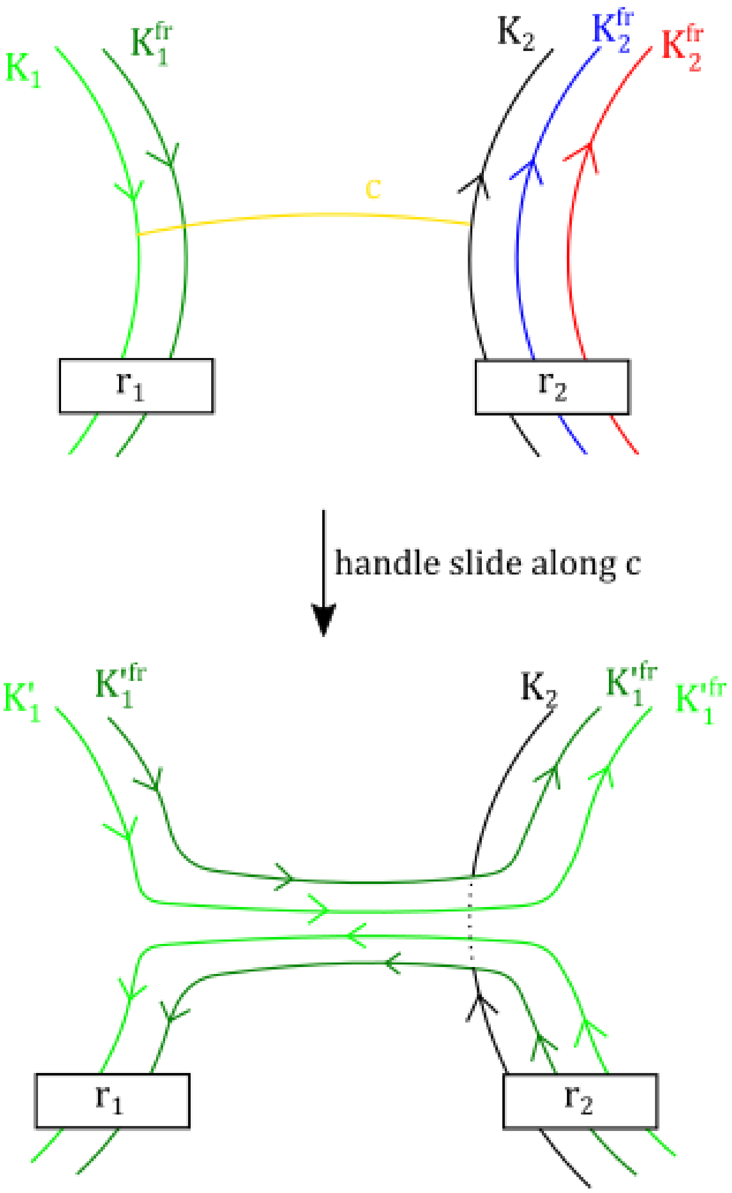

Figure 26: Performing a 2-handle slide along c using parallel knots that encode the framing

building blocks and every edge represents the process of putting the elementary building blocks together, see again Figure 30. Note that most plumbings do not define closed 4-manifolds.

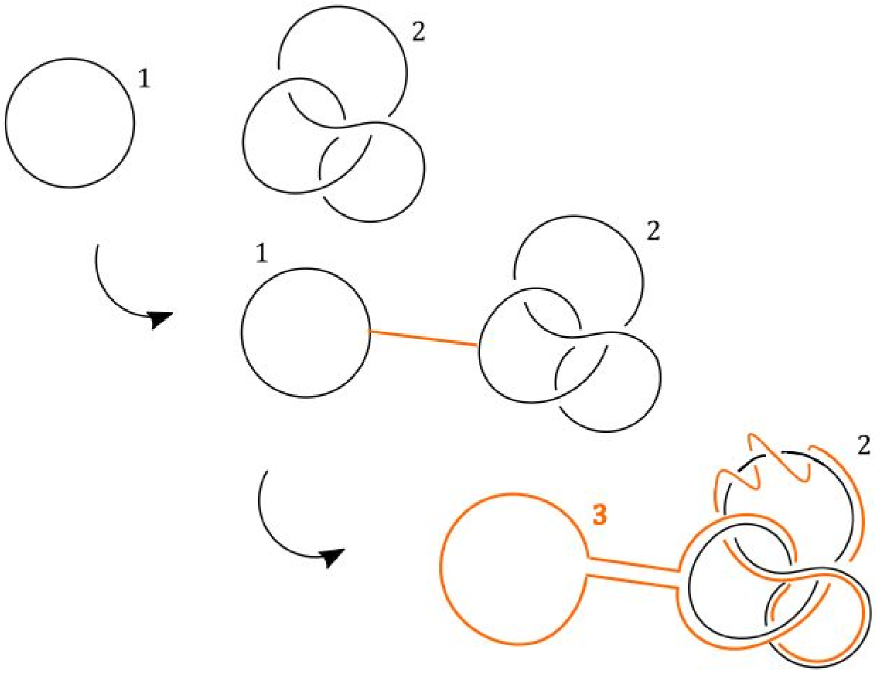

Figure 27: A concrete example of a 2-handle slide without the parallel knots

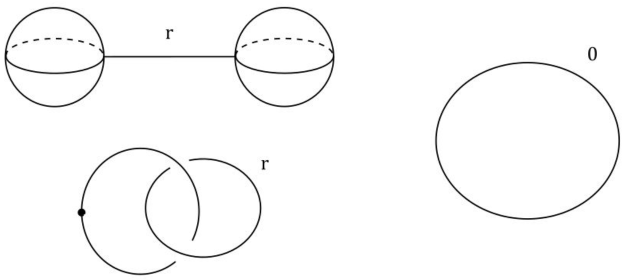

Figure 28: Cancelling $1-2$ and $2-3$ pairs. The Kirby diagram on the left is drawn in both the standard notation and the dotted circle notation.

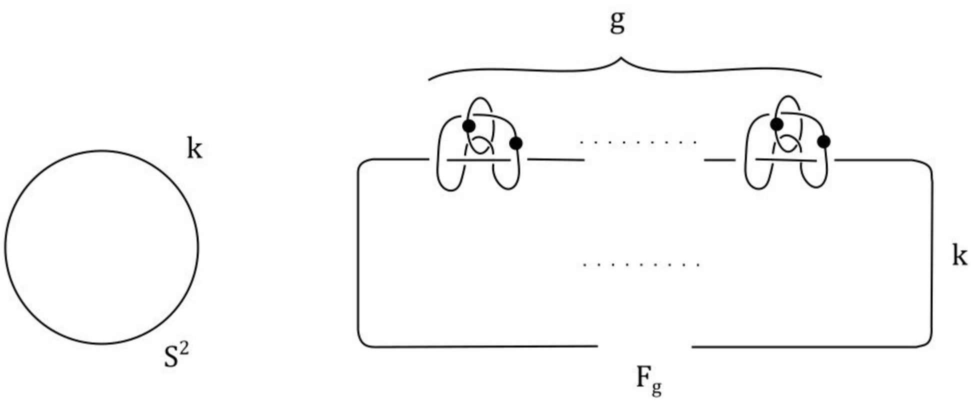

Figure 29: The elementary building blocks for plumbings (associated to orientable surfaces)

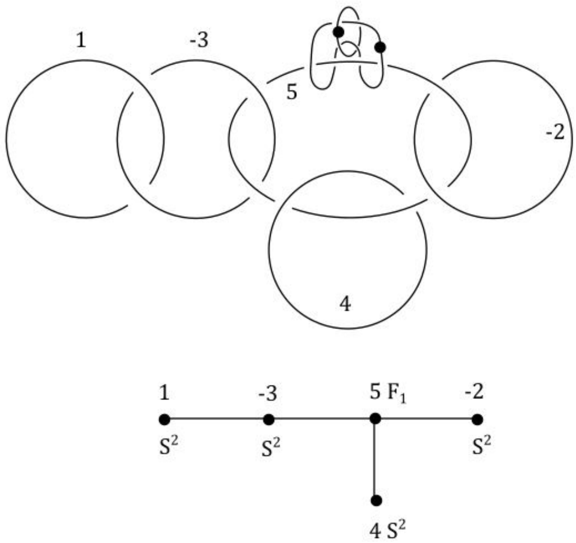

Figure 30: An example of how the elementary building blocks for plumbings are strung together and how to associate a so called plumbing graph to the resulting Kirby diagram.

3 Trisections of closed oriented 4-Manifolds

Trisections are the natural 4-dimensional analogue of Heegaard splittings in that they decompose the manifold into handlebodies glued along their boundary. This decomposition is unique up to a simple set of moves.

3.1 Basic definitions and notions

We start with a definition of trisections. We use the notation H^d_k for a d-dimensional genus k handlebody.

Definition 3.1.1. *[Trisections] Let X be a closed oriented 4-manifold. A $(g;k_1,k_2,k_3)$-trisection of X is a decomposition*

$$X = X_1 \cup X_2 \cup X_3$$

where $X_i \cong H^4_{k_i}$, $H_{i,i+1} = X_i \cap X_{i+1} \cong H^3_g$, and $\Sigma = X_1 \cap X_2 \cap X_3 \cong \Sigma_g$. We call the tuple (X, X_1, X_2, X_3) a trisected manifold.

For a helpful schematic of trisections, see Figure 4.

Remark 3.1.1. *[balanced trisections] Trisections for which $k_1 = k_2 = k_3$ are called balanced trisections.*

There is a natural notion of structure preserving maps for trisections which is

Definition 3.1.2. *[trisected diffeomorphism] Let (X, X_1, X_2, X_3) and (X', X'_1, X'_2, X'_3) be two trisected manifolds. An oriented diffeomorphism $h : X \longrightarrow X'$ is called trisected if it maps X_i to X'_i for all i. If such a diffeomorphism exists we say that (X, X_1, X_2, X_3) and (X', X'_1, X'_2, X'_3) are trisected-diffeomorphic to one another.*

Consequently if (X, X_1, X_2, X_3) and (X', X'_1, X'_2, X'_3) are trisected-diffeomorphic to one another then $h(X_i \cap X_j) = X'_i \cap X'_j$ for $i, j \in \{1,2,3\}$, and $h(X_1 \cap X_2 \cap X_3) = X'_1 \cap X_2\prime \cap X'_3$ and in particular $g = g'$ and $k_i = k'_i$ for all $i \in \{1,2,3\}$.

Remark 3.1.2. *Suppose we have a trisected manifold $(X, X_1.X_2, X_3)$ and a self-diffeomorphism $h : X \longrightarrow X$. We can make h into a trisected diffeomorphism from (X, X_1, X_2, X_3) to $(X, h(X_1), h(X_2), h(X_3))$. Similarly, if we have an isotopy $f : I \times X \longrightarrow X$ we automatically have a 1-parameter family of trisections $(X, f_t(X_1), f_t(X_2), f_t(X_3))$ that are all trisected-diffeomorphic to one another.*

Thinking back to the 3-dimensional case we have that while every 3-manifold admits a Heegaard splitting, there is only a one to one correspondence once we factor out some moves on the set of Heegaard splittings. These moves are isotopies of the gluing map, which do not change the split-diffeomorphism type of the manifold, and stabilizations. The set of moves is more or less the same for trisections and the appropriate notion of stabilization works as follows.

Definition 3.1.3. *[Stabilization of trisections] Given a $(g;k_1,k_2,k_3)$-trisection $X = X_1 \cup X_2 \cup X_3$ with 3-dimensional handlebodies $H_{i,i+1}$ we perform an i-stabilization of the trisection by introducing a properly embedded boundary parallel (to Σ_g) arc c into $H_{i+1,i+2}$ such that the endpoints of c are disjoint in Σ_g. We now define a four dimensional tubular neighbourhood $N(c)$ of c and set*

- $X'_i = X_i \cup N(c)$
- $X'_{i+1} = X_{i+1} \setminus \mathring{N}(c)$

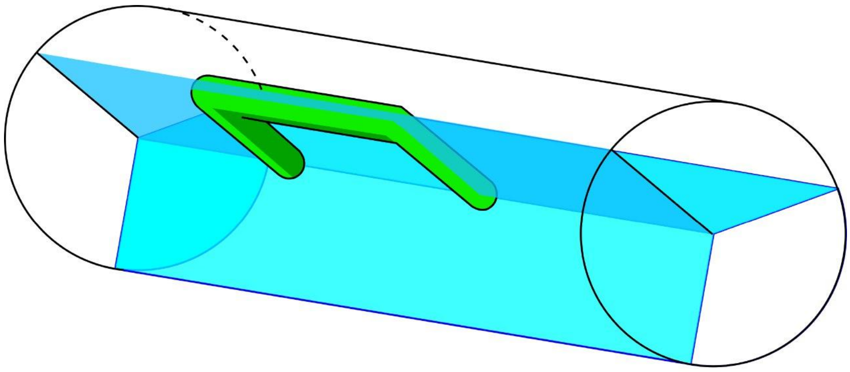

Figure 31: Stabilizing trisections

- $X'_{i+2} = X_{i+2} \setminus \mathring{N}(c)$

Proposition 3.1.1. *An i-stabilized $(g; k_1, k_2, k_3)$-trisection is a $(g'; k'_1, k'_2, k'_3)$-trisection where $g' = g+1$, $k'_i = k_i + 1$, $k'_j = k_j$ for $j \neq i$. This operation is clearly well defined on trisections modulo trisected diffeomorphisms.*

Remark 3.1.3. *This means that we can always pass from an unbalanced trisection to a balanced one. After all, given a $(g; k_1, k_2, k_3)$-trisection we can set $k = max\{k_1, k_2, k_3\}$ and i-stabilize the trisection $k - k_i$ times for all i. Also note that the existence of a trisection of genus g implies the existence of trisections of genus $g + k$, $k \in \mathbb{N}$.*

Proof. Let us for the sake of clarity of notation assume that we are 1-stabilizing a $(g;\, k_1,\, k_2,\, k_3)$-trisection $X = X_1 \cup X_2 \cup X_3$. We need to check that
$\underline{X'_1 = X_1 \cup N(c) \cong \natural_{k_1+1} \mathbb{S}^1 \times B^3}$: We can interpret $N(c) \cong I \times B^3$ as a 1-handle that we attach along $N(c) \cap \partial X_1$, a pair of 3-dimensional balls, to ∂X_1 and so we see that we increase the genus of the handlebody by 1.
$\underline{X'_2 = X_2 \setminus \mathring{N}(c) \cong \natural_{k_2} \mathbb{S}^1 \times B^3 \text{ and } X'_3 = X_3 \setminus \mathring{N}(c) \cong \natural_{k_3} \mathbb{S}^1 \times B^3}$: The key idea is to realize that $N(c) \cap X_2$ is half of a 4-dimensional cylinder $I \times B^3$ which is diffeomorphic to B^4 after smoothing (the other half lies in X_3). In particular the boundary of this half-cylinder is just a disk after smoothing and so we see that if we attach B^4 along a disk to X'_2, a process that produces a manifold which is diffeomorphic to X'_2, we obtain X_2 meaning that $X'_2 \cong X_2 \cong \natural_{k_2} \mathbb{S}^1 \times B^3$. The same argument works for X'_3.
The only thing left to do is to work out the intersections:
$\underline{X'_1 \cap X'_2} = \big(X_1 \cup N(c)\big) \cap \big(X_2 \setminus \mathring{N}(c)\big) = \big(X_1 \cap X_2\big) \cup \big(\partial N(c) \cap X_2\big) \cong X_1 \cap X_2$ where in the last step we used that $\partial N(c) \cap X_2$ is half of the cylinder $\partial N(c)$ and can be pushed over the interior of this cylinder onto $X_1 \cap X_2$ without changing the diffeomorphism type of this manifold with boundary. The same argument can be made for $X'_1 \cap X'_3$.
$\underline{X'_2 \cap X'_3} = \big(X_2 \setminus \mathring{N}(c)\big) \cap \big(X_3 \setminus \mathring{N}(c)\big) = \big(X_2 \cap X_3\big) \setminus \mathring{N}(c) \cong H^3_g \cup$ 3-dimensional 1-handle $\cong H^3_{g+1}$.
The cumulative intersection is not hard to work out either:

$$\begin{aligned} X'_1 \cap X'_2 \cap X'_3 &= \big(X_1 \cup N(c)\big) \cap \big(X_2 \setminus \mathring{N}(c)\big) \cap \big(X_3 \setminus \mathring{N}(c)\big) \\ &= \big(X_1 \cup N(c)\big) \cap \big(\big(X_2 \cap X_3\big) \setminus \mathring{N}(c)\big) \\ &= \big(X_1 \cap X_2 \cap X_3\big) \cup \big(X_2 \cap X_3 \ \cap \partial N(c)\big) \\ &\cong \Sigma_{g+1} \end{aligned}$$

□

This notion of stabilization is crucial for correctly formulating the uniqueness of trisections, the proof of the following theorem can be found in [GK16, Theorem 11].

Theorem 3.1.1. *[Uniqueness of trisections] Given two trisections $X_1 \cup X_2 \cup X_3 = X = X_1' \cup X_2' \cup X_3'$ of X, then after stabilizing each trisection some number of times, there is a trisected diffeomorphism $h : X \longrightarrow X$ that is isotopic to the identity.*

3.2 Trisections and trisection diagrams

The goal of this chapter is to derive a diagrammatic description that, once one mods out some moves, is fully equivalent to trisections mod stabilizations and trisected diffeomorphisms. This very much carries the spirit of the relationship between Heegaard diagrams and Heegaard splittings of closed oriented 3-manifolds. In a sense this means that the entire information of closed oriented 4-manifolds can be captured in a surface plus some curves, so in a codimension 2 manifold! First we need to make some definitions.

Similarly we say that $(\Sigma_g, \alpha, \beta)$ is slide-diffeomorphic to $(\Sigma_g', \alpha', \beta')$ if there exists a diffeomorphism $\phi : \Sigma \longrightarrow \Sigma'$ such that $\phi(\alpha)$ and $\phi(\beta)$ are slide equivalent to α' and β' respectively (where we only slide amongst α's and β's but don't allow mixed slides e.g. an α over a β curve).

We say that $(\Sigma_g, \alpha, \beta, \gamma)$ is slide-diffeomorphic to $(\Sigma_g', \alpha', \beta', \gamma')$ if there exists a diffeomorphism $\phi : \Sigma \longrightarrow \Sigma'$ such that $\phi(\alpha)$, $\phi(\beta)$, and $\phi(\gamma)$ are slide equivalent to α', β', and γ', respectively.

Definition 3.2.1. *[trisection diagrams] A $(g; k_1, k_2, k_3)$-trisection diagram $(\Sigma_g, \alpha, \beta, \gamma)$ is a genus g surfaces Σ_g together with three sets of g curves $\alpha = (\alpha_1, \dots, \alpha_g)$, $\beta = (\beta_1, \dots, \beta_g)$, and $\gamma = (\gamma_1, \dots, \gamma_g)$ satisfying that*

- *$(\Sigma_g, \alpha, \beta)$ is slide diffeomorphic to the $g - k_1$ times stabilized standard Heegaard diagram of $\sharp_{k_1} \mathbb{S}^1 \times \mathbb{S}^2$, see Figure 20,*
- *$(\Sigma_g, \beta, \gamma)$ is slide diffeomorphic to the $g - k_2$ times stabilized standard Heegaard diagram of $\sharp_{k_2} \mathbb{S}^1 \times \mathbb{S}^2$,*
- *$(\Sigma_g, \gamma, \alpha)$ is slide diffeomorphic to the $g - k_3$ times stabilized standard Heegaard diagram of $\sharp_{k_3} \mathbb{S}^1 \times \mathbb{S}^2$.*

We say that a trisection diagram is in X_1-standard form if $(\Sigma_g, \alpha, \beta)$ looks like Figure 20, meaning that $(\Sigma_g, \alpha, \beta, \gamma)$ are as in Figure 5. X_2- and X_3-standard form are defined similarly only with (k_2, β, γ) and (k_3, γ, α) instead of (k_1, α, β), respectively.

Note that the above definition simplifies for balanced trisections.

Moving from trisections to trisection diagrams and vice versa involves an intermediate structure, $H_{12} \cup H_{23} \cup H_{31}$ which we call the spine of the trisection. Recall that $H_{i-1,i} \cup_{\Sigma_g} H_{i,i+1} \cong \sharp_{k_i} \mathbb{S}^1 \times \mathbb{S}^2$ for all i taken mod 3. After thickening up the spine we can use Laudenbach-Poenaru to uniquely (up to diffeomorphism) cap off each pair of the resulting boundary components with one copy of $\natural_{k_i} \mathbb{S}^1 \times B^3$ each and we have thus obtained a trisection.

Proposition 3.2.1. *A $(g; k_1, k_2, k_3)$-trisection diagram $(\Sigma_g, \alpha, \beta, \gamma)$ determines a $(g; k_1, k_2, k_3)$-trisected closed oriented 4-manifold (X, X_1, X_2, X_3) with spine obtained by attaching 3-dimensional 2-handles along the α-, β-, and γ-curves (and capping this off with a 3-dimensional 3-handle). Furthermore any trisection diagram $(\Sigma'_g, \alpha', \beta', \gamma')$ that differs from $(\Sigma_g, \alpha, \beta, \gamma)$ by a slide-diffeomorphism determines a trisected manifold (X', X'_1, X'_2, X'_3) that is trisected diffeomorphic to (X, X_1, X_2, X_3).*

Proof. For this one realizes that given a trisection diagram any of the three systems of g curves determines a handlebody filling, meaning we have three 3-dimensional genus g handlebodies H_{12}, H_{23}, and H_{31} that intersect in Σ_g, so we have the spine of a trisection.
$(\Sigma'_g, \alpha', \beta', \gamma')$ differs from $(\Sigma_g, \alpha, \beta, \gamma)$ by a diffeomorphism of the surface, which extends to the entire manifold, or by handleslides, which extend to isotopies of the entire manifold, so by remark 3.1.2 the resulting trisected manifold is trisected-diffeomorphic to the one we get from $(\Sigma_g, \alpha, \beta, \gamma)$. □

There is a natural way of going the other direction as well:

Proposition 3.2.2. *Let X be a closed oriented 4-manifold. A $(g; k_1, k_2, k_3)$ trisection (X, X_1, X_2, X_3) determines a $(g; k_1, k_2, k_3)$-trisection diagram describing X. Furthermore, a trisection (X', X'_1, X'_2, X'_3) that is trisected-diffeomorphic to (X, X_1, X_2, X_3) leads to a slide-diffeomorphic trisection diagram.*

Proof. Constructing a trisection diagram from a trisection is quite easy as we just have to realize that $\partial X_i \cong \sharp_{k_i} \mathbb{S}^1 \times \mathbb{S}^2$, $g - k_i$ times stabilized, has a standard Heegaard diagram that looks like Figure 20. The other set of g curves is obtained by mapping the corresponding curves of the third leg of the spine to Σ_g. We know that Σ_g together with any two of the sets of curves is slide-diffeomorphic to the desired X_i-standard form because they describe $\sharp_{k_i} \mathbb{S}^1 \times \mathbb{S}^2$. One easily sees that all choices of curves that come from this third leg lead to the same trisection diagram since this third set of curves together with one of the other two sets of curves can be transformed into X_j-standard form with $i \neq j$. Using this we see that there is a sequence of allowed moves that takes us from one choice to any other. This means that we have indeed defined a trisection diagram this way and that all choices lead to the same diagram, this process is thus well defined. □

To actually get a bijection between closed oriented 4-manifolds (i.e. trisected manifolds modulo stabilizations) and trisection diagrams modulo some moves we need to analyze what stabilizations of trisection mean on the level of diagrams. Suppose that we start out with a trisected manifold (X, X_1, X_2, X_3) and i-stabilize it to obtain a different trisected manifold (X, X'_1, X'_2, X'_3). Then the genus of the trisection diagram associated to (X, X_1, X_2, X_3) has to increase by one as we transition to the one associated to (X, X'_1, X'_2, X'_3), and the boundary ∂X_i gets one more $\mathbb{S}^1 \times \mathbb{S}^2$-factor, resulting in a change of the Heegaard diagram by a standard factor with two meridinal curves. The third curve has to be such that the other two boundaries are only changed trivially since they merely get stabilized. We conclude that the corresponding change of the trisection diagram in X_i-standard position has to look like a connected sum of the original diagram with the i-th trisection diagram depicted in Figure 32. The process of connect-summing the original diagram in X_i-standard position with the i stabilizing element is called an i-stabilization of the trisection diagram.
We have thus constructed a notion of stabilization for trisection diagrams so that the following diagram, in which s_i^T and s_i^{TD} denote i-stabilization of trisections and trisection diagrams respectively and the downward arrows represent the process described in proposition 3.2.2, commutes.

$$\begin{tikzcd}
Trisections \arrow[r, "s_i^T"] \arrow[d] & Trisections \arrow[d] \\
Trisection\ Diagrams \arrow[r, "s_i^{TD}"] & Trisection\ Diagrams
\end{tikzcd}$$

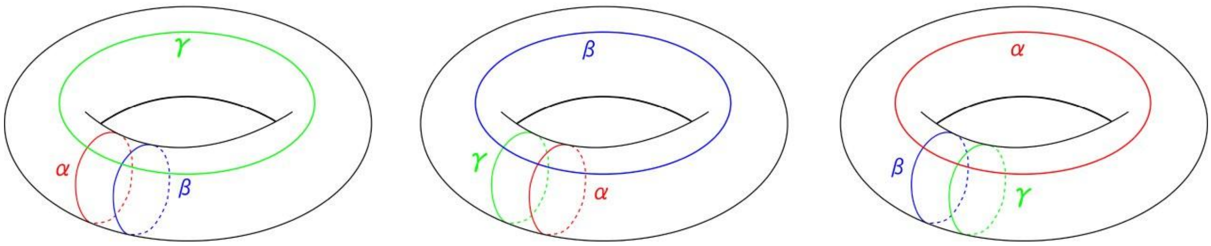

Figure 32: The genus 1 trisections of $\mathbb{S}^4$. Stabilizations on the level of trisection diagrams means connect-summing the diagram with one of these.

Denote now by $\mathcal{T}$ the collection of trisected closed oriented 4-manifolds modulo stabilizations and trisected diffeomorphisms, which is in one-to-one correspondence to closed oriented 4-manifolds by Theorem 3.1.1, and by $\mathcal{TD}$ the collection of trisection diagrams modulo stabilizations and slide-diffeomorphisms. In the following proposition the map (1) and (2) will denote the processes described in Propositions 3.2.2 and 3.2.1 respectively, only on the level of equivalence classes.

Proposition 3.2.3. *$\mathcal{T}$ is in bijective correspondence to $\mathcal{TD}$.*

Proof. $\underline{\mathcal{T} \xrightarrow{(1)} \mathcal{TD} \xrightarrow{(2)} \mathcal{T}}$**:** We have already checked that the map (1) is well defined, so up to stabilizations it gives a unique trisection diagram. For (2) we have to notice that if $(\Sigma_g, \alpha, \beta, \gamma) =: D_1$ and $(\Sigma', \alpha', \beta', \gamma') =: D_2$ differ by a slide-diffeomorphism and stabilizations they will give two different trisections T_1 and T_2. However we stabilize T_1 into another trisection T_1' which has D_2 as its trisection diagram, we conclude that $T_1 = T_1'$ because a trisection diagram uniquely determines a trisection. Thus the map (2) is well defined.
Applying the composition of (1) and (2) to a trisection T gives $T \mapsto D \mapsto T'$. We know that T and T' describe the same 4-manifold so uniqueness of trisections, Theorem 3.1.1, tells us that they only differ by a sequence of stabilizations, so $(2) \circ (1) = id_{\mathcal{T}}$.

$\underline{\mathcal{DT} \xrightarrow{(2)} \mathcal{T} \xrightarrow{(1)} \mathcal{TD}}$**:** We just checked that the maps are well defined and by applying the maps to representants we see that their composition is the identity up to slide-diffeomorphisms. □

Together with the uniqueness theorem of trisections we have shown that closed oriented 4-manifolds are in one-to-one correspondence with trisection diagrams modulo stabilizations.

Similar to how one considers flat Heegaard diagrams one can also represent trisection diagram in a flat way. This happens analogously to flat Heegaard diagrams as we are again on a genus g surface. The resulting diagram is called a flat trisection diagram and it is evident that they are merely a different way of looking at a trisection diagram, so they contain equivalent information. See Figure 6 for some examples. Again one can define the X_i-standard forms for flat trisection diagrams.

3.3 Existence of trisections using handle decompositions

In this chapter we want to derive from the fact that every closed oriented 4-manifold can be described by means of a Kirby diagram the existence of trisections for the same class of manifolds. We largely follow [GK16, Theorem 4] but provide additional details in some places.
The setup is that we already have a handle decomposition of a closed oriented 4-manifold X with one 0-handle, k 1-handles, l 2-handles, m 3-handles, and one 4-handle. One can easily see two pieces of the trisection by grouping together the 0-handle with the 1-handles and the 4-handle with the 3-handles into a 4-dimensional handlebody. The difficulty, then, lies in showing how one can use everything that is left to construct another 4-dimensional handlebody and that all these handlebodies intersect so as to define an unbalanced trisection. We can then stabilize the unbalanced trisection to pass to a balanced one if we wish to.
We begin by collecting the 0-handle and the 1-handles into X_1 and manipulate the framed attaching link L of the 2-handles associated to the handle decomposition that lives in the 3-manifold $\partial X_1 = \sharp_k \mathbb{S}^1 \times \mathbb{S}^2$. The following proposition will allow us to more effectively think about and play with this attaching link.

Proposition 3.3.1. *Given a framed link $L = \{K_1, \dots, K_l\}$ in $\sharp_k \mathbb{S}^1 \times \mathbb{S}^2$ there exists a Heegaard decomposition $\sharp_k \mathbb{S}^1 \times \mathbb{S}^2 = H_g^+ \cup_{\Sigma_g} H_g^-$ with a link $\overline{L} \subset \Sigma_g$ and a complete system of compression disks $\{D_1, \dots, D_g\} \subset H_g^+$ with boundaries $\alpha_i = \partial D_i \subset \Sigma_g$ transversal to $\overline{L}$ such that*

1. *$\overline{L}$ is isotopic to L in $\sharp_k \mathbb{S}^1 \times \mathbb{S}^2$;*
2. *the framing of $\overline{L}$ induced by Σ_g coincides with the framing of L;*
3. *$|\alpha_i \cap \overline{K}_j| = \delta_{ij}$ $\forall i \in \{1, \dots, g\}$ and $j \in \{1, \dots, l\}$.*

Note that attaching 4-dimensional 2-handles onto X_1 along L and along $\overline{L}$ leads to isotopic 4-manifolds. The most important consequence of the proposition for the purposes of proving the existence of trisections is that we can interpret the link $\overline{L} \subset \Sigma_g$ as the attaching link of both 3- and 4-dimensional 2-handles in H_g^+ and X_1, respectively, which will be used when constructing the third piece of the trisection. This idea will be crucial in constructing X_2.

Proof. We first introduce two notion crucial to the proof: let K be an immersed knot in $\mathbb{R}^2 \subset \mathbb{R}^3$. We say that we add a kink to K when we apply a Reidemeister 1 move to K. Note that these come in two flavours, one where the undercrossing comes first and one where the overcrossing comes first, see Figure 43. Suppose $K \subset \mathbb{R}^2$ intersects another immersed knot $K' \subset \mathbb{R}^2$ transversely. Resolving this intersection by a stabilization means that we attach the ends of a cylinder onto the plane $\mathbb{R}^2$ so that one of the knots can run over the cylinder, thus avoiding the other knot. These two concepts are illustrated in Figures 33 and 34.

The proof proceeds in four steps:

(i) Using Lemma 2.2.2 we flow the attaching link L onto Σ_k, an (arbitrary) unstabilized Heegaard-splitting surface of $\natural_k \mathbb{S}^1 \times \mathbb{S}^2 = \partial X_1$, which is in general impossible without incurring some double points. We call the image of L under this flow $\overline{L}$ and flow L in such a way that every $\overline{K} \subset \overline{L}$ that does not have an overcrossing gets a kink. That is to say no component $\overline{K}$ of $\overline{L}$ is intersection-free because in particular every overcrossing is an intersection. Once we resolve these intersections we will have proven item 1. of the proposition.

(ii) In the end we want the surface framing of the link $\overline{L}$ to coincide with the framing of L. We do this by realizing that, once we have resolved all intersections, every knot $\overline{K}_i$ has some framing induced by the surface and this framing can be changed by introducing transverse self-intersections (which we shall call kinks) and resolving them. Every such kink with subsequent resolution corresponds to a change in framing by ±1.

(iii) Both the kinks and the intersections of knots will be resolved by stabilizing the Heegaard-splitting in such a way that the corresponding Heegaard surface develops new holes near the intersection points. The intersections can thus be avoided by sending the "overstrand" of the intersection, which is well defined, over the new handle of the surface. By equipping $\overline{L}$ with the framing induced by the surface we actually retrieve the original framing. L is thus isotopic to $\overline{L}$ as a framed link, i.e. attachment of two-handles defined by them yield isotopic manifolds. This proves items 1. and 2. of the proposition. Note that we now have another Heegaard-splitting $\partial X_1 = H_g^+ \cup_{\Sigma_g} H_g^-$ which is a stabilization of the original one.

(iv) Since no knot $\overline{K}_i$ is intersection free (when two different knots intersect then intersections come in pairs) there is a compression disk D_i, corresponding to the stabilization that resolved said intersection, such that $|\overline{K_i} \cap \partial D_i| = 1$. It might still be true that $\overline{K}_i$ intersects more than just this one disk and we will use handle slides among the compression disk to avoid these superfluous intersections after which item 3. of the proposition will be proven.

Ad (i):
We first want to argue why we can move the link around in the way we want. To this end we remember Lemma 2.2.2 which told us that that, given a Heegaard surface, as long as one avoids two bouquets of circles, one can always flow onto the Heegaard surface.

More precisely, by a dimension count we can assume that the attaching link avoids these bouquets of circles, then we flow them onto Σ. We can do this because the bouquets of circles are a collection of 1-dimensional objects, as is the attaching link L, so by Lemma 2.1.1 we can flow onto Σ. One can now rescale this flow so that we define a smooth projection $p : \mathcal{N}(\Sigma) \longrightarrow \Sigma$ that can in turn be used to define a homotopy $(1-t)id_{\mathcal{N}(\Sigma)}|_L + tp|_L$ that moves the link along the flow lines from L to $\overline{L} := p(L)$ and that even is an isotopy as long as we avoid vertical tangencies (L being tangent to the flow lines at any point) and self intersections which we will always do eventually, that is after all the necessary stabilizations.
At this point it might seem somewhat mysterious that we require each knot in $\overline{L}$ to have one intersection either with another knot or with itself (before stabilizations). The reason for this is that we want each knot to have a hole in the Heegaard surface that is encircled only by this knot and by no other knot since this will allow us to cancel some handles later on.

Ad (ii):
We only introduce the kinks in order to resolve them in step (iii) where it will become apparent how the kinks change the framing.

Ad (iii):
The resolution of intersections and kinks, which are just purposefully introduced transverse self-intersections, can be seen in Figure 33 and Figure 34. An effective way of thinking about what is happening is that we want build bridges or dig tunnels in order for streets to not hit one another.

To see how to change the framing of a knot $\overline{K}$ with respect to the surface one draws a pushoff of $\overline{K}$ along a vectorfield orthogonal to K before and after introducing and resolving a kink. One notices that the intersection number of the pushoff and $\overline{K}$ has changed by ±2, meaning that the framing has changed by ±1.

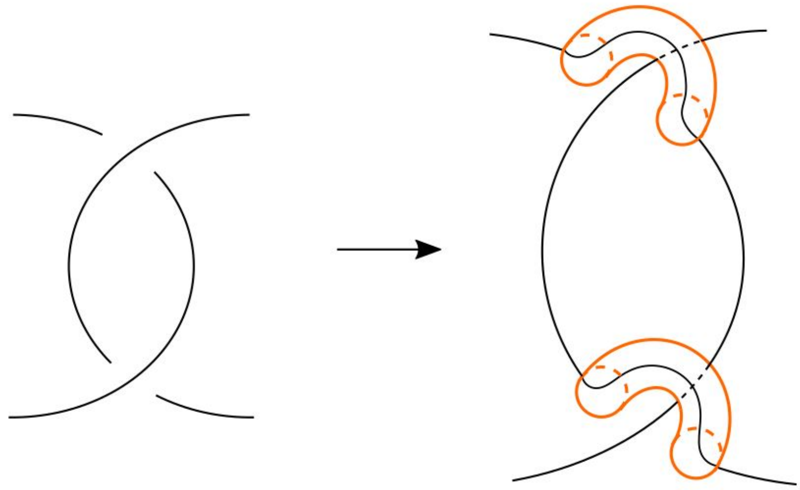

Figure 33: Resolving intersections

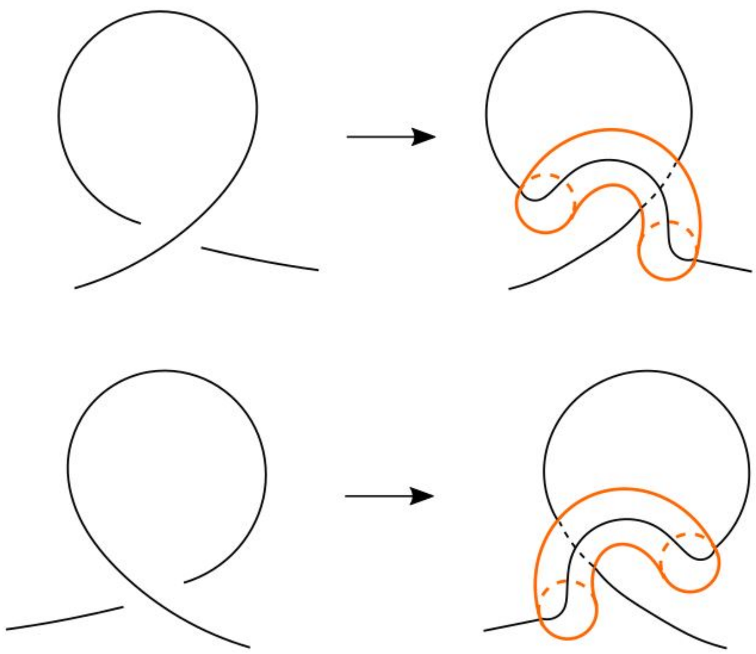

Figure 34: Resolving kinks to change the surface framing

Ad (iv):

As mentioned before the only thing that might go wrong is that there is more than one compression disk which is intersected by some knot $\overline{K}_j$. Among these compression disks there has to be one which is intersected only by the knot $\overline{K}_j$ since it has an overcrossing, call this disk D_j. To get rid of the intersection with another compression disk D_i we slide it over D_j in such a way that $|D_i \cap \overline{K}_j| = 0$ after the slide, we repeat this process until D_j is the only compression disk still intersecting $\overline{K}_j$. In terms of the curves bounding the disks this means that we slide all the curves bounding the disks that $\overline{K}$ intersects over $\partial D_j = \alpha_j$ along the segment of $\overline{K}_j$ that connects them. This way we ensure that $\overline{K}$ intersects precisely one boundary of a compression disk. After this process is repeated for all j we are done. See Figure 35 to get an idea of how the sliding works.

□

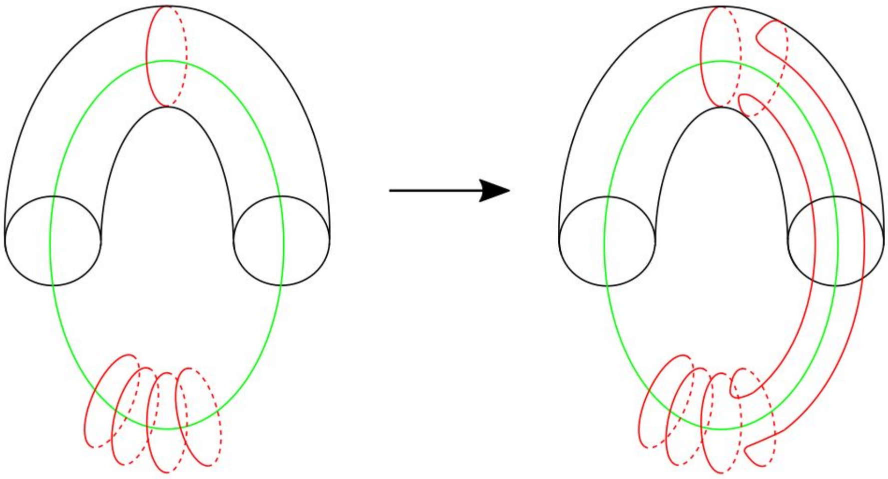

Figure 35: Preparing for the handle cancellations. This process can be repeated for as many red curves as necessary. Caution: even though this simplified image seems to suggest it, "independent" curves never become contractible on the surface through handle slides

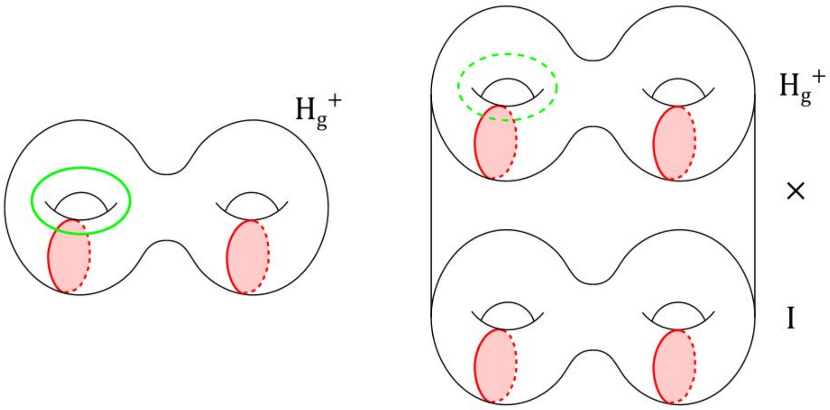

Figure 36: Going from H_g^+ to $H_g^+ \times I$

We are now ready to construct the trisection: we set X_1 = 0-handle ∪ 1-handles, $X_2 = \mathcal{C}(H_g^+) \cup_{\overline{L}}$ (4-dimensional 2-handles), meaning that we take a collar neighbourhood $\mathcal{C}(H_g^+) \cong H_g^+ \times I$ of H_g^+ inside X such that $H_g^+ \times \{0\} \subset \partial X_1$ and attach the 4-dimensional 2-handles of the handle decomposition of X along $\overline{L} \times \{1\}$, and $X_3 = X \setminus X_1 \cup X_2$, so we already know that X_1 is a handlebody and that $X = X_1 \cup X_2 \cup X_3$, see 37. Using all the previous propositions we are now in a position to prove

Theorem 3.3.1. *[Existence of trisections of closed oriented 4-manifolds] $X = X_1 \cup X_2 \cup X_3$ is a trisection.*

Proof. X_3 is a handlebody of genus m because the only handles that have not been used in X_1 and X_3 are the 3-handles and the 4-handle.

The more difficult step is checking that X_2 is a 4-dimensional handlebody. For this notice that since the surface framing equals the framing of L we can think of $H_g^+ \times I \cup_{\overline{L}}$ (4-dimensional 2-handles with the framing coming from the surface framing of Σ_g) as ($H_g^+ \cup_{\overline{L}}$ 3-dimensional 2-handles) $\times I$.

Proposition 3.3.1 tells us that attaching 3-dimensional 2-handles along $\overline{L} \subset \Sigma_g = \partial H_g^+$ leads to these l 2-handles canceling with l 3-dimensional 1-handles of H_g^+. This is because for each $\overline{K}_i \in \overline{L}$, the attaching sphere of the corresponding 3-dimensional 2-handle, intersects the belt sphere of precisely one 3-dimensional 1-handle transversely and exactly once, so l-many of the g 1-handles are cancelled. Cross-

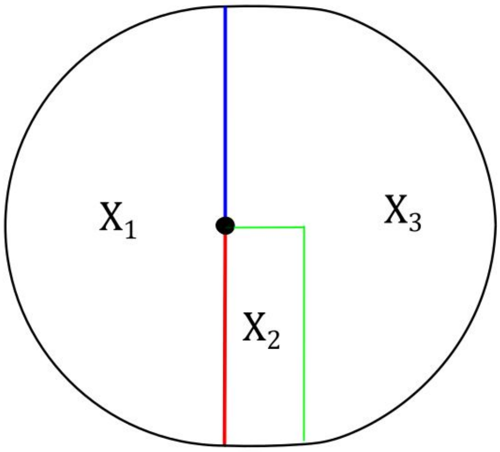

Figure 37: Schematic of trisection

ing the three dimensional handlebody with I we get $H_g^+ \times I \cong H_g^4$ and see that the cocore $D_{\overline{K}}$ of the 3-dimensional 1-handles becomes a compression ball $B_{\overline{K}} = D_{\overline{K}} \times I$ with boundary $\mathbb{S}^2_{\overline{K}}$ that corresponds to the belt sphere of a 4-dimensional 1-handle. We interpret $\overline{L}$ as the attaching link of 4-dimensional 2 handles (with the framing induced by the surface Σ_g), push it into $H_g^+ \times \{1\}$ so that it intersects the compression disk in there transversely and exactly once. Since this compression disk is part of the boundary of the compression ball $B_{\overline{K}}$ and is the only place in which the attaching sphere $\overline{K}$ intersects said belt sphere we again end up with a cancellation, see Figure 36. Thus X_2 is a 4-dimensional handlebody of genus $g-l$.

Next, a quick look at Figure 37 tells us that

- $X_1 \cap X_2 = H_g^+ \cong \natural_g \mathbb{S}^1 \times B^2$
- $X_3 \cap X_1 = H_g^- \cong \natural_g \mathbb{S}^1 \times B^2$

and so the only intersection left to check is $X_2 \cap X_3 = \partial \overline{X_2 \setminus H_g^+}$. Denote by $H_{\overline{L}}$ the 3-dimensional handlebody defined by $\overline{L}$, considered as a union of a 0-handle and l 1-handles, and denote by $H^*_{\overline{L}}$ its dual handlebody, considered as a union of 2-handles and a 3-handle. Let furthermore $g' = g - l$ and note that using this notation $H_g^+ \cup_{\overline{L}} H^*_{\overline{L}}$ is the handlebody obtained by attaching 3-dimensional 2-handles to H_g^+ along $\overline{L}$ and that we may write $X_2 = \left(H_g^+ \cup_{\overline{L}} H^*_{\overline{L}}\right) \times I$ with boundary as depicted in Figure 38.

Since X_2 is glued to X_1 along H_g^+, $X_2 \cap X_3 = H^*_{\overline{L}} \cup \left(H^*_{\overline{L}} \cup H_g^+\right)^*$, which is just a 3-handle with a bunch of 2-handles, so a handlebody. □

3.4 Relating Kirby and trisection diagrams

We have now seen that a handle decomposition of a closed oriented 4-manifold X gives a trisection of X and it is shown in [GK16, Lemma 13] that, conversely, a trisection of X gives a handle decomposition of X. Our aim here is to relate the structures algorithmically on a diagrammatic level, meaning that we want to write a pseudo-code for a program that constructs from a Kirby diagram describing X a trisection diagram describing X and vice versa.

In the following we will sometimes use dotted-circle notation for Kirby diagrams. All Kirby diagrams in this section will be framed using the blackboard framing.

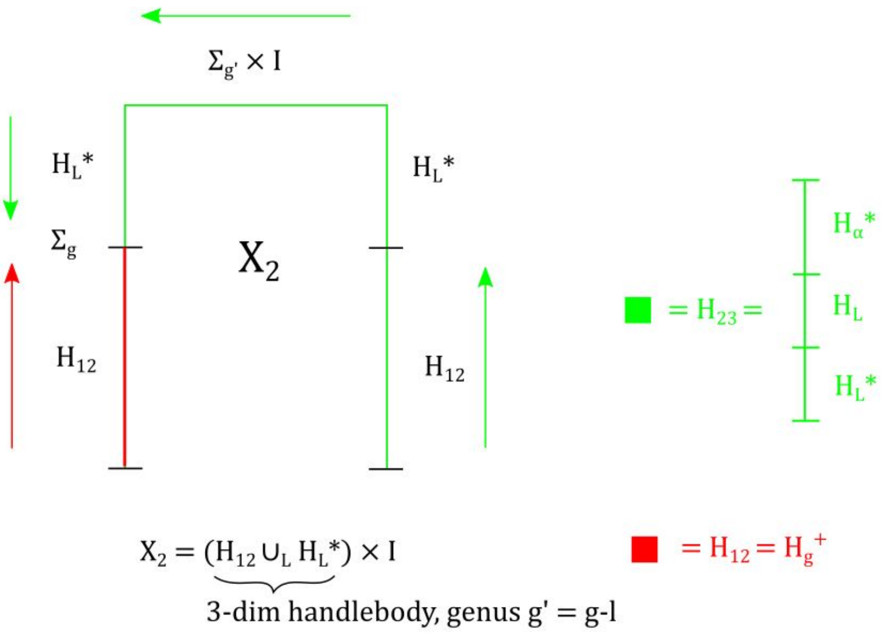

Figure 38: Finding $X_2 \cap X_3 = H_\gamma$

We will start with the easier direction in the Kirby-trisection diagram correspondence: going from a trisection diagram to a Kirby diagram is not too hard. Given a trisection diagram $(\Sigma_g, \alpha, \beta\gamma)$ of a closed oriented 4-manifold X in X_1-standard form (i.e. we look at the standard Heegaard diagram for α and β, the curves describing ∂X_1) we have the g-many $\gamma = (\gamma_1, \dots, \gamma_g)$ curves that in general are in not in standard form, meaning the trisection diagram can in general not be in different X_i-standard positions at once. Together with the framing induced by the surface γ is a framed link. By "lifting off" γ, i.e. replacing every parallel $\alpha - \beta$ pair with a dotted circle and deleting all other $\alpha - \beta$ pairs as well as the surface while remembering the induced framing, we obtain a framed link diagram that can be capped off uniquely to give a 4-manifold X' with genus of X_3 many 3-handles. We will show shortly that a Kirby diagram obtained this way is actually closable. The process of "lifting off" γ can be seen in Figure 40.

Proposition 3.4.1. *X' is a closed oriented 4-manifold. Moreover $X \cong X'$.*

We have thus described a process that takes a trisection diagram for X and produces a Kirby diagram for X.

Proof. Given an abstract trisection diagram $(\Sigma_g, \alpha, \beta, \gamma)$ describing a closed 4-manifold X in X_1-standard form (so α and β are standard while γ is not) we can build up the manifold X in the following way:

We use the α and β curves to build $\natural_{k_1} \mathbb{S}^1 \times \mathbb{S}^2$ and then uniquely glue to this $\sharp_{k_1} \mathbb{S}^1 \times B^3$ which we then call X_1. X_1 can be interpreted as the collection of the 0-handle and 1-handles. Now we attach 4-dimensional 2-handles along γ with the induced surface framing. This is the same as attaching 3-dimensional 2-handles and crossing the result with I. Since the union of Σ and the 3-dimensional 2-handles is $H_{23} \smallsetminus B^3$ the union of $\Sigma \times I$ and the 4-dimensional 2-handles is $\left(H_{23} \smallsetminus B^3\right) \times I$. See the result of this construction in Figure 39. So far we have used only the 0-, the 1-, and the 2-handles so the complement is the union of the 3-handles and the 4-handle. The structure which we have built up so far exactly coincides with the data provided by the Kirby diagram defined by the process outlined above, so this Kirby diagram describes a closed oriented 4-manifold X' that clearly equals X, so we are done. □

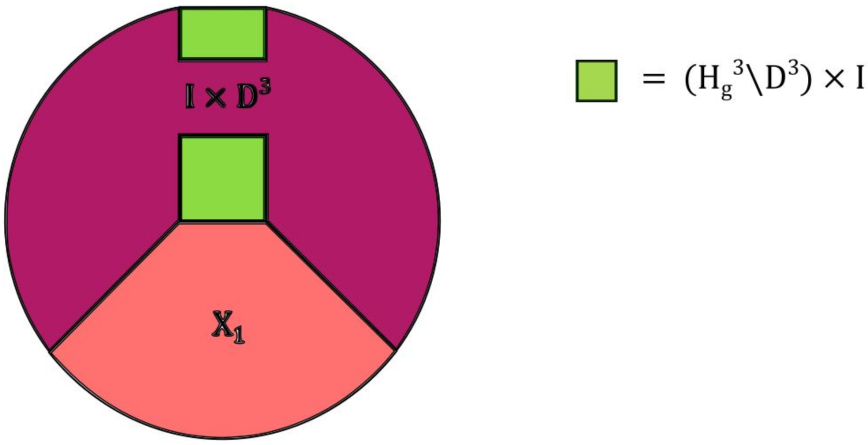

Figure 39: X_1 with attached 4-dimensional 2-handles

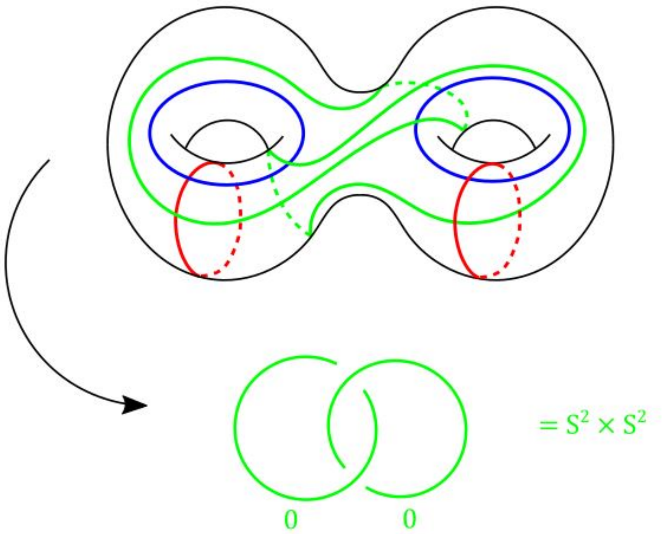

Figure 40: An example of going from a trisection diagram to a Kirby diagram

Remark 3.4.1. *The previous proposition tells us that we can often guess the trisection diagram of a manifold. We just need to find a trisection diagram such that, after applying this process, we end up with the desired Kirby diagram. The difficulty in guessing the correct trisection diagram is checking that the surface plus collection of curves one has guessed really is a trisection diagram. See Figure 41 for examples where it is easy to do so.*

One (hard to compute) invariant of 4-manifolds that is associated with trisections is the notion of the **genus of a 4-manifold**: we define it as the minimal genus of all trisections of a manifold. Keeping this notion in mind, the "guessing-method" we just introduced is enough to prove

Corollary 3.4.1. *[Classification of genus 0 and 1 closed oriented 4-manifolds] The only closed oriented 4-manifold that has genus 0 is the standard $\mathbb{S}^4$. The only closed oriented 4-manifolds that have genus 1 are $\mathbb{S}^1 \times \mathbb{S}^3$, $\mathbb{C}P^2$, and $\overline{\mathbb{C}P^2}$.*

Proof. We know that if X has genus 0, it admits a trisection that has $k_1 = k_2 = k_3 = g = 0$, meaning that the associated Kirby diagram is the empty diagram. Therefore $X = \mathbb{S}^4$ which settles the genus 0 part of the proposition.

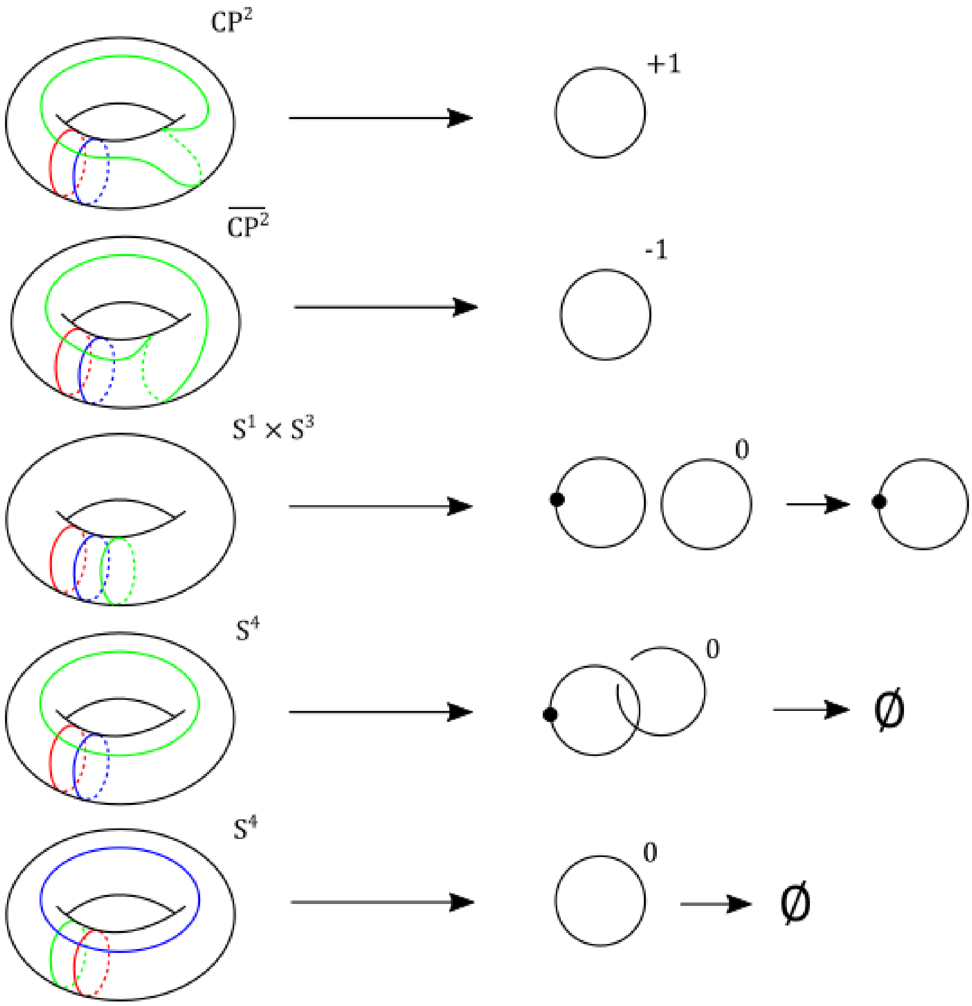

Figure 41: Guessing some trisection diagrams. Connected sums of those are also easily obtained.

If X has genus 1 then there exists a trisection diagram on a surface of genus 1, so wlog let it be in X_1-standard form. We now try all the different possible candidates for a γ-curve under the constraint that we end up with a trisection diagram. This constraint allows us to restrict our attention to γ that, after an isotopy, satisfy that $|\gamma \cap \alpha| \in \{0,1\}$ and $|\gamma \cap \beta| \in \{0,1\}$. We may do this because, in the absence of possible partners for handle slides, we may only use diffeomorphisms to get to X_2- and X_3-standard position, and diffeomorphisms do not change the intersection numbers. Isotopies might change the intersection numbers but we can take the representative of the isotopy class of the γ curve that has minimal intersections with α and β.

Case 1: α and β are a parallel pair: Then either γ intersects both or it intersects neither α and β. If the former is true, meaning that γ has a nontrivial longitudinal component, then we obtain a cancelling $1-2$ handle pair and end up with $X = \mathbb{S}^4$, which does not have genus 1. If the latter is true, meaning that λ has trivial longitudinal component, then we have that α, β, and γ are all parallel to one another. After "lifting off" γ we recognize that $X = \mathbb{S}^1 \times \mathbb{S}^3$.

Case 2: α and β are a cancelling pair: In this case the $\alpha - \beta$ pair will cancel after "lifting off" γ, so we are looking for all the framed unknots that define a closed 4-manifold. The only possible choices are the $0, \pm 1$-framed unknot, giving us $\mathbb{S}^4$, which has genus 0, and $\mathbb{C}P^2$ as well as $\overline{\mathbb{C}P^2}$, respectively.

□

All of the closed oriented 4-manifolds of genus 1 are depicted in Figure 41.

We want to build up to the algorithm described in Theorem 3.4.1. The next proposition tells us how to construct a trisection diagram from a Kirby diagram, but it is not yet algorithmic because it relies on a choice of flow that moves the attaching link of the 2-handles in a Kirby diagram onto a Heegaard surface of X_1. Still once we have proven it we are almost at our goal.
Recall that in the proof of the existence of trisections we just picked some Heegaard surface onto which we flowed the attaching link. The idea is that we construct one concretely and then work from there.

Let us start by describing the construction:
We are given a closable Kirby diagram consisting of k 1-handles and a link L with l knots $K_1, \dots, K_l \subset \mathbb{R}^3$ that describes a closed oriented 4-manifold X'. We construct meridians for all the 1-handles and use paths that connect all of them to an arbitrary point in ∂X_1. This gives a bouquet of circles which we thicken up to a 3-dimensional genus k handlebody. We keep thickening it up until the dotted circles representing the 1-handles lie on the boundary Σ_k of this handlebody, see 42. We claim that the boundary Σ_k of this torus is a Heegaard surface of ∂X_1. The next step is to flow the attaching link L onto Σ_k exactly as in the existence proof. Remember that we need to do some stabilizations in order for this argument to work, the stabilized surface will be called Σ_g. We get the α and β curves by drawing a parallel $\alpha - \beta$ pair wherever a dotted circle is and a cancelling $\alpha - \beta$ pair at every hole that was created by a stabilization. Now do handle slides among the α-curves so that $|\alpha_i \cap \overline{K_j}| = \delta_{ij}$. We get the γ-curves by taking $\overline{L}$ as $\gamma_1, \dots, \gamma_l$ and for $\gamma_{l+1}, \dots, \gamma_g$ we take parallel pushoffs of $\alpha_{l+1}, \dots, \alpha_g$, the α-curves that do not intersect any of the the $\overline{K}_j$. We have thus obtained a collection $(\Sigma_g, \alpha, \beta, \gamma)$ of a genus g surface Σ_g and three sets α, β, γ of g simple closed curves in Σ_g.

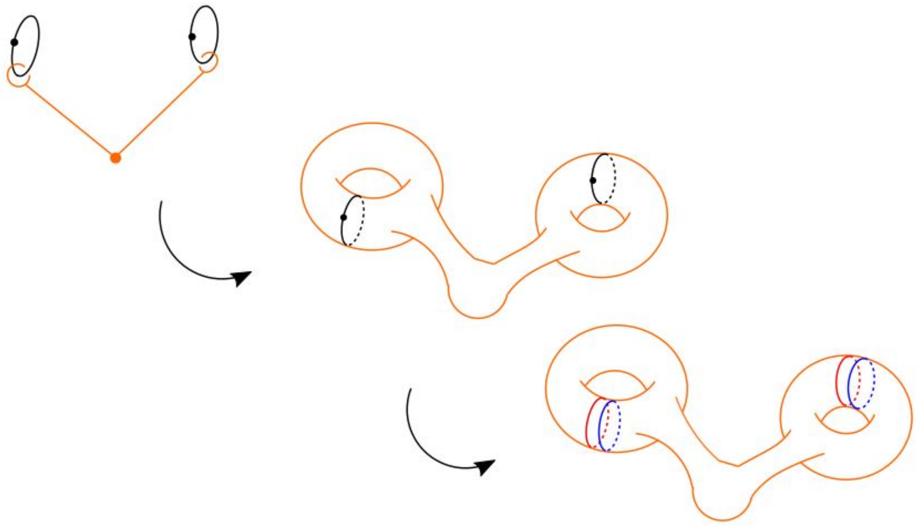

Figure 42: The process of constructing the Heegaard-surface for 2 dotted circles.

Proposition 3.4.2. $(\Sigma_g, \alpha, \beta, \gamma)$ *is a trisection diagram. Moreover this trisection diagram describes the manifold* X.

Proof. The first thing we want to show is that Σ actually is a Heegaard-splitting surface for X_1. The argument can be reduced to a single dotted circle with a ring c around it by looking at just one of the connect summands in ∂X_1. Now thicken up the ring and call the resulting genus 1 handlebody H. The boundary of the 0-handle with a single attached 1-handle is $\mathbb{S}^1 \times \mathbb{S}^2$, with c running along the $\mathbb{S}^1$-factor. The boundary of H, being nothing but $\partial(\mathbb{S}^1 \times B^2) = \mathbb{S}^1 \times \mathbb{S}^1$, separates every S^2 into an inner and an

outer disk, meaning that it bounds a solid torus both inside and outside. The boundary of this torus is thus a Heegaard surface of $\mathbb{S}^1 \times \mathbb{S}^2$.
After flowing the attaching link L of the 2-handles onto Σ_g, with all the steps like introducing and resolving intersections and kinks if necessary, resulting in $\overline{L} \subset \Sigma_g$, the only thing we still have to worry about is finding the γ-curves. Recall that we already analyzed $H_\gamma = X_2 \cap X_3 = H^*_{\overline{L}} \cup \left(H^+_g \cup H^*_{\overline{L}}\right)^*$ briefly in the proof of Theorem 3.3.1. The first handlebody we attach, $H^*_{\overline{L}}$, is precisely represented by the attaching link $\overline{L} \subset \Sigma_g$. Now we do handleslides among the α-curves such that every $\overline{K} \in \overline{L}$ intersects exactly one of the α-curves. To get to the missing γ-curves we now inspect $\left(H^+_g \cup H^*_{\overline{L}}\right)^*$. To avoid getting lost in the following argument, consult figure 38. We recall that $g' = g - l$ and that $\partial X_2 = \sharp_{g'} \mathbb{S}^1 \times \mathbb{S}^2 = \left(H^+_g \cup H^*_{\overline{L}}\right) \cup_{\Sigma_{g'}} \left(H^+_g \cup H^*_{\overline{L}}\right)^*$ is a Heegaard splitting. The genus g' handlebody $\left(H^+_g \cup H^*_{\overline{L}}\right)$ is the handlebody that remains after taking H^+_g and filling in the 2-handles defined by $\overline{L}$. To get ∂X_2 we simply double this handlebody by attaching its dual to its boundary. Diagrammatically this means that we have to attach 2-handles to curves parallel to the remaining α-curves. These are then precisely the missing γ-curves.
Putting these new curves together with the original l curves we now have a complete system of g curves in Σ_g. Since all the curves actually are the attaching link of the spine of a trisection they, together with the surface Σ_g, define a trisection diagram. Furthermore this trisection is a trisection of the manifold X by construction. The sceptical reader is invited to "lift off" γ and see that we retrieve the original Kirby diagram plus some 0-framed unknots which represent cancelling $2-3$ handle pairs. $\square$

A consequence of the equivalence of trisection- and Kirby diagrams is that we can port ideas from one to the other. Note for example that while trisection diagrams are mostly complicated (the only manifolds that have trisection diagrams that are in X_i-standard position for every i are $\natural_k \mathbb{S}^1 \times \mathbb{S}^3$ and $\mathbb{S}^4$), they can be tidied up in the following sense, where we call an element of the γ-curves an unknot if it is lifted to an unknot under the process that takes trisection diagrams to Kirby diagrams:

Corollary 3.4.2. *[Kirby moves in trisection diagrams] Given a trisection diagram $(\Sigma_g, \alpha, \beta, \gamma)$ of a closed oriented 4-manifold in X_1-standard form we can always use handle slides to transform γ into a collection of unlinked zero-framed unknots after a number of connected sums with the trisection diagrams of $\mathbb{C}P^2$ and $\overline{\mathbb{C}P^2}$ and a number of stabilizations.*

Proof. Recall that the moves in Theorem 2.3.1 comprise blowing up and down, which can be interpreted as connect-summing with $\mathbb{C}P^2$ and $\overline{\mathbb{C}P^2}$, as well as handle slides. The plan here is to mirror these moves on the level of trisection diagrams.
Connected sums have a natural counterpart on the level of diagrams as we can just connect sum the diagrams. The handle slides can be translated as well since they are performed using band-connect sums of pushoffs of the knots and we may flow the band onto the trisection surface together with the knots. The ensuing intersections and framing issues can be dealt with by stabilizations of the surface and so we have moved the handle slide from the Kirby diagram level to the trisection diagram level. $\square$

While this shows that Kirby moves can in principle be transported to the setting of trisections diagrams this is awkward to do in practice since one really just does the moves in three dimensions and then projects them to the surface, creating intersection and framing issues which need to be resolved by stabilizations.

Using Proposition 3.4.2 we are in a position to prove the main result of this thesis. The trick of the proof is to regard the blackboard on which the Kirby diagram is defined as (part of) the Heegaard surface which serves two functions: first it means that we do not have to choose any flows and second it means that the notions of blackboard framing and surface framing coincide. So far we only ever argued that the surface framing can be made to coincide with the framing of the attaching link but this was not enough to get an algorithm since the algorithm has to prescribe exactly how many kinks one has to introduce, not just that there is a number of kinks that makes the framings coincide.

Theorem 3.4.1. *Given a Kirby diagram describing a closed oriented* 4*-manifold* X *we obtain a flat trisection diagram describing* X *by going through the following steps (many of which are illustrated in Figure 44):*

1. *Replace the projection of the attaching regions of the* 4*-dimensional* 1*-handles by attaching regions of* 3*-dimensional* 1*-handles with a parallel* $\alpha - \beta$ *curve pair around one of the disks.*

2. *If a* 2*-handle does not have an overcrossing, introduce first a* $+1$*- then a* -1*-kink, see Figure 43. The order or place where the kinks are put does not matter.*

3. *Introduce* ± 1 *kinks matching the sign and the number specifying the framing.*

4. *Replace intersections (including the ones introduced in step* 1*) in the fashion shown in Figure 44.*

The process so far ensures that every knot K *in the* 2*-handle attaching link has an* α*-curve that it intersects once and transversely and that no other knot intersects. It might still be the case that* K *intersects more than just one* α*-curve which we want to avoid and so the next step is:*

5. *Do handle slides among the* α*-curves such that* $|\alpha_i \cap K_j| = \delta_{ij}$ *for* $i = 1, \ldots, g$ *and* $j = 1, \ldots, l$*, where the* K_j *are the knots constituting the link* L *in the Kirby diagram, of which there are* l *many, and* g *is the number of pairs of attaching regions of* 3*-dimensional* 1*-handles. This process works as follows: Recall that, because of step* 2*, every* K_j *has an overcrossing. In particular every* K_1 *therefore has some* α*-curve that it intersects only once, if there are several pick one and call it* α_1*. Every other* α*-curve that* K_1 *intersects can now be slid over* α_1 *along the segment of* K_1 *that connects them. We have thus achieved that* $|\alpha_i \cap K_1| = \delta_{i1}$*. Repeat this process for all* j *and note that doing this process for other knots does not destroy this intersection property for the knots that came before it.*

6. *Color* L *green.*

7. *Draw a green curve parallel to every unpaired* α*-curve, i.e. all the* α*-curves that do not intersect one of the* K_j *(precisely* $\alpha_{l+1}, \ldots, \alpha_g$*), and call all of the green curves* γ*.*

Proof. Remember that we create a Kirby diagram from the collection of handle attachments by appropriately projecting onto the coordinate plane $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3 \subset \mathbb{S}^3$. If we can show that the resulting Kirby diagram can be interpreted as laying on a Heegaard surface then we are done since all the steps we do afterwards just precisely mirror those of 3.4.2. This actually is not hard to see since we can pick 3-dimensional 0-handle whose boundary is $\mathbb{R}^2 \cup \{\infty\} \cong \mathbb{S}^2$ with $\{\infty\}$ being the point $\mathbb{S}^3 \setminus \mathbb{R}^3$. Now we use the images of the one-handle in the Kirby diagram to attach 3-dimensional 1-handles that are attached to the 3-dimensional 0-handle at said images of 1-handles and run through the 1-handles of the Kirby diagram. By an argument precisely as in 3.4.2, this gives a Heegaard-surface and the projection can be used to define a flow onto it.
The only reason why step 2 introduces 2 self intersections is to ensure that the framing does not change which it would if we only introduced one self intersection. ☐

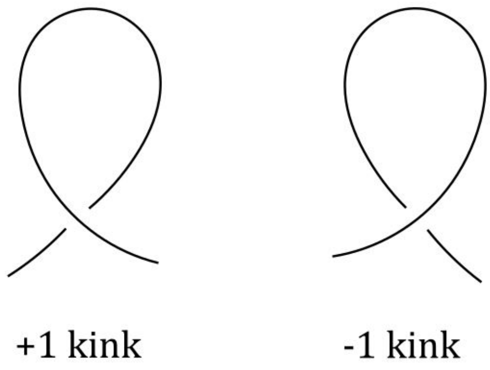

Figure 43: ±1-kinks

3.5 Some concrete examples and simplifications of the algorithm in special cases

We apply the algorithm of theorem 3.4.1 to a simple example, namely a cancelling $1-2$ pair, so a Kirby diagram describing $\mathbb{S}^4$, see Figure 28. The process is carried out step by step in Figure 45.

At this point one is forced to recognize that the algorithm is far from efficient as the trisection diagram the algorithm produced in the example is much more complicated than it needs to be. In special cases some simplifications can be made.

The first one stems from the observation that in the existence proof of trisection, when we took care for the surface framing of the attaching link to coincide with the initial framing, we were wasteful in the sense that we introduced more stabilizations than necessary: if the framing was r then we introduced r kinks leading to r stabilizations. The following lemma tells us that we do not need to introduce any new stabilizations in order to take care of the framing of $\overline{K}$ because we can just use one of the bridges that $\overline{K}$ goes over and that no other knot goes over (remember that by construction there always is at least one such bridge for every knot $\overline{K} \subset \overline{L}$) and wind it around this bridge as many times as the framing of K requires.

Lemma 3.5.1. *[economic framing] Let Y be an oriented closed 3-manifold with Heegaard surface Σ_g, let furthermore $c \subset \Sigma_g$ be a simple closed curve that does not bound a disk in Σ_g (i.e. goes around at least one hole) and has framing m with respect to the surface Σ_g. Then the surface framing can be changed to $m+1$ or $m-1$ simply by increasing the winding number of c around an arm of the Σ_g by $+1$ or -1, as shown in figure 46.*

Proof. As was the case with the change in framing by the introduction and resolution of kinks in the previous section one simply has to notice that the vectorfield determined by the surface, when interpreted as framing, will have a winding number higher (or lower) by one when compared to the original one. □

It is easy to carry this idea over to flat trisection diagrams as one just has to introduce the convention that a number next to a bridge refers to the number of times the knot twists around said bridge. Alternatively one can just associate the number to the curve itself since every curve has at least one bridge it can use to wind around this number of times.

Note that this massively simplifies the trisection diagram produced in Figure 45, especially when we combine it with the realization that we can now replace step 2 of the algorithm by doing a +1-kink

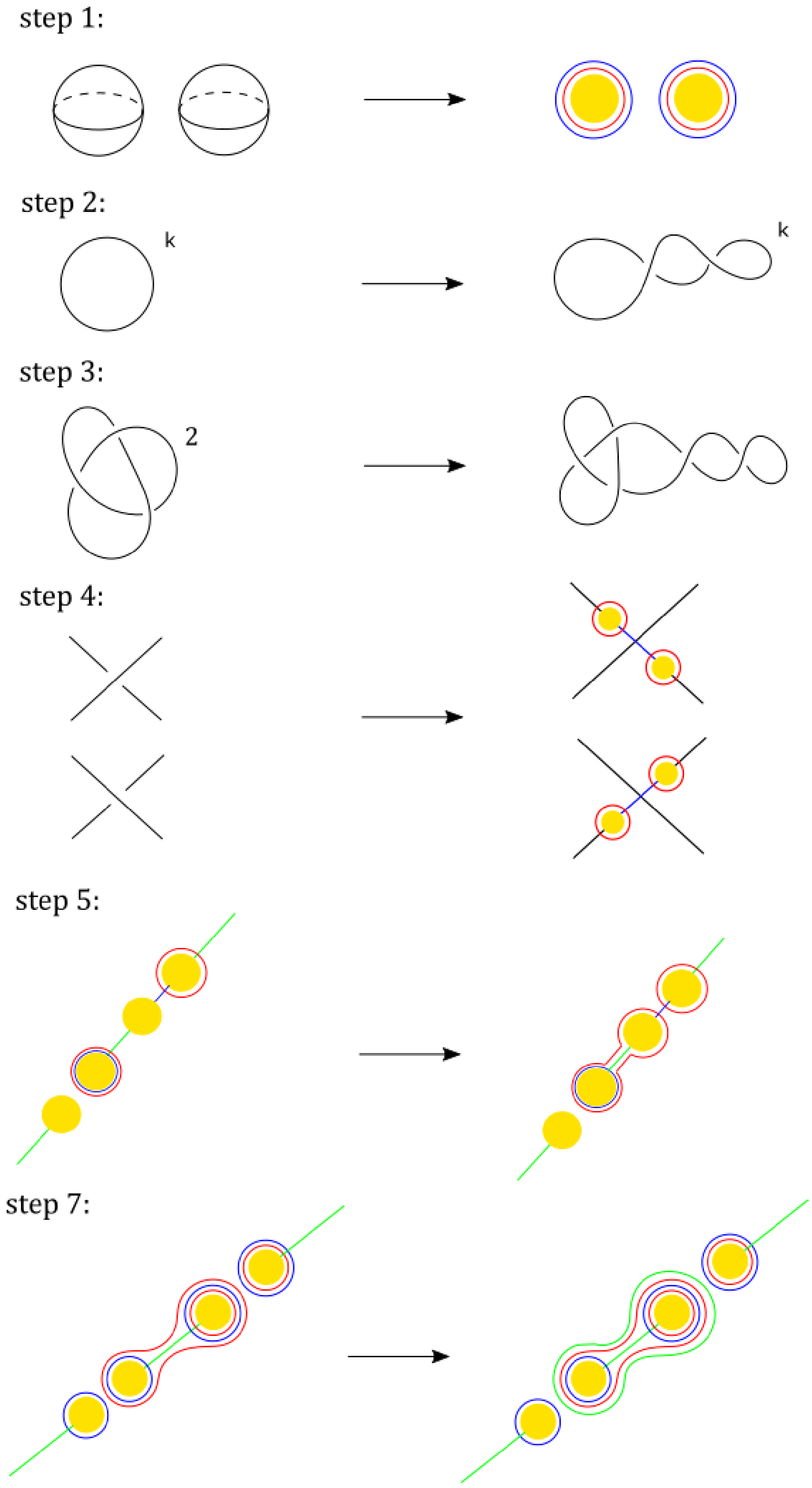

Figure 44: Illustrating some of the steps described in Theorem 3.4.1

but writing a -1 over the introduced intersection to have no net change in framing. Taking these two simplifications together means that the example in Figure 45 produces a genus 2 surface instead of a genus 5 surface, see Figure 47.

We introduce the notion of the **intersection number** of a link L as the minimum of the number of

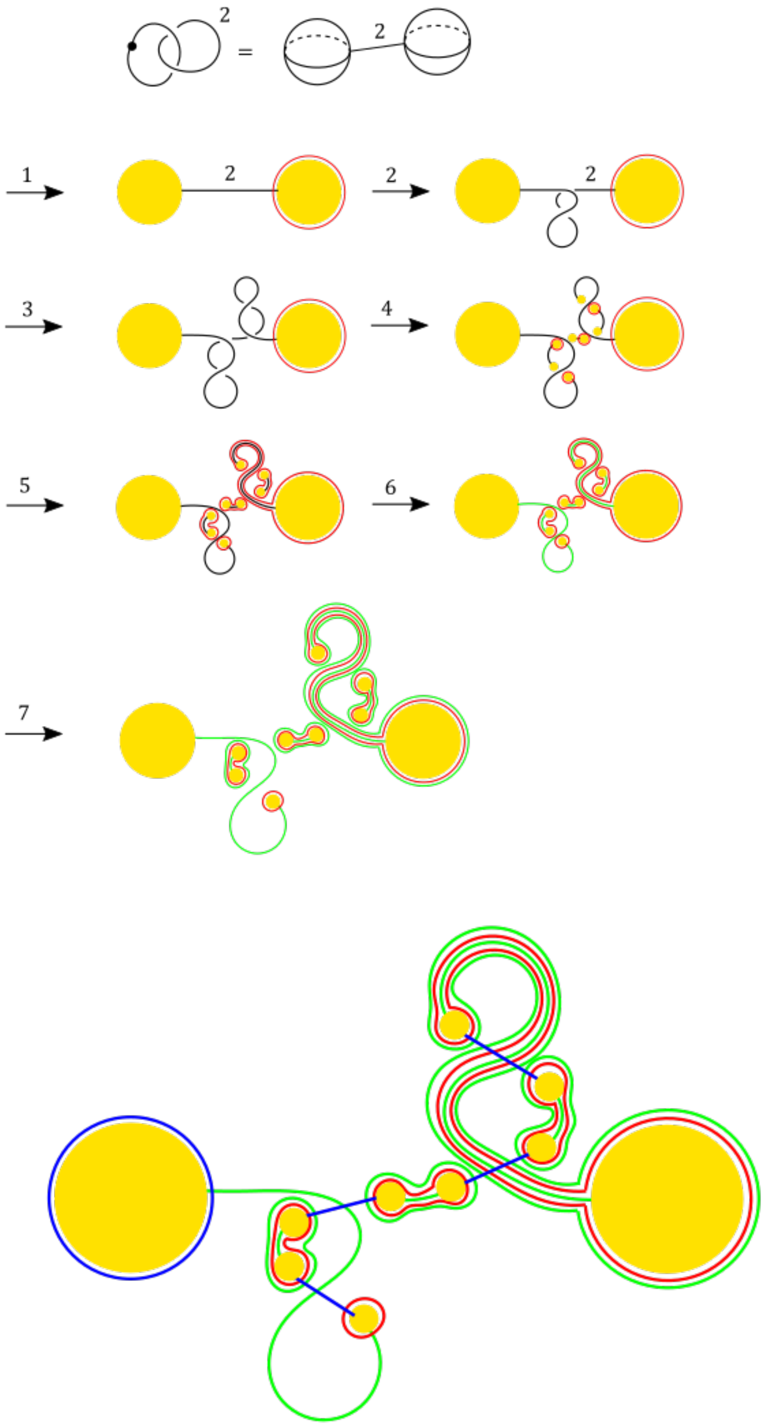

Figure 45: Applying the algorithm to a simple example. The blue curves were suppressed in the steps (since they are all standard anyways and do not feature in the algorithm) and were only added in the end.

intersections over all link diagrams of L. This definition can easily be generalized for link diagrams on a plane in which disks are identified with one another. Using this we prove

Corollary 3.5.1. *[Estimate from above for genus of* 4*-manifolds] Let* X *be a closed oriented* 4*-manifold.*

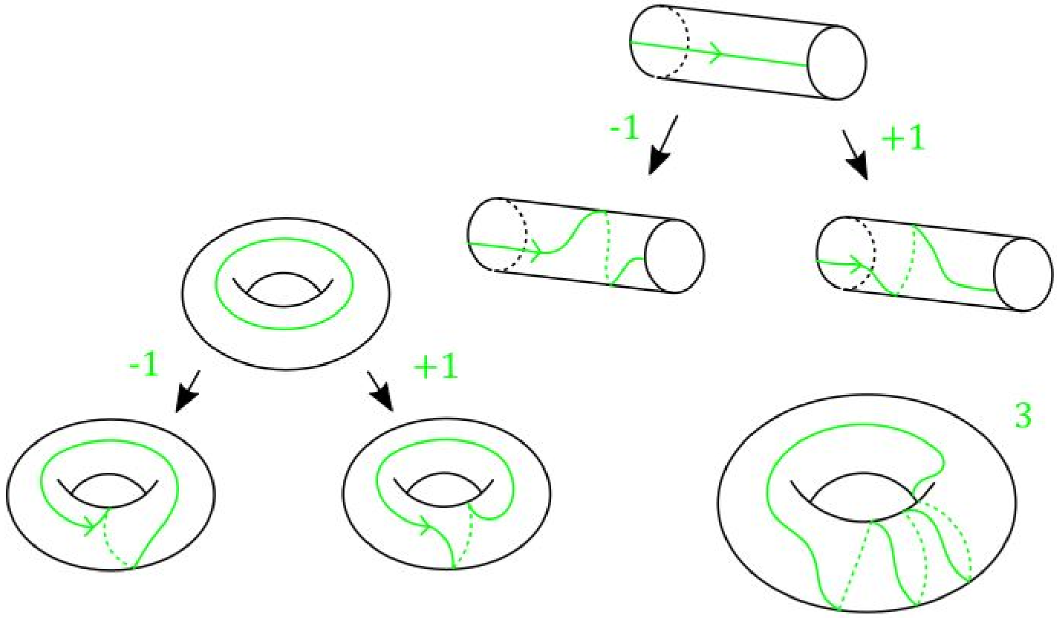

Figure 46: Changing the framing by coiling up curve

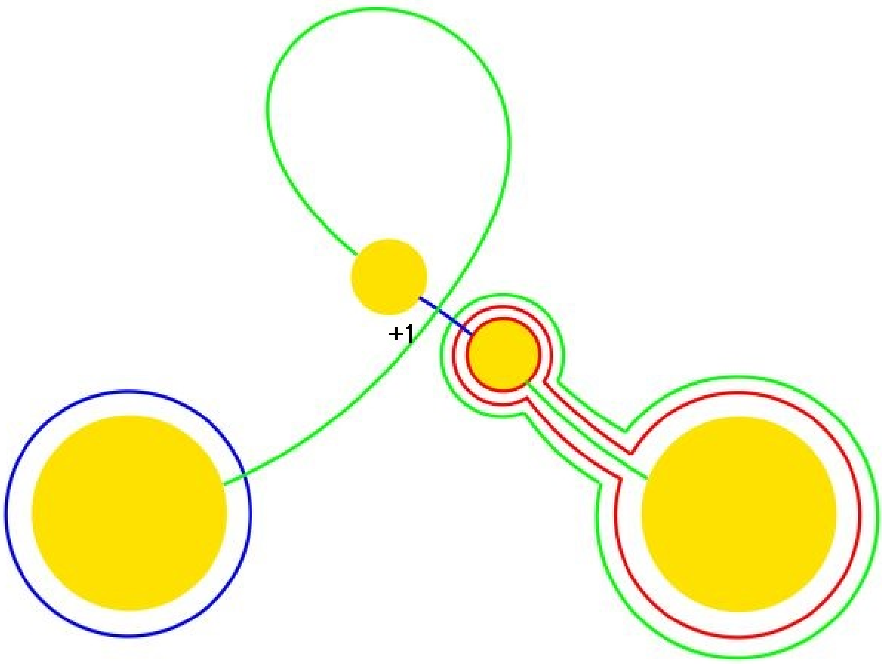

Figure 47: The result of applying the simplified algorithm to the same Kirby diagram as in Figure 45.

Given a Kirby diagram describing X we obtain an estimate from above for the genus of X, namely $g(X) \leq k + s + l$, where k is the number of 1*-handles in the diagram, s is the intersection number of the attaching link, and l is the number of knots in the attaching link.*

Proof. We start with a link diagram (on a plane that has pairs of identified disks) that has minimal intersection number. We then embed the plane on which this diagram lies into ∂X_1 and construct from this a Heegaard splitting of ∂X_1 much like in the proof of 3.4.1. Now we are in a position to apply the improved algorithm to get a trisection diagram. The genus of the diagram will be the number of

1-handles k plus the number of intersections s. The algorithm introduces a bunch of overcrossings that we need to account for and the worst case is that this happens for every knot in the link, so we add the number of knots to the estimate. □

The next simplification is one that is particularly useful for plumbings: In the existence proof, when we tried to avoid intersections of knots using stabilizations, we only ever considered pairs of linked knots but in general more than just two knots will be linked. In these cases it will often happen that two knots can actually share the same bridge, meaning that we have to stabilize less often, resulting in a far easier trisection diagram. See Figure 48 for clarification. Note that this cannot always be done since every bridge allows only one framing to be imparted on the knots that run over it as we do not want the knots to intersect. In the next few paragraphs we assume that the framing-numerology works out, otherwise this particular simplification cannot be done. The restriction just described is anyways only a small one that fixes the framing of very few knots in plumbings as one can use the fact that almost all knots intersect with at least 2 other knots and so one can push the unwanted framing to the leaves of the graph.

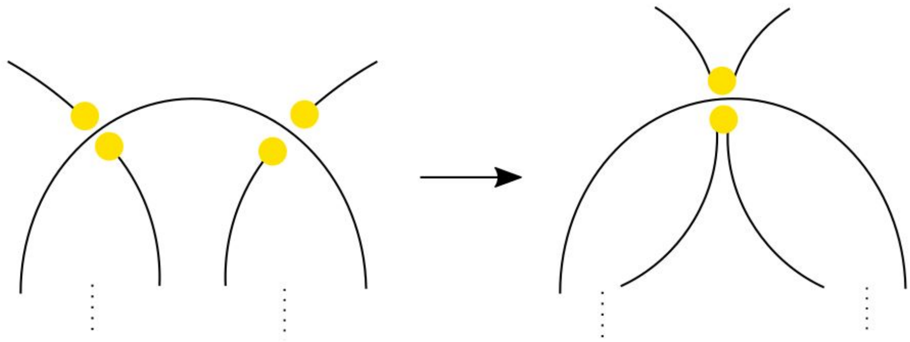

Figure 48: Allowing knots to share the same stabilization to reduce the genus of the resulting trisection.

We want to apply these two simplifications by outlining what they do for plumbings. When reading through the following paragraphs one has to keep in mind that our algorithm only works for Kirby diagrams that define closed oriented 4-manifolds, so if the plumbing cannot be closed then one cannot use it though it can still be used for the double of the plumbing graph which is a plumbing graph as well. We therefore still cover a large class of closed 4-manifolds this way.

First consider a linear plumbing. We can arrange for every nth building block to share a bridge with the $n+2$nd one simply by turning every element starting with the 2nd, then turning every element starting with the 3rd and so on until we arrive at the last one. The process is shown for a linear plumbing with $\mathbb{S}^2$s in Figure 49. This procedure reduces the number of stabilizations one needs for a linear plumbing with n $\mathbb{S}^2$s from 2^{n-1} to n and it works similarly for plumbings of disk bundles over other surfaces.

To go from linear plumbings to plumbing trees one needs to deal with junctions where more than two knots interlink with a central knot. There is a nice symmetric way to do this by using a new type of structural element in flat trisection diagrams which can be seen in Figure 50. In this way we can reduce the number of stabilizations for a junction with $n-1$ outgoing edges (involving n knots counting the central one) from 2^{n-1} to $(n-1)+1 = n$. Note that the "turning" process can easily be extend over junctions.

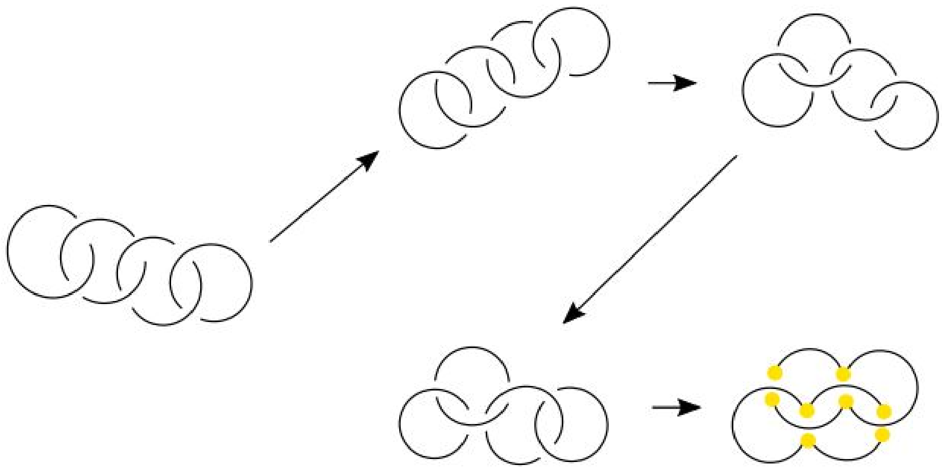

Figure 49: Repeated flipping arranges linear plumbings in the desired form

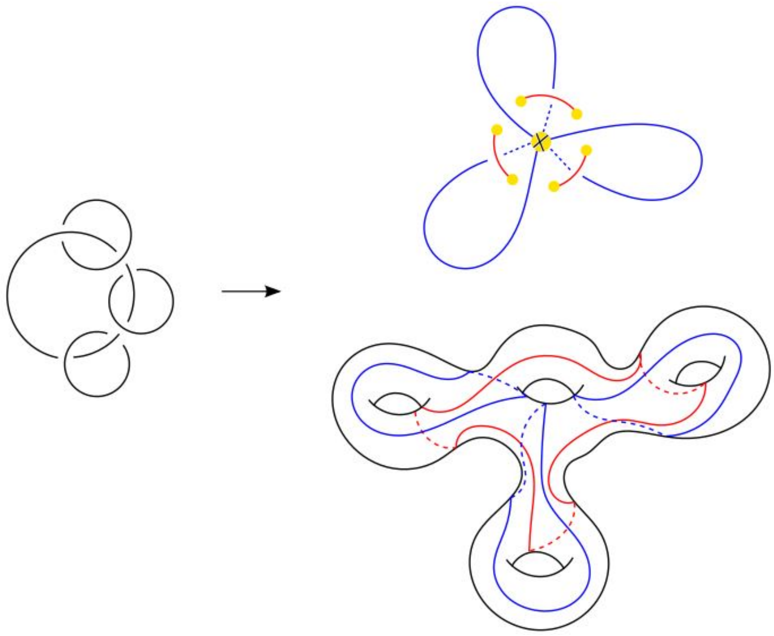

Figure 50: Introducing 3-junctions. The new symbol means that the second end is somehow "on the other side" of the diagram (it lives on an $\mathbb{S}^2$) and will not be drawn. This diagram generalizes in an obvious way to k-junctions.

For general plumbings (without self plumbings) one has to deal with loops introduced in the plumbing graph. The only difference to the case of plumbings that correspond to trees is that it is in general no longer possible to consistently "turn over" all the rings so that the nth ring can share a bridge with the $n+2$nd ring, because in the first place we can no longer impose a well defined order if the graph is not a tree.

This simplification scheme clearly works for all plumbings (that do not involve self plumbings and that respect the framing condition), we just have to replace the individual building blocks by the ones corresponding to the desired genus g surface.

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