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„Splitting Theorems in Riemannian and Lorentzian Geometry“

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Preface

This thesis deals with the splitting theorems which appear in Riemannian and Lorentzian geometry. Upon collecting some background material in Chapter 0, we turn to the Riemannian Splitting Theorem in Chapter 1. Busemann functions and their properties are the main ingredient for the proof. In Chapter 2, we discuss the Lorentzian Splitting Theorem in all of its variants, and then go on to present the proof of a slightly weaker version of the theorem.
The prerequisites for reading this thesis are a solid understanding of modern differential geometry. In particular, some advanced aspects of Riemannian and Lorentzian geometry (comparison theory, causality theory, etc.) will come into play. These requirements are fully met by a thorough study of the texts [15], [16] and [19]. For comparison theory in particular, [20] is very useful. Most standard results from these sources will be used without reference.
I hope to have written a convenient and easy-to-follow entry point/reference to/on splitting theorems that can be used to quickly get a grasp of the goings-on of this subject.

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Chapter 0

Preliminaries

In this preliminary chapter, we first fix the notation we will use throughout the thesis and then go on to discuss various topics from Riemannian and Lorentzian geometry that will play a role in the proofs of the splitting theorems.

0.1 Notation and Conventions

In this section, we fix the notation and the conventions that will be used throughout the thesis. For the most part, we will be in adherence to standard modern differential geometry in the spirit of [15], [16] and [19]. Topological notions are in line with [14]. Only the more nonstandard conventions and/or those differing from the mentioned standard sources will be addressed.

A *manifold* will always be a 2nd countable Hausdorff space that is locally Euclidean and that possesses a smooth ($= C^\infty$) structure. More often than not in this thesis (and generally in most global studies of Riemannian and Lorentzian manifolds), connectedness will play no role and will be assumed even when not mentioned explicitly.
Any appearing map (provided it is possible) will be assumed smooth (or piecewise smooth in the context of curves) unless otherwise specified. Given a smooth map $f : M \to N$ between manifolds, the tangent map at $p \in M$ will be denoted $T_p f$.
Submanifold without further specifiers will always mean embedded submanifold (this will also be the case for more specialized notions, e.g. hypersurface = embedded hypersurface). A subtler differentiation between immersed and embedded submanifolds will play no role for us.
For a vector field $X \in \mathfrak{X}(M)$, Fl^X will denote its flow map, and $\mathrm{Fl}^X_t(p) := \mathrm{Fl}^X(t,p)$, where t is a parameter and $p \in M$.
Given a semi-Riemannian manifold (M, g) and a smooth function $f : M \to \mathbb{R}$, $\operatorname{grad} f \in \mathfrak{X}(M)$ will denote the vector field corresponding to $df \in \Omega^1(M)$

by the g-musical isomorphisms. If ∇ denotes the Levi-Civita connection on (M,g) (which it always will throughout the thesis), then $df = \nabla f$.
For notions concerning semi-Riemannian submanifolds of a semi-Riemannian manifold, we will stick to the notation in [19]. On the other hand, we will adopt the conventions in [16] concerning curvature quantities, e.g. $R_{XY}Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ for the Riemannian curvature tensor, which would be $-R_{XY}Z$ in the convention of [19].
A *Lorentzian metric* will always be of signature $(-1,+1,\dots,+1)$. A *spacetime* is a connected Lorentzian manifold that possesses a time orientation, i.e. a timelike vector field $X \in \mathfrak{X}(M)$. X determines forward and backward lightcones as follows: If $v \in T_pM$ is timelike, then v is (by definition) future-directed if and only if $g(v, X_p) < 0$. The causality theory and notation we will employ is in accordance with [19].
We shall not use the box notation for the d'Alembertian (= Lorentzian Laplacian) of a Lorentzian manifold (M,g), but write Δ instead and call it Laplacian.
If $f : M \to \mathbb{R}$ is a smooth function, then $\mathrm{Hess}(f) \in \mathcal{T}^0_2(M)$ (the space of 2-covariant tensors on M; other tensor spaces will be denoted similarly) will always denote the 2-*covariant Hessian* of f, and will be referred to as the Hessian of f.
A *convex set (or convex neighborhood)* in a semi-Riemannian manifold M is an open set that is a normal neighborhood of each of its points, i.e. given any point p in it, it is diffeomorphic via $\exp_p$ to some star-shaped neighborhood of 0 in T_pM.
The notation $K \subset\subset M$ means that K is compact and contained in M.

0.2 Calabi's Maximum Principle

Calabi's Maximum Principle is a generalization of the usual maximum principle to continuous, sub- resp. superharmonic functions (in a generalized sense) on Riemannian manifolds. It will play a fundamental role in the last stages of the proof of the Riemannian Splitting theorem, cf. 1.3.5. Throughout this section, we follow [20], Sec. 7.1.3.

Remark 0.2.1. (Support functions)
Let $f : M \to \mathbb{R}$ be a continuous function, where M is a semi-Riemannian manifold. Let $p \in M$. We say that a (smooth resp. continuous) function g defined in a neighborhood of p is a *smooth resp. continuous upper support function* for f at p if g is continuous resp. smooth, $g(p) = f(p)$ and $g \geq f$ in that neighborhood of p. *Lower support functions* are defined in the obvious way.
Let now M be a Riemannian manifold. Support functions are useful because they allow us to define inequalities for differential operators of not necessarily differentiable functions. For example, for a continuous function f as above

we say that $\operatorname{Hess} f|_p \geq 0$ if for every $\varepsilon > 0$ there exists a smooth lower support function f_ε for f at p such that $\operatorname{Hess} f_\varepsilon|_p \geq -\varepsilon g_p$, where g is the metric. Similar inequalities can also be defined for the Laplacian Δf. We say that these statements hold *in the support sense*.
The key observation that guarantees consistency in these definitions is the following lemma.

Lemma 0.2.2. (Motivation for support functions)
Let $f, g : M \to \mathbb{R}$ be C^2 functions on a Riemannian manifold M with $f(p) = g(p)$ and $f \geq g$ near p. Then:

$$\begin{aligned}
&\operatorname{grad} f(p) = \operatorname{grad} g(p),\\
&\operatorname{Hess} f|_p \geq \operatorname{Hess} g|_p,\\
&\Delta f(p) \geq \Delta g(p).
\end{aligned}$$

Proof. If $(M, g) \subset (\mathbb{R}, dt^2)$, then this is a result from elementary calculus. In the general case, consider a smooth curve $c : (-\varepsilon, \varepsilon) \to M$ with $c(0) = p$. Applying the elementary result to $f \circ c$ and $g \circ c$, we get

$$\begin{aligned}
&df_p(c'(0)) = dg_p(c'(0)),\\
&\operatorname{Hess} f|_p(c'(0), c'(0)) \geq \operatorname{Hess} g|_p(c'(0), c'(0)).
\end{aligned}$$

Since T_pM is exhausted by such $c'(0)$, we have shown the first two claims. The inequality on the Laplacians follows trivially, since the Laplacian is the trace of the Hessian. □

Definition 0.2.3. (Sub- and superharmonic functions)
A continuous function $f : M \to \mathbb{R}$ on a Riemannian manifold is called *subharmonic (in the support sense)* if $\Delta f \geq 0$ everywhere on M in the support sense. *Superharmonicity (in the support sense)* is defined analogously. f is called *harmonic (in the support sense)* if it is both sub- and superharmonic (in the support sense).

We are now ready to state and prove Calabi's generalized maximum principle. In the proof, we take some inspiration from [5], Lemma on p. 150.

Theorem 0.2.4. (Calabi's Maximum Principle)
Let (M, g) be a connected Riemannian manifold and $f : M \to \mathbb{R}$ a continuous function that is subharmonic in the support sense. Then f is constant near every local maximum. In particular, if f has a global maximum, then f is constant on M.

Proof. Suppose first that $\Delta f > 0$ everywhere. Then we claim that f cannot have any local maxima. Because if $p \in M$ were a local maximum of f, then, since $\Delta f(p) > 0$ in the support sense, we may find smooth lower support functions f_ε for f at p such that

$$\Delta f_\varepsilon(p) > 0$$

for $\varepsilon > 0$ sufficiently small. But by definition of lower support functions, $f_\varepsilon(p) = f(p)$ and $f_\varepsilon(q) \leq f(q)$ for all q near p, which implies that p is also a local maximum of f_ε. But then $\operatorname{Hess} f_\varepsilon|_p \leq 0$, which implies $\Delta f_\varepsilon(p) \leq 0$, giving a contradiction.
Now, suppose that $\Delta f \geq 0$ and let $p \in M$ be a local maximum of f. Then there is a regular coordinate ball B around p (i.e. B is a coordinate ball such that $\overline{B}$ is compactly contained in a bigger coordinate ball) such that $f|_B$ attains a global maximum on B at p. If f is constant on B, we are finished. Otherwise we may assume $\partial B \neq V$, where

$$V := \{x \in \partial B \mid f(x) = f(p)\}.$$

We want to construct a smooth function $h := e^{a\phi} - 1$ such that $h < 0$ on V, $h(p) = 0$, and $\Delta h > 0$ on $\overline{B}$. This may be done as follows: Choose an open disc $U \subset \partial B$ containing V. Then, choose ϕ such that $\phi(p) = 0$, $\phi < 0$ on U and $\operatorname{grad} \phi \neq 0$ on $\overline{B}$ (e.g. by choosing $\phi = x^1$ in a coordinate system centered at p where U is mapped to the lower half plane $x^1 < 0$).
Finally, choose $a \in \mathbb{R}$ so large that

$$\Delta h = a e^{a\phi}(a|\operatorname{grad} \phi|^2 + \Delta \phi) > 0$$

on $\overline{B}$ (this can be done since $\overline{B}$ is compact). Consider now $\tilde{f} := f + \delta h$ for small $\delta > 0$, then

$$\tilde{f}(p) = f(p) > \max\{\tilde{f}(x) \mid x \in \partial B\}$$

by choice of p and since $h < 0$ on V. Thus, $\tilde{f}$ achieves a local maximum inside B. Using the first part of the proof, we shall achieve a contradiction by showing that $\Delta \tilde{f} > 0$ everywhere (in the support sense).
To this end, let $q \in B$ and let f_ε be a smooth lower support function for f at q, then $f_\varepsilon + \delta h$ is a smooth lower support function for $\tilde{f}$ at q. Furthermore,

$$\Delta(f_\varepsilon + \delta h)(q) \geq -\varepsilon + \delta \Delta h(q),$$

hence using the fact that $\Delta h > 0$ on $\overline{B}$, we get $\Delta \tilde{f}(q) \geq \delta \Delta h(q) > 0$. Since q was arbitrary, we get the desired contradiction.
For the last claim, note that if f has a global maximum, it follows from above that the set of points where it attains that maximum is open and closed, hence it equals all of M by connectedness. $\square$

As a corollary, we get the following result guaranteeing the smoothness of harmonic functions, which is well known in the case of $\mathbb{R}^n$.

Corollary 0.2.5. (Regularity of harmonic functions)
Let M be a connected Riemannian manifold. If $f : M \to \mathbb{R}$ is continuous and harmonic in the support sense, then f is smooth.

Proof. Let $p \in M$ and let Ω be a coordinate neighborhood of p, with compact closure and smooth boundary, such that Ω is contained in another coordinate neighborhood (e.g. take a coordinate ball). From PDE theory we know that the Dirichlet boundary value problem

$$\Delta u = 0, \quad u|_{\partial\Omega} = f|_{\Omega}$$

has a solution u that is smooth in the interior of Ω.
Consider now the functions $f-u, u-f$ on Ω. If $f = u$, we are done. If $f \neq u$, then one of these functions is positive somewhere in Ω. But this function vanishes on $\partial\Omega$, hence by continuity it must have a global maximum in the interior of Ω. By 0.2.4, it is constant in Ω. Using again that f and u agree on $\partial\Omega$, they must thus agree on Ω, i.e. $f = u$ on Ω and f is smooth near p. Since $p \in M$ was arbitrary, we are done. $\square$

0.3 Limits of Curves in Lorentzian Geometry

There are many notions of limits of curves for spacetimes (quasilimits, limit curves, C^0-limits, etc.). In this section, we will treat limits of curves such that the convergence is uniform on compact sets with respect to some fixed complete Riemannian metric. This is the only case relevant for us in the proof of the Lorentzian Splitting Theorem in Chapter 2.
Throughout this section, let (M, g) be a globally hyperbolic spacetime. Let h be a complete Riemannian metric on M. All geometric objects associated with h (distance, arc-length, etc.) will be indexed by h. Anything unindexed pertains to the Lorentzian metric g. The developments of this section closely follow [21], Sec. 3.2.3. We refer to [17] for a thorough discussion of the various notions of limits of curves in a much more general setting.

Definition 0.3.1. (h-uniform convergence)
Let $I_n \subset \mathbb{R}$ be intervals (of any type) and $\gamma_n : I_n \to M$ a sequence of (not necessarily continuous) curves.

1. The curves γ_n *converge h-uniformly* to a curve $\gamma : I \to M$ (I again an interval) if $I_n \to I$ and

   $$\sup_{t \in I_n \cap I} d_h(\gamma_n(t), \gamma(t)) \to 0,$$

   where the notation $I_n \to I$ means that the boundary points of I_n converge to the boundary points of I (irrespective of their containment in I_n resp. I).

2. We say that the γ_n *converge h-uniformly on compact sets* if for every compact interval $J \subset I$, there are intervals $J_n \subset I_n$ such that $\gamma_n|_{J_n} \to \gamma|_J$ h-uniformly.

Very often (but not always), the intervals appearing will be equal (i.e. $I_n = I$). Apparently, 0.3.1 depends on the parametrization of the curves. This shall pose no problem for us, as we will mostly consider causal curves with h-arc length parametrization (this can always be achieved; cf. [21], Appendix B).
Since h-uniform limits of a sequence of smooth curves will in general not be smooth, let us recall: A continuous curve is called *future-directed causal* if for any convex set and any part of the curve lying in that set, any two points on that part are causally related (the direction being given by the parameter values). *Past-directed* and other similar notions are defined analogously.

The following result demonstrates that the notion of h-uniform convergence is rather powerful, especially in the context of causal curves.

Proposition 0.3.2. (Properties of h-uniform convergence)
Let $\gamma_n : I_n \to M$ be continuous future-directed causal curves parametrized by h-arc length that converge h-uniformly to a curve $\gamma : I \to M$. Then:

1. γ is a continuous, future-directed causal curve.

2. For $t_n \in I_n$ with $t_n \to t \in I$, it holds that $\gamma_n(t_n) \to \gamma(t)$.

Proof. Item (2) is immediate from h-uniform convergence, as is continuity of γ.
Let now $t_0 \in I$ be fixed and take a convex neighborhood U of $\gamma(t_0)$ such that $\gamma([t_0, t_1]) \subset U$ for some $t_1 > t_0$. Choose now $t_{1,n}, t_{0,n} \in I_n$ such that $t_{i,n} \to t_i$. Then $t_{0,n} < t_{1,n}$ for large n and, by h-uniform convergence, $\gamma_n([t_{0,n}, t_{1,n}]) \subset U$. Since the γ_n are future directed causal, we have $\gamma_n(t_{0,n}) \leq_U \gamma_n(t_{1,n})$, hence $\gamma(t_0) \leq_U \gamma(t_1)$ by closedness of $\leq_U$. To finish this proof, we note that it is impossible that $\gamma(t_0) = \gamma(t_1)$ by the proof of [17], Lem. 2.7. $\square$

We turn now to the existence of h-uniform limits. An important tool that will be employed is the general topological version of the Arzela-Ascoli theorem (cf. [13], Thm. 12.3.6).

Proposition 0.3.3. (Limit Curve Lemma)
Let $\gamma_n : \mathbb{R} \to M$ be continuous future directed causal curves parametrized by h-arc length. Let $p \in M$ be an accumulation point of $\{\gamma_n(0)\}$. Then there exists a continuous future directed causal curve $\gamma : \mathbb{R} \to M$ with $\gamma(0) = p$ and a subsequence of $\{\gamma_n\}$ that converges h-uniformly on compact sets to γ.

Proof. Since the parametrization of the γ_n is with respect to h-arc length,

$$d_h(\gamma_n(t_0), \gamma_n(t_1)) \leq L_h(\gamma_n|_{[t_0,t_1]}) = |t_1 - t_0|, \quad (*)$$

thus the γ_n are uniformly Lipschitz continuous with common constant 1 (as maps between the metric spaces (M, d_h) and $\mathbb{R}$ with the usual metric). In particular, $\{\gamma_n\}$ is equicontinuous.
We may assume that $\gamma_n(0) \to p$. Thus, there is some $R > 0$ such that $\gamma_n(0) \in B_h(p, R)$ for all n. Using $(*)$, we have $\gamma_n(t) \in B_h(p, R+t)$ for all $t \in \mathbb{R}$, hence $\cup_n\{\gamma_n(t)\}$ is relatively compact for each $t \in \mathbb{R}$ (since, by Hopf-Rinow, $B_h(p, R+t)$ is). Thus, Arzela-Ascoli may be applied to give the desired limit $\gamma : \mathbb{R} \to M$ such that the convergence $\gamma_n \to \gamma$ is h-uniform on compact sets. The other properties of γ follow from 0.3.2. $\square$

Sometimes, we want a result of the above form for curves defined on more general domains than just $\mathbb{R}$. The next result gives us just that. Note that future directed may be replaced by past directed to get analogous results.

Proposition 0.3.4. (Limit Curve Lemma for compact intervals)
Let $p_n \to p$, $q_n \to q$ and let $\gamma_n : [0, b_n] \to M$ be continuous future directed causal curves from p_n to q_n parametrized by h-arc length. Then either $p = q$ or a subsequence of the γ_n converges to a continuous future-directed causal curve $\gamma : [0, b] \to M$ from p to q.

Proof. For each n, take the maximal past-directed geodesic starting at p_n and parametrize it by h-arc length to get a future-directed causal curve $\alpha_n : (-\infty, 0] \to M$ with $\alpha_n(0) = p_n$. Similarly, we obtain a future-directed h-arc length parametrized causal curve $\beta_n : [b_n, \infty) \to M$ with $\beta_n(b_n) = q_n$. Define now $\tilde{\gamma}_n : \mathbb{R} \to M$ as the concatenation of α_n, γ_n and β_n. Clearly, $\tilde{\gamma}_n$ is future directed causal and parametrized by h-arc length. Also, $\tilde{\gamma}_n(0) = p_n \to p$, hence 0.3.3 applies to give a future directed causal curve $\tilde{\gamma} : \mathbb{R} \to M$ such that $\tilde{\gamma}_n \to \tilde{\gamma}$ h-uniformly on compact sets. Our goal is to extract the curve γ whose existence we claimed out of $\tilde{\gamma}$.
To this end, choose $p', q' \in M$ such that $p \in I^+(p')$ and $q \in I^-(q')$. Then $J(p', q')$ is a neighborhood of p and q. Hence we may assume that $p_n, q_n \in J(p', q')$ and thus $\gamma_n \subset J(p', q')$ for all n. By compactness of $J(p', q')$ (guaranteed by global hyperbolicity), we may cover it by finitely many convex neighborhoods $U_1, \ldots, U_k$ of its points such that no causal curve leaving U_i ever returns. Let $C > 0$ be such that each causal curve contained in $J(p', q')$ has h-arc length of at most C. In particular, $b_n = L_h(\gamma_n) \in [0, C]$, so we may assume that $b_n \to b \in [0, C]$. If $b = 0$, then $p = q$. If $b > 0$, then γ_n converge h-uniformly to $\tilde{\gamma}|_{[0,b]}$, and this is the desired curve γ. $\square$

0.4 Local Lorentzian Distances

In this section, following p. 160 - 161 from [2], we show that in globally hyperbolic spacetimes (M, g), the local distance on a convex set agrees with the global Lorentzian distance on M.

Definition 0.4.1. (Local distance)
Let $U \subset M$ be a convex open set. Then the *local (Lorentzian) distance* on U is the Lorentzian distance D of the spacetime $(U, g|_U)$.

Note that $D(p, q) = 0$ if there is no future directed timelike geodesic segment from p to q entirely in U. Otherwise, $D(p, q)$ is the g-arc length of the unique future directed timelike geodesic segment in U from p to q. Since U is convex, we get from the properties of the exponential map that D is continuous on $U \times U$ and differentiable on $U^+ := \{(p, q) \in U \times U \mid q \in I_U^+(p)\}$.

Proposition 0.4.2. (Local distance = restriction of global distance)
Any $p \in M$ has a convex neighborhood U such that the local Lorentzian distance D on U agrees with the restriction of the global Lorentzian distance d to $U \times U$.

Proof. By strong causality, there exists a convex neighborhood U of p such that no causal curve leaving U ever returns. Clearly, for any $q, q' \in U$, $D(q, q') \leq d(q, q')$. If $d(q, q') = 0$, we are done. Otherwise, there is a maximizing future directed timelike geodesic segment from q to q' in M (by global hyperbolicity). If this geodesic segment were to leave U somewhere between q and q', it could not return, thus it must stay entirely in U. But U is convex, hence this geodesic must be the unique future-directed timelike geodesic segment connecting q to q'. Thus $D(q, q') = d(q, q')$. $\square$

0.5 Upper Semicontinuity of the Lorentzian Arc Length Functional

Throughout this section, fix a spacetime (M, g) and a complete Riemannian metric h on M. The goal of this section is to prove that the Lorentzian arc length functional L is upper semicontinuous with respect to h-uniform convergence of causal curves. The reference for this section is [21], p.109 - 111.
Before we can tackle this, we need to extend the Lorentzian arc length functional to continuous, not necessarily piecewise smooth, causal curves. This is, however, only of relevance in this section and not in the proof of the Lorentzian Splitting Theorem, because there, the limits we consider will turn out to be maximizing and thus smooth.

Definition 0.5.1. (Lorentzian arc length)
Let $\gamma : [a, b] \to M$ be a future-directed, continuous causal curve. Then its *Lorentzian arc length* $L(\gamma)$ is the infimum of all sums

$$\sum_{i=1}^{k} D_{U_i}(\gamma(t_{i-1}), \gamma(t_i)),$$

where the infimum is taken over all $k \in \mathbb{N}$ and all partitions $a = t_0 < \cdots < t_k = b$ of $[a, b]$ such that there are convex sets $U_1, \ldots, U_k \subset M$ with $\gamma([t_{i-1}, t_i]) \subset U_i$. Here D_{U_i} are the local Lorentzian distances, cf. 0.4.1.

One can show that for piecewise smooth causal curves, the above definition agrees with the usual integral definition (cf. [21], Appendix B.2), because it essentially boils down to representing the integral as a limit of Riemann sums. So even though we do not technically need this definition in this thesis, using it simplifies the proof of upper semicontinuity.

Proposition 0.5.2. (Upper semicontinuity of Lorentzian arc length)
Let $\gamma_n : [a_n, b_n] \to M$ be a sequence of continuous future-directed causal curves parametrized by h-arc length. Suppose $\gamma_n \to \gamma : [a, b] \to M$ h-uniformly, then

$$L(\gamma) \geq \limsup_{n \to \infty} L(\gamma_n).$$

Proof. We first assume that $[a_n, b_n] = [a, b]$ for all n. For given $\varepsilon > 0$ we have to show that $L(\gamma_n) \leq L(\gamma) + \varepsilon$ for sufficiently large n. By 0.5.1, we find a partition $a = t_0 < \cdots < t_k = b$ of $[a, b]$ and convex sets U_i such that $\gamma([t_{i-1}, t_i]) \subset U_i$, and if α denotes the geodesic polygon that is defined as the concatenation of the unique future-directed causal geodesics α_i connecting $\gamma(t_{i-1})$ to $\gamma(t_i)$ in U_i, then

$$L(\alpha) \leq L(\gamma) + \frac{\varepsilon}{2}.$$

For each $i = 1, \ldots, k-1$, choose $p_i, q_i \in U_i \cap U_{i+1}$ such that

$$p_i <<_{U_{i+1}} \gamma(t_i) <<_{U_i} q_i. \quad (*)$$

Choose also $p_0, q_0 \in U_1$ with $p_0 <<_{U_1} \gamma(t_0) <<_{U_1} q_0$ and $p_k, q_k \in U_k$ with $p_k <<_{U_k} \gamma(t_k) <<_{U_k} q_k$. By continuity of the D_{U_i}, these points may be chosen such that for each $i = 1, \ldots, k$,

$$D_{U_i}(p_{i-1}, q_i) < D_{U_i}(\gamma(t_{i-1}), \gamma(t_i)) + \frac{\varepsilon}{2k}. \quad (**)$$

By h-uniform convergence, there is some N such that $\gamma_n([t_{i-1}, t_i]) \subset U_i$ for all $n \geq N$. By $(*)$, by increasing N if necessary, we have that for $n \geq N$

$$\gamma_n(t_i) \in I^+_{U_{i+1}}(p_i) \cap I^-_{U_i}(q_i)$$

for all $i = 1, \dots, k-1$, and also $\gamma_n(t_0) \in I^+_{U_1}(p_0)$ and $\gamma_n(t_k) \in I^-_{U_k}(q_k)$. Thus, after concatenating with causal curves emanating from p_{i-1} and ending in q_i that are entirely in U_i, we see that $L(\gamma_n|_{[t_{i-1},t_i]}) \leq D_{U_i}(p_{i-1}, q_i)$ by definition of the Lorentzian distance. Using this, $(**)$ and summing from $i = 1$ to k, we get

$$L(\gamma_n) \overset{(**)}{<} L(\alpha) + \frac{\varepsilon}{2} < L(\gamma) + \varepsilon.$$

This concludes the proof if $[a_n, b_n] = [a, b]$ for all n.

Consider now the general case. Given $\varepsilon > 0$, we may find convex neighborhoods U, V of $\gamma(a)$ and $\gamma(b)$, respectively, that are sufficiently small so that their local Lorentzian distances D_U, D_V are bounded by $\varepsilon/4$. By definition, also the Lorentzian arc lengths of causal curves lying entirely in U or V are bounded by $\varepsilon/4$. Let $c, d \in (a, b)$ be such that $\gamma([a, c]) \subset U$, $\gamma([d, b]) \subset V$. Since $a_n \to a$, $b_n \to b$, we have that $a_n \leq c$ and $b_n \geq d$ for large n. Hence, it makes sense to consider $\gamma_n|_{[c,d]}$, and these restrictions converge h-uniformly to $\gamma|_{[c,d]}$. Using the special case of equal intervals that we already proved, we may find an $N \in \mathbb{N}$ such that $L(\gamma_n|_{[c,d]}) \leq L(\gamma|_{[c,d]}) + \varepsilon/2$ for any $n \geq N$. By h-uniform convergence, we may further assume that $\gamma_n([a_n, c]) \subset U$ and $\gamma_n([d, b_n]) \subset V$ for $n \geq N$. Altogether, we have for $n \geq N$

$$\begin{aligned}
L(\gamma_n) &= L(\gamma_n|_{[a_n,c]}) + L(\gamma_n|_{[c,d]}) + L(\gamma_n|_{[d,b_n]}) \\
&\leq \frac{\varepsilon}{4} + L(\gamma|_{[c,d]}) + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} \\
&\leq L(\gamma) + \varepsilon,
\end{aligned}$$

which concludes the proof of the general case. □

0.6 The Lorentzian Cut Locus

The goal of this section is to introduce some basics about the Lorentzian version of cut locus theory, which is similar to the Riemannian case, but additional causality issues have to be dealt with. We do not intend to present a detailed exposition here, that can be found in [2], Chapter 9, or [21], Section 3.2.5. Here, we only want to establish that (roughly) the Lorentzian distance function $x \mapsto d(x, p)$ is smooth outside the cut locus of p. This will be of great importance in the proof of the Lorentzian Splitting theorem.

Our exposition follows [21], Section 3.2.5, but is much more simplified and less general. We refer there for any details not sufficiently expounded upon here. Throughout this section, we fix a globally hyperbolic spacetime (M, g) and a complete Riemannian metric h on M. In the following, we only discuss the future cut locus, but it is clear that the past cut locus can be defined in an analogous manner and the dual variants of all of the results hold for past cut points.

Definition 0.6.1. (h-unit future tangent bundle)
The subbundle

$$UTM_h^+ := \{v \in TM \mid h(v,v) = 1, v \text{ is future-directed causal}\}$$

of TM is called *h-unit future tangent bundle* of M.

Observe that UTM_h^+ has compact fibers.

Definition 0.6.2. (Cut function)
The function $s_h^+ : UTM_h^+ \to [0, \infty]$ defined by

$$s_h^+(v) := \sup\{t > 0 \mid d(\gamma_v(0), \gamma_v(t)) = L(\gamma_v|_{[0,t]})\}$$

is called *future cut function* with respect to the complete Riemannian metric h. Here, $\gamma_v : I_v \to M$ is the maximally defined geodesic with $\gamma_v'(0) = v$. If $s_h^+(v) \in I_v$, then $\gamma_v(s_h^+(v)) \in M$ is called the *(future) cut point* of $\gamma_v(0)$ along γ_v.

Note that the notion of cut point along a geodesic is independent of the choice of complete Riemannian metric h. Also, note that $s_h^+(v) > 0$ for any $v \in UTM_h^+$, since γ_v is initially maximizing.
The situation is more involved that what one is usually confronted with in the complete Riemannian case. E.g., $s_h^+(v) < \infty$ may happen because γ_v may not be defined for arbitrarily high parameter values.
It may be shown that s_h^+ is lower semicontinuous and continuous at each $v \in UTM_h^+$ for which either $s_h^+(v) = \infty$ or $s_h^+(v)v$ is in the maximal domain of definition of $(\pi, \exp)$, cf. [21], Prop. 3.2.29.

Definition 0.6.3. (Future causal cut locus)
Let $p \in M$. Set

$$\Gamma_T^+(p) := \{s_h^+(v)v \mid v \in UTM_h^+|_p, s_h^+(v)v \in \mathcal{D}_p\} \subset T_pM,$$

where $\mathcal{D}_p$ is the maximal domain of definition of $\exp_p$. Then $\mathrm{Cut}^+(p) := \exp_p(\Gamma_T^+(p)) \subset M$ is called the *future causal cut locus* of p.

Analogously, the *past causal cut locus*, the *future timelike cut locus*, etc. are defined. The following result is proven exactly as the corresponding Riemannian result, upon replacing "minimizing" with "maximizing".

Proposition 0.6.4. (Cut point $\Leftrightarrow$ conjugate or 2 max. geod.)
Let $p \in M$. For any $q \in M$, it holds that $q \in \mathrm{Cut}^+(p)$ if and only if there is more than one maximizing future directed causal geodesic from p to q or if q is conjugate to p along a maximizing future-directed causal geodesic emanating from p.

Proof. For the direction "$\Rightarrow$", see [21], Sec. 3.2.5. For "$\Leftarrow$", see Prop. 3.2.28 from the same reference. $\square$

Proposition 0.6.5. (Regularity outside the cut locus)
Let $p \in M$. Set

$$\mathcal{J}_T^+(p) := \{tv \mid v \in UTM_h^+|_p,\, t \in [0, s_h^+(v))\} \subset T_pM$$

and $\mathcal{I}_T^+(p) := \mathcal{J}_T^+(p)^\circ$. Then:

1. $\exp_p$ restricts to a diffeomorphism from $\mathcal{I}_T^+(p)$ onto the open set $\mathcal{I}^+(p) := \exp_p(I_T^+(p)) \subset M$.

2. $\mathcal{I}^+(p)$ is the largest open subset of $I^+(p)$ such that for each $q \in \mathcal{I}^+(p)$, there exists a unique maximizing future directed timelike geodesic from p to q.

3. The Lorentzian distance function $x \mapsto d(p, x)$ is smooth on $\mathcal{I}^+(p)$.

Proof. 1. Since a geodesic stops being maximizing upon intersecting another maximizing geodesic, we have that $\exp_p$ is injective on $\mathcal{I}^+(p)$. Since a geodesic also stops being maximizing after its first conjugate point, it follows that $\exp_p$ is nonsingular on $\mathcal{I}^+(p)$.

2. This is immediate from 0.6.4.

3. From the above points, it follows that

$$d(p, x) = \left| \exp_p|_{\mathcal{I}^+(p)}^{-1}(x) \right|_g ,$$

which is smooth.

□

Chapter 1

The Riemannian Splitting Theorem

1.1 Motivation for the Riemannian Splitting Problem

Motivation 1.1.1. (Rigidity in geometry)
To give meaningful context to the splitting theorems, let us first consider the idea of rigidity in geometry. Suppose a result holds on a Riemannian manifold given a certain curvature bound, say $\mathrm{Ric} > 0$. Then we cannot expect the result to hold if we assume a weaker bound on the curvature, say $\mathrm{Ric} \geq 0$ (e.g. Myers' Theorem, where $\mathrm{Ric} \geq \lambda > 0$ cannot be weakened). Now, the idea of *rigidity* is that the result, under these weakened assumptions, does not fail generally, but only in very special cases, i.e. the manifold with only $\mathrm{Ric} \geq 0$ for which the result fails must be very special.

To motivate this in the context of the Riemannian Splitting Theorem, let us first start with some terminology. Note: In the entirety of the first chapter, all Riemannian manifolds will be assumed complete, noncompact and connected (unless explicitly stated otherwise). Connectedness is (as for most questions in Riemannian geometry) not a real restriction.

Definition 1.1.2. (Rays, lines, connectedness at infinity)
Let (M, g) be a Riemannian manifold.

1. M is *connected at infinity* if for every compact set $K \subset M$ there is a compact set $C \subset M$ with $K \subset C$ such that any two points in $M \setminus C$ can be joined by a curve in $M \setminus K$.

2. A *ray* is a unit-speed geodesic $\gamma : [0, \infty) \to M$ minimizing the distance between any of its points, i.e.

$$d(\gamma(s), \gamma(t)) = |s - t|.$$

3. A *line* is a ray in both directions, i.e. a unit-speed geodesic $\gamma : \mathbb{R} \to M$ s.t. $d(\gamma(s), \gamma(t)) = |s - t|$.

Now let us consider the following classical result requiring $\mathrm{Ric} > 0$ to give a topological restriction on the manifold, cf. [12].

Theorem 1.1.3. (Gromoll-Meyer)
A Riemannian manifold M with $\dim M \geq 2$ and $\mathrm{Ric} > 0$ is connected at infinity.

Here, rigidity comes into play: Suppose we have a Riemannian manifold M with $\mathrm{Ric} \geq 0$ that is *not* connected at infinity, i.e. it has two ends. Then, as we shall see, M is isometric to a product. So manifolds with $\mathrm{Ric} \geq 0$ that are disconnected at infinity are very special!
Let us close this section with a preparatory result, cf. [20], Lemma 7.3.1.

Lemma 1.1.4. (Existence of rays and lines)

1. For any $p \in M$, there is a ray emanating from p.

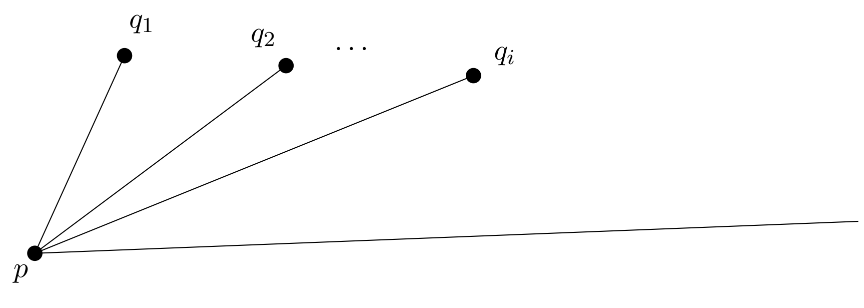
2. If M is disconnected at infinity, then M contains a line.

Proof. 1. Let $q_i \in M$, $q_i \to \infty$ (i.e. the sequence leaves every compact set). Since M is complete, we can connect p to q_i by minimizing geodesics

$$\sigma_i(t) := \exp_p(tv_i), \quad t \in [0, d(p, q_i)],$$

where $v_i \in T_pM$ are unit vectors. By compactness, we may assume $v_i \to v$ for some unit vector $v \in T_pM$. Now set $\sigma(t) := \exp_p(tv)$. This is a ray, since by continuity of the exponential map we have

$$d(\sigma(s), \sigma(t)) = \lim_{i\to\infty} d(\sigma_i(t), \sigma_i(s)) = |s - t|.$$



2. If M is disconnected at infinity, there exists a compact set $K \subset M$ and sequences $p_i \to \infty$ and $q_i \to \infty$ in $M \setminus K$ such that every curve from p_i to q_i passes through K. As in (1), join p_i to q_i via a segment $\sigma_i : (-a_i, b_i) \to M$, where $a_i, b_i \to \infty$ (one sees that $d(p_i, q_i) \to \infty$

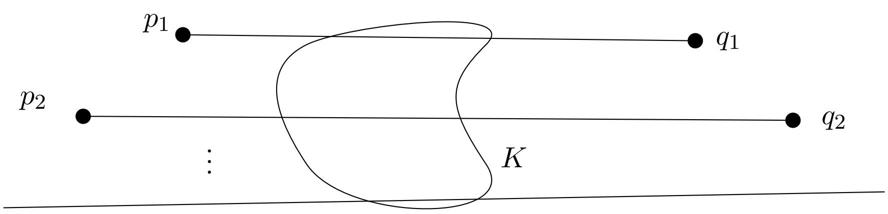
by considering a compact exhaustion of M by compact sets containing K), such that $\sigma_i(0) \in K$ for every i. Again by compactness, we may assume that $\sigma_i(0) \to x \in K$. Moreover, note that the unit tangent bundle over K,

$$UTM|_K := \{v_p \in TM : p \in K, |v_p| = 1\} \subset TM$$

is compact: To see this, consider a convering $\{U_i\}$ of K by TM-trivializing open sets in M such that $\overline{U}_i \subset\subset V_i$, where the V_i are also trivializing for TM. Then by compactness, K is covered by $U_{i_1}, \ldots, U_{i_m}$. Now, $UTM|_{\overline{U}_i} \cong \overline{U}_i \times S^{n-1}$ is compact for every i, and

$$UTM|_K \subset \bigcup_{j=1}^{m} UTM|_{\overline{U}_{i_j}}.$$

Since $UTM|_K$ is closed in this compact set, it is itself compact. Let us now use this: The $\sigma_i'(0) \in UTM|_K$ thus converge to a unit vector $v \in UTM|_K$, in fact $v \in T_xM$, and the same continuity argument as in (1) (noting that exp is continuous in both arguments simultaneously) shows that $\exp_x(tv)$ is a line in M.



□

Example 1.1.5. (Cylinder)
Let $M = S^1 \times \mathbb{R}$. Consider the strip $K := S^1 \times [a, b]$, then any two points on opposite sides of K can only be joined by passing through K, so we see that $S^1 \times \mathbb{R}$ is disconnected at infinity. Hence every complete metric on it contains a line. For the standard complete metric, this is just a spiral.

1.2 Statement of the Riemannian Splitting Theorem

Let us now state the main result that we already motivated. For the original reference, cf. [4].

Theorem 1.2.1. (Cheeger-Gromoll, Riemannian Splitting Theorem)
Let (M, g) be a complete, noncompact, connected Riemannian manifold with $\mathrm{Ric} \geq 0$. If (M, g) contains a line, then it is isometric to a product

$$(M, g) \cong (H \times \mathbb{R}, g_0 \oplus dt^2),$$

for some complete (possibly compact) Riemannian manifold (H, g_0).

The following general version is then immediate:

Corollary 1.2.2. (Riemannian Splitting Theorem, general version)
Let (M, g) be a complete, noncompact, connected Riemannian Manifold with $\mathrm{Ric} \geq 0$. If (M, g) contains a line, then

$$(M, g) \cong (M_1 \times \mathbb{R}^k, g_1 \oplus g_{Eucl}),$$

where (M_1, g_1) is a complete (possibly compact) Riemannian manifold that contains no lines (or is not there, i.e. $(M, g) \cong (\mathbb{R}^n, g_{Eucl})$).

Note that compact manifolds cannot contain lines.

1.3 Proof of the Riemannian Splitting Theorem

Rather than following the original proof by Cheeger and Gromoll [4] which heavily uses the theory of elliptic operators, we will instead give the proof along the lines of Eschenburg and Heintze, cf. [5] for the original reference. This has the advantage that the use of elliptic operator theory is minimized, and it can be adapted to the proof of the Lorentzian Splitting Theorem, see Chapter 2 (note that the Laplacian in a Riemannian manifold is elliptic, while the Laplacian in a Lorentzian manifold is hyperbolic). We stick to the presentation provided in [8] and [20], Section 7.3.

We assume throughout that M is a complete, noncompact, connected Riemannian manifold with $\mathrm{Ric} \geq 0$. The key to the proof is an analysis of the "distance from infinity".

Definition and Lemma 1.3.1. (Busemann functions)
Let $c : [0, \infty) \to M$ be a ray and $b_{c,t}(x) := t - d(x, c(t))$.

1. $t \mapsto b_{c,t}(x)$ is nondecreasing and bounded above by $d(x, c(0))$.
2. $|b_{c,t}(x) - b_{c,t}(y)| \leq d(x, y)$.
3. $\Delta b_{c,t}(x) \geq \frac{n-1}{b_{c,t}(x)-t}$ for all $x \in M$ (in the support sense).
4. The limit $\lim_{t\to\infty} b_{c,t}(x)$ exists for every $x \in M$ and defines a continuous function

   $$b_c : M \to \mathbb{R}, \quad b_c(x) := \lim_{t\to\infty} b_{c,t}(x),$$

   called the *Busemann function* associated with the ray c.

Proof. 1. The bound follows from

$$|b_{c,t}(x)| = |t - d(x, c(t))| = |d(c(0), c(t)) - d(x, c(t))| \le d(c(0), x).$$

Moreover for $s < t$, we have

$$\begin{aligned} & t - s = d(c(t), c(s)) \ge d(x, c(t)) - d(x, c(s)) \\ \Rightarrow \quad & t - d(x, c(t)) \ge s - d(x, c(s)), \end{aligned}$$

so $t \mapsto b_{c,t}(x)$ is nondecreasing.

2. $|b_{c,t}(x) - b_{c,t}(y)| = |t - d(x, c(t)) - t + d(y, c(t))| \le d(x, y)$.

3. Since $t - b_{c,t}(x) = d(x, c(t))$ and $\mathrm{Ric} \ge 0$, a standard inequality on the Laplacian of the distance (cf. [20], Lem. 7.1.2. and 7.1.9 with $k = 0$) yields

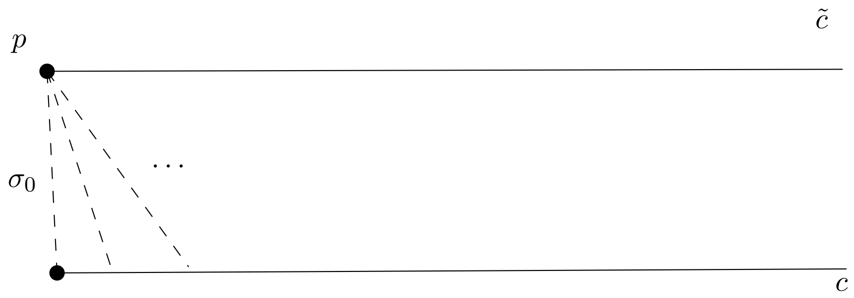
$$\Delta(t - b_{c,t}(x)) \le \frac{n-1}{t - b_{c,t}(x)} \quad \Leftrightarrow \quad \Delta b_{c,t}(x) \ge \frac{n-1}{b_{c,t}(x) - t}.$$

4. The limit exists by (1) and b_c is continuous by (2).

□

$b_c(x)$ is interpreted as the renormalized distance of x from $c(\infty)$.

Remark 1.3.2. (Asymptotes)
Given a ray $c : [0, \infty) \to M$ as before, and $p \in M$, consider unit speed segments $\sigma_t : [0, d(p, c(t))] \to M$ connecting p to $c(t)$. As in 1.1.4, $(\sigma_t)_t$ converges (in the precise sense as in that lemma) to a ray $\tilde{c} : [0, \infty) \to M$ with $\tilde{c}(0) = p$. Such a ray $\tilde{c}$ is called an *asymptote* to c at p. It need not be unique.



Lemma 1.3.3. (Relating Busemann functions)
Let $\tilde{c}$ be an asymptote of the ray c starting at p. Then:

1. $b_c(x) \ge b_c(p) + b_{\tilde{c}}(x)$.

2. $b_c(\tilde{c}(s)) = b_c(p) + b_{\tilde{c}}(\tilde{c}(s)) = b_c(p) + s$.

Proof. Let us first prove (2): Note that by construction

$$d(p, c(t_i)) = d(p, \sigma_i(s)) + d(\sigma_i(s), c(t_i)),$$

where $t_i \to \infty$. Using this, we may express the Busemann function $b_c(p)$ at p as follows:

$$\begin{aligned} b_c(p) &= \lim_{i\to\infty} (t_i - d(p, c(t_i))) \\ &= \lim_{i\to\infty} (t_i - \underbrace{d(p, \sigma_j(s))}_{=s} - d(\sigma_j(s), c(t_i))) \\ &= b_c(\sigma_j(s)) - s = b_c(\sigma_j(s)) - b_{\tilde{c}}(\tilde{c}(s)). \end{aligned}$$

Using the continuity of b_c and taking $j \to \infty$, we get (2). To establish (1), observe that

$$\begin{aligned} t - d(x, c(t)) &\geq t - d(x, \tilde{c}(s)) - d(\tilde{c}(s), c(t)) \\ &= t + \underbrace{s - d(p, \tilde{c}(s))}_{=0} - d(x, \tilde{c}(s)) - d(\tilde{c}(s), c(t)) \\ &= (t - d(\tilde{c}(s), c(t))) + (s - d(x, \tilde{c}(s))) - d(p, \tilde{c}(s)). \end{aligned}$$

Hence by $t \to \infty$ and using (2), we get

$$\begin{aligned} b_c(x) &\geq b_c(\tilde{c}(s)) + (s - d(x, \tilde{c}(s))) - d(p, \tilde{c}(s)) \\ &= b_c(p) + \underbrace{(s - d(p, \tilde{c}(s)))}_{=0} + (s - d(x, \tilde{c}(s))) \\ &\overset{s\to\infty}{\longrightarrow} b_c(p) + b_{\tilde{c}}(x). \end{aligned}$$

□

Proposition 1.3.4. (Subharmonicity of the Busemann functions)
The Busemann functions are subharmonic, i.e.

$$\Delta b_c \geq 0 \quad \text{everywhere (in the support sense).}$$

Proof. By 1.3.3, $b_c(p) + b_{\tilde{c}}$ is a support function for b_c at p from below, so we only need to check that $\Delta b_{\tilde{c}} \geq 0$ at p (again in the support sense). For this, observe that the $b_{\tilde{c},t}$ are support functions for $b_{\tilde{c}}$ at p from below by 1.3.1 and the fact that $b_{\tilde{c},t}(p) = 0 = b_{\tilde{c}}(p)$, since p is the starting point of $\tilde{c}$. Now, $b_{\tilde{c},t}$ is smooth near p (because for every r, we may find a neighborhood of $\tilde{c}([0, r])$ not meeting the cut locus of p; we prove this in the Lorentzian case in 2.2.22; the proof of the Riemannian version can more or less be extracted from there by leaving out all the causality notions). Furthermore, by 1.3.1,

$$\Delta b_{\tilde{c},t}(p) \geq \frac{n-1}{b_{\tilde{c},t}(p) - t} = -\frac{n-1}{t} \overset{t\to\infty}{\longrightarrow} 0.$$

Thus $\Delta b_{\tilde{c}}(p) \geq 0$ in the support sense, and this in turn gives $\Delta b_c(p) \geq 0$ in the support sense. Since p was arbitrary, we conclude that b_c is subharmonic. (Note that this double support sense argument works because the $b_{\tilde{c},t}$ converge in a monotonically increasing fashion to $b_{\tilde{c}}$.) $\square$

1.3.5 (Proof of the Riemannian Splitting Theorem)**.**
Let $c : \mathbb{R} \to M$ be a line in M. Then $c|_{[0,\infty)}$ and $c|_{(-\infty,0]}$ are rays. Let $b^{\pm}$ be the corresponding Busemann functions, i.e.

$$\begin{aligned} b^+(x) &= \lim_{t\to\infty}(t - d(x, c(t))), \\ b^-(x) &= \lim_{t\to\infty}(t - d(x, c(-t))). \end{aligned}$$

Observe now that

$$b^+(x) + b^-(x) = \lim_{t\to\infty}(2t \underbrace{-d(x, c(t)) - d(x, c(-t))}_{\leq -d(c(t),c(-t))=-2t}) \leq 0.$$

Moreover, for every s,

$$b^+(c(s)) + b^-(c(s)) = \lim_{t\to\infty}(2t - (t-s) - (s+t)) = 0.$$

Since $b^+ + b^-$ is a subharmonic function assuming a global maximum (at every point of c), we have $b^+ + b^- \equiv 0$ on M by Calabi's Maximum Principle, cf. 0.2.4. In particular, since $b^+ = -b^-$, we see that $\Delta b^+ = 0 = \Delta b^-$ everywhere (in the support sense). Thus, b^+ and b^- are harmonic and hence smooth everywhere on M (cf. 0.2.5).
Next, we show that $b^{\pm}$ are distance functions, meaning $|\operatorname{grad} b^{\pm}| \equiv 1$. For this, let $p \in M$ and consider asymptotes $\tilde{c}^{\pm}$ for the rays $c^{\pm}$ from p, and let $b_t^{\pm}(x) := t - d(x, \tilde{c}^{\pm}(t))$. Observe that, by 1.3.3 (1) and its proof, we get for every $t \geq 0$:

$$b_t^+(x) \leq b^+(x) - b^+(p) = -b^-(x) + b^-(p) \leq -b_t^-(x)$$

with equality holding for $x = p$, since then $b_t^{\pm}(p) = 0$. Now, the distance is a distance function (cf. [20], Section 3.2.2. for that notion), and the $b_t^{\pm}$ are (up to addition of a constant t) just the distance and they are smooth around p, so $b_t^+ + b^+(p)$ is a support function for b^+ at p from below and $b_t^- + b^-(p)$ is a support functions for b^- from below at p, with $|\operatorname{grad}(b_t^{\pm} + b^{\pm}(p))|_p| = |\operatorname{grad} b_t^{\pm}|_p| = 1$. Thus $|\operatorname{grad} b^{\pm}|_p| = 1$ by 0.2.2.
In conclusion: $b^{\pm}$ are smooth, harmonic functions on M with unit gradient everywhere. Now consider the following inequality on distance functions r (here $\partial_r \equiv \nabla_{\operatorname{grad} r}$, cf. [20], Prop. 7.1.1.)

$$\partial_r \Delta r + \frac{(\Delta r)^2}{n-1} \leq \partial_r \Delta r + Tr(S_r^2) = -\operatorname{Ric}(\partial_r, \partial_r)$$

where $\mathrm{Hess}(r)(X,Y) = \langle S_r(X), Y\rangle = \langle X, S_r(Y)\rangle$, i.e. $S_r(X) = \nabla_X(\mathrm{grad}\ r)$. Obviously, $\mathrm{Hess}(r) = 0$ if and only if $S_r = 0$. Since S_r is symmetric and hence diagonalizable, we have that $S_r^2 = 0$ if and only if $S_r = 0$, and again by diagonalizability $Tr(S_r^2) = 0$ if and only if $S_r^2 = 0$. Now consider this for $r = b^{\pm}$, then harmonicity and the assumption on M that $\mathrm{Ric} \geq 0$ give $\mathrm{Hess}(b^{\pm}) = 0$, hence $b^{\pm}$ are linear distance functions. The result follows from the next Proposition, which is a collection and generalization of results from [11].

Proposition 1.3.6. (Splitting induced by a linear distance function)
Let $r : M \to \mathbb{R}$ be a *linear distance function*, i.e. a smooth function with $|N| \equiv 1$, where $N := \mathrm{grad}\ r$, and $\mathrm{Hess}(r) \equiv 0$. Then:

1. N is a parallel vector field on M.
2. The integral curves of N are geodesics.
3. N is complete.
4. The level sets $r^{-1}(\lambda) \subset M$ are totally geodesic, embedded hypersurfaces in M for any $\lambda \in \mathbb{R}$.
5. For any $t \in \mathbb{R}$ and $p \in r^{-1}(0)$
$$r(\mathrm{Fl}_t^N(p)) = t.$$
(Note that Fl^N is defined on all of $\mathbb{R} \times M$). As a consequence,
$$\mathrm{Fl}_t^N(r^{-1}(0)) = r^{-1}(t).$$
6. *(Main result)*
Let $M' := r^{-1}(0)$. Then
$$\Phi : \mathbb{R} \times M' \to M, \quad (t,p) \mapsto \mathrm{Fl}_t^N(p),$$
is an isometry.

Proof. 1. Since $\mathrm{Hess}(r) = 0$, and $\mathrm{Hess}(r)(X,Y) = \langle \nabla_X \,\mathrm{grad}\ r, Y\rangle$, we conclude $\nabla_X \,\mathrm{grad}\ r = 0$ for any $X \in \mathfrak{X}(M)$.

2. Follows immediately from (1).
3. Follows directly from (2) since M is complete.
4. Since $|\,\mathrm{grad}\ r| = 1$ on all of M, $r^{-1}(\lambda)$ are embedded hypersurfaces. Let us show that they are totally geodesic: Since N is parallel, the shape operator of $r^{-1}(\lambda)$ vanishes, which means that the second fundamental form also vanishes, but this by definition implies that the $(b^{\pm})^{-1}(\lambda)$ are totally geodesic (cf. [19], 4.12 and 4.19).

5. Consider the function $h : \mathbb{R} \to \mathbb{R}$, $h(t) := r(\mathrm{Fl}_t^N(p))$. Then

$$h'(t) = T_{\mathrm{Fl}_t^N(p)} r(N(\mathrm{Fl}_t^N(p))) = \langle N(\mathrm{Fl}_t^N(p)), N(\mathrm{Fl}_t^N(p))\rangle = 1$$

and $h(0) = r(p) = 0$ by choice of p, so the result follows by integration. Thus we have "$\subset$" in the claim after. "$=$" follows from the fact that one can similarly show $\mathrm{Fl}_{-t}^N(r^{-1}(t)) \subset r^{-1}(0)$.

6. Injectivity: Suppose $\mathrm{Fl}_t^N(p) = \mathrm{Fl}_s^N(q)$. Then $q = \mathrm{Fl}_{-s+t}^N(p)$. Since Fl_{-s+t}^N maps $r^{-1}(0)$ into $r^{-1}(-s+t)$, and level sets are disjoint, we must have $s = t$, and since Fl_t^N is a diffeomorphism, we have $p = q$.

   Surjectivity: Any $q \in M$ is in some $r^{-1}(t)$, so surjectivity follows from (5).

   Let us now show that the tangent map of Φ is everywhere invertible.

   Let $(t, p) \in \mathbb{R} \times M'$, consider the curve $\alpha(s) := (as+t, p)$, then $\alpha(0) = (t, p)$ and $\alpha'(0) = (a, 0)$. Hence

$$T_{(t,p)}\Phi(a, 0) = \left.\frac{d}{ds}\right|_0 \Phi \circ \alpha = \left.\frac{d}{ds}\right|_0 \mathrm{Fl}_{as+t}^N(p) = aN(\mathrm{Fl}_t^N(p)).$$

   Now, since M' is totally geodesic, $\exp_p(sv)$ lies in M' (initially) for $p \in M'$ and $v \in T_pM'$. Thus

$$T_{(0,p)}\Phi(0, v) = \left.\frac{d}{ds}\right|_0 \Phi(0, \exp_p(sv)) = \left.\frac{d}{ds}\right|_0 \exp_p(sv) = v.$$

   Assuming now that $|T_{(t,p)}\Phi(0, v)|^2$ is independent of t (let $(*)$ be this claim), and noting that

$$T_{(t,p)}\Phi(0, v) = \left.\frac{d}{ds}\right|_0 \mathrm{Fl}_t^N(\underbrace{\exp_p(sv)}_{\in r^{-1}(0)}) \in T_{\mathrm{Fl}_t^N(p)} r^{-1}(t),$$

   we have that

$$\begin{aligned}|T_{(t,p)}\Phi(a, v)|^2 &= \underbrace{|T_{(t,p)}\Phi(a, 0)|^2}_{=a^2|N|^2=a^2} + \underbrace{|T_{(0,p)}\Phi(0, v)|^2}_{=|v|^2} \\ &\quad + 2\langle \underbrace{T_{(t,p)}\Phi(a, 0)}_{=aN\perp r^{-1}(t)}, \underbrace{T_{(t,p)}\Phi(0, v)}_{\in Tr^{-1}(t)}\rangle \\ &= a^2 + |v|^2,\end{aligned}$$

   from which we conclude that Φ is a local diffeomorphism, thus a diffeomorphism since it is bijective, and an isometry.

It remains to prove $(*)$, which is a simple calculation using the interchangeability of differentials of two-parameter maps and the fact that N is a parallel vector field:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\langle T_{(t,p)}\Phi(0,v), T_{(t,p)}\Phi(0,v)\rangle \\
&= \langle \frac{\nabla}{dt} T_{(t,p)}\Phi(0,v), T_{(t,p)}\Phi(0,v)\rangle \\
&= \langle \frac{\nabla}{dt} \frac{d}{ds}\Big|_0 \mathrm{Fl}^N_t(\exp_p(sv)), T_{(t,p)}\Phi(0,v)\rangle \\
&= \langle \frac{\nabla}{ds}\Big|_0 \frac{d}{dt} \mathrm{Fl}^N_t(\exp_p(sv)), T_{(t,p)}\Phi(0,v)\rangle \\
&= \langle \frac{\nabla}{ds}\Big|_0 N(\mathrm{Fl}^N_t(\exp_p(sv))), T_{(t,p)}\Phi(0,v)\rangle \\
&= \langle \underbrace{\nabla_{\frac{d}{ds}|_0 \mathrm{Fl}^N_t(\exp_p(sv))} N}_{=0}, T_{(t,p)}\Phi(0,v)\rangle = 0.
\end{aligned}$$

$\square$

Chapter 2

The Lorentzian Splitting Theorem

2.1 Formulations of the Lorentzian Splitting Theorem and some History

Since the developments surrounding the Lorentzian Splitting Theorem are rather recent (around the 90s), let us put them into historical context, following [8]. This section is rather informal and uses concepts that have not yet been defined. We will restate the to-be-proven theorems and define everything in a rigorous manner in the following sections.

In 1982, inspired by the Riemannian Splitting Theorem and its significance and usefulness in Riemannian geometry, Yau conjectured the following analogue for Lorentzian manifolds:

Conjecture 2.1.1. (Yau's Conjecture [22])
Let (M, g) be a spacetime of dimension > 2 satisfying the following conditions:

1. (M, g) is timelike geodesically complete.
2. (M, g) satisfies the *strong energy condition* (SEC), i.e.
$$\mathrm{Ric}(X, X) \geq 0 \quad \text{for all timelike } X \in TM.$$
3. There exists a complete timelike line in M.

Then (M, g) is isometric to a product $(M, g) \cong (\mathbb{R} \times S, -dt^2 \oplus h)$, where (S, h) is a complete Riemannian manifold.

By 1990, a more general result had been proven which contained Yau's Conjecture as a special case. Let us give a timeline of the most important contributions:

1. In 1988, Eschenburg [7] proved Yau's Conjecture under the additional assumption that the spacetime is globally hyperbolic.
2. In 1989, Galloway [10] proved the splitting for globally hyperbolic spacetimes without the timelike geodesic completeness condition.
3. In 1990, Newman [18] proved Yau's original formulation, that is without the assumption of global hyperbolicity.
4. In 1995, Galloway and Horta [9] simplified many of the rather technical arguments given in the prior papers. Based on this, [2] systematically treats the timelike geodesically complete case in a very understandable manner.

Galloway's and Newmans's works built upon the general foundation laid by Eschenburg, which in turn was inspired by the his and Heinze's simpler version of the proof of the Riemannian Splitting Theorem, as discussed in Chapter 1.
The general result, which encompasses Yau's conjecture, is one of the most important results in Lorentzian geometry and may now be stated as follows:

Theorem 2.1.2. (Lorentzian Splitting Theorem (LST))
Let (M,g) be a spacetime of dimension > 2 that satisfies the following conditions:

1. (M, g) is globally hyperbolic or timelike geodesically complete.
2. (M, g) satisfies the SEC, i.e.
$$\mathrm{Ric}(X, X) \geq 0 \quad \text{for all timelike } X \in TM.$$
3. There exists a complete timelike line in M.

Then M is isometric to a product $(M, g) \cong (\mathbb{R} \times S, -dt^2 \oplus h)$, where (S, h) is a complete Riemannian manifold.

In many ways, especially with view towards general relativity, global hyperbolicity is a more natural assumption than timelike geodesic completeness.

2.2 Proof of the Weak Lorentzian Splitting Theorem

2.2.1 Preliminary Discussion

Our aim in this section is to prove the LST while assuming both global hyperbolicity and timelike geodesic completeness (this is known as the Weak Lorentzian Splitting Theorem, or WLST). This simplifies the presentation

considerably, and we are still able to demonstrate most of the important techniques that go into the proofs of the general versions of the LST. We will follow the presentation of the timelike geodesically complete case in [2], Chapter 14, while simplifying some steps due to our additional assumption of global hyperbolicity.
Let us first establish some terminology.

Definition 2.2.1. (Timelike Rays, Lines)
Let (M, g) be a spacetime.

1. A *future directed timelike ray* in M is a future directed, future inextendible unit speed timelike geodesic $\gamma : [a, b) \to M$ which realizes the (Lorentzian) distance between any of its points. γ is *future complete* if $b = \infty$. *Past directed rays*, *causal rays* etc. are defined analogously.

2. A *timelike line* in M is an inextendible (past and future) timelike geodesic $\gamma : (a, b) \to M$ realizing the distance between any of its points. γ is *complete* if $a = -\infty$, $b = \infty$.

The precise formulation of the weak LST is:

Theorem 2.2.2. (Weak Lorentzian Splitting Theorem (WLST))
Let (M, g) be a spacetime of dimension $n > 2$ satisfying the following conditions:

1. (M, g) is globally hyperbolic and timelike geodesically complete.

2. (M, g) satisfies the strong energy condition (SEC), i.e.

$$\mathrm{Ric}(X, X) \geq 0 \quad \text{for all } X \in TM \text{ timelike.}$$

3. There exists a complete timelike line in M.

Then M is isometric to a product $(M, g) \cong (\mathbb{R} \times S, -dt^2 \oplus h)$, where (S, h) is a complete Riemannian manifold.

Throughout this section, we fix (M, g) with the properties in 2.2.2. Denote by $d : M \times M \to \mathbb{R}$ the Lorentzian distance. We also fix an auxiliary complete Riemannian metric h on M. Given a future inextentible causal curve, assume it is h-unit speed parametrized unless explicitly stated otherwise, similarly for past inextendible causal curves. Note that properties such as being a ray are not dependent on the parametrization, as the Lorentzian arc length (like the Riemannian one) is invariant under reparametrizations.
Let us restate some of the important tools from Chapter 0 for easier readability.

Lemma 2.2.3. (Limit Curve Lemma)

1. Let $\gamma_n : (-\infty, \infty) \to M$ be a sequence of causal, future directed, h-unit speed parametrized curves. Let $p \in M$ be any accumulation point of $\{\gamma_n(0)\}$. Then there exists an inextendible, future directed causal curve $\gamma : (-\infty, \infty) \to M$ such that $\gamma(0) = p$ and a subsequence of $\{\gamma_n\}$ converges to γ h-uniformly on compact sets.

2. Let $p_n \to p$, $q_n \to q$ and let $\gamma_n : [0, b_n] \to M$ be future directed causal curves from p_n to q_n parametrized by h-arc length. Then either $p = q$ or a subsequence of the γ_n converges to a future-directed causal curve $\gamma : [0, b] \to M$ from p to q.

Proof. See 0.3.3 and 0.3.4. □

Lemma 2.2.4. (Upper semicontinuity of the Lorentzian arc length)
Let $\gamma_n : [a_n, b_n] \to M$ be a sequence of continuous future-directed causal curves parametrized by h-arc length. Suppose $\gamma_n \to \gamma : [a, b] \to M$ h-uniformly, then

$$L(\gamma) \geq \limsup_{n\to\infty} L(\gamma_n).$$

Proof. See 0.5.2. □

2.2.2 The Lorentzian Busemann Function

Analogously to the Riemannian case, we now turn to an analysis of Busemann functions associated with rays.

Definition 2.2.5. (Lorentzian Busemann function)
Let $\gamma : [0, \infty) \to M$ be a future directed ray parametrized by h-arc length. We define the *(Lorentzian) Busemann function associated with* γ by

$$b_\gamma(q) \equiv b(q) := \lim_{r\to\infty} b_r(q),$$

where $b_r(q) := d(\gamma(0), \gamma(r)) - d(q, \gamma(r))$.

We will address well-definedness and domain questions shortly. Note that if $q \in M \setminus I^-(\gamma)$, then $d(q, \gamma(r)) = 0$. Hence, for γ future complete (which will be the case in our proof by assumption) we have $b(q) = \infty$, since γ is a ray. Thus, if we consider the Busemann function on all of M, then it will generally not be finite-valued.
It is helpful to consider the following generalization of rays.

Definition 2.2.6. (S-ray)
Let $S \subset M$ be any subset. A future directed, future inextendible causal curve $\gamma : [0, \infty) \to M$ is said to be an S*-ray* if $\gamma(0) \in S$ and

$$L(\gamma|_{[0,a]}) = d(S, \gamma(a)) \quad \text{for every } a \geq 0.$$

A similar definition applies if γ is past inextendible/past directed. This will generally be true for most of the notions we consider, and we shall not mention this explicitly every time. If a future-directed, causal geodesic γ is a ray in the sense of 2.2.1, then we may reparametrize it with respect to h-arc length so that it is an S-ray with $S = \{\gamma(0)\}$. Conversely, if γ is an S-ray, then

$$d(\gamma(0), \gamma(t)) \leq d(S, \gamma(t)) = L(\gamma|_{[0,t]}),$$

and thus equality holds above. This means that γ is maximizing between any of its points and may thus be reparametrized to be a causal geodesic ray.
Let us now collect some basic facts about Busemann functions.

Lemma 2.2.7. (Busemann functions: The basics)
Let $\gamma : [0, \infty) \to M$ be a future directed, future complete S-ray parametrized by h-arc length. Then:

1. For all $q \in I^-(\gamma) \cap I^+(S)$, $0 < d(S, q) < \infty$.
2. The Busemann function $b : I^-(\gamma) \to [-\infty, \infty]$ associated with γ is well-defined and upper semicontinuous.
3. On $I^-(\gamma) \cap I^+(S)$, $0 < b < \infty$.
4. For all $p, q \in I^-(\gamma) \cap I^+(S)$ with $p \leq q$,

$$b(q) \geq b(p) + d(p, q).$$

Proof. 1. Let $p \in S \cap J^-(q)$. For r large we have $q << \gamma(r)$, thus the reverse triangle inequality gives

$$d(p, q) + d(q, \gamma(r)) \leq d(p, \gamma(r)) \leq d(S, \gamma(r)) = L(\gamma|_{[0,r]}) < \infty.$$

Taking the supremum over such p gives $d(S, q) < \infty$. Also, since $q \in I^+(S)$, $d(S, q) > 0$.

2. Let $x \in I^-(\gamma)$, hence $x << \gamma(r)$ for some large r. For $s > r$, we again have by the reverse triangle inequality that

$$d(x, \gamma(s)) \geq d(x, \gamma(r)) + d(\gamma(r), \gamma(s)).$$

Since γ is a ray, we have

$$d(\gamma(0), \gamma(s)) = d(\gamma(0), \gamma(r)) + d(\gamma(r), \gamma(s)).$$

Thus,

$$\begin{aligned} b_s(x) &= d(\gamma(0),\gamma(s)) - d(x,\gamma(s)) \\ &\leq d(\gamma(0),\gamma(r)) + d(\gamma(r),\gamma(s)) \\ &- [d(x,\gamma(r)) + d(\gamma(r),\gamma(s))] \\ &= d(\gamma(0),\gamma(r)) - d(x,\gamma(r)) \\ &= b_r(x). \end{aligned}$$

Hence for $x \in I^-(\gamma)$ fixed, $r \mapsto b_r(x)$ is monotonically decreasing, thus b defined on $I^-(\gamma)$ exists in $[-\infty,\infty]$ (in fact, $b < \infty$ since each $b_r < \infty$ by global hyperbolicity). Since each b_r is continuous, b is upper semicontinuous as the infimum of continuous functions.

3. Let $x \in I^-(\gamma) \cap I^+(S)$, and $s > 0$ such that $x << \gamma(s)$. Then $b(x) < \infty$ as just discussed. Let now $m \in S \cap J^-(x)$, then

$$\begin{aligned} d(m,x) + d(x,\gamma(r)) \leq d(m,\gamma(r)) \leq d(S,\gamma(r)) &= d(S,\gamma(r)) \\ &= L(\gamma|_{[0,r]}) \\ &= d(\gamma(0),\gamma(r)). \end{aligned}$$

By taking the supremum over all such m, we arrive at

$$\begin{aligned} & d(S,x) + d(x,\gamma(r)) \leq d(\gamma(0),\gamma(r)) \\ \overset{(1)}{\Rightarrow}\quad & 0 < d(S,x) \leq d(\gamma(0),\gamma(r)) - d(x,\gamma(r)) = b_r(x). \end{aligned}$$

By letting $r \to \infty$, we get the result $0 < d(S,x) \leq b(x)$.

4. We have $p \leq q << \gamma(r)$ for $r \geq r_0$, where r_0 is suitable and fixed. Then

$$\begin{aligned} b_r(p) &= d(\gamma(0),\gamma(r)) - d(p,\gamma(r)) \\ &\leq d(\gamma(0),\gamma(r)) - d(p,q) - d(q,\gamma(r)) \\ &= b_r(q) - d(p,q) \end{aligned}$$

The claim follows by letting $r \to \infty$.

□

2.2.3 Limit-Maximizing Sequences, (Generalized) Co-rays, Asymptotes

Recall that curves are always h-unit speed parametrized unless stated otherwise. We consider now a generalization of the notion of maximizing curves, which, while not strictly necessary in our case, is helpful in the proofs of the more general versions of the LST.

Definition 2.2.8. (Limit maximizing sequences)
A sequence $\gamma_n : [a_n, b_n] \to M$ of future directed causal curves that are h-unit speed parametrized is called *limit maximizing* if there is $\varepsilon_n \to 0$ such that

$$L(\gamma_n) \geq d(\gamma_n(a_n), \gamma_n(b_n)) - \varepsilon_n.$$

Even on general spacetimes, for every $p << q$ there exists a limit maximizing sequence such that $\gamma_n(a_n) \to p$ and $\gamma_n(b_n) \to q$ by definition of the Lorentzian distance.
The following result shows that h-uniform limits of limit maximizing sequences are maximizing.

Proposition 2.2.9. (Uniform limits of limit maximizing sequences)
Let $\gamma_n : [a_n, b_n] \to M$ be a limit maximizing sequence of h-arc length parametrized future directed causal curves converging h-uniformly to a future directed causal curve $\gamma : [a, b] \to M$ on some $[a, b] \subset \bigcap [a_n, b_n]$. Then $L(\gamma) = d(\gamma(a), \gamma(b))$, i.e. γ is maximizing and may thus be reparametrized as a future directed maximal causal geodesic from $\gamma(a)$ to $\gamma(b)$.

Proof. For convenience, we just write $\gamma_n \equiv \gamma_n|_{[a,b]}$. Then by upper semicontinuity of arc length (cf. 2.2.4) and limit maximality,

$$\begin{aligned}\limsup L(\gamma_n) \leq L(\gamma) \leq d(\gamma(a), \gamma(b)) &\leq \liminf d(\gamma_n(a), \gamma_n(b)) \\ &\leq \liminf (L(\gamma_n) + \varepsilon_n) = \liminf L(\gamma_n).\end{aligned}$$

Thus $\lim_n L(\gamma_n)$ exists and equals $\limsup_n L(\gamma_n)$ and $\liminf_n L(\gamma_n)$, and equality holds everywhere above, giving

$$L(\gamma) = d(\gamma(a), \gamma(b)).$$

□

We would now like to make sense of the notion of *asymptote* to a ray as we did in the Riemannian case, cf. 1.3.2. Let us establish the following auxiliary result:

Lemma 2.2.10. (Divergent curves in complete RMF)
Let (N, h) be a complete, noncompact Riemannian manifold and $\alpha_n : [0, a_n] \to N$ be a sequence of piecewise smooth curve segments such that for every compact set $K \subset\subset M$ there is some α_n which leaves K. Assume further that $\alpha_n(0) =: z_n \to z \in M$. Then $L_h(\alpha_n) \to \infty$.

Proof. By the Hopf-Rinow theorem, for every closed n-ball $B_h(z, n)$ there is some α_m leaving it, say at $\alpha_m(t_k)$. Then

$$L_h(\alpha_m) \geq d_h(z, \alpha_m(t_k)) > n.$$

Clearly, $m \to \infty$ as $n \to \infty$, yielding $L_h(\alpha_m) \to \infty$, as claimed. □

Proposition 2.2.11. (Maximality of co-rays)
Let $z_n \to z$ in M. Let $p_n \in I^+(z_n)$ and let $\gamma_n : [0, a_n] \to M$ be a limit maximizing sequence of future directed causal curves, $\gamma_n(0) = z_n$ and $\gamma_n(a_n) = p_n$, with some choice of future inextendible extensions $\tilde{\gamma}_n : [0, \infty) \to M$. If $d(z_n, p_n) \to \infty$, then there exists a limit curve $\gamma : [0, \infty) \to M$ of $\{\tilde{\gamma}_n\}$ that is a causal geodesic ray starting at z.

Proof. First, (the obvious variant of) 2.2.3 guarantees the existence of a future directed, future inextendible causal curve $\gamma : [0, \infty) \to M$ starting at z such that $\{\tilde{\gamma}_n\}$ converges h-uniformly on compact sets to γ. Thus, if we show that $a_n \to \infty$, γ is a geodesic ray by 2.2.9.
Suppose that $a_n \not\to \infty$. By considering a subsequence if necessary, we may assume that $a_n \to a < \infty$. Since the γ_n are h-unit speed parametrized and h is a complete Riemannian metric, $L_h(\gamma_n) = a_n \to a < \infty$ implies that all γ_n are contained in some compact set $K \subset\subset M$ by 2.2.10.
Now, using our assumption $d(z_n, p_n) \to \infty$, we will show that $L_h(\gamma_n) \to \infty$, which is a contradiction.
For this, let $T \in \mathfrak{X}(M)$ be a unit timelike vector field (whose existence is guaranteed by the time orientation), and denote $T^\flat \in \Omega^1(M)$ the one-form corresponding to T by the g-musical isomorphisms. Consider now

$$h_0 := g + 2T^\flat \otimes T^\flat \in \mathcal{T}^0_2(M).$$

We claim that h_0 is a Riemannian metric: Only positive definiteness needs to be argued for. We proceed case by case: If $v \in TM \setminus TM_0$ is null, then $h_0(v, v) = 2g(T, v)^2 > 0$, since the g-orthogonal complement of a timelike vector is spacelike. If v is spacelike, then $g(v, v) > 0$ and hence $h_0(v, v) > 0$. Finally, if v is timelike, then using the inverse Cauchy-Schwarz inequality we establish

$$h_0(v, v) = g(v, v) + 2g(T, v)^2 \geq g(v, v) + 2\underbrace{g(T, T)}_{=-1} g(v, v) = -g(v, v) > 0.$$

Now, for any causal curve segment $c : [a, b] \to M$, the Lorentzian g-arc length and Riemannian h_0-arc length are related by

$$L_{h_0}(c) \geq L_g(c). \quad (*)$$

This follows from the following simple calculation, using again the inverse Cauchy-Schwarz inequality:

$$\begin{aligned} L_{h_0}(c) &= \int_a^b \sqrt{h_0(c'(t), c'(t)}\ dt \\ &= \int_a^b \big(g(c'(t), c'(t)) + 2 \quad \underbrace{g(T(c(t)), c'(t))^2}_{\geq g(T(c(t)), T(c(t))) g(c'(t), c'(t))} \quad \big)^{1/2}\ dt \\ &\geq \int_a^b \sqrt{-g(c'(t), c'(t))}\ dt = L_g(c). \end{aligned}$$

Since h, h_0 are Riemannian metrics and K is a compact set, there is some constant $\lambda > 0$ such that $h(v, v) \geq \lambda^2 h_0(v, v)$ for any $v \in TM|_K$, which then implies

$$L_h(c) \geq \lambda L_{h_0}(c) \quad (**)$$

for any curve c contained in K. For $c = \gamma_n$ and using limit maximality, we finally arrive at

$$L_h(\gamma_n) \overset{(**)}{\geq} \lambda L_{h_0}(\gamma_n) \overset{(*)}{\geq} \lambda L_g(\gamma_n) \geq \lambda d(z_n, p_n) - \lambda \varepsilon_n \to \infty,$$

which is the desired contradiction. □

Let now $\gamma : [0, \infty)$ be a future directed, future complete timelike S-ray (in particular, its Lorentzian arc length is infinite) and fix $z \in I^-(\gamma) \cap I^+(S)$. Let z_n be a sequence such that $z_n \to z$ and set $p_n := \gamma(r_n)$, where $r_n \to \infty$. It is then easily seen that $d(z_n, p_n) \to \infty$ by using that γ is an S-ray. Thus, the requirements of 2.2.11 are satisfied, and we can use that result: Let μ_n be any sequence of limit maximizing curves from z_n to p_n, and let $\mu : [0, \infty) \to M$ be the limit of the extensions $\{\tilde{\mu}\}$ whose existence we just proved in 2.2.11. In particular, μ is a causal geodesic ray.

Definition 2.2.12. ((Generalized) co-rays, asymptotes)
Any ray μ constructed as above is called a *generalized co-ray* of γ. If the μ_n are chosen to be maximizing between z_n and p_n (which is always possible in our case since M is globally hyperbolic), then μ will be called a *co-ray*. If, in addition to μ_n being maximizing, $z_n = z$ for every n, then μ will be called an *asymptote* to γ at z.

2.2.4 Support Functions for the Busemann Function, the Generalized Timelike Co-ray Condition

Let us note at this point that all constructions and definitions involving a "future-directed" requirement can also be obtained by requiring "past-directedness", upon making the obvious modifications. We will not explicitly mention that something is future-directed from here on out, unless it comes into play in an important way (which will be the case in the actual proof of the WLST).
We want to proceed to study regularity properties of Busemann functions. For this, we shall employ support function methods, and to do this properly we will try to reduce everything to considerations made on spacelike hypersurfaces.
Motivated by 1.3.3, let $\gamma : [0, \infty) \to M$ be a future complete S-ray, and b the associated Busemann function. Let $\alpha : [0, \infty) \to M$ be a timelike asymptote to γ at $\alpha(0) =: p \in I^-(\gamma) \cap I^+(S)$. For $s > 0$, set

$$b_{p,s} : M \to \mathbb{R}, \quad b_{p,s}(x) := b(p) + d(p, \alpha(s)) - d(x, \alpha(s)). \tag{2.1}$$

Lemma 2.2.13. (Upper support functions for the Busemann function)
For any fixed $s > 0$, $b_{p,s}$ is a continuous upper support function for b at p, i.e. $b_{p,s}$ is continuous, $b_{p,s}(x) \geq b(x)$ for x near p and equality holds for $x = p$.

Proof. Clearly, $b_{p,s}$ is continuous (even on all of M), since the Lorentzian distance is. Evidently, $b_{p,s}(p) = b(p)$.
Let us show $b_{p,s}(x) \geq b(x)$ for x close to p: By definition of asymptotes, there are maximizing timelike geodesic segments $\alpha_n : [0, a_n] \to M$, $\alpha_n(0) = p$, $\alpha_n(a_n) = \gamma(r_n)$, where $r_n \to \infty$. This implies then that also $a_n \to \infty$, since

$$a_n = d(p, \alpha_n(a_n)) = d(p, \gamma(r_n)) \geq d(p, \gamma(s)) + d(\gamma(s), \gamma(r_n)) \to \infty.$$

Now, $U := I^-(\alpha(s)) \cap I^+(S)$ is an open neighborhood of p. Let $x \in U$, then $x << \alpha_n(s)$ for large n. By continuity, $d(., \alpha_n(s)) \to d(., \alpha(s))$. The result follows from the next calculation which uses (again) the inverse triangle inequality and the fact that α_n is maximizing:

$$\begin{aligned} b_{r_n}(x) - b_{r_n}(p) &= d(p, \gamma(r_n)) - d(x, \gamma(r_n)) \\ &\leq d(p, \underbrace{\gamma(r_n)}_{=\alpha_n(a_n)}) - d(x, \alpha_n(s)) - d(\alpha_n(s), \underbrace{\gamma(r_n)}_{=\alpha_n(a_n)}) \\ &= d(p, \alpha_n(s)) - d(x, \alpha_n(s)) \end{aligned}$$

$$\overset{n\to\infty}{\Longrightarrow} \quad b(x) \leq b(p) + d(p, \alpha(s)) - d(x, \alpha(s)) \overset{(2.1)}{=} b_{p,s}(x).$$

□

We now show that if we consider the Busemann function along an asymptotic ray, then equality holds in 2.2.7 (4).

Corollary 2.2.14. (Busemann function along an asymptote)
Let $\gamma : [0, \infty) \to M$ be a future directed, future complete S-ray and b its associated Busemann function. Let $\alpha : [0, \infty) \to M$ be a timelike asymptote to γ at $p \in I^-(\gamma) \cap I^+(S)$. Then

$$b(\alpha(t)) = d(p, \alpha(t)) + b(p) \quad \text{for all } t \geq 0.$$

Proof. As already mentioned, "$\geq$" is just 2.2.7 (4) applied to p and $q = \alpha(t)$ (noting that $\alpha(t) \in I^-(\gamma) \cap I^+(S)$ by construction of asymptotes). "$\leq$" follows by maximality of α: Fix $s > t$, then

$$b(\alpha(t)) \overset{2.2.13}{\leq} b(p) + d(p, \alpha(s)) - d(\alpha(t), \alpha(s)) = b(p) + d(p, \alpha(t)).$$

□

Up until now, we always assumed the existence of some timelike asymptote at some point. Of course, at any $p \in I^-(\gamma) \cap I^+(S)$ there will certainly be a causal asymptote to γ. Now, we study this and show that near γ, asymptotes will always be timelike.

Definition 2.2.15. (Generalized Timelike Co-ray Condition (GTCRC))
Let $\gamma : [0, \infty) \to M$ be a future directed, future complete timelike S-ray and let $p \in I^-(\gamma) \cap I^+(S)$. Then the *generalized timelike co-ray condition* (GTCRC) holds at p if every generalized co-ray to γ at p is timelike.

The investigation and systematic use of the GTCRC was one of the main simplifications obtained in [9].

Lemma 2.2.16. (GTCRC is open)
The GTCRC is open, i.e. if it holds at some $p \in I^-(\gamma) \cap I^+(S)$, then it also holds in some neighborhood of p.

Proof. Suppose this is false. Then there are $p_n \in I^-(\gamma) \cap I^+(S)$, $p_n \to p$ such that the GTCRC does not hold at any p_n. By definition, this means that there exist $p_{nk} \to p_n$ and $r_{nk} \to \infty$ as $k \to \infty$, limit maximizing sequences $\mu_{nk} : [0, a_{nk}] \to M$ with $\mu_{nk}(0) = p_{nk}$, $\mu_{nk}(a_{nk}) = \gamma(r_{nk})$, such that the generalized co-rays $\mu_n : [0, \infty) \to M$ that are obtained from these sequences satisfy $\mu_n(0) = p_n$ and are null geodesic rays (cf. 2.2.11). By 2.2.9, we have for any $s > 0$ that

$$0 = L(\mu_n|_{[0,s]}) = \lim_{k \to \infty} L(\mu_{nk}|_{[0,s]}).$$

It follows that, by a diagonalization argument, we may find $k(n) \to \infty$ such that for $\eta_n := \mu_{nk(n)}$, $b_n := a_{nk(n)}$, we have that

1. $b_n \to \infty$,
2. $L(\eta_n|_{[0,1]}) < \frac{1}{n}$,
3. $L(\eta_n|_{[0,b_n]}) \geq d(\eta_n(0), \eta_n(b_n)) - \frac{1}{n}$.

In particular, η_n is again a limit maximizing sequence and, by 2.2.11, converges to a causal geodesic ray $\eta : [0, \infty) \to M$ with $\eta(0) = p$. By construction, η is a generalized co-ray to γ at p, and by (2) above, η must be a null ray. This contradicts our assumption that the GTCRC holds at p. $\square$

With the help of the GTCRC, which holds near γ as we will show, our goal is now to prove (Lipschitz) continuity and, ultimately, smoothness of the Busemann function of a timelike line near the line. To this end, we need several preparatory results.
First, define the *unit observer bundle*

$$T_{-1}M := \{v \in TM \mid g(v, v) = -1, v \text{ future directed}\}$$

and set

$$K_C(p) := \{v \in T_{-1}M|_p \mid h(v,v) \leq C\}$$

where h is the complete Riemannian metric on M which we fixed at the beginning of this section, $C > 0$ and $p \in M$. By the Heine-Borel theorem, each $K_C(p)$ is compact and

$$\bigcup_{C>0} K_C(p) = T_{-1}M|_p.$$

Lemma 2.2.17. (GTCRC and boundedness of maximal geodesics)
Let $\gamma : [0,\infty) \to M$ be a future-directed, timelike S-ray. Let $p \in I^-(\gamma) \cap I^+(S)$ such that the GTCRC holds at p. Then there is a neighborhood U of p and constants $R, C > 0$ such that for every $q \in U$ and every $r > R$, the following holds: If $\alpha : [0,a] \to M$ is a maximal timelike geodesic segment from q to $\gamma(r)$ parametrized by g-arc length, then

$$\alpha'(0) \in K_C(p).$$

Proof. Suppose this is false. Then there exist $p_n \to p$, $r_n \to \infty$ and maximal timelike geodesics $\alpha_n : [0,a_n] \to M$ from p_n to $\gamma(r_n)$ that are g-unit speed parametrized such that $h(\alpha_n'(0), \alpha_n'(0)) \to \infty$.
We reparametrize now the α_n to h-unit speed curves, and denote them by $\tilde{\alpha}_n$. Using 2.2.11 (on some future inextendible extensions of the $\tilde{\alpha}_n$) and the terminology from 2.2.12, we get a co-ray $\tilde{\alpha}$ to γ at p, and we may assume that $\tilde{\alpha}_n \to \tilde{\alpha}$ h-uniformly on compact sets. Since we assumed that the GTCRC holds at p, $\tilde{\alpha}$ is timelike. Reparametrizing it by g-arc length, we get $\alpha : [0,\infty) \to M$ by timelike geodesic completeness.
Choose now $\delta > 0$ such that $\tilde{\alpha}([0,\delta]) \subset W$, where $W \ni p$ is a convex neighborhood that is contained in $I^-(\gamma)$. Let $\varepsilon := d(\tilde{\alpha}(0), \tilde{\alpha}(\delta)) > 0$ and $\varepsilon_n := d(\tilde{\alpha}_n(0), \tilde{\alpha}_n(\delta))$. Then $\varepsilon_n \to \varepsilon$ by continuity of the Lorentzian distance. Since $\exp_p$ is everywhere defined for timelike vectors and we are working in the convex set W, we can use the fact that the endpoints are the same after reparametrizing to calculate (we use only the g-exponential map here!)

$$\varepsilon_n \alpha_n'(0) = \exp_{p_n}^{-1}(\tilde{\alpha}_n(\delta)) \to \exp_p^{-1}(\tilde{\alpha}(\delta)) = \varepsilon \alpha'(0),$$

thus $\alpha_n'(0) \to \alpha'(0)$ and

$$h(\alpha_n'(0), \alpha_n'(0)) \to h(\alpha'(0), \alpha'(0)) < \infty,$$

contradicting $h(\alpha_n'(0), \alpha_n'(0)) \to \infty$. □

The following is an immediate consequence of (the proof of) 2.2.17.

Corollary 2.2.18. (GTCRC $\Rightarrow$ boundedness of co-rays)
Let $\gamma : [0, \infty) \to M$ be a future-directed, timelike S-ray. Let $p \in I^-(\gamma) \cap I^+(S)$ such that the GTCRC holds at p. Then there exists a neighborhood U of p and a constant $C > 0$ such that for all $q \in U$, if $\alpha : [0, \infty) \to M$ is a co-ray to γ at q, parametrized by g-arc length, then

$$\alpha'(0) \in K_C(p).$$

We come now to the result on the local continuity of the Busemann function.

Theorem 2.2.19. (Local Lipschitz continuity of the Busemann function)
Let $\gamma : [0, \infty) \to M$ be a future directed, timelike S-ray and $p \in I^-(\gamma) \cap I^+(S)$ such that the GTCRC holds at p. Then the Busemann function b associated with γ is Lipschitz continuous near p.

Proof. In the proof, we shall make use of the following result from elementary analysis:

Let $U \subset \mathbb{R}^n$ be an open, convex set and $f : U \to \mathbb{R}$ continuous. Suppose that for all $q \in U$ there is a smooth lower support function f_q defined near q such that $||\,\mathrm{grad}\ f_q(q)|| < L$ (in the standard Euclidean norm), where $L > 0$ is some constant that is independent of q. Then f is Lipschitz continuous with Lipschitz constant L.
A proof of this can be found in the Appendix of [7] or in [2], Sublemma 14.20.

Let now γ be parametrized by g-arc length and let U, R, C be as in 2.2.17. By shrinking U if necessary, we may assume that $\overline{U}$ is compact and contained in some convex neighborhood N of p, in particular we find coordinates $\phi = (x^i)$ on N such that $\phi(N) \subset \mathbb{R}^n$ is a convex open set. Let $h_0 = \sum_i dx^i \otimes dx^i$ denote the induced Euclidean metric on N. Since $\overline{U}$ is compact and h, h_0 are Riemannian metrics, there is some $K > 0$ such that $h_0 \leq Kh$ on TU.
From now on, we argue locally in U. Set $d_r(x) := d(x, \gamma(r))$. Then $b_r = r - d_r$ since γ is g-unit speed parametrized. Note that $b = \inf_r b_r$. Hence, if we can show that all b_r, $r > R$, are Lipschitz continuous with the same Lipschitz constant, b will also be Lipschitz continuous with the same Lipschitz constant. In order to prove Lipschitz continuity of the d_r (which is obviously equivalent to that of the b_r since $d_r(x) - d_r(y) = b_r(y) - b_r(x)$), $r > R$, with uniform Lipschitz bound, we make use of the result from analysis that we mentioned at the beginning.
To this end, let $D : N \times N \to \mathbb{R}$ be the Lorentzian distance of the spacetime (N, g_N). Fix now a point $q \in U$ and $r > R$. Let α be a maximal timelike geodesic segment from q to $\gamma(r)$, parametrized by g-arc length. Fix $q' \in U$ lying on α, $q \neq q'$. Define now

$$f_{q,r}(x) := D(x, q') + d(q', \gamma(r)).$$

$f_{q,r}$ is smooth near q since N is convex, and thus, D can be expressed as the inverse of some exponential map (which is a diffeomorphism there). By 0.4.2, $D(x, q') = d(x, q')$ for any $x, q' \in N$. Thus,

$$f_{q,r}(x) = D(x, q') + d(q', \gamma(r)) = d(x, q') + d(q', \gamma(r)) \leq d(x, \gamma(r)) = d_r(x)$$

and, by the maximality of α and the fact that the α-segment from q to q' lies entirely in U,

$$f_{q,r}(q) = D(q, q') + d(q', \gamma(r)) = d(q, q') + d(q', \gamma(r)) = d(q, \gamma(r)) = d_r(q).$$

Thus, we have established that the $f_{q,r}$ are smooth lower support functions for d_r. To obtain the desired uniform Lipschitz estimate, consider

$$G := \sup_{x \in U} |g_{ij}(x)|.$$

Denote by $||v||_0 = \sqrt{h_0(v, v)}$, $v \in T_qM$, the Euclidean norm induced by h_0. Clearly, $G > 0$. Since U has compact closure that is contained in W, $G < \infty$. By a result we shall prove a little later (cf. 2.2.24),

$$df_{q,r}(v) = v(f_{q,r}) = g(v, \alpha'(0)), \quad v \in T_qM.$$

With this, we can calculate

$$\begin{aligned} |df_{q,r}(v)| &= |g(v, \alpha'(0))| \leq G||\alpha'(0)||_0||v||_0 \\ &\leq GK\sqrt{h(\alpha'(0), \alpha'(0))}||v||_0 \overset{2.2.17}{\leq} GK\sqrt{C}||v||_0, \end{aligned}$$

where $GK\sqrt{C} =: L$ is independent of q and of $r > R$. Thus, considering the induced Euclidean norm and considering $df_{q,r}$ to be equivalent to the Euclidean gradient of $f_{q,r}$, we get

$$||df_{q,r}||_0 = \sup_{||v||_0 = 1} |d(f_{q,r})(v)| \leq L.$$

By the preliminary discussions above, we are finished. $\square$

The next uniqueness result tells us that (generalized) co-rays, thus in particular asymptotes, to a timelike ray at a point of the ray itself (away from the origin) must necessarily coincide with the ray (which is intuitively quite obvious).

Proposition 2.2.20. (Generalized co-rays issuing from points on γ)
Let $\gamma : [0, \infty) \to M$ be a future directed, timelike ray. Then any generalized co-ray to γ at any $p = \gamma(a)$, $a > 0$, coincides with (a part of) γ.

Proof. During the course of this proof, let all curves be given an h-unit speed parametrization. Let $\alpha : [0, \infty) \to M$ be a generalized co-ray to γ at p. By definition, there is a limit maximizing sequence $\alpha_n : [0, a_n] \to M$ of future directed causal curves from $p_n \to \gamma(r_n)$, where $p_n \to p$, $r_n \to \infty$ and thus $a_n \to \infty$, such that α is a limit curve of the α_n and the convergence is h-uniform on compact sets. Let U be a convex neighborhood of p. Let $q \in U \cap \gamma \cap I^-(p)$, q_n on γ between q and p such that $q_n \to p$. We may assume $q_n << p_n$ and $p_n \in U$ for all n. Since $p_n, q_n \in U$ with $p_n, q_n \to p$ and U is convex, q_n can be connected to p_n by timelike geodesic segments η_n lying entirely in U such that $L(\eta_n) \to 0$ as $n \to \infty$.
Define now a sequence $\sigma_n : [0, s_n] \to M$ of causal curves as follows: σ_n connects q to $\gamma(r_n)$ by first going from q to q_n via γ, then taking η_n from q_n to p_n, and then following α_n from p_n to $\gamma(r_n)$. Let a_n, b_n be such that

$$q_n = \sigma_n(a_n), \quad p_n = \sigma_n(b_n).$$

Fix now $r > a$. Since $p = \gamma(a)$, $p_n \in I^-(\gamma(r))$ and $r_n > r$ for large n. Our goal is now to show that the sequence $\{\sigma_n\}$ is limit maximizing.
Since γ is a ray and the α_n are limit maximizing,

$$\begin{aligned} L(\sigma|_{[0,s_n]}) &= \underbrace{L(\sigma|_{[0,a_n]})}_{=d(q,q_n)} + L(\sigma|_{[a_n,b_n]}) + \underbrace{L(\sigma_{[b_n,s_n]})}_{\geq d(p_n,\gamma(r_n))-\varepsilon_n} \\ &\geq \underbrace{d(q,q_n)}_{=d(q,p)-d(q_n,p)} + L(\sigma_n|_{[a_n,b_n]}) + d(p_n,\gamma(r_n)) - \varepsilon_n \\ &= d(q,p) - d(q_n,p) + L(\sigma|_{[a_n,b_n]}) + d(p_n,\gamma(r_n)) - \varepsilon_n. \end{aligned}$$

On the other hand, using the inverse triangle inequality, the fact that γ is a ray, and the fact that the Lorentzian distance is continuous and hence, in particular, lower semicontinuous, there exists $\delta_n \downarrow 0$ such that

$$\begin{aligned} d(p_n, \gamma(r_n)) &\geq d(p_n, \gamma(r)) + d(\gamma(r), \gamma(r_n)) \\ &= d(p_n, \gamma(r)) + d(p, \gamma(r_n)) - d(p, \gamma(r)) \\ &\geq d(p, \gamma(r)) - \delta_n + d(p, \gamma(r_n)) - d(p, \gamma(r)) \\ &= d(p, \gamma(r_n)) - \delta_n. \end{aligned}$$

Combining these two calculations and remembering that $L(\sigma_n|_{[a_n,b_n]}) = L(\eta_n) \to 0$, we get

$$\begin{aligned} L(\sigma_n|_{[0,s_n]}) &\geq d(q,p) - d(q_n,p) + L(\eta_n) + d(p,\gamma(r_n)) - \delta_n - \varepsilon_n \\ &= d(q,\gamma(r_n)) - d(q_n,p) + L(\eta_n) - \delta_n - \varepsilon_n, \end{aligned}$$

which shows that the σ_n are limit maximizing since $d(q_n, p) \to 0$. By 2.2.9, the sequence σ_n has a limit curve σ that is a maximal geodesic ray with $\sigma(0) = q$. By construction, $\sigma = \gamma$ on the portion from q to p, followed by α. Since σ is everywhere maximizing, a standard "round corner" argument thus gives $\alpha \subset \gamma$. □

Corollary 2.2.21. (GTCRC holds near $\gamma \setminus \{\gamma(0)\}$)
Let $\gamma : [0, \infty) \to M$ be a future directed timelike ray. Then the GTCRC holds in a neighborhood of $\gamma \setminus \{\gamma(0)\}$.

Proof. By 2.2.20, any generalized co-ray to γ at $\gamma(a)$ for any $a > 0$, coincides with γ and is thus timelike (since γ is). By openness of the GTCRC (cf. 2.2.16), we are finished. $\square$

2.2.5 Smoothness of the Lorentzian Distance, Riccati Estimate

Next, we intend to show that the support functions $b_{p,s}$ (cf. (2.1)) are smooth near p. Here, the timelike cut locus comes into play.

Proposition 2.2.22. (Asymptote avoids cut locus of its points)
Let $\gamma : [0, \infty) \to M$ be a future directed, timelike S-ray. Suppose that the GTCRC holds at $p \in I^-(\gamma) \cap I^+(S)$, and let $\alpha : [0, \infty) \to M$ be a timelike asymptote to γ at p. Then for each $r > 0$, there is a neighborhood U of $\alpha([0, r])$ that does not meet the past timelike cut locus of $\alpha(r)$.

Proof. By taking unions, it obviously suffices to show that for every $s \in [0, r]$ there is a neighborhood U of $\alpha(s)$ not meeting the past timelike cut locus of $\alpha(r)$. We consider the cases $s = r, 0 < s < r$ and $s = 0$ separately.

1. $s = r$: Take a convex neighborhood U of $\alpha(r)$, then clearly no causal geodesic emanating from $\alpha(r)$ has a cut point in U.

2. $0 < s < r$: This is proven exactly in the same way as the case $s = 0$.

3. $s = 0$: Let all appearing timelike geodesics be parametrized by g-arc length. Suppose the proposition does not hold in this case. Then there are $r > 0$ and $p_n \to p = \alpha(0)$ such that every p_n is a past timelike cut point of $\alpha(r)$. By definition, this means that there exist past directed, timelike geodesics

$$c_n : [0, d(p_n, \alpha(r)) + \delta_n) \to M$$

with $c_n(0) = \alpha(r)$, $c_n(d(p_n, \alpha(r))) = p_n$, c_n is maximizing up to p_n, but not past it. Since α is maximizing, p and $\alpha(r)$ are not conjugate along α. Hence, some neighborhood $V \subset T_{\alpha(r)}M$ of $-\alpha'(r)$ (the minus sign is due to the reversed orientation that we consider) is diffeomorphic via $\exp_{\alpha(r)}$ to some neighborhood U of p. We may assume $p_n \in U$ for every n and $U \subset I^-(\alpha(r))$.

Let now c_n be as above. If $c_n'(0) \in V$, then we can go a little further than p_n on c_n to reach $q_n \in U$ such that $q_n << p_n$ and c_n is not maximizing up to q_n. Connect now $\alpha(r)$ to q_n via a past directed, maximal timelike geodesic segment γ_n. Since $\exp_{\alpha(r)}$ is a diffeomorphism $V \to U$, $\gamma_n(0) = \alpha(r) = c_n(0)$ and $c_n'(0) \in V$ by assumption, we necessarily have $\gamma_n'(0) \notin V$. Replace then the p_n by the q_n.

All in all, we may assume that we have past directed, maximal timelike geodesic segments c_n from $\alpha(r)$ to p_n with $c_n'(0) \notin V$ for any n. In the limit (cf. 2.2.3), we get a maximizing, past directed timelike (since we already know $p << \alpha(r)$) geodesic c connecting $\alpha(r)$ to p. Since $c'(0)$ is an accumulation point of $\{c_n'(0)\}$ (by construction) and $c_n'(0) \notin V$ for any n, we have $c'(0) \notin V$. By choice of V, this then means that $c \neq \alpha$ and both are maximizing timelike geodesics from p to $\alpha(r)$. But by 0.6.4, this means that $\alpha(r)$ is a cut point of p along α, which is impossible since α is a ray.

□

Corollary 2.2.23. (Local smoothness of the Lorentzian distance)
Let the assumptions be as in 2.2.22. Consider $d_r : U \to \mathbb{R}, d_r(x) = d(x, \alpha(r))$, where U is a neighborhood of $\alpha([0, r])$ that avoids the past timelike cut locus of $\alpha(r)$. Then d_r is smooth on $U \cap I^-(\alpha(r))$.

Proof. This immediately follows from 2.2.22 and 0.6.5 (3). □

Next, let us give a formula for the gradient of the distance function which we have already used in a proof.

Lemma 2.2.24. (Gradient of the Lorentzian distance)
Let U be as in 2.2.23. Let $q \in U$ and $d(q, \alpha(r)) = d_r(q) =: t_0$. Let $c : [0, t_0] \to M$ be the (unique due to 0.6.5 (2)) past directed unit speed maximal timelike geodesic segment with $c(0) = \alpha(r)$ and $c(t_0) = q$. Then

$$(\operatorname{grad} d_r)(q) = -c'(t_0).$$

In particular, $\langle \operatorname{grad} d_r, \operatorname{grad} d_r \rangle = -1$ and the past directed maximal timelike geodesic segments from $\alpha(r)$ to points in U are integral curves of $-\operatorname{grad} d_r$ in U.

Proof. Let $v \in T_qM$ and write $c_v(s) := \exp_q(sv)$. For small s, $c_v(s) \in U$ and we may consider the unique past directed maximal unit speed geodesic segment γ_s in U from $\alpha(r)$ to $c_v(s)$. Then $\gamma(s, t) := \gamma_s(t)$ defines a geodesic variation of the geodesic c with $\gamma(s, 0) = \alpha(r)$ for all s. By standard variational calculus, cf. [2], Cor. 12.24,

$$\langle \operatorname{grad} d_r|_q, v \rangle = v(d_r) = \frac{d}{ds}\bigg|_0 d_r(c_v(s)) = \frac{d}{ds}\bigg|_0 L(\gamma_s) = -\langle v, c'(t_0) \rangle,$$

because $\frac{\partial}{\partial s}\gamma(s, t_0)|_{s=0} = \frac{\partial}{\partial s}c_v(s)|_{s=0} = v$ and $\frac{\partial}{\partial s}\gamma(s, 0)|_{s=0} = \frac{\partial}{\partial s}\alpha(r)|_{s=0} = 0$. □

Remark 2.2.25. (Preparations for curvature calculations)
Let us consider the result of 2.2.24 in a more general setting: Let $U_1 \subset M$ be open, $f : U_1 \to \mathbb{R}$ smooth such that $\langle \operatorname{grad} f, \operatorname{grad} f\rangle = -1$ on U_1. Similarly to the Riemannian case, cf. 1.3.6, one shows that the integral curves of $\operatorname{grad} f$ are timelike geodesics that are maximal in U_1 (cf. [2], Appendix B, p. 575ff). Furthermore,

$$\operatorname{Hess}(f)(V, \operatorname{grad} f) = 0$$

for any $V \in \mathfrak{X}(U_1)$, because

$$\operatorname{Hess}(f)(V, \operatorname{grad} f) = \langle \nabla_V \operatorname{grad} f, \operatorname{grad} f\rangle = \frac{1}{2} V \underbrace{\langle \operatorname{grad} f, \operatorname{grad} f\rangle}_{\equiv -1} = 0.$$

Since Hess is symmetric, we also have $0 = \operatorname{Hess}(f)(\operatorname{grad} f, V) = \langle \nabla_{\operatorname{grad} f} \operatorname{grad} f, V\rangle$, thus $\nabla_{\operatorname{grad} f} \operatorname{grad} f = 0$ on U_1.

Lemma 2.2.26. (Riccati inequality)
Let $f : U_1 \to \mathbb{R}$ be as in 2.2.25. Let c be an integral curve of $X := -\operatorname{grad} f$ in U_1. Then

$$(\Delta f \circ c(t))' \geq \operatorname{Ric}(c'(t), c'(t)) + \frac{1}{n-1}(\Delta f \circ c(t))^2.$$

Proof. Let $\{E_1, \ldots, E_n := X \circ c\}$ be an ON-frame along c, consider the E_i extended locally along neighboring integral curves. Let $V \in \mathfrak{X}(c)$ be parallel along c, also extended as a parallel field along neighboring integral curves of c. We define $\nabla_{c'} Y$ pointwise via local extensions. Then, since V is parallel along c,

$$[V, X] = \nabla_V X - \nabla_X V = \nabla_V X - \nabla_{c'} V = \nabla_V X.$$

Thus,

$$\begin{aligned} R(V, c')c' &= \nabla_V \underbrace{\nabla_X X}_{=0} - \nabla_{c'} \nabla_V X - \nabla_{[V,X]} X \\ &= -\nabla_{c'} \nabla_V X - \nabla_{\nabla_V X} X. \end{aligned}$$

Using that $\nabla_V X = \sum_{j=1}^{n-1} \langle \nabla_V X, E_j\rangle E_j$ since $\langle \nabla_V X, X\rangle = 0$ (cf. 2.2.25), $\nabla_{c'} V = 0$ and $X = -\operatorname{grad} f$, we get (extending all the vector fields along curves locally)

$$\begin{aligned}
\langle R(V,c')c',V\rangle &= -\langle \nabla_{c'}\nabla_V X, V\rangle - \langle \nabla_{\nabla_V X} X, V\rangle \\
&= -c'\langle \nabla_V X, V\rangle + \langle \nabla_V X, \underbrace{\nabla_{c'}V}_{=0}\rangle - \langle \nabla_{\sum_{j=1}^{n-1}\langle \nabla_V X, E_j\rangle E_j} X, V\rangle \\
&= -c'\langle \nabla_V X, V\rangle - \sum_{j=1}^{n-1} \underbrace{\langle \nabla_V X, E_j\rangle}_{=-\operatorname{Hess}(f)(V,E_j)} \cdot \overbrace{\langle \nabla_{E_j} X, V\rangle}^{=-\operatorname{Hess}(f)(V,E_j)} \\
&= \frac{d}{dt}(\operatorname{Hess}(f)(V,V)\circ c) - \sum_{j=1}^{n-1} \operatorname{Hess}(f)(V,E_j)^2
\end{aligned}$$

Multiply now by -1, set $V = E_i$ and sum over i to get

$$\begin{aligned}
-\operatorname{Ric}(c',c') &= -(\Delta f\circ c)' + \sum_{i,j=1}^{n-1} \operatorname{Hess}(f)(E_i,E_j)^2 \\
&\geq -(\Delta f\circ c)' + \sum_{i=1}^{n-1} \operatorname{Hess}(f)(E_i,E_i)^2. \quad (*)
\end{aligned}$$

With the help of the elementary inequality $ab \leq (a^2+b^2)/2$ for real numbers, we can estimate

$$\begin{aligned}
\left(\sum_{i=1}^{n-1} \operatorname{Hess}(f)(E_i,E_i)\right)^2 &= \sum_{i,j=1}^{n-1} \operatorname{Hess}(f)(E_i,E_i)\operatorname{Hess}(f)(E_j,E_j) \\
&\leq \sum_{i,j=1}^{n-1} \frac{\operatorname{Hess}(f)(E_i,E_i)^2 + \operatorname{Hess}(f)(E_j,E_j)^2}{2} \\
&= (n-1)\sum_{i=1}^{n-1} \operatorname{Hess}(f)(E_i,E_i)^2,
\end{aligned}$$

or, equivalently,

$$\sum_{i=1}^{n-1} \operatorname{Hess}(f)(E_i,E_i)^2 \geq \frac{1}{n-1}\left(\sum_{i=1}^{n-1} \operatorname{Hess}(f)(E_i,E_i)\right)^2 = \frac{1}{n-1}(\Delta f\circ c)^2.$$

Inserting this into $(*)$ finishes the proof. □

Proposition 2.2.27. (Laplace estimate for the Lorentzian distance)
Let U be a neighborhood of $p = \alpha(0)$ so that $d_r = d(.,\alpha(r))$ is smooth on U, cf. 2.2.23. Then for any $q \in U$,

$$\Delta d_r(q) \geq -\frac{n-1}{d_r(q)}.$$

Proof. By 2.2.24, we may take $f = d_r$ in 2.2.26, with $c : [0, d(q, \alpha(r)) =: t_0] \to M$ the past directed timelike geodesic segment from $\alpha(r)$ to q, which is an integral curve of $-\operatorname{grad} d_r$ again by 2.2.24. Set $\phi(t) := \Delta d_r(c(t))$. By 2.2.26 and our assumption that $\mathrm{Ric} \geq 0$ for timelike vectors,

$$\phi'(t) \geq \frac{\phi(t)^2}{n-1}. \quad (*)$$

One can show that $\phi(t) \to -\infty$ for $t \downarrow 0$ (cf. [3], the proof of Prop. 4.4). Next, observe that

$$\frac{d}{dt}\frac{1}{\phi(t)} = -\frac{\phi'(t)}{\phi(t)^2} \overset{(*)}{\leq} -\frac{1}{n-1}.$$

Now integrate this from 0 to t_0:

$$\frac{1}{\phi(t_0)} \leq -\frac{t_0}{n-1} \quad \Leftrightarrow \quad \phi(t_0) \geq \frac{n-1}{t_0}.$$

Since $t_0 = d(q, \alpha(r))$ and $\phi(t_0) = \Delta d_r(c(t_0)) = \Delta d_r(q)$, we are done. $\square$

Observe how similar the general procedure of the proof so far is to the Riemannian case (where we also considered cut loci, Riccati inequalities, the smoothness of the distance, etc.)

2.2.6 Level Sets of the Busemann Function, Maximum Principle

Next, we turn to an analysis of the level sets of the Busemann function (the "horospheres"). Recall that a subset V of a spacetime is called *achronal* resp. *acausal* if there is no pair of timelike resp. causally related points in V. The subset V is called *edgeless* if $\mathrm{edge}(V) = \emptyset$. Let us now establish some terminology.

Definition 2.2.28. (Topological spacelike HSF, partial Cauchy surfaces)
A subset $S \subset M$ is called *topological (or C^0-) spacelike hypersurface* if for every $p \in S$ there is a neighborhood U of p in M such that $S \cap U$ is acausal and edgeless in U. Furthermore, if S itself is acausal and edgeless in M, then S is called *partial Cauchy surface.*

Remark 2.2.29. (Comments on 2.2.28)
By [19], Cor. 14.26 (applied locally), a topological spacelike hypersurface in the sense of 2.2.28 is a topological hypersurface of M.
Conversely, suppose $S \subset M$ is a smooth, spacelike hypersurface of M. Then S is also a topological spacelike hypersurface in the above-defined sense, see [19] p. 426-427 for local acausality; local edgelessness follows by basic edge theory, cf. p. 413-415 from the same reference.

Proposition 2.2.30. (Horospheres are locally partial Cauchy surfaces)
Let $\gamma : [0, \infty) \to M$ be a timelike S-ray. Let $p \in I^-(\gamma) \cap I^+(S) \cap \{b = c\}$, where $c \in \mathbb{R}$ and b is the Busemann function associated with γ. Suppose the GTCRC holds at p. Then there exists a neighborhood U of p such that $\Sigma_c := U \cap \{b = c\}$ is acausal in M and edgeless in U. In particular, Σ_c is a partial Cauchy surface in U.

Proof. Let U be an open neighborhood of p such that b is continuous on U (cf. 2.2.19) such that, in addition, 2.2.17, 2.2.18 and the GTCRC hold on U. Since $b(q) \geq b(p) + d(p, q)$ for $q \in I^-(\gamma) \cap I^+(S)$ with $p \leq q$ by 2.2.7, we see that b is strictly increasing along timelike curves (and increasing along causal curves), implying that Σ_c is achronal in M.
Suppose $p \in \mathrm{edge}_U(\Sigma_c)$. By definition, there exist $x, y \in U \setminus \Sigma_c$, $x <<_U \Sigma_c <<_U y$ and future directed timelike curves $c_1, c_2 : [0, 1] \to U$ from x to y such that c_1 meets Σ_c at p (and only at p, since Σ_c is achronal) and c_2 does not meet Σ_c (exists, since $p \in \mathrm{edge}_U(\Sigma_c)$). Since b is strictly increasing along c_1, we get

$$\begin{aligned} & b(x) < c = b(p) < b(y) \\ \Leftrightarrow \quad & b(c_2(0)) < c < b(c_2(1)). \end{aligned}$$

But then the mean value theorem says that there is some $t_0 \in (0, 1)$ such that $b(c_2(t_0)) = c$, contradicting the assumption that c_2 does not meet Σ_c. Thus, $\mathrm{edge}_U(\Sigma_c) = \emptyset$.
It remains to show that Σ_c is acausal in M. Suppose it is not. By achronality, there must then exist $x, y \in \Sigma_c$ and a future directed, maximal null geodesic segment η from x to y. Let $r_n \to \infty$ and let α_n be a maximal timelike geodesic segment connecting y to $\gamma(r_n)$. These converge to a timelike asymptotic ray α to γ at $\alpha(0) = y$. By a standard corner argument, using the "boundedness of initial tangents" property that U has by assumption, we see that there is some fixed $\varepsilon > 0$ such that

$$b_{r_n}(y) - b_{r_n}(x) = d(x, \gamma(r_n)) - d(y, \gamma(r_n)) \geq \varepsilon > 0$$

for large n. In the limit $n \to \infty$, we get $b(y) - b(x) \geq \varepsilon > 0$. But since $x, y \in \Sigma_c$, we must have $b(x) = b(y)$, which is a contradiction. Thus, Σ_c is acausal. □

Because we use a different convention than [19], let us fix the following terminology.

Definition 2.2.31. (Second fundamental form)
Let $N \subset M$ be a nondegenerate hypersurface. Let n be some local choice of unit normal on N. Let $p \in N$, $x, y \in T_pN$ and X, Y local extensions of x, y to vector fields tangent to N. Then the covariant and symmetric 2-tensor

$$II_p(x, y) := g(\nabla_X Y|_p, n_p)$$

is called *second fundamental form* of N (associated to the choice of normal n).

We shall need the following result which gives a kind of bound on the Hessian of the Lorentzian distance. In this result, all timelike geodesics are g-unit speed parametrized.

Lemma 2.2.32. (Bound on the Hessian of the Lorentzian distance)
Let γ be a timelike S-ray. Let $p \in I^-(\gamma) \cap I^+(S)$ such that the GTCRC holds at p. Then there exists a neighborhood U of p and constants $t_0, A > 0$ such that for all $q \in U$ and all timelike asymptotes α to γ at q,

$$\mathrm{Hess}(d_t)_q(v,v) \geq -A\langle v^\perp, v^\perp \rangle$$

for every $v \in T_qM$ and $t \geq t_0$, where $d_t = d(.,\alpha(t))$ and $v^\perp$ is the projection of v onto $\alpha'(0)^\perp \subset T_qM$.

Proof. Let U be a neighborhood of p that has compact closure such that the GTCRC, 2.2.17 and 2.2.18 hold on U. By 2.2.18, the set of initial tangents to asymptotes to γ at points in U is contained in some compact subset of the h-unit tangent bundle UTM_h. By shrinking U and taking t_0 sufficiently small, we may assume that

$$\{\alpha'(t) \mid 0 \leq t \leq t_0,\ \alpha \text{ asymptote to } \gamma \text{ at some point in } U\}$$

is also contained in some compact subset of UTM_h. Hence, the set of timelike 2-planes Π containing these initial tangents is contained in some compact subset of the 2-Grassmannian bundle $G_2(M)$. Since the sectional curvature K is continuous, there exists some $k > 0$ such that $K(\Pi) \leq k$ for all such Π. Consider now $B_s := B_{\alpha(s)} := -\mathrm{Hess}(d_{t_0})_{\alpha(s)}$. By "shifting X over to n" in 2.2.31 and the definition of the Hessian, we see that B_s is the second fundamental form of

$$\{q \in I^-(\alpha(t_0)) \mid d(q,\alpha(t_0)) = s\}$$

at $\alpha(t_0 - s)$, $0 \leq s \leq t_0$. Then, precisely as in the Riemannian case, $u \mapsto B_u$ satisfies the Riccati equation $B_u' + B_u^2 + R(.,\alpha')\alpha' = 0$ (cf. [16], Thm. 11.4 for a proof in the Riemannian case. The Lorentzian case is almost identical, cf. [6], Eq. (3)). Given the sectional curvature bound that we established above, the Riccati Comparison Theorem ([6], Prop. 2.4) applies and gives the existence of a constant A (depending on k) such that

$$\mathrm{Hess}(d_{t_0})_q(v,v) \geq -A\langle v^\perp, v^\perp \rangle$$

for all $v \in T_qM$. The claim follows from the fact that $t \mapsto \mathrm{Hess}(d_t)_q(v,v)$ is increasing. This is proven using Lagrange tensor methods, cf. [2], the proof of Lem 12.13. $\square$

Before going on to make a maximum principle type argument for the Busemann functions, we need to establish some relations between quantities on M and those on a spacelike hypersurface Σ of M.

Definition 2.2.33. (Mean curvature)
Let $\Sigma \subset M$ be a spacelike hypersurface, i.e. it has an everywhere timelike normal N. Let $p \in \Sigma$ and let $\{e_1, \ldots, e_{n-1}\}$ be an orthonormal basis of $T_p\Sigma$. Then the *mean curvature* of Σ at p is

$$H_\Sigma(p) := -\sum_{i=1}^{n-1} II_p(e_i, e_i) = \sum_{i=1}^{n-1} \langle \nabla_{e_i} N, e_i \rangle.$$

Lemma 2.2.34. (Comparing grad and Δ in M and in $\Sigma \subset M$)
Let $\Sigma \subset M$ be a spacelike hypersurface, and let $f \in C^\infty(M)$. Let N denote a local timelike future directed normal to Σ around p. Then for any $p \in \Sigma$, we have

1. $\operatorname{grad}_\Sigma f|_\Sigma(p) = \operatorname{grad} f(p) + N(f)|_p N(p)$.
2. $\Delta_\Sigma f|_\Sigma(p) = \Delta f(p) + \operatorname{Hess}(f)(N, N)|_p + \langle \operatorname{grad} f, N \rangle|_p H_\Sigma(p)$.

Proof. Let $\{e_1, \ldots, e_{n-1}\} \subset T_p\Sigma$ be an ONB.

1. Since $\{e_1, \ldots, e_n, N(p)\}$ is an ONB of T_pM, we have

$$\begin{aligned}
\operatorname{grad} f(p) &= \sum_{i=1}^{n-1} \langle \operatorname{grad} f(p), e_i \rangle e_i - \langle \operatorname{grad} f(p), N(p) \rangle N(p) \\
&= \sum_{i=1}^{n-1} e_i(f) e_i - N(f)|_p N(p) \\
&= \operatorname{grad}_\Sigma f|_\Sigma(p) - N(f)|_p N(p).
\end{aligned}$$

2. Using again the ONB $\{e_1, \ldots, e_{n-1}, N(p)\}$ of T_pM, we find

$$\Delta f(p) = \sum_{i=1}^{n-1} \operatorname{Hess}(f)_p(e_i, e_i) - \operatorname{Hess}(f)_p(N(p), N(p)). \quad (*)$$

Furthermore, using (1) and the connection comparison results from [19], Ch. 4,

$$\begin{aligned}
\operatorname{Hess}(f)_p(e_i, e_i) &= \langle \nabla_{e_i} \operatorname{grad} f, e_i \rangle \\
&= \langle \nabla_{e_i} \operatorname{grad}_\Sigma f|_\Sigma, e_i \rangle - \langle \nabla_{e_i} N(f)|_p N(p), e_i \rangle \\
&= \langle \nabla^\Sigma_{e_i} \operatorname{grad}_\Sigma f|_\Sigma, e_i \rangle - \langle e_i(N(f)) N(p) + N(f)|_p \nabla_{e_i} N, e_i \rangle \\
&= \operatorname{Hess}_\Sigma(f|_\Sigma)(e_i, e_i) - N(f)|_p \langle \nabla_{e_i} N, e_i \rangle,
\end{aligned}$$

since $N(p) \perp e_i$. Inserting this into $(*)$ gives

$$\Delta f(p) = \Delta_\Sigma f|_\Sigma(p) - N(f)|_p H_\Sigma(p) - \mathrm{Hess}(f)_p(N(p), N(p)),$$

as claimed.

□

Now we come to the announced maximum principle type result for the Busemann function of a timelike S-ray.

Proposition 2.2.35. (Maximum principle for b_γ)
Let γ be a timelike S-ray. Let $W \subset I^-(\gamma) \cap I^+(S)$ be an open set on which the GTCRC holds. Let $\Sigma \subset W$ be a connected smooth spacelike hypersurface with $H_\Sigma \leq 0$. If the restriction $b|_\Sigma$ of the Busemann function b associated with γ attains a minimum on Σ, then it is constant on Σ.

Proof. Let $q \in \Sigma$ such that b achieves a minimum at q in Σ, $b(q) =: a$. Let U be a neighborhood of q such that the GTCRC and 2.2.17, 2.2.18 hold on U. By continuity of $b|_\Sigma$ and connectedness of Σ, it suffices to show $b \equiv a$ in a neighborhood of q.
Suppose this does not hold. Then there exists a Σ-coordinate ball B with $\overline{B} \subset \Sigma \cap U$ (which can be chosen arbitrarily small) centered at q such that $\partial B \neq \partial^0 B$, where

$$\partial^0 B := \{x \in \partial B \mid b(x) = a\}.$$

Since a is the minimum of b on Σ, we have $b > a$ on $\partial B \setminus \partial^0 B$. Now, 2.2.32 implies that there exists $C > 0$ such that

$$\mathrm{Hess}(d_t)_x(v, v) \geq -C \quad (\square\square)$$

for all asymptotes at x for any $x \in B$ and for all $v \in T_x\Sigma$ with $\langle v, v\rangle \leq 1$ (which then obviously implies $\langle v^\perp, v^\perp\rangle \leq 1$), and for all large enough t.
For B small enough, we can construct a function $h \in C^\infty(\Sigma)$ (not to be confused with our choice of complete Riemannian metric h, but that does not appear in this proof anyway!) satisfying the following properties (cf. [5], Lemma on p.150 for a proof):

1. $h(q) = 0$.
2. $\|\operatorname{grad}_\Sigma h\| \leq 1$ on B, where $\|.\|$ is the Riemannian norm on $T\Sigma$ induced by $g|_\Sigma$.
3. $\Delta_\Sigma h \leq -D$ on B, where $D > 0$ is some constant.
4. $h > 0$ on $\partial^0 B$.

Consider $f_\varepsilon = b + \varepsilon h$. Then $f_\varepsilon(q) = a$ and for small enough $\varepsilon > 0$, $f_\varepsilon(x) > a$ for all $x \in \partial B$ by the properties of h. Thus, f_ε attains a minimum at some $p = p(\varepsilon) \in B$. Let $\alpha : [0, \infty) \to M$ be a timelike asymptote to γ at $p = \alpha(0)$ (exists due to our GTCRC assumption). Then for $t > 0$,

$$b_{p,t}(x) = b(p) + t - d(x, \alpha(t))$$

(cf. (2.1) and 2.2.13) is a smooth upper support function for b at p. Thus,

$$f_{\varepsilon,t} := b_{p,t} + \varepsilon h$$

is a smooth upper support function for f_ε at p (again due to the properties of h), which was chosen to be the minimum of f_ε on B. Thus, $f_{\varepsilon,t}$ also has a minimum at p. This implies that $\Delta_\Sigma f_{\varepsilon,t}(p) \geq 0$ for every relevant ε and t.

However, we shall derive now a contradiction by showing that $\Delta_\Sigma f_{\varepsilon,t}(p) < 0$ for sufficiently small ε and sufficiently large t.
First, observe that

$$\begin{aligned} \Delta_\Sigma f_{\varepsilon,t}(p) &= \Delta_\Sigma b_{p,t}(p) + \varepsilon \Delta_\Sigma h(p) \\ &\leq \Delta_\Sigma b_{p,t}(p) - D\varepsilon \qquad (5*) \end{aligned}$$

by property (3) of h. Using 2.2.34 (2), we find

$$\begin{aligned} \Delta_\Sigma b_{p,t}(p) &= \Delta b_{p,t}(p) + \langle \operatorname{grad} b_{p,t}(p), N(p)\rangle H_\Sigma(p) \\ &\quad + \operatorname{Hess}(b_{p,t})_p(N(p), N(p)). \qquad (\Box) \end{aligned}$$

By 2.2.27,

$$\Delta b_{p,t}(p) = -\Delta d_t(p) \leq \frac{n-1}{t}. \quad (4*)$$

Since $f_{\varepsilon,t}$ attains a minimum (on $B \subset \Sigma$) at p, $\operatorname{grad}_\Sigma f_{\varepsilon,t}(p) = 0$. Using all of this, 2.2.34 (1) and the fact that $\operatorname{grad} b_{t,p}(p) = -\alpha'(0)$ (cf. 2.2.24 with "reversed orientations"), we get

$$\begin{aligned} 0 = \operatorname{grad}_\Sigma f_{\varepsilon,t}(p) &= \operatorname{grad}_\Sigma b_{p,t}(p) + \varepsilon \operatorname{grad}_\Sigma h(p) \\ &= \operatorname{grad} b_{p,t}(p) + \langle \operatorname{grad} b_{p,t}(p), N(p)\rangle N(p) + \varepsilon \operatorname{grad}_\Sigma h(p) \\ &= -\alpha'(0) - \langle \alpha'(0), N(p)\rangle N(p) + \varepsilon \operatorname{grad}_\Sigma h(p). \end{aligned}$$

Thus,

$$N(p) = \frac{1}{\langle \alpha'(0), N(p)\rangle}(-\alpha'(0) + \varepsilon \operatorname{grad}_\Sigma h(p)). \quad (*)$$

By 2.2.25 and 2.2.24, $\mathrm{Hess}(d_t)(\mathrm{grad}\, d_t, .) \equiv 0$, so in particular it holds that $\mathrm{Hess}(d_t)(\alpha'(0), .) = 0$. Using this while inserting the expression $(*)$ for $N(p)$ into $\mathrm{Hess}(b_{p,t})(.,.)$, we find

$$\begin{aligned}&\mathrm{Hess}(b_{p,t})(N(p), N(p)) \\ =&\varepsilon^2 \frac{1}{\langle \alpha'(0), N(p)\rangle^2} \mathrm{Hess}(b_{p,t})(\mathrm{grad}_\Sigma h(p), \mathrm{grad}_\Sigma h(p)). \quad (**)\end{aligned}$$

Next, we observe that $|\langle \alpha'(0), N(p)\rangle| \geq \|N(p)\| \cdot \|\alpha'(0)\| = 1$ by the inverse Cauchy Schwarz inequality. Furthermore, it follows from $(\Box\Box)$ that $\mathrm{Hess}(b_{p,t})(\mathrm{grad}_\Sigma h(p), \mathrm{grad}_\Sigma h(p)) \leq C$. Using these results, we estimate $(**)$ as

$$\mathrm{Hess}(b_{p,t})(N(p), N(p)) \leq C\varepsilon^2. \quad (3*)$$

Since $\mathrm{grad}\, b_{p,t}(p) = -\alpha'(0)$, it is past directed. Due to our choice of future directed normal N, we thus have that $\langle \mathrm{grad}\, b_{p,t}(p), N(p)\rangle > 0$. With this, $(3*)$, $(4*)$, and our assumption that $H_\Sigma \leq 0$, we can estimate $(\Box)$ as

$$\Delta_\Sigma b_{p,t}(p) \leq \frac{n-1}{t} + C\varepsilon^2.$$

Inserting this in $(5*)$ then gives

$$\Delta_\Sigma f_{\varepsilon,t}(p) \leq \frac{n-1}{t} + C\varepsilon^2 - D\varepsilon,$$

which establishes the desired contradiction $\Delta_\Sigma f_{\varepsilon,t}(p) < 0$ for large t and small ε. $\Box$

We note the following corollaries that will be important in the proof of the WLST.

Corollary 2.2.36. (HSF intersecting horospheres)
Let γ be a timelike S-ray. Let $W \subset I^-(\gamma) \cap I^+(S)$ be an open set on which the GTCRC holds. Let $\Sigma \subset W$ be a connected acausal smooth spacelike hypersurface with $H_\Sigma \leq 0$. Suppose Σ and $\Sigma_c := W \cap \{b_\gamma = c\}$ intersect nontrivially and that $\Sigma \subset J^+_W(\Sigma_c)$. Then $\Sigma \subset \Sigma_c$.

Proof. Let $p \in \Sigma \cap \Sigma_c$, so $b(p) = c$. Now, given any $q \in \Sigma$, there exists a future directed causal curve $\eta : [0,1] \to W$ connecting $\eta(0) \in \Sigma_c$ and $q \in \Sigma$, since we assume that $\Sigma \subset J^+_W(\Sigma_c)$. By 2.2.7 (4), b is increasing along causal curves. Thus $b(q) = b(\eta(1)) \geq b(\eta(0)) = c$. Thus, b attains a minimum on Σ at p. 2.2.35 now implies that $b \equiv c$ on Σ, i.e. $\Sigma \subset \Sigma_c$. $\Box$

Corollary 2.2.37. (HSF contained in superlevel sets)
Let γ, W be as in 2.2.36. Let Σ be a smooth spacelike hypersurface with $H_\Sigma \leq 0$ such that $\overline{\Sigma} \subset\subset W$. Suppose that Σ is acausal in W. If $\mathrm{edge}(\Sigma) \subset \{b_\gamma \geq c\}$, then $\Sigma \subset \{b_\gamma \geq c\}$.

Proof. Since $\overline{\Sigma}$ is compact, $b \equiv b|_{\overline{\Sigma}} \equiv b_\gamma|_{\overline{\Sigma}}$ assumes a minimum on it. If that minimum is $\geq c$, then trivially $\Sigma \subset \{b \geq c\}$.
Suppose b assumes a minimum $\tilde{c} < c$ on Σ. Then by 2.2.35, b is constant on Σ (and equals $\tilde{c}$). By continuity, b is then also constant (and equal to $\tilde{c}$) on $\overline{\Sigma}$, contradicting the assumption that $\mathrm{edge}(\Sigma) \subset \{b \geq c\}$.
Since Σ is achronal in W, it is easily seen that $\overline{\Sigma} \setminus \Sigma \subset \mathrm{edge}(\Sigma)$, hence there are no more cases to be covered and we have proven the claim. □

2.2.7 Establishing a Local Splitting

At this point, we are finished with the preliminary results and shall proceed to prove the WLST, i.e. Theorem 2.2.2. For easier readability, we split the proof into several parts.

2.2.38 (Proof of WLST: Local Splitting I)**.**
Let $\gamma : \mathbb{R} \to M$ be a complete, timelike line in M, which we assume to be future directed and parametrized by g-arc length, i.e. $\langle \gamma'(t), \gamma'(t) \rangle = -1$. Let $\gamma^-(t) := \gamma(-t)$ be γ with the opposite orientation, so γ^- is past directed. Define

$$\begin{aligned} b_r^+(x) &:= r - d(x, \gamma(r)), \\ b_r^-(x) &:= r - d(\gamma^-(r), x) = r - d(\gamma(-r), x) \end{aligned}$$

and $b^\pm(x) := \lim_{r\to\infty} b_r^\pm(x)$. Since $\gamma|_{[a,\infty)}$ and $\gamma^-|_{[a,\infty)}$ are (future resp. past directed) timelike rays for any $a \in \mathbb{R}$, b^+ and b^- are well-defined and finite-valued on $I(\gamma) := I^+(\gamma) \cap I^-(\gamma)$ by 2.2.7. Note that all the regularity results (resp. their time duals) we established up to this point hold in a neighborhood of every point of the line γ. Also, by 2.2.7 (4) we have for $p, q \in I(\gamma)$, $p \leq q$,

$$b^+(q) \geq b^+(p) + d(p, q), \tag{2.2}$$
$$b^-(p) \geq b^-(q) + d(p, q). \tag{2.3}$$

Let now $q \in I(\gamma)$ and $s, t > 0$ such that $\gamma(-s) << q << \gamma(t)$. Then, using the inverse triangle inequality and the fact that γ is a line,

$$t + s = d(\gamma(-s), \gamma(t)) \geq d(\gamma(-s), q) + d(q, \gamma(t)),$$

and hence

$$b_t^+(q) + b_s^-(q) = t - d(q, \gamma(t)) + s - d(\gamma(-s), q) \geq 0.$$

Setting $s = t$ and then $t \to \infty$ to get

$$B := b^+ + b^- \geq 0 \text{ on } I(\gamma). \tag{2.4}$$

Similarly, one sees that $B(\gamma(t)) = 0$ for any $t \in \mathbb{R}$.

Let now U be a convex neighborhood of $\gamma(0)$ such that all the previously established regularity results (GTCRC, boundedness of initial tangents of asymptotes, Lipschitz continuity of the Busemann function, etc.) hold on U. Consider the horospheres

$$S^{\pm} := \{b^{\pm} = 0\} \cap U.$$

By 2.2.30, S^+ and S^- are partial Cauchy surfaces in U containing $\gamma(0)$ (since $b_r^+(\gamma(0)) = r - d(\gamma(0), \gamma(r)) = r - r = 0$ and hence $b^+(\gamma(0)) = 0$; similarly $b^-(\gamma(0)) = 0$). Thus, in particular, S^+ is a topological hypersurface of M, so we may find a small coordinate ball $W \subset S^+$ centered at $\gamma(0)$ with $\overline{W} \subset S^+$. At this point, we may invoke Bartnik's fundamental existence result on maximal hypersurfaces ([1], Thm. 4.1) to get a smooth spacelike hypersurface $\Sigma \subset U$ that is *maximal* ($:\Leftrightarrow H_\Sigma = 0$), acausal in U with compact closure, $\mathrm{edge}(\Sigma) = \mathrm{edge}(W) \subset S^+$, and Σ meets γ. We may assume

$$\Sigma \cap \mathrm{edge}(\Sigma) \neq \emptyset.$$

Since $\mathrm{edge}(\Sigma) \subset S^+ \subset \{b^+ \geq 0\}$ and $H_\Sigma = 0 \leq 0$, we may apply 2.2.37 to conclude $\Sigma \subset \{b^+ \geq 0\} \cap U$. Now, since $\mathrm{edge}(\Sigma) \subset S^+ \subset \{b^+ = 0\}$ and $b^- \geq -b^+$ on $I(\gamma)$ by (2.4), we also have that

$$\mathrm{edge}(\Sigma) \subset \{b^- \geq 0\} \cap U$$

and again by (the dual of) 2.2.37,

$$\Sigma \subset \{b^- \geq 0\} \cap U.$$

Putting both of these results together, we have

$$\Sigma \subset \{b^+ \geq 0\} \cap \{b^- \geq 0\}.$$

Now, because $\Sigma \subset \{b^+ \geq 0\} \cap U = J_U^+(\{b^+ = 0\})$, $\mathrm{edge}(\Sigma) \subset \{b^+ = 0\}$ and Σ intersects $\mathrm{edge}(\Sigma)$ nontrivially, 2.2.36 applies to give $\Sigma \subset \{b^+ = 0\} \cap U = S^+$, i.e. $b^+ \equiv 0$ on Σ.
Let now $x = \gamma(t_0)$ be a point of intersection of Σ with γ, then $b^+(x) = 0$ and, since $b^+ + b^- \equiv 0$ on γ, also $b^-(x) = 0$. Using additionally the fact that $\Sigma \subset \{b^- \geq 0\} \cap U = J_U^-(\{b^- = 0\})$, the time dual of 2.2.36 applies to give $\Sigma \subset \{b^- = 0\} \cap U = S^-$, i.e. $b^- \equiv 0$ on Σ. In conclusion, we have shown that

$$b^+ \equiv b^- \equiv 0 \text{ on } \Sigma. \tag{2.5}$$

2.2.39. (Proof of WLST: Local Splitting II)
Let now $x \in \Sigma$, and let $\alpha_x^+, \alpha_x^- : [0, \infty) \to M$ denote any timelike asymptote (future resp. past directed, parametrized by g-arc length) to $\gamma|_{[0,\infty)}$ resp.

$\gamma^-|_{[0,\infty)}$ at x. Using (2.5), we will show that α_x^- and α_x^+ fit at x to give a smooth timelike geodesic line α_x, so in particular $(\alpha_x^+)'(0) = -(\alpha_x^-)'(0)$.

To this end and unsurprisingly, we define the (possibly broken at $t = 0$) geodesic

$$\alpha_x(t) := \begin{cases} \alpha_x^+(t), & t \geq 0, \\ \alpha_x^-(-t), & t < 0. \end{cases}$$

By 2.2.14, for all $t \geq 0$ it holds that

$$b^+(\alpha_x(t)) = b^+(x) + d(x, \alpha_x(t)) = 0 + t = t \tag{2.6}$$

and similarly

$$b^-(\alpha_x(t)) = -t, \quad t < 0. \tag{2.7}$$

Let now $t < 0$ and use (2.2) with $p = \alpha_x(t) = \alpha_x^-(-t)$ and $q = x = \alpha_x(0)$:

$$0 = b^+(x) \overset{(2.2)}{\geq} b^+(\alpha_x(t)) + d(\alpha_x(t), x) = b^+(\alpha_x(t)) - t,$$

which implies that

$$b^+(\alpha_x(t)) \leq t \quad \text{for any } t < 0. \tag{2.8}$$

For $t < 0$ we also have that

$$0 \overset{(2.4)}{\leq} B(\alpha_x(t)) = b^+(\alpha_x(t)) + b^-(\alpha_x(t)) \overset{(2.7)}{=} b^+(\alpha_x(t)) - t,$$

giving $b^+(\alpha_x(t)) \geq t$ for $t < 0$, hence altogether $b^+(\alpha_x(t)) = t$ for $t < 0$, but using (2.6) we see that this holds for all $t \in \mathbb{R}$. Similarly, one shows that $b^-(\alpha_x(t)) = -t$ for all $t \in \mathbb{R}$. Let us recapitulate:

$$b^+(\alpha_x(t)) = t, \, b^-(\alpha_x(t)) = -t \quad \text{for all } t \in \mathbb{R}. \tag{2.9}$$

Let us now show that α_x maximizes the distance between any of its points. By the usual "maximizing $\Rightarrow$ unbroken" argument, it follows that α_x is a smooth maximal timelike geodesic line.
For this, take any $t_1 < 0 < t_2$ (this is the only case to be considered, as $\alpha_x^\pm$ are rays). Using (2.2) with $p = \alpha_x(t_1)$ and $q = \alpha_x(t_2)$, we get that

$$b^+(\alpha_x(t_2)) \geq b^+(\alpha_x(t_1)) + d(\alpha_x(t_1), \alpha_x(t_2))$$

and hence

$$d(\alpha_x(t_1), \alpha_x(t_2)) \leq t_2 - t_1$$

by (2.9).

Since $\alpha_x^{\pm}$ are parametrized by g-arc length, we have $d(\alpha_x(t_1), \alpha_x(t_2)) \geq L(\alpha_x|_{[t_1,t_2]}) = t_2 - t_1$, thus α_x is everywhere maximizing. By the discussion above, it follows that α_x is a smooth, maximal, timelike geodesic line.

Conclusion: For any $x \in \Sigma$, any future directed asymptote at x to $\gamma|_{[0,\infty)}$ and any past directed asymptote at x to $\gamma^-|_{[0,\infty)}$ fit together at x to give a smooth, maximal timelike geodesic line. This will be crucial for establishing the splitting near γ.

2.2.40 (Proof of WLST: Local Splitting III)**.**
Let us now show that for each $x \in \Sigma$ there is precisely one timelike line α_x passing through x that is asymptotic (in the sense of the line constructed in 2.2.39) to γ.

First, we show that α_x^+, α_x^- are Σ-rays for any $x \in \Sigma$. Indeed, for $q \in I^-(\gamma|_{[0,\infty)}) \cap I^+(\Sigma)$, we have that $0 < d(\Sigma, q) < b^+(q)$ by the proof of 2.2.7 (3). Taking $q = \alpha_x(t)$ for any $t > 0$, we obtain

$$d(\Sigma, \alpha_x(t)) \leq b^+(\alpha_x(t)) \overset{(2.9)}{=} t = L(\alpha_x|_{[0,t]}),$$

thus $L(\alpha_x|_{[0,t]}) = d(\Sigma, \alpha_x(t))$, showing that $\alpha_x^+ = \alpha_x|_{[0,\infty)}$ is a Σ-ray. Similarly, using time dual arguments, one sees that α_x^- is also a Σ-ray.
By focal point theory (cf. [2], Prop. 12.25 and Prop. 12.29, and generally [19], p. 281 - 299), it follows that Σ has no focal points along $\alpha_x^{\pm}$ and that α_x meets Σ orthogonally (i.e. that $\alpha_x'(0) \perp T_x\Sigma$).

Now we prove that for any $p, q \in \Sigma$, $p \neq q$, $\gamma_1 := \alpha_p^+$ and $\gamma_2 := \alpha_q^+$ do not meet.
Suppose that $m = \gamma_1(s) = \gamma_2(t)$ is an intersection point of γ_1 and γ_2. Then $s, t > 0$ since $p \neq q$. Since γ_1, γ_2 are g-unit speed Σ-rays, we have $s = d(\Sigma, m) = t$. Set $t_0 := t \equiv s > 0$, then

$$L(\gamma_1|_{[0,t_0]}) = d(\Sigma, m) = L(\gamma_2|_{[0,t_0]}).$$

Let now $\delta > 0$ and set $t_1 := t_0 + \delta$, $n := \gamma_1(t_1)$. Then, since γ_1 is a Σ-ray, we have (using the equality of lengths up to t_0 we just established)

$$d(\Sigma, n) = L(\gamma_1|_{[0,t_1]}) = L(\gamma_1|_{[0,t_0]}) + L(\gamma_1|_{[t_0,t_1]}) = L(\gamma_2|_{[0,t_0]}) + L(\gamma_1|_{[t_0,t_1]}).$$

Thus, if c is the curve via concatenation by first following γ_2 up to m and then following γ_1 up to n, then $L(c) = d(\Sigma, n)$. But since $p \neq q$, $\gamma_1'(t_0) \neq \gamma_2'(t_0)$ (otherwise $p = q$ by the unique solvability of the geodesic equation since γ_1, γ_2 meet at $m = \gamma_1(t_0) = \gamma_2(t_0)$). Thus, c has a "true" corner and therefore it cannot be maximizing, contradicting the fact that $d(\Sigma, n) = L(c)$.
Similarly, one shows that if $p, q \in \Sigma, p \neq q$, then α_p^- and α_q^- do not meet.

Next, we show that past and future asymptotes do not intersect, except maybe at the point of origin. More explicitly, let $p, q \in \Sigma$. We show that $\gamma_1 := \alpha_p^+$ and $\gamma_2 := \alpha_q^-$ do not intersect except possibly at $p = q$.
For the sake of contradiction, suppose $p \neq q$ and $m = \gamma_1(s) = \gamma_2(t)$ for $s, t > 0$. Then

$$b^+(m) = b^+(\gamma_1(s)) = b^+(\alpha_p^+(s)) = b^+(\alpha_p(s)) \overset{(2.9)}{=} s > 0.$$

On the other hand,

$$b^+(m) = b^+(\gamma_2(t)) = b^+(\alpha_q^-(t)) = b^+(\alpha_q(-t)) \overset{(2.9)}{=} -t < 0,$$

which is a contradiction.

2.2.41 (Proof of WLST: Local Splitting IV)**.**
We have made all the necessary preparations to establish the local splitting: Let N be a future pointing smooth timelike normal field to Σ, and consider

$$E : \mathbb{R} \times \Sigma \to M,\ E(t, q) := \exp_q(tN(q)). \tag{2.10}$$

Since the asymptotic lines α_q, $q \in \Sigma$, meet Σ orthogonally at q (and only at q), it holds that $E(t, q) = \alpha_q(t)$. By the nonintersection results from 2.2.40, E is injective. Since Σ has no focal points along any α_q, E is nonsingular. Thus, E is a diffeomorphism onto its image.
In this notation, (2.9) can be written as

$$b^\pm(E(t, q)) = \pm t \quad \text{for all } t \in \mathbb{R}.$$

Since E is a diffeomorphism onto its image, this shows that the Busemann functions $b^\pm$ are smooth on the open set $U_1 := E(\mathbb{R} \times \Sigma)$.
Let now $x \in \Sigma$ and let $\alpha_{x,r}$ denote the (unique, cf. 2.2.22 and 0.6.5 (2)) timelike geodesic from x to $\gamma(r)$, for sufficiently large r. Recalling the notation $b_r^+(x) = r - d(x, \gamma(r))$, we have $\operatorname{grad} b_r^+(x) = -\alpha'_{x,r}(0)$ by 2.2.24. Since the construction of asymptotes guarantees that $\alpha_{x,r} \to \alpha_x$ locally uniformly (cf. the proof of 2.2.9), and the tangents are in some compact set (cf 2.2.17 and 2.2.18) we have that

$$\alpha'_x(0) = \lim_{r \to \infty} \alpha'_{x,r}(0),$$

and thus

$$\operatorname{grad} b^+(x) = \lim_{r \to \infty} \operatorname{grad} b_r^+(x) = -\alpha'_x(0).$$

By starting at some later point on α_x, one can analogously show

$$\operatorname{grad} b^+(\alpha_x(t)) = -\alpha'_x(t) \quad \text{for any } t \in \mathbb{R}.$$

Let us summarize what has been accomplished so far: b^+ is smooth on U_1 and satisfies $\langle \operatorname{grad} b^+, \operatorname{grad} b^+ \rangle = -1$ on U_1 (since α_x is g-unit speed parametrized). Asymptotes to γ have tangents parallel to $\operatorname{grad} b^+$. Furthermore, for any $q \in U_1$ there is precisely one unit speed maximal timelike line asymptotic to γ through q. Note also that since Σ meets γ at some point x, it necessarily (cf. 2.2.20) holds that $\alpha_x = \gamma$. In particular, U_1 is a neighborhood of $\gamma(\mathbb{R})$.
Consider now

$$\Sigma_t := E(\{t\} \times \Sigma).$$

Since $b^+(E(t,q)) = t$, $\Sigma_t = U_1 \cap \{b^+ = t\}$. Hence $N = -\operatorname{grad} b^+$ is a smooth, future directed unit normal to Σ_t (globally).
Let us argue that N is parallel in U_1: Let $H(t)$ denote the mean curvature of Σ_t and let c be any timelike geodesic asymptotic to γ in U_1, then the vorticity-free Raychaudhuri equation (cf. [2], eq. (12.2)) translates into the following equation for $H(t)$:

$$H'(t) = -\operatorname{Ric}(N(c(t)), N(c(t))) - \operatorname{Tr}(\nabla N)^2 - \frac{1}{n-1} H(t)^2.$$

Suppose now that $N \circ c$ is not parallel along c. Since $\operatorname{Ric}(N,N) \geq 0$, we can estimate

$$H'(t) \leq -\frac{1}{n-1} H(t)^2.$$

From here, usual ODE arguments (cf. [2], Prop. 12.9) show that $H(t)$ blows up in finite time, which is absurd because the Σ_t are well-defined hypersurfaces for every $t \in \mathbb{R}$. Since U_1 can be covered by such c, it follows that N is globally parallel in U_1.
Having established that N is parallel in U_1, it follows precisely as in the Riemannian case (cf. 1.3.6) that the Σ_t are totally geodesic submanifolds of $\mathbb{R} \times \Sigma$ and that $E : (\mathbb{R} \times \Sigma, -dt^2 \oplus g|_\Sigma) \to (U_1 = E(\mathbb{R} \times \Sigma), g|_{U_1})$ is an isometry.

2.2.42 (Proof of WLST: What happened so far)**.**
Let us summarize what we have accomplished so far: There exists a neighborhood $U \supset \gamma(\mathbb{R})$ (this U was denoted U_1 so far) so that

1. $\Sigma = \{b^+ = 0 = b^-\} \cap U$ is a smooth, embedded, acausal hypersurface of M,
2. $(U, g|_U)$ is isometric to $(\mathbb{R} \times \Sigma, -dt^2 \oplus g|_\Sigma)$,
3. for each $q \in \Sigma$, the curve $c_q : \mathbb{R} \to U$, $c_q(t) := \exp_q(tN(q))$ is a maximal timelike line through q asymptotic to γ. Up to parametrization, any timelike asymptote to $\gamma|_{[a,\infty)}$ or $\gamma^-|_{[a,\infty)}$ (for any $a \in \mathbb{R}$) coincides with one of the c_q, which are precisely the flow lines of $N = -\operatorname{grad} b^+$, which is the future directed timelike unit normal on Σ.

2.2.8 Extending the Local Splitting to a Global Splitting

We shall now proceed to extend the local isometric splitting around γ that we just obtained to a global splitting on all of M. For this, we recall some details about parallel transport.

Remark 2.2.43. (Parallel translation of a parallel vector field)
(The notation used here is of general nature and has nothing to do with the specific notation used during the course of the proof so far).
Let $U \subset M$ be a connected open set and let $N \in \mathfrak{X}(M)$ be parallel on U, i.e. for any $q \in U$ and any $v \in T_qM$, $\nabla_v N = 0$. Fix now $q_1 \in U$ and let $v_1 := N(q_1) \in T_{q_1}M$. Then for any $q_2 \in U$, parallel translating v_1 along any path from q_1 to q_2 in U gives the same result; more precisely:

Lemma: Let $U \subset M$ be a connected open set and let $N \in \mathfrak{X}(M)$ be a vector field on M that is parallel on U. If $c, \tilde{c} : [0,1] \to U$ are smooth curves connecting $q_1 = c(0) = \tilde{c}(0)$ and $q_2 = c(1) = \tilde{c}(1)$, and $v_2, \tilde{v}_2 \in T_{q_2}M$ are the parallel translates of v_1 along $c, \tilde{c}$, then $v_2 = \tilde{v}_2$.

Proof. By uniqueness, since $N(c(0)) = N(q_1) = v_1 = N(\tilde{c}(0))$, $N \circ c$ and $N \circ \tilde{c}$ are precisely the parallel vector fields along c resp. $\tilde{c}$ starting at v_1. Thus

$$v_2 = N(c(1)) = N(q_2) = N(\tilde{c}(1)) = \tilde{v}_2.$$

$\square$

2.2.44 (Proof of WLST: Global Splitting I)**.**
Let now $\gamma_0 : \mathbb{R} \to M$ be a timelike line, and let W be a convex neighborhood of $p_0 = \gamma_0(0)$ such that all of our regularity results hold on W, in particular the Busemann functions b_0^+, b_0^- are continuous on W. Let W_0 be a smaller convex neighborhood of p_0 with $\overline{W}_0 \subset W$, such that there is an isometry $E : (\mathbb{R} \times \Sigma_0, -dt^2 \oplus h_0) \to (U_0, g|_{U_0})$, where $\Sigma_0 \subset W_0$, $h_0 = g|_{\Sigma_0}$ is the induced Riemannian metric, $E(t,q) = \exp_q(tN_0(q))$, $U_0 = E(\mathbb{R} \times \Sigma_0)$ and $N_0 = -\operatorname{grad} b_0^+$ is the future directed, timelike unit normal on Σ_0 which is globally parallel on U_0. All of this was established in 2.2.41.
Let now $p_1 \in \operatorname{edge}(\Sigma_0)$ and take $q_n \in \Sigma_0$ with $q_n = E(0, q_n) \to p_1$. Then by continuity

$$b_0^+(p_1) = \lim_{n\to\infty} \underbrace{b_0^+(E(0,q_n))}_{=0 \text{ by } (2.9)} = 0.$$

Similarly, $b_0^-(p_1) = 0$. Since Σ_0 is totally geodesic and W is convex, there exist h_0-unit speed parametrized h_0-geodesics $c_n : [0, a_n] \to \Sigma_0$ with $c_n(0) = p_0$ and $c_n(a_n) = q_n$. Since Σ_0 is a spacelike submanifold (i.e. Riemannian),

$\{c_n'(0)\} \subset T_{p_0}\Sigma_0$ have a limit w (up to a subsequence) that is also an h_0-unit vector. Put $c(t) := \exp_{p_0}(tw)$, where $c : [0, a) \to \Sigma_0$ is defined on its maximal interval $[0, a)$. Note that c is parametrized by h_0-arc length.
In the local splitting, $c_2(t) := (\sqrt{2}t, c(t)) \in \mathbb{R} \times \Sigma_0$ is a unit speed timelike geodesic (considered in M), because

$$(-dt^2 \oplus h_0)((\sqrt{2}, c'(t)), (\sqrt{2}, c'(t))) = -2 + h_0(c'(t), c'(t)) = -2 + 1 = -1.$$

Since M is timelike geodesically complete, c_2 and thus c extend to a, i.e. c extends to a geodesic $c_1 : [0, a] \to (M, g)$ with $c_1(a) = p_1$ (by continuity).

Let now $P \in \mathfrak{X}(c_1)$ be the unique parallel vector field along c_1 with $P(0) = \gamma_0'(0)$. Set $v := P(a) \in T_{p_1}M$ and $\gamma_1 : \mathbb{R} \to M$, $\gamma_1(t) := \exp_{p_1}(tv)$ (this is possible due to timelike geodesic completeness). Since γ_1 is the limit of $\sigma_n(t) := \exp_{c_1(s_n)}(tN_0(c_1(s_n)))$, where $s_n \to a$, $0 < s_n < a$, and by construction of the local splitting each σ_n is a maximal timelike line asymptotic to γ_0, it follows that γ_1 is also a maximal timelike line by 2.2.9.

Applying the local splitting to γ_1 (cf. 2.2.42), we get an isometry $E : (\mathbb{R} \times \Sigma_1, -dt^2 \oplus h_1) \to (U_1, g|_{U_1})$, where $U_1 \supset \gamma(\mathbb{R})$, $E(t, q) = \exp_q(tN_1(q))$, where $N_1 = -\operatorname{grad} b_1^+$ is globally parallel on U_1. Furthermore, $b_1^+(p_1) = b_1^-(p_1) = 0$ (this is generally true: The Busemann function of a given ray vanishes at the starting point of that ray).

Our goal is now to show that N_0 and N_1 agree on $U_0 \cap U_1$ (equivalently, b_0^+ and b_1^+ and hence also b_0^- and b_1^- agree on $U_0 \cap U_1$).
Let now a' be a real number such that $0 < a' < a$ and $q := c_1(a') \in \Sigma_0 \cap \Sigma_1$. By construction, $\gamma_1'(0) \in T_{p_1}M$ is the parallel translate of $\gamma_0'(0) \in T_{p_0}M$ along the curve c_1 from p_0 to p_1. Also, $N_0(q) = -\operatorname{grad} b_0^+(q)$ is obtained by parallel translation of $\gamma_0'(0)$ along c_1 from $t = 0$ to $t = a'$ (by definition, since N is parallel along c_1). Similarly, $N_1(q) = -\operatorname{grad} b_1^+(q)$ is obtained by parallel translation of $\gamma_1'(0) = N_1(p_1)$ along c_1 (going the opposite way) from $t = a$ to $t = a'$. Thus, by uniqueness of parallel transport (cf. 2.2.43), $N_0(q) = N_1(q)$.
Given any other $m \in \Sigma_0 \cap \Sigma_1$, take a smooth curve $\sigma : [0, 1] \to \Sigma_0 \cap \Sigma_1$ connecting $q = \sigma(0)$ to $m = \sigma(1)$ (exists, since Σ_0 and Σ_1 are totally geodesic). The same parallel translation argument gives that $N_1(m) = N_0(m)$ (since $N_0 \circ \sigma, N_1 \circ \sigma$ are precisely the parallel vector fields along σ). Thus, $N_0 = N_1$ on $\Sigma_0 \cap \Sigma_1$.
Now if $n \in U_0 \cap U_1$, we can use either splitting to find a smooth curve $\mu(t) = E(t, m)$ from some $m \in \Sigma_0 \cap \Sigma_1$ to n. Since $N_0(m) = N_1(m)$, the same parallel translation argument applied yet another time gives $N_0(n) = N_1(n)$. Thus, $N_0 = N_1$ on $U_0 \cap U_1$.
We have shown that $\operatorname{grad} b_0^+ = -N_0 = -N_1 = \operatorname{grad} b_1^+$ on $U_0 \cap U_1$. Since $\Sigma_0 \cap \Sigma_1 = \{b_0^+ = b_0^- = b_1^+ = b_1^- = 0\} \cap U_0 \cap U_1$ (in particular, b_0^+, b_1^+ agree on $\Sigma_0 \cap \Sigma_1$), we get that $b_0^+ = b_1^+$ on all of $U_0 \cap U_1$.

This allows us to combine our splittings to get a new, larger splitting: Set $\Sigma := \Sigma_0 \cup \Sigma_1$, $U := U_0 \cup U_1$ and $N := N_0 \cup N_1$ (this is well-defined since N_0 and N_1 agree on $U_0 \cap U_1$). Define the extended splitting $E : (\mathbb{R} \times \Sigma, -dt^2 \oplus h) \to (U, g|_U)$ by $E(t, q) := \exp_q(tN(q))$.

Let us now show how to extend a given local splitting $\mathbb{R} \times \Sigma_0 \to U_0$ in all directions. Choose enough points $\{p_1, \dots, p_k\} \subset \mathrm{edge}(\Sigma_0)$, convex neighborhoods W_i for p_i (as was done above), hypersurfaces Σ_i and splitting neighborhoods U_i, where we require that if $U_i \cap U_j \neq \emptyset$, then $\Sigma_i \cap \Sigma_j$ contains a point of the starting hypersurface Σ_0. With this, the local splittings $E_j : \mathbb{R} \times \Sigma_j \to U_j$ can be combined just like above to give a well-defined isometry $E : \mathbb{R} \times \Sigma \to U$, $E(t, q) = \exp_q(tN(q))$, where $\Sigma = \bigcup_i \Sigma_i$, $N = \bigcup_i N_i$ and $U = \bigcup_i U_i$.

2.2.45 (Proof of WLST: Global Splitting II)**.**
Our goal is to iterate the procedure in 2.2.44 to maximally extend a given local splitting so that the appearing hypersurface is maximal in the sense that it is edgeless, i.e. the splitting can no longer be continued and thus has to be global.
Let us reintroduce the scene: Suppose Σ is a smooth, spacelike hypersurface contained in U with a (global on U) timelike, future pointing unit normal N that is parallel in U, so that

$$E : (\mathbb{R}, \times\Sigma, -dt^2 \oplus h) \to (U, g|_U), \quad E(t, q) = \exp_q(tN(q))$$

is an isometry. Suppose $\mathrm{edge}(\Sigma) \neq \emptyset$ and pick some $p \in \mathrm{edge}(\Sigma)$. Let $p \in \overline{W}$, where $W \subset U$ is a convex open set contained in a larger convex open set. If we choose $p_0 \in W$ close to p such that p_0 is in some convex open set W_0 and $p \in \partial W_0$, then applying the local splitting to p_0 (which sits on a timelike line, since it is encompassed by the given splitting!) we get a hypersurface $\Sigma_0 \subset \Sigma \cap W_0$ with $p_0 \in \Sigma_0$ and $p \in \mathrm{edge}(\Sigma_0)$. As before, we may find a geodesic $c_1 : [0, 1] \to M$ with $c_1(0) = p_0$, $c_1(1) = p$ and $c_1([0, 1)) \subset \Sigma_0$. Parallel translating $N(p_0)$ along c_1 gives a timelike g-unit vector $w_1 \in T_pM$, so that $c(t) = \exp_p(tw_1)$ is a complete timelike line (this c was denoted γ_1 during the extension of the local splitting in 2.2.44).
We may now apply the local splitting to the line $c(t)$, get a hypersurface $\Sigma_1 \subset W_1$ where W_1 is a convex neighborhood of $c(0)$, and extend the given normals N_0 and N_1 to a normal N on $\Sigma_0 \cup \Sigma_1$, and thus extend the splitting.

It remains to show that this method of extending is independent of the choice of the point p_0. Thus let $q_0 \in W$ be another point in W close enough to p, Σ_1 the appropriate hypersurface, and $c_2 : [0, 1] \to M$ a geodesic connecting $c_2(0) = q_0$ and $c_2(1) = p$, and $c_2([0, 1)) \subset \Sigma_1$. Let $w_2 \in T_pM$ be the parallel translate of $N(q_0)$ along c_2. We need to show $w_1 = w_2$.
Suppose $w_1 \neq w_2$. Let P_j be the parallel vector field along c_j with $P_1(0) = N(p_0)$ resp. $P_2(0) = N(q_0)$. By the requirements on the c_j, $P_j(t) = N \circ c_j(t)$

for $0 \leq t < 1$. We may now replace p_0, q_0 by points p_0', q_0' on c_1 resp. c_2 close enough to p, such that after obtaining the associated splittings $E_j : (\mathbb{R} \times \Sigma_j', -dt^2 \oplus h_j) \to (U_j', g)$, we have $p \in \Sigma_1' \cap \Sigma_2'$. Thus, by our study of the compatibility of splittings, we have

$$w_1 = P_1(1) = -\operatorname{grad} b_1^+(p) = -\operatorname{grad} b_2^+(p) = P_2(1) = w_2.$$

This finishes the argument that the given local splitting can be extended to a global splitting

$$E : (\mathbb{R} \times S, -dt^2 \oplus h) \to (M, g).$$

2.2.46 (Proof of WLST: Completeness of the Riemannian factor)**.**
The only remaining issue that must be taken care of is the completeness of the Riemannian factor (S, h).
We have used a similar type of argument before, cf. 2.2.44: Suppose (S, h) is not complete. Then there is some h-unit speed geodesic $c : (a, b) \to S$ such that either $a < \infty$ or $b < \infty$, and c is inextendible to a or to b. Then $\gamma : (a, b) \to (M, g)$, $\gamma(t) := (\sqrt{2}t, c(t))$ (considered in the global splitting of (M, g)) is a g-unit speed timelike geodesic that is inextendible to either a or b, contradicting the fact that (M, g) is timelike geodesically complete.

Remark 2.2.47. (On the Proof presented here)
The proof we presented is the one given in [2], Chapter 14. We were able to simplify numerous technical steps due to our additional assumption of global hyperbolicity. However, most of the technicalities appear no matter the assumptions one makes. The bulk of the proof, as given here, applies to either case of the general LST, with (minor) modifications. It can, with very little extra work, be adapted to a proof of the variant of the LST where timelike geodesic completeness is assumed. To my knowledge, this simplified approach (first given in [9] for the timelike geodesically complete case) has not yet been rigorously adapted to the globally hyperbolic case (without the completeness assumption). This may be an interesting project for the future.

Abstract

The aim of this thesis is a discussion of the splitting theorems that appear in Riemannian and Lorentzian geometry.

After an initial preparatory chapter where we collect various necessary background topics for convenience, we turn to the Cheeger-Gromoll Splitting Theorem from Riemannian geometry in Chapter 1. Instead of following the original proof [4] of Cheeger and Gromoll that relies heavily on exploiting the ellipticity of the Riemannian Laplace operator, we instead present the proof along the lines of [5]. This turns out to be a major simplification over [4] and it avoids much of the heavy use of elliptic operator methods, making it adaptable to the Lorentzian situation, where the Laplacian is not elliptic. The main tools that come into play are support functions and a generalized maximum principle due to Calabi.

In Chapter 2, which represents the bulk of this thesis, we turn to the Lorentzian Splitting Theorem. It turns out that one can assume either timelike geodesic completeness or global hyperbolicity of the spacetime, and the techniques involved in the proofs of these two variants of the theorem differ. We discuss the relevant literature where the proofs of these various versions of the theorem may be found. Afterwards, we turn to the proof of the splitting theorem, assuming both timelike geodesic completeness and global hyperbolicity for the sake of a simplified presentation. The exposition is given along the lines of [2] Chapter 14, where global hyperbolicity is not assumed, hence that generalization can be obtained quite easily. With some effort, it should also be possible to adapt this proof to the globally hyperbolic case (without the completeness assumption), as was outlined in the original publication [9] of this proof. To my knowledge, this has not yet been worked out rigorously. It should be noted that this proof was a major simplification over any of the proofs that were available up to then in the literature for the various versions of the theorem, the key new idea being the "generalized timelike co-ray condition". The main procedure is quite similar to that of the Riemannian Splitting Theorem (rays, lines, Busemann functions, level sets thereof, inequalities of distance functions, etc.), but the use of elliptic/maximum principle methods is much more subtle and can only be done upon reducing the given situation to a suitable spacelike (= Riemannian) hypersurface. In this context, a fundamental result due to Bartnik [1] on the existence of nice hypersurfaces given rough boundary data plays a crucial role. Another striking difference is that one initially only obtains a local splitting around the given line, and then that splitting has to be extended by standard methods to give the desired global splitting of the entire spacetime.

Zusammenfassung

Das Ziel dieser Arbeit ist eine Darstellung der Spaltungstheoreme, die in der Riemann- und Lorentzgeometrie von Relevanz sind.

Nach einem vorbereitenden Kapitel, in dem wir notwendiges Hintergrundmaterial sammeln, wenden wir uns in Kapitel 1 dem Cheeger-Gromoll Spaltungstheorem aus der Riemanngeometrie zu. Dabei orientieren wir uns nicht am Originalbeweis [4] von Cheeger und Gromoll, in dem ganz essenziell Methoden aus der Theorie der elliptischen Operatoren zum Einsatz kommen. Stattdessen präsentieren wir eine abgeänderte Variante des Beweises aus [5], was sich als starke Vereinfachung herausstellt und wo die Techniken aus der Theorie der elliptischen Operatoren keine so fundamentale Rolle spielen wie in [4]. Das ist der Hauptgrund, warum sich der Beweis auf den Lorentzschen Fall anpassen lässt, denn der Laplace-Operator in einer Lorentzschen Mannigfaltigkeit ist hyperbolisch. Die Haupttechniken, die in unserer Version des Beweises zum Einsatz kommen, sind Barrierefunktionen und ein verallgemeinertes Maximumsprinzip von Calabi.

Kapitel 2 ist dem Lorentzschen Spaltungstheorem gewidmet und stellt den Hauptteil dieser Arbeit dar. Es zeigt sich, dass man im Rahmen dieses Theorems von der betrachteten Raumzeit entweder zeitartige geodätische Vollständigkeit oder globale Hyperbolizität annehmen kann. Dabei unterscheiden sich die Techniken, die in den Beweisen dieser beiden Versionen zum Einsatz kommen. In diesem Zusammenhang besprechen wir die relevante Fachliteratur. Danach widmen wir uns dem Beweis des Spaltungstheorem, machen aber beide der oben genannten Annahmen, um eine vereinfachte Präsentation zu ermöglichen. Hierbei folgen wir [2] Kapitel 14, wo die Annahme der globalen Hyperbolizität nicht gemacht wird, daher sollte es relativ einfach sein, aus unserer Darstellung auf diese allgemeine Version des Theorems zu schließen. Mit etwas Mühe sollte es ebenfalls nach [9] möglich sein, den hier präsentierten Beweis auf die Version des Theorems anzupassen, die nur mit der globalen Hyperbolizität auskommt. Nach meinem Wissen wurde das bisher noch nicht rigoros ausgeführt. An dieser Stelle sollte erwähnt werden, dass der hier präsentierte Beweis bei seiner Publikation in [9] eine deutliche Vereinfachung darstellte. Die allgemeine Vorgehensweise ist dem Riemannschen Fall sehr ähnlich, jedoch ist der Einsatz von elliptischen und Maximumsprinzipmethoden viel subtiler. Man schränkt etwa oft gegebene Objekte auf eine geeignete raumartige (= Riemannsche) Hyperfläche ein und arbeitet dort weiter. In diesem Kontext sei das fundamentale Existenzresultat von Bartnik [1] erwähnt, welches uns gerade solche geeigneten Hyperflächen liefert. Ein weiterer Unterschied zum Riemannschen Fall ist, dass man die Spaltung zuerst lokal beweist, und dann diese lokale Spaltung zu einer globalen ausdehnen muss.

Bibliography

[1] Bartnik, R. Regularity of variational maximal surfaces. Acta Mathematica 161, 1 (1988), 145–181.

[2] Beem, J. K., Ehrlich, P. E., Easley, K. L., et al. Global Lorentzian geometry. Marcel Dekker, Inc., 1996.

[3] Beem, J. K., Ehrlich, P. E., Markvorsen, S., Galloway, G. J., et al. Decomposition theorems for Lorentzian manifolds with nonpositive curvature. Journal of differential geometry 22, 1 (1985), 29–42.

[4] Cheeger, J., Gromoll, D., et al. The splitting theorem for manifolds of nonnegative Ricci curvature. Journal of Differential Geometry 6, 1 (1971), 119–128.

[5] Eschenburg, J., and Heintze, E. An elementary proof of the Cheeger-Gromoll splitting theorem. Annals of Global Analysis and Geometry 2, 2 (1984), 141–151.

[6] Eschenburg, J.-H. Comparison theorems and hypersurfaces. manuscripta mathematica 59, 3 (1987), 295–323.

[7] Eschenburg, J.-H., et al. The splitting theorem for space-times with strong energy condition. Journal of differential geometry 27, 3 (1988), 477–491.

[8] Flores, J. L. The Riemannian and Lorentzian splitting theorems. In Topics in Modern Differential Geometry. Springer, 2017, pp. 1–20.

[9] Galloway, G., and Horta, A. Regularity of Lorentzian Busemann functions. Transactions of the American Mathematical Society 348, 5 (1996), 2063–2084.

[10] Galloway, G. J., et al. The Lorentzian splitting theorem without the completeness assumption. Journal of differential geometry 29, 2 (1989), 373–387.

[11] Gomez, J. A. T. Teorema de descomposicion de Cheeger-Gromoll. Thesis, 2019.

[12] Gromoll, D., and Meyer, W. On complete open manifolds of positive curvature. Annals of Mathematics (1969), 75–90.

[13] Kunzinger, M. General Topology. Lecture Notes, `https://www.mat.univie.ac.at/~mike/teaching/ss16/general_topology.pdf`, 2016. Accessed: 17.09.2020.

[14] Lee, J. Introduction to topological manifolds, vol. 202. Springer Science & Business Media, 2010.

[15] Lee, J. M. Introduction to Smooth manifolds. Springer, 2013.

[16] Lee, J. M. Introduction to Riemannian manifolds. Springer, 2018.

[17] Minguzzi, E. Limit curve theorems in lorentzian geometry. Journal of mathematical physics 49, 9 (2008), 092501.

[18] Newman, R. P., et al. A proof of the splitting conjecture of s.-t. yau. Journal of differential geometry 31, 1 (1990), 163–184.

[19] O'neill, B. Semi-Riemannian geometry with applications to relativity. Academic press, 1983.

[20] Petersen, P. Riemannian Geometry, Third ed. Springer, 2016.

[21] Treude, J.-H. Ricci curvature comparison in Riemannian and Lorentzian geometry. Diploma Thesis, `https://freidok.uni-freiburg.de/data/8405`, 2011. Accessed: 17.09.2020.

[22] Yau, S.-T. Seminar on differential geometry. No. 102. Princeton University Press, 1982.