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DISSERTATION

Dissertation

Short-Time Fourier Transform and Modulation Spaces in Algebras of Generalized Functions

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Abstract

The thesis is organized as follows: In the first chapter we recall some well-established notions and relations from both the theory of modulation spaces and of Colombeau algebras. They are presented in a form suitable for the further course of the thesis.

In the second chapter we consider basic properties of convolution of elements of modulation spaces with a mollifier, as well as connections between multiplication and regularity properties. We construct suitable Colombeau algebras and corresponding embedding maps for modulation spaces. We define the notion of association as a tool to determine classical counterparts (if available) to the generalized elements/solutions.

In the third chapter we ensure that it is possible to construct generalized algebras corresponding to Wiener-Amalgam spaces as well. We show that all the "usual" properties hold in this setting, too. Additionally we construct an algebra of generalized functions which simultaneously carries the information on decay properties of the nets measured in the L^q-norm. We show that the fundamental characterization of the ideal holds, too, but only when $p \leq q$.

In the fourth chapter we introduce the basic notions of "classical" theory of pseudo-differential operators in the setting of time-frequency analysis and investigate mapping properties of operators corresponding to generalized symbols.

In the fifth chapter the STFT itself is generalized in the Colombeau sense - we investigate the properties of the generalized STFT $(\big(V_g f\big)_\varepsilon(x,\omega))$ with generalized function f and generalized window function g. We ensure that the basic classical regularity properties carry over to the generalized case.

Inhalt

Die vorliegende Dissertation ist wie folgt aufgebaut: Im ersten Kapitel werden einige bekannte Begriffe und Resultate aus der Theorie der Modulationsräume sowie aus der Theorie der Colombeau-Algebren wiederholt. Diese werden in einer Form eingeführt, die für den weiteren Gang der Dissertation am günstigsten erscheint.

Im zweiten Kapitel betrachten wir grundlegende Eigenschaften der Faltung von Elementen von Modulationsräumen mit einem Glätter und beleuchten die Zusammenhänge zwischen Multiplikation und Regularitätseigenschaften. Wir konstruieren entsprechende Colombeau Algebren sowie die zugehörigen Einbettungen von Modulationsräumen. Wir führen den Begriff der Assoziation ein um die klassischen Entsprechungen (falls vorhanden) der verallgemeinerten Funktionen/Lösungen bestimmen zu können.

Im dritten Kapitel zeigen wir, dass es möglich ist, analoge Verallgemeinerungen von Wiener-Amalgam Räumen zu konstruieren. Wir zeigen, dass die wichtigsten Eigenschaften auch in diesem Setting erhalten bleiben. Zusätzlich konstruieren wir eine Algebra verallgemeinerter Funktionen, die gleichzeitig die Abfalleigenschaften der entsprechnden Netze in der L^q-Norm widerspiegelt. Wir zeigen, dass die fundamentale Characterisierung des Ideals auch hier gültig bleibt, allerding nur für $p \leq q$.

Im vierten Kapitel wiederholen wir zunächst die grundlegenden Begriffe der Theorie der Pseudodifferentialoperatoren in der Theorie der Zeit-Frequenz-Analyse und untersuchen dann Abbildungseigenschaften von Operatoren mit verallgemeinerten Symbolen.

Im abschließenden fünften Kapitel wird die Kurzzeit-Fouriertransformation (STFT) selbst im Sinn Colombeaus verallgemeinert. Wir untersuchen die grundlegenden Eigenschaften der STFT $((V_g f)_\varepsilon(x,\omega))$ mit verallgemeinerter Funktion f und verallgemeinertem Fenster g. Wir zeigen, dass die fundamentalen klassischen Regularitätseigenschaften auch in diesem Setting ihre Gültigkeit behalten.

Preface

Within the "Initiativkolleg" on MODERN MATHEMATICAL ANALYSIS and APPLICATIONS, Time-frequency analysis and microlocal analysis, the interdisciplinary PhD program provided at the University of Vienna, Faculty of Mathematics, as a joint project of two mathematical research groups, DIANA and NuHAG, several questions were raised concerning the connections between both the theory of time-frequency analysis and the theory of generalized functions. Firstly, it was unclear whether it would be possible to give a generalized notion (in the sense of Colombeau) of the short-time Fourier transform as of a single object which carries both properties of the function and its Fourier transform; and, if yes, would it be possible to obtain a generalized version of the regularity properties of 'classical' short-time Fourier transform. Secondly, a more important segment of the intersection of these two diverse mathematical fields was the question of an "improvement" of modulation spaces, as the most common function spaces in the time-frequency analysis, by means of Colombeau theory of generalized algebras. In the present thesis we give answers to these questions. Additionally, we have introduced a corresponding theory of pseudo-differential operators, a topic of increasing popularity in both the theory of generalized functions and the theory of modulation spaces. We establish a number of mapping properties of generalized pseudo-differential operators in this new setting of generalized modulation spaces.

The most significant results of my research contained in the present thesis have been written in Innsbruck, Austria, where I have been given an excellent opportunity to work under the supervision of Prof. Claudia Garetto and Prof. Michael Oberguggenberger at the Institute of Basic Sciences in Civil Engineering, University of Innsbruck, Faculty of Civil Engineering.

Prof. Garetto patiently and carefully nurtured and cherished my fragile scientific mind of a young mathematician through numerous discussions and thoughtful advising. At the same time she had not missed any opportunity to unobtrusively and softly plant a methodical and systematic scientific approach in me as well as to recognize, encourage and help me develop my own style.

On many occasions Prof. Oberguggenberger managed to provide me important feedback on important issues and questions concerning the topic of my work, regardless of how busy he was at the time. Additionally, Elisabeth and Michael Oberguggenberger made sure my stay in Innsbruck was a pleasant friend/tourist visit as well as it was an academic one, which certainly affected the quality of my work at the time.

The DIANA members Prof. Roland Steinbauer and Prof. Guenther Hörmann, have been involved in my research from the very beginning of my doctoral studies and surely strongly influenced and had considerable impact on my work. They have been involved in the most technical issues I dealt with as well as the more general mathematical ideas and aspects concerning my research.

Last in the group of people who were directly and scientifically involved in my research, but surely not least, my thesis adviser, Prof. Michael Kunzinger was the one who made all of this possible. He has been continuously providing me, before and during my doctoral study, with all the things the people mentioned above provided me separately; and above all, Prof. Kunzinger has been certainly most responsible for me coming any closer to a mature and complete scientist capable of conducting independent research, which has been a crucial process in the making of the present thesis. I am very happy to have had a role-model of a true mathematician, and equally important, a role-model of a great person for my thesis adviser, collaborator, friend.

My brother Mirsad, my parents, Besim and Senada Šahbegović and my grandparents Orhan and Zlata Kurtić have been very caring and understanding throughout my whole doctoral study. They have been patient and unfailingly supportive; their approval and belief in me has been very motivating and confidence-building which greatly influenced my determination in pursuing a scientific career. My grandfather Orhan has always been playing a special role in my life, and immensely influenced my overall perception of everything since I was a child. He has been encouraging my curiosity, while at the same time trying to teach me everything he could, and most of all he has thought me how to think. In a certain way I owe him my systematical, methodical and mathematical mind, and therefore I wish to dedicate this thesis to him, to my grandfather Orhan Kurtić.

There is a whole bunch of people who indirectly, yet significantly contributed to the making of this thesis. Those are my friends, who I spent most of my spare time with. I will list them in alphabetical order: Enes Bajrović (a.k.a. "Dilatacija", a.k.a. "Jogurt-Je-Dobar-Za-Mamurluka", a.k.a. "Es-Muss-Ordnung-Sein"), Philipp David (a.k.a. "King Deee"), M.Sc. Ajdin Halilović (a.k.a. "Fakat-Možda-Ozbiljno"), B.Sc.M. Suad Kapidžija (a.k.a. "Direktor"), Enis Maksumić (a.k.a. "Enis Maksumić"), B.Sc.(Econ) Hajrudin Osmanović (a.k.a. "Dino Pkš."), Džemal Sarić (a.k.a. "Šaban Sakić"), Selma Tvrtković, Dr.Sc. Ognjen Vukadin (a.k.a. "Džejlo"), Amela Zukić.

to my grandfather Orhan Kurtić

Contents

Chapter 1

Introduction

One of the central objects in the theory of time-frequency analysis is short-time Fourier transform (STFT). The STFT contains simultaneous information both on time and frequency. However, it does not carry 'instantaneous' information both on time and frequency due to the uncertainty principle (cf. [Grö01]).

In his fundamental paper [Fei83b], Feichtinger introduced $M^{p,q}$-norms (and corresponding modulation spaces) as a tool to quantify the time-frequency concentration (smoothness and decay properties) of functions via STFT in $L^{p,q}$-norms. This way of measuring the time-frequency concentration has often been regarded as the most suitable one, hence, modulation spaces are considered as the most suitable function spaces for time-frequency analysis. However, $M^{p,q}$ spaces are not algebras in general. Additionally, another 'obstacle' for multiplication is the Schwartz impossibility result (cf. [Sch54]). Nevertheless, there are relations ([Tof04a, Tof04b]) which tell us when it is allowed to multiply elements (of different modulation spaces) and ensure that the product lies in some modulation space (cf. Thm. 2.1.14). Our task here is to allow multiplication between arbitrary elements of the same $M^{p,q}$ space by embedding it in a differential algebra (in the sense of Colombeau). However, by allowing all the derivatives of a distribution to belong to the same $M^{p,q}$ space we 'lose' the information measured by L^q - which makes sense since this information measures the smoothness of the distribution.

An important consequence of the resulting theory is that we have an appropriate tool to cope with 'strongly nonlinear' situations in time-frequency analysis, in particular, in modulation spaces. The paper of Okoudjou, [OB] exemplifies such a situation where 'classical' modulation spaces theory enters the realm of nonlinear PDE's.

Colombeau algebras (usually denoted by the letter $\mathscr{G}$) are differential (quotient) algebras with unit, and were introduced by J. F. Colombeau (cf. [Col84, Col85]) as a nonlinear extension of distribution theory to deal with nonlinearities and singularities in PDE theory. These algebras contain the space of distributions $\mathscr{D}'$ (or its subspace) as a subspace with an embedding realized through convolution with a suitable mollifier. Elements of these algebras are classes of nets of smooth functions.

The subalgebra $\mathscr{G}^\infty$ of regular elements of $\mathscr{G}$ was constructed by Oberguggenberger (cf. [Obe92b]) and plays the role of a suitable analog for the subspace C^∞ of the space $\mathscr{D}'$.

Of particular significance is $\mathscr{G}_\tau$, the algebra of generalized temperate distributions (as a generalization of the subspace $\mathscr{S}'$ of distributions wherein the Fourier transform operator is an isomorphism). It has been investigated in various instances ([Gar05a, Gar04a, GGO05, Hör99]).

Typically, Colombeau algebras are constructed by utilizing the underlying inner structure of some subspace of $\mathscr{D}'$. As a general construction principle, Garetto introduced (cf. [Gar05b]) *Colombeau algebras based on locally convex topological vector spaces.* ([DHVP02]).

In order to keep notations simple, in what follows C will denote a generic constant whose value may change from line to line. Also, for practical reasons we shall sometimes simply omit the phrase *for ε small enough* as well as other similar qualifications.

We will mainly adopt notations from time-frequency analysis from [Grö01] and notations from Colombeau theory from [GKOS01].

Definition 1.0.1. *Let $x \in \mathbb{R}^n$ and $s \in \mathbb{R}$. We set*

$$\langle x\rangle^s := (1+|x|^2)^{\frac{s}{2}}.$$

Lemma 1.0.2 (Peetre's Inequality)**.** *For every $s \in \mathbb{R}$ there exists a constant $C_s \geq 0$ such that*

$$\langle x\rangle^s \leq C_s \cdot \langle x-y\rangle^{|s|}\langle y\rangle^s$$

and

$$(1+|x|)^s \leq C_s \cdot (1+|x-y|)^{|s|}(1+|y|)^s$$

for every x, $y \in \mathbb{R}^n$.

Definition 1.0.3. *For any $p \in [1,\infty]$ and $s \in [-\infty,\infty]$, the Sobolev space $H^p_s(\mathbb{R}^d)$ is the Banach space of all $f \in \mathscr{S}'(\mathbb{R}^d)$ such that $\|f\|_{H^p_s} := \|\langle D\rangle^s f\|_{L^p}$ is finite.*

1.1 STFT - the Short-Time Fourier Transform.

Definition 1.1.1. *Let x, y, $\omega \in \mathbb{R}^d$. The time-frequency shifts are the operators of translation (time shift) and modulation (frequency shift), given by:*

$$T_x f(y) = f(x-y),$$

and

$$M_\omega f(x) = f(x) \cdot e^{2\pi i x\cdot\omega}.$$

The action of time-frequency shifts can be extended to distributions in the obvious way.

Note that we may write $V_g f(x,\omega) = \langle f, M_\omega T_x g\rangle$ (for f, $g \in L^2$). If we strengthen the condition on f and choose $f \in \mathscr{S}$ ($g \in \mathscr{S}$), then we may write $V_g f(x,\omega) = \overline{\langle M_\omega T_x g, f\rangle} = \overline{\langle g, T_{-x}M_{-\omega}f\rangle}$. If we take $g_\varepsilon(y) = 1/\varepsilon^d g(y/\varepsilon)$ we may consider $V_{g_\varepsilon} f(x,\omega)$ as the action of distribution g_ε on the Schwartz function $\overline{T_{-x}M_{-\omega}f}$ and by distribution theory (cf. [Fri98]) we have

$$\lim_{\varepsilon\to 0} V_{g_\varepsilon} f(x,\omega) = \overline{\langle \delta, T_{-x}M_{-\omega}f\rangle} = f(x)e^{2\pi i x\omega}.$$

which yields $|\lim_{\varepsilon\to 0} V_{g_\varepsilon} f(x,\omega)| = |f(x)|$.

Similarly, if we choose $g_\varepsilon(y) := g(\varepsilon \cdot y)$ where g is again real-valued, even, compactly supported in $\{|y| \leq 1\}$ such that $g(x) = 1$ for $|y| \leq 1/2$, we conclude

$$\lim_{\varepsilon\to 0} V_{g_\varepsilon} f(x,\omega) = \int f(y)g(0)e^{-2\pi i y\omega} dy = \hat{f}(\omega).$$

This implies that the STFT contains simultaneous information both on time and frequency.

It is important to emphasize that the definition of the STFT can be extended in various ways (cf. [Grö01]), however, for our purpose we shall consider 'only' the distributional extension, i.e. we define the STFT in the following way:

Definition 1.1.2. *Let $f \in \mathscr{S}'(\mathbb{R}^d)$ and $0 \neq g \in \mathscr{S}(\mathbb{R}^d)$. We define the* Short-Time Fourier Transform (STFT) *as an action of the distribution f on a (time-frequency-shifted) test function g in the following way:*

$$V_g f(x,\omega) := \langle f, M_\omega T_x g\rangle. \tag{1.1}$$

The function g is called window function.

As seen before, if f is rapidly decreasing, then the STFT of f i.e. (1.1) takes the form

$$V_g f(x,\omega) := \int_{\mathbb{R}^n} f(y)\overline{g(y-x)}e^{-2\pi i y\omega} dy.$$

$V_g f$ is continuous and of moderate growth ([Grö01, Thm. 11.2.3]), i.e. there exist constants $C \geq 0$ and $N \in \mathbb{R}^d$ such that for all $(x,\omega) \in \mathbb{R}^{2d}$

$$|V_g f(x,\omega)| \leq C \cdot (1 + |x| + |\omega|)^N.$$

Moreover, the following statement holds:

Proposition 1.1.3. *Let $f \in \mathscr{S}'$ and $g \in \mathscr{S}$. Then $V_g f \in \mathscr{O}(\mathbb{R}^n)$.*

Proof. In order to show that $V_g f$ is $\mathscr{O}(\mathbb{R}^n)$ it suffices to prove that $\partial_{x_j}\partial_{\omega_k} V_g f(x,\omega)$ is continuous and moderate.

$$\begin{aligned}
\partial_{x_j}\partial_{\omega_k} V_g f(x,\omega) &= \partial_{x_j}\partial_{\omega_k}\langle f, M_\omega T_x g\rangle \\
&= \langle f, \partial_{x_j}\partial_{\omega_k}(g(y-x)\cdot e^{2\pi i y\omega})\rangle \\
&= \langle(-2\pi i y_k)\cdot f(y), -\partial_j g(y-x)\cdot e^{2\pi i y\omega}\rangle \\
&= C\cdot\langle y_k \cdot f(y), M_\omega T_x \partial_j g(y)\rangle \\
&= C\cdot V_{\partial_j g}(y_k \cdot f(y))(x,\omega).
\end{aligned}$$

Since $y_k \cdot f(y)$ is again in $\mathscr{S}'$ and $\partial_j g \in \mathscr{S}$, we have that $V_{\partial_j g}(y_k \cdot f(y))(x,\omega)$ is continuous and moderate, thus $\partial_{x_j}\partial_{\omega_k} V_g f(x,\omega)$ is continuous and moderate, too. □

If f is rapidly decreasing, then the STFT of f is rapidly decreasing, too. Moreover, the following statement holds ([Grö01]):

Theorem 1.1.4. *Let $f \in \mathscr{S}'(\mathbb{R}^d)$ and $g \in \mathscr{S}(\mathbb{R}^d)$ with $\|g\|_{L^2} \neq 0$. Then for the STFT $V_g f$ the following conditions are equivalent:*

(i) $f \in \mathscr{S}(\mathbb{R}^d)$,

(ii) $V_g f \in \mathscr{S}(\mathbb{R}^{2d})$,

(iii) *for every* $N \in \mathbb{N}$ *there exists a non-negative constant* $C \geq 0$ *such that*

$$|V_g f(x,\omega)| \leq C \cdot (1 + |x| + |\omega|)^{-N},$$

(iv) *for every* $\alpha \in \mathbb{N}_0^d$ *and* $k \in \mathbb{N}_0$ *we have that* $\partial^\alpha f$, $\langle\cdot\rangle^k f \in L^\infty$.

Remark 1.1.5. *The condition* $\|g\|_{L^2} \neq 0$ *in the above theorem is necessary since in proving it we use the inversion formula for the STFT (see Corollary 3.2.3 in [Grö01]):*

Let $f \in \mathscr{S}(\mathbb{R}^d)$. *Then for any* γ *such that* $\langle \gamma, g \rangle \neq 0$ *(in particular, for* $\gamma = g$ *and* $\|g\|_{L^2} \neq 0$*)*

$$f(\xi) = \frac{1}{\langle \gamma, g \rangle} \iint V_g f(x,\omega) M_\omega T_x \gamma(\xi) dx d\omega.$$

This holds for $f \in \mathscr{S}'(\mathbb{R}^d)$, *too, in the weak sense.*

Actually, it is possible to construct a family of seminorms for rapidly decreasing functions, by means of STFT, which generates the usual topology on $\mathscr{S}(\mathbb{R}^d)$. We obtain this family by "weighting" the STFT's by polynomials (i.e. by powers of $\langle\cdot\rangle$) with different orders. Later on we shall investigate $L^{p,q}$-norms of such weighted objects and show that $\mathscr{S}$ ($\mathscr{S}'$) equals the intersection (resp. union) of the corresponding weighted spaces, see Proposition 1.2.6.

Theorem 1.1.6. *[Grö01, Cor. 11.2.6.] Let* $0 \neq g \in \mathscr{S}(\mathbb{R}^d)$ *be a fixed window function. Set* $z := (x,\omega) \in \mathbb{R}^{2d}$. *The family of seminorms* ($m \in \mathbb{N}_0$)

$$\sup_{z \in \mathbb{R}^{2d}} |V_g f(z)| \cdot \langle z \rangle^m$$

generates the usual topology on $\mathscr{S}(\mathbb{R}^d)$.

Lemma 1.1.7. *[Grö01, Prop. 11.2.4.] Fix* $g \in \mathscr{S}(\mathbb{R}^d)$ *and assume that* $F(x,\omega)$ *is of rapid decay, that is, for all* $N \in \mathbb{N}_0$ *there is a constant* $C \geq 0$ *such that*

$$|F(x,\omega)| \leq C \cdot (1 + |x| + |\omega|)^{-N}.$$

Then the integral

$$f(\xi) = \iint F(x,\omega) M_\omega T_x g(\xi) \; dx d\omega$$

defines a function $f \in \mathscr{S}(\mathbb{R}^d)$.

1.2 Modulation Spaces.

Modulation spaces are Banach spaces whose norms allow to measure the time-frequency concentration (smoothness and decay properties of a temperate distribution). This is done via short-time Fourier transform, in $L^{p,q}$-norms. We shall state here several definitions and assertions which we will make use of later on. Most of them can be found in [Grö01] and [Tof04c]. More on modulation spaces can be found in [Tof04a] and [Tof04b].

Definition 1.2.1. *Let $0 \neq g \in \mathscr{S}(\mathbb{R}^d)$. Let $p,\, q \in [1,\infty]$. The modulation space $M^{p,q}(\mathbb{R}^d)$ is defined as the subspace of those $f \in \mathscr{S}'(\mathbb{R}^d)$ such that*

$$\|f\|_{M^{p,q}} := \|V_g f\|_{L^{p,q}} = \Big(\int \Big|\int \Big|V_g f(x,\omega)\Big|^p dx\Big|^{\frac{q}{p}} d\omega\Big)^{\frac{1}{q}} < \infty, \tag{1.1}$$

with an obvious modification in the norm $\|\cdot\|_{M^{p,q}}$ for $p=\infty$ and/or $q=\infty$. We will simply denote $M^{p,p}(\mathbb{R}^d)$ by $M^{\infty}(\mathbb{R}^d)$.

We may write the condition (1.1) in a slightly different form:

$$\|f\|_{M^{p,q}} = \|\|(\widehat{f\cdot T_x g})(\omega)\|_{L^p_x}\|_{L^q_\omega} = \|\|(\widehat{\hat{f}\cdot T_\omega \bar{\hat{g}}})(x)\|_{L^p_x}\|_{L^q_\omega},$$

which displays the correspondence of measuring decay in the L^p-sense and of measuring smoothness in L^q terms more clearly.

Definition 1.2.2. *Let $g \in \mathscr{S}(\mathbb{R}^d)$. Let m be a v-moderate function on $\mathbb{R}^{2d}$ (i.e., $m(x+y) \leq m(x)\cdot v(y)$, and v submultiplicative).The weighted modulation space $M^{p,q}_m(\mathbb{R}^d)$ is defined as the subspace of those $f \in \mathscr{S}'(\mathbb{R}^d)$ such that*

$$\|f\|_{M^{p,q}_m} := \|V_g f\|_{L^{p,q}_m} = \Big(\int \Big|\int \Big|V_g f(x,\omega) m(x,\omega)\Big|^p dx\Big|^{\frac{q}{p}} d\omega\Big)^{\frac{1}{q}} < \infty.$$

Proposition 1.2.3. *[Grö01, Thm. 11.3.5.]*

(i) $M^{p,q}_m(\mathbb{R}^d)$ is a Banach space for $p,\, q \in [1,\infty]$,

(ii) $M^{p,q}_m(\mathbb{R}^d)$ is invariant under time-frequency shifts and

$$\|T_x M_\omega f\|_{M^{p,q}_m} \leq C\cdot m(x,\omega)\cdot \|f\|_{M^{p,q}_m},$$

(iii) If $p=q$ and $m(\omega,-x) \leq C\cdot m(x,\omega)$ then M^p_m $(= M^{p,p}_m)$ is invariant under the Fourier transform.

We shall deal with modulation spaces weighted by 'standard' (polynomial) weights of the form $m(x,\omega) = \langle x\rangle^t \langle \omega\rangle^s$ (these spaces are denoted by $M^{p,q}_{s,t}(\mathbb{R}^d)$).

Proposition 1.2.4. *[Tof04c] Let $p_1 \leq p_2$, $q_1 \leq q_2$ and $m_2 \prec m_1$ (i.e., $\exists C \geq 0$ s.t. $m_2 \leq C\cdot m_1$). Then $M^{p_1,q_1}_{m_1}(\mathbb{R}^d)$ is continuously embedded in $M^{p_2,q_2}_{m_2}(\mathbb{R}^d)$, i.e.*

$$M^{p_1,q_1}_{m_1}(\mathbb{R}^d) \hookrightarrow M^{p_2,q_2}_{m_2}(\mathbb{R}^d).$$

In particular, for $s,\, t > 0$

$$M^{p,q}_{s,t}(\mathbb{R}^d) \hookrightarrow M^{p,q}(\mathbb{R}^d) \hookrightarrow M^{p,q}_{-s,-t}(\mathbb{R}^d).$$

Definition 1.2.5. *The cone $\mathscr{P}(\mathbb{R}^{2d})$ consists of all positive $m \in L^\infty_{loc}$ which are v-moderate, for some polynomial v on $\mathbb{R}$.*

Proposition 1.2.6. *[Tof04b, Remark 1.7] Assume $\Omega \subset \mathscr{P}(\mathbb{R}^{2d})$ such that for any polynomial P on $\mathbb{R}^{2d}$ there exists an element $Q \in \Omega$ such that $\frac{P}{Q}$ is bounded. Then*

$$\mathscr{S}(\mathbb{R}^d) = \bigcap_{Q\in\Omega} M^{p,q}_Q(\mathbb{R}^d)$$

and

$$\mathscr{S}'(\mathbb{R}^d) = \bigcup_{Q\in\Omega} M^{p,q}_{\frac{1}{Q}}(\mathbb{R}^d).$$

The space of rapidly decreasing functions $\mathscr{S}(\mathbb{R}^d)$ is dense in all $M^{p,q}_{s,t}(\mathbb{R}^d)$ (for finite p and q, weakly dense if $p=\infty$ and/or $q=\infty$). We have also that $M^{p,q}_{s,t}(\mathbb{R}^d)$ is isomorphic to $M^{p,q}(\mathbb{R}^d)$ for any $p,\ q\in[1,\infty]$ and any $s,\ t\in\mathbb{R}$. More precisely, in [Tof04c] Toft proved the following:

Theorem 1.2.7. *Assume that $p,\ q\in[1,\infty]$, $m_1\in\mathscr{P}(\mathbb{R}^d)$ and $m_2\in\mathscr{P}_0(\mathbb{R}^d)$ such that $m_2(x,\omega)=m_2(x)$ or $m_2(x,\omega)=m_2(\omega)$. Then $u\to m_2(x,D)u$ is a (linear) homeomorphism from $M^{p,q}_{m_1\cdot m_2}(\mathbb{R}^d)$ to $M^{p,q}_{m_1}(\mathbb{R}^d)$.*

Theorem 1.2.8. *[Tof04c] Assume that $s,\ t\in\mathbb{R}$ and that $p,q\in[1,\infty]$. Then*

$$\begin{aligned}M^{p,q}_{s,t}(\mathbb{R}^d)&=\{f\in\mathscr{S}'(\mathbb{R}^d)|\ \langle x\rangle^t\langle D\rangle^s f\in M^{p,q}(\mathbb{R}^d)\}\\&=\{f\in\mathscr{S}'(\mathbb{R}^d)|\ \langle D\rangle^s\langle x\rangle^t f\in M^{p,q}(\mathbb{R}^d)\}.\end{aligned}$$

Corollary 1.2.9. *[Tof04c] Let $s\geq 0$. Then*

$$M^{p,q}_{s,s}(\mathbb{R}^d)=M^{p,q}_{s,0}(\mathbb{R}^d)\cap M^{p,q}_{0,s}(\mathbb{R}^d).$$

Lemma 1.2.10. *Let $p,\ q\in[1,\infty)$, $s,\ t\in\mathbb{R}$. The following equivalences hold:*

$$f\in M^{p,q}_{s+1,t}\Leftrightarrow\partial^\alpha f\in M^{p,q}_{s,t}\ \textit{for all}\ |\alpha|\leq 1,$$

$$f\in M^{p,q}_{s,t+1}\Leftrightarrow(\cdot)^\alpha f\in M^{p,q}_{s,t}\ \textit{for all}\ |\alpha|\leq 1.$$

Proof. Let $f\in M^{p,q}_{s+1,t}$. We have

$$|V_gf(x,\omega)|\in L^{p,q}_{s+1,t}\Leftrightarrow|V_gf(x,\omega)|\langle\omega\rangle^{s+1}\in L^{p,q}_{0,t}.$$

$$\big(|V_gf(x,\omega)|\langle\omega\rangle^{s+1}\big)^2=\big(|V_gf(x,\omega)|\langle\omega\rangle^{s}\big)^2+\sum_{j=1}^d\big(|V_gf(x,\omega)|\langle\omega\rangle^{s}\cdot\omega_j\big)^2.\quad(1.2)$$

By Proposition 1.2.4 we have that $f(x,\omega)\in M^{p,q}_{s,t}$.

$$\begin{aligned}|V_gf(x,\omega)\omega_j|&=|\langle f(y),g(y-x)\cdot e^{2\pi iy\omega}\omega_j\rangle|\\&=|\langle f(y)\cdot g(y-x),e^{2\pi iy\omega}\omega_j\rangle|=\frac{1}{2\pi}|\langle f(y)\cdot g(y-x),\partial_{y_j}e^{2\pi iy\omega}\rangle|\\&=\frac{1}{2\pi}\Big(|\langle\partial_jf(y)\cdot g(y-x),e^{2\pi iy\omega}\rangle+\langle f(y)\cdot\partial_jg(y-x),e^{2\pi iy\omega}\rangle|\Big)\\&\qquad=\frac{1}{2\pi}\Big(|V_g\partial_jf(x,\omega)+V_{\partial_jg}f(x,\omega)|\Big).\end{aligned}$$

Hence, by (1.2)

$$\|\partial_jf\|_{M^{p,q}_{s,t}}\leq C\cdot(\|f\|_{M^{p,q}_{s+1,t}}+\|f\|_{M^{p,q}_{s,t}})<\infty.$$

We immediately conclude from

$$\big(|V_gf(x,\omega)|\langle x\rangle^{s+1}\big)^2=\big(|V_gf(x,\omega)|\langle x\rangle^{s}\big)^2+\sum_{j=1}^d\big(|V_gf(x,\omega)|\langle x\rangle^{s}\cdot x_j\big)^2$$

that the second assertion holds true as well. $\square$

Example 1. *Let p, $q \in [1, \infty]$ and $t \in \mathbb{R}$ and $s < -1/q$. Then*

$$\delta \in M^{p,q}_{s,t}(\mathbb{R}).$$

Proof. Let $0 \neq g \in \mathscr{S}$. We have

$$\langle \delta, M_\omega T_x g \rangle = \langle \delta(y), e^{2\pi i y \omega} g(y-x) \rangle = g(-x).$$

Thus, we have

$$\|\delta\|_{M^{p,q}_{s,t}} = \Big(\int \Big| \int |g(-x)|^p \langle x \rangle^{pt} dx \Big|^{\frac{q}{p}} \langle \omega \rangle^{qs} d\omega \Big)^{\frac{1}{q}} = \|g(-x)\langle x \rangle^t\|_p \cdot \|\langle \omega \rangle^s\|_q$$

□

Example 2. *Let p, $q \in [1, \infty]$ and $t \in \mathbb{R}$ and $s < -1/q - 1$. Then*

$$H \in M^{p,q}_{s,t}(\mathbb{R}).$$

Proof. Since we have $\partial H = \delta$, the claim follows from Lemma 1.2.10. □

Example 3. *Let p, $q \in [1, \infty]$ and $s \in \mathbb{R}$ and $t < -(1/p+1)$. Let $f(x) = \mathrm{const}$. Then*

$$f \in M^{p,q}_{s,t}(\mathbb{R}).$$

Proof.

$$\Big| \int f(y) \cdot \overline{g(y-x)} e^{-2\pi i y \omega} dy \Big| = C \cdot \Big| \int \overline{g(y-x)} \cdot \frac{(1 + \triangle_y)^k (e^{-2\pi i y \omega})}{\langle \omega \rangle^{2k}} dy \Big|$$

$$\leq C \cdot \sum_{|\alpha| \leq 2k} \Big| \int \overline{\partial^\alpha g(y-x)} \cdot \frac{e^{-2\pi i y \omega}}{\langle \omega \rangle^{2k}} dy \Big| \leq C \cdot \sum_{|\alpha| \leq 2k} \int \langle y \rangle^{-m} \langle x \rangle^m \cdot \langle \omega \rangle^{-2k} dy,$$

and thus,

$$\|f\|_{M^{p,q}_{s,t}} \leq C \cdot \sum_{|\alpha| \leq 2k} \|\langle x \rangle^{m+t}\|_p \cdot \|\langle \omega \rangle^{-2k+s}\|_q.$$

□

Proposition 1.2.11. *[Tof04c] Let p, $q \in [1, \infty]$ and m a weight function. The inner product $\langle \cdot, \cdot \rangle$ on L^2 extends to a bilinear, continuous mapping*

$$M^{p,q}_m \times M^{p',q'}_{1/m} \to \mathbb{C}.$$

Let us now consider the set $C^\infty_{M^{p,q}}(\mathbb{R}^d)$ of all $u \in M^{p,q}(\mathbb{R}^d)$ such that for every multi-index α, $\partial^\alpha u \in M^{p,q}(\mathbb{R}^d)$. The derivatives are distributional, but it will be shown that they are actually classical and that this set is equal to $H^p_\infty(\mathbb{R}^d)$. We denote by $C^k_B(\mathbb{R}^d)$ the set of all k-times continuously differential functions all of whose partial derivatives (up to order k) are bounded.

Theorem 1.2.12. *Let p, $q \in [1, \infty]$, $s_0 \in \mathbb{R}^+$, $k \in \mathbb{N}$ and $s_0 - k > \frac{3}{2}d$. Then $M^{p,q}_{s_0,0}(\mathbb{R}^d) \subset C^{k-1}_B(\mathbb{R}^d)$. In particular, $C^\infty_{M^{p,q}}(\mathbb{R}^d) = H^p_\infty(\mathbb{R}^d) \subset C^\infty_B(\mathbb{R}^d)$.*

Proof. By [Tof04c] we have

$$H^p_{s+\mu d\theta_1(p,q)}(\mathbb{R}^d) \subset M^{p,q}_{s,0}(\mathbb{R}^d) \subset H^p_{s+\mu d\theta_2(p,q)}(\mathbb{R}^d), \quad \mu > 1,$$

where

$$\theta_1(p,q) = max\Big(0, \frac{1}{q} - min\Big(\frac{1}{p}, \frac{1}{p'}\Big)\Big),$$
$$\theta_2(p,q) = min\Big(0, \frac{1}{q} - max\Big(\frac{1}{p}, \frac{1}{p'}\Big)\Big), \quad p, q \in [1, \infty].$$

On the other hand for all p, $q \in [1,\infty]$ we have that $\theta_i(p,q) \in [-1,1]$ $(i = 1,2)$. Hence,

$$M^{p,q}_{s,0}(\mathbb{R}^d) \subset H^p_{s+\mu d\theta_2(p,q)}(\mathbb{R}^d) \subset H^p_{s-\mu d}(\mathbb{R}^d) \subset C_B^{\lfloor s-(\mu+\frac{1}{2})d\rfloor}(\mathbb{R}^d).$$

In particular

$$\bigcap_{s\geq s_0} H^p_{s+\mu d\theta_1(p,q)}(\mathbb{R}^d) \subset \bigcap_{s\geq s_0} M^{p,q}_{s,0}(\mathbb{R}^d) \subset \bigcap_{s\geq s_0} H^p_{s+\mu d\theta_2(p,q)}(\mathbb{R}^d) \subset C^\infty_B(\mathbb{R}^d).$$

□

1.3 Colombeau Algebras.

Definition 1.3.1 (Moderate Functions)**.**

$$\mathcal{O}_M(\mathbb{R}^d) := \{f \in C^\infty(\mathbb{R}^d) |\ \forall\alpha \in \mathbb{N}_0^d\ \exists N \in \mathbb{N}:\ \sup\langle x\rangle^{-N}|\partial^\alpha f(x)| < \infty\}$$

Definition 1.3.2. *The Colombeau algebra of tempered generalized functions is defined as the quotient space*

$$\mathcal{G}_\tau(\mathbb{R}^d) := \mathcal{E}_\tau(\mathbb{R}^d)/\mathcal{N}_\tau(\mathbb{R}^d)$$

where

$$\mathcal{E}_\tau(\mathbb{R}^d) := \{(u_\varepsilon)_\varepsilon \in \mathcal{O}_M(\mathbb{R}^d)^{(0,1]} \big|\ \forall\alpha \in \mathbb{N}_0^n\ \exists N \in \mathbb{N}\ \sup_{x\in\mathbb{R}^d} \langle x\rangle^{-N}\big|\partial^\alpha u_\varepsilon\big| = \mathcal{O}(\varepsilon^{-N})\ \text{as}\ \varepsilon \to 0\}$$

and

$$\mathcal{N}_\tau(\mathbb{R}^d) := \{(u_\varepsilon)_\varepsilon \in \mathcal{O}_M(\mathbb{R}^d)^{(0,1]} \big|\ \forall\alpha \in \mathbb{N}_0^d\ \exists N \in \mathbb{N}\ \forall m \in \mathbb{N}_0\ \sup_{x\in\mathbb{R}^d} \langle x\rangle^{-N}\big|\partial^\alpha u_\varepsilon\big| = \mathcal{O}(\varepsilon^{m})\ \text{as}\ \varepsilon \to 0\}.$$

Definition 1.3.3. *Let E be a locally convex topological vector space topologized through the family of seminorms $\{p_j\}_{j\in J}$. The elements of*

$$\mathcal{M}_\mathcal{E} := \{(u_\varepsilon)_\varepsilon \in E^{(0,1]} \big|\ \forall j \in J\ \exists N \in \mathbb{N}\ p_j(u_\varepsilon) = \mathcal{O}(\varepsilon^{-N})\ \text{as}\ \varepsilon \to 0\}$$

and

$$\mathcal{N}_\mathcal{E} := \{(u_\varepsilon)_\varepsilon \in E^{(0,1]} \big|\ \forall j \in J\ \forall m \in \mathbb{N}\ p_j(u_\varepsilon) = \mathcal{O}(\varepsilon^{m})\ \text{as}\ \varepsilon \to 0\},$$

are called E-moderate and E-negligible respectively (J denotes an index-set). We define the space of generalized functions based on E as the factor space $\mathcal{G}_E = \mathcal{M}_E/\mathcal{N}_E$.

The space of Schwartz functions is a locally convex space topologized through seminorms $\{p_j\}_{j\in\mathbb{N}}$, where $p_j(f) := \max_{|\alpha+\beta|\le j} \sup_{x\in\mathbb{R}^d} |x^\alpha \partial^\beta f(x)|$. Therefore:

Definition 1.3.4. *The Colombeau algebra of generalized functions based on $\mathscr{S}$ is defined as the quotient space*

$$\mathscr{G}_{\mathscr{S}}(\mathbb{R}^d) := \mathscr{E}_{\mathscr{S}}(\mathbb{R}^d)/\mathscr{N}_{\mathscr{S}}(\mathbb{R}^d)$$

where

$$\mathscr{E}_{\mathscr{S}}(\mathbb{R}^d) := \{(u_\varepsilon)_\varepsilon \in \mathscr{S}(\mathbb{R}^d)^{(0,1]} \big| \ \forall \alpha,\beta \in \mathbb{N}_0^d \ \exists N \in \mathbb{N} \\ \sup_{x\in\mathbb{R}^d} \left|x^\alpha \partial^\beta u_\varepsilon\right| = \mathcal{O}(\varepsilon^{-N}) \ \textit{as } \varepsilon \to 0\}$$

and

$$\mathscr{N}_{\mathscr{S}}(\mathbb{R}^d) := \{(u_\varepsilon)_\varepsilon \in \mathscr{S}(\mathbb{R}^d)^{(0,1]} \big| \ \forall \alpha,\beta \in \mathbb{N}_0^d \ \forall m \in \mathbb{N}_0 \\ \sup_{x\in\mathbb{R}^d} \left|x^\alpha \partial^\beta u_\varepsilon\right| = \mathcal{O}(\varepsilon^{m}) \ \textit{as } \varepsilon \to 0\}.$$

The following results provide an important characterization of Colombeau ideals as subspaces of the respective space of moderate nets.

Proposition 1.3.5. *[Gar05a]. $(u_\varepsilon)_\varepsilon \in \mathscr{E}_{\mathscr{S}}(\mathbb{R}^d)$ is $\mathscr{S}$-negligible if and only if the following condition is satisfied:*

$$\forall m \in \mathbb{N}_0 \quad \sup_{x\in\mathbb{R}^d} |u_\varepsilon(x)| = \mathcal{O}(\varepsilon^m) \qquad \textit{as } \varepsilon \to 0.$$

Proposition 1.3.6. *$(u_\varepsilon)_\varepsilon \in \mathscr{E}_\tau(\mathbb{R}^d)$ is τ-negligible, i.e., if and only if:*

$$\exists N \in \mathbb{N} \ \forall m \in \mathbb{N}_0 \quad \sup_{x\in\mathbb{R}^d} \langle x\rangle^{-N} |u_\varepsilon(x)| = \mathcal{O}(\varepsilon^m) \qquad \textit{as } \varepsilon \to 0.$$

For a proof see [GKOS01, p. 27].

An important issue in the theory of generalized functions is a notion of regularity. We shall make use of the regular algebra corresponding to Colombeau algebras based on $\mathscr{S}$:

Definition 1.3.7. *The Colombeau algebra of $\mathscr{S}$-regular generalized functions is defined as the quotient space*

$$\mathscr{G}^\infty_{\mathscr{S}}(\mathbb{R}^d) := \mathscr{E}^\infty_{\mathscr{S}}(\mathbb{R}^d)/\mathscr{N}_{\mathscr{S}}(\mathbb{R}^d)$$

where

$$\mathscr{E}^\infty_{\mathscr{S}}(\mathbb{R}^d) := \{(u_\varepsilon)_\varepsilon \in \mathscr{S}(\mathbb{R}^d)^{(0,1]} \big| \ \exists N \in \mathbb{N} \ \forall \alpha,\beta \in \mathbb{N}_0^d \\ \sup_{x\in\mathbb{R}^d} \left|x^\alpha \partial^\beta u_\varepsilon\right| = \mathcal{O}(\varepsilon^{-N}) \ \textit{as } \varepsilon \to 0\}.$$

Garetto investigated relations between $\mathscr{G}^\infty_{\mathscr{S}}(\mathbb{R}^d)$ and $\mathscr{S}'(\mathbb{R}^d)$ and concluded the following (cf. [Gar04a]) important relation (analog to the one in [Obe92b]), which justifies that the subalgebra $\mathscr{G}^\infty_{\mathscr{S}}(\mathbb{R}^d)$ is the right algebra for detecting regularity on the level of generalized functions based on $\mathscr{S}(\mathbb{R}^d)$:

Theorem 1.3.8.

$$\mathscr{G}^\infty_{\mathscr{S}}(\mathbb{R}^d) \cap \mathscr{S}'(\mathbb{R}^d) = \mathscr{S}(\mathbb{R}^d).$$

Chapter 2

Modulation spaces in the Colombeau algebra

2.1 Preparation for constructing a differential algebra

Definition 2.1.1. *Let $\rho \in \mathscr{S}(\mathbb{R}^d)$ such that $\int \rho(x)dx = 1$ and $\int \rho(x) \cdot x^\alpha = 0$ (vanishing moments) for all multi-indices $0 \neq \alpha \in \mathbb{N}_0^d$. Then ρ is called a* mollifier with vanishing moments.

Remark 2.1.2. *The definition of the mollifier varies (i.e. depending on the space we generalize and on the constructing algebra, we may have that there is no condition on vanishing moments and/or $\rho \in \mathscr{D}$) - in our case the standard mollifier will be a mollifier with vanishing moments.*

Our goal is to find an embedding of any modulation space $M^{p,q}$ (p, $q \in [1,\infty)$) into a differential (quotient) algebra in the sense of Colombeau theory. The standard procedure would be to convolve elements of $M^{p,q}$ with the function $\rho_\varepsilon(x) := \frac{1}{\varepsilon^d}\rho(\frac{x}{\varepsilon})$, ($\rho$ is a mollifier) in order to obtain nets of smooth functions which satisfy some moderateness conditions. These conditions involve estimating the growth of derivatives of nets $(u_\varepsilon)_\varepsilon$ in $M^{p,q}$-norms by powers of ε.

Let ρ_ε be as above. Let $g \in \mathscr{S}(\mathbb{R}^d)$ be a window function. Set $D_j := \frac{1}{2\pi i}\partial_j$ and $\triangle = \sum_{j=1}^d D_j^2$. Then

$$(1+\triangle_y)^k(e^{-2\pi i y\omega}) = \langle\omega\rangle^{2k} \cdot e^{-2\pi i y\omega}, \text{ so}$$

$$e^{-2\pi i y\omega} = \frac{(1+\triangle_y)^k(e^{-2\pi i y\omega})}{\langle\omega\rangle^{2k}}.$$

By integration by parts we obtain

$$
\begin{aligned}
\big|V_g\rho_\varepsilon(x,\omega)\big| &= \Big|\int \rho_\varepsilon(y)\overline{g(y-x)}e^{-2\pi i y\omega}\;dy\Big| \\
&= \Big|\int \rho_\varepsilon(y)\overline{g(y-x)}\frac{(1+\triangle_y)^k(e^{-2\pi i y\omega})}{\langle\omega\rangle^{2k}}\;dy\Big| \\
&= \Big|\int (1+\triangle_y)^k\big(\rho_\varepsilon(y)\overline{g(y-x)}\big)\frac{(e^{-2\pi i y\omega})}{\langle\omega\rangle^{2k}}\;dy\Big| \\
&\le C\cdot \sum_{|\alpha|,|\beta|\le 2k}\int |\partial^\alpha\rho_\varepsilon(y)|\cdot|\partial_y^\beta g(y-x)|\;dy\cdot\frac{1}{\langle\omega\rangle^{2k}}. \qquad (2.1)
\end{aligned}
$$

For every $\beta\in\mathbb{N}_0^d$, $m\in\mathbb{N}_0$ there exists a constant $C\ge 0$ such that

$$\big|\partial^\beta g(y-x)\big|\le\frac{C}{\langle y-x\rangle^m}.$$

This together with Peetre's inequality gives

$$\big|\partial^\beta g(y-x)\big|\le C\cdot\langle y\rangle^m\langle x\rangle^{-m}.$$

Therefore, we obtain

$$\big|V_g\rho_\varepsilon(x,\omega)\big|\le C\cdot\sum_{|\alpha|\le 2k}\int\Big|\partial^\alpha\rho_\varepsilon(y)\Big|\langle y\rangle^m dy\cdot\langle\omega\rangle^{-2k}\langle x\rangle^{-m}.$$

Also,

$$\int\Big|\partial^\alpha\rho_\varepsilon(y)\Big|\langle y\rangle^m dy=\varepsilon^{-|\alpha|}\int\Big|\partial^\alpha\rho(y)\Big|\langle\varepsilon\cdot y\rangle^m dy\le C\cdot\varepsilon^{-2k},$$

since $g\in\mathscr{S}(\mathbb{R}^d)$ and $\langle\varepsilon\cdot y\rangle\le\langle y\rangle$. Of course we could have examined derivatives of ρ in a similar way. We conclude the following:

Lemma 2.1.3. *For each m, $k\in\mathbb{N}_0$ and any multi-index $\alpha\in\mathbb{N}_0^d$ there exists a constant $C\ge 0$ such that*

$$|V_g\partial^\alpha\rho_\varepsilon(x,\omega)|\le C\cdot\langle x\rangle^{-m}\langle\omega\rangle^{-2k-|\alpha|}\cdot\varepsilon^{-2k-|\alpha|}$$

holds for all $(x,\omega)\in\mathbb{R}^{2d}$.

In particular, for each p, $q\in[1,\infty]$ and each s, $t\in\mathbb{R}$ there exists $k\in\mathbb{N}$ such that

$$\|\partial^\alpha\rho_\varepsilon\|_{M^{p,q}_{s,t}}\le C\cdot\varepsilon^{-2k-|\alpha|},$$

for some $C\ge 0$.

Remark 2.1.4. *Note that for any $p\in[1,\infty]$ and $t\in\mathbb{R}$ we have that $\|\rho_\varepsilon\|_{M^{p,\infty}_{0,t}}$ is bounded by a constant C independent of ε.*

Lemma 2.1.5. *For any $f\in\mathscr{S}(\mathbb{R}^d)$ and every multi-index α; k, l, $m\in\mathbb{N}_0$ there exists a constant $C\ge 0$ such that*

$$|V_g(\partial^\alpha f*\rho_\varepsilon-\partial^\alpha f)(x,\omega)|\le C\cdot\varepsilon^m\langle x\rangle^{-l}\langle\omega\rangle^{-2k} \qquad (2.2)$$

holds for all $(x,\omega)\in\mathbb{R}^{2d}$ (ε small enough).

Proof.

$$
\begin{aligned}
|V_g(f * \rho_\varepsilon - f)(x,\omega)| &= \Big| \int (f * \rho_\varepsilon - f)(y) \cdot \overline{g(y-x)} \cdot e^{-2\pi i y\omega} dy \Big| \\
&\leq C \cdot \Big| \int (1+\triangle_y)^k [(f * \rho_\varepsilon - f)(y) \cdot \overline{g(y-x)}] \cdot \frac{e^{-2\pi i y\omega}}{\langle\omega\rangle^{2k}} dy \Big| \\
&\leq C \cdot \sum_{|\alpha+\beta|\leq 2k} \int |\partial^\alpha (f * \rho_\varepsilon - f)(y) \cdot \overline{\partial^\beta g(y-x)} \cdot \langle\omega\rangle^{-2k}| dy \qquad (2.3)
\end{aligned}
$$

It therefore suffices to estimate

$$
\int |(f * \rho_\varepsilon - f)(y)| \cdot |g(y-x)| dy.
$$

The fact that $g \in \mathscr{S}(\mathbb{R}^d)$ and Peetre's inequality ensure that for every $l \in \mathbb{N}$ there exists a constant $C \geq 0$ such that

$$
|g(y-x)| \leq C \cdot \langle y\rangle^l \langle x\rangle^{-l}.
$$

Hence,

$$
\int |(f * \rho_\varepsilon - f)(y)| \cdot |g(y-x)| dy \leq C \cdot \langle x\rangle^{-l} \int |(f * \rho_\varepsilon - f)(y)| \langle y\rangle^l dy. \qquad (2.4)
$$

By change of variables,

$$
\begin{aligned}
(f * \rho_\varepsilon - f)(y) &= \int f(y-\eta)\rho_\varepsilon(\eta) d\eta - f(y) \\
&= \int f(y-\eta)\rho_\varepsilon(\eta) d\eta - f(y) \cdot \int \rho_\varepsilon(\eta)\ d\eta \\
&= \int \big(f(y-\eta) - f(y)\big) \cdot \rho_\varepsilon(\eta)\ d\eta \\
&= \int \big(f(y-\varepsilon\eta) - f(y)\big) \cdot \rho(\eta)\ d\eta.
\end{aligned}
$$

Further, by Taylor expansion we have

$$
\begin{aligned}
(f * \rho_\varepsilon - f)(y) = &\sum_{0<|\alpha|\leq m-1} \int \frac{\partial^\alpha f(y)}{\alpha!} \cdot (-\varepsilon\eta)^\alpha \rho(\eta) d\eta \\
&+ \sum_{|\alpha|=m} \int \int_0^1 \frac{\partial^\alpha f(y-\varepsilon\theta\eta)}{(m-1)!} \cdot (1-\theta)^{m-1} d\theta \cdot (-\varepsilon\eta)^\alpha \rho(\eta) d\eta.
\end{aligned}
$$

Since $\int \rho(\eta)\eta^\alpha = 0$ for every multi-index α with $|\alpha| > 0$ we have

$$
(f * \rho_\varepsilon - f)(y) = \sum_{|\alpha|=m} \int \int_0^1 \frac{\partial^\alpha f(y-\varepsilon\theta\eta)}{(m-1)!} \cdot (1-\theta)^{m-1} d\theta \cdot (-\varepsilon\eta)^\alpha \rho(\eta) d\eta. \qquad (2.5)
$$

Now we estimate $\partial^\alpha f(y-\varepsilon\theta\eta)$ similarly as we did $\partial^\beta g$ before (Peetre's inequality): for every multi-index α there exists constant $C \geq 0$ such that

$$|\partial^\alpha f(y-\varepsilon\theta\eta)| \leq C\cdot\langle y\rangle^{-l-d-1}\langle\varepsilon\theta\eta\rangle^{l+d+1} \leq C\cdot\langle y\rangle^{-l-d-1}\langle\eta\rangle^{l+d+1}.$$

Consequently,

$$|(f*\rho_\varepsilon - f)(y)| \leq C\cdot\varepsilon^m\cdot\langle y\rangle^{-l-d-1}|\int\langle\eta\rangle^{l+d+1+m}\rho(\eta)d\eta|. \tag{2.6}$$

By (2.4) and (2.6) we have

$$\int|(f*\rho_\varepsilon-f)(y)|\cdot|g(y-x)|dy \leq C\cdot\varepsilon^m\cdot\langle x\rangle^{-l}\cdot\int\langle y\rangle^{-d-1}dy\cdot\int\langle\eta\rangle^{m+l+d+1}\rho(\eta)d\eta\,.$$

We have $|\int\langle y\rangle^{-d-1}dy| < \infty$, as well as $|\int\langle\eta\rangle^{m+l+d+1}\rho(\eta)d\eta| < \infty$ (due to the vanishing higher moments of ρ). This yields the estimate (2.2), thus the claim holds. □

Before we make sure that $\|f*\rho_\varepsilon - f\|_{M^{p,q}_{s,t}} \to 0$ for $f \in M^{p,q}_{s,t}$, we state a useful theorem.

Theorem 2.1.6. *[Tof04a] Assume that p_j, $q_j \in [1,\infty]$, $m_j \in \mathscr{P}$ $(0 \leq j \leq N)$ satisfy*

$$\frac{1}{p_1}+\ldots+\frac{1}{p_N} = N-1+\frac{1}{p_0}, \tag{2.7}$$

$$\frac{1}{q_1}+\ldots+\frac{1}{q_N} = \frac{1}{q_0}, \tag{2.8}$$

and

$$m_0(x_1+\ldots+x_N,\omega) \leq C\cdot m_1(x_1,\omega)\cdot\ldots\cdot m_N(x_N,\omega).$$

Then $(u_1,\ldots,u_N) \to u_1*\ldots*u_N$ is a continuous, symmetric and associative map from $M^{p_1,q_1}_{m_1}\times\ldots\times M^{p_N,q_N}_{m_N}$ to $M^{p_0,q_0}_{m_0}$. There exists a constant $C \geq 0$ (depending only on the window function, the number of factors and the dimension) such that

$$\|u_1*\ldots*u_N\|_{M^{p_0,q_0}_{m_0}} \leq C\cdot\prod_{1\leq j\leq N}\|u_j\|_{M^{p_j,q_j}_{m_j}}.$$

Corollary 2.1.7. *Let p, p', q, $q' \in [1,\infty]$ such that $1/p+1/p' = 1/q+1/q' = 1$ and s, $t \in \mathbb{R}$. There exists a constant $C \geq 0$ such that*

$$\|u*v\|_{M^{p,q}_{s,t}} \leq C\cdot\|u\|_{M^{p,q}_{s,t}}\cdot\|v\|_{M^{1,\infty}_{0,|t|}}$$

and

$$\|u*v\|_{M^{\infty,1}_{s,t}} \leq C\cdot\|u\|_{M^{p,q}_{s,t}}\cdot\|v\|_{M^{p',q'}_{0,|t|}}$$

for any $u \in M^{p,q}_{s,t}$, $v \in M^{1,\infty}_{0,|t|}$ resp. $u \in M^{p,q}_{s,t}$, $v \in M^{p',q'}_{0,|t|}$.

Lemma 2.1.8. *Let p, $q \in [1,\infty)$, s, $t \in \mathbb{R}$ and $\rho \in \mathscr{S}(\mathbb{R}^d)$ as above. Then for every $f \in M^{p,q}_{s,t}(\mathbb{R}^d)$ we have that*

$$\|f*\rho_\varepsilon - f\|_{M^{p,q}_{s,t}(\mathbb{R}^d)} \to 0, \quad (\varepsilon\to 0).$$

Proof. By Proposition 11.3.4 in [Grö01], for any $f \in M^{p,q}_{s,t}(\mathbb{R}^d)$ there exists a sequence $(f_k)_{k\in\mathbb{N}} \in \mathscr{S}(\mathbb{R}^d)$ such that

$$\|f_k - f\|_{M^{p,q}_{s,t}(\mathbb{R}^d)} \to 0. \tag{2.9}$$

We may write now the following estimate:

$$\begin{aligned}
\|f * \rho_\varepsilon - f\|_{M^{p,q}_{s,t}} &\le \|f * \rho_\varepsilon - f_k * \rho_\varepsilon\|_{M^{p,q}_{s,t}} \\
&\quad + \|f_k * \rho_\varepsilon - f_k\|_{M^{p,q}_{s,t}} + \|f_k - f\|_{M^{p,q}_{s,t}} \\
&\le C \cdot \|f - f_k\|_{M^{p,q}_{s,t}} \cdot \|\rho_\varepsilon\|_{M^{1,\infty}_{0,|t|}} \\
&\quad + \|f_k * \rho_\varepsilon - f_k\|_{M^{p,q}_{s,t}} + \|f_k - f\|_{M^{p,q}_{s,t}}.
\end{aligned}$$

Since $\|\rho_\varepsilon\|_{M^{1,\infty}_{0,|t|}}$ is bounded by a constant independent of ε, we have

$$\|f * \rho_\varepsilon - f\|_{M^{p,q}_{s,t}} \le C \cdot \|f - f_k\|_{M^{p,q}_{s,t}} + \|f_k * \rho_\varepsilon - f_k\|_{M^{p,q}_{s,t}}.$$

The claim follows now from (2.9) and Lemma 2.1.5. □

Lemma 2.1.3 and Corollary 2.1.7 together imply:

Theorem 2.1.9. *Let $u \in M^{p,q}_{s,t}$, s, $t \in \mathbb{R}$, α a multi-index. Then $\partial^\alpha u * \rho_\varepsilon \in M^{p,q}_{s,t} \cap M^{\infty,1}_{s,t}$ and there exist constants $C \ge 0$ and $k \in \mathbb{N}^d$ such that*

$$max\{\|\partial^\alpha u * \rho_\varepsilon\|_{M^{\infty,1}_{s,t}}, \|\partial^\alpha u * \rho_\varepsilon\|_{M^{p,q}_{s,t}}\} \le C \cdot \varepsilon^{-N} \|u\|_{M^{p,q}_{s,t}}.$$

2.1.1 Embedding theorems for modulation spaces

Let us now consider the $M^{\infty,1}$-norm of a distribution f and its connection to the $M^{p,q}$-norm of the derivatives of f. As in (2.1) we calculate (now for $f \in \mathscr{S}'(\mathbb{R}^d)$):

$$\begin{aligned}
\big|V_g f(x,\omega)\big| &\le C \cdot \sum_{|\alpha|,|\beta|\le 2k} \big|\langle \partial^\alpha f(y), \partial_y^\beta\big(g(y-x)\big)e^{2\pi i y\omega}\ \rangle\big| \cdot \frac{1}{\langle\omega\rangle^{2k}} \\
&= C \cdot \sum_{|\alpha|,|\beta|\le 2k} \big|V_{\partial^\beta g}(\partial^\alpha f)(x,\omega)\big| \cdot \frac{1}{\langle\omega\rangle^{2k}}.
\end{aligned}$$

Hence the estimate

$$\big|V_g f(x,\omega)\cdot m(x,\omega)\big| \le C \cdot \sum_{|\alpha|,|\beta|\le 2k} \big|V_{\partial^\beta g}\partial^\alpha f(x,\omega)\big| \cdot \frac{m(x,\omega)}{\langle\omega\rangle^{2k}} \tag{2.10}$$

holds for any weight function $m \in \mathscr{P}$. Further we have

$$\begin{aligned}
\sup_x \big|V_g f(x,\omega)\cdot m(x,\omega)\big| &\le C \cdot \sum_{|\alpha|,|\beta|\le 2k} \sup_x \big|V_{\partial^\beta g}\partial^\alpha f(x,\omega)\cdot m(x,\omega)\big| \cdot \frac{1}{\langle\omega\rangle^{2k}} \\
&\le C \cdot \sum_{|\alpha|,|\beta|\le 2k} \|\partial^\alpha f\|_{M^\infty_m} \cdot \frac{1}{\langle\omega\rangle^{2k}}.
\end{aligned}$$

After integrating with respect to ω we obtain

$$\|f\|_{M_m^{\infty,1}} \leq C \cdot \sum_{|\alpha|,|\beta|\leq 2k} \|\partial^\alpha f\|_{M_m^\infty} \cdot \int \frac{1}{\langle\omega\rangle^{2k}} d\omega.$$

We may also write

$$\|f\|_{M_m^{\infty,1}} \leq C \cdot \sum_{|\alpha|,|\beta|\leq 2k} \|\partial^\alpha f\|_{M_{m\cdot n}^\infty} \cdot \int \frac{1}{\langle\omega\rangle^{2k-l}} d\omega,$$

where $n(x,\omega) = \langle\omega\rangle^{-l}$, $l \in \mathbb{N}$ arbitrary and k is large enough ($2k > d+l$). Since for $p \leq p'$ and $q \leq q'$ we have $M^{p,q} \hookrightarrow M^{p',q'}$, the following claim holds:

Lemma 2.1.10. *Let $m \in \mathscr{P}$. For any p, $q \in [1,\infty]$ and k, $l \in \mathbb{N}$ ($2k > d+l$) there exists a constant $C \geq 0$ such that*

$$\|f\|_{M_m^{\infty,1}} \leq C \cdot \sum_{|\alpha|\leq 2k} \|\partial^\alpha f\|_{M_{m\cdot n}^{p,q}}$$

holds for every $f \in \mathscr{S}(\mathbb{R}^d)$, where $n(x,\omega) := \langle\omega\rangle^{-l}$.

Corollary 2.1.11. *Let p, $q \in [1,\infty]$ and $s \in \mathbb{R}$, $t \geq 0$. Then there exist constants $C \geq 0$ and $k \in \mathbb{N}$ such that*

$$\|f\|_{M^{\infty,1}} \leq C \cdot \sum_{|\alpha|\leq 2k} \|\partial^\alpha f\|_{M_{s,t}^{p,q}}$$

holds for every $f \in \mathscr{S}(\mathbb{R}^d)$.

By using the same methods as in Lemma 2.1.10 we now give a proof of the following:

Lemma 2.1.12. *Let $m \in \mathscr{P}$. For any p, $q \in [1,\infty]$, k, $l \in \mathbb{N}$ ($2k > l + d/q$) there exists a constant $C \geq 0$ such that*

$$\|f\|_{M_m^{p,q}} \leq C \cdot \sum_{|\alpha|\leq 2k} \|\partial^\alpha f\|_{M_{m\cdot n}^{p,\infty}} \tag{2.11}$$

holds for every $f \in \mathscr{S}$, where $n(x,\omega) := \langle\omega\rangle^{-l}$.

Proof. Take the L^p-norm with respect to x on both sides of the estimate (2.10) to obtain

$$\begin{aligned}
\|V_g f(\cdot,\omega)\cdot m(\cdot,\omega)\|_{L^p} &\leq C \cdot \Big\| \sum_{|\alpha|,|\beta|\leq 2k} \left|V_{\partial^\beta g}\partial^\alpha f(\cdot,\omega)\right| \cdot m(\cdot,\omega)\Big\|_{L^p} \cdot \frac{1}{\langle\omega\rangle^{2k}} \\
&\leq C \cdot \sum_{|\alpha|,|\beta|\leq 2k} \|V_{\partial^\beta g}\partial^\alpha f(\cdot,\omega)\cdot m(\cdot,\omega)\|_{L^p} \cdot \frac{1}{\langle\omega\rangle^{2k}} \\
&\leq C \cdot \sum_{|\alpha|,|\beta|\leq 2k} \sup_{\omega\in\mathbb{R}^d} \|V_{\partial^\beta g}\partial^\alpha f(\cdot,\omega)\cdot m(\cdot,\omega)\cdot\langle\omega\rangle^{-l}\|_{L^p} \cdot \frac{1}{\langle\omega\rangle^{2k-l}} \\
&\leq C \cdot \sum_{|\alpha|,|\beta|\leq 2k} \|\partial^\alpha f\|_{M_{m\cdot n}^{p,\infty}} \cdot \frac{1}{\langle\omega\rangle^{2k-l}}.
\end{aligned}$$

Finally, taking the L^q-norm with respect to ω on both sides of the latter estimate we obtain the estimate (2.11). □

Corollary 2.1.13.

$$C^\infty_{M^{p,q}_{s,t}} = C^\infty_{M^{p,1}_{0,t}}$$

Proof. This follows from Proposition 1.2.4 and Lemma 2.1.12. □

Theorem 2.1.14. *[Tof04a] The multiplication $(u_1 \cdot u_2 \cdot \ldots \cdot u_n)$ on $\mathscr{S}(\mathbb{R}^d)$ extends in a natural way to a continuous mapping from $M^{p_1,q_1} \times \ldots \times M^{p_N,q_N}$ to M^{p_0,q_0}, provided p_j, $q_j \in [1,\infty]$ satisfy*

$$\frac{1}{p_1} + \ldots + \frac{1}{p_N} = \frac{1}{p_0}, \tag{2.12}$$

$$\frac{1}{q_1} + \ldots + \frac{1}{q_N} = N - 1 + \frac{1}{q_0}. \tag{2.13}$$

There exists a constant $C \geq 0$ (depending only on the window function, the number of factors and the dimension) such that

$$\|u_1 \cdot \ldots \cdot u_N\|_{M^{p_0,q_0}} \leq C \cdot \prod_{1 \leq j \leq N} \|u_j\|_{M^{p_j,q_j}}$$

holds.

Corollary 2.1.15. *Let p, $q \in [1,\infty]$ and s, $t \in \mathbb{R}$, $u \in M^{p,q}_{s,t}$, $v \in M^{\infty,1}$. There exists a constant $C \geq 0$ such that*

$$\|u \cdot v\|_{M^{p,q}_{s,t}} \leq C \cdot \|u\|_{M^{p,q}_{s,t}} \cdot \|v\|_{M^{\infty,1}} \tag{2.14}$$

holds for any $u \in M^{p,q}$, $v \in M^{\infty,1}$. In particular, we have (for $p = \infty$ and $q = 1$)

$$\|u \cdot v\|_{M^{\infty,1}_{s,t}} \leq C \cdot \|u\|_{M^{\infty,1}_{s,t}} \cdot \|v\|_{M^{\infty,1}}.$$

Corollary 2.1.16. *Let $u \in M^{p_1,q_1}_{s_1,t_1}$, $v \in C^\infty_{M^{p_2,q_2}_{s_2,t_2}}$, where p_1, p_2, q_1, $q_2 \in [1,\infty]$ and s_1, s_2, t_1, $t_2 \in \mathbb{R}$ with $t_2 \geq 0$. Then $u \cdot v \in M^{p_1,q_1}_{s_1,t_1}$. In particular, for $u \in C^\infty_{M^{p_1,q_1}_{s_1,t_1}}$ we obtain $u \cdot v \in C^\infty_{M^{p_1,q_1}_{s_1,t_1}} \cap C^\infty_{M^{p_2,q_2}_{s_2,t_2}}$.*

Proof. By Corollaries 2.1.11 and 2.1.15 there exists a constant $C \geq 0$ such that

$$\|u \cdot v\|_{M^{p_1,q_1}_{s_1,t_1}} \leq C \cdot \sum_\alpha \|u\|_{M^{p_1,q_1}_{s_1,t_1}} \cdot \|\partial^\alpha v\|_{M^{p_2,q_2}_{s_2,t_2}}.$$

For the second assertion we apply the Leibnitz rule. □

2.2 The generalized modulation space $\mathscr{G}_{C^\infty_{M^{p,q}_{s,t}}}(\mathbb{R}^d)$

Let $t \geq 0$ from now on, unless otherwise stated. We are now ready to define $\mathscr{G}_{C^\infty_{M^{p,q}_{s,t}}}(\mathbb{R}^d)$ as a quotient algebra of the (differential) algebra $\mathscr{E}_{C^\infty_{M^{p,q}_{s,t}}}(\mathbb{R}^d)$ and its (differential) ideal $\mathscr{N}_{C^\infty_{M^{p,q}_{s,t}}}(\mathbb{R}^d)$ (resp. $M^{p,q}_{s,t,\mathscr{N}}(\mathbb{R}^d)$). In order to specify the space which our nets should belong to, we will take into account the following observation:

Remark 2.2.1. *Let $s_0 \in \mathbb{R}$. By Theorem 1.2.12 (for $t \geq 0$) $\bigcap_{s \geq s_0} M^{p,q}_{s,t} \subset \bigcap_{s \geq s_0} M^{p,q}_{s,0} = H^p_\infty \subset C^\infty_B \subset C^{2d+1}_B \subset M^{\infty,1}$. Thus, Theorem 2.1.9 leads us to the conclusion that the natural space of nets to be considered is $H^p_\infty(\mathbb{R}^d)^{(0,1]}$.*

Definition 2.2.2.

$$\mathscr{E}_{C^\infty_{M^{p,q}_{s,t}}}(\mathbb{R}^d) := \{(u_\varepsilon)_\varepsilon \in H^p_\infty(\mathbb{R}^d)^{(0,1]} \big| \, \forall \alpha \in \mathbb{N}^d_0 \, \exists N \in \mathbb{N} : \, \|\partial^\alpha u_\varepsilon\|_{M^{p,q}_{s,t}} = \mathcal{O}(\varepsilon^{-N})\},$$

$$\mathscr{N}_{C^\infty_{M^{p,q}_{s,t}}}(\mathbb{R}^d) := \{(u_\varepsilon)_\varepsilon \in H^p_\infty(\mathbb{R}^d)^{(0,1]} \big| \, \forall \alpha \in \mathbb{N}^d_0 \, \forall m \in \mathbb{N}_0 : \, \|\partial^\alpha u_\varepsilon\|_{M^{p,q}_{s,t}} = \mathcal{O}(\varepsilon^{m})\}.$$

Theorem 2.2.3. *$\mathscr{E}_{C^\infty_{M^{p,q}_{s,t}}}(\mathbb{R}^d)$is a (differential) algebra and $\mathscr{N}_{C^\infty_{M^{p,q}_{s,t}}}(\mathbb{R}^d)$ is an ideal in $\mathscr{E}_{C^\infty_{M^{p,q}_{s,t}}}(\mathbb{R}^d)$.*

Proof. Take $(u_\varepsilon)_\varepsilon$, $(v_\varepsilon)_\varepsilon \in \mathscr{E}_{C^\infty_{M^{p,q}_{s,t}}}(\mathbb{R}^d)$. By Corollaries 2.1.11 and 2.1.15 we have

$$\|u_\varepsilon \cdot v_\varepsilon\|_{M^{p,q}_{s,t}} \leq C \cdot \|u_\varepsilon\|_{M^{p,q}_{s,t}} \cdot \|v_\varepsilon\|_{M^{\infty,1}} \leq C \cdot \sum_{|\alpha| \leq 2k} \|u_\varepsilon\|_{M^{p,q}_{s,t}} \cdot \|\partial^\alpha v_\varepsilon\|_{M^{p,q}_{s,t}},$$

which implies

$$\|u_\varepsilon \cdot v_\varepsilon\|_{M^{p,q}_{s,t}} \leq C \cdot \varepsilon^{-N_1} \sum_{|\alpha| \leq 2k} \varepsilon^{-N_\alpha} \leq C \cdot \varepsilon^{-\max_{|\alpha| \leq 2k}\{N_1, N_\alpha\}}.$$

Thus, $u_\varepsilon \cdot v_\varepsilon$ (in norm) is bounded by some ε^{-N}. Applying the Leibniz rule we deduce that $(u_\varepsilon \cdot v_\varepsilon)_\varepsilon \in \mathscr{E}_{C^\infty_{M^{p,q}_{s,t}}}(\mathbb{R}^d)$ indeed. Similarly, one establishes that $\mathscr{N}_{C^\infty_{M^{p,q}_{s,t}}}(\mathbb{R}^d)$ is a (differential) ideal in (the differential algebra) $\mathscr{E}_{C^\infty_{M^{p,q}_{s,t}}}(\mathbb{R}^d)$. □

Remark 2.2.4. *On the other hand $C^\infty_{M^{p,q}_{s,t}}$ is a locally convex topological vector space topologized through the family of seminorms (see Definition 1.3.3):*

$$p_j(u) := \max_{|\alpha| \leq j} \sup_{x \in \mathbb{R}^d} \|\partial^\alpha u\|_{M^{p,q}_{s,t}}.$$

Therefore, we may define the algebra $\mathscr{G}_{C^\infty_{M^{p,q}_{s,t}}}$ as an algebra based on a locally convex space.

Remark 2.2.5. *In order to construct algebras for $p = \infty$ and/or $q = \infty$ one might use the notion of* narrow convergence *(see [Tof04c], [Tof04a], [Tof04b]) - since $\mathscr{S}$ is dense in any modulation space with respect to this type of convergence; however, we will not investigate this in the present thesis. On the other hand we can also construct corresponding algebras of generalized functions for spaces*

$$\mathscr{M}^{p,q}_{s,t}(\mathbb{R}^d) := \{u \in \mathscr{S}'(\mathbb{R}^d) | \, \exists (u_k)_{k \in \mathbb{N}} \subset \mathscr{S} : \|u_k - u\|_{M^{p,q}_{s,t}} \to 0, k \to \infty\},$$

for all p and $q \in [1, \infty]$. Note that for finite p and q we have $\mathscr{M}^{p,q}_{s,t}(\mathbb{R}^d) = M^{p,q}_{s,t}(\mathbb{R}^d)$.

Remark 2.2.6. *Algebra $\mathscr{G}_{C^\infty_{M^{p,q}_{s,0}}}$ coincides with Oberguggenberger's algebra $\mathscr{G}_{p,p}$ (cf. [Obe92a]; with obvious modification for the case of the special algebra) for $p \in [1, \infty)$.*

Theorem 2.2.7.
$$\mathscr{G}_{C^\infty_{M^{p,q}_{s,t}}}(\mathbb{R}^d) = \mathscr{G}_{C^\infty_{M^{p,1}_{0,t}}}(\mathbb{R}^d),$$
holds for all p, $q \in [1, \infty)$, $s \in \mathbb{R}$ and $t \geq 0$.

Proof. The assertion of the theorem follows directly from Corollary 2.1.13. □

Remark 2.2.8. *For $p_1 \leq p_2$, $q \in [1, \infty)$, $s \in \mathbb{R}$, $t_1 \geq t_2 \geq 0$ the canonical map*
$$j : \mathscr{G}_{C^\infty_{M^{p_1,q}_{s,t_1}}}(\mathbb{R}^d) \to \mathscr{G}_{C^\infty_{M^{p_2,q}_{s,t_2}}}(\mathbb{R}^d), \quad j(u_\varepsilon + \mathscr{N}_{C^\infty_{M^{p_1,q}_{s,t_1}}}(\mathbb{R}^d)) := (u_\varepsilon)_\varepsilon + \mathscr{N}_{C^\infty_{M^{p_2,q}_{s,t_2}}}(\mathbb{R}^d)$$
is well defined. Namely, it is easy to see that
$$\mathscr{E}_{C^\infty_{M^{p_1,q}_{s,t_1}}}(\mathbb{R}^d) \subset \mathscr{E}_{C^\infty_{M^{p_2,q}_{s,t_2}}}(\mathbb{R}^d) \text{ and } \mathscr{N}_{C^\infty_{M^{p_1,q}_{s,t_1}}}(\mathbb{R}^d) \subset \mathscr{N}_{C^\infty_{M^{p_2,q}_{s,t_2}}}(\mathbb{R}^d)$$
for $t_1 \geq t_2$, $p_1 \leq p_2$.

We conjecture that this map is not an inclusion map (not injective).

Theorem 2.2.9. *Let ρ be a standard mollifier. The linear mapping*
$$\iota(u) := (u * \rho_\varepsilon)_\varepsilon + \mathscr{N}_{C^\infty_{M^{p,q}_{s,t}}}(\mathbb{R}^d)$$
of $M^{p,q}_{s,t}(\mathbb{R}^d)$ into $\mathscr{G}_{C^\infty_{M^{p,q}_{s,t}}}(\mathbb{R}^d)$ is an embedding.

Proof. By Theorem 2.1.9 $(u * \rho_\varepsilon)_\varepsilon$ is $C^\infty_{M^{p,q}_{s,t}}$-moderate. Let $(u * \rho_\varepsilon)_\varepsilon$ be negligible and let $\varphi \in \mathscr{S}(\mathbb{R}^d)$ be arbitrary. By Proposition 1.2.11 we have that
$$|\langle u * \rho_\varepsilon, \varphi\rangle| \leq C \cdot \|u * \rho_\varepsilon\|_{M^{p,q}_{s,t}} \cdot \|\varphi\|_{M^{p',q'}_{-s,-t}} \to 0, \quad (\varepsilon \to 0).$$
Thus, $u = 0$ in $\mathscr{S}'(\mathbb{R}^d)$ and so $u = 0$ in $M^{p,q}_{s,t}(\mathbb{R}^d)$. □

Next we give a characterization of the ideal in $\mathscr{G}_{C^\infty_{M^{p,q}_{s,t}}}(\mathbb{R}^d)$. This result is important since it suffices to consider only the 0-th order derivative estimate of the representative of a moderate net in order to show that it is negligible.

Theorem 2.2.10 (Characterization of the Ideal)**.** *$(u_\varepsilon)_\varepsilon \in \mathscr{E}_{C^\infty_{M^{p,q}_{s,t}}}(\mathbb{R}^d)$ is $C^\infty_{M^{p,q}_{s,t}}(\mathbb{R}^d)$-negligible if and only if the following condition is satisfied:*
$$\forall m \in \mathbb{N}_0 \quad \|u_\varepsilon\|_{M^{p,q}_{s,t}} = \mathcal{O}(\varepsilon^m). \tag{2.1}$$

Proof. The necessity of condition (2.1) is obvious. Let now $(u_\varepsilon)_\varepsilon \in \mathscr{E}_{C^\infty_{M^{p,q}_{s,t}}}(\mathbb{R}^d)$ such that (2.1) holds. It suffices to prove that for any $1 \leq j \leq d$, $\partial_j u_\varepsilon$ satisfies the 0-th negligibility estimate.

By our assumption we may choose $N \in \mathbb{N}$ such that for every $m \in \mathbb{N}_0$ both of the following estimates hold:
$$\|u_\varepsilon\|_{M^{p,q}_{s,t}} = \mathcal{O}(\varepsilon^{2m+N}), \tag{2.2}$$
$$\|\partial_j^2 u_\varepsilon\|_{M^{p,q}_{s,t}} = \mathcal{O}(\varepsilon^{-N}). \tag{2.3}$$

By Taylor expansion we have

$$u_\varepsilon(x+he_j) = u_\varepsilon(x) + \partial_j u_\varepsilon(x)\cdot h + h^2 \cdot \int_0^1 (1-\theta)\partial_j^2 u_\varepsilon(x+h\theta e_j)d\theta.$$

We extract the first-order derivative term,

$$\partial_j u_\varepsilon(x) = \big(u_\varepsilon(x+he_j) - u_\varepsilon(x)\big)\cdot h^{-1} - h\cdot \int_0^1 (1-\theta)\cdot \partial_j^2 u_\varepsilon(x+\theta h e_j)d\theta.$$

By taking norms and inserting $h = \varepsilon^{m+N}$ we obtain

$$\begin{aligned}\|\partial_j u_\varepsilon\|_{M^{p,q}_{s,t}} \leq \underbrace{(\langle \varepsilon^{m+N} e_j\rangle^t + 1)}_{\leq 2^{t+1}} \cdot \|u_\varepsilon\|_{M^{p,q}_{s,t}} \cdot \varepsilon^{-m-N} \\ + \|\int_0^1 (1-\theta)\cdot \partial_j^2 u_\varepsilon(\cdot + \theta\varepsilon^{m+N}e_j)d\theta\|_{M^{p,q}_{s,t}} \cdot \varepsilon^{m+N}.\end{aligned}$$

Note that we have used here the translation invariance of modulation spaces (Prop. (1.2.3)):

$$\|u_\varepsilon(\cdot + \varepsilon^{m+N}e_j)\|_{M^{p,q}_{s,t}} \leq \langle \varepsilon^{m+N}e_j\rangle^t \cdot \|u_\varepsilon\|_{M^{p,q}_{s,t}}.$$

Let us now estimate the second term on the right hand side of the above estimate:

$$\begin{aligned}&|V_g\Big(\int_0^1 (1-\theta)\cdot\partial_j^2 u_\varepsilon(\cdot+\theta\varepsilon^{m+N}e_j)d\theta\Big)(x,\omega)| \\ &\quad = \Big|\int\Big(\int_0^1 (1-\theta)\cdot \partial_j^2 u_\varepsilon(y+\theta\varepsilon^{m+N}e_j)d\theta\Big)\cdot \overline{g(y-x)}\cdot e^{2\pi i y\omega}dy\Big| \\ &\quad = \Big|\int_0^1 (1-\theta)\cdot\Big(\int \partial_j^2 u_\varepsilon(y+\theta\varepsilon^{m+N}e_j \cdot \overline{g(y-x)}\cdot e^{2\pi i y\omega}dy\Big)d\theta\Big| \\ &\qquad\qquad \leq \int_0^1 (1-\theta)\cdot\Big|V_g\Big(\partial_j^2 u_\varepsilon(\cdot+\theta\varepsilon^{m+N}e_j)\Big)(x,\omega)\Big|d\theta.\end{aligned}$$

Thus by the generalized Minkowski inequality, we have

$$\|\int_0^1 (1-\theta)\cdot\partial_j^2 u_\varepsilon(\cdot+\theta\varepsilon^{m+N}e_j)d\theta\|_{M^{p,q}_{s,t}} \leq C\cdot\sup_\theta \|\partial_j^2 u_\varepsilon(\cdot+\theta\varepsilon^{m+N}e_j)\|_{M^{p,q}_{s,t}}.$$

Again, since modulation spaces are translation invariant we have:

$$\sup_\theta \|\partial_j^2 u_\varepsilon(\cdot+\theta\varepsilon^{m+N}e_j)\|_{M^{p,q}_{s,t}} \leq C\cdot\sup_\theta\langle\theta\varepsilon^{m+N}e_j\rangle^t\cdot\|\partial_j^2 u_\varepsilon\|_{M^{p,q}_{s,t}} \leq C\cdot\|\partial_j^2 u_\varepsilon\|_{M^{p,q}_{s,t}},$$

we have (for $t\geq 0$, $\langle\theta\varepsilon^{m+N}e_j\rangle^t \leq 2^t$, for $t<0$, $\langle\theta\varepsilon^{m+N}e_j\rangle^t\leq 1$)

$$\|\partial_j u_\varepsilon\|_{M^{p,q}_{s,t}} \leq 2^{t+1}\cdot \underbrace{\|u_\varepsilon\|_{M^{p,q}_{s,t}}}_{\leq C\cdot\varepsilon^{2m+N},\text{ by (2.2)}} \cdot\varepsilon^{-m-N} + C\cdot \underbrace{\|\partial_j^2 u_\varepsilon\|_{M^{p,q}_{s,t}}}_{\leq C\cdot\varepsilon^{-N},\text{ by (2.3)}} \cdot\varepsilon^{m+N}.$$

Thus,

$$\|\partial_j u_\varepsilon\|_{M^{p,q}_{s,t}} \leq C\cdot\varepsilon^{2m+N}\cdot\varepsilon^{-m-N} + C\cdot\varepsilon^{-N}\cdot\varepsilon^{m+N} \leq C\cdot\varepsilon^m.$$

$\square$

Now we show that we can identify the constant map with our embedding on the space $C^{\infty}_{M^{p,q}_{s,t}}$:

Theorem 2.2.11. *Let $\sigma : C^{\infty}_{M^{p,q}_{s,t}} \to \mathscr{G}_{C^{\infty}_{M^{p,q}_{s,t}}}$ given by $\sigma : f \to [(f)_{\epsilon}]$. Then*

$$\iota|_{C^{\infty}_{M^{p,q}_{s,t}}} = \sigma.$$

Proof. Since any $f \in C^{\infty}_{M^{p,q}_{s,t}}$ satisfies the equality (2.5), we have

$$V_g(f * \rho_\varepsilon - f)(x,\omega) = \int \sum_{|\alpha|=m} \int \int_0^1 \frac{\partial^\alpha f(y - \varepsilon\theta\eta)}{(m-1)!} \cdot (1-\theta)^{(m-1)} d\theta \\ \cdot (-\varepsilon\eta)^\alpha \rho(\eta) d\eta \cdot \overline{g(t-x)} e^{-2\pi i y\omega} dy \ .$$

$$\|f * \rho_\varepsilon - f\|_{M^{p,q}_{s,t}} = \Big(\int \Big(\int \Big| \int \sum_{|\alpha|=m} \int \int_0^1 \frac{\partial^\alpha f(y + \varepsilon\theta\eta)}{(m-1)!} \cdot (1-\theta)^{m-1} d\theta \\ \cdot (-\varepsilon\eta)^\alpha \rho(\eta) d\eta \cdot \overline{g(y-x)} e^{-2\pi i y\omega} dy \ \langle x\rangle^t \langle\omega\rangle^s \Big|^p dx \ \Big)^{\frac{q}{p}} d\omega \Big)^{\frac{1}{q}}.$$

By the generalized Minkowski inequality

$$\|f * \rho_\varepsilon - f\|_{M^{p,q}_{s,t}} \le C \cdot \varepsilon^m \cdot \sum_{|\alpha|=m} \int \int_0^1 \Big(\int \Big(\int \Big| \int \frac{\partial^\alpha f(y + \varepsilon\theta\eta)}{(m-1)!} \\ \cdot \overline{g(y-x)} e^{-2\pi i y\omega} dy \ \langle x\rangle^t \langle\omega\rangle^s \Big|^p dx \ \Big)^{\frac{q}{p}} d\omega \Big)^{\frac{1}{q}} \cdot |\eta^\alpha| \cdot |\rho(\eta)| d\eta \cdot (1-\theta)^{m-1} d\theta.$$

Again, by invariance under time-frequency shifts, we may write

$$\|f * \rho_\varepsilon - f\|_{M^{p,q}_{s,t}} \le C \cdot \varepsilon^m \cdot \sum_{|\alpha|=m} \|\partial^\alpha f\|_{M^{p,q}_{s,t}} \int \int_0^1 |\eta^\alpha| \cdot \langle \varepsilon\theta\eta\rangle^t \cdot |\rho(\eta)| d\eta \ d\theta,$$

thus

$$\|f * \rho_\varepsilon - f\|_{M^{p,q}_{s,t}} \le C \cdot \varepsilon^m \sum_{|\alpha|\le m} \|\partial^\alpha f\|_{M^{p,q}_{s,t}}.$$

□

The following two theorems provide us sufficient conditions for the classical product to remain invariant (under the action of embedding map ι) in the algebra of generalized functions. Hence ι renders $C^{\infty}_{M^{p,q}_{s,t}}$ a subalgebra of $\mathscr{G}_{C^{\infty}_{M^{p,q}_{s,t}}}$.

Theorem 2.2.12. *Let $u \in C^{\infty}_{M^{p_1,q_1}_{s_1,t_1}}(\mathbb{R}^d)$, $v \in C^{\infty}_{M^{p_2,q_2}_{s_2,t_2}}(\mathbb{R}^d)$, where p_1, p_2, q_1, $q_2 \in [1,\infty]$ and s_1, s_2, t_1, $t_2 \in \mathbb{R}$ with t_1, $t_2 \ge 0$. Then*

$$\iota(u)_\varepsilon \cdot \iota(v)_\varepsilon - \iota(uv)_\varepsilon \in \mathscr{N}_{C^{\infty}_{M^{p_1,q_1}_{s_1,t_1}}}(\mathbb{R}^d) \cap \mathscr{N}_{C^{\infty}_{M^{p_2,q_2}_{s_2,t_2}}}(\mathbb{R}^d),$$

where $\iota(u)_\varepsilon := (u * \rho_\varepsilon)_\varepsilon$.

Proof. By Theorem 2.2.10 it suffices to show that for every $m \in N$ the estimate

$$\|(u * \rho_\varepsilon) \cdot (v * \rho_\varepsilon) - (u \cdot v) * \rho_\varepsilon\|_{M^{p_j,q_j}_{s_j,t_j}} < \varepsilon^m$$

for $j = 1$ or $j = 2$, holds for ε small enough. By Corollary 2.1.16 we have $u \cdot v \in C^\infty_{M^{p_1,q_1}_{s_1,t_1}}$. By using Corollaries 2.1.11 and 2.1.15 we calculate

$$\begin{aligned}
&\|(u * \rho_\varepsilon) \cdot (v * \rho_\varepsilon) - (u \cdot v) * \rho_\varepsilon\|_{M^{p_1,q_1}_{s_1,t_1}} \\
&\quad \le \|(u * \rho_\varepsilon - u) \cdot (v * \rho_\varepsilon - v)\|_{M^{p_1,q_1}_{s_1,t_1}} + \|u \cdot (v * \rho_\varepsilon - v)\|_{M^{p_1,q_1}_{s_1,t_1}} \\
&\qquad + \|(u * \rho_\varepsilon - u) \cdot v\|_{M^{p_1,q_1}_{s_1,t_1}} + \|u \cdot v - (u \cdot v) * \rho_\varepsilon\|_{M^{p_1,q_1}_{s_1,t_1}} \\
&\qquad \le C_1 \cdot \underbrace{\|u * \rho_\varepsilon - u\|_{M^{p_1,q_1}_{s_1,t_1}}}_{\text{negligible}} \cdot \sum_\alpha \underbrace{\|\partial^\alpha v * \rho_\varepsilon - \partial^\alpha v\|_{M^{p_2,q_2}_{s_2,t_2}}}_{\text{negligible}} \\
&\qquad\quad + C_2 \cdot \|u\|_{M^{p_1,q_1}_{s_1,t_1}} \cdot \sum_\alpha \underbrace{\|\partial^\alpha v * \rho_\varepsilon - \partial^\alpha v\|_{M^{p_2,q_2}_{s_2,t_2}}}_{\text{negligible}} \\
&\qquad\quad + C_3 \cdot \underbrace{\|u * \rho_\varepsilon - u\|_{M^{p_1,q_1}_{s_1,t_1}}}_{\text{negligible}} \cdot \sum_\alpha \|\partial^\alpha v\|_{M^{p_2,q_2}_{s_2,t_2}} \\
&\qquad\qquad + C_4 \cdot \underbrace{\|uv - (uv) * \rho_\varepsilon\|_{M^{p_1,q_1}_{s_1,t_1}}}_{\text{negligible}} .
\end{aligned}$$

□

Note that we can slightly weaken the conditions on u in the theorem above, namely, the following holds:

Theorem 2.2.13. *Let $u \in M^{p_1,q_1}_{s_1,t_1}$, $v \in C^\infty_{M^{p_2,q_2}_{s_2,t_2}}$ such that $u \cdot v \in C^\infty_{M^{p_1,q_1}_{s_1,t_1}}$ and for every $m \in \mathbb{N}_0$, $\|\iota(u)_\varepsilon - u\|_{M^{p_1,q_1}_{s_1,t_1}} < \varepsilon^m$, where p_1, p_2, q_1, $q_2 \in [1, \infty]$ and s_1, s_2, t_1, $t_2 \in \mathbb{R}$ with only $t_2 \ge 0$. Then*

$$\iota(u)_\varepsilon \cdot \iota(v)_\varepsilon - \iota(uv)_\varepsilon \in \mathscr{N}_{C^\infty_{M^{p_1,q_1}_{s_1,t_1}}}(\mathbb{R}^d),$$

where $\iota(u)_\varepsilon := (u * \rho_\varepsilon)_\varepsilon$.

2.3 Association in $\mathscr{G}_{C^\infty_{M^{p,q}_{s,t}}}(\mathbb{R}^d)$

We note that two elements of Colombeau algebra which differ on the level of the algebra (their difference does not belong to the algebra of negligible nets) may have the same distributional limit. This phenomenon is explained by the fact that in the process of regularizing we have added to the new generalized element nonlinear information, which did not exist in the 'classical' space. In other words, in the algebra of generalized functions we have now elements which differ from each other by the additional information on the 'nonlinear' nature they carry, but they describe one and the same element on the 'linear' (classical) level. We will therefore consider an equivalence relation for such elements of the algebra whose representatives have the same limit (cast the same shadow), and its properties.

For a more detailed analysis (in the setting of the 'classical' special Colombeau algebra) we refer the interested reader to chapter 1.2.6 of an excellent book by Michael Oberguggenberger, Michael Grosser, Michael Kunzinger and Roland Steinbauer, ([GKOS01]).

Definition 2.3.1. *Let $s \in \mathbb{R}$, $t \geq 0$. An element $u \in \mathscr{G}_{C^\infty_{M^{p,q}_{s,t}}}(\mathbb{R}^d)$ is called* associated *with 0 (denoted by $u \overset{M^{p,q}_{s,t}}{\approx} 0$) if*

$$\lim_{\varepsilon \to 0} \|u_\varepsilon\|_{M^{p,q}_{s,t}} = 0.$$

Two elements u, v of $\mathscr{G}_{C^\infty_{M^{p,q}_{s,t}}}(\mathbb{R}^d)$ are associated with each other (and we write $u \overset{M^{p,q}_{s,t}}{\approx} v$) if

$$u - v \overset{M^{p,q}_{s,t}}{\approx} 0.$$

Definition 2.3.1 is independent of the chosen representative(s). See the proof of Proposition 2.3.3 below.

Definition 2.3.2. *Let $u \in \mathscr{G}_{C^\infty_{M^{p,q}_{s,t}}}(\mathbb{R}^d)$, $v \in M^{p,q}_{s,t}$. Suppose that $u \overset{M^{p,q}_{s,t}}{\approx} \iota(v)$. Then u is said to admit v as associated element of $M^{p,q}_{s,t}$ and v is also called the* shadow *of u. In this case we simply write $u \overset{M^{p,q}_{s,t}}{\approx} v$.*

Proposition 2.3.3. *Let $u \in M^{p,q}_{s,t}$, p, $q \in [1,\infty)$. If $\iota(u) \overset{M^{p,q}_{s,t}}{\approx} 0$, then $u = 0$.*

Proof. Let $\varphi \in \mathscr{S}(\mathbb{R}^d)$. Then there exists a constant $C \geq 0$ such that

$$|\langle u * \rho_\varepsilon, \varphi \rangle| \leq C \cdot \|u * \rho_\varepsilon\|_{M^{p,q}_{s,t}} \cdot \|\varphi\|_{M^{p',q'}_{-s,-t}} \to 0, \quad (\varepsilon \to 0).$$

□

Proposition 2.3.4. *The following assertions hold true:*

(i) Let u, $v \in M^{p,q}_{s+s_0,t}$, $s_0 \in \mathbb{N}_0$. Then

$$u \overset{M^{p,q}_{s+s_0,t}}{\approx} v \Leftrightarrow \partial^\alpha u \overset{M^{p,q}_{s,t}}{\approx} \partial^\alpha v$$

for all $|\alpha| \leq s_0$.

(ii) Let u, $v \in \mathscr{G}_{C^\infty_{M^{p_1,q_1}_{s_1,t_1}}}(\mathbb{R}^d)$, $u \overset{M^{p_1,q_1}_{s_1,t_1}}{\approx} v$, $f \in C^\infty_{M^{p_2,q_2}_{s_2,t_2}}$, $t_2 \geq 0$ for any p_2, q_2, s_2 and t_2). Then we have

$$f \cdot u \overset{M^{p_1,q_1}_{s_1,t_1}}{\approx} f \cdot v.$$

(iii) Let $u \in M^{p_1,q_1}_{s_1,t_1}$ *and* $v \in M^{p_0,1}$ *(or* $u \in M^{p_1,q_1}_{s_1,t_1}$ *and* $v \in C^\infty_{M^{p_2,q_2}_{s_2,t_2}}$ *as in Corollary 2.1.16). Then we have*

$$\iota(u) \cdot \iota(v) \overset{M^{p_1,q_1}_{s_1,t_1}}{\approx} \iota(u \cdot v).$$

Proof. It is easy to prove (i) by using a calculation similar to the proof of Lemma 1.2.10.

(ii) Let $f \in C^\infty_{M^{p_2,q_2}_{s_2,t_2}}$. We use the following estimate (cf. Corollary 2.1.16):

$$\|f \cdot (u_\varepsilon - v_\varepsilon)\|_{M^{p_1,q_1}_{s_1,t_1}} \le C \cdot \sum_\alpha \|u_\varepsilon - v_\varepsilon\|_{M^{p_1,q_1}_{s_1,t_1}} \cdot \|\partial^\alpha f\|_{M^{p_2,q_2}_{s_2,t_2}}.$$

(iii) Let $u \in M^{p_1,q_1}_{s_1,t_1}$ and $v \in M^{p_0,1}$. We may write

$$\begin{aligned}
&\|(u * \rho_\varepsilon) \cdot (v * \rho_\varepsilon) - (u \cdot v) * \rho_\varepsilon\|_{M^{p_1,q_1}_{s_1,t_1}} \\
&\quad \le \|(u * \rho_\varepsilon - u) \cdot (v * \rho_\varepsilon - v)\|_{M^{p_1,q_1}_{s_1,t_1}} + \|u \cdot (v * \rho_\varepsilon - v)\|_{M^{p_1,q_1}_{s_1,t_1}} \\
&\quad\quad + \|(u * \rho_\varepsilon - u) \cdot v\|_{M^{p_1,q_1}_{s_1,t_1}} + \|u \cdot v - (u \cdot v) * \rho_\varepsilon\|_{M^{p_1,q_1}_{s_1,t_1}} \\
&\le C_1 \cdot \underbrace{\|u * \rho_\varepsilon - u\|_{M^{p_1,q_1}_{s_1,t_1}}}_{\to 0} \cdot \underbrace{\|v * \rho_\varepsilon - v\|_{M^{p_0,1}}}_{\to 0} + C_2 \cdot \|u\|_{M^{p_1,q_1}_{s_1,t_1}} \cdot \underbrace{\|v * \rho_\varepsilon - v\|_{M^{p_0,1}}}_{\to 0} \\
&\quad + C_3 \cdot \underbrace{\|u * \rho_\varepsilon - u\|_{M^{p_1,q_1}_{s_1,t_1}}}_{\to 0} \cdot \|v\|_{M^{p_0,1}} + C_4 \cdot \underbrace{\|uv - (uv) * \rho_\varepsilon\|_{M^{p_1,q_1}_{s_1,t_1}}}_{\to 0}.
\end{aligned}$$

Similarly, for the second assertion we have:

$$\begin{aligned}
&\|(u * \rho_\varepsilon) \cdot (v * \rho_\varepsilon) - (u \cdot v) * \rho_\varepsilon\|_{M^{p_1,q_1}_{s_1,t_1}} \\
&\quad \le \|(u * \rho_\varepsilon - u) \cdot (v * \rho_\varepsilon - v)\|_{M^{p,q}_{s,t}} + \|u \cdot (v * \rho_\varepsilon - v)\|_{M^{p_1,q_1}_{s_1,t_1}} \\
&\quad\quad + \|(u * \rho_\varepsilon - u) \cdot v\|_{M^{p_1,q_1}_{s_1,t_1}} + \|u \cdot v - (u \cdot v) * \rho_\varepsilon\|_{M^{p_1,q_1}_{s_1,t_1}} \\
&\quad\quad \le C_1 \cdot \underbrace{\|u * \rho_\varepsilon - u\|_{M^{p_1,q_1}_{s_1,t_1}}}_{\to 0} \cdot \sum_\alpha \underbrace{\|\partial^\alpha v * \rho_\varepsilon - \partial^\alpha v\|_{M^{p_2,q_2}_{s_2,t_2}}}_{\to 0} \\
&\quad\quad\quad + C_2 \cdot \|u\|_{M^{p_1,q_1}_{s_1,t_1}} \cdot \sum_\alpha \underbrace{\|\partial^\alpha v * \rho_\varepsilon - \partial^\alpha v\|_{M^{p_2,q_2}_{s_2,t_2}}}_{\to 0} \\
&\quad\quad\quad + C_3 \cdot \underbrace{\|u * \rho_\varepsilon - u\|_{M^{p_1,q_1}_{s_1,t_1}}}_{\to 0} \cdot \sum_\alpha \|\partial^\alpha v\|_{M^{p_2,q_2}_{s_2,t_2}} \\
&\quad\quad\quad\quad + C_4 \cdot \underbrace{\|uv - (uv) * \rho_\varepsilon\|_{M^{p_1,q_1}_{s_1,t_1}}}_{\to 0}.
\end{aligned}$$

□

2.4 Regular Colombeau generalized functions

One of the important notions in the classical theory of PDE is the notion of regularity. Regularity within the classical framework is expressed in terms of smoothness. The algebra of regular Colombeau generalized functions arose as an object that generalizes C^∞ (and its corresponding subspaces). The relation between the algebra of regular generalized functions $\mathscr{G}^\infty$ and a space of distributions (in this case $\mathscr{D}'$) is given by (cf. [Obe92b]):

$$\mathscr{G}^\infty \cap \mathscr{D}' = C^\infty.$$

We define the space of regular Colombeau generalized functions in the present context by

$$\mathscr{G}^\infty_{C^\infty_{M^{p,q}_{s,t}}}(\mathbb{R}^d) := \mathscr{E}^\infty_{C^\infty_{M^{p,q}_{s,t}}}(\mathbb{R}^d)/\mathscr{N}_{C^\infty_{M^{p,q}_{s,t}}}(\mathbb{R}^d),$$

where

$$\mathscr{E}^\infty_{C^\infty_{M^{p,q}_{s,t}}}(\mathbb{R}^d) := \{(u_\varepsilon)_\varepsilon \in H^p_\infty \,\big|\, \exists N \in \mathbb{N}: \ \forall \alpha \in \mathbb{N}_0^d \ \|\partial^\alpha u_\varepsilon\|_{M^{p,q}} = \mathcal{O}(\varepsilon^{-N})\}.$$

Conjecture 2.4.1.

$$\mathscr{G}^\infty_{C^\infty_{M^{p,q}_{s,0}}}(\mathbb{R}^d) \cap \mathscr{S}'(\mathbb{R}^d) = C^\infty_{M^{p,q}}(\mathbb{R}^d) = H^p_\infty(\mathbb{R}^d).$$

Chapter 3

Wiener-Amalgam spaces in the Colombeau framework

An important class of spaces in the theory of modulation spaces are Wiener-Amalgam spaces, as their Fourier image. We shall now recall the definition of Wiener-Amalgam spaces ([Fei83a], [Fei81], [Hei03]). Let B (e.g. L^p, $\mathcal{F}L^p$) and C (e.g. L^p, L^q) be Banach spaces, where $1 \leq p, q \leq \infty$. Let f be a distribution which is locally in B, i.e. $g \cdot f \in B$, $\forall g \in \mathscr{D}$. We define

$$f_B(x) := \|f \cdot T_x g\|_B.$$

The Wiener amalgam space $W(B, C)$ is defined as the space of all temperate distributions f (locally in B) such that $f_B \in C$. We call B and C local and global component, respectively. $W(B, C)$ is a Banach space, with the norm given by

$$\|f\|_{W(B,C)} := \|f_B\|_C.$$

As in modulation spaces, different windows $g \in \mathscr{D}$ generate the same space and yield equivalent norms.

For $B = L^q$ and $C = L^p$ we denote the Wiener-Amalgam space $W(B, C)$ by $W^{p,q}(\mathbb{R}^d)$. Now, the Wiener-Amalgam space $W^{p,q}(\mathbb{R}^d)$ can be also seen through the modulation spaces "glasses", by the following relation:

$$W^{p,q}(\mathbb{R}^d) = \mathcal{F}M^{q,p}(\mathbb{R}^d).$$

Set $\| \cdot \|_{\mathcal{F}M^{q,p}} = \|\hat{\cdot}\|_{M^{q,p}}$ In this case, we have that the norms $\| \cdot \|_{W^{p,q}}$ and $\| \cdot \|_{\mathcal{F}M^{q,p}}$ are equivalent. This follows from the relation

$$\left|V_{\hat{g}}\hat{f}(x, \omega)\right| = \left|V_g f(-\omega, x)\right|.$$

For the sake of convenience, we shall further work with the spaces $\mathcal{F}M^{p,q}(\mathbb{R}^d)$ rather than $W^{p,q}(\mathbb{R}^d)$. We will show that it is possible to construct an algebra of generalized functions based on these spaces (Wiener-Amalgam spaces) in a similar way to that for the modulation spaces. Such a generalization is important since it will provide us with a sort of remedy for the loss of the q-information (mentioned in the Introduction). In particular, we will show that it is possible to define an algebra of generalized functions whose factors are the algebra of moderate nets corresponding to modulation spaces, and whose ideal, algebra of negligible nets corresponds to the Wiener-Amalgam spaces $\mathcal{F}M^{p,q}(\mathbb{R}^d)$.

3.1 Basic setup

We will mainly follow the same path and line of arguments in constructing the corresponding algebras as done in the previous chapter. This means that we will present the statements in the same order as before. We will provide proofs only when the claim does not follow obviously from the case of modulation spaces; in some proofs we will point out only the missing links in the argument for the new claim to follow.

Lemma 3.1.1. *For each m, $k \in \mathbb{N}_0$ and any multi-index $\alpha \in \mathbb{N}_0^d$ there exists a constant $C \geq 0$ such that*

$$|V_g \widehat{\partial^\alpha \rho_\varepsilon}(x,\omega)| \leq C \cdot \langle x \rangle^{-2k-|\alpha|} \langle \omega \rangle^{-m} \cdot \varepsilon^{-2k-|\alpha|}$$

holds for all $(x,\omega) \in \mathbb{R}^{2d}$.

In particular, for each p, $q \in [1,\infty]$ and each s, $t \in \mathbb{R}$ there exists $k \in \mathbb{N}$ such that

$$\|\partial^\alpha \rho_\varepsilon\|_{\mathcal{F}M^{p,q}_{s,t}} \leq C \cdot \varepsilon^{-2k-|\alpha|},$$

for some $C \geq 0$.

Proof. By lemma 2.1.3 we have

$$\left|V_{\hat{g}} \hat{\rho_\varepsilon}(x,\omega)\right| = \left|V_g \rho_\varepsilon(-\omega, x)\right| \leq C \cdot \langle x \rangle^{-2k-|\alpha|} \langle \omega \rangle^{-m} \cdot \varepsilon^{-2k-|\alpha|}.$$

□

Remark 3.1.2. *Note that for any $q \in [1,\infty]$ and $s \in \mathbb{R}$ we have that $\|\rho_\varepsilon\|_{\mathcal{F}M^{\infty,q}_{s,0}}$ is bounded by a constant C independent of ε.*

Lemma 3.1.1 and Corollary 2.1.15 together imply:

Theorem 3.1.3. *Let $u \in \mathcal{F}M^{p,q}_{s,t}$, s, $t \in \mathbb{R}$, α a multi-index. Then $\partial^\alpha u * \rho_\varepsilon \in \mathcal{F}M^{p,q}_{s,t} \cap \mathcal{F}M^{1,\infty}_{s,t}$ and there exist constants $C \geq 0$ and $k \in \mathbb{N}^d$ such that*

$$max\{\|\partial^\alpha u * \rho_\varepsilon\|_{\mathcal{F}M^{1,\infty}_{s,t}}, \|\partial^\alpha u * \rho_\varepsilon\|_{\mathcal{F}M^{p,q}_{s,t}}\} \leq C \cdot \varepsilon^{-N} \|u\|_{\mathcal{F}M^{p,q}_{s,t}}.$$

Lemma 3.1.4. *For any $f \in \mathscr{S}(\mathbb{R}^d)$ and every multi-index α; k, l, $m \in \mathbb{N}_0$ there exists a constant $C \geq 0$ such that*

$$|V_g(\mathcal{F}(\partial^\alpha f * \rho_\varepsilon - \partial^\alpha f))(x,\omega)| \leq C \cdot \varepsilon^m \langle x \rangle^{-l} \langle \omega \rangle^{-2k} \tag{3.1}$$

holds for all $(x,\omega) \in \mathbb{R}^{2d}$ (ε small enough).

Proof. The proof follows from lemma 2.1.5 and

$$|V_g(\mathcal{F}(\partial^\alpha f * \rho_\varepsilon - \partial^\alpha f))(x,\omega)| = |V_{\hat{g}}(\partial^\alpha f * \rho_\varepsilon - \partial^\alpha f)(-\omega, x)|.$$

□

Lemma 3.1.5. *Let p, $q \in [1,\infty)$, s, $t \in \mathbb{R}$ and $\rho \in \mathscr{S}(\mathbb{R}^d)$ as above. Then for every $f \in \mathcal{F}M^{p,q}_{s,t}(\mathbb{R}^d)$ we have that*

$$\|\mathcal{F}(f * \rho_\varepsilon - f)\|_{M^{p,q}_{s,t}(\mathbb{R}^d)} \to 0, \quad (\varepsilon \to 0).$$

Proof. By Proposition 11.3.4 in [Grö01], for any $\hat{f} \in M^{p,q}_{s,t}(\mathbb{R}^d)$ there exists a sequence $(\hat{f}_k)_{k\in\mathbb{N}} \in \mathscr{S}(\mathbb{R}^d)$ such that

$$\|\mathcal{F}(f_k - f)\|_{M^{p,q}_{s,t}(\mathbb{R}^d)} = \|f_k - f\|_{\mathcal{F}M^{p,q}_{s,t}(\mathbb{R}^d)} \to 0. \tag{3.2}$$

We now obtain the following estimate:

$$\begin{aligned}
\|\mathcal{F}(f * \rho_\varepsilon - f)\|_{M^{p,q}_{s,t}} &\le \|\mathcal{F}(f * \rho_\varepsilon - f_k * \rho_\varepsilon)\|_{M^{p,q}_{s,t}} \\
&\quad + \|\mathcal{F}(f_k * \rho_\varepsilon - f_k)\|_{M^{p,q}_{s,t}} + \|\mathcal{F}(f_k - f)\|_{M^{p,q}_{s,t}} \\
&\le C \cdot \|\mathcal{F}(f - f_k)\|_{M^{p,q}_{s,t}} \cdot \|\hat{\rho}_\varepsilon\|_{M^{1,\infty}_{0,|t|}} \\
&\qquad + \|\mathcal{F}(f_k * \rho_\varepsilon - f_k)\|_{M^{p,q}_{s,t}} + \|\mathcal{F}(f_k - f)\|_{M^{p,q}_{s,t}}.
\end{aligned}$$

Since $\|\hat{\rho}_\varepsilon\|_{M^{1,\infty}_{0,|t|}}$ is bounded by a constant independent of ε, we have

$$\|\mathcal{F}(f * \rho_\varepsilon - f)\|_{M^{p,q}_{s,t}} \le C \cdot \|\mathcal{F}(f - f_k)\|_{M^{p,q}_{s,t}} + \|\mathcal{F}(f_k * \rho_\varepsilon - f_k)\|_{M^{p,q}_{s,t}}.$$

The claim follows now from (3.2) and Lemma 3.1.4. □

3.1.1 Embedding theorems for Wiener-Amalgam spaces

As in Subsection 2.1.1, we have

$$\begin{aligned}
\left|V_{\hat{g}}\hat{f}(x,\omega)\right| = \left|V_g f(-\omega, x)\right| &\le C \cdot \sum_{|\alpha|,|\beta|\le 2k} \left|V_{\partial^\beta g}(\partial^\alpha f)(-\omega, x)\right| \cdot \frac{1}{\langle x\rangle^{2k}} \\
&= C \cdot \sum_{|\alpha|,|\beta|\le 2k} \left|V_{\widehat{\partial^\beta g}}(\widehat{\partial^\alpha f})(x,\omega)\right| \cdot \frac{1}{\langle x\rangle^{2k}}.
\end{aligned}$$

Hence the estimate

$$\begin{aligned}
\left|V_{\hat{g}}\hat{f}(x,\omega) \cdot m(x,\omega)\right| &\le C \cdot \sum_{|\alpha|,|\beta|\le 2k} \left|V_{\widehat{\partial^\beta g}}\widehat{\partial^\alpha f}(x,\omega)\right| \cdot \frac{m(x,\omega)}{\langle x\rangle^{2k}} \\
&\le C \cdot \sum_{|\alpha|,|\beta|\le 2k} \sup_{x\in\mathbb{R}^d} \left|V_{\widehat{\partial^\beta g}}\widehat{\partial^\alpha f}(x,\omega)\right| \cdot \frac{m(x,\omega)}{\langle x\rangle^{2k}} \qquad (3.3)
\end{aligned}$$

holds for any weight function $m \in \mathscr{P}$. After integrating with respect to x we obtain

$$\begin{aligned}
\int \left|V_{\hat{g}}\hat{f}(x,\omega) \cdot m(x,\omega)\right| dx &\le C \cdot \sum_{|\alpha|,|\beta|\le 2k} \sup_{x\in\mathbb{R}^d} \left|V_{\widehat{\partial^\beta g}}\widehat{\partial^\alpha f}(x,\omega) \cdot m(x,\omega)\right| \cdot \int \frac{1}{\langle x\rangle^{2k}}\, dx \\
&\le C \cdot \sum_{|\alpha|,|\beta|\le 2k} \|\widehat{\partial^\alpha f}\|_{M^\infty_m}.
\end{aligned}$$

Further we have

$$\|\hat{f}\|_{M^{1,\infty}_m} \le C \cdot \sum_{|\alpha|,|\beta|\le 2k} \|\widehat{\partial^\alpha f}\|_{M^\infty_m}.$$

We may also write

$$\|\hat{f}\|_{M^{1,\infty}_m} \le C \cdot \sum_{|\alpha|,|\beta|\le 2k} \|\widehat{\partial^\alpha f}\|_{M^{\infty}_{m\cdot n}} \cdot \int \frac{1}{\langle x\rangle^{2k-l}} d\omega,$$

where $n(x,\omega) = \langle x\rangle^{-l}$, $l \in \mathbb{N}$ arbitrary and k is large enough $(2k > d + l)$. Since for $p \le p'$ and $q \le q'$ we have $M^{p,q} \hookrightarrow M^{p',q'}$, the following claim holds:

Lemma 3.1.6. *Let $m \in \mathscr{P}$. For any p, $q \in [1,\infty]$ and k, $l \in \mathbb{N}$ $(2k > d + l)$ there exists a constant $C \ge 0$ such that*

$$\|\hat{f}\|_{M^{1,\infty}_m} \le C \cdot \sum_{|\alpha|\le 2k} \|\widehat{\partial^\alpha f}\|_{M^{p,q}_{m\cdot n}}$$

holds for every $f \in \mathscr{S}(\mathbb{R}^d)$, where $n(x,\omega) := \langle \omega\rangle^{-l}$.

Corollary 3.1.7. *Let p, $q \in [1,\infty]$ and $s \ge 0$, $t \in \mathbb{R}$. Then there exist constants $C \ge 0$ and $k \in \mathbb{N}$ such that*

$$\|\hat{f}\|_{M^{1,\infty}} \le C \cdot \sum_{|\alpha|\le 2k} \|\widehat{\partial^\alpha f}\|_{M^{p,q}_{s,t}}$$

holds for every $f \in \mathscr{S}(\mathbb{R}^d)$.

Lemma 3.1.8. *Let $m \in \mathscr{P}$. For any p, $q \in [1,\infty]$, k, $l \in \mathbb{N}$ $(2k > l + d/p)$ there exists a constant $C \ge 0$ such that*

$$\|\hat{f}\|_{M^{p,q}_m} \le C \cdot \sum_{|\alpha|\le 2k} \|\widehat{\partial^\alpha f}\|_{M^{\infty,q}_{m\cdot n}} \tag{3.4}$$

holds for every $f \in \mathscr{S}$, where $n(x,\omega) := \langle x\rangle^{-l}$.

Proof. The estimate (3.3) yields

$$\begin{aligned}\left|V_{\hat{g}}\hat{f}(x,\omega)\cdot m(x,\omega)\right| &\le C \cdot \sum_{|\alpha|,|\beta|\le 2k} \left|V_{\widehat{\partial^\beta g}}\widehat{\partial^\alpha f}(x,\omega)\right| \cdot \frac{m(x,\omega)}{\langle x\rangle^{l}} \cdot \frac{1}{\langle x\rangle^{2k-l}} \\ &\le C \cdot \sum_{|\alpha|,|\beta|\le 2k} \sup_{x\in\mathbb{R}^d}\Big[\left|V_{\widehat{\partial^\beta g}}\widehat{\partial^\alpha f}(x,\omega)\right| \cdot \frac{m(x,\omega)}{\langle x\rangle^{l}}\Big] \cdot \frac{1}{\langle x\rangle^{2k-l}}.\end{aligned}$$

By taking the L^p-norm with respect to x and L^q-norm with respect to ω respectively, we obtain the estimate (3.4).

□

Corollary 3.1.9.

$$C^\infty_{\mathcal{F}M^{p,q}_{s,t}} = C^\infty_{\mathcal{F}M^{1,q}_{s,0}}$$

Proof. This follows from Proposition 1.2.4 and Lemma 3.1.8. □

Corollary 3.1.10. *Let $u \in \mathcal{F}M^{p_1,q_1}_{s_1,t_1}$, $v \in C^\infty_{\mathcal{F}M^{p_2,q_2}_{s_2,t_2}}$, where p_1, p_2, q_1, $q_2 \in [1,\infty]$ and s_1, s_2, t_1, $t_2 \in \mathbb{R}$ with $s_2 \ge 0$. Then $u \cdot v \in \mathcal{F}M^{p_1,q_1}_{s_1,t_1}$. In particular, for $u \in C^\infty_{\mathcal{F}M^{p_1,q_1}_{s_1,t_1}}$ we obtain $u \cdot v \in C^\infty_{\mathcal{F}M^{p_1,q_1}_{s_1,t_1}} \cap C^\infty_{\mathcal{F}M^{p_2,q_2}_{s_2,t_2}}$.*

Proof. By Corollaries 3.1.7 and 2.1.7 there exists a constant $C \geq 0$ such that

$$\|u \cdot v\|_{\mathcal{F}M^{p_1,q_1}_{s_1,t_1}} = C \cdot \|\hat{u} * \hat{v}\|_{M^{p_1,q_1}_{s_1,t_1}} \leq C \cdot \sum_\alpha \|u\|_{\mathcal{F}M^{p_1,q_1}_{s_1,t_1}} \cdot \|\partial^\alpha v\|_{\mathcal{F}M^{p_2,q_2}_{s_2,t_2}}.$$

For the second assertion we apply the Leibniz rule. □

3.2 The generalized Wiener-Amalgam space $\mathscr{G}_{C^\infty_{\mathcal{F}M^{p,q}_{s,t}}}(\mathbb{R}^d)$

We are now ready to define $\mathscr{G}_{C^\infty_{\mathcal{F}M^{p,q}_{s,t}}}(\mathbb{R}^d)$ as a quotient algebra of the (differential) algebra $\mathscr{E}_{C^\infty_{\mathcal{F}M^{p,q}_{s,t}}}(\mathbb{R}^d)$ and its (differential) ideal $\mathscr{N}_{C^\infty_{\mathcal{F}M^{p,q}_{s,t}}}(\mathbb{R}^d)$.

Throughout this section we will assume $s \geq 0$, unless otherwise stated.

Definition 3.2.1.

$$\mathscr{E}_{C^\infty_{\mathcal{F}M^{p,q}_{s,t}}}(\mathbb{R}^d) := \{(u_\varepsilon)_\varepsilon \in H^q_\infty(\mathbb{R}^d)^{(0,1]} \big| \, \forall \alpha \in \mathbb{N}_0^d \; \exists N \in \mathbb{N} : \; \|\partial^\alpha u_\varepsilon\|_{\mathcal{F}M^{p,q}_{s,t}} = \mathcal{O}(\varepsilon^{-N})\},$$

$$\mathscr{N}_{C^\infty_{\mathcal{F}M^{p,q}_{s,t}}}(\mathbb{R}^d) := \{(u_\varepsilon)_\varepsilon \in H^q_\infty(\mathbb{R}^d)^{(0,1]} \big| \, \forall \alpha \in \mathbb{N}_0^d \; \forall m \in \mathbb{N}_0 : \; \|\partial^\alpha u_\varepsilon\|_{\mathcal{F}M^{p,q}_{s,t}} = \mathcal{O}(\varepsilon^{m})\}.$$

Theorem 3.2.2. *$\mathscr{E}_{C^\infty_{\mathcal{F}M^{p,q}_{s,t}}}(\mathbb{R}^d)$ is a (differential) algebra and $\mathscr{N}_{C^\infty_{\mathcal{F}M^{p,q}_{s,t}}}(\mathbb{R}^d)$ is an ideal in $\mathscr{E}_{C^\infty_{\mathcal{F}M^{p,q}_{s,t}}}(\mathbb{R}^d)$.*

Proof. Take $(u_\varepsilon)_\varepsilon$, $(v_\varepsilon)_\varepsilon \in \mathscr{E}_{C^\infty_{\mathcal{F}M^{p,q}_{s,t}}}(\mathbb{R}^d)$. By Corollaries 2.1.7 and 3.1.7 we have

$$\|u_\varepsilon \cdot v_\varepsilon\|_{\mathcal{F}M^{p,q}_{s,t}} \leq C \cdot \|u_\varepsilon\|_{\mathcal{F}M^{p,q}_{s,t}} \cdot \|v_\varepsilon\|_{\mathcal{F}M^{1,\infty}} \leq C \cdot \sum_{|\alpha| \leq 2k} \|u_\varepsilon\|_{\mathcal{F}M^{p,q}_{s,t}} \cdot \|\partial^\alpha v_\varepsilon\|_{\mathcal{F}M^{p,q}_{s,t}},$$

which implies

$$\|u_\varepsilon \cdot v_\varepsilon\|_{\mathcal{F}M^{p,q}_{s,t}} \leq C \cdot \varepsilon^{-N_1} \sum_{|\alpha| \leq 2k} \varepsilon^{-N_\alpha} \leq C \cdot \varepsilon^{-\max_{|\alpha| \leq 2k}\{N_1, N_\alpha\}}.$$

Thus, $u_\varepsilon \cdot v_\varepsilon$ (in norm) is bounded by some ε^{-N}. Applying the Leibniz rule we deduce that $(u_\varepsilon \cdot v_\varepsilon)_\varepsilon \in \mathscr{E}_{C^\infty_{\mathcal{F}M^{p,q}_{s,t}}}(\mathbb{R}^d)$ indeed. Similarly, one establishes that $\mathscr{N}_{C^\infty_{\mathcal{F}M^{p,q}_{s,t}}}(\mathbb{R}^d)$ is a (differential) ideal in (the differential algebra) $\mathscr{E}_{C^\infty_{\mathcal{F}M^{p,q}_{s,t}}}(\mathbb{R}^d)$. □

Remark 3.2.3. *We note that $C^\infty_{\mathcal{F}M^{p,q}_{s,t}}$ is a locally convex topological vector space topologized through the family of seminorms:*

$$p_j(u) := \max_{|\alpha| \leq j} \sup_{x \in \mathbb{R}^d} \|\partial^\alpha u\|_{\mathcal{F}M^{p,q}_{s,t}}.$$

Therefore, we may define the algebra $\mathscr{G}_{C^\infty_{\mathcal{F}M^{p,q}_{s,t}}}$ as an algebra based on a locally convex space (see Definition 1.3.3).

Theorem 3.2.4.

$$\mathscr{G}_{C^\infty_{\mathcal{F}M^{p,q}_{s,t}}}(\mathbb{R}^d) = \mathscr{G}_{C^\infty_{\mathcal{F}M^{1,q}_{s,0}}}(\mathbb{R}^d),$$

holds for all p, $q \in [1, \infty)$, $s \geq 0$ and $t \in \mathbb{R}$.

Proof. The assertion of the theorem follows directly from Corollary 3.1.9. $\square$

Remark 3.2.5. *For $p \in [1,\infty)$, $q_1 \leq q_2$, $s_1 \geq s_2 \geq 0$, $t \in \mathbb{R}$ the canonical map*

$$j : \mathscr{G}_{C^\infty_{\mathcal{F}M^{p,q_1}_{s_1,t}}}(\mathbb{R}^d) \to \mathscr{G}_{C^\infty_{\mathcal{F}M^{p,q_2}_{s_2,t}}}(\mathbb{R}^d), \quad j(u_\varepsilon + \mathscr{N}_{C^\infty_{\mathcal{F}M^{p,q_1}_{s_1,t}}}(\mathbb{R}^d)) := (u_\varepsilon)_\varepsilon + \mathscr{N}_{C^\infty_{\mathcal{F}M^{p,q_2}_{s_2,t}}}(\mathbb{R}^d)$$

is well defined. Namely, it is easy to see that

$$\mathscr{E}_{C^\infty_{\mathcal{F}M^{p,q_1}_{s_1,t}}}(\mathbb{R}^d) \subset \mathscr{E}_{C^\infty_{\mathcal{F}M^{p,q_2}_{s_2,t}}}(\mathbb{R}^d) \text{ and } \mathscr{N}_{C^\infty_{\mathcal{F}M^{p,q_1}_{s_1,t}}}(\mathbb{R}^d) \subset \mathscr{N}_{C^\infty_{\mathcal{F}M^{p,q_2}_{s_2,t}}}(\mathbb{R}^d)$$

for $s_1 \geq s_2$, $q_1 \leq q_2$.

We conjecture that this map is not an inclusion map (not injective).

Theorem 3.2.6. *Let ρ be a standard mollifier. The linear mapping*

$$\widetilde{\iota}(u) := (u * \rho_\varepsilon)_\varepsilon + \mathscr{N}_{C^\infty_{\mathcal{F}M^{p,q}_{s,t}}}(\mathbb{R}^d)$$

of $\mathcal{F}M^{p,q}_{s,t}(\mathbb{R}^d)$ into $\mathscr{G}_{C^\infty_{\mathcal{F}M^{p,q}_{s,t}}}(\mathbb{R}^d)$ is an embedding.

Proof. By Theorem 3.1.3 $(u*\rho_\varepsilon)_\varepsilon$ is $C^\infty_{\mathcal{F}M^{p,q}_{s,t}}$-moderate. Let $(u*\rho_\varepsilon)_\varepsilon$ be negligible and let $\varphi \in \mathscr{S}(\mathbb{R}^d)$ be arbitrary. By Proposition 1.2.11 and Parseval's formula we have that

$$|\langle u * \rho_\varepsilon, \varphi\rangle| = |\langle \widehat{u * \rho_\varepsilon}, \hat{\varphi}\rangle| \leq C \cdot \|u * \rho_\varepsilon\|_{\mathcal{F}M^{p,q}_{s,t}} \cdot \|\varphi\|_{\mathcal{F}M^{p',q'}_{-s,-t}} \to 0, \quad (\varepsilon \to 0).$$

Thus, $u = 0$ in $\mathscr{S}'(\mathbb{R}^d)$ and so $u = 0$ in $\mathcal{F}M^{p,q}_{s,t}(\mathbb{R}^d)$. $\square$

Next we give a characterization of the ideal in $\mathscr{G}_{C^\infty_{\mathcal{F}M^{p,q}_{s,t}}}(\mathbb{R}^d)$. This result is important since it shows that it suffices to consider only the 0-th order derivative estimate of the representative of a moderate net in order to show that it is negligible.

Theorem 3.2.7 (Characterization of the Ideal)**.** $(u_\varepsilon)_\varepsilon \in \mathscr{E}_{C^\infty_{\mathcal{F}M^{p,q}_{s,t}}}(\mathbb{R}^d)$ *is $C^\infty_{\mathcal{F}M^{p,q}_{s,t}}(\mathbb{R}^d)$-negligible if and only if the following condition is satisfied:*

$$\forall m \in \mathbb{N}_0 \quad \|u_\varepsilon\|_{\mathcal{F}M^{p,q}_{s,t}} = \mathcal{O}(\varepsilon^m). \tag{3.1}$$

Proof. The necessity of condition (3.1) is obvious. Let now $(u_\varepsilon)_\varepsilon \in \mathscr{E}_{C^\infty_{\mathcal{F}M^{p,q}_{s,t}}}(\mathbb{R}^d)$ such that (3.1) holds. It suffices to prove that for any $1 \leq j \leq d$, $\partial_j u_\varepsilon$ satisfies the 0-th negligibility estimate.

Since $(u_\varepsilon)_\varepsilon$ is negligible and $(\partial_j^2 u_\varepsilon)_\varepsilon$ is moderate, we may choose $N \in \mathbb{N}$ such that for every $m \in \mathbb{N}_0$ both of the following estimates hold:

$$\|u_\varepsilon\|_{\mathcal{F}M^{p,q}_{s,t}} = \mathcal{O}(\varepsilon^{2m+N}), \tag{3.2}$$

$$\|\partial_j^2 u_\varepsilon\|_{\mathcal{F}M^{p,q}_{s,t}} = \mathcal{O}(\varepsilon^{-N}). \tag{3.3}$$

By Taylor expansion we have

$$u_\varepsilon(x + he_j) = u_\varepsilon(x) + \partial_j u_\varepsilon(x) \cdot h + h^2 \cdot \int_0^1 (1-\theta)\partial_j^2 u_\varepsilon(x + h\theta e_j)d\theta,$$

so the first-order derivative term is

$$\partial_j u_\varepsilon(x) = \big(u_\varepsilon(x+he_j) - u_\varepsilon(x)\big) \cdot h^{-1} - h \cdot \int_0^1 (1-\theta) \cdot \partial_j^2 u_\varepsilon(x+\theta h e_j) d\theta.$$

Applying Fourier transform (with respect to x) we arrive at:

$$\begin{aligned}\widehat{\partial_j u_\varepsilon}(\xi) &= \big(\hat{u_\varepsilon}(\xi) e^{2\pi i h e_j \xi} - \hat{u_\varepsilon}(\xi)\big) \cdot h^{-1} - h \cdot \int_0^1 (1-\theta) \cdot \widehat{\partial_j^2 u_\varepsilon}(\xi) e^{2\pi i \theta h e_j \xi}\, d\theta \\ &= \big(M_{he_j} \hat{u_\varepsilon}(\xi) - \hat{u_\varepsilon}(\xi)\big) \cdot h^{-1} - h \cdot \int_0^1 (1-\theta) \cdot M_{\theta h e_j} \widehat{\partial_j^2 u_\varepsilon}(\xi)\, d\theta.\end{aligned}$$

Setting $h = \varepsilon^{m+N}$ we obtain

$$\begin{aligned}\|\widehat{\partial_j u_\varepsilon}\|_{M^{p,q}_{s,t}} \le \underbrace{(\langle \varepsilon^{m+N} e_j\rangle^s + 1)}_{\le 2^{s+1}} \cdot \|\hat{u_\varepsilon}\|_{M^{p,q}_{s,t}} \cdot \varepsilon^{-m-N} \\ + \|\int_0^1 (1-\theta) \cdot M_{\theta \varepsilon^{m+N} e_j} \widehat{\partial_j^2 u_\varepsilon}\, d\theta\|_{M^{p,q}_{s,t}} \cdot \varepsilon^{m+N}.\end{aligned}$$

The above calculation make use of the translation invariance of modulation spaces (Prop. (1.2.3)), this time in the form:

$$\|M_{\varepsilon^{m+N} e_j} \hat{u_\varepsilon}\|_{M^{p,q}_{s,t}} \le \langle \varepsilon^{m+N} e_j\rangle^s \cdot \|\hat{u_\varepsilon}\|_{M^{p,q}_{s,t}}.$$

The second term on the right hand side of the above estimate gives

$$\begin{aligned}&|V_g\Big(\int_0^1 (1-\theta)\cdot M_{\theta\varepsilon^{m+N} e_j} \widehat{\partial_j^2 u_\varepsilon} d\theta\Big)(x,\omega)| \\ &\quad = \Big| \int \Big(\int_0^1 (1-\theta) \cdot M_{\theta\varepsilon^{m+N} e_j} \widehat{\partial_j^2 u_\varepsilon} d\theta\Big) \cdot \overline{g(y-x)} \cdot e^{2\pi i y\omega} dy \Big| \\ &\quad = \Big| \int_0^1 (1-\theta) \cdot \Big(\int M_{\theta\varepsilon^{m+N} e_j} \widehat{\partial_j^2 u_\varepsilon} \cdot \overline{g(y-x)} \cdot e^{2\pi i y\omega} dy\Big) d\theta \Big| \\ &\quad\le \int_0^1 (1-\theta)\cdot \Big| V_g\Big(M_{\theta\varepsilon^{m+N} e_j} \widehat{\partial_j^2 u_\varepsilon}\Big)(x,\omega)\Big| d\theta.\end{aligned}$$

Thus by the generalized Minkowski inequality, we have

$$\|\int_0^1 (1-\theta)\cdot M_{\theta\varepsilon^{m+N} e_j} \widehat{\partial_j^2 u_\varepsilon}\, d\theta\|_{M^{p,q}_{s,t}} \le C \cdot \sup_\theta \|M_{\theta\varepsilon^{m+N} e_j} \widehat{\partial_j^2 u_\varepsilon}\|_{M^{p,q}_{s,t}}.$$

Therefore,

$$\sup_\theta \|M_{\theta\varepsilon^{m+N} e_j} \widehat{\partial_j^2 u_\varepsilon}\|_{M^{p,q}_{s,t}} \le C \cdot \sup_\theta \langle \theta \varepsilon^{m+N} e_j\rangle^s \cdot \|\widehat{\partial_j^2 u_\varepsilon}\|_{M^{p,q}_{s,t}} \le C \cdot \|\widehat{\partial_j^2 u_\varepsilon}\|_{M^{p,q}_{s,t}},$$

we have (for $s \ge 0$, $\langle\theta\varepsilon^{m+N} e_j\rangle^s \le 2^s$, for $s < 0$, $\langle\theta\varepsilon^{m+N} e_j\rangle^s \le 1$)

$$\|\widehat{\partial_j u_\varepsilon}\|_{M^{p,q}_{s,t}} \le 2^{s+1} \cdot \underbrace{\|\hat{u_\varepsilon}\|_{M^{p,q}_{s,t}}}_{\le C\cdot\varepsilon^{2m+N},\ \text{by (3.2)}} \cdot \varepsilon^{-m-N} + C \cdot \underbrace{\|\widehat{\partial_j^2 u_\varepsilon}\|_{M^{p,q}_{s,t}}}_{\le C\cdot\varepsilon^{-N},\ \text{by (3.3)}} \cdot \varepsilon^{m+N}.$$

Finally,

$$\|\widehat{\partial_j u_\varepsilon}\|_{M^{p,q}_{s,t}} \le C \cdot \varepsilon^{2m+N} \cdot \varepsilon^{-m-N} + C \cdot \varepsilon^{-N} \cdot \varepsilon^{m+N} \le C \cdot \varepsilon^m.$$

□

We will now take the constant embedding into consideration and show that it actually coincides (up to negligible nets) with our standard embedding (via convolution with a mollifier) on the level of smooth functions in $C^{\infty}_{\mathcal{F}M^{p,q}_{s,t}}$. This means that instead of convolving functions in $C^{\infty}_{\mathcal{F}M^{p,q}_{s,t}}$, it is enough to consider the constant net corresponding to the function itself. This also implies the invariance of the product on the level of smooth functions in $C^{\infty}_{\mathcal{F}M^{p,q}_{s,t}}$.

Theorem 3.2.8. *Let* $\widetilde{\sigma} : C^{\infty}_{\mathcal{F}M^{p,q}_{s,t}} \to \mathscr{G}_{C^{\infty}_{\mathcal{F}M^{p,q}_{s,t}}}$ *given by* $\widetilde{\sigma} : f \to [(f)_\epsilon]$. *Then*

$$\widetilde{\iota}_{|C^{\infty}_{\mathcal{F}M^{p,q}_{s,t}}} = \widetilde{\sigma}.$$

Proof. Since any $f \in C^{\infty}_{\mathcal{F}M^{p,q}_{s,t}}$ satisfies the equality (2.5), we have

$$V_g(f * \rho_\varepsilon - f)(x,\omega) = \int \sum_{|\alpha|=m} \int \int_0^1 \frac{\partial^\alpha f(y-\varepsilon\theta\eta)}{(m-1)!} \cdot (1-\theta)^{(m-1)} d\theta \cdot (-\varepsilon\eta)^\alpha \rho(\eta) d\eta \cdot \overline{g(t-x)} e^{-2\pi i y\omega} dy \ .$$

After applying Fourier transform, we obtain:

$$\|\mathcal{F}(f * \rho_\varepsilon - f)\|_{M^{p,q}_{s,t}} = \Big(\int \Big(\int \Big| \int \sum_{|\alpha|=m} \int \int_0^1 \frac{M_{\varepsilon\theta\eta}\widehat{\partial^\alpha f}(y)}{(m-1)!} \cdot (1-\theta)^{m-1} d\theta \cdot (-\varepsilon\eta)^\alpha \rho(\eta) d\eta \cdot \overline{g(y-x)} e^{-2\pi i y\omega} dy \ \langle x\rangle^t \langle\omega\rangle^s \Big|^p dx \ \Big)^{\frac{q}{p}} d\omega\Big)^{\frac{1}{q}}.$$

By the generalized Minkowski inequality

$$\|f * \rho_\varepsilon - f\|_{\mathcal{F}M^{p,q}_{s,t}} \le C \cdot \varepsilon^m \cdot \sum_{|\alpha|=m} \int \int_0^1 \Big(\int \Big(\int \Big| \int \frac{M_{\varepsilon\theta\eta}\widehat{\partial^\alpha f}(y)}{(m-1)!} \cdot \overline{g(y-x)} e^{-2\pi i y\omega} dy \ \langle x\rangle^t \langle\omega\rangle^s \Big|^p dx \ \Big)^{\frac{q}{p}} d\omega\Big)^{\frac{1}{q}} \cdot |\eta^\alpha| \cdot |\rho(\eta)| d\eta \cdot (1-\theta)^{m-1} d\theta.$$

Finally, invariance under time-frequency shifts gives

$$\|f * \rho_\varepsilon - f\|_{\mathcal{F}M^{p,q}_{s,t}} \le C \cdot \varepsilon^m \cdot \sum_{|\alpha|=m} \|\partial^\alpha f\|_{\mathcal{F}M^{p,q}_{s,t}} \int \int_0^1 |\eta^\alpha| \cdot \langle\varepsilon\theta\eta\rangle^s \cdot |\rho(\eta)| d\eta \ d\theta,$$

thus

$$\|f * \rho_\varepsilon - f\|_{\mathcal{F}M^{p,q}_{s,t}} \le C \cdot \varepsilon^m \sum_{|\alpha|\le m} \|\partial^\alpha f\|_{\mathcal{F}M^{p,q}_{s,t}}.$$

□

As hinted, now we will show that our embedding map does preserve the product of two functions in $C^{\infty}_{\mathcal{F}M^{p,q}_{s,t}}$. Additionally, we show that this invariance remains valid under slightly less restrictive assumptions.

Theorem 3.2.9. *Let $u \in C^\infty_{\mathcal{F}M^{p_1,q_1}_{s_1,t_1}}(\mathbb{R}^d)$, $v \in C^\infty_{\mathcal{F}M^{p_2,q_2}_{s_2,t_2}}(\mathbb{R}^d)$, where p_1, p_2, q_1, $q_2 \in [1,\infty]$ and s_1, s_2, t_1, $t_2 \in \mathbb{R}$ with s_1, $s_2 \geq 0$. Then*

$$\widetilde{\iota}(u)_\varepsilon \cdot \widetilde{\iota}(v)_\varepsilon - \widetilde{\iota}(uv)_\varepsilon \in \mathscr{N}_{C^\infty_{\mathcal{F}M^{p_1,q_1}_{s_1,t_1}}}(\mathbb{R}^d) \cap \mathscr{N}_{C^\infty_{\mathcal{F}M^{p_2,q_2}_{s_2,t_2}}}(\mathbb{R}^d),$$

where $\widetilde{\iota}(u)_\varepsilon := (u * \rho_\varepsilon)_\varepsilon$.

Proof. By Theorem 3.2.7 it suffices to show that for every $m \in N$ the estimate

$$\|(u * \rho_\varepsilon) \cdot (v * \rho_\varepsilon) - (u \cdot v) * \rho_\varepsilon\|_{\mathcal{F}M^{p_j,q_j}_{s_j,t_j}} < \varepsilon^m$$

for $j = 1$ or $j = 2$, holds for ε small enough. By Corollary 3.1.10 we have $u \cdot v \in C^\infty_{\mathcal{F}M^{p_1,q_1}_{s_1,t_1}}$. By using Corollaries 3.1.7 and 2.1.7 we calculate

$$\begin{aligned}
&\|(u * \rho_\varepsilon) \cdot (v * \rho_\varepsilon) - (u \cdot v) * \rho_\varepsilon\|_{\mathcal{F}M^{p_1,q_1}_{s_1,t_1}} \\
&\quad \leq \|(u * \rho_\varepsilon - u) \cdot (v * \rho_\varepsilon - v)\|_{\mathcal{F}M^{p_1,q_1}_{s_1,t_1}} + \|u \cdot (v * \rho_\varepsilon - v)\|_{\mathcal{F}M^{p_1,q_1}_{s_1,t_1}} \\
&\quad\quad + \|(u * \rho_\varepsilon - u) \cdot v\|_{\mathcal{F}M^{p_1,q_1}_{s_1,t_1}} + \|u \cdot v - (u \cdot v) * \rho_\varepsilon\|_{\mathcal{F}M^{p_1,q_1}_{s_1,t_1}} \\
&\quad\quad \leq C_1 \cdot \underbrace{\|u * \rho_\varepsilon - u\|_{\mathcal{F}M^{p_1,q_1}_{s_1,t_1}}}_{\text{negligible}} \cdot \sum_\alpha \underbrace{\|\partial^\alpha v * \rho_\varepsilon - \partial^\alpha v\|_{\mathcal{F}M^{p_2,q_2}_{s_2,t_2}}}_{\text{negligible}} \\
&\quad\quad\quad + C_2 \cdot \|u\|_{\mathcal{F}M^{p_1,q_1}_{s_1,t_1}} \cdot \sum_\alpha \underbrace{\|\partial^\alpha v * \rho_\varepsilon - \partial^\alpha v\|_{\mathcal{F}M^{p_2,q_2}_{s_2,t_2}}}_{\text{negligible}} \\
&\quad\quad\quad + C_3 \cdot \underbrace{\|u * \rho_\varepsilon - u\|_{\mathcal{F}M^{p_1,q_1}_{s_1,t_1}}}_{\text{negligible}} \cdot \sum_\alpha \|\partial^\alpha v\|_{\mathcal{F}M^{p_2,q_2}_{s_2,t_2}} \\
&\quad\quad\quad\quad + C_4 \cdot \underbrace{\|uv - (uv) * \rho_\varepsilon\|_{\mathcal{F}M^{p_1,q_1}_{s_1,t_1}}}_{\text{negligible}}.
\end{aligned}$$

□

Theorem 3.2.10. *Let $u \in \mathcal{F}M^{p_1,q_1}_{s_1,t_1}$, $v \in C^\infty_{\mathcal{F}M^{p_2,q_2}_{s_2,t_2}}$ such that $u \cdot v \in C^\infty_{\mathcal{F}M^{p_1,q_1}_{s_1,t_1}}$ and for every $m \in \mathbb{N}_0$, $\|\widetilde{\iota}(u)_\varepsilon - u\|_{\mathcal{F}M^{p_1,q_1}_{s_1,t_1}} < \varepsilon^m$, where p_1, p_2, q_1, $q_2 \in [1,\infty]$ and s_1, s_2, t_1, $t_2 \in \mathbb{R}$ with only $s_2 \geq 0$. Then*

$$\widetilde{\iota}(u)_\varepsilon \cdot \widetilde{\iota}(v)_\varepsilon - \widetilde{\iota}(uv)_\varepsilon \in \mathscr{N}_{C^\infty_{\mathcal{F}M^{p_1,q_1}_{s_1,t_1}}}(\mathbb{R}^d),$$

where $\widetilde{\iota}(u)_\varepsilon := (u * \rho_\varepsilon)_\varepsilon$.

3.3 Association in $\mathscr{G}_{C^\infty_{\mathcal{F}M^{p,q}_{s,t}}}(\mathbb{R}^d)$

We shall now introduce a relation that generates classes of generalized functions (which carry nonlinear information) that correspond to a single 'linear' element in modulation space, and where the correspondence is realized through a limiting procedure. This procedure is important since it provides the link between a generalized object and a distribution in modulation space (e.g. a generalized solution of a differential equation associated to a distribution in modulation space, also in cases when a classical distributional solution does not exist).

Definition 3.3.1. *Let $s \geq 0$, $t \in \mathbb{R}$. An element $u \in \mathscr{G}_{C^\infty_{\mathcal{F}M^{p,q}_{s,t}}}(\mathbb{R}^d)$ is called* associated *with 0 (denoted by $u \overset{\mathcal{F}M^{p,q}_{s,t}}{\approx} 0$) if*

$$\lim_{\varepsilon\to 0} \|u_\varepsilon\|_{\mathcal{F}M^{p,q}_{s,t}} = 0. \tag{3.1}$$

Two elements u, v of $\mathscr{G}_{C^\infty_{\mathcal{F}M^{p,q}_{s,t}}}(\mathbb{R}^d)$ are associated with each other (and we write $u \overset{\mathcal{F}M^{p,q}_{s,t}}{\approx} v$) if

$$u - v \overset{\mathcal{F}M^{p,q}_{s,t}}{\approx} 0.$$

If a representative (u^1_ε) of element u satisfies (3.1), than any other representative , (u^2_ε) of element u satisfies (3.1), too. This follows from the proof of Proposition 3.3.3.

Definition 3.3.2. *Let $u \in \mathscr{G}_{C^\infty_{\mathcal{F}M^{p,q}_{s,t}}}(\mathbb{R}^d)$, $v \in \mathcal{F}M^{p,q}_{s,t}$. Suppose that $u \overset{\mathcal{F}M^{p,q}_{s,t}}{\approx} \widetilde{\iota}(v)$. Then u is said to admit v as associated element of $\mathcal{F}M^{p,q}_{s,t}$ and v is also called the* shadow *of u. In this case we simply write $u \overset{\mathcal{F}M^{p,q}_{s,t}}{\approx} v$.*

Proposition 3.3.3. *Let $u \in \mathcal{F}M^{p,q}_{s,t}$, p, $q \in [1,\infty)$. If $\widetilde{\iota}(u) \overset{\mathcal{F}M^{p,q}_{s,t}}{\approx} 0$, then $u = 0$.*

Proof. Let $\varphi \in \mathscr{S}(\mathbb{R}^d)$. Then there exists a constant $C \geq 0$ such that

$$|\langle u * \rho_\varepsilon, \varphi\rangle| = |\langle \widehat{u * \rho_\varepsilon}, \hat{\varphi}\rangle| \leq C \cdot \|u * \rho_\varepsilon\|_{\mathcal{F}M^{p,q}_{s,t}} \cdot \|\varphi\|_{\mathcal{F}M^{p',q'}_{-s,-t}} \to 0, \quad (\varepsilon \to 0).$$

□

Proposition 3.3.4. *The following properties hold:*

(i) *Let u, $v \in \mathcal{F}M^{p,q}_{s,t+t_0}$, $t_0 \in \mathbb{N}_0$. Then*

$$u \overset{\mathcal{F}M^{p,q}_{s,t+t_0}}{\approx} v \Leftrightarrow \partial^\alpha u \overset{\mathcal{F}M^{p,q}_{s,t}}{\approx} \partial^\alpha v$$

for all $|\alpha| \leq t_0$.

(ii) *Let u, $v \in \mathscr{G}_{C^\infty_{\mathcal{F}M^{p_1,q_1}_{s_1,t_1}}}(\mathbb{R}^d)$, $u \overset{\mathcal{F}M^{p_1,q_1}_{s_1,t_1}}{\approx} v$, $f \in C^\infty_{\mathcal{F}M^{p_2,q_2}_{s_2,t_2}}$, $s_2 \geq 0$ for any p_2, q_2, s_2 and t_2). Then we have*

$$f \cdot u \overset{\mathcal{F}M^{p_1,q_1}_{s_1,t_1}}{\approx} f \cdot v.$$

(iii) *Let $u \in \mathcal{F}M^{p_1,q_1}_{s_1,t_1}$ and $v \in \mathcal{F}M^{1,q_0}$ (or $u \in \mathcal{F}M^{p_1,q_1}_{s_1,t_1}$ and $v \in C^\infty_{\mathcal{F}M^{p_2,q_2}_{s_2,t_2}}$ as in Corollary 3.1.10). Then we have*

$$\widetilde{\iota}(u) \cdot \widetilde{\iota}(v) \overset{\mathcal{F}M^{p_1,q_1}_{s_1,t_1}}{\approx} \widetilde{\iota}(u \cdot v).$$

Proof. The proof (i) easily follows from the proof of Lemma 1.2.10.

(ii) Let $f \in C^\infty_{\mathcal{F}M^{p_2,q_2}_{s_2,t_2}}$. We use the following estimate (cf. Corollary 3.1.10):

$$\|f \cdot (u_\varepsilon - v_\varepsilon)\|_{\mathcal{F}M^{p_1,q_1}_{s_1,t_1}} \le C \cdot \sum_\alpha \|u_\varepsilon - v_\varepsilon\|_{\mathcal{F}M^{p_1,q_1}_{s_1,t_1}} \cdot \|\partial^\alpha f\|_{\mathcal{F}M^{p_2,q_2}_{s_2,t_2}}.$$

(iii) Let $u \in \mathcal{F}M^{p_1,q_1}_{s_1,t_1}$ and $v \in \mathcal{F}M^{1,q_0}$. We may write

$$\begin{aligned}
&\|(u*\rho_\varepsilon)\cdot(v*\rho_\varepsilon) - (u\cdot v)*\rho_\varepsilon\|_{\mathcal{F}M^{p_1,q_1}_{s_1,t_1}} \\
&\quad \le \|(u*\rho_\varepsilon - u)\cdot(v*\rho_\varepsilon - v)\|_{\mathcal{F}M^{p_1,q_1}_{s_1,t_1}} + \|u\cdot(v*\rho_\varepsilon - v)\|_{\mathcal{F}M^{p_1,q_1}_{s_1,t_1}} \\
&\qquad + \|(u*\rho_\varepsilon - u)\cdot v\|_{\mathcal{F}M^{p_1,q_1}_{s_1,t_1}} + \|u\cdot v - (u\cdot v)*\rho_\varepsilon\|_{\mathcal{F}M^{p_1,q_1}_{s_1,t_1}} \\
&\le C_1 \cdot \underbrace{\|u*\rho_\varepsilon - u\|_{\mathcal{F}M^{p_1,q_1}_{s_1,t_1}}}_{\to 0} \cdot \underbrace{\|v*\rho_\varepsilon - v\|_{\mathcal{F}M^{1,q_0}}}_{\to 0} + C_2 \cdot \|u\|_{\mathcal{F}M^{p_1,q_1}_{s_1,t_1}} \cdot \underbrace{\|v*\rho_\varepsilon - v\|_{\mathcal{F}M^{1,q_0}}}_{\to 0} \\
&\qquad + C_3 \cdot \underbrace{\|u*\rho_\varepsilon - u\|_{\mathcal{F}M^{p_1,q_1}_{s_1,t_1}}}_{\to 0} \cdot \|v\|_{\mathcal{F}M^{1,q_0}} + C_4 \cdot \underbrace{\|uv - (uv)*\rho_\varepsilon\|_{\mathcal{F}M^{p_1,q_1}_{s_1,t_1}}}_{\to 0}.
\end{aligned}$$

The second part can be proved in a similar fashion:

$$\begin{aligned}
&\|(u*\rho_\varepsilon)\cdot(v*\rho_\varepsilon) - (u\cdot v)*\rho_\varepsilon\|_{\mathcal{F}M^{p_1,q_1}_{s_1,t_1}} \\
&\quad \le \|(u*\rho_\varepsilon - u)\cdot(v*\rho_\varepsilon - v)\|_{\mathcal{F}M^{p,q}_{s,t}} + \|u\cdot(v*\rho_\varepsilon - v)\|_{\mathcal{F}M^{p_1,q_1}_{s_1,t_1}} \\
&\qquad + \|(u*\rho_\varepsilon - u)\cdot v\|_{\mathcal{F}M^{p_1,q_1}_{s_1,t_1}} + \|u\cdot v - (u\cdot v)*\rho_\varepsilon\|_{\mathcal{F}M^{p_1,q_1}_{s_1,t_1}} \\
&\qquad \le C_1 \cdot \underbrace{\|u*\rho_\varepsilon - u\|_{\mathcal{F}M^{p_1,q_1}_{s_1,t_1}}}_{\to 0} \cdot \sum_\alpha \underbrace{\|\partial^\alpha v*\rho_\varepsilon - \partial^\alpha v\|_{\mathcal{F}M^{p_2,q_2}_{s_2,t_2}}}_{\to 0} \\
&\qquad\quad + C_2 \cdot \|u\|_{\mathcal{F}M^{p_1,q_1}_{s_1,t_1}} \cdot \sum_\alpha \underbrace{\|\partial^\alpha v*\rho_\varepsilon - \partial^\alpha v\|_{\mathcal{F}M^{p_2,q_2}_{s_2,t_2}}}_{\to 0} \\
&\qquad\quad + C_3 \cdot \underbrace{\|u*\rho_\varepsilon - u\|_{\mathcal{F}M^{p_1,q_1}_{s_1,t_1}}}_{\to 0} \cdot \sum_\alpha \|\partial^\alpha v\|_{\mathcal{F}M^{p_2,q_2}_{s_2,t_2}} \\
&\qquad\qquad + C_4 \cdot \underbrace{\|uv - (uv)*\rho_\varepsilon\|_{\mathcal{F}M^{p_1,q_1}_{s_1,t_1}}}_{\to 0}.
\end{aligned}$$

□

3.4 Regular elements in Wiener-Amalgam generalized functions

We define the space of regular Colombeau generalized functions in the present context by

$$\mathscr{G}^\infty_{C^\infty_{\mathcal{F}M^{p,q}_{s,t}}}(\mathbb{R}^d) := \mathscr{E}^\infty_{C^\infty_{\mathcal{F}M^{p,q}_{s,t}}}(\mathbb{R}^d)/\mathscr{N}_{C^\infty_{\mathcal{F}M^{p,q}_{s,t}}}(\mathbb{R}^d),$$

where

$$\mathscr{E}^\infty_{C^\infty_{\mathcal{F}M^{p,q}_{s,t}}}(\mathbb{R}^d) := \{(u_\varepsilon)_\varepsilon \in H^q_\infty \big| \ \exists N \in \mathbb{N}: \ \forall \alpha \in \mathbb{N}_0^d \ \|\partial^\alpha u_\varepsilon\|_{\mathcal{F}M^{p,q}} = \mathcal{O}(\varepsilon^{-N})\}.$$

Conjecture 3.4.1.

$$\mathscr{G}^{\infty}_{C^{\infty}_{\mathcal{F}M^{p,q}_{0,t}}}(\mathbb{R}^d) \cap \mathscr{S}'(\mathbb{R}^d) = C^{\infty}_{\mathcal{F}M^{p,q}}(\mathbb{R}^d) = H^q_{\infty}(\mathbb{R}^d).$$

3.5 The generalized modulation space $M^{p,q}_{\mathscr{G}}(\mathbb{R}^d)$

In this section we will define the generalized modulation space $M^{p,q}_{\mathscr{G}}(\mathbb{R}^d)$ as a quotient algebra of the (differential) algebra $\mathscr{E}_{C^{\infty}_{M^{p,q}}}(\mathbb{R}^d)$ and its (differential) ideal $\mathcal{F}M^{p,q}_{\mathscr{N}}(\mathbb{R}^d)$. Such an algebra is important, since it represents a remedy for the phenomena of the loss of information provided by the L^q-integral (in the case of generalized modulation spaces) or of the loss of information provided by the L^p-integral (in the case of generalized Wiener-Amalgam spaces); the algebra $M^{p,q}_{\mathscr{G}}(\mathbb{R}^d)$ consists of algebras of moderate nets carrying the p-information and algebra of negligible nets, carrying q-information.

Definition 3.5.1.

$$\mathcal{F}M^{p,q}_{\mathscr{N}}(\mathbb{R}^d) := \{(u_\varepsilon)_\varepsilon \in \mathscr{E}_{C^{\infty}_{M^{p,q}}}(\mathbb{R}^d) \big| \; \forall \alpha \in \mathbb{N}_0^d \; \forall m \in \mathbb{N}_0 : \; \|\partial^\alpha u_\varepsilon\|_{\mathcal{F}M^{p,q}} = \mathcal{O}(\varepsilon^m)\}.$$

Theorem 3.5.2. *$\mathcal{F}M^{p,q}_{\mathscr{N}}(\mathbb{R}^d)$ is an ideal in $\mathscr{E}_{C^{\infty}_{M^{p,q}}}(\mathbb{R}^d)$.*

Proof. Take $(u_\varepsilon)_\varepsilon \in \mathcal{F}M^{p,q}_{\mathscr{N}}(\mathbb{R}^d)$, $(v_\varepsilon)_\varepsilon \in \mathscr{E}_{C^{\infty}_{M^{p,q}}}(\mathbb{R}^d)$. By Corollaries 2.1.7 and 3.1.7 we have

$$\begin{aligned}
\|u_\varepsilon \cdot v_\varepsilon\|_{\mathcal{F}M^{p,q}} &\le C \cdot \|u_\varepsilon\|_{\mathcal{F}M^{p,q}} \cdot \|v_\varepsilon\|_{\mathcal{F}M^{1,\infty}} \\
&\le C \cdot \sum_{|\alpha|\le 2k} \|u_\varepsilon\|_{\mathcal{F}M^{p,q}} \cdot \|\partial^\alpha v_\varepsilon\|_{\mathcal{F}M^{p,p}} \\
&\le C \cdot \sum_{|\alpha|\le 2k} \|u_\varepsilon\|_{\mathcal{F}M^{p,q}} \cdot \|\partial^\alpha v_\varepsilon\|_{M^{p,p}} \\
&\le C \cdot \sum_{|\alpha|\le 2k} \sum_{|\beta|\le 2k} \|u_\varepsilon\|_{\mathcal{F}M^{p,q}} \cdot \|\partial^{\alpha+\beta} v_\varepsilon\|_{M^{p,\infty}} \\
&\le C \cdot \sum_{|\alpha|\le 2k} \sum_{|\beta|\le 2k} \|u_\varepsilon\|_{\mathcal{F}M^{p,q}} \cdot \|\partial^{\alpha+\beta} v_\varepsilon\|_{M^{p,q}}.
\end{aligned}$$

Thus, $u_\varepsilon \cdot v_\varepsilon$ (in norm) is bounded by ε^m, for every $m \in \mathbb{N}_0$. Applying the Leibniz rule we deduce that $(u_\varepsilon \cdot v_\varepsilon)_\varepsilon \in \mathcal{F}M^{p,q}_{\mathscr{N}}(\mathbb{R}^d)$ indeed. □

Theorem 3.5.3. *Let ρ be a standard mollifier. The linear mapping*

$$\widetilde{\iota}(u) := (u * \rho_\varepsilon)_\varepsilon + \mathcal{F}M^{p,q}_{\mathscr{N}}(\mathbb{R}^d)$$

of $M^{p,q}(\mathbb{R}^d)$ into $M^{p,q}_{\mathscr{G}}(\mathbb{R}^d)$ is an embedding.

Proof. By Theorem 2.1.9 $(u*\rho_\varepsilon)_\varepsilon$ is $C^{\infty}_{M^{p,q}_{s,t}}$-moderate. Let $(u*\rho_\varepsilon)_\varepsilon$ be $M^{p,q}_{\mathscr{G}}(\mathbb{R}^d)$-negligible and let $\varphi \in \mathscr{S}(\mathbb{R}^d)$ be arbitrary. By Proposition 1.2.11 we have that

$$|\langle u * \rho_\varepsilon, \varphi\rangle| = |\langle \widehat{u * \rho_\varepsilon}, \hat{\varphi}\rangle| \le C \cdot \|u * \rho_\varepsilon\|_{\mathcal{F}M^{p,q}} \cdot \|\varphi\|_{\mathcal{F}M^{p',q'}} \to 0, \quad (\varepsilon \to 0).$$

Thus, $u = 0$ in $\mathscr{S}'(\mathbb{R}^d)$ and so $u = 0$ in $M^{p,q}(\mathbb{R}^d)$. □

Theorem 3.5.4 (Characterization of the Ideal)**.** *Let $p \le q$. $(u_\varepsilon)_\varepsilon \in \mathscr{E}_{C^\infty_{M^{p,q}}}(\mathbb{R}^d)$ is $M^{p,q}_{\mathscr{G}}(\mathbb{R}^d)$-negligible if and only if the following condition is satisfied:*

$$\forall m \in \mathbb{N}_0 \quad \|u_\varepsilon\|_{\mathcal{F}M^{p,q}} = \mathcal{O}(\varepsilon^m). \tag{3.1}$$

Proof. The relation (3.1) follows from the definition of negligible nets. Let us consider now the moderate net $(u_\varepsilon)_\varepsilon \in \mathscr{E}_{C^\infty_{\mathcal{M}^{p,q}}}(\mathbb{R}^d)$ that satisfies condition (3.1). We will show that $\partial_j u_\varepsilon$ satisfies the 0-th negligibility estimate, for any $1 \le j \le d$.

Let us now choose $N \in \mathbb{N}$ such that for every $m \in \mathbb{N}_0$ both of the following conditions are satisfied:

$$\|u_\varepsilon\|_{\mathcal{F}M^{p,q}} = \mathcal{O}(\varepsilon^{2m+N}), \tag{3.2}$$

$$\sum_{|\beta| \le 2k} \|\partial^{2e_j+\beta} u_\varepsilon\|_{M^{p,q}} = \mathcal{O}(\varepsilon^{-N}). \tag{3.3}$$

Taylor expansion implies

$$u_\varepsilon(x+he_j) = u_\varepsilon(x) + \partial_j u_\varepsilon(x)\cdot h + h^2 \cdot \int_0^1 (1-\theta)\partial_j^2 u_\varepsilon(x+h\theta e_j)d\theta.$$

We extract the first-order derivative term,

$$\partial_j u_\varepsilon(x) = \big(u_\varepsilon(x+he_j) - u_\varepsilon(x)\big)\cdot h^{-1} - h\cdot \int_0^1 (1-\theta)\cdot \partial_j^2 u_\varepsilon(x+\theta h e_j)d\theta.$$

Next we apply Fourier transform (with respect to x):

$$\begin{aligned}\widehat{\partial_j u_\varepsilon}(\xi) &= \big(\hat{u_\varepsilon}(\xi)e^{2\pi i h e_j \xi} - \hat{u_\varepsilon}(\xi)\big)\cdot h^{-1} - h\cdot \int_0^1 (1-\theta)\cdot \widehat{\partial_j^2 u_\varepsilon}(\xi) e^{2\pi i\theta h e_j\xi}d\theta \\ &= \big(M_{he_j}\hat{u_\varepsilon}(\xi) - \hat{u_\varepsilon}(\xi)\big)\cdot h^{-1} - h\cdot \int_0^1 (1-\theta)\cdot M_{\theta h e_j}\widehat{\partial_j^2 u_\varepsilon}(\xi)d\theta.\end{aligned}$$

By taking norms and inserting $h = \varepsilon^{m+N}$ we obtain

$$\begin{aligned}\|\widehat{\partial_j u_\varepsilon}\|_{M^{p,q}} \le{}& (\|M_{\varepsilon^{m+N}e_j}\hat{u_\varepsilon}\|_{M^{p,q}} + \|\hat{u_\varepsilon}\|_{M^{p,q}})\cdot \varepsilon^{-m-N} \\ &+ \|\int_0^1 (1-\theta)\cdot M_{\theta\varepsilon^{m+N}e_j}\widehat{\partial_j^2 u_\varepsilon}d\theta\|_{M^{p,q}}\cdot \varepsilon^{m+N}.\end{aligned}$$

We may estimate the norm of frequency shifted distribution by using the multiplication properties of modulation spaces:

$$\|M_{\varepsilon^{m+N}e_j}\hat{u_\varepsilon}\|_{M^{p,q}} \le \|\hat{u_\varepsilon}\|_{M^{p,q}}\cdot \|e^{2\pi i\varepsilon^{m+N}e_j(\cdot)}\|_{M^{\infty,1}} \le C\cdot \|\hat{u_\varepsilon}\|_{M^{p,q}}.$$

Let us now consider the second term on the right hand side of the above estimate:

$$
\begin{aligned}
&|V_g\Big(\int_0^1 (1-\theta)\cdot M_{\theta\varepsilon^{m+N}e_j}\widehat{\partial_j^2 u_\varepsilon} d\theta\Big)(x,\omega)| \\
&\qquad = \Big|\int\Big(\int_0^1 (1-\theta)\cdot M_{\theta\varepsilon^{m+N}e_j}\widehat{\partial_j^2 u_\varepsilon}\, d\theta\Big)\cdot\overline{g(y-x)}\cdot e^{2\pi i y\omega}dy\Big| \\
&\qquad = \Big|\int_0^1 (1-\theta)\cdot\Big(\int M_{\theta\varepsilon^{m+N}e_j}\widehat{\partial_j^2 u_\varepsilon}\cdot\overline{g(y-x)}\cdot e^{2\pi i y\omega}dy\Big)d\theta\Big| \\
&\qquad\qquad\qquad \le \int_0^1 (1-\theta)\cdot\Big|V_g\Big(M_{\theta\varepsilon^{m+N}e_j}\widehat{\partial_j^2 u_\varepsilon}\Big)(x,\omega)\Big|d\theta.
\end{aligned}
$$

Thus by the generalized Minkowski inequality, we have

$$
\|\int_0^1 (1-\theta)\cdot M_{\theta\varepsilon^{m+N}e_j}\widehat{\partial_j^2 u_\varepsilon} d\theta\|_{M^{p,q}} \le C\cdot\sup_\theta \|M_{\theta\varepsilon^{m+N}e_j}\widehat{\partial_j^2 u_\varepsilon}\|_{M^{p,q}}.
$$

We treat the second derivative a bit differently:

$$
\begin{aligned}
&\|M_{\varepsilon^{m+N}e_j}\widehat{\partial_j^2 u_\varepsilon}\|_{M^{p,q}} \le \|\widehat{\partial_j^2 u_\varepsilon}\|_{M^{p,p}}\cdot\|e^{2\pi i\varepsilon^{m+N}e_j(\cdot)}\|_{M^{\infty,q_0}} \\
&\le C\cdot\|\partial_j^2 u_\varepsilon\|_{M^{p,p}} \le C\cdot\sum_{|\beta|\le 2k}\|\partial^{2e_j+\beta}u_\varepsilon\|_{M^{p,\infty}} \le C\cdot\sum_{|\beta|\le 2k}\|\partial^{2e_j+\beta}u_\varepsilon\|_{M^{p,q}}.
\end{aligned}
$$

$1/q_0 = 1+1/q-1/p$ (we may choose such a q_0 since $p\le q$).

$$
\|\widehat{\partial_j u_\varepsilon}\|_{M^{p,q}} \le C\cdot \underbrace{\|\hat{u}_\varepsilon\|_{M^{p,q}}}_{\le C\cdot\varepsilon^{2m+N},\text{ by (3.2)}} \cdot\varepsilon^{-m-N} + C\cdot\underbrace{\sum_{|\beta|\le 2k}\|\partial^{2e_j+\beta}u_\varepsilon\|_{M^{p,q}}}_{\le C\cdot\varepsilon^{-N},\text{ by (3.3)}}\cdot\varepsilon^{m+N}.
$$

Thus,

$$
\|\widehat{\partial_j u_\varepsilon}\|_{M^{p,q}} \le C\cdot\varepsilon^{2m+N}\cdot\varepsilon^{-m-N} + C\cdot\varepsilon^{-N}\cdot\varepsilon^{m+N} \le C\cdot\varepsilon^m.
$$

□

Now we show that we can identify the constant map with our embedding on the space $C^\infty_{M^{p,q}}$ (for $p\le q$). This follows directly from the proof of Theorem 3.2.8 ($s=t=0$) for the space $C^\infty_{\mathcal{F}M^{p,q}}$ (p, $q\in[1,\infty)$):

Theorem 3.5.5. *Let* $\widetilde{\sigma}: C^\infty_{\mathcal{F}M^{p,q}} \to M^{p,q}_{\mathscr{G}}$ *given by* $\widetilde{\sigma}: f\to[(f)_\epsilon]$. *Then*

$$
\widetilde{\iota}_{|C^\infty_{\mathcal{F}M^{p,q}}} = \widetilde{\sigma}.
$$

Theorem 3.5.6. *Let* $p\le q$, *and let* $\widetilde{\sigma}: C^\infty_{M^{p,q}} \to M^{p,q}_{\mathscr{G}}$ *given by* $\widetilde{\sigma}: f\to[(f)_\epsilon]$. *Then*

$$
\widetilde{\iota}_{|C^\infty_{M^{p,q}}} = \widetilde{\sigma}.
$$

Proof. The proof follows from the fact (for $p\le q$):

$$
C^\infty_{M^{p,q}} \subset C^\infty_{\mathcal{F}M^{p,q}},
$$

which is justified by:

$$\|\hat{f}\|_{M^{p,q}} \leq C \cdot \|\hat{f}\|_{M^{p,p}} \leq C \cdot \|f\|_{M^{p,p}} \\ \leq C \cdot \sum_{\alpha} \|\partial^{\alpha} f\|_{M^{p,\infty}} \leq C \cdot \sum_{\alpha} \|\partial^{\alpha} f\|_{M^{p,q}}.$$

□

The following theorems provide us with sufficient conditions for the classical product to remain invariant (under the action of the embedding map $\widetilde{\iota}$) in the algebra of generalized functions. Hence $\widetilde{\iota}$ renders $C^{\infty}_{M^{p,q}}$ ($C^{\infty}_{\mathcal{F}M^{p,q}}$), a subalgebra of $M_{\mathscr{G}}^{p,q}$.

Corollary 3.5.7. *Let $p \leq q$; let $u \in C^{\infty}_{M^{p_1,q_1}}(\mathbb{R}^d)$, $v \in C^{\infty}_{M^{p_2,q_2}}(\mathbb{R}^d)$, where p_1, p_2, q_1, $q_2 \in [1,\infty]$. Then*

$$\widetilde{\iota}(u)_\varepsilon \cdot \widetilde{\iota}(v)_\varepsilon - \widetilde{\iota}(uv)_\varepsilon \in \mathscr{N}_{C^{\infty}_{\mathcal{F}M^{p_1,q_1}}}(\mathbb{R}^d) \cap \mathscr{N}_{C^{\infty}_{\mathcal{F}M^{p_2,q_2}}}(\mathbb{R}^d),$$

where $\widetilde{\iota}(u)_\varepsilon := (u * \rho_\varepsilon)_\varepsilon$.

Theorem 3.5.8. *Let $u \in C^{\infty}_{\mathcal{F}M^{p_1,q_1}}(\mathbb{R}^d)$, $v \in C^{\infty}_{\mathcal{F}M^{p_2,q_2}}(\mathbb{R}^d)$, where p_1, p_2, q_1, $q_2 \in [1,\infty]$. Then*

$$\widetilde{\iota}(u)_\varepsilon \cdot \widetilde{\iota}(v)_\varepsilon - \widetilde{\iota}(uv)_\varepsilon \in \mathscr{N}_{C^{\infty}_{\mathcal{F}M^{p_1,q_1}}}(\mathbb{R}^d) \cap \mathscr{N}_{C^{\infty}_{\mathcal{F}M^{p_2,q_2}}}(\mathbb{R}^d),$$

where $\widetilde{\iota}(u)_\varepsilon := (u * \rho_\varepsilon)_\varepsilon$.

Theorem 3.5.9. *Let $u \in \mathcal{F}M^{p_1,q_1}$, $v \in C^{\infty}_{\mathcal{F}M^{p_2,q_2}}$ such that $u \cdot v \in C^{\infty}_{\mathcal{F}M^{p_1,q_1}}$ and for every $m \in \mathbb{N}_0$, $\|\widetilde{\iota}(u)_\varepsilon - u\|_{\mathcal{F}M^{p_1,q_1}} < \varepsilon^m$, where p_1, p_2, q_1, $q_2 \in [1,\infty]$. Then*

$$\widetilde{\iota}(u)_\varepsilon \cdot \widetilde{\iota}(v)_\varepsilon - \widetilde{\iota}(uv)_\varepsilon \in \mathscr{N}_{C^{\infty}_{\mathcal{F}M^{p_1,q_1}}}(\mathbb{R}^d),$$

where $\widetilde{\iota}(u)_\varepsilon := (u * \rho_\varepsilon)_\varepsilon$.

Chapter 4

Generalized pseudo-differential operators

In this chapter we shall investigate mapping properties of generalized pseudo-differential operators acting on elements of modulation spaces $M^{p,q}$ as well as on elements of the algebra $\mathscr{G}_{C^\infty_{M^{p,q}}}$ of generalized functions.

4.1 Basic facts and definitions

In mathematical analysis, pseudo-differential operators (henceforth abbreviated by ΨDO) arise as a generalization of partial-differential operators (henceforth abbreviated by PDO). Namely, let $P = P(x, D) := \sum_{|\alpha|\leq N} a_\alpha(x) D_x^\alpha$ be a PDO (of order N) and let f be some "nice" function (Schwartz function, for example); by the Fourier inversion formula we may write:

$$
\begin{aligned}
P(x,D)f(x) &= \sum_{|\alpha|\leq N} a_\alpha(x) D_x^\alpha f(x) \\
&= \sum_{|\alpha|\leq N} a_\alpha(x) D_x^\alpha \Big(\int \hat{f}(\xi) \cdot e^{2\pi i \xi x}\, d\xi\Big) \\
&= \sum_{|\alpha|\leq N} a_\alpha(x) \cdot \int \hat{f}(\xi) \cdot (2\pi i \xi)^\alpha \cdot e^{2\pi i \xi x}\, d\xi \\
&= \int \sum_{|\alpha|\leq N} a_\alpha(x) \cdot (2\pi i \xi)^\alpha \cdot \hat{f}(\xi) \cdot e^{2\pi i \xi x}\, d\xi \\
&= \int P(x,\xi) \cdot \hat{f}(\xi) \cdot e^{2\pi i \xi x}\, d\xi.
\end{aligned}
$$

Then it comes naturally to consider more general classes of functions (so called symbols of ΨDO) $P(x, \xi)$ like Hörmander classes [Hör85], SG classes [Cor95], etc. In the time-frequency analysis framework, operators corresponding to symbols in modulation spaces are considered. In order to define such classes of symbols we shall provide some basic notions and assertions first. For the omitted proofs and complementary facts we refer to [Grö] and references therein.

Definition 4.1.1. *Let $\sigma \in \mathscr{S}'(\mathbb{R}^{2d})$ and $f \in \mathscr{S}(\mathbb{R}^d)$. Then we define the ΨDO σ^{KN} by the action of the distribution $\sigma(x,\xi)$ on a function $\hat{f}(\xi) \cdot e^{2\pi i x\xi}$ (with respect to ξ):*

$$\sigma^{KN} f(x) = \sigma(x, D) f(x) := \langle \sigma(x,\xi), \bar{\hat{f}}(\xi) \cdot e^{-2\pi i x\xi} \rangle_\xi .$$

The mapping $\sigma \to \sigma^{KN}$ is called Kohn-Nirenberg correspondence *and σ is called the* Kohn-Nirenberg symbol *of the operator.*

It is clear that for $\sigma \in \mathscr{S}'(\mathbb{R}^{2d})$ we have that $\sigma^{KN} f \in \mathscr{S}'(\mathbb{R}^d)$, and therefore we may consider

$$\langle \sigma^{KN} f, g \rangle = \langle \sigma(x,\xi), \bar{\hat{f}}(\xi) \cdot g(x) \cdot e^{-2\pi i x\xi} \rangle_{x,\xi},$$

which motivates the following:

Definition 4.1.2. *Let $f,\ g \in \mathscr{S}(\mathbb{R}^d)$. We define the (cross) Rihaczek distribution of f and g as the time-shifted tensor product*

$$R(f,g)(x,\xi) := f(x)\bar{\hat{g}}(\xi) e^{-2\pi i x\xi}.$$

Moreover, we may define σ^{KN} via the Rihaczek distribution:

Lemma 4.1.3. *If $\sigma \in \mathscr{S}'(\mathbb{R}^{2d})$, then the sesquilinear form*

$$\langle \sigma^{KN} f, g \rangle := \langle \sigma, R(g,f) \rangle.$$

defines a continuous operator σ^{KN} from $\mathscr{S}(\mathbb{R}^d)$ to $\mathscr{S}'(\mathbb{R}^d)$.

This is exactly the form which is used in extending the notion of Kohn-Nirenberg symbols to modulation spaces. The following lemma will be of use later:

Lemma 4.1.4. *The two-dimensional Fourier transform of the Rihaczek distribution is a rotated short-time Fourier transform. If $f,\ g \in \mathscr{S}(\mathbb{R}^d)$, then*

$$\widehat{R(f,g)}(x,\xi) = V_g f(-\xi, x) \quad \text{for all } (x,\xi) \in \mathbb{R}^{2d}. \tag{4.1}$$

If $f,\ g \in \mathscr{S}'(\mathbb{R}^d)$ then (4.1) holds in the distributional sense as $\widehat{R(f,g)} = \widetilde{V_f g}$, where $\widetilde{F}(x,\xi) := F(-\xi, x)$.

4.1.1 Boundedness of ΨDO

In the theory of ΨDO (in the setting of modulation spaces), the so called *Sjöstrand class $M^{\infty,1}$* is of particular importance. Its significance stems from the fact that symbols in $M^{\infty,1}$ are non-smooth and the corresponding operators are bounded on $M^{p,q}$ for any $p,\ q \in [1,\infty]$. This is not the case for symbols in other $M^{p,q}$ spaces, moreover, not all operators (with symbols in modulation spaces) are bounded on L^2 spaces. The following characterization is well-known:

Theorem 4.1.5 (Complete picture of L^2-boundedness)**.**

- *If $q \le 2$ and either $1 \le p \le 2$ or $p \le q'$, then σ^{KN} is a bounded operator on $L^2(\mathbb{R}^d)$.*
- *If $q > 2$ or if $p \ge 2$ and $p > q'$, then there exist $\sigma \in M^{p,q}$ such that σ^{KN} is unbounded on $L^2(\mathbb{R}^d)$.*

Lemma 4.1.6 (Magic formula)**.** *Let $\psi_1,\ \psi_2,\ f,\ g \in L^2(\mathbb{R}^d)$. Set $\Psi = R(\psi_1,\psi_2) \in L^2(\mathbb{R}^{2d})$, $z = (z_1,z_2) \in \mathbb{R}^d \times \mathbb{R}^d$, $\zeta = (\zeta_1,\zeta_2) \in \mathbb{R}^d \times \mathbb{R}^d$. Then*

$$V_\Psi(R(f,g))(z,\zeta) = e^{-2\pi i z_2\zeta_2} \cdot V_{\psi_1} f(z_1, z_2+\zeta_1) \cdot \overline{V_{\psi_2} g(z_1+\zeta_2, z_2)}.$$

4.2 Pseudodifferential operators with generalized symbols

In this section we will introduce a construction principle for the basic framework of the theory of ΨDO with generalized symbols. In this we will mainly follow the work of Claudia Garetto ([Gar04a, Gar04b] etc.). Substantial contributions to this theory have been made by Claudia Garetto, Michael Oberguggenberger, Todor Gramchev, Günther Hörmann, Stevan Pilipović (cf. [GGO05], [Hör04] and [Pil94]).

PDO (PDE) play the main role in the mathematical modeling in physics. Often it is convenient to take into an account singularities, e.g. 'jumps', usually described by objects such as the delta (Dirac) distribution. In the mathematical models describing such occurrences, one needs to work with singular (non-smooth) data, i.e. non-smooth coefficients in the PDO and in the initial data. Since the nonlinear nature of the problem exceeds the capabilities of the toolkit of the classical theory of distributions, the framework of ΨDO with generalized symbols arises as a natural mathematical realm to describe such nonlinear models. In several instances the question of the existence and uniqueness of solutions of a generalized PDE have been carried out in this framework, by different authors, cf. [Hör04], [HO04] (for more check the references therein).

The presence of singularities in a PDE automatically involves the issue of the regularity of solutions, thus, there has also been introduced a generalized notion of amplitudes of a ΨDO as well as generalized wave front sets.

We will present here the simplest setting of generalized ΨDO, developed by Garetto. As a preparation, we introduce a suitable Colombeau framework corresponding to Hörmander symbol classes.

4.2.1 Basic notions

In this subsection we introduce the definitions and results from the theory of Colombeau generalized functions (cf. [GKOS01], [Col84], [?] and [Obe92b]) we shall further make use of.

Definition 4.2.1. *Let Ω an open subset of $\mathbb{R}^n$. Then $\mathscr{E}[\Omega]$ denotes the algebra of all the sequences $(u_\varepsilon)_{\varepsilon\in(0,1]}$ of smooth functions $u_\varepsilon \in C^\infty(\Omega)$.*

Definition 4.2.2. *$\mathscr{E}_M(\Omega)$ is the differential subalgebra of the elements $(u_\varepsilon)_\varepsilon \in \mathscr{E}[\Omega]$ such that for all $K \subset\subset \Omega$, for all $\alpha \in \mathbb{N}^n$ there exists $N \in \mathbb{N}$ with the following property:*

$$\sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = \mathcal{O}(\varepsilon^{-N}), \quad (\varepsilon \to 0).$$

Definition 4.2.3. *$\mathscr{N}(\Omega)$ is the differential ideal in $\mathscr{E}_M(\Omega)$, consisting of the elements $(u_\varepsilon)_\varepsilon$ in $\mathscr{E}[\Omega]$ such that for all $K \subset\subset \Omega$, for all $\alpha \in \mathbb{N}^n$ and $q \in \mathbb{N}$ the following condition is satisfied:*

$$\sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = \mathcal{O}(\varepsilon^{q}), \quad (\varepsilon \to 0).$$

The algebras $\mathscr{E}_M(\Omega)$ and $\mathscr{N}(\Omega)$ are called algebra of moderate nets and algebra of negligible nets, respectively.

We define the (factor) algebra $\mathscr{G}(\Omega) := \mathscr{E}_M(\Omega)/\mathscr{N}(\Omega)$. The so called 'sheaf' nature of the algebra $\mathscr{G}(\Omega)$ allows to define an embedding ι of $\mathscr{D}'(\Omega)$ into $\mathscr{G}(\Omega)$. This embedding coincides with the constant embedding $\sigma : f \to (f)_\varepsilon + \mathscr{N}(\Omega)$ of $C^\infty(\Omega)$ into $\mathscr{G}(\Omega)$.

Definition 4.2.4. *Let $\mathbb{K}$ be either the field $\mathbb{R}$ or $\mathbb{C}$. We set*

$$\begin{aligned} \mathscr{E}_M &= \{(r_\varepsilon)_\varepsilon \in \mathbb{K}^{(0,1]} |\ \exists N \in \mathbb{N}\ |r_\varepsilon| = O(\varepsilon^{-N}), \quad \varepsilon \to 0\}, \\ \mathscr{N} &= \{(r_\varepsilon)_\varepsilon \in \mathbb{K}^{(0,1]} |\ \forall q \in \mathbb{N}\ |r_\varepsilon| = O(\varepsilon^{q}), \quad \varepsilon \to 0\}. \end{aligned}$$

$\widetilde{\mathbb{K}} := \mathscr{E}_M/\mathscr{N}$ *is called the ring of generalized numbers.*

For $\mathbb{K} = \mathbb{R}$ one obtains the algebra $\widetilde{\mathbb{R}}$ of real generalized numbers and in case of $\mathbb{K} = \mathbb{C}$, the algebra $\widetilde{\mathbb{C}}$ of complex generalized numbers. $\widetilde{\mathbb{R}}$ can be enriched with the structure of a partially ordered ring i.e., for $r, s \in \widetilde{\mathbb{R}}$, we define $r \leq s$ if and only if there exist corresponding representatives $(r_\varepsilon)_\varepsilon$ and $(s_\varepsilon)_\varepsilon$ such that $r_\varepsilon \leq s_\varepsilon$ for all $\varepsilon \in (0, 1]$. $\widetilde{\mathbb{C}}$ is canonically embedded in $\mathscr{G}(\Omega)$. If Ω is a connected set, then $\widetilde{\mathbb{C}}$ can be considered as the ring of constants of $\mathscr{G}(\Omega)$.

Prior to defining a concept of generalized point value for the generalized functions of $\mathscr{G}(\Omega)$ (via $\widetilde{\mathbb{C}}$), we shall introduce generalized points.

Definition 4.2.5. *Let*

$$\Omega_M = \{(x_\varepsilon)_\varepsilon \in \Omega^{(0,1]} : \exists N \in \mathbb{N}, \quad |x_\varepsilon| = O(\varepsilon^{-N}), \quad \varepsilon \to 0\}.$$

We define an equivalence relation on Ω_M:

$$(x_\varepsilon)_\varepsilon \sim (y_\varepsilon)_\varepsilon \ \Leftrightarrow\ \forall q \in \mathbb{N},\ |x_\varepsilon - y_\varepsilon| = O(\varepsilon^q), \quad \varepsilon \to 0$$

and denote by $\widetilde{\Omega} := \Omega_M/\sim$ the set of generalized points. Moreover, if $[(x_\varepsilon)_\varepsilon]$ is the class of $(x_\varepsilon)_\varepsilon$ in $\widetilde{\Omega}$ then the set of compactly supported generalized points is

$$\widetilde{\Omega}_c = \{\tilde{x} = [(x_\varepsilon)_\varepsilon] \in \widetilde{\Omega} : \exists K \subset\subset \Omega,\ \exists \eta > 0 : \forall \varepsilon \in (0, \eta],\ x_\varepsilon \in K\}.$$

Note that for $\Omega = \mathbb{R}$ we have that the factor $\mathbb{R}_M/\sim$ is the usual algebra of real generalized numbers.

Let $(u_\varepsilon)_\varepsilon$ and $(x_\varepsilon)_\varepsilon$ be representatives of $u \in \mathscr{G}(\Omega)$ and $\tilde{x} \in \widetilde{\Omega}_c$, respectively. The generalized point value of u at $\tilde{x}$,

$$u(\tilde{x}) := (u_\varepsilon(x_\varepsilon))_\varepsilon + \mathscr{N} \tag{4.1}$$

is a well-defined element of $\widetilde{\mathbb{C}}$.

Proposition 4.2.6. *Let $u \in \mathscr{G}(\Omega)$. Then $u = 0$ if and only if $u(\tilde{x}) = 0$ for all $\tilde{x} \in \widetilde{\Omega}_c$.*

Before we introduce now the notion of generalized singular support we shall first define the algebra of compactly supported generalized functions.

Definition 4.2.7. *We denote by $\mathscr{E}_{c,M}(\Omega)$ the set of all the elements $(u_\varepsilon)_\varepsilon \in \mathscr{E}_M(\Omega)$ such that there exists $K \subset\subset \Omega$ with* supp $u_\varepsilon \subseteq K$ *for all $\varepsilon \in (0, 1]$.*

Definition 4.2.8. *We denote by* $\mathscr{N}_c(\Omega)$ *the set of all the elements* $(u_\varepsilon)_\varepsilon \in \mathscr{N}(\Omega)$ *such that there exists* $K \subset\subset \Omega$ *with* $\mathrm{supp}\, u_\varepsilon \subseteq K$ *for all* $\varepsilon \in (0,1]$.

The factor algebra $\mathscr{G}_c(\Omega) := \mathscr{E}_{c,M}(\Omega)/\mathscr{N}_c(\Omega)$ is called the algebra of compactly supported generalized functions.

Additionally, $\mathscr{G}_c(\Omega)$ is a subalgebra of $\mathscr{G}(\Omega)$ containing $\mathscr{E}'(\Omega)$ as a subspace and $C_c^\infty(\Omega)$ as a subalgebra, due to the fact that the map $l : \mathscr{G}_c(\Omega) \to \mathscr{G}(\Omega)$: $(u_\varepsilon)_\varepsilon + \mathscr{N}_c(\Omega) \to (u_\varepsilon)_\varepsilon + \mathscr{N}(\Omega)$ is injective. Since for $u \in \mathscr{G}(\Omega)$ and Ω' an open subset of Ω, we have that $u|_{\Omega'}$ is the generalized function in $\mathscr{G}(\Omega')$ having as representative $(u_\varepsilon|_{\Omega'})_\varepsilon$, it is possible to define the support of u in the following way

$$\Omega \setminus \mathrm{supp}\, u = \{x \in \Omega : \ \exists V(x) \subset \Omega, \ \text{open}, x \in V(x) : \ u|_{V(x)} = 0\}.$$

The map l identifies $\mathscr{G}_c(\Omega)$ with the set of generalized functions in $\mathscr{G}(\Omega)$ with compact support. If we consider an open subset Ω' of Ω, the map $\mathscr{G}_c(\Omega') \to \mathscr{G}_c(\Omega)$: $(u_\varepsilon)_\varepsilon + \mathscr{N}_c(\Omega') \to (u_\varepsilon)_\varepsilon + \mathscr{N}_c(\Omega)$ allows us to embed $\mathscr{G}_c(\Omega')$ into $\mathscr{G}_c(\Omega)$.

A notion of integral of a generalized function $u \in \mathscr{G}(\Omega)$ over a compact subset of Ω is given by

$$\int_K u(x)dx := \Big(\int_K u_\varepsilon(x)dx\Big)_\varepsilon + \mathscr{N};$$

in particular, we calculate the integral of a generalized function $u \in \mathscr{G}_c(\Omega)$ over Ω by the formula

$$\int_\Omega u(x)dx := \Big(\int_V u_\varepsilon(x)dx\Big)_\varepsilon + \mathscr{N},$$

where V is any compact set containing $\mathrm{supp}\, u$ in its interior.

4.2.2 Generalized oscillatory integrals

In order to define a ΨDO with generalized symbols, we need first to define and investigate properties of the integral

$$\int_{K\times\mathbb{R}^p} e^{i\phi(y,\xi)} a_\varepsilon(y,\xi)\, dy\, d\xi,$$

where $K \subset\subset \Omega$, Ω an open subset of $\mathbb{R}^n$. The phase function $\phi(y,\xi)$ is smooth on $\Omega \times \mathbb{R}^p \setminus 0$, real valued, positively homogeneous of degree 1 in the second variable (ξ) and $\nabla\phi(y,\xi) \neq 0$ for all $y \in \Omega$, $\xi \in \mathbb{R}^p \setminus 0$. We will denote by $\mathcal{S}^m_{\rho,\delta}[\Omega \times \mathbb{R}^p]$ the space of nets $S^m_{\rho,\delta}(\Omega \times \mathbb{R}^p)^{(0,1]}$ where $S^m_{\rho,\delta}(\Omega \times \mathbb{R}^p)$, $m \in \mathbb{R}$, $\rho, \delta \in [0,1]$, represents the usual space of Hörmander symbols.

Definition 4.2.9. *An element of* $\mathcal{S}^m_{\rho,\delta,\mathrm{M}}(\Omega \times \mathbb{R}^p)$ *is a net* $(a_\varepsilon)_\varepsilon \in \mathcal{S}^m_{\rho,\delta}[\Omega \times \mathbb{R}^p]$ *such that*

$$\begin{aligned}&\forall\alpha \in \mathbb{R}^p, \ \forall\beta \in \mathbb{N}^n, \ \forall K \subset\subset \Omega, \ \exists N \in \mathbb{N}, \ \exists\eta \in (0,1], \ \exists c > 0 : \\ &\qquad \forall y \in K, \ \forall\xi \in \mathbb{R}^p, \ \forall\varepsilon \in (0,\eta], \ |\partial_\xi^\alpha \partial_y^\beta a_\varepsilon(y,\xi)| \le c\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}\varepsilon^{-N}.\end{aligned}$$

Definition 4.2.10. *An element of* $\mathscr{N}^m_{\rho,\delta}(\Omega \times \mathbb{R}^p)$ *is a net* $(a_\varepsilon)_\varepsilon \in \mathcal{S}^m_{\rho,\delta}[\Omega \times \mathbb{R}^p]$ *satisfying the following requirement:*

$$\begin{aligned}&\forall\alpha \in \mathbb{N}^p, \ \forall\beta \in \mathbb{N}^n, \ \forall K \subset\subset \Omega, \ \forall q \in \mathbb{N}, \ \exists\eta \in (0,1], \ \exists c > 0 : \\ &\qquad \forall y \in K, \ \forall\xi \in \mathbb{R}^p, \ \forall\varepsilon \in (0,\eta], \ |\partial_\xi^\alpha \partial_y^\beta a_\varepsilon(y,\xi)| \le c\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}\varepsilon^{q}.\end{aligned}$$

Nets of this type are called negligible.

Definition 4.2.11. *A (generalized) symbol of order m and type (ρ,δ) is an element of the factor space $\widetilde{\mathcal{S}}^m_{\rho,\delta}(\Omega\times\mathbb{R}^p) := \mathcal{S}^m_{\rho,\delta,\mathrm{M}}(\Omega\times\mathbb{R}^p)/\mathcal{N}^m_{\rho,\delta}(\Omega\times\mathbb{R}^p)$.*

Proposition 4.2.12. *Let K be a compact set contained in Ω. Let ϕ be a phase function on $\Omega\times\mathbb{R}^p$ and a an element of $\widetilde{\mathcal{S}}^m_{\rho,\delta}(\Omega\times\mathbb{R}^p)$. The oscillatory integral*

$$I_{\phi,K}(a) := \int_{K\times\mathbb{R}^p} e^{i\phi(y,\xi)} a(y,\xi)\,dy\,d\xi := (I_{\phi,K}(a_\varepsilon))_\varepsilon + \mathcal{N}$$

is a well-defined element of $\widetilde{\mathbb{C}}$.

Definition 4.2.13. *Let $a\in\widetilde{\mathcal{S}}^m_{\rho,\delta}(\Omega\times\mathbb{R}^p)$ with $\operatorname{supp}_y a \subset\subset \Omega$. We define the (generalized) oscillatory integral*

$$I_\phi(a) := \int_{\Omega\times\mathbb{R}^p} e^{i\phi(y,\xi)} a(y,\xi)\,dy\,d\xi := \int_{K\times\mathbb{R}^p} e^{i\phi(y,\xi)} a(y,\xi)\,dy\,d\xi,$$

where K is any compact subset of Ω containing $\operatorname{supp}_y a$ in its interior.

This definition does not depend on the choice of K, of course.

Proposition 4.2.14. *Let $a\in\widetilde{\mathcal{S}}^m_{\rho,\delta}(\Omega\times\Omega\times\mathbb{R}^n)$. The oscillatory integral*

$$Au := \int_{\Omega\times\mathbb{R}^n} e^{i(x-y)\xi} a(x,y,\xi) u(y)\,dy\,\bar{d}\xi = \big(A_\epsilon u_\epsilon(x)\big)_\epsilon + \mathcal{N}(\Omega) \tag{4.2}$$

where

$$A_\epsilon u_\epsilon(x) = \int_{\Omega\times\mathbb{R}^n} e^{i(x-y)\xi} a_\epsilon(x,y,\xi) u_\epsilon(y)\,dy\,\bar{d}\xi$$

defines a linear map from $\mathcal{G}_c(\Omega)$ into $\mathcal{G}(\Omega)$.

Definition 4.2.15. *Let $a\in\widetilde{\mathcal{S}}^m_{\rho,\delta}(\Omega\times\Omega\times\mathbb{R}^n)$. The linear map defined by*

$$A:\mathcal{G}_c(\Omega)\to\mathcal{G}(\Omega) : u\to Au(x) := \int_{\Omega\times\mathbb{R}^n} e^{i(x-y)\xi} a(x,y,\xi)u(y)\,dy\,\bar{d}\xi$$

is called a (generalized) pseudodifferential operator with amplitude a.

4.3 Generalized modulation spaces as symbol classes for ΨDO

In this section we will define pseudodifferential operators with symbols in generalized modulation spaces. Since the nature of the classical pseudodifferential operators with Hörmander symbols differs from their counterpart in time-frequency analysis, we will choose a path that essentially differs from the 'usual' approach (described in section 4.2), when defining generalized pseudodifferential operators in our setting.

Theorem 4.3.1. *Let $\sigma\in\mathcal{G}_{C^\infty_{M^{p,q}}}(\mathbb{R}^{2d})$ and let $f\in\mathcal{G}_{C^\infty_{M^{r_1,s_1}}}(\mathbb{R}^d)$. Then for $1\le r_1,\, s_1,\, r_2,\, s_2<\infty$ and every multi-index α there exists a constant C_α such that*

$$\|\partial^\alpha(\sigma^{KN}_\varepsilon f_\varepsilon)\|_{M^{r_2,s_2}} \le C_\alpha\cdot\sum_{\beta\le\alpha}\|\partial_x^{\alpha-\beta}\sigma_\varepsilon\|_{M^{p,q}}\cdot\|\partial^\beta f_\varepsilon\|_{M^{r_1,s_1}},$$

where σ_ε^{KN} *is a* ΨDO *corresponding to the symbol* σ_ε *(for fixed* ε*).*

In particular,

$$\sigma^{KN} : \mathscr{G}_{C^\infty_{M^{r_1,s_1}}}(\mathbb{R}^d) \to \mathscr{G}_{C^\infty_{M^{r_2,s_2}}}(\mathbb{R}^d),$$

where σ^{KN} *represents the mapping given by*

$$\sigma^{KN} f := [(\sigma_\varepsilon^{KN} f_\varepsilon)_\varepsilon],$$

for $f \in \mathscr{G}_{C^\infty_{M^{r_1,s_1}}}(\mathbb{R}^d)$.

Proof. Let $g \in \mathscr{S}$. Then

$$\begin{aligned}|\langle \partial^\alpha(\sigma_\varepsilon^{KN} f_\varepsilon), g\rangle| &= |\langle \sigma_\varepsilon^{KN} f_\varepsilon, \partial^\alpha g\rangle| = |\langle \sigma_\varepsilon, R(\partial^\alpha g, f_\varepsilon)\rangle| \\ &= |\langle \sigma_\varepsilon(x,\xi), \partial^\alpha g(x) \cdot \bar{\hat{f}}_\varepsilon(\xi) \cdot e^{-2\pi i x\xi}\rangle| \\ &= |\langle \sigma_\varepsilon(x,\xi) \cdot e^{2\pi i x\xi}, \partial^\alpha g(x) \cdot \bar{\hat{f}}_\varepsilon(\xi)\rangle|.\end{aligned}$$

By integration by parts we have:

$$\begin{aligned}|\langle \partial^\alpha(\sigma_\varepsilon^{KN} f_\varepsilon), g\rangle| &= |\langle \partial_x^\alpha(\sigma_\varepsilon(x,\xi) \cdot e^{2\pi i x\xi}), g(x) \cdot \bar{\hat{f}}_\varepsilon(\xi)\rangle| \\ &= |\sum_{\beta\le\alpha} \binom{\alpha}{\beta} \langle \partial_x^{\alpha-\beta}\sigma_\varepsilon(x,\xi) \cdot (2\pi i\xi)^\beta \cdot e^{2\pi i x\xi}, g(x) \cdot \bar{\hat{f}}_\varepsilon(\xi)\rangle| \\ &\le C_\alpha \cdot \sum_{\beta\le\alpha} |\langle \partial_x^{\alpha-\beta}\sigma_\varepsilon(x,\xi), g(x) \cdot \bar{\hat{f}}_\varepsilon(\xi) \cdot (2\pi i\xi)^\beta \cdot e^{-2\pi i x\xi}\rangle| \\ &\le C_\alpha \cdot \sum_{\beta\le\alpha} |\langle \partial_x^{\alpha-\beta}\sigma_\varepsilon(x,\xi), g(x) \cdot \overline{\widehat{\partial^\beta f_\varepsilon}}(\xi) \cdot e^{-2\pi i x\xi}\rangle| \\ &= C_\alpha \cdot \sum_{\beta\le\alpha} |\langle \partial_x^{\alpha-\beta}\sigma_\varepsilon(x,\xi), R(g, \partial^\beta f_\varepsilon)\rangle| \\ &= C_\alpha \cdot \sum_{\beta\le\alpha} |\langle (\partial_x^{\alpha-\beta}\sigma_\varepsilon)^{KN} \partial^\beta f_\varepsilon, g\rangle|.\end{aligned}$$

By Hölder's inequality we have:

$$\begin{aligned}|\langle \partial^\alpha(\sigma_\varepsilon^{KN} f_\varepsilon), g\rangle| &\le C_\alpha \cdot \sum_{\beta\le\alpha} |\langle \partial_x^{\alpha-\beta}\sigma_\varepsilon(x,\xi), R(g, \partial^\beta f_\varepsilon)\rangle| \\ &\le C_\alpha \cdot \sum_{\beta\le\alpha} \|\partial_x^{\alpha-\beta}\sigma_\varepsilon\|_{M^{p,q}} \cdot \|R(g, \partial^\beta f_\varepsilon)\|_{M^{p',q'}}.\end{aligned}$$

Let us now estimate $\|R(g, \partial^\beta f_\varepsilon)\|_{M^{p',q'}}$. Since different windows yield equivalent norms, we will consider short-time Fourier transform with the Rihaczek distribution $\Phi = R(\varphi, \varphi)$ as a window. By the *magic formula* we have

$$\begin{aligned}&\|R(g, \partial^\beta f_\varepsilon)\|_{M^{p',q'}} = \|V_\Phi R(g, \partial^\beta f_\varepsilon)\|_{L^{p',q'}} = \|V_{R(\varphi,\varphi)} R(g, \partial^\beta f_\varepsilon)\|_{L^{p',q'}} \\ &= (\int_{\mathbb{R}^{2d}} (\int_{\mathbb{R}^{2d}} |V_\varphi g(z_1, z_2+\zeta_1)|^{p'} \cdot |V_\varphi \partial^\beta f_\varepsilon(z_1+\zeta_2, z_2)|^{p'}\, dz_1 dz_2)^{q'/p'}\, d\zeta_1 d\zeta_2)^{1/q'}.\end{aligned}$$

Set now $F_\varepsilon(z) := |V_\varphi f_\varepsilon(z)|$, $G(z) := |V_\varphi g(-z)|$ and $\tilde{\zeta} = (\zeta_2, -\zeta_1)$. Then we have

$$\begin{aligned}\|R(g, \partial^\beta f_\varepsilon)\|_{M^{p',q'}} &= (\int_{\mathbb{R}^{2d}} (\int_{\mathbb{R}^{2d}} F_\varepsilon(z)^{p'} \cdot G(\tilde{\zeta} - z)^{p'} \, dz)^{q'/p'} \, d\tilde{\zeta})^{1/q'} \\ &= [\int_{\mathbb{R}^{2d}} (F_\varepsilon^{p'} * G^{p'})^{q'/p'}(\tilde{\zeta}) \, d\tilde{\zeta}]^{p'/q' \cdot 1/p'} = \|F_\varepsilon^{p'} * G^{p'}\|_{L^{q'/p'}}^{1/p'}.\end{aligned}$$

Set now $t = q'/p'$. Then we have $t \geq 1 \Leftrightarrow q \geq p$. Since $\mathscr{G}_{C^\infty_{M^{p,q}}}(\mathbb{R}^{2d}) = \mathscr{G}_{C^\infty_{M^{p,1}}}(\mathbb{R}^{2d})$ we may assume that $t \geq 1$ automatically. Now by Young's inequality $(1/\mu_1 + 1/\mu_2 = 1/\nu + 1/\nu_2 = 1 + 1/t)$ we obtain:

$$\|V_\Phi R(g, \partial^\beta f_\varepsilon)\|_{L^{p',q'}} \leq \|F_\varepsilon^{p'}\|_{L^{\mu_1,\nu_1}}^{1/p'} \cdot \|G^{p'}\|_{L^{\mu_2,\nu_2}}^{1/p'}.$$

Additionally, we calculate

$$\|F_\varepsilon^{p'}\|_{L^{\mu_1,\nu_1}}^{1/p'} = \|\partial^\beta f_\varepsilon\|_{M^{\mu_1 \cdot p', \nu_1 \cdot p'}},$$

$$\|G^{p'}\|_{L^{\mu_2,\nu_2}}^{1/p'} = \|g\|_{M^{\mu_2 \cdot p', \nu_2 \cdot p'}}.$$

If we set $r_1 := \mu_1 \cdot p'$, $s_1 := \nu_1 \cdot p'$, $r_2' := \mu_2 \cdot p'$, $s_2' := \nu_2 \cdot p'$ then we may write:

$$|\langle \partial^\alpha(\sigma_\varepsilon^{KN} f_\varepsilon), g \rangle| \leq C_\alpha \cdot \sum_{\beta \leq \alpha} \|\partial_x^{\alpha-\beta} \sigma_\varepsilon\|_{M^{p,q}} \cdot \|\partial^\beta f_\varepsilon\|_{M^{r_1,s_1}} \cdot \|g\|_{M^{r_2',s_2'}} \qquad (4.1)$$

where r_1, $s_1 \geq p'$; r_2, $s_2 \leq p$ and

$$\frac{1}{r_1} - \frac{1}{r_2} = \frac{1}{s_1} - \frac{1}{s_2} = \frac{1}{q'} - \frac{1}{p}.$$

Estimate (4.1) can be extended to all $g \in M^{r_2',s_2'}$ (for finite r_2' and s_2') by a standard density argument. Then

$$\|\partial^\alpha(\sigma_\varepsilon^{KN} f_\varepsilon)\|_{M^{r_2,s_2}} \leq C_\alpha \cdot \sum_{\beta \leq \alpha} \|\partial_x^{\alpha-\beta} \sigma_\varepsilon\|_{M^{p,q}} \cdot \|\partial^\beta f_\varepsilon\|_{M^{r_1,s_1}}. \qquad (4.2)$$

Note that there exists some $k \in \mathbb{N}$ (by Lemma 2.1.12) such that

$$\|\partial^\alpha(\sigma_\varepsilon^{KN} f_\varepsilon)\|_{M^{r_2,s_2}} \leq C_{\alpha,k} \cdot \sum_{\gamma \leq 2k} \sum_{\beta \leq \alpha} \|\partial_x^{\alpha-\beta} \sigma_\varepsilon\|_{M^{p,q}} \cdot \|\partial^{\beta+\gamma} f_\varepsilon\|_{M^{r_1,\infty}},$$

thus, s_2' (s_2) can be chosen arbitrarily. This means that for any finite r_1 and r_2 we have

$$\sigma^{KN} : \mathscr{E}_{C^\infty_{M^{r_1,s_1}}}(\mathbb{R}^d) \to \cap_{1 \leq r_2, s_2 < \infty} \mathscr{E}_{C^\infty_{M^{r_2,s_2}}}(\mathbb{R}^d) = \mathscr{E}_{C^\infty_{M^{1,1}}}(\mathbb{R}^d).$$

The map

$$\sigma^{KN} : \mathscr{G}_{C^\infty_{M^{r_1,s_1}}}(\mathbb{R}^d) \to \mathscr{G}_{C^\infty_{M^{r_2,s_2}}}(\mathbb{R}^d).$$

is well-defined since for negligible $(\sigma_\varepsilon)_\varepsilon$ (or negligible $(f_\varepsilon)_\varepsilon$) we have that $(\sigma_\varepsilon^{KN} f_\varepsilon)_\varepsilon$ is negligible again. This follows from the estimate (4.2). $\square$

Definition 4.3.2. *Let $(\sigma^1_\varepsilon)_\varepsilon \in \mathscr{E}_{C^\infty_{M^{p_1,q_1}}}(\mathbb{R}^d)$ and $(\sigma^2_\varepsilon)_\varepsilon \in \mathscr{E}_{C^\infty_{M^{p_2,q_2}}}(\mathbb{R}^d)$. Then the net $R(\sigma^1_\varepsilon, \sigma^2_\varepsilon)_\varepsilon$ is called the generalized Rihaczek distribution and is given by*

$$R(\sigma^1_\varepsilon, \sigma^2_\varepsilon)(x,\xi) := \sigma^1_\varepsilon(x) \cdot \overline{\widehat{\sigma^2_\varepsilon}}(\xi) \cdot e^{-2\pi i x\xi}.$$

$$\begin{aligned}\|\partial^\alpha((R(\sigma^1_\varepsilon, \sigma^2_\varepsilon))^{KN} f)\| &\le \|\partial^\alpha \sigma^1_\varepsilon\|_{M^1} \cdot \|\sigma^2_\varepsilon\|_{M^\infty} \cdot \|f\|_{M^1} \\ &\le C \cdot \|\partial^\alpha \sigma^1_\varepsilon\|_{M^1} \cdot \|\sigma^2_\varepsilon\|_{M^1} \cdot \|f\|_{M^1}.\end{aligned}$$

Theorem 4.3.3. *Let $R(\sigma^1_\varepsilon, \sigma^2_\varepsilon)_\varepsilon \in R(\mathscr{E}_{C^\infty_{M^{p,q}}}(\mathbb{R}^d), \mathscr{E}_{C^\infty_{M^{p',q'}}}(\mathbb{R}^d))$. Then*

$$(R(\sigma^1_\varepsilon, \sigma^2_\varepsilon)_\varepsilon)^{KN} : M^{p,q}(\mathbb{R}^d) \to \mathscr{E}_{C^\infty_{M^{p,q}}}(\mathbb{R}^d).$$

Proof.

$$\begin{aligned}|\langle \partial^\alpha((R(\sigma^1_\varepsilon, \sigma^2_\varepsilon))^{KN} f), g\rangle| &= |\langle (R(\sigma^1_\varepsilon, \sigma^2_\varepsilon))^{KN} f, \partial^\alpha g\rangle| \\ &= |\langle R(\sigma^1_\varepsilon, \sigma^2_\varepsilon), R(\partial^\alpha g, f)\rangle| = |\langle \widehat{R(\sigma^1_\varepsilon, \sigma^2_\varepsilon)}, \widehat{R(\partial^\alpha g, f)}\rangle| \\ &= |\langle \widetilde{V_{\sigma^2_\varepsilon}\sigma^1_\varepsilon}, \widetilde{V_f \partial^\alpha}\rangle| = |\langle V_{\sigma^2_\varepsilon}\sigma^1_\varepsilon, V_f \partial^\alpha\rangle| \\ &= |\langle \sigma^1_\varepsilon, \partial^\alpha g\rangle| \cdot |\langle f, \sigma^2_\varepsilon\rangle| = |\langle \partial^\alpha \sigma^1_\varepsilon, g\rangle| \cdot |\langle f, \sigma^2_\varepsilon\rangle|.\end{aligned}$$

Thus,

$$|\langle \partial^\alpha((R(\sigma^1_\varepsilon, \sigma^2_\varepsilon))^{KN} f), g\rangle| \le \|\partial^\alpha \sigma^1_\varepsilon\|_{M^{p,q}} \cdot \|g\|_{M^{p',q'}} \cdot \|f\|_{M^{p,q}} \cdot \|\sigma^2_\varepsilon\|_{M^{p',q'}}.$$

Hence, we conclude

$$\|\langle \partial^\alpha((R(\sigma^1_\varepsilon, \sigma^2_\varepsilon))^{KN} f)\|_{M^{p,q}} \le \|\partial^\alpha \sigma^1_\varepsilon\|_{M^{p,q}} \cdot \|\sigma^2_\varepsilon\|_{M^{p',q'}} \cdot \|f\|_{M^{p,q}}.$$

The claim follows now from the moderateness of the nets $(\sigma^1_\varepsilon)_\varepsilon$ and $(\sigma^2_\varepsilon)_\varepsilon$. $\square$

Theorem 4.3.4. *Let $R(\sigma^1_\varepsilon, \sigma^2)_\varepsilon \in R(\mathscr{E}_{C^\infty_{M^{p,q}}}(\mathbb{R}^d), M^{p',q'}(\mathbb{R}^d))$. Then*

$$(R(\sigma^1_\varepsilon, \sigma^2)_\varepsilon)^{KN} : M^{p,q}(\mathbb{R}^d) \to \mathscr{E}_{C^\infty_{M^{p,q}}}(\mathbb{R}^d).$$

Proof. The claim follows from the estimate

$$\|\langle \partial^\alpha((R(\sigma^1_\varepsilon, \sigma^2))^{KN} f)\|_{M^{p,q}} \le \|\partial^\alpha \sigma^1_\varepsilon\|_{M^{p,q}} \cdot \|\sigma^2\|_{M^{p',q'}} \cdot \|f\|_{M^{p,q}}.$$

$\square$

Theorem 4.3.5. *Let $(\sigma_\varepsilon)_\varepsilon = (\sigma^1_\varepsilon \otimes \hat{\sigma^2_\varepsilon})_\varepsilon \in \mathscr{E}_{C^\infty_{M^{p,q}}}(\mathbb{R}^d) \times \mathcal{F}\mathscr{E}_{C^\infty_{\mathcal{F}M^{p,q}}}(\mathbb{R}^d)$ and let $f \in M^{p,q}$. Then for $1 \le r_1,\, s_1,\, r_2,\, s_2 < \infty$ and every multi-index α there exists a constant C_α such that*

$$\|\partial^\alpha(\sigma^{KN}_\varepsilon f)\|_{M^{r_2,s_2}} \le C_\alpha \cdot \sum_{\beta\le\alpha} \|\partial^{\alpha-\beta}\sigma^1_\varepsilon\|_{M^{p,q}} \cdot \|\partial^\beta \sigma^2_\varepsilon\|_{\mathcal{F}M^{p,q}} \cdot \|f\|_{M^{r_1,s_1}}.$$

In particular,

$$\sigma^{KN} : M^{r_1,s_1}(\mathbb{R}^d) \to \cap_{1\ge r_2,s_2<\infty} \mathscr{E}_{C^\infty_{M^{r_2,s_2}}}(\mathbb{R}^d)) = \mathscr{E}_{C^\infty_{M^{1,1}}}(\mathbb{R}^d)).$$

Proof. Similarly to the proof of Theorem 4.3.1, for $g \in \mathscr{S}$ we have:

$$\begin{aligned} |\langle \partial^\alpha(\sigma_\varepsilon^{KN} f, g\rangle| &= |\langle \sigma_\varepsilon^{KN} f, \partial^\alpha g\rangle| = |\langle \sigma_\varepsilon, R(\partial^\alpha g, f)\rangle| \\ &= |\langle \sigma_\varepsilon(x,\xi), \partial^\alpha g(x) \cdot \bar{\hat{f}}(\xi) \cdot e^{-2\pi i x\xi}\rangle| \\ &= |\langle \sigma_\varepsilon(x,\xi) \cdot e^{2\pi i x\xi}, \partial^\alpha g(x) \cdot \bar{\hat{f}}(\xi)\rangle|. \end{aligned}$$

By integration by parts we obtain:

$$\begin{aligned} |\langle \partial^\alpha(\sigma_\varepsilon^{KN} f), g\rangle| &= |\langle \partial_x^\alpha(\sigma_\varepsilon(x,\xi) \cdot e^{2\pi i x\xi}), g(x) \cdot \bar{\hat{f}}(\xi)\rangle| \\ &= |\sum_{\beta\le\alpha} \binom{\alpha}{\beta} \langle \partial_x^{\alpha-\beta}\sigma_\varepsilon(x,\xi) \cdot (2\pi i\xi)^\beta \cdot e^{2\pi i x\xi}, g(x) \cdot \bar{\hat{f}}(\xi)\rangle| \\ &\le C_\alpha \cdot \sum_{\beta\le\alpha} |\langle (\xi)^\beta \cdot \partial_x^{\alpha-\beta}\sigma_\varepsilon(x,\xi), g(x) \cdot \bar{\hat{f}}(\xi) \cdot e^{-2\pi i x\xi}\rangle| \\ &= C_\alpha \cdot \sum_{\beta\le\alpha} |\langle (\xi)^\beta \cdot \partial_x^{\alpha-\beta}\sigma_\varepsilon(x,\xi), R(g,f)\rangle| \\ &= C_\alpha \cdot \sum_{\beta\le\alpha} |\langle ((\xi)^\beta \cdot \partial_x^{\alpha-\beta}\sigma_\varepsilon)^{KN} f, g\rangle| \end{aligned}$$

Again, by following the line of argument in the proof of Theorem 4.3.1, we conclude that for any finite r_1 and r_2 we have

$$\begin{aligned} \|\partial^\alpha(\sigma_\varepsilon^{KN} f)\|_{M^{r_2,s_2}} &\le C_\alpha \cdot \sum_{\beta\le\alpha} \|(\xi)^\beta \partial_x^{\alpha-\beta}\sigma_\varepsilon\|_{M^{p,q}(\mathbb{R}^{2d})} \cdot \|f\|_{M^{r_1,s_1}} \\ &\le C_\alpha \cdot \sum_{\beta\le\alpha} \|\partial^{\alpha-\beta}\sigma_\varepsilon^1\|_{M^{p,q}(\mathbb{R}^d)} \cdot \|\partial^\beta\sigma_\varepsilon^2\|_{\mathcal{F}M^{p,q}(\mathbb{R}^d)} \cdot \|f\|_{M^{r_1,s_1}}. \end{aligned}$$

which implies

$$\sigma^{KN} : M^{r_1,s_1}(\mathbb{R}^d) \to \cap_{1\le r_2,s_2<\infty} \mathscr{E}_{C^\infty_{M^{r_2,s_2}}}(\mathbb{R}^d) = \mathscr{E}_{C^\infty_{M^{1,1}}}(\mathbb{R}^d).$$

□

Chapter 5

Short-time Fourier transform in $\mathscr{G}_{\mathscr{S}}$ and $\mathscr{G}_{\tau}$

5.1 STFT, well-defined in $\mathscr{G}_{\mathscr{S}}$ and $\mathscr{G}_{\tau}$

Proposition 5.1.1. *Let $f = [(f_\varepsilon)_\varepsilon] \in \mathscr{G}_{\mathscr{S}}(\mathbb{R}^d)$, respectively in $\mathscr{G}_{\tau}(\mathbb{R}^d)$, $g \in \mathscr{S}(\mathbb{R}^d)$. Then the class of*

$$\left(V_g f\right)_\varepsilon(x,\omega) := \int_{\mathbb{R}^d} f_\varepsilon(y)\overline{g(y-x)}e^{-2\pi i y\omega}\ dy$$

is a well-defined element of $\mathscr{G}_{\mathscr{S}}(\mathbb{R}^{2d})$, respectively $\mathscr{G}_{\tau}(\mathbb{R}^{2d})$.

Proof. We shall first show the moderateness of $\left(V_g f_\varepsilon(x,\omega)\right)_\varepsilon$, and then the independence of the chosen representatives in the mapping $V_g : \mathscr{G}_{\mathscr{S}}(\mathbb{R}^d) \to \mathscr{G}_{\mathscr{S}}(\mathbb{R}^{2d})$, respectively $\mathscr{G}_{\tau}(\mathbb{R}^d) \to \mathscr{G}_{\tau}(\mathbb{R}^{2d})$. For the τ - moderateness it is enough to check that for every α, $\beta \in \mathbb{N}^d$ there exist $N \in \mathbb{N}$, $C \geq 0$ such that

$$\left|\partial_x^\alpha \partial_\omega^\beta V_g f_\varepsilon(x,\omega)\right| \leq C \cdot \langle x\rangle^N \langle\omega\rangle^N \varepsilon^{-N}$$

holds. For the $\mathscr{S}$ - moderateness it is enough to check that for every m_1, $m_2 \in \mathbb{N}_0$, and for every α, $\beta \in \mathbb{N}^d$ there exists $N \in \mathbb{N}$ such that

$$\left|\partial_x^\alpha \partial_\omega^\beta V_g f_\varepsilon(x,\omega)\right| \leq C \cdot \langle x\rangle^{-m_1} \langle\omega\rangle^{-m_2} \varepsilon^{-N}$$

holds.

Either way we have

$$\begin{aligned}\left|\partial_x^\alpha \partial_\omega^\beta V_g f_\varepsilon(x,\omega)\right| &= \left|\partial_x^\alpha \partial_\omega^\beta \int f_\varepsilon(y)\overline{g(y-x)}e^{-2\pi i y\omega}\ dy\right| \\ &= (2\pi)^{|\beta|}\left|\int f_\varepsilon(y)\overline{\partial^\alpha g(y-x)}y^\beta e^{-2\pi i y\omega}\ dy\right|.\end{aligned}$$

For later use we observe that for every $\rho \in \mathbb{R}$, $m \in \mathbb{N}_0$ there exists a constant $C \geq 0$ such that

$$\left|\partial^\rho g(y-x)\right| \leq \frac{C}{\langle y-x\rangle^m}.$$

This together with Peetre's inequality gives

$$\left|\partial^{\rho} g(y-x)\right| \leq C \cdot \langle y\rangle^{\pm m}\langle x\rangle^{\mp m}. \tag{5.1}$$

$\mathscr{G}_{\tau}$ **- case:** For $(f_\varepsilon)_\varepsilon$ we have that there exists $N_1 \in \mathbb{N}$, $C \geq 0$ such that

$$|f_\varepsilon(y)| \leq C \cdot \langle y\rangle^{N_1}\varepsilon^{-N_1},$$

holds for ε small enough. Now we may write

$$\begin{aligned}\left|\partial_x^\alpha \partial_\omega^\beta V_g f_\varepsilon(x,\omega)\right| &\leq C \cdot \int |f_\varepsilon(y)||\partial^\alpha g(y-x)|\langle y\rangle^{|\beta|}\, dy \\ &\leq C \cdot \langle x\rangle^m \varepsilon^{-N_1} \int \langle y\rangle^{|\beta|+N_1-m}\, dy.\end{aligned}$$

Here we have chosen the negative sign in the power of $\langle y\rangle$ in order to ensure the convergence of integral. Take $m = |\beta| + N_1 + d + 1$. Thus we have that for every α, $\beta \in \mathbb{N}_0^d$ there exists $N := m$ and $C \geq 0$ such that

$$\left|\partial_x^\alpha \partial_\omega^\beta V_g f_\varepsilon(x,\omega)\right| \leq C \cdot \langle x\rangle^N \langle \omega\rangle^N \varepsilon^{-N}$$

holds.

$\mathscr{G}_{\mathscr{S}}$ **- case:** By integration by parts we obtain:

$$\begin{aligned}\left|\partial_x^\alpha \partial_\omega^\beta V_g f_\varepsilon(x,\omega)\right| &= (2\pi)^{|\beta|}\left|\int (1+\triangle_y)^k \Big(f_\varepsilon(y)\overline{\partial^\alpha g(y-x)}y^\beta\Big)\frac{e^{-2\pi i y\omega}}{\langle\omega\rangle^{2k}}\, dy\right| \\ &\leq C \cdot \frac{1}{\langle\omega\rangle^{2k}} \cdot \sum_{\substack{|\gamma_1|\leq 2k \\ |\gamma_2|\leq 2k}} \int |\partial^{\gamma_1} f_\varepsilon(y)| \cdot |\partial_y^{\gamma_2}(\partial^\alpha g(y-x)\cdot y^\beta)|\, dy\end{aligned}$$

$$\left|\partial_x^\alpha \partial_\omega^\beta V_g f_\varepsilon(x,\omega)\right| \leq C\cdot \frac{1}{\langle\omega\rangle^{2k}} \cdot \sum_{\substack{|\gamma_1|\leq 2k \\ |\gamma_2|\leq 2k \\ \gamma_3\leq\gamma_2}} \int |\partial^{\gamma_1} f_\varepsilon(y)|\cdot|\partial^{\alpha+\gamma_2-\gamma_3} g(y-x)|\cdot|y^{\beta-\gamma_3}|\, dy. \tag{5.2}$$

With estimate (5.1) for $\rho = \alpha + \gamma_2 - \gamma_3$ (and for '$-$' in the power of $\langle x\rangle$) we obtain the following inequality:

$$\left|\partial_x^\alpha \partial_\omega^\beta V_g f_\varepsilon(x,\omega)\right| \leq C \cdot \frac{1}{\langle\omega\rangle^{2k}\langle x\rangle^m} \sum_{\substack{|\gamma_1|\leq 2k \\ |\gamma_2|\leq 2k \\ \gamma_3\leq\gamma_2}} \int |\partial^{\gamma_1} f_\varepsilon(y)|\langle y\rangle^{m+|\beta-\gamma_3|}\, dy. \tag{5.3}$$

The moderateness of $(f_\varepsilon)_\varepsilon$ ensures the existence of $N \in \mathbb{N}$ and a non-negative constant C such that

$$\langle y\rangle^{d+1+m+|\beta-\gamma_3|}\left|\partial^{\gamma_1} f_\varepsilon(y)\right| \leq C \cdot \varepsilon^{-N}$$

holds for all $y \in \mathbb{R}^d$, so the integrals $\int |\partial^{\gamma_1} f_\varepsilon(y)|\langle y\rangle^{m+|\beta-\gamma_3|}\, dy$ converge. We have that the estimate

$$\left|\partial_x^\alpha \partial_\omega^\beta V_g f_\varepsilon(x,\omega)\right| \leq \frac{C}{\langle\omega\rangle^{2k}\langle x\rangle^m}\varepsilon^{-N},$$

holds for all k and $m \in \mathbb{N}$.

Let us now suppose that f_ε^1 and f_ε^2 are representatives of an element $f \in \mathscr{G}_\tau(\mathbb{R}^d)$, respectively $\mathscr{G}_{\mathscr{S}}(\mathbb{R}^d)$. We have to show that $[(V_g f_\varepsilon^1)_\varepsilon] = [(V_g f_\varepsilon^2)_\varepsilon]$, i.e.,

$$V_g f_\varepsilon^1 - V_g f_\varepsilon^2 = \int (f_\varepsilon^1 - f_\varepsilon^2)(y)\cdot \overline{g(y-x)}e^{2\pi i y\omega}\, dy \in \mathscr{N}_\tau(\mathbb{R}^{2d}),\ \text{resp. } \mathscr{N}_{\mathscr{S}}(\mathbb{R}^{2d}).$$

Set $f_\varepsilon := f_\varepsilon^1 - f_\varepsilon^2$.

$\mathscr{G}_\tau$ **- case:** Let $(f_\varepsilon)_\varepsilon \in \mathscr{N}_{\mathscr{S}}$. It is enough to show (see Proposition 1.3.6) that there exists $N \in \mathbb{N}$ such that for every $m \in \mathbb{N}_0$

$$\sup_{x\in\mathbb{R}^d} \langle x\rangle^{-N}\langle\omega\rangle^{-N}\left|\left(V_g f\right)_\varepsilon(x,\omega)\right| = \mathcal{O}(\varepsilon^m).$$

There exists $N_1 \in \mathbb{N}$ such that for every $m \in \mathbb{N}_0$

$$|\left(V_g f\right)_\varepsilon(x,\omega)| \leq \int |f_\varepsilon(y)||g(y-x)|\, dy \leq C\cdot\langle x\rangle^{m_1}\varepsilon^m \int \langle y\rangle^{N_1-m_1}\, dy \leq C\cdot\langle x\rangle^{m_1}\varepsilon^m,$$

where $m_1 = N_1+d+1$ is chosen such that the integral $\int\langle y\rangle^{N_1-m_1}\, dy$ converges. For $N := m_1$, x, $\omega \in \mathbb{R}^d$ and ε small enough we may write

$$|\left(V_g f\right)_\varepsilon(x,\omega)| \leq C\cdot\langle x\rangle^N\langle\omega\rangle^N\varepsilon^m.$$

$\mathscr{G}_{\mathscr{S}}$ **- case:** Let $(f_\varepsilon)_\varepsilon \in \mathscr{N}_{\mathscr{S}}$. It is enough to show (see Proposition 1.3.5) that for every $m \in \mathbb{N}_0$

$$\sup_{x,\omega\in\mathbb{R}^d}\left|\left(V_g f\right)_\varepsilon(x,\omega)\right| = \mathcal{O}(\varepsilon^m).$$

For any m, $m_1 \in \mathbb{N}_0$,

$$|f_\varepsilon(y)| \leq \langle y\rangle^{-m_1}\cdot\varepsilon^m.$$

Since $g \in \mathscr{S}$ there exists some constant C such that $\|g\|_\infty < C$. Thus

$$\left|\left(V_g f\right)_\varepsilon(x,\omega)\right| \leq \int |f_\varepsilon(y)||g(y-x)|\, dy \leq C\cdot\varepsilon^m \int \langle y\rangle^{-m_1}\, dy \leq C\cdot\varepsilon^m,$$

for $m_1 = d+1$. □

Proposition 5.1.2. *Let $f = [(f_\varepsilon)_\varepsilon] \in \mathscr{G}_{\mathscr{S}}(\mathbb{R}^d)$, $g = [(g_\varepsilon)_\varepsilon] \in \mathscr{G}_\tau(\mathbb{R}^d)$. Then the class of*

$$V_{g_\varepsilon} f_\varepsilon(x,\omega) := \int_{\mathbb{R}^d} f_\varepsilon(y)\overline{g_\varepsilon(y-x)}e^{-2\pi i y\omega}\, dy \tag{5.4}$$

is a well-defined element of $\mathscr{G}_\tau(\mathbb{R}^{2d})$.

Proof. Let us adapt the calculation from the proof of Proposition 5.1.1 ($g = g_\varepsilon$):

$$\begin{aligned}\left|\partial_x^\alpha\partial_\omega^\beta V_{g_\varepsilon} f_\varepsilon(x,\omega)\right| &= (2\pi)^{|\beta|}\left|\int f_\varepsilon(y)\overline{\partial^\alpha g_\varepsilon(y-x)}y^\beta e^{-2\pi i y\omega}\, dy\right| \\ &\leq C\cdot\int |f_\varepsilon(y)||\partial^\alpha g_\varepsilon(y-x)||y^\beta|\, dy. \qquad (5.5)\end{aligned}$$

On the other hand, (since $g \in \mathscr{G}_\tau(\mathbb{R}^d)$) there exist $N_1 \in \mathbb{N}$ and $C \geq 0$ such that

$$\left|\partial^\alpha g_\varepsilon(y-x)\right| \leq C \cdot \langle y-x\rangle^{N_1}\varepsilon^{-N_1}.$$

This together with Peetre's inequality gives

$$\left|\partial^\alpha g_\varepsilon(y-x)\right| \leq C \cdot \langle y\rangle^{N_1}\langle x\rangle^{N_1}\varepsilon^{-N_1}.$$

Next we may write

$$\left|\partial_x^\alpha \partial_\omega^\beta V_g f_\varepsilon(x,\omega)\right| \leq C \cdot \langle x\rangle^{N_1}\varepsilon^{-N_1}\int \left|f_\varepsilon(y)\right|\langle y\rangle^{N_1+|\beta|}\,dy.$$

The $\mathscr{S}$-moderateness of f_ε ensures the existence of some $N_2 \in \mathbb{N}$ and a non-negative constant C such that

$$\langle y\rangle^{d+1+N_1+|\beta|}\left|f_\varepsilon(y)\right| \leq C \cdot \varepsilon^{-N_2}$$

holds for all $y \in \mathbb{R}^d$, so the integral $\int \left|f_\varepsilon(y)\right|\langle y\rangle^{N_1+|\beta|}\,dy$ converges and we obtain the estimate

$$\left|\partial_x^\alpha \partial_\omega^\beta V_g f_\varepsilon(x,\omega)\right| \leq C \cdot \langle x\rangle^{N_1}\varepsilon^{-N} \leq C \cdot \langle x\rangle^{N_1}\langle \omega\rangle^{N_1}\varepsilon^{-N},$$

where $N = N_1 + N_2$ and ε small enough. Thus we have shown that for every $\alpha, \beta \in \mathbb{N}_0^d$ there exists $N \in \mathbb{N}$ such that

$$\left|\partial_x^\alpha \partial_\omega^\beta V_g f_\varepsilon(x,\omega)\right| \leq C \cdot \langle x\rangle^{N}\langle \omega\rangle^{N}\varepsilon^{-N}.$$

Now we shall check that the class of (5.4) is independent of the representatives of g and f. It suffices to show that $(g_\varepsilon)_\varepsilon \in \mathscr{N}_\tau(\mathbb{R}^d)$ yields $(V_{g_\varepsilon} f_\varepsilon)_\varepsilon \in \mathscr{N}_\tau(\mathbb{R}^{2d})$ and $(f_\varepsilon)_\varepsilon \in \mathscr{N}_{\mathscr{S}}(\mathbb{R}^d)$ yields $(V_{g_\varepsilon} f_\varepsilon)_\varepsilon \in \mathscr{N}_\tau(\mathbb{R}^{2d})$. For the first part it is enough to prove (see Proposition 1.3.6) that there exists $N \in \mathbb{N}$ such that for every $m \in \mathbb{N}_0$

$$\sup_{x,\omega\in\mathbb{R}^d} \langle x\rangle^N\langle \omega\rangle^N\left|V_{g_\varepsilon} f_\varepsilon(x,\omega)\right| = \mathcal{O}(\varepsilon^m).$$

$$\left|V_{g_\varepsilon} f_\varepsilon(x,\omega)\right| = \left|\int_{\mathbb{R}^d} f_\varepsilon(y)\overline{g_\varepsilon(y-x)}e^{-2\pi i y\omega}\,dy\right| \leq \int_{\mathbb{R}^d} |f_\varepsilon(y)|\cdot|g_\varepsilon(y-x)|\,dy.$$

The τ-neglibility of $(g_\varepsilon)_\varepsilon$ and Peetre's inequality imply that there exists $N \in \mathbb{N}$ such that for every $m_1 \in \mathbb{N}_0$

$$|g_\varepsilon(y-x)| \leq C \cdot \langle y-x\rangle^N\varepsilon^{m_1} \leq C \cdot \langle y\rangle^N\langle x\rangle^N\varepsilon^{m_1}.$$

The $\mathscr{S}$-moderateness condition of $(f_\varepsilon)_\varepsilon$ implies that for every $m_2 \in \mathbb{N}$ there exist $N_1 \in \mathbb{N}$ and a non-negative constant C such that

$$|f_\varepsilon(y)| \leq C \cdot \varepsilon^{-N_1}\langle y\rangle^{-m_2}.$$

Now we may write

$$\left|V_{g_\varepsilon} f_\varepsilon(x,\omega)\right| \leq C \cdot \langle x\rangle^N\varepsilon^{m_1-N_1}\int \langle y\rangle^{-m_2+N}\,dy.$$

Set $m := m_1 - N_1 \in \mathbb{N}_0$. We have

$$\left|V_{g_\varepsilon} f_\varepsilon(x,\omega)\right| \le C \cdot \langle x\rangle^N \langle\omega\rangle^N \varepsilon^m \int \langle y\rangle^{-m_2+N}\, dy.$$

Take $m_2 = N+d+1$. The convergence of the latter integral is then ensured. Thus we have shown that there exists $N \in \mathbb{N}$ such that for every $m \in N$ the inequality

$$\left|V_{g_\varepsilon} f_\varepsilon(x,\omega)\right| \le C \cdot \langle x\rangle^N \langle\omega\rangle^N \varepsilon^m$$

holds. In case $(f_\varepsilon)_\varepsilon \in \mathscr{N}_{\mathscr{S}}(\mathbb{R}^d)$, $(g_\varepsilon)_\varepsilon \in \mathscr{E}_\tau(\mathbb{R}^d)$ we have

$$\left|V_{g_\varepsilon} f_\varepsilon(x,\omega)\right| \le C \cdot \int |f_\varepsilon(y)||g_\varepsilon(y-x)|\, dy = C \cdot \int |f_\varepsilon(y+x)||g_\varepsilon(y)|\, dy$$

which shows that we may carry out a proof similar to the one above. Hence, we have proved that $\left[\left(V_{g_\varepsilon} f_\varepsilon(x,\omega)\right)_\varepsilon\right]$ is a well-defined element of $\mathscr{G}_\tau(\mathbb{R}^{2d})$. □

Proposition 5.1.3. *Let $f = [(f_\varepsilon)_\varepsilon] \in \mathscr{G}_\tau(\mathbb{R}^d)$, $g = [(g_\varepsilon)_\varepsilon] \in \mathscr{G}_{\mathscr{S}}(\mathbb{R}^d)$. Then*

$$\left(V_{g_\varepsilon} f_\varepsilon\right)(x,\omega) := \int_{\mathbb{R}^d} f_\varepsilon(y)\overline{g_\varepsilon(y-x)} e^{-2\pi i y\omega}\, dy \tag{5.6}$$

is a well-defined element of $\mathscr{G}_\tau(\mathbb{R}^{2d})$.

Proof. Take over the previous calculation i.e. start with the following estimate (see (5.5)):

$$\left|\partial_x^\alpha \partial_\omega^\beta V_{g_\varepsilon} f_\varepsilon(x,\omega)\right| \le C \cdot \int |f_\varepsilon(y)||\partial^\alpha g_\varepsilon(y-x)||y^\beta|\, dy.$$

Since $g \in \mathscr{G}_{\mathscr{S}}(\mathbb{R}^d)$ we have that for every $\alpha \in \mathbb{N}_0^d$ and for every $m \in \mathbb{N}_0$ there exist $N_1 \in \mathbb{N}$, $C \ge 0$ such that

$$|\partial^\alpha g_\varepsilon(y-x)| \le C \cdot \langle y-x\rangle^{-m} \varepsilon^{-N_1} \le C \cdot \langle y\rangle^{-m} \langle x\rangle^m \varepsilon^{-N_1}.$$

On the other hand, $f \in \mathscr{G}_\tau(\mathbb{R}^d)$ yields that there exists $N_2 \in \mathbb{N}$ and $C \ge 0$ with

$$|f_\varepsilon(y)| \le C \cdot \langle y\rangle^{N_2} \varepsilon^{-N_2}.$$

Thus we may write

$$\left|\partial_x^\alpha \partial_\omega^\beta V_{g_\varepsilon} f_\varepsilon(x,\omega)\right| \le C \cdot \langle x\rangle^m \varepsilon^{-N_1-N_2} \int \langle y\rangle^{-m+|\beta|+N_2}\, dy.$$

Choose $m = |\beta| + N_2 + d + 1$ to make the latter integral converge. Thus we have proved that for every α, $\beta \in \mathbb{N}_0^d$ there exists $N = N_1 + N_2$ such that

$$\left|\partial_x^\alpha \partial_\omega^\beta V_{g_\varepsilon} f_\varepsilon(x,\omega)\right| \le C \cdot \langle x\rangle^m \varepsilon^{-N} \le C \cdot \langle x\rangle^m \langle\omega\rangle^m \varepsilon^{-N}.$$

Similar to the argument in the corresponding part of the proof of Proposition 5.1.2 we may check that the class of (5.6) is independent of the representatives of f and g.

□

Remark 5.1.4. *An alternative way to prove Proposition 5.1.3 is to check whether $k(\xi,x,\omega)=\overline{g(\xi-x)}e^{2\pi i\xi\omega}$ is an element of $\mathscr{G}_\tau(\mathbb{R}^{3d})$ and has a representative $k_\varepsilon(\xi,x,\omega)=\overline{g_\varepsilon(\xi-x)}e^{2\pi i\xi\omega}$ that satisfies the following condition:*
($*$) For every α, β, $\gamma\in\mathbb{N}_0^d$ and every $s\in\mathbb{N}$ there exists $N\in\mathbb{N}$ such that

$$\sup_{x,\xi\in\mathbb{R}^d,|\alpha|\le s}\{\langle x\rangle^{-N}\langle\omega\rangle^{-N}\langle\xi\rangle^{s}|\partial_\xi^\alpha\partial_x^\beta\partial_\omega^\gamma k_\varepsilon(\xi,x,\omega)|\}=\mathcal{O}(\varepsilon^{-N}).$$

Then K $\big(Ku(x):=\int k(\xi,x,\omega)u(\xi)\,d\xi\big)$ maps $\mathscr{G}_\tau(\mathbb{R}^d)$ into $\mathscr{G}_\tau(\mathbb{R}^{2d})$.

Proof. We will proceed in analogy to [Gar04b, Prop. 1.2.25, (vii)]. Let $g=[(g_\varepsilon)_\varepsilon]\in\mathscr{G}_{\mathscr{S}}(\mathbb{R}^d)$. For $k_\varepsilon(\xi,x,\omega)=\overline{g_\varepsilon(\xi-x)}e^{2\pi i\xi\omega}$ we have

$$\begin{aligned}
|\partial_\xi^\alpha\partial_x^\beta\partial_\omega^\gamma k_\varepsilon(\xi,x,\omega)| &= |\partial_\xi^\alpha\partial_x^\beta\partial_\omega^\gamma(\overline{g_\varepsilon(\xi-x)}\cdot e^{2\pi i\xi\omega})|\\
&= |\partial_\xi^\alpha\big(\overline{\partial^\beta g_\varepsilon(\xi-x)}\cdot\partial_\omega^\gamma(e^{2\pi i\xi\omega})\big)|\le C\cdot|\partial_\xi^\alpha\big(\overline{\partial^\beta g_\varepsilon(\xi-x)}\cdot\xi^\gamma\cdot e^{2\pi i\xi\omega}\big)|\\
&\le C\cdot|\sum_{\gamma_1\le\alpha}\binom{\alpha}{\gamma_1}\partial_\xi^{\alpha-\gamma_1}(\overline{\partial^\beta g_\varepsilon(\xi-x)})\cdot\partial_\xi^{\gamma_1}(\xi^\gamma\cdot e^{2\pi i\xi\omega})|\\
&\le C\cdot\sum_{\gamma_1\le\alpha}|\partial^{\alpha+\beta-\gamma_1}g_\varepsilon(\xi-x)|\cdot\sum_{\gamma_2\le\gamma_1}|\xi^{\gamma-\gamma_2}\omega^{\gamma_1-\gamma_2}|\\
&\le C\cdot\langle\xi\rangle^{|\gamma|}\cdot\langle\omega\rangle^{|\alpha|}\sum_{\gamma_1\le\alpha}|\partial^{\alpha+\beta-\gamma_1}g_\varepsilon(\xi-x)|.
\end{aligned}$$

$g\in\mathscr{G}_{\mathscr{S}}(\mathbb{R}^d)$ implies that for every $m\in\mathbb{N}_0$ there exists $N_1\in\mathbb{N}$ such that

$$|\partial^{\alpha+\beta-\gamma_1}g_\varepsilon(\xi-x)|\le C\cdot\langle\xi-x\rangle^{-m}\cdot\varepsilon^{-N_1}\le C\cdot\langle\xi\rangle^{-m}\cdot\langle x\rangle^{m}\cdot\varepsilon^{-N_1},$$

(for ε small enough). This yields

$$|\partial_\xi^\alpha\partial_x^\beta\partial_\omega^\gamma k_\varepsilon(\xi,x,\omega)|\le C\cdot\langle\xi\rangle^{-m+|\gamma|}\cdot\langle x\rangle^{m}\cdot\langle\omega\rangle^{|\alpha|}\cdot\varepsilon^{-N_1}. \tag{5.7}$$

So we have proved that for every α, β, $\gamma\in\mathbb{N}_0^d$ there exists $N:=m+N_1+|\alpha|+|\gamma|$ and a non-negative constant C such that

$$|\partial_\xi^\alpha\partial_x^\beta\partial_\omega^\gamma k_\varepsilon(\xi,x,\omega)|\le C\cdot\langle\xi\rangle^{N}\cdot\langle x\rangle^{N}\cdot\langle\omega\rangle^{N}\cdot\varepsilon^{-N}$$

holds (for ε small enough). Hence $k(\xi,x,\omega)=\overline{g(\xi-x)}e^{2\pi i\xi\omega}\in\mathscr{G}_\tau(\mathbb{R}^{3d})$. Yet it remains to check if condition ($*$) is satisfied as well. Estimate (5.7) implies that for every $s:=m-|\gamma|\in\mathbb{N}$, α, β, $\gamma\in\mathbb{N}_0^d$ ($|\alpha|\le s$) there exists $N:=m+N_1+|\alpha|$ and a non-negative constant C such that

$$|\partial_\xi^\alpha\partial_x^\beta\partial_\omega^\gamma k_\varepsilon(\xi,x,\omega)|\le C\cdot\langle\xi\rangle^{-s}\cdot\langle x\rangle^{N}\langle\omega\rangle^{N}\cdot\varepsilon^{-N}$$

holds (for ε small enough). $\square$

It is possible to prove Proposition 5.1.2 along the same lines. Note that

$$V_gf(x,\omega):=\int f(\xi)\overline{g(\xi-x)}e^{2\pi i\xi\omega}\,d\xi=\int f(\xi+x)\overline{g(\xi)}e^{2\pi i(\xi+x)\omega}\,d\xi,$$

and consider the map V_f^* *defined on* $\mathscr{G}_\tau(\mathbb{R}^d)$ *by*

$$V_{f_\varepsilon}^* g_\varepsilon(x,\omega) := V_{g_\varepsilon} f_\varepsilon(x,\omega) = \int f_\varepsilon(\xi+x)\overline{g_\varepsilon(\xi)} e^{2\pi i(\xi+x)\omega}\, d\xi.$$

Then we may apply the same method for the case $f \in \mathscr{G}_{\mathscr{S}}(\mathbb{R}^d)$, $g \in \mathscr{G}_\tau(\mathbb{R}^d)$ $\big(k_\varepsilon(\xi,x,\omega) := f_\varepsilon(\xi+x)e^{2\pi i(\xi+x)\omega}\big)$ *to the one for the case* $f \in \mathscr{G}_\tau(\mathbb{R}^d)$, $g \in \mathscr{G}_{\mathscr{S}}(\mathbb{R}^d)$, *with small differences in calculations:*

Proof.

$$\begin{aligned}
|\partial_\xi^\alpha \partial_x^\beta \partial_\omega^\gamma k_\varepsilon(\xi,x,\omega)| &= |\partial_\xi^\alpha \partial_x^\beta \partial_\omega^\gamma (f_\varepsilon(\xi+x)\cdot e^{2\pi i(\xi+x)\omega})| \\
&= C\cdot |\partial_\xi^\alpha \partial_x^\beta \big(f_\varepsilon(\xi+x)\cdot(\xi+x)^\gamma \cdot e^{2\pi i(\xi+x)\omega}\big)| \\
&= C\cdot|\partial_\xi^\alpha\Big(\sum_{\gamma_1\le\beta}\binom{\beta}{\gamma_1}\partial^{\beta-\gamma_1} f_\varepsilon(\xi+x)\cdot\sum_{\gamma_2\le\gamma_1}\binom{\gamma_1}{\gamma_2}(\xi+x)^{\gamma-\gamma_1+\gamma_2}\cdot\omega^{\gamma_2}\cdot e^{2\pi i(\xi+x)\omega}\Big)| \\
&\le C\cdot\sum_{\substack{\gamma_1\le\beta\\ \gamma_2\le\alpha}} |\partial_\xi^\alpha\big(\partial^{\beta-\gamma_1} f_\varepsilon(\xi+x)\cdot(\xi+x)^{\gamma-\gamma_1+\gamma_2}\cdot\omega^{\gamma_2}\cdot e^{2\pi i(\xi+x)\omega}\big)| \\
&= C\cdot\sum_{\substack{\gamma_1\le\beta\\ \gamma_2\le\alpha}} |\omega^{\gamma_2}|\cdot|\partial_\xi^\alpha\big(\partial^{\beta-\gamma_1} f_\varepsilon(\xi+x)\cdot(\xi+x)^{\gamma-\gamma_1+\gamma_2}\cdot e^{2\pi i(\xi+x)\omega}\big)| \\
&\le C\cdot\sum_{\substack{\gamma_1\le\beta\\ \gamma_2\le\alpha}} |\omega^{\gamma_2}|\cdot|\sum_{\gamma_3\le\alpha}\binom{\alpha}{\gamma_3}\partial_\xi^{\gamma_3}\big(\partial^{\beta-\gamma_1} f_\varepsilon(\xi+x)\cdot(\xi+x)^{\gamma-\gamma_1+\gamma_2}\big)\cdot\partial_\xi^{\alpha-\gamma_3}(e^{2\pi i(\xi+x)\omega})| \\
&\le C\cdot\sum_{\substack{\gamma_1\le\beta\\ \gamma_2\le\alpha}} |\omega^{\gamma_2}|\cdot\sum_{\substack{\gamma_3\le\alpha\\ \gamma_4\le\gamma_3}} |\partial^{\beta-\gamma_1+\gamma_3-\gamma_4} f_\varepsilon(\xi+x)|\cdot|(\xi+x)^{\gamma-\gamma_1+\gamma_2-\gamma_4}\big)|\cdot|\omega^{\alpha-\gamma_3}| \\
&\le C\cdot\sum_{\substack{\gamma_1\le\beta\\ \gamma_2\le\alpha}}\cdot\sum_{\substack{\gamma_3\le\alpha\\ \gamma_4\le\gamma_3}} |\partial^{\beta+\gamma_3-\gamma_2-\gamma_4} f_\varepsilon(\xi+x)|\cdot|(\xi+x)^{\gamma+\gamma_2-\gamma_1-\gamma_4}|\cdot|\omega^{\alpha+\gamma_2-\gamma_3}|.
\end{aligned}$$

One can easily show by a similar argument as in the previous case that such a k_ε defines an element of $\mathscr{G}_\tau(\mathbb{R}^{3d})$ and satisfies condition $(*)$. Thus $V_f^* : \mathscr{G}_\tau(\mathbb{R}^d) \to \mathscr{G}_\tau(\mathbb{R}^{2d})$. □

5.2 Properties of the generalized STFT

Now we shall state and prove some generalizations of basic classical relations of the STFT which we will use to prove the central theorem of this section (Thm 5.2.7). This (main) theorem represents a characterization of the regular generalized moderate nets via STFT.

We shall make use of the following lemma:

Lemma 5.2.1. *[Grö01, Lemma 11.2.1] If* $g \in \mathscr{S}(\mathbb{R}^n)$, *then*

$$X^\alpha\partial^\beta(M_\omega T_x g) = \sum_{\gamma_1\le\alpha}\sum_{\gamma_2\le\beta}\binom{\alpha}{\gamma_1}\binom{\beta}{\gamma_2}x^{\beta-\gamma_2}(2\pi i\omega)^{\alpha-\gamma_1}M_\omega T_x(X^{\gamma_1}\partial^{\gamma_2}g)$$

for all $(x,\omega)\in\mathbb{R}^{2n}$ *and multi-indices* α, β.

Proposition 5.2.2 (Orthogonality relations in $\mathscr{G}_{\mathscr{S}}$). *Let $f_1 = [(f^1_\varepsilon)_\varepsilon]$, $f_2 = [(f^2_\varepsilon)_\varepsilon] \in \mathscr{G}_{\mathscr{S}}(\mathbb{R}^d)$ and g_1, $g_2 \in \mathscr{S}(\mathbb{R}^d)$. Then we have:*

$$\iint V_{g_1} f^1_\varepsilon(x,\omega)\overline{V_{g_2} f^2_\varepsilon(x,\omega)}\ dxd\omega = \langle f^1_\varepsilon, f^2_\varepsilon\rangle\overline{\langle g_1, g_2\rangle}.$$

Remark 5.2.3. *The above equality holds for elements of the space L^2 ([Grö01, p. 42]) so it holds for elements in $\mathscr{S}(\mathbb{R}^d)$ (f^1_ε, f^2_ε, g_1, g_2) componentwise (for each fixed ε). For the same reason, we have that the following inversion formula is true for the elements of the Colombeau algebra $\mathscr{G}_{\mathscr{S}}(\mathbb{R}^d)$:*

Theorem 5.2.4 (Inversion Formula). *Suppose that g, $h \in \mathscr{S}(\mathbb{R}^d)$ and $\langle g, h\rangle \neq 0$. Then for all $f = [(f_\varepsilon)_\varepsilon] \in \mathscr{G}_{\mathscr{S}}(\mathbb{R}^d)$,*

$$f = [(f_\varepsilon)_\varepsilon] = \frac{1}{\langle g, h\rangle}\left[\left(\iint_{\mathbb{R}^{2d}} V_g f_\varepsilon(x,\omega) M_\omega T_x h\ dxd\omega\right)_\varepsilon\right]. \tag{5.1}$$

Remark 5.2.5. *The classical Inversion Formula actually holds for $f \in \mathscr{S}'(\mathbb{R}^d)$ (g, $h \in \mathscr{S}(\mathbb{R}^d)$), thus the equation (5.1) holds for representatives of $\mathscr{G}_\tau(\mathbb{R}^d)$, since they belong to $\mathcal{O}_M(\mathbb{R}^d) \subset \mathscr{S}'(\mathbb{R}^d)$ for fixed ε.*

Before we proceed to the next theorem we will state and prove a useful Lemma.

Lemma 5.2.6. *Let $g \in \mathscr{S}(\mathbb{R}^d)$ fixed and $F_\varepsilon : \mathbb{R}^{2d} \to \mathbb{C}$, $\varepsilon \in (0,1]$, such that there exists $N \in \mathbb{N}$, $C \geq 0$ such that for every $m \in \mathbb{N}_0$*

$$|F_\varepsilon(x,\omega)| \leq C \cdot \langle x\rangle^{-m}\langle\omega\rangle^{-m}\varepsilon^{-N}.$$

Then

$$\big(f_\varepsilon(y)\big)_\varepsilon = \left(\iint F_\varepsilon(x,\omega) M_\omega T_x g(y)\ dxd\omega\right)_\varepsilon \in \mathscr{E}^\infty_{\mathscr{S}}(\mathbb{R}^d).$$

Proof. The estimate for F_ε ensures the convergence of the integral

$$f_\varepsilon(y) = \iint F_\varepsilon(x,\omega) M_\omega T_x g(y)\ dxd\omega.$$

and allows to differentiate under the integral. Thus in view of Lemma 5.2.1 we may write

$$\begin{aligned} X^\alpha\partial^\beta f_\varepsilon(y) &= \iint F_\varepsilon(x,\omega) X^\alpha\partial^\beta\big(M_\omega T_x g(y)\big)\ dxd\omega \\ &= \sum_{\gamma_1\leq\alpha}\sum_{\gamma_2\leq\beta}\binom{\alpha}{\gamma_1}\binom{\beta}{\gamma_2}\iint F_\varepsilon(x,\omega) x^{\beta-\gamma_2}(2\pi i\omega)^{\alpha-\gamma_1} M_\omega T_x\big(X^{\gamma_1}\partial^{\gamma_2} g(y)\big). \end{aligned}$$

Since $g \in \mathscr{S}(\mathbb{R}^d)$ we may find a non-negative constant C such that

$$\|M_\omega T_x\big(X^{\gamma_1}\partial^{\gamma_2} g\big)\|_{L^\infty} \leq C,$$

hence, we have

$$\|X^\alpha\partial^\beta f_\varepsilon\|_{L^\infty} \leq C \cdot \sum_{\gamma_1\leq\alpha}\sum_{\gamma_2\leq\beta}\binom{\alpha}{\gamma_1}\binom{\beta}{\gamma_2}\iint |F_\varepsilon(x,\omega)||x^{\beta-\gamma_2}||\omega^{\alpha-\gamma_1}|\ dxd\omega.$$

$$|x^{\beta-\gamma_2}||\omega^{\alpha-\gamma_1}| \le \langle x\rangle^{|\beta-\gamma_2|}\langle\omega\rangle^{|\alpha-\gamma_1|} \le \langle x\rangle^{|\beta|}\langle\omega\rangle^{|\alpha|} \le \langle x\rangle^{|\alpha+\beta|}\langle\omega\rangle^{|\alpha+\beta|}.$$

$$\|X^\alpha\partial^\beta f_\varepsilon\|_{L^\infty} \le C\cdot\varepsilon^{-N}\iint \langle x\rangle^{-m+|\alpha+\beta|}\langle\omega\rangle^{-m+|\alpha+\beta|}\,dxd\omega.$$

Now choose $m = |\alpha+\beta|+d+1$. □

Theorem 5.2.7. *Let* $f \in \mathscr{G}_{\mathscr{S}}(\mathbb{R}^d)$ *or* $\mathscr{G}_\tau(\mathbb{R}^d)$. *Set* $(V_g f)_\varepsilon := \int_{\mathbb{R}^d} f_\varepsilon(y)\overline{g(y-x)}e^{-2\pi i y\omega}\,dy$, $0 \neq g \in \mathscr{S}(\mathbb{R}^d)$. *Then the following conditions are equivalent:*

(i) $(f_\varepsilon)_\varepsilon \in \mathscr{E}^\infty_{\mathscr{S}}(\mathbb{R}^d)$;

(ii) $(V_g f)_\varepsilon \in \mathscr{E}^\infty_{\mathscr{S}}(\mathbb{R}^{2d})$;

(iii) there exists $N \in \mathbb{N}$ *such that for every* $m \in \mathbb{N}_0$ *there exists a non-negative constant* C *such that*

$$|\big(V_g f\big)_\varepsilon(x,\omega)| \le C\cdot\langle x\rangle^{-m}\langle\omega\rangle^{-m}\varepsilon^{-N}.$$

Proof.

(i)⇒(ii) Let $(f_\varepsilon)_\varepsilon \in \mathscr{E}^\infty_{\mathscr{S}}(\mathbb{R}^d)$. By integration by parts we obtain (see (5.2)):

$$\left|\partial_x^\alpha\partial_\omega^\beta V_g f_\varepsilon(x,\omega)\right| \le C\cdot\frac{1}{\langle\omega\rangle^{2k}}\cdot\sum_{\substack{|\gamma_1|\le 2k\\|\gamma_2|\le 2k\\\gamma_3\le\gamma_2}}\int|\partial^{\gamma_1}f_\varepsilon(y)|\cdot|\partial^{\alpha+\gamma_2-\gamma_3}g(y-x)|\cdot|y^{\beta-\gamma_3}|\,dy.$$

Since $(f_\varepsilon)_\varepsilon \in \mathscr{E}^\infty_{\mathscr{S}}(\mathbb{R}^d)$ we have that there exists $N \in \mathbb{N}$ such that for all $\gamma_1 \in \mathbb{N}_0^d$ and every $m_1 \in \mathbb{N}_0$ there exists a non-negative constant C such that

$$|\partial^{\gamma_1}f_\varepsilon(y)| \le C\cdot\langle y\rangle^{-m_1}\varepsilon^{-N}.$$

For every $\alpha+\gamma_2-\gamma_3$ and every $m_2 \in \mathbb{N}_0$ there exists a non-negative constant C such that

$$|\partial^{\alpha+\gamma_2-\gamma_3}g(y-x)| \le C\cdot\langle y-x\rangle^{-m_2} \le C\cdot\langle y\rangle^{m_2}\langle x\rangle^{-m_2}.$$

So we may write

$$\left|\partial_x^\alpha\partial_\omega^\beta V_g f_\varepsilon(x,\omega)\right| \le C\cdot\frac{1}{\langle\omega\rangle^{2k}}\frac{1}{\langle x\rangle^{m_2}}\varepsilon^{-N}\sum_{|\gamma_3|\le 2k}\int\langle y\rangle^{-m_1+m_2+|\beta-\gamma_3|}\,dy.$$

It follows that for the chosen N we have that for every $\alpha, \beta \in \mathbb{N}_0^d$ and $k, m_2 \in \mathbb{N}_0$

$$\left|\partial_x^\alpha\partial_\omega^\beta V_g f_\varepsilon(x,\omega)\right| \le C\cdot\frac{1}{\langle\omega\rangle^{2k}}\frac{1}{\langle x\rangle^{m_2}}\varepsilon^{-N}$$

holds (since the latter integrals converge for $m_1 = m_2+|\beta-\gamma_3|+d+1$).

(ii)⇒(iii) The estimate (iii) follows directly from the $\mathscr{E}^\infty_{\mathscr{S}}$-estimate for $(V_g f)_\varepsilon$ when $\alpha=\beta=0$.

(iii)⇒(i) Let now $f \in \mathscr{G}_{\mathscr{S}}(\mathbb{R}^d)$ or $\mathscr{G}_{\tau}(\mathbb{R}^d)$ and $0 \neq g \in \mathscr{S}(\mathbb{R}^d)$. According to the Inversion formula in Theorem 5.2.4 (for $h = g$) we have

$$f_\varepsilon(y) = \frac{1}{\|g\|_{L^2}^2} \iint_{\mathbb{R}^{2d}} V_g f_\varepsilon(x, \omega) M_\omega T_x g \; dx d\omega.$$

Set $F_\varepsilon := V_g f_\varepsilon$. Lemma 5.2.6 together with condition (iii) ensures that $\big(f_\varepsilon\big)_\varepsilon \in \mathscr{E}^\infty_{\mathscr{S}}(\mathbb{R}^d)$ indeed.

□

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