MASTERARBEIT / MASTER'S THESIS

Titel der Masterarbeit / Title of the Master's Thesis

"The Wave Equation in Hyperboloidal Similarity Coordinates"

verfasst von / submitted by

Matthias Ostermann, BSc BSc

angestrebter akademischer Grad / in partial fulfilment of the requirements for the degree of

Master of Science (MSc)

Wien, 2020 / Vienna, 2020

Studienkennzahl lt. Studienblatt / degree programme code as it appears on the student record sheet:	UA 066 821
Studienrichtung lt. Studienblatt / degree programme as it appears on the student record sheet:	Masterstudium Mathematik
Betreut von / Supervisor:	Assoz. Prof. Mag. Dr. Roland Donninger

Abriss

Wir präsentieren eine Konstruktion von Energienormen für die radiale Wellengleichung in ungeraden Raumdimensionen in einem kürzlich entwickelten Koordinatensystem, genannt *hyperboloidal similarity coordinates*. Unsere Strategie verwendet die bekannte Transformation der radialen Wellengleichung in ungeraden Raumdimensionen auf die eindimensionale Wellengleichung. Der Schlüssel dabei ist das Verständnis dieser Reduktion in den neuen Koordinaten in Schritten von zwei Dimensionen. Wir beweisen Abbildungseigenschaften der zugehörigen Operatoren und erhalten zu Sobolevnormen äquivalente Energienormen, welche zugleich gute Wachstumsabschätzungen für die freie Wellenevolution geben. Daraus folgen Existenz und Energieabschätzungen für Lösungen der freien Wellengleichung in hyperboloidal similarity coordinates. Dies liefert für weitere Anwendungen passende Topologien, welche neue Möglichkeiten zur Untersuchung der Stabilität von selbstähnlichem Blowup in Wellengleichungen auf der globalen Raumzeit durch hyperboloidal similarity coordinates eröffnen.

Abstract

We present a construction of energy norms for the radial wave equation in odd space dimensions in a recently developed coordinate system called *hyperboloidal similarity coordinates*. Our strategy adopts the well-known transformation of the radial wave equation in odd space dimensions to the one-dimensional wave equation. The key is to understand this reduction in the new coordinate system as an operation in steps of two space dimensions. We prove mapping properties for the associated operators and obtain energy norms that are equivalent to Sobolev norms and at the same time exhibit good growth estimates for the free wave evolution. As a result, existence of solutions and energy bounds for the free wave evolution in hyperboloidal similarity coordinates follow. As an outline for further applications, this provides suitable topologies which open the door for a rigorous study of stability of self-similar blowup in wave equations on the whole spacetime through hyperboloidal similarity coordinates.

Contents

1 Introduction

The free wave equation in $(1+d)$-dimensional Minkowski space $\mathbb{R}^{1,d}$ reads

$$\partial_t^2 u(t,x) - \Delta_x u(t,x) = 0\,. \tag{1.1}$$

If $u \in C^\infty(\mathbb{R}^{1,d})$ is radial there exists $\widehat{u} \in C^\infty(\mathbb{R}^{1,1})$ such that $\widehat{u}(t,\,.\,)$ is even and $u(t,\,.\,) = \widehat{u}(t,|\,.\,|)$. Under this symmetry assumption the wave equation takes the form of the **d-dimensional radial wave equation**

$$\partial_t^2 \widehat{u}(t,r) - \partial_r^2 \widehat{u}(t,r) - \tfrac{d-1}{r}\widehat{u}(t,r) = 0\,. \tag{1.2}$$

This type of equation appears in many interesting situations in some perturbed form, e.g., in the study of stability of self-similar blowup in wave equations or as models in the mathematical foundations of physics. Self-similar blowup at a finite blowup time $T \in \mathbb{R}$ is conveniently studied in coordinates that are compatible with self-similarity, meaning that the fraction $\frac{r}{T-t}$ is independent of the new time variable in such coordinates. One such novel system, that has been introduced in [1], are so-called **hyperboloidal similarity coordinates**. These coordinates are capable of covering the whole exterior region of the future light cone of the blowup event $(T,0)$ and thus make it possible to study the stability of blowup solutions on the global space, rather than only in the backward light cone. They are defined through

$$(t,x) = (T + \mathrm{e}^{-s} h(y), \mathrm{e}^{-s} y)$$

where $h(y) = \sqrt{2+|y|^2} - 2$. In order to deal with a perturbation theory for the radial wave equation in hyperboloidal similarity coordinates one has to develop a solid understanding of the free wave evolution in these coordinates. As the wave equation transforms to a geometric wave equation, this becomes algebraically rather evolved so that knowledge of the special structure of the equation will be essential. In particular, to control the wave evolution in these coordinates one is led to come up with energy norms that incorporate enough regularity and allow to prove energy bounds. In one space dimension the full range of energy bounds with exponential decay has been derived by Biernat-Donninger-Schörkhuber in [1] and extended to the wave evolution in five space dimensions where it was used to study wave maps on $(1+3)$-dimensional Minkowski spacetime. However, it was unclear how to generalize this to higher dimensions, as there are models to be analysed such as equivariant Yang-Mills fields on $(1+5)$-dimensional Minkowski space.

The set-up employed by Biernat-Donninger-Schörkhuber is based on the reduction of the five-dimensional radial wave equation to the one-dimensional wave equation that manifests itself as a commutation relation for the radial Laplace operator $r^{-(d-1)}\partial_r\big(r^{d-1}\partial_r \widehat{u}(t,r)\big)$. Such transformations exist and are well-known, provided that the space dimension d is odd, cf. [13, p. 74, Lemma 2]. More concretely, if u solves (1.2) we get by virtue of

$$\widetilde{u}(t,r) = \big(r^{-1}\partial_r\big)^{\frac{d-3}{2}} \big(r^{d-2}\widehat{u}(t,r)\big) \tag{1.3}$$

a solution to $\partial_t^2 \widetilde{u}(t,r) - \partial_r^2 \widetilde{u}(t,r) = 0$. If one tries to transfer this transformation directly to another coordinate system, the resulting algebraic expressions do become inaccessibly complicated. Apart from that, the transferred field will contain in general derivatives that are higher order in the new time variable which makes a systematic analysis practically impossible.

The key observation in overcoming this difficulty is that (1.3) arises from repeated applications of transformations of the type

$$r^{-(d-3)}\partial_r\big(r^{d-2}\widehat{u}(t,r)\big) \tag{1.4}$$

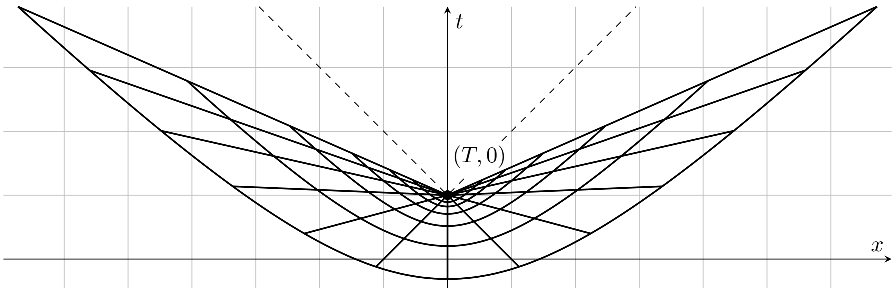

Figure 1. Hyperboloidal similarity coordinates [1]. The hyperboloids correspond to the level sets of s and are asymptotic to the boundary of the future light cone of $(T, 0)$ which is sent to $\{s = \infty\}$. The radial lines emerging from $(T, 0)$ are sets of constant y.

that map solutions of the radial wave equation in d dimensions to corresponding solutions in $d-2$ dimensions. Throughout this thesis we will refer to this as the **descent method**. As it turns out, the transformations in steps of two dimensions have very convenient mapping properties in hyperboloidal similarity coordinates that are the main subject of this thesis.

1.1 Notation

In order to deal with constants in inequalities we define on $\mathbb{R}$ the relation

$$x \lesssim y \quad :\Leftrightarrow \quad \exists C > 0 : x \leq Cy\,.$$

With this $x \gtrsim y :\Leftrightarrow y \lesssim x$ as well as $x \simeq y :\Leftrightarrow x \lesssim y$ and $x \gtrsim y$. We express uniformity in such inequalities by the attachment

$$x_\lambda \lesssim y_\lambda \quad \text{for all } \lambda\,,$$

meaning that there is a constant $C > 0$, independent of λ, such that for all λ the inequality $x_\lambda \leq C y_\lambda$ holds uniformly in λ.

We say $f \in C^\infty(\mathbb{R}^d)$ is radial if there exists an even function $\widehat{f} \in C^\infty(\mathbb{R})$ such that $f(x) = \widehat{f}(|x|)$ for all $x \in \mathbb{R}^d$.

If $\Omega \subseteq \mathbb{R}^d$ is open and $k \in \mathbb{N}_0$ we define for $f \in C^\infty_{\mathrm{c}}(\Omega)$ the Sobolev norm $\|\,.\,\|_{H^k(\Omega)}$ and homogeneous Sobolev norm $\|\,.\,\|_{\dot{H}^k(\Omega)}$ by

$$\|f\|_{H^k(\Omega)} = \sum_{|\alpha| \leq k} \|\partial^\alpha f\|_{L^2(\Omega)}\,, \qquad \|f\|_{\dot{H}^k(\Omega)} = \sum_{|\alpha| = k} \|\partial^\alpha f\|_{L^2(\Omega)}\,,$$

respectively. The corresponding Sobolev spaces $H^k(\Omega)$ are the completions of $C^\infty_{\mathrm{c}}(\Omega)$ with respect to $\|\,.\,\|_{H^k(\Omega)}$. Moreover, we have the closed subspace $H^k_{\mathrm{rad}}(\Omega)$ of radial H^k-functions that is the completion of smooth radial functions in the Sobolev norm. On intervals $\mathbb{B}_R \subseteq \mathbb{R}$ that are symmetric around the origin we will frequently consider the spaces $H^k_{\mathrm{even}}(\mathbb{B}_R)$, $H^k_{\mathrm{odd}}(\mathbb{B}_R)$ that arise from the completion of smooth functions that are even or odd, respectively.

We denote by $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space on $\mathbb{R}^d$ where we have the Fourier transform available. We introduce it conveniently by

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}\xi \cdot x} f(x)\, \mathrm{d}^d x$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$. The Sobolev norms on $\mathbb{R}^d$ may be equivalently introduced with the Fourier transform via

$$\|f\|_{H^s(\Omega)} = \|\langle\,.\,\rangle^s \mathcal{F} f\|_{L^2(\mathbb{R}^d)}\,, \qquad \|f\|_{\dot{H}^s(\Omega)} = \||\,.\,|^s \mathcal{F} f\|_{L^2(\mathbb{R}^d)}\,,$$

where $s > 0$ and $\langle x \rangle = \sqrt{1+|x|^2}$ denotes the Japanese bracket.

Throughout this thesis we make use of Einstein summation convention.

1.2 The result

Whenever u denotes a function on $\mathbb{R}^{1,d}$ in Cartesian coordinates (t,x), then v will denote this function on $\mathbb{R}^{1,d}$ in hyperboloidal similarity coordinates (s,y), i.e.,

$$v(s,y) = u(T + \mathrm{e}^{-s} h(y), \mathrm{e}^{-s} y)\,.$$

In particular, if $\partial^\mu \partial_\mu u = 0$ then u transforms to a function v that solves the geometric wave equation

$$\frac{1}{\sqrt{|\,\mathrm{g}\,|}} \partial_\mu \left(\sqrt{|\,\mathrm{g}\,|} \mathrm{g}^{\mu\nu} \partial_\nu v \right) = 0\,,$$

where g denotes the Minkowski metric expressed in hyperboloidal similarity coordinates. In our approach we formulate the wave equation within the notions of strongly continuous semigroups. We introduce $\mathbf{v}(s,\,.\,) = [\,v(s,\,.\,), \partial_s v(s,\,.\,)\,]$, then the latter equation is equivalent to a first order system

$$\partial_s \mathbf{v}(s,\,.\,) = \mathbf{L}_d \mathbf{v}(s,\,.\,) \tag{1.5}$$

for some spatial differential operators $\mathbf{L}_d$ that are properly introduced in section 2. As indicated earlier, in order to deal with perturbations of (1.5) one has to provide a sufficiently regular norm that controls the non-linearity and gives energy bounds on the free equation. This is one of the main tasks in this master's thesis that we solve by reducing it to a one-dimensional problem.

Proposition 1.1. *Let $R \in \mathbb{R}$, $d, k \in \mathbb{N}_0$ such that $R > 0$, $d \geq 3$ is odd and $k \geq \frac{d-3}{2}$. There is a bijective mapping*

$$\mathbf{D}_d : H^{k+1}_{\mathrm{rad}}(\mathbb{B}^d_R) \times H^k_{\mathrm{rad}}(\mathbb{B}^d_R) \to H^{k+1-\frac{d-3}{2}}_{\mathrm{odd}}(\mathbb{B}_R) \times H^{k-\frac{d-3}{2}}_{\mathrm{odd}}(\mathbb{B}_R)$$

with

$$\|\mathbf{D}_d \mathbf{f}\|_{H^{k+1-\frac{d-3}{2}}(\mathbb{B}_R) \times H^{k-\frac{d-3}{2}}(\mathbb{B}_R)} \simeq \|\mathbf{f}\|_{H^{k+1}(\mathbb{B}^d_R) \times H^k(\mathbb{B}^d_R)} \tag{1.6}$$

such that

$$\mathbf{D}_d \mathbf{L}_d \mathbf{f} - \mathbf{D}_d \mathbf{f} = \mathbf{L}_1 \mathbf{D}_d \mathbf{f} \tag{1.7}$$

for all $\mathbf{f} \in C^\infty_{\mathrm{rad}}(\overline{\mathbb{B}^d_R})^2$.

This result is constructive in the way that for an arbitrary but fixed odd space dimension d, the operator $\mathbf{D}_d$ is given explicitly. It enables us to reduce problems to the one-dimensional wave equation and transfer the solutions back to the original higher dimension. For instance, the equivalence of Sobolev norms (1.6) will yield energy bounds and behind the intertwining identity (1.7) is the reduction of the radial wave equation from d dimensions to one dimension from which we will obtain existence of solutions. This is based on the corresponding results in [1], reviewed in subsection 3.2, about the semigroup for the one-dimensional wave evolution in hyperboloidal similarity coordinates that is generated by $\mathbf{L}_1$. With this we arrive at our main result that is a systematic generalization of [1, Theorem 3.12] therein to odd space dimensions.

Theorem 1.1. *Let $R \in \mathbb{R}$, $d, k \in \mathbb{N}_0$ such that $R \geq \frac{1}{2}$, $d \geq 3$ is odd and $k \geq \frac{d-3}{2}$.*

1.) *There is a strongly continuous one-parameter semigroup*

$$\mathbf{S}_d : [0, \infty) \to \mathfrak{L}\big(H^{k+1}_{\text{rad}}(\mathbb{B}^d_R) \times H^k_{\text{rad}}(\mathbb{B}^d_R)\big)$$

generated by

$$\mathbf{L}_d = \mathbf{D}_d^{-1}(\mathbf{L}_1 + \mathbf{I})\mathbf{D}_d \tag{1.8}$$

such that

$$\|\mathbf{S}_d(s)\mathbf{f}\|_{H^{k+1}(\mathbb{B}^d_R) \times H^k(\mathbb{B}^d_R)} \lesssim \mathrm{e}^{s/2} \|\mathbf{f}\|_{H^{k+1}(\mathbb{B}^d_R) \times H^k(\mathbb{B}^d_R)} \tag{1.9}$$

for all $s \geq 0$ and $\mathbf{f} \in H^{k+1}_{\text{rad}}(\mathbb{B}^d_R) \times H^k_{\text{rad}}(\mathbb{B}^d_R)$.

2.) *Furthermore, if $k \geq \frac{d-3}{2} + 2$ and $\mathbf{f} \in C^\infty_{\text{rad}}(\overline{\mathbb{B}^d_R})^2$ let $\mathbf{v}(s, \,.\,) = \mathbf{S}_d(s)\mathbf{f}$ for $s \geq 0$. Then*

$$\partial_s \mathbf{v}(s, \,.\,) = \mathbf{L}_d \mathbf{v}(s, \,.\,)\,.$$

and $v = [\mathbf{v}]_1 \in C^2([0, \infty) \times \overline{\mathbb{B}_R})$ is a classical solution to the wave equation in hyperboloidal similarity coordinates.

1.3 Outline of applications

Our result paves the way to study stability of self-similar blowup in equivariant wave maps and Yang-Mills models on the spacetime region outside the future light cone of the blowup.

Wave maps equation. The semilinear wave equation

$$\partial_t^2 \psi(t, r) - \partial_r^2 \psi(t, r) - \frac{d-1}{r} \partial_r \psi(t, r) + \frac{d-1}{2} \frac{\sin(2\psi(t, r))}{r^2} = 0$$

arises for example in the study of equivariant wave maps $\mathbb{R}^{1,d} \to \mathbb{S}^d$ of Minkowski space into the d-sphere, cf. [4]. This model exhibits self-similar blowup solutions of the form

$$\psi_T(t, r) = 2 \arctan\left(\frac{1}{\sqrt{d-2}} \frac{r}{T-t}\right)\,.$$

Stability of this solution on the whole space has recently been solved for $d = 3$ in [1] with the aid of hyperboloidal similarity coordinates.

Yang-Mills models. Another instance where a semilinear wave equation of the form

$$\partial_t^2 \psi(t, r) - \partial_r^2 \psi(t, r) - \frac{d-3}{r} \partial_r \psi(t, r) + \frac{d-2}{r^2} F(\psi(t, r)) = 0\,,$$

with a nonlinearity $F(\psi) = \psi(\psi + 1)(\psi + 2)$, appears is in Yang-Mills theory, cf. [12]. In the energy-supercritical dimension $d = 5$ there is an explicit blowup solution

$$\psi_T(t, r) = W_0(\tfrac{r}{T-t}) - 1\,, \qquad W_0(\varrho) = \frac{1 - \varrho^2}{1 + \frac{3}{5}\varrho^2}\,.$$

Stability of this blowup solution in the backwards lightcone of the blowup event has been demonstrated by Donninger [12]. Stability on the whole space may be tackled with the results of this master's thesis.

Let us remark that the nonlinearity is singular at $r = 0$, so one considers instead the radial field $u(t, \,.\,) = \widehat{u}(t, |\,.\,|)$ such that $\widehat{u}(t, r) = r^2\psi(t, r)$. We end up with a semilinear-wave equation

$$\partial_t^2\widehat{u}(t,r) - \partial_r^2\widehat{u}(t,r) - \frac{d+1}{r}\widehat{u}(t,r) + \big(d-2\big)\big(3\widehat{u}(t,r)^2 + r^2\widehat{u}(t,r)^3\big)$$

in effectively $d+2$ dimensions. This field scales like e^{2s}. If we take this factor in the energy bounds from theorem 1.1 into account, we have in this sense indeed an exponential decay estimate.

1.4 Structure of the thesis

The goal of this thesis is to develop tools for a perturbation theory of the radial wave equation in hyperboloidal similarity coordinates, as outlined in the introduction.

In section 2 we introduce hyperboloidal similarity coordinates and study their geometry. We prepare quantities such as the Jacobian, metric, Christoffel symbols and the Laplace-Beltrami operator. This serves as a reference and background for computations made throughout this thesis.

In section 3 we discuss how a standard energy estimate can be obtained for the radial wave equation in hyperboloidal similarity coordinates. We also review the results for the one-dimensional wave evolution that have been established exhaustively by Biernat-Donninger-Schörkhuber [1]. This case can be viewed as a base case to which, as we will ultimately show, problems in higher dimensions can be reduced to. One exploits that there are vector fields that commute with the half-wave equation and lead to energy bounds that feature exponential decay for the free wave evolution. However, it is not clear how the technique that leads in one dimension to the energy bounds generalizes to higher dimensions, if at all. The next attempt challenges us on three different levels. First of all, we have to incorporate higher derivatives in our energy. As the standard energy only provides a seminorm on open balls, a further crucial step is to come up with a full norm with sufficient regularity. At the same time, all of this has to be compatible with the free wave evolution so that the topology of the norm does not obstruct decay estimates in hyperboloidal similarity coordinates. Therefore we consider transformations that preserve radial-type wave equations and this will lead us to a re-discovery of the reduction of dimension for the wave equation. We refer to this as the descent method.

In section 4 we develop the descent method in hyperboloidal similarity coordinates. This is formulated appropriately within the picture of strongly continuous semigroups. We construct descent operators that govern the reduction of the dimension in the radial wave equation and show that they are bounded and bijective with bounded inverse. The inverse arises as an explicit solution to a Sturm-Liouville problem. The bounds come from estimates of a Duhamel integral. To some significant extent, we rely on the subtle structure of the coefficients of the descent operators for this technical proof to work out. The successive application of these descent operators gives the mapping that is anticipated in proposition 1.1. This gives a solution to the problem of finding a compatible norm with the desired properties that were set out previously. We conclude with the proof of theorem 1.1 which will be the main part of this thesis.

2 Geometry of hyperboloidal similarity coordinates

We denote by $\mathbb{R}^{1,d}$ the $(1+d)$-dimensional Minkowski space $(\mathbb{R}\times\mathbb{R}^d,\eta)$ where $d\in\mathbb{N}$ is the space dimension. Let $T\in\mathbb{R}$ be a parameter. **Hyperboloidal similarity coordinates** are defined by

$$\eta_T:\mathbb{R}^{1,d}\to\mathbb{R}^{1,d}\,,\quad (s,y)\mapsto(T+\mathrm{e}^{-s}h(y),\mathrm{e}^{-s}y)\,,$$

where

$$h(y):=\sqrt{2+|y|^2}-2\,,\quad y\in\mathbb{R}^d\,,$$

is called **height function**, throughout all dimensions. These coordinates define a diffeomorphism onto their image $\Omega_T=\{(t,x)\in\mathbb{R}\times\mathbb{R}^d\mid -(T-t)<|x|\}$ which is the complement of the future light cone at $(T,0)$. The inverse is given by

$$\eta_T^{-1}:\Omega_T\to\mathbb{R}^{1,d}\,,\quad (t,x)\mapsto\big(\log g_T(t,x),g_T(t,x)x\big)\,,$$

where

$$g_T(t,x)=\frac{2(T-t)-\sqrt{2}\sqrt{(T-t)^2+|x|^2}}{(T-t)^2-|x|^2}\,,\quad (t,x)\in\{-(T-t)<|x|\}\,.$$

Unless the specific form of h is fixed, hyperboloidal similarity coordinates can be viewed as a general family of coordinate systems. Their essential property is that the set $\{(h(y),y)\in\mathbb{R}^{1,d}\mid y\in\mathbb{R}^d\}$ is a Cauchy surface asymptotic to the forward light cone, cf. [1]. Note, for example, for the choice $h(y)=-1$ resp. $g_T(t,x)=(T-t)^{-1}$ we retrieve **standard similarity coordinates**

$$\sigma_T:\mathbb{R}^{1,d}\to\mathbb{R}^{1,d}\,,\quad (\tau,\xi)\mapsto(T-\mathrm{e}^{-\tau},\mathrm{e}^{-\tau}\xi)\,.$$

They only cover the portion $\{(t,x)\in\mathbb{R}^{1,d}\mid t<T\}$ of Minkowski space.

2.1 Jacobian of hyperboloidal similarity coordinates

Let us compute the Jacobi matrices for hyperboloidal similarity coordinates. We find

$$[\partial\eta_T](s,y)=\mathrm{e}^{-s}\begin{bmatrix}-h(y) & \partial_y h(y)\\ -y & \mathrm{I}_d\end{bmatrix}$$

and

$$[\partial\eta_T^{-1}](t,x)=\begin{bmatrix}\frac{\partial_t g(t,x)}{g(t,x)} & \frac{1}{g(t,x)}\partial_x g(t,x)\\ \partial_t g(t,x)x & x\partial_x g(t,x)+g(t,x)\mathrm{I}_d\end{bmatrix}\,.$$

For $i,j=1,\dots,d$ we read off the components

$$\partial_0\eta_T^0(s,y)=-\mathrm{e}^{-s}h(y)\,,\qquad \partial_i\eta_T^0(s,y)=\mathrm{e}^{-s}\partial_{y^i}h(y)\,,$$
$$\partial_0\eta_T^i(s,y)=-\mathrm{e}^{-s}y^i\,,\qquad \partial_j\eta_T^i(s,y)=\mathrm{e}^{-s}\delta^i_j\,,$$

and

$$\partial_0{\eta_T^{-1}}^0(t,x)=\frac{\partial_t g(t,x)}{g(t,x)}\,,\qquad \partial_i{\eta_T^{-1}}^0(t,x)=\frac{\partial_{x^i}g(t,x)}{g(t,x)}\,,$$
$$\partial_0{\eta_T^{-1}}^i(t,x)=x^i\partial_t g(t,x)\,,\qquad \partial_j{\eta_T^{-1}}^i(t,x)=x^j\partial_{x^i}g(t,x)+g(t,x)\delta^i_j\,.$$

By block matrix inversion, or direct computation, the inverse matrices

$$[\partial\eta_T]^{-1}(s,y) = \mathrm{e}^s \begin{bmatrix} \frac{1}{\partial_y h(y) y - h(y)} & -\frac{\partial_y h(y)}{\partial_y h(y) y - h(y)} \\ \frac{y}{\partial_y h(y) y - h(y)} & \mathrm{I}_d - \frac{y\partial_y h(y)}{\partial_y h(y) y - h(y)} \end{bmatrix},$$

and

$$[\partial\eta_T^{-1}]^{-1}(t,x) = \begin{bmatrix} (T-t) & -\frac{1}{g(t,x)} \frac{\partial_x g(t,x)}{\partial_t g(t,x)} \\ -x & \frac{1}{g(t,x)}\mathrm{I}_d \end{bmatrix},$$

are obtained. For $i,j = 1,\dots,d$ the components are

$$(\partial_0 \eta_T^{-1\,0} \circ \eta_T)(s,y) = \frac{\mathrm{e}^s}{y^k\partial_{y^k}h(y) - h(y)}, \quad (\partial_i \eta_T^{-1\,0} \circ \eta_T)(s,y) = -\mathrm{e}^s \frac{\partial_{y^i}h(y)}{y^k\partial_{y^k}h(y) - h(y)},$$
$$(\partial_0 \eta_T^{-1\,i} \circ \eta_T)(s,y) = \frac{\mathrm{e}^s y^i}{y^k\partial_{y^k}h(y) - h(y)}, \quad (\partial_j \eta_T^{-1\,i} \circ \eta_T)(s,y) = \mathrm{e}^s \left(\delta^i_j - \frac{y^i\partial_{y^j}h(y)}{y^k\partial_{y^k}h(y) - h(y)}\right),$$

and

$$(\partial_0 \eta_T^0 \circ \eta_T^{-1})(t,x) = T - t\,, \qquad (\partial_i \eta_T^0 \circ \eta_T^{-1})(t,x) = \frac{1}{g(t,x)} \frac{\partial_{x^i} g(t,x)}{\partial_t g(t,x)},$$
$$(\partial_0 \eta_T^i \circ \eta_T^{-1})(t,x) = -x^i\,, \qquad (\partial_j \eta_T^i \circ \eta_T^{-1})(t,x) = \frac{1}{g(t,x)}\delta^i_j\,.$$

It is easy to read off the determinant from the Jacobian, note that we can simply add a y^i-multiple of the $(i+1)$-st column to the first column to see

$$\det[\,\partial\eta_T\,](s,y) = \mathrm{e}^{-(d+1)s} \det \begin{bmatrix} \partial_y h(y) y - h(y) & \partial_y h(y) \\ 0 & \mathrm{I}_d \end{bmatrix} = \mathrm{e}^{-(d+1)s}\big(\partial_y h(y) y - h(y)\big)\,. \tag{2.1}$$

In the following set $v = u \circ \eta_T$ for $u \in C^\infty(\mathbb{R}^{1,d})$. As an immediate consequence we obtain that partial derivatives transform to

$$(\partial_0 u \circ \eta_T)(s,y) = \frac{\mathrm{e}^s}{y^k\partial_{y^k}h(y) - h(y)}\big(\partial_s + y^k\partial_{y^k}\big)v(s,y)\,, \tag{2.2}$$

$$(\partial_j u \circ \eta_T)(s,y) = \mathrm{e}^s\partial_j v(s,y) - \frac{\mathrm{e}^s\partial_{y^j}h(y)}{y^k\partial_{y^k}h(y) - h(y)}\big(\partial_s + y^k\partial_{y^k}\big)v(s,y)\,. \tag{2.3}$$

2.2 Metric associated with hyperboloidal similarity coordinates

The metric associated with hyperboloidal similarity coordinates is

$$[\,\mathrm{g}\,] = [\,\partial\eta_T\,]^{\mathrm{t}}[\,\eta\,][\,\partial\eta_T\,]\,,$$

in components,

$$\mathrm{g}_{00}(s,y) = \mathrm{e}^{-2s}\left(-h(y)^2 + |y|^2\right)\,, \tag{2.4}$$
$$\mathrm{g}_{i0}(s,y) = \mathrm{g}_{0i}(s,y) = \mathrm{e}^{-2s}\left(h(y)\partial_{y^i}h(y) - y_i\right)\,, \tag{2.5}$$
$$\mathrm{g}_{ij}(s,y) = \mathrm{e}^{-2s}\left(\delta_{ij} - \partial_{y_i}h(y)\partial_{y_j}h(y)\right)\,. \tag{2.6}$$

The inverse metric is given by

$$[\,\mathrm{g}\,]^{-1} = [\,\partial\eta_T\,]^{-1}[\,\eta\,]^{-1}[\,\partial\eta_T\,]^{-\mathrm{t}}\,,$$

so the components read

$$\mathrm{g}^{00}(s,y) = \mathrm{e}^{2s}\frac{-1+\partial_{y^k}h(y)\partial_{y_k}h(y)}{\big(y^k\partial_{y^k}h(y)-h(y)\big)^2}\,, \tag{2.7}$$

$$\mathrm{g}^{i0}(s,y) = \mathrm{g}^{0i}(s,y) = \mathrm{e}^{2s}\left(\frac{-1+\partial_{y^k}h(y)\partial_{y_k}h(y)}{\big(y^k\partial_{y^k}h(y)-h(y)\big)^2}y^i - \frac{\partial_{y_i}h(y)}{y^k\partial_{y^k}h(y)-h(y)}\right)\,, \tag{2.8}$$

$$\mathrm{g}^{ij}(s,y) = \mathrm{e}^{2s}\left(\delta^{ij} + \frac{-1+\partial_{y^k}h(y)\partial_{y_k}h(y)}{\big(y^k\partial_{y^k}h(y)-h(y)\big)^2}y^iy^j - \frac{y^j\partial_{y_i}h(y)+y^i\partial_{y_j}h(y)}{y^k\partial_{y^k}h(y)-h(y)}\right)\,. \tag{2.9}$$

The metric determinant $\sqrt{|\,\mathrm{g}\,|} = |\det[\,\partial\eta_T\,]|$ reads

$$\sqrt{|\,\mathrm{g}\,|}(s,y) = \mathrm{e}^{-(d+1)s}\left(y^k\partial_{y^k}h(y)-h(y)\right)\,.$$

2.3 Christoffel symbols of the metric

The Christoffel symbols of the Minkowski metric transform in hyperboloidal similarity coordinates to

$$\Gamma^{\lambda}{}_{\mu\nu} = (\partial_\mu\partial_\nu\eta_T^\kappa)(\partial_\kappa{\eta_T^{-1}}^\lambda\circ\eta_T)\,,$$

where $\lambda,\mu,\nu = 0,1,\dots,d$. This gives the expressions

$$\Gamma^0{}_{00}(s,y) = -1\,,\quad \Gamma^0{}_{i0}(s,y) = \Gamma^0{}_{0i}(s,y) = 0\,,\qquad \Gamma^0{}_{ij}(s,y) = \frac{\partial_{y^i}\partial_{y^j}h(y)}{y^k\partial_{y^k}h(y)-h(y)}\,, \tag{2.10}$$

$$\Gamma^k{}_{00}(s,y) = 0\,,\qquad \Gamma^k{}_{i0}(s,y) = \Gamma^k{}_{0i}(s,y) = -\delta^k_i\,,\quad \Gamma^k{}_{ij}(s,y) = \frac{\partial_{y^i}\partial_{y^j}h(y)}{y^k\partial_{y^k}h(y)-h(y)}y^k\,, \tag{2.11}$$

where $i,j,k = 1,\dots,d$. Also recall the identity

$$\frac{1}{\sqrt{|\mathrm{g}|}}\partial_\mu\big(\mathrm{g}^{\mu\nu}\sqrt{|\mathrm{g}|}\big) = -\mathrm{g}^{\alpha\beta}\Gamma^{\nu}{}_{\alpha\beta}\,. \tag{2.12}$$

2.4 Laplace-Beltrami operator and the wave equation

Let $u\in C^\infty(\mathbb{R}^{1,d})$ and $v = u\circ\eta_T$, i.e.,

$$v(s,y) = u(T+\mathrm{e}^{-s}h(y),\mathrm{e}^{-s}y)\,.$$

The wave operator $\partial^\mu\partial_\mu u = \eta^{\mu\nu}\partial_\mu\partial_\nu u$ is given in hyperboloidal similarity coordinates by the Laplace-Beltrami operator

$$(\partial^\mu\partial_\mu u)\circ\eta_T = \frac{1}{\sqrt{|\,\mathrm{g}\,|}}\partial_\mu\left(\sqrt{|\,\mathrm{g}\,|}\mathrm{g}^{\mu\nu}\partial_\nu v\right) = \mathrm{g}^{\mu\nu}\partial_\mu\partial_\nu v - \mathrm{g}^{\mu\nu}\Gamma^\lambda{}_{\mu\nu}\partial_\lambda v\,.$$

If u solves the free wave equation $\partial^\mu\partial_\mu u = 0$ then this implies for v the equation

$$\begin{aligned}\partial_0^2 v = &-\frac{\mathrm{g}^{ij}}{\mathrm{g}^{00}}\partial_i\partial_j v - 2\frac{\mathrm{g}^{i0}}{\mathrm{g}^{00}}\partial_i\partial_0 v\\ &+\left(\Gamma^0{}_{00} + 2\frac{\mathrm{g}^{0i}}{\mathrm{g}^{00}}\Gamma^0{}_{0i} + \frac{\mathrm{g}^{ij}}{\mathrm{g}^{00}}\Gamma^0{}_{ij}\right)\partial_0 v + \left(\Gamma^k{}_{00} + 2\frac{\mathrm{g}^{0i}}{\mathrm{g}^{00}}\Gamma^k{}_{0i} + \frac{\mathrm{g}^{ij}}{\mathrm{g}^{00}}\Gamma^k{}_{ij}\right)\partial_k v\,.\end{aligned}$$

Explicitly,

$$\partial_0^2 v = c^{ij}\partial_i\partial_j v + c^{i0}\partial_i\partial_0 v + c^0\partial_0 v + c^i\partial_i v \tag{2.13}$$

with

$$\begin{aligned}c^{ij}(y) &= -\frac{y^k\partial_{y^k}h(y) - h(y)}{-1+|\partial_y h(y)|^2}\big((y^k\partial_{y^k}h(y) - h(y))\delta^{ij} - y^i\partial_{y_j}h(y) - y^j\partial_{y_i}h(y)\big) - y^iy^j\,,\\ c^{i0}(y) &= -2y^i + 2\frac{y^k\partial_{y^k}h(y) - h(y)}{-1+|\partial_y h(y)|^2}\partial_{y_i}h(y)\,,\\ c^0(y) &= -1 + \frac{y^k\partial_{y^k}h(y) - h(y)}{-1+|\partial_y h(y)|^2}\partial_{y^k}\partial_{y_k}h(y) + \frac{y^iy^j}{y^k\partial_{y^k}h(y) - h(y)}\partial_{y^i}\partial_{y^j}h(y)\\ &\quad - \frac{y^i\partial_{y_j}h(y) + y^j\partial_{y_i}h(y)}{-1+|\partial_y h(y)|^2}\partial_{y^i}\partial_{y^j}h(y)\,,\\ c^i(y) &= c^{i0}(y) + (c^0(y)+1)y^i\,.\end{aligned}$$

From here we get the wave operator in hyperboloidal similarity coordinates in semigroup formulation. If we denote $\mathbf{v}(s,\,.\,) = [\,v(s,\,.\,), \partial_s v(s,\,.\,)\,]$, then equation (2.13) is equivalent to

$$\partial_s\mathbf{v}(s,\,.\,) = \mathbf{L}_d\mathbf{v}(s,\,.\,)$$

with the spatial differential operator

$$\mathbf{L}_d\mathbf{f} := \begin{bmatrix} f_2 \\ c^{ij}\partial_i\partial_j f_1 + c^i\partial_i f_1 + c^0 f_2 + c^{i0}\partial_i f_2\end{bmatrix}$$

defined for $\mathbf{f} = [\,f_1, f_2\,] \in C^\infty(\mathbb{R}^d)^2$.

3 Energy norm for the radial wave equation

In this section we denote by $u \in C^\infty(\mathbb{R}^{1,d})$ a generic smooth radial function and by $\widehat{u} \in C^\infty(\mathbb{R}^{1,1})$ its radial representative. That is, $\widehat{u}(t,\,.\,)$ is an even function such that $u(t,\,.\,) = \widehat{u}(t,|\,.\,|)$. Moreover, in hyperboloidal similarity coordinates we denote

$$v(s,y) = u(T + \mathrm{e}^{-s}h(y), \mathrm{e}^{-s}h(y)) \quad\text{and}\quad \widehat{v}(s,\eta) := \widehat{u}(T + \mathrm{e}^{-s}h(\eta), \mathrm{e}^{-s}\eta)\,.$$

Note the relation

$$v(s,y) = u(T + \mathrm{e}^{-s}h(y), \mathrm{e}^{-s}y) = \widehat{u}(T + \mathrm{e}^{-s}h(y), \mathrm{e}^{-s}|y|) = \widehat{v}(s,|y|)\,.$$

The first important step is to provide a suitable norm in which the radial wave evolution is studied. This norm will be derived from conserved quantities of the free wave equation. As the wave equation in hyperboloidal similarity coordinates is structurally rather complicated as a geometric wave equation we rely on observations in radial Cartesian coordinates that carry over to general coordinate systems. In order to see what can be expected we consider the standard energy of the radial d-dimensional wave equation

$$\partial_t^2 \widehat{u}(t,r) - r^{-(d-1)}\partial_r \left(r^{d-1}\partial_r \widehat{u}(t,r)\right) = 0\,. \tag{3.1}$$

An important domain is the backward light cone, so we want to adapt our quantities to the geometry of this region eventually. For this purpose, the latter equation is considered in terms of the vector fields $\partial_t \pm \partial_r$. Adding a smart zero then gives

$$\big(\partial_t \mp \partial_r\big)\big(r^{d-1}(\partial_t \pm \partial_r)\widehat{u}(t,r)\big) \pm (d-1)r^{d-2}\partial_t \widehat{u}(t,r) = 0\,. \tag{3.2}$$

We introduce the forward and backward half-wave

$$\widehat{u}_\pm(t,r) \coloneqq \partial_t \widehat{u}(t,r) \pm \partial_r \widehat{u}(t,r) \tag{3.3}$$

so that (3.2) becomes

$$\big(\partial_t \mp \partial_r\big)\big(\widehat{u}_\pm(t,r)r^{d-1}\big) \pm \tfrac{d-1}{2r}\big(\widehat{u}_+(t,r)r^{d-1} + \widehat{u}_-(t,r)r^{d-1}\big) = 0\,. \tag{3.4}$$

This is the starting point for the derivation of an energy estimate for which it is clear how it translates to hyperboloidal similarity coordinates. Testing with $\widehat{u}_\pm(t,r)$ yields

$$\big(\partial_t \mp \partial_r\big)\big(\widehat{u}_\pm(t,r)^2 r^{d-1}\big) \pm \tfrac{d-1}{r}\widehat{u}_+(t,r)\widehat{u}_-(t,r)r^{d-1} = 0\,. \tag{3.5}$$

This gives the differential energy conservation

$$\big(\partial_t - \partial_r\big)\big(\widehat{u}_+(t,r)^2 r^{d-1}\big) + \big(\partial_t + \partial_r\big)\big(\widehat{u}_-(t,r)^2 r^{d-1}\big) = 0\,. \tag{3.6}$$

Remark 3.1. Incidentally, we arrive at this conservation law by just considering linear combinations of the energy- and momentum-conservation. In dimension $d = 1$ the picture is somehow different as the equations for the fields $\widehat{u}_\pm$ decouple and make a separate analysis possible. It seems that this is not present in higher dimensions and causes obstructions.

3.1 Standard energy in hyperboloidal similarity coordinates

Now we transform the equations from above to hyperboloidal similarity coordinates. The half-waves are

$$\widehat{v}_\pm(s,\eta) \coloneqq \widehat{u}_\pm(T + \mathrm{e}^{-s}h(\eta), \mathrm{e}^{-s}\eta)\,.$$

Equation (3.6) implies

$$\begin{aligned}
0 = \partial_s\Big(&\mathrm{e}^{-(d-1)s}\big((1+h'(\eta))\widehat{v}_+(s,\eta)^2 + (1-h'(\eta))\widehat{v}_-(s,\eta)^2\big)\eta^{d-1}\Big)\\
+\partial_\eta\Big(&\mathrm{e}^{-(d-1)s}\big((\eta+h(\eta))\widehat{v}_+(s,\eta)^2 + (\eta-h(\eta))\widehat{v}_-(s,\eta)^2\big)\eta^{d-1}\Big)\\
&-\mathrm{e}^{-(d-1)s}\big((1+h'(\eta))\widehat{v}_+(s,\eta)^2 + (1-h'(\eta))\widehat{v}_-(s,\eta)^2\big)\eta^{d-1}
\end{aligned}$$

which gives multiplied with e^{-s}

$$\begin{aligned}
0 = \partial_s\Big(&\mathrm{e}^{-ds}\big((1+h'(\eta))\widehat{v}_+(s,\eta)^2 + (1-h'(\eta))\widehat{v}_-(s,\eta)^2\big)\eta^{d-1}\Big)\\
+\partial_\eta\Big(&\mathrm{e}^{-ds}\big((\eta+h(\eta))\widehat{v}_+(s,\eta)^2 + (\eta-h(\eta))\widehat{v}_-(s,\eta)^2\big)\eta^{d-1}\Big)\,.
\end{aligned}$$

This gives integrated across the set $\{0 \le \eta \le R\}$ for some fixed $R \ge 1/2$

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}s}\Big(\mathrm{e}^{-ds}\int_0^R \big((1+h'(\eta))\widehat{v}_+(s,\eta)^2 + (1-h'(\eta))\widehat{v}_-(s,\eta)^2\big)\eta^{d-1}\,\mathrm{d}\eta\Big)\\ &= -\Big[\mathrm{e}^{-ds}\big((\eta+h(\eta))\widehat{v}_+(s,\eta)^2 + (\eta-h(\eta))\widehat{v}_-(s,\eta)^2\big)\eta^{d-1}\Big]_0^R \le 0\,.\end{aligned} \tag{3.7}$$

Because $1 \pm h' \simeq 1$ on $\mathbb{B}_R$ we have $(1+h')\widehat{v}_+^2 + (1-h')\widehat{v}_-^2 \simeq \widehat{v}_+^2 + \widehat{v}_-^2$. This implies

$$\int_0^R \big(\widehat{v}_+(s,\eta)^2 + \widehat{v}_-(s,\eta)^2\big)\eta^{d-1}\,\mathrm{d}\eta \lesssim \mathrm{e}^{ds}\int_0^R \big(\widehat{v}_+(s_0,\eta)^2 + \widehat{v}_-(s_0,\eta)^2\big)\eta^{d-1}\,\mathrm{d}\eta\,. \tag{3.8}$$

Taking the transformations (2.2), (2.3) for the partial derivatives into account, we have

$$\begin{aligned}\widehat{v}_\pm(s,\eta) &= \frac{\mathrm{e}^s}{\eta h'(\eta)-h(\eta)}\big((1\mp h'(\eta))\partial_s\widehat{v}(s,\eta) + (\eta\mp h(\eta))\partial_\eta\widehat{v}(s,\eta)\big)\,,\\ \partial_s\widehat{v}(s,\eta) &= \frac{\mathrm{e}^{-s}}{2}\big((\eta-h(\eta))\widehat{v}_-(s,\eta) - (\eta+h(\eta))\widehat{v}_+(s,\eta)\big)\,,\\ \partial_\eta\widehat{v}(s,\eta) &= \frac{\mathrm{e}^{-s}}{2}\big(-(1-h'(\eta))\widehat{v}_-(s,\eta) + (1+h'(\eta))\widehat{v}_+(s,\eta)\big)\,.\end{aligned}$$

The occuring coefficients are bounded on $\mathbb{B}_R$, so we get immediately

$$\int_0^R \big(\partial_s\widehat{v}(s,\eta)^2 + \partial_\eta\widehat{v}(s,\eta)^2\big)\,\eta^{d-1}\,\mathrm{d}\eta \simeq \mathrm{e}^{-2s}\int_0^R \big(\widehat{v}_+(s,\eta)^2 + \widehat{v}_-(s,\eta)^2\big)\eta^{d-1}\,\mathrm{d}\eta\,. \tag{3.9}$$

Using that $\widehat{v}(s,\,.\,)$ is even, this implies with (3.8)

$$\int_{-R}^R \big(\partial_s\widehat{v}(s,\eta)^2 + \partial_\eta\widehat{v}(s,\eta)^2\big)\,|\eta|^{d-1}\,\mathrm{d}\eta \lesssim \mathrm{e}^{(d-2)s}\int_{-R}^R \big(\partial_0\widehat{v}(s_0,\eta)^2 + \partial_\eta\widehat{v}(s_0,\eta)^2\big)\,|\eta|^{d-1}\,\mathrm{d}\eta\,.$$

This is precisely the **standard energy estimate**

$$\|v(s,\,.\,)\|_{\dot{H}^1(\mathbb{B}^d_R)} + \|\partial_0 v(s,\,.\,)\|_{L^2(\mathbb{B}^d_R)} \lesssim \mathrm{e}^{(\frac{d}{2}-1)s}\Big(\|v(s_0,\,.\,)\|_{\dot{H}^1(\mathbb{B}^d_R)} + \|\partial_0 v(s_0,\,.\,)\|_{L^2(\mathbb{B}^d_R)}\Big)\,.$$

However, this quantity only provides a seminorm. In order to implement control on the field itself besides its derivatives, we can make a Gronwall-type argument to improve the above estimate to $H^1(\mathbb{B}^d_R) \times L^2(\mathbb{B}^d_R)$, with the expense that the exponential scaling factor increases by $\mathrm{e}^{\frac{1}{2}s}$. Already we make the observation that with this approach we get a norm at the cost of decay. On the other hand, we will see at the end of this section that seminorms with the desired exponential decay may be easily produced.

3.2 The one-dimensional wave evolution

It is worth to review the one-dimensional wave evolution in hyperboloidal similarity coordinates from [1, section 3], for if $d = 1$ the situation seems to be exceptional.

Half waves. We observe that the half-wave equations (3.4) decouple, that is,

$$\big(\partial_t \mp \partial_x\big)\big(u_\pm(t,x)\big) = 0\,.$$

In hyperboloidal similarity coordinates, this reads

$$(1 \pm h'(y))\partial_s v_\pm(s,y) + (y \pm h(y))\partial_y v_\pm(s,y) = 0\,.$$

Testing with $\mathrm{e}^{-s} v_\pm(s,y)$ yields

$$\partial_s\big(\mathrm{e}^{-s} v_\pm(s,y)^2(1 \pm h'(y))\big) + \partial_y\big(\mathrm{e}^{-s} v_\pm(s,y)^2(y \pm h(y))\big) = 0$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big(\mathrm{e}^{-s}\int_{-R}^{R} v_\pm(s,y)^2(1 \pm h'(y))\,\mathrm{d}y\Big) = \Big[\mathrm{e}^{-s} v_\pm(s,y)^2(y \pm h(y))\Big]_{-R}^{R} \leq 0$$

if $R \geq \frac{1}{2}$. After integrating this over $[s_0, s]$ we get

$$\|v_\pm(s,\,.\,)\|_\pm \lesssim \mathrm{e}^{s/2}\|v_\pm(s_0,\,.\,)\|_\pm\,, \tag{3.10}$$

where we introduce the norm $\|\,.\,\|_\pm$ that is induced by the inner product

$$(f\,|\,g)_\pm = \int_{-R}^{R} f(y)\overline{g(y)}(1 \pm h'(y))\,\mathrm{d}y$$

on $L^2(\mathbb{B}_R)$. This is the energy conservation for the half-wave. Let us make some definitions for the further study.

Definition 3.1. Let $R \geq \frac{1}{2}$, $k \in \mathbb{N}_0$ and $f \in C^\infty(\overline{\mathbb{B}_R})$.

1.) We define the vector fields

$$L_\pm f(y) = -\frac{1 \pm h'(y)}{y \pm h(y)} f'(y)\,, \qquad D_\pm f(y) = \frac{1}{1 \pm h'(y)} f'(y)\,.$$

2.) With this we introduce on $C^\infty(\overline{\mathbb{B}_R})$ an inner product $(\,.\,|\,..\,)_{\pm,k}$ that is given by

$$(f\,|\,g)_{\pm,k} = \sum_{j=0}^{k}(D_\pm^j f\,|\,D_\pm^j g)_\pm\,.$$

Let $\|\,.\,\|_{\pm,k}$ be the induced norms.

Note that the half-wave equations take the form

$$\partial_s v_\pm(s,\,.\,) = L_\pm v_\pm(s,\,.\,)\,.$$

Also, if

$$\mathrm{P}\,f(y) = f(-y) \tag{3.11}$$

defines the parity operator on $C^\infty(\overline{\mathbb{B}_R})$, we have

$$\mathrm{P}\,L_\pm f = L_\mp\,\mathrm{P}\,f \tag{3.12}$$

for all $f \in C^\infty(\overline{\mathbb{B}_R})$.

The point of the vector field $D_\pm$ is that it satisfies a crucial commutator relation

$$[D_\pm, L_\pm]f = -D_\pm f \tag{3.13}$$

for all $f \in C^\infty(\overline{\mathbb{B}_R})$. By induction we have $[D^j_\pm, L_\pm] = -jD^j_\pm$ for all $j \in \mathbb{N}$. We will see soon that the commutator provides a decay inducing term in the half-wave equation. Moreover, this vector field controls Sobolev norms.

Lemma 3.1. *Let $k \in \mathbb{N}_0$. Then*

$$\|f\|_{\pm,k} \simeq \|f\|_{H^k(\mathbb{B}_R)}$$

for all $f \in C^\infty(\overline{\mathbb{B}_R})$.

Proof. **"$\lesssim$":** By the Leibniz rule we find

$$D^j_\pm f = \sum_{i=0}^{j-1} \varphi_{\pm,i} f^{(i)} + \frac{1}{1 \pm h'} f^{(j)}$$

with $\varphi_{\pm,i} \in C^\infty(\mathbb{R})$. Hence $\|f\|_{\pm,k} \lesssim \|f\|_{H^k(\mathbb{B}_R)}$.

"$\gtrsim$": From rearranging the Leibniz rule above we also get

$$\|f^{(j)}\|_{L^2(\mathbb{B}_R)} \lesssim \sum_{i=0}^{j-1} \|f^{(i)}\|_{L^2(\mathbb{R})} + \|D^j_\pm f\|_\pm \lesssim \|f\|_{H^{j-1}(\mathbb{B}_R)} + \|D^j_\pm f\|_\pm \,.$$

Since $\|f\|_\pm \simeq \|f\|_{L^2(\mathbb{B}_R)}$ the bound follows by induction. □

An application of the standard energy estimate (3.10) then yields together with the commutator relation corresponding estimates in higher regularity.

Lemma 3.2. *Let $R, s_0 \in \mathbb{R}$, $k \in \mathbb{N}_0$ such that $R \geq \frac{1}{2}$. Suppose $v_\pm \in C^\infty(\overline{\mathbb{B}_R})$ solve $\partial_s v_\pm(s, \,.\,) = L_\pm v_\pm(s, \,.\,)$, respectively. Then*

$$\|v_\pm(s, \,.\,)\|_{\pm,k} \lesssim e^{s/2} \|v_\pm(s_0, \,.\,)\|_{\pm,k} \,.$$

Proof. This is the point where we use the commutator relation (3.13) to see

$$\partial_s \big(e^{js} D^j_\pm \widehat{v}_\pm(s, \,.\,)\big) = L_\pm \big(e^{js} D^j_\pm \widehat{v}_\pm(s, \,.\,)\big) \,.$$

That is, given that $v_\pm(s, \,.\,)$ solves the half-wave equation, $e^{js} D^j_\pm v(s, \,.\,)$ is again a solution to the half-wave equation, for any $0 \leq j \leq k$, and we can apply the half-wave energy estimate (3.10) to get

$$\|v_\pm(s, \,.\,)\|_{\pm,k} = \sum_{j=0}^{k} \|D^j_\pm v_\pm(s, \,.\,)\|_\pm \lesssim \sum_{j=0}^{k} e^{(1/2-j)s} \|D^j_\pm v_\pm(s_0, \,.\,)\|_\pm \lesssim e^{s/2} \|v_\pm(s, \,.\,)\|_{\pm,k} \,. \quad □$$

Next we turn to the existence of solutions in hyperboloidal similarity coordinates.

Proposition 3.1. *Let $R \geq \frac{1}{2}$, $k \in \mathbb{N}$.*

1.) *The operator* $L_\pm : C^\infty(\overline{\mathbb{B}_R}) \subset H^{k-1}(\mathbb{B}_R) \to H^{k-1}(\mathbb{B}_R)$ *is closable and its closure* $\overline{L_\pm}$ *is the generator of a strongly continuous one-parameter semigroup* $S_\pm : [0,\infty) \to \mathfrak{L}\big(H^{k-1}(\mathbb{B}_R)\big)$ *with the bound*

$$\|S_\pm(s)f_\pm\|_{H^{k-1}(\mathbb{B}_R)} \lesssim \mathrm{e}^{s/2}\|f_\pm\|_{H^{k-1}(\mathbb{B}_R)} \tag{3.14}$$

for all $f_\pm \in H^{k-1}(\mathbb{B}_R)$.

2.) *If* $k \geq 3$ *then* $\mathfrak{D}(\overline{L_\pm}) \subseteq C^{k-2}(\overline{\mathbb{B}_R})$ *and* $\overline{L}_\pm f_\pm = L_\pm f_\pm$ *for all* $f_\pm \in \mathfrak{D}(\overline{L_\pm})$.

3.) *Moreover, let* $f_\pm \in C^\infty(\overline{\mathbb{B}_R})$ *and set* $v_\pm(s,\,.\,) = S_\pm(s)f_\pm$. *Then* $v_\pm \in C^1([0,\infty)\times\overline{\mathbb{B}_R})$ *and we have in this case a classical solution of the half-wave equation* $\partial_s v_\pm(s\,.\,) = L_\pm v_\pm(s,\,.\,)$.

Proof. 1.) We have

$$(L_\pm f_\pm \mid f_\pm)_\pm = -\Big[(y \pm h(y))|f_\pm(y)|^2\Big]_{-R}^{R} - \overline{(L_\pm f_\pm \mid f_\pm)_\pm} + \|f_\pm\|_\pm^2\,,$$

so $\mathrm{Re}(L_\pm f_\pm \mid f_\pm)_\pm \leq \frac{1}{2}\|f_\pm\|_\pm^2$ for all $f_\pm \in C^\infty(\overline{\mathbb{B}_R})$. This implies

$$\begin{aligned}\mathrm{Re}(L_\pm f_\pm \mid f_\pm)_{\pm,k} &= \sum_{j=0}^{k}\mathrm{Re}(D_\pm^j L_\pm f_\pm \mid D_\pm^j f_\pm)_\pm \\ &= \sum_{j=0}^{k}\Big(\mathrm{Re}(L_\pm D_\pm^j f_\pm \mid D_\pm^j f_\pm)_\pm - j\|D_\pm^j f_\pm\|_\pm^2\Big) \\ &\leq \frac{1}{2}\|f_\pm\|_{\pm,k}^2\,.\end{aligned}$$

Hence $L_\pm - \frac{1}{2}\mathrm{I}$ is dissipative on $H^k(\mathbb{B}_R)$ endowed with $(\,.\mid..\,)_{\pm,k}$. Moreover, let $F_\pm \in C^\infty(\overline{\mathbb{B}_R})$ and consider the equation $(\mathrm{I} - L_\pm)f_\pm = F_\pm$. The differential equation

$$\frac{y \pm h(y)}{1 \pm h'(y)} f_\pm'(y) + f_\pm(y) = F_\pm(y)$$

has the solution

$$f_\pm(y) = \frac{y \mp \frac{1}{2}}{y \pm h(y)} \int_0^1 \big(1 \pm h'(\tfrac{1}{2} + y'(y \mp \tfrac{1}{2}))\big) F_\pm(\tfrac{1}{2} + y'(y \mp \tfrac{1}{2}))\,\mathrm{d}y'\,.$$

As $y \pm h(y)$ is smooth with a zero of first order at $\mp 1/2$ we see with the fundamental theorem of calculus that the coefficient in front of the integral is smooth. Thus $f_\pm \in C^\infty(\overline{\mathbb{B}_R})$ and the range of $\mathrm{I} - L_\pm$ is dense in $H^k(\mathbb{B}_R)$. Now the proposition follows from the Lumer-Phillips theorem, cf. [17] for a reference.

2.) If $f_\pm \in \mathfrak{D}(\overline{L_\pm})$ there is $\{f_{\pm n}\}_{n\in\mathbb{N}} \subseteq C^\infty(\overline{\mathbb{B}_R})$ such that $f_{\pm n} \to f_\pm$ and $L_\pm f_{\pm n} \to \overline{L_\pm} f_\pm$ in $H^{k-1}(\mathbb{B}_R)$ as $n \to \infty$. By Sobolev embedding $H^{k-1}(\mathbb{B}_R) \subseteq C^{k-2}(\overline{\mathbb{B}_R})$ and thus

$$\left| L_\pm f_{\pm n} - \frac{y \pm h(y)}{1 \pm h'(y)} f_\pm'(y) \right| \lesssim \|f_{\pm n} - f_\pm\|_{L^\infty(\mathbb{B}_R)} \leq \|f_{\pm n} - f_\pm\|_{H^{k-1}(\mathbb{B}_R)} \to 0 \quad \text{as } n \to \infty\,.$$

3.) We have $v_\pm(s,\,.\,) \in \mathfrak{D}(\overline{L_\pm}) \subseteq C^{k-1}([0,\infty)\times\overline{\mathbb{B}_R})$ and $\partial_s v_\pm(s,\,.\,) = \overline{L_\pm} v_\pm(s,\,.\,) = L_\pm v_\pm(s,\,.\,)$ by the first and second part. $\square$

Remark 3.2. Let us investigate whether the exponential growth bound (3.14) is sharp. We do this via considering **mode solutions** for the half-wave equation in lowest regularity, i.e., suppose

$$S_\pm(s) f_\pm = \mathrm{e}^{\lambda s} f_\pm$$

for some non-trivial $f_\pm \in L^2(\mathbb{B}_R)$ and $\lambda \in \mathbb{C}$. In this respect, let us consider the spectral equation $(L_\pm - \lambda \mathrm{I}) f_\pm = 0$, i.e.,

$$\frac{y \pm h(y)}{1 \pm h'(y)} f_\pm'(y) + \lambda f_\pm(y) = 0 \,.$$

There is an explicit solution

$$f_\pm(y) = (y \pm h(y))^{-\lambda} \,.$$

The zero of $y \pm h(y)$ is precisely $y = \mp\frac{1}{2}$, respectively. The condition $f_\pm \in L^2(\mathbb{B}_R)$, where $R \geq \frac{1}{2}$ forces λ such that a possible pole at $\mp\frac{1}{2}$ is integrable. This is precisely the case when $\operatorname{Re}\lambda < \frac{1}{2}$. This demonstrates that the bound is sharp.

Is the bound sharp also in higher regularity spaces? Since the half-wave equation admits constant solutions, the evolution certainly cannot decay exponentially, that is, the optimal exponential factor is between 0 and $\frac{1}{2}$. The exact bound may be determined by a full spectral analysis of the operator.

It will be convenient to introduce on $H^{k-1}(\mathbb{B}_R) \times H^{k-1}(\mathbb{B}_R)$ the operators given by

$$\mathbf{L}_\mp \mathbf{f}_\mp = \begin{bmatrix} L_- f_- \\ L_+ f_+ \end{bmatrix} \,, \qquad \overline{\mathbf{L}_\mp} \mathbf{f}_\mp = \begin{bmatrix} \overline{L_-} f_- \\ \overline{L_+} f_+ \end{bmatrix} \,, \qquad \mathbf{S}_\mp \mathbf{f}_\mp = \begin{bmatrix} S_- f_- \\ S_+ f_+ \end{bmatrix} \,,$$

for $\mathbf{f}_\mp = [\, f_-, f_+ \,]$ in the respective domains.

Existence on the half-line. Having the results for the half-waves at hand, we can construct solutions to the one-dimensional wave equation in hyperboloidal similarity coordinates with a Dirichlet condition at the origin. As before we begin with the energy estimates. The symmetry assumption in the following is a consequence of the descent method that we will explore afterwards.

Lemma 3.3. *Let $R, s_0, T \in \mathbb{R}$, $k \in \mathbb{N}$ with $R \geq \frac{1}{2}$, $k \geq 2$. Assume that $u \in C^k(\mathbb{R}^{1,1})$ and $u(t,\,.\,)$ is odd for all times and satisfies the one-dimensional wave equation $\partial_0^2 u - \partial_1^2 u = 0$. Let $v(s,y) = u(T + \mathrm{e}^{-s} h(y), \mathrm{e}^{-s} y)$. Then we have*

$$\|v(s,\,.\,)\|_{H^\ell(\mathbb{B}_R)} + \|\partial_s v(s,\,.\,)\|_{H^{\ell-1}(\mathbb{B}_R)} \lesssim \mathrm{e}^{-\frac{s}{2}} \big(\|v(s_0,\,.\,)\|_{H^\ell(\mathbb{B}_R)} + \|\partial_0 v(s_0,\,.\,)\|_{H^{\ell-1}(\mathbb{B}_R)} \big)$$

for all $s \geq s_0$ and $\ell \in \mathbb{N}_0$ with $\ell \leq k$.

Proof. Recall that we have

$$\begin{aligned}
v_\pm(s,y) &= \frac{\mathrm{e}^s}{y h'(y) - h(y)} \big((1 \mp h'(y)) \partial_s v(s,y) + (y \mp h(y)) \partial_y v(s,y) \big) \,, \\
\partial_s v(s,y) &= \frac{\mathrm{e}^{-s}}{2} \big((y - h(y)) v_-(s,y) - (y + h(y)) v_+(s,y) \big) \,, \\
\partial_y v(s,y) &= \frac{\mathrm{e}^{-s}}{2} \big(-(1 - h'(y)) v_-(s,y) + (1 + h'(y)) v_+(s,y) \big) \,.
\end{aligned}$$

Hence

$$\|v(s,\,.\,)\|_{\dot{H}^\ell(\mathbb{B}_R)} + \|\partial_s v(s,\,.\,)\|_{H^{\ell-1}(\mathbb{B}_R)} \simeq \mathrm{e}^{-s}\big(\|v_+(s,\,.\,)\|_{H^{\ell-1}(\mathbb{B}_R)} + \|v_-(s,\,.\,)\|_{H^{\ell-1}(\mathbb{B}_R)}\big)\,.$$

Since $v(s,0)=0$ the fundamental theorem of calculus implies

$$|v(s,y)|^2 = \left|\int_0^y \partial_{y'} v(s,y')\,\mathrm{d}y'\right|^2 \le \int_0^y |\partial_{y'} v(s,y')|^2\,\mathrm{d}y'\,,$$

which implies $\|v(s,\,.\,)\|_{\dot{H}^1(\mathbb{B}_R)} \simeq \|v(s,\,.\,)\|_{H^1(\mathbb{B}_R)}$. Now the energy estimate follows from lemma 3.2. □

We put the transformations between the half-waves and partial derivatives from the previous proof into operators.

Definition 3.2. Let $R>0$, $k\in\mathbb{N}$. We introduce densely defined bounded linear operators

$$\begin{aligned}\mathbf{A} &: C^\infty(\overline{\mathbb{B}_R})^2 \subseteq H^k(\mathbb{B}_R)\times H^{k-1}(\mathbb{B}_R) \to H^{k-1}(\mathbb{B}_R)\times H^{k-1}(\mathbb{B}_R)\,,\\ \mathbf{B} &: C^\infty(\overline{\mathbb{B}_R})^2 \subseteq H^{k-1}(\mathbb{B}_R)\times H^{k-1}(\mathbb{B}_R) \to H^{k}(\mathbb{B}_R)\times H^{k-1}(\mathbb{B}_R)\,,\end{aligned}$$

by

$$\begin{aligned}\mathbf{A}\mathbf{f}(y) &= \frac{1}{yh'(y)-h(y)}\begin{bmatrix}(y+h(y))f_1'(y) + (1+h'(y))f_2(y)\\ (y-h(y))f_1'(y) + (1-h'(y))f_2(y)\end{bmatrix}\,,\\ \mathbf{B}\mathbf{f}_\mp(y) &= \frac{1}{2}\begin{bmatrix}-\int_0^y (1-h'(y'))f_-(y')\,\mathrm{d}y' + \int_0^y (1+h'(y'))f_+(y')\,\mathrm{d}y'\\ (y-h(y))f_-(y) - (y+h(y))f_+(y)\end{bmatrix}\,.\end{aligned}$$

The operator $\mathbf{A}$ forms half-waves and $\mathbf{B}$ undoes this transformation and produces a Dirichlet condition at the origin. We have that $\mathbf{AB}=\mathbf{I}$ and $\mathbf{BAf}(y)=\mathbf{f}(y)-\mathbf{f}(0)$.

So we have with

$$\mathbf{S}_1(s) = \mathrm{e}^{-s}\mathbf{B}\mathbf{S}_\mp(s)\mathbf{A} \tag{3.15}$$

the solution operator to the one-dimensional wave equation in hyperboloidal similarity coordinates on the half-line.

Proposition 3.2. *Let $R\ge\frac{1}{2}$, $k\in\mathbb{N}$.*

1.) *The mapping $\mathbf{S}_1$ forms a strongly continuous semigroup of $\mathfrak{L}\big(H^k_{\text{odd}}(\mathbb{B}_R)\times H^{k-1}_{\text{odd}}(\mathbb{B}_R)\big)$ with generator*

$$\mathbf{L}_1 = \mathbf{B}\overline{\mathbf{L}_\mp}\mathbf{A} - \mathbf{I} \tag{3.16}$$

such that

$$\|\mathbf{S}_1(s)\mathbf{f}\|_{H^k(\mathbb{B}_R)\times H^{k-1}(\mathbb{B}_R)} \lesssim \mathrm{e}^{-s/2}\|\mathbf{f}\|_{H^k(\mathbb{B}_R)\times H^{k-1}(\mathbb{B}_R)} \tag{3.17}$$

for all $\mathbf{f}\in H^k_{\text{odd}}(\mathbb{B}_R)\times H^{k-1}_{\text{odd}}(\mathbb{B}_R)$.

2.) *If $k\ge 3$ and $\mathbf{f}\in C^\infty_{\text{odd}}(\overline{\mathbb{B}_R})^2$, let $v(s,\,.\,)=[\mathbf{S}_1(s)\mathbf{f}]_1$. Then $v\in C^2([0,\infty)\times\overline{\mathbb{B}_R})$, $v(s,\,.\,)$ is odd, $\partial_s v(s,\,.\,)=[\mathbf{S}_1(s)\mathbf{f}]_2$ and a classical solution of*

$$\partial_s\mathbf{v}(s,\,.\,) = \mathbf{L}_1\mathbf{v}(s,\,.\,)\,.$$

Proof. 1.) Let $\mathbf{f} \in C^\infty_{\text{odd}}(\overline{\mathbb{B}_R})^2$ and set $\mathbf{v}_\mp(s,\,.\,) = \mathbf{S}_\mp(s)\mathbf{A}\mathbf{f}$ for $s \geq 0$ so that $\partial_s \mathbf{v}_\mp(s,\,.\,) = \overline{\mathbf{L}_\mp}\mathbf{v}_\mp(s,\,.\,)$. Put $v_-(s,-\,.\,) = \mathrm{P}\, v_-(s,\,.\,)$, where P is the parity operator (3.11), and observe

$$\begin{aligned} v_-(0,-y) &= \frac{1}{-yh'(-y)-h(-y)}\big((-y+h(-y))f_1'(-y) + (1+h'(-y))f_2(-y)\big) \\ &= \frac{1}{yh'(y)-h(y)}\big((-y+h(y))f_1'(y) - (1-h'(y))f_2(y)\big) \\ &= -v_+(0,y)\,. \end{aligned}$$

Identity (3.12) remains true for the closure and we conclude

$$\partial_s v_-(s,-\,.\,) = \mathrm{P}\,\partial_s v_-(s,\,.\,) = \mathrm{P}\,\overline{L_-}v_-(s,\,.\,) = \overline{L_+}\,\mathrm{P}\,v_-(s,\,.\,) = \overline{L_+}v_-(s,-\,.\,)\,,$$

Since $v_-(0,-\,.\,) = -v_+(0,\,.\,)$ we infer from uniqueness of solutions to the Cauchy problem $v_-(s,-\,.\,) = -v_+(s,\,.\,)$. It follows that

$$-(1-h')v_-(s,\,.\,)+(1+h')v_+(s,\,.\,) \in H^{k-1}_{\text{even}}(\mathbb{B}_R)\,,\quad (y-h)v_-(s,\,.\,)-(y+h)v_+(s,\,.\,) \in H^{k-1}_{\text{odd}}(\mathbb{B}_R)\,.$$

This shows that $\mathbf{S}_1(s)\mathbf{f} = \mathbf{B}\mathbf{v}_\mp(s,\,.\,) \in H^k_{\text{odd}}(\mathbb{B}_R)\times H^{k-1}_{\text{odd}}(\mathbb{B}_R)$.

Since $\mathbf{S}_\mp$ is strongly continuous, so is $\mathbf{S}_1$. The semigroup property follows from the semigroup property $\mathbf{S}_\mp(s+t) = \mathbf{S}_\mp(s)\mathbf{S}_\mp(t)$ and the fact that $\mathbf{BAf} = \mathbf{f}$ for odd $\mathbf{f}$,

$$\mathbf{S}_1(s+t) = \mathrm{e}^{-(s+t)}\mathbf{B}\mathbf{S}_\mp(s+t)\mathbf{A} = \mathrm{e}^{-s}\mathbf{B}\mathbf{S}_\mp(s)\mathbf{A}\mathrm{e}^{-t}\mathbf{B}\mathbf{S}_\mp(t)\mathbf{A} = \mathbf{S}_1(s)\mathbf{S}_1(t)\,.$$

Differentiating $\mathbf{S}_1(s)\mathbf{f}$ and inserting the generator of $\mathbf{S}_\mp$ yields (3.16). The semigroup estimate follows from (3.14) since $\mathbf{A}$, $\mathbf{B}$ are bounded.

2.) The second part of the proposition follows with the third part of proposition 3.1 and the relation between the half-waves and partial derivatives that are encoded in $\mathbf{A}$, $\mathbf{B}$. □

3.3 Higher energy norms

Given the standard energy, we have to incorporate the field and its derivatives to produce a full norm of higher regularity. At the same time this norm should permit suitable growth estimates. We build up our way with two types of transformations that map certain semilinear wave equations onto themselves. For the beginning we orient ourself towards the strategy in [11], [12].

Descent by multiplication. We begin by considering an equation of the type

$$\partial_t^2 u(t,r) - \partial_r^2 u(t,r) - \tfrac{d_1}{r}\partial_r u(t,r) = 0$$

for some constant $d_1 \in \mathbb{R}$. The field u_1 is introduced by $u(t,r) = r^\ell u_1(t,r)$ for some $\ell \in \mathbb{R}$. Then

$$\begin{aligned} \partial_r u(t,r) &= r^\ell\big(\tfrac{\ell}{r}u_1(t,r) + \partial_r u_1(t,r)\big)\,, \\ \partial_r^2 u(t,r) &= r^\ell\big(\tfrac{\ell(\ell-1)}{r^2}u_1(t,r) + \tfrac{2\ell}{r}\partial_r u_1(t,r) + \partial_r^2 u_1(t,r)\big)\,, \end{aligned}$$

and hence

$$\partial_t^2 u_1(t,r) - \partial_r^2 u_1(t,r) - \tfrac{2\ell+d_1}{r}\partial_r u_1(t,r) - \tfrac{\ell(\ell-1+d_1)}{r^2}u_1(t,r) = 0\,.$$

By fixing ℓ we can decide which term we want to cancel. Since our goal is to transform between radial wave equations, let us try to make the zeroth order term disappear. This amounts to the choice $\ell = 0$, in which case the transformation is just the identitiy and the original equation remains invariant, or $\ell = -d_1 + 1$. The field $u_1(t,r) = r^{d_1-1}u(t,r)$ solves

$$\partial_t^2 u_1(t,r) - \partial_r^2 u_1(t,r) - \tfrac{-d_1+2}{r}\partial_r u_1(t,r) = 0\,.$$

Depending on the sign of d_1, we end up with a radial wave equation in negative dimensions.

Descent by differentiation. Instead we could first differentiate the field and then apply a multiplication by some power. That is, suppose

$$\partial_t^2 u(t,r) - \partial_r^2 u(t,r) - \tfrac{d_2}{r}\partial_r u(t,r) = 0$$

for some constant $d_2 \in \mathbb{R}$ and consider the equation

$$\partial_t^2 \partial_r u(t,r) - \partial_r^2 \partial_r u(t,r) - \tfrac{d_2}{r}\partial_r\partial_r u(t,r) + \tfrac{d_2}{r^2}\partial_r u(t,r) = 0$$

satisfied by the spatial derivative of the field. Now set $\partial_r u(t,r) = r^\ell u_2(t,r)$ for some $\ell \in \mathbb{R}$ which implies

$$\begin{aligned}\partial_r\partial_r u(t,r) &= r^\ell\big(\tfrac{\ell}{r}u_2(t,r) + \partial_r u_2(t,r)\big)\,,\\ \partial_r^2\partial_r u(t,r) &= r^\ell\big(\tfrac{\ell(\ell-1)}{r^2}u_2(t,r) + \tfrac{2\ell}{r}\partial_r u_2(t,r) + \partial_r^2 u_2(t,r)\big)\,.\end{aligned}$$

Therefore

$$\partial_t^2 u_2(t,r) - \partial_r^2 u_2(t,r) - \tfrac{2\ell+d_2}{r}\partial_r u_2(t,r) - \tfrac{\ell(\ell-1+d_2)-d_2}{r^2}u_2(t,r) = 0\,.$$

Again we want to preserve the radial character of the equation. So, the zeroth order term vanishes if $\ell = 1$ or $\ell = -d_2$. Then we have

$$u_2(t,r) = r^{-1}\partial_r u(t,r) \quad \text{solves} \quad \partial_t^2 u_2(t,r) - \partial_r^2 u_2(t,r) - \tfrac{d_2+2}{r}\partial_r u_2(t,r) = 0$$

and

$$u_2(t,r) = r^{d_2}\partial_r u(t,r) \quad \text{solves} \quad \partial_t^2 u_2(t,r) - \partial_r^2 u_2(t,r) - \tfrac{-d_2}{r}\partial_r u_2(t,r) = 0\,.$$

The first transformation increases the effective dimension by two. This may be helpful if $d_2 < 0$ but is useless otherwise. The second transformation makes a negative dimension positive and vice versa.

Combined descent. We have to combine both previous transformations to end up with a wave equation. Some comments on the effective dimension are in order.

When we start from a radial d-dimensional wave equation and apply the differentiation operation first, an ensuing multiplication has no effect. A second application of the differentiation operation increases the dimension or leaves it unchanged. The latter approach yields that the radial Laplace operator solves again the wave equation in the original dimension. This is obvious from the equation by applying the radial Laplace operator $r^{-d+1}\partial_r(r^{d-1}\partial_r)$. However, it is not constructive to persue this route because the Laplacian only controls seminorms. Even worse, the corresponding seminorm obtained in this way in hyperboloidal similarity coordinates contains higher order s-derivatives, which rule it out as a candidate for an energy norm. This may be resolved by substituting higher s-derivatives with the wave equation but the resulting expressions are algebraically extremely difficult to handle and elude a systematic analysis.

The remaining combinations lead to the **descent method**. Starting with a multiplication and then combining it with differentiation, there are two types of transformations that yield solutions to a radial wave equation in a lower positive dimension. One has

$$r^{-(d-3)}\partial_r\big(r^{d-2}\widehat{u}(t,r)\big) = (d-2)\widehat{u}(t,r) + r\partial_r\widehat{u}(t,r) \tag{3.18}$$

to map solutions of the radial d-dimensional wave equation in any given spatial dimension d into a solution in $d-2$ dimensions, and

$$\big(r^{-1}\partial_r\big)^{\frac{d-3}{2}}\big(r^{d-2}\widehat{u}(t,r)\big) \tag{3.19}$$

to end up with a solution of the one-dimensional wave equation. Note that repeated application of transformations of the type (3.18) yield (3.19).

If $d=3$ and u is a solution of the three-dimensional radial wave equation, the field $ru(t,r)$ solves the one-dimensional wave equation.

Incidentally, if one makes the first order term vanish, one ends up with a standard energy plus a zero order term evaluated at the light cone boundary.

We capture the crucial transformations from the discussion above in a lemma.

Lemma 3.4. *Let $d \geq 3$, $f \in C^\infty(\mathbb{R})$. Then we have the commutation relations*

$$\big((d-2)+r\partial_r\big)\big(\partial_r^2 + \tfrac{d-1}{r}\partial_r\big)f(r) = \big(\partial_r^2 + \tfrac{d-3}{r}\partial_r\big)\big((d-2)+r\partial_r\big)f(r)\,, \tag{3.20}$$

$$r\big(\partial_r^2 + \tfrac{2}{r}\partial_r\big)f(r) = \partial_r^2\big(rf(r)\big)\,. \tag{3.21}$$

4 Descent method in hyperboloidal similarity coordinates

We continue with an anlaysis of the previously established descent method in hyperboloidal similarity coordinates. To get to that, let $u \in C^\infty(\mathbb{R}^{1,d})$ such that $u(t,\,.\,)$ is radial. Then there is $\widehat{u} \in C^\infty(\mathbb{R}^{1,1})$ such that $\widehat{u}(t,\,.\,)$ is even and $u(t,\,.\,) = \widehat{u}(t,|\,.\,|)$. Let $v \in C^\infty(\mathbb{R}^{1,d})$ and $\widehat{v} \in C^\infty(\mathbb{R}^{1,1})$ be defined by $v(s,y) = u(T+\mathrm{e}^{-s}h(y),\mathrm{e}^{-s}y)$ and $\widehat{v}(s,\eta) = \widehat{u}(T+\mathrm{e}^{-s}h(\eta),\mathrm{e}^{-s}\eta)$. Then $v(s,\,.\,)$ is radial since $v(s,|\,.\,|) = \widehat{v}(s,\,.\,)$. If u satisifes $\partial_t^2 u(t,\,.\,) - \Delta u(t,\,.\,) = 0$ then equation (2.13) satisfied by v implies for $\widehat{v}$ the radial wave equation in hyperboloidal similarity coordinates,

$$\partial_s^2\widehat{v}(s,\eta) = \big(c_{11}^d(\eta)\partial_\eta + c_{12}(\eta)\partial_\eta^2 + c_{20}^d(\eta)\partial_s + c_{21}(\eta)\partial_\eta\partial_s\big)\widehat{v}(s,\eta) \tag{4.1}$$

with coefficients

$$c_{11}^d(\eta) := -\frac{d-1}{\eta}\frac{\big(\eta h'(\eta)-h(\eta)\big)h(\eta)}{1-h'(\eta)^2} + \frac{\eta^2-h(\eta)^2}{1-h'(\eta)^2}\frac{\eta h''(\eta)}{\eta h'(\eta)-h(\eta)} + 2\frac{h(\eta)h'(\eta)-\eta}{1-h'(\eta)^2}\,, \tag{4.2}$$

$$c_{12}(\eta) := \frac{h(\eta)^2-\eta^2}{1-h'(\eta)^2}\,, \tag{4.3}$$

$$c_{20}^d(\eta) := -1 - \frac{d-1}{\eta}\frac{\big(\eta h'(\eta)-h(\eta)\big)h'(\eta)}{1-h'(\eta)^2} + \frac{\eta^2-h(\eta)^2}{1-h'(\eta)^2}\frac{h''(\eta)}{\eta h'(\eta)-h(\eta)}\,, \tag{4.4}$$

$$c_{21}(\eta) := 2\frac{h(\eta)h'(\eta)-\eta}{1-h'(\eta)^2}\,. \tag{4.5}$$

Formulated as a first order system this takes the form

$$\partial_s \widehat{\mathbf{v}}(s, \,.\,) = \widehat{\mathbf{L}}_d \widehat{\mathbf{v}}(s, \,.\,)$$

with the field $\widehat{\mathbf{v}}(s, \,.\,) = [\,\widehat{v}(s, \,.\,), \partial_s \widehat{v}(s, \,.\,)\,]$ and the spatial differential operator

$$\widehat{\mathbf{L}}_d \widehat{\mathbf{f}} = \begin{bmatrix} \widehat{f}_2 \\ c_{11}^d \widehat{f}_1' + c_{12} \widehat{f}_1'' + c_{20}^d \widehat{f}_2 + c_{21} \widehat{f}_2' \end{bmatrix}.$$

The transformation (3.18) given by $\widetilde{u}(t,r) = (d-2)\widehat{u}(t,r) + r\partial_r \widehat{u}(t,r)$ reads in hyperboloidal similarity coordinates

$$\begin{aligned} \widetilde{v}(s,\eta) &= (d-2)\widehat{v}(s,\eta) - \tfrac{\eta h'(\eta)}{\eta h'(\eta) - h(\eta)} \partial_s \widehat{v}(s,\eta) - \tfrac{\eta h(\eta)}{\eta h'(\eta) - h(\eta)} \partial_\eta \widehat{v}(s,\eta) \\ &=: \big(a_{10}^d(\eta)\widehat{v}(s,\eta) + a_{11}(\eta)\partial_\eta + a_{20}(\eta)\partial_s\big)\widehat{v}(s,\eta) \end{aligned} \tag{4.6}$$

with coefficients

$$a_{10}^d(\eta) := d-2\,, \qquad a_{11}(\eta) := -\frac{\eta h(\eta)}{\eta h'(\eta) - h(\eta)}\,, \qquad a_{20}(\eta) := -\frac{\eta h'(\eta)}{\eta h'(\eta) - h(\eta)}\,. \tag{4.7}$$

Moreover,

$$\partial_s \widetilde{v}(s,\eta) = \big(a_{10}^d(\eta)\partial_s + a_{11}(\eta)\partial_\eta\partial_s + a_{20}(\eta)\partial_s^2\big)\widehat{v}(s,\eta)\,.$$

Equation (4.1) allows to replace the second order s-derivatives by lower order derivatives, in particular,

$$\partial_s \widetilde{v}(s,\eta) = \big(b_{11}(\eta)\partial_\eta + b_{12}(\eta)\partial_\eta^2 + b_{21}(\eta)\partial_\eta\partial_s + b_{21}(\eta)\partial_s\big)\widehat{v}(s,\eta)\,, \tag{4.8}$$

where

$$b_{11}^d := a_{20} c_{11}^d\,, \qquad b_{12} := a_{20} c_{12}\,, \qquad b_{20}^d := a_{10}^d + a_{20} c_{20}^d\,, \qquad b_{21} := a_{11} + a_{20} c_{21}\,. \tag{4.9}$$

In the case $d = 3$ we have the transformation $\widetilde{u}(t,r) = r\widehat{u}(t,r)$ at hand that reads in hyperboloidal similarity coordinates

$$\widetilde{v}(s,\eta) = \mathrm{e}^{-s}\eta\widehat{v}(s,\eta)\,, \qquad \partial_s \widetilde{v}(s,\eta) = -\mathrm{e}^{-s}\eta\widehat{v}(s,\eta) + \mathrm{e}^{-s}\eta\partial_s\widehat{v}(s,\eta)\,. \tag{4.10}$$

4.1 Analysis of the descent operators

To formulate the method of descent in hyperboloidal similarity coordinates we introduce the **descent operators** that are obtained by combining equations (4.6), (4.8) and (4.10), respectively, into a first order system.

Definition 4.1. Let $d \in \mathbb{N}$, $R > 0$ and $\widehat{\mathbf{f}} \in C^\infty(\overline{\mathbb{B}_R})^2$.

1.) For $d > 3$ we set

$$\widehat{\mathbf{D}}_d^{\downarrow} \widehat{\mathbf{f}} := \begin{bmatrix} a_{10}^d \widehat{f}_1 + a_{11} \widehat{f}_1' + a_{20} \widehat{f}_2 \\ b_{11}^d \widehat{f}_1' + b_{12} \widehat{f}_1'' + b_{20}^d \widehat{f}_2 + b_{21} \widehat{f}_2' \end{bmatrix}. \tag{4.11}$$

The radial descent operator from three to one dimension is given by the multiplication operator

$$\widehat{\mathbf{D}}_3^{\downarrow} \widehat{\mathbf{f}} = \begin{bmatrix} \eta \widehat{f}_1 \\ -\eta \widehat{f}_1 + \eta \widehat{f}_2 \end{bmatrix}.$$

2.) Let $d \geq 3$, $R > 0$ and $\mathbf{f} \in C^\infty_{\text{rad}}(\overline{\mathbb{B}^d_R})^2$. We define $\mathbf{D}^\downarrow_d \mathbf{f}$ such that

$$\mathbf{D}^\downarrow_d \mathbf{f}(y) = \widehat{\mathbf{D}}^\downarrow_d \widehat{\mathbf{f}}(|y|)\,, \quad y \in \mathbb{R}^{d-2}\,.$$

That is, equations (4.6), (4.8) read

$$\widetilde{\mathbf{v}}(s,\,.\,) = \widehat{\mathbf{D}}^\downarrow_d \widehat{\mathbf{v}}(s,\,.\,)\,, \qquad \widetilde{\mathbf{v}}(s,\,.\,) = \mathrm{e}^{-s} \widehat{\mathbf{D}}^\downarrow_3 \widehat{\mathbf{v}}(s,\,.\,)\,,$$

respectively, and this suggests to consider the commutation relation presented in lemma 3.4 as the following **intertwining identities**.

Lemma 4.1. *Let $d > 3$ odd.*

1.) *The intertwining identity (3.18) reads*

$$\widehat{\mathbf{D}}^\downarrow_d \widehat{\mathbf{L}}_d \widehat{\mathbf{f}} = \widehat{\mathbf{L}}_{d-2} \widehat{\mathbf{D}}^\downarrow_d \widehat{\mathbf{f}} \tag{4.12}$$

for all $\mathbf{f} \in C^\infty_{\text{rad}}(\overline{\mathbb{B}^d_R})^2$.

2.) *The descent from three dimensions to one dimension manifests itself in hyperboloidal similarity coordinates as*

$$\widehat{\mathbf{D}}^\downarrow_3 \widehat{\mathbf{L}}_3 \widehat{\mathbf{f}} = \widehat{\mathbf{L}}_1 \widehat{\mathbf{D}}^\downarrow_3 \widehat{\mathbf{f}} + \widehat{\mathbf{D}}^\downarrow_3 \widehat{\mathbf{f}} \tag{4.13}$$

for all $\mathbf{f} \in C^\infty_{\text{rad}}(\overline{\mathbb{B}^3_R})^2$.

Proof. 1.) We compute the first component for the left-hand side

$$\begin{aligned}
[\widehat{\mathbf{D}}^\downarrow_d \widehat{\mathbf{L}}_d \widehat{\mathbf{f}}]_1 &= a^d_{10} [\widehat{\mathbf{L}}_d \widehat{\mathbf{f}}]_1 + a_{11} [\widehat{\mathbf{L}}_d \widehat{\mathbf{f}}]_1' + a_{20} [\widehat{\mathbf{L}}_d \widehat{\mathbf{f}}]_2 \\
&= a_{20} c^d_{11} \widehat{f}_1' + a_{20} c_{12} \widehat{f}_1'' + (a^d_{10} + a_{20} c^d_{20}) \widehat{f}_2 + (a_{11} + a_{20} c_{21}) \widehat{f}_2'
\end{aligned}$$

and for the right-hand side

$$[\widehat{\mathbf{L}}_{d-2} \widehat{\mathbf{D}}^\downarrow_d \widehat{\mathbf{f}}]_1 = [\widehat{\mathbf{D}}^\downarrow_d \widehat{\mathbf{f}}]_2 = b^d_{11} \widehat{f}_1' + b_{12} \widehat{f}_1'' + b^d_{20} \widehat{f}_2 + b_{21} \widehat{f}_2'\,,$$

respectively, as well as the second component on the left-hand side

$$\begin{aligned}
[\widehat{\mathbf{D}}^\downarrow_d \widehat{\mathbf{L}}_d \widehat{\mathbf{f}}]_2 &= b^d_{11} [\widehat{\mathbf{L}}_d \widehat{\mathbf{f}}]_1' + b_{12} [\widehat{\mathbf{L}}_d \widehat{\mathbf{f}}]_1'' + b^d_{20} [\widehat{\mathbf{L}}_d \widehat{\mathbf{f}}]_2 + b_{21} [\widehat{\mathbf{L}}_d \widehat{\mathbf{f}}]_2' \\
&= \big(b^d_{20} c^d_{11} + b_{21} c^{d\,\prime}_{11}\big) \widehat{f}_1' + \big(b^d_{20} c_{12} + b_{21} (c^d_{11} + c_{12}')\big) \widehat{f}_1'' + b_{21} c_{12} \widehat{f}_1''' \\
&\quad + \big(b^d_{20} c^d_{20} + b_{21} c^{d\,\prime}_{20}\big) \widehat{f}_2 + \big(b^d_{11} + b^d_{20} c_{21} + b_{21} (c^d_{20} + c_{21}')\big) \widehat{f}_2' + \big(b_{12} + b_{21} c_{21}\big) \widehat{f}_2''\,.
\end{aligned}$$

and on the right-hand side

$$\begin{aligned}
[\widehat{\mathbf{L}}_{d-2} \widehat{\mathbf{D}}^\downarrow_d \widehat{\mathbf{f}}]_2 &= c^{d-2}_{11} [\widehat{\mathbf{D}}^\downarrow_d \widehat{\mathbf{f}}]_1' + c_{12} [\widehat{\mathbf{D}}^\downarrow_d \widehat{\mathbf{f}}]_1'' + c^{d-2}_{20} [\widehat{\mathbf{D}}^\downarrow_d \widehat{\mathbf{f}}]_2 + c_{21} [\widehat{\mathbf{D}}^\downarrow_d \widehat{\mathbf{f}}]_2' \\
&= \big(c^{d-2}_{11} (a^d_{10} + a_{11}') + c_{12} a_{11}'' + c^{d-2}_{20} b^d_{11} + c_{21} b^{d\,\prime}_{11}\big) \widehat{f}_1' \\
&\quad + \big(c^{d-2}_{11} a_{11} + c_{12} (a^d_{10} + 2a_{11}') + c^{d-2}_{20} b_{12} + c_{21} (b^d_{11} + b_{12}')\big) \widehat{f}_1'' \\
&\quad + \big(c_{12} a_{11} + c_{21} b_{12}\big) \widehat{f}_1''' \\
&\quad + \big(c^{d-2}_{11} a_{20}' + c_{12} a_{20}'' + c^{d-2}_{20} b^d_{20} + c_{21} b^{d\,\prime}_{20}\big) \widehat{f}_2 \\
&\quad + \big(c^{d-2}_{11} a_{20} + 2c_{12} a_{20}' + c^{d-2}_{20} b_{21} + c_{21} (b^d_{20} + b_{21}')\big) \widehat{f}_2' + \big(c_{12} a_{20} + c_{21} b_{21}\big) \widehat{f}_2''\,,
\end{aligned}$$

respectively. By (4.9) we have the equality $[\widehat{\mathbf{D}}^\downarrow_d \widehat{\mathbf{L}}_d \widehat{\mathbf{f}}]_1 = [\widehat{\mathbf{L}}_{d-2} \widehat{\mathbf{D}}^\downarrow_d \widehat{\mathbf{f}}]_1$. The rest follows by inspection.

2.) We compute the first components

$$[\widehat{\mathbf{D}}_3^{\downarrow}\widehat{\mathbf{L}}_3\widehat{\mathbf{f}}]_1 = \eta[\widehat{\mathbf{L}}_3\widehat{\mathbf{f}}]_2 = \eta\widehat{f}_2$$

and

$$[\widehat{\mathbf{L}}_1\widehat{\mathbf{D}}_3^{\downarrow}\widehat{\mathbf{f}} + \widehat{\mathbf{D}}_3^{\downarrow}\widehat{\mathbf{f}}]_1 = [\widehat{\mathbf{D}}_3^{\downarrow}\widehat{\mathbf{f}}]_2 + \eta\widehat{f}_1 = \eta\widehat{f}_2\,.$$

For the second componentes we find

$$[\widehat{\mathbf{D}}_3^{\downarrow}\widehat{\mathbf{L}}_3\widehat{\mathbf{f}}]_2 = -\eta[\widehat{\mathbf{L}}_3\widehat{\mathbf{f}}]_1 + \eta[\widehat{\mathbf{L}}_3\widehat{\mathbf{f}}]_2 = \eta c_{11}^3\widehat{f}_1' + \eta c_{12}\widehat{f}_1'' + \eta(c_{20}^3 - 1)\widehat{f}_2 + \eta c_{21}\widehat{f}_2$$

and

$$\begin{aligned}[\widehat{\mathbf{L}}_1\widehat{\mathbf{D}}_3^{\downarrow}\widehat{\mathbf{f}} + \widehat{\mathbf{D}}_3^{\downarrow}\widehat{\mathbf{f}}]_2 &= (c_{11}^1 - \eta c_{20}^1 - c_{21} - \eta)\widehat{f}_1 + (\eta c_{11}^1 + 2c_{12} - \eta c_{21})\widehat{f}_1' + \eta c_{12}\widehat{f}_1'' \\ &\quad + (\eta c_{20}^1 + c_{21} + \eta)\widehat{f}_2 + \eta c_{21}\widehat{f}_2'(\eta)\,.\end{aligned}$$

Comparing coefficients on both sides reveals the identity. □

The crucial part is now to understand the mapping properties of the descent operators and link them with the radial wave evolutions in appropriate energy spaces.

Proposition 4.1. *Let $R \in \mathbb{R}$, $d, k \in \mathbb{N}$ such that $R > 0$ and $d > 3$ is odd.*

1.) *The operator $\mathbf{D}_d^{\downarrow}$ extends to a bijective map*

$$\mathbf{D}_d^{\downarrow} : H^{k+1}_{\text{rad}}(\mathbb{B}^d_R) \times H^k_{\text{rad}}(\mathbb{B}^d_R) \to H^k_{\text{rad}}(\mathbb{B}^{d-2}_R) \times H^{k-1}_{\text{rad}}(\mathbb{B}^{d-2}_R)$$

and

$$\|\mathbf{D}_d^{\downarrow}\mathbf{f}\|_{H^k(\mathbb{B}^{d-2}_R)\times H^{k-1}(\mathbb{B}^{d-2}_R)} \simeq \|\mathbf{f}\|_{H^{k+1}(\mathbb{B}^d_R)\times H^k(\mathbb{B}^d_R)} \tag{4.14}$$

for all $\mathbf{f} \in H^{k+1}_{\text{rad}}(\mathbb{B}^d_R) \times H^k_{\text{rad}}(\mathbb{B}^d_R)$.

2.) *The operator $\mathbf{D}_3^{\downarrow}$ extends to a bijective map*

$$\mathbf{D}_3^{\downarrow} : H^k_{\text{rad}}(\mathbb{B}^3_R) \times H^{k-1}_{\text{rad}}(\mathbb{B}^3_R) \to H^k_{\text{odd}}(\mathbb{B}_R) \times H^{k-1}_{\text{odd}}(\mathbb{B}_R)$$

and

$$\|\mathbf{D}_3^{\downarrow}\mathbf{f}\|_{H^k(\mathbb{B}_R)\times H^{k-1}(\mathbb{B}_R)} \simeq \|\mathbf{f}\|_{H^k(\mathbb{B}^3_R)\times H^{k-1}(\mathbb{B}^3_R)} \tag{4.15}$$

for all $\mathbf{f} \in H^k_{\text{rad}}(\mathbb{B}^3_R) \times H^{k-1}_{\text{rad}}(\mathbb{B}^3_R)$.

Proof. 1.) "$\lesssim$": Note that the coefficients (4.7), (4.9) in the descent operator (4.11) are elements of $C^\infty(\overline{\mathbb{B}_R})$ that are of the form

$$\begin{aligned}&a_{10}^d(\eta) \simeq 1\,, && a_{11}(\eta) \simeq \eta\,, && a_{20}(\eta) \simeq \eta^2\,, && \\ &b_{11}^d(\eta) \simeq \eta\,, && b_{12}(\eta) \simeq \eta^2\,, && b_{20}^d(\eta) \simeq 1\,, && b_{21}(\eta) \simeq \eta\,,\end{aligned}$$

modulo a smooth function that is non-zero at the origin with all its derivatives bounded on $\mathbb{B}_R$. As derivatives are controlled by corresponding Sobolev norms, we get

$$\begin{aligned}\||\,.\,|^{\frac{d-3}{2}}[\widehat{\mathbf{D}}_d^{\downarrow}\widehat{\mathbf{f}}]_1\|_{H^k(\mathbb{B}_R)} &\lesssim \||\,.\,|^{\frac{d-3}{2}}\widehat{f}_1\|_{H^k(\mathbb{B}_R)} + \||\,.\,|^{\frac{d-3}{2}}|\,.\,|\widehat{f}_1'\|_{H^k(\mathbb{B}_R)} + \||\,.\,|^{\frac{d-3}{2}}|\,.\,|^2\widehat{f}_2\|_{H^k(\mathbb{B}_R)} \\ &= \||\,.\,|^{-1}|\,.\,|^{\frac{d-1}{2}}\widehat{f}_1\|_{H^k(\mathbb{B}_R)} + \||\,.\,|^{\frac{d-1}{2}}\widehat{f}_1'\|_{H^k(\mathbb{B}_R)} + \||\,.\,||\,.\,|^{\frac{d-1}{2}}\widehat{f}_2\|_{H^k(\mathbb{B}_R)} \\ &\lesssim \||\,.\,|^{\frac{d-1}{2}}\widehat{f}_1\|_{H^{k+1}(\mathbb{B}_R)} + \||\,.\,|^{\frac{d-1}{2}}\widehat{f}_2\|_{H^k(\mathbb{B}_R)}\end{aligned}$$

and

$$
\begin{aligned}
\||\,.\,|^{\frac{d-3}{2}}[\widehat{\mathbf{D}}_d^{\downarrow}\widehat{\mathbf{f}}]_2\|_{H^{k-1}(\mathbb{B}_R)} &\lesssim \||\,.\,|^{\frac{d-3}{2}}|\,.\,|\widehat{f}_1'\|_{H^{k-1}(\mathbb{B}_R)} + \||\,.\,|^{\frac{d-3}{2}}|\,.\,|^2\widehat{f}_1''\|_{H^{k-1}(\mathbb{B}_R)} \\
&\quad + \||\,.\,|^{\frac{d-3}{2}}\widehat{f}_2\|_{H^{k-1}(\mathbb{B}_R)} + \||\,.\,|^{\frac{d-3}{2}}|\,.\,|\widehat{f}_2'\|_{H^{k-1}(\mathbb{B}_R)} \\
&= \||\,.\,|^{\frac{d-1}{2}}\widehat{f}_1'\|_{H^{k-1}(\mathbb{B}_R)} + \||\,.\,||\,.\,|^{\frac{d-1}{2}}\widehat{f}_1''\|_{H^{k-1}(\mathbb{B}_R)} \\
&\quad + \||\,.\,|^{-1}|\,.\,|^{\frac{d-1}{2}}\widehat{f}_2\|_{H^{k-1}(\mathbb{B}_R)} + \||\,.\,|^{\frac{d-1}{2}}\widehat{f}_2'\|_{H^{k-1}(\mathbb{B}_R)} \\
&\lesssim \||\,.\,|^{\frac{d-1}{2}}\widehat{f}_1\|_{H^{k+1}(\mathbb{B}_R)} + \||\,.\,|^{\frac{d-1}{2}}\widehat{f}_2\|_{H^{k}(\mathbb{B}_R)}\,.
\end{aligned}
$$

In combination with corollary A.1 this implies the bound.

"$\gtrsim$": We compute the inverse operator and prove that it is bounded. By definition of $\mathbf{D}_d^{\downarrow}$ it suffices to show that $\widehat{\mathbf{D}}_d^{\downarrow}$ is bijective. Let $\widehat{\mathbf{g}} \in C^\infty(\mathbb{B}_R)^2$ and consider solving $\widehat{\mathbf{D}}_d^{\downarrow}\widehat{\mathbf{f}} = \widehat{\mathbf{g}}$, i.e.,

$$
\begin{cases}
a_{10}^d\widehat{f}_1 + a_{11}\widehat{f}_1' + a_{20}\widehat{f}_2 &= \widehat{g}_1\,, \\
b_{11}^d\widehat{f}_1' + b_{12}\widehat{f}_1'' + b_{20}^d\widehat{f}_2 + b_{21}\widehat{f}_2' &= \widehat{g}_2\,.
\end{cases}
$$

From the first equation we get

$$
\widehat{f}_2 = \tfrac{1}{a_{20}}\widehat{g}_1 - \tfrac{a_{10}^d}{a_{20}}\widehat{f}_1 - \tfrac{a_{11}}{a_{20}}\widehat{f}_1'\,. \tag{4.16}
$$

Putting this in the second equation we see that the first order system is indeed equivalent to the second order ordinary differential equation

$$
\widehat{f}_1'' + p\widehat{f}_1' + q\widehat{f}_1 = r_{10}\widehat{g}_1 + r_{11}\widehat{g}_1' + r_{20}\widehat{g}_2 =: g
$$

with

$$
p(\eta) = \frac{2(d-2)h'(\eta) - \eta h''(\eta)}{\eta h'(\eta)}\,, \qquad q(\eta) = (d-2)\frac{(d-3)h'(\eta) - \eta h''(\eta)}{\eta^2 h'(\eta)}\,,
$$

and on the right-hand side

$$
\begin{aligned}
r_{10}(\eta) &= \frac{(d-3)h'(\eta) - \eta h''(\eta)}{\eta^2 h'(\eta)} \simeq \frac{1}{\eta^2}\,, \qquad r_{11}(\eta) = \frac{2\eta h'(\eta) - h(\eta) - h(\eta)h'(\eta)^2}{\eta\big(\eta h'(\eta) - h(\eta)\big)} \simeq \frac{1}{\eta}\,, \\
r_{20}(\eta) &= \frac{1 - h'(\eta)^2}{\eta h'(\eta) - h(\eta)}\,\frac{h'(\eta)}{\eta} \simeq 1\,.
\end{aligned}
$$

The Sturm-Liouville problem with these coefficients has two fundamental solutions that are given explicitly by

$$
\phi_1(\eta) = \frac{h(\eta) - h(0)}{\eta^{d-2}} \simeq \frac{1}{\eta^{d-4}}\,, \qquad \phi_2(\eta) = \frac{1}{\eta^{d-2}}\,.
$$

Our foresighted choice of ϕ_1 will turn out to be convenient for the analysis of the inhomogeneous problem whose solution is given by Duhamel's formula

$$
\widehat{f}_1(\eta) = \alpha_1\phi_1(\eta) + \alpha_2\phi_2 - \phi_1(\eta)\int_0^\eta \frac{\phi_2(\eta')}{W(\eta')}g(\eta')\,\mathrm{d}\eta' + \phi_2(\eta)\int_0^\eta \frac{\phi_1(\eta')}{W(\eta')}g(\eta')\,\mathrm{d}\eta'\,,
$$

where

$$
W(\eta) = \phi_1(\eta)\phi_2'(\eta) - \phi_1'(\eta)\phi_2(\eta) = -\frac{h'(\eta)}{\eta^{2(d-2)}} \simeq \frac{1}{\eta^{2d-5}}
$$

is the Wronskian of ϕ_1, ϕ_2. The condition $f_1 \in H^{k+1}(\mathbb{B}^d_R)$ implies $\alpha_1 = \alpha_2 = 0$. The next important task is to group the solution advantageously as integral operators and exploit lemma A.4. We have

$$\widehat{f}_1 = T_1\widehat{g}_1 + T_2\widehat{g}_2 \tag{4.17}$$

where we set

$$\begin{aligned}
T_1\widehat{g}_1(\eta) &= -\phi_1(\eta)\int_0^\eta \frac{\phi_2(\eta')r_{10}(\eta')}{W(\eta')}\widehat{g}_1(\eta')\,\mathrm{d}\eta' + \phi_2(\eta)\int_0^\eta \frac{\phi_1(\eta')r_{10}(\eta')}{W(\eta')}\widehat{g}_1(\eta')\,\mathrm{d}\eta' \\
&\quad - \phi_1(\eta)\int_0^\eta \frac{\phi_2(\eta')r_{11}(\eta')}{W(\eta')}\widehat{g}_1'(\eta')\,\mathrm{d}\eta' + \phi_2(\eta)\int_0^\eta \frac{\phi_1(\eta')r_{11}(\eta')}{W(\eta')}\widehat{g}_1'(\eta')\,\mathrm{d}\eta' \\
&= -\phi_1(\eta)\int_0^\eta \left(\frac{\phi_2(\eta')r_{10}(\eta')}{W(\eta')} - \partial_{\eta'}\frac{\phi_2(\eta')r_{11}(\eta')}{W(\eta')}\right)\widehat{g}_1(\eta')\,\mathrm{d}\eta' \\
&\quad + \phi_2(\eta)\int_0^\eta \left(\frac{\phi_1(\eta')r_{10}(\eta')}{W(\eta')} - \partial_{\eta'}\frac{\phi_1(\eta')r_{11}(\eta')}{W(\eta')}\right)\widehat{g}_1(\eta')\,\mathrm{d}\eta' \\
&= -\phi_1(\eta)\int_0^\eta t_{11}(\eta'){\eta'}^{d-3}\widehat{g}_1(\eta')\,\mathrm{d}\eta' + \phi_2(\eta)\int_0^\eta t_{12}(\eta'){\eta'}^{d-3}\widehat{g}_1(\eta')\,\mathrm{d}\eta' \,,
\end{aligned}$$

as it follows by integration by parts and the fact that the integrand vanishes at the origin, and

$$\begin{aligned}
T_2\widehat{g}_2(\eta) &= -\phi_1(\eta)\int_0^\eta \frac{\phi_2(\eta')r_{20}(\eta')}{W(\eta')}\widehat{g}_2(\eta')\,\mathrm{d}\eta' + \phi_2(\eta)\int_0^\eta \frac{\phi_1(\eta')r_{20}(\eta')}{W(\eta')}\widehat{g}_2(\eta')\,\mathrm{d}\eta' \\
&= -\phi_1(\eta)\int_0^\eta t_{21}(\eta'){\eta'}^{d-3}\widehat{g}_1(\eta')\,\mathrm{d}\eta' + \phi_2(\eta)\int_0^\eta t_{22}(\eta'){\eta'}^{d-1}\widehat{g}_1(\eta')\,\mathrm{d}\eta'
\end{aligned}$$

with $t_{11}, t_{12}, t_{21}, t_{22} \in C^\infty(\overline{\mathbb{B}_R})$, all non-zero at the origin, and given by

$$\begin{aligned}
t_{11}(\eta) &= \frac{1-h'(\eta)^2}{\eta h'(\eta)-h(\eta)} + (d-3)\frac{\eta - h(\eta)h'(\eta)}{\eta\big(\eta h'(\eta)-h(\eta)\big)} - \frac{\eta^2-h(\eta)^2}{\big(\eta h'(\eta)-h(\eta)\big)^2}h''(\eta)\,, \\
t_{12}(\eta) &= \big(h(\eta)-h(0)\big)t_{11}(\eta) + \eta r_{11}(\eta)\,, \\
t_{21}(\eta) &= -\frac{1-h'(\eta)^2}{\eta h'(\eta)-h(\eta)}\,, \\
t_{22}(\eta) &= \frac{h(\eta)-h(0)}{\eta^2}t_{21}(\eta)\,.
\end{aligned}$$

According to (4.17) we have

$$\||\,.\,|^{\frac{d-1}{2}}\widehat{f}_1\|_{H^{k+1}(\mathbb{B}_R)} \le \||\,.\,|^{\frac{d-1}{2}}T_1\widehat{g}_1\|_{H^{k+1}(\mathbb{B}_R)} + \||\,.\,|^{\frac{d-1}{2}}T_2\widehat{g}_2\|_{H^{k+1}(\mathbb{B}_R)}\,,$$

so let us bound the norms on the right-hand side. Consider

$$\||\,.\,|^{\frac{d-1}{2}}T_1\widehat{g}_1\|_{H^{k+1}(\mathbb{B}_R)} \lesssim \||\,.\,|^{\frac{d-1}{2}}T_1\widehat{g}_1\|_{L^2(\mathbb{B}_R)} + \||\,.\,|^{\frac{d-3}{2}}T_1\widehat{g}_1\|_{H^k(\mathbb{B}_R)} + \||\,.\,|^{\frac{d-3}{2}}(T_1\widehat{g}_1)'\|_{H^k(\mathbb{B}_R)}\,.$$

In the following we denote by φ_∞ a generic smooth function with $\varphi_\infty(0) \ne 0$. With this

$$\begin{aligned}
\eta^{\frac{d-1}{2}}T_1\widehat{g}_1(\eta) &= \frac{\varphi_\infty(\eta)}{\eta^{\frac{d-7}{2}}}\int_0^\eta {\eta'}^{\frac{d-3}{2}}t_{11}(\eta'){\eta'}^{\frac{d-3}{2}}\widehat{g}_1(\eta')\,\mathrm{d}\eta' + \frac{\varphi_\infty(\eta)}{\eta^{\frac{d-3}{2}}}\int_0^\eta {\eta'}^{\frac{d-3}{2}}t_{12}(\eta'){\eta'}^{\frac{d-3}{2}}\widehat{g}_1(\eta')\,\mathrm{d}\eta'\,, \\
\eta^{\frac{d-3}{2}}T_1\widehat{g}_1(\eta) &= \frac{\varphi_\infty(\eta)}{\eta^{\frac{d-5}{2}}}\int_0^\eta {\eta'}^{\frac{d-3}{2}}t_{11}(\eta'){\eta'}^{\frac{d-3}{2}}\widehat{g}_1(\eta')\,\mathrm{d}\eta' + \frac{\varphi_\infty(\eta)}{\eta^{\frac{d-1}{2}}}\int_0^\eta {\eta'}^{\frac{d-3}{2}}t_{12}(\eta'){\eta'}^{\frac{d-3}{2}}\widehat{g}_1(\eta')\,\mathrm{d}\eta'\,, \\
\eta^{\frac{d-1}{2}}(T_1\widehat{g}_1)'(\eta) &= \frac{\varphi_\infty(\eta)}{\eta^{\frac{d-5}{2}}}\int_0^\eta {\eta'}^{\frac{d-3}{2}}t_{11}(\eta'){\eta'}^{\frac{d-3}{2}}\widehat{g}_1(\eta')\,\mathrm{d}\eta' + \frac{\varphi_\infty(\eta)}{\eta^{\frac{d-1}{2}}}\int_0^\eta {\eta'}^{\frac{d-3}{2}}t_{12}(\eta'){\eta'}^{\frac{d-3}{2}}\widehat{g}_1(\eta')\,\mathrm{d}\eta'\,.
\end{aligned}$$

By an application of lemma A.4 we conclude immediately

$$\||\,.\,|^{\frac{d-1}{2}} T_1 \widehat{g}_1\|_{H^{k+1}(\mathbb{B}_R)} \lesssim \||\,.\,|^{\frac{d-3}{2}} \widehat{g}_1\|_{H^k(\mathbb{B}_R)} \,. \tag{4.18}$$

Now we turn to the norm

$$\begin{aligned}
&\||\,.\,|^{\frac{d-1}{2}} T_2 \widehat{g}_2\|_{H^{k+1}(\mathbb{B}_R)} \lesssim \||\,.\,|^{\frac{d-1}{2}} T_2 \widehat{g}_2\|_{L^2(\mathbb{B}_R)} + \||\,.\,|^{\frac{d-3}{2}} T_2 \widehat{g}_2\|_{H^k(\mathbb{B}_R)} + \||\,.\,|^{\frac{d-1}{2}} (T_2 \widehat{g}_2)'\|_{H^k(\mathbb{B}_R)} \\
&\lesssim \||\,.\,|^{\frac{d-1}{2}} T_2 \widehat{g}_2\|_{L^2(\mathbb{B}_R)} \\
&+ \||\,.\,|^{\frac{d-5}{2}} T_2 \widehat{g}_2\|_{H^{k-1}(\mathbb{B}_R)} + \||\,.\,|^{\frac{d-3}{2}} (T_2 \widehat{g}_2)'\|_{H^{k-1}(\mathbb{B}_R)} + \||\,.\,|^{\frac{d-1}{2}} (T_2 \widehat{g}_2)''\|_{H^{k-1}(\mathbb{B}_R)} \,.
\end{aligned}$$

To bound each term we compute

$$\begin{aligned}
\eta^{\frac{d-1}{2}} T_2 \widehat{g}_2(\eta) &= \frac{\varphi_\infty(\eta)}{\eta^{\frac{d-7}{2}}} \int_0^\eta \eta'^{\frac{d-3}{2}} t_{21}(\eta') \eta'^{\frac{d-3}{2}} \widehat{g}_2(\eta') \, \mathrm{d}\eta' + \frac{\varphi_\infty(\eta)}{\eta^{\frac{d-3}{2}}} \int_0^\eta \eta'^{\frac{d+1}{2}} t_{22}(\eta') \eta'^{\frac{d-3}{2}} \widehat{g}_2(\eta') \, \mathrm{d}\eta' \,, \\
\eta^{\frac{d-5}{2}} T_2 \widehat{g}_2(\eta) &= \frac{\varphi_\infty(\eta)}{\eta^{\frac{d-3}{2}}} \int_0^\eta \eta'^{\frac{d-3}{2}} t_{21}(\eta') \eta'^{\frac{d-3}{2}} \widehat{g}_2(\eta') \, \mathrm{d}\eta' + \frac{\varphi_\infty(\eta)}{\eta^{\frac{d+1}{2}}} \int_0^\eta \eta'^{\frac{d+1}{2}} t_{22}(\eta') \eta'^{\frac{d-3}{2}} \widehat{g}_2(\eta') \, \mathrm{d}\eta' \,, \\
\eta^{\frac{d-3}{2}} (T_2 \widehat{g}_2)'(\eta) &= \frac{\varphi_\infty(\eta)}{\eta^{\frac{d-3}{2}}} \int_0^\eta \eta'^{\frac{d-3}{2}} t_{21}(\eta') \eta'^{\frac{d-3}{2}} \widehat{g}_2(\eta') \, \mathrm{d}\eta' + \frac{\varphi_\infty(\eta)}{\eta^{\frac{d+1}{2}}} \int_0^\eta \eta'^{\frac{d+1}{2}} t_{22}(\eta') \eta'^{\frac{d-3}{2}} \widehat{g}_2(\eta') \, \mathrm{d}\eta' \,, \\
\eta^{\frac{d-1}{2}} (T_2 \widehat{g}_2)''(\eta) &= \frac{\varphi_\infty(\eta)}{\eta^{\frac{d-3}{2}}} \int_0^\eta \eta'^{\frac{d-3}{2}} t_{21}(\eta') \eta'^{\frac{d-3}{2}} \widehat{g}_2(\eta') \, \mathrm{d}\eta' + \frac{\varphi_\infty(\eta)}{\eta^{\frac{d+1}{2}}} \int_0^\eta \eta'^{\frac{d+1}{2}} t_{22}(\eta') \eta'^{\frac{d-3}{2}} \widehat{g}_2(\eta') \, \mathrm{d}\eta' \\
&\quad + \eta \eta^{\frac{d-3}{2}} \widehat{g}_2(\eta) \,.
\end{aligned}$$

This implies again with lemma A.4

$$\||\,.\,|^{\frac{d-1}{2}} T_2 \widehat{g}_2\|_{H^{k+1}(\mathbb{B}_R)} \lesssim \||\,.\,|^{\frac{d-3}{2}} \widehat{g}_2\|_{H^{k-1}(\mathbb{B}_R)} \tag{4.19}$$

as desired.
To bound $\widehat{f}_2$ we insert (4.17) in (4.16) and obtain

$$\widehat{f}_2 = S_1 \widehat{g}_1 + S_2 \widehat{g}_2 \,,$$

where

$$S_1 \widehat{g}_1 = \tfrac{1}{a_{20}} \widehat{g}_1 - \tfrac{a_{10}^d}{a_{20}} T_1 \widehat{g}_1 - \tfrac{a_{11}}{a_{20}} (T_1 \widehat{g}_1)' \,, \qquad S_2 \widehat{g}_2 = -\tfrac{a_{10}^d}{a_{20}} T_2 \widehat{g}_2 - \tfrac{a_{11}}{a_{20}} (T_2 \widehat{g}_2)' \,.$$

In particular,

$$\begin{aligned}
S_1 \widehat{g}_1(\eta) &= \frac{1 - a_{11}(\eta) r_{11}(\eta)}{a_{20}(\eta)} \widehat{g}_1(\eta) + \frac{a_{10}^d(\eta)\phi_1(\eta) + a_{11}(\eta)\phi_1'(\eta)}{a_{20}(\eta)} \int_0^\eta t_{11}(\eta') \eta'^{d-3} \widehat{g}_1(\eta') \, \mathrm{d}\eta' \\
&\quad - \frac{a_{10}^d(\eta)\phi_2(\eta) + a_{11}(\eta)\phi_2'(\eta)}{a_{20}(\eta)} \int_0^\eta t_{12}(\eta') \eta'^{d-3} \widehat{g}_1(\eta') \, \mathrm{d}\eta' \\
&= s_0(\eta) \widehat{g}_1(\eta) + s_1(\eta) \int_0^\eta t_{11}(\eta') \eta'^{d-3} \widehat{g}_1(\eta') \, \mathrm{d}\eta' + s_2(\eta) \int_0^\eta t_{12}(\eta') \eta'^{d-3} \widehat{g}_1(\eta') \, \mathrm{d}\eta'
\end{aligned}$$

and

$$\begin{aligned}
S_2 \widehat{g}_2(\eta) &= \frac{a_{10}^d(\eta)\phi_1(\eta) + a_{11}(\eta)\phi_1'(\eta)}{a_{20}(\eta)} \int_0^\eta t_{21}(\eta') \eta'^{d-3} \widehat{g}_1(\eta') \, \mathrm{d}\eta' \\
&\quad - \frac{a_{10}^d(\eta)\phi_2(\eta) + a_{11}(\eta)\phi_2'(\eta)}{a_{20}(\eta)} \int_0^\eta t_{22}(\eta') \eta'^{d-3} \widehat{g}_1(\eta') \, \mathrm{d}\eta' \\
&= s_1(\eta) \int_0^\eta t_{21}(\eta') \eta'^{d-3} \widehat{g}_1(\eta') \, \mathrm{d}\eta' + s_2(\eta) \int_0^\eta t_{22}(\eta') \eta'^{d-3} \widehat{g}_1(\eta') \, \mathrm{d}\eta'
\end{aligned}$$

with

$$s_0(\eta) = -\frac{\eta^2 - h(\eta^2)}{\eta h'(\eta) - h(\eta)} \frac{h'(\eta)}{\eta} \simeq 1 \,, \quad s_1(\eta) = -\frac{(d-3)h(\eta) - (d-2)h(0)}{\eta^{d-2}} \simeq \frac{1}{\eta^{d-2}} \,,$$
$$s_2(\eta) = -\frac{d-2}{\eta^{d-2}} \simeq \frac{1}{\eta^{d-2}} \,.$$

Let us turn to the estimate

$$\begin{aligned} &\||\,.\,|^{\frac{d-1}{2}} \widehat{f}_2\|_{H^k(\mathbb{B}_R)} \leq \||\,.\,|^{\frac{d-1}{2}} S_1 \widehat{g}_1\|_{H^k(\mathbb{B}_R)} + \||\,.\,|^{\frac{d-1}{2}} S_2 \widehat{g}_2\|_{H^k(\mathbb{B}_R)} \\ \lesssim\ &\||\,.\,|^{\frac{d-1}{2}} S_1 \widehat{g}_1\|_{H^k(\mathbb{B}_R)} + \||\,.\,|^{\frac{d-1}{2}} S_2 \widehat{g}_2\|_{L^2(\mathbb{B}_R)} + \||\,.\,|^{\frac{d-3}{2}} S_2 \widehat{g}_2\|_{H^{k-1}(\mathbb{B}_R)} + \||\,.\,|^{\frac{d-1}{2}} (S_2 \widehat{g}_2)'\|_{H^{k-1}(\mathbb{B}_R)} \,. \end{aligned}$$

We get

$$\begin{aligned} \eta^{\frac{d-1}{2}} S_1 \widehat{g}_1(\eta) = &\frac{\varphi_\infty(\eta)}{\eta^{\frac{d-3}{2}}} \int_0^\eta \eta'^{\frac{d-3}{2}} t_{11}(\eta') \eta'^{\frac{d-3}{2}} \widehat{g}_1(\eta') \,\mathrm{d}\eta' + \frac{\varphi_\infty(\eta)}{\eta^{\frac{d-3}{2}}} \int_0^\eta \eta'^{\frac{d-3}{2}} t_{12}(\eta') \eta'^{\frac{d-3}{2}} \widehat{g}_1(\eta') \,\mathrm{d}\eta' \\ &+ \eta \varphi_\infty(\eta) \eta^{\frac{d-3}{2}} \widehat{g}_1(\eta) \end{aligned}$$

and infer immediately from lemma A.4

$$\||\,.\,|^{\frac{d-1}{2}} S_1 \widehat{g}_1\|_{H^k(\mathbb{B}_R)} \lesssim \||\,.\,|^{\frac{d-3}{2}} \widehat{g}_1\|_{H^k(\mathbb{B}_R)} \,. \tag{4.20}$$

Finally,

$$\begin{aligned} \eta^{\frac{d-1}{2}} S_2 \widehat{g}_2(\eta) &= \frac{\varphi_\infty(\eta)}{\eta^{\frac{d-3}{2}}} \int_0^\eta \eta'^{\frac{d-3}{2}} t_{21}(\eta') \eta'^{\frac{d-3}{2}} \widehat{g}_2(\eta') \,\mathrm{d}\eta' + \frac{\varphi_\infty(\eta)}{\eta^{\frac{d-3}{2}}} \int_0^\eta \eta'^{\frac{d+1}{2}} t_{22}(\eta') \eta'^{\frac{d-3}{2}} \widehat{g}_2(\eta') \,\mathrm{d}\eta' \,, \\ \eta^{\frac{d-3}{2}} S_2 \widehat{g}_2(\eta) &= \frac{\varphi_\infty(\eta)}{\eta^{\frac{d-1}{2}}} \int_0^\eta \eta'^{\frac{d-3}{2}} t_{21}(\eta') \eta'^{\frac{d-3}{2}} \widehat{g}_2(\eta') \,\mathrm{d}\eta' + \frac{\varphi_\infty(\eta)}{\eta^{\frac{d-1}{2}}} \int_0^\eta \eta'^{\frac{d+1}{2}} t_{22} \eta'^{\frac{d-3}{2}} \widehat{g}_2(\eta') \,\mathrm{d}\eta' \,, \\ \eta^{\frac{d-1}{2}} (S_2 \widehat{g}_2)'(\eta) &= \frac{\varphi_\infty(\eta)}{\eta^{\frac{d-1}{2}}} \int_0^\eta \eta'^{\frac{d-3}{2}} t_{21}(\eta') \eta'^{\frac{d-3}{2}} \widehat{g}_2(\eta') \,\mathrm{d}\eta' + \frac{\varphi_\infty(\eta)}{\eta^{\frac{d-1}{2}}} \int_0^\eta \eta'^{\frac{d+1}{2}} t_{22}(\eta') \eta'^{\frac{d-3}{2}} \widehat{g}_2(\eta') \,\mathrm{d}\eta' \\ &\quad + \varphi_\infty(\eta) \eta^{\frac{d-3}{2}} \widehat{g}_2(\eta) \,. \end{aligned}$$

Altogether, this implies

$$\|S_2 \widehat{g}_2\|_{H^k(\mathbb{B}_R)} \lesssim \|\widehat{g}_2\|_{H^{k-1}(\mathbb{B}_R)} \tag{4.21}$$

The estimates (4.18) - (4.21) imply together with corollary A.1 the desired result. This finishes the proof of the first part.

2.) "$\lesssim$": We get directly for the first component

$$\|[\widehat{\mathbf{D}}_3^{\downarrow} \widehat{\mathbf{f}}]_1\|_{H^k(\mathbb{B}_R)} = \||\,.\,| \widehat{f}_1\|_{H^k(\mathbb{B}_R)}$$

and for the second component

$$\|[\widehat{\mathbf{D}}_3^{\downarrow} \widehat{\mathbf{f}}]_2\|_{H^{k-1}(\mathbb{B}_R)} \leq \||\,.\,| \widehat{f}_1\|_{H^{k-1}(\mathbb{B}_R)} + \||\,.\,| \widehat{f}_2\|_{H^{k-1}(\mathbb{B}_R)} \leq \||\,.\,| \widehat{f}_1\|_{H^k(\mathbb{B}_R)} + \||\,.\,| \widehat{f}_2\|_{H^{k-1}(\mathbb{B}_R)}$$

With corollary A.1 this gives precisely

$$\|\widehat{\mathbf{D}}_3^{\downarrow} \widehat{\mathbf{f}}\|_{H^k(\mathbb{B}_R) \times H^{k-1}(\mathbb{B}_R)} \lesssim \|\widehat{\mathbf{f}}\|_{H^k(\mathbb{B}_R^3) \times H^{k-1}(\mathbb{B}_R^3)} \,.$$

"$\gtrsim$": As before we invert $\mathbf{D}_3^{\downarrow}$. Let $\widehat{\mathbf{g}} \in C^\infty_{\text{odd}}(\mathbb{B}_R)^2$ and consider $\widehat{\mathbf{D}}_d^{\downarrow}\widehat{\mathbf{f}} = \widehat{\mathbf{g}}$, that is,

$$\begin{cases} \eta \widehat{f}_1 &= \widehat{g}_1\,, \\ -\eta \widehat{f}_1 + \eta \widehat{f}_2 &= \widehat{g}_2\,. \end{cases}$$

Since $\widehat{g}_1$ is odd we have $\widehat{g}_1(0) = 0$ and with the fundamental theorem of calculus we infer $\widehat{f}_1, \widehat{f}_2 \in C^\infty_{\text{even}}(\mathbb{B}_R)^2$ given by

$$\widehat{f}_1(\eta) = \int_0^1 \widehat{g}_1'(t\eta)\,\mathrm{d}t\,, \qquad \widehat{f}_2(\eta) = \int_0^1 \big(\widehat{g}_1'(t\eta) + \widehat{g}_2'(t\eta)\big)\,\mathrm{d}t\,.$$

Now the bound follows similarly from

$$\||\,.\,|\widehat{f}_1\|_{H^k(\mathbb{B}_R)} = \|\widehat{g}_1\|_{H^k(\mathbb{B}_R)}$$

and

$$\||\,.\,|\widehat{f}_2\|_{H^{k-1}(\mathbb{B}_R)} \leq \|\widehat{g}_1\|_{H^{k-1}(\mathbb{B}_R)} + \|\widehat{g}_2\|_{H^{k-1}(\mathbb{B}_R)} \leq \|\widehat{g}_1\|_{H^k(\mathbb{B}_R)} + \|\widehat{g}_2\|_{H^{k-1}(\mathbb{B}_R)}\,.$$

This translates with corollary A.1 to the inequality

$$\|\widehat{\mathbf{f}}\|_{H^k(\mathbb{B}_R^3)\times H^{k-1}(\mathbb{B}_R^3)} \lesssim \|\widehat{\mathbf{D}}_3^{\downarrow}\widehat{\mathbf{f}}\|_{H^k(\mathbb{B}_R)\times H^{k-1}(\mathbb{B}_R)}\,.$$

From density and boundedness we infer the bounded and bijective extension. □

4.2 Construction of the semigroup

Having proved this major result, we can answer now how to derive an energy norm directly from the wave equation that exhibits growth estimates, namely via the descent method in steps.

Proof of proposition 1.1. By successive application of the descent operators we obtain an operator

$$\begin{aligned} \mathbf{D}_d : H^{k+1}_{\text{rad}}(\mathbb{B}_R^d) \times H^k_{\text{rad}}(\mathbb{B}_R^d) &\to H^{k+1-\frac{d-3}{2}}_{\text{odd}}(\mathbb{B}_R) \times H^{k-\frac{d-3}{2}}_{\text{odd}}(\mathbb{B}_R) \\ \mathbf{f} &\mapsto \mathbf{D}_d\mathbf{f} \coloneqq (\mathbf{D}_3^{\downarrow} \circ \ldots \circ \mathbf{D}_d^{\downarrow})\mathbf{f}\,. \end{aligned}$$

From proposition 4.1 we infer the bound

$$\|\mathbf{D}_d\mathbf{f}\|_{H^{k+1-\frac{d-3}{2}}(\mathbb{B}_R)\times H^{k-\frac{d-3}{2}}(\mathbb{B}_R)} \simeq \|\mathbf{f}\|_{H^{k+1}(\mathbb{B}_R^d)\times H^k(\mathbb{B}_R^d)}$$

for all $\mathbf{f} \in H^{k+1}_{\text{rad}}(\mathbb{B}_R^d) \times H^k_{\text{rad}}(\mathbb{B}_R^d)$. Repeated application of the intertwining identites from lemma 4.1 yields

$$\mathbf{D}_d\mathbf{L}_d\mathbf{f} - \mathbf{D}_d\mathbf{f} = \mathbf{L}_1\mathbf{D}_d\mathbf{f}$$

for all $\mathbf{f} \in C^\infty(\mathbb{B}_R^d)^2$. □

The results gathered in subsection 3.2 about the one-dimensional wave evolution imply the anticipated theorem.

Proof of theorem 1.1. **1.)** Let $\mathbf{S}_1$ be the strongly continuous semigroup on $\mathfrak{L}\big(H^{k+1-\frac{d-3}{2}}_{\mathrm{odd}}(\mathbb{B}_R) \times H^{k-\frac{d-3}{2}}_{\mathrm{odd}}(\mathbb{B}_R)\big)$ from proposition 3.2. Then

$$\mathbf{S}_d(s) = \mathrm{e}^s \mathbf{D}_d^{-1} \mathbf{S}_1(s) \mathbf{D}_d$$

forms a strongly continuous semigroup on $\mathfrak{L}\big(H^{k+1}_{\mathrm{rad}}(\mathbb{B}^d_R) \times H^k_{\mathrm{rad}}(\mathbb{B}^d_R)\big)$ with the growth bound (1.9) and generator (1.8).

2.) The second part is furnished by the second part of proposition 3.2 and the intertwining identity,

$$\begin{aligned}
\partial_s \mathbf{S}_d(s)\mathbf{f} &= \mathrm{e}^s \mathbf{D}_d^{-1} \mathbf{S}_1(s) \mathbf{D}_d \mathbf{f} + \mathrm{e}^s \mathbf{D}_d^{-1} \mathbf{L}_1 \mathbf{S}_1(s) \mathbf{D}_d \mathbf{f} \\
&= \mathbf{S}_d(s) + \mathrm{e}^s \mathbf{D}_d^{-1} \mathbf{L}_1 \mathbf{D}_d \mathbf{D}_d^{-1} \mathbf{S}_1(s) \mathbf{D}_d \mathbf{f} \\
&= \mathbf{S}_d(s) + \mathbf{D}_d^{-1} \mathbf{L}_1 \mathbf{D}_d \mathbf{S}_d(s) \mathbf{f} \\
&= \mathbf{S}_d(s) + \mathbf{D}_d^{-1} (\mathbf{D}_d \mathbf{L}_d - \mathbf{D}_d) \mathbf{S}_d(s) \mathbf{f} \\
&= \mathbf{L}_d \mathbf{S}_d(s) \mathbf{f}\,.
\end{aligned}$$

□

A Appendix

The following inequality is known as Hardy's inequality.

Lemma A.1. *Let $d \in \mathbb{N}$ and $s \in \left[0, \frac{d}{2}\right)$. Then*

$$\| |\,.\,|^{-s} f\|_{L^2(\mathbb{R}^d)} \lesssim \|f\|_{\dot{H}^s(\mathbb{R}^d)}$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$.

In order to make good use of the Fourier transform, we construct ourselves some Sobolev extensions.

Lemma A.2. *Let $R > 0$ and $k \in \mathbb{N}_0$. There is a map $\mathcal{E} : C^k(\overline{\mathbb{B}_R}) \to C^k(\mathbb{R})$ such that*

$$(\mathcal{E} f)|_{\mathbb{B}_R} = f\,, \qquad \operatorname{supp}(\mathcal{E} f) \subseteq \mathbb{B}_{2R}\,, \qquad \|\mathcal{E} f\|_{H^k(\mathbb{R}\setminus\mathbb{B}_R)} \lesssim \|f\|_{H^k(\mathbb{B}_R\setminus\mathbb{B}_{R/2})}$$

for all $f \in C^k(\mathbb{B}_R)$.

Proof. Cf. [1, Appendix A, Lemma A.4]. □

We rely on a convenient description of the Sobolev norm for radial functions in odd space dimensions.

Lemma A.3. *Let $d, k \in \mathbb{N}_0$, d odd. We have*

$$\|f\|_{H^k(\mathbb{R}^d)} \simeq \| |\,.\,|^{\frac{d-1}{2}} \widehat{f}\|_{H^k(\mathbb{R})} \quad \textit{for all } f \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d).$$

Proof. "$\gtrsim$": After a change of variables we see

$$\|f\|^2_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |f(x)|^2 \,\mathrm{d}x = \int_{\mathbb{R}^d} |\widehat{f}(|x|)|^2 \,\mathrm{d}x \simeq \int_{\mathbb{R}} |\widehat{f}(r)|^2 r^{d-1} \,\mathrm{d}r = \| |\,.\,|^{\frac{d-1}{2}} \widehat{f}\|^2_{L^2(\mathbb{R})}\,.$$

Since d is odd, a derivative ∂_r^ℓ hitting $r^{\frac{d-1}{2}}$ contributes to the Sobolev norm only when $\ell \leq \frac{d-1}{2}$, i.e., $\ell < \frac{d}{2}$. In this case Hardy's inequality yields

$$\| |\,.\,|^{-\ell} |\,.\,|^{\frac{d-1}{2}} \widehat{f}^{(k-\ell)}\|_{L^2(\mathbb{R})} \simeq \| |\,.\,|^{-\ell} \partial_d^{k-\ell} f\|_{L^2(\mathbb{R}^d)} \lesssim \|\partial_d^{k-\ell} f\|_{\dot{H}^\ell(\mathbb{R}^d)} \lesssim \|f\|_{\dot{H}^k(\mathbb{R}^d)}\,.$$

Using this in the classical definition of the homogeneous Sobolev seminorm we get one inequality.

"$\lesssim$": To see the other inequality, we introduce the Fourier transform on $\mathcal{S}(\mathbb{R}^d)$ by

$$\mathcal{F}_d f(\xi) = \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}\xi\cdot x} f(x) \,\mathrm{d}^d x\,.$$

If f is radial then $\mathcal{F}_d f$ is also radial and

$$\widehat{\mathcal{F}_d f}(\rho) = (2\pi)^{\frac{d}{2}} \rho^{-\frac{d}{2}+1} \int_0^\infty \widehat{f}(r) J_{\frac{d}{2}-1}(\rho r) r^{\frac{d}{2}} \,\mathrm{d}r\,,$$

cf. [14] for a reference. The Bessel function of non-negative half-integer order is given explicitly by

$$\begin{aligned} J_{\frac{d}{2}-1}(z) &= \sqrt{\frac{2}{\pi}} z^{\frac{d-1}{2}} \left(-\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z}\right)^{\frac{d}{2}-1} \frac{\sin z}{z} \\ &= \sqrt{\frac{2}{\pi}} z^{-\frac{1}{2}} \left(P_{\frac{d-3}{2}}(z^{-1})\sin(z) - Q_{\frac{d-5}{2}}(z^{-1})\cos(z)\right) , \end{aligned}$$

where P_n, Q_n are polynomials of degree n with $P_n(-x) = (-1)^n P_n(x)$ and $Q_n(-x) = (-1)^n Q_n(x)$. It follows that

$$\begin{aligned} \widehat{\mathcal{F}_d f}(\rho) &= \mathrm{i}(2\pi)^{\frac{d-1}{2}} \rho^{\frac{-d+1}{2}} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\rho r} \widehat{f}(r) r^{\frac{d-1}{2}} P_{\frac{d-3}{2}}((\rho r)^{-1})\,\mathrm{d}r \\ &\quad -(2\pi)^{\frac{d-1}{2}} \rho^{\frac{-d+1}{2}} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\rho r} \widehat{f}(r) r^{\frac{d-1}{2}} Q_{\frac{d-5}{2}}((\rho r)^{-1})\,\mathrm{d}r \\ &= (2\pi)^{\frac{d-1}{2}} \rho^{\frac{-d+1}{2}} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\rho r} \widehat{f}(r) r^{\frac{d-1}{2}} P((\rho r)^{-1})\,\mathrm{d}r \end{aligned}$$

for some polynomial P. Note that by Hardy's inequality [1, Appendix A, Lemma A.1] we have

$$\begin{aligned} \||\,.\,|^k |\,.\,|^{-j} \mathcal{F}((\,.\,)^{\frac{d-1}{2}-j}\widehat{f})\|_{L^2(\mathbb{R})} &\lesssim \||\,.\,|^k |\,.\,|^{-j+1} \mathcal{F}((\,.\,)^{\frac{d-1}{2}-j}\widehat{f})'\|_{L^2(\mathbb{R})} \\ &\lesssim \||\,.\,|^k |\,.\,|^{-j+1} \mathcal{F}((\,.\,)^{\frac{d-1}{2}-j+1}\widehat{f})\|_{L^2(\mathbb{R})} \\ &\lesssim \||\,.\,|^k \mathcal{F}((\,.\,)^{\frac{d-1}{2}}\widehat{f})\|_{L^2(\mathbb{R})} \\ &= \||\,.\,|^{\frac{d-1}{2}}\widehat{f}\|_{\dot{H}^k(\mathbb{R})} . \end{aligned}$$

This yields the other inequality. ☐

Using a Sobolev extension we transfer this to open balls.

Corollary A.1. *Let $R > 0$, $d, k \in \mathbb{N}_0$, d odd. We have*

$$\|f\|_{H^k(\mathbb{B}^d_R)} \simeq \||\,.\,|^{\frac{d-1}{2}}\widehat{f}\|_{H^k(\mathbb{B}_R)} \quad \textit{for all } f \in C^\infty_{\mathrm{rad}}(\overline{\mathbb{B}^d_R}).$$

Proof. The proof of [1, Appendix A, Corollary A.5] can be literally adopted. ☐

Also, we need to collect some estimates for integral operators of the following type.

Lemma A.4. *Let $R \in \mathbb{R}$, $k, m, n \in \mathbb{N}_0$ such that $R > 0$, $n + 1 - m \geq 0$. Let $\varphi \in C^\infty(\overline{\mathbb{B}_R})$ and*

$$T : C^\infty(\overline{\mathbb{B}_R}) \to C^\infty(\overline{\mathbb{B}_R}) \quad \textit{such that} \quad Tf(x) = \frac{1}{x^m}\int_0^x y^n \varphi(y) f(y)\,\mathrm{d}y\,.$$

Then there is $C_k > 0$ such that

$$\|Tf\|_{H^k(\mathbb{B}_R)} \leq C_k \|f\|_{H^k(\mathbb{B}_R)} \quad \textit{for all } f \in C^\infty(\overline{\mathbb{B}_R})\,.$$

In particular, T extends to a bounded operator on $H^k(\mathbb{B}_R)$.

Proof. Cf. [1, Appendix A, Lemma A.6]. ☐

References

[1] Paweł Biernat, Roland Donninger, and Birgit Schörkhuber. *Hyperboloidal similarity coordinates and a globally stable blowup profile for supercritical wave maps*. 2017. arXiv: 1707.09812 [math.AP].

[2] Paweł Biernat, Roland Donninger, and Birgit Schörkhuber. *Stable self-similar blowup in the supercritical heat flow of harmonic maps*. 2016. arXiv: 1610.09497 [math.AP].

[3] Athanasios Chatzikaleas and Roland Donninger. *Stable blowup for the cubic wave equation in higher dimensions*. 2018. arXiv: 1712.03833 [math.AP].

[4] Athanasios Chatzikaleas, Roland Donninger, and Irfan Glogić. "On blowup of co-rotational wave maps in odd space dimensions". In: *Journal of Differential Equations* 263.8 (Oct. 2017), pp. 5090–5119. URL: http://dx.doi.org/10.1016/J.JDE.2017.06.011.

[5] Roland Donninger. "Stable self-similar blowup in energy supercritical Yang–Mills theory". In: *Mathematische Zeitschrift* 278.3-4 (June 2014), pp. 1005–1032. URL: http://dx.doi.org/10.1007/S00209-014-1344-0.

[6] Roland Donninger. "The radial wave operator in similarity coordinates". In: *Journal of Mathematical Physics* 51.2 (Feb. 2010), p. 023527. URL: http://dx.doi.org/10.1063/1.3299302.

[7] Roland Donninger and Irfan Glogić. *Strichartz estimates for the one-dimensional wave equation*. 2019. arXiv: 1908.02157 [math.AP].

[8] Roland Donninger and Birgit Schörkhuber. "On Blowup in Supercritical Wave Equations". In: *Communications in Mathematical Physics* 346.3 (Mar. 2016), pp. 907–943. URL: http://dx.doi.org/10.1007/s00220-016-2610-2.

[9] Roland Donninger and Birgit Schörkhuber. *Stable blowup for the supercritical Yang-Mills heat flow*. 2016. arXiv: 1604.07737 [math.AP].

[10] Roland Donninger and Birgit Schörkhuber. *Stable blowup for wave equations in odd space dimensions*. 2015. arXiv: 1504.00808 [math.AP].

[11] Roland Donninger, Birgit Schörkhuber, and Peter C. Aichelburg. "On Stable Self-Similar Blow up for Equivariant Wave Maps: The Linearized Problem". In: *Annales Henri Poincaré* 13.1 (July 2011), pp. 103–144. URL: http://dx.doi.org/10.1007/s00023-011-0125-0.

[12] Roland Donninger and Anil Zenginoğlu. "Nondispersive decay for the cubic wave equation". In: *Analysis & PDE* 7.2 (May 2014), pp. 461–495. URL: http://dx.doi.org/10.2140/apde.2014.7.461.

[13] Lawrence C. Evans. *Partial differential equations*. Vol. 19. American Mathematical Society, 2010.

[14] Elias M. Stein and Rami Shakarchi. *Fourier analysis*. Princeton University Press, Princeton, NJ, 2003.

[15] Walter A. Strauss. *Partial differential equations*. Vol. 2. John Wiley & Sons New York, NY, USA, 1992.

[16] Gerald Teschl. "Mathematical Methods in Quantum Mechanics". In: *Graduate Studies in Mathematics* 157 (2014).

[17] Gerald Teschl. *Topics in Real and Functional Analysis*. 2018. URL: http://www.mat.univie.ac.at/~gerald/ftp/book-fa/.

Glossary of notation

$\mathbb{B}_R^d(x)$	ball of radius R centred around x in $\mathbb{R}^d$
$\mathbb{B}_R^d, \mathbb{B}_R$	ball of radius R centred around 0 in $\mathbb{R}^d$, $\mathbb{R}$, respectively
$C^\infty(\Omega)$	set of smooth functions on some open set $\Omega \subset \mathbb{R}^d$
$C^\infty_{\text{even}}(\mathbb{B}_R), C^\infty_{\text{odd}}(\mathbb{B}_R)$	set of smooth even or odd functions, respectively
$C^\infty_{\text{rad}}(\mathbb{B}_R^d)$	set of smooth radial functions
$\mathbb{C}$	set of complex numbers
∂f	gradient or Jacobian of a function
Δf	Laplacian of a function on $\mathbb{R}^d$
$\mathbf{D}_d, \mathbf{D}_d^\downarrow$	descent operators
$\mathfrak{D}(L)$	domain of a linear operator L
e	Euler's number
$\mathcal{F}f$	Fourier transform of f
$H^k(\Omega)$	L^2-based Sobolev space of order k on some open set $\Omega \subset \mathbb{R}^d$
$H^k_{\text{even}}(\mathbb{B}_R), H^k_{\text{odd}}(\mathbb{B}_R)$	Sobolev spaces of even, odd functions, respectivley
$H^k_{\text{rad}}(\mathbb{B}_R^d)$	Sobolev spaces of radial functions
I_d, I	identity matrix on $\mathbb{R}^d$, identitiy operator on a Banach space X
i	imaginary unit
$\mathfrak{L}(X)$	space of bounded linear operators on a Banach space X
$\mathbb{N}, \mathbb{N}_0$	set of natural numbers, set of natural numbers with zero included
$\mathbb{R}^d$	d-dimensional Euclidean space
$\mathbb{R}^{1,d}$	$(1+d)$-dimensional Minkowski spacetime
$\mathbb{R}$	set of real numbers