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"On the decrease of velocity with depth in periodic water waves"

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Luigi Roberti, BSc BSc

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Abstract / Zusammenfassung

Abstract (in English)

The main objective of this thesis is to give an alternative proof for the classical result in [27] that provides an estimate for the decay rate with depth of the velocity beneath two-dimensional, spatially periodic, irrotational gravity water waves over a flat bed. We start with the derivation of the governing equations for the full gravity water wave problem in three dimensions, thereby explaining the physical interpretation of the quantities that are introduced. Next we discuss the main features of the model for the case of two-dimensional, irrotational flows. Finally we turn to the main result, also providing an improvement to the same estimate, and we show how this can be generalized to flows with constant non-zero vorticity. An overview of the question of well-posedness of the governing equations is briefly addressed in the Appendix.

Deutsche Zusammenfassung

Das Hauptziel von dieser Arbeit ist, einen alternativen Beweis für das klassische Resultat in [27] vorzustellen, welches eine Abschätzung für die Abfallrate von der Geschwindigkeit unterhalb einer zweidimensionalen, Ort-periodischen, irrotationalen Wasserschwerewelle oberhalb eines flachen Bodens bei zunehmender Tiefe angibt. Zunächst einmal leiten wir die Gleichungen her, die das allgemeine Wasserschwerewellenproblem in drei räumlichen Dimensionen beschreiben, wobei die physikalische Interpretation der eingeführten Größen erklärt wird. Danach besprechen wir die wichtigsten Eigenschaften des Modells für den Fall von zweidimensionalen irrotationalen Strömungen. Schließlich befassen wir uns mit dem Hauptresultat, wobei zusätzlich eine verbesserte Version derselben Abschätzung angegeben wird, und wir zeigen, wie diese Ergebnisse auf Strömungen mit konstanter nicht-verschwindender Wirbelstärke verallgemeinert werden können. Ein Überblick über die Frage, ob das allgemeine Wasserwellenproblem korrekt gestellt ist, wird im Anhang kurz geschildert.

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Contents

Introduction

> "On the Nautilus men's hearts never fail them. [...] no tempest to brave, for when it dives below the water it reaches absolute tranquillity."
>
> J. Verne, *Twenty Thousand Leagues under the Sea*

Water waves are an ubiquitous, yet tremendously diverse natural phenomenon. The amount of different ways they can viewed is quite extraordinary: sea or ocean waves, for instance, can please the eye of the observer with the beautiful patterns they create, and may even serve for recreational purposes, such as when surfers ride waves that are about to plunge into the water ahead in a roaring cloud of spray—however, under certain circumstances they can also reach immense proportions that turn them into an utterly frightening spectacle, worthy of being immortalized in a painting by Caspar David Friedrich, and, in the case of tsunami waves, be so forceful to be able to annihilate entire settlements near the shore line, claiming at times a toll of hundreds of human lives by means of their destructive fury.

Despite everyone's familiarity and direct experience with water waves, obtaining a satisfactory quantitative mathematical description thereof is surprisingly difficult. This phenomenon being so important for everyday life, efforts to gain mathematical understanding of it are relatively ancient. The equations that are most commonly used for the modelling of water flows, the Euler equations (which we will describe in much detail below), are known since the mid-18th century [16], and first attempts to solve them rigorously can be dated back to the early 19th century with the remarkable work of Gerstner [20, 21], later rediscovered by Rankine [32], although the actual initiation of the field of research was primarily due to Stokes [37] in the mid-19th century. From then on the dynamics of water waves has been studied by numerous authors and has evolved parallel to general fluid mechanics, although the answers to many essential questions remain elusive. For example, our understanding is basically limited to two-dimensional flows—a thorough quantitative analysis of the full three-dimensional case being way out of the reach of the mathematical tools we have at our disposal at the moment—and even in this lower dimensional case we are far from having a full picture of the different aspects of the theory.

Historically one first looked at the linearized equations in two dimensions for irrotational flows (the meaning of these concepts is made clear below) and investigated properties such as the particles trajectories or the pressure fluctuations beneath the surface wave in the case of a water layer of infinite depth; later, more general situations where gradually discussed (for instance, by introducing a bottom of finite depth, or looking at different approximations of the governing equations in several regimes, e.g. the *shallow-water* regime, cf. [6]), until it was clear that many key properties of water waves, such as solitons propagation [6], can be captured only by nonlinear theory. Considering the presence of non-zero vorticity is also an improvement towards a more general understanding, and several steps forward have been made in recent years in this direction. Moreover, models for particular situations, for instance geophysical flows, equatorial flows, tsunamis, etc., have been developed and investigated. The issue of well-posedness is also of paramount importance and has been addressed accordingly (see Appendix A).

In this thesis we restrict our attention to two-dimensional, spatially periodic water waves over a flat bed and investigate how the motion of the water below the surface wave decays with increasing depth. This particular aspect is not merely of sterile academic interest, because wind-generated waves usually show a spatially periodic profile after having travelled away from the place where they have been originated, and are approximately two-dimensional, meaning that they propagate uniformly in the direction of the wind that has generated them [1, 35]. In such cases, it is intuitively expected that the effects that the surface wave induces on the motion of the water below should be less pronounced at higher depths, a fact that, to take a concrete example, is very useful for submarines: if a wind storm causes the formation of waves of several metres of elevation, the submarine can escape the danger by diving at an adequate depth beneath the surface. In fact, the classical "rule of thumb" in oceanography is that at the so-called *wave base* (corresponding to a depth of half the wavelength of the surface wave) the effects of the wave on the underlying motion of the water should already be negligible [12].

The thesis is organized as follows. In Chapter 1 we go through the derivation of the governing equations for water waves (which are comprised of the equation of mass conservation, the Euler equations, suitable boundary conditions and some assumptions on the vorticity of the flow), thereby focussing on the physical meaning and relevance of the introduced quantities. In Chapter 2 we discuss the important special case of two-dimensional irrotational periodic water waves, providing some general considerations on the role of irrotationality and its consequences. In Chapter 3 we prove the main result for irrotational flows, which is an alternative proof for (a slight generalization of) a classical result (proven in [27]) on the decrease of velocity with depth for such water waves, also providing in § 3.2 a quantitative estimate for the velocity decay at the wave base. Chapter 4 is devoted to the discussion of the main properties of flows with vorticity and the most striking differences between this and the irrotational case, also providing a generalization of the results from Chapter 3 to flows with constant vorticity. Finally, a brief overview of the main results about the fundamental issue of well-posedness is offered in Appendix A.

1 The governing equations

In the present chapter we derive the model for gravity water waves in three dimensions using physical reasoning and making reasonable simplifying assumptions whenever needed. In the last part of the chapter we also discuss some of the relevant aspects of the role of vorticity in the description of water flows. A thorough answer to the question of well-posedness of the model—which builds an entire, broad field of research of its own—is well beyond the scope of this thesis; nonetheless, the most important results and the essential methods used in that context are outlined in Appendix A.

The material presented in this chapter, especially the first three sections, is standard and can be found in basically every general textbook about fluid mechanics and/or water waves, such as [6, 12, 23, 29, 36]; we mainly follow the book [6], although with several adaptations.

1.1 The continuum assumption

In classical mechanics, the most realistic and "straightforward" way of describing a fluid is to view it as an ensemble of, say, N discrete molecules, interacting with each other and possibly subjected to some external forces. Supposing that the N molecules are each located at $\mathbf{x}_1(t), \dots, \mathbf{x}_N(t) \in \mathbb{R}^3$ at time t and have the (positive) masses $m_1, \dots, m_N$, respectively, and denoting the total force acting on each of them (as a sum of external forces and interaction forces with all surrounding molecules) by $\mathbf{F}_k : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^3$, $k = 1, \dots, N$, all we would need to do in order to get a full picture of the situation we are interested in would be to write down Newton's second law for each molecule,

$$m_k \frac{\mathrm{d}^2 \mathbf{x}_k}{\mathrm{d}t^2}(t) = \mathbf{F}_k(\mathbf{x}_1(t), \dots, \mathbf{x}_N(t), t), \quad k = 1, \dots, N, \tag{1.1}$$

and solve the resulting system of equations. However, this turns out to be utterly hopeless, because we usually deal (as in the study of water waves) with a number of molecules approximately of the order $N \approx 10^{24}$, which would make treating the system (1.1) impossible, even for the most advanced computers available. Instead, we make use of the simplifying assumption, upon which continuum mechanics is founded, that the fluid is continuously distributed (*continuum assumption*); this hypothesis, albeit an approximation, still yields (in the macroscopic case) a remarkably accurate description, in thorough agreement with our everyday-life experience. In particular, this means that one can attach a meaning to the value properties of the fluid at a point: this is what we are referring to whenever we talk of a "particle" of fluid.

In general, the continuum assumption comes together with the introduction of a function $\rho : \mathbb{R}^3 \times \mathbb{R} \to [0, \infty)$, called the *mass density*, defined via the property that the mass contained inside of a volume $V \subset \mathbb{R}^3$ at time t should be given by

$$\int_V \rho(\mathbf{x},t)\,\mathrm{d}\mathbf{x}.$$

It will be convenient for our purposes to assume further properties on the mass density, as well as on other quantities that we will introduce shortly, as we shall see in the upcoming sections.

1.2 The equation of mass conservation

We now begin the derivation of the basic equations that, together with suitable boundary conditions, will form the model for water waves; we begin with the equation of *mass conservation*. For each $t \in \mathbb{R}_+ = (0,\infty)$, let $\mathcal{D}(t) \subset \mathbb{R}^3$ denote the fluid domain at time t. Suppose that $V \subset \mathcal{D}(t)$ is a volume with C^1-boundary ∂V, and let $\mathbf{n}$ denote the outward unit normal vector on ∂V. If $\mathbf{u} \in C^1(\mathbb{R}_+; C^1(\overline{\mathcal{D}(t)}))$ is the velocity field in the water, i.e. $\mathbf{u}(\mathbf{x},t)$ is the velocity of the water at the point $\mathbf{x} = (x,y,z)^T \in \mathcal{D}(t)$, then the rate at which mass flows out of V is

$$\int_{\partial V} \rho(\mathbf{x},t)\,\mathbf{u}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x})\,\mathrm{d}S(\mathbf{x}),$$

where $\mathrm{d}S$ denotes the surface measure on ∂V. On the other hand, the rate of change of mass in V is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho(\mathbf{x},t)\,\mathrm{d}\mathbf{x} = \int_V \frac{\partial\rho}{\partial t}(\mathbf{x},t)\,\mathrm{d}\mathbf{x}.$$

Now, one of the axioms of continuum mechanics is *conservation of mass*, which, in our case, means that mass can neither be created nor destroyed in the water: in other words, the rate of change of mass in V can only be due to the rate of mass flowing into V across its boundary ∂V. Therefore

$$\int_V \frac{\partial\rho}{\partial t}(\mathbf{x},t)\,\mathrm{d}\mathbf{x} = -\int_{\partial V} \rho(\mathbf{x},t)\,\mathbf{u}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x})\,\mathrm{d}S(\mathbf{x}) = -\int_V \nabla\cdot[\rho(\mathbf{x},t)\,\mathbf{u}(\mathbf{x},t)]\,\mathrm{d}\mathbf{x}$$

(where in the last step we used the divergence theorem), or equivalently

$$\int_V \left(\frac{\partial\rho}{\partial t}(\mathbf{x},t) + \nabla\cdot[\rho(\mathbf{x},t)\,\mathbf{u}(\mathbf{x},t)]\right)\mathrm{d}\mathbf{x} = 0.$$

Since this has to hold true for arbitrary $V \subset \mathcal{D}(t)$, we obtain the *equation of mass conservation* (or *continuity equation*)

$$\frac{\partial\rho}{\partial t}(\mathbf{x},t) + \nabla\cdot[\rho(\mathbf{x},t)\,\mathbf{u}(\mathbf{x},t)] = 0, \quad \mathbf{x}\in\mathcal{D}(t). \tag{1.2}$$

Writing $\mathbf{u}$ in components as $\mathbf{u} = (u_1,u_2,u_3)^T$, we may introduce the *material derivative* of ρ by

$$\begin{aligned}\frac{\mathrm{D}\rho}{\mathrm{D}t}(\mathbf{x},t) &:= \frac{\partial\rho}{\partial t}(\mathbf{x},t) + (\mathbf{u}(\mathbf{x},t)\cdot\nabla)\rho(\mathbf{x},t)\\ &= \frac{\partial\rho}{\partial t}(\mathbf{x},t) + u_1(\mathbf{x},t)\frac{\partial\rho}{\partial x}(\mathbf{x},t) + u_2(\mathbf{x},t)\frac{\partial\rho}{\partial y}(\mathbf{x},t) + u_3(\mathbf{x},t)\frac{\partial\rho}{\partial z}(\mathbf{x},t),\end{aligned}$$

which by the chain rule can be physically interpreted as the time derivative of ρ along a particle path $\mathbf{x}(t)$. Equation (1.2) can therefore be rewritten as

$$\frac{\mathrm{D}\rho}{\mathrm{D}t}(\mathbf{x},t) + \rho(\mathbf{x},t)\,\nabla\cdot\mathbf{u}(\mathbf{x},t) = 0, \quad \mathbf{x}\in\mathcal{D}(t).$$

A fluid satisfying

$$\frac{\mathrm{D}\rho}{\mathrm{D}t}(\mathbf{x},t) = 0, \quad \mathbf{x}\in\mathcal{D}(t) \tag{1.3}$$

for all times t is called *incompressible*; in this case, the equation of mass conservation (1.2) reduces to

$$\nabla\cdot\mathbf{u}(\mathbf{x},t) = 0, \quad \mathbf{x}\in\mathcal{D}(t), \tag{1.4}$$

valid for each $t > 0$. For water in seas and oceans, it turns out (see [6]) that the mass density ρ is subject to negligible changes due to depth or temperature, so that it is an excellent approximation to assume that ρ is in fact a constant, both in space and in time, so that the incompressibility condition (1.3) is trivially satisfied.

1.3 The Euler equations

As briefly mentioned above, we distinguish between two types of forces that are relevant in fluid mechanics: external forces (whose source is external to the fluid and which are the same for each fluid "particle") and internal forces (exerted on a fluid element by other elements nearby). We consider *gravity water waves*, i.e. we assume gravity to be the only external force acting on the fluid bulk. If, as above, $V\subset\mathcal{D}(t)$ is a volume with C^1-boundary ∂V, then the total external force acting on the water volume V is

$$\int_V \rho\,\mathbf{g}\,\mathrm{d}\mathbf{x},$$

where, in Cartesian coordinates $\mathbf{x} = (x,y,z)^T$ with z measured upwards, we have $\mathbf{g} = (0,0,-g)^T$, with the constant gravitational acceleration $g \approx 9.8\,\mathrm{m\,s^{-2}}$. Moreover, we assume water to be an *ideal fluid*—a concept supposing the matter to be continuously distributed and the fluid to be inviscid[1]; this means, heuristically, that the force that is exerted on a small water element by the water elements nearby is always perpendicular to the boundary of the element (i.e. there are no friction effects that would account for non-zero tangential components of the internal forces), so that the internal forces on the volume V at time t are given by

$$-\int_{\partial V} p(\mathbf{x},t)\mathbf{n}(\mathbf{x})\,\mathrm{d}S(\mathbf{x}),$$

where the function p, for which we assume $p\in C(\mathbb{R}_+; C^1(\overline{\mathcal{D}(t)}))$, is called the *hydrodynamical pressure.* Therefore the total force acting on V at time t is

$$\int_V \rho\,\mathbf{g}\,\mathrm{d}\mathbf{x} - \int_{\partial V} p(\mathbf{x},t)\mathbf{n}(\mathbf{x})\,\mathrm{d}S(\mathbf{x}) = \int_V (\rho\,\mathbf{g} - \nabla p(\mathbf{x},t))\,\mathrm{d}\mathbf{x},$$

where we used the divergence theorem once again.

[1] This is a good approximation for water (see [13]), especially for gravity water waves where the dissipation effects caused by viscosity are negligible over long periods of time (cf. [6]).

Now notice that the velocity vector of a particle at the point $\mathbf{x}$ at time t is $\mathbf{u}(\mathbf{x}, t)$, so that the particle will move along the path $\mathbf{x}(t)$ satisfying the equation

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}(t) = \mathbf{u}(\mathbf{x}(t), t),$$

hence, by the chain rule, the acceleration of the particle is

$$\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2}(t) = \frac{\partial \mathbf{u}}{\partial t}(\mathbf{x}(t), t) + (\mathbf{u}(\mathbf{x}(t), t) \cdot \nabla)\mathbf{u}(\mathbf{x}(t), t) = \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}(\mathbf{x}(t), t),$$

with the material derivative $\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}$ of $\mathbf{u}$. Therefore Newton's second law applied to the body of water surrounded by ∂V yields

$$\int_V \rho \, \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}(\mathbf{x}, t) \, \mathrm{d}\mathbf{x} = \int_V (\rho \, \mathbf{g} - \nabla p(\mathbf{x}, t)) \, \mathrm{d}\mathbf{x}.$$

Since V is arbitrary, we obtain the *Euler equations of motion*

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}(\mathbf{x}, t) = -\frac{1}{\rho} \, \nabla p(\mathbf{x}, t) + \mathbf{g}, \quad \mathbf{x} \in \mathcal{D}(t),$$

or, in other words,

$$\frac{\partial \mathbf{u}}{\partial t}(\mathbf{x}, t) + (\mathbf{u}(\mathbf{x}, t) \cdot \nabla)\mathbf{u}(\mathbf{x}, t) = -\frac{1}{\rho} \, \nabla p(\mathbf{x}, t) + \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix}, \quad \mathbf{x} \in \mathcal{D}(t) \tag{1.5}$$

for all $t > 0$. The system of equations (1.5) together with the equation of mass conservation (1.4) is often referred to as the *incompressible Euler equations*.

1.4 Boundary conditions

When dealing with water waves, the boundary of the fluid region consists of two distinct parts: a lower boundary, which we assume to be a rigid flat bed, and the free surface—the fluid region is supposed to be unbounded in the horizontal (x and y) directions. We restrict ourselves to the case where the free water surface at time t can be written as

$$\{\mathbf{x} = (x, y, z)^T \in \mathbb{R}^3 : z = \eta(x, y, t)\} \tag{1.6}$$

for a function $\eta \in C^1(\mathbb{R}_+; C^1(\mathbb{R}^2))$ such that $\eta(\cdot, \cdot, t)$ is a perturbation of the mean surface level $z = 0$ for each time t. Notice that η is unknown and determining it is part of the problem. On the other hand, the rigid flat bed is supposed to be the horizontal plane

$$\{\mathbf{x} = (x, y, z)^T \in \mathbb{R}^3 : z = -d\},$$

situated at the mean depth $d > 0$.

We distinguish between *kinematic* and *dynamic boundary conditions*. The kinematic boundary conditions express the fact that a particle that is initially situated on the boundary remains there at all later times. Clearly, the kinematic boundary condition on the flat rigid bed reads simply

$$u_3 = 0 \quad \text{on } \{z = -d\}; \tag{1.7}$$

in the case of the free water surface, however, this condition takes the more complicated form

$$u_3 = \eta_t + u_1\eta_x + u_2\eta_y \quad \text{on } \{z = \eta(x,y,t)\}, \tag{1.8}$$

which can be justified as follows. First, notice that the free surface can be identified with the set of solutions to the equation

$$S(\mathbf{x}, t) = 0,$$

with $S : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$, $S(\mathbf{x}, t) = z - \eta(x, y, t)$. Therefore, taking an arbitrary point $\mathbf{x}_0$ situated on the free surface at time $t = 0$, its subsequent trajectory $\mathbf{x}(t)$ is clearly given by the solution to the differential equation $\mathbf{x}'(t) = \mathbf{u}(\mathbf{x}(t), t)$ satisfying the constraint $S(\mathbf{x}(t), t) = 0$ and having the initial data $\mathbf{x}(0) = \mathbf{x}_0$. Thus it follows by differentiation that the condition

$$\frac{\mathrm{D}S}{\mathrm{D}t} = 0$$

along the surface, which is precisely (1.8), is necessary. To see that this is also a sufficient condition we need the following result concerning flow-invariant sets:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a closed C^1-hypersurface in $\mathbb{R}^n$, and let $F \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. If $\nu(Y) \cdot F(Y) = 0$ whenever $\nu(Y)$ is normal to Ω at $Y \in \Omega$, then Ω is flow-invariant for F, i.e. every solution $X : [t_0, T) \to \mathbb{R}^n$ of the differential equation*

$$X'(t) = F(X(t))$$

with $X(t_0) \in \Omega$ remains in Ω for all $t \in [t_0, T)$.

For a proof of this statement, see [33]. In our case, taking

$$\Omega = \{(x, y, z, t)^T \in \mathbb{R}^4 : z - \eta(x, y, t) = 0\}$$

and noticing that the differential equation $\mathbf{x}'(t) = \mathbf{u}(\mathbf{x}(t), t)$ can be rewritten as

$$X'(t) = F(X(t))$$

with $X(t) = (\mathbf{x}(t), t)^T$ and $F(X(t)) = (u_1(\mathbf{x}(t), t), u_2(\mathbf{x}(t), t), u_3(\mathbf{x}(t), t), 1)^T$, we see that the condition $\nu(X) \cdot F(X) = 0$ is precisely $\frac{\mathrm{D}S}{\mathrm{D}t} = 0$, and applying Theorem 1.1 concludes the argument.

The dynamic boundary condition expresses the fact that the motion of the air above the water surface is decoupled from that of the water below. For gravity water waves we ignore the effects of surface tension (cf. [6]), so that the atmosphere above the water exerts an influence only in the form of pressure acting on the surface; this pressure is taken to be equal to the (constant) atmospheric pressure p_{atm} (its common reference value being $p_{\mathrm{atm}} = 101.325\,\mathrm{Pa}$). Thus the dynamic boundary condition for gravity water waves is

$$p = p_{\mathrm{atm}} \quad \text{on } \{z = \eta(x, y, t)\}. \tag{1.9}$$

1.5 Vorticity

The *vorticity* of a fluid flow $\mathbf{u} \in C^1(\mathbb{R}_+; C^1(\overline{\mathcal{D}(t)}))$ is defined as the curl of the velocity field:

$$\boldsymbol{\omega}(\mathbf{x}, t) \coloneqq \operatorname{curl} \mathbf{u}(\mathbf{x}, t) = \nabla \wedge \mathbf{u}(\mathbf{x}, t), \quad \mathbf{x} \in \mathcal{D}(t),\ t > 0.$$

The vorticity is a very essential property of a fluid. Physically, one can interpret it as a measure of the local spin or rotation of a fluid element, as we describe next (following the discussion in [6]).

1.5.1 Physical interpretation

Suppose for simplicity that we are considering a time-independent smooth flow $\mathbf{u}(\mathbf{x})$, and look at a fixed point $\mathbf{x}_0$ of the fluid domain. Also, let $\mathbf{h} \in \mathbb{R}^3$ with $|\mathbf{h}|$ "small". Then the Taylor expansion of $\mathbf{u}$ around $\mathbf{x}_0$ is

$$\mathbf{u}(\mathbf{x}_0 + \mathbf{h}) = \mathbf{u}(\mathbf{x}_0) + (\mathrm{D}\mathbf{u})(\mathbf{x}_0)\,\mathbf{h} + O(|\mathbf{h}|^2),$$

where $(\mathrm{D}\mathbf{u})(\mathbf{x}_0)$ denotes the Jacobian of $\mathbf{u}$ at $\mathbf{x}_0$. If we then split $(\mathrm{D}\mathbf{u})(\mathbf{x}_0)$ into its symmetric and antisymmetric part,

$$(\mathrm{D}\mathbf{u})(\mathbf{x}_0) = \mathrm{D_S}(\mathbf{x}_0) + \mathrm{D_A}(\mathbf{x}_0)$$

with

$$\mathrm{D_S}(\mathbf{x}_0) = \frac{(\mathrm{D}\mathbf{u})(\mathbf{x}_0) + (\mathrm{D}\mathbf{u})(\mathbf{x}_0)^T}{2} \quad \text{and} \quad \mathrm{D_A}(\mathbf{x}_0) = \frac{(\mathrm{D}\mathbf{u})(\mathbf{x}_0) - (\mathrm{D}\mathbf{u})(\mathbf{x}_0)^T}{2},$$

and performing a straightforward calculation to see that

$$2\,\mathrm{D_A}(\mathbf{x}_0)\,\mathbf{h} = \boldsymbol{\omega}(\mathbf{x}_0) \wedge \mathbf{h},$$

we obtain

$$\mathbf{u}(\mathbf{x}_0 + \mathbf{h}) = \mathbf{u}(\mathbf{x}_0) + \mathrm{D_S}(\mathbf{x}_0)\,\mathbf{h} + \frac{1}{2}\,\boldsymbol{\omega}(\mathbf{x}_0) \wedge \mathbf{h} + O(|\mathbf{h}|^2).$$

If we are interested in a first-order approximation of the motion of a particle near $\mathbf{x}_0$, whose trajectory can be written as $\mathbf{x}(t) = \mathbf{x}_0 + \mathbf{h}(t)$, we need to solve the differential equation

$$\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}t} = \mathbf{u}(\mathbf{x}_0) + \mathrm{D_S}(\mathbf{x}_0)\,\mathbf{h} + \frac{1}{2}\,\boldsymbol{\omega}(\mathbf{x}_0) \wedge \mathbf{h}. \tag{1.10}$$

We can give an independent physical interpretation to the flow generated by each of the three addends on the right-hand side.

- Clearly, the differential equation

  $$\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}t} = \mathbf{u}(\mathbf{x}_0)$$

  has the solution $\mathbf{h}(t) = \mathbf{h}(0) + \mathbf{u}(\mathbf{x}_0)t$, which describes an *infinitesimal translation* (keep in mind that we are working only at first-order level).
- The flow generated by the equation

  $$\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}t} = \mathrm{D_S}(\mathbf{x}_0)\,\mathbf{h} \tag{1.11}$$

  represents an *infinitesimal stretching* along the eigenspaces of the symmetric matrix $\mathrm{D_S}(\mathbf{x}_0)$. Indeed, if $\lambda_1 \le \lambda_2 \le \lambda_3$ are the (not necessarily distinct) real eigenvalues of the matrix $\mathrm{D_S}(\mathbf{x}_0)$ (which, being symmetric, is diagonalizable) with a corresponding basis of eigenvectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, so that $\mathrm{D_S}(\mathbf{x}_0)\,\mathbf{e}_k = \lambda_k\,\mathbf{e}_k$ for each $k = 1, 2, 3$, we may write $\mathbf{h}(t) = h_1(t)\,\mathbf{e}_1 + h_2(t)\,\mathbf{e}_2 + h_3(t)\,\mathbf{e}_3$, with the consequence that (1.11) reduces to

  $$\frac{\mathrm{d}h_k}{\mathrm{d}t} = \lambda_k\,h_k, \quad k = 1, 2, 3,$$

  with solutions

  $$h_k(t) = h_k(0)\,\mathrm{e}^{\lambda_k t}, \quad k = 1, 2, 3.$$

Moreover, if $\mathbf{u}$ satisfies the equation of mass conservation (1.4), then

$$0 = \mathrm{tr}((\mathrm{D}\mathbf{u})(\mathbf{x}_0)) = \mathrm{tr}(\mathrm{D_S}(\mathbf{x}_0)) = \lambda_1 + \lambda_2 + \lambda_3,$$

so that the deformation due to $\mathrm{D_S}(\mathbf{x}_0)$ is volume-preserving (see the discussion in [6]).

- Finally, we look at the equation

$$\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}t} = \frac{1}{2}\,\boldsymbol{\omega}(\mathbf{x}_0) \wedge \mathbf{h} \tag{1.12}$$

and convince ourselves that its solution defines a *rigid rotation* with angular velocity $\frac{1}{2}\,\boldsymbol{\omega}(\mathbf{x}_0)$. Let $\mathbf{f}, \mathbf{j} \in \mathbb{R}^3$ be unit vectors such that $\{\mathbf{e} = \frac{\boldsymbol{\omega}(\mathbf{x}_0)}{|\boldsymbol{\omega}(\mathbf{x}_0)|}, \mathbf{f}, \mathbf{j}\}$ is an orthonormal basis for $\mathbb{R}^3$. Furthermore, let $\ell, L \geq 0$ and $\varphi \in [0, 2\pi)$ with

$$\mathbf{h}(0) = \ell\,\mathbf{e} + L\cos(\varphi)\,\mathbf{f} + L\sin(\varphi)\,\mathbf{j},$$

and observe that

$$\mathbf{h}(t) = \ell\,\mathbf{e} + L\cos(\varphi + \theta t)\,\mathbf{f} + L\sin(\varphi + \theta t)\,\mathbf{j},$$

where $\theta = \frac{|\boldsymbol{\omega}(\mathbf{x}_0)|}{2}$, satisfies

$$\mathbf{h}'(t) = -L\theta\sin(\varphi + \theta t)\,\mathbf{f} + L\theta\cos(\varphi + \theta t)\,\mathbf{j} = \frac{1}{2}\,\boldsymbol{\omega}(\mathbf{x}_0) \wedge \mathbf{h}(t),$$

so that (by uniqueness theory for ODEs, cf. [5, 39]) it must be the unique solution to (1.12) with initial data $\mathbf{h}(0)$. Thus the flow associated to (1.12) is a rotation with constant angular speed θ and axis of rotation $\boldsymbol{\omega}(\mathbf{x}_0)$, as claimed.

Nevertheless, from this separate analysis we cannot conclude that the solution of (1.10) at leading order can be written as a combination of a translation, a deformation and a rigid rotation. In fact, by Duhamel's formula (see [39]), the solution to (1.10) is

$$\mathbf{h}(t) = \mathrm{e}^{(\mathrm{D_S}(\mathbf{x}_0)+\mathrm{D_A}(\mathbf{x}_0))t}\,\mathbf{h}(0) + \left(\int_0^t \mathrm{e}^{(\mathrm{D_S}(\mathbf{x}_0)+\mathrm{D_A}(\mathbf{x}_0))(t-s)}\,\mathrm{d}s\right)\mathbf{u}(\mathbf{x}_0),$$

and, as we have seen, the maps $t \mapsto \mathrm{e}^{\mathrm{D_S}(\mathbf{x}_0)t}\,\mathbf{h}(0)$ and $t \mapsto \mathrm{e}^{\mathrm{D_A}(\mathbf{x}_0)t}\,\mathbf{h}(0)$ determine a deformation along three fixed axes and a rotation about a fixed axis, respectively. Thus, if the matrices $\mathrm{D_S}(\mathbf{x}_0)$ and $\mathrm{D_A}(\mathbf{x}_0)$ commuted, then $\mathbf{h}(t)$ would indeed consist of a deformation and a rotation (not necessarily in this order, due to commutativity) and a succeeding translation by the vector

$$\left(\int_0^t \mathrm{e}^{(\mathrm{D_S}(\mathbf{x}_0)+\mathrm{D_A}(\mathbf{x}_0))(t-s)}\,\mathrm{d}s\right)\mathbf{u}(\mathbf{x}_0).$$

However, it is easily shown that if the matrices $\mathrm{D_S}(\mathbf{x}_0)$ and $\mathrm{D_A}(\mathbf{x}_0)$ commute if and only if $(\mathrm{D}\mathbf{u})(\mathbf{x}_0)$ commutes with $(\mathrm{D}\mathbf{u})(\mathbf{x}_0)^T$, which is in general not true, not even under the incompressibility condition (1.4), so an actual decomposition of the flow into a translation, a deformation and a rotation cannot be achieved in general. Still, in view of Trotter's product formula

$$\mathrm{e}^{(A+B)t} = \lim_{n\to\infty}\left(\mathrm{e}^{At/n}\,\mathrm{e}^{Bt/n}\right)^n \quad \text{uniformly on compact intervals,}$$

valid also for non-commuting matrices A and B (cf. [5]), we may get an approximation of the flow by performing n combinations of such rotations and deformations, followed by a translation, provided that n be large enough.

1.5.2 The vorticity equation

One of the main features of the vorticity of a flow is the so-called *vorticity equation*, which we briefly derive here[2]. Plugging the identity

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right) + (\nabla\wedge\mathbf{u})\wedge\mathbf{u}$$

into Euler's equation (1.5) we get

$$\frac{\partial\mathbf{u}}{\partial t} + (\nabla\wedge\mathbf{u})\wedge\mathbf{u} = -\nabla\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u} + p + gz\right).$$

Taking the curl and using the identity $\nabla\wedge(\nabla f) = \mathbf{0}$, valid for all $f \in C^2(\mathbb{R}^3,\mathbb{R})$, we obtain

$$\frac{\partial\boldsymbol{\omega}}{\partial t} + \nabla\wedge(\boldsymbol{\omega}\wedge\mathbf{u}) = \mathbf{0}.$$

If we now apply the vector identity

$$\nabla\wedge(\mathbf{F}\wedge\mathbf{G}) = (\mathbf{G}\cdot\nabla)\mathbf{F} - (\mathbf{F}\cdot\nabla)\mathbf{G} + \mathbf{F}(\nabla\cdot\mathbf{G}) - \mathbf{G}(\nabla\cdot\mathbf{F}),$$

which holds true for all $\mathbf{F},\mathbf{G} \in C^2(\mathbb{R}^3,\mathbb{R}^3)$, this becomes

$$\frac{\partial\boldsymbol{\omega}}{\partial t} + (\mathbf{u}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\mathbf{u} + \boldsymbol{\omega}(\nabla\cdot\mathbf{u}) - \mathbf{u}(\nabla\cdot\boldsymbol{\omega}) = \mathbf{0};$$

in this equality the fourth term vanishes because of incompressibility (1.4), whereas the fifth term vanishes because of the vector identity $\nabla\cdot(\nabla\wedge\mathbf{F}) = 0$, $\mathbf{F} \in C^2(\mathbb{R}^3,\mathbb{R}^3)$. Thus

$$\frac{\partial\boldsymbol{\omega}}{\partial t} + (\mathbf{u}\cdot\nabla)\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\mathbf{u},$$

or, in other words,

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = (\boldsymbol{\omega}\cdot\nabla)\mathbf{u},$$

which is the anticipated *vorticity equation*; notice that the hydrodynamical pressure is not involved in this equation. In particular, *for a two dimensional flow* (for instance, a flow that is independent of the y-coordinate) *the vorticity of each individual water particle is constant along the particle's path*: indeed, if we write $\boldsymbol{\omega} = (\omega_1,\omega_2,\omega_3)^T$, for such a flow we have $\omega_1 = \omega_3 = 0$, thus

$$(\boldsymbol{\omega}\cdot\nabla)\mathbf{u} = \omega_2\frac{\partial\mathbf{u}}{\partial y} = 0,$$

and therefore

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = 0. \tag{1.13}$$

Notice that, in such a two-dimensional flow,

$$\boldsymbol{\omega} = \left(0, \frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}, 0\right)^T,$$

[2] For the sake of notational simplicity, we omit all arguments in the following calculations. Furthermore, we assume that everything is sufficiently smooth for each step to be justified.

so that we could identify the vorticity with the scalar

$$\omega := \frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}, \tag{1.14}$$

which, by a slight abuse of language and notation, we call again the vorticity of the flow. A two-dimensional flow whose vorticity, in view of the vorticity equation (1.13), vanishes throughout the fluid domain at each time is called *irrotational*.

For a brief outline of the main properties of rotational flows and the differences to irrotational ones, we refer to the later discussion in Chapter 4.

2 The two-dimensional periodic irrotational problem

In the previous chapter we derived the general model for gravity water waves in three dimensions. For the rest of this thesis we restrict ourselves to the most investigated case, the two dimensional one, modelling a water wave which propagates in only one direction, perpendicular to what me may call the "wave front". We start by introducing some notation and terminology, before analysing some immediate consequences of the governing equations.

2.1 The model

As anticipated, we consider a two-dimensional water flow, with a horizontal direction, labelled by x, and a vertical direction, measured upwards and labelled by y (instead of z, in contrast to the previous section). To further simplify the notation, we will denote the components of the velocity field by $\mathbf{u} = (u, v)^T$. The free water surface is assumed to be described, in analogy with (1.6), by the equation $y = \eta(x,t)$, for some (unknown) function $\eta \in C^1(\mathbb{R}_+; C^1(\mathbb{R}))$, which is assumed to be periodic of period $\lambda > 0$ (called the *wavelength*) in the x-direction. The fluid domain at time $t > 0$ is thus

$$\mathcal{D}(t) = \{(x,y) \in \mathbb{R}^2 : -d < y < \eta(x,t)\}.$$

Let us fix at this point some more terminology for the case of a two-dimensional periodic wave (see Figure 2.1). The maximal elevation of the wave from the mean water level $y = 0$ is called the *crest* of the wave; the deepest point of the free water surface is called the *trough* of the wave. The vertical distance between the wave crest and the wave trough is called the *height* of the wave, whereas the *amplitude* is defined as the maximal deviation of the free water surface above the mean level: this can occur either at the crest or at the trough, although periodic sea waves usually have higher, sharper crests and flatter, less pronounced depressions, such as in Figure 2.1 (see [6]). For travelling waves, in which the (x,t)-dependence of η, u, v and p is of the form $(x - ct)$, the time necessary for two successive crests (or troughs) to pass a fixed point in space is called the *wave period*; the ratio of wavelength and wave period is called the *wave speed* $c > 0$.

Now let us go back to the governing equations. As in the previous chapter, we are dealing with an (unknown) hydrodynamical pressure $p \in C(\mathbb{R}_+; C^1(\overline{\mathcal{D}(t)}))$ and a constant (known) mass density $\rho > 0$. The Euler equations (1.5) take then the following form:

$$\begin{cases} u_t + uu_x + vu_y = -\dfrac{1}{\rho}\,p_x \\ v_t + uv_x + vv_y = -\dfrac{1}{\rho}\,p_y - g \end{cases} \quad \text{in } \mathcal{D}(t),\ t > 0, \tag{2.1}$$

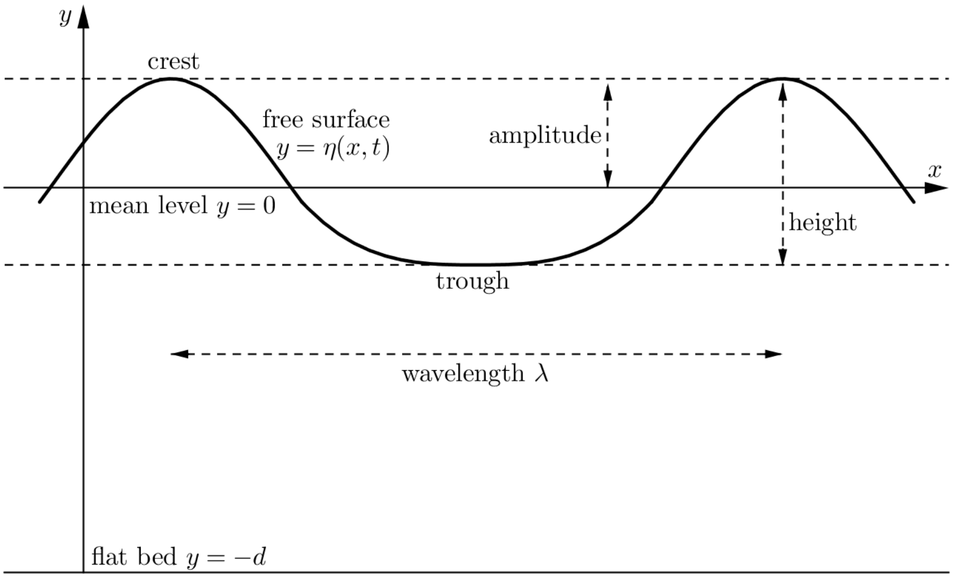

Figure 2.1: Schematic depiction of a two-dimensional space-periodic water wave.

whereas the mass conservation (1.4) is

$$u_x + v_y = 0 \quad \text{in } \mathcal{D}(t),\ t > 0. \tag{2.2}$$

The boundary conditions (1.8), (1.7) and (1.9) are then

$$v = \eta_t + u\eta_x \quad \text{on } \{(x,y) \in \mathbb{R}^2 : y = \eta(x,t)\},\ t > 0, \tag{2.3}$$

$$v = 0 \quad \text{on } \{(x,y) \in \mathbb{R}^2 : y = -d\},\ t > 0, \tag{2.4}$$

$$p = p_{\mathrm{atm}} \quad \text{on } \{(x,y) \in \mathbb{R}^2 : y = \eta(x,t)\},\ t > 0, \tag{2.5}$$

respectively. All variables u, v, p, η depend periodically on the horizontal variable x, with period λ; moreover, the free surface is supposed to be a disturbance of the mean level $y = 0$, so that

$$\int_0^\lambda \eta(x,t)\,\mathrm{d}x = 0 \quad \text{at all times } t > 0. \tag{2.6}$$

Finally, the flow is supposed to be irrotational, so that

$$u_y - v_x = 0 \quad \text{in } \mathcal{D}(t),\ t > 0, \tag{2.7}$$

according to (1.14) and in view of the vorticity equation (1.13). The system of equations is completed by suitable initial conditions.

2.2 Velocity potential and stream function

Since the fluid domain $\mathcal{D}(t)$ is simply connected, some standard results from calculus allow us to infer some very important consequences from Equations (2.2) and (2.7).

2.2.1 General considerations

Firstly, from (2.7) it follows the existence of a smooth function $\phi(\cdot,\cdot,t)$, called the *velocity potential*, with the property

$$\phi_x = u \quad \text{and} \quad \phi_y = v \quad \text{in } \mathcal{D}(t) \tag{2.8}$$

for every $t > 0$. This function $\phi(\cdot,\cdot,t)$ is determined up to an additive function of t by the line integral[1]

$$\phi(\mathbf{x},t) = \int_{\gamma(\mathbf{x})} (u\,\mathrm{d}x + v\,\mathrm{d}y), \quad \mathbf{x} = (x,y) \in \mathcal{D}(t),$$

where $\gamma(\mathbf{x})$ is a path connecting $\mathbf{x} \in \mathcal{D}(t)$ to an arbitrary, fixed point $\mathbf{x}_0 \in \mathcal{D}(t)$; Stokes' theorem (see for example [25]) guarantees that the line integral does not depend on the particular choice of the path $\gamma(\mathbf{x})$. For instance, a possible explicit formula for $\phi(\cdot,\cdot,t)$ is

$$\phi(x,y,t) = \int_0^x u(s,-d,t)\,\mathrm{d}s + \int_{-d}^y v(x,s,t)\,\mathrm{d}s, \quad (x,y) \in \mathcal{D}(t),$$

which can be easily verified by differentiation. Notice that from (2.2) and (2.8) it follows

$$\Delta\phi = 0 \quad \text{in } \mathcal{D}(t),$$

which has also the consequence that ϕ (and therefore u and v) is in fact real-analytic throughout the fluid domain at all times.

On the other hand, (2.2) implies for each time $t > 0$ the existence of a smooth function $\psi(\cdot,\cdot,t)$, called the *stream function*, such that

$$\psi_y = u \quad \text{and} \quad \psi_x = -v \quad \text{in } \mathcal{D}(t). \tag{2.9}$$

Similarly as before, the stream function $\psi(\cdot,\cdot,t)$ is prescribed up to an additive function of t by the line integral

$$\psi(\mathbf{x},t) = \int_{\gamma(\mathbf{x})} (u\,\mathrm{d}y - v\,\mathrm{d}x), \quad \mathbf{x} = (x,y) \in \mathcal{D}(t),$$

where $\gamma(\mathbf{x})$ is a path as above. The incompressibility condition (2.2) accounts for the independence of this definition of the chosen integration path. Moreover, at every instant $t > 0$ the function $\psi(\cdot,\cdot,t)$ is uniquely determined by requiring that

$$\psi(x,-d,t) = 0, \quad x \in \mathbb{R},$$

thus

$$\psi(x,y,t) = \int_{-d}^y u(x,s,t)\,\mathrm{d}s, \quad (x,y) \in \mathcal{D}(t);$$

in particular, we see that $\psi(\cdot,\cdot,t)$ has period λ in the first variable for each $t > 0$. Also, plugging (2.9) into (2.7) yields

$$\Delta\psi = 0 \quad \text{in } \mathcal{D}(t). \tag{2.10}$$

[1] The velocity potential for an irrotational flow can be introduced in an analogous way also in the three-dimensional setting. This is not true for the stream function, defined below, which can exist only in two dimensions.

2.2.2 Travelling waves

In the special case of travelling waves, where the (x,t)-dependence of all variables is of the form $(x-ct)$ for a constant propagation speed $c>0$, it is more convenient to consider a stream function that satisfies

$$\psi_y = u - c \quad \text{and} \quad \psi_x = -v \quad \text{in } \mathcal{D}(t). \tag{2.11}$$

We can eliminate the time variable time via the transformation $x-ct \to x$ and $y \to y$, so that the governing equations read now

$$(u-c)u_x + vu_y = -\frac{1}{\rho}\,p_x \qquad \text{in } \mathcal{D}, \tag{2.12}$$

$$(u-c)v_x + vv_y = -\frac{1}{\rho}\,p_y - g \quad \text{in } \mathcal{D}, \tag{2.13}$$

$$u_x + v_y = 0 \qquad \text{in } \mathcal{D}, \tag{2.14}$$

$$u_y - v_x = 0 \qquad \text{in } \mathcal{D}, \tag{2.15}$$

$$v = (u-c)\eta' \qquad \text{on } \{(x,y)\in\mathbb{R}^2 : y = \eta(x)\}, \tag{2.16}$$

$$v = 0 \qquad \text{on } \{(x,y)\in\mathbb{R}^2 : y = -d\}, \tag{2.17}$$

$$p = p_{\text{atm}} \qquad \text{on } \{(x,y)\in\mathbb{R}^2 : y = \eta(x)\}, \tag{2.18}$$

where $\mathcal{D}$ is the fluid domain in the new time-independent transformed coordinates,

$$\mathcal{D} = \{(x,y)\in\mathbb{R}^2 : -d < y < \eta(x)\}.$$

Let us introduce the *relative mass flux* through x as

$$m_0 = \int_{-d}^{\eta(x)} (u(x,s)-c)\,\mathrm{d}s, \tag{2.19}$$

which is independent of x, because

$$\begin{aligned}
\frac{\mathrm{d}m_0}{\mathrm{d}x} &= (u(x,\eta(x))-c)\eta'(x) + \int_{-d}^{\eta(x)} u_x(x,s)\,\mathrm{d}s \\
&= (u(x,\eta(x))-c)\eta'(x) + \int_{-d}^{\eta(x)} \psi_{yx}(x,s)\,\mathrm{d}s \\
&= (u(x,\eta(x))-c)\eta'(x) + [\psi_x(x,s)]_{s=-d}^{s=\eta(x)} \\
&= (u(x,\eta(x))-c)\eta'(x) - v(x,\eta(x)) + v(x,-d) \\
&= 0,
\end{aligned}$$

where we used (2.11) and the boundary conditions (2.16) and (2.17). Thus we may choose $\psi = m_0$ on the free surface, which implies that the flat bottom is the streamline $\psi = 0$. Furthermore, from (2.12) and (2.13) it follows that the expression

$$E = \frac{(c-u)^2 + v^2}{2} + gy + p,$$

sometimes called the *hydraulic head*, is constant throughout the flow (*Bernoulli's law*), so that the boundary condition (2.18) can be restated as

$$\psi_x^2 + \psi_y^2 + 2g(y+d) = Q,$$

where $Q := 2(E - p_{\text{atm}} + gd)$ is a (known) constant. Thus the whole system (2.12)–(2.18) of equations and boundary conditions can be reformulated in terms of ψ as the boundary-value problem

$$\begin{cases} \Delta\psi = 0 & \text{in } \mathcal{D}, \\ |\nabla\psi|^2 + 2g(y+d) = Q & \text{on } \{(x,y) \in \mathbb{R}^2 : y = \eta(x)\}, \\ \psi = m_0 & \text{on } \{(x,y) \in \mathbb{R}^2 : y = \eta(x)\}, \\ \psi = 0 & \text{on } \{(x,y) \in \mathbb{R}^2 : y = -d\} \end{cases} \tag{2.20}$$

for the Laplace equation. This fact should highlight the importance of the theory of harmonic functions (and also of holomorphic functions in the two-dimensional case, as we will see in the next chapter) in the study of the water wave problem; see [6, 8] for further information on this topic.

2.3 The mean flow beneath the surface waves

If for each $t > 0$ we have

$$\eta_0(t) := \min_{x\in[0,\lambda]} \eta(x,t) > -d$$

for the depth of the wave trough at time t, from (2.9) we obtain

$$\frac{1}{\lambda}\int_0^\lambda v(x,y,t)\,\mathrm{d}x = -\frac{1}{\lambda}\int_0^\lambda \psi_x(x,y,t)\,\mathrm{d}x = 0, \quad -d \le y \le \eta_0(t), \tag{2.21}$$

and

$$\begin{aligned} \frac{\partial}{\partial y}\left(\frac{1}{\lambda}\int_0^\lambda u(x,y,t)\,\mathrm{d}x\right) &= \frac{1}{\lambda}\int_0^\lambda u_y(x,y,t)\,\mathrm{d}x \\ &= \frac{1}{\lambda}\int_0^\lambda \psi_{yy}(x,y,t)\,\mathrm{d}x \\ &= \frac{1}{\lambda}\int_0^\lambda \Delta\psi(x,y,t)\,\mathrm{d}x = 0, \quad -d \le y \le \eta_0(t); \end{aligned}$$

the latter implies that

$$\frac{1}{\lambda}\int_0^\lambda u(x,y,t)\,\mathrm{d}x = f(t), \quad -d \le y \le \eta_0(t),$$

for some smooth function $f : \mathbb{R}_+ \to \mathbb{R}$. Moreover, from (2.4), the first equation in (2.1) evaluated on $y = -d$, and the fact that both u and p are periodic in the first variable, we obtain

$$\begin{aligned} f'(t) &= \frac{\partial}{\partial t}\left(\frac{1}{\lambda}\int_0^\lambda u(x,y,t)\,\mathrm{d}x\right)\Bigg|_{\{y=-d\}} \\ &= \frac{1}{\lambda}\int_0^\lambda u_t(x,-d,t)\,\mathrm{d}x \\ &= -\frac{1}{\lambda}\int_0^\lambda \left(\frac{1}{\rho}p_x(x,-d,t) + u(x,-d,t)u_x(x,-d,t)\right)\mathrm{d}x \\ &= -\frac{1}{\rho\lambda}\int_0^\lambda p_x(x,-d,t)\,\mathrm{d}x - \frac{1}{2\lambda}\int_0^\lambda \frac{\partial}{\partial x}(u(x,-d,t))^2\,\mathrm{d}x \\ &= 0, \end{aligned}$$

and so

$$\frac{1}{\lambda}\int_0^\lambda u(x,y,t)\,\mathrm{d}x = u_0, \quad -d \le y \le \eta_0(t), \tag{2.22}$$

for some constant $u_0 \in \mathbb{R}$; this means that we have the constant mean flow u_0 for all depths between the flat bed and the trough level. Thus the water flow is experiencing a constant horizontal "drift" with speed u_0, that is, the average behaviour of the flow in the region $\{(x,y) : -d < y < \eta_0(t)\}$ is that of a constant current with velocity field $(u_0, 0)$, so that we may interpret u_0 as the underlying (constant) current to the flow.

3 Main result for irrotational flows

In this chapter we consider a two-dimensional, irrotational water flow that is spatially periodic with period λ, thus subjected to the governing equations (2.1)–(2.7) from § 2.1. Notice, however, that we do not require that the flow be a travelling wave—the wave profile and the velocity field may evolve in a non-periodic way in time, although maintaining at every instant the λ-periodicity in the horizontal direction x. In the subsequent discussion we investigate how the influence of the motion of the free surface on the water layer below attenuates with increasing depth.

3.1 Proof of the main result for irrotational flows

Our goal is to recover, using a different method (adapted from [1]), the classical result of [27], which, rewritten in the notation introduced above, can be stated as follows:

Theorem 3.1. *The flow underneath any two-dimensional, spatially periodic, irrotational water wave with no underlying current tends exponentially fast to zero: if*

$$\mathfrak{s}(x,y,t) \coloneqq \sqrt{u(x,y,t)^2 + v(x,y,t)^2}, \quad (x,y) \in \mathcal{D}(t),$$

then at any point $(x,y) \in \mathcal{D}(t)$ *with* $-d \le y \le \eta_0(t) - \frac{\lambda}{4}$ *we have*

$$\mathfrak{s}(x,y,t) \le 2\,\mathrm{e}^{-k\eta_0(t)}\mathrm{e}^{ky}\left(1 + \mathrm{e}^{-2k(d+y)}\right) \max_{\xi\in[0,\lambda]} \mathfrak{s}(\xi, \eta_0(t), t), \tag{3.1}$$

where $k = 2\pi/\lambda$ *is the wave number.*

This we do (in a slightly more general setting) in the next theorem; then, in Corollary 3.4 below, we show how the bound can be further improved by conveniently modifying the proof of the theorem.

Theorem 3.2. *The flow underneath any two-dimensional, spatially periodic, irrotational water wave tends exponentially fast to the underlying mean flow* u_0*: if*

$$\mathfrak{s}(x,y,t) \coloneqq \sqrt{(u(x,y,t) - u_0)^2 + v(x,y,t)^2}, \quad (x,y) \in \mathcal{D}(t),$$

then the maximum $\mathfrak{S}(t)$ *of the map* $(x,y) \mapsto \mathfrak{s}(x,y,t)$ *in* $\mathcal{D}_0(t) \coloneqq \{(x,y) : -d < y < \eta_0(t)\}$ *is attained on the line* $y = \eta_0(t)$ *and for all* $-d \le y \le \eta_0(t) - \frac{\lambda}{4}$ *we have*

$$\mathfrak{s}(x,y,t) \le \frac{\mathrm{e}^{\pi/4}}{2\sinh(\frac{\pi}{4})}\,\mathrm{e}^{-k\eta_0(t)}\mathrm{e}^{ky}\left(1 + \mathrm{e}^{-2k(d+y)}\right)\mathfrak{S}(t), \tag{3.2}$$

where $k = 2\pi/\lambda$ *is the wave number.*

Proof. At any fixed $t>0$ the function defined by

$$f(x+\mathrm{i}y,t) \coloneqq -\psi_x(x,y,t)+\mathrm{i}(\psi_y(x,y,t)-u_0) = v(x,y,t)+\mathrm{i}(u(x,y,t)-u_0) \tag{3.3}$$

is holomorphic in $\{x+\mathrm{i}y \in \mathbb{C} : -d < y < \eta_0(t)\}$ (the validity of the Cauchy–Riemann equations being an immediate consequence of (2.9) and (2.10)) and continuous on its closure; moreover, (2.21) and (2.22) imply that

$$\frac{1}{\lambda}\int_0^\lambda f(x+\mathrm{i}y,t)\,\mathrm{d}x = \frac{1}{\lambda}\int_0^\lambda v(x,y,t)\,\mathrm{d}x + \frac{\mathrm{i}}{\lambda}\int_0^\lambda (u(x,y,t)-u_0)\,\mathrm{d}x = 0, \quad -d \le y \le \eta_0(t). \tag{3.4}$$

By (2.5), the real part of f vanishes on the line $y=-d$, thus we can apply the Schwarz reflection principle (see [34]) to obtain a holomorphic extension of f to the rectangular domain

$$\mathcal{B}(t) \coloneqq \{x+\mathrm{i}y \in \mathbb{C} : 0 < x < \lambda, -2d-\eta_0(t) < y < \eta_0(t)\}$$

given by

$$f(x+\mathrm{i}y,t) \coloneqq -\overline{f(x-\mathrm{i}(2d+y),t)}, \quad -2d-\eta_0(t) < y < -d. \tag{3.5}$$

Now we introduce the function

$$F(z,t) \coloneqq -\mathrm{i}k\,\mathrm{e}^{\mathrm{i}kz} f(z,t), \quad z \in \mathcal{B}(t);$$

the residue theorem yields

$$f(z_0,t) = \frac{1}{2\pi\mathrm{i}}\int_{\partial\mathcal{B}(t)} \frac{F(z,t)}{\mathrm{e}^{-\mathrm{i}kz}-\mathrm{e}^{-\mathrm{i}kz_0}}\,\mathrm{d}z, \quad z_0 \in \mathcal{B}(t). \tag{3.6}$$

Indeed,

$$\frac{1}{2\pi\mathrm{i}}\int_{\partial\mathcal{B}(t)} \frac{F(z,t)}{\mathrm{e}^{-\mathrm{i}kz}-\mathrm{e}^{-\mathrm{i}kz_0}}\,\mathrm{d}z = \operatorname{Res}\left(-\frac{\mathrm{i}k\,\mathrm{e}^{\mathrm{i}kz} f(z,t)}{\mathrm{e}^{-\mathrm{i}kz}-\mathrm{e}^{-\mathrm{i}kz_0}}; z_0\right) = \operatorname{Res}\left(\frac{\mathrm{i}kf(z,t)}{\mathrm{e}^{\mathrm{i}k(z-z_0)}-1}; z_0\right),$$

and applying the generalized binomial theorem we compute

$$\begin{aligned}
\frac{1}{\mathrm{e}^{\mathrm{i}k(z-z_0)}-1} &= \left(\sum_{n=1}^\infty \frac{(\mathrm{i}k)^n}{n!}(z-z_0)^n\right)^{-1} \\
&= \left(\mathrm{i}k(z-z_0) + \sum_{n=2}^\infty \frac{(\mathrm{i}k)^n}{n!}(z-z_0)^n\right)^{-1} \\
&= \sum_{m=0}^\infty (-1)^m (\mathrm{i}k(z-z_0))^{-1-m}\left(\sum_{n=2}^\infty \frac{(\mathrm{i}k)^n}{n!}(z-z_0)^n\right)^m \\
&= \frac{1}{\mathrm{i}k(z-z_0)} - \frac{1}{(\mathrm{i}k)^2(z-z_0)^2}\cdot\sum_{n=2}^\infty \frac{(\mathrm{i}k)^n}{n!}(z-z_0)^n + \dots,
\end{aligned}$$

hence

$$\operatorname{Res}\left(\frac{\mathrm{i}kf(z,t)}{\mathrm{e}^{\mathrm{i}k(z-z_0)}-1}; z_0\right) = \mathrm{i}kf(z_0,t)\operatorname{Res}\left(\frac{1}{\mathrm{e}^{\mathrm{i}k(z-z_0)}-1}; z_0\right) = \mathrm{i}kf(z_0,t)\frac{1}{\mathrm{i}k} = f(z_0,t).$$

We split the boundary $\partial\mathcal{B}(t)$ into the segments

$$\begin{aligned}
L^r(t) &\coloneqq \{\lambda+\mathrm{i}y : -2d-\eta_0(t) \le y \le \eta_0(t)\}, & C^-(t) &\coloneqq \{x+\mathrm{i}(-2d-\eta_0(t)) : 0 \le x \le \lambda\}, \\
L^\ell(t) &\coloneqq \{\mathrm{i}y : -2d-\eta_0(t) \le y \le \eta_0(t)\}, & C^+(t) &\coloneqq \{x+\mathrm{i}\eta_0(t) : 0 \le x \le \lambda\}.
\end{aligned}$$

Since the function ψ is λ-periodic in the x-direction, so is $v = -\psi_x$, thus, keeping in mind that $k = 2\pi/\lambda$, we get

$$\begin{aligned} F(x+\lambda+\mathrm{i}y,t) &= -\mathrm{i}k\,\mathrm{e}^{-\mathrm{i}k(x+\lambda)}\mathrm{e}^{y}[v(x+\lambda,y,t)+\mathrm{i}(u(x+\lambda,y,t)-u_0)] \\ &= -\mathrm{i}k\,\mathrm{e}^{-\mathrm{i}kx}\mathrm{e}^{y}[v(x,y,t)+\mathrm{i}(u(x,y,t)-u_0)] \\ &= F(x+\mathrm{i}y,t), \end{aligned}$$

i.e. F is λ-periodic in the x-direction as well; hence

$$\int_{L^r(t)} \frac{F(z,t)}{\mathrm{e}^{-\mathrm{i}kz}-\mathrm{e}^{-\mathrm{i}kz_0}}\,\mathrm{d}z = \int_{L^\ell(t)} \frac{F(z,t)}{\mathrm{e}^{-\mathrm{i}kz}-\mathrm{e}^{-\mathrm{i}kz_0}}\,\mathrm{d}z.$$

Therefore, by means of (3.4), we see that (3.6) reduces to

$$\begin{aligned} f(z_0,t) &= \frac{1}{2\pi\mathrm{i}}\int_{C^-(t)} \frac{F(z,t)}{\mathrm{e}^{-\mathrm{i}kz}-\mathrm{e}^{-\mathrm{i}kz_0}}\,\mathrm{d}z - \frac{1}{2\pi\mathrm{i}}\int_{C^+(t)} \frac{F(z,t)}{\mathrm{e}^{-\mathrm{i}kz}-\mathrm{e}^{-\mathrm{i}kz_0}}\,\mathrm{d}z \\ &= \frac{1}{2\pi\mathrm{i}}\int_{C^-(t)} \frac{F(z,t)}{\mathrm{e}^{-\mathrm{i}kz}-\mathrm{e}^{-\mathrm{i}kz_0}}\,\mathrm{d}z - \frac{1}{2\pi\mathrm{i}}\int_{C^+(t)} \mathrm{e}^{\mathrm{i}kz}F(z,t)\,\mathrm{d}z \\ &\quad - \frac{\mathrm{e}^{-\mathrm{i}kz_0}}{2\pi\mathrm{i}}\int_{C^+(t)} \frac{\mathrm{e}^{\mathrm{i}kz}F(z,t)}{\mathrm{e}^{-\mathrm{i}kz}-\mathrm{e}^{-\mathrm{i}kz_0}}\,\mathrm{d}z \\ &= \frac{1}{2\pi\mathrm{i}}\int_0^\lambda \frac{\mathrm{i}k\overline{f(x+\mathrm{i}\eta_0(t),t)}\,\mathrm{e}^{-\mathrm{i}k[x-2\mathrm{i}d-\mathrm{i}\eta_0(t)]}}{\mathrm{e}^{-\mathrm{i}k[x+\mathrm{i}\eta_0(t)]}-\mathrm{e}^{-\mathrm{i}kz_0}}\,\mathrm{d}x + \frac{1}{2\pi\mathrm{i}}\int_0^\lambda \mathrm{i}kf(x+\mathrm{i}\eta_0(t),t)\,\mathrm{d}x \\ &\quad + \frac{\mathrm{e}^{-\mathrm{i}kz_0}}{2\pi\mathrm{i}}\int_0^\lambda \frac{\mathrm{i}kf(x+\mathrm{i}\eta_0(t),t)}{\mathrm{e}^{-\mathrm{i}k[x+\mathrm{i}\eta_0(t)]}-\mathrm{e}^{-\mathrm{i}kz_0}}\,\mathrm{d}x \\ &= \frac{1}{\lambda}\int_0^\lambda \frac{\overline{f(x+\mathrm{i}\eta_0(t),t)}\,\mathrm{e}^{-\mathrm{i}k[x-2\mathrm{i}d-\mathrm{i}\eta_0(t)]}}{\mathrm{e}^{-\mathrm{i}k[x+\mathrm{i}\eta_0(t)]}-\mathrm{e}^{-\mathrm{i}kz_0}}\,\mathrm{d}x + \frac{\mathrm{e}^{-\mathrm{i}kz_0}}{\lambda}\int_0^\lambda \frac{f(x+\mathrm{i}\eta_0(t),t)}{\mathrm{e}^{-\mathrm{i}k[x+\mathrm{i}\eta_0(t)]}-\mathrm{e}^{-\mathrm{i}kz_0}}\,\mathrm{d}x, \end{aligned}$$

or, in other words,

$$\begin{aligned} f(x_0+\mathrm{i}y_0,t) &= \frac{1}{\lambda}\int_0^\lambda \frac{\overline{f(x+\mathrm{i}\eta_0(t),t)}\,\mathrm{e}^{-\mathrm{i}k[(x-x_0)-\mathrm{i}(2d+\eta_0(t)+y_0)]}}{\mathrm{e}^{-\mathrm{i}k[(x-x_0)-\mathrm{i}(2d+\eta_0(t)+y_0)]}-1}\,\mathrm{d}x \\ &\quad + \frac{1}{\lambda}\int_0^\lambda \frac{f(x+\mathrm{i}\eta_0(t),t)\,\mathrm{e}^{\mathrm{i}k[(x-x_0)+\mathrm{i}(\eta_0(t)-y_0)]}}{1-\mathrm{e}^{\mathrm{i}k[(x-x_0)+\mathrm{i}(\eta_0(t)-y_0)]}}\,\mathrm{d}x \end{aligned} \tag{3.7}$$

for all $x_0, y_0 \in \mathbb{R}$ with $-d \le y_0 \le \eta_0(t)$.

Now suppose that $-d \le y_0 \le \eta_0(t) - \frac{\lambda}{4}$. Then

$$k[\eta_0(t)-y_0] \ge \frac{2\pi}{\lambda}\cdot\frac{\lambda}{4} = \frac{\pi}{2} \quad \text{and} \quad k[2d+\eta_0(t)+y_0] \ge k[\eta_0(t)+d] \ge \frac{2\pi}{\lambda}\left(-d+\frac{\lambda}{4}+d\right) = \frac{\pi}{2},$$

so that

$$\left|\frac{\mathrm{e}^{\mathrm{i}k[(x-x_0)+\mathrm{i}(\eta_0(t)-y_0)]}}{1-\mathrm{e}^{-\mathrm{i}k[(x-x_0)+\mathrm{i}(\eta_0(t)-y_0)]}}\right| \le \frac{\mathrm{e}^{k[y_0-\eta_0(t)]}}{1-\mathrm{e}^{k[y_0-\eta_0(t)]}} \le \frac{\mathrm{e}^{k[y_0-\eta_0(t)]}}{1-\mathrm{e}^{-\pi/2}} \tag{3.8}$$

and

$$\left|\frac{\mathrm{e}^{-\mathrm{i}k[(x-x_0)-\mathrm{i}(2d+\eta_0(t)+y_0)]}}{\mathrm{e}^{-\mathrm{i}k[(x-x_0)-\mathrm{i}(2d+\eta_0(t)+y_0)]}-1}\right| \le \frac{\mathrm{e}^{-k[2d+y_0+\eta_0(t)]}}{1-\mathrm{e}^{-k[2d+y_0+\eta_0(t)]}} \le \frac{\mathrm{e}^{-k[2d+y_0+\eta_0(t)]}}{1-\mathrm{e}^{-\pi/2}}. \tag{3.9}$$

Applying these estimates to (3.7) yields

$$|f(x_0+\mathrm{i}y_0,t)| \le \left(\max_{x\in[0,\lambda]}|f(x+\mathrm{i}\eta_0(t),t)|\right)\frac{\mathrm{e}^{-k[2d+y_0+\eta_0(t)]}+\mathrm{e}^{k[y_0-\eta_0(t)]}}{1-\mathrm{e}^{-\pi/2}}\frac{1}{\lambda}\int_0^\lambda \mathrm{d}x$$
$$= \left(\max_{x\in[0,\lambda]}|f(x+\mathrm{i}\eta_0(t),t)|\right)\mathrm{e}^{-k\eta_0(t)}\mathrm{e}^{ky_0}\left(1+\mathrm{e}^{-2k(d+y_0)}\right)\frac{\mathrm{e}^{\pi/4}}{2\sinh(\frac{\pi}{4})}, \quad (3.10)$$

or, equivalently,

$$\mathfrak{s}(x_0,y_0,t) \le \frac{\mathrm{e}^{\pi/4}}{2\sinh(\frac{\pi}{4})}\mathrm{e}^{-k\eta_0(t)}\mathrm{e}^{ky_0}\left(1+\mathrm{e}^{-2k(d+y_0)}\right)\max_{x\in[0,\lambda]}\mathfrak{s}(x,\eta_0(t),t).$$

The domain $\mathcal{B}(t)$ is a periodicity cell of the region $\{x+\mathrm{i}y\in\mathbb{C} : -2d-\eta_0(t)<y<\eta_0(t)\}$, therefore by the maximum modulus principle applied to the holomorphic function f the maximum $\mathfrak{S}(t)$ is attained on the boundary of the region, and exploiting the symmetry that derives from the Schwarz reflection principle (Equation (3.5)) we can choose a location on the upper boundary, i.e.

$$\mathfrak{S}(t) = \max_{x\in[0,\lambda]}\mathfrak{s}(x,\eta_0(t),t).$$

Thus

$$\mathfrak{s}(x_0,y_0,t) \le \frac{\mathrm{e}^{\pi/4}}{2\sinh(\frac{\pi}{4})}\mathrm{e}^{-k\eta_0(t)}\mathrm{e}^{ky_0}\left(1+\mathrm{e}^{-2k(d+y_0)}\right)\mathfrak{S}(t), \quad -d\le y_0\le\eta_0(t)-\frac{\lambda}{4},$$

which concludes the proof. □

Remark 3.3. In the case of $u_0 = 0$, we recover from (3.2) the estimate (3.1) of [27], although with a different (better) coefficient in front: in (3.1) this is namely 2, which is replaced by $\mathrm{e}^{\pi/4}/(2\sinh(\frac{\pi}{4})) \approx 1.262$ in (3.2). However, a sharper estimate can be achieved, as we show next.

Corollary 3.4. *Under the same assumptions as in Theorem 3.2, for all* $-d\le y\le\eta_0(t)-\frac{\lambda}{4}$ *we have*

$$\mathfrak{s}(x,y,t) \le \mathrm{e}^{-k\eta_0(t)}\mathrm{e}^{ky}\left[1+\mathrm{e}^{-2k(d+y)}+\frac{\mathrm{e}^{\pi/4}}{2\sinh(\frac{\pi}{4})}\mathrm{e}^{-k\eta_0(t)}\mathrm{e}^{ky}\left(1+\mathrm{e}^{-4k(d+y)}\right)\right]\mathfrak{S}(t). \quad (3.11)$$

Proof. We would like to derive better estimates for the expressions

$$\frac{\mathrm{e}^{k[y_0-\eta_0(t)]}}{1-\mathrm{e}^{k[y_0-\eta_0(t)]}} \quad \text{and} \quad \frac{\mathrm{e}^{-k[2d+y_0+\eta_0(t)]}}{1-\mathrm{e}^{-k[2d+y_0+\eta_0(t)]}}$$

than those in (3.8) and (3.9), respectively. To do so, we use the following trick. Let us look at the first fraction. For $-d\le y_0\le\eta_0(t)-\frac{\lambda}{4}$ we know that

$$0<\mathrm{e}^{k[y_0-\eta_0(t)]}\le\mathrm{e}^{-\pi/2}<1 \quad \text{and} \quad 0<\mathrm{e}^{-k[2d+y_0+\eta_0(t)]}\le\mathrm{e}^{-\pi/2}<1,$$

with sharp inequalities if $y_0<\eta_0(t)-\frac{\lambda}{4}$. Thus

$$\frac{\mathrm{e}^{k[y_0-\eta_0(t)]}}{1-\mathrm{e}^{k[y_0-\eta_0(t)]}} = \mathrm{e}^{k[y_0-\eta_0(t)]}\left(1+\frac{\mathrm{e}^{k[y_0-\eta_0(t)]}}{1-\mathrm{e}^{k[y_0-\eta_0(t)]}}\right)$$
$$\le \mathrm{e}^{k[y_0-\eta_0(t)]}+\frac{\mathrm{e}^{2k[y_0-\eta_0(t)]}}{1-\mathrm{e}^{-\pi/2}},$$

which is an improvement to the bound in (3.8) because

$$\mathrm{e}^{k[y_0-\eta_0(t)]}+\frac{\mathrm{e}^{2k[y_0-\eta_0(t)]}}{1-\mathrm{e}^{-\pi/2}}=\mathrm{e}^{k[y_0-\eta_0(t)]}\left(1+\frac{\mathrm{e}^{k[y_0-\eta_0(t)]}}{1-\mathrm{e}^{-\pi/2}}\right)$$
$$\leq \mathrm{e}^{k[y_0-\eta_0(t)]}\left(1+\frac{\mathrm{e}^{-\pi/2}}{1-\mathrm{e}^{-\pi/2}}\right)=\frac{\mathrm{e}^{k[y_0-\eta_0(t)]}}{1-\mathrm{e}^{-\pi/2}},$$

with sharp inequality if $y_0<\eta_0(t)-\frac{\lambda}{4}$. Analogously we obtain for the second fraction the improved bound

$$\frac{\mathrm{e}^{-k[2d+y_0+\eta_0(t)]}}{1-\mathrm{e}^{-k[2d+y_0+\eta_0(t)]}}\leq \mathrm{e}^{-k[2d+y_0+\eta_0(t)]}+\frac{\mathrm{e}^{-2k[2d+y_0+\eta_0(t)]}}{1-\mathrm{e}^{-\pi/2}}.$$

Thus

$$\frac{\mathrm{e}^{k[y_0-\eta_0(t)]}}{1-\mathrm{e}^{k[y_0-\eta_0(t)]}}+\frac{\mathrm{e}^{-k[2d+y_0+\eta_0(t)]}}{1-\mathrm{e}^{-k[2d+y_0+\eta_0(t)]}}$$
$$\leq \mathrm{e}^{-k\eta_0(t)}\left(\mathrm{e}^{ky_0}+\mathrm{e}^{-k(2d+y_0)}+\frac{\mathrm{e}^{2ky_0}+\mathrm{e}^{-2k(2d+y_0)}}{1-\mathrm{e}^{-\pi/2}}\mathrm{e}^{-k\eta_0(t)}\right)$$
$$=\mathrm{e}^{-k\eta_0(t)}\mathrm{e}^{ky_0}\left(1+\mathrm{e}^{-2k(d+y_0)}+\mathrm{e}^{ky_0}\mathrm{e}^{-k\eta_0(t)}\frac{1+\mathrm{e}^{-4k(d+y_0)}}{1-\mathrm{e}^{-\pi/2}}\right)$$
$$=\mathrm{e}^{-k\eta_0(t)}\mathrm{e}^{ky_0}\left[1+\mathrm{e}^{-2k(d+y_0)}+\frac{\mathrm{e}^{\pi/4}}{2\sinh(\frac{\pi}{4})}\mathrm{e}^{ky_0}\mathrm{e}^{-k\eta_0(t)}\left(1+\mathrm{e}^{-4k(d+y_0)}\right)\right];$$

inserting this instead of (3.8) and (3.9) in (3.10) and completing the argument exactly as in the proof of Theorem 3.2 yields the claim. □

3.2 Decay at the wave base

Typically, below a depth of one half the wavelength of the surface wave (also called the *wave base* in oceanography) one expects the effects of the water flow to be negligible compared to those of the underlying current (see [12]), so let us check this basing on the bounds we derived. For typical surface wind-waves, we have that $\eta_0(t)>-\frac{\lambda}{10}$ (see [6]). The bound in (3.11) is non-decreasing on $-d\leq y\leq -\frac{\lambda}{2}$, thus with $\mathrm{e}^{-k\eta_0(t)}\leq \mathrm{e}^{\pi/4}$ we see that

$$\mathfrak{s}\left(x,-\frac{\lambda}{2},t\right)\leq \mathrm{e}^{-\pi}\mathrm{e}^{\pi/4}\left(2+\frac{\mathrm{e}^{-\pi/2}}{2\sinh(\frac{\pi}{4})}\cdot 2\right)\mathfrak{S}(t)\approx 0.212\,\mathfrak{S}(t),$$

which means that the deviation of the mean velocity of the flow from the underlying current at a depth of a half wavelength is reduced approximately fivefold from its (maximal) value on the trough level, in accordance to our expectations. Also note that using the original bound (3.2) yields

$$\mathfrak{s}\left(x,-\frac{\lambda}{2},t\right)\leq \frac{\mathrm{e}^{-\pi/2}}{2\sinh(\frac{\pi}{4})}\mathfrak{S}(t)\approx 0.239\,\mathfrak{S}(t),$$

so that the estimate (3.11) provides us with an improvement of approximately 11% on the bound at the wave base.

4 Rotational flows

The purpose of this chapter is to outline the essential properties of flows with non-zero vorticity, highlighting the main differences from irrotational flows. Obviously, the presence of a non-vanishing vorticity has the immediate consequence that there is no velocity potential as in (2.8); however, for two-dimensional flows, it is still possible to introduce a stream function ψ analogously to (2.9), with the *caveat* that this time ψ is not harmonic, but instead it satisfies the Poisson equation

$$\Delta\psi = \omega \quad \text{in } \mathcal{D}(t),$$

where ω is the vorticity of the flow, according to (1.14). For the rest of this chapter we will restrict our attention to two-dimensional flows, as in Chapter 1, using the same notation introduced there.

4.1 Currents

A flow with a flat surface (typically pre-existing before the arrival of a surface wind wave) is called a *current*. It turns out that vorticity is a very adequate tool for describing currents, as the next example (borrowed from [6]) shows in the case of a two-dimensional, steady flow.

Example 1. Suppose that we are looking at a current of the form

$$\mathbf{u}(x,y,t) = (u(x-ct,y), v(x-ct,y))^T,$$

with $c > 0$, in the layer $\mathcal{D} = \{(x,y) \in \mathbb{R}^2 : -d < y < 0\}$ delimited by the flat surface $\{y = 0\}$ and the flat bed $\{y = -d\}$, where $d > 0$ is the average depth. As in (2.11) we may introduce a stream function $\psi(x-ct, y)$ with

$$\begin{cases} \psi_x = -v \\ \psi_y = u - c \end{cases} \quad \text{in } \mathcal{D}.$$

In analogy with (2.19) we also define the relative mass flux

$$m_0 = \int_{-d}^{0} (u(x-ct,s) - c)\,\mathrm{d}s = \psi(x-ct,0) - \psi(x-ct,-d),$$

which is a constant because the kinematic boundary conditions (1.7) and (1.8) in this context simply mean that $v = 0$ on the lines $\{y = -d\}$ and $\{y = 0\}$, so that both $\psi(x-ct,0)$ and $\psi(x-ct,-d)$ are constants. The stream function ψ, which is constant on both parts of the boundary, is determined up to an additive constant, so that we may choose $\psi = m_0$ on $\{y = 0\}$ and $\psi = 0$ on $\{y = -d\}$. The vorticity is $\omega(x - ct, y)$, therefore the governing equations can be rewritten in the form

$$\begin{cases} \psi(x, 0) = m_0, & x \in \mathbb{R}, \\ \psi(x, -d) = 0, & x \in \mathbb{R}, \\ \Delta\psi(x, y) = \omega(x, y), & (x, y) \in \mathcal{D}, \end{cases}$$

with a periodicity condition in the x-variable. It can be shown that if $\omega \in C^1(\overline{\mathcal{D}})$ is 2π-periodic in x, then this problem possesses a unique solution $\psi \in C^2(\overline{\mathcal{D}})$. Moreover, the solution can be written out only in terms of the relative mass flux and the vorticity: in fact, expanding ω and ψ into the Fourier series

$$\omega(x, y) = \sum_{n=0}^{\infty} \omega_n(y)\mathrm{e}^{\mathrm{i}nx},$$
$$\psi(x, y) = \sum_{n=0}^{\infty} \psi_n(y)\mathrm{e}^{\mathrm{i}nx},$$

a calculation shows that the Fourier coefficients $(\psi_n)_{n=0}^{\infty}$ of ψ can be written as

$$\psi_0(y) = m_0 + y\left(\frac{1}{d}\int_{-d}^{0} s\,\omega_0(s)\,\mathrm{d}s + \int_{-d}^{y} \omega_0(s)\,\mathrm{d}s\right) + \int_{y}^{0} s\,\omega_0(s)\,\mathrm{d}s,$$
$$\psi_n(y) = \frac{1}{n}\frac{\sinh(ny)}{\sinh(nd)}\int_{-d}^{0} \sinh(n(s + d))\omega_n(s)\,\mathrm{d}s + \frac{1}{n}\int_{0}^{y} \sinh(n(y - s))\omega_n(s)\,\mathrm{d}s, \quad n \geq 1.$$

In particular we see that if ω does not depend on x (that is, it is time-independent), and thus $\omega_n = \psi_n = 0$ for every $n \geq 1$, then this applies to ψ and u as well, whereas $v = 0$ throughout $\mathcal{D}$. ♢

4.2 Constant vorticity

As pointed out in [13], in important cases such as the propagation in the ocean or at sea of a long surface wave over a layer of water with a flat bed the mere presence of a non-vanishing vorticity is more important than the actual distribution thereof, so that it is of great interest to study flows with constant non-zero vorticity.

4.2.1 Tidal currents: flood and ebb

A constant non-zero vorticity is often used to the describe the effect of *tidal currents*, as illustrated by the next simple example.

Example 2. Let us consider the flow

$$\mathbf{u}(x, y, t) = (u_0 + \gamma y, 0), \quad (x, y) \in \{(x_0, y_0) \in \mathbb{R}^2 : -d < y_0 < 0\}$$

for some constants $u_0, \gamma \in \mathbb{R}$ with $\gamma \neq 0$. It is easily verified that this flow has the constant non-zero vorticity $\omega = \gamma$; u_0 can be interpreted as the constant underlying current (cf. § 2.3). Commonly, the alternating horizontal movements of water associated with the rise and fall of the tide (which, on the other hand, refers to the vertical motion of the water caused by the varying gravitational forces due to the relative motions of earth, Moon and

Sun) are called tidal currents; the case of a rising tide is referred to as the *flood*, whereas the current associated to a falling tide is called the *ebb*. In our example, a positive vorticity $\gamma > 0$ describes a flow whose (horizontal) velocity is greater on the surface than on the bottom, hence this case is adequate for modelling the ebb current; on the contrary, a negative vorticity $\gamma < 0$, for which the velocity on the flat bed exceeds that on the surface, is appropriate for flood currents.

Notice that, in the case of an adverse underlying constant current (that is, $u_0\gamma > 0$) with $u_0/\gamma < d$, the horizontal velocity vanishes at the depth $y = -u_0/\gamma \in (-d, 0)$ and changes sign when crossing this horizontal line (called a *critical level*, a feature that cannot occur in the irrotational context). ♢

4.2.2 Main result for constant vorticity

Now let us turn to the problem (2.1)–(2.7) from Chapter 2 for a spatially periodic flow, although with Equation (2.7) replaced by

$$u_y - v_x = \gamma \quad \text{in } \mathcal{D}(t) \tag{4.1}$$

for some $\gamma \neq 0$. As in (2.8) we introduce the stream function ψ and normalize it to

$$\psi(x, y, t) = \int_{-d}^{y} u(x, s, t)\,\mathrm{d}s, \quad (x, y) \in \mathcal{D}(t);$$

with (4.1) it also follows

$$\Delta\psi = \gamma \quad \text{in } \mathcal{D}(t).$$

Modifying accordingly the calculations done in § 2.3, we see that if

$$\eta_0(t) \coloneqq \min_{x\in[0,\lambda]} \eta(x, t) > -d$$

for the depth of the wave trough at all times $t > 0$, we have

$$\frac{1}{\lambda}\int_0^{\lambda} v(x, y, t)\,\mathrm{d}x = 0, \quad -d \le y \le \eta_0(t), \tag{4.2}$$

and

$$\frac{\partial}{\partial y}\left(\frac{1}{\lambda}\int_0^{\lambda} u(x, y, t)\,\mathrm{d}x\right) = \gamma, \quad -d \le y \le \eta_0(t);$$

the latter implies that

$$\frac{1}{\lambda}\int_0^{\lambda} u(x, y, t)\,\mathrm{d}x = \gamma(y + d) + f(t), \quad -d \le y \le \eta_0(t),$$

for some smooth function $f : \mathbb{R}_+ \to \mathbb{R}$ that describes the mean flow on the flat bed. Moreover, from (2.4), the first equation in (2.1) evaluated on $y = -d$, and the fact that both u and p are periodic in the first variable, we obtain as above

$$f'(t) = \frac{\partial}{\partial t}\left(\frac{1}{\lambda}\int_0^{\lambda} u(x, y, t)\,\mathrm{d}x\right)\Bigg|_{\{y=-d\}} = 0,$$

and so

$$\frac{1}{\lambda}\int_0^{\lambda} u(x, y, t)\,\mathrm{d}x = \gamma(y + d) + u_0, \quad -d \le y \le \eta_0(t), \tag{4.3}$$

for some constant $u_0 \in \mathbb{R}$.

Using this considerations, one can extend the result of Theorem 3.2 to flows with constant vorticity:

Theorem 4.1. *The flow underneath any two-dimensional, spatially periodic water wave with constant vorticity tends exponentially fast to the underlying mean flow: if*

$$\mathfrak{s}(x,y,t) := \sqrt{[u(x,y,t)-\gamma(y+d)-u_0]^2+v(x,y,t)^2}, \quad (x,y)\in\mathcal{D}(t),$$

then the maximum $\mathfrak{S}(t)$ of the map $(x,y)\mapsto\mathfrak{s}(x,y,t)$ in $\mathcal{D}_0(t) := \{(x,y) : -d < y < \eta_0(t)\}$ is attained on the line $y=\eta_0(t)$ and for all $-d\le y\le\eta_0(t)-\frac{\lambda}{4}$ we have

$$\mathfrak{s}(x,y,t) \le \frac{e^{\pi/4}}{2\sinh(\frac{\pi}{4})}\,e^{-k\eta_0(t)}e^{ky}\left(1+e^{-2k(d+y)}\right)\mathfrak{S}(t),$$

where $k=2\pi/\lambda$ is the wave number.

Clearly, Theorem 3.2 is contained in Theorem 4.1 as the special case $\gamma=0$. The main advantage of the proof given for Theorem 3.2 on the original one in [27] is that it can be very easily generalized to the case of non-zero vorticity: instead of introducing the holomorphic function f via (3.3), one defines for every instant t the harmonic function

$$\Psi(x,y,t) := \psi(x,y,t) - \frac{\gamma}{2}(y+d)^2 - u_0(y+d), \quad (x,y)\in\mathcal{D}(t),$$

and considers the function

$$f(x+iy,t) := -\Psi_x(x,y,t) + i\Psi_y(x,y,t) = v(x,y,t) + i[u(x,y,t)-\gamma(y+d)-u_0],$$

which is easliy shown to be holomorphic in the region $\{x+iy\in\mathbb{C} : -d<y<\eta_0(t)\}$, continuous on its closure, and such that

$$\frac{1}{\lambda}\int_0^\lambda f(x+iy,t)\,dx = 0, \quad -d\le y\le\eta_0(t);$$

from here the proof proceeds exactly as that for Theorem 3.2.

Obviously, an analogous generalization of Corollary 3.4 holds as well:

Corollary 4.2. *Under the same assumptions as in Theorem 4.1, for all $-d\le y\le\eta_0(t)-\frac{\lambda}{4}$ we have*

$$\mathfrak{s}(x,y,t) \le e^{-k\eta_0(t)}e^{ky}\left[1+e^{-2k(d+y)} + \frac{e^{\pi/4}}{2\sinh(\frac{\pi}{4})}\,e^{-k\eta_0(t)}e^{ky}\left(1+e^{-4k(d+y)}\right)\right]\mathfrak{S}(t).$$

A Well-posedness

Any reasonable model which is supposed to describe the evolution of a physical system should posses certain basic properties, whose validity needs to be checked in the first place. First of all, clearly one expects the model to be solvable (i.e. a solution *exists*); moreover, since two copies of the same system that (ideally) start from the same initial conditions are expected to evolve identically, it is also required that the solution be *unique*. However, when performing two experiments that are supposed to be "identical", in practice one has to deal with slight, yet unavoidable differences in the initial data; still, if those initial data are "close" to each other, it should be reasonable to expect that the behaviours of the two system will be "similar" as well, at least up to a certain time. These considerations lead us to the most generally accepted concept of *well-posedness* [15]: a (time-evolution) model is called well-posed if there is a time $T > 0$ such that

(i) there exists a solution defined for all $t \in [0, T)$,
(ii) the solution is unique on $[0, T)$, and
(iii) there is continuous dependence of the solution on the initial data (in some reasonable topology).

Obviously, as is clear from point (iii), the notion of "continuous dependence on the initial data" is very much problem-specific, and the choice of different topologies can lead to different notions of well-posedness for the same problem (as is the case, for instance, of the H^s-(un)conditional well-posedness for the Schrödinger equation, cf. [38]). The fact that this definition is just local in time, i.e. only up to a (possibly small) existence time T, is a consequence of the fact that it may very well happen that the solution ceases to exist in finite time; examples for this phenomenon are the blow-up of solutions of the Schrödinger equation (again, see [38]) or, in the case of water waves that are of interest for us, the breaking of the surface wave.

The well-posedness theory for the water wave equations is highly technical and relies on very deep results from many areas of mathematics, such as functional analysis, operator theory and differential geometry. Very little insight into the qualitative properties of the model is gained from this analysis, nevertheless it is of paramount importance, because lack of well-posedness invalidates the model. Moreover, there are still several open questions in this fascinating field, especially in the case of flows with vorticity—this is not at all surprising, given the notorious difficulty of the equations of fluid mechanics. After all it is well-known that the problem of well-posedness for the Navier–Stokes equations (where the effects of *viscosity* are taken into account by adding the term $\nu\Delta\mathbf{u}$ to the right-hand side of Equation (1.5), with the viscosity $\nu > 0$) is one of the renowned Millennium Problems assigned by the Clay Mathematics Institute in the early 2000s.[1]

For irrotational water waves, the issue of well-posedness was treated by many authors

[1] For more information, visit the dedicated website `https://www.claymath.org/millennium-problems/navier%E2%80%93stokes-equation` of the Clay Mathematics Institute.

over several decades, first for linearized surface waves [30, 42, 10], then for the linearization of the water wave equations around an exact solution that is supposed to satisfy a generalized Taylor criterion [2], until the nonlinear problem for overhanging, non-self-intersecting surface profiles (albeit only for the case of infinite depth) was solved in [40, 41]; finally, the problem for the general nonlinear irrotational water wave equations over a (not necessarily flat) bed of finite depth in arbitrary dimensions was settled in [24]. Written out in the notation of Chapter 1 for the 3-D case over a flat bed, the result in [24] is the following. As in § 2.2.1, we can find a velocity potential ϕ with $\mathbf{u} = \nabla_{\mathbf{x}}\phi$ and

$$\Delta\phi = 0 \quad \text{in } \mathcal{D}(t),\ t \geq 0.$$

As in Chapter 1, the free surface is described by (1.6), and we write $\mathbf{x} = (x, y, z)^T \in \mathcal{D}(t)$; the gradient ∇ is always taken with respect to the spatial components only, but for clarity we specify this by writing $\nabla_{\mathbf{x}}$ (respectively $\nabla_{(x,y)}$ for functions on $\mathbb{R}^2$ such as $\eta(\cdot, \cdot, t)$). Moreover, we introduce the notation

$$\mathbf{n} := \frac{1}{\sqrt{1 + \left|\nabla_{(x,y)}\eta\right|^2}} \left(-\nabla_{(x,y)}\eta, 1\right)^T \quad \text{and} \quad \partial_{\mathbf{n}} := \mathbf{n} \cdot \nabla_{\mathbf{x}},$$

and we set

$$\xi(x, y, t) := \phi(x, y, \eta(x, y, t), t) = \phi(\mathbf{x}, t)|_{\{z=\eta(x,y,t)\}}.$$

A calculation (similar to that in § 2.2.2, cf. also [11]) shows that the irrotational water wave problem

$$\begin{cases} \dfrac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\dfrac{1}{\rho}\nabla_{\mathbf{x}}p + \mathbf{g} & \text{in } \mathcal{D}(t), \\ \nabla_{\mathbf{x}} \cdot \mathbf{u} = 0 & \text{in } \mathcal{D}(t), \\ \operatorname{curl}\mathbf{u} = 0 & \text{in } \mathcal{D}(t), \\ u_3 = 0 & \text{on } \{z = -d\}, \\ u_3 = \eta_t + u_1\eta_x + u_2\eta_y & \text{on } \{z = \eta(x, y, t)\}, \\ p = 0 & \text{on } \{z = \eta(x, y, t)\} \end{cases} \tag{A.1}$$

(where the last equation is (1.9) after a renormalization) is equivalent to the system

$$\begin{cases} \eta_t - G(\eta)\xi = 0, \\ \xi_t + g\eta + \dfrac{1}{2}\left|\nabla_{(x,y)}\xi\right|^2 - \dfrac{1}{2\left(1 + \left|\nabla_{(x,y)}\eta\right|^2\right)}\left(G(\eta)\xi + \nabla_{(x,y)}\eta \cdot \nabla_{(x,y)}\xi\right) = 0, \end{cases} \tag{A.2}$$

where

$$G(\eta)\xi := \sqrt{1 + \left|\nabla_{(x,y)}\eta\right|^2}\, \partial_{\mathbf{n}}\phi|_{\{z=\eta(x,y,t)\}}$$

is called the *Dirichlet–Neumann operator* (see for instance [31]). Then the main result of [24] for the special case of a flat bed situated at the mean depth $z = -d$ can be stated as follows:

Theorem A.1. *Let $\eta_0 \in H^{s+1}(\mathbb{R}^2)$ and ξ_0 be such that $\nabla_{(x,y)}\xi_0 \in H^s(\mathbb{R}^2)^2$, with Sobolev index $s \geq 3$. Assume moreover that*

$$\eta_0 + d > 2h_0 \quad \textit{in } \mathbb{R}^2 \textit{ for some } h_0 > 0.$$

Then there exists $T > 0$ and a unique solution (η, ξ) to the water wave equations (A.2) with initial conditions (η_0, ξ_0) and such that

$$(\eta, \xi - \xi_0) \in C^1\left([0, T]; H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)\right).$$

Theorem A.1 can be restated in terms of the velocity field $\mathbf{u}$:

Theorem A.2. *Let $\eta_0 \in H^{s+1}(\mathbb{R}^2)$ and $\mathbf{u}_0 \in H^{s+1}(\mathbb{R}^3)^3$, with Sobolev index $s \geq 3$. Assume moreover that*

$$\eta_0 + d > 2h_0 \quad \text{in } \mathbb{R}^2 \text{ for some } h_0 > 0.$$

Then there exists $T > 0$ and a unique solution $(\eta, \mathbf{u})$ to the water wave equations (A.1) with initial conditions $(\eta_0, \mathbf{u}_0)$ and such that

$$(\eta, \mathbf{u}) \in C^1\left([0,T]; H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^3)^3\right).$$

Notice that, by Sobolev embedding (cf. [3]), the condition $s \geq 3$ implies that η and $\mathbf{u}$ are in fact of class C^1.

The first step towards the proof of Theorem A.1 consists of an in-depth analysis of the Dirichlet–Neumann operator: after showing a collection of estimates for its norm, culminating in the estimate

$$\|G(\eta)\xi\|_{H^{k+1/2}} \leq C(k, \|\eta\|_{H^{s_0}})\left(\|\eta\|_{H^{k+3/2}}\|\nabla_{(x,y)}\xi\|_{H^{s_0-1}} + \|\nabla_{(x,y)}\xi\|_{H^{k+1/2}}\right)$$

for all $k \in \mathbb{N}$, where s_0 is a fixed positive real number, one proceeds to investigate further properties of the operator $G(\eta)\cdot$, such as its principal symbol, its commutator with derivatives (both in space and in time) and its shape derivative (i.e. the derivative of the map $\eta \mapsto G(\eta)\cdot$), also providing more estimates for the norm of this and higher-order derivatives. Next, all this gained information is used to solve the water wave equations (A.2). First one looks at the linearization around a reference state $\underline{U} = (\underline{\eta}, \underline{\xi})$, giving an explicit expression of the linearized operator $\underline{\mathcal{L}}$ (in view of the formula found for the shape derivative of the Dirichlet–Neumann operator) and discovering that $\underline{\mathcal{L}}$ is hyperbolic. Next, the operator $\underline{\mathcal{L}}$ is transformed into an operator $\underline{\mathcal{M}}$ whose principal part exhibits the Jordan block structure inherent to the water wave equations; a careful study of the well-posedness of the Cauchy problem for the new operator $\underline{\mathcal{M}}$ is then pursued, the main technical tool therein being the Nash–Moser theory (see [22]), and the proof is finally completed by the subsequent solution of the nonlinear system (A.2) via a Nash–Moser iterative scheme. For all technical details we refer the reader once again to [24].

In the general case of flows with non-zero vorticity (cf. Chapter 4), the issue of well-posedness becomes even trickier, and there is still plenty of room for improvement in our understanding of this problem. Nevertheless, it has been shown [9, 26] that the governing equations are well-posed if the following Taylor sign condition is satisfied: the exterior normal derivative $\frac{\partial p}{\partial \mathbf{n}} = \mathbf{n} \cdot \nabla p$ has to be uniformly negative all along the initial free surface, i.e. there has to exist some $c_0 > 0$ such that

$$\frac{\partial p}{\partial \mathbf{n}}(\mathbf{x}, 0) \leq -c_0 < 0 \quad \text{for all } \mathbf{x} = (x,y,z)^T \text{ with } z = \eta(x,y,0).$$

Moreover, this condition is necessary, because if one fails to impose it, then the governing equations turn out to be ill-posed (see [14]). More recently, a reformulation of the water wave equations in Hamiltonian terms has also provided some new insight [4].

We conclude this chapter by pointing out that one of the current most interesting directions of research regarding this topic is the investigation of how solutions cease to exist in finite time, in which case we say the surface wave *breaks*. It is known that if the initial solutions satisfy particular smallness assumptions in certain weighted Sobolev spaces, then the solution exists for all times (cf. [19]); nevertheless, experiments (and everyday experience) suggest that wave breaking is a key ingredient of the nature of water waves, although the mechanisms that lead to this fascinating phenomenon are to a huge extent still unclear, both in the irrotational and in the rotational case, despite some recent developments (see for instance [7]).

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