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Local limit theorems for hitting and return times

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Contents

Abstract (German)

Ein wichtiger und oft untersuchter Aspekt im Studium von dynamischen Systemen ist die erste Rückkehrzeit zu gewissen Teilmengen. Eine typische Situation hierbei ist eine Folge von Mengen $(A_l)_{l\geq 1}$, die zu einem Punkt schrumpfen. Es ist bekannt das die Rückkehrzeiten in diesem Fall üblicherweise in Verteilung gegen die Standard-Exponentailverteilung konvergiert. Hier untersuchen wir ein verwandtes Konzept der punktweisen Grenzwertsätzen, das von Saussol, B. und Zweimüller, R. vorgeschlagen wurde. Wir zeigen, dass, unter stärkeren Vorrausetzungen an das Misch-Verhalten des Systems, solche Grenzwertsätze immer noch existieren. Zusätzlich geben wir einige Beispiele von Systemen, die solche Grenzwertsätze erfüllen. Prominente Beispiele beinhalten Rychlik Abbildungen und Collet-Eckmann unimodale Abbildungen.

Abstract

An important tool in the study of dynamical systems is the first return time to a given set. Usually, one considers a sequence of sets shrinking to a single point. It is well known that for many natural systems and sufficiently nice sets $(A_l)_{l\geq 1}$ the return times, normalized by the measure of the set, will converge in distribution, the most prominent case being that of a standard exponential. We consider a stronger notion suggested by Saussol, B. and Zweimüller, R. of local limit theorems for return times. As it turns out, those limit theorems, though much more elusive, are still present in various systems with strong mixing properties. Along with some abstract criteria, we give examples of prominent systems having these limit theorems. Examples include piecewise expanding interval maps, as introduced by Rychlik, M., and Collet-Eckmann unimodal maps.

Chapter 1

Introduction

1.1 Central notions and notation

The set-up in this thesis will be a very classical one. Let $(X, \mathcal{A}, \mu)$ be a probability space, and $T : X \to X$ be a μ-preserving, ergodic transformation. One important consequence of this is Birkhoff's ergodic theorem. It tells you that for any observable [1] $f : X \to \mathbb{R}$ with $f \in L^1(\mu)$ we have that

$$\frac{1}{n}\sum_{i=0}^{n-1} f \circ T^i \to \int_X f \, \mathrm{d}\mu \quad \textit{as } n \to \infty$$

pointwise a.e and in $L^1(\mu)$.
For any $A \in \mathcal{A}$ with $\mu(A) > 0$ we will be interested in the *(first) hitting time* of A, $\varphi_A : X \to \bar{\mathbb{N}} := \{1, 2, ..., \infty\}$ given by $\varphi_A(x) := \min\{n \geq 1 \mid T^n(x) \in A\}$. When restricted to the set A we will also call it (without changing notation otherwise) the *(first) return time* of A. By the Poincaré recurrence theorem and ergodicity, φ_A is indeed finite a.e on X.
In order to talk about successive returns to A, we first define the *(first) entrance (resp. return) map* of A, $T_A : X(\textit{resp. } A) \to A$ given by $T_A(x) := T^{\varphi_A(x)}(x)$. It is known that in this situation the *induced* system $(A, \mathcal{A} \cap A, \mu_A, T_A)$ [2], where $\mu_A(B) := \frac{\mu(A \cap B)}{\mu(A)}$ for $B \in \mathcal{A}$, is also measure preserving and ergodic. We can define the *nth hitting (resp. return) time* $\varphi_A^{(n)} : X(\textit{resp. } A) \to \bar{\mathbb{N}}$ recursively by $\varphi_A^{(1)} := \varphi_A$ and $\varphi_A^{(n+1)} := \varphi_A^{(n)} \circ T_A$.

For $f \in L^1(\mu)$ we will write $f \odot \mu$ for the measure with density f wrt. μ, i.e the measure given by $f \odot \mu(A) := \int_A f \, \mathrm{d}\mu$ for $A \in \mathcal{A}$. A useful tool to investigate dynamical systems is the *(Perron-Frobenius) transfer operator* $\hat{T} : L^1(\mu) \to L^1(\mu)$, given by $\hat{T}(f) := \frac{d((f \odot \mu) \circ T^{-1})}{d\mu}$ for $f \in L^1(\mu)$,

[1] Observables here are measurable functions into $\mathbb{R}$. All functions mentioned from here on out are assumed to be measurable.

[2] We will adapt this style of notation (statespace, σ-algebra, measure, transformation) throughout.

where $\frac{d\cdot}{d\cdot}$ denotes the Radon-Nikodym derivative. When there is ambiguity about which measure to use, we will denote the measure as a subscript, e.g $\hat{T} = \hat{T}_\mu$. We will usually use this operator to push-forward densities by the transformation T, therefore notice that for $f, g \in L^1(\mu)$ we have

$$\int_X g \cdot (f \circ T)\,\mathrm{d}\mu = \int_X \hat{T}(g) \cdot f\,\mathrm{d}\mu \tag{1.1}$$

as long as T is μ-preserving. In particular, if $f \geq 0$ we have $||f||_{L^1(\mu)} = ||\hat{T}(f)||_{L^1(\mu)}$. [3]

In this thesis we will talk about asymptotic behaviour, so let us first fix some notation.

Definition 1.1.1. *For two sequences $(f_n)_{n\geq 1}$ and $(g_n)_{n\geq 1}$ of positive real numbers we will say*

(i) f_n has the same asymptotics as g_n if

$$\lim_{n\to\infty} |f_n - g_n| = 0 \tag{1.2}$$

and write $f_n \approx g_n$,

(ii) f_n is asymptotically equivalent to g_n if

$$\lim_{n\to\infty} \frac{f_n}{g_n} = 1 \tag{1.3}$$

and write $f_n \sim g_n$,

(iii) f_n is of order $o(g_n)$ if

$$\limsup_{n\to\infty} \frac{f_n}{g_n} = 0 \tag{1.4}$$

and write $f_n = o(g_n)$,

(iv) f_n is of order $O(g_n)$ if

$$\limsup_{l\to\infty} \frac{f_n}{g_n} < \infty \tag{1.5}$$

and write $f_n = O(g_n)$.

(v) and say f_n is of order $\Theta(g_n)$ if

$$0 < \liminf_{l\to\infty} \frac{f_n}{g_n} \leq \limsup_{l\to\infty} \frac{f_n}{g_n} < \infty \tag{1.6}$$

and write $f_n = \Theta(g_n)$.

[3] When there is no ambiguity, we will write L^1 instead of $L^1(\mu)$.

1.2 Convergence in law

We will henceforth consider the hitting and return times for a sequence $(A_l)_{l\geq 1}$ of *asymptotically rare events*, i.e $A_l \in \mathcal{A}$ for $l \geq 1$ and $0 < \mu(A_l) \to 0$ as $l \to \infty$. If the A_l are genuinely decreasing, i.e $A_{l+1} \subset A_l$, the hitting and return times are increasing in l, $\varphi_{A_{l+1}} \geq \varphi_{A_l}$. So in order to properly describe the asymptotics *as* $l \to \infty$, we will have to find a normalising factor.

Theorem 1.2.1 (Kac's formula)**.** *Let $(X, \mathcal{A}, \mu)$ be a probability space, and $T : X \to X$ be a μ-preserving, ergodic transformation. Then for $A \in \mathcal{A}$ with $\mu(A) > 0$, the first return time of A has expectation $\frac{1}{\mu(A)}$*

$$\int_A \varphi_A \, \mathrm{d}\mu_A = \frac{1}{\mu(A)}. \tag{1.7}$$

Proof. For $n \geq 1$ we can split the set $\{\varphi_A = n\}$ into its part in A and the complement A^c. By μ-invariance we obtain

$$\begin{aligned} \mu(\varphi_A = n) &= \mu(A \cap \{\varphi_A = n\}) + \mu(A^c \cap \{\varphi_A = n\}) \\ &= \mu(A \cap \{\varphi_A = n\}) + \mu(\{\varphi_A = n + 1\}). \end{aligned}$$

Iterating this and recalling that $\mu(\{\varphi_A = n+1\}) \to 0$ *as* $n \to \infty$, we obtain

$$\mu(\varphi_A = n) = \mu(A \cap \{\varphi_A \geq n\}). \tag{1.8}$$

Therefore rewriting the integral we get

$$\begin{aligned} \int_A \varphi_A \, \mathrm{d}\mu &= \sum_{n\geq 1} \mu(A \cap \{\varphi_A \geq n\}) \\ &= \sum_{n\geq 1} \mu(\varphi_A = n) \\ &= 1. \end{aligned}$$

□

Therefore the *normalised return times* $\mu(A_l)\varphi_{A_l}$ have expectation 1 on A_l. There is a well-developed theory of convergence in law of the variables $\mu(A_l)\varphi_{A_l}$ in this situation. We are interested in *convergence in law of (normalised) hitting times*

$$\mu(\mu(A_l)\varphi_{A_l} \leq t) \Rightarrow F(t) \quad \textit{as } l \to \infty, \tag{1.9}$$

and *convergence in law of (normalised) return times*

$$\mu_{A_l}(\mu(A_l)\varphi_{A_l} \leq t) \Rightarrow \tilde{F}(t) \quad \textit{as } l \to \infty, \tag{1.10}$$

where we write $F_l(t) \Rightarrow F(t)$ to indicate that $F_l(t) \to F(t)$ at every continuity point t of F. In (1.9) and (1.10), F and $\tilde{F}$ are sub-probability distributions on $[0, \infty)$.
From the next theorem we will see that in the above situation $\tilde{F}$ is a probability distribution satisfying $\int_0^\infty (1 - \tilde{F}(t))\,\mathrm{d}t \le 1$, while F is continuous, concave and has $F(t) \le t$ for $t \ge 0$. It is furthermore known ([KL05], [Lac02], etc.) that under the (relatively mild) assumption that μ is aperiodic, for any such F resp. $\tilde{F}$ there is a sequence $(A_l)_{l\ge1}$ of asymptotically rare events such that (1.9) resp. (1.10) is satisfied.

There are several different approaches to finding convergence in law of hitting and return times. We only highlight one here.

One of the main ingredients to find good sequences for convergence in law of the hitting and return times is the relation between F and $\tilde{F}$ in (1.9) and (1.10). This has first been established in [HLV05], for an alternative approach see [AS11].

Theorem 1.2.2. *Let $(X, \mathcal{A}, \mu, T)$ be an ergodic, probability-preserving system, and $(A_l)_{l\ge1}$ be a sequence of asymptotically rare events. Then*

$$\mu(\mu(A_l)\varphi_{A_l} \le t) \Rightarrow F(t) \quad \textit{for some sub-probability } F \textit{ on } [0,\infty) \tag{1.11}$$

iff

$$\mu_{A_l}(\mu(A_l)\varphi_{A_l} \le t) \Rightarrow \tilde{F}(t) \quad \textit{for some sub-probability } \tilde{F} \textit{ on } [0,\infty). \tag{1.12}$$

In this case F and $\tilde{F}$ are related by

$$F(t) = \int_0^t (1 - \tilde{F}(t))\,\mathrm{d}t \quad \textit{for } t \ge 0. \tag{1.13}$$

An especially interesting situation is if $F(t) = \tilde{F}(t) = 1 - e^{-t}$, as various results in probability theory suggest that such limits should be exponential. Indeed, the above result ensures that $F = \tilde{F}$ is already sufficient for $F(t) = \tilde{F}(t) = 1 - e^{-t}$. Together with Helly's selection theorem we get to following result.

Corollary 1.2.1. *Let $(X, \mathcal{A}, \mu, T)$ be an ergodic, probability-preserving system, and $(A_l)_{l\ge1}$ be a sequence of asymptotically rare events. Then the following are equivalent*

1. *for $(A_l)_{l\ge1}$ we have convergence of the hitting times to the exponential law,*
$$\mu(\mu(A_l)\varphi_{A_l} \le t) \Rightarrow 1 - e^{-t} \quad \textit{as } l \to \infty, \tag{1.14}$$

2. *for $(A_l)_{l\geq 1}$ we have convergence of the return times to the exponential law,*
$$\mu_{A_l}(\mu(A_l)\varphi_{A_l} \leq t) \Rightarrow 1 - e^{-t} \quad \textit{as } l \to \infty, \tag{1.15}$$

3. *the difference between the hitting and return time distribution functions goes to zero,*
$$|\mu(\mu(A_l)\varphi_{A_l} \leq t) - \mu_{A_l}(\mu(A_l)\varphi_{A_l} \leq t)| \to 0 \quad \textit{as } l \to \infty, \tag{1.16}$$

4. *hitting and return times have the same asymptotic along converging subsequences, that is if $(l_k)_{k\geq 1}$ is a strictly increasing sequence of natural numbers and F is a sub-probability distribution such that*
$$\mu(\mu(A_{l_k})\varphi_{A_{l_k}} \leq t) \Rightarrow F(t) \quad \textit{as } k \to \infty, \tag{1.17}$$
then
$$\mu_{A_{l_k}}(\mu(A_{l_k})\varphi_{A_{l_k}} \leq t) \Rightarrow F(t) \quad \textit{as } k \to \infty. \tag{1.18}$$

Recall that ergodicity of the system implies that, for any collection of densities $\mathcal{U} \subset \mathcal{D}(\mu)$ [4] compact in L^1,
$$||\frac{1}{n}\sum_{i=0}^{n-1} \hat{T}(u) - 1||_{L^1} \to 0 \quad \textit{as } n \to \infty \textit{ uniformly in } u \in \mathcal{U}. \tag{1.19}$$

Using this, Theorem 6.3 of [Zweb] guarantees that
$$\mu(\mu(A_l)\varphi_{A_l} \leq t) \Rightarrow F(t) \quad \textit{as } l \to \infty \tag{1.20}$$
implies
$$u_l \odot \mu(\mu(A_l)\varphi_{A_l} \leq t) \Rightarrow F(t) \quad \textit{as } l \to \infty, \tag{1.21}$$
provided that $u_l \in \mathcal{U}$ for some $\mathcal{U} \subset \mathcal{D}(\mu)$ compact in L^1 and $l \geq 1$.

We want to apply this result to the situation $u_l = \frac{1_{A_l}}{\mu(A_l)}$, unfortunately, as $\frac{1}{\mu(A_l)} \to \infty$, there is no chance of finding a compact set as required. However we can try a little trick, which we apply plenty often in this thesis; We go a small (compared to $\frac{1}{\mu(A_l)}$) amount of steps, say τ_l, such that $\hat{T}^{\tau_l}\left(\frac{1_{A_l}}{\mu(A_l)}\right)$ is in a compact set. This was used in Theorem 3.1 of [Zweb].

Theorem 1.2.3. *Let $(X, \mathcal{A}, \mu, T)$ be an ergodic, probability-preserving system, and $(A_l)_{l\geq 1}$ be a sequence of asymptotically rare events. Suppose there is a sequence of natural numbers $(\tau_l)_{l\geq 1}$ such that*

(i)
$$\mu(A_l)\tau_l \to 0 \quad \textit{as } l \to \infty, \tag{1.22}$$

[4] Define $\mathcal{D}(\mu) := \{u : X \to [0, \infty) \mid \int_X u \, \mathrm{d}\mu = 1\}$ as the collection of all densities.

(ii)

$$\mu_{A_l}(\varphi_{A_l} \leq \tau_l) \to 0 \quad as\ l \to \infty, \tag{1.23}$$

(iii) and there is a $\mathcal{U} \subset \mathcal{D}(\mu)$ compact in $L^1(\mu)$ such that

$$\hat{T}^{\tau_l}\left(\frac{1_{A_l}}{\mu(A_l)}\right) \in \mathcal{U} \quad for\ l \geq 1. \tag{1.24}$$

Then

$$\mu(\mu(A_l)\varphi_{A_l} \leq t) \Rightarrow 1 - e^{-t} \quad as\ l \to \infty \tag{1.25}$$

and

$$\mu_{A_l}(\mu(A_l)\varphi_{A_l} \leq t) \Rightarrow 1 - e^{-t} \quad as\ l \to \infty. \tag{1.26}$$

Remark 1.2.1. *[Zweb] even shows that under the assumptions of Theorem 1.2.3, we also have convergence in law of consecutive hitting and return times. That is for $\Phi_l = (\varphi^{(1)}_{A_l}, \varphi^{(2)}_{A_l}, ...)$ then*

$$\mu(A_l)\Phi_l \overset{\mu}{\Rightarrow} \mathcal{E} \quad and \quad \mu(A_l)\Phi_l \overset{\mu_{A_l}}{\Rightarrow} \mathcal{E}, \tag{1.27}$$

where $\mathcal{E}$ is a iid sequence of exponentially distributed random variables.

The same paper also gives a class of system for which this result can be applied.

Definition 1.2.1 (Gibbs-Markov maps)**.** *Let (X, d) be a metric space with* $\mathrm{diam}(X) < \infty$*, $\mathcal{A}$ be the usual Borel-σ-algebra on X and μ be a probability measure on $(X, \mathcal{A})$. We call a μ-preserving transformation $T : X \to X$ a Gibbs-Markov (GM) map if there is a countable partition (mod μ) ζ of X such that*

(i) $TZ \in \sigma(\zeta)$ for $Z \in \zeta$,

(ii) T has a uniformly expanding iterate T^N for some $N \geq 1$,

(iii) $T|_Z$ is a homeomorphism, we denote its inverse by $v_Z := (T|_Z)^{-1}$, for $Z \in \zeta$,

(iv) T has the big image property

$$b := \inf_{Z \in \zeta} \mu(TZ) > 0, \tag{1.28}$$

(v) the Radon-Nikodym derivatives $v'_Z := \frac{d(\mu \circ v_Z)}{d\mu}$ have well behaved versions, there exists a $r > 0$ such that

$$\left|\frac{v'_Z(x)}{v'_Z(y)} - 1\right| \leq rd(x, y) \quad for\ Z \in \zeta\ and\ x, y \in TZ. \tag{1.29}$$

For $n \geq 1$ we will denote the set of all n-cylinder by

$$\zeta_n := \{[Z_0, ..., Z_{n-1}] := \bigcap_{i=0}^{n-1} T^{-i} Z_i \mid Z_i \in \zeta \textit{ for } 0 \leq i \leq n-1\}. \tag{1.30}$$

The inverses of T are analogously denoted by $v_Z := (T^n|_Z)^{-1}$ for $n \geq 1$ and $Z \in \zeta_n$. In this case (see [AD01]) $r > 0$ can be chosen such that

$$\left| \frac{v'_Z(x)}{v'_Z(y)} - 1 \right| \leq r d(x,y) \quad \textit{for } n \geq 1, Z \in \zeta_n \textit{ and } x, y \in T^n Z. \tag{1.31}$$

In the following v'_Z will always denote such a version.

We recall some known properties of such systems. Denote by β the partition generated by $T\zeta$, and $\mathrm{Lip}_\beta(f) := \sup_{B \in \beta} \mathrm{Lip}_B(f)$ where $\mathrm{Lip}_B(f) := \sup_{x,y \in B, x \neq y} \left| \frac{f(x)-f(y)}{d(x,y)} \right|$ is the usual Lipschitz-coefficient. Define $\mathcal{U}'_r := \{u \in \mathcal{D}(\mu) \mid \mathrm{Lip}_\beta \leq r\}$. Then for $u \in \mathcal{U}'_r$, $n \geq 1$ and $Z \in \zeta_n$

$$\hat{T}^n \left(\frac{1_Z u}{u \odot \mu(Z)} \right) \in \mathcal{U}'_r. \tag{1.32}$$

It is also known that there is a constant $c > 0$ such that

$$\left|\left| \hat{T}^n \left(\frac{1_Z u}{u \odot \mu(Z)} \right) \right|\right|_{L^\infty} \leq c \tag{1.33}$$

for $u \in \mathcal{U}'_r$, $n \geq 1$ and $Z \in \zeta_n$. Therefore the set $\mathcal{U}_r := \{u \in \mathcal{U}'_r \mid ||u|| \leq \max(1, c)\}$ is convex, bounded in L^∞, compact in L^1 and furthermore has the property that

$$\hat{T}^n \left(\frac{1_Z u}{u \odot \mu(Z)} \right) \in \mathcal{U}_r \tag{1.34}$$

for $u \in \mathcal{U}_r$, $n \geq 1$ and $Z \in \zeta_n$. Furthermore T has *bounded distortion*, i.e

$$\mu(Z \cap T^{-n} A) \leq \frac{e^r}{b} \mu(Z) \mu(A) \quad \textit{for } n \geq 1, Z \in \zeta_n \textit{ and } A \in \mathcal{A}. \tag{1.35}$$

An easy argument then gives $\kappa \geq 1$ and $q \in (0,1)$ such that

$$\mu(Z) \leq \kappa q^n \quad \textit{for } n \geq 1 \textit{ and } Z \in \zeta_n. \tag{1.36}$$

Lastly, for mixing GM-maps, densities in $\mathcal{U}_r$ are smoothed out exponentially fast by the transfer operator, i.e

$$||\hat{T}^n(u) - 1||_{L^\infty} \leq \kappa q^n \quad \textit{for } n \geq 1 \textit{ and } u \in \mathcal{U}_r. \tag{1.37}$$

We therefore see that we can use Theorem 1.2.3 on sequences of cylinders converging to a non-periodic point x^*, this is to avoid return after a short amount of time as in (1.23).

Theorem 1.2.4. *Let $(X, \mathcal{A}, \mu, T, \zeta)$ be an ergodic, μ-preserving GM-map. Let $(A_l)_{l\geq 1}$ be asymptotically rare events such that either **a)** $A_l \in \sigma(\zeta)$ is a union of cylinder or **b)** there is a non-periodic point $x^* \in X$ such that $A_l \in \zeta_l$ is the unique l-cylinder containing x^*. Then $(A_l)_{l\geq 1}$ has convergence in law to an exponential distribution*

$$\mu(\mu(A_l)\varphi_{A_l} \leq t) \Rightarrow 1 - e^{-t} \quad \text{as } l \to \infty \tag{1.38}$$

and

$$\mu_{A_l}(\mu(A_l)\varphi_{A_l} \leq t) \Rightarrow 1 - e^{-t} \quad \text{as } l \to \infty. \tag{1.39}$$

For a more probabilistic statement, this kind of convergence can also be shown for irreducible, positively recurrent, aperiodic Markov-chains.

Example 1.2.1. *Let M be an irreducible, positively recurrent, aperiodic Markov-chain with transition matrix $P = (p_{i,j})_{i,j\in S}$ on a countable state space S. Let us, for the sake of this argument, assume that there is a $\delta < 1$ such that $p_{i,j} \leq \delta$ for $i, j \in S$. It is well known that positive recurrence implies the existence of an invariant probability distribution π on S. We look at the usual dynamical representation of M. Let $\Sigma := \{x = (x_0, x_1, ...) \in S^{\mathbb{N}} \mid p_{x_i,x_{i+1}} > 0 \text{ for } i \geq 0\}$, σ be the usual shift operator and d the product metric defined by*

$$d(x, y) := 2^{-\min(i\geq 0 \mid x_i \neq y_i)} \quad \text{for } x, y \in \Sigma. \tag{1.40}$$

Let $\zeta_n := \{[i_0, ..., i_{n-1}] \mid i_j \in S \text{ for } 0 \leq j \leq n-1\}$ be the collection of n-cylinders, where

$$[i_0, ..., i_{n-1}] := \{x \in \Sigma \mid x_j = i_j \text{ for } 0 \leq j \leq n-1\}, \tag{1.41}$$

and $\mathcal{A}$ be the σ-Algebra generated by all cylinder. Lastly define the Markov measure μ on $\mathcal{A}$ by

$$\mu([i_0, ...i_{n-1}]) = \pi(i_0) \prod_{j=0}^{n-2} p_{i_j,i_{j+1}} \quad \text{for } n \geq 1 \text{ and } [i_0, ..., i_{n-1}] \in \zeta_n. \tag{1.42}$$

Let $i \in S$, then the inverse branches are given by

$$v_{[i]}(x) := (\sigma|_{[i]})^{-1}(x) = ix \quad \text{for } x \in \Sigma \tag{1.43}$$

and are also continuous. Furthermore we get

$$v'_{[i]}(x) := \frac{d(\mu \circ v_{[i]})}{d\mu}(x) = \frac{\pi(i)}{\pi(x_0)} p_{i,x_0} \tag{1.44}$$

for $x = (x_0, x_1, ...) \in \Sigma$. So our transfer operator is given by

$$\hat{\sigma}(f)(x) = \sum_{i\in S} f(ix) \frac{\pi(i)}{\pi(x_0)} p_{i,x_0} \tag{1.45}$$

for $f : \Sigma \to \mathbb{R}$ *bounded and* $x = (x_0, x_1, ...) \in \Sigma$*. Let* $(i_l)_{l\geq 0}$ *be a non-periodic sequence of elements of* S*, we claim that the corresponding* l*-cylinders* $A_l = [i_0, ..., i_{l-1}]$ *for* $l \geq 1$ *satisfy the assumptions of Theorem 1.2.3. Set*

$$\mathcal{U} := \{f \in L^1(\mu) \mid f \geq 0, \int_\Sigma f \, \mathrm{d}\mu = 1, \ f(x) \text{ only depends on the first letter of } x\} \tag{1.46}$$

be the set of locally constant functions with mean 1*. Clearly* $\mathcal{U} \subset L^1(\mu)$ *is compact.*[5] *Now*

$$\begin{aligned}\hat{\sigma}^l(1_{A_l})(x) &= \sum_{j_0,...,j_{l-1} \in S} 1_{A_l}(j_0...j_{l-1}x) \cdot v'_{[j_0,...,j_{l-1}]}(x) \\ &= v'_{[i_0,...,i_{l-1}]}(x) = \frac{\pi(i_0)}{\pi(x_0)} p_{i_0,i_1}...p_{i_{l-2},i_{l-1}} p_{i_{l-1},x_0}\end{aligned}$$

for $l \geq 1$ *and* $x = (x_0, x_1, ...) \in \Sigma$*. So* $\hat{\sigma}^l\left(\frac{1_{A_l}}{\mu(A_l)}\right) \in \mathcal{U}$*. Clearly* $l\mu(A_l) \to 0$ *as* $l \to \infty$*. So all that is left to check is* $\mu_{A_l}(\varphi_{A_l} \leq l) \to 0$*.*
Note first that $\mu_{A_l}(T^{-k}(A_l)) \leq \delta^k$ *for* $k \leq l$*, this is because* $A_l \cap T^{-k}(A_l) = [i_0, ..., i_{l-1}, i_{l-k}, ..., i_{l-1}]$ *if* $i_j = i_{j+k}$ *for* $0 \leq j \leq l-k-1$*, and empty otherwise. So for* $k_l = \min{(k \geq 1 \mid i_j = i_{j+k} \text{ for } 0 \leq j \leq l-k-1)}$ *we have* $\mu_{A_l}(\varphi_{A_l} \leq l) \leq \frac{\delta^{k_l}}{1-\delta}$*. Clearly* k_l *increases in* l*, it also cannot be bounded as otherwise* $(i_l)_{l\geq 0}$ *would have a period of* $\max_{l\geq 1} k_l$*. Therefore* $\mu_{A_l}(\varphi_{A_l} \leq l) \to 0$ *and we can apply Theorem 1.2.3.*

[5] To see this we could identify $\mathcal{U}$ with a subset of $\prod_{i\in S}[0, \pi(i)^{-1}]$ with norm $||x|| = \sum_{i\in S} x_i \pi(i)$ which is the product of compact intervals.

Chapter 2

Abstract local limit theorems

2.1 First hitting and return time

Much in this chapter is closely related to the analysis in [AZ], for more information we refer the reader there and to the sources given there.

The main part of this thesis will concern itself with finer properties of hitting and return times. We saw in the last chapter that in many situations we have convergence in law to some limit law

$$\mu(\mu(A_l)\varphi_{A_l} \leq t) \Rightarrow F(t) \quad as\ l \to \infty, \tag{2.1}$$

and

$$\mu_{A_l}(\mu(A_l)\varphi_{A_l} \leq t) \Rightarrow \tilde{F}(t) \quad as\ l \to \infty. \tag{2.2}$$

However sometimes estimates for the law are not good enough and we want to know about the precise time of the first return to small set. So instead of $\mu(\varphi_{A_l} \leq \frac{t}{\mu(A_l)})$ (resp. $\mu_{A_l}(\varphi_{A_l} \leq \frac{t}{\mu(A_l)})$) we want to control $\mu(\varphi_{A_l} = k_l)$ (resp. $\mu_{A_l}(\varphi_{A_l} = k_l)$) for suitable costants $(k_l)_{l\geq 1}$. Note that this question is only interesting if the k_l are natural numbers (otherwise these quantities are 0) they should also be of the same order as the expectation, therefore $k_l = \Theta(\frac{1}{\mu(A_l)})$.

So what should $\mu(\varphi_{A_l} = k_l)$ behave like? Let $t > 0$ and k_l be such that $k_l\mu(A_l) \to t$ and F be C^1 and strictly monotone increasing in the situation of (2.1).

$$\mu(\mu(A_l)\varphi_{A_l} \leq \mu(A_l)k_l) \sim F(\mu(A_l)k_l) \quad as\ l \to \infty. \tag{2.3}$$

Notice that by virtue of F being strictly monotone increasing, both sides are strictly positive, so it is enough to show

$$\mu(\mu(A_l)\varphi_{A_l} \leq \mu(A_l)k_l) \to F(t) \quad as\ l \to \infty. \tag{2.4}$$

Lemma 2.1.1. *Let $F_n, F : \mathbb{R} \to [0,1]$ be sub-probability distribution functions for $n \geq 1$ such that*

$$F_n(t) \Rightarrow F(t) \quad \text{as } n \to \infty. \tag{2.5}$$

Then for all continuity points $t \geq 0$ and sequences $(t_n)_{n\geq 1}$ in $[0,\infty)$ such that $t_n \to t$ as $n \to \infty$ we have

$$F_n(t_n) \to F(t) \quad \text{as } n \to \infty. \tag{2.6}$$

Proof. Let $\epsilon > 0$ and $s > t$ be a continuity point of F such that $F(t) > F(s) - \epsilon$. For n large enough such that $s > t_n$ we have $F_n(t_n) \leq F_n(s)$ and therefore

$$\limsup_{n\to\infty} F_n(t_n) \leq \lim_{n\to\infty} F_n(s) = F(s) < F(t) + \epsilon. \tag{2.7}$$

Letting $\epsilon \to 0$ yields $\limsup_{n\to\infty} F_n(t_n) \leq F(t)$.
In the case $t > 0$ we get $\liminf_{n\to\infty} F_n(t_n) \geq F(t)$ analogously. If $t = 0$ then

$$\liminf_{n\to\infty} F_n(t_n) \geq \lim_{n\to\infty} F_n(0) = F(0). \tag{2.8}$$

$\square$

Using the mean value theorem, we could try

$$\begin{aligned}
\mu(\varphi_{A_l} = k_l) &= \mu(\varphi_{A_l} \leq k_l) - \mu(\varphi_{A_l} \leq k_l - 1) \\
&= \underbrace{\mu(\mu(A_l)\varphi_{A_l} \leq \mu(A_l)k_l)}_{\sim F(\mu(A_l)k_l)} - \underbrace{\mu(\mu(A_l)\varphi_{A_l} \leq \mu(A_l)k_l - \mu(A_l))}_{\sim F(\mu(A_l)k_l - \mu(A_l))} \\
&\overset{?}{\sim} \mu(A_l)F'(\xi_l) \quad \textit{for some } \xi_l \in (\mu(A_l)k_l - \mu(A_l), \mu(A_l)k_l) \\
&\sim \mu(A_l)F'(t)
\end{aligned}$$

similar for μ_{A_l}. Unfortunately $\sim$ is not stable under addition so this analysis is not valid. As the most important case is $F(t) = \tilde{F}(t) = 1 - e^{-t}$ anyway, we will assume throughout that our distribution function is C^1 and strictly monotone increasing. It is possible to assume this kind of regularity only near the sequence $\mu(A_l)k_l$ and in fact the proofs stay the same, but for convenience of notation we will stick with that.

In the situation of (2.1) (resp. (2.2)) we will say that we have a *local limit theorem (LLT) for the (first) hitting (resp. return) time* if

$$\mu(\varphi_{A_l} = k_l) \sim \mu(A_l)F'(t) \quad \text{as } l \to \infty \tag{2.9}$$

resp.

$$\mu_{A_l}(\varphi_{A_l} = k_l) \sim \mu(A_l)\tilde{F}'(t) \quad \text{as } l \to \infty \tag{2.10}$$

whenever $t > 0$ and $(k_l)_{l\geq 1}$ is a sequence of natural numbers such that $\mu(A_l)k_l \to t$. Note that, in our set-up, by the subsequence principle these conditions are equivalent to

$$\mu(\varphi_{A_l} = k_l) \sim \mu(A_l)F'(\mu(A_l)k_l) \quad \textit{as } l \to \infty \tag{2.11}$$

resp.

$$\mu_{A_l}(\varphi_{A_l} = k_l) \sim \mu(A_l)\tilde{F}'(\mu(A_l)k_l) \quad \textit{as } l \to \infty \tag{2.12}$$

whenever $(k_l)_{l\geq 1}$ is a sequence of natural numbers such that $(\mu(A_l)k_l)_{l\geq 1}$ is bounded away from 0 and ∞.

As there are many different sources about convergence in law of hitting times, we will for the rest of this thesis concentrate on the question, when convergence in law implies local limit theorems.

Theorem 2.1.1. *Let $(X, \mathcal{A}, \mu, T)$ be an ergodic, probability-preserving system, and $(A_l)_{l\geq 1}$ be a sequence of asymptotically rare events. Let $F : [0, \infty) \to [0, 1]$ be a strictly monotone increasing sub-probability function of class C^1. Then the following are equivalent*

(a) we have convergence in law to F

$$\mu(\mu(A_l)\varphi_{A_l} \leq t) \Rightarrow F(t) \quad \textit{as } l \to \infty, \tag{2.13}$$

(b) for all $t > 0$ and sequences $(k_l)_{l\geq 1}$ of natural numbers such that $\mu(A_l)k_l \to t$ we have

$$\mu(\varphi_{A_l} = k_l) \sim \mu(A_l)F'(t) \quad \textit{as } l \to \infty. \tag{2.14}$$

Proof. By Theorem 1.2.2, **a)** is equivalent to

$$\mu_{A_l}(\mu(A_l)\varphi_{A_l} \leq t) \Rightarrow \tilde{F}(t) \quad \textit{as } l \to \infty, \tag{2.15}$$

with $F'(t) = 1 - \tilde{F}(t)$. Using (1.8) from the proof of Kac's formula and Lemma 2.1.1, for any $t > 0$ and sequences $(k_l)_{l\geq 1}$ of natural numbers such that $\mu(A_l)k_l \to t$ we obtain

$$\begin{aligned} \frac{1}{\mu(A_l)}\mu(\varphi_{A_l} = k_l) &= \mu_{A_l}(\varphi_{A_l} \geq k_l) \\ &\to 1 - \tilde{F}(t) \end{aligned}$$

as $l \to \infty$. Therefore

$$\mu(\varphi_{A_l} = k_l) \sim \mu(A_l)F'(t) \quad \textit{as } l \to \infty. \tag{2.16}$$

as required. The same argument gives the converse direction. $\square$

A useful fact when investigating convergence in law of (first) hitting times is the fact that we can (within reason) change the initial distribution, i.e if

$$\mu(\mu(A_l)\varphi_{A_l} \leq t) \Rightarrow F(t) \quad \textit{as } l \to \infty, \tag{2.17}$$

then also

$$\nu(\mu(A_l)\varphi_{A_l} \leq t) \Rightarrow F(t) \quad \textit{as } l \to \infty, \tag{2.18}$$

for any probability measure $\nu \ll \mu$. We would like an analogous result for return times. But this unfortunately does not hold in full generality.
For example if the system has a cyclic structure in that $C, T^{-1}C, ..., T^{-p+1}C$ are pairwise disjoint and $C = T^{-p}C$ for some $C \in \mathcal{A}$ with $\mu(C) > 0$. If $A_l \subset C$ for $l \geq 1$, then for $\nu := \mu_C$ we would always have $\nu(\varphi_{A_l} = k_l) = 0$ unless k_l is a multiple of p. In fact, even without periodicity a fairly simple argument shows that there are always initial distributions ν that fail (2.18) for a given sequence $(k_l)_{l\geq 1}$. The key idea to constructing such a ν is that for $B_l = A_l \setminus \{\varphi_{A_l} = k_l\}$, its return times will never be k_l, i.e $\varphi_{B_l}(x) \neq k_l$ for $x \in B_l$. For more details see [AZ].

Looking at the transfer operator $\hat{T}$ will give us a good class of initial distributions for (2.18).
First we need a technical lemma, which we will use at various points throughout this thesis.

Lemma 2.1.2. *Let $(X, \mathcal{A}, \mu, T)$ be an ergodic, probability-preserving system, and $A \in \mathcal{A}$ with $\mu(A) > 0$. Then*

$$1_{\{\varphi_A = k\}} = 1_{\{\varphi_A = k-j\}} \circ T^j - 1_{\{\varphi_A \leq j\} \cap \{\varphi_A \circ T^j = k-j\}} \quad \textit{for } 0 \leq j < k, \tag{2.19}$$

and

$$\mu(\{\varphi_A \leq j\} \cap \{\varphi_A \circ T^j = m\} = \mu(A \cap \{m \leq \varphi_A < m + j\} \quad \textit{for } j, m \geq 1. \tag{2.20}$$

Proof. For the first assertion note that

$$\{\varphi_A = k\} = \{\varphi_A \circ T^j = k - j\} \setminus \{\varphi_A \leq j\}. \tag{2.21}$$

The second follows from

$$\{\varphi_A \leq j\} \cap \{\varphi_A \circ T^j = m\} = \biguplus_{i=1}^{j} T^{-i}(A \cap \{\varphi_A = m + j - i\}) \tag{2.22}$$

and μ-invariance. □

Theorem 2.1.2. *Let $(X, \mathcal{A}, \mu, T)$ be an ergodic, probability-preserving system, and $(A_l)_{l\geq 1}$ be a sequence of asymptotically rare events. Let F :*

$[0,\infty) \to [0,1]$ be a strictly monotone increasing sub-probability function of class C^1 such that we have convergence in law of the first hitting times to F

$$\mu(\mu(A_l)\varphi_{A_l} \le t) \Rightarrow F(t) \quad \textit{as } l \to \infty. \tag{2.23}$$

Let $\mathcal{U} \subset \mathcal{D}(\mu) \cap L^\infty$ be bounded in L^∞ and such that there is a $p \in [1,\infty]$ and a sequence $(g_n)_{n\ge 1}$ with $\frac{g_n}{n} \to \infty$ as $n \to \infty$ such that

$$||\hat{T}^n(u) - 1||_{L^p} = o\left(\left(\frac{1}{g_n}\right)^{\frac{1}{p}}\right) \quad \textit{as } n \to \infty \textit{ uniformly in } u \in \mathcal{U}, \tag{2.24}$$

where for convenience $\frac{1}{\infty} := 0$. Then for all sequences $(k_l)_{l\ge 1}$ of natural numbers such that $(\mu(A_l)k_l)_{l\ge 1}$ is bounded away from 0 and ∞ we have

$$u \odot \mu(\varphi_{A_l} = k_l) \sim \mu(A_l)F'(\mu(A_l)k_l) \quad \textit{as } l \to \infty \textit{ uniformly in } u \in \mathcal{U}. \tag{2.25}$$

Proof. (i) Fix a sequence $(k_l)_{l\ge 1}$ as in the statement of the Theorem. By Theorem 2.1.1 we obtain

$$\mu(\varphi_{A_l} = k_l) \sim \mu(A_l)F'(\mu(A_l)k_l) \quad \textit{as } l \to \infty. \tag{2.26}$$

(ii) Define $j_l = \min(n > 0 \mid g_n \ge \frac{1}{\mu(A_l)})$, then $j_l \to \infty$ as $l \to \infty$, while $j_l = o(\frac{1}{\mu(A_l)})$, we may assume that $j_l < k_l$ for $l \ge 1$. Then $\kappa_l = k_l - j_l$ satisfies $\kappa_l \sim k_l$. As $(\mu(A_l)k_l)_{l\ge 1}$ and $(\mu(A_l)\kappa_l)_{l\ge 1}$ are bounded away from 0 and ∞ and $\tilde{F}$ is uniformly continuous on compact sets we have $|\tilde{F}(\mu(A_l)k_l) - \tilde{F}(\mu(A_l)\kappa_l)| \to 0$ as $l \to \infty$.

(iii) We first show

$$\left|\int_X (1_{\{\varphi_{A_l}=\kappa_l\}} \circ T^{j_l} - 1_{\{\varphi_{A_l}=k_l\}})u \,\mathrm{d}\mu\right| = o(\mu(A_l)) \quad \substack{\textit{as } l\to\infty \\ \textit{uniformly in } u\in\mathcal{U}.} \tag{2.27}$$

Using Lemmata 2.1.1 and 2.1.2 and letting $C := \sup_{u\in\mathcal{U}} ||u||_{L^\infty}$ we get

$$\begin{aligned}
&\left|\int_X (1_{\{\varphi_{A_l}=\kappa_l\}} \circ T^{j_l} - 1_{\{\varphi_{A_l}=k_l\}})u \,\mathrm{d}\mu\right| \\
&\le C\mu(\{\varphi_{A_l} \le j_l\} \cap \{\varphi_{A_l} \circ T^{j_l} = \kappa_l\}) \\
&= C\mu(A_l)\mu_{A_l}(\kappa_l \le \varphi_{A_l} < k_l) \\
&= C\mu(A_l)\underbrace{\big(\underbrace{\mu_{A_l}(\varphi_{A_l} < k_l)}_{\approx \tilde{F}(\mu(A_l)k_l))} - \underbrace{\mu_{A_l}(\varphi_{A_l} < \kappa_l)}_{\approx \tilde{F}(\mu(A_l)\kappa_l)}\big)}_{\to 0} \\
&= o(\mu(A_l))
\end{aligned}$$

as $l \to \infty$ uniformly in $u \in \mathcal{U}$.

(iv) Next we check that

$$\left|\int_X (1_{\{\varphi_{A_l}=\kappa_l\}} \circ T^{j_l})(1-u)\,\mathrm{d}\mu\right| = o(\mu(A_l)) \quad \underset{\text{uniformly in } u\in\mathcal{U}.}{\text{as } l\to\infty} \tag{2.28}$$

Let q be the Hölder-conjugate of p, i.e the unique real number such that $\frac{1}{p}+\frac{1}{q}=1$ (by our convention, the Hölder-conjugate to 1 is ∞). Using Theorem 2.1.1, and Hölder's inequality if $p\in(1,\infty)$, we get

$$\begin{aligned}
&\left|\int_X (1_{\{\varphi_{A_l}=\kappa_l\}} \circ T^{j_l})(1-u)\,\mathrm{d}\mu\right| \\
&= \left|\int_X 1_{\{\varphi_{A_l}=\kappa_l\}}(1-\hat{T}^{j_l}(u))\,\mathrm{d}\mu\right| \\
&\le \underbrace{||1_{\{\varphi_{A_l}=\kappa_l\}}||_{L^q}}_{=O\left(\mu(A_l)^{\frac{1}{q}}\right)} \quad \underbrace{||1-\hat{T}^{j_l}(u)||_{L^p}}_{=o\left(\frac{1}{g_{l_j}^{\frac{1}{p}}}\right)=o\left(\mu(A_l)^{\frac{1}{p}}\right)} \\
&= o(\mu(A_l))
\end{aligned}$$

as $l\to\infty$ uniformly in $u\in\mathcal{U}$.

(v) Altogether we need to check

$$|u\odot\mu(\varphi_{A_l}=k_l)-\mu(\varphi_{A_l}=k_l)| = o\left(\mu(A_l)\right) \tag{2.29}$$

as $l\to\infty$ uniformly in $u\in\mathcal{U}$. But the left hand side is bounded above by

$$\begin{aligned}
&\left|u\odot\mu(\varphi_{A_l}=k_l)-u\odot\mu(\varphi_{A_l}\circ T^{j_l}=\kappa_l)\right| \\
&+\left|u\odot\mu(\varphi_{A_l}\circ T^{j_l}=\kappa_l)-\mu(\varphi_{A_l}\circ T^{j_l}=\kappa_l)\right| \\
&+\left|\mu(\varphi_{A_l}\circ T^{j_l}=\kappa_l)-\mu(\varphi_{A_l}=k_l)\right|,
\end{aligned}$$

and step (iii) confirms that the first and the third term are of order $o(\mu(A_l))$ as $l\to\infty$ uniformly in $u\in\mathcal{U}$, while by step (iv) the same is true for the middle term.

□

Remark 2.1.1. *The condition that there is a sequence $(g_n)_{n\ge1}$ with $\frac{g_n}{n}\to\infty$ as $n\to\infty$ such that*

$$||\hat{T}^n(u)-1||_{L^p} = o\left(\left(\frac{1}{g_n}\right)^{\frac{1}{p}}\right) \tag{2.30}$$

was only chosen for technical purposes. We can without much work substitute it by

$$||\hat{T}^n(u)-1||_{L^p} = o\left(\left(\frac{1}{n}\right)^{\frac{1}{p}}\right) \tag{2.31}$$

by the next lemma.

Lemma 2.1.3. *Let $(f_n)_{n\geq 1}$ and $(g_n)_{n\geq 1}$ be sequences of positive real numbers such that $f_n = o(g_n)$ as $n \to \infty$. Then there is a sequence of positive real numbers $(g'_n)_{n\geq 1}$ with $g'_n = o(g_n)$ and $f_n = o(g'_n)$ as $n \to \infty$.*

Proof. Since $f_n = o(g_n)$ as $n \to \infty$, there is a sequence of positive real numbers $(a_n)_{n\geq 1}$ such that $f_n = a_n g_n$ and $a_n \to 0$ as $n \to \infty$. Therefore setting $g'_n := \sqrt{a_n} g_n$ we obtain such a sequence. □

Now that we know that local limit theorems for the first hitting time are automatic from convergence in law, it would be nice if the same were true for the first return time. However, it turns out that the situation there is much more involved, and we cannot expect local limit theorems for the return times as easily. For example in periodic situations as before, counterexamples are easily constructed. Even more, for any sequence $(A_l)_{l\geq 1}$ as above, there are sequences of asymptotically rare events $(B_l)_{l\geq 1}$ that fail local limit theorems for return times.

Proposition 2.1.1. *Let $(X, \mathcal{A}, \mu, T)$ be an ergodic, probability-preserving system, and $(A_l)_{l\geq 1}$ be a sequence of asymptotically rare events. Let $F : [0, \infty) \to [0, 1]$ be a strictly monotone increasing sub-probability function of class C^1 such that we have convergence in law of the first hitting times to F*

$$\mu(\mu(A_l)\varphi_{A_l} \leq t) \Rightarrow F(t) \quad \textit{as } l \to \infty. \tag{2.32}$$

Then for every sequence $(k_l)_{l\geq 1}$ of natural numbers such that $(\mu(A_l)k_l)_{l\geq 1}$ is bounded away from 0 and ∞ there are measurable $B_l \subset A_l$ for $l \geq 1$ such that $\mu(A_l \setminus B_l) = o(\frac{1}{\mu(A_l)}$ and

$$\mu(\mu(B_l)\varphi_{B_l} \leq t) \Rightarrow F(t) \quad \textit{as } l \to \infty, \tag{2.33}$$

while

$$\mu_{B_l}(\varphi_{B_l} = k_l) = 0 \quad \textit{for } l \geq 1. \tag{2.34}$$

Proof. For $l \geq 1$ we set $B_l := A_l \setminus \{\varphi_{A_l} = k_l\}$. Then $\varphi_{B_l} \neq k_l$ on B_l, furthermore $\varphi_{B_l} = \varphi_{A_l}$ on $B_l \setminus T_{A_l}^{-1}\{\varphi_{A_l} = k_l\}$. Since

$$\mu_{A_l}(\varphi_{A_l} = k_l) \to 0 \quad \textit{as } l \to \infty \tag{2.35}$$

all claims follow easily. (see Remark 2.3 in [Zwe16]) □

Nevertheless, we can still prove that LLTs for all suitable limits exist. Therefore define

$$\mathcal{F} := \left\{ F : \mathbb{R} \to [0,1) \;\middle|\; \begin{array}{l} F \textit{ is a distribution function with } F|_{(-\infty,0)} \equiv 0 \\ F|_{(0,\infty)} \textit{ is strictly increasing, of class } C^1 \\ \int_0^\infty 1 - F(t)\,\mathrm{d}t \leq 1 \end{array} \right\} \tag{2.36}$$

Our goal, for $F \in \mathcal{F}$, will be to construct a sequence of asymptotically rare events $(A_l)_{l\geq 1}$ such that for all sequences $(k_l)_{l\geq 1}$ in $\mathbb{N}$ such that $(\mu(A_l)k_l)_{l\geq 1}$ is bounded away from 0 and ∞, we have

$$\mu_{A_l}(\varphi_{A_l} = k_l) \sim F'(\mu(A_l)k_l)\mu(A_l) \quad \text{as } l \to \infty. \tag{2.37}$$

The core of the argument will be a slight variation of the construction in Proposition 0.6 of [Zwea]. We call a distribution function (d.f) $G : [0, \infty) \to [0, 1]$ rational if it only has finitely many jumps of rational height at rational times, and G is locally constant at all other times. Also recall the Lèvy-distance of two d.f on $[0, \infty)$

$$d_{\mathcal{L}}(F, G) := \inf\{\epsilon > 0 \mid G(t-\epsilon)-\epsilon \leq F(t) \leq G(t+\epsilon)+\epsilon \text{ for } t \in \geq 0\}. \tag{2.38}$$

Lastly denote by F_A the d.f of the (normalized) return times to A, i.e

$$F_A(t) = \mu_A(\mu(A)\varphi_A \leq t) \tag{2.39}$$

for $t \geq 0$. A variation of Proposition 0.6 of [Zwea] shows the following.

Proposition 2.1.2. *Let $(X, \mathcal{A}, \mu, T)$ be an aperiodic, ergodic, probability-preserving system. Let G be a rational d.f on $[0, \infty)$ with $\int_0^\infty 1-G(t)\,\mathrm{d}t = 1$, and let $\epsilon > 0$. Then, for any $M \in \mathbb{N}$ such that the jump times of G can be written as rational numbers with denominator M, there is a measurable $A \in \mathcal{A}$ such that*

$$\frac{1}{M}(1-\epsilon) < \mu(A) < \frac{1}{M}(1+\epsilon) \tag{2.40}$$

and

$$d_{\mathcal{L}}(F_A, G) < M^{-3}. \tag{2.41}$$

More precisely, if G has jumps at times $\frac{r(k)}{M}$ for $k = 1, ..., K$ and some natural numbers $K, r(1), ..., r(K)$, then F_A has jumps at times $\frac{r(k)}{M}$ for $k = 1, ..., K$ (it might have more). The heights of corresponding jumps differs by at most M^{-3}.

With this we can show the following (compare also [KL05] and [Lac02]).

Theorem 2.1.3. *Let $(X, \mathcal{A}, \mu, T)$ be an aperiodic, ergodic, probability-preserving system, and $F \in \mathcal{F}$. Then there is a sequence of asymptotically rare events $(A_l)_{l\geq 1}$ such that*

$$\mu_{A_l}(\mu(A_l)\varphi_{A_l} \leq t) \Rightarrow \tilde{F}(t) \quad \text{as } l \to \infty, \tag{2.42}$$

and for all sequences $(k_l)_{l\geq 1}$ in $\mathbb{N}$ such that $(\mu(A_l)k_l)_{l\geq 1}$ is bounded away from 0 and ∞, we have

$$\mu_{A_l}(\varphi_{A_l} = k_l) \sim F'(-\mu(A_l)k_l)\mu(A_l) \quad \text{as } l \to \infty. \tag{2.43}$$

Proof. (i) We first approximate F by rational distribution functions. For $n \geq 1$ define

$$s_n = \min\left(n, F^{-1}\left(1 - \frac{2}{n^2}\right), \sup\{s > 0 \mid \int_0^s 1 - F(t)\,\mathrm{d}t \leq 1 - \frac{1}{n}\}\right), \tag{2.44}$$

and note that $s_n \to \infty$ as $n \to \infty$. For $n \geq 1$ big enough such that $ns_n > 2$ define rational d.f $G_n : [0, \infty) \to [0, 1]$ by $G_n(0) = 0$ and

- for $t = \frac{k}{n}$ and $k = 1, ..., \lfloor ns_n \rfloor - 1$ set

$$G_n(\frac{k}{n}) := n^{-2}\left\lceil n^2 F\left(\frac{k+1}{n}\right)\right\rceil, \tag{2.45}$$

- for $k = 1, ..., \lfloor ns_n \rfloor - 1$ and $\frac{k}{n} < t < \frac{k+1}{n}$ set $G_n(t) = G_n(\frac{k}{n})$.
- Since $G_n \geq F$ on $[\frac{1}{n}, \frac{\lfloor ns_n \rfloor}{n}]$ we have

$$\int_0^{\frac{\lfloor ns_n \rfloor}{n}} 1 - G_n(t)\,\mathrm{d}t \leq \frac{1}{n} + \int_0^{s_n} 1 - F(t) \leq 1.$$

At the same time $\int_0^{\frac{\lfloor ns_n \rfloor}{n}} 1 - G_n(t)\,\mathrm{d}t$ is a rational number with (not necessarily least) denominator n^3. So we can write $\int_0^{\frac{\lfloor ns_n \rfloor}{n}} 1 - G_n(t)\,\mathrm{d}t = 1 - \frac{L}{n^3}$ for some $L \in \{0, ..., n^3\}$. Finally set $G_n(t) = 1 - \frac{1}{n^2}$ for $\frac{\lfloor ns_n \rfloor}{n} \leq t < \frac{\lfloor ns_n \rfloor}{n} + \frac{L}{n}$, and $G_n(t) = 1$ for $t \geq \frac{\lfloor ns_n \rfloor}{n} + \frac{L}{n}$.

For $n \geq 1$ it can be easily checked that G_n is rational, $G_n \Rightarrow F$ as $n \to \infty$, and $\int_0^\infty 1 - G_n(t)\,\mathrm{d}t = 1$.

(ii) We can apply Proposition 2.1.2 to find $A_n \in \mathcal{A}$ such that

$$\frac{1}{n}(1 - \frac{1}{n^3}) < \mu(A_n) < \frac{1}{n}(1 + \frac{1}{n^3}) \tag{2.46}$$

and

$$d_{\mathcal{L}}(F_{A_n}, G) < n^{-3}. \tag{2.47}$$

F_{A_n} has jumps at times $k\mu(A_n)$ for $k = 2, ..., \lfloor ns_n \rfloor - 1$. The height J_n^k of the jump at time $k\mu(A_n)$ satisfies

$$n^{-2}\left(\left\lceil n^2 F\left(\frac{k+1}{n}\right)\right\rceil - \left\lceil n^2 F\left(\frac{k}{n}\right)\right\rceil\right) - n^{-3} < J_n^k$$
$$n^{-2}\left(\left\lceil n^2 F\left(\frac{k+1}{n}\right)\right\rceil - \left\lceil n^2 F\left(\frac{k}{n}\right)\right\rceil\right) + n^{-3} > J_n^k$$

It follows that $F_{A_n} \Rightarrow F$ as $n \to \infty$.

(iii) Now we check (2.43) for our sequence $(A_n)_{n\geq 1}$. By considering a subsequence, we may assume that $\mu(A_n)k_n \to t$ as $n \to \infty$, for some $t > 0$. Choose $n \geq 1$ big enough such that $\lfloor ns_n \rfloor - 1 \geq k_n \geq 2$. Then we get

$$\begin{aligned}
&\frac{1}{\mu(A_n)}\mu_{A_n}(\varphi_{A_n} = k_n) = \frac{1}{\mu(A_n)}(F_{A_n}(\mu(A_n)k_n) - F_{A_n}(\mu(A_n)(k_n-1))) \\
&= \frac{1}{\mu(A_n)}J_n^{k_n} \leq \frac{1}{\mu(A_n)}\left(n^{-2}\left(\left\lceil n^2 F\left(\frac{k_n+1}{n}\right)\right\rceil - \left\lceil n^2 F\left(\frac{k_n}{n}\right)\right\rceil\right) + \frac{1}{n^3}\right) \\
&\leq \frac{1}{\mu(A_n)}\left(F\left(\frac{k_n+1}{n}\right) - F\left(\frac{k_n}{n}\right) + \frac{1}{n^2} + \frac{1}{n^3}\right) \\
&\leq \frac{1}{\mu(A_n)}\left(\frac{1}{n}F'(\xi_n) + \frac{1}{n^2} + \frac{1}{n^3}\right) \leq \left(\frac{1}{1-\frac{1}{n}}\right)\left(F'(\xi_n) + \frac{1}{n} + \frac{1}{n^2}\right)
\end{aligned}$$

for some $\xi_n \in \left(\frac{k_n}{n}, \frac{k_n+1}{n}\right)$, where we used the Mean Value Theorem in the penultimate inequality. Taking $n \to \infty$ on both sides, we get $\limsup_{n\to\infty} \frac{1}{\mu(A_n)}\mu_{A_n}(\varphi_{A_n} = k_n) \leq F'(t)$. A similar argument gives $\liminf_{n\to\infty} \frac{1}{\mu(A_n)}\mu_{A_n}(\varphi_{A_n} = k_n) \geq F'(t)$. As both sides are positive, this shows (2.43).

□

Despite this cautioning result, we can still give sufficient conditions for local limit theorems the the first return time. Much like in our short analysis of convergence in law, if the densities of the conditioned measures $\frac{1_{A_l}}{\mu(A_l)}$ are in a good set as described in Theorem 2.1.2 we can relate the local behaviour of return times to that of hitting times. From now on we will mainly consider the case of exponential limit laws $F(t) = 1 - e^{-t}$.

Theorem 2.1.4. *Let $(X, \mathcal{A}, \mu, T)$ be an ergodic, probability-preserving system, and $(A_l)_{l\geq 1}$ be a sequence of asymptotically rare events such that*

$$\mu(\mu(A_l)\varphi_{A_l} \leq t) \Rightarrow 1 - e^{-t} \quad \text{as } l \to \infty. \tag{2.48}$$

Let $\mathcal{U} \subset \mathcal{D}(\mu) \cap L^\infty$ be bounded in L^∞ and such that there is a $p \in [1, \infty]$ such that

$$||\hat{T}^n(u) - 1||_{L^p} = o\left(\left(\frac{1}{n}\right)^{\frac{1}{p}}\right) \quad \text{as } n \to \infty \text{ uniformly in } u \in \mathcal{U}, \tag{2.49}$$

where for convenience $\frac{1}{\infty} := 0$. Assume that there are constants $\tau_l \in \mathbb{N}_0$ such that

$$\mu(A_l)\tau_l \to 0 \quad \text{as } l \to \infty \tag{2.50}$$

while

$$\hat{T}^{\tau_l}(\mu(A_l)^{-1}1_{A_l}) \in \mathcal{U} \quad \text{for } l \geq 1. \tag{2.51}$$

Let $(k_l)_{l\geq 1}$ *be a sequence in* $\mathbb{N}$ *such that* $(\mu(A_l)k_l)_{l\geq 1}$ *is bounded away from* 0 *and* ∞*. If*

$$\mu_{A_l}(\{\varphi_{A_l} \leq \tau_l\} \cap T^{-k_l} A_l) = o(\mu(A_l)) \quad \text{as } l \to \infty \tag{2.52}$$

then

$$\mu_{A_l}(\varphi_{A_l} = k_l) \sim e^{-\mu(A_l)k_l}\mu(A_l) \quad \text{as } l \to \infty. \tag{2.53}$$

Proof. (i) Define $\kappa_l := k_l - \tau_l$, then we have that $\kappa_l \sim k_l$ and therefore $e^{-\mu(A_l)k_l} \sim e^{-\mu(A_l)\kappa_l}$. As $(\mu(A_l)\kappa_l)_{l\geq 1}$ is still bounded away from 0 and ∞, Theorem 2.1.2 gives us

$$u \odot \mu(\mu(A_l)\varphi_{A_l} = \kappa_l) \sim \mu(A_l)e^{-(\mu(A_l)k_l} \quad \substack{\text{as } l\to\infty \\ \text{uniformly in } u\in\mathcal{U}.} \tag{2.54}$$

(ii) Setting $u_l := \hat{T}^{\tau_l}(\mu(A_l)^{-1}1_{A_l})$, by (2.54) we only need to show

$$|\mu_{A_l}(\varphi_{A_l} = k_l) - u_l \odot \mu(\varphi_{A_l} = \kappa_l)| = o(\mu(A_l)) \quad \text{as } l \to \infty. \tag{2.55}$$

By Lemma 2.1.2 the left-hand side coincides with

$$\begin{aligned} &\left| \int_X (1_{\{\varphi_{A_l}=\kappa_l\}} \circ T^{\tau_l} - 1_{\{\varphi_{A_l}=k_l\}} \,\mathrm{d}\mu_{A_l} \right| \\ &= \mu_{A_l}(\{\varphi_{A_l} \leq \tau_l\} \cap \{\varphi_{A_l} \circ T^{\tau_l} = \kappa_l\}) \\ &\leq \mu_{A_l}(\{\varphi_{A_l} \leq \tau_l\} \cap T^{-k_l} A_l) \\ &= o(\mu(A_l)). \end{aligned}$$

□

2.2 Consecutive times

We now look for local limit theorems for consecutive hitting or return times of the form

$$\mu\left(\bigcap_{j=0}^{d-1} T_{A_l}^{-j}\{\varphi_{A_l} = k_l^{(j)}\}\right) \sim e^{-\mu(A_l)\sum_{j=0}^{d-1} k_l^{(j)}}\mu(A_l)^d \quad \text{as } l \to \infty \tag{2.56}$$

or

$$\mu_{A_l}\left(\bigcap_{j=0}^{d-1} T_{A_l}^{-j}\{\varphi_{A_l} = k_l^{(j)}\}\right) \sim e^{-\mu(A_l)\sum_{j=0}^{d-1} k_l^{(j)}}\mu(A_l)^d \quad \text{as } l \to \infty \tag{2.57}$$

for suitable $A_l \in \mathcal{A}$ and $k_l \in \mathbb{N}$. First notice that for any $f \in L^1(\mu)$, $g \in L^\infty(\mu)$ and $j \geq 1$ we have

$$\hat{T}^j(f \cdot g \circ T^j) = \hat{T}^j(f) \cdot g. \tag{2.58}$$

This is because

$$\begin{aligned}(\hat{T}^j(f)\cdot g)\odot\mu &= g\odot(\hat{T}^j(f)\odot\mu)\\ &= g\odot((f\odot\mu)\circ T^{-j})\\ &= ((g\circ T^j)\odot(f\odot\mu))\circ T^{-j}\\ &= ((f\cdot g\circ T^j)\odot\mu)\circ T^{-j}.\end{aligned}$$

Theorem 2.2.1. *Let T be an ergodic measure-preserving map on the probability space $(X,\mathcal{A},\mu)$ and $(A_l)_{l\geq 1}$ be a sequence in $\mathcal{A}$ such that $0<\mu(A_l)\to 0$ such that the normalised return times converge to the exponential law*

$$\mu_{A_l}(\varphi_{A_l}\leq t)\Rightarrow 1-e^{-t}. \tag{2.59}$$

Suppose $\mathcal{U}\subset\mathcal{D}(\mu)\cap L_\infty(\mu)$ is bounded in $L_\infty(\mu)$ such that there is a $p\in[1,\infty]$ such that

$$||\hat{T}^n(u)-1||_{L^p}=o\left(\left(\frac{1}{n}\right)^{\frac{1}{p}}\right)\quad \textit{as } n\to\infty \textit{ uniformly in } u\in\mathcal{U}, \tag{2.60}$$

where for convenience $\frac{1}{\infty}:=0$.
Assume there are families $\bar{\mathcal{U}}_l\subset\mathcal{D}(\mu_{A_l})\cap L_\infty(\mu)$ are such that $\bar{u}=\bar{u}\cdot 1_{A_l}$ for $\bar{u}\in\bar{\mathcal{U}}_l$, $1_{A_l}\in\bar{\mathcal{U}}_l$ and $\bigcup_{l\geq 1}\bar{\mathcal{U}}_l$ is bounded in $L_\infty(\mu)$. Assume that there are constants $\tau_l\in\mathbb{N}_0$ such that

$$\mu(A_l)\tau_l\to 0\quad \textit{as } l\to\infty \tag{2.61}$$

while

$$\hat{T}^{\tau_l}(\mu(A_l)^{-1}\bar{u}_l)\in\mathcal{U}\quad \textit{for } l\geq 1 \textit{ and } \bar{u}_l\in\bar{\mathcal{U}}_l. \tag{2.62}$$

Furthermore assume that

$$\mu_{A_l}(\{\varphi_{A_l}\leq\tau_l\}\cap T^{-k_l}B_l)=o(\mu(B_l))\quad \textit{as } l\to\infty \tag{2.63}$$

for any sequences $B_l\in A_l\cap\mathcal{A}$ for $l\geq 1$ and $(k_l)_{l\geq 1}$ such that $(\mu(A_l)k_l)_{l\geq 1}$ is bounded away from 0 and ∞. Last assume that for all $k,l\geq 1$ we have

$$\hat{T}^k\left(\frac{\mu(A_l)1_{\{\varphi_{A_l}=k\}}u}{u\odot\mu(\varphi_{A_l}=k)}\right)\in\bar{\mathcal{U}}_l\quad \textit{for } u\in\mathcal{U}. \tag{2.64}$$

Then, for $d\geq 1$ and sequences $(k_l^{(0)})_{l\geq 1},\ldots,(k_l^{(d-1)})_{l\geq 1}$ of natural numbers such that $(\mu(A_l)k_l^{(j)})_{l\geq 1}$ is bounded away from 0 and ∞, we have that

$$\mu_{A_l}\left(\bigcap_{j=0}^{d-1}T_{A_l}^{-j}\{\varphi_{A_l}=k_l^{(j)}\}\right)\sim e^{-\mu(A_l)\sum_{j=0}^{d-1}k_l^{(j)}}\mu(A_l)^d\quad \textit{as } l\to\infty. \tag{2.65}$$

In (2.65) the sequence (μ_{A_l}) of measures on the left can be replaced by $(\bar{u}_l\odot\mu(A_l))$ for any $\bar{u}_l\in\bar{\mathcal{U}}_l$ or by $(u_l\odot\mu)$ for any $u_l\in\mathcal{U}$.

Proof. We will focus on the case of $(\bar{u}_l \odot \mu(A_l))_{l\geq 1}$, the proof for $(u_l \odot \mu)_{l\geq 1}$ goes analogous. We henceforth fix $d \geq 1$ (but not $\bar{u}_l \in \bar{\mathcal{U}}_l$ for $l \geq 1$ and $(k_l^{(0)})_{l\geq 1}, ..., (k_l^{(d-1)})_{l\geq 1}$).

(i) Theorem 2.1.2 shows that

$$u \odot \mu(\varphi_{A_l} = k_l) \sim \mu(A_l)e^{-\mu(A_l)k_l} \quad as\ l \to \infty \tag{2.66}$$

uniformly in $u \in \mathcal{U}$ and for (k_l) as above. An argument similar to the proof of Theorem 2.1.4 shows that whenever $\bar{u}_l \in \bar{\mathcal{U}}_l$ for $l \geq 1$ we get

$$\bar{u}_l \odot \mu_{A_l}(\varphi_{A_l} = k_l) \sim \mu(A_l)e^{-\mu(A_l)k_l} \quad as\ l \to \infty. \tag{2.67}$$

In the following, without saying it, statements including the $k_l^{(j)}$ and $\bar{u}_l$ are to be understood as statements about all choices of $k_l^{(j)}$ and $\bar{u}_l$ as above.

(ii) We set $s_l^{(m)} := \sum_{j=0}^{m-1} k_l^{(j)}$ for $1 \leq m \leq d$ and $s_l^{(0)} := 0$ the time of the m-th return to A_l (on the set we are interested in), $\kappa_l^{(m)} := k_l^{(m)} - \tau_l$ and

$$W_l^{(m)} := A_l \cap \bigcap_{j=0}^{m-1} T_{A_l}^{-j}\{\varphi_{A_l} = k_l^{(j)}\} = A_l \cap \bigcap_{j=0}^{m-1} T^{-s_l^{(j)}}\{\varphi_{A_l} = k_l^{(j)}\} \tag{2.68}$$

for $0 \leq m \leq d$ and $W_l^{(0)} := A_l$.

(iii) The main goal is to prove

$$\bar{u}_l \odot \mu_{A_l}(W_l^{(m)}) \sim e^{-\mu(A_l)s_l^{(m)}}\mu(A_l)^m \tag{$\clubsuit_m$}$$

for $0 \leq m \leq d$, $\bar{u}_l \in \bar{\mathcal{U}}_l$ for $l \geq 1$ and $(k_l^{(0)})_{l\geq 1}, ..., (k_l^{(d-1)})_{l\geq 1}$ as above. For $m = 1$ this is done by (2.67), the case $m = 0$ is trivial.

(iv) To see what we will need for the induction step, note that since $W_l^{(m+1)} = W_l^{(m)} \cap T^{s_l^{(m)}}\{\varphi_{A_l} = k_l^{(m)}\}$ we get

$$\bar{u}_l \odot \mu_{A_l}(W_l^{(m+1)}) = \bar{u}_l \odot \mu_{A_l}(W_l^{(m)})\eta_l^{(m)}(\varphi_{A_l} = k_l^{(m)}) \tag{2.69}$$

where

$$\begin{aligned}\eta_l^{(m)} &= (\bar{u}_l \odot \mu_{A_l})_{W_l^{(m)}} \circ T^{-s_l^{(m)}} \\ &= \left(\left(\mu(A_l)^{-1}\bar{u}_l \frac{1_{W_l^{(m)}}}{\bar{u}_l \odot \mu_{A_l}(W_l^{(m)})}\right) \odot \mu\right) \circ T^{-s_l^{(m)}} \\ &= \left(\mu(A_l)^{-1}\hat{T}^{s_l^{(m)}}\left(\underbrace{\bar{u}_l \frac{1_{W_l^{(m)}}}{\bar{u}_l \odot \mu_{A_l}(W_l^{(m)})}}_{g_l^{(m)}}\right)\right) \odot \mu\end{aligned}$$

So we need to show

$$\eta_l^{(m)}(\varphi_{A_l} = k_l^{(m)}) \sim \mu(A_l)e^{-\mu(A_l)k_l^{(m)}} \tag{2.70}$$

that is, as $\operatorname{supp}\left(\hat{T}^{s_l^{(m)}}\left(\bar{u}_l \frac{1_{W_l^{(m)}}}{\bar{u}_l \odot \mu_{A_l}(W_l^{(m)})}\right)\right) \subset A_l$, by (2.67) we need

$$\hat{T}^{s_l^{(m)}} g_l^{(m)} = c_l^{(m)} \bar{f}_l^{(m)} - \bar{h}_l^{(m)} \tag{$\spadesuit_m$}$$

where $\bar{f}_l^{(m)} \in \bar{\mathcal{U}}_l$, the $c_l^{(m)}$ are constants $c_l^{(m)} \to 1$ as $l \to \infty$ and $\bar{h}_l^{(m)} \in L^1(\mu_{A_l})$ is such that $\bar{h}_l^{(m)} \odot \mu_{A_l}(\varphi_{A_l} = k_l^{(m)}) = o(\mu(A_l))$.

(v) As $g_l^{(0)} = \bar{u}_l$ and $s_l^{(0)} = 0$ this is trivial for $m = 0$, moreover by Lemma 2.1.2 and (2.58)

$$\begin{aligned}&\hat{T}^{k_l^{(0)}} g_l^{(1)} = \hat{T}^{k_l^{(0)}}\left(\bar{u}_l \frac{1_{\{\varphi_{A_l}=k_l^{(0)}\}}}{\bar{u}_l \odot \mu_{A_l}(\varphi_{A_l} = k_l^{(0)})}\right) \\ &= \hat{T}^{k_l^{(0)}}\left(\mu(A_l)^{-1}\bar{u}_l \frac{\mu(A_l)1_{\{\varphi_{A_l}=\kappa_l^{(0)}\}} \circ T^{\tau_l}}{\bar{u}_l \odot \mu_{A_l}(\varphi_{A_l} = k_l^{(0)})} - \bar{u}_l \frac{1_{\{\varphi_{A_l}\le\tau_l\}\cap T^{-\tau_l}\{\varphi_{A_l}=\kappa_l^{(0)}\}}}{\bar{u}_l \odot \mu_{A_l}(\varphi_{A_l} = k_l^{(0)})}\right) \\ &= \underbrace{\frac{\bar{u}_l \odot \mu_{A_l}(\varphi_{A_l} = \kappa_l^{(0)})}{\bar{u}_l \odot \mu_{A_l}(\varphi_{A_l} = k_l^{(0)})}}_{\to 1 \quad as\ l\to\infty} \underbrace{\hat{T}^{\kappa_l^{(0)}}\left(\underbrace{\hat{T}^{\tau_l}\left(\mu(A_l)^{-1}\bar{u}_l\right)}_{\in\mathcal{U}} \frac{\mu(A_l)1_{\{\varphi_{A_l}=\kappa_l^{(0)}\}}}{\bar{u}_l \odot \mu_{A_l}(\varphi_{A_l} = \kappa_l^{(0)})}\right)}_{\in\bar{\mathcal{U}}_l} \\ &\quad \underbrace{- \hat{T}^{k_l^{(0)}}\left(\bar{u}_l \frac{1_{\{\varphi_{A_l}\le\tau_l\}\cap T^{-\tau_l}\{\varphi_{A_l}=\kappa_l^{(0)}\}}}{\bar{u}_l \odot \mu_{A_l}(\varphi_{A_l} = k_l^{(0)})}\right)}_{\bar{h}_l^{(1)}}.\end{aligned}$$

Now $\bar{h}_l^{(1)} \le \frac{M}{\bar{u}_l \odot \mu_{A_l}(\varphi_{A_l}=k_l^{(0)})}\hat{T}^{k_l^{(0)}}(1_{\{\varphi_{A_l}\le\tau_l\}\cap T^{-\tau_l}\{\varphi_{A_l}=\kappa_l^{(0)}\}})$, where $M > 0$ is a constant such that $||\bar{u}_l||_{L^\infty} \le M$ for $\bar{u}_l \in \bar{U}_l$ and $l \ge 1$. So by

($\clubsuit_1$) it is enough to show that
$\hat{T}^{k_l^{(0)}}(\mu(A_l)^{-1}1_{\{\varphi_{A_l}\le\tau_l\}\cap T^{-\tau_l}\{\varphi_{A_l}=\kappa_l^{(0)}\}})\odot\mu(A_l\cap\{\varphi_{A_l}=k_l^{(1)}\})=o(\mu(A_l)^2)$.
But

$$\begin{aligned}
&\hat{T}^{k_l^{(0)}}(\mu(A_l)^{-1}1_{\{\varphi_{A_l}\le\tau_l\}\cap T^{-\tau_l}\{\varphi_{A_l}=\kappa_l^{(0)}\}})\odot\mu(A_l\cap\{\varphi_{A_l}=k_l^{(1)}\})\\
&=\mu_{A_l}(\{\varphi_{A_l}\le\tau_l\}\cap T^{-\tau_l}\{\varphi_{A_l}=\kappa_l^{(0)}\}\cap T^{-k_l^{(0)}}(A_l\cap\{\varphi_{A_l}=k_l^{(1)}\}))\\
&\le\mu_{A_l}(\{\varphi_{A_l}\le\tau_l\}\cap T^{-k_l^{(0)}}(A_l\cap\{\varphi_{A_l}=k_l^{(1)}\})).
\end{aligned}$$

So by (2.63) it is enough to show $\mu(A_l\cap\{\varphi_{A_l}=k_l^{(1)}\})=O(\mu(A_l)^2)$, but this holds once again by ($\clubsuit_1$) for $\bar{u}_l=1_{A_l}$ and $k_l^{(0)}=k_l^{(1)}$.

(vi) We need one more auxiliary condition, namely

$$\bar{h}_l^{(m)}\odot\mu_{A_l}\left(A_l\cap\bigcap_{i=0}^{j}T^{-s_{l,i}^{(m)}}\{\varphi_{A_l}=k_l^{(m+i)}\}\right)=o(\mu(A_l)^{j+1})\quad(\diamondsuit_{m,j})$$

for $s_{l,i}^{(m)}=\sum_{r=0}^{i-1}k_l^{(m+r)}$ and $j\ge 0$ with $j+m\le d-1$. Let us write ($\heartsuit_m$) for all of ($\diamondsuit_{i,j}$) for $0\le i\le m$, $j\ge 0$ and $i+j=m$. First we note that $\bar{h}_l^{(0)}=0$ by the above, so ($\diamondsuit_{0,j}$) is always true. Additionally ($\diamondsuit_{m,0}$) follows from ($\spadesuit_m$). So ($\heartsuit_0$) and ($\heartsuit_1$) hold.

(vii) Finally we are ready to formulate our induction step. Let $1\le m\le d-1$ and ($\spadesuit_0$), .., ($\spadesuit_m$), ($\clubsuit_0$), ..., ($\clubsuit_m$) and ($\heartsuit_0$), ..., ($\heartsuit_m$) hold. First (2.69) and (2.70) show that ($\spadesuit_m$) and ($\clubsuit_m$) imply ($\clubsuit_{m+1}$). We also claim ($\diamondsuit_{1,m}$). Following the analysis of $\bar{h}_l^{(1)}$ in (v), we see that we only need to show

$$\mu\left(A_l\cap\bigcap_{i=0}^{m}T^{-s_{l,i}^{(1)}}\{\varphi_{A_l}=k_l^{(1+i)}\}\right)=O(\mu(A_l)^{m+2}).\tag{2.71}$$

But as above this follows from (2.63) and ($\clubsuit_{m+1}$) (for $\bar{u}_l=1_{A_l}$ and $((k_l^{(0)}),...,(k_l^{(m)}))=((k_l^{(1)}),...,(k_l^{(m+1)})))$. For $1\le j\le m$ we have that ($\diamondsuit_{j,m-j+1}$) implies ($\diamondsuit_{j+1,m-j}$). Indeed

$$g_l^{(j+1)}=g_l^{(j)}1_{\{\varphi_{A_l}=k_l^{(j)}\}}\circ T^{s_l^{(j)}}C_l^{(j)}\tag{2.72}$$

where $C_l^{(j)}=\frac{\bar{u}_l\odot\mu_{A_l}(W_l^{(j)})}{\bar{u}_l\odot\mu_{A_l}(W_l^{(j+1)})}\sim e^{\mu(A_l)k_l^{(j)}}\mu(A_l)^{-1}$ by ($\clubsuit_j$) and ($\clubsuit_{j+1}$).

Therefore by (2.58), Lemma 2.1.2 and ($\spadesuit_j$) we get

$$\begin{aligned}
&\hat{T}^{s_l^{(j+1)}}(g_l^{(j+1)}) = C_l^{(j)}\hat{T}^{k_l^{(j)}}\left(\hat{T}^{s_l^{(j)}}\left(g_l^{(j)}1_{\{\varphi_{A_l}=k_l^{(j)}\}}\circ T^{s_l^{(j)}}\right)\right)\\
&= C_l^{(j)}\hat{T}^{k_l^{(j)}}\left(\left(c_l^{(j)}\bar{f}_l^{(j)} - \bar{h}_l^{(j)}\right)1_{\{\varphi_{A_l}=k_l^{(j)}\}}\right)\\
&= C_l^{(j)}\hat{T}^{k_l^{(j)}}\left(c_l^{(j)}\bar{f}_l^{(j)}1_{\{\varphi_{A_l}\circ T^{\tau_l}=\kappa_l^{(j)}\}}\right)\\
&- C_l^{(j)}\hat{T}^{k_l^{(j)}}\left(c_l^{(j)}\bar{f}_l^{(j)}1_{\{\varphi_{A_l}\le\tau_l\}}1_{\{\varphi_{A_l}\circ T^{\tau_l}=\kappa_l^{(j)}\}} + \bar{h}_l^{(j)}1_{\{\varphi_{A_l}=k_l^{(j)}\}}\right)\\
&= D_l^{(j)}\underbrace{\hat{T}^{\kappa_l^{(j)}}\left(\underbrace{\hat{T}^{\tau_l}\left(\bar{f}_l^{(j)}\mu(A_l)^{-1}\right)}_{\in\mathcal{U}}\frac{1_{\{\varphi_{A_l}=\kappa_l^{(j)}\}}\mu(A_l)}{\hat{T}^{\tau_l}\left(\bar{f}_l^{(j)}\mu(A_l)^{-1}\right)\odot\mu(\varphi_{A_l}=\kappa_j^{(j)})}\right)}_{\in\bar{\mathcal{U}}_l}\\
&\underbrace{- C_l^{(j)}\hat{T}^{k_l^{(j)}}\left(c_l^{(j)}\bar{f}_l^{(j)}1_{\{\varphi_{A_l}\le\tau_l\}}1_{\{\varphi_{A_l}\circ T^{\tau_l}=\kappa_l^{(j)}\}}\right)}_{h_{l,1}^{(j+1)}} + \underbrace{C_l^{(j)}\hat{T}^{k_l^{(j)}}\left(\bar{h}_l^{(j)}1_{\{\varphi_{A_l}=k_l^{(j)}\}}\right)}_{h_{l,2}^{(j+1)}}
\end{aligned}$$

where $D_l^{(j)} = C_l^{(j)}c_l^{(j)}\hat{T}^{\tau_l}\left(\bar{f}_l^{(j)}\mu(A_l)^{-1}\right)\odot\mu(\varphi_{A_l}=\kappa_j^{(j)}) \to 1$ by Theorem 2.1.2. Furthermore [1]

$$\begin{aligned}
&h_{l,1}^{(j+1)}\odot\mu_{A_l}\left(A_l\cap\bigcap_{i=0}^{m-j}T^{-s_{l,i}^{(m)}}\{\varphi_{A_l}=k_l^{(m+i)}\}\right)\\
&= C_l^{(j)}c_l^{(j)}\bar{f}_l^{(j)}\odot\mu_{A_l}\left(\begin{aligned}&A_l\cap\{\varphi_{A_l}\le\tau_l\}\cap\{\varphi_{A_l}\circ T^{\tau_l}=\kappa_l^{(j)}\}\\ &\cap T^{-k_l^{(j)}}\left(A_l\cap\bigcap_{i=0}^{m-j}T^{-s_{l,i}^{(m)}}\{\varphi_{A_l}=k_l^{(m+i)}\}\right)\end{aligned}\right)\\
&\le M\underbrace{C_l^{(j)}c_l^{(j)}}_{O(\mu(A_l)^{-1})}\underbrace{\mu_{A_l}\left(A_l\cap\{\varphi_{A_l}\le\tau_l\}\cap T^{-k_l^{(j)}}\left(\underbrace{A_l\cap\bigcap_{i=0}^{m-j}T^{-s_{l,i}^{(m)}}\{\varphi_{A_l}=k_l^{(m+i)}\}}_{\mu(\cdot)=O(\mu(A_l)^{m-j+2})}\right)\right)}_{o(\mu(A_l)^{m-j+2})}\\
&= o(\mu(A_l)^{m-j+1})
\end{aligned}$$

[1]In the third equation we use ($\clubsuit_{m-j+1}$) for $\bar{u}_l = 1_{A_l}$ and $(k_l^{(i)})_{l\ge1} = (k_l^{(m+i)})_{l\ge1}$ for $0\le i\le m-j+1$.

and

$$
\begin{aligned}
&h_{l,2}^{(j+1)} \odot \mu_{A_l}\left(A_l \cap \bigcap_{i=0}^{m-j} T^{-s_{l,i}^{(j+1)}}\{\varphi_{A_l} = k_l^{(j+1+i)}\}\right) \\
&= C_l^{(j)} \bar{h}_l^{(j)} \odot \mu_{A_l}\left(A_l \cap \{\varphi_{A_l} = k_l^{(j)}\} \cap T^{-k_l^{(j)}}\left(\bigcap_{i=0}^{m-j} T^{-s_{l,i}^{(j+1)}}\{\varphi_{A_l} = k_l^{(j+1+i)}\}\right)\right) \\
&= C_l^{(j)} \underbrace{\bar{h}_l^{(j)} \odot \mu_{A_l}\left(A_l \cap \bigcap_{i=0}^{m-j+1} T^{-s_{l,i}^{(j)}}\{\varphi_{A_l} = k_l^{(j+i)}\}\right)}_{o(\mu(A_l))^{m-j+2}} \\
&= o(\mu(A_l)^{m-j+1})
\end{aligned}
$$

by $(\diamondsuit_{j,m-j+1})$. Iterating this argument, we arrive at $(\heartsuit_{m+1})$. The same argument for $j = m$ shows $(\spadesuit_{m+1})$ finishing the proof.

□

2.3 An Example: Gibbs-Markov maps

We can apply our abstract analysis to the type of systems introduced in Chapter 1, Gibbs-Markov maps. As we will deal with a more general class (Rychlik-maps) later on, we will not give complete proofs. For a more thorough argument see Theorems 3.1 and 3.2 of [AZ].

Theorem 2.3.1. *Let $(X, \mathcal{A}, \mu, T, \zeta)$ be a mixing, probability preserving GM-map. Let $(A_l)_{l\geq 1}$ be asymptotically rare events such that either* ***a)*** *$A_l \in \sigma(\zeta)$ is a union of cylinder or* ***b)*** *there is a non-periodic point $x^* \in X$ such that $A_l \in \zeta_l$ is the unique l-cylinder containing x^*. Then, for $d \geq 1$ and sequences $(k_l^{(0)})_{l\geq 1}, ..., (k_l^{(d-1)})_{l\geq 1}$ of natural numbers such that $(\mu(A_l)k_l^{(j)})_{l\geq 1}$ is bounded away from 0 and ∞, we have that*

$$
\mu\left(\bigcap_{j=0}^{d-1} T_{A_l}^{-j}\{\varphi_{A_l} = k_l^{(j)}\}\right) \sim e^{-\mu(A_l)\sum_{j=0}^{d-1} k_l^{(j)}} \mu(A_l)^d \quad \text{as } l \to \infty \tag{2.73}
$$

and

$$
\mu_{A_l}\left(\bigcap_{j=0}^{d-1} T_{A_l}^{-j}\{\varphi_{A_l} = k_l^{(j)}\}\right) \sim e^{-\mu(A_l)\sum_{j=0}^{d-1} k_l^{(j)}} \mu(A_l)^d \quad \text{as } l \to \infty. \tag{2.74}
$$

Proof. Take $\mathcal{U} = \mathcal{U}_r$ and $\mathcal{U}_l = \{(\int_{A_l} u \, \mathrm{d}\mu_{A_l})^{-1} 1_{A_l} u \mid u \in \mathcal{U}\}$, then in case **a)** we can take $\tau_l = 1$, while in case **b)** $\tau_l = l$ fulfils all the conditions.
Bounded distortion together with non-periodicity of x^* (in case **b)** guarantees that (2.63) holds. □

2.4 Conjugate systems

One important aspect of the study of dynamical systems, is recognising when two systems are similar. We can then, instead of working in the original system, look at easier representations (as one often does with systems that have a cylinder structure).

Therefore we define the notion of (measurable) conjugacy.

Definition 2.4.1. *Let $(X, \mathcal{A}, \mu, T)$ and $(Y, \mathcal{B}, \nu, S)$ be two dynamical systems, with normalized measure such that $\mu(X) = \nu(Y)$.*

(i) We will say that $(X, \mathcal{A}, \mu, T)$ is semiconjugate to $(Y, \mathcal{B}, \nu, S)$ (or shorter T is a factor of S) if there are measurable subsets $X' \subset X$ and $Y' \subset Y$ of full measure, i.e $\mu(X'^c) = 0$ and $\nu(Y'^c) = 0$, invariant with respect to T (resp. S), and a map $\pi : X' \to Y'$, measurable with respect to $\mathcal{A} \cap X'$ and $\mathcal{B} \cap Y'$, called the conjugation map, such that $\pi \circ T|_{X'} = S \circ \pi|_{X'}$. In addition we require that ν is the push-forward of μ under π, i.e for $B \in \mathcal{B} \cap Y'$ we have $\nu(B) = \mu(\pi^{-1}(B))$.

(ii) If in addition π is invertible with measurable inverse $\pi^{-1} : Y' \to X'$, then we say that the systems are conjugate.

Remark 2.4.1. *The nomenclature "factor" comes from the fact that if $(X, \mathcal{A}, \mu, T)$ is semiconjugate to $(Y, \mathcal{B}, \nu, S)$ we can identify Y with $X/_{\equiv}$, where $x_1 \equiv x_2$ if $\pi(x_1) = \pi(x_2)$ for $x_1, x_2 \in X$, and S acts much like T on the equivalence classes i.e $S([x]) = [T(x)]$ for $x \in X$.*
The usual assumption that factor maps should be surjective is valid almost everywhere (in the case of the measures being finite).

By slight abuse of notation we will denote hitting (resp. return) times with φ regardless of the transformation and ambient space. [2] We also usually drop X' and Y' from the notation. [3]

For (semi)conjugate systems we have that hitting and return times of corresponding set behave the same. That means if $B \in \mathcal{B}$ is a measurable subset Y, then for $A = \pi^{-1}(B)$ and $n \in \mathbb{N}$ we have

$$\{\varphi_A = n\} = \pi^{-1}(\{\varphi_B = n\}) \tag{2.75}$$

and

$$A \cap \{\varphi_A = n\} = \pi^{-1}(B \cap \{\varphi_B = n\}). \tag{2.76}$$

[2] It will always be clear from the context.
[3] We might for example write $\pi^{-1}(B)$ for $(\pi|_{X'})^{-1}(B \cap Y')$ for $B \subset Y$.

Indeed

$$\begin{aligned}\pi^{-1}(\{\varphi_B = n\}) &= \pi^{-1}\left(\bigcap_{i=1}^{n-1} S^{-i}(B^c) \cap S^{-n}(B)\right)\\ &= \bigcap_{i=1}^{n-1} \pi^{-1} \circ S^{-i}(B^c) \cap \pi^{-1} S^{-n}(B)\\ &= \bigcap_{i=1}^{n-1} T^{-i}(A^c) \cap T^{-n}(A).\end{aligned}$$

It follows that also convergence in law and LLTs for hitting and return times behave in the same way. We will record this in the next proposition.

Proposition 2.4.1. *Let $(X, \mathcal{A}, \mu, T)$ and $(Y, \mathcal{B}, \nu, S)$ be two dynamical systems such that $(X, \mathcal{A}, \mu, T)$ is semiconjugate to $(Y, \mathcal{B}, \nu, S)$. Let $(B_l)_{l\geq 1}$ be a sequence in $\mathcal{B}$ such that $\nu(B_l) \to 0$ as $l \to \infty$. Set $A_l := \pi^{-1}(B_l)$ and let $F : [0, \infty) \to [0, 1]$ be a sub-probability distribution. Then*

(a)

$$\mu(\mu(A_l)\varphi_{A_l} \leq t) \Rightarrow F(t) \quad \text{as } l \to \infty \tag{2.77}$$

iff

$$\nu(\nu(B_l)\varphi_{B_l} \leq t) \Rightarrow F(t) \quad \text{as } l \to \infty, \tag{2.78}$$

(b) and for any sequence $(k_l)_{l\geq 1}$ of natural numbers

$$\mu_{A_l}(\varphi_{A_l} = k_l) \sim \mu(A_l)F'(\mu(A_l)k_l) \quad \text{as } l \to \infty \tag{2.79}$$

iff

$$\nu_{B_l}(\varphi_{B_l} = k_l) \sim \nu(B_l)F'(\nu(B_l)k_l) \quad \text{as } l \to \infty. \tag{2.80}$$

Proof. Straightforward from the relation of μ and ν. $\square$

As we saw above, a useful tool is the transfer operator. We therefore wish to relate $\hat{T}_\mu$ to $\hat{S}_\nu$. Therefore assume $(X, \mathcal{A}, \mu, T)$ and $(Y, \mathcal{B}, \nu, S)$ are conjugate by π. Since $\pi \circ T = S \circ \pi$ and $\nu = \mu \circ \pi^{-1}$, for $g \in L^1(\nu)$ we have

$$\begin{aligned}(g \odot \nu) \circ S^{-1} &= (g \odot (\mu \circ \pi^{-1})) \circ S^{-1} = ((g \circ \pi) \odot \mu) \circ \pi^{-1} \circ S^{-1}\\ &= ((g \circ \pi) \odot \mu) \circ T^{-1} \circ \pi^{-1} = (\hat{T}_\mu(g \circ \pi) \odot \mu) \circ \pi^{-1}\end{aligned}$$

and on the other hand

$$\begin{aligned}(g \odot \nu) \circ S^{-1} &= \hat{S}_\nu(g) \odot \nu = \hat{S}_\nu(g) \odot (\mu \circ \pi^{-1})\\ &= ((\hat{S}_\nu(g) \circ \pi) \odot \mu) \circ \pi^{-1}.\end{aligned}$$

As $\mathcal{A} = \pi^{-1}\mathcal{B}$ it follows that $\hat{T}_\mu(g \circ \pi) \odot \mu = (\hat{S}_\nu(g) \circ \pi) \odot \mu$ and therefore

$$\hat{T}_\mu(g \circ \pi) = \hat{S}_\nu(g) \circ \pi. \tag{2.81}$$

Suppose there is a function $\Phi : X \to [0, \infty)$ such that for all $f \in L^1(\mu)$ we have $\hat{T}_\mu(f)(x) = \sum_{y \in T^{-1}(x)} f(y)\Phi(y)$ for μ-a.e x. For $g \in L^1(\nu)$ we get

$$\begin{aligned}\hat{S}_\nu(g)(y) = \hat{T}_\mu(g \circ \pi)(\pi^{-1}(y)) &= \sum_{z \in T^{-1}(\pi^{-1}(y))} g(\pi(z))\Phi(z) \\ = \sum_{\pi(z) \in S^{-1}(y)} g(\pi(z))\Phi(z) &= \sum_{x \in S^{-1}(y)} g(x)\Phi(\pi^{-1}(x))\end{aligned}$$

for ν-a.e y. We will record this in the next lemma.

Lemma 2.4.1. *Let $(X, \mathcal{A}, \mu, T)$ and $(Y, \mathcal{B}, \nu, S)$ be conjugate with conjugation map π.*

(a) For $g \in L^1(\nu)$ we have

$$\hat{S}_\nu(g) \circ \pi = \hat{T}_\mu(g \circ \pi). \tag{2.82}$$

(b) Suppose there is a function $\Phi : X \to [0, \infty)$ such that for all $f \in L^1(\mu)$ we have $\hat{T}_\mu(f)(x) = \sum_{y \in T^{-1}(x)} f(y)\Phi(y)$ for μ-a.e x. Then for $g \in L^1(\nu)$ we have

$$\hat{S}_\nu(g)(y) = \sum_{x \in S^{-1}(y)} g(x)\Phi \circ \pi^{-1}(x). \tag{2.83}$$

We will demonstrate how this can be use with one example.

Example 2.4.1. *Let $I = [0, 1]$ and $T : I \to I$ be measurable. Assume that there is a finite partition ζ such that $T|_Z$ is of class C^1 with uniformly Lipschitz continuous derivative. Assume that the almost everywhere defined derivative T' is bounded below. Also assume that the n-cylinders are uniformly exponentially small, i.e there is a $C > 0$ and a $r \in (0, 1)$ such that for any $n \geq 1$ and $Z_0, ..., Z_{n-1} \in \zeta$ we have*

$$\operatorname{diam}([Z_0, ..., Z_{n-1}]) = \operatorname{diam}\Big(\bigcap_{i=0}^{n-1} T^{-i}(Z_i)\Big) \leq Cr^{n-1}. \tag{2.84}$$

We are going to show that $(I, \mathcal{A}, \lambda, T)$ is conjugate to a SFT of the type discussed in Section 3.5. If $|\zeta| = M$, define the $M \times M$ matrix $B = (b_{i,j})_{1 \leq i,j \leq M}$ by $b_{i,j} = 1 \iff T(Z_i) \supset Z_j$ and set

$$\Sigma := \{x = (x_1, x_2, ...) \in \{1, ..., M\}^{\mathbb{N}} \mid b_{x_i, x_{i+1}} = 1 \text{ for } i \geq 1\}. \tag{2.85}$$

The conjugation map $\pi : \Sigma \to I$ given by

$$(i_1, i_2, ...) \mapsto \bigcap_{j \geq 1} T^{-j}(Z_{i_j}) \tag{2.86}$$

is almost everywhere well defined and bijective. The potential for $\hat{T}_\lambda$ *is given by* $\Phi(y) = -\log(T'(y))$ *(well defined on the image of* π*) is uniformly Lipschitz continuous. One glance at Lemma 2.4.1 reveals that all that is left to show is that* $\Phi \circ \pi$ *is uniformly Lipschitz continuous. Choosing the metric on* Σ *as*

$$d(x,y) = r^{\min\{i\geq 1 \mid x_i \neq y_i\}-1} \quad \text{for } x,y \in \Sigma. \tag{2.87}$$

We see that for $x, y \in \Sigma$ *with* $x \neq y$ *we have*

$$|\Phi(\pi(x)) - \Phi(\pi(y))| \leq \mathrm{Lip}(\Phi)\mathrm{diam}(Z) \leq C\mathrm{Lip}(\Phi) r^{n-1} = C\mathrm{Lip}(\Phi) d(x,y) \tag{2.88}$$

where $n = \min\{i \geq 1 \mid x_i \neq y_i\}$ *and* $Z = \pi([x_1, ..., x_n])$*.*

We can therefore apply Proposition 2.4.1 to see that for sequences of asymptotically rare events $(A_l)_{l\geq 1}$ *such that either* ***a)*** A_l *is* ζ *measurable, i.e* A_l *is a union of* 1*-cylinders or* ***b)*** A_l *is a* l*-cylinder, convergence in law of the normalised hitting times to the exponential distribution implies LLTs for the return times.*

Chapter 3

Examples

3.1 Mixing System

Thinking back to the abstract setting of section 2, we saw that the general ergodic system will fail local limit theorems for various sequences of sets. Here we will explore the stronger version, mixing, meaning that for any to measurable sets A and B, $\mu(A \cap T^{-n}(B))$ will converge to $\mu(A)\mu(B)$. We will use some slightly different notions here.

Definition 3.1.1. *Let $(X, \mathcal{A}, \mu, T)$ be a dynamical system and ζ be a countable partition of X (mod μ). For $n \geq 1$ define the collection of n-cylinders as $\zeta_n := \{\bigcap_{j=0}^{n-1} T^{-j} Z_j \mid Z_j \in \zeta \text{ for } 0 \leq j \leq n-1\}$.*

(a) We will say that the partition ζ is uniform mixing of rate $\gamma(n)$, if

$$\gamma(n) = \sup_{k,l \geq 1} \sup_{\substack{A \in \sigma(\zeta_k) \\ B \in T^{-(n+k)}(\sigma(\zeta_l))}} |\mu(A \cap B) - \mu(A)\mu(B)| \tag{3.1}$$

with $\gamma(n) \to 0$ as $n \to \infty$.

(b) We will say that the partition ζ is α mixing of rate $\alpha(n)$, if

$$\alpha(n) = \sup_{k,l \geq 1} \sup_{\substack{A \in \zeta_k \\ B \in T^{-(n+k)}(\sigma(\zeta_l))}} \left| \frac{\mu(A \cap B)}{\mu(A)} - \mu(B) \right| \tag{3.2}$$

with $\alpha(n) \to 0$ as $n \to \infty$.

(c) We will say that the partition ζ is ϕ mixing of rate $\phi(n)$, if

$$\phi(n) = \sup_{k,l \geq 1} \sup_{\substack{A \in \zeta_k \\ B \in T^{-(n+k)}(\zeta_l)}} \left| \frac{\mu(A \cap B)}{\mu(A)\mu(B)} - 1 \right| \tag{3.3}$$

with $\phi(n) \to 0$ as $n \to \infty$.

Remark 3.1.1.

- *Note that ϕ mixing implies α mixing implies uniform mixing. More precisely $\gamma(n) \leq \alpha(n) \leq \phi(n)$.*
- *α mixing tells us that for $A \in \zeta_k$ and $B \in T^{-(n+k)}(\sigma(\zeta_l))$ we have*

$$|\mu(A \cap B) - \mu(A)\mu(B)| \leq \alpha(n)\mu(A). \tag{3.4}$$

As ζ_k is a partition, for $A \in \sigma(\zeta_k)$ there are disjoint $A_i \in \zeta_k$ for $i \geq 1$ such that $A = \bigcup_{i\geq 1} A_i$. Therefore

$$\begin{aligned}|\mu(A \cap B) - \mu(A)\mu(B)| &\leq \sum_{i\geq 1} |\mu(A_i \cap B) - \mu(A_i)\mu(B)| \\ &\leq \alpha(n) \sum_{i\geq 1} \mu(A_i) \\ &\leq \alpha(n)\mu(A).\end{aligned}$$

- *Analogously, for ϕ mixing, we can also take $A \in \sigma(\zeta_k)$ and $B \in T^{-(n+k)}(\sigma(\zeta_l))$ instead.*

It turns out that these mixing conditions (with appropriate rates) are already enough to guarantee local limit theorems for return times. We will mention results for uniform mixing, α mixing and ϕ mixing separately, as we need to require different rates for the three types.

As a warm-up, let us consider the question of convergence in distribution for hitting times in such systems. A similar version of this (using different methods) was attained in [AS11].

Theorem 3.1.1. *Let $(X, \mathcal{A}, \mu, T)$ be an ergodic, probability-preserving system. Let ζ be a measurable partition of X (mod μ) and let $(A_l)_{l\geq 1}$ be a sequence of asymptotically rare events such that for some sequence $(n_l)_{l\geq 1}$ of natural numbers we have $A_l \in \sigma(\zeta_{n_l})$ and $n_l = o\left(\frac{1}{\mu(A_l)}\right)$. Suppose there is a sequence of natural numbers $(\tau_l)_{l\geq 1}$ such that*

$$\mu(A_l)\tau_l \to 0 \quad \text{as } l \to \infty, \tag{3.5}$$

and

$$\mu_{A_l}(\varphi_{A_l} \leq \tau_l) \to 0 \quad \text{as } l \to \infty, \tag{3.6}$$

as well as $\tau_l - n_l \to \infty$ as $l \to \infty$. Furthermore assume that one of the following holds

a) *the partition ζ is uniformly mixing with respect to T of rate $\gamma(n)$ and $\gamma(\tau_l - n_l) = o(\mu(A_l))$ as $l \to \infty$,*

b) *or the partition ζ is α mixing with respect to T of rate $\alpha(n)$.*

Then we have convergence of the hitting times to the exponential law, i.e

$$\mu(\mu(A_l)\varphi_{A_l} \leq t) \Rightarrow 1 - e^{-t} \quad \text{as } l \to \infty. \tag{3.7}$$

Proof. (i) Due to Corollary 1.2.1, it is sufficient to show that

$$|\mu(A_l \cap \{\mu(A_l)\varphi_{A_l} \leq t\}) - \mu(A_l)\mu(\mu(A_l)\varphi_{A_l} \leq t)| = o(\mu(A_l)) \tag{3.8}$$

as $l \to \infty$. Note that this is trivially true for $t = 0$. So fix some $t > 0$ and denote $t_l = \frac{t}{\mu(A_l)}$, we may assume that $t_l > \tau_l > n_l$.

(ii) We claim that

$$|\mu(A_l \cap T^{-\tau_l}\{\varphi_{A_l} \leq t_l - \tau_l\}) - \mu(A_l)\mu(\varphi_{A_l} \leq t_l - \tau_l)| = o(\mu(A_l)) \tag{3.9}$$

as $l \to \infty$. As $A_l \in \sigma(\zeta_{n_l})$ and $T^{-\tau_l}\{\varphi_{A_l} \leq t_l - \tau_l\} \in T^{-(\tau_l - n_l + n_l)}\sigma(\zeta_{\tau_l + n_l})$. So we can estimate the left side of (3.9), in case **a)** by $\gamma(\tau_l - n_l)$ and in case **b)** by $\mu(A_l)\alpha(\tau_l - n_l)$. In either case (3.9) follows.

(iii) Note that

$$\{\varphi_{A_l} \circ T^{\tau_l} \leq t_l - \tau_l\} \setminus \{\varphi_{A_l} \leq t_l\} \subset \{\varphi_{A_l} \leq \tau_l\}. \tag{3.10}$$

Therefore we can estimate the left side of (3.8) by

$$\begin{aligned}
&|\mu(A_l \cap \{\mu(A_l)\varphi_{A_l} \leq t\}) - \mu(A_l)\mu(\mu(A_l)\varphi_{A_l} \leq t)| \\
&\leq \mu(A_l \cap (\{\varphi_{A_l} \circ T^{\tau_l} \leq t_l - \tau_l\} \setminus \{\varphi_{A_l} \leq t_l\})) \\
&+ |\mu(A_l \cap T^{-\tau_l}\{\varphi_{A_l} \leq t_l - \tau_l\}) - \mu(A_l)\mu(\varphi_{A_l} \leq t_l - \tau_l)| \\
&+ \mu(A_l)\mu(\{\varphi_{A_l} \circ T^{\tau_l} \leq t_l - \tau_l\} \setminus \{\varphi_{A_l} \leq t_l\}) \\
&\leq \mu(A_l)\mu_{A_l}(\varphi_{A_l} \leq \tau_l) \\
&+ |\mu(A_l \cap T^{-\tau_l}\{\varphi_{A_l} \leq t_l - \tau_l\}) - \mu(A_l)\mu(\varphi_{A_l} \leq t_l - \tau_l)| \\
&+ \mu(A_l)\mu(\varphi_{A_l} \leq \tau_l).
\end{aligned}$$

Due to (3.6), the third term of the last expression are of order $o(\mu(A_l))$ as $l \to \infty$. By (3.9) the same is true for the middle term. The first term can be bounded by $\mu(A_l)^2\tau_l$.

□

Remark 3.1.2.

- *Due to (3.5), in case **a)** we must have $\gamma(n) = o(\frac{1}{n})$,*
- *if $(A_l)_{l\geq 1}$ has convergence of the hitting times to the exponential law, (3.6) holds for any sequence of positive real numbers $(\tau_l)_{l\geq 1}$ with*

$$\mu(A_l)\tau_l \to 0 \quad \text{as } l \to \infty. \tag{3.11}$$

3.1.1 Uniformly mixing

Theorem 3.1.2. *Let $(X, \mathcal{A}, \mu, T)$ be an ergodic, probability-preserving system and $d \geq 1$ be a natural number. Let ζ be a measurable partition of X (mod μ), which is uniformly mixing with respect to T of order $\gamma(n) = o\left(\left(\frac{1}{n}\right)^{d+1}\right)$. Let $(A_l)_{l\geq 1}$ be a sequence of asymptotically rare events such that for some sequence $(n_l)_{l\geq 1}$ of natural numbers we have $A_l \in \sigma(\zeta_{n_l})$ and $n_l = o\left(\frac{1}{\mu(A_l)}\right)$. Assume that the normalised hitting times converge in law to the exponential distribution*

$$\mu(\mu(A_l)\varphi_{A_l} \leq t) \Rightarrow 1 - e^{-t} \quad \textit{as } l \to \infty. \tag{3.12}$$

Then, for any $1 \leq m \leq d$ and sequences $(k_l^{(0)})_{l\geq 1}, ..., (k_l^{(m-1)})_{l\geq 1}$ of natural numbers such that $(\mu(A_l)k_l^{(j)})_{l\geq 1}$ is bounded away from 0 and ∞, we have that

$$\mu_{A_l}\left(\bigcap_{j=0}^{m-1} T_{A_l}^{-j}\{\varphi_{A_l} = k_l^{(j)}\}\right) \sim e^{-\mu(A_l)\sum_{j=0}^{m-1} k_l^{(j)}} \mu(A_l)^m \quad \textit{as } l \to \infty. \tag{3.13}$$

In (3.13) the $(\mu_{A_l})_{l\geq 1}$ of measures on the left can be replaced by μ.
In particular if $\gamma(n)$ decays faster than (the multiplicative inverse of) any polynomial, (3.13) holds for any $m \geq 1$.

Proof. We will show (3.13) only for $(\mu_{A_l})_{l\geq 1}$, the proof for μ goes analogous.

(i) As in Section 2.2 define $s_l^{(m)} := \sum_{j=0}^{m-1} k_l^{(j)}$ for $1 \leq m \leq d$ and $s_l^{(0)} := 0$ the time of the m-th return to A_l (on the set we are interested in), $\kappa_l^{(m)} = k_l^{(m)} - \tau_l$ and

$$W_l^{(m)} := A_l \cap \bigcap_{j=0}^{m-1} T_{A_l}^{-j}\{\varphi_{A_l} = k_l^{(j)}\} = A_l \cap \bigcap_{j=0}^{m-1} T^{-s_l^{(j)}}\{\varphi_{A_l} = k_l^{(j)}\} \tag{3.14}$$

for $0 \leq m \leq d$ and $W_l^{(0)} := A_l$. Note that $W_l^{(m)} \in \sigma(\zeta_{s_l^{(m)}+n_l})$. We will show

$$\mu_{A_l}(W_l^{(m)}) \sim e^{-\mu(A_l)\sum_{j=0}^{m-1} k_l^{(j)}} \mu(A_l)^m \quad \textit{as } l \to \infty. \tag{$\clubsuit_m$}$$

by induction on m. Note that $(\clubsuit_0)$ is trivial.

(ii) By (a version of) Lemma 2.1.3 there is a sequence $(g_n)_{n\geq 1}$ of positive real numbers such that $\frac{g_n}{n} \to \infty$ and $\gamma(n) = o\left(\left(\frac{1}{g_n}\right)^{d+1}\right)$. Define $\tilde{\tau}_l = \min(n > 0 \mid g_n \geq \frac{1}{\mu(A_l)})$ and $\tau_l := \tilde{\tau}_l + n_l$, then $\tau_l \to \infty$ as $l \to \infty$, while $\tau_l = o(\frac{1}{\mu(A_l)})$. For any sequence $(k_l)_{l\geq 1}$ of natural numbers such that $(\mu(A_l)k_l)_{l\geq 1}$ is bounded away from 0 and ∞ define $\kappa_l := k_l - \tau_l$. (We may just assume that $k_l > \tau_l$). As above, by Theorem 2.1.1 we have that

$$\mu(\varphi_{A_l} = \kappa_l) \sim \mu(A_l)e^{-\mu(A_l)k_l} \quad \textit{as } l \to \infty. \tag{3.15}$$

(iii) We will also show the auxiliary condition

$$\mu(W_l^{(m)} \cap T^{-s_l^{(m)}}\{\varphi_{A_l} \leq \tau_l\}) = o(\mu(A_l)^{m+1}) \quad \textit{as } l \to \infty \tag{$\spadesuit_m$}$$

by induction. Note that ($\spadesuit_0$) follows from convergence in law of the normalised return times.

(iv) For ($\clubsuit_1$) fix a sequence $(k_l)_{l\geq 1}$ as above. It will be sufficient so show

$$|\mu(\varphi_{A_l} = \kappa_l) - \mu_{A_l}(\varphi_{A_l} = k_l)| = o(\mu(A_l)) \tag{3.16}$$

as $l \to \infty$. We do this by first splitting up

$$\begin{aligned}|\mu(\varphi_{A_l} = \kappa_l) - \mu_{A_l}(\varphi_{A_l} = k_l)| \leq{}& \left|\mu(\varphi_{A_l} = \kappa_l) - \mu_{A_l}(T^{-\tau_l}\{\varphi_{A_l} = \kappa_l\})\right| \\ &+ \left|\mu_{A_l}(T^{-\tau_l}\{\varphi_{A_l} = \kappa_l\}) - \mu_{A_l}(\varphi_{A_l} = k_l)\right|\end{aligned}$$

The first term we can rewrite as

$$\begin{aligned}&\left|\mu_{A_l}(T^{-\tau_l}\{\varphi_{A_l} = \kappa_l\}) - \mu(\varphi_{A_l} = \kappa_l)\right| \\ &= \frac{1}{\mu(A_l)}\left|\mu(A_l \cap T^{-\tau_l}\{\varphi_{A_l} = \kappa_l\}) - \mu(A_l)\mu(\varphi_{A_l} = \kappa_l)\right| \\ &\leq \frac{1}{\mu(A_l)}\gamma(\tilde{\tau}_l) \\ &\leq o(\mu(A_l)),\end{aligned}$$

since $A_l \in \zeta_{n_l}$ and $T^{-\tau_l}\{\varphi_{A_l} = \kappa_l) \in T^{-(\tilde{\tau}_l + n_l)}\sigma((\zeta_{n_l+\kappa_l}))$. By Lemma 2.1.2 and ($\spadesuit_0$) the second term is

$$\begin{aligned}&\left|\mu_{A_l}(T^{-\tau_l}\{\varphi_{A_l} = \kappa_l\}) - \mu_{A_l}(\varphi_{A_l} = k_l)\right| \\ &\leq \frac{1}{\mu(A_l)}\mu(A_l \cap \{\varphi_{A_l} \leq \tau_l\} \cap T^{-k_l}(A_l)) \\ &\leq \frac{1}{\mu(A_l)}\left|\mu(A_l \cap \{\varphi_{A_l} \leq \tau_l\} \cap T^{-k_l}(A_l)) - \mu(A_l \cap \{\varphi_{A_l} \leq \tau_l\})\mu(A_l)\right| \\ &+ \frac{1}{\mu(A_l)}\mu(A_l \cap \{\varphi_{A_l} \leq \tau_l\})\mu(A_l) \\ &\leq \frac{1}{\mu(A_l)}\gamma(\tilde{\tau}_l) + o(\mu(A_l)) \\ &= o(\mu(A_l)),\end{aligned}$$

since $A_l \cap \{\varphi_{A_l} \leq \tau_l\} \in \sigma(\zeta_{n_l+\tau_l})$ and $T^{-k_l}(A_l) \in T^{-k_l}(\zeta_{n_l})$, and as soon as $k_l \geq n_l + \tau_l + \tilde{\tau}_l$.

(v) Assume that for some $1 \le m \le d-1$ we have $(\clubsuit_0), ..., (\clubsuit_m)$ and $(\spadesuit_0), ..., (\spadesuit_{m-1})$. We first claim $(\spadesuit_m)$. Indeed, since

$$\begin{aligned}
&W_l^{(m)} \cap T^{-s_l^{(m)}}\{\varphi_{A_l} \le \tau_l\} \\
&= W_l^{(m-1)} \cap T^{-s_l^{(m-1)}}(\{\varphi_{A_l} = k_l^{(m-1)}\} \cap T^{-k_l^{(m-1)}}\{\varphi_{A_l} \le \tau_l\}) \\
&\subset W_l^{(m-1)} \cap T^{-s_l^{(m)}}(A_l \cap \{\varphi_{A_l} \le \tau_l\})
\end{aligned}$$

we get, using $(\clubsuit_{m-1})$ and $(\spadesuit_0)$,

$$\begin{aligned}
&\mu(W_l^{(m)} \cap T^{-s_l^{(m)}}\{\varphi_{A_l} \le \tau_l\}) \\
&\le \mu\left(W_l^{(m-1)} \cap T^{-s_l^{(m)}}(A_l \cap \{\varphi_{A_l} \le \tau_l\})\right) \\
&\le \left|\mu(W_l^{(m-1)} \cap T^{-s_l^{(m)}}(A_l \cap \{\varphi_{A_l} \le \tau_l\}) - \mu(W_l^{(m-1)})\mu(A_l \cap \{\varphi_{A_l} \le \tau_l\})\right| \\
&+ \mu(W_l^{(m-1)})\mu(A_l \cap \{\varphi_{A_l} \le \tau_l\}) \\
&\le \gamma(\tilde{\tau}_l) + o(\mu(A_l)^{m+1}) \\
&= o(\mu(A_l)^{m+1}).
\end{aligned}$$

since $W_l^{(m-1)} \in \sigma(\zeta_{s_l^{(m-1)}+n_l})$
and $T^{-s_l^{(m)}}(A_l \cap \{\varphi_{A_l} \le \tau_l\}) \in T^{-(s_l^{(m-1)}+k_l^{(m-1)})}\sigma(\zeta_{\tau_l+n_l})$, and as soon as $k_l^{(m-1)} \ge \tilde{\tau}_l + n_l$.

(vi) For $(\clubsuit_{m+1})$ it will be enough to show

$$\begin{aligned}
&\frac{1}{\mu(A_l)}\left|\mu(W_l^{(m+1)}) - \mu(W_l^{(m)})\mu(\{\varphi_{A_l} = \kappa_l^{(m)}\})\right| \\
&= o(\mu(A_l)^{m+1}).
\end{aligned}$$

We again first split it up into

$$\begin{aligned}
&\frac{1}{\mu(A_l)}\left|\mu(W_l^{(m+1)}) - \mu(W_l^{(m)})\mu(\{\varphi_{A_l} = \kappa_l^{(m)}\})\right| \\
&\le \frac{1}{\mu(A_l)}\left|\mu(W_l^{(m+1)}) - \mu(W_l^{(m)} \cap T^{-(s_l^{(m)}+\tau_l)}\{\varphi_{A_l} = \kappa_l^{(m)}\})\right| \\
&+ \frac{1}{\mu(A_l)}\left|\mu(W_l^{(m)} \cap T^{-(s_l^{(m)}+\tau_l)}\{\varphi_{A_l} = \kappa_l^{(m)}\}) - \mu(W_l^{(m)})\mu(\{\varphi_{A_l} = \kappa_l^{(m)}\})\right|.
\end{aligned}$$

(vii) For the first term note that by Lemma 2.1.2 we have

$$\begin{aligned}
&\left(W_l^{(m)} \cap T^{-(s_l^{(m)}+\tau_l)}\{\varphi_{A_l} = \kappa_l^{(m)}\}\right) \setminus W_l^{(m+1)} \\
&= W_l^{(m)} \cap T^{-s_l^{(m)}}\left(\left(T^{-\tau_l}\{\varphi_{A_l} = \kappa_l^{(m)}\}\right) \setminus \{\varphi_{A_l} = k_l^{(m)}\}\right) \\
&= W_l^{(m)} \cap T^{-s_l^{(m)}}\left(\{\varphi_{A_l} \le \tau_l\} \cap T^{-\tau_l}\{\varphi_{A_l} = \kappa_l^{(m)}\}\right) \\
&\subset W_l^{(m)} \cap T^{-s_l^{(m)}}\{\varphi_{A_l} \le \tau_l\} \cap T^{-s_l^{(m+1)}}(A_l).
\end{aligned}$$

An elementary computation reveals that $\mu(\{\varphi_{A_l} \leq \tau_l\} \cap T^{-k_l^{(m)}}(A_l)) = o(\mu(A_l))$. Therefore using $(\spadesuit_m)$ we get

$$\begin{aligned}
&\frac{1}{\mu(A_l)}\left|\mu(W_l^{(m+1)}) - \mu(W_l^{(m)} \cap T^{-(s_l^{(m)}+\tau_l)}\{\varphi_{A_l} = \kappa_l^{(m)}\})\right| \\
&\leq \frac{1}{\mu(A_l)}\mu\left(W_l^{(m)} \cap T^{-s_l^{(m)}}\{\varphi_{A_l} \leq \tau_l\} \cap T^{-s_l^{(m+1)}}(A_l)\right) \\
&\leq \frac{1}{\mu(A_l)}\left|\begin{array}{l} \mu\left(W_l^{(m)} \cap T^{-s_l^{(m)}}\{\varphi_{A_l} \leq \tau_l\} \cap T^{-s_l^{(m+1)}}(A_l)\right) \\ -\,\mu\left(W_l^{(m)} \cap T^{-s_l^{(m)}}\{\varphi_{A_l} \leq \tau_l\}\right)\mu((A_l)) \end{array}\right| \\
&+ \frac{1}{\mu(A_l)}\mu\left(W_l^{(m)} \cap T^{-s_l^{(m)}}\{\varphi_{A_l} \leq \tau_l\}\right)\mu((A_l)) \\
&\leq \frac{1}{\mu(A_l)}\gamma(\tilde{\tau}_l) + o(\mu(A_l)^{m+1}) \\
&= o(\mu(A_l)^{m+1}),
\end{aligned}$$

as $W_l^{(m)} \cap T^{-s_l^{(m)}}\{\varphi_{A_l} \leq \tau_l\} \in \sigma(\zeta_{s_l^{(m)}+\tau_l+n_l})$ and $T^{-s_l^{(m+1)}}(A_l) \in T^{-(s_l^{(m)}+k_l^{(m)})}\sigma(\zeta_{n_l})$ and as soon as $k_l^{(m)} \geq \tilde{\tau}_l + \tau_l + n_l$.

(viii) For the second term note that $W_l^{(m)} \in \sigma(\zeta_{s_l^{(m)}+n_l})$ and $T^{-(s_l^{(m)}+\tau_l)}\{\varphi_{A_l} = \kappa_l^{(m)}\} \in T^{-(s_l^{(m)}+n_l+\tilde{\tau}_l)}\sigma(\zeta_{\kappa_l^{(m)}+n_l})$, and therefore

$$\begin{aligned}
&\left|\mu(W_l^{(m)} \cap T^{-(s_l^{(m)}+\tau_l)}\{\varphi_{A_l} = \kappa_l^{(m)}\}) - \mu(W_l^{(m)})\mu(\varphi_{A_l} = \kappa_l^{(m)})\right| \\
&= \gamma(\tilde{\tau}_l) = o(\mu(A_l)^{m+2}).
\end{aligned}$$

□

Now for instance we can provide LLTs for Markov chains on a countable state space.

Example 3.1.1. *Let X be an irreducible, positively recurrent, aperiodic Markov-chain with transition matrix $P = (p_{i,j})_{i,j\in S}$ on a countable state space S. Assume furthermore that either S is in fact finite or Doeblin's condition holds, i.e there is a $j \in S$ such that $\inf_{i\in S} p_{i,j} > 0$. In addition assume for convenience as in Example 1.2.1 that there is a $\alpha < 1$ such that $p_{i,j} \leq \alpha$ for $i, j \in S$. It is standard theory that in both cases there is a $c > 0$ and a $\delta \in (0, 1)$ such that for all distributions q on S we have*

$$\sup_{i\in S} |q^{(n)}(i) - \pi(i)| \leq c\delta^n \tag{3.17}$$

where $q^{(n)}(i) = \sum_{j_1,\ldots,j_{n-1}\in S} q(j_1)p_{j_1,j_2}\ldots p_{j_{n-2},j_{n-1}}p_{j_{n-1},i}$ for $n \geq 2$ and $p^(1)(i) = q(i)$. We look at the dynamical representation of X as in Example 1.2.1. We claim that the partition $\zeta = \zeta_1$ is uniformly mixing with *exponential rate. For $A = [a_0, ..., a_{j-1}] \in \zeta_j$ and $B = [b_0, .., b_{k-1}] \in \zeta_k$ we have*

$$\begin{aligned}\left|\mu(A \cap T^{-(n+j)}(B) - \mu(A)\mu(B)\right| &\leq \left|p^{(n)}_{a_j,b_1} - \pi(b_1)\right| \\ &\leq \sup_{i,j\in S} |p^{(n)}_{j,i} - \pi(i)| \leq c\delta^n\end{aligned}$$

independent of A and B. Using Example 1.2.1 we obtain return time LLTs for cylinder, which could be phrased in the following way. Let $(x_l)_{l\geq 0}$ be a non-periodic sequence of elements of S, then the probability of a long part of $(x_l)_{l\geq 0}$ first appearing at time $(k_l)_{l\geq 1}$ (for some $k_l = \Theta(\frac{1}{\mathbb{P}((X_0,...,X_{l-1})=(x_0,...,x_{l-1}))})$ is of the same order as it appearing at the start. More precisely

$$\begin{aligned}&\mathbb{P}\left(\min(k \geq 1 \mid (X_k, ..., X_{k+l}) = (x_k, ..., x_{k+l})) = k_l\right) \\ &\sim \mathbb{P}\left((X_0, ..., X_{l-1}) = (x_0, ..., x_{l-1})\right) e^{k_l \mathbb{P}((X_0,...,X_{l-1})=(x_0,...,x_{l-1}))} \quad \textit{as } l \to \infty\end{aligned}$$

and

$$\begin{aligned}&\mathbb{P}\left(\min(k \geq 1 \mid (X_k, ..., X_{k+l}) = (x_k, ..., x_{k+l})) = k_l \mid (X_0, ..., X_{l-1}) = (x_0, ..., x_{l-1})\right) \\ &\sim \mathbb{P}\left((X_0, ..., X_{l-1}) = (x_0, ..., x_{l-1})\right) e^{k_l \mathbb{P}((X_0,...,X_{l-1})=(x_0,...,x_{l-1}))} \quad \textit{as } l \to \infty.\end{aligned}$$

3.1.2 α mixing

Theorem 3.1.3. *Let $(X, \mathcal{A}, \mu, T)$ be an ergodic, probability-preserving system. Let ζ be a measurable partition of X (mod μ), which is α mixing with respect to T of order $\alpha(n) = o\left(\frac{1}{n}\right)$. Let $(A_l)_{l\geq 1}$ be a sequence of asymptotically rare events such that for some sequence $(n_l)_{l\geq 1}$ of natural numbers we have $A_l \in \sigma(\zeta_{n_l})$ and $n_l = o\left(\frac{1}{\mu(A_l)}\right)$. Assume that the normalised hitting times converge in law to the exponential distribution*

$$\mu(\mu(A_l)\varphi_{A_l} \leq t) \Rightarrow 1 - e^{-t} \quad \textit{as } l \to \infty. \tag{3.18}$$

Then, for any $d \geq 1$ and sequences $(k^{(0)}_l)_{l\geq 1}, ..., (k^{(d-1)}_l)_{l\geq 1}$ of natural numbers such that $(\mu(A_l)k^{(j)}_l)_{l\geq 1}$ is bounded away from 0 and ∞, we have that

$$\mu_{A_l}\left(\bigcap_{j=0}^{d-1} T^{-j}_{A_l}\{\varphi_{A_l} = k^{(j)}_l\}\right) \sim e^{-\mu(A_l)\sum_{j=0}^{d-1} k^{(j)}_l} \mu(A_l)^d \quad \textit{as } l \to \infty. \tag{3.19}$$

In (3.19) the $(\mu_{A_l})_{l\geq 1}$ of measures on the left can be replaced by μ.

Proof. We follow the proof of Theorem 3.1.2, where g_n is such that $\alpha(n) = o(\frac{1}{g_n})$ and $\frac{g_n}{n} \to \infty$. Otherwise all estimates can be done in much the same

way, there are no hidden tricks.
For example

$$\begin{aligned}
&\left|\mu(W_l^{(d)} \cap T^{-(s_l^{(d)}+\tau_l)}\{\varphi_{A_l} = \kappa_l^{(d)}\}) - \mu(W_l^{(d)})\mu(\varphi_{A_l} = \kappa_l^{(d)})\right| \\
&= \mu(W_l^{(d)})\alpha(\tilde{\tau}_l) \\
&= o(\mu(A_l)^{d+2}).
\end{aligned}$$

□

3.1.3 ϕ mixing

Theorem 3.1.4. *Let $(X, \mathcal{A}, \mu, T)$ be an ergodic, probability-preserving system. Let ζ be a measurable partition of X (mod μ), which is ϕ mixing with respect to T. Let $(A_l)_{l\geq 1}$ be a sequence of asymptotically rare events such that for some sequence $(n_l)_{l\geq 1}$ of natural numbers we have $A_l \in \sigma(\zeta_{n_l})$ and $n_l = o\left(\frac{1}{\mu(A_l)}\right)$. Assume that the normalised hitting times converge in law to the exponential distribution*

$$\mu(\mu(A_l)\varphi_{A_l} \leq t) \Rightarrow 1 - e^{-t} \quad \text{as } l \to \infty. \tag{3.20}$$

Then, for any $d \geq 1$ and sequences $(k_l^{(0)})_{l\geq 1}, ..., (k_l^{(d-1)})_{l\geq 1}$ of natural numbers such that $(\mu(A_l)k_l^{(j)})_{l\geq 1}$ is bounded away from 0 and ∞, we have that

$$\mu_{A_l}\left(\bigcap_{j=0}^{d-1} T_{A_l}^{-}j\{\varphi_{A_l} = k_l^{(j)}\}\right) \sim e^{-\mu(A_l)\sum_{j=0}^{d-1} k_l^{(j)}} \mu(A_l)^d \quad \text{as } l \to \infty. \tag{3.21}$$

In (3.21) the $(\mu_{A_l})_{l\geq 1}$ of measures on the left can be replaced by μ.

Proof. We follow the proof of Theorem 3.1.2, where $\tilde{\tau}_l$ is any sequence of natural numbers such that $\tilde{\tau}_l = o\left(\frac{1}{\mu(A_l)}\right)$ and $\tilde{\tau}_l \to \infty$ as $l \to \infty$. Otherwise all estimates can be done in much the same way, there are no hidden tricks.
For example

$$\begin{aligned}
&\left|\mu(W_l^{(d)} \cap T^{-(s_l^{(d)}+\tau_l)}\{\varphi_{A_l} = \kappa_l^{(d)}\}) - \mu(W_l^{(d)})\mu(\varphi_{A_l} = \kappa_l^{(d)})\right| \\
&= \mu(W_l^{(d)})\mu(\varphi_{A_l} = \kappa_l^{(d)})\phi(\tilde{\tau}_l) \\
&= o(\mu(A_l)^{d+2}).
\end{aligned}$$

□

3.1.4 Non-exponential limits

There are, however, situations where cylinders do not enjoy exponential behaviour for the hitting times, or even convergence in law at all. However [AS11] still gives us a characterisation for all "reasonable" sequences of cylinders. The resulting behaviour is comparable to that of GM-maps around periodic points.
Let $(X, \mathcal{A}, \mu, T)$ be a probability preserving dynamical system, ζ be a uniformly mixing partition of X and $(A_l)_{l\geq 1}$ be a sequence of asymptotically rare events such that $A_l \in \sigma(\zeta_l)$. Suppose that

$$\mu(\varphi_{A_l} \leq l) \to 0 \quad as\ l \to \infty, \tag{3.22}$$

this is in particular the case if $\mu(A_l) = o\left(\frac{1}{l}\right)$.
Then there are renormalising constants $\lambda_l > 0$ with $\limsup_{l\to\infty} \lambda_l \leq 1$ such that

$$\sup_{t\geq 0} \left|\mu(\lambda_l \mu(A_l)\phi_{A_l} > t) - e^{-t}\right| \to 0 \quad as\ l \to \infty, \tag{3.23}$$

and

$$\sup_{t\geq s} \left|\frac{1}{\lambda_l}\mu_{A_l}(\lambda_l \mu(A_l)\phi_{A_l} > t) - e^{-t}\right| \to 0 \quad as\ l \to \infty\ for\ all\ s > 0. \tag{3.24}$$

This say that the distribution of the return times can be approximated by a convex combination of a Dirac-mass at 0 and a (scaled) exponential distribution.

Theorem 3.1.5. *Let $(X, \mathcal{A}, \mu, T)$ be a probability preserving dynamical system and ζ be a partition of X. Let $(A_l)_{l\geq 1}$ be a sequence of asymptotically rare events such that $A_l \in \sigma(\zeta_l)$ and suppose that $\mu(A_l) = o\left(\frac{1}{l}\right)$ as $l \to \infty$. Suppose that the partition ζ is either*

a) uniformly mixing of super-polynomial rate $\gamma(n)$, i.e for any $d \geq 1$ we have $\gamma(n) = o\left(\frac{1}{n^d}\right)$ as $n \to \infty$,

b) or α-mixing of rate $\alpha(n) = o\left(\frac{1}{n}\right)$ as $n \to \infty$,

c) or ϕ-mixing.

Then there are renormalising constants $\lambda_l > 0$ with $\limsup_{l\to\infty} \lambda_l \leq 1$ such that, for $d \geq 1$ and sequences $(k_l^{(0)})_{l\geq 1}, ..., (k_l^{(d-1)})_{l\geq 1}$ of natural numbers such that $(\lambda_l \mu(A_l) k_l^{(j)})_{l\geq 1}$ is bounded away from 0 and ∞, we have that

$$\mu\left(\bigcap_{j=0}^{d-1} T_{A_l}^{-j}\{\varphi_{A_l} = k_l^{(j)}\}\right) \sim e^{-\lambda_l \mu(A_l) \sum_{j=0}^{d-1} k_l^{(j)}} \lambda_l^d \mu(A_l)^d \quad as\ l \to \infty, \tag{3.25}$$

and

$$\mu_{A_l}\left(\bigcap_{j=0}^{d-1} T_{A_l}^{-j}\{\varphi_{A_l}=k_l^{(j)}\}\right) \sim e^{-\lambda_l\mu(A_l)\sum_{j=0}^{d-1}k_l^{(j)}}\lambda_l^{d+1}\mu(A_l)^d \quad \textit{as } l\to\infty. \tag{3.26}$$

Proof. (i) Use Theorem 1 of [AS11] to obtain such $\lambda_l > 0$ such that (3.23) and (3.24) hold. We may assume that $\lambda_l \le 1$ for all $l \ge 1$, otherwise we replace λ_l by $\min(\lambda_l, 1)$.

(ii) The arguments, used in Theorem 2.1.1 with (3.24) instead of Lemma 2.1.1, guarantee that, for any sequence $(k_l)_{l\ge 1}$ of natural numbers such that $(\lambda_l\mu(A_l)k_l)_{l\ge1}$ is bounded away form 0 and ∞, we have

$$\mu(\varphi_{A_l}=k_l) \sim \lambda_l\mu(A_l)e^{-\lambda_l\mu(A_l)k_l} \quad \textit{as } l\to\infty. \tag{3.27}$$

(iii) We first identify (compare Theorem 3.4.1) subsets A_l^e on which the distribution of the return times has approximately exponential behaviour. Therefore define

$$\begin{aligned} m_l' &:= \max\{m\ge 0 \mid \mu_{A_l}(\varphi_{A_l}>m)\ge\lambda_l\},\\ m_l &:= \min\{m\ge 0 \mid \mu_{A_l}(\varphi_{A_l}>m)\le\lambda_l\},\\ A_l^e &:= A_l\cap\{\varphi_{A_l}>m_l\}, \end{aligned}$$

where all of the quantities are well-defined because of $\lambda_l \le 1$. Clearly we have $m_l \in \{m_l', m_l'+1\}$.

(iv) We first show $m_l = o\left(\frac{1}{\lambda_l\mu(A_l)}\right)$ as $l\to\infty$, which is equivalent to $m_l' = o\left(\frac{1}{\lambda_l\mu(A_l)}\right)$. Suppose this is not the case, then there are a constant $\delta > 0$ and a sequence of indices $(l_k)_{k\ge1}$ with $l_k\to\infty$ as $k\to\infty$ such that $m_{l_k}'\lambda_{l_k}\mu(A_{l_k}) \ge \delta$. Therefore

$$\begin{aligned} \frac{1}{\lambda_{l_k}}\mu_{A_{l_k}}(\varphi_{A_{l_k}}>m_{l_k}') &= \frac{1}{\lambda_{l_k}}\mu_{A_{l_k}}(\lambda_{l_k}\mu(A_{l_k}\varphi_{A_{l_k}}>\lambda_{l_k}\mu(A_{l_k})m_{l_k}')\\ &\le \frac{1}{\lambda_{l_k}}\mu_{A_{l_k}}(\lambda_{l_k}\mu(A_{l_k}\varphi_{A_{l_k}}>\delta). \end{aligned}$$

It follows that $\limsup_{k\to\infty}\frac{1}{\lambda_{l_k}}\mu_{A_{l_k}}(\varphi_{A_{l_k}}>m_{l_k}') \le e^{-\delta}$. But this contradicts $\mu_{A_l}(\varphi_{A_l}>m_l') \ge \lambda_l$ for all $l\ge 1$.

(v) Next we show that

$$\mu(A_l^e) \sim \lambda_l\mu(A_l) \quad \textit{as } l\to\infty. \tag{3.28}$$

On one hand we have $\mu(A_l^e) \leq \lambda_l \mu(A_l)$ for all l. On the other, by the previous step, for any $\delta > 0$, we have $m_l \lambda_l \mu(A_l) \leq \delta$ for l big enough. For such l we have

$$\begin{aligned}\frac{1}{\lambda_l}\mu_{A_l}(A_l^e) &= \frac{1}{\lambda_l}\mu_{A_l}(\varphi_{A_l} > m_l)\\ &\geq \frac{1}{\lambda_l}\mu_{A_l}(\lambda_l\mu(A_l)\varphi_{A_l} > \delta),\end{aligned}$$

and therefore $\liminf_{l\to\infty} \frac{1}{\lambda_l}\mu_{A_l}(A_l^e) \geq e^{-\delta}$. Combining these (3.28) follows.

(vi) Before we apply the arguments from Theorem 3.1.2, we still need to check that, for any sequence $(\tau_l)_{l\geq 1}$ of natural numbers such that $\tau_l = o\left(\frac{1}{\lambda_l\mu(A_l)}\right)$ as $l \to \infty$, we have

$$\mu(A_l^e) \cap \{\varphi_{A_l} \leq m_l + \tau_l\}) = o(\lambda_l \mu(A_l)) \quad \textit{as } l \to \infty. \tag{3.29}$$

For any $\delta > 0$, for l big enough such that $\lambda_l\mu(A_l)(m_l + \tau_l) \leq \delta$, we have

$$\begin{aligned}&\frac{1}{\lambda_l}\mu_{A_l}(A_l^e) \cap \{\varphi_{A_l} \leq m_l + \tau_l\})\\ &= \frac{1}{\lambda_l}\mu_{A_l}(\varphi_{A_l} > m_l) - \frac{1}{\lambda_l}\mu_{A_l}(\varphi_{A_l} > m_l + \tau_l)\\ &\leq 1 - \frac{1}{\lambda_l}\mu_{A_l}(\lambda_l\mu(A_l)\varphi_{A_l} > \delta),\end{aligned}$$

and therefore $\limsup_{l\to\infty} \frac{1}{\lambda_l}\mu_{A_l}(A_l^e) \cap \{\varphi_{A_l} \leq m_l + \tau_l\}) \leq 1 - e^{-\delta}$. Taking $\delta \to 0$ yields the claim.

(vii) Now, for $1 \leq m \leq d$, define

$$W_l^{(m)} := A_l^e \cap \bigcap_{j=0}^{m-1} T_{A_l}^{-j}(A_l^e \cap \{\varphi_{A_l} = k_l^{(j)}\}). \tag{3.30}$$

With the arguments in the proof of Theorems 3.1.2, 3.1.3 and 3.1.4, we show

$$\left|\mu(W_l^{(m+1)}) - \mu(W_l^{(m)})\mu(\{\varphi_{A_l} = \kappa_l^{(m)}\})\right| = o((\lambda_l\mu(A_l))^{m+2}). \tag{3.31}$$

This yields (3.26), we can show (3.25) analogously.

□

3.2 Rychlik maps

We will investigate the behaviour of first return and hitting times for certain classes of monotonic interval maps. Let therefore $X \subset \mathbb{R}$ be compact. For a function $g : X \to \mathbb{R}$ and $A \subset X$ we define its variation over A as

$$\mathrm{Var}_A g := \sup\{\sum_{j=0}^{n-1} |g(x_{j+1}) - g(x_j)|\}$$

where the supremum goes over all finite ordered sequences $x_j \in A$ for $j = 0, ..., n$. We will in the following omit the subscript X whenever we use the variation over the whole space. The variation of a function in L^1 is the minimum of the variations of any version. With the norm $||g||_{BV} := ||g||_{L^\infty} + \mathrm{Var} g$, the linear space $BV := \{g : X \to \mathbb{R} \mid ||g||_{BV} < \infty\}$ is a Banach space. We will call $I \subset X$ an X-Interval if there is an interval $J \subset \mathbb{R}$ such that $I = J \cap X$. In the following $\mathcal{A}$ will be the Borel-σ-Algebra on X and λ will denote a regular Borel probability measure on X.

Definition 3.2.1 ($\mathcal{R}$-maps)**.** *Let $Y \subset X$ be open and dense in X with $\lambda(Y) = 1$, set $S = X \setminus Y$. As in [BSTV03] we call a map $T : Y \to X$ an $\mathcal{R}$-map if*

- *there is a countable family ζ of closed X-intervals with pairwise disjoint interiors such that $Y \subset \bigcup_{Z\in\zeta} Z$ and for all $Z \in \zeta$ the sets $Z \cap S$ are exactly the endpoints of Z,*
- *for all $Z \in \zeta$, $T|_{Z\cap Y}$ can be extended to a homeomorphism from Z to some X-interval I_Z,*
- *there exists a function $g \in BV$, $g \geq 0$, $g|_S \equiv 0$ such that the operator $P : L^1(\lambda) \to L^1(\lambda)$ given by*

$$Pf(x) = \sum_{y\in T^{-1}(x)} f(y)g(y)$$

  preserves λ. This means for each $f \in L^1(\lambda)$ we have $\int_X f \ \mathrm{d}\lambda = \int_X Pf \ \mathrm{d}\lambda$.

- *T is uniformly expanding i.e $\alpha := ||g||_\infty < 1$.*

For $n \geq 1$ we will denote the set of all n-cylinders by

$$\zeta_n := \{[Z_0, ..., Z_{n-1}] := \bigcap_{i=0}^{n-1} T^{-i} Z_i \mid Z_i \in \zeta \text{ for } 0 \leq i \leq n-1\}. \tag{3.32}$$

Note that in this notation for $n \geq 1$ and $f \in L^1(\lambda)$ we have

$$P^n(f) = \sum_{Z\in\zeta_n} (fg_n) \circ (T|_Z)^{-n} \tag{3.33}$$

for $g_n = \prod_{i=0}^{n-1} g \circ T^n$. The $(fg_n) \circ (T|_Z)^{-n}$ are taken to be 0 where not defined. Clearly we have

$$||g_n||_\infty \le ||g||_\infty^n. \tag{3.34}$$

For any point $x \in X \setminus \left(\bigcup_{i\ge 0} T^{-i}S\right)$ and $n \ge 1$, we define $\zeta_n(x)$ to be the n-cylinder containing x. By Remark 2 in [Ryc83] P is the Perron-Frobenius-Operator for T w.r.t λ

$$Pf = \hat{T}_\lambda f = \frac{d(f \odot \lambda) \circ T^{-1}}{\mathrm{d}\lambda}.$$

[Ryc83] also shows

Theorem 3.2.1. *If T is an $\mathcal{R}$-map then there exists an invariant measure μ which is absolutely continuous with respect to the reference measure λ with density $h = \frac{\mathrm{d}\mu}{\mathrm{d}\lambda} \in BV$. Furthermore there are $L_i \in \mathbb{N}$ and there is a partition (mod λ) into disjoint open sets $X_{i,j}$ where $i = 1, ..., k$ and $j = 1, ..., L_i$ for some $k \in \mathbb{N}$ such that the system $(X_{i,j}, \mathcal{A} \cap X_{i,j}, \mu_{X_{i,,j}}, T^{L_i}|_{X_{i,j}})$ is mixing and has exponential decay of correlations for bounded variation observables, i.e there are constants $C_{i,j} > 0$ and $\theta_{i,j} \in (0,1)$ such that*

$$\left| \int_{X_{i,j}} \varphi \circ T^{nL_i} \psi \ \mathrm{d}\mu_{X_{i,j}} - \int_{X_{i,j}} \varphi \ \mathrm{d}\mu_{X_{i,j}} \int_{X_{i,j}} \psi \ \mathrm{d}\mu_{X_{i,j}} \right| \le C_{i,j} ||\varphi||_{L^1(\lambda)} ||\psi||_{BV} \theta_{i,j}^n \tag{3.35}$$

for $\varphi \in L^1(\lambda)$, $\psi \in BV$ and $n \in \mathbb{N}$.

It is known (see e.g [Ryc83]) that if $(X, \mathcal{A}, \mu, T)$ is weakly mixing then (3.35) simplifies to

$$\left| \int_X \varphi \circ T^n \psi \, \mathrm{d}\mu - \int_X \varphi \, \mathrm{d}\mu \int_X \psi \, \mathrm{d}\mu \right| \le C ||\varphi||_{L^1(\lambda)} ||\psi||_{BV} \theta^n \tag{3.36}$$

for some $C > 0$, $\theta \in (0,1)$ and all $\varphi \in L^1(\lambda)$, $\psi \in BV$ and $n \in \mathbb{N}$. We will assume weak mixing from now on. Furthermore we will as before have to deal with the Perron-Frobenius Operator $\hat{T}_\mu$. To relate this to the much nicer operator $\hat{T}_\lambda$ we will use the fact that $h\hat{T}_\mu(f) = \hat{T}_\lambda(fh)$ for $f \in L^1(\mu)$ and all $n \ge 1$. [1] Therefore we see that we need $\frac{1}{h} \in BV$, so we will in the following assume assume $\inf_{\{h>0\}} h > 0$. In this case we redefine $h = \max(h, \inf_{h>0} h)$, then the well-defined function $\frac{1}{h}$ is of bounded variation (We compose h with the Lipschitz-function $x \mapsto \frac{1}{x}$). In this case we also define $\hat{T}_\mu(f)$ as the bona fide function $\frac{1}{h}\hat{T}_\lambda(hf)$, so we can just take the variation and not worry about any identifications in L^1. Now by Proposition 1 in [Ryc83] there are constants $\hat{K} > 0$ and $0 < r < 1$ such that for any $f \in BV$ and $n \in \mathbb{N}$

$$\mathrm{Var}(\hat{T}_\lambda^n(f)) \le \hat{K}(||f||_{L^1(\lambda)} + r^n \mathrm{Var}(f)). \tag{3.37}$$

[1] To verify this note that $\hat{T}_\mu(f)h \odot \lambda = \hat{T}_\mu(f) \odot \mu = (f \odot \mu) \circ T^{-1} = (fh \odot \lambda) \circ T^{-1}$.

Therefore

$$\begin{aligned}
\mathrm{Var}(\hat{T}_\mu^n(f)) &\le \left|\left|\frac{1}{h}\right|\right|_{BV} \hat{K}(||f||_{L^1(\mu)} + ||h||_{BV} r^n ||f||_{BV}) \\
&\le \left|\left|\frac{1}{h}\right|\right|_{BV} \hat{K}((1+||h||_{BV})||f||_{L^1(\mu)} + 2||h||_{BV} r^n \mathrm{Var} f) \\
&\le \underbrace{\left|\left|\frac{1}{h}\right|\right|_{BV} \hat{K}(1+||h||_{BV})}_{=:K}(||f||_{L^1(\mu)} + \underbrace{\frac{2||h||_{BV}}{(1+||h||_{BV})}}_{=:c} r^n \mathrm{Var} f).
\end{aligned}$$

We used that for all $0 \le f \in BV$ we have $||f||_{L^1_\mu} \ge ||f||_\infty - \mathrm{Var} f$. [2] In the following we will use $\hat{T}_\mu$ only and denote it by $\hat{T}$.

Also note that by the assumption $\inf_{\{h>0\}} h > 0$, in (3.36) we can use $L^1(\mu)$ instead of $L^1(\lambda)$.

First we look at a version of Theorem 2.1.2 for these maps.

Proposition 3.2.1. *Let $(X, \mathcal{A}, \lambda, T)$ be an $\mathcal{R}$-map such that $(X, \mathcal{A}, \mu, T)$ is weakly mixing. Let $\mathcal{U} \subset \mathcal{D}(\mu)$ be bounded in BV and $(A_l)_{l\ge1}$ be a sequence in $\mathcal{A}$ such that $\mu(A_l) \to 0$. Suppose that F is a continuously differentiable sub-probability distribution on $[0,\infty)$, such that*

$$\mu(\mu(A_l)\varphi_{A_l} \le t) \Rightarrow F(t) \quad \text{as } l \to \infty. \tag{3.38}$$

Then

$$u \odot \mu(\varphi_{A_l} = k_l) \sim F'(t)\mu(A_l) \quad \text{as } l \to \infty, \text{ uniformly in } u \in \mathcal{U} \tag{3.39}$$

for any sequence $(k_l)_{l\ge1}$ of integers such that $(\mu(A_l)k_l)_{l\ge1}$ is bounded away from 0 and ∞.

Proof. We proceed in the same steps as in the proof of Theorem 2.1.2. Steps (i) to (iii) stay the same as $\mathcal{U}$ is also bounded in L^∞. For (iv) say $||u||_{BV} \le K$ for $u \in \mathcal{U}$. Then since $\int_X 1_X - u \, d\mu = 0$ using (3.36) we get

$$\begin{aligned}
|\int_X 1_{\{\varphi_{A_l}=\kappa_l\}} \circ T^{j_l}(1_X - u)\, d\mu| &\le C||1_{\{\varphi_{A_l}=\kappa_l\}}||_{L^1(\lambda)}||1_X - u||_{BV}\theta^{j_l} \\
&\le C\left(\frac{1}{\inf_{h>0} h}\right)(K+1)\theta^{j_l}\mu(\varphi_{A_l} = k_l).
\end{aligned}$$

Under present assumptions we have $\mu(\varphi_{A_l} = k_l) \sim F'(t)\mu(A_l)$. (v) again can be done as in Theorem 2.1.2 concluding the proof. □

[2] The "worst" thing that can happen is that $f = ||f||_\infty$ at one of the endpoints, then it makes a jump downwards by $\mathrm{Var} f$ and is constant from then on.

Remark 3.2.1. *The proof shows that similar results hold for systems with similar decay of correlation as in (3.36). More details on this can be found later in this chapter.*

From now on we only use $L^1(\mu)$ and just denote it by L^1.

Proposition 3.2.2. *Let $(X, \mathcal{A}, \lambda, T)$ be an $\mathcal{R}$-map such that $(X, \mathcal{A}, \mu, T)$ is weakly mixing. Let $\mathcal{U} \subset \mathcal{D}(\mu)$ be bounded in BV and $(A_l)_{l\geq 1}$ be a sequence in $\mathcal{A}$ such that $\mu(A_l) \to 0$ such that*

$$\mu(\mu(A_l)\varphi_{A_l} \leq t) \Rightarrow 1 - e^{-t} \quad \text{as } l \to \infty. \tag{3.40}$$

Assume that there are constants $\tau_l \in \mathbb{N}_0$ such that

$$\mu(A_l)\tau_l \to 0 \quad \text{as } l \to \infty \tag{3.41}$$

while

$$\hat{T}^{\tau_l}(\mu(A_l)^{-1}1_{A_l}) \in \mathcal{U} \quad \text{for } l \geq 1. \tag{3.42}$$

Let $(k_l)_{l\geq 1}$ be a sequence in $\mathbb{N}$ such that $(\mu(A_l)k_l)_{l\geq 1}$ is bounded away from 0 and ∞. If

$$\mu_{A_l}(\{\varphi_{A_l} \leq \tau_l\} \cap T^{-k_l}A_l) = o(\mu(A_l)) \quad \text{as } l \to \infty. \tag{3.43}$$

then

$$\mu_{A_l}(\varphi_{A_l} = k_l) \sim e^{-\mu(A_l)k_l}\mu(A_l) \quad \text{as } l \to \infty. \tag{3.44}$$

Proof. We can use the same proof as for Theorem 2.1.4 (using theorem 3.2.1 instead of Theorem 2.1.2). □

Now we can formulate a result about "good" sequences for $\mathcal{R}$-maps.

Theorem 3.2.2. *Let $(X, \mathcal{A}, \lambda, T)$ be an $\mathcal{R}$-map such that $(X, \mathcal{A}, \mu, T)$ is weakly mixing, in addition assume $\inf_{\{h>0\}} h > 0$. Let $x^* \in X \setminus \left(\bigcup_{j\geq 0} T^{-j}S\right)$ be a non-periodic point and $A_l \in \mathcal{A}$ be a sequence of intervals shrinking to x^*. Then for all sequences $(k_l)_{l\geq 1}$ such that $(\mu(A_l)k_l)_{l\geq 1}$ is bounded away from 0 and ∞ we have*

$$\mu_{A_l}(\varphi_{A_l} = k_l) \sim \mu(A_l)e^{-\mu(A_l)k_l} \quad \text{as } l \to \infty. \tag{3.45}$$

Proof.

(i) Theorem 3.2 in [BSTV03] shows that A_l have exponential first return times (the extra assumption $\inf_{\{h>0\}} h > 0$ guarantees that this holds for all non-periodic points in $X \setminus \left(\bigcup_{j\geq 0} T^{-j}S\right)$ and such sequences of intervals instead of only μ-a.e non-periodic point and sequences of balls). We will ignore points in $\bigcup_{j\geq 0} T^{-j}S$ from now on, as this is a nullset anyway.

(ii) We now identify a sequence $(\tau_l)_{l\geq 1}$ in $\mathbb{N}$ such that

$$\mu(A_l)\tau_l \to 0 \quad \textit{as } l \to \infty \tag{3.46}$$

and

$$\hat{T}^{\tau_l}(\mu(A_l)^{-1}1_{A_l}\hat{T}^i(1_{A_l})) \in \tilde{\mathcal{U}} \quad \textit{for } l \geq 1 \textit{ and } 0 \leq i \leq \tau_l \tag{3.47}$$

for some $\tilde{\mathcal{U}} \subset BV$ which is bounded in BV.
We will not bother with the $||\cdot||_\infty$-part of the BV-norm as $||\cdot||_\infty \leq ||\cdot||_{L^1} + \mathrm{Var}(\cdot)$.
Recall that there are constants $K > 0$, $c > 0$ and $0 < r < 1$ such that for any $f \in BV$ and $n \in \mathbb{N}$

$$\mathrm{Var}(\hat{T}^n(f)) \leq K(||f||_{L^1} + cr^n\mathrm{Var}(f)). \tag{3.48}$$

So we get

$$\begin{aligned}
&\mathrm{Var}\left(\hat{T}^{\tau_l}\left(\frac{1_{A_l}}{\mu(A_l)}\hat{T}^i(1_{A_l})\right)\right)\\
&\leq K\left(\underbrace{||\frac{1_{A_l}}{\mu(A_l)}\hat{T}^i(1_{A_l})||_{L^1}}_{\leq 1} + cr^{\tau_l}\mathrm{Var}\left(\frac{1_{A_l}}{\mu(A_l)}\hat{T}^i(1_{A_l})\right)\right)\\
&\leq K + K\left(cr^{\tau_l}\left(\underbrace{\mathrm{Var}\left(\frac{1_{A_l}}{\mu(A_l)}\right)||\hat{T}^i(1_{A_l})||_\infty}_{\leq\frac{2}{\mu(A_l)}} + \underbrace{||\frac{1_{A_l}}{\mu(A_l)}||_\infty \underbrace{\mathrm{Var}(\hat{T}^i(1_{A_l}))}_{K(1+2c)}}_{\leq\frac{K(1+2c)}{\mu(A_l)}}\right)\right).
\end{aligned}$$

So for $\tau_l = \lceil\frac{\log(\mu(A_l))}{\log(r)}\rceil$ we get

$$\hat{T}\left(\frac{1_{A_l}}{\mu(A_l)}\hat{T}^i(1_{A_l})\right) \in \tilde{\mathcal{U}}, \tag{3.49}$$

where $\tilde{\mathcal{U}} = \{f \in BV \cap \mathcal{D}(\mu) \mid \mathrm{Var}f \leq K(1+Kc)(1+2c)\}$.

(iii) Denote $n_l := \min_{x\in A_l}\varphi_{A_l}(x)$. We first claim that $n_l \nearrow \infty$ as $l \to \infty$. Indeed for any $n \in \mathbb{N}$ let $\epsilon > 0$ be such that $d(T^j(x^*), x^*) > \epsilon$ for $1 \leq j \leq n$ (this is possible because x^* is non-periodic). As $T, ..., T^n$ are continuous at x^* there is a $\delta > 0$ such that $d(x, x^*) < \delta$ implies $d(T^j(x), T^j(x^*)) < \frac{\epsilon}{3}$ for $1 \leq j \leq n$. Therefore if l is big enough such that $\mathrm{diam}(A_l) < \min(\delta, \frac{\epsilon}{3})$, then $\varphi_{A_l} > n$ on A_l.

(iv) Next we check that (3.43) holds for the sequence $(\tau_l)_{l\geq 1}$. For l big enough such that $k_l > 2\tau_l$ we get

$$\begin{aligned}
&\mu_{A_l}(\{\varphi_{A_l} \leq \tau_l\} \cap T^{-k_l} A_l) \\
&\leq \sum_{i=n_l}^{\tau_l} \mu_{A_l}(T^{-i} A_l \cap T^{-k_l} A_l) \\
&= \sum_{i=n_l}^{\tau_l} \int_X \frac{1_{A_l}}{\mu(A_l)} \left(1_{A_l} \circ T^i\right) \left(1_{A_l} \circ T^{k_l}\right) \mathrm{d}\mu \\
&= \sum_{i=n_l}^{\tau_l} \int_X \hat{T}^{\tau_l}(1_{A_l} \hat{T}^i (\frac{1_{A_l}}{\mu(A_l)})) \left(1_{A_l} \circ T^{k_l - \tau_l - i}\right) \mathrm{d}\mu.
\end{aligned}$$

We now estimate the term in the last sum using (3.36)

$$\begin{aligned}
&\int_X \hat{T}^{\tau_l}(1_{A_l} \hat{T}^i (\frac{1_{A_l}}{\mu(A_l)})) \left(1_{A_l} \circ T^{k_l - \tau_l - i}\right) \mathrm{d}\mu \\
&\leq \int_X 1_{A_l} \,\mathrm{d}\mu \int_X \hat{T}^{\tau_l}(1_{A_l} \hat{T}^i (\frac{1_{A_l}}{\mu(A_l)})) \,\mathrm{d}\mu \\
&+ C\theta^{k_l - \tau_l - i} ||1_{A_l}||_{L^1} ||\hat{T}^{\tau_l}(1_{A_l} \hat{T}^i (\frac{1_{A_l}}{\mu(A_l)}))||_{BV} \\
&\leq \mu(A_l) \left(\int_X 1_{A_l} \circ T^i \frac{1_{A_l}}{\mu(A_l)} \,\mathrm{d}\mu + C\theta^{k_l - 2\tau_l} L \right) \\
&\leq \mu(A_l) \left(\frac{1}{\mu(A_l)} \left(\left(\int_X 1_{A_l} \right)^2 + C\theta^i ||1_{A_l}||_{L^1} ||1_{A_l}||_{BV} \right) + C\theta^{k_l - 2\tau_l} L \right) \\
&\leq \mu(A_l) \left(\mu(A_l) + 3C\theta^i + C\theta^{k_l - 2\tau_l} L \right)
\end{aligned}$$

Where L is such that $||\tilde{u}||_{BV} \leq L$ for $\tilde{u} \in \tilde{\mathcal{U}}$. So altogether we get

$$\begin{aligned}
\mu_{A_l}(\{\varphi_{A_l} \leq \tau_l\} \cap T^{-k_l} A_l) &\leq \mu(A_l) \sum_{i=n_l}^{\tau_l} \left(\mu(A_l) + 3C\theta^i + C\theta^{k_l - 2\tau_l} L \right) \\
&\leq \mu(A_l) \left(\tau_l \mu(A_l) \tau_l C\theta^{k_l - 2\tau_l} L + 3C \sum_{i \geq n_l} \theta^i \right) \\
&= \mu(A_l) \left(\tau_l \mu(A_l) + \tau_l C\theta^{k_l - 2\tau_l} L + 3C \frac{\theta^{n_l}}{1-\theta} \right) \\
&= o(\mu(A_l))
\end{aligned}$$

as $l \to \infty$. Therefore we can use Proposition 3.2.2.

□

Let us also find a statement about consecutive hitting-times for $\mathcal{R}$-maps. We first need to reformulate our assumptions in Theorem 2.2.1.

Proposition 3.2.3. *Let $(X, \mathcal{A}, \lambda, T)$ be an $\mathcal{R}$-map such that $(X, \mathcal{A}, \mu, T)$ is weakly mixing. Let $(A_l)_{l\geq 1}$ be a sequence in $\mathcal{A}$ such that $\mu(A_l) \to 0$ such that the normalized hitting times converge to an exponential law,*

$$\mu(\mu(A_l)\varphi_{A_l} \leq t) \Rightarrow 1 - e^{-t} \quad \text{as } l \to \infty. \tag{3.50}$$

Suppose that for every sequence $(K_l)_{l\geq 1}$ of natural numbers such that $(\mu_{A_l} K_l)_{l\geq 1}$ is bounded away form 0 and ∞ there are $\mathcal{U} \subset \mathcal{D}(\mu)$ bounded in BV and a family $\bar{\mathcal{U}}_l \subset \mathcal{D}(\mu_{A_l}) \cap BV$ such that $\bar{u} = \bar{u} 1_{A_l}$ for $\bar{u} \in \bar{\mathcal{U}}_l$, $1_{A_l} \in \bar{\mathcal{U}}_l$ and $\bigcup_{l\geq 1} \bar{\mathcal{U}}_l$ is bounded in BV. Assume that there are constants $\tau_l \in \mathbb{N}_0$ such that

$$\mu(A_l)\tau_l \to 0 \quad \text{as } l \to \infty \tag{3.51}$$

while

$$\hat{T}^{\tau_l}(\mu(A_l)^{-1}\bar{u}_l) \in \mathcal{U} \quad \text{for } l \geq 1 \text{ and } \bar{u}_l \in \bar{\mathcal{U}}_l. \tag{3.52}$$

Furthermore assume that

$$\mu_{A_l}(\{\varphi_{A_l} \leq \tau_l\} \cap T^{-k_l} B_l) = o(\mu(B_l)) \quad \text{as } l \to \infty \tag{3.53}$$

for any sequences $B_l \in A_l \cap \mathcal{A}$ for $l \geq 1$ and $(k_l)_{l\geq 1}$ such that $k_l \leq K_l$ for $l \geq 1$ and $(\mu(A_l)k_l)_{l\geq 1}$ is bounded away from 0 and ∞. Furthermore for all such $(k_l)_{l\geq 1}$ we have

$$\hat{T}^{k_l}\left(\frac{\mu(A_l)1_{\{\varphi_{A_l}=k_l\}}u}{u \odot \mu(\varphi_{A_l} = k_l}\right) \in \bar{\mathcal{U}}_l \quad \text{for } u \in \mathcal{U} \tag{3.54}$$

for l sufficiently large. Then, for $d \geq 1$ and sequences $(k_l^{(0)})_{l\geq 1}, ..., (k_l^{(d-1)})_{l\geq 1}$ of natural numbers such that $(\mu(A_l)k_l^{(j)})_{l\geq 1}$ is bounded away from 0 and ∞, we have that

$$\mu_{A_l}\left(\bigcap_{j=0}^{d-1} T_{A_l}^{-j}\{\varphi_{A_l} = k_l^{(j)}\}\right) \sim e^{-\mu(A_l)\sum_{j=0}^{d-1} k_l^{(j)}} \mu(A_l)^d \quad \text{as } l \to \infty. \tag{3.55}$$

In (3.55) the sequence (μ_{A_l}) of measures on the left can be replaced by μ.

Proof. Analogous to the proof of Theorem 2.2.1 for fixed sequences $(k_l^{(0)})_{l\geq 1}, ..., (k_l^{(d-1)})_{l\geq 1}$ as above we take $K_l = \max_{0\leq j\leq d-1} k_l^{(j)}$, then we can use the same arguments (with Proposition 3.2.1 instead of Theorem 2.1.2). □

For finding "good" sequences for $\mathcal{R}$-maps we cannot hope for all the sequences indicated in Theorem 3.2.2.

Theorem 3.2.3. *Let $(X, \mathcal{A}, \lambda, T)$ be an $\mathcal{R}$-map such that $(X, \mathcal{A}, \mu, T)$ is weakly mixing, in addition assume* $\inf_{\{h>0\}} h > 0$. *Let $x^* \in X \setminus \left(\bigcup_{j \geq 0} T^{-j} S \right)$ be a non-periodic point and $A_l \in \mathcal{A}$ be either **a)** the l-cylinder containing x^*, $A_l = \zeta_l(x)$ or **b)** a union of cylinders converging to x^* i.e A_l is ζ-measurable. Then $(A_l)_{l \geq 1}$ has LLTs for consecutive hitting- or return-times i.e for $d \geq 1$ and sequences $(k_l^{(0)})_{l \geq 1}, ..., (k_l^{(d-1)})_{l \geq 1}$ of natural numbers such that $(\mu(A_l) k_l^{(j)})_{l \geq 1}$ is bounded away from 0 and ∞, we have that*

$$\mu_{A_l} \left(\bigcap_{j=0}^{d-1} T_{A_l}^{-j} \{ \varphi_{A_l} = k_l^{(j)} \} \right) \sim e^{-\mu(A_l) \sum_{j=0}^{d-1} k_l^{(j)}} \mu(A_l)^d \quad \textit{as } l \to \infty, \tag{3.56}$$

where the sequence (μ_{A_l}) on the left hand side can be replaced by μ.

Proof. We check all the assumptions of Proposition 3.2.3. The fact that

$$\mu(\mu(A_l) \varphi_{A_l} \leq t) \Rightarrow 1 - e^{-t} \quad \textit{as } l \to \infty \tag{3.57}$$

follows from Theorem 3.2.2. Fix a sequence $(K_l)_{l \geq 1}$ of natural numbers such that $(\mu_{A_l} K_l)_{l \geq 1}$ is bounded away form 0 and ∞.

(i) By (ii), (iii) and (iv) of Theorem 3.2.2 we see that

$$\mu_{A_l}(\{\varphi_{A_l} \leq \tilde{\tau}_l\} \cap T^{-k_l} B_l) = o(\mu(B_l)) \quad \textit{as } l \to \infty \tag{3.58}$$

for any sequences $B_l \in A_l \cap \mathcal{A}$ for $l \geq 1$ and $(k_l)_{l \geq 1}$ such that $k_l \leq K_l$ for $l \geq 1$ and $(\mu(A_l) k_l)_{l \geq 1}$ is bounded away from 0 and ∞. As long as $\mu(A_l) \tilde{\tau}_l \to 0$ as $l \to \infty$ and $\tilde{\tau}_l \geq \lceil \frac{\log(\mu(A_l))}{\log(r)} \rceil$.

(ii) As

$$||f||_{BV} = \mathrm{Var}(f) + ||f||_\infty \leq 2\mathrm{Var}(f) + ||f||_{L^1}$$

for $f \geq 0 \in BV$, it will be enough to bound $\mathrm{Var}(u)$ for $u \in \mathcal{U}$ or $u \in \bar{\mathcal{U}}_l$. Recall that there are constants $K > 0$, $c > 0$ and $0 < r < 1$ such that for any $f \in BV$ and $n \in \mathbb{N}$

$$\mathrm{Var}(\hat{T}^n(f)) \leq K(||f||_{L^1} + c r^n \mathrm{Var}(f)). \tag{3.59}$$

By Corollary 4 of [Ryc83]

$$L := \sup_{n \geq 1} \mathrm{Var}(g_n) < \infty. \tag{3.60}$$

(iii) First let $D > 0$ be such that $e^{\mu(A_l) K_l} \leq \frac{D}{2}$ for all $l \leq 1$. Take $N > 0$ so big that for $\tau_l = \lceil \frac{\log(\mu(A_l))}{\log(r)} \rceil + N$ we have $7 \frac{c r^{\tau_l}}{\mu(A_l)} D K (3L + 1) \leq 1$. For convenience assume that all of D, K and L are greater than 1. Let

$$\mathcal{U} := \{u \in \mathcal{D}(\mu) \mid \mathrm{Var}(f) \leq 2K\} \tag{3.61}$$

and

$$\bar{\mathcal{U}}_l := \{u \in \mathcal{D}(\mu_{A_l}) \mid \mathrm{supp} u \subset A_l \textit{ and } \mathrm{Var}(u) \leq 7DK(3L+1)\} \quad (3.62)$$

for $l \geq 1$.

(iv) We first check (3.52), so let $\bar{u}_l \in \bar{\mathcal{U}}_l$. First we have

$$\int_X \hat{T}^{\tau_l}\left(\mu(A_l)^{-1}\bar{u}_l\right) \mathrm{d}\mu = \int_{A_l} \bar{u}_l \, \mathrm{d}\mu_{A_l} = 1 \quad (3.63)$$

therefore $0 \leq \hat{T}\left(\mu(A_l)^{-1}\bar{u}_l\right) \in \mathcal{D}(\mu)$. Furthermore by (3.59)

$$\begin{aligned}\mathrm{Var}\left(\hat{T}^{\tau_l}\left(\mu(A_l)^{-1}\bar{u}_l\right)\right) &\leq K\left(1 + \frac{cr^{\tau_l}}{\mu(A_l)}\mathrm{Var}(\bar{u}_l)\right) \\ &\leq K\left(1 + \underbrace{7\frac{cr^{\tau_l}}{\mu(A_l)}DK(3L+1)}_{\leq 1}\right) \\ &\leq 2K.\end{aligned}$$

Therefore $\hat{T}^{\tau_l}\left(\mu(A_l)^{-1}\bar{u}_l\right) \in \mathcal{U}$.

(v) Now to check (3.54) let $u \in \mathcal{U}$ and $(k_l)_{l\geq 1}$ be a sequence of natural numbers such that $k_l \leq K_l$ for $l \geq 1$ and $(\mu(A_l)k_l)_{l\geq 1}$ is bounded away from 0 and ∞. As $\mathrm{supp}\left(1_{\{\varphi_{A_l}=k_l\}}\right) \subset T^{-k_l}A_l$, it follows that $\mathrm{supp}\left(\hat{T}^{k_l}\left(\frac{\mu(A_l)1_{\{\varphi_{A_l}=k_l\}}u}{u\odot\mu(\varphi_{A_l}=k_l)}\right)\right) \subset A_l$. Therefore

$$\begin{aligned}\int_{A_l} \hat{T}^{k_l}\left(\frac{\mu(A_l)1_{\{\varphi_{A_l}=k_l\}}u}{u\odot\mu(\varphi_{A_l}=k_l)}\right) \mathrm{d}\mu_{A_l} &= \int_X \hat{T}^{k_l}\left(\frac{1_{\{\varphi_{A_l}=k_l\}}u}{u\odot\mu(\varphi_{A_l}=k_l)}\right) \mathrm{d}\mu \\ &= \int_X \frac{1_{\{\varphi_{A_l}=k_l\}}u}{u\odot\mu(\varphi_{A_l}=k_l)} \mathrm{d}\mu = 1,\end{aligned}$$

so $\hat{T}^{k_l}\left(\frac{1_{\{\varphi_{A_l}=k_l\}}u}{u\odot\mu(\varphi_{A_l}=k_l)}\right) \in \mathcal{D}(\mu_{A_l})$. Note that by Theorem 3.2.2 $u \odot \mu(\varphi_{A_l} = k_l) \sim \mu(A_l)e^{-\mu(A_l)k_l}$, and therefore $\frac{\mu(A_l)}{u\odot\mu(\varphi_{A_l}=k_l)} \sim e^{\mu(A_l)k_l} \leq e^{\mu(A_l)K_l} \leq D$ for l sufficiently large.

(vi) In the case **a)** that $A_l = \zeta_{n_l}(x^*)$ define

$$W_l := \{[Z_0, ..., Z_{k_l-1}] \in \zeta_{k_l} \mid [Z_i, ..., Z_{i+l-1}] = A_l \textit{ iff } i = k_l\} \quad (3.64)$$

where $A_l = [Z_{k_l}, ..., Z_{k_l+l-1}]$. Then $1_{\{\varphi_{A_l}=k_l\}} = 1_{A_l} \circ T^{k_l} \sum_{Z\in W_l} 1_Z$ [3], additionally all the Zs are intervals. Therefore

$$\hat{T}^{k_l}\left(1_{\{\varphi_{A_l}=k_l\}}u\right) = 1_{A_l} \sum_{Z\in W_l} (g_{k_l} 1_Z u) \circ (T|_Z)^{-k_l}. \tag{3.65}$$

note that $(T|_Z)^{-k_l}$ is strictly monotone and precomposition with a strictly monotone function can not change the variation.
As $\mathrm{Var}(f) \le \sum_{B\in\beta} \mathrm{Var}_B(f)$ for any $f \in BV$ and partition β of X into closed intervals with overlapping boundaries, it follows that

$$\begin{aligned}
&\mathrm{Var}\left(\hat{T}^{k_l}\left(\frac{\mu(A_l)1_{\{\varphi_{A_l}=k_l\}}u}{u\odot\mu(\varphi_{A_l}=k_l)}\right)\right) \le D\cdot\mathrm{Var}\left(\hat{T}^{k_l}\left(1_{\{\varphi_{A_l}=k_l\}}u\right)\right)\\
&\le D\cdot\mathrm{Var}\left(1_{A_l}\sum_{Z\in W_l}(g_{k_l}1_Z u)\circ(T|_Z)^{-k_l}\right)\\
&\le D\left(1+3\sum_{Z\in W_l}\mathrm{Var}_Z(g_{k_l}1_Z u)\right)\\
&\le D\left(1+3\sum_{Z\in W_l}\left(\mathrm{Var}_Z(u)||g_{k_l}||_\infty+\mathrm{Var}_Z(g_{k_l})||u||_\infty\right)\right)\\
&\le D\left(1+3\mathrm{Var}(u)+3(1+4K)\mathrm{Var}(g_{k_l})\right) \le D(1+6K+3(1+4K)L)\\
&\le 7DK(1+3L)
\end{aligned}$$

(vii) In case **b)** A_l is a union of cylinders, say $A_l = \bigcup_{B\in\beta_l} B$ for some family $\beta_l \subset \zeta$ define

$$W_l := \{[Z_0, ..., Z_{k_l-1}] \mid Z_i \in \zeta\setminus\beta_l \textit{ for } 0 \le i \le k_l - 1\}. \tag{3.66}$$

We therefore have

$$\hat{T}^{k_l}\left(1_{\{\varphi_{A_l}=k_l\}}u\right) = \sum_{Z\in W_l}(g_{k_l}1_Z u)\circ(T|_Z)^{-k_l} \tag{3.67}$$

[3] The equality is to be understood as an eqality in L^1, since all the Zs are intervals with overlapping boundaries, we might get 2 instead of 1 there. Luckily, the boudaries are a nullset anyway.

and by the same argument

$$
\begin{aligned}
&\mathrm{Var}\left(\hat{T}^{k_l}\left(\frac{\mu(A_l)1_{\{\varphi_{A_l}=k_l\}}u}{u\odot\mu(\varphi_{A_l}=k_l)}\right)\right) \leq D\cdot\mathrm{Var}\left(\hat{T}^{k_l}\left(1_{\{\varphi_{A_l}=k_l\}}u\right)\right)\\
&\leq D\cdot\mathrm{Var}\left(\sum_{Z\in W_l}(g_{k_l}1_Z u)\circ(T|_Z)^{-k_l}\right)\\
&\leq D\left(\sum_{Z\in W_l}\mathrm{Var}_Z(g_{k_l}1_Z u)\right)\\
&\leq D\left(\sum_{Z\in W_l}(\mathrm{Var}_Z(u)||g_{k_l}||_\infty+\mathrm{Var}_Z(g_{k_l})||u||_\infty)\right)\\
&\leq D\left(\mathrm{Var}(u)+3(1+4K)\mathrm{Var}(g_{k_l})\right)\leq D(2K+3(1+4K)L)\\
&\leq 7DK(1+3L).
\end{aligned}
$$

In both cases $\hat{T}^{k_l}\left(\frac{\mu(A_l)1_{\{\varphi_{A_l}=k_l\}}u}{u\odot\mu(\varphi_{A_l}=k_l}\right)\in\bar{\mathcal{U}}_l$.

□

3.3 Unimodal maps

We turn to another widely regarded class of dynamical systems, unimodal maps.

Definition 3.3.1. *We will call a function $T:[0,1]\to[0,1]$ unimodal if it has a nondegenerate critical point $c\in[0,1]$ of order m such that*

1. *$T|_{[0,c)}$ is strictly increasing,*
2. *$T|_{(c,1]}$ is strictly decreasing,*

and $T(0)=T(1)=0$. If furthermore T is of class C^3 and the Schwarzian-derivative $\frac{T'''(x)}{T'(x)}-\frac{3}{2}\left(\frac{T''(x)}{T'(x)}\right)^2$ is negative at all points $x\neq c$ we call the map S-unimodal.

Definition 3.3.2. *We call an S-unimodal map T a Collet-Eckmann-map (CE-map) if it satisfies the Collet-Eckmann condition $\lambda_c>1$, where*

$$\lambda_c:=\liminf_{n\to\infty}\sqrt[n]{DT^n(T(C))}. \tag{3.68}$$

In [BC91] it is shown that, for a set of parametrs a which has positive Lebesgue-measure, the quadratic family $T_a(x)=ax(1-x)$ has property

(3.68). Furthermore by [CE83] a CE-map has an invariant probability measure μ with density h with respect to the Lebesgue measure λ. To apply a result by [KN92] we furthermore have to assume that for each $\beta_0 > 0$ there is a constant $M > 0$ such that for all $\beta \in [0, \beta_0]$

(R1) $M^{-1} < \left(\frac{|x-c|^{m-1}}{|DT(x)|}\right)^{\beta} < M$ for all $x \neq c$,

(R2) $\mathrm{Var}\left(\left(\frac{|x-c|^{m-1}}{|DT(x)|}\right)^{\beta}\right) < M$ and

(R3) $\max\left(\mathrm{Var}_{[0,u)}\left(\left(\frac{|T(x)-T(u)|}{|x-u||DT(x)|}\right)^{\beta}\right), \mathrm{Var}_{(v,1]}\left(\left(\frac{|T(x)-T(v)|}{|x-v||DT(x)|}\right)^{\beta}\right)\right) < M$ whenever $u < c < v$.

These conditions are satisfied for example for polynomial T with vanishing derivatives at c up to order $m-1$. We will call a CE-map satisfying (R1)-(R3) an RCE-map.
By Theorem 1.1 of [KN92] such maps have exponential decay of correlations, that is

Theorem 3.3.1. *Let $(X, \mathcal{A}, T, \mu)$ be a RCE-map, where μ is an invariant probability measure as above. Then there are $C > 0$ and $0 < r < 1$ such that for all measurable $\varphi, \psi : X \to \mathbb{R}$ with $\varphi \in BV$ and $\psi \in L^{m+\delta}(\mu)$ for some $\delta > 0$, and $n \in \mathbb{N}$ we have*

$$\left|\int_X \varphi\left(\psi \circ T^n\right) \mathrm{d}\mu - \int_X \varphi \,\mathrm{d}\mu \int_X \psi \,\mathrm{d}\mu\right| \leq C||\varphi||_{BV}||\psi||_{L^{m+\delta}(\lambda)} r^n. \tag{3.69}$$

So as before we have to investigate the behaviour of the total variation under the composition with T.

Lemma 3.3.1. *Let $T : [0,1] \to [0,1]$ be a unimodal map and $f : [0,1] \to [0,1]$ be of bounded variation. Then*

$$\mathrm{Var}\left(f \circ T\right) \leq 2\mathrm{Var}(f). \tag{3.70}$$

Iterating this we get $\mathrm{Var}\left(f \circ T^n\right) \leq 2^n \mathrm{Var}(f)$.

Proof. Let c be the critical point of T. Note that $f \circ T = 1_{[0,c]}(f \circ (T_{[0,c]})) + 1_{(c,1]}(f \circ (T_{(c,1]}))$. As composition with a monotone function leaves the variation invariant, we have

$$\mathrm{Var}(f \circ T) \leq 2\mathrm{Var}_{[0,T(c)]}(f) \leq 2\mathrm{Var}(f). \tag{3.71}$$

Although both $1_{[0,c]}(f \circ (T_{[0,c]}))$ and $1_{(c,1]}(f \circ (T_{(c,1]}))$ have a jump of height $f(T(c))$ at c, their sum does not. We can therefore ignore their jump at c, we use this for the first inequality. $\square$

In particular if $A \subset X$ is an interval then for all $n \in \mathbb{N}$

$$\mathrm{Var}(1_{T^{-n}A}) \le 2^{n+1}. \tag{3.72}$$

Corollary 3.3.1. *Let T be a unimodal map and $A \subset X$ an interval. Then for $n \in \mathbb{N}$*

$$\mathrm{Var}(1_{\{\varphi_A \le n\}}) \le n2^{n+1}. \tag{3.73}$$

Proof. Since $1_{\{\varphi_A \le n\}} = \sum_{i=1}^{n} \left(\left(\prod_{j=1}^{i-1} 1_{A^c} \circ T^j \right) 1_A \circ T^i \right)$, using (3.72) we get

$$\begin{aligned} \mathrm{Var}(1_{\{\varphi_A \le n\}}) &\le \sum_{i=1}^{n} \sum_{j=1}^{i} 2^j \\ &\le \sum_{i=1}^{n} i2^i \le n2^{n+1}. \end{aligned}$$

□

Remark 3.3.1. *A version of the previous Corollary can also be proven for all piecewise monotone interval-maps only finitely many cylinders.* [4] *The factor 2 is then to be replaced by the number of cylinders.*

We now proof that return time LLTs are "automatic" for intervals in RCE-maps

Theorem 3.3.2. *Let $(X, \mathcal{A}, T, \mu)$ be an RCE-map , where μ is an invariant probability measure as above. Let $(A_l)_{l \ge 1}$ be a sequence of intervals such that $\mu(A_l) \to 0$ as $l \to \infty$. Suppose that the normalised return times converge to the exponential law,*

$$\mu_{A_l}(\mu(A_l)\varphi_{A_l} \le t) \Rightarrow 1 - e^{-t} \quad \text{as } l \to \infty. \tag{3.74}$$

Then for all sequences $(k_l)_{l \ge 1}$ of natural numbers such that $(\mu(A_l)k_l)_{l \ge 1}$ is bounded away from 0 and ∞ we have

$$\mu_{A_l}(\varphi_{A_l} = k_l) \sim \mu(A_l)e^{-\mu(A_l)k_l} \quad \text{as } l \to \infty. \tag{3.75}$$

Proof. (i) Fix such a sequence $(k_l)_{l \ge 1}$. Theorem 2.1.1 shows that in this situation

$$\mu(\varphi_{A_l} = k_l) \sim \mu(A_l)e^{-\mu(A_l)k_l} \quad \text{as } l \to \infty. \tag{3.76}$$

For a sequence $(\tau_l)_{l \ge 1}$ of natural numbers with $\mu(A_l)\tau_l \to 0$ and $\tau_l \le k_l$ (which we can just assume as $k_l = \Theta(\frac{1}{\mu(A_l)})$) define $\kappa_l := k_l - \tau_l$ and $u_l := \hat{T}^{\tau_l}\left(\frac{1_{A_l}}{\mu(A_l)}\right)$ for $l \ge 1$. As $(\kappa_l)_{l \ge 1}$ fulfils the assumptions on $(k_l)_{l \ge 1}$ in the statement of the Theorem, and furthermore $\mu(A_l)e^{-\mu(A_l)k_l} \sim \mu(A_l)e^{-\mu(A_l)\kappa_l}$, we have

$$\mu(\varphi_{A_l} = \kappa_l) \sim \mu(A_l)e^{-\mu(A_l)k_l} \quad \text{as } l \to \infty. \tag{3.77}$$

[4] In this context, we call maximal intervals of monotonicity cylinders.

(ii) We first show that for a suitable choice of $(\tau_l)_{l\geq 1}$ we have

$$|\mu(\varphi_{A_l} = \kappa_l) - u_l \odot \mu(\varphi_{A_l} = \kappa_l)| = o(\mu(A_l)) \quad \textit{as } l \to \infty. \tag{3.78}$$

Indeed by (3.69) and (3.72)

$$\begin{aligned}
&|\mu(\varphi_{A_l} = \kappa_l) - u_l \odot \mu(\varphi_{A_l} = \kappa_l)| \\
&= \left| \int_X 1_{\{\varphi_{A_l}=\kappa_l\}} \left(1 - \hat{T}^{\tau_l}\left(\frac{1_{A_l}}{\mu(A_l)}\right)\right) \mathrm{d}\mu \right| \\
&= \frac{1}{\mu(A_l)} \left| \int_X 1_{\{\varphi_{A_l}=\kappa_l\}} \circ T^{\tau_l} \left(\mu(A_l) - 1_{A_l}\right) \mathrm{d}\mu \right| \\
&\leq \frac{C}{\mu(A_l)} r^{\tau_l} ||1_{\{\varphi_{A_l}=\kappa_l\}}||_{L^{m+1}(\lambda)} ||1_{A_l}||_{BV} \\
&\leq 3C\mu(A_l)^2 = o(\mu(A_l))
\end{aligned}$$

as $l \to \infty$, for $\tau_l = 3\lceil \frac{\log(\mu(A_l))}{\log(r)} \rceil$.

(iii) Next we check that

$$|\mu_{A_l}(\varphi_{A_l} = k_l) - u_l \odot \mu(\varphi_{A_l} = \kappa_l)| = o(\mu(A_l)) \quad \textit{as } l \to \infty. \tag{3.79}$$

By Lemma 2.1.2 and (3.73) the left-hand side coincides with

$$\begin{aligned}
&\left| \int_X (1_{\{\varphi_{A_l}=\kappa_l\}} \circ T^{\tau_l} - 1_{\{\varphi_{A_l}=k_l\}}) \, \mathrm{d}\mu_{A_l} \right| \\
&= \frac{1}{\mu(A_l)} \int_X 1_{A_l} 1_{\{\varphi_{A_l}\leq\tau_l\}} 1_{\{\varphi_{A_l}\circ T^{\tau_l}=\kappa_l\}} \, \mathrm{d}\mu \\
&\leq \frac{1}{\mu(A_l)} \int_X 1_{A_l} 1_{\{\varphi_{A_l}\leq\tau_l\}} (1_{A_l} \circ T^{k_l}) \, \mathrm{d}\mu \\
&\leq \frac{C}{\mu(A_l)} r^{k_l} ||1_{A_l}||_{L^{m+1}(\lambda)} \underbrace{||1_{A_l} 1_{\{\varphi_{A_l}\leq\tau_l\}}||_{BV}}_{\leq 3\tau_l 2^{\tau_l+3}} \\
&+ \mu(A_l)\mu_{A_l}(\varphi_{A_l} \leq \tau_l)
\end{aligned}$$

the first part of the last term is bounded by $6C\mu(A_l)^2$ as soon as $k_l \geq 3\frac{\log(\mu(A_l))}{\log(r)} + \tau_l \frac{\log(\frac{1}{2})}{\log(r)} + \frac{\log(\tau_l)}{\log(\frac{1}{r})}$, which it eventually is, and is therefore of order $o(\mu(A_l))$. Furthermore the assumption $\mu_{A_l}(\mu(A_l)\varphi_{A_l} \leq t) \Rightarrow 1 - e^{-t}$ implies $\mu_{A_l}(\varphi_{A_l} \leq \tau_l) \to 0$ as $l \to \infty$. [5]

(iv) By (i) we only need to show

$$|\mu(\varphi_{A_l} = \kappa_l) - \mu_{A_l}(\varphi_{A_l} = k_l)| = o(\mu(A_l)). \tag{3.80}$$

[5] Indeed, for any $\delta > 0$ we have $\tau_l \leq \delta \frac{1}{\mu(A_l)}$ for large enough l, it follows that $\mu_{A_l}(\varphi_{A_l} \leq \tau_l) \leq \mu_{A_l}(\mu(A_l)\varphi_{A_l} \leq \delta) \to 1 - e^{-\delta}$.

But this holds by (ii) and (iii)

$$\begin{aligned}
&|\mu(\varphi_{A_l} = \kappa_l) - \mu_{A_l}(\varphi_{A_l} = k_l)| \\
&\leq |\mu(\varphi_{A_l} = \kappa_l) - u_l \odot \mu(\varphi_{A_l} = \kappa_l)| \\
&+ |u_l \odot \mu(\varphi_{A_l} = \kappa_l) - \mu_{A_l}(\varphi_{A_l} = k_l)| \,.
\end{aligned}$$

□

Remark 3.3.2. *This proof together with Remark 3.3.1, show that, for $\mathcal{R}$ - maps with finitely many cylinders, in Theorem 3.2.2, the condition* $\inf_{h>0} h >$ *can be omitted.*

3.4 Periodic points in Rychlik maps

Up until now we (mostly) dealt with exponential limit laws. The reason being that **a)** our assumptions were already (almost) strong enough to imply convergence in law to the exponential law and **b)** we always showed equivalence of LLTs for hitting and return times, which gives the "correct" rate (as per motivation in chapter 2) only in the case of the exponential law. Here we shall investigate a similar case, which appears for example near periodic points in chaotic interval maps. In typical situations one usually gets

$$\mu(\mu(A_l)\varphi_{A_l} \leq t) \Rightarrow 1 - e^{-\theta t} \quad \text{as } l \to \infty, \tag{3.81}$$

when the A_l are intervals shrinking to a point x^* of period p and where $\theta = 1 - \frac{1}{(T^p)'(x^*)}$ (see [Kel12] or [FFT12]). We will prove LLTs for return times in such cases by modifying some of the arguments above[6].

Theorem 3.4.1. *Let $X = [0,1]$, $\mathcal{A}$ the Borel-σ-algebra on X and $T : X \to X$ be a piecewise C^1 map on X such that for some density $h : X \to [0,\infty)$ and $\mu = h \odot \lambda$ - where λ is the Lebesgue-measure on X - the dynamical system $(X, \mathcal{A}, \mu, T)$ is probability preserving and weakly mixing. Assume furthermore that $(X, \mathcal{A}, \lambda, T)$ is a $\mathcal{R}$-map with $M < \infty$ many cylinders. Let $x^* \in X \setminus \left(\bigcup_{j\geq 0} T^{-j} S\right)$ be a point of period p such that h is continuous at x^* and $h(x^*) > 0$. and $(A_l)_{l\geq 1}$ be a sequence of balls shrinking to x^*, i.e for $l \geq 0$ let $r_l > 0$ such that $r_l \to 0$ as $l \to \infty$ and set $A_l = (x^* - r_l, x^* + r_l) \cap X$. Suppose*

$$\mu(\mu(A_l)\varphi_{A_l} \leq t) \Rightarrow 1 - e^{-\theta t} \quad \text{as } l \to \infty, \tag{3.82}$$

where $\theta = 1 - \frac{1}{(T^p)'(x^*)}$. Then for all sequences $(k_l)_{l\geq 1}$ of natural numbers such that $(\mu(A_l)k_l)_{l\geq 1}$ is bounded away from 0 and ∞ we have

$$\mu_{A_l}(\varphi_{A_l} = k_l) \sim \theta^2 \mu(A_l) e^{-\theta\mu(A_l)k_l} \quad \text{as } l \to \infty. \tag{3.83}$$

[6] One point where we have to be careful is that in the analysis above we regularly used $\mu_{A_l}(\varphi_{A_l} \leq \tau_l) \to 0$ whenever $\tau_l = o\left(\frac{1}{\mu(A_l)}\right)$. This is not the case here.

Proof. (i) Recall that in the present situation

$$\left|\int_X \varphi \circ T^n \psi \, \mathrm{d}\mu - \int_X \varphi \, \mathrm{d}\mu \int_X \psi \, \mathrm{d}\mu\right| \leq C||\varphi||_{L^1(\lambda)}||\psi||_{BV} r^n \quad (3.84)$$

for some $C > 0$, $r \in (0,1)$ and all $\varphi \in L^1(\lambda)$, $\psi \in BV$ and $n \in \mathbb{N}$. Furthermore

$$\mathrm{Var}(1_{\{\varphi_A \leq n\}}) \leq n^2 M^n. \quad (3.85)$$

Also Theorem 2.1.1 implies that for all sequences $(k_l')_{l\geq 1}$ of natural numbers such that $(\mu(A_l)k_l')_{l\geq 1}$ is bounded away from 0 and ∞ we have

$$\mu(\varphi_{A_l} = k_l') \sim \theta\mu(A_l)e^{-\theta\mu(A_l)k_l'} \quad \textit{as } l \to \infty. \quad (3.86)$$

(ii) Assume that x^* is not on the boundary of X and $A_l = (x^* - r_l, x^* + r_l)$, the proof for x^* on the boundary goes analogously. Then $T^{-p}(A_l)$ is a disjoint union of at most M^p intervals, one containing x^*, which we call $A_l^p = (a_l, b_l)$. Taking l big enough we can assume $A_l \cap B_l \supset A_l^p$, on the other hand A_l^p has length $\frac{1}{(T^p)'(\xi_l)}\lambda(A_l)$ for some $\xi_l \in A_l^p$ and therefore, as T is expanding, shorter than A_l and therefore[7] $A_l^p \subset A_l$. Furthermore denote $A_l^e = A_l \setminus A_l^p$, which is a disjoint union of at most two intervals.

(iii) Using the MVT for integrals we get

$$\mu(A_l) = \int_{x^*-r_l}^{x^*+r_l} h(y)\, \mathrm{d}y = 2r_l h(\xi_l^{(1)}) \quad (3.87)$$

and

$$\mu(A_l^p) = \int_{a_l}^{b_l} h(y)\, \mathrm{d}y = (a_l - b_l)h(\xi_l^{(2)}) = 2r_l \frac{1}{(T^p)'(\xi_l)} h(\xi_l^{(2)}), \quad (3.88)$$

for some $\xi_l^{(1)} \in A_l$ and $\xi_l^{(2)} \in A_l^p$. By continuity of $(T^p)'$and h at x^* together with $h(x^*) > 0$, we get $\mu(A_l^p) \sim (1-\theta)\mu(A_l)$.

(iv) Using that $\varphi_{A_l} \leq p$ on A_l^p, for any sequence $(\tau_l')_{l\geq 1}$ of positive real numbers such that $\tau_l' \to \infty$ as $l \to \infty$, but $\tau_l' = o\left(\frac{1}{\mu(A_l)}\right)$, as soon as $\tau_l' \geq p$ we get

$$\begin{aligned}\mu_{A_l}(A_l^e \cap \{\varphi_{A_l} \leq \tau_l'\}) &= \mu_{A_l}(\varphi_{A_l} \leq \tau_l') - \mu_{A_l}(A_l^p \cap \{\varphi_{A_l} \leq \tau_l'\}) \\ &= \mu_{A_l}(\varphi_{A_l} \leq \tau_l') - \frac{\mu(A_l^p)}{\mu(A_l)} \to 0\end{aligned}$$

as $l \to \infty$.

[7] More precisely the part of A_l^p to the left (resp. right) side of x^* has length $< r_l$. A subtlety here is that this argument requires that A_l is a ball around x^* as opposed to, say, just any interval. One could imagine the situation if A_l to the left of x^* is much shorter than to the right, if T^p is then monotone decreasing around x^* the given argument fails.

(v) For a sequence $(\tau_l)_{l\geq 1}$ of positive real numbers such that $\tau_l \to \infty$ as $l \to \infty$ and $\tau_l = o\left(\frac{1}{\mu(A_l)}\right)$, to be determined later, set $\kappa_l = k_l - \tau_l$ (assuming $k_l > \tau_l$). The result follows as soon as we show

$$|\mu(A_l \cap \{\varphi_{A_l} = k_l\}) - \mu(A_l^e)\mu(\varphi_{A_l} = \kappa_l)| = o(\mu(A_l)^2) \tag{3.89}$$

as $l \to \infty$. Note that for $n > p$ we have $A_l \cap \{\varphi_{A_l} = n\} = A_l^e \cap \{\varphi_{A_l} = n\}$. We can estimate the left side of (3.89) by

$$\begin{aligned}
&|\mu(A_l^e \cap \{\varphi_{A_l} = k_l\}) - \mu(A_l^e)\mu(\varphi_{A_l} = \kappa_l)| \\
&\leq |\mu(A_l^e \cap \{\varphi_{A_l} = k_l\}) - \mu(A_l^e \cap T^{-\tau_l}\{\varphi_{A_l} = \kappa_l\})| \\
&+ |\mu(A_l^e \cap T^{-\tau_l}\{\varphi_{A_l} = \kappa_l\}) - \mu(A_l^e)\mu(\varphi_{A_l} = \kappa_l)|
\end{aligned}$$

(vi) We first show

$$|\mu(A_l^e \cap T^{-\tau_l}\{\varphi_{A_l} = \kappa_l\}) - \mu(A_l^e)\mu(\varphi_{A_l} = \kappa_l)| = o(\mu(A_l)^2) \tag{3.90}$$

We can estimate the left side of (3.90) by

$$\begin{aligned}
&\left|\int_X 1_{A_l^e}\left(1_{\{\varphi_{A_l}=\kappa_l\}} \circ T^{\tau_l} - \int_X 1_{\{\varphi_{A_l}=\kappa_l\}}\, \mathrm{d}\mu\right) \mathrm{d}\mu\right| \\
&\leq 3Cr^{\tau_l} \leq 3C\mu(A_l)^3 \leq o(\mu(A_l)^2)
\end{aligned}$$

as $l \to \infty$, and choosing $\tau_l = 3\left\lceil \frac{\log(\mu(A_l)}{\log(r)} \right\rceil$.

(vii) Now we show

$$|\mu(A_l^e \cap \{\varphi_{A_l} = k_l\}) - \mu(A_l^e \cap T^{-\tau_l}\{\varphi_{A_l} = \kappa_l\})| = o(\mu(A_l)^2) \tag{3.91}$$

By Lemma 2.1.2 we can estimate the left side of (3.91) by

$$\begin{aligned}
&\left|\int_X 1_{A_l^e}\left(1_{\{\varphi_{A_l}=\kappa_l\}} \circ T^{\tau_l} - 1_{\{\varphi_{A_l}=k_l\}}\right) \mathrm{d}\mu\right| \\
&\leq \int_X 1_{A_l^e} 1_{\{\varphi_{A_l}\leq\tau-l\}}\left(1_{A_l} \circ T^{k_l}\right) \mathrm{d}\mu \\
&\leq \left|\int_X 1_{A_l^e} 1_{\{\varphi_{A_l}\leq\tau-l\}}\left(1_{A_l} \circ T^{k_l} - \int_X 1_{A_l}\, \mathrm{d}\mu\right) \mathrm{d}\mu\right| + \mu(A_l)\mu(A_l^e \cap \{\varphi_{A_l} \leq \tau - l\}) \\
&\leq 5C\tau_l^2 r^{k_l} M^{\tau_l} + o(\mu(A_l)^2) \leq 5C\mu(A_l)^3 + o(\mu(A_l)^2) = o(\mu(A_l)^2)
\end{aligned}$$

as soon as $k_l > 3\frac{\log(\mu(A_l))}{\log(r)} - 2\frac{\log(\tau_l)}{\log(r)} - \tau_l\frac{\log(M)}{\log(r)}$, and as $l \to \infty$.

□

The moral of this proof is that A_l^p is the part of weight $1-\theta$ that returns after p steps, while all the exponential behaviour is in A_l^e, which, by being far away from x^*, does behave aperiodically. In fact even the strong aperiodicity condition used in Theorem 3.2.2 holds.

Lemma 3.4.1. *In the situation of the previous Theorem if* ***a)*** A_l *is an* n_l *cylinder for some* $(n_l)_{l\geq 1}$ *or* ***b)*** T^p *is monotone increasing near* x^* *we have*

$$\min_{x\in A_l^e} \varphi_{A_l} \to \infty \tag{3.92}$$

as $l \to \infty$.

Proof. (i) The same analysis as for aperiodic points in Theorem 3.2.2 shows that for any $n \geq 1$ which is not a multiple of p there is a $L_n \geq 1$ such that

$$\varphi_{A_l}(x) \neq n \quad \forall x \in A_l \tag{3.93}$$

for $l \geq L_n$.

(ii) Now for $k \geq 1$ choose L_{kp} so big that A_l is contained in one interval where T^{kp} is continuous, say I, for $l \geq L_{kp}$. We show that $A_l^{kp} = T^{-kp}(A_l) \cap A_l = T^{-kp}(A_l) \cap I$ is contained in A_l^p. Then it follows that

$$\varphi_{A_l}(x) \neq kp \quad \forall x \in A_l^e \tag{3.94}$$

for $l \geq L_{kp}$.

(iii) In case **a)** this follows from $A_l^p = \zeta_{n_l+p}(x^*)$ and $A_l^{kp} = \zeta_{n_l+kp}(x^*)$.

(iv) In case **b)** $T^{(k-1)p}(A_l^{kp})$ is an interval containing x^* and also a subset of $T^{-p}(A_l)$, and therefore a subset of A_l^p. It follows that A_l^{kp} itself is an interval containing x^* of length at most $\lambda(A_l^p)\frac{1}{(T^{(k-1)p})'(\xi_l)}$ for some $\xi_l \in A_l^{kp}$. This together with $T^{(k-1)p}$ being monotone increasing near x^* implies (see argument above) $A_l^{kp} \subset A_l^p$.

□

Remark 3.4.1.

- *Note that we only used the fact that the* A_l *are balls when showing* $A_l^p \subset A_l$. *Therefore, as seen above, this can be replaced by either*

  a) for some natural numbers $(n_l)_{l\geq 1}$ *with* $n_l \to \infty$ *as* $l \to \infty$ *we have* $A_l = \zeta_{n_l}(x^*)$, *in that case* $A_l^p = \zeta_{n_l+p}(x^*)$, *or*

  b) $(A_l)_{l\geq 1}$ *are some intervals shrinking to* x^*, *and* T^p *is monotone increasing in some neighbourhood of* x^*.

- *Using previous results together with Lemma 3.4.1, we see that we can also make statements analogous to Theorem 3.4.1 if* $(X, \mathcal{A}, \lambda, T)$ *is an* $\mathcal{R}$*-map with countably many cylinders and* $\inf_{h>0} h > 0$.

3.5 Subshifts of finite type

We now look at a certain class of SFTs described in [MD01].
Therefore let A be a countable set (for convenience $A = \mathbb{N}$), $B = (b_{i,j})_{i,j \in A \times A}$ a matrix of 0 and 1 and $\Sigma := \{x \in A^{\mathbb{N}} \mid x_{i,i+1} = 1 \text{ } for \text{ } i \geq 1\}$. On Σ define the shift operator σ as usual by $\sigma((x_0, x_1, x_2, ...)) = (x_1, x_2, ...)$. We equip Σ with the product topology, for any $r \in (0,1)$ this can be described the metric

$$d(x,y) = r^{\min\{i \geq 0 \mid x_i \neq y_i\}} \quad for \ x,y \in \Sigma. \tag{3.95}$$

As in our example of Markov-chains, define for $n \geq 1$ the n-cylinders by

$$[i_0, ..., i_{n-1}] := \{x \in \Sigma \mid x_j = i_j \ for \ 0 \leq j \leq n-1\} \quad for \ i_0, ..., i_{n-1} \in A, \tag{3.96}$$

and let $\mathcal{A}$ be the σ-Algebra generated by all cylinders. Denote the collection of all 1-cylinders by ζ. And λ be some Borel probability measure on Σ with support Σ.
We will assume that Σ is aperiodic, i.e for all $i, j \in A$ there is a $n_0 \geq 1$ such that

$$\sigma^{-n}[i] \cap [j] \neq \emptyset \quad for \ n \geq n_0. \tag{3.97}$$

Define the Lipschitz-constant of a function $f : \Sigma \to \mathbb{R}$ as the smallest $K > 0$ such that

$$|f(x) - f(y)| \leq K|x - y| \quad for \ all \ x, y \in \Sigma, \tag{3.98}$$

denoted by $\mathrm{Lip}(f)$, and define $||f|| := \max(||f||_\infty, \mathrm{Lip}(f))$. Note that due to Σ having "gaps", i.e different n-cylinders have a distance of r^{n-1}, even some indicators are Lipschitz continuous, e.g if Z is a n-cylinder then $||1_Z|| \leq r^{-(n-1)}$.

Following thermodynamic formalism we assume there is a function $\Phi : \Sigma \to \mathbb{R}$ called a potential such that

- $C := ||\Phi|| < \infty$,
- $\sup_{x \in \Sigma} \sum_{y \in \sigma^{-1}(\{x\})} e^{\Phi(y)}$, so the operator P given by

$$P(f)(x) := \sup_{x \in \Sigma} \sum_{y \in \sigma^{-1} x} e^{\Phi(y)} f(y) \quad for \ x \in \Sigma \tag{3.99}$$

  for $f : \Sigma \to \mathbb{R}$ bounded, is well defined.
- P coincides with the transfer operator $\hat{\sigma}_\lambda$, that is for $f, g \in L^1\lambda$ we have

$$\int_\Sigma f \circ \sigma \cdot g \, \mathrm{d}\lambda = \int_\Sigma f P(g) \, \mathrm{d}\lambda. \tag{3.100}$$

Last we assume that cylinder close to infinity do not contribute too much to P, that is there are $k_1 \geq 1$ and $n_1 \geq 1$ such that for all $k > k_1$ there is a constant $p_k < 1$ such that

$$\sup_{x\in[n]} |P^k(1)(x)| \leq p_k \quad \textit{for } n > n_1. \tag{3.101}$$

Note that if A is finite, the last assumption is superfluous and the well definiteness of P on bounded functions follows from boundedness of the potential.

[MD01] shows that in this case there is an invariant probability measure μ on Σ such that $(\Sigma, \mathcal{A}, \mu, \sigma)$ enjoys exponential decay of correlations. More precisely there are constants $C > 0$ and $r \in (0,1)$ such that for $f, g \in L_\infty$ and $n \geq 1$ we have

$$\left|\int_\Sigma f(g \circ \sigma^n)\,\mathrm{d}\mu - \int_\Sigma f\,\mathrm{d}\mu \int_\Sigma g\,\mathrm{d}\mu\right| \leq Cr^n ||f||\,||g||_\infty. \tag{3.102}$$

We now proceed in a similar fashion as we did before. First we need to estimate the Lipschitz-coefficients for some indicator functions.

Lemma 3.5.1. *Let $A \subset \Sigma$ be a union of n-cylinders for some $n \geq 1$, and $k \geq 1$ be a natural number. Then*

$$\mathrm{Lip}(1_{\{\varphi_A = k\}}) \leq 2^{n+k}, \tag{3.103}$$

and

$$\mathrm{Lip}(1_{\{\varphi_A \leq k\}}) \leq 2^{n+k}. \tag{3.104}$$

Proof. Since $\{\varphi_A = k\}$ is the disjoint union of some $n+k$-cylinders, say W is the collection of all those $n+k$- cylinders, we get

$$\mathrm{Lip}(1_{\{\varphi_A = k\}}) \leq \max_{Z\in W} \mathrm{Lip}(1_Z) = 2^{n+k}. \tag{3.105}$$

Likewise $\{\varphi_A \leq k\} = \bigcup_{j=1}^k \{\varphi_A = j\}$, and this union is also disjoint. □

Theorem 3.5.1. *Let $(\Sigma, \mathcal{A}, \sigma, \mu)$ be as above. Let $(A_l)_{l\geq 1}$ be a sequence of asymptotically rare events such that either **a)** A_l is ζ measurable, i.e A_l is a union of 1-cylinders or **b)** A_l is a l-cylinder. Suppose in case **b)** that $l = o\left(\frac{1}{\mu(A_l)}\right)$.* [8] *Suppose that the normalised return times converge to the exponential law,*

$$\mu_{A_l}(\mu(A_l)\varphi_{A_l} \leq t) \Rightarrow 1 - e^{-t} \quad \textit{as } l \to \infty. \tag{3.106}$$

Then for all sequences $(k_l)_{l\geq 1}$ of natural numbers such that $(\mu(A_l)k_l)_{l\geq 1}$ is bounded away from 0 and ∞ we have

$$\mu_{A_l}(\varphi_{A_l} = k_l) \sim \mu(A_l)e^{-\mu(A_l)k_l} \quad \textit{as } l \to \infty. \tag{3.107}$$

[8] Which is true in all interesting cases.

Proof. (i) Fix such a sequence $(k_l)_{l\geq 1}$. Theorem 2.1.1 shows that in this situation

$$\mu(\varphi_{A_l} = k_l) \sim \mu(A_l)e^{-\mu(A_l)k_l} \quad \textit{as } l \to \infty. \tag{3.108}$$

For a sequence $(\tau_l)_{l\geq 1}$ of natural numbers with $\mu(A_l)\tau_l \to 0$ and $\tau_l \leq k_l$ (which we can just assume as $k_l = \Theta(\frac{1}{\mu(A_l)})$) define $\kappa_l := k_l - \tau_l$ and $u_l := \hat{T}^{\tau_l}\left(\frac{1_{A_l}}{\mu(A_l)}\right)$ for $l \geq 1$. As $(\kappa_l)_{l\geq 1}$ fulfils the assumptions on $(k_l)_{l\geq 1}$ in the statement of the Theorem and furthermore $\mu(A_l)e^{-\mu(A_l)k_l} \sim \mu(A_l)e^{-\mu(A_l)\kappa_l}$, we have

$$\mu(\varphi_{A_l} = \kappa_l) \sim \mu(A_l)e^{-\mu(A_l)k_l} \quad \textit{as } l \to \infty. \tag{3.109}$$

(ii) We first show that for a suitable choice of $(\tau_l)_{l\geq 1}$ we have

$$|\mu(\varphi_{A_l} = \kappa_l) - u_l \odot \mu(\varphi_{A_l} = \kappa_l)| = o(\mu(A_l)) \quad \textit{as } l \to \infty. \tag{3.110}$$

Indeed by Lemma 3.5.1

$$\begin{aligned}
&|\mu(\varphi_{A_l} = \kappa_l) - u_l \odot \mu(\varphi_{A_l} = \kappa_l)| \\
&= \left|\int_X 1_{\{\varphi_{A_l}=\kappa_l\}}\left(1 - \hat{T}^{\tau_l}\left(\frac{1_{A_l}}{\mu(A_l)}\right)\right) \mathrm{d}\mu\right| \\
&= \frac{1}{\mu(A_l)}\left|\int_X 1_{\{\varphi_{A_l}=\kappa_l\}} \circ T^{\tau_l}\left(\mu(A_l) - 1_{A_l}\right) \mathrm{d}\mu\right| \\
&\leq \frac{C}{\mu(A_l)} r^{\tau_l} ||1_{\{\varphi_{A_l}=\kappa_l\}}||_{L^\infty} \mathrm{Lip}(1_{A_l}) \\
&\leq 2C\mu(A_l)^2 = o(\mu(A_l))
\end{aligned}$$

for $\tau_l = 3\lceil\frac{\log(\mu(A_l))}{\log(r)}\rceil$.

(iii) Next we check that

$$|\mu_{A_l}(\varphi_{A_l} = k_l) - u_l \odot \mu(\varphi_{A_l} = \kappa_l)| = o(\mu(A_l)) \quad \textit{as } l \to \infty. \tag{3.111}$$

By Lemma 3.5.1 the left-hand side coincides with

$$\begin{aligned}
&\left|\int_X (1_{\{\varphi_{A_l}=\kappa_l\}} \circ T^{\tau_l} - 1_{\{\varphi_{A_l}=k_l\}}) \, \mathrm{d}\mu_{A_l}\right| \\
&= \frac{1}{\mu(A_l)}\int_X 1_{A_l} 1_{\{\varphi_{A_l}\leq\tau_l\}} 1_{\{\varphi_{A_l}\circ T^{\tau_l}=\kappa_l\}} \, \mathrm{d}\mu \\
&\leq \frac{1}{\mu(A_l)}\int_X 1_{A_l} 1_{\{\varphi_{A_l}\leq\tau_l\}}(1_{A_l} \circ T^{k_l}) \, \mathrm{d}\mu \\
&\leq \frac{C}{\mu(A_l)} r^{k_l} ||1_{A_l}||_{L^\infty} \underbrace{\mathrm{Lip}(1_{A_l}1_{\{\varphi_{A_l}\leq\tau_l\}})}_{\substack{\leq 2^{\tau_l+2} \quad \textit{in case } \mathbf{a)} \\ \leq 2^{\tau_l+2l} \quad \textit{in case } \mathbf{b)}}} \\
&+ \mu(A_l)\mu_{A_l}(\varphi_{A_l} \leq \tau_l)
\end{aligned}$$

the first part of the last term is bounded by $4C\mu(A_l)^2$ as soon as $k_l \geq 2\frac{\log(\mu(A_l))}{\log(r)} + l\frac{\log(\frac{1}{2})}{\log(r)}$, which it eventually is, and is therefore of order $o(\mu(A_l))$. Furthermore the assumption $\mu_{A_l}(\mu(A_l)\varphi_{A_l} \leq t) \Rightarrow 1 - e^{-t}$ implies $\mu_{A_l}(\varphi_{A_l} \leq \tau_l) \to 0$ as $l \to \infty$. [9]

(iv) By (i) we only need to show

$$|\mu(\varphi_{A_l} = \kappa_l) - \mu_{A_l}(\varphi_{A_l} = k_l)| = o(\mu(A_l)). \tag{3.112}$$

But this holds by (ii) and (iii)

$$\begin{aligned} &|\mu(\varphi_{A_l} = \kappa_l) - \mu_{A_l}(\varphi_{A_l} = k_l)| \\ &\leq |\mu(\varphi_{A_l} = \kappa_l) - u_l \odot \mu(\varphi_{A_l} = \kappa_l)| \\ &+ |u_l \odot \mu(\varphi_{A_l} = \kappa_l) - \mu_{A_l}(\varphi_{A_l} = k_l)|\,. \end{aligned}$$

□

3.6 General discussion

In the last three examples we used an estimate on the decay of correlations on certain classes of functions. Let us make some comments on how to apply the same techniques to systems with similar estimates. We will discuss some necessary and sufficient conditions and formulate a version of this. The arguments in this section are rather loose, precise arguments are collect in Theorem 3.6.1.

Let us say we are working on some ergodic, probability-preserving system $(X, \mathcal{A}, \mu, T)$. Assume an estimate on the decay of correlations of following form; There is some decreasing function $c : \mathbb{N} \to [0, \infty)$ with $c(n) \to 0$ as $n \to \infty$ and Banach algebras of real valued functions on X, $(\mathcal{B}, ||\cdot||_{\mathcal{B}})$ and $(\mathcal{B}', ||\cdot||_{\mathcal{B}'})$ such that for $n \in \mathbb{N}$, $f \in \mathcal{B}$ and $g \in \mathcal{B}'$ we have

$$\left| \int_X f\,(g \circ T^n)\,\mathrm{d}\mu - \int_X f\,\mathrm{d}\mu \int_X g\,\mathrm{d}\mu \right| \leq c(n)||f||_{\mathcal{B}}||g||_{\mathcal{B}'}. \tag{3.113}$$

For our LLTs to follow from convergence in law, we needed

$$\left| \int_X 1_{\{\varphi_{A_l}=\kappa_l\}} \circ T^{\tau_l} \cdot 1_{A_l}\,\mathrm{d}\mu - \int_X 1_{\{\varphi_{A_l}=\kappa_l\}}\,\mathrm{d}\mu \int_X 1_{A_l}\,\mathrm{d}\mu \right| \stackrel{!}{=} o(\mu(A_l)^2) \tag{3.114}$$

and

$$\left| \int_X 1_{A_l} \cdot 1_{\{\varphi_{A_l}\leq\tau_l\}}(1_{A_l} \circ T^{k_l})\,\mathrm{d}\mu - \int_X 1_{A_l} \cdot 1_{\{\varphi_{A_l}\leq\tau_l\}}\,\mathrm{d}\mu \int_X 1_{A_l}\,\mathrm{d}\mu \right| \stackrel{!}{=} o(\mu(A_l)^2), \tag{3.115}$$

[9] Indeed, for any $\delta > 0$ we have $\tau_l \leq \delta\frac{1}{\mu(A_l)}$ for large enough l, it follows that $\mu_{A_l}(\varphi_{A_l} \leq \tau_l) \leq \mu_{A_l}(\mu(A_l)\varphi_{A_l} \leq \delta) \to 1 - e^{-\delta}$.

as well as (following notation from above) $\tilde{F}(0) = 0$. We first remark that usually $||\cdot||_{\mathcal{B}'} = ||\cdot||_{L^\infty}$ or a similar norm, which is bounded on indicators. Therefore we will omit this part of the estimate and in addition write $||\cdot||$ instead of $||\cdot||_{\mathcal{B}}$.
If this is not the case, e.g for smooth T on a Riemannian manifold in some situations we have that $||\cdot||_{\mathcal{B}'} = ||\cdot||_{C^\alpha}$ for some $\alpha > 0$, there is not much hope of acquiring the desired estimates this way, as will be apparent soon.

In our examples we had that $c(n)$ decays exponentially. However such a strong decay is not necessary. We first give a rough criterion on how fast the decay should be.
In many common situations we will have that $||\cdot||$ is the bounded variation norm, some (symbolic) Lipschitz or Hölder norm or some C^k norm. In all of these cases (usually) composition with T is bounded with respect to $||\cdot||$ i.e there is a constant $K > 1$ such that [10]

$$||f \circ T|| \leq K||f|| \tag{3.116}$$

for $f \in \mathcal{B}$. Suppose first that $(A_l)_{l\geq 1}$ is chosen in a way that $1_{A_l} \in \mathcal{B}$ and $||1_{A_l}||$ is bounded, say by $L > 0$. Then the left side of (3.114) can be estimated by $Lc(\tau_l)$. Denoting $c^{-1}(x) =: \min(n \mid c(n) \leq x)$ for $x > 0$, we see that we will at least need $\tau_l \geq c^{-1}(\mu(A_l)^2)$.
On the other hand the left side of (3.115) can be roughly estimated by $L(L+1)^{\tau_l} K^{\tau_l^2} c(k_l) \leq \hat{K}^{\tau_l^2} c(k_l)$, for some constant $\hat{K} \geq K$, this should be eventually bounded by $\mu(A_l)^2$. So taking $x = \mu(A_l)$ and assuming $k_l = \frac{1}{x}$ we get

$$e^{\log(\hat{K})(c^{-1}(x^2))^2} c\left(\frac{1}{x}\right) \leq x^2 \tag{3.117}$$

rewriting we see that c should have the property that for x small enough

$$x^2 \geq c\left(\sqrt{\frac{\log\left(\frac{x^2}{c\left(\frac{1}{x}\right)}\right)}{\log(\hat{K})}}\right). \tag{3.118}$$

Taking some faster decaying quantity than x^2 e.g x^3 gives the sufficient condition

$$x^3 \geq c\left(\sqrt{\frac{\log\left(\frac{x^3}{c\left(\frac{1}{x}\right)}\right)}{\log(\hat{K})}}\right). \tag{3.119}$$

[10] The condition $K \geq 1$ comes from the fact that constant functions are invariant under compositon with T. The case $K = 1$ is much easier and left to the reader.

We note that as a consequence polynomial decay is not enough, e.g if $c(n) = n^{-\alpha}$ for some $\alpha > 2$ then (3.118) becomes

$$x^2 \geq \left(\frac{\alpha - 2}{\log(\hat{K})}\right)^{-\frac{\alpha}{2}} (-\log(x))^{-\frac{\alpha}{2}}, \tag{3.120}$$

which is not true for x sufficiently small. Another common situation, that arises e.g in [BLVS03], is that of a stretched exponential i.e $c(n) = e^{-\beta n^\alpha}$ for some $\alpha \in (0,1)$ and $\beta > 0$. In that case the right side of (3.119) becomes

$$e^{-\frac{\beta}{\sqrt{\log(\hat{K})}}\left(-\log\left(\frac{1}{x^3}\right)+\beta x^{-\alpha}\right)^{\frac{\alpha}{2}}} \geq e^{-\tilde{K}x^{-\frac{\alpha^2}{2}}} \tag{3.121}$$

for some constant $\tilde{K} > 0$, which decays much faster than x^3. Therefore if the norm in the estimates on correlations are as above, we can conclude that LlTs for hitting times follow from convergence in distribution.
Note that LLTs for consecutive hitting times are much more difficult to obtain using this method. One obstacle (following notation from before) is that we would need to estimate $\left|\left|1_{W_l^{(m)} \cap T^{-s_l^{(m)}}\{\varphi_{A_l} \leq \tau_l\}}\right|\right|$, which, using the estimates described above, we get a bound that grows much faster than $K^{\frac{1}{\mu(A_l)}}$.
This also explains why we would want to assume $||\cdot||_{\mathcal{B}'}$ bounded on indicators. If it behaved in the same way as $||\cdot||$, we would have the same problem estimating $||1_{\{\varphi_{A_l} = \kappa_l\}}||_{\mathcal{B}'}$.

Another obstacle is that 1_{A_l} might not be an element of $\mathcal{B}$. We have to still be able to approximate the indicator functions in question well with functions of $\mathcal{B}$. We shall demonstrate what we mean by that in a typical situation.

Assume $X = [0,1]$, A_l is a closed interval for each $l \geq 1$ and $||\cdot|| = \mathrm{Lip}(\cdot)$ is the (genuine, not a symbolic) Lipschitz norm. We still assume that composition with T is bounded with respect to the Lipschitz norm, which is satisfied e.g if T itself is Lipschitz continuous. For simplicity assume exponential decay of correlation $c(n) = e^{-\alpha n}$ for some $\alpha > 0$, stretched exponential or similar decay would also do.
For $l \geq 1$ we choose small $\epsilon_l > 0$, to be determined later, and approximate 1_{A_l} by $f_l : [0,1] \to [0,1]$ given by

$$f_l(x) := \max(0, 1 - \epsilon_l^{-1} d(x, A_l)). \tag{3.122}$$

Note that $\mathrm{Lip}(f_l) = \epsilon_l^{-1}$, while $||f_l - 1_{A_l}||_{L^1} = \epsilon_l$. We estimate the right side of (3.114) by

$$\begin{aligned}
&\left|\int_X 1_{\{\varphi_{A_l}=\kappa_l\}} \circ T^{\tau_l} \left(1_{A_l} - f_l\right) \mathrm{d}\mu\right| \\
&+ \left|\int_X 1_{\{\varphi_{A_l}=\kappa_l\}} \circ T^{\tau_l} f_l \,\mathrm{d}\mu - \int_X 1_{\{\varphi_{A_l}=\kappa_l\}} \,\mathrm{d}\mu \int_X f_l \,\mathrm{d}\mu\right| \\
&+ \left|\int_X 1_{\{\varphi_{A_l}=\kappa_l\}} \,\mathrm{d}\mu \left(\int_X 1_{A_l} - f_l \,\mathrm{d}\mu\right)\right| \\
&\leq 2\epsilon_l + e^{-\alpha\tau_l}\epsilon_l^{-1} = o(\mu(A_l)^2)
\end{aligned}$$

by setting $\epsilon_l = \mu(A_l)^3$ and $\tau_l = \lceil \frac{-6\log(\mu(A_l))}{\alpha} \rceil$.
In order to estimate (3.115) in the same fashion first note that

$$\begin{aligned}
1_{A_l\cap\{\varphi_{A_l}\leq\tau_l\}} &= \max_{1\leq i\leq\tau_l} (1_{A_l\cap\{\varphi_{A_l}=i})\\
&= \max_{1\leq i\leq\tau_l} \left(1_{A_l} \prod_{j=1}^{i-1}((1-1_{A_l}) \circ T^j)(1_{A_l} \circ T^i)\right).
\end{aligned}$$

So set

$$g_l(x) := \max_{1\leq i\leq\tau_l} \left(f_l \prod_{j=1}^{i-1}((1-f_l) \circ T^j)(f_l \circ T^i)\right). \tag{3.123}$$

A straightforward calculation (see Theorem 3.6.1) reveals that $\mathrm{Lip}(g_l) \leq K^{\tau_l^2}\epsilon_l^{-1}$ and
$||g_l - 1_{A_l\cap\{\varphi_{A_l}\leq\tau_l\}}||_{L^1} \leq \tau_l^2\epsilon_l$. So the right side of (3.115) can be bounded by

$$\begin{aligned}
&\left|\int_X 1_{A_l} \circ T^{k_l} \left(1_{A_l\cap\{\varphi_{A_l}\leq\tau_l\}} - g_l\right) \mathrm{d}\mu\right| \\
&+ \left|\int_X 1_{A_l\cap\{\varphi_{A_l}\leq\tau_l\}}(1_{A_l} \circ T^{k_l}) \,\mathrm{d}\mu - \int_X 1_{A_l\cap\{\varphi_{A_l}\leq\tau_l\}} \,\mathrm{d}\mu \int_X 1_{A_l} \,\mathrm{d}\mu\right| \\
&+ \left|\int_X 1_{A_l} \,\mathrm{d}\mu \left(\int_X 1_{A_l\cap\{\varphi_{A_l}\leq\tau_l\}} - g_l \,\mathrm{d}\mu\right)\right| \\
&\leq 2\tau_l^2\epsilon_l + K^{\tau_l^2}\epsilon_l^{-1}e^{-\alpha k_l} = o(\mu(A_l)^2)
\end{aligned}$$

as soon as $k_l \geq \tau_l + \frac{\log(K)\tau_l^2}{\alpha}$.

Our analysis so far relied on the fact that composition with T is bounded with respect to $||\cdot||$. But in some situations, e.g Section 3.2, this might not be the case. In that case assume instead that the transfer operator $\hat{T}$ is bounded with respect to $||\cdot||$, and $||\cdot||_{\mathcal{B}'} = ||\cdot||_{L^1}$. Furthermore assume that there are natural numbers $(n_l)_{l\geq 1}$ such that $\varphi_{A_l} \geq n_l$ for $l \geq 1$, and

$n_l \to \infty$ as $l \to \infty$, which happens e.g if A_l converges to an aperiodic point and T is (locally) continuous. In the situation of Section 3.2 all of these are fulfilled.
First note that we only used this fact for (3.115), for (3.114) we only needed $||\cdot||$ to be bounded on indicators and $c(n) = o(\frac{1}{n^2})$. So assume there are constants $K > 1$ such that $||\hat{T}(f)|| \leq K||f||$ for $f \in \mathcal{B}$. Then we can estimate

$$\begin{aligned}
&\int_X 1_{A_l} 1_{\{\varphi_{A_l} \leq \tau_l\}} (1_{A_l} \circ T^{k_l}) \, \mathrm{d}\mu \\
&\leq \sum_{i=n_l}^{\tau_l} \int_X 1_{A_l} (1_{A_l} \circ T^i)(1_{A_l} \circ T^{k_l}) \, \mathrm{d}\mu \\
&\leq \sum_{i=n_l}^{\tau_l} \int_X \hat{T}^i(1_{A_l}) 1_{A_l} (1_{A_l} \circ T^{k_l - i}) \, \mathrm{d}\mu \\
&\leq \mu(A_l) \sum_{i=n_l}^{\tau_l} \left(L^2 c(k_l - i) K^i + \int_X 1_{A_l} \circ T^i 1_{A_l} \, \mathrm{d}\mu \right) \\
&\leq \mu(A_l)\tau_l (L^2 c(k_l - \tau_l) K^{\tau_l} + \mu(A_l)^2) + L\mu(A_l)^2 \sum_{i=n_l}^{\infty} c(i).
\end{aligned}$$

So we will need $c(k_l - \tau_l)K^{\tau_l} = O(\mu(A_l)^2)$ as $l \to \infty$. Proceeding in the same fashion as for deriving (3.115), we get the sufficient condition

$$x^3 \geq c \left(\frac{\log\left(\frac{x^2}{c(\frac{1}{x})}\right)}{\log(K)} \right), \tag{3.124}$$

for $x > 0$ sufficiently small. As before if e.g c is stretched exponential, this holds true.
Note that in the case $K = 1$, the analysis above reveals that we only need $c(n) = o(\frac{1}{n^2})$ (compare Subsection 3.1.1 on unform mixing).

We formulate our findings in a Theorem. Note that we do not include the general conditions on the rate of decay as this would only make the formulations unpleasant.

Theorem 3.6.1. *Let $(X, \mathcal{A}, \mu, T)$ be an ergodic, probability-preserving system. Assume that there are constants $\alpha \in (0,1)$, $\beta > 0$ and $C > 0$, and Banach algebras of real valued functions on X, $(\mathcal{B}, ||\cdot||_{\mathcal{B}})$ and $(\mathcal{B}', ||\cdot||_{\mathcal{B}'})$ such that for $n \in \mathbb{N}$, $f \in \mathcal{B}$ and $g \in \mathcal{B}'$ we have*

$$\left| \int_X f\,(g \circ T^n) \, \mathrm{d}\mu - \int_X f \, \mathrm{d}\mu \int_X g \, \mathrm{d}\mu \right| \leq Ce^{-\beta n^\alpha} ||f||_{\mathcal{B}} ||g||_{\mathcal{B}'}. \tag{3.125}$$

Assume that $||\cdot||_{\mathcal{B}'}$ *is bounded on indicators, i.e there is a constant* C_1 *such that for any measurable set* $B \in \mathcal{A}$ *we have*

$$||1_B||_{\mathcal{B}'} \leq C_1. \tag{3.126}$$

Furthermore assume that composition with T *is bounded with respect to* $||\cdot||$ *i.e there is a constant* $K > 0$ *such that*

$$||f \circ T||_{\mathcal{B}} \leq K||f||_{\mathcal{B}} \tag{3.127}$$

for $f \in \mathcal{B}$. *Let* $(A_l)_{l\geq 1}$ *be a sequence of asymptotically rare events such that*

- *the indicators of* A_l *are well approximated by functions it* $\mathcal{B}$, *i.e there are constants* $C_2, C_3 > 0$ *such that for each* $l \geq 1$ *and* $\epsilon > 0$ *there is a function* $f \in \mathcal{B}$ *with* $1 - f \in \mathcal{B}$ *and* $||f||_{L^\infty} \leq 1$ *such that*

$$||f - 1_{A_l}||_{L^1} \leq \epsilon, \tag{3.128}$$

  while

$$||f||_{\mathcal{B}} \leq C_2\epsilon^{-1} \textit{ and } ||1-f||_{\mathcal{B}} \leq C_3\epsilon^{-1} \tag{3.129}$$

- *and we have convergence in law of the first hitting times to the exponential distribution*

$$\mu(\mu(A_l)\varphi_{A_l} \leq t) \Rightarrow 1 - e^{-t} \quad \textit{as } l \to \infty. \tag{3.130}$$

Then for all sequences $(k_l)_{l\geq 1}$ *of natural numbers such that* $(\mu(A_l)k_l)_{l\geq 1}$ *is bounded away from* 0 *and* ∞ *we have*

$$\mu_{A_l}(\varphi_{A_l} = k_l) \sim \mu(A_l)e^{-\mu(A_l)k_l} \quad \textit{as } l \to \infty. \tag{3.131}$$

Proof. (i) Without loss of generality (changing C if necessary) we may assume that $C_1 = C_2 = C_3 = 1$. We abbreviate $||\cdot||_{\mathcal{B}} = ||\cdot||$. For $\tau_l = \left\lceil \left(\frac{-6\log(\mu(A_l))}{\beta}\right)^{\frac{1}{\alpha}} \right\rceil$ let $\kappa_l := k_l - \tau_l$. By Theorem 2.1.1 and Lemma 2.1.2 we only need to verify (3.114) and (3.115).

(ii) For $l \geq 1$ set $\epsilon_l = \mu(A_l)^3$ and choose $f_l \in \mathcal{B}$ such that $||f_l||_{L^\infty} \leq 1$ and

$$||f_l - 1_{A_l}||_{L^1} \leq \epsilon_l, \tag{3.132}$$

while

$$||f_l|| \leq \epsilon_l^{-1} \textit{ and } ||1-f_l|| \leq \epsilon_l^{-1} \tag{3.133}$$

Furthermore define $g_l \in \mathcal{B}$ by

$$g_l(x) := \sum_{i=1}^{\tau_l} \left(f_l \prod_{j=1}^{i-1} ((1-f_l)\circ T^j)(f_l \circ T^i) \right). \tag{3.134}$$

Then we have

$$
\begin{aligned}
&||g_l - 1_{A_l \cap \{\varphi_{A_l} \leq \tau_l\}}||_{L^1} \\
&\leq \sum_{i=1}^{\tau_l} ||f_l \prod_{j=1}^{i-1} ((1-f_l)\circ T^j)(f_l \circ T^i) - 1_{A_l \cap \{\varphi_{A_l} = i\}}||_{L^1} \\
&\leq \sum_{i=1}^{\tau_l} ||f_l \prod_{j=1}^{i-1} ((1-f_l)\circ T^j)((f_l - 1_{A_l}) \circ T^i)||_{L^1} \\
&+ \sum_{i=1}^{\tau_l} \sum_{k=1}^{i-1} \left\| \begin{array}{l} f_l \prod_{j=1}^{k-1} ((1-f_l)\circ T^j) \\ ((1-1_{A_l}) - (1-f_l)) \circ T^k \\ \prod_{j=k+1}^{i-1} ((1_{A_l^c}) \circ T^j)(1_{A_l} \circ T^i) \end{array} \right\|_{L^1} \\
&+ \sum_{i=1}^{\tau_l} ||(f_l - 1_{A_l}) \prod_{j=1}^{i-1} ((1-1_{A_l})\circ T^j)((1_{A_l}) \circ T^i)||_{L^1} \\
&\leq \binom{\tau_l + 1}{2} \epsilon_l \leq \tau_l^2 \epsilon_l,
\end{aligned}
$$

while

$$
\begin{aligned}
||g_l|| &\leq || \sum_{i=1}^{\tau_l} \left(f_l \prod_{j=1}^{i-1} ((1-f_l)\circ T^j)(f_l \circ T^i) \right) \\
&\leq \sum_{i=1}^{\tau_l} \left(||f_l|| \prod_{j=1}^{i-1} ||(1-f_l)\circ T^j|| ||f_l \circ T^i|| \right) \\
&\leq K^{\binom{\tau_l+1}{2}} \epsilon_l^{-1} \leq K^{\tau_l^2} \epsilon_l^{-1}.
\end{aligned}
$$

(iii) Now we the calculations done above verify (3.114) and (3.115).

□

Likewise we formulate an analogue in the situation that the transfer operator is bounded.

Theorem 3.6.2. *Let $(X, \mathcal{A}, \mu, T)$ be an ergodic, probability-preserving system. Assume that there are constants $\alpha \in (0,1)$, $\beta > 0$ and $C > 0$, and a Banach algebra of real valued functions on X, $(\mathcal{B}, ||\cdot||_{\mathcal{B}})$ such that for $n \in \mathbb{N}$, $f \in \mathcal{B}$ and $g \in L^1$ we have*

$$
\left| \int_X f\,(g \circ T^n)\,\mathrm{d}\mu - \int_X f\,\mathrm{d}\mu \int_X g\,\mathrm{d}\mu \right| \leq Ce^{-\beta n^\alpha} ||f||_{\mathcal{B}} ||g||_{L^1}. \tag{3.135}
$$

Assume that the transfer operator is bounded with respect to $||\cdot||$ i.e there is a constant $K > 0$ such that

$$||\hat{T}f||_{\mathcal{B}} \leq K||f||_{\mathcal{B}} \tag{3.136}$$

for $f \in \mathcal{B}$. Let $(A_l)_{l\geq 1}$ be a sequence of asymptotically rare events such that

- *the indicators of A_l are bounded with respect to $||\cdot||_{\mathcal{B}}$, i.e $1_{A_l} \in \mathcal{B}$ for $l \geq 1$, and there is a $C_2 > 0$ such that*

$$||1_{A_l}||_{\mathcal{B}} \leq C_2 \tag{3.137}$$

  for $l \geq 1$,

- *there are natural numbers $(n_l)_{l\geq 1}$ such that $\varphi_{A_l} \geq n_l$ for $l \geq 1$, and $n_l \to \infty$ as $l \to \infty$,*

- *and we have convergence in law of the first hitting times to the exponential law*

$$\mu(\mu(A_l)\varphi_{A_l} \leq t) \Rightarrow 1 - e^{-t} \quad \text{as } l \to \infty. \tag{3.138}$$

Then for all sequences $(k_l)_{l\geq 1}$ of natural numbers such that $(\mu(A_l)k_l)_{l\geq 1}$ is bounded away from 0 and ∞ we have

$$\mu_{A_l}(\varphi_{A_l} = k_l) \sim \mu(A_l)e^{-\mu(A_l)k_l} \quad \text{as } l \to \infty. \tag{3.139}$$

Proof. Without loss of generality $C_1 = C_2 = 1$. We again only need to verify (3.114) and (3.115). Letting $\tau_l = \left\lceil \left(\frac{-3\log(\mu(A_l))}{\beta}\right)^{\frac{1}{\alpha}} \right\rceil$, a simple use of the decorrelation reveals (3.114). Using the above calculation and abbreviating $c(n) = e^{-\beta n^\alpha}$ we get

$$\begin{aligned}
&\int_X 1_{A_l} 1_{\{\varphi_{A_l} \leq \tau_l\}} (1_{A_l} \circ T^{k_l})\, \mathrm{d}\mu \\
&\leq \mu(A_l)\tau_l(c(k_l - \tau_l)K^{\tau_l} + \mu(A_l)^2) + \mu(A_l)^2 \sum_{i=n_l}^{\infty} c(i) \\
&\leq \mu(A_l)^2 \left(\mu(A_l) + \sum_{i=n_l}^{\infty} c(i)\right) + \mu(A_l) \underbrace{K^{\tau_l} e^{-\beta(\frac{k_l}{2})^\alpha}}_{\leq \mu(A_l)^2} \\
&= o(\mu(A_l)^2)
\end{aligned}$$

as soon as $k_l > 2\tau_l$ and $k_l > 2\left(\log(K)\tau_l - \frac{2\log(\mu(A_l))}{\beta}\right)^{\frac{1}{\alpha}}$. □

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