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Characterizations of Symmetry in Banach $\ast$ -Algebras

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Chapter 1

Introduction

In general Banach $*$ -algebras there is relatively little connection between the underlying Banach algebra structure on the one hand and the $*$ -algebra structure on the other hand. One can study different concepts, such as positivity, representations and radicals, either in the algebraic or in the $*$ -algebraic fashion.

Regarding positivity, there are at least two natural definitions. An element a in a Banach $*$ -algebra $\mathcal{A}$ may be defined to be positive if the spectrum of a is nonnegative. Another equally natural definition is to declare an element a in $\mathcal{A}$ to be positive if it is of the form $a = b^*b$ for some b in $\mathcal{A}$.

In order to avoid disagreement we may insist that the spectrum of every element of the form a^*a is nonnegative. This motivates the following definition:

A Banach $*$ -algebra $\mathcal{A}$ is *symmetric* if $Sp(a^*a) \subseteq \mathbb{R}_+$ for all $a \in \mathcal{A}$.

Whereas the distinct notions of positivity then agree per definition on hermitian elements in symmetric Banach $*$ -algebras, the symmetry axiom also secures a closer connection of the Banach algebra structure and the $*$ -algebra structure in several other respects and thereby reveals many interesting results.

The starting point of the theory of Banach $*$ -algebras was the groundbreaking paper of Gelfand and Naimark in 1943 where they introduced C^* -algebras [8]. To obtain their main results they had to assume, in addition to the classical C^* -axioms, that each element of the form $e + a^*a$ was invertible, or to put it differently, that $Sp(a^*a) \subseteq \mathbb{R}_+$ for all $a \in \mathcal{A}$. They conjectured that this assumption was redundant. Indeed, their conjecture was shown to be true years later after considerable work. Further it turned out that it was precisely the symmetry axiom that was essential for many results to hold.

The first mathematician who studied symmetry in its own right was Raikov. In his 1946 paper on normed rings with involution [24] he extended some results of Gelfand and Naimark under the purely algebraic assumption of symmetry. He already developed a remarkable portion of the theory of symmetric Banach $*$ -algebras including a criterion for symmetry that relates the spectral radius to positive linear functionals.

Another important algebraic notion, today called hermiticity, requires the spectrum of every hermitian element to be real. This concept was introduced by Rickart [25] and was soon conjectured to be equivalent to symmetry by Kaplansky [12].

It was quite easy to see that symmetric Banach $*$ -algebras are hermitian whereas the converse implication remained an unresolved problem for more than 20 years. It was finally resolved by Shirali and Ford in 1970 [27].

Also several other concepts similar to symmetry, such as C-symmetry [12] and complete symmetry [30, 31], were studied.

Around 1970 further contributions were made by Ptak [22, 23]. He characterized symmetry by various properties of the spectral radius, e.g., by the fundamental inequality

$$r(a) \leq r(a^*a)^{\frac{1}{2}}.$$

This inequality was crucial for the development of the theory of symmetric Banach $*$ -algebras and led to considerable simplifications for many proofs. Ptak also noted that the function $a \mapsto \rho(a) := r(a^*a)^{\frac{1}{2}}$ is an algebra seminorm exactly in symmetric Banach $*$ -algebras.

Ptak's work led Palmer [18] to formulate other characterizations of symmetry in terms of the Gelfand-Naimark seminorm γ , which is a remarkable algebra seminorm defined via positive linear functionals. Also the unitary seminorm ν , defined as the Minkowski functional of the convex hull of the unitaries, was studied with regard to symmetry by Palmer [17] and Ptak [23] independently.

A connection to representation theory and a corresponding characterization of symmetry were observed by Leptin [16].

Over the years many equivalent descriptions of symmetry have accumulated and still new characterizations emerge [1].

The aim of this thesis is to collect and systematically study some of the characterizations of symmetry.

1.1 Theorem. *For a unital Banach $*$ -algebra $\mathcal{A}$ with continuous involution the following conditions are equivalent:*

- (1) $\mathcal{A}$ is hermitian: $Sp(h) \subseteq \mathbb{R}$ for all hermitian elements $h \in \mathcal{A}$.
- (2) $\mathcal{A}$ is symmetric: $Sp(a^*a) \subseteq \mathbb{R}_+$ for all $a \in \mathcal{A}$.
- (3) $\mathcal{A}$ is completely symmetric: $Sp(p) \subseteq \mathbb{R}_+$ for all elements of the form $p = \sum_{k=1}^n a_k^* a_k$ for $a_k \in \mathcal{A}$, $n \in \mathbb{N}$.
- (4) $i \notin Sp(h)$ for all hermitian elements $h \in \mathcal{A}$.
- (5) $e + a^*a$ is invertible for all $a \in \mathcal{A}$.
- (6) The Ptak function ρ is an algebra seminorm on $\mathcal{A}$.
- (7) $\rho(a + b) \leq \rho(a) + \rho(b)$ for all $a, b \in \mathcal{A}$.
- (8) $r(\frac{1}{2}(a^* + a)) \leq \rho(a)$ for all $a \in \mathcal{A}$.
- (9) $r(a) \leq \rho(a)$ for all $a \in \mathcal{A}$.
- (10) $r(a) = \rho(a)$ for all normal elements $a \in \mathcal{A}$.
- (11) $r(a) \leq \|a^*a\|^{1/2}$ for all normal elements $a \in \mathcal{A}$.
- (12) The Gelfand-Naimark seminorm γ is a spectral seminorm on $\mathcal{A}$.
- (13) There exists a constant C with $r(a^*a) \leq C\gamma(a^*a)$ for all $a \in \mathcal{A}$.
- (14) Raikov's criterion: $\gamma(a) = \rho(a)$ for all $a \in \mathcal{A}$.
- (15) Every maximal left ideal in $\mathcal{A}$ is closed with respect to γ .
- (16) For every maximal left ideal $\mathcal{I}$ in $\mathcal{A}$ there exists a state f on $\mathcal{A}$ such that $\mathcal{I} = \{a \in \mathcal{A} : f(a^*a) = 0\}$.
- (17) For every maximal left ideal $\mathcal{I}$ in $\mathcal{A}$ there exists a pure state g on $\mathcal{A}$ such that $\mathcal{I} = \{a \in \mathcal{A} : g(a^*a) = 0\}$.
- (18) $Sp(a) \subseteq \{g(a) : g \text{ is a pure state on } \mathcal{A}\}$ for all normal elements $a \in \mathcal{A}$.
- (19) If $T : \mathcal{A} \rightarrow \mathcal{L}(X)$ is any irreducible representation, then there is an inner product for X relative to which T is a pre- $*$ -representation.
- (20) $Rad(\mathcal{A}) = R^*(\mathcal{A})$ and $\mathcal{A}/R^*(\mathcal{A})$ is a spectral $*$ -subalgebra of its enveloping C^* -algebra.
- (21) The spectrum of every unitary element of $\mathcal{A}$ is contained in the unit circle.
- (22) $\rho(a) = v(a)$ for all $a \in \mathcal{A}$.
- (23) $r(h) \leq v(h)$ for all hermitian elements $h \in \mathcal{A}$.

Some of these characterizations also hold in general $*$ -algebras whereas others rely crucially on the specific setting of Banach $*$ -algebras. We will deal exclusively with Banach $*$ -algebras and direct the reader interested in generalizations to Palmer's comprehensive account of the general theory of $*$ -algebras [20].

Throughout this work we will assume that the involution is continuous. One could drop this assumption by some occasional modifications in the proofs. For example Ford's square root lemma [7] could be used to extract square roots of hermitian elements. Another useful method in this context would be passing to the semisimple Banach $*$ -algebra $\mathcal{A}/\text{Rad}(\mathcal{A})$, where the involution is always continuous.

For reasons of exposition we restrict our discussion in the first chapters to unital Banach $*$ -algebras, that is, Banach $*$ -algebras that contain an identity element e . We explain the necessary modifications for a treatment of non-unital Banach $*$ -algebras separately.

Necessary prerequisites are usually explained in the beginning of each chapter or section.

This survey is organized as follows:

Chapter 2 presents characterizations of symmetry by conditions on the spectrum of certain elements. It includes Ptak's theory on the properties of the spectral radius that characterize symmetry (Conditions (1)-(11)).

Chapter 3 comprises those characterizations of symmetry that involve certain linear functionals and representations, sometimes in a disguised form via the Gelfand-Naimark seminorm (Conditions (12)-(19)).

In Chapter 4 the spectrum of unitary elements is studied and the unitary seminorm is used to characterize symmetry (Conditions (20)-(22)).

Chapter 5 is devoted to the modifications needed for an extension of symmetry and its characterizations to non-unital Banach $*$ -algebras.

Finally in chapter 6 we observe that symmetry is stable under various canonical constructions, such as quotient algebras, matrix algebras or direct sums.

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Chapter 2

Characterizations in Terms of Spectrum, Spectral Radius and Ptak Function

Throughout this chapter $\mathcal{A}$ will denote a unital Banach $*$ -algebra over $\mathbb{C}$ with continuous involution.

2.1 Definitions and preliminary results

For the reader's convenience we recall the definitions of the spectrum and the spectral radius as well as some important properties. This can be found in any introductory book on functional analysis or Banach algebras such as [2], [4].

2.1 Definition. (i) The *spectrum* of an element $a \in \mathcal{A}$ is the set $Sp_{\mathcal{A}}(a) = \{\lambda \in \mathbb{C} : a - \lambda e \text{ is not invertible in } \mathcal{A}\}.$

(ii) The *spectral radius* of an element $a \in \mathcal{A}$ is defined by $r_{\mathcal{A}}(a) = \max\{|\lambda| : \lambda \in Sp_{\mathcal{A}}(a)\}.$

Notation. When no confusion can occur, we write $Sp(a)$ instead of $Sp_{\mathcal{A}}(a)$ and $r(a)$ instead of $r_{\mathcal{A}}(a)$.

In a complex Banach $*$ -algebra $\mathcal{A}$ the spectrum of each element $a \in \mathcal{A}$ is a nonempty compact subset of $\mathbb{C}$ included in the closed ball of radius $\|a\|$, in particular the spectral radius is dominated by the norm.

The resolvent $z \mapsto (ze - a)^{-1}$ is an analytic function on $\mathbb{C} \setminus Sp_{\mathcal{A}}(a)$ with values in $\mathcal{A}$.

2.2 Holomorphic functional calculus. Let $a \in \mathcal{A}$ be fixed and let $Hol(a)$ denote the set of all analytic functions $f : G \rightarrow \mathcal{A}$ defined on some open neighbourhood G of $Sp(a)$. For $f \in Hol(a)$ with domain G we can choose a contour Γ of $Sp(a)$ in G and define

$$f(a) := \frac{1}{2\pi i} \int_{\Gamma} f(z)(ze - a)^{-1} dz$$

as an $\mathcal{A}$ -valued integral.

Since $z \mapsto f(z)(ze - a)^{-1}$ is analytic on $G \setminus Sp(a)$, the element $f(a) \in \mathcal{A}$ is well-defined and we can invoke Cauchy's theorem to guarantee that the definition does not depend on the choice of the contour Γ .

The main theorem in this context states that for fixed $a \in \mathcal{A}$ the mapping $f \mapsto f(a)$ is an algebra homomorphism from $Hol(a)$ into $\mathcal{A}$.

For further information and detailed proofs consult [2], [4] or [6].

The next theorem will be of considerable importance for our further development and shows how to compute the spectrum of $f(a)$.

2.3 Theorem (Spectral Mapping Theorem). *Let $a \in \mathcal{A}$, $f \in Hol(a)$. Then $Sp(f(a)) = f(Sp(a))$.*

Proof. Let $\lambda \in Sp(a)$. Choose $g \in Hol(a)$ such that $f(z) - f(\lambda) = (z - \lambda)g(z)$. Then we have

$$f(a) - f(\lambda)e = (a - \lambda e)g(a)$$

by the holomorphic functional calculus.

Since $a - \lambda e$ is not invertible, it follows that $f(a) - f(\lambda)e$ is not invertible either. Hence $f(\lambda) \in Sp(f(a))$, that is, $f(Sp(a)) \subseteq Sp(f(a))$.

For the converse suppose that $\mu \notin f(Sp(a))$. Then $g(z) = (f(z) - \mu)^{-1} \in Hol(a)$ and so

$$g(a)(f(a) - \mu e) = e.$$

Thus $\mu \notin Sp(f(a))$, that is, $Sp(f(a)) \subseteq f(Sp(a))$. □

We will particularly use the Spectral Mapping Theorem to extract square roots of elements in $\mathcal{A}$.

2.4 Corollary. *Let $a \in \mathcal{A}$ with $Sp(a) \subseteq \mathbb{C} \setminus (-\infty, 0]$. Then there exists an invertible element $b \in \mathcal{A}$ such that $b^2 = a$ and $ab = ba$.*

If a is hermitian, then also b can be chosen to be hermitian.

Next we collect some further results on the behaviour of the spectrum and the spectral radius following [28].

2.5 Proposition. *For $a, b \in \mathcal{A}$ we have $Sp(ab) \setminus \{0\} = Sp(ba) \setminus \{0\}$. In particular, $r(ab) = r(ba)$.*

Proof. Let $\lambda \in \mathbb{C} \setminus \{0\}$, $\lambda \notin Sp(ab)$.

Then there exists some $c \in \mathcal{A}$ such that $c(\lambda e - ab) = (\lambda e - ab)c = e$.

We claim that $\lambda^{-1}(e + bca)$ is the inverse of $\lambda e - ba$. Indeed,

$$\begin{aligned} \lambda^{-1}(e + bca)(\lambda e - ba) &= e - \lambda^{-1}ba + bca - \lambda^{-1}bcaba \\ &= e - \lambda^{-1}ba + \lambda^{-1}bc(\lambda e - ab)a \\ &= e - \lambda^{-1}ba + \lambda^{-1}ba = e, \end{aligned}$$

and similarly $(\lambda e - ba)\lambda^{-1}(e + bca) = e$.

Hence we have $\lambda \notin Sp(ba)$, that is, $Sp(ba) \setminus \{0\} \subseteq Sp(ab) \setminus \{0\}$.

The converse inclusion follows in the same way. □

2.6 Proposition. *For $a \in \mathcal{A}$ we have $Sp(a^*) = \overline{Sp(a)}$. In particular, $r(a^*) = r(a)$.*

Proof. This follows from $(\lambda e - a)^* = \bar{\lambda}e - a^*$. □

2.7 Theorem (Spectral Radius Formula). *The spectral radius of any element $a \in \mathcal{A}$ satisfies*

$$r(a) = \inf_{n \geq 0} \|a^n\|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \|a^n\|^{\frac{1}{n}}.$$

In particular, this limit exists.

Proof. [4] □

2.8 Corollary. *For all $a \in \mathcal{A}$ and $k \in \mathbb{N}$ we have $r(a^k) = r(a)^k$.*

Proof. Using the Spectral Radius Formula 2.7 we can calculate

$$r(a^k) = \lim_{n \rightarrow \infty} \|(a^k)^n\|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} (\|a^{kn}\|^{\frac{1}{kn}})^k = r(a)^k.$$

□

2.9 Proposition. *Let a, b be commuting elements in $\mathcal{A}$. Then*

(1) $r(ab) \leq r(a)r(b)$;

(2) $r(a + b) \leq r(a) + r(b)$.

Proof. (1) Using the Spectral Radius Formula 2.7 and $ab = ba$ we obtain

$$r(ab) = \lim_{n \rightarrow \infty} \|(ab)^n\|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \|a^n b^n\|^{\frac{1}{n}} \leq \lim_{n \rightarrow \infty} \|a^n\|^{\frac{1}{n}} \|b^n\|^{\frac{1}{n}} = r(a)r(b).$$

(2) Let s, t with $r(a) < s, r(b) < t$ be arbitrary and define $c := s^{-1}a, d := t^{-1}b$. Then $r(c), r(d) < 1$, so there exists $0 < \alpha < \infty$ such that $\|c^n\|, \|d^n\| \leq \alpha$ for all $n \geq 1$. It follows from commutativity of a and b that

$$\|(a+b)^n\| \leq \sum_{k=1}^n \binom{n}{k} s^k t^{n-k} \|c^k\| \|d^{n-k}\| \leq \alpha^2 (s+t)^n$$

and further, by the Spectral Radius Formula 2.7,

$$r(a+b) = \lim_{n \rightarrow \infty} \|(a+b)^n\|^{\frac{1}{n}} \leq \lim_{n \rightarrow \infty} \alpha^{\frac{2}{n}} (s+t) = s+t.$$

Since s, t with $r(a) < s, r(b) < t$ were arbitrary, we can infer that $r(a+b) \leq r(a) + r(b)$. □

We end this section with a first glance at the Ptak function [22].

2.10 Definition. The *Ptak function* $\rho : \mathcal{A} \rightarrow [0, \infty)$ is the function defined by $\rho(a) = r(a^*a)^{\frac{1}{2}}$ for $a \in \mathcal{A}$.

2.11 Proposition. The *Ptak function* ρ has the following properties:

- (1) $\rho(a^*) = \rho(a)$ for all $a \in \mathcal{A}$.
- (2) $\rho(a^*a) = \rho(a)^2$ for all $a \in \mathcal{A}$.
- (3) There exists some $C > 0$ such that $\rho(a) \leq C\|a\|$ for all $a \in \mathcal{A}$.
- (4) $\rho(h) = r(h)$ for all hermitian elements $h \in \mathcal{A}$.
- (5) $\rho(a) \leq r(a)$ for all normal elements $a \in \mathcal{A}$.

Proof. Let a be an arbitrary element in $\mathcal{A}$.

$$(1) \rho(a^*) = r(aa^*)^{\frac{1}{2}} = r(a^*a)^{\frac{1}{2}} = \rho(a) \text{ by Proposition 2.5.}$$

$$(2) \rho(a^*a) = r(a^*aa^*a)^{\frac{1}{2}} = r((a^*a)^2)^{\frac{1}{2}} = r(a^*a) = \rho(a)^2 \text{ by Corollary 2.8.}$$

$$(3) \rho(a)^2 = r(a^*a) \leq \|a^*a\| \leq \|a^*\| \|a\| \leq C\|a\|^2 \text{ for some } C > 0.$$

(4) Let h be a hermitian element in $\mathcal{A}$. Then, by Corollary 2.8, we obtain

$$\rho(h) = r(h^*h)^{\frac{1}{2}} = r(h^2)^{\frac{1}{2}} = r(h).$$

(5) Let a be a normal element in $\mathcal{A}$, that is, an element $a \in \mathcal{A}$ satisfying $a^*a = aa^*$. Since the spectral radius is submultiplicative on commuting elements (Proposition 2.5) and satisfies $r(a^*) = r(a)$ (Proposition 2.6), we have

$$\rho(a) = r(a^*a)^{\frac{1}{2}} \leq r(a^*)^{\frac{1}{2}}r(a)^{\frac{1}{2}} = r(a)$$

for all normal elements $a \in \mathcal{A}$. □

2.2 Symmetry and spectral characterizations

We are now ready to give the definition of symmetry in Banach $*$ -algebras and some first characterizations.

2.12 Definition. $\mathcal{A}$ is *symmetric* if $Sp(a^*a) \subseteq \mathbb{R}_+$ for all $a \in \mathcal{A}$.

Our goal in this section is to collect those characterizations of symmetric Banach $*$ -algebras that are expressed by conditions on the spectrum of certain elements. Among these are the concepts of hermiticity and complete symmetry.

2.13 Definition. (i) $\mathcal{A}$ is *hermitian* if $Sp(h) \subseteq \mathbb{R}$ for all hermitian $h \in \mathcal{A}$.
(ii) $\mathcal{A}$ is *completely symmetric* if $Sp(p) \subseteq \mathbb{R}_+$ for all elements of the form $p = \sum_{k=1}^n a_k^* a_k$ for some $a_k \in \mathcal{A}, n \in \mathbb{N}$.

It is immediate from the definition that complete symmetry implies symmetry.

Furthermore, we can easily deduce hermiticity from symmetry:

$Sp(h)^2 = Sp(h^2) = Sp(h^*h) \subseteq \mathbb{R}_+$ implies $Sp(h) \subseteq \mathbb{R}$ for any hermitian element h in $\mathcal{A}$.

On the other hand, it is quite difficult to show that hermitian Banach $*$ -algebras are symmetric. This question is known as the Shirali-Ford Lemma and constitutes the essence of the present section. It had been an outstanding conjecture for many years and was finally proved in 1970 by Shirali and Ford [27]. Since then alternative and refined proofs have been established by many authors, see, e.g., [13] or [23].

We will mainly follow the explanations in [2] and [23] and begin with some preparations.

First we will establish that in hermitian Banach $*$ -algebras the spectral radius is dominated by the Ptak function, that is, $r(a) \leq \rho(a)$ for all $a \in \mathcal{A}$. This fundamental inequality was observed by Ptak in [22] and is crucial for the further development.

2.14 Lemma (Ptak's inequality). *If $\mathcal{A}$ is hermitian, then $r(a) \leq \rho(a)$ for all $a \in \mathcal{A}$.*

Proof. By definition of the spectral radius the desired inequality holds if and only if $\lambda e - b$ is invertible for any $b \in \mathcal{A}$, $\lambda \in \mathbb{C}$ with $|\lambda| > \rho(b)$.

Considering $a = \lambda^{-1}b$, it suffices to show that $e - a$ is invertible whenever $\rho(a) < 1$.

Let $a \in \mathcal{A}$, $\rho(a) < 1$, then $Sp(a^*a) \subseteq (-1, 1)$ and, by the Spectral Mapping Theorem 2.3, $Sp(e - a^*a) \subseteq (0, 2)$. In view of Corollary 2.4 we may use the existence of a hermitian invertible square root b of $e - a^*a$ to write

$$(e + a^*)(e - a) = e - a^*a + a^* - a = b^2 + a^* - a = b(e + b^{-1}(a^* - a)b^{-1})b.$$

The element $ib^{-1}(a^* - a)b^{-1}$ is hermitian and therefore has real spectrum by assumption on $\mathcal{A}$. We can infer that $Sp(e + b^{-1}(a^* - a)b^{-1}) \subseteq 1 + i\mathbb{R}$ and in particular that $e + b^{-1}(a^* - a)b^{-1}$ is invertible.

Now $(e + a^*)(e - a) = b(e + b^{-1}(a^* - a)b^{-1})b$ is invertible as a product of invertible elements. Thus $e - a$ is left invertible.

Since $\rho(a^*) = \rho(a) < 1$, we can carry out a similar argument for $(e - a)(e + a^*)$ to obtain that $e - a$ is also right invertible.

This finishes the proof since an element is invertible if and only if it has both a left inverse and a right inverse. □

The next observation shows that in hermitian Banach $*$ -algebras the spectral radius is both submultiplicative and subadditive on hermitian elements and that the sum of positive elements is again positive.

2.15 Definition. An element $a \in \mathcal{A}$ is *positive* if $a^* = a$ and $Sp(a) \subseteq \mathbb{R}_+$. The set of positive elements in $\mathcal{A}$ is denoted by $\mathcal{A}_+$.

2.16 Lemma. *Let $\mathcal{A}$ be hermitian, let h, k be hermitian elements in $\mathcal{A}$. Then*

- (1) $r(hk) \leq r(h)r(k)$;
- (2) *If $h, k \in \mathcal{A}_+$, then $h + k \in \mathcal{A}_+$;*
- (3) $r(h + k) \leq r(h) + r(k)$.

Proof. (1) Using Ptak's inequality of Lemma 2.14 and Proposition 2.5 we have

$$r(hk) \leq \rho(hk) = r(khkh)^{\frac{1}{2}} = r(h^2k^2)^{\frac{1}{2}}.$$

It follows by induction that for all $n \in \mathbb{N}$

$$r(hk) \leq r(h^{2^n} k^{2^n})^{\frac{1}{2^n}} \leq \|h^{2^n} k^{2^n}\|^{\frac{1}{2^n}} \leq \|h^{2^n}\|^{\frac{1}{2^n}} \|k^{2^n}\|^{\frac{1}{2^n}}.$$

Passing to the limit as n approaches infinity yields $r(hk) \leq r(h)r(k)$.

(2) Given $h, k \in \mathcal{A}_+$ we have to show that $Sp(h+k) \subseteq \mathbb{R}_+$. Since $h+k$ is a hermitian element in a hermitian Banach $*$ -algebra this amounts to showing invertibility of $h+k - (-\lambda)e = h+k + \lambda e$ for any positive real number λ . We can further reduce our task to showing invertibility of $a+b+e$ if we consider suitable multiples $a = \lambda^{-1}h, b = \lambda^{-1}k$.

Since both $e+a$ and $e+b$ are invertible, we may define $c := (e+a)^{-1}a, d := b(e+b)^{-1}$.

The Spectral Mapping Theorem 2.3 applied to the function $f(x) = \frac{x}{1+x}$ implies that $Sp(c) = f(Sp(a)) \subseteq [0, 1)$ and $Sp(d) = f(Sp(b)) \subseteq [0, 1)$.

In particular, $r(c), r(d) < 1$, and hence, by (1), $r(cd) \leq r(c)r(d) < 1$.

So $e - cd$ is invertible and we can write $e+a+b$ as a product of invertible elements:

$$\begin{aligned} e+a+b &= (e+a)(e+b) - ab \\ &= (e+a)(e - (e+a)^{-1}ab(e+b)^{-1})(e+b) \\ &= (e+a)(e - cd)(e+b). \end{aligned}$$

(3) We have $Sp(h) \subseteq [-r(h), r(h)]$.

Hence, by the Spectral Mapping Theorem 2.3, $Sp(r(h)e \pm h) \subseteq [0, 2r(h)]$, that is, $r(h)e \pm h \in \mathcal{A}_+$.

Similarly $r(k)e \pm k \in \mathcal{A}_+$ and, by (2), $(r(h) + r(k))e \pm (h+k) \in \mathcal{A}_+$.

Therefore $r(h+k) \leq r(h) + r(k)$. □

We have now collected the necessary tools to prove the Shirali-Ford Lemma. We will follow the approach of Thill [28] who simplified previous proofs by using a polynomial rather than a rational function.

2.17 Theorem (Shirali-Ford). *Let $\mathcal{A}$ be a hermitian Banach $*$ -algebra. Then $\mathcal{A}$ is symmetric.*

Proof. Let $\delta := \sup\{-\lambda : \lambda \in Sp(a^*a), a \in \mathcal{A}, \rho(a) \leq 1\}$. Seeking a contradiction we assume that $\delta > 0$.

Then there exist $a \in \mathcal{A}, \mu \in Sp(a^*a)$ such that $\rho(a) \leq 1$ and $\mu < -\frac{4}{9}\delta$.

Consider the polynomial $p(x) = \frac{x}{4}(3-x)^2$. We have $p(-1) = -4, p(1) = 1$

and p is strictly increasing on $[-1, 1]$.

Let $b := \frac{1}{2}a(3e - a^*a)$, then $b^* = \frac{1}{2}(3e - a^*a)a^*$ and hence

$$b^*b = \frac{1}{4}(3e - a^*a)a^*a(3e - a^*a) = p(a^*a).$$

The Spectral Mapping Theorem 2.3 yields that

$$Sp(b^*b) = p(Sp(a^*a)) \subseteq [-4, 1].$$

Further we have $Sp(e - b^*b) \subseteq [0, 5]$, that is, $e - b^*b$ is positive.

Now we can write $b = h + ik$ with hermitian elements $h, k \in \mathcal{A}$.

Then $bb^* + b^*b = 2h^2 + 2k^2$. By Lemma 2.16 (2), the element

$$bb^* + e = 2h^2 + 2k^2 + (e - b^*b)$$

is positive because it is a sum of positive elements.

Therefore $Sp(bb^*) \subseteq [-1, \infty)$.

From Proposition 2.5 we know that $Sp(b^*b) \cup \{0\} = Sp(bb^*) \cup \{0\}$.

Hence $Sp(b^*b) \subseteq [-1, 1]$. But this means that $\rho(b) \leq 1$.

Consequently $\lambda \geq -\delta$ for all $\lambda \in Sp(b^*b)$, in particular $p(\mu) \geq -\delta$.

On the other hand, $p(\mu) < p(-\frac{4}{9}\delta) < -\delta$, a contradiction.

□

Remark. The Shirali-Ford Lemma is only true for Banach $*$ -algebras. In arbitrary $*$ -algebras the notions of symmetry and hermiticity are not the same. J. Wichmann [29] constructed examples of hermitian $*$ -algebras which are not symmetric.

We conclude this section by summarizing equivalent conditions concerning the spectra of certain elements in $\mathcal{A}$.

2.18 Theorem. *For a unital Banach $*$ -algebra $\mathcal{A}$ with continuous involution the following conditions are equivalent:*

- (1) $\mathcal{A}$ is completely symmetric.
- (2) $\mathcal{A}$ is symmetric.
- (3) $-1 \notin Sp(a^*a)$ for all $a \in \mathcal{A}$.
- (4) $\mathcal{A}$ is hermitian.
- (5) $i \notin Sp(h)$ for all hermitian elements $h \in \mathcal{A}$.

Proof. The implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are immediate from the definitions.

(3) $\Rightarrow$ (5): Suppose on the contrary that there exists a hermitian element $h \in \mathcal{A}$ such that $i \in Sp(h)$. Using the Spectral Mapping Theorem 2.3 we can infer that

$$-1 \in Sp(h^2) = Sp(h^*h).$$

But this contradicts the assumption in (3).

(5) $\Rightarrow$ (4): Again we argue by contradiction and suppose that $\mathcal{A}$ is not hermitian. Then there exists a hermitian element $h \in \mathcal{A}$ such that $\alpha + i\beta \in Sp(h)$ for some $\alpha, \beta \in \mathbb{R}, \beta \neq 0$. Consider the polynomial $p(\lambda) = \beta^{-1}(\lambda - \alpha)$ for $\lambda \in \mathbb{C}$ and set $k = p(h) = \beta^{-1}(h - \alpha e)$. Then k is hermitian. The Spectral Mapping Theorem 2.3 implies that

$$i = p(\alpha + i\beta) \in p(Sp(h)) = Sp(p(h)) = Sp(k).$$

This yields the desired contradiction.

(4) $\Rightarrow$ (2): Theorem 2.17

(2) $\Rightarrow$ (1): Note that the above implications already imply the equivalence of symmetry and hermiticity of $\mathcal{A}$. Thus the fact that the sum of positive elements is positive (Lemma 2.16 (2)) can also be applied to symmetric Banach *-algebras and immediately yields the complete symmetry. $\square$

Notation. By virtue of the preceding Theorem there is no need to distinguish between hermitian and symmetric Banach *-algebras. In the sequel we will drop this distinction and call a Banach *-algebra symmetric if it satisfies any of the above properties.

2.3 Characterizations via spectral radius and Ptak function

In the previous section we have established that in symmetric Banach *-algebras Ptak's inequality holds, that is, $r(a) \leq \rho(a)$ for all $a \in \mathcal{A}$.

Indeed, Ptak's inequality already characterizes the symmetry of a Banach *-algebra [23].

This result together with similar characterizations of symmetry by properties of the spectral radius will be topic of the present section.

2.19 Theorem. *For a unital Banach $*$ -algebra $\mathcal{A}$ with continuous involution the following conditions are equivalent:*

- (1) $\mathcal{A}$ is symmetric.
- (2) $r(a) \leq \rho(a)$ for all $a \in \mathcal{A}$.
- (3) $r(a) = \rho(a)$ for all normal elements $a \in \mathcal{A}$.
- (4) $r(a) \leq \|a^*a\|^{\frac{1}{2}}$ for all normal elements $a \in \mathcal{A}$.
- (5) $r(\frac{1}{2}(a^* + a)) \leq \rho(a)$ for all $a \in \mathcal{A}$.
- (6) $\rho(a + b) \leq \rho(a) + \rho(b)$ for all $a, b \in \mathcal{A}$.
- (7) ρ is an algebra seminorm on $\mathcal{A}$.

Proof. The proof is divided in several steps. First we show the chain of implications

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1).$$

(1) $\Rightarrow$ (2): This is Lemma 2.14.

(2) $\Rightarrow$ (3): By Proposition 2.11 (5) we have $\rho(a) \leq r(a)$ for all normal elements $a \in \mathcal{A}$. Thus (2) implies (3).

(3) $\Rightarrow$ (4): Since the spectral radius is always dominated by the complete norm, this implication follows immediately from the definition of ρ .

(4) $\Rightarrow$ (1): Let h be a hermitian element in $\mathcal{A}$ and let $\lambda \in Sp(h)$.

We need to show that λ is real.

For arbitrary $\mu \in \mathbb{R}$ and $n \in \mathbb{N}$ define $b := (h + i\mu e)^n$. Then the Spectral Mapping Theorem 2.3 implies that $(\lambda + i\mu)^n \in Sp(b)$.

We have $b^*b = bb^* = (h^2 + \mu^2 e)^n$, so we can invoke the assumption to deduce

$$|\lambda + i\mu|^{2n} \leq r(b)^2 \leq \|b^*b\| = \|(h^2 + \mu^2 e)^n\|$$

and further

$$|\lambda + i\mu|^2 \leq \|(h^2 + \mu^2 e)^n\|^{\frac{1}{n}}$$

for all $n \in \mathbb{N}$. Passing to the limit as n approaches infinity gives

$$|\lambda + i\mu|^2 \leq r(h^2 + \mu^2 e) \leq r(h^2) + \mu^2,$$

where the last inequality follows from Proposition 2.9 (2).

If we now decompose $\lambda = \alpha + i\beta$ with real α, β , we get $\alpha^2 + \beta^2 + 2\beta\mu \leq r(h^2)$.

But μ was an arbitrary real number, so β has to be zero.

In the next chain of implications we prove that

$$(1) \Rightarrow (5) \Rightarrow (3) [\Rightarrow (1)].$$

(1) $\Rightarrow$ (5): Write $a = h + ik$ with hermitian elements $h, k \in \mathcal{A}$.

Then $\frac{1}{2}(a^* + a) = h$ and $a^*a + aa^* = 2(h^2 + k^2)$.

We will first show

$$r(h^2) \leq r(h^2 + k^2). \quad (2.1)$$

Note that $r(h^2 + k^2)e - (h^2 + k^2) \in \mathcal{A}_+$ and $k^2 \in \mathcal{A}_+$.

Once again we use that in symmetric Banach $*$ -algebras the sum of positive elements is positive (Lemma 2.16 (2)) to obtain that

$$r(h^2 + k^2)e - h^2 = (r(h^2 + k^2)e - (h^2 + k^2)) + k^2 \in \mathcal{A}_+. \quad (2.2)$$

Since $h^2 \in \mathcal{A}_+$, formula (2.2) shows that $r(h^2) \leq r(h^2 + k^2)$.

By symmetry of $\mathcal{A}$, we may apply subadditivity of the spectral radius on hermitian elements (Lemma 2.16 (3)) to deduce the desired inequality from formula (2.1) as follows:

$$\begin{aligned} r\left(\frac{1}{2}(a^* + a)\right)^2 = r(h^2) &\leq r(h^2 + k^2) \\ &= \frac{1}{2}r(a^*a + aa^*) \\ &\leq \frac{1}{2}r(a^*a) + \frac{1}{2}r(aa^*) \\ &= \frac{1}{2}\rho(a)^2 + \frac{1}{2}\rho(a^*)^2 = \rho(a)^2 \end{aligned}$$

(5) $\Rightarrow$ (3): Let a be a normal element in $\mathcal{A}$. Again we write $a = h + ik$ with hermitian elements $h, k \in \mathcal{A}$. Since a is normal, we have $hk = kh$. By Proposition 2.9 the spectral radius is subadditive on commuting elements, so we obtain $r(a) \leq r(h) + r(k)$.

Note that $h = \frac{1}{2}(a^* + a)$, $k = -\frac{1}{2}((ia)^* + ia)$. Then the assumption in (5) gives

$$r(a) \leq r(h) + r(k) \leq \rho(a) + \rho(ia) = 2\rho(a).$$

We apply this inequality to the normal element a^n and use Corollary 2.8 to deduce that, for all $n \in \mathbb{N}$,

$$r(a)^n = r(a^n) \leq 2\rho(a^n) = 2r((a^n)^*a^n)^{\frac{1}{2}} = 2r((a^*)^na^n)^{\frac{1}{2}} = 2r((a^*a)^n)^{\frac{1}{2}} = 2\rho(a)^n,$$

and further

$$r(a) \leq 2^{\frac{1}{n}}\rho(a).$$

Taking the limit gives $r(a) \leq \rho(a)$.

By Proposition 2.11 (5) the converse inequality $\rho(a) \leq r(a)$ always holds for normal elements. Thus we get equality of the spectral radius and the Ptak function on normal elements, which was to be shown.

Finally we treat the chain of implications

$$(1) \Rightarrow (7) \Rightarrow (6) \Rightarrow (5)[\Rightarrow (1)].$$

(1) $\Rightarrow$ (7): Let $\mathcal{A}$ be symmetric and let $a, b \in \mathcal{A}$.

Recall from Lemma 2.16 that in symmetric Banach $*$ -algebras the spectral radius is both submultiplicative and subadditive on hermitian elements. Hence we have

$$\begin{aligned} \rho(ab)^2 &= r(b^*a^*ab) = r(a^*abb^*) \\ &\leq r(a^*a)r(bb^*) = \rho(a)^2\rho(b^*)^2 \\ &= (\rho(a)\rho(b))^2 \end{aligned} \tag{2.3}$$

and also

$$\begin{aligned} \rho(a+b)^2 &= r((a^*+b^*)(a+b)) \\ &\leq r(a^*a) + r(b^*b) + r(a^*b + b^*a) \\ &= \rho(a)^2 + \rho(b)^2 + r(a^*b + b^*a). \end{aligned} \tag{2.4}$$

In view of the preceding chain of implications we already know that condition (5) holds in symmetric Banach $*$ -algebras. Now combining condition (5) with formula (2.3) gives

$$r(a^*b + b^*a) \leq 2\rho(a^*b) \leq 2\rho(a^*)\rho(b) = 2\rho(a)\rho(b).$$

Putting things together we obtain

$$\rho(a+b)^2 \leq \rho(a)^2 + \rho(b)^2 + 2\rho(a)\rho(b) = (\rho(a) + \rho(b))^2.$$

(7) $\Rightarrow$ (6): clear

(6) $\Rightarrow$ (5): Using Proposition 2.11 (4) and the subadditivity of the Ptak function we can easily deduce

$$r\left(\frac{1}{2}(a^* + a)\right) = \rho\left(\frac{1}{2}(a^* + a)\right) \leq \frac{1}{2}\rho(a^*) + \frac{1}{2}\rho(a) = \rho(a)$$

for all $a \in \mathcal{A}$.

Since the equivalence of all conditions is now established, the proof is complete. $\square$

This section closes with a corollary on the Ptak function that we will need later on [23].

2.20 Corollary. *If $\mathcal{A}$ is symmetric, then the Ptak function ρ is continuous on $\mathcal{A}$.*

Proof. Recall from Proposition 2.11 (3) that there exists some $C > 0$ such that $\rho(a) \leq C\|a\|$ for all $a \in \mathcal{A}$.

Since $\mathcal{A}$ is symmetric, the Ptak function ρ is subadditive by the preceding theorem. Hence we have

$$|\rho(a) - \rho(b)| \leq \rho(a - b) \leq C\|a - b\|,$$

which yields continuity of ρ . $\square$

2.4 C^* -algebras are symmetric

When Gelfand and Naimark introduced C^* -algebras in their famous paper [8] they had to add symmetry to their axioms for obtaining the main results. They already suspected that this assumption was redundant. Nevertheless their conjecture remained an unresolved problem for several years.

However, with the preceding characterizations of symmetry established, it is now quite easy to show that C^* -algebras are indeed symmetric.

2.21 Definition. A Banach $*$ -algebra $\mathcal{A}$ is called a C^* -algebra, if the norm satisfies the C^* -condition $\|a^*a\| = \|a\|^2$ for all $a \in \mathcal{A}$.

2.22 Proposition. *Let $\mathcal{A}$ be a unital C^* -algebra. Then $\rho(a) = \|a\|$ for all $a \in \mathcal{A}$.*

Proof. Let $a \in \mathcal{A}$.

Using the C^* -condition we have

$$\|(a^*a)^2\| = \|(a^*a)^*(a^*a)\| = \|a^*a\|^2.$$

By induction $\|(a^*a)^{2^n}\| = \|a^*a\|^{2^n}$ for all $n \in \mathbb{N}$.

Hence we can invoke the spectral radius formula 2.7 to obtain

$$\rho(a)^2 = r(a^*a) = \lim_{n \rightarrow \infty} \|(a^*a)^{2^n}\|^{\frac{1}{2^n}} = \|a^*a\| = \|a\|^2.$$

□

2.23 Theorem. *Every unital C^* -algebra is symmetric.*

Proof. The preceding proposition yields $r(a) \leq \|a\| = \rho(a)$ for all $a \in \mathcal{A}$. Hence $\mathcal{A}$ is symmetric by Theorem 2.19.

□

Further we give another implication which we will need in the next chapter.

2.24 Corollary. *Let $\mathcal{A}$ be a C^* -algebra. Then $r(a) = \|a\|$ for all normal elements $a \in \mathcal{A}$.*

Proof. Since C^* -algebras are symmetric, we can invoke Theorem 2.19 (3) to observe that $r(a) = \rho(a)$ for all normal elements a in $\mathcal{A}$. Combined with Proposition 2.22 this already yields the assertion.

□

Chapter 3

Characterizations concerning Ideals, Positive Functionals and Representations

Throughout this chapter $\mathcal{A}$ will denote a unital Banach $*$ -algebra over $\mathbb{C}$ with continuous involution.

3.1 Characterizations in terms of ideals and positive functionals

In this section we show that in a symmetric Banach $*$ -algebra $\mathcal{A}$ all maximal left ideals are annihilated by certain linear functionals. Further we use this property to relate the spectrum of normal elements to linear functionals.

Definitions and Preparations

We start with a brief discussion of ideals following [11].

Recall that a linear subspace $\mathcal{I}$ of $\mathcal{A}$ is a *left ideal* if $ax \in \mathcal{I}$ for all $a \in \mathcal{A}$ and $x \in \mathcal{I}$. Analogously a linear subspace $\mathcal{I}$ of $\mathcal{A}$ is a *right ideal* if $xa \in \mathcal{I}$ for all $a \in \mathcal{A}$ and $x \in \mathcal{I}$. A linear subspace $\mathcal{I}$ of $\mathcal{A}$ is a (*two-sided*) *ideal* if it is both a left ideal and a right ideal. An ideal $\mathcal{I}$ is called *$*$ -ideal* if $\mathcal{I}^* = \mathcal{I}$.

A left ideal $\mathcal{I}$ is *proper* if $\mathcal{I} \neq \mathcal{A}$. It is *maximal* if it is proper and not properly included in any other proper left ideal.

Similar definitions apply to right ideals and ideals.

The following proposition forms a link between ideals and invertibility:

3.1 Proposition. *An element $a \in \mathcal{A}$ is left invertible if and only if a is not contained in any proper left ideal of $\mathcal{A}$.*

Proof. Suppose that $a \in \mathcal{A}$ is left invertible and let $\mathcal{I}$ be a left ideal of $\mathcal{A}$ containing a . Let c denote the left inverse of a .

Then $b = b(ca) = (bc)a \in \mathcal{I}$ for all $b \in \mathcal{A}$. Hence $\mathcal{I} = \mathcal{A}$, so $\mathcal{I}$ is not proper. Conversely, assume that a is not left invertible and consider the left ideal $\mathcal{I} := \mathcal{A}a$.

As $e \notin \mathcal{I}$ we have constructed a proper left ideal containing a . □

Krull's lemma guarantees the existence of maximal ideals:

3.2 Proposition (Krull). *Every proper left ideal is included in a maximal left ideal.*

Proof. The proof is a standard application of Zorn's lemma.

Let $\mathcal{E}$ be the set of proper left ideals which include the given left ideal, ordered by inclusion. Let $\mathcal{E}'$ be any linearly ordered subset of $\mathcal{E}$ and define $\mathcal{J}$ to be the union of all left ideals in $\mathcal{E}'$. Since $e \notin \mathcal{I}$ for any $\mathcal{I} \in \mathcal{E}'$, we have $e \notin \mathcal{J}$. Therefore $\mathcal{J}$ is a proper left ideal and hence an upper bound for $\mathcal{E}'$ in $\mathcal{E}$. Now, by Zorn's lemma, $\mathcal{E}$ has a maximal element which is a maximal left ideal including the given left ideal. □

Remark. Respective results also hold for right ideals and ideals.

Next we introduce positive linear functionals and present some elementary properties which can be found in [6] or [23].

3.3 Definition. (i) A linear functional f on $\mathcal{A}$ is *positive* if $f(a^*a) \geq 0$ for all $a \in \mathcal{A}$.

(ii) A linear functional f on $\mathcal{A}$ is *hermitian* if $f(a^*) = \overline{f(a)}$ for all $a \in \mathcal{A}$.

3.4 Proposition. *Let f be a positive functional on $\mathcal{A}$. Then, for $a, b \in \mathcal{A}$ and hermitian $h \in \mathcal{A}$, we have*

(1) $f(b^*a) = \overline{f(a^*b)}$, in particular, f is hermitian;

(2) $|f(b^*a)|^2 \leq f(a^*a)f(b^*b)$, in particular $|f(a)|^2 \leq f(e)f(a^*a)$;

- (3) $|f(h)| \leq f(e)r(h)$;
(4) $f(b^*a^*ab) \leq r(a^*a)f(b^*b)$;
(5) If $\mathcal{I}_f := \{a \in \mathcal{A} : f(a^*a) = 0\}$, then $\mathcal{I}_f = \{a \in \mathcal{A} : f(b^*a) = 0 \text{ for all } b \in \mathcal{A}\}$ and $\mathcal{I}_f$ is a proper left ideal in $\mathcal{A}$.

Proof. (1) For each $\mu \in \mathbb{C}$ we have

$$0 \leq f((\mu a + b)^*(\mu a + b)) = |\mu|^2 f(a^*a) + \mu f(b^*a) + \bar{\mu} f(a^*b) + f(b^*b).$$

As $|\mu|^2 f(a^*a) + f(b^*b)$ is real, it follows that $\mu f(b^*a) + \bar{\mu} f(a^*b)$ also has to be real for each $\mu \in \mathbb{C}$.

Setting $\mu = 1$ shows that $f(b^*a) + f(a^*b)$ is real, hence $\Im f(b^*a) = -\Im f(a^*b)$.

Setting $\mu = i$ yields $\Re f(b^*a) = \Re f(a^*b)$.

Therefore $f(b^*a) = \overline{f(a^*b)}$.

(2) We carry out the same proof as for the Cauchy-Schwarz inequality.

If $f(b^*a) = 0$, then property (2) is obvious. Suppose that $f(b^*a) \neq 0$.

Let t be any real number and set $\mu = t \frac{\overline{f(b^*a)}}{|f(b^*a)|}$.

Using part (1) we obtain the following quadratic inequality in t :

$$0 \leq t^2 f(a^*a) + 2t |f(b^*a)| + f(b^*b)$$

Since this inequality holds for all real t , the discriminant must satisfy

$$4|f(b^*a)|^2 - 4f(a^*a)f(b^*b) \leq 0,$$

that is, $|f(b^*a)|^2 \leq f(a^*a)f(b^*b)$.

(3) Assume first that $r(h) < 1$. Then $Sp(e \pm h) \subseteq \{z \in \mathbb{C} : \Re z > 0\}$ and thus Corollary 2.4 yields the existence of hermitian elements $u, v \in \mathcal{A}$ such that $u^2 = e - h$ and $v^2 = e + h$.

It follows that $f(e) - f(h) = f(u^2) = f(u^*u) \geq 0$ and also

$f(e) + f(h) = f(v^2) = f(v^*v) \geq 0$, that is, $|f(h)| \leq f(e)$.

Next, for arbitrary hermitian $h \in \mathcal{A}$ and $\epsilon > 0$, set $h_\epsilon = (r(h) + \epsilon)^{-1}h$.

Then $r(h_\epsilon) < 1$, so that we have $|f(h_\epsilon)| \leq f(e)$ or $|f(h)| \leq f(e)(r(h) + \epsilon)$.

Since ϵ was arbitrary, we obtain $|f(h)| \leq f(e)r(h)$.

(4) If $f(b^*b) = 0$, we have by (2) that $|f(b^*a^*ab)|^2 \leq f(w^*w)f(b^*b) = 0$ for $w = a^*ab$ and so the desired inequality is satisfied.

If $f(b^*b) > 0$, define the linear functional g on $\mathcal{A}$ by

$$g(z) = \frac{f(b^*zb)}{f(b^*b)}.$$

Then g is positive, so property (3) is applicable to g . We obtain

$$g(a^*a) \leq g(e)r(a^*a) = r(a^*a),$$

that is, $f(b^*a^*ab) \leq r(a^*a)f(b^*b)$.

(5) Clearly $\mathcal{I}_f \supseteq \{a \in \mathcal{A} : f(b^*a) = 0 \text{ for all } b \in \mathcal{A}\}$. For the converse inclusion we use (2) to obtain that, for all $a \in \mathcal{I}_f$, $b \in \mathcal{A}$,

$$|f(b^*a)| \leq f(a^*a)^{\frac{1}{2}}f(b^*b)^{\frac{1}{2}} = 0.$$

Thus $\mathcal{I}_f = \{a \in \mathcal{A} : f(b^*a) = 0 \text{ for all } b \in \mathcal{A}\}$. From this expression it is now immediate that $\mathcal{I}_f$ is a proper left ideal in $\mathcal{A}$. □

We single out an important class of positive linear functionals, the so-called states [6, 23].

3.5 Definition. A *state* on $\mathcal{A}$ is a positive linear functional f on $\mathcal{A}$ such that $f(e) = 1$. The set of all states on $\mathcal{A}$ is denoted by $S(\mathcal{A})$.

3.6 Lemma. *The set $S(\mathcal{A})$ of states on $\mathcal{A}$ is a weak*-compact convex subset of the dual space $\mathcal{A}'$ of $\mathcal{A}$.*

Proof. Convexity is clear. To see that $S(\mathcal{A})$ is weak*-compact we may assume that the involution on $\mathcal{A}$ is isometric (consider the equivalent norm $\|a\|' = \max\{\|a\|, \|a^*\|\}$).

For $f \in S(\mathcal{A})$ and $a \in \mathcal{A}$ Proposition 3.4 gives

$$|f(a)|^2 \leq f(a^*a) \leq r(a^*a) \leq \|a^*a\| \leq \|a^*\|\|a\| = \|a\|^2.$$

Hence $S(\mathcal{A})$ is a subset of the closed unit ball B_1 in the dual space $\mathcal{A}'$.

Further $S(\mathcal{A})$ is weak*-closed in B_1 , because the pointwise properties $f(a^*a) \geq 0$ for all $a \in \mathcal{A}$ and $f(e) = 1$ determine that a functional $f \in B_1$ belongs to $S(\mathcal{A})$. Since B_1 is weak*-compact by Alaoglu's theorem, $S(\mathcal{A})$ is also weak*-compact. □

In view of Lemma 3.6 the Krein-Milman Theorem yields the existence of extreme points of $S(\mathcal{A})$.

3.7 Definition. A *pure state* on $\mathcal{A}$ is an extreme point of the convex set $S(\mathcal{A})$ of states on $\mathcal{A}$.

In the proof of the next characterization we will need that a linear functional on $\mathcal{A}$ is positive whenever it is dominated by a C^* -seminorm on $\mathcal{A}$.

3.8 Definition. A C^* -seminorm on $\mathcal{A}$ is an algebra seminorm σ satisfying $\sigma(a^*a) = \sigma(a)^2$ for all $a \in \mathcal{A}$.

We first treat this topic in C^* -algebras [5, 28] and then extend the result to arbitrary Banach $*$ -algebras [13].

3.9 Lemma. Let $\mathcal{A}$ be a unital C^* -algebra and let f be a linear functional on $\mathcal{A}$ satisfying $\|f\| = f(e) = 1$. Then f is a state on $\mathcal{A}$.

Proof. Seeking a contradiction we suppose that there is an $a \in \mathcal{A}$ such that $f(a^*a) \not\geq 0$.

Since C^* -algebras are symmetric by Theorem 2.23, we have that $Sp(a^*a) \subseteq \mathbb{R}_+$. Thus we can find a closed disc $D := \{z \in \mathbb{C} : |z - \mu| \leq M\}$ which includes $Sp(a^*a)$, but does not contain $f(a^*a)$. Now the spectrum of the normal element $\mu e - a^*a$ is included in the disc $\{z \in \mathbb{C} : |z| \leq M\}$ and so, by Corollary 2.24, $\|\mu e - a^*a\| = r(\mu e - a^*a) \leq M$. Because $f(e) = \|f\| = 1$ it follows that

$$|\mu - f(a^*a)| = |f(\mu e - a^*a)| \leq \|f\| \|\mu e - a^*a\| \leq M,$$

which is a contradiction to the choice of D . □

3.10 Proposition. Let σ be a C^* -seminorm on $\mathcal{A}$ and let f be a linear functional satisfying $f(e) = 1$ and $|f(a)| \leq \sigma(a)$ for all $a \in \mathcal{A}$. Then f is a state on $\mathcal{A}$.

Proof. As already mentioned the proof is a reduction to the case of C^* -algebras.

Since the involution is an isometry with respect to C^* -seminorms, the set $\mathcal{N} = \{a \in \mathcal{A} : \sigma(a) = 0\}$ is a $*$ -ideal in $\mathcal{A}$. Consider the quotient algebra $\mathcal{A}/\mathcal{N}$ with the induced involution and with the norm induced by σ , i.e., $(a')^* := (a^*)'$ and $\|a'\| := \sigma(a)$ where $a' = a + \mathcal{N}$.

Then $\|\cdot\|$ is a C^* -norm on $\mathcal{A}/\mathcal{N}$ and thus the completion, denoted by $(\mathcal{C}, \|\cdot\|)$, is a C^* -algebra.

As f vanishes on $\mathcal{N}$, we may define a linear functional f' on $\mathcal{A}/\mathcal{N}$ by $f'(a') := f(a)$. By assumption on f , $|f'(a')| \leq \|a'\|$, so f' has a unique continuous extension to a linear functional $\tilde{f}$ on $\mathcal{C}$. We have $|\tilde{f}(c)| \leq \|c\|$ for all $c \in \mathcal{C}$ and $\tilde{f}(e') = f(e) = 1$, hence $\|\tilde{f}\| = \tilde{f}(e') = 1$. By Lemma 3.9 $\tilde{f}$ is positive. So, for $a \in \mathcal{A}$, we have $f(a^*a) = \tilde{f}((a')^*(a')) \geq 0$. □

Characterizations of symmetry

In the next theorem we state some characterizations of symmetry that involve positive linear functionals [18], [24], [26].

3.11 Theorem. *For a unital Banach $*$ -algebra $\mathcal{A}$ with continuous involution the following conditions are equivalent:*

- (1) $\mathcal{A}$ is symmetric.
- (2) For every maximal left ideal $\mathcal{I}$ in $\mathcal{A}$ there exists a state f on $\mathcal{A}$ such that $\mathcal{I} = \{a \in \mathcal{A} : f(a^*a) = 0\}$.
- (3) For every maximal left ideal $\mathcal{I}$ in $\mathcal{A}$ there exists a pure state g on $\mathcal{A}$ such that $\mathcal{I} = \{a \in \mathcal{A} : g(a^*a) = 0\}$.
- (4) $Sp(a) \subseteq \{g(a) : g \text{ is a pure state on } \mathcal{A}\}$ for all normal elements $a \in \mathcal{A}$.

Proof. (1) $\Rightarrow$ (2): Let $\mathcal{I}$ be a maximal left ideal in $\mathcal{A}$.

Then $\mathcal{I}_o := \{b + \lambda e : b \in \mathcal{I}, \lambda \in \mathbb{C}\}$ is a linear subspace of $\mathcal{A}$. Define

$$f_o(b + \lambda e) := \lambda$$

for all $b + \lambda e \in \mathcal{I}_o$. Then f_o is a linear functional on $\mathcal{I}_o$ with $f_o(e) = 1$.

If $b + \lambda e \in \mathcal{I}_o$, then $\lambda e - (b + \lambda e) \in \mathcal{I}$. Since $\mathcal{I}$ is a proper left ideal, it follows from Proposition 3.1 that $\lambda e - (b + \lambda e)$ is not left invertible. In particular, $\lambda \in Sp(b + \lambda e)$ and therefore $|\lambda| \leq r(b + \lambda e)$.

Since $\mathcal{A}$ is symmetric, we know from Theorem 2.19 and Proposition 2.11 that the Ptak function ρ is a C^* -seminorm on $\mathcal{A}$ which dominates the spectral radius.

Hence we have $|f_o(b + \lambda e)| \leq r(b + \lambda e) \leq \rho(b + \lambda e)$ for all $b + \lambda e \in \mathcal{I}_o$.

Further, the Hahn-Banach Theorem (see, e.g., [4], p.81) furnishes a linear extension $f : \mathcal{A} \rightarrow \mathbb{C}$ of f_o preserving the property $|f(a)| \leq \rho(a)$ for all $a \in \mathcal{A}$. According to Proposition 3.10 f is a state on $\mathcal{A}$.

As $a^*a \in \mathcal{I}$ for $a \in \mathcal{I}$ we have $f(a^*a) = 0$ by construction of f . This means that $\mathcal{I} \subseteq \{a \in \mathcal{A} : f(a^*a) = 0\}$.

By Proposition 3.4 (5) the set $\{a \in \mathcal{A} : f(a^*a) = 0\}$ is a proper left ideal in $\mathcal{A}$. Now maximality of $\mathcal{I}$ yields the assertion.

(2) $\Rightarrow$ (3): Consider the set $\mathcal{E}$ of all states f on $\mathcal{A}$ with $\mathcal{I} = \{a \in \mathcal{A} : f(a^*a) = 0\}$. By assumption $\mathcal{E}$ is nonempty. Further, $\mathcal{E}$ is a weak*-closed convex subset of $S(\mathcal{A})$ and hence weak*-compact. By the Krein-Milman Theorem there exists some extreme point g of $\mathcal{E}$. It remains to show that g is a pure state on $\mathcal{A}$.

Let $g = \alpha_1 f_1 + \alpha_2 f_2$ where $\alpha_1, \alpha_2 > 0$, $\alpha_1 + \alpha_2 = 1$ and $f_1, f_2 \in S(\mathcal{A})$. Then $g(a^*a) = 0$ implies $f_1(a^*a) = f_2(a^*a) = 0$ and thus $f_1, f_2 \in \mathcal{E}$. We infer that $g = f_1 = f_2$ and therefore g is also an extreme point of $S(\mathcal{A})$, that is, g is a

pure state on $\mathcal{A}$.

(3) $\Rightarrow$ (4): Let a be a normal element in $\mathcal{A}$, let $\lambda \in Sp(a)$ and set $b := a - \lambda e$. Then b is a normal element in $\mathcal{A}$. For any pure state g on $\mathcal{A}$ we compute

$$\begin{aligned}
g(b^*b) &= g((a - \lambda e)^*(a - \lambda e)) \\
&= g(a^*a - \bar{\lambda}a - \lambda a^* + |\lambda|^2 e) \\
&= g(a^*a) - \bar{\lambda}g(a) - \lambda g(a^*) + |\lambda|^2 \\
&= g(a^*a) - \bar{\lambda}g(a) - \lambda \overline{g(a)} + |\lambda|^2 \\
&= g(a^*a) - |g(a)|^2 + (|g(a)|^2 - \bar{\lambda}g(a) - \lambda \overline{g(a)} + |\lambda|^2) \\
&= g(a^*a) - |g(a)|^2 + (g(a) - \lambda)(\overline{g(a)} - \bar{\lambda}) \\
&= g(a^*a) - |g(a)|^2 + (g(a) - \lambda)(\overline{g(a)} - \bar{\lambda}) \\
&= g(a^*a) - |g(a)|^2 + |g(a) - \lambda|^2.
\end{aligned}$$

Since $\lambda \in Sp(a)$, the element $b = a - \lambda e$ is not invertible in $\mathcal{A}$. We first assume that b is not left invertible. Then b is contained in a proper left ideal $\mathcal{I}$ (Proposition 3.1) which is further included in some maximal left ideal $\tilde{\mathcal{I}}$ (Proposition 3.2). By assumption there exists a pure state g_1 on $\mathcal{A}$ such that $\tilde{\mathcal{I}} = \{a \in \mathcal{A} : g_1(a^*a) = 0\}$. In particular,

$$0 = g_1(b^*b) = g_1(a^*a) - |g_1(a)|^2 + |g_1(a) - \lambda|^2. \quad (3.1)$$

Note that $g_1(a^*a) - |g_1(a)|^2 \geq 0$ by Proposition 3.4 (2). Hence formula (3.1) implies that $g_1(a^*a) = |g_1(a)|^2$ and $\lambda = g_1(a)$.

If otherwise b is left invertible but not right invertible, then b^* is not left invertible. We can repeat the above argument with b^* instead of b to obtain a pure state g_2 on $\mathcal{A}$ such that $g_2(bb^*) = 0$. Since b is normal, we have

$$0 = g_2(bb^*) = g_2(b^*b) = g_2(a^*a) - |g_2(a)|^2 + |g_2(a) - \lambda|^2,$$

which implies $\lambda = g_2(a)$.

Therefore $Sp(a) \subseteq \{g(a) : g \text{ is a pure state on } \mathcal{A}\}$.

(4) $\Rightarrow$ (1): Let a be an arbitrary element in $\mathcal{A}$. By assumption,

$$Sp(a^*a) \subseteq \{g(a^*a) : g \text{ is a pure state on } \mathcal{A}\} \subseteq \mathbb{R}_+.$$

Hence $\mathcal{A}$ is symmetric. □

3.2 A characterization by means of representations

Definitions and Preparations

We start with the algebraic notion of representations [6, 19].

- 3.12 Definition.** (i) A *representation* T of $\mathcal{A}$ on a vector space $\mathcal{X}$ is a homomorphism $a \mapsto T_a$ of $\mathcal{A}$ into the algebra $\mathcal{L}(\mathcal{X})$ of linear operators on $\mathcal{X}$. The vector space $\mathcal{X}$ is called the *representation space* of T .
- (ii) A linear subspace $\mathcal{M}$ of $\mathcal{X}$ is said to be *invariant* under T if $T_a(\mathcal{M}) \subseteq \mathcal{M}$ for all $a \in \mathcal{A}$.

3.13 Definition. Let T be a representation of $\mathcal{A}$ on a vector space $\mathcal{X}$.

- (i) T is *cyclic* if there exists a vector $z \in \mathcal{X}$ such that $\mathcal{X} = \{T_a z : a \in \mathcal{A}\}$. The vector z is called a *cyclic vector* for T .
- (ii) T is *irreducible* if T is not trivial and $\{0\}$ and $\mathcal{X}$ are the only subspaces of $\mathcal{X}$ invariant under T .
- (iii) A representation S of $\mathcal{A}$ on a vector space $\mathcal{Y}$ is *equivalent* to T if there exists a linear bijective mapping $U : \mathcal{X} \rightarrow \mathcal{Y}$ such that $S_a U = U T_a$ for all $a \in \mathcal{A}$. Then U is called an *intertwining operator* between S and T .

Remark. In the literature the above properties are often referred to as *algebraically cyclic*, *algebraically irreducible* and *algebraically equivalent* in contrast to their topological analogues. We have omitted these adjectives because in our discussion the respective topological properties do not occur.

3.14 Example (The left regular representation). For each $a \in \mathcal{A}$ define

$$L_a : \mathcal{A} \rightarrow \mathcal{A}, L_a b := ab.$$

Then $L_a \in \mathcal{L}(\mathcal{A})$ for each $a \in \mathcal{A}$ and the mapping $L : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{A}), a \mapsto L_a$ is a representation of $\mathcal{A}$ on $\mathcal{A}$, called the *left regular representation on $\mathcal{A}$* .

If $\mathcal{I}$ is a left ideal of $\mathcal{A}$, then $L_a(\mathcal{I}) \subseteq \mathcal{I}$ for all $a \in \mathcal{A}$. Hence we may define a linear mapping $L_a^{\mathcal{I}}$ on the quotient space $\mathcal{A}/\mathcal{I}$ by

$$L_a^{\mathcal{I}}(b + \mathcal{I}) := ab + \mathcal{I}.$$

Then $L^{\mathcal{I}} : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{A}/\mathcal{I})$, $a \mapsto L_a^{\mathcal{I}}$, is a representation of $\mathcal{A}$ on $\mathcal{A}/\mathcal{I}$, called the *left regular representation on $\mathcal{A}/\mathcal{I}$* .

Notation. If there is no danger of confusion, we will simply write L instead of $L^{\mathcal{I}}$.

3.15 Lemma. *The left regular representation L of $\mathcal{A}$ on $\mathcal{A}/\mathcal{I}$ is irreducible if and only if $\mathcal{I}$ is a maximal left ideal in $\mathcal{A}$.*

Proof. It suffices to establish a one-to-one correspondence between left ideals $\mathcal{J}$ in $\mathcal{A}$ with $\mathcal{J} \supseteq \mathcal{I}$ and subspaces of $\mathcal{A}/\mathcal{I}$ invariant under L .

If $\mathcal{J}$ is a left ideal in $\mathcal{A}$ such that $\mathcal{J} \supseteq \mathcal{I}$, let $\mathcal{J}' := \{a' \in \mathcal{A}/\mathcal{I} : a \in \mathcal{J}\}$. Then $L_a(\mathcal{J}') \subseteq \mathcal{J}'$ for all $a \in \mathcal{A}$, hence $\mathcal{J}'$ is a subspace of $\mathcal{A}/\mathcal{I}$ invariant under L . Since $\mathcal{I} \subseteq \mathcal{J}$ we also have $\mathcal{J} = \{a \in \mathcal{A} : a' \in \mathcal{J}'\}$. Therefore the correspondence is one-to-one.

Conversely, let $\mathcal{M}'$ be a subspace of $\mathcal{A}/\mathcal{I}$ invariant under L and define $\mathcal{M} := \{a \in \mathcal{A} : a' \in \mathcal{M}'\}$. Then $\mathcal{M}$ is a left ideal in $\mathcal{A}$ satisfying $\mathcal{M} \supseteq \mathcal{I}$. □

The importance of the previous example is indicated in the next proposition [19].

3.16 Proposition. *Let $T : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{X})$ be a cyclic representation of $\mathcal{A}$ on a vector space $\mathcal{X}$. Then T is equivalent to the left regular representation $L : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{A}/\mathcal{I})$ for some left ideal $\mathcal{I}$ in $\mathcal{A}$.*

If z is a cyclic vector for T , then $\mathcal{I}$ can be chosen as $\mathcal{I} = \{a \in \mathcal{A} : T_a z = 0\}$.

Proof. Let z be a cyclic vector for T in $\mathcal{X}$. Then $\mathcal{I} = \{a \in \mathcal{A} : T_a z = 0\}$ is a left ideal in $\mathcal{A}$.

Define a linear mapping $U : \mathcal{A}/\mathcal{I} \rightarrow \mathcal{X}$ by $Ua' := T_a z$ for $a' = a + \mathcal{I}$.

By the choice of $\mathcal{I}$, the mapping U is well-defined and injective. Since z is a cyclic vector, U is also surjective.

Since $UL_a b' = U(ab)' = T_{ab} z = T_a T_b z = T_a U b'$ for all $a \in \mathcal{A}, b' \in \mathcal{A}/\mathcal{I}$ we obtain

$$UL_a = T_a U$$

for all $a \in \mathcal{A}$ and thus U is an intertwining operator between L and T . □

Next we observe that irreducible representations are automatically cyclic [19].

3.17 Proposition. *Let $T : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{X})$ be an irreducible representation of $\mathcal{A}$ on a vector space $\mathcal{X}$. Then every non-zero vector in $\mathcal{X}$ is a cyclic vector for T .*

Proof. Let $z \in \mathcal{X}$ and consider $T_{\mathcal{A}}z = \{T_a z : a \in \mathcal{A}\}$. Then $T_{\mathcal{A}}z$ is a subspace of $\mathcal{X}$ invariant under T and thus, by irreducibility, equals $\{0\}$ or $\mathcal{X}$. Now the set $\{z \in \mathcal{X} : T_{\mathcal{A}}z = \{0\}\}$ is also a T -invariant subspace and hence equals $\{0\}$ or $\mathcal{X}$. However, this subspace is $\{0\}$ since T is not trivial. Accordingly, for each non-zero vector $z \in \mathcal{X}$ we have $T_{\mathcal{A}}z = \mathcal{X}$, that is, z is cyclic. □

3.18 Corollary. *Let $T : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{X})$ be an irreducible representation of $\mathcal{A}$ on a vector space $\mathcal{X}$. Then T is equivalent to $L : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{A}/\mathcal{I})$ for some maximal left ideal $\mathcal{I}$ in $\mathcal{A}$.*

If z is a non-zero vector in $\mathcal{X}$, then $\mathcal{I}$ can be chosen as $\mathcal{I} = \{a \in \mathcal{A} : T_a z = 0\}$.

Proof. This follows from the preceding propositions combined with Lemma 3.15. □

Up to now our discussion on representations of $\mathcal{A}$ does not take into account the $*$ -algebraic structure of $\mathcal{A}$ and can be carried out for general algebras as well. In order to bring the involution into the theory we want to employ $*$ -homomorphisms instead of homomorphisms. For that to work out we must assume additional structure on the representation space $\mathcal{X}$, so that the target space of the representation, in general a subalgebra of $\mathcal{L}(\mathcal{X})$, can be equipped with a $*$ -algebraic structure.

Thus one usually considers Hilbert spaces $\mathcal{H}$ and $*$ -homomorphisms into $\mathcal{B}(\mathcal{H})$ where the involution is given by passing to the adjoint operator.

3.19 Definition. A (pre-) $*$ -representation T of $\mathcal{A}$ on a (pre-) Hilbert space $\mathcal{H}$ is a $*$ -homomorphism $a \mapsto T_a$ of $\mathcal{A}$ into the $*$ -algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators on $\mathcal{H}$.

Notation. Throughout this work all $*$ -representations T are supposed to be non-degenerate, that is, to satisfy $\overline{T_{\mathcal{A}}(\mathcal{H})} = \mathcal{H}$.

This is no serious restriction since every $*$ -representation is a direct sum of a non-degenerate $*$ -representation and a trivial $*$ -representation (see, e.g., [28], p.86).

Remark. In the literature a (pre-)*-representation is sometimes defined more generally as a *-homomorphism T of a general *-algebra into a subalgebra of the *-algebra of linear operators on a (pre-) Hilbert space $\mathcal{X}$ that possess an adjoint operator; the representing operators T_a are not required to be bounded (see, e.g., [20], [26], [28]).

For a Hilbert space $\mathcal{H}$ the Hellinger-Toeplitz Theorem implies that a linear operator on $\mathcal{H}$ is bounded if and only if it possesses an adjoint. Hence the definitions agree.

For *-homomorphisms of a general *-algebra into a subalgebra of linear operators on a pre-Hilbert space this is not true. However, if the investigation is restricted to Banach *-algebras, one can show that the representing operators are automatically bounded (see [20], p.1162, [28], p.83).

Characterizations of symmetry

We are now prepared to give a further characterization of symmetry. Indeed, $\mathcal{A}$ is symmetric if and only if representations and *-representations of $\mathcal{A}$ correlate in a certain way.

This was first observed by Leptin [16]. In the proof we will follow [20].

3.20 Theorem. *For a unital Banach *-algebra $\mathcal{A}$ with continuous involution the following condition is equivalent to symmetry of $\mathcal{A}$:*

If T is an irreducible representation of $\mathcal{A}$ on a linear space $\mathcal{X}$, then there is an inner product for $\mathcal{X}$ relative to which T is a pre--representation.*

Proof. The proof is based on the characterization with positive functionals and ideals established in the previous section (Theorem 3.11).

Let $\mathcal{A}$ be symmetric and let $T : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{X})$ be an irreducible representation.

Choose a non-zero vector $z \in \mathcal{X}$, then z is a cyclic vector for T and

$\mathcal{I} := \{a \in \mathcal{A} : T_a z = 0\}$ is a maximal left ideal in $\mathcal{A}$ by Corollary 3.18.

Now we apply Theorem 3.11 to obtain a state f on $\mathcal{A}$ satisfying

$\mathcal{I} = \{a \in \mathcal{A} : f(a^*a) = 0\}$.

Since z is cyclic, every $x \in \mathcal{X}$ is of the form $x = T_b z$ for some $b \in \mathcal{A}$. We show that

$$(x, y) = (T_b z, T_c z) := f(c^*b)$$

is an inner product on $\mathcal{X}$.

It is skew linear according to the linearity of f , hermitian since $f(c^*b) = \overline{f(b^*c)}$ (see Proposition 3.4), and positive definite because $\mathcal{I} = \{a \in \mathcal{A} : f(a^*a) = 0\}$.

Next we show that T is a pre- $*$ -representation, i.e., $T_{a^*} = (T_a)^*$ and $\|T_a\| < \infty$ for all $a \in \mathcal{A}$.

Given $a \in \mathcal{A}$ and $x, y \in \mathcal{X}$ we have to show that $(T_ax, y) = (x, T_{a^*}y)$.

Again, since $\mathcal{X} = \{T_az : a \in \mathcal{A}\}$, there exist $b, c \in \mathcal{A}$ such that $x = T_bz, y = T_cz$ and we have

$$(T_ax, y) = (T_aT_bz, T_cz) = f(c^*ab) = f((a^*c)^*b) = (T_bz, T_{a^*}zT_cz) = (x, T_{a^*}y).$$

It remains to show that T_a is bounded for all $a \in \mathcal{A}$, i.e.,

$$\|T_a\| = \sup\left\{\frac{\|T_ax\|}{\|x\|} : x \neq 0\right\} < \infty$$

for all $a \in \mathcal{A}$.

Using inequality (4) in Proposition 3.4 we obtain for $x \neq 0$

$$\frac{\|T_ax\|^2}{\|x\|^2} = \frac{\|T_aT_bz\|^2}{\|T_bz\|^2} = \frac{f(b^*a^*ab)}{f(b^*b)} \leq \frac{r(a^*a)f(b^*b)}{f(b^*b)} = r(a^*a).$$

Thus T is a pre- $*$ -representation with respect to $(.,.)$.

To prove the converse we also make use of the characterization in Theorem 3.11.

For every maximal left ideal $\mathcal{I}$ in $\mathcal{A}$ we construct a state f on $\mathcal{A}$ such that $\mathcal{I} = \{a \in \mathcal{A} : f(a^*a) = 0\}$. Then Theorem 3.11 yields symmetry of $\mathcal{A}$.

Let $\mathcal{I}$ be a maximal left ideal in $\mathcal{A}$.

From Lemma 3.15 we know that the corresponding left regular representation $L : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{A}/\mathcal{I})$ is irreducible.

Now, by hypothesis, the representation space $\mathcal{A}/\mathcal{I}$ can be given an inner product $(.,.)$ relative to which L is a pre- $*$ -representation.

Define $f(a) := (L_a e', e') = (a', e')$ for all $a \in \mathcal{A}$.

Then the equation

$$f(a^*a) = (L_{a^*a} e', e') = (L_{a^*} a', e') = (a', L_a e') = (a', a')$$

shows that f is a positive linear functional on $\mathcal{A}$. Furthermore we infer that

$$f(a^*a) = 0 \Leftrightarrow (a', a') = 0 \Leftrightarrow a' = 0' \Leftrightarrow a \in \mathcal{I}.$$

Set $\mu = (e', e')$, then $\tilde{f} = \mu^{-1}f$ is a state on $\mathcal{A}$ satisfying $\mathcal{I} = \{a \in \mathcal{A} : \tilde{f}(a^*a) = 0\}$.

□

3.3 Characterizations via the Gelfand-Naimark seminorm

In this section we discuss characterizations of symmetry via the Gelfand-Naimark seminorm γ . This important seminorm is defined in terms of states on $\mathcal{A}$, but can alternatively be expressed by means of $*$ -representations of $\mathcal{A}$.

Definitions and Preparations

As a preparation we further explore the connection between states and $*$ -representations.

First we observe that states on $\mathcal{A}$ naturally arise from given $*$ -representations of $\mathcal{A}$.

Indeed, let T be a $*$ -representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ and let $x \in \mathcal{H}$ be a unit vector. Define a linear functional f on $\mathcal{A}$ by

$$f(a) := (T_a x, x).$$

Then, for $a \in \mathcal{A}$, we have

$$f(a^*a) = (T_{a^*a}x, x) = (T_{a^*}T_ax, x) = (T_ax, T_ax) \geq 0$$

and $f(e) = \|x\|^2 = 1$. So f is a state on $\mathcal{A}$.

Conversely, given a state f on $\mathcal{A}$ we can construct a Hilbert space $\mathcal{H}_f$ and a corresponding $*$ -representation T_f of $\mathcal{A}$ on $\mathcal{H}_f$. This construction is often referred to as the *Gelfand-Naimark-Segal construction* and uses arguments similar to the proof of Theorem 3.20.

Consider the left ideal $\mathcal{I}_f := \{a \in \mathcal{A} : f(a^*a) = 0\}$. On the quotient space $\mathcal{A}/\mathcal{I}_f$ define an inner product by

$$(a', b') := f(b^*a)$$

for $a', b' \in \mathcal{A}/\mathcal{I}_f$. First we observe that $(\cdot, \cdot)$ is well-defined on $\mathcal{A}/\mathcal{I}_f$: Suppose that $a'_1 = a'_2$ and $b'_1 = b'_2$ in $\mathcal{A}/\mathcal{I}_f$. Then

$$\begin{aligned} (a'_1, b'_1) - (a'_2, b'_2) &= f(b_1^*a_1) - f(b_2^*a_2) \\ &= f(b_1^*(a_1 - a_2) + (b_1 - b_2)^*a_2) \\ &= f(b_1^*(a_1 - a_2)) + f((b_1 - b_2)^*a_2) \\ &= f(b_1^*(a_1 - a_2)) + \overline{f(a_2^*(b_1 - b_2))} = 0, \end{aligned}$$

since $a_1 - a_2, b_1 - b_2 \in \mathcal{I}_f$ and $\mathcal{I}_f = \{a \in \mathcal{A} : f(b^*a) = 0 \text{ for all } b \in \mathcal{A}\}$ by Proposition 3.4 (5).

The properties required for $(\cdot, \cdot)$ to be an inner product follow from the linearity of f , from $f(b^*a) = \overline{f(a^*b)}$ (see Proposition 3.4) and from the definition of $\mathcal{I}_f$.

Now consider the left regular representation L of $\mathcal{A}$ on $\mathcal{A}/\mathcal{I}_f$. Then L is a pre- $*$ -representation of $\mathcal{A}$ on $\mathcal{A}/\mathcal{I}_f$:

For $a \in \mathcal{A}$ and $b', c' \in \mathcal{A}/\mathcal{I}_f$ we have

$$(L_a b', c') = ((ab)', c') = f(c^*ab) = f((a^*c)^*b) = (b', (a^*c)') = (b', L_{a^*} c')$$

and further, by Proposition 3.4 (4),

$$\frac{\|L_a b'\|^2}{\|b'\|^2} = \frac{\|(ab)'\|^2}{\|b'\|^2} = \frac{f(b^*a^*ab)}{f(b^*b)} \leq \frac{r(a^*a)f(b^*b)}{f(b^*b)} = r(a^*a)$$

for $b' \neq 0'$. Denote by $\mathcal{H}_f$ the Hilbert space completion of $\mathcal{A}/\mathcal{I}_f$ with respect to $(\cdot, \cdot)$. Then L extends to a $*$ -representation T_f of $\mathcal{A}$ on the Hilbert space $\mathcal{H}_f$.

Next we give the definition of the Gelfand-Naimark seminorm.

3.21 Definition. The *Gelfand-Naimark seminorm* γ is defined by $\gamma(a) := \sup\{f(a^*a)^{\frac{1}{2}} : f \in S(\mathcal{A})\}$ for all $a \in \mathcal{A}$.

If $S(\mathcal{A}) = \emptyset$ we set $\gamma(a) = 0$ for all $a \in \mathcal{A}$.

Using the above correspondence between states and $*$ -representations, we can give an alternative formula for γ .

3.22 Proposition. For all $a \in \mathcal{A}$ we have $\gamma(a) = \sup\{\|T_a\| : T \text{ is a } * \text{-representation of } \mathcal{A}\}$.

Proof. Choose a state f on $\mathcal{A}$ and consider the associated $*$ -representation $T = T_f$ of $\mathcal{A}$ on $\mathcal{H}_f$. By construction of T we have, for $a \in \mathcal{A}$,

$$\begin{aligned} \|T_a\|^2 &= \sup\left\{\frac{f(b^*a^*ab)}{f(b^*b)} : b' \neq 0'\right\} \\ &\geq \frac{f(e^*a^*ae)}{f(e^*e)} = f(a^*a). \end{aligned}$$

Now let T be an arbitrary $*$ -representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ and let $x \in \mathcal{H}$ be a unit vector.

Then $f(a) := (T_a x, x)$ defines a state f on $\mathcal{A}$.

Hence,

$$\|T_a x\|^2 = (T_a x, T_a x) = (T_{a^* a} x, x) = f(a^* a) \leq \gamma(a)^2,$$

that is, $\|T_a\| \leq \gamma(a)$ for all $a \in \mathcal{A}$.

Putting things together we obtain, for all $a \in \mathcal{A}$,

$$\begin{aligned} \gamma(a)^2 &= \sup\{f(a^* a) : f \in S(\mathcal{A})\} \\ &\leq \sup\{\|T_a\|^2 : T = T_f \text{ for some } f \in S(\mathcal{A})\} \\ &\leq \sup\{\|T_a\|^2 : T \text{ is a }^*\text{-representation of } \mathcal{A}\} \leq \gamma(a)^2. \end{aligned}$$

□

3.23 Corollary. *The Gelfand-Naimark seminorm γ is an algebra seminorm on $\mathcal{A}$ such that $\gamma(a^*) = \gamma(a)$ and $\gamma(a^* a) = \gamma(a)^2$ for all $a \in \mathcal{A}$.*

Proof. By Proposition 3.22 the desired properties of γ follow from the respective properties of the operator norm on Hilbert spaces.

□

Another useful fact is that γ is always dominated by the Ptak function ρ .

3.24 Lemma. *For all $a \in \mathcal{A}$ we have $\gamma(a) \leq \rho(a)$.*

Proof. Let $a \in \mathcal{A}$. By Proposition 3.4 (3), we have $f(a^* a) \leq r(a^* a)$ for all states $f \in S(\mathcal{A})$ and hence

$$\gamma(a) = \sup\{f(a^* a)^{\frac{1}{2}} : f \in S(\mathcal{A})\} \leq r(a^* a)^{\frac{1}{2}} = \rho(a).$$

□

Now we introduce the notion of spectral seminorms following [19].

3.25 Definition. A *spectral seminorm* on $\mathcal{A}$ is an algebra seminorm σ which satisfies $r(a) \leq \sigma(a)$ for all $a \in \mathcal{A}$.

The complete norm on $\mathcal{A}$ is a first example of a spectral seminorm. Further we have seen in Theorem 2.19 that on symmetric Banach * -algebras the Ptak function ρ is a spectral seminorm.

3.26 Proposition. *An algebra seminorm σ on $\mathcal{A}$ is spectral if and only if for $a \in \mathcal{A}$ with $\sigma(a) < 1$ the element $e - a$ is invertible in $\mathcal{A}$.*

Proof. ($\Rightarrow$) Let σ be a spectral seminorm on $\mathcal{A}$ and let $a \in \mathcal{A}$ with $\sigma(a) < 1$. Then $r(a) \leq \sigma(a) < 1$. This inequality implies that $1 \notin Sp(a)$, and thus $e - a$ is invertible in $\mathcal{A}$.

($\Leftarrow$) Let $a \in \mathcal{A}$ with $\sigma(a) < 1$.

By assumption, $e - \lambda^{-1}a$ is invertible for all $\lambda \in \mathbb{C}$, $|\lambda| \geq 1$. In other words, $\lambda \notin Sp(a)$ for all $\lambda \in \mathbb{C}$, $|\lambda| \geq 1$. Therefore $r(a) < 1$.

Now, for arbitrary $a \in \mathcal{A}$ and $\epsilon > 0$, set $a_\epsilon = (\sigma(a) + \epsilon)^{-1}a$. Then $\sigma(a_\epsilon) < 1$. By the above we have $r(a_\epsilon) < 1$ which means $r(a) < \sigma(a) + \epsilon$. Since ϵ was arbitrary, $r(a) \leq \sigma(a)$ and so σ is spectral.

□

Characterizations of symmetry

In the next theorem we collect some characterizations of symmetry involving properties of the Gelfand-Naimark seminorm γ . Most of these conditions were derived by Palmer in [18] and [20]. Criterion (5), however, can be traced back to the observations of Raikov in [24] where the notion of symmetric $*$ -algebras was first introduced.

3.27 Theorem. *For a unital Banach $*$ -algebra $\mathcal{A}$ with continuous involution the following conditions are equivalent:*

- (1) $\mathcal{A}$ is symmetric.
- (2) Every maximal left ideal in $\mathcal{A}$ is closed with respect to γ .
- (3) γ is a spectral seminorm on $\mathcal{A}$.
- (4) There exists a constant C with $r(a^*a) \leq C\gamma(a^*a)$ for all $a \in \mathcal{A}$.
- (5) Raikov's criterion: $\gamma(a) = \rho(a)$ for all $a \in \mathcal{A}$.

Proof. (1) $\Rightarrow$ (2): Let $\mathcal{I}$ be a maximal left ideal in $\mathcal{A}$.

Then Theorem 3.11 yields a state f on $\mathcal{A}$ such that $\mathcal{I} = \{a \in \mathcal{A} : f(a^*a) = 0\}$.

Recall from Proposition 3.4 (2) that $|f(a)| \leq f(a^*a)^{\frac{1}{2}}$ for all $a \in \mathcal{A}$ and thus, by definition of γ , $|f(a)| \leq \gamma(a)$ for all $a \in \mathcal{A}$.

In other words, f is continuous with respect to γ .

Moreover, both involution and multiplication on $\mathcal{A}$ are continuous with respect to γ by Corollary 3.23.

Now define $F(a) := f(a^*a)$. Then F is continuous with respect to γ and $\mathcal{I} = F^{-1}(\{0\})$. Thus $\mathcal{I}$ is closed with respect to γ .

(2) $\Rightarrow$ (3): Assume on the contrary that γ is not spectral.

By Proposition 3.26, there is some $b \in \mathcal{A}$ such that $\gamma(b) < 1$, but $e - b$ is not

invertible. We may assume that $e - b$ is not left invertible, otherwise consider b^* .

Then $\mathcal{I} := \mathcal{A}(e - b)$ is a proper left ideal in $\mathcal{A}$ and hence is included in some maximal left ideal $\tilde{\mathcal{I}}$ (Proposition 3.2).

To achieve a contradiction we need to find an element in $\mathcal{A}$ which is contained in the γ -closure of $\tilde{\mathcal{I}}$ but not in $\tilde{\mathcal{I}}$ itself.

We claim that b fulfills these requirements:

$\tilde{\mathcal{I}}$ does not contain b , because otherwise $e = (e - b) + b \in \tilde{\mathcal{I}}$, which contradicts the properness of $\tilde{\mathcal{I}}$.

On the other hand b belongs to the γ -closure of $\tilde{\mathcal{I}}$, because

$$b - b^n = \left(\sum_{k=1}^{n-1} b^k \right) (e - b) \in \tilde{\mathcal{I}}$$

for all $n \in \mathbb{N}$ and $\gamma(b - (b - b^n)) \leq \gamma(b)^n$ tends to zero as n approaches infinity.

Thus we have created the desired contradiction to the hypothesis that all maximal left ideals are closed with respect to γ .

(3) $\Rightarrow$ (4): clear

(4) $\Rightarrow$ (5): By Lemma 3.24 we always have $\gamma(a) \leq \rho(a)$ for all $a \in \mathcal{A}$.

For the converse inequality we first assume that h is hermitian. We use $r(a^k) = r(a)^k$ (Corollary 2.8) and the assumption (4) to deduce that

$$r(h) = r(h^2)^{\frac{1}{2}} = r(h^*h)^{\frac{1}{2}} \leq (C\gamma(h^*h))^{\frac{1}{2}} = \tilde{C}\gamma(h). \quad (3.2)$$

By a similar argument with formula (3.2) applied to the hermitian element $h = (a^*a)^n$ we further establish that

$$\rho(a) = r(a^*a)^{\frac{1}{2}} = r((a^*a)^n)^{\frac{1}{2n}} \leq (\tilde{C}\gamma((a^*a)^n))^{\frac{1}{2n}} \leq \tilde{C}^{\frac{1}{2n}}\gamma(a^*a)^{\frac{1}{2}} = \tilde{C}^{\frac{1}{2n}}\gamma(a)$$

for all $a \in \mathcal{A}$. Taking the limit yields $\rho(a) \leq \gamma(a)$ for all $a \in \mathcal{A}$.

(5) $\Rightarrow$ (1): By equality of the Ptak function ρ and the Gelfand-Naimark seminorm γ , it follows that ρ is an algebra seminorm on $\mathcal{A}$. Now Theorem 2.19 yields symmetry of $\mathcal{A}$.

□

3.4 A characterization involving radicals and the enveloping C^* -algebra

Definitions and Preparations

To begin this section we briefly discuss the Jacobson radical. There are several equivalent ways of defining it. We prefer the definition in terms of irreducible representations following [19], [26].

Note that the Jacobson radical is a purely algebraic object and as such can be defined and studied in general algebras.

3.28 Definition. The *Jacobson radical* $Rad(\mathcal{A})$ of $\mathcal{A}$ is the intersection of the kernels of all irreducible representations of $\mathcal{A}$.

Clearly $Rad(\mathcal{A})$ is an ideal in $\mathcal{A}$. Thus we may form the quotient algebra $\mathcal{A}/Rad(\mathcal{A})$ and consider algebraic concepts such as invertibility and spectrum in the algebra $\mathcal{A}/Rad(\mathcal{A})$.

3.29 Lemma. *An element $b \in \mathcal{A}$ is invertible in $\mathcal{A}$ if and only if $b' = b + Rad(\mathcal{A})$ is invertible in $\mathcal{A}/Rad(\mathcal{A})$.*

Proof. ($\Rightarrow$) This implication is true for any ideal $\mathcal{I}$ in $\mathcal{A}$, since $cb = bc = e$ implies $c'b' = (cb)' = e'$ and $b'c' = (bc)' = e'$ where $a \mapsto a'$ denotes the canonical mapping of $\mathcal{A}$ onto $\mathcal{A}/\mathcal{I}$.

($\Leftarrow$) Let $b \in \mathcal{A}$ such that b' is invertible in $\mathcal{A}/Rad(\mathcal{A})$. First we show that b has a left inverse in $\mathcal{A}$.

Seeking a contradiction, we suppose that b is not left invertible in $\mathcal{A}$. Then $\{ab : a \in \mathcal{A}\}$ is a proper left ideal in $\mathcal{A}$ and hence, by Proposition 3.2, is included in some maximal left ideal $\mathcal{I}$ in $\mathcal{A}$.

Consider the left regular representation L of $\mathcal{A}$ on $\mathcal{A}/\mathcal{I}$ defined by $L_a c' = (ac)'$. By maximality of $\mathcal{I}$, this is an irreducible representation (Lemma 3.15). Hence $Rad(\mathcal{A}) \subseteq \ker(L)$. Furthermore any element $c \in \ker(L)$ belongs to $\mathcal{I}$, because $L_c e' = 0'$ means $c \in \mathcal{I}$. Thus $Rad(\mathcal{A}) \subseteq \ker(L) \subseteq \mathcal{I}$.

Since b' is invertible in $\mathcal{A}/Rad(\mathcal{A})$ by assumption, there exists an element $a \in \mathcal{A}$ such that $ab - e \in Rad(\mathcal{A}) \subseteq \mathcal{I}$. But then we have $e = ab - (ab - e) \in \mathcal{I}$, which contradicts the properness of $\mathcal{I}$.

Thus we have shown that each element $b \in \mathcal{A}$ which becomes invertible in $\mathcal{A}/Rad(\mathcal{A})$ has a left inverse a in $\mathcal{A}$. Clearly a' is the inverse of b' in $\mathcal{A}/Rad(\mathcal{A})$. So a' is also invertible in $\mathcal{A}/Rad(\mathcal{A})$ and therefore has a left

inverse in $\mathcal{A}$ which must be b .

Hence b is invertible in $\mathcal{A}$ as was to be shown. □

It is useful to note some alternative descriptions of the Jacobson radical [19].

3.30 Proposition. *The following sets in $\mathcal{A}$ are equal:*

- (1) *The Jacobson radical $Rad(\mathcal{A})$ of $\mathcal{A}$.*
- (2) *$\{b \in \mathcal{A} : r(ab) = 0 \text{ for all } a \in \mathcal{A}\}$.*
- (3) *The largest ideal $\mathcal{I}$ in $\mathcal{A}$ satisfying $Sp_{\mathcal{A}}(a) = Sp_{\mathcal{A}/\mathcal{I}}(a + \mathcal{I})$ for all $a \in \mathcal{A}$.*

Proof. We will temporarily denote the set in (2) by $\mathcal{B}$.

First we note that, by Lemma 3.29, the Jacobson radical satisfies

$$Sp_{\mathcal{A}}(a) = Sp_{\mathcal{A}/Rad(\mathcal{A})}(a + Rad(\mathcal{A}))$$

for all $a \in \mathcal{A}$.

Now let $\mathcal{I}$ be any ideal satisfying $Sp_{\mathcal{A}}(a) = Sp_{\mathcal{A}/\mathcal{I}}(a')$ where $a' = a + \mathcal{I}$. Let $b \in \mathcal{I}$, then

$$Sp_{\mathcal{A}}(b) = Sp_{\mathcal{A}/\mathcal{I}}(b') = Sp_{\mathcal{A}/\mathcal{I}}(0') = \{0\}$$

and further $r(b) = 0$. Since $\mathcal{I}$ is an ideal, we also have $r(ab) = 0$ for all $a \in \mathcal{A}$. It follows that $b \in \mathcal{B}$ and therefore $\mathcal{I} \subseteq \mathcal{B}$.

In particular, this implies $Rad(\mathcal{A}) \subseteq \mathcal{B}$.

Next we will show that every element $b \in \mathcal{B}$ is in the kernel of every irreducible representation of $\mathcal{A}$, which implies $\mathcal{B} \subseteq Rad(\mathcal{A})$.

Let $T : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{X})$ be an irreducible representation of $\mathcal{A}$ and let $b \in \mathcal{B}$.

Suppose on the contrary that b does not belong to the kernel of T . Then we can choose a non-zero vector $z \in \mathcal{X}$ such that $T_b z \neq 0$. Since for irreducible representations every non-zero vector is cyclic (Proposition 3.17), there exists some $a \in \mathcal{A}$ satisfying $T_a T_b z = z$. Since $r(ab) = 0$, there exists an inverse $c \in \mathcal{A}$ for $e - ab$. But this gives a contradiction:

$$0 = T_c z - T_c z = T_c z - T_c T_{ab} z = T_{c(e-ab)} z = T_e z = z.$$

We may infer that $\mathcal{B} \subseteq Rad(\mathcal{A})$ and hence $\mathcal{B} = Rad(\mathcal{A})$.

We have shown that any ideal $\mathcal{I}$ satisfying $Sp_{\mathcal{A}}(a) = Sp_{\mathcal{A}/\mathcal{I}}(a')$ is included in $\mathcal{B} = Rad(\mathcal{A})$. So $Rad(\mathcal{A})$ is the largest ideal with this property. □

3.31 Lemma. $Rad(\mathcal{A}/Rad(\mathcal{A})) = \{0'\}$.

Proof. If $a \notin Rad(\mathcal{A})$, then there exists an irreducible representation $T : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{X})$ such that $T_a \neq 0$. Since $Rad(\mathcal{A}) \subseteq \ker(T)$, we may define an irreducible representation $T' : \mathcal{A}/Rad(\mathcal{A}) \rightarrow \mathcal{L}(\mathcal{X})$ by $T'_{a'} := T_a$. Then $T'_{a'} \neq 0$ and hence $a' \notin Rad(\mathcal{A}/Rad(\mathcal{A}))$. Therefore $Rad(\mathcal{A}/Rad(\mathcal{A})) = \{0'\}$. $\square$

To guarantee that the quotient algebra $\mathcal{A}/Rad(\mathcal{A})$ is a Banach $*$ -algebra we need to show that $Rad(\mathcal{A})$ is a closed $*$ -ideal of $\mathcal{A}$.

3.32 Lemma. *The Jacobson radical $Rad(\mathcal{A})$ of a Banach $*$ -algebra $\mathcal{A}$ is a closed $*$ -ideal of $\mathcal{A}$.*

Proof. Using elementary properties of the spectral radius (Proposition 2.6 and Proposition 2.5) we calculate that, for $a, b \in \mathcal{A}$,

$$r(ab^*) = r((ab^*)^*) = r(ba^*) = r(a^*b).$$

Hence, by Proposition 3.30 (2), $b \in Rad(\mathcal{A})$ if and only if $b^* \in Rad(\mathcal{A})$. Therefore $Rad(\mathcal{A})$ is a $*$ -ideal.

Let $\mathcal{I}$ be an ideal in $\mathcal{A}$ and consider the quotient algebra $\mathcal{A}/\mathcal{I}$. Recall that the canonical seminorm on $\mathcal{A}/\mathcal{I}$ is defined by

$$\|a + \mathcal{I}\|_{\mathcal{A}/\mathcal{I}} := \inf\{\|a + b\|_{\mathcal{A}} : b \in \mathcal{I}\}. \quad (3.3)$$

Then $\|\cdot\|_{\mathcal{A}/\mathcal{I}}$ is a norm on $\mathcal{A}/\mathcal{I}$ if and only if $\mathcal{I}$ is closed (see, e.g., [19], p.14).

We thus need to show that $\|\cdot\|_{\mathcal{A}/Rad(\mathcal{A})}$, defined as in (3.3), is a norm on $\mathcal{A}/Rad(\mathcal{A})$. First we estimate

$$\begin{aligned} \|a + Rad(\mathcal{A})\|_{\mathcal{A}/Rad(\mathcal{A})} &:= \inf\{\|a + b\|_{\mathcal{A}} : b \in Rad(\mathcal{A})\} \\ &\geq \inf\{r_{\mathcal{A}}(a + b) : b \in Rad(\mathcal{A})\} \\ &\geq \inf\{r_{\mathcal{A}/Rad(\mathcal{A})}(a + b + Rad(\mathcal{A})) : b \in Rad(\mathcal{A})\} \\ &= r_{\mathcal{A}/Rad(\mathcal{A})}(a + Rad(\mathcal{A})) \end{aligned}$$

for all $a \in \mathcal{A}$. Now let $b' \in \mathcal{A}/Rad(\mathcal{A})$ such that $\|b'\|_{\mathcal{A}/Rad(\mathcal{A})} = 0$. Then

$$r_{\mathcal{A}/Rad(\mathcal{A})}(a'b') \leq \|a'b'\|_{\mathcal{A}/Rad(\mathcal{A})} \leq \|a'\|_{\mathcal{A}/Rad(\mathcal{A})} \|b'\|_{\mathcal{A}/Rad(\mathcal{A})} = 0$$

for all $a' \in \mathcal{A}/Rad(\mathcal{A})$. By Proposition 3.30, $b' \in Rad(\mathcal{A}/Rad(\mathcal{A})) = \{0'\}$. Therefore $\|\cdot\|_{\mathcal{A}/Rad(\mathcal{A})}$ is a norm on $\mathcal{A}/Rad(\mathcal{A})$ and thus $Rad(\mathcal{A})$ is closed. $\square$

Next we consider the $*$ -algebraic version of a radical [20].

3.33 Definition. The $*$ -radical $R^*(\mathcal{A})$ of $\mathcal{A}$ is the intersection of the kernels of all $*$ -representations of $\mathcal{A}$.

Clearly $R^*(\mathcal{A})$ is a $*$ -ideal in $\mathcal{A}$.

From the expression of the Gelfand-Naimark seminorm γ in terms of $*$ -representations (Proposition 3.22) we have

$$R^*(\mathcal{A}) = \{a \in \mathcal{A} : \gamma(a) = 0\}. \quad (3.4)$$

We use this description of the $*$ -radical to clarify its relation to the Jacobson radical.

3.34 Lemma. $Rad(\mathcal{A}) \subseteq \rho^{-1}(\{0\}) \subseteq R^*(\mathcal{A})$.

Proof. If $a \in Rad(\mathcal{A})$, then $\rho(a) = r(a^*a)^{\frac{1}{2}} = 0$ by Proposition 3.30 (2). Thus $Rad(\mathcal{A}) \subseteq \rho^{-1}(\{0\})$.

If now $a \in \mathcal{A}$ with $\rho(a) = 0$, we use Lemma 3.24 to obtain $\gamma(a) \leq \rho(a) = 0$. Hence also $\rho^{-1}(\{0\}) \subseteq R^*(\mathcal{A})$ by formula (3.4). □

In view of formula (3.4) we may define a C^* -norm $\|\cdot\|$ on $\mathcal{A}/R^*(\mathcal{A})$ by

$$\|a + R^*(\mathcal{A})\| := \gamma(a)$$

for all $a \in \mathcal{A}$.

The completion of $\mathcal{A}/R^*(\mathcal{A})$ with respect to $\|\cdot\|$ is an important C^* -algebra.

3.35 Definition. The *enveloping C^* -algebra* $C^*(\mathcal{A})$ of $\mathcal{A}$ is defined to be the completion of $\mathcal{A}/R^*(\mathcal{A})$ in the C^* -norm induced by γ .

In the subsequent theorem we will see that the spectrum in $\mathcal{A}/R^*(\mathcal{A})$ is the same as in the completion $C^*(\mathcal{A})$, provided that $\mathcal{A}$ is symmetric.

In general, if $\mathcal{B}$ is a unital subalgebra of $\mathcal{A}$, we have $Sp_{\mathcal{A}}(b) \subseteq Sp_{\mathcal{B}}(b)$ for all $b \in \mathcal{B}$:

If $\lambda \notin Sp_{\mathcal{B}}(b)$, then $(b - \lambda e)^{-1}$ exists in $\mathcal{B} \subseteq \mathcal{A}$. Hence $\lambda \notin Sp_{\mathcal{A}}(b)$.

We now introduce a class of subalgebras $\mathcal{B}$ of $\mathcal{A}$ where the spectrum is the same no matter whether it is calculated in $\mathcal{A}$ or in $\mathcal{B}$.

3.36 Definition. A unital subalgebra $\mathcal{B}$ of $\mathcal{A}$ is called a *spectral subalgebra* if $b \in \mathcal{B}$ and $b^{-1} \in \mathcal{A}$ implies $b^{-1} \in \mathcal{B}$.

3.37 Lemma. Let $\mathcal{B}$ be a unital subalgebra of $\mathcal{A}$.

Then $\mathcal{B}$ is a spectral subalgebra of $\mathcal{A}$ if and only if $Sp_{\mathcal{A}}(b) = Sp_{\mathcal{B}}(b)$ for all $b \in \mathcal{B}$.

Proof. ($\Rightarrow$) Let $b \in \mathcal{B}$. If $\lambda \notin Sp_{\mathcal{A}}(b)$, then $(b - \lambda e)^{-1}$ exists in $\mathcal{A}$. Since $\mathcal{B}$ is a spectral subalgebra of $\mathcal{A}$, we have $(b - \lambda e)^{-1} \in \mathcal{B}$ and hence $\lambda \notin Sp_{\mathcal{B}}(b)$. Therefore $Sp_{\mathcal{B}}(b) \subseteq Sp_{\mathcal{A}}(b)$ for all $b \in \mathcal{B}$.

In combination with the above observation for general subalgebras we obtain $Sp_{\mathcal{A}}(b) = Sp_{\mathcal{B}}(b)$ for all $b \in \mathcal{B}$.

($\Leftarrow$) Let $b \in \mathcal{B}$ with $b^{-1} \in \mathcal{A}$. Then we have $0 \notin Sp_{\mathcal{A}}(b) = Sp_{\mathcal{B}}(b)$. Hence $b^{-1} \in \mathcal{B}$ and so $\mathcal{B}$ is a spectral subalgebra of $\mathcal{A}$. □

We can give a simple criterion for a normed algebra $\mathcal{A}$ to be a spectral subalgebra of its completion [19].

3.38 Proposition. Let $\|\cdot\|$ be an algebra norm on $\mathcal{A}$ and let $\overline{\mathcal{A}}$ denote the completion of $\mathcal{A}$ with respect to $\|\cdot\|$.

The norm $\|\cdot\|$ is a spectral norm on $\mathcal{A}$ if and only if $\mathcal{A}$ is a spectral subalgebra of $\overline{\mathcal{A}}$.

Proof. ($\Rightarrow$): Suppose that $a \in \mathcal{A}$ has an inverse $b \in \overline{\mathcal{A}}$. Choose a sequence $\{b_n\}_{n \in \mathbb{N}}$ in $\mathcal{A}$ converging to b . Then $ab_n \rightarrow ab = e$ and $b_na \rightarrow ba = e$. Using the assumption that $\|\cdot\|$ is a spectral norm on $\mathcal{A}$ we have

$$r_{\mathcal{A}}(e - ab_n) \leq \|e - ab_n\| < 1 \text{ and } r_{\mathcal{A}}(e - b_na) \leq \|e - b_na\| < 1$$

for sufficiently large $n \in \mathbb{N}$.

Hence ab_n and b_na are eventually invertible in $\mathcal{A}$. But this implies that a is also invertible in $\mathcal{A}$.

($\Leftarrow$): Recall that a complete norm always dominates the spectral radius. In view of Lemma 3.37 we may thus infer that, for all $a \in \mathcal{A}$,

$$r_{\mathcal{A}}(a) = r_{\overline{\mathcal{A}}}(a) \leq \|a\|.$$

□

Characterizations of symmetry

Now we are ready for the characterization of symmetry in terms of radicals and the enveloping C^* -algebra. In the proof we will follow [9].

3.39 Theorem. *For a unital Banach $*$ -algebra $\mathcal{A}$ with continuous involution the following condition is equivalent to symmetry of $\mathcal{A}$:*

$Rad(\mathcal{A}) = R^(\mathcal{A})$ and $\mathcal{A}/R^*(\mathcal{A})$ is a spectral $*$ -subalgebra of its enveloping C^* -algebra $C^*(\mathcal{A})$.*

Proof. Let $\mathcal{A}$ be symmetric.

Then, by Theorem 3.27, the Gelfand-Naimark seminorm γ is a spectral seminorm on $\mathcal{A}$. Hence we have, for $b \in R^*(\mathcal{A})$,

$$r(ab) \leq \gamma(ab) \leq \gamma(a)\gamma(b) = 0.$$

Proposition 3.30 (2) implies that $b \in Rad(\mathcal{A})$ and thus $R^*(\mathcal{A}) \subseteq Rad(\mathcal{A})$.

In the light of Lemma 3.34 this gives $Rad(\mathcal{A}) = R^*(\mathcal{A})$.

To show the second assertion we first observe that the norm $\|\cdot\|$ induced by γ is a spectral norm on $\mathcal{A}/R^*(\mathcal{A})$. Indeed equality of Jacobson radical and $*$ -radical combined with Proposition 3.30 (3) gives

$$r_{\mathcal{A}/R^*(\mathcal{A})}(a') = r_{\mathcal{A}}(a) \leq \gamma(a) = \|a'\|$$

for all $a' \in \mathcal{A}/R^*(\mathcal{A})$. Now we apply Proposition 3.38 to deduce that $\mathcal{A}/R^*(\mathcal{A})$ is a spectral $*$ -subalgebra of $C^*(\mathcal{A})$.

Conversely, assume that $Rad(\mathcal{A}) = R^*(\mathcal{A})$ and that $\mathcal{A}/R^*(\mathcal{A})$ is a spectral $*$ -subalgebra of $C^*(\mathcal{A})$.

Using Lemma 3.37 and again Proposition 3.30 (3) we have

$$Sp_{\mathcal{A}}(a) = Sp_{\mathcal{A}/R^*(\mathcal{A})}(a') = Sp_{C^*(\mathcal{A})}(a')$$

for all $a \in \mathcal{A}$.

Since C^* -algebras are symmetric by Theorem 2.23, we in particular obtain

$$Sp_{\mathcal{A}}(a^*a) = Sp_{C^*(\mathcal{A})}((a^*a)') = Sp_{C^*(\mathcal{A})}((a')^*(a')) \subseteq \mathbb{R}_+$$

for all $a \in \mathcal{A}$. Hence $\mathcal{A}$ is symmetric. □

Chapter 4

Characterizations involving Unitary Elements

Throughout this chapter $\mathcal{A}$ will denote a unital Banach $*$ -algebra over $\mathbb{C}$ with continuous involution.

4.1 A characterization concerning the spectrum of unitary elements

In this short section we present the requirements on the spectra of unitary elements in a Banach $*$ -algebra $\mathcal{A}$ for $\mathcal{A}$ to be symmetric [28].

4.1 Definition. An element u in $\mathcal{A}$ is called *unitary* if $u^*u = uu^* = e$. The set of all unitary elements in $\mathcal{A}$ is denoted by $U(\mathcal{A})$.

4.2 Theorem. *A unital Banach $*$ -algebra $\mathcal{A}$ with continuous involution is symmetric if and only if the spectrum of every unitary element in $\mathcal{A}$ is a subset of the unit circle.*

Proof. Assume that $\mathcal{A}$ is symmetric and let u be a unitary element in $\mathcal{A}$. Then $\rho(u) = r(u^*u)^{\frac{1}{2}} = r(e)^{\frac{1}{2}} = 1$ and, by Theorem 2.19, $r(u) = \rho(u) = 1$. Hence the spectrum of u is included in the unit disc. Since also $u^{-1} = u^*$ is unitary, it follows from the Spectral Mapping Theorem 2.3 that $Sp(u)^{-1} = Sp(u^{-1})$ is included in the unit disc as well. Thus the spectrum of u is a subset of the unit circle.

Conversely, let h be a hermitian element in $\mathcal{A}$ and let $\mu > r(h)$. Define $u := (h - i\mu e)(h + i\mu e)^{-1}$. Then u is a unitary element in $\mathcal{A}$, because

$$u^* = ((h + i\mu e)^{-1})^*(h - i\mu e)^* = (h - i\mu e)^{-1}(h + i\mu e) = (h + i\mu e)(h - i\mu e)^{-1}$$

and hence $u^*u = uu^* = e$.

Now define $r(z) := (z - i\mu)(z + i\mu)^{-1}$, then $Sp(u) = r(Sp(h))$ by the Spectral Mapping Theorem 2.3. Since the spectrum of u is included in the unit circle, the spectrum of h must be real. Hence $\mathcal{A}$ is symmetric. $\square$

4.2 Characterizations in terms of the unitary seminorm

In this section we will introduce the unitary seminorm v and give several characterizations of symmetry involving v .

Definitions and Preparations

First we note that every element $a \in \mathcal{A}$ can be written as a linear combination of unitary elements.

It suffices to consider hermitian elements h in $\mathcal{A}$ with $r(h) < 1$. Then also $r(h^2) = r(h)^2 < 1$ and thus $Sp(e - h^2) \subseteq \{z \in \mathbb{C} : \Re z > 0\}$. According to Corollary 2.4 there exists some hermitian element $k \in \mathcal{A}$ such that $hk = kh$ and $k^2 = e - h^2$. Define

$$u := h + ik. \tag{4.1}$$

Since $u^*u = uu^* = h^2 + k^2 = e$, the element u is unitary and $h = \frac{1}{2}u + \frac{1}{2}u^*$.

4.3 Definition. The *unitary seminorm* v is defined by
$$v(a) := \inf \left\{ \sum_{k=1}^n |\lambda_k| : a = \sum_{k=1}^n \lambda_k u_k, n \in \mathbb{N}, \lambda_k \in \mathbb{C}, u_k \in U(\mathcal{A}) \right\}.$$

By the above remarks v is well-defined and finite for all $a \in \mathcal{A}$. Clearly v is subadditive and, since $U(\mathcal{A})$ is a group, v is also submultiplicative. Thus v is an algebra seminorm on $\mathcal{A}$.

Remark. Suppose $a = \sum_{k=1}^n \lambda_k u_k$ with $\lambda_k \in \mathbb{C}$ and $u_k \in U(\mathcal{A})$. We may consider the polar form of the coefficients and write

$$a = \sum_{k=1}^n |\lambda_k| \exp(i\theta_k) u_k.$$

Note that $\exp(i\theta_k) u_k \in U(\mathcal{A})$. Hence it suffices to consider linear combinations of unitary elements with *positive coefficients* when computing $v(a)$, that is,

$$v(a) = \inf \left\{ \sum_{k=1}^n \lambda_k : a = \sum_{k=1}^n \lambda_k u_k, n \in \mathbb{N}, \lambda_k > 0, u_k \in U(\mathcal{A}) \right\}. \quad (4.2)$$

Remark. In view of formula (4.2) the unitary seminorm v is just the Minkowski functional of the convex hull of the unitaries.

4.4 Lemma. *For all hermitian elements $h \in \mathcal{A}$ we have $v(h) \leq r(h)$.*

Proof. We first assume that h is a hermitian element in $\mathcal{A}$ satisfying $r(h) < 1$. Again we write $h = \frac{1}{2}u + \frac{1}{2}u^*$, where u is a unitary element defined as in formula (4.1). This implies $v(h) \leq 1$.

Now, for arbitrary hermitian $h \in \mathcal{A}$ and $\epsilon > 0$, set $h_\epsilon = (r(h) + \epsilon)^{-1}h$. Then $r(h_\epsilon) < 1$, so that we have $v(h_\epsilon) \leq 1$ or $v(h) \leq r(h) + \epsilon$. Since ϵ was arbitrary, we obtain $v(h) \leq r(h)$. □

Before presenting the next characterizations we need to establish some preparatory results including a generalization of the Russo-Dye Theorem to symmetric Banach *-algebras (Proposition 4.6) due to L. A. Harris [10].

We will mainly follow the exposition in [23].

4.5 Lemma. *Let $\mathcal{A}$ be symmetric. Let $a \in \mathcal{A}$ with $\rho(a) < 1$ and define $F(\lambda) := (e - aa^*)^{-\frac{1}{2}}(\lambda e + a)(e + \lambda a^*)^{-1}(e - a^*a)^{\frac{1}{2}}$.*

(1) F is defined and holomorphic in a neighbourhood of the closed unit disc in $\mathbb{C}$.

(2) $F(\lambda) \in U(\mathcal{A})$ for all $\lambda \in \mathbb{C}$, $|\lambda| = 1$.

(3) For each $\epsilon > 0$ there exists some $N \in \mathbb{N}$ such that $\|a - \frac{1}{n} \sum_{k=1}^n F(\exp(k \frac{2\pi i}{n}))\| < \epsilon$ for all $n \geq N$.

This means that an element $a \in \mathcal{A}$ with $\rho(a) < 1$ is in the closed convex hull of the set $U(\mathcal{A})$ of unitary elements in $\mathcal{A}$.

Proof. (1) Since $\mathcal{A}$ is symmetric, we have $r(a^*) = r(a) \leq \rho(a) < 1$ by Lemma 2.14. Hence $(e + \lambda a^*)^{-1}$ exists in a neighbourhood of the closed unit disc and, by holomorphy of the resolvent, we can infer that $F(\lambda)$ is defined and holomorphic in a neighbourhood of the closed unit disc.

(2) Now consider $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. First we establish two elementary identities:

We write

$$\lambda e + a = (e + \lambda a^*)a + \lambda(e - a^*a)$$

to obtain that

$$(e + \lambda a^*)^{-1}(\lambda e + a) = a + \lambda(e + \lambda a^*)^{-1}(e - a^*a). \quad (4.3)$$

Similarly, from

$$\lambda e + a = a(e + \lambda a^*) + \lambda(e - aa^*)$$

we deduce

$$(\lambda e + a)(e + \lambda a^*)^{-1} = a + \lambda(e - aa^*)(e + \lambda a^*)^{-1}. \quad (4.4)$$

Since $r(a) < 1$, $(\lambda e + a)^{-1}$ and hence also $F(\lambda)^{-1}$ exist. Using the identities (4.3) and (4.4) we compute

$$\begin{aligned} (F(\lambda)^{-1})^* &= (e - aa^*)^{\frac{1}{2}}(\bar{\lambda}e + a^*)^{-1}(e + \bar{\lambda}a)(e - a^*a)^{-\frac{1}{2}} \\ &= (e - aa^*)^{\frac{1}{2}}(e + \lambda a^*)^{-1}(\lambda e + a)(e - a^*a)^{-\frac{1}{2}} \\ &= (e - aa^*)^{-\frac{1}{2}}b(e - a^*a)^{\frac{1}{2}}, \end{aligned}$$

where

$$\begin{aligned} b &= (e - aa^*)(e + \lambda a^*)^{-1}(\lambda e + a)(e - a^*a)^{-1} \\ &= (e - aa^*)[a + \lambda(e + \lambda a^*)^{-1}(e - a^*a)](e - a^*a)^{-1} \\ &= a + \lambda(e - aa^*)(e + \lambda a^*)^{-1} \\ &= (\lambda e + a)(e + \lambda a^*)^{-1}. \end{aligned}$$

This shows that $F(\lambda)$ is unitary.

(3) Next we observe that $F(0) = a$. Indeed, consideration of the power series expansion of the square root gives

$$a(e - a^*a)^{\frac{1}{2}} = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} a(-a^*a)^k = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} (-aa^*)^k a = (e - aa^*)^{\frac{1}{2}} a$$

and we can thus conclude that

$$F(0) = (e - aa^*)^{-\frac{1}{2}} a (e - a^*a)^{\frac{1}{2}} = a.$$

Since F is holomorphic in a neighbourhood of the closed unit disc, we apply Cauchy's integral formula to deduce that

$$a = F(0) = \frac{1}{2\pi} \int_0^{2\pi} F(\exp(it)) dt.$$

Now we approximate the integral by Riemann sums of the form

$$\frac{1}{n} \sum_{k=1}^n F(\exp(k \frac{2\pi i}{n}))$$

to obtain the desired result. □

4.6 Proposition. *Let $\mathcal{A}$ be symmetric. Then the closed unit ball of $\mathcal{A}$ with respect to the seminorm ρ is the closed convex hull of $U(\mathcal{A})$, that is, $\{a \in \mathcal{A} : \rho(a) \leq 1\} = \overline{co} U(\mathcal{A})$.*

Proof. Denote $U(\rho) = \{a \in \mathcal{A} : \rho(a) \leq 1\}$. Clearly $U(\mathcal{A}) \subseteq U(\rho)$. Since ρ is a seminorm on $\mathcal{A}$ (Theorem 2.19) and since $U(\rho)$ is closed by continuity of ρ (Corollary 2.20), we also have $\overline{co} U(\mathcal{A}) \subseteq U(\rho)$.

Conversely, suppose that $a \in U(\rho)$. For any $n \in \mathbb{N}$ we have $\rho((1 - \frac{1}{n})a) < 1$ and hence $(1 - \frac{1}{n})a \in \overline{co} U(\mathcal{A})$ by the preceding lemma. Since $\overline{co} U(\mathcal{A})$ is closed, it follows that also $a = \lim_{n \rightarrow \infty} (1 - \frac{1}{n})a \in \overline{co} U(\mathcal{A})$.

Thus $U(\rho) \subseteq \overline{co} U(\mathcal{A})$. □

4.7 Corollary. *Let $\mathcal{A}$ be symmetric and let $a \in \mathcal{A}$ with $\rho(a) \neq 0$. For each $\epsilon > 0$ there exist positive numbers $\lambda_1, \dots, \lambda_n$ and unitary elements $u_1, \dots, u_n$ such that $\sum_{k=1}^n \lambda_k = \rho(a)$ and $\|a - \sum_{k=1}^n \lambda_k u_k\| < \epsilon$.*

Proof. Define $b := \frac{a}{\rho(a)}$. Then $b \in \overline{co} U(\mathcal{A})$ by the previous corollary. Hence, for given $\epsilon > 0$, there exists some $w \in co U(\mathcal{A})$ such that $\|b - w\| < \frac{\epsilon}{\rho(a)}$. Therefore $\|a - \rho(a)w\| < \epsilon$. □

Characterizations of symmetry

We have now collected the necessary tools for a treatment of the following characterizations [23].

4.8 Theorem. *For a unital Banach $*$ -algebra $\mathcal{A}$ with continuous involution the following conditions are equivalent:*

- (1) $\mathcal{A}$ is symmetric.
- (2) $\rho(a) = v(a)$ for all $a \in \mathcal{A}$.
- (3) $r(h) \leq v(h)$ for all hermitian elements $h \in \mathcal{A}$.

Proof. (1) $\Rightarrow$ (2): First we establish the auxiliary inequality $v(a) \leq 2\rho(a)$ for all $a \in \mathcal{A}$. Let $a \in \mathcal{A}$ and write $a = h + ik$ with hermitian elements h and k . Then Lemma 4.4 combined with the fact that ρ is a spectral seminorm (Theorem 2.19) yields

$$v(a) \leq v(h) + v(k) \leq r(h) + r(k) \leq \rho(h) + \rho(k) \leq 2\rho(a).$$

Next we show $v(a) \leq \rho(a)$ for all $a \in \mathcal{A}$. Let $a \in \mathcal{A}$ and $\epsilon > 0$. By Proposition 2.11 (3) we have $\rho(a) \leq C\|a\|$ for some $C > 0$. Corollary 4.7 gives positive numbers $\lambda_1, \dots, \lambda_n$ and unitary elements $u_1, \dots, u_n$ such that $\sum_{k=1}^n \lambda_k = \rho(a)$ and

$$\rho(a - \sum_{k=1}^n \lambda_k u_k) \leq C\|a - \sum_{k=1}^n \lambda_k u_k\| < \epsilon.$$

Hence

$$\begin{aligned} v(a) &\leq v\left(\sum_{k=1}^n \lambda_k u_k\right) + v\left(a - \sum_{k=1}^n \lambda_k u_k\right) \\ &\leq \underbrace{v\left(\sum_{k=1}^n \lambda_k u_k\right)}_{\leq \sum_{k=1}^n \lambda_k = \rho(a)} + 2\rho\left(a - \sum_{k=1}^n \lambda_k u_k\right) < \rho(a) + 2\epsilon. \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, this proves $v(a) \leq \rho(a)$.

To complete the proof, consider any decomposition $a = \sum_{k=1}^n \lambda_k u_k$ with positive numbers $\lambda_1, \dots, \lambda_n$ and unitary elements $u_1, \dots, u_n$. Since ρ is subadditive by Theorem 2.19, it follows that

$$\rho(a) \leq \sum_{k=1}^n \lambda_k \rho(u_k) = \sum_{k=1}^n \lambda_k \leq v(a).$$

Therefore $\rho(a) = v(a)$.

(2) $\Rightarrow$ (3): clear

(3) $\Rightarrow$ (1): Assume on the contrary that $\mathcal{A}$ is not symmetric. Then, by Theorem 2.18, there exists some hermitian element $h \in \mathcal{A}$ with $i \in Sp(h)$. Set $b := \exp(ih)$, then $b \in U(\mathcal{A})$. In virtue of the Spectral Mapping Theorem 2.3 we have $\exp(-1) \in Sp(b)$ and further

$$\exp(-1) + \exp(1) \in Sp(b + b^{-1}) = Sp(b + b^*).$$

Hence

$$r(b + b^*) > 2. \tag{4.5}$$

Using assumption (3) on the other hand yields

$$r(b + b^*) \leq v(b + b^*) \leq 2v(b) \leq 2,$$

which is a contradiction to (4.5). Therefore $\mathcal{A}$ must be symmetric.

□

The proof of Theorem 1.1 is now complete.

Chapter 5

Treatment of non-unital Banach *-Algebras

There are many important examples of Banach *-algebras without identity element. In this chapter we extend our discussion on symmetry to non-unital Banach *-algebras and explain the necessary modifications.

The treatment of non-unital Banach *-algebras is a bit more technical, but does not require any substantial new ideas. In the standard references discussion is usually for non-unital Banach *-algebras from the beginning. We prefer a separated discussion for reasons of exposition to separate essential ideas from technical overload.

One way to deal with the absence of a unit element is to adjoin one. More precisely, we embed the non-unital Banach *-algebra into a larger Banach *-algebra which contains an identity element.

5.1 Definition. Let $\mathcal{A}$ be a Banach *-algebra over $\mathbb{C}$ without identity. The *unitization* of $\mathcal{A}$ is the linear space $\tilde{\mathcal{A}} := \mathcal{A} \times \mathbb{C}$ with addition, scalar multiplication and product defined by

$$(a, \lambda) + (b, \mu) := (a + b, \lambda + \mu),$$

$$\mu(a, \lambda) := (\mu a, \mu \lambda),$$

$$(a, \lambda)(b, \mu) := (ab + \lambda b + \mu a, \mu \lambda).$$

The norm is defined by

$$\|(a, \lambda)\| := \|a\| + |\lambda|,$$

and the involution is

$$(a, \lambda)^* := (a^*, \bar{\lambda})$$

for all $a, b \in \mathcal{A}$ and $\lambda, \mu \in \mathbb{C}$.

With these definitions $\tilde{\mathcal{A}}$ becomes a Banach $*$ -algebra with unit $e = (0, 1)$. The mapping $a \mapsto (a, 0)$ embeds $\mathcal{A}$ isometrically into $\tilde{\mathcal{A}}$.

Notation. If $\mathcal{A}$ is unital, we set $\tilde{\mathcal{A}} := \mathcal{A}$.

Now one possibility to define symmetry of a non-unital Banach $*$ -algebra $\mathcal{A}$ is to declare that $\mathcal{A}$ is symmetric if its unitization $\tilde{\mathcal{A}}$ is symmetric. Then, according to our previous explanations, $\mathcal{A}$ is symmetric if and only if any of the equivalent conditions in Theorem 1.1 holds in $\tilde{\mathcal{A}}$.

However, it is also desirable to obtain an intrinsic formulation of symmetry that does not refer to the unitization. Thus our goal in this chapter is to formulate corresponding conditions entirely in terms of the non-unital Banach $*$ -algebra $\mathcal{A}$ rather than its unitization $\tilde{\mathcal{A}}$.

5.1 Spectrum, spectral radius and Ptak function

Our first task in this section is to extend the notion of spectrum and spectral radius to a non-unital Banach $*$ -algebra $\mathcal{A}$. Again it is possible to switch to the unitization $\tilde{\mathcal{A}}$ and define the spectrum of an element $a \in \mathcal{A}$ as the spectrum of its image $(a, 0)$ in $\tilde{\mathcal{A}}$. Seeking a logically equivalent definition of spectrum which is intrinsic to the given Banach $*$ -algebra $\mathcal{A}$ leads to the concept of quasi-product and quasi-inversion which takes the place of invertibility in non-unital Banach $*$ -algebras.

This concept was first used by Perlis [21] and can be found in the standard books on Banach $*$ -algebras such as [2] and [26].

5.2 Definition. The *quasi-product* $a \circ b$ of two elements a and b in a Banach $*$ -algebra $\mathcal{A}$ is defined by $a \circ b := a + b - ab$.

Note that the quasi-product is an associative operation with identity element $e = 0$.

However, the quasi-product does not behave well with respect to the linear structure. Instead of the distributive law we have the following relations.

5.3 Lemma. *The quasi-product in a Banach *-algebra $\mathcal{A}$ satisfies, for all $a, b, c \in \mathcal{A}$,*

$$\begin{aligned}(a + b) \circ c &= a \circ c + b \circ c - c, \\ c \circ (a + b) &= c \circ a + c \circ b - c.\end{aligned}$$

Proof. Let $a, b, c \in \mathcal{A}$. Then

$$\begin{aligned}(a + b) \circ c &= a + b + c - (a + b)c \\ &= a + b + c - ac - bc \\ &= (a + c - ac) + (b + c - bc) - c \\ &= a \circ c + b \circ c - c.\end{aligned}$$

The second equation follows similarly. □

5.4 Definition. Let a be an element in a Banach *-algebra $\mathcal{A}$.

- (i) a is *left quasi-invertible* if there exists some $b \in \mathcal{A}$ such that $b \circ a = 0$. Then b is called a *left quasi-inverse* of a .
- (ii) a is *right quasi-invertible* if there exists some $b \in \mathcal{A}$ such that $a \circ b = 0$. Then b is called a *right quasi-inverse* of a .
- (iii) a is *quasi-invertible* if it is both left and right quasi-invertible. The the unique element $b \in \mathcal{A}$ satisfying $a \circ b = b \circ a = 0$ is called the *quasi-inverse* of a and is denoted by a^q .

The following proposition relates quasi-inversion in a non-unital Banach *-algebra $\mathcal{A}$ to inversion in the unitization $\tilde{\mathcal{A}}$ of $\mathcal{A}$.

5.5 Proposition. *An element a in a non-unital Banach *-algebra $\mathcal{A}$ has the quasi-inverse b in $\mathcal{A}$ if and only if $(0, 1) - (a, 0)$ has the inverse $(0, 1) - (b, 0)$ in $\tilde{\mathcal{A}}$.*

Proof. Consider the identity

$$(-a, 1)(-b, 1) = (-a - b + ab, 1) = (-a \circ b, 1).$$

Hence $a \circ b = b \circ a = 0$ if and only if $(-a, 1)$ is invertible in $\tilde{\mathcal{A}}$. □

If $\mathcal{A}$ is unital, an element $a \in \mathcal{A}$ has the quasi-inverse $b \in \mathcal{A}$ if and only if $e - a$ has the inverse $e - b$. This follows from $(e - a)(e - b) = e - (a \circ b)$. For $\lambda \neq 0$, the relation $e - \lambda^{-1}a = \lambda^{-1}(\lambda e - a)$ further shows that λ will be in the spectrum of a if and only if $\lambda^{-1}a$ is not quasi-invertible.

This observation suggests an appropriate definition of spectrum in non-unital Banach $*$ -algebras.

5.6 Definition. Let $\mathcal{A}$ be a non-unital Banach $*$ -algebra. The *spectrum* of an element $a \in \mathcal{A}$ is the set

$$Sp_{\mathcal{A}}(a) = \{\lambda \in \mathbb{C} \setminus \{0\} : \lambda^{-1}a \text{ is not quasi-invertible in } \mathcal{A}\} \cup \{0\}.$$

Spectral radius and Ptak function of an element $a \in \mathcal{A}$ can be defined as before by

$$\begin{aligned} r_{\mathcal{A}}(a) &= \max\{|\lambda| : \lambda \in Sp_{\mathcal{A}}(a)\}, \\ \rho_{\mathcal{A}}(a) &= r_{\mathcal{A}}(a^*a)^{\frac{1}{2}}. \end{aligned}$$

5.7 Proposition. Let $\mathcal{A}$ be a non-unital Banach $*$ -algebra. Then $Sp_{\mathcal{A}}(a) = Sp_{\tilde{\mathcal{A}}}((a, 0))$ for all $a \in \mathcal{A}$.

Proof. Let $a \in \mathcal{A}$ and $\lambda \in \mathbb{C} \setminus \{0\}$. Using Proposition 5.5 we deduce

$$\begin{aligned} \lambda \notin Sp_{\tilde{\mathcal{A}}}((a, 0)) &\Leftrightarrow \lambda(0, 1) - (a, 0) \text{ is invertible in } \tilde{\mathcal{A}} \\ &\Leftrightarrow (0, 1) - (\lambda^{-1}a, 0) \text{ is invertible in } \tilde{\mathcal{A}} \\ &\Leftrightarrow \lambda^{-1}a \text{ is quasi-invertible in } \mathcal{A} \\ &\Leftrightarrow \lambda \notin Sp_{\mathcal{A}}(a). \end{aligned}$$

Since $(a, 0)$ is not invertible in $\tilde{\mathcal{A}}$, we further have $0 \in Sp_{\tilde{\mathcal{A}}}((a, 0))$ and thus $Sp_{\mathcal{A}}(a) = Sp_{\tilde{\mathcal{A}}}((a, 0))$. □

5.8 Corollary. Let $\mathcal{A}$ be a non-unital Banach $*$ -algebra. Then $r_{\mathcal{A}}(a) = r_{\tilde{\mathcal{A}}}((a, 0))$ and $\rho_{\mathcal{A}}(a) = \rho_{\tilde{\mathcal{A}}}((a, 0))$ for all $a \in \mathcal{A}$.

It is also worth noting that in non-unital Banach $*$ -algebras the Spectral Mapping Theorem still holds for polynomials where the constant coefficient is zero. Then $a \in \mathcal{A}$ implies $p(a) \in \mathcal{A}$ and we can infer that

$$\begin{aligned} Sp_{\mathcal{A}}(p(a)) &= Sp_{\tilde{\mathcal{A}}}((p(a), 0)) = Sp_{\tilde{\mathcal{A}}}(p((a, 0))) \\ &= p(Sp_{\tilde{\mathcal{A}}}((a, 0))) = p(Sp_{\mathcal{A}}(a)). \end{aligned} \tag{5.1}$$

Now, with the notion of spectrum established, the definitions of symmetry, hermiticity and complete symmetry by spectral properties of certain elements as in Definition 2.12 and Definition 2.13 carry over without change to non-unital Banach $*$ -algebras.

For example, a non-unital Banach $*$ -algebra is *hermitian* if the spectrum of every hermitian element is real.

This intrinsic description is equivalent to the corresponding property in the unitization.

5.9 Proposition. *A non-unital Banach $*$ -algebra $\mathcal{A}$ is hermitian if and only if its unitization $\tilde{\mathcal{A}}$ is hermitian.*

Proof. Suppose that $\tilde{\mathcal{A}}$ is hermitian and let h be a hermitian element in $\mathcal{A}$. Then, by Proposition 5.7, we have

$$Sp_{\mathcal{A}}(h) = Sp_{\tilde{\mathcal{A}}}((h, 0)) \subseteq \mathbb{R}.$$

Conversely, assume that $\mathcal{A}$ is hermitian. If (a, λ) is a hermitian element in $\tilde{\mathcal{A}}$, then a is a hermitian element in $\mathcal{A}$ and λ is real. Using the Spectral Mapping Theorem 2.3 for $\tilde{\mathcal{A}}$ and again Proposition 5.7 we obtain

$$Sp_{\tilde{\mathcal{A}}}((a, \lambda)) = Sp_{\tilde{\mathcal{A}}}((a, 0) + \lambda(0, 1)) = Sp_{\tilde{\mathcal{A}}}((a, 0)) + \lambda = Sp_{\mathcal{A}}(a) + \lambda \subseteq \mathbb{R}.$$

Hence $\tilde{\mathcal{A}}$ is hermitian. □

Now we extend the spectral characterizations of Theorem 2.18 to non-unital Banach $*$ -algebras.

5.10 Theorem. *Let $\mathcal{A}$ be a non-unital Banach $*$ -algebra with continuous involution. Then the following conditions are equivalent:*

- (1) $\mathcal{A}$ is completely symmetric.
- (2) $\mathcal{A}$ is symmetric.
- (3) $-1 \notin Sp_{\mathcal{A}}(a^*a)$ for all $a \in \mathcal{A}$.
- (4) $\mathcal{A}$ is hermitian.
- (5) $i \notin Sp_{\mathcal{A}}(h)$ for all hermitian elements $h \in \mathcal{A}$.

Proof. (4) $\Rightarrow$ (1): Suppose that $\mathcal{A}$ is hermitian. By Proposition 5.9 the unitization $\tilde{\mathcal{A}}$ is also hermitian. Theorem 2.18 further implies that $\tilde{\mathcal{A}}$ is completely symmetric. If now $p = \sum_{k=1}^n a_k^* a_k$ for some $a_k \in \mathcal{A}$, $n \in \mathbb{N}$, then

$$Sp_{\mathcal{A}}(p) = Sp_{\tilde{\mathcal{A}}}((p, 0)) = Sp_{\tilde{\mathcal{A}}}(\sum_{k=1}^n (a_k, 0)^*(a_k, 0)) \subseteq \mathbb{R}_+.$$

Therefore $\mathcal{A}$ is completely symmetric.

The implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (5): In virtue of formula (5.1), which extends the Spectral Mapping Theorem for polynomials to non-unital Banach $*$ -algebras, we can carry out exactly the same proof as in Theorem 2.18, (3) $\Rightarrow$ (5).

(5) $\Rightarrow$ (4): We modify the proof of Theorem 2.18, (5) $\Rightarrow$ (4). We consider the slightly more complicated polynomial

$$p(\lambda) = \beta^{-1}(\alpha^2 + \beta^2)^{-1}(\alpha\lambda^2 + (\beta^2 - \alpha^2)\lambda)$$

for $\lambda \in \mathbb{C}$ to ensure that $k = p(h)$ is still an element in $\mathcal{A}$. Then $p(\alpha + i\beta) = i$ and we can repeat the argument on p. 13. □

Remark. The preceding Theorem combined with Proposition 5.9 and Theorem 2.18 further yields that all these to symmetry equivalent conditions in $\mathcal{A}$ are also equivalent to the respective conditions in $\tilde{\mathcal{A}}$. In particular, $\mathcal{A}$ is symmetric if and only if its unitization $\tilde{\mathcal{A}}$ is symmetric.

Next we observe that also the characterizations by properties of the spectral radius carry over to non-unital Banach $*$ -algebras.

5.11 Theorem. *Let $\mathcal{A}$ be a non-unital Banach $*$ -algebra with continuous involution. Then the following conditions are equivalent:*

- (1) $\mathcal{A}$ is symmetric.
- (2) $r_{\mathcal{A}}(a) \leq \rho_{\mathcal{A}}(a)$ for all $a \in \mathcal{A}$.
- (3) $r_{\mathcal{A}}(a) = \rho_{\mathcal{A}}(a)$ for all normal elements $a \in \mathcal{A}$.
- (4) $r_{\mathcal{A}}(a) \leq \|a^*a\|_{\mathcal{A}}^{\frac{1}{2}}$ for all normal elements $a \in \mathcal{A}$.
- (5) $r_{\mathcal{A}}(\frac{1}{2}(a^* + a)) \leq \rho_{\mathcal{A}}(a)$ for all $a \in \mathcal{A}$.
- (6) $\rho_{\mathcal{A}}(a + b) \leq \rho_{\mathcal{A}}(a) + \rho_{\mathcal{A}}(b)$ for all $a, b \in \mathcal{A}$.
- (7) $\rho_{\mathcal{A}}$ is an algebra seminorm on $\mathcal{A}$.

Proof. (1) $\Rightarrow$ (2): If $\mathcal{A}$ is symmetric, then also the unitization $\tilde{\mathcal{A}}$ is symmetric. Hence Theorem 2.19 applied to $\tilde{\mathcal{A}}$ gives

$$r_{\tilde{\mathcal{A}}}((a, \lambda)) \leq \rho_{\tilde{\mathcal{A}}}((a, \lambda))$$

for all elements (a, λ) in $\tilde{\mathcal{A}}$. By Corollary 5.8 we obtain

$$r_{\mathcal{A}}(a) = r_{\tilde{\mathcal{A}}}((a, 0)) \leq \rho_{\tilde{\mathcal{A}}}((a, 0)) = \rho_{\mathcal{A}}(a)$$

for all $a \in \mathcal{A}$.

(2) $\Rightarrow$ (3): By Corollary 5.8 and Proposition 2.11 (5) we always have

$$\rho_{\mathcal{A}}(a) = \rho_{\tilde{\mathcal{A}}}((a, 0)) \leq r_{\tilde{\mathcal{A}}}((a, 0)) = r_{\mathcal{A}}(a)$$

for all normal elements a in $\mathcal{A}$. Thus (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (4): For a normal element a in $\mathcal{A}$ we calculate

$$r_{\mathcal{A}}(a)^2 = \rho_{\mathcal{A}}(a)^2 = \rho_{\tilde{\mathcal{A}}}((a, 0))^2 \leq \|(a^*a, 0)\|_{\tilde{\mathcal{A}}} = \|a^*a\|_{\mathcal{A}}.$$

(4) $\Rightarrow$ (1): We can carry out a similar proof as in Theorem 2.19, (4) $\Rightarrow$ (1). We only need to redefine $b = (b, 0) = ((h, 0) + i\mu(0, 1))^n(\lambda^{-1}h, 0)$.

(1) $\Rightarrow$ (7): With $\mathcal{A}$ also the unitization $\tilde{\mathcal{A}}$ is symmetric. Hence, by Theorem 2.19, the Ptak function $\rho_{\tilde{\mathcal{A}}}$ is an algebra seminorm on $\tilde{\mathcal{A}}$. Now Corollary 5.8 implies that $\rho_{\mathcal{A}}$ is an algebra seminorm on $\mathcal{A}$.

(7) $\Rightarrow$ (6): clear

(6) $\Rightarrow$ (5) and (5) $\Rightarrow$ (3) follow exactly as in the proof of Theorem 2.19

□

5.2 Ideals, positive functionals and representations

In this section we will adapt the equivalent conditions on ideals, positive functionals, and representations to non-unital Banach *-algebras.

5.12 Definition. (i) A left ideal $\mathcal{I}$ in $\mathcal{A}$ is called a *modular left ideal* if there exists some $u \in \mathcal{A}$ such that $a - au \in \mathcal{I}$ for all $a \in \mathcal{A}$. The element u is called a *right modular unit* for $\mathcal{I}$.

(ii) A left ideal is called *maximal modular* if it is proper and modular and not properly included in any other proper modular left ideal.

Similar definitions apply to right ideals and ideals.

Remark. Note that in a unital Banach *-algebra every left ideal is modular.

5.13 Proposition. (1) A left ideal that includes a modular left ideal $\mathcal{I}$ is also modular.

(2) If u is a right modular unit for a proper left ideal $\mathcal{I}$, then $u \notin \mathcal{I}$.

(3) Every maximal modular left ideal is a maximal left ideal.

(4) Every proper modular left ideal $\mathcal{I}$ is included in a maximal (modular) left ideal.

Proof. (1) The right modular unit for $\mathcal{I}$ is also a right modular unit for the larger left ideal.

(2) If the right modular unit u for $\mathcal{I}$ belongs to $\mathcal{I}$, then any $a \in \mathcal{A}$ satisfies $a = (a - au) + au \in \mathcal{I}$, a contradiction to the properness of $\mathcal{I}$.

(3) This is an immediate consequence of (1).

(4) Let $\mathcal{I}$ be a proper modular left ideal with right modular unit u . Then $u \notin \mathcal{J}$ for any proper left ideal $\mathcal{J} \supseteq \mathcal{I}$. Therefore we can repeat the proof of Proposition 3.2 where the unit element e is replaced by u . □

5.14 Definition. A positive linear functional f on a Banach $*$ -algebra $\mathcal{A}$ is said to have *finite variation* if

$$|f(a)|^2 \leq Cf(a^*a)$$

for all $a \in \mathcal{A}$ and some $C > 0$. In this case we call the quantity

$$V(f) := \min\{C > 0 : |f(a)|^2 \leq Cf(a^*a) \text{ for all } a \in \mathcal{A}\}$$

the *variation* of f .

5.15 Proposition. Let $\mathcal{A}$ be a non-unital Banach $*$ -algebra. A positive linear functional f on $\mathcal{A}$ can be extended to a positive linear functional $\tilde{f}$ on the unitization $\tilde{\mathcal{A}}$ if and only if f is hermitian and has finite variation.

Proof. ($\Rightarrow$) Suppose that f admits an extension to a positive linear functional $\tilde{f}$ on $\tilde{\mathcal{A}}$. Then, by Proposition 3.4, f is hermitian and has finite variation $V(f) = \tilde{f}((0, 1))$.

($\Leftarrow$) Conversely, assume that f is hermitian and has finite variation. Define $\tilde{f}((a, \lambda)) := f(a) + C\lambda$ for some $C > V(f)$. Then $\tilde{f}$ is linear and coincides

with f on $\mathcal{A}$. The functional $\tilde{f}$ is also positive since

$$\begin{aligned}
\tilde{f}((a, \lambda)^*(a, \lambda)) &= \tilde{f}((a^*a + \lambda a^* + \bar{\lambda}a, |\lambda|^2)) \\
&= f(a^*a) + \lambda f(a^*) + \bar{\lambda}f(a) + C|\lambda|^2 \\
&= f(a^*a) + \overline{\lambda f(a)} + \bar{\lambda}f(a) + C|\lambda|^2 \\
&= f(a^*a) + 2\Re(\bar{\lambda}f(a)) + C|\lambda|^2 \\
&\geq f(a^*a) - 2|\lambda||f(a)| + C|\lambda|^2 \\
&\geq f(a^*a) - 2|\lambda|C^{\frac{1}{2}}f(a^*a)^{\frac{1}{2}} + C|\lambda|^2 \\
&= (f(a^*a)^{\frac{1}{2}} - C^{\frac{1}{2}}|\lambda|)^2 \geq 0.
\end{aligned}$$

□

5.16 Definition. Let $\mathcal{A}$ be a non-unital Banach $*$ -algebra.

A *state* on $\mathcal{A}$ is a positive linear functional f which is hermitian and has finite variation $V(f) = 1$.

The set of all states on $\mathcal{A}$ is denoted by $S(\mathcal{A})$. It is again a convex set and its extreme points shall be called *pure states*.

In the light of Proposition 5.15 the restrictions to $\mathcal{A}$ of the states on the unitization $\tilde{\mathcal{A}}$ are exactly the union of the set of states on $\mathcal{A}$ and the trivial functional $f_o \equiv 0$, that is,

$$S(\mathcal{A}) \cup \{f_o\} = \{\tilde{f}|_{\mathcal{A}} : \tilde{f} \in S(\tilde{\mathcal{A}})\}.$$

Hence the set $S(\mathcal{A}) \cup \{f_o\}$ is a weak*-compact convex set.

Note that Proposition 3.26 extends to non-unital Banach $*$ -algebras in the following way:

5.17 Proposition. *An algebra seminorm σ on a non-unital Banach $*$ -algebra $\mathcal{A}$ is spectral if and only if every $a \in \mathcal{A}$ with $\sigma(a) < 1$ is quasi-invertible in $\mathcal{A}$.*

Proof. We can carry out the same proof as in Proposition 3.26, with the only difference that $\lambda \notin Sp_{\mathcal{A}}(a)$ now corresponds to $\lambda^{-1}a$ being quasi-invertible in $\mathcal{A}$.

□

In the next theorem we rephrase the characterizations of symmetry regarding ideals, positive functionals and representations in the setting of non-unital Banach $*$ -algebras.

5.18 Theorem. *Let $\mathcal{A}$ be a non-unital Banach $*$ -algebra with continuous involution. Then the following conditions are equivalent:*

- (1) $\mathcal{A}$ is symmetric.
- (2) If $T : \mathcal{A} \rightarrow \mathcal{L}(X)$ is an irreducible representation, then there is an inner product for X relative to which T is a pre- $*$ -representation.
- (3) For every maximal modular left ideal $\mathcal{I}$ in $\mathcal{A}$ there exists a state f on $\mathcal{A}$ such that $\mathcal{I} = \{a \in \mathcal{A} : f(a^*a) = 0\}$.
- (4) For every maximal modular left ideal $\mathcal{I}$ in $\mathcal{A}$ there exists a pure state g on $\mathcal{A}$ such that $\mathcal{I} = \{a \in \mathcal{A} : g(a^*a) = 0\}$.
- (5) Every maximal modular left ideal in $\mathcal{A}$ is closed with respect to γ .
- (6) γ is a spectral seminorm on $\mathcal{A}$.
- (7) There exists a constant C with $r(a^*a) \leq C\gamma(a^*a)$ for all $a \in \mathcal{A}$.
- (8) Raikov's criterion: $\gamma(a) = \rho(a)$ for all $a \in \mathcal{A}$.
- (9) $Sp_{\mathcal{A}}(a) \subseteq \{g(a) : g \text{ is a pure state on } \mathcal{A}\} \cup \{0\}$ for all normal elements $a \in \mathcal{A}$.

Proof. (1) $\Rightarrow$ (2): Suppose that T is an irreducible representation of $\mathcal{A}$ on a linear space $\mathcal{X}$. Define $\tilde{T}_{(a,\lambda)} := T_a + \lambda I$ for all $(a, \lambda) \in \tilde{\mathcal{A}}$, then $\tilde{T}$ is a representation of $\tilde{\mathcal{A}}$ on $\mathcal{X}$. Moreover $\tilde{T}$ is again irreducible since any $\tilde{T}$ -invariant subspace of $\mathcal{X}$ is in particular a T -invariant subspace and hence equals $\{0\}$ or $\mathcal{X}$.

Now recall that $\mathcal{A}$ is symmetric if and only if $\tilde{\mathcal{A}}$ is symmetric. Hence Theorem 3.20 applied to $\tilde{\mathcal{A}}$ yields the existence of an inner product $(\cdot, \cdot)$ on $\mathcal{X}$ relative to which $\tilde{T}$ is a pre- $*$ -representation. This means that $(\tilde{T}_{(a,\lambda)})^* = \tilde{T}_{(a,\lambda)}^*$ and $\|\tilde{T}_{(a,\lambda)}\| < \infty$ for all $(a, \lambda) \in \tilde{\mathcal{A}}$, where the adjoint operator and the norm are defined with respect to the inner product on $\mathcal{X}$. In particular, we have $T_a^* = T_a^*$ and $\|T_a\| < \infty$ for all $a \in \mathcal{A}$, so T is also a pre- $*$ -representation with respect to $(\cdot, \cdot)$.

(2) $\Rightarrow$ (3): Let $\mathcal{I}$ be a maximal modular left ideal in $\mathcal{A}$. Replace the unit element e by a right modular unit u for $\mathcal{I}$ in the proof of Theorem 3.20 when defining the linear functional f . Because $a - au \in \mathcal{I}$ we have $(au)' = a'$ for all $a \in \mathcal{A}$ and thus

$$f(a) := (L_a u', u') = ((au)', u') = (a', u').$$

We can proceed as in the proof of Theorem 3.20 to derive that f is a positive linear functional satisfying $\mathcal{I} = \{a \in \mathcal{A} : f(a^*a) = 0\}$. Further we deduce that, for all $a \in \mathcal{A}$,

$$|f(a)|^2 \leq \|L_a u'\|^2 \|u'\|^2 = (L_a u', L_a u') \|u'\|^2 = (L_{a^*a} u', u') \|u'\|^2 = \|u'\|^2 f(a^*a)$$

$$\text{and } f(a^*) = (L_{a^*}u', u') = (u', L_a u') = \overline{(L_a u', u')} = \overline{f(a)}.$$

Hence f is hermitian and has finite variation $V(f) \leq \|u'\|^2$.

Defining $\tilde{f} := V(f)^{-1}f$ gives the required state on $\mathcal{A}$.

(3) $\Rightarrow$ (4): We can essentially repeat the proof of Theorem 3.11, (2) $\Rightarrow$ (3). However, we have to consider $\mathcal{E}$ as a weak*-closed subset of the weak*-compact set $S(\mathcal{A}) \cup \{f_o\}$ to deduce that $\mathcal{E}$ is also weak*-compact and thus has some extreme point.

(4) $\Rightarrow$ (5): Note that a state f in a non-unital Banach *-algebra $\mathcal{A}$ satisfies $|f(a)| \leq f(a^*a)^{\frac{1}{2}}$ per definition. This implies that states are again continuous with respect to γ , and we can repeat the proof of Theorem 3.27, (1) $\Rightarrow$ (2).

(5) $\Rightarrow$ (6): Suppose on the contrary that γ is not spectral. By Proposition 5.17, there exists an element $u \in \mathcal{A}$ which satisfies $\gamma(u) < 1$, but is not quasi-invertible in $\mathcal{A}$. We can assume that u is not left quasi-invertible (otherwise consider u^*). This means that $a \circ u = a + u - au \neq 0$ for all $a \in \mathcal{A}$. Then $\mathcal{I} = \{a - au : a \in \mathcal{A}\}$ is a proper modular left ideal in $\mathcal{A}$ with u as a right modular unit. By Proposition 5.13, $\mathcal{I}$ is included in some maximal modular left ideal $\tilde{\mathcal{I}}$ which does not contain u . To derive a contradiction we can now proceed as in the proof of Theorem 3.27, (2) $\Rightarrow$ (3), and show that u belongs to the γ -closure of $\tilde{\mathcal{I}}$.

(6) $\Rightarrow$ (7): clear

(7) $\Rightarrow$ (8) and (8) $\Rightarrow$ (1) follow exactly as in the proof of Theorem 3.27, because the arguments do not use a unit e .

(1) $\Rightarrow$ (9): Let a be a normal element in $\mathcal{A}$. Since $\mathcal{A}$ is symmetric, also the unitization $\tilde{\mathcal{A}}$ is symmetric. By Theorem 3.11,

$$Sp_{\mathcal{A}}(a) = Sp_{\tilde{\mathcal{A}}}((a, 0)) \subseteq \{\tilde{g}((a, 0)) : \tilde{g} \text{ is a pure state on } \tilde{\mathcal{A}}\}.$$

Note that the restrictions to $\mathcal{A}$ of pure states on $\tilde{\mathcal{A}}$ result in pure states on $\mathcal{A}$ or the trivial functional $f_o \equiv 0$. Hence we may infer that

$$\begin{aligned} Sp_{\mathcal{A}}(a) &\subseteq \{g(a) : g = \tilde{g}|_{\mathcal{A}} \text{ for some pure state } \tilde{g} \text{ on } \tilde{\mathcal{A}}\} \\ &= \{g(a) : g \text{ is a pure state on } \mathcal{A}\} \cup \{0\}. \end{aligned}$$

(9) $\Rightarrow$ (1): Let a be an arbitrary element in $\mathcal{A}$. By assumption,

$$Sp(a^*a) \subseteq \{g(a^*a) : g \text{ is a pure state on } \mathcal{A}\} \cup \{0\} \subseteq \mathbb{R}_+.$$

Hence $\mathcal{A}$ is symmetric. $\square$

We are still missing the characterization of symmetry in terms of the Jacobson radical, the $*$ -radical and the enveloping C^* -algebra in non-unital Banach $*$ -algebras. The definitions of these concepts do not refer to a unit element and can be defined without change in non-unital Banach $*$ -algebras.

For the $*$ -radical $R^*(\mathcal{A})$ we again have $R^*(\mathcal{A}) = \{a \in \mathcal{A} : \gamma_{\mathcal{A}}(a) = 0\}$.

The alternative descriptions of the Jacobson radical $Rad(\mathcal{A})$ extend to non-unital Banach $*$ -algebras in the following way:

5.19 Proposition. *The following sets in $\mathcal{A}$ are equal:*

- (1) *The Jacobson radical $Rad(\mathcal{A})$ of $\mathcal{A}$.*
- (2) *$\{b \in \mathcal{A} : r(ab) = 0 \text{ for all } a \in \mathcal{A}\}$.*
- (3) *The largest ideal $\mathcal{I}$ in $\mathcal{A}$ satisfying, for all $a \in \mathcal{A}$,*

$$Sp_{\mathcal{A}/\mathcal{I}}(a + \mathcal{I}) \subseteq Sp_{\mathcal{A}}(a) \subseteq Sp_{\mathcal{A}/\mathcal{I}}(a + \mathcal{I}) \cup \{0\}.$$

The proof is similar to the unital case. For details consult [19].

Also $Rad(\mathcal{A}) \subseteq R^*(\mathcal{A})$ follows exactly as in the case of unital Banach $*$ -algebras.

In possibly non-unital Banach $*$ -algebras we can now define spectral subalgebras via quasi-invertible elements [19].

5.20 Definition. A subalgebra $\mathcal{B}$ of a Banach $*$ -algebra $\mathcal{A}$ is called a *spectral subalgebra* if $b \in \mathcal{B}$ and $b^q \in \mathcal{A}$ implies $b^q \in \mathcal{B}$.

Then we can rephrase Lemma 3.37 as follows:

5.21 Lemma. *Let $\mathcal{B}$ be a subalgebra of a Banach $*$ -algebra $\mathcal{A}$.*

Then $\mathcal{B}$ is a spectral subalgebra of $\mathcal{A}$ if and only if $Sp_{\mathcal{A}}(b) \cup \{0\} = Sp_{\mathcal{B}}(b) \cup \{0\}$ for all $b \in \mathcal{B}$.

The criterion for a normed algebra $\mathcal{A}$ to be a spectral subalgebra of its completion (Lemma 3.38) remains valid for non-unital Banach $*$ -algebras.

So we can finally observe that the characterization of symmetry in terms of radicals and the enveloping C^* -algebra still holds in non-unital Banach $*$ -algebras.

5.22 Theorem. *A non-unital Banach $*$ -algebra $\mathcal{A}$ with continuous involution is symmetric if and only if $\text{Rad}(\mathcal{A}) = R^*(\mathcal{A})$ and $\mathcal{A}/R^*(\mathcal{A})$ is a spectral $*$ -subalgebra of its enveloping C^* -algebra.*

Proof. $(\Rightarrow)$ By the above remarks, we can repeat the proof of Theorem 3.39.

$(\Leftarrow)$ The assumption combined with Lemma 5.21 in this case gives

$$Sp_{\mathcal{A}}(a) \subseteq Sp_{\mathcal{A}/R^*(\mathcal{A})}(a') \cup \{0\} = Sp_{C^*(\mathcal{A})}(a') \cup \{0\}$$

for all $a \in \mathcal{A}$. However, this relation suffices to deduce symmetry of $\mathcal{A}$ as in the proof of Theorem 3.39. □

5.3 (Quasi-) unitary elements

In non-unital Banach $*$ -algebras the notion of unitary elements is inapplicable. However, once again we can consider the respective concept for the quasi-product [17], [20].

5.23 Definition. Let $\mathcal{A}$ be a non-unital Banach $*$ -algebra.

An element $w \in \mathcal{A}$ is called *quasi-unitary* if $w \circ w^* = w^* \circ w = 0$.

The set of all quasi-unitary elements is denoted by $U_q(\mathcal{A})$.

By the definition of the quasi-product, an element $w \in \mathcal{A}$ is quasi-unitary if and only if $w^*w = ww^* = w + w^*$.

We first explore the relation between quasi-unitary elements in a non-unital Banach $*$ -algebra $\mathcal{A}$ and unitary elements in its unitization $\tilde{\mathcal{A}}$.

5.24 Lemma. $U(\tilde{\mathcal{A}}) = \{\zeta((0, 1) - (w, 0)) : \zeta \in \mathbb{C}, |\zeta| = 1, w \in U_q(\mathcal{A})\}$

Proof. Let w be a quasi-unitary element in $\mathcal{A}$. This means that w has the quasi-inverse w^* in $\mathcal{A}$. Then, by Proposition 5.5, the element $(0, 1) - (w, 0)$ has the inverse $(0, 1) - (w^*, 0) = ((0, 1) - (w, 0))^*$ in $\tilde{\mathcal{A}}$.

Thus $(0, 1) - (w, 0)$ is a unitary element in $\tilde{\mathcal{A}}$ and so is $\zeta((0, 1) - (w, 0))$ for any $\zeta \in \mathbb{C}, |\zeta| = 1$.

Conversely, let (u, ζ) be a unitary element in $\tilde{\mathcal{A}}$. Then

$$(u, \zeta)^*(u, \zeta) = (u^*u + \bar{\zeta}u + \zeta u^*, |\zeta|^2) = (0, 1) = (u, \zeta)(u, \zeta)^* = (uu^* + \bar{\zeta}u + \zeta u^*, |\zeta|^2).$$

Hence we must have $|\zeta| = 1$ and

$$0 = u^*u + \bar{\zeta}u + \zeta u^* = |\zeta|^2 u^*u - (-\bar{\zeta}u) - (-\zeta u^*) = -(-\bar{\zeta}u)^* \circ (-\bar{\zeta}u),$$

$$0 = uu^* + \bar{\zeta}u + \zeta u^* = |\zeta|^2 uu^* - (-\bar{\zeta}u) - (-\zeta u^*) = -(-\bar{\zeta}u) \circ (-\zeta u^*).$$

This means that $-\bar{\zeta}u$ is quasi-unitary in $\mathcal{A}$. We can now write (u, ζ) in the form

$$(u, \zeta) = \zeta(\bar{\zeta}u, 1) = \zeta((0, 1) - (-\bar{\zeta}u, 0)) = \zeta((0, 1) - (w, 0))$$

where $\zeta \in \mathbb{C}$, $|\zeta| = 1$ and $w := -\bar{\zeta}u$ is a quasi-unitary element in $\mathcal{A}$. □

5.25 Theorem. *A non-unital Banach *-algebra $\mathcal{A}$ is symmetric if and only if the spectrum of every quasi-unitary element is included in the unit circle with centre at 1.*

Proof. The proof relies on the corresponding characterization of symmetry for unital Banach *-algebras (Theorem 4.2).

Suppose that $\mathcal{A}$ is symmetric and let w be a quasi-unitary element in $\mathcal{A}$. By Lemma 5.24, the element $(0, 1) - (w, 0)$ is unitary in $\tilde{\mathcal{A}}$. Let $\mathbb{T}$ denote the unit circle in $\mathbb{C}$. Since $\mathcal{A}$ is symmetric, $\tilde{\mathcal{A}}$ is symmetric, too, and hence Theorem 4.2 yields $Sp_{\tilde{\mathcal{A}}}((0, 1) - (w, 0)) \subseteq \mathbb{T}$. Now we can invoke the Spectral Mapping Theorem 2.3 and Proposition 5.7 and obtain that

$$1 - Sp_{\mathcal{A}}(w) = 1 - Sp_{\tilde{\mathcal{A}}}((w, 0)) = Sp_{\tilde{\mathcal{A}}}((0, 1) - (w, 0)) \subseteq \mathbb{T}$$

and hence $Sp_{\mathcal{A}}(w) \subseteq \mathbb{T} + 1$.

Conversely, suppose that $Sp_{\mathcal{A}}(w) \subseteq \mathbb{T} + 1$ for all elements $w \in U_q(\mathcal{A})$. We will show that the spectrum of every unitary element in $\tilde{\mathcal{A}}$ is included in the unit circle $\mathbb{T}$ and again use Theorem 4.2.

By Lemma 5.24, any unitary element (u, ζ) in $\tilde{\mathcal{A}}$ can be written in the form

$$(u, \zeta) = \zeta((0, 1) - (w, 0))$$

where $|\zeta| = 1$ and $w \in U_q(\mathcal{A})$. Using the Spectral Mapping Theorem 2.3, Proposition 5.7 and the assumption we can infer that

$$Sp_{\tilde{\mathcal{A}}}((u, \zeta)) = Sp_{\tilde{\mathcal{A}}}(\zeta((0, 1) - (w, 0))) = \zeta(1 - \underbrace{Sp_{\mathcal{A}}(w)}_{\subseteq \mathbb{T}+1}) \subseteq \zeta\mathbb{T} = \mathbb{T}.$$

Now Theorem 4.2 implies that $\tilde{\mathcal{A}}$ is symmetric and hence also $\mathcal{A}$ is symmetric. □

Next we treat the unitary seminorm $v_{\mathcal{A}}$ in a non-unital Banach $*$ -algebra $\mathcal{A}$. Once again we have two options. We may define $v_{\mathcal{A}}$ to be the restriction to $\mathcal{A}$ of the unitary seminorm $v_{\tilde{\mathcal{A}}}$ on the unitization, or give an intrinsic definition in terms of quasi-unitary elements in $\mathcal{A}$ [17], [20].

5.26 Definition. The *unitary seminorm* $v_{\mathcal{A}}$ is defined, for all $a \in \mathcal{A}$, by

$$v_{\mathcal{A}}(a) := \inf \left\{ \sum_{k=1}^n |\lambda_k| : a = \sum_{k=1}^n \lambda_k w_k, 0 = \sum_{k=1}^n \lambda_k, n \in \mathbb{N}, \lambda_k \in \mathbb{C}, w_k \in U_q(\mathcal{A}) \right\}.$$

5.27 Proposition. Let $\mathcal{A}$ be a non-unital Banach $*$ -algebra.

Then $v_{\mathcal{A}}(a) = v_{\tilde{\mathcal{A}}}((a, 0))$ for all $a \in \mathcal{A}$.

Proof. Let $a = \sum_{k=1}^n \lambda_k w_k$ with coefficients $\lambda_k \in \mathbb{C}$ satisfying $0 = \sum_{k=1}^n \lambda_k$ and $w_k \in U_q(\mathcal{A})$. Then we can write

$$(a, 0) = \sum_{k=1}^n (-\lambda_k)((0, 1) - (w_k, 0)),$$

where $(0, 1) - (w_k, 0) \in U(\tilde{\mathcal{A}})$ by Lemma 5.24. Thus $v_{\tilde{\mathcal{A}}}((a, 0)) \leq v_{\mathcal{A}}(a)$.

Conversely, suppose $(a, 0) = \sum_{k=1}^n \lambda_k (u_k, \zeta_k)$ for $\lambda_k \in \mathbb{C}$ and $(u_k, \zeta_k) \in U(\tilde{\mathcal{A}})$. By Lemma 5.24, we have $(u_k, \zeta_k) = \zeta_k((0, 1) - (w_k, 0))$ with $|\zeta_k| = 1$ and $w_k \in U_q(\mathcal{A})$ and hence we can write

$$(a, 0) = \sum_{k=1}^n \lambda_k \zeta_k ((0, 1) - (w_k, 0)).$$

But this implies that $0 = \sum_{k=1}^n \lambda_k \zeta_k$ and $a = \sum_{k=1}^n (-\lambda_k \zeta_k) w_k$. Therefore also $v_{\mathcal{A}}(a) \leq v_{\tilde{\mathcal{A}}}((a, 0))$. □

We can now discuss the characterizations of symmetry concerning the unitary seminorm in non-unital Banach $*$ -algebras.

5.28 Theorem. Let $\mathcal{A}$ be a non-unital Banach $*$ -algebra with continuous involution. Then the following conditions are equivalent:

- (1) $\mathcal{A}$ is symmetric.
- (2) $\rho_{\mathcal{A}}(a) = v_{\mathcal{A}}(a)$ for all $a \in \mathcal{A}$.
- (3) $r_{\mathcal{A}}(h) \leq v_{\mathcal{A}}(h)$ for all hermitian elements $h \in \mathcal{A}$.

Proof. (1) $\Rightarrow$ (2): If $\mathcal{A}$ is symmetric, then also $\tilde{\mathcal{A}}$ is symmetric. Hence we can invoke Theorem 4.8 to obtain

$$\rho_{\tilde{\mathcal{A}}}((a, \lambda)) = v_{\tilde{\mathcal{A}}}((a, \lambda))$$

for all $(a, \lambda) \in \tilde{\mathcal{A}}$. Combined with Corollary 5.8 and the preceding proposition this implies

$$\rho_{\mathcal{A}}(a) = \rho_{\tilde{\mathcal{A}}}((a, 0)) = v_{\tilde{\mathcal{A}}}((a, 0)) = v_{\mathcal{A}}(a)$$

for all $a \in \mathcal{A}$.

(2) $\Rightarrow$ (3): clear

(3) $\Rightarrow$ (1): We can essentially adopt the proof for the unital case (Theorem 4.8).

Assume that $\mathcal{A}$ is not symmetric and choose a hermitian element $h \in \mathcal{A}$ with $i \in Sp_{\mathcal{A}}(h) = Sp_{\tilde{\mathcal{A}}}((h, 0))$. Set $\exp(i(h, 0)) =: (-b, 1)$, then $(-b, 1) \in U(\tilde{\mathcal{A}})$ and so, by Lemma 5.24, $b \in U_q(\mathcal{A})$.

Using the Spectral Mapping Theorem 2.3 we can observe that $\exp(-1) \in Sp_{\tilde{\mathcal{A}}}((-b, 1))$, $\exp(1) \in Sp_{\tilde{\mathcal{A}}}((-b, 1)^{-1}) = Sp_{\tilde{\mathcal{A}}}((-b^*, 1))$ and hence

$$\exp(1) - \exp(-1) \in Sp_{\tilde{\mathcal{A}}}((-b^*, 1) - (-b, 1)) = Sp_{\tilde{\mathcal{A}}}((-b^* + b, 0)) = Sp_{\mathcal{A}}(-b^* + b).$$

Consequently, $r_{\mathcal{A}}(-b^* + b) > 2$.

Using assumption (3) on the other hand yields

$$r(-b^* + b) \leq v(-b^* + b) \leq 2v(b) \leq 2,$$

which is a contradiction. Therefore $\mathcal{A}$ must be symmetric.

□

5.4 Summary

Concluding this chapter we restate Theorem 1.1 for arbitrary Banach $*$ -algebras. Note that the slight modifications needed for the treatment of non-unital Banach $*$ -algebras also make sense in unital Banach $*$ -algebras and in this case reduce to the simpler formulation of Theorem 1.1.

5.29 Theorem. *For a Banach $*$ -algebra $\mathcal{A}$ with continuous involution the following conditions are equivalent:*

- (1) $\mathcal{A}$ is hermitian: $Sp(h) \subseteq \mathbb{R}$ for all hermitian elements $h \in \mathcal{A}$.
- (2) $\mathcal{A}$ is symmetric: $Sp(a^*a) \subseteq \mathbb{R}_+$ for all $a \in \mathcal{A}$.
- (3) $\mathcal{A}$ is completely symmetric: $Sp(p) \subseteq \mathbb{R}_+$ for all elements of the form $p = \sum_{k=1}^n a_k^* a_k$ for $a_k \in \mathcal{A}$, $n \in \mathbb{N}$.
- (4) $i \notin Sp(h)$ for all hermitian elements $h \in \mathcal{A}$.
- (5) $-a^*a$ is quasi-invertible for all $a \in \mathcal{A}$.
- (6) The Pták functional ρ is an algebra seminorm on $\mathcal{A}$.
- (7) $\rho(a+b) \leq \rho(a) + \rho(b)$ for all $a, b \in \mathcal{A}$.
- (8) $r(\frac{1}{2}(a^* + a)) \leq \rho(a)$ for all $a \in \mathcal{A}$.
- (9) $r(a) \leq \rho(a)$ for all $a \in \mathcal{A}$.
- (10) $r(a) = \rho(a)$ for all normal elements $a \in \mathcal{A}$.
- (11) $r(a) \leq \|a^*a\|^{1/2}$ for all normal elements $a \in \mathcal{A}$.
- (12) The Gelfand-Naimark seminorm γ is a spectral seminorm on $\mathcal{A}$.
- (13) There exists a constant C with $r(a^*a) \leq C\gamma(a^*a)$ for all $a \in \mathcal{A}$.
- (14) Raikov's criterion: $\gamma(a) = \rho(a)$ for all $a \in \mathcal{A}$.
- (15) Every maximal modular left ideal in $\mathcal{A}$ is closed with respect to γ .
- (16) For every maximal modular left ideal $\mathcal{I}$ in $\mathcal{A}$ there exists a state f on $\mathcal{A}$ such that $\mathcal{I} = \{a \in \mathcal{A} : f(a^*a) = 0\}$.
- (17) For every maximal modular left ideal $\mathcal{I}$ in $\mathcal{A}$ there exists a pure state g on $\mathcal{A}$ such that $\mathcal{I} = \{a \in \mathcal{A} : g(a^*a) = 0\}$.
- (18) $Sp_{\mathcal{A}}(a) \subseteq \{g(a) : g \text{ is a pure state on } \mathcal{A}\} \cup \{0\}$ for all normal elements $a \in \mathcal{A}$.
- (19) If $T : \mathcal{A} \rightarrow \mathcal{L}(X)$ is any algebraically irreducible representation, then there is an inner product for X relative to which T is a pre- $*$ -representation.
- (20) $Rad(\mathcal{A}) = R^*(\mathcal{A})$ and $\mathcal{A}/R^*(\mathcal{A})$ is a spectral $*$ -subalgebra of its enveloping C^* -algebra.
- (21) The spectrum of every quasi-unitary element of $\mathcal{A}$ is contained in the unit circle with centre at 1.
- (22) $\rho(a) = v(a)$ for all $a \in \mathcal{A}$.
- (23) $r(h) \leq v(h)$ for all hermitian elements $h \in \mathcal{A}$.

Chapter 6

Stability Properties

There are several ways to construct new Banach $*$ -algebras from a given Banach $*$ -algebra $\mathcal{A}$. Every closed $*$ -subalgebra and every closed $*$ -ideal of $\mathcal{A}$ can be regarded as a Banach $*$ -algebra, a quotient algebra $\mathcal{A}/\mathcal{I}$ is a Banach $*$ -algebra whenever $\mathcal{I}$ is a closed $*$ -ideal of $\mathcal{A}$, and one may construct a direct sum or tensor product with some other Banach $*$ -algebra.

In this chapter we will show that the property of being a symmetric Banach $*$ -algebra is stable under various such constructions [20].

Definitions and Preparations

The Direct Sum of two Banach $*$ -algebras

Let $\mathcal{A}$ and $\mathcal{B}$ be Banach $*$ -algebras. The *direct sum* $\mathcal{A} \oplus \mathcal{B}$ of $\mathcal{A}$ and $\mathcal{B}$ is the cartesian product $\mathcal{A} \times \mathcal{B}$ with pointwise algebra operations, with the norm

$$\|(a, b)\|_{\mathcal{A} \oplus \mathcal{B}} := \|a\|_{\mathcal{A}} + \|b\|_{\mathcal{B}}$$

and with the involution

$$(a, b)^* := (a^*, b^*)$$

for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

Then $\mathcal{A} \oplus \mathcal{B}$ is again a Banach $*$ -algebra and $\mathcal{A}, \mathcal{B}$ can be identified with closed $*$ -ideals of $\mathcal{A} \oplus \mathcal{B}$.

The Banach *-algebra $M_n(\mathcal{A})$ of $n \times n$ -matrices over $\mathcal{A}$

Let $\mathcal{A}$ be a Banach *-algebra. For each i and j satisfying $1 \leq i, j \leq n$ let a_{ij} be an element of $\mathcal{A}$. Then $A = (a_{ij})_{n \times n}$ represents the matrix

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}.$$

The set of all $n \times n$ -matrices with entries from $\mathcal{A}$ is denoted by $M_n(\mathcal{A})$.

Let $A = (a_{ij})_{n \times n}$ and $B = (b_{ij})_{n \times n}$ be matrices in $M_n(\mathcal{A})$ and let $\lambda \in \mathbb{C}$. Then the addition, the scalar multiplication and the product are defined by

$$A + B := (a_{ij} + b_{ij})_{n \times n},$$

$$\lambda A := (\lambda a_{ij})_{n \times n},$$

$$AB := \left(\sum_{k=1}^n a_{ik} b_{kj} \right)_{n \times n}.$$

A norm for $A = (a_{ij})_{n \times n} \in M_n(\mathcal{A})$ can be defined by

$$\|A\|_{M_n(\mathcal{A})} := \max_{i=1, \dots, n} \sum_{j=1}^n \|a_{ij}\|_{\mathcal{A}},$$

and an involution by

$$A^* := (a_{ji}^*)_{n \times n}.$$

With these definitions $M_n(\mathcal{A})$ becomes a Banach *-algebra.

If $\mathcal{A}$ is unital, so is $M_n(\mathcal{A})$, where the unit element is given by

$$E := \begin{pmatrix} e & 0 & \dots & 0 \\ 0 & e & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e \end{pmatrix}.$$

Remark. $M_n(\mathcal{A})$ is often interpreted as the tensor product $\mathcal{A} \otimes M_n(\mathbb{C})$, see, e.g., [19], p. 165.

Symmetry of certain constructions

We now list several Banach $*$ -algebras constructed from $\mathcal{A}$, which inherit the property of being symmetric. In particular we treat $*$ -ideals and $*$ -subalgebras of $\mathcal{A}$ which may be non-unital even if $\mathcal{A}$ is unital. For that reason we do not separate the discussion of unital and non-unital Banach $*$ -algebras and let $\mathcal{A}$ denote an arbitrary Banach $*$ -algebra for the remainder of this chapter.

Conditions (5) and (9) in the following theorem are due to Leptin [14, 15], and condition (7) was first observed by Civin and Yood [3]. We mainly follow the presentation in [20].

6.1 Theorem. *For a Banach $*$ -algebra $\mathcal{A}$ with continuous involution the following conditions are equivalent:*

- (1) $\mathcal{A}$ is a symmetric Banach $*$ -algebra.
- (2) Every closed $*$ -subalgebra of $\mathcal{A}$ is a symmetric Banach $*$ -algebra.
- (3) Every closed $*$ -ideal of $\mathcal{A}$ is a symmetric Banach $*$ -algebra.
- (4) $\mathcal{A}/\mathcal{I}$ is a symmetric Banach $*$ -algebra for each closed $*$ -ideal $\mathcal{I}$ of $\mathcal{A}$.
- (5) There is some closed $*$ -ideal $\mathcal{I}$ of $\mathcal{A}$ for which both $\mathcal{I}$ and $\mathcal{A}/\mathcal{I}$ are symmetric Banach $*$ -algebras.
- (6) $\mathcal{A} \oplus \mathcal{B}$ is a symmetric Banach $*$ -algebra for each symmetric Banach $*$ -algebra $\mathcal{B}$.
- (7) The unitization $\tilde{\mathcal{A}}$ of $\mathcal{A}$ is a symmetric Banach $*$ -algebra.
- (8) $\mathcal{A}/\text{Rad}(\mathcal{A})$ is a symmetric Banach $*$ -algebra.
- (9) $M_n(\mathcal{A})$ is a symmetric Banach $*$ -algebra for all $n \in \mathbb{N}$.
- (10) $M_n(\mathcal{A})$ is a symmetric Banach $*$ -algebra for some $n \in \mathbb{N}$.

Proof. (1) $\Leftrightarrow$ (2): Let $\mathcal{A}$ be symmetric and let $\mathcal{B}$ be a closed $*$ -subalgebra of $\mathcal{A}$. Then $\mathcal{B}$ is a Banach $*$ -algebra with the norm and the involution in $\mathcal{A}$ restricted to $\mathcal{B}$. We apply the Spectral Radius Formula 2.7 to infer that

$$r_{\mathcal{B}}(b) = \lim_{n \rightarrow \infty} \|b^n\|^{\frac{1}{n}} = r_{\mathcal{A}}(b)$$

for all $b \in \mathcal{B}$. Since $\mathcal{A}$ is symmetric, Ptak's inequality (Theorem 5.29 (9)) holds in $\mathcal{A}$ and hence

$$r_{\mathcal{B}}(b) = r_{\mathcal{A}}(b) \leq \rho_{\mathcal{A}}(b) = \rho_{\mathcal{B}}(b)$$

for all $b \in \mathcal{B}$. Therefore $\mathcal{B}$ is symmetric, because Ptak's inequality characterizes symmetry by Theorem 5.29 (9).

For the converse implication choose the closed $*$ -subalgebra $\mathcal{B} = \mathcal{A}$.

(1) $\Leftrightarrow$ (3): This is immediate from (1) $\Leftrightarrow$ (2) since every closed $*$ -ideal of $\mathcal{A}$ is in particular a closed $*$ -subalgebra of $\mathcal{A}$.

(1) $\Leftrightarrow$ (4): Let $\mathcal{I}$ be a closed $*$ -ideal of $\mathcal{A}$. First we show that, for all $a \in \mathcal{A}$,

$$Sp_{\mathcal{A}/\mathcal{I}}(a') \subseteq Sp_{\mathcal{A}}(a).$$

Let $a \in \mathcal{A}$, let $\lambda \in \mathbb{C}$, $\lambda \neq 0$ and suppose that $\lambda \notin Sp_{\mathcal{A}}(a)$. Then there exists an element $b \in \mathcal{A}$ such that

$$b \circ (\lambda^{-1}a) = (\lambda^{-1}a) \circ b = 0.$$

This implies

$$b' \circ (\lambda^{-1}a') = (\lambda^{-1}a') \circ b' = 0'$$

and hence $\lambda \notin Sp_{\mathcal{A}/\mathcal{I}}(a')$, where $a \mapsto a'$ denotes the canonical mapping of $\mathcal{A}$ onto $\mathcal{A}/\mathcal{I}$.

If $0 \notin Sp_{\mathcal{A}}(a)$, then $\mathcal{A}$ is unital and a is invertible in $\mathcal{A}$. In this case also $\mathcal{A}/\mathcal{I}$ is unital and a' is invertible in $\mathcal{A}/\mathcal{I}$, that is, $0 \notin Sp_{\mathcal{A}/\mathcal{I}}(a')$.

Thus $Sp_{\mathcal{A}/\mathcal{I}}(a') \subseteq Sp_{\mathcal{A}}(a)$.

Since $\mathcal{A}$ is symmetric, we deduce that

$$Sp_{\mathcal{A}/\mathcal{I}}(h') \subseteq Sp_{\mathcal{A}}(h) \subseteq \mathbb{R}$$

for all hermitian elements h in $\mathcal{A}$. Therefore $\mathcal{A}/\mathcal{I}$ is symmetric.

For the converse implication, let $\mathcal{I}$ be $\{0\}$.

(1) $\Leftrightarrow$ (5): If $\mathcal{A}$ is symmetric, then $\mathcal{I}$ and $\mathcal{A}/\mathcal{I}$ are both symmetric for any closed $*$ -ideal $\mathcal{I}$ of $\mathcal{A}$ by the implications (1) $\Rightarrow$ (3) and (1) $\Rightarrow$ (4).

Conversely, suppose that $\mathcal{I}$ is a closed $*$ -ideal of $\mathcal{A}$ for which $\mathcal{I}$ and $\mathcal{A}/\mathcal{I}$ are symmetric. Let a be an arbitrary element in $\mathcal{A}$. We have to show that $-1 \notin Sp_{\mathcal{A}}(a^*a)$ or equivalently, $-a^*a$ is quasi-invertible in $\mathcal{A}$.

By symmetry of $\mathcal{A}/\mathcal{I}$, the element $-(a')^*(a') = -(a^*a)'$ is quasi-invertible in $\mathcal{A}/\mathcal{I}$. Hence there exists some $b \in \mathcal{A}$ such that $b' \circ (-a^*a)' = 0'$. As a consequence $h := (-a^*a) \circ b^* \circ b \circ (-a^*a)$ is a hermitian element in $\mathcal{I}$. Since $\mathcal{I}$ is a $*$ -ideal, $-ha^*ah \in \mathcal{I}$. By symmetry of $\mathcal{I}$, the element $-(ah)^*(ah) = -ha^*ah$ is quasi-invertible in $\mathcal{I}$. Recall from Lemma 5.3 that the quasi-product satisfies $(u + v) \circ w = u \circ w + v \circ w - w$ for all $u, v, w \in \mathcal{A}$. Using this relation we calculate

$$\begin{aligned} & (-h + (ha^*a) \circ (-a^*a) \circ b^* \circ b) \circ (-a^*a) \\ &= (-h) \circ (-a^*a) + (ha^*a) \circ (-a^*a) \circ b^* \circ b \circ (-a^*a) + a^*a \\ &= (-h) \circ (-a^*a) + (ha^*a) \circ h + a^*a \\ &= -h - a^*a - ha^*a + ha^*a + h - ha^*ah + a^*a \\ &= -ha^*ah. \end{aligned}$$

If c is a left quasi-inverse of $-ha^*ah$, then $d := c \circ (-h + (ha^*a) \circ (-a^*a) \circ b^* \circ b)$ is a left quasi-inverse of $-a^*a$, that is, $d \circ (-a^*a) = 0$. Since $-a^*a$ is hermitian, we also have $(-a^*a) \circ d^* = (d \circ (-a^*a))^* = 0^* = 0$. This shows that $-a^*a$ is quasi-invertible. Therefore $\mathcal{A}$ is symmetric.

(1) $\Leftrightarrow$ (6): This follows from (1) $\Leftrightarrow$ (5) with $\mathcal{A} = \mathcal{A} \oplus \mathcal{B}$ and $\mathcal{I} = \mathcal{B}$.

(1) $\Leftrightarrow$ (7): In Chapter 5 we already noted that $\mathcal{A}$ is symmetric if and only if its unitization $\tilde{\mathcal{A}}$ is symmetric (Remark after Theorem 5.10). Alternatively, this result is now a consequence of (1) $\Leftrightarrow$ (5).

(1) $\Leftrightarrow$ (8): By Proposition 5.19, we have, for all $a \in \mathcal{A}$,

$$Sp_{\mathcal{A}/Rad(\mathcal{A})}(a + Rad(\mathcal{A})) \subseteq Sp_{\mathcal{A}}(a) \subseteq Sp_{\mathcal{A}/Rad(\mathcal{A})}(a + Rad(\mathcal{A})) \cup \{0\}.$$

Now the assertion is immediate.

(1) $\Rightarrow$ (9): Without loss of generality we may assume that $\mathcal{A}$ is unital. Otherwise consider the algebra $M_n(\tilde{\mathcal{A}})$ of $n \times n$ -matrices over the unitization $\tilde{\mathcal{A}}$ and note that $M_n(\mathcal{A})$ is $*$ -isomorphic to a closed $*$ -subalgebra of $M_n(\tilde{\mathcal{A}})$. By (1) $\Rightarrow$ (7) the unitization $\tilde{\mathcal{A}}$ is symmetric. If we now prove (1) $\Rightarrow$ (9) in the unital case, we may apply the result to $\tilde{\mathcal{A}}$ and infer that $M_n(\tilde{\mathcal{A}})$ is symmetric. Then $M_n(\mathcal{A})$ is also symmetric by (1) $\Rightarrow$ (2).

First we suppose $n = 2$ and prove that $M_2(\mathcal{A})$ is symmetric. Let $A = (a_{ij})_{2 \times 2}$ be an arbitrary matrix in $M_2(\mathcal{A})$. We need to show that $E + A^*A$ is invertible. We write

$$E + A^*A = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} + \begin{pmatrix} a_{11}^* & a_{21}^* \\ a_{12}^* & a_{22}^* \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} e + p & b \\ b^* & e + q \end{pmatrix}$$

where $p = a_{11}^*a_{11} + a_{21}^*a_{21}$, $q = a_{12}^*a_{12} + a_{22}^*a_{22}$ and $b = a_{11}^*a_{12} + a_{21}^*a_{22}$. Since $\mathcal{A}$ is symmetric, the element $e + p$ is invertible and we can thus define

$$B := \begin{pmatrix} e & -(e + p)^{-1}b \\ 0 & e \end{pmatrix}.$$

Then B is invertible. We calculate

$$\begin{aligned} B^*(E + A^*A)B &= \begin{pmatrix} e & 0 \\ -b^*(e + p)^{-1} & e \end{pmatrix} \begin{pmatrix} e + p & b \\ b^* & e + q \end{pmatrix} \begin{pmatrix} e & -(e + p)^{-1}b \\ 0 & e \end{pmatrix} \\ &= \begin{pmatrix} e + p & 0 \\ 0 & e + q - b^*(e + p)^{-1}b \end{pmatrix} \end{aligned}$$

and denote the resulting diagonal matrix by D . Further we compute

$$\begin{aligned} B^*B &= \begin{pmatrix} e & 0 \\ -b^*(e+p)^{-1} & e \end{pmatrix} \begin{pmatrix} e & -(e+p)^{-1}b \\ 0 & e \end{pmatrix} \\ &= \begin{pmatrix} e & -(e+p)^{-1}b \\ -b^*(e+p)^{-1} & e + ((e+p)^{-1}b)^*(e+p)^{-1}b \end{pmatrix}. \end{aligned} \quad (6.1)$$

If we define $C := AB$, then we can write D in the form

$$D = B^*(E + A^*A)B = B^*B + (AB)^*(AB) = B^*B + C^*C. \quad (6.2)$$

In view of (6.1) and (6.2) the $(2, 2)$ -entry of D can be written as

$$e + ((e+p)^{-1}b)^*(e+p)^{-1}b + c_{12}^*c_{12} + c_{22}^*c_{22}$$

and thus is invertible by symmetry of $\mathcal{A}$. This shows the invertibility of D , which in turn implies that also $E + A^*A = (B^*)^{-1}DB^{-1}$ is invertible as a product of invertible matrices. Therefore $M_2(\mathcal{A})$ is symmetric.

Next we consider $n = 2^m$ for some $m \in \mathbb{N}$.

Since $M_{2^m}(\mathcal{A})$ is $*$ -isomorphic to $M_2(M_{2^{m-1}}(\mathcal{A}))$, the symmetry of $M_{2^m}(\mathcal{A})$ follows from the previous step by induction.

Now let $n \in \mathbb{N}$ be arbitrary. Choose $m \in \mathbb{N}$ such that $n \leq 2^m$. Then $M_n(\mathcal{A})$ is $*$ -isomorphic to the closed $*$ -subalgebra of $M_{2^m}(\mathcal{A})$ consisting of all matrices with zero in the last $2^m - n$ rows and columns. Hence $M_n(\mathcal{A})$ is symmetric by (1) $\Rightarrow$ (2).

(9) $\Rightarrow$ (10): clear

(10) $\Rightarrow$ (1): Let $a \in \mathcal{A}$ be arbitrary and consider the matrix $A = (a_{ij})_{n \times n}$ with $a_{11} = a$ and $a_{ij} = 0$ otherwise. Since $M_n(\mathcal{A})$ is symmetric, the matrix $-A^*A$ is quasi-invertible in $M_n(\mathcal{A})$. Hence there exists some $B \in M_n(\mathcal{A})$ such that

$$(-A^*A) \circ B = -A^*A + B + A^*AB = 0, \quad (6.3)$$

$$B \circ (-A^*A) = B - A^*A + BA^*A = 0. \quad (6.4)$$

Considering the $(1, 1)$ -entries in (6.3) and (6.4), we can infer that

$$-a^*a + b_{11} + a^*ab_{11} = (-a^*a) \circ b_{11} = 0,$$

$$b_{11} - a^*a + b_{11}a^*a = b_{11} \circ (-a^*a) = 0.$$

This means that $-a^*a$ is quasi-invertible in $\mathcal{A}$ with quasi-inverse b_{11} .

Thus $\mathcal{A}$ is symmetric. □

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Zusammenfassung

Symmetrische Banach $*$ -Algebren bilden einen wichtigen Zwischenschritt von Banach $*$ -Algebren zu C^* -Algebren.

Dabei bezeichnet man eine Banach $*$ -Algebra $\mathcal{A}$ als *symmetrisch*, falls alle Elemente der Form a^*a positives Spektrum haben, das heißt,

$$Sp(a^*a) \subseteq \mathbb{R}_+ \quad \forall a \in \mathcal{A}.$$

Diese Bedingung wurde ursprünglich unter den Axiomen für C^* -Algebren aufgelistet. Es wurde jedoch vermutet, dass sie überflüssig sei und bereits aus den übrigen Axiomen folgen würde, wie Jahre später auch bewiesen werden konnte. Weiters stellte sich heraus, dass oft gerade diese Symmetrie-Bedingung für viele interessante Resultate ausschlaggebend ist.

Ziel dieser Diplomarbeit ist eine systematische Diskussion verschiedener Charakterisierungen von symmetrischen Banach $*$ -Algebren.

Das erste Kapitel gibt eine kurze Einführung und einen Überblick über die behandelten Charakterisierungen.

In Kapitel 2 wird Symmetrie durch Eigenschaften von Spektrum und Spektralradius bestimmter Elemente charakterisiert.

Kapitel 3 beinhaltet jene Charakterisierungen von Symmetrie, die durch positive lineare Funktionale oder Darstellungen ausgedrückt sind.

In Kapitel 4 wird das Spektrum unitärer Elemente behandelt und weiters die unitäre Seminorm zur Charakterisierung von Symmetrie herangezogen.

In Kapitel 5 werden die besprochenen Charakterisierungen dann auf Banach $*$ -Algebren ohne Einselement erweitert.

Kapitel 6 fasst einige Konstruktionen von Banach $*$ -Algebren zusammen, unter denen die Eigenschaft der Symmetrie erhalten bleibt.

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