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"Behind the scenes of the Prime Number Theorem"

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Abstract

One of the most prominent results of the 19th century in the field of analytical number theory is the Prime Number Theorem, which describes the asymptotic distribution of prime numbers. Its original proof, like the overwhelming majority of modern proofs, uses properties of the L-series associated to a certain number theoretic function Λ to draw conclusions from the investigation of the series $L_\Lambda = \zeta'/\zeta$, where ζ is the famous Riemann zeta function. This approach follows the general philosophy that consists in extracting information about the growth of $F(x) = \sum_{n \leq x} f(n)$ from the analytic properties of L_f.

There are several ways of establishing this connection and thus proving the Prime Number Theorem, two of which are presented in this thesis. The first proof which is introduced, is very similar to the historically first proof and uses complex analysis. The second one utilizes Tauberian theory, namely the Wiener-Ikehara Theorem. Both approaches differ greatly in their underlying ideas, thus each of them has its own advantages.

However, these two proofs do not cover the entire relationship between the asymptotic behaviour of $F(x) = \sum_{n \leq x} f(n)$ and L_f, but rather only indicate it. Therefore, in the thesis this connection is further developed, in order to present the full picture of this interaction. For example, the importance of the poles of L_f, which not necessarily lie on the real axis, as it is the case when proving the Prime Number Theorem, is examined. It is also shown to what extent the Prime Number Theorem is equivalent to the absence of zeros of the zeta function on the line with real part 1. This information then allows a precise comparison of the two presented proofs, whereby an astonishing number of parallels can be found.

Zusammenfassung

Eines der wohl prominentesten Theoreme des 19. Jahrhunderts aus dem Bereich der analytischen Zahlentheorie ist der Primzahlsatz, welcher die asymptotische Häufigkeit der Primzahlen beschreibt. Sein ursprünglicher Beweis, ebenso wie die überwältigende Mehrheit der modernen Beweise, nutzt Eigenschaften der L-Reihe, die einer speziellen zahlentheoretischen Funktion Λ zugeordnet ist, um Rückschlüsse aus der Untersuchung der Reihe $L_\Lambda = \zeta'/\zeta$ zu ziehen, wobei ζ die berühmte Riemannsche Zeta-Funktion ist. Dieser Ansatz folgt der Philosophie, Informationen über das Wachstum von $F(x) = \sum_{n \leq x} f(n)$ aus analytischen Eigenschaften von L_f zu gewinnen.
Es gibt mehrere Arten diese Verbindung herzustellen und damit den Primzahlsatz zu beweisen, wobei in dieser Arbeit zwei präsentiert werden. Zunächst wird ein Beweis vorgestellt, der sehr ähnlich dem historisch ersten Beweis ist und auf Methoden der komplexen Analysis zurückgreift. Der zweite benutzt den sogenannten Taubersatz von Wiener-Ikehara. Beide Ansätze unterscheiden sich stark in den zugrunde liegenden Ideen, wodurch jeder seine Vorzüge besitzt.
Jedoch wird durch diese beiden Beweise der gesamte Zusammenhang zwischen dem asymptotischen Verhalten von $F(x) = \sum_{n \leq x} f(n)$ und L_f nicht geliefert, sondern vielmehr erst angedeutet. Deshalb wird in dieser Arbeit diese Verbindung genauestens erarbeitet, um ein möglichst vollständiges Bild dieser Wechselwirkung zu bieten. Beispielsweise wird die Bedeutung der Pole von L_f untersucht, die nicht notwendigerweise auf der reellen Achse liegen, wie es etwa beim Beweis des Primzahlsatzes der Fall ist. Weiters wird gezeigt, inwiefern der Primzahlsatz äquivalent zur Nicht-Existenz von Nullstellen der Zeta-Funktion auf der Geraden mit Realteil 1 ist. Diese Informationen erlauben anschließend einen präzisen Vergleich der beiden vorgestellten Beweise, wodurch eine erstaunliche Anzahl von Parallelen gefunden werden kann.

Chapter 1

Prime numbers: basic tools and first results

Everyone who has examined the sequence of Prime Numbers knows that it seems to be rather chaotic. For instance, it is hard to predict how large the gap between a given prime number and the next prime is. A related natural question to ask is how many primes there are up to some number x, a quantity that is usually denoted by $\pi(x)$. This seemingly easy question is rather hard to answer. After examining the distribution of primes, it is easy to conjecture inequalities like $\pi(x) > \sqrt{x}$ or $\pi(x) = o(x)$. Still, showing them is everything but trivial. The first breakthrough on estimating $\pi(x)$ was made by Chebyshev who proved the following theorem [Har08, p. 9].

Theorem (Chebyshev)**.** *There exist positive constants c_1, c_2 such that for x sufficiently large*

$$c_1 \frac{x}{\log x} < \pi(x) < c_2 \frac{x}{\log x}.$$

Hence, the order of magnitude of $\pi(x)$ is $x/\log(x)$. This result leads to the more delicate question, whether the limit

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log(x)}$$

exists and if yes, what it is. As Hadamard and de la Vallée Poussin proved independently from each other in 1896, the limit exists and equals 1, or in other words

$$\pi(x) \sim \frac{x}{\ln(x)}.$$

This famous result is called the **Prime Number Theorem (PNT)**. In this chapter we build the foundation for proving this theorem and along the way prove Chebyshev's theorem.

1.1 Getting things started

In order to prove Chebyshev's theorem as well as the Prime Number Theorem, one usually works with some specific functions connected to prime numbers. They will follow us for the rest of this thesis.

Definition. *The so-called* ***von Mangoldt function*** $\Lambda : \mathbb{N} \to \mathbb{R}$*, named after the German mathematician Hans von Mangoldt, is defined as*

$$\Lambda(n) := \begin{cases} \log(p) & \text{for } n = p^m \\ 0 & \text{otherwise.} \end{cases}$$

This function can be viewed as assigning a positive weight only to those positive integers, which only have one distinct prime factor. It will be very convenient to examine this function, as it satisfies the following elementary identity.

Proposition 1.1.1. *For any* $n \in \mathbb{N}$ *one has*

$$\sum_{d|n} \Lambda(d) = \log(n).$$

Proof. First note that for $n = 1$ the statement is true by inspection. For $n \geq 2$ let $p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ be the prime decomposition of n. Since Λ is zero for non-prime powers, we get

$$\sum_{d|n} \Lambda(d) = \sum_{p^\alpha|n} \Lambda(p^\alpha) = \alpha_1 \log(p_1) + ... + \alpha_r \log(p_r) = \log(n).$$

□

Definition. *For* $x \in \mathbb{R}^+$ *define the* ***first Chebyshev function*** *by*

$$\theta(x) := \sum_{p \leq x} \log(p)$$

and the ***second Chebyshev function*** *as*

$$\psi(x) := \sum_{p^k \leq x} \log(p) = \sum_{n \leq x} \Lambda(n).$$

It will turn out that the asymptotic behaviour of $\pi(x)$ stands in close relation with those of $\theta(x)$ and $\psi(x)$ as we will show in a moment. They express the sum over all primes and prime powers respectively with the weight $\log(p)$. In Chapter 2 and 3 we will provide two different proofs of the Prime Number Theorem, which both rely on examining the asymptotic behaviour of ψ.
One approach for bounding ψ, which will lead to Chebyshev's theorem, goes as follows.

Theorem 1.1.2. *There exist fixed positive constants* c_1, c_2*, such that for sufficiently large* x *the inequality*

$$c_1 x < \psi(x) < c_2 x$$

holds.

Proof. The idea of the proof is to carefully examine the prime factorization of the binomial coefficient $\binom{2m}{m}$. We begin by defining

$$S(x) := \sum_{n \leq x} \log(n) \quad \text{and} \quad D(x) := S(x) - 2S(x/2)$$

for any $x \geq 2$. Note that when $x = 2m$, where m is a positive integer, D can be expressed as

$$D(2m) = \log \binom{2m}{m}.$$

We will now estimate $D(x)$ in two different ways. First, bounds on $S(x)$ can be found easily by applying partial summation [see Appendix A.1]. This leads to

$$\begin{aligned} S(x) = \sum_{n \leq x} 1 \cdot \log(n) &= \lfloor x \rfloor \log(x) - \int_1^x \lfloor t \rfloor \frac{1}{t} \mathrm{d}t \\ &= \lfloor x \rfloor \log(x) - \int_1^x \frac{t - \{t\}}{t} \mathrm{d}t \\ &= \lfloor x \rfloor \log(x) - (x-1) + \int_1^x \frac{\{t\}}{t} \mathrm{d}t, \end{aligned}$$

where $\{t\}$ denotes the fractional part of t. Since $\frac{\{t\}}{t} < \frac{1}{t}$ we get

$$S(x) = x\big(\log(x) - 1\big) + O\big(\log(x)\big).$$

Applying this estimate to $D(x)$ yields

$$\begin{aligned} D(x) = S(x) - 2S(x/2) =& x\big(\log(x) - 1\big) + O\big(\log(x)\big) \\ & - 2 \cdot \frac{x}{2}\big(\log(x/2) - 1\big) + O\big(\log(x/2)\big) \\ =& x\big(\log(x) - 1\big) - x\big(\log(x) - \log(2) - 1\big) + O\big(\log(x)\big) \\ =& x\log(2) + O\big(\log(x)\big). \end{aligned}$$

Since $\log(2) \approx 0.693$ there exists some constant x_0 such that for every $x > x_0$ the inequality

$$x/2 < D(x) < x$$

holds, which completes the first bound on $D(x)$.

Alternatively, $S(x)$ can be computed using Proposition 1.1.1 as

$$S(x) = \sum_{n \leq x} \log(n) = \sum_{n \leq x} \sum_{d \mid n} \Lambda(d) = \sum_{d \leq x} \Lambda(d) \lfloor x/d \rfloor.$$

Applying this identity to $S(x)$ and $S(x/2)$ gives

$$D(x) = S(x) - 2S(x/2) = \sum_{d \leq x/2} \Lambda(d) \left(\left\lfloor \frac{x}{d} \right\rfloor - 2 \left\lfloor \frac{x}{2d} \right\rfloor \right) + \sum_{x/2 < d \leq x} \Lambda(d). \tag{1.1}$$

Here the last sum is simplified using $\lfloor x/d \rfloor = 1$ for $d \in (x/2, x]$. As it turns out, the expression $\left\lfloor \frac{x}{d} \right\rfloor - 2\left\lfloor \frac{x}{2d} \right\rfloor$ in the first sum is either 0 or 1:

Using the inequalities $s-1 < \lfloor s \rfloor \le s$ gives

$$\left\lfloor \frac{x}{d} \right\rfloor - 2\left\lfloor \frac{x}{2d} \right\rfloor < \frac{x}{d} - 2\left(\frac{x}{2d} - 1\right) = 2$$
$$\left\lfloor \frac{x}{d} \right\rfloor - 2\left\lfloor \frac{x}{2d} \right\rfloor > \frac{x}{d} - 1 - 2\frac{x}{2d} = -1$$

Since $\lfloor \frac{x}{d} \rfloor - 2\lfloor \frac{x}{2d} \rfloor$ is also an integer, it is indeed either 0 or 1. Using this fact in (1.1) establishes the bounds

$$\psi(x) - \psi(x/2) \le D(x) \le \psi(x). \tag{1.2}$$

Combining the second inequality with the already established bounds on $D(x)$ already gives a lower bound on $\psi(x)$ by

$$\psi(x) \ge D(x) > x/2 \quad \text{for } x > x_0.$$

Moreover, combining the first inequality of (1.2) with the inequality $D(x) < x$ gives $\psi(x) \le x + \psi(x/2)$. Applying this estimate iteratively k times yields

$$\psi(x) \le \sum_{i=0}^{k-1} x 2^{-i} + \psi(x/2^k)$$

as long as $x/2^{k-1}$ is greater than x_0. Now let k be the greatest such number, i.e. $x/2^{k-1} \ge x_0 > x/2^k$. Using the trivial inequality $\psi(x/2^k) \le \psi(x_0)$ we obtain

$$\psi(x) \le \sum_{i=0}^{k-1} x 2^{-i} + \psi(x/2^k) < 2x + \psi(x_0).$$

Hence for every $\epsilon > 0$ we have $\psi(x) < (2+\epsilon)x$ for sufficiently large x. □

Next, we want to show that the growth of π, θ and ψ are closely related. As it turns out, their connection can be expressed in a simple way:

Theorem 1.1.3. *As $x \to \infty$, we have*

a) $\theta(x) = \psi(x) + O\big(\sqrt{x}\log(x)\big)$,

b) $\pi(x) = \frac{\theta(x)}{\log(x)} + O\big(\frac{x}{\log^2(x)}\big)$ *and*

c) $\pi(x) \sim \frac{x}{\ln(x)} \Leftrightarrow \psi(x) \sim x$.

Proof. Each of these statements can be proven by careful estimation.

a) One simply computes

$$\begin{aligned}\psi(x)-\theta(x) &= \sum_{p^m\le x}\log(p)-\sum_{p\le x}\log(p)\\ &= \sum_{\substack{p\le\sqrt{x}\\ 2\le m\le\lfloor\log(x)/\log(p)\rfloor}}\log(p)\\ &< \sum_{p\le\sqrt{x}}\lfloor\log(x)/\log(p)\rfloor\log(p)\le\sqrt{x}\log(x).\end{aligned}$$

b) We can rewrite $\pi(x)$ as

$$\pi(x)=\sum_{p\le x}1=\sum_{n\le x}f(n)\frac{1}{\log(n)}$$

where

$$f(n)=\begin{cases}\log(p) & \text{if } n=p,\\ 0 & \text{else.}\end{cases}$$

By applying partial summation to $\pi(x)$ one gets

$$\pi(x)=\frac{\theta(x)}{\log(x)}+\int_2^x\frac{\theta(t)}{t\log^2(t)}\mathrm{d}t.$$

Hence, we are left to bound the growth of the integral. Splitting the integral in two parts and using the inequality $\theta(x)\le\psi(x)<cx$ from Theorem 1.1.2 for some positive constant c gives

$$\begin{aligned}\int_2^x\frac{\theta(t)}{t\log^2(t)}\mathrm{d}t &< \int_2^{\sqrt{x}}\underbrace{\frac{c}{\log^2(t)}}_{\le c/\log^2(2)}\mathrm{d}t+\int_{\sqrt{x}}^x\underbrace{\frac{c}{\log^2(t)}}_{\le c/\log^2(\sqrt{x})}\mathrm{d}t\\ &< \frac{c\sqrt{x}}{\log(2)}+\frac{cx}{\log^2(\sqrt{x})}=O\Big(\frac{x}{\log^2(x)}\Big).\end{aligned}$$

This gives the desired relation

$$\pi(x)=\frac{\theta(x)}{\log(x)}+O\Big(\frac{x}{\log^2(x)}\Big).$$

c) To prove the last part and conclude the theorem, note that *a*) and *b*) imply $\pi(x)\sim\psi(x)/\log(x)$ and hence

$$\text{PNT:}\quad \pi(x)\sim\frac{x}{\log(x)}\quad\Leftrightarrow\quad\psi(x)\sim x.$$

$\square$

By combining the results in the previous theorem we can relate the growth of π and ψ by $\pi(x)=\frac{\psi(x)}{\log(x)}+O\big(\frac{x}{\log^2(x)}\big)$. We already established lower and upper bounds for

$\psi(x)$, namely that there exist c_1, c_2 positive real constants, such that

$$c_1 x < \psi(x) < c_2 x$$

as $x \to \infty$. These yield

$$(c_1 - \epsilon)\frac{x}{\log(x)} < \pi(x) < (c_2 + \epsilon)\frac{x}{\log(x)}$$

for any $\epsilon > 0$ and x sufficiently large, proving Chebyshev's theorem.

1.2 The Riemann zeta function

The starting point of the two proofs we will cover, is the so-called Riemann zeta function. It is defined as

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

for $\{s \in \mathbb{C} : \mathrm{Re}(s) > 1\}$. Later we will extend its domain of definition.

Theorem 1.2.1. *The above sum converges and defines a holomorphic function on* $\{s : \mathrm{Re}(s) > 1\}$.

Proof. To show that $\zeta(s)$ converges when $\mathrm{Re}(s) > 1$ and hence is well defined, we write s as the sum of its real and imaginary part $s = \sigma + it$. Since

$$|n^s| = |n^{\sigma+it}| = |n^\sigma| \cdot \underbrace{|(n^t)^i|}_{=1} = n^\sigma$$

we get

$$\sum_{n=1}^{\infty} \left|\frac{1}{n^s}\right| \leq \sum_{n=1}^{\infty} \frac{1}{n^\sigma} \leq 1 + \int_1^\infty x^{-\sigma} \mathrm{d}x = 1 + \frac{1}{\sigma - 1}.$$

Hence, the initial sum converges absolutely for $\sigma = \mathrm{Re}(s) > 1$.
Fix now some real number $a > 1$ and let $\mathrm{Re}(s) \geq a$. The infinite series $\zeta(s)$ is dominated by the convergent series

$$\sum_{n=1}^{\infty} \frac{1}{n^a}.$$

Hence, $\zeta(s)$ converges uniformly when $\mathrm{Re}(s) > a$ and therefore by the Weierstrass' criterion is a holomorphic function there. As a can be chosen arbitrarily close to 1, $\zeta(s)$ is a holomorphic function in the half-plane $\{s : \mathrm{Re}(s) > 1\}$.

□

A similar computation implies that the zeta function has a pole at $s = 1$:

$$\lim_{s \to 1^+} \zeta(s) = \lim_{s \to 1^+} \sum_{n=1}^{\infty} \frac{1}{n^s} \geq \lim_{s \to 1^+} \int_1^\infty x^{-s} \mathrm{d}x = \lim_{s \to 1^+} \frac{1}{s-1} = +\infty$$

Therefore $\zeta(s)$ is holomorphic when $\text{Re}(s) > 1$ and has at least one pole on the boundary. As we will discuss later, it is possible to extend the zeta function to a meromorphic function on $\mathbb{C}$ with the only pole at $s = 1$.

1.3 Dirichlet series and exploring the connection to prime numbers

A function is called **arithmetic** if its domain is the positive integers and its range is a subset of the complex numbers. Further, to each arithmetic function f we can associate its **summatory function**, usually denoted by $F(x)$, defined by

$$F(x) := \sum_{n \leq x} f(n).$$

As an example, ψ is the summatory function of Λ.

Last, for each arithmetic function we define its associated **Dirichlet series** or **L-series** by the formal expression

$$L_f(s) := \sum_{n \geq 1} \frac{f(n)}{n^s}.$$

In Chapter 4 will we be able to characterize the set of points on which a given L-series converges depending on the underlying arithmetic function. Such series often only converge on a half-plane. A meromorphic extension to the complex plane, if it exists, is called **L-function** and will also be denoted by L_f. The Riemann zeta function can now be viewed as the L-series of $f \equiv 1$. When the image of f is a subset of the real numbers, then $\overline{L_f(s)} = L_f(\bar{s})$. As an example, one has $\overline{\zeta(s)} = \zeta(\bar{s})$.

An arithmetic function $f \not\equiv 0$ is called **multiplicative**, if $f(ab) = f(a)f(b)$ holds for all coprime $a, b \in \mathbb{N}$. If $f(ab) = f(a)f(b)$ holds for all positive integers a, b we call it **strongly multiplicative**.

For any multiplicative arithmetic function f there exists an n such that $f(n) \neq 0$. As $\gcd(n, 1) = 1$ we get $f(n) = f(n)f(1)$ which leads to $f(1) = 1$.

Note that any multiplicative arithmetic function is uniquely defined by its values at powers of primes. Similarly, a strongly multiplicative function is characterized by its values at primes. This allows us to express its L-series as a product, called Euler-Product, as we will show in the following two lemmas.

Lemma 1.3.1. *Let f be a multiplicative arithmetic function such that $\sum_{n \geq 1} f(n)$ converges absolutely. Then $\sum_{n \geq 1} f(n)$ can be expressed as the product taken over all primes*

$$\sum_{n \geq 1} f(n) = \prod_p \left(1 + f(p) + f(p^2) + \dots\right).$$

Furthermore, the infinite series equals 0 if and only if at least one term on the right *is* 0*. Finally, the product can be simplified to*

$$\prod_p \left(\frac{1}{1-f(p)}\right)$$

whenever f is strongly multiplicative.

Proof. For any $k \geq 2$ we define

$$P(k) := \prod_{p\leq k} \big(1 + f(p) + f(p^2) + \dots\big).$$

The sum $1 + f(p) + f(p^2) + \dots$ is a subseries of the absolutely converging series $\sum_{n\geq 1} f(n)$ and thus is itself absolutely converging. Multiplying out and using the fact that f is multiplicative, we get

$$P(k) = \sum_{n\in T(k)} f(n),$$

where $T(k)$ is set of all natural numbers which do not have a prime factor greater than k. Let $S(k) := \mathbb{N} \setminus T(k)$, the set of natural numbers with at least one prime factor exceeding k. Then

$$\Big|\sum_{n\geq 1} f(n) - P(k)\Big| = \Big|\sum_{n\geq 1} f(n) - \sum_{n\in T(k)} f(n)\Big| = \Big|\sum_{n\in S(k)} f(n)\Big| \leq \sum_{n>k} |f(n)| \to 0,$$

as $k \to \infty$, thus $P(k) \to \sum_{n\geq 1} f(n)$.
We are left to show that the infinite product can only equal zero, when at least one factor vanishes. Choose K such that $\sum_{n\geq K} |f(n)| < 1/2$. Then

$$\Big|\prod_{p\geq K} \big(1 + f(p) + f(p^2) + \dots\big) - 1\Big| \leq \sum_{n\in T(K)} |f(n)| \leq \sum_{n\geq K} |f(n)| < \frac{1}{2}$$

and therefore

$$\prod_{p\geq K} \big(1 + f(p) + f(p^2) + ...\big) \neq 0.$$

Hence, if $\sum_{n\geq 1} f(n) = 0$, we have that $\prod_{p<K} \big(1+f(p)+f(p^2)+...\big) = 0$ and therefore at least one of the terms is zero. Finally, assuming that f is strongly multiplicative the sums can be transformed into a geometric series, which completes the proof. □

Next up, we can apply this lemma to get a result about the Euler-Product of L-series of a multiplicative function:

Corollary 1.3.2. *Let g be a multiplicative arithmetic function which additionally satisfies $g(n) = O(n^c)$ for some real constant c. Then for* $\mathrm{Re}(s) > c + 1$ *we have*

$$L_g(s) = \sum_{n\geq 1} \frac{g(n)}{n^s} = \prod_p \left(1 + \frac{g(p)}{g^p} + \frac{g(p^2)}{g^{2p}} + ...\right)$$

which can be simplified in the case of strong multiplicativity of g to

$$\prod_p \left(1 - \frac{g(p)}{p^s}\right)^{-1}.$$

Proof. Fix some $s \in \mathbb{C}$ with $\mathrm{Re}(s) > c + 1$ and note that the function $f(n) = \frac{g(n)}{n^s}$ is multiplicative. Further, $f(n) = O(n^d)$ with $d := c - \mathrm{Re}(s) < -1$ which ensures that the sum $\sum_{n \geq 1} f(n)$ converges absolutely. Now apply the previous lemma. □

We will now apply this knowledge to the zeta function. With f being the (strongly multiplicative) function 1, we interpret $\zeta(s)$ as the L-series of f. Then the previous corollary yields the Euler-Product

$$\zeta(s) = \prod_p \left(1 - p^{-s}\right)^{-1}$$

which is valid for $\mathrm{Re}(s) > 1$. Applying the second part of Lemma 1.3.1, we get $\zeta(s) \neq 0$ when $\mathrm{Re}(s) > 1$, as all factors $(1 - p^{-s})^{-1}$ are non-zero.

This new identity already indicates a connection between $\zeta(s)$ and the prime numbers. In order to get a more appealing and useful formula involving primes and the zeta function, we apply the logarithm to the product representation. Using the expansion $\log(1 + x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} \mp ...$ for $|x| < 1$ yields

$$\log\left(\zeta(s)\right) = -\sum_p \log\left(1 - \frac{1}{p^s}\right) = \sum_p \sum_{n \geq 1} \frac{1}{np^{ns}}.$$

Taking the derivative with respect to s on both sides gives rise to the important identity

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_p \sum_{n \geq 1} \frac{\log(p)}{p^{ns}}$$

which we will use in many ways. Note that ζ'/ζ is still an analytic function. This Dirichlet series will play a central role in both proofs of the Prime Number Theorem. Finally, the von Mangoldt function allows us to rewrite the identity as

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n \geq 1} \frac{\Lambda(n)}{n^s} = L_\Lambda(s).$$

1.4 Zeta function: To be continued ...

In this section, we will extend the zeta function and L_Λ beyond their initial domain of convergence. In 1859 Bernhard Riemann noticed the connection between the zeta function and the distribution of prime numbers. More than 30 years later, in 1895 the German mathematician Hans von Mangoldt was able to prove the conjecture of Riemann that the Prime Number Theorem is equivalent to L_Λ having no poles on the line $\mathrm{Re}(s) = 1$ except at $s = 1$ [DM13, pp. 59-60]. As we will show, this is equivalent to ζ being non-zero on this line. The next year, both Hadamard and de la Vallée Poussin were able to prove this statement independently from each other and thus established the Prime Number Theorem.

We will now establish basic properties of the zeta function needed for the proof of the Prime Number Theorem. The following lemma will ultimately be critical for proving $\zeta(s) \neq 0$ when $\mathrm{Re}(s) = 1$:

Lemma 1.4.1. *a) For any real φ we have*

$$3 + 4\cos(\varphi) + \cos(2\varphi) \geq 0.$$

b) For $\sigma > 1, t \in \mathbb{R}$ the inequality

$$\left|\zeta(\sigma)^3\zeta(\sigma+it)^4\zeta(\sigma+2it)\right| \geq 1$$

holds.

Proof. a) Using the trigonometric identity $\cos^2(\varphi) = \frac{1+\cos(2\varphi)}{2}$ we indeed get

$$\begin{aligned} 0 \leq (1+\cos(\varphi))^2 &= 1 + 2\cos(\varphi) + \cos^2(\varphi) \\ &= \frac{3+4\cos(\varphi)+\cos(2\varphi)}{2}. \end{aligned}$$

b) For $s = \sigma + it$ with $\sigma > 1$, we apply the principal branch of the logarithm to $|\zeta(s)|$. Using $\log\left(|z|\right) = \mathrm{Re}\big(\log(z)\big)$ for $z \in \mathbb{C} \setminus 0$ we get

$$\begin{aligned} \log(|\zeta(s)|) = \mathrm{Re}\big(\log(\zeta(s)\big) &= \mathrm{Re}\Big(\sum_p \sum_{n\geq 1} \frac{1}{np^{ns}}\Big) \\ &= \sum_p \sum_{n\geq 1} \frac{\cos\left(t\log(p^n)\right)}{np^{n\sigma}}. \end{aligned}$$

Applying the same procedure to $\left|\zeta(\sigma)\right|, \left|\zeta(\sigma+it)\right|$ and $\left|\zeta(\sigma+2it)\right|$ we get

$$\begin{aligned} &\log\left(\left|\zeta(\sigma)^3 \cdot \zeta(\sigma+it)^4 \cdot \zeta(\sigma+2it)\right|\right) \\ =&3\log(|\zeta(s)|) + 4\log\left(|\zeta(\sigma+it)|\right) + \log\left(|\zeta(\sigma+2it)|\right) \\ =&\sum_p \sum_{n\geq 1} \frac{3+4\cos\left(t\log(p^n)\right)+\cos\left(2t\log(p^n)\right)}{np^{n\sigma}}. \end{aligned}$$

As proven in part a), the nominator is always non-negative which implies the claim.

□

Now we are ready to extend ζ and L_Λ beyond their initial domain and characterize their poles:

Theorem 1.4.2. *a) The previously defined function $\zeta(s)$ extends meromorphically to $\mathbb{C}$. In this region it only has one (simple) pole at $s = 1$ with residue 1.*

b) The zeta function has no zeros in the closed half-plane $\{s : \mathrm{Re}(s) \geq 1\}$.

c) The Dirichlet series $L_\Lambda = -\zeta'/\zeta$ can be extended to an L-function on $\mathbb{C}$. In the closed half-plane $\{s : \mathrm{Re}(s) \geq 1\}$ it has only one (simple) pole at $s = 1$ with residue 1. Any poles in the strip $0 < \mathrm{Re}(s) < 1$ of L_Λ are at the zeros of ζ.

Proof. We will restrict our proof of part *a*) and *c*) to the half-plane $\{s : \mathrm{Re}(s) > 0\}$, as it suffices for all purposes of this theses. For a complete proof, see for instance [Pat88, p. 19].
We start by noting that for $\mathrm{Re}(s) > 1$

$$\frac{1}{n^s} = s\int_n^\infty \frac{1}{t^{s+1}}\mathrm{d}t = s\sum_{k=n}^{\infty}\int_k^{k+1}\frac{\mathrm{d}t}{t^{s+1}}.$$

By using this formula and reordering the summands one gets

$$\zeta(s) = s\sum_{n\geq 1}\sum_{k=n}^{\infty}\int_k^{k+1}\frac{\mathrm{d}t}{t^{s+1}} = s\sum_{k\geq 1}\Big(\sum_{n\leq k}\int_k^{k+1}\frac{\mathrm{d}t}{t^{s+1}}\Big) = s\sum_{k\geq 1}k\int_k^{k+1}\frac{\mathrm{d}t}{t^{s+1}}.$$

Note that we are allowed to change the order of summation because of absolute convergence. Finally, the last expression can also be written as

$$\zeta(s) = s\int_1^\infty \frac{\lfloor t\rfloor}{t^{s+1}}\mathrm{d}t = s\int_1^\infty \frac{t-\{t\}}{t^{s+1}}\mathrm{d}t = 1 + \frac{1}{s-1} - s\int_1^\infty \frac{\{t\}}{t^{s+1}}\mathrm{d}t. \tag{1.3}$$

Let σ be the real part of s. Then the last integral converges absolutely for $\sigma > 0$ as

$$\int_1^\infty \left|\frac{\{t\}}{t^{s+1}}\right|\mathrm{d}t < \int_1^\infty \frac{1}{t^{\sigma+1}}\mathrm{d}t = \frac{1}{\sigma} < \infty.$$

Therefore (1.3) gives a meromorphic continuation of ζ to the half-plane $\{s : \mathrm{Re}(s) > 0\}$, with the only pole at $s = 1$ and residue 1.

Regarding *b*), the product representation of $\zeta(s)$ already showed that $\zeta(s) \neq 0$ for $\mathrm{Re}(s) > 1$. Hence, we are left to prove that there are no zeros on the line $\mathrm{Re}(s) = 1$. Assume that ζ has a zero at $s = 1 + it$ for some $t \in \mathbb{R} \setminus 0$. We will examine the limit of

$$\lim_{\sigma\to 1^+}\left(\left|\frac{\zeta(\sigma+it)}{\sigma-1}\right|^4 \cdot \big|\zeta(\sigma+2it)\big| \cdot \big|\zeta(\sigma)(\sigma-1)\big|^3\right) \tag{1.4}$$

$$= \lim_{\sigma\to 1^+}\Big(\big|\zeta(\sigma+it)\big| \cdot \big|\zeta(\sigma+2it)\big| \cdot \big|\zeta(\sigma)\big|^3\Big)\frac{1}{\sigma-1} \tag{1.5}$$

in two different ways. On the one hand, we know from Lemma 1.4.1 that the expression (1.5) is greater than $\frac{1}{\sigma-1}$ and hence the limit is $+\infty$. On the other hand, using

$\zeta(1+it) = 0$ one has

$$\lim_{\sigma\to 1^+} \left|\frac{\zeta(\sigma+it)}{\sigma-1}\right| = \lim_{\sigma\to 1^+} \left|\frac{\zeta(\sigma+it)-\zeta(1+it)}{\sigma-1}\right| = \left|\zeta'(1+it)\right|$$

and also

$$\lim_{\sigma\to 1^+} \zeta(\sigma)(\sigma-1) = 1$$

because of the pole at $s=1$ with residue 1. The zeta function has no pole at $1+it$ and hence neither does ζ'. As a result, the limit (1.4) evaluates to

$$|\zeta'(1+it)|^4 \cdot |\zeta(1+2it)| < \infty,$$

hence a contradiction.

Addressing part c), we now consider $\zeta'(s)/\zeta(s)$. As ζ is meromorphic on $\mathrm{Re}(s) > 0$, so is ζ' and hence ζ'/ζ. This gives a meromorphic extension of L_Λ.
Because of the simple pole of ζ at $s=1$, it can be written as

$$\zeta(s) = \frac{1}{s-1}h(s)$$

where $h(s)$ is a holomorphic function on $\{s : \mathrm{Re}(s) > 0\}$ with no zero at $s=1$. We get

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{h'(s)}{h(s)} - \frac{1}{s-1},$$

hence $L_\Lambda = -\zeta'/\zeta$ has a simple pole at $s=1$ with residue 1. As $\zeta(s)$ does not have poles in the strip $0 < \mathrm{Re}(s) \leq 1$, $s \neq 1$, this also holds true for $\zeta'(s)$. Hence, any poles of $L_\Lambda = -\zeta'/\zeta$ are at the zeros of the zeta function in this strip. As $\zeta(1+it) \neq 0$ for $t \in \mathbb{R}$, we conclude that $L_\Lambda(s)$ indeed has no additional poles on the line $\mathrm{Re}(s) = 1$.

□

1.5 Connecting L_Λ to the Prime Number Theorem

We will now briefly address the role of the zeta function and L_Λ for proving the Prime Number Theorem.

In some sense, $L_\Lambda(s)$ can be viewed as the average of all $\Lambda(p^n) = \log(p)$, each with the weight $1/p^{ns}$. When the real part of s is large, the small prime powers become the leading terms of $L_\Lambda(s)$. Hence in order to extract information about all prime numbers and therefore their asymptotic distribution, it is far more interesting to consider $L_\Lambda(s)$ when $\mathrm{Re}(s)$ is small, say close to or equal to 1.
As already explained, $L_\Lambda(s)$ has only one pole on the line $\mathrm{Re}(s) = 1$, namely at $s=1$. Its existence implies that there are infinitely many primes:
Recall that for $\mathrm{Re}(s) > 1$ we have

$$L_\Lambda(s) = \sum_{p\in\mathbb{P}} \log(p) \sum_{k\geq 1} \frac{1}{p^{ks}} = \sum_{p\in\mathbb{P}} \frac{\log(p)}{p^s - 1}.$$

Note that as $s \to 1^+$ one has $L_\Lambda(s) \to \infty$ while each of the summands on the right stays bounded. Therefore, we can conclude that the sum has to have infinitely many summands, i.e. there are infinitely many prime numbers.

Recall that the Prime Number Theorem is equivalent to $\psi(x) = \sum_{n \leq x} \Lambda(n) \sim x$. This is a statement about the partial sums of the numerators of the series $L_\Lambda(s)$. For any given arithmetic function f, there is a duality between the poles of L_f and the asymptotic behaviour of the summatory function $F(x) := \sum_{n \leq x} f(n)$. The methods described in the next two chapters will underline this connection.
Before Hadamard and de la Vallée Poussin proved in 1896 that $\psi(x) \sim x$, the best bound on ψ had been of the form

$$c_1 x < \psi(x) < c_2 x$$

for some positive constants c_1, c_2 as $x \to \infty$. The sharpest bounds were given by James Joseph Sylvester in 1892 who showed

$$0.95695x < \psi(x) < 1.04423x,$$

meaning he could estimate ψ with a relative error of about 5% [Yan99, p. 69]. The Prime Number Theorem implies that the constants can be taken arbitrarily close to 1. One related result that is surprisingly simple to show, is once again due to Chebyshev:

Theorem 1.5.1. *If there exists a constant c, such that $\psi(x) \sim cx$, then c has to be equal to* 1.

Proof. We start with an alternate expression for $L_\Lambda(s)$ when $\mathrm{Re}(s) > 1$ given by

$$\begin{aligned} s\int_1^\infty \psi(x)x^{-s-1}\mathrm{d}x &= s\int_1^\infty \sum_{n\leq x}\Lambda(n)x^{-s-1}\mathrm{d}x \\ &= s\sum_{n\geq 1}\Lambda(n)\int_n^\infty x^{-s-1}\mathrm{d}x \\ &= \sum_{n\geq 1}\Lambda(n)n^{-s} = L_\Lambda(s). \end{aligned}$$

Interchanging the order of summation and integration is justified by absolute convergence. As $\psi(x) \sim cx$, assume now that for any $\epsilon > 0$ one has $\psi(x) < (c+\epsilon)x$ where $x > x_\epsilon$. Then $L_\Lambda(s)$ can be bounded for $s \in (1, \infty)$ from above by

$$\begin{aligned} L_\Lambda(s) = s\int_1^\infty \psi(x)x^{-s-1}\mathrm{d}x &< s\int_{x_\epsilon}^\infty \psi(x)x^{-s-1}\mathrm{d}x \\ &< s\int_{x_\epsilon}^\infty (c+\epsilon)x^{-s}\mathrm{d}x = h(s) + (c+\epsilon)\frac{1}{s-1}, \end{aligned}$$

where $h(s)$ is a bounded, continuous functions. Taking the limit $s \to 1^+$ on both sides gives us $1 < c + \epsilon$, using the fact that $L_\Lambda(s) \sim \frac{1}{s-1}$. Analogously, one can show that $1 > c - \epsilon$ for any ϵ, which proves $c = 1$. □

Even though this last theorem already feels remarkably close to the Prime Number Theorem, we are still quite far away. The two proofs we present will need a more careful examination of the function L_Λ.

1.6 Manipulating L-series

We will now quickly address some operations defined on L-series. The goal is to develop a basic understanding of them which makes the coming chapters easier to follow.
First, we want to mention that multiplying an arithmetic function f by some $c \in \mathbb{C}$ not only translates to multiplying the summatory function $F(x) := \sum_{n \leq x} f(n)$ with this constant, but also its L-series, i.e. $L_{c \cdot f}(s) = cL_f(s)$. As we already pointed out, there is a strong connection between poles of L-functions and the asymptotic growth of the corresponding summatory function. With this idea in mind, we will without loss of generality mostly focus on L-functions which have a pole with residue 1 at some specific point (usually $s = 1$), i.e. they behave like $L_f(s) \sim \frac{1}{s-1}$ as $s \to 1^+$.
Next, taking the sum of two arithmetic functions f, g translates to summing their L-series, i.e. $L_{f+g}(s) = L_f(s) + L_g(s)$. As before, also their summatory functions add up. This motivates examining the growth of the sum of two summatory functions. For instance, as both L_Λ and $\zeta(s)$ have a pole of order 1 at $s = 1$, we might consider the function

$$L_{\Lambda-1}(s) = \sum_{n \geq 1} \frac{\Lambda(n) - 1}{n^s} = L_\Lambda(s) - \zeta(s)$$

where the arithmetic function denoted by 1 is the constant 1 function. This new L-function has no pole at $s = 1$ and the Prime Number Theorem is equivalent to this new function satisfying $\sum_{n \leq x} (\Lambda - 1)(x) = o(x)$ as $x \to \infty$.

Another aspect of $L_{\Lambda-1}$ is that while it does not have a pole at $s = 1$, it is still not evident that the series converges there, i.e. whether the infinite sum

$$\sum_{n \geq 1} \frac{\Lambda(n) - 1}{n}$$

converges. Indeed, we will later discover the incredible result that the convergence of this series directly implies the Prime Number Theorem.

A last remark before starting the proof of the Prime Number Theorem is that while we will be focused on the distribution of prime numbers and therefore on the corresponding series L_Λ, we will actually use very little facts about this function. Indeed, we will almost only be interested in the position of its poles (or rather lack of their existence), as well as growth conditions near the line $\{s : \text{Re}(s) = 1\}$. Hence both methods can be used on a variety of arithmetic functions, for which one is able to establish a similar behaviour.

Chapter 2

Proof using Complex Analysis

In this chapter we present our first proof of the Prime Number Theorem. The ideas and techniques used are very similar to the historically first proof. An outline of the proof can be given as follows:

Instead of showing $\psi(x) \sim x$, we will turn to its integrated and therefore smoothened form $\psi_1(x) := \int_1^x \psi(t)\mathrm{d}t$. As it turns out, the Prime Number Theorem is then a consequence of $\psi_1(x) \sim x^2/2$. Miraculously, we will be able to get a closed form for $\psi_1(x)$ using the L-function $L_\Lambda(s) = (-\zeta'/\zeta)(s)$. Using Perron's formula we will get

$$\frac{\psi_1(x)}{x^2} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^{s-1}}{s(s+1)}\mathrm{d}s$$

for $c > 1$ and $x \geq 1$. The integrand has a simple pole at $s = 1$ which can be subtracted from the integral to establish the similar identity

$$\frac{\psi_1(x)}{x^2} - \frac{1}{2}\left(1 - \frac{1}{x}\right)^2 = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}\right)\frac{x^{s-1}}{s(s+1)}\mathrm{d}s.$$

The left-hand side of this equation converges to $\lim_{x\to\infty}\psi_1(x)/x^2 - 1/2$ assuming this limit exists. Hence, we are left to show that the right-hand side converges to 0 as $x \to \infty$.

This will be our main task and will involve the most effort. Along the path from $c - i\infty$ to $c + i\infty$ where $c > 1$, one has $|x^{s-1}| = x^{c-1}$. As this term does not converge as $x \to \infty$, we will need to change the path of integration to the vertical line with real part 1. In order to do so, we will first need to establish bounds on the integrand and hence on $\zeta'(s)/\zeta(s)$, where the real part of s is close to 1 and the imaginary part tends to infinity. This will be a rather technical part which unfortunately will not provide greater insight into the proof. Thereafter, we will let $x \to \infty$ on the right-hand side which by the Riemann-Lebesgue Lemma tends to 0, concluding the proof of the Prime Number Theorem.

We will now proceed with the execution of the outlined plan.

2.1 The function ψ_1

In the first chapter it was established that proving the Prime Number Theorem is equivalent to showing $\psi(x) \sim x$ as $x \to \infty$. The proof we present here will involve a closely related function.

Definition. *For $x \geq 1$ we define*

$$\psi_1(x) := \int_1^x \psi(t)\mathrm{d}t.$$

Note that an alternative way of writing $\psi_1(x)$ is

$$\psi_1(x) = \int_1^x \psi(t)\mathrm{d}t = \int_1^x \sum_{n \leq t} \Lambda(n)\mathrm{d}t = \sum_{n \leq x} \Lambda(n) \int_n^x 1\mathrm{d}t = \sum_{n \leq x} (x-n)\Lambda(n). \quad (2.1)$$

This function is an integrated version of ψ and hence smoother, as ψ_1 is continuous while ψ is not. The asymptotic growth of these two functions are closely linked, as described in the following proposition.

Proposition 2.1.1. *As $x \to \infty$ the relation $\psi_1(x) \sim \frac{x^2}{2}$ implies $\psi(x) \sim x$.*

Proof. Since $\psi(x)$ is monotone increasing, we have for any $\alpha < 1 < \beta$

$$\frac{1}{(1-\alpha)x} \int_{\alpha x}^x \psi(t)\mathrm{d}t \leq \psi(x) \leq \frac{1}{(\beta-1)x} \int_x^{\beta x} \psi(t)\mathrm{d}t.$$

The first inequality can be written as

$$\psi(x) \geq \frac{1}{(1-\alpha)x}\big(\psi_1(x) - \psi_1(\alpha x)\big)$$

or equivalently as

$$\frac{\psi(x)}{x} \geq \frac{1}{1-\alpha}\Big(\frac{\psi_1(x)}{x^2} - \frac{\psi_1(\alpha x)}{(\alpha x)^2}\alpha^2\Big).$$

Taking the limit $x \to \infty$ and using $\psi_1(x)/x^2 \to 1/2$ gives

$$\liminf_{x\to\infty} \frac{\psi(x)}{x} \geq \frac{1}{1-\alpha}\Big(\frac{1}{2} - \frac{\alpha^2}{2}\Big) = \frac{1+\alpha}{2}.$$

Since we can choose α arbitrarily close to 1, we get $\liminf_{x\to\infty} \psi(x)/x \geq 1$. A similar argument for β yields $\limsup_{x\to\infty} \psi(x)/x \leq 1$ and therefore proves the proposition. □

Here the only property of ψ we used is its monotonicity. Hence this proposition also applies to any summatory function of a non-negative arithmetic function.

2.2 Perron's formula

Proposition 2.1.1 established that the Prime Number Theorem is a consequence of $\psi_1(x) \sim x^2/2$. This next proposition will later be used as a tool to get an expression for $\psi_1(x)$ via an integral involving L_Λ. It is called Perron's formula named after German mathematician Oskar Perron.

Proposition 2.2.1 (Perron's formula)**.** *Let c and u be two positive real numbers. Then for every $n \geq 1$ we have*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{u^{-z}}{z(z+1)\cdots(z+n)}\mathrm{d}z = \begin{cases}\frac{1}{n!}(1-u)^n & \text{if } 0 < u \leq 1,\\ 0 & \text{if } u > 1\end{cases}$$

and the integral is absolutely convergent.

Proof. We will evaluate this path integral as a part of a contour integral along two different paths, depending on the size of u. Combining this with Cauchy's residue theorem will yield the result.
Depending on whether $u \in (0,1]$ or $u \in (1,\infty)$ we will consider the integral along $C_1(R)$ or $C_2(R)$ respectively. These paths are defined in Figure 2.1 where R is a parameter that we will let go to ∞, so we may only consider the case when this radius R of the two arcs is greater than $2n+c$. In this way the path C_1 contains all the poles of the integrand located at $z = 0, -1, \ldots, -n$.

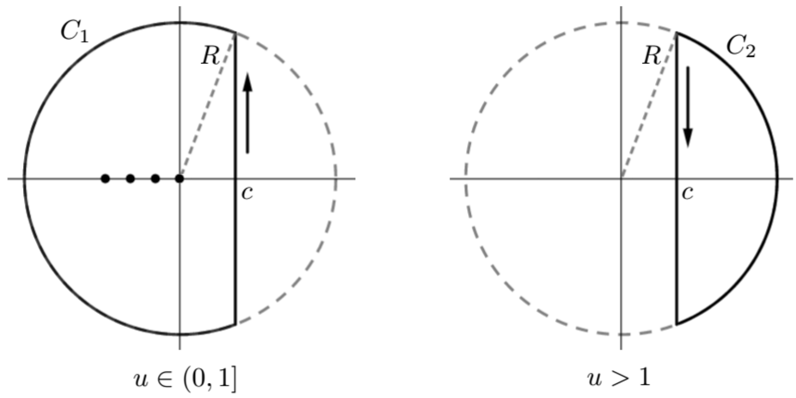

Figure 2.1: The contours C_1 and C_2

First, we focus on the integral along the two different arcs and will show that they tend to 0 as $R \to \infty$. We write $z = x + iy$ and get

$$u \in (0,1] \Rightarrow z \in C_1 \Rightarrow x \leq c \Rightarrow u^{-x} \leq u^{-c},$$
$$u \in (1,\infty) \Rightarrow z \in C_2 \Rightarrow x \geq c \Rightarrow u^{-x} \leq u^{-c},$$

hence one has $u^{-x} \leq u^{-c}$ independently from u. As $|z| = R$ on the arcs, the integrand is bounded by

$$\left|\frac{u^{-z}}{z(z+1)\cdots(z+n)}\right| = \frac{u^{-x}}{|z||z+1|\cdots|z+n|} \leq \frac{u^{-c}}{R|z+1|\cdots|z+n|}.$$

Furthermore, for $1 \leq k \leq n$ the reversed triangle inequality gives

$$|z+k| \geq |z| - k = R - k \geq R/2$$

since $k \leq n \leq R/2$. Using these estimates, the integral along each arc is dominated by

$$2R\pi \frac{u^{-c}}{R(R/2)^n} = O(R^{-n})$$

and thus tends to 0 as $R \to \infty$.
We are left to compute the integrals using Cauchy's residue theorem. In the case $u > 1$, using the path C_2, there are no poles inside the contour, therefore the integral along $C_2(R)$ is 0. As the integral along the arc tends to zero as well, we get the desired result.
For $u \in (0, 1]$ the integrand has poles inside $C_1(R)$ at $k = 0, -1, \ldots, -n$. Notice that the integrand can be written as $u^{-z}\Gamma(z)/\Gamma(z+n+1)$. Applying Cauchy's residue theorem gives

$$\begin{aligned}\frac{1}{2\pi i}\int_{C_1(R)} \frac{u^{-z}\Gamma(z)}{\Gamma(z+n+1)}\mathrm{d}z &= \sum_{k=0}^{n} \mathrm{Res}_{z=-k}\frac{u^{-z}\Gamma(z)}{\Gamma(z+n+1)} \\ &= \sum_{k=0}^{n} \frac{u^k}{\Gamma(n+1-k)}\mathrm{Res}_{z=-k}\Gamma(z) \\ &= \sum_{k=0}^{n} \frac{u^k(-1)^k}{(n-k)!k!} \\ &= \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}(-u)^k = \frac{(1-u)^n}{n!}.\end{aligned}$$

Here we used the fact that $\mathrm{Res}_{z=-k}\Gamma(z) = \frac{(-1)^k}{k!}$. Again, the integral over the arcs does not contribute to the total integral as $R \to \infty$, hence we have proven Perron's formula. □

This formula allows us to express $\psi_1(x)$ by means of $L_\Lambda = -\zeta'/\zeta$. We get the wonderful result:

Proposition 2.2.2. *For $c > 1$ and $x \geq 1$ we have*

$$\frac{\psi_1(x)}{x^2} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Big(-\frac{\zeta'(s)}{\zeta(s)}\Big)\frac{x^{s-1}}{s(s+1)}\mathrm{d}s. \tag{2.2}$$

Proof. Recall that we have from (2.1)

$$\frac{\psi_1(x)}{x} = \sum_{n\leq x}\Big(1 - \frac{n}{x}\Big)\Lambda(n) \tag{2.3}$$

for $x \geq 1$. Perron's formula yields

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{(x/n)^s}{s(s+1)}\mathrm{d}s = \begin{cases} 1 - n/x & \text{if } x/n \leq 1 \Leftrightarrow n \leq x, \\ 0 & \text{if } x/n > 1 \Leftrightarrow n > x. \end{cases}$$

This fits perfectly with (2.3) as it allows us to write $\psi_1(x)/x$ as the infinite sum

$$\frac{\psi_1(x)}{x} = \sum_{n\geq 1} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{(x/n)^s}{s(s+1)} \Lambda(n) \mathrm{d}s. \tag{2.4}$$

We would now like to change the order of summation and integration. To do so, it suffices to show that the infinite sum over the integrals converges absolutely:

$$\begin{aligned} \sum_{n=1}^{N} \Big| \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{(x/n)^s}{s(s+1)} \Lambda(n) \mathrm{d}s \Big| &\leq \frac{1}{2\pi} \sum_{n=1}^{N} \int_{c-i\infty}^{c+i\infty} \frac{|x^s|}{|s||s+1|} \cdot \frac{\Lambda(n)}{|n^s|} \mathrm{d}s \\ &= \frac{1}{2\pi} \sum_{n=1}^{N} \frac{\Lambda(n)}{n^c} \int_{c-i\infty}^{c+i\infty} \frac{|x|^c}{|s||s+1|} \mathrm{d}s. \end{aligned}$$

Using $c > 1$ and therefore $\sum_{n=1}^{N} \frac{\Lambda(n)}{n^c} < L_\Lambda(c) < \infty$, we indeed have absolute convergence in (2.4) and can rewrite it as

$$\frac{\psi_1(x)}{x} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \sum_{n\geq 1} \frac{x^s}{s(s+1)} \frac{\Lambda(n)}{n^s} \mathrm{d}s = \frac{1}{2\pi i} \sum_{n\geq 1} \int_{c-i\infty}^{c+i\infty} \Big(-\frac{\zeta'(s)}{\zeta(s)} \Big) \frac{x^s}{s(s+1)} \mathrm{d}s.$$

Dividing both sides by x proves the proposition. □

The above proposition links ψ_1 with an integral involving ζ'/ζ. The following steps to the Prime Number Theorem will focus on the right-hand side of (2.2). At first, we turn our focus to the integrand. Notice that it has a simple pole with residue 1 at $s = 1$ as shown in Theorem 1.4.2. As it turns out, one can pull this pole outside the integral using once again Perron's formula.

Proposition 2.2.3. *For $c > 1$ and $x \geq 1$ we have*

$$\frac{\psi_1(x)}{x^2} - \frac{1}{2}\Big(1 - \frac{1}{x}\Big)^2 = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{s-1} h(s) \mathrm{d}s \tag{2.5}$$

where

$$h(s) = \frac{1}{s(s+1)} \Big(-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} \Big).$$

Note that $h(s)$ is holomorphic on a region containing $\{s : \mathrm{Re}(s) \geq 1\}$.

Proof. By Perron's formula with $n = 2$ and $u = 1/x$, we get

$$\frac{1}{2}\Big(1 - \frac{1}{x}\Big)^2 = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^s}{s(s+1)(s+2)} \mathrm{d}s$$

for $c > 0$ and $x \geq 1$. Replacing s by $s-1$ in this equation, introducing $c > 1$ for the new path of integration, followed by subtracting it from (2.2) yields the desired equation. □

Under the assumption that the limit $x \to \infty$ in equation (2.5) exists, the left-hand side converges to

$$\lim_{x\to\infty} \frac{\psi_1(x)}{x^2} - \frac{1}{2}.$$

Hence, in order to prove the Prime Number Theorem, we are left to show that the right-hand side of (2.5) converges to 0 as $x \to \infty$. There remains a problem with the term x^{s-1} where $\mathrm{Re}(s) = c > 1$, as it is unbounded for $x \to \infty$. Hence our goal will be to shift the path of integration to have real part 1. In order to do so, we will have to bound the integrand $h(s)$ and therefore in particular ζ'/ζ.

2.3 Establishing bounds

In this section, our goal is to establish an upper bound on $|\zeta'(s)/\zeta(s)|$ when $\mathrm{Re}(s)$ is close to 1 and when $|\mathrm{Im}(s)|$ is large. As $\zeta(\bar{s}) = \overline{\zeta(s)}$, we can focus on bounds when $\mathrm{Im}(s) > 0$.

We will start with the following lemma, which already gives upper bounds on $\zeta(s)$ and $\zeta'(s)$. Notice that the upper bound on $\zeta(s)$ for now seems useless, since we eventually need a bound from below. Still, this upper bound on ζ can be used to establish a lower bound in the upcoming lemmas.

Lemma 2.3.1. *For every $A > 0$ there exists a constant M_A such that*

$$\big|\zeta(s)\big| \leq M_A \log(t)$$

and

$$\big|\zeta'(s)\big| \leq M_A \log^2(t)$$

for all $s = \sigma + it$ with $\sigma \geq 1/2$ satisfying $\sigma > 1 - \frac{A}{\log(t)}$ and $t \geq e$.

Proof. First, for $\sigma \geq 2$ we have $|\zeta(s)| \leq \zeta(2)$. Similar for ζ'

$$\big|\zeta'(s)\big| = \Big|\sum_{n\geq 1} \frac{\ln(n)}{n^s}\Big| \leq \sum_{n\geq 1} \frac{\ln(n)}{n^\sigma} \leq \sum_{n\geq 1} \frac{\ln(n)}{n^2} = \zeta'(2),$$

so both inequalities hold by choosing M_A large enough.

Hence, it remains to prove the claim for $\sigma < 2$, $t \geq e$. We start by introducing an alternative formula for the zeta function by

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} - s\int_N^\infty \frac{\{x\}}{x^{s+1}}\mathrm{d}x + \frac{N^{1-s}}{s-1} \tag{2.6}$$

which is valid when $\mathrm{Re}(s) > 1$, $N \in \mathbb{N}$. It can be obtained in the same manner as equation (1.3). (Note that $N = 1$ precisely gives (1.3).) Differentiating both sides yields

$$\zeta'(s) = -\sum_{n=1}^{N} \frac{\log(n)}{n^s} + s\int_N^\infty \frac{\{x\}\log(x)}{x^{s+1}}\mathrm{d}x - \int_N^\infty \frac{\{x\}}{x^{s+1}}\mathrm{d}x \tag{2.7}$$

$$-\frac{N^{1-s}\log(N)}{s-1} - \frac{N^{1-s}}{(1-s)^2}. \tag{2.8}$$

Now as $\sigma < 2$ and $t \geq e$ we have

$$|s| \leq \sigma + t \leq 2 + t < 2t \text{ and } |s-1| \geq t \quad \Leftrightarrow \quad \frac{1}{|s-1|} \leq \frac{1}{t}.$$

Using these inequalities and our new formula (2.6) we find

$$\big|\zeta(s)\big| \leq \sum_{n=1}^{N} \frac{1}{n^\sigma} + 2t\int_N^\infty \frac{1}{x^{\sigma+1}}\mathrm{d}x + \frac{N^{1-\sigma}}{t} = \sum_{n=1}^{N} \frac{1}{n^\sigma} + \frac{2t}{\sigma N^\sigma} + \frac{N^{1-\sigma}}{t} =: B + C + D.$$

Hence, we want to bound each of the last three summands. Let now N depend on t by setting $N = \lfloor t \rfloor$. Therefore, we have $N \leq t < N+1$ and $\log(n) \leq \log(t)$ for $n \leq N$. Then the assumption $1 - \sigma < A/\log(t)$ of this lemma implies

$$\frac{1}{n^\sigma} = \frac{n^{1-\sigma}}{n} = \frac{e^{(1-\sigma)\log(n)}}{n} < \frac{e^{A\log(n)/\log(t)}}{n} \leq \frac{e^A}{n}.$$

Also, for C there are the bounds

$$\frac{2t}{\sigma N^\sigma} < \frac{2\cdot(N+1)e^A}{2\cdot N} = O(1)$$

as t and thereby N tend to infinity. Regarding D

$$\frac{N^{1-\sigma}}{t} = \frac{N}{t}\cdot\frac{1}{N^\sigma} \leq \frac{N}{t}\cdot\frac{e^A}{N} = O(1).$$

Putting these three estimates for B, C and D together one gets

$$\big|\zeta(s)\big| \leq \sum_{n=1}^{N} \frac{e^A}{n} + O(1) = O\big(\log(N)\big) = O\big(\log(t)\big)$$

proving the bound for $|\zeta(s)|$.

The estimate for $|\zeta'(s)|$ follows from the same type of argument. This time an extra factor $\log(N)$ appears and as $\log(N) = O\big(\log(t)\big)$ we get $|\zeta'(s)| = O\big(\log^2(t)\big)$.

□

Now we are ready to prove the following lemma, which gives the desired upper bound on $|\zeta'/\zeta|$.

Lemma 2.3.2. *Let $s = \sigma + it$ with $\sigma \geq 1$ and $t \geq e$. Then there exists a positive constant C such that*

$$\left|\frac{1}{\zeta(s)}\right| < C\log^7(t)$$

and
$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| < C\log^9(t).$$

Proof. First note that when $\sigma \geq 2$ the reversed triangle inequality yields
$$\big|\zeta(s)\big| \geq 1 - \sum_{n\geq 2} |n^{-s}| \geq 1 - \sum_{n\geq 2} n^{-2} = 2 - \zeta(2) = 0.355...$$
and
$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| \leq \sum_{n\geq 1} \frac{\Lambda(n)}{n^2} < \infty,$$
hence both inequalities of the lemma hold trivially when C is sufficiently large. Suppose for the rest of the proof $1 \leq \sigma < 2$ and $t \geq e$.
Rewriting the inequality from Lemma 1.4.1 b), which was used to prove that the zeta function has no zeros on the line with real part 1, gives us
$$\frac{1}{|\zeta(\sigma+it)|} \leq \zeta(\sigma)^{3/4}|\zeta(\sigma+2it)|^{1/4}.$$
The function $(\sigma-1)\zeta(\sigma)$ is bounded for $\sigma \in (1,2]$, hence there exists an absolute constant M with $(\sigma-1)\zeta(\sigma) < M$, so that
$$\zeta(\sigma) \leq \frac{M}{\sigma-1}$$
for $1 < \sigma \leq 2$. Also $\zeta(\sigma+2it) \leq M_1\big(\log(2t)\big) < 2M_1\log(t)$ where M_1 is the constant implied by the previous Lemma 2.3.1 by setting $A = 1$. This yields
$$\frac{1}{|\zeta(\sigma+it)|} \leq \frac{M^{3/4}\big(2M_1\log(t)\big)^{1/4}}{(\sigma-1)^{3/4}}.$$
Taking reciprocals and setting $B := M^{-3/4}(2M_1)^{-1/4}$ gives
$$\big|\zeta(\sigma+it)\big| \geq \frac{B(\sigma-1)^{3/4}}{\log(t)^{1/4}} \tag{2.9}$$
for $1 < \sigma \leq 2$ and $t \geq e$. The bound also holds for $\sigma = 1$ by inspection.
Let α be any number satisfying $1 < \alpha < 2$. We want to show
$$|\zeta(\sigma+it)| \geq \frac{B(\alpha-1)^{3/4}}{\log(t)^{1/4}} - (\alpha-1)M_1\log^2(t) \tag{2.10}$$
for all $\sigma \in [1,2]$ where M_1 is again the constant from the previous lemma. We split this claim in two parts:

- Case 1: $\sigma \in [\alpha, 2]$. One has $(\sigma-1)^{3/4} \geq (\alpha-1)^{3/4}$, hence the claim is a consequence of (2.9).

- Case 2: $\sigma \in [1, \alpha]$. By the last lemma we get

$$|\zeta(\sigma + it) - \zeta(\alpha + it)| \leq \int_\sigma^\alpha |\zeta'(u + it)| \mathrm{d}u \leq (\alpha - \sigma) M_1 \log^2(t) \\ \leq (\alpha - 1) M_1 \log^2(t).$$

  The triangle inequality yields further

$$|\zeta(\sigma + it)| \geq |\zeta(\alpha + it)| - |\zeta(\sigma + it) - \zeta(\alpha + it)| \\ \geq \frac{B(\alpha - 1)^{3/4}}{\log(t)^{1/4}} - (\alpha - 1) M_1 \log^2(t).$$

We now want to optimize the choice of α in (2.10), such that the right-hand side of the inequality is rather big. In order to make the first term equal twice the second term, we set

$$\alpha_0 = 1 + \Big(\frac{B}{2M_1}\Big)^4 \frac{1}{\log^9(t)}.$$

Note that α_0 is indeed in the interval $(1, 2)$ for sufficiently large t, say $t \geq t_0$. Finally, for $1 \leq \sigma \leq 2$ and $t \geq t_0$ we get with $\alpha = \alpha_0$

$$|\zeta(\sigma + it)| \geq (\alpha_0 - 1) M_1 \log^2(t) = \frac{1}{C \log^7(t)}$$

where C is again some constant. In the interval $e \leq t \leq t_0$ the same inequality holds with possibly a larger C. This shows that there exists a positive constant C such that $|\zeta(s)|^{-1} < C \log^7(t)$ for all $\sigma \geq 1$ and $t \geq e$. Combining this result with Lemma 2.3.1, which states that $|\zeta'(s)| = O\big(\log^2(t)\big)$, gives $|\zeta'(s)/\zeta(s)| < C \log^9(t)$ by increasing C once more if necessary.

□

2.4 Completing the proof of the Prime Number Theorem

Now we are able to move the path of integration in (2.5) to the line $\sigma = 1$.

Proposition 2.4.1. *For $x \geq 1$ we have*

$$\frac{\psi_1(x)}{x^2} - \frac{1}{2}\Big(1 - \frac{1}{x}\Big)^2 = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} x^{s-1} h(s) \mathrm{d}s = \\ = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(1 + it) e^{it \log(x)} \mathrm{d}t$$

where the integral is absolutely convergent and once again

$$h(s) = \frac{1}{s(s+1)}\Big(- \frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}\Big).$$

Proof. In Proposition 2.2.3 we showed that

$$\frac{\psi_1(x)}{x^2} - \frac{1}{2}\Big(1 - \frac{1}{x}\Big)^2 = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{s-1} h(s) \mathrm{d}s$$

for $c > 1$. Let us now consider the integral along the rectangle with vertices at $1-iT$, $c-iT$, $c+iT$ and $1+iT$ of $x^{s-1}h(s)$, as shown in Figure 2.2.

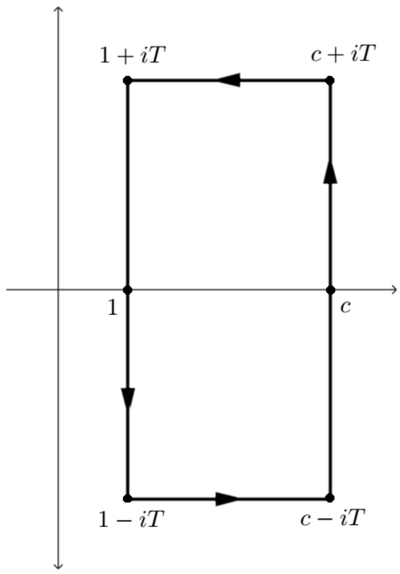

Figure 2.2: Path of integration

Since $x^{s-1}h(s)$ is holomorphic on $\mathrm{Re}(s) \geq 1$, the integral equals 0. For some fixed c we want to show that the integrals along the horizontal strips do not contribute to the total integral as $T \to \infty$. Note that since the integrand has conjugated values at conjugated arguments, it suffices to consider the strip at $t = +T$, i.e. s lies on the line between $1+iT$ and $c+iT$. On this segment we have

$$\Big|\frac{1}{s(s+1)}\Big| \leq \frac{1}{T^2} \quad \text{and} \quad \Big|\frac{1}{s-1}\Big| \leq \frac{1}{T}. \tag{2.11}$$

Lemma 2.3.2 assures the existence of some constant C, for which $|\zeta'(s)/\zeta(s)| \leq C\log^9(t)$ for $\sigma \geq 1$ and $t \geq e$. Since we are considering $T \to \infty$, we have

$$\begin{aligned} |h(s)| &\leq \Big|\frac{1}{s(s+1)}\Big| \cdot \left(\Big|\frac{\zeta'(s)}{\zeta(s)}\Big| + \Big|\frac{1}{s-1}\Big|\right) \\ &\leq \frac{1}{T^2} \cdot \left(C\log^9(T) + \frac{1}{T}\right) < \frac{2C\log^9(T)}{T^2} \end{aligned}$$

and consequently

$$\Big|\int_{c+iT}^{1+iT} x^{s-1}h(s)\mathrm{d}s\Big| \leq \int_1^c x^{\sigma-1}\underbrace{\frac{2C\log^9(T)}{T^2}}_{\to 0}\mathrm{d}\sigma \to 0.$$

Therefore we have shown that the paths along the two vertical strips have the same

value
$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{s-1}h(s)\mathrm{d}s = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} x^{s-1}h(s)\mathrm{d}s.$$
Further, the last expression can be written as
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} h(1+it)e^{it\log(x)}\mathrm{d}t$$
by parameterizing $s = 1+it$, which proves the proposition.
To justify the absolute convergence of the integral, note that $t \mapsto h(1+it)$ is a continuous function and $|h(1+it)| \leq \frac{2C\log^9(|t|)}{t^2}$ when $|t|$ is large.

□

To conclude the proof of the Prime Number Theorem, consider the equation proven in the last proposition
$$\frac{\psi_1(x)}{x^2} - \frac{1}{2}\Big(1-\frac{1}{x}\Big)^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(1+it)e^{it\log(x)}\mathrm{d}t.$$
Since the integral convergences absolutely, we can apply the Riemann-Lebesgue Lemma [see Appendix A.2] as we take the limit $x \to \infty$, which implies that the right-hand side converges to 0. Similar the expression $\frac{1}{2}(1-1/x)^2$ tends to $1/2$. This results in
$$\frac{\psi_1(x)}{x^2} \to \frac{1}{2}$$
as $x \to \infty$, completing the proof of the Prime Number Theorem.

2.5 Additional remarks

In this section, we briefly want to reflect on the given proof. Specifically, we want to discuss variations on the proof, the assumptions we had to use and a generalization of the proof.
Even though the theorem is a statement about the distribution of prime numbers, the main part of the proof appeals to techniques from complex analysis with the focus on properties of the Dirichlet series L_Λ. We want to list the 3 assumptions we used to show that
$$\psi(x) = \sum_{n\leq x}\Lambda(n) \sim x.$$

- In order to conclude that $\psi_1(x) \sim x^2/2$ implies $\psi(x) \sim x$, we had to use that Λ is a non-negative function.

- The series L_Λ converges on $\{s : \mathrm{Re}(s) > 1\}$, has a simple pole of order 1 at $s = 1$, can be extended to a meromorphic function including the half-plane $\{s : \mathrm{Re}(s) \geq 1\}$ and has no other pole there.

- There exists a positive constant ϵ such that $|L_\Lambda(s)| = o(|\sigma|^{1-\epsilon})$ as $|\sigma| \to \infty$.

This was used to ensure that the integral

$$\int_{-\infty}^{\infty} h(1+it)e^{it\log(x)}\mathrm{d}t$$

converges absolutely.

Hence, our proof can also be applied to other arithmetic functions instead of Λ where these conditions are met.

In the proof, we applied Perron's formula to L_Λ to get the identity

$$\frac{\psi_1(x)}{x^2} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Big(-\frac{\zeta'(s)}{\zeta(s)}\Big)\frac{x^{s-1}}{s(s+1)}\mathrm{d}s.$$

As our goal was to set the path of integration to have real part 1, even though the integrand has a pole at $s = 1$, we wrote the integrand as

$$\frac{x^{s-1}}{(s-1)s(s+1)} + \frac{x^{s-1}}{s(s+1)}\Big(-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}\Big)$$

where the first term has a pole and the second term is holomorphic and was denoted by $h(s)\cdot x^{s-1}$. Luckily, the integral involving the pole could then be evaluated using Perron's formula again.
One idea to circumvent this step, is to apply Perron's formula not to $L_\Lambda = -\zeta'/\zeta$ but instead to

$$L_{\Lambda-1}(s) = -\frac{\zeta'(s)}{\zeta(s)} - \zeta(s).$$

This function is more convenient, as it does not have a pole at $s = 1$. All the established inequalities for the integrand do still hold true, hence one can again shift the path of integration to have real part 1. This way one would deal with the function

$$\widehat{\psi}(x) := \sum_{n\le x}\big(\Lambda(n)-1\big) = -\lfloor x\rfloor + \sum_{n\le x}\Lambda(n)$$

and again its integrated form $\widehat{\psi_1}(x) := \int_1^x \widehat{\psi}(t)\mathrm{d}t$. For $\psi_1(x)$ this method yielded $\psi_1(x) \sim x^2/2$, while now it would show

$$\widehat{\psi_1}(x) = o(x^2).$$

Here we would like to conclude from $\widehat{\psi_1}(x) = o(x^2)$ that $\widehat{\psi}(x) = o(x)$. However this is not as straightforward as in Proposition 2.1.1, as $\Lambda(n) - 1$ is not a non-negative function. Still, the idea of subtracting the pole first is worth mentioning and will appear in the following chapter in a similar form.

Chapter 3

Proof using Tauberian theory

In this chapter we will introduce a quite different approach for proving the Prime Number Theorem, namely by using Tauberian theory. More precisely, we will prove the Wiener-Ikehara Theorem, which is named after Norbert Wiener and Shikao Ikehara. Then applying this theorem to L-functions proves the Prime Number Theorem.
When doing so, we will only use the fact that L_Λ has no poles on $\{s : \mathrm{Re}(s) \geq 1\}$, except for the simple pole at $s = 1$ as well as its meromorphic extension to a domain including this half-plane. Hence, one avoids some technical computations needed to bound the zeta function. All historically previous proofs required at least some growth condition of L_Λ [BD04, p. 181]. Also, it can be used to prove that the Prime Number Theorem is "equivalent" to ζ having no zeros on the line $\{s : \mathrm{Re}(s) = 1\}$, as we will address in Chapter 4 in detail.
This chapter starts with an introduction to Tauberian theory and gives some first examples. Then we will examine certain properties of the Feyèr kernel, which will be heavily used for proving the Wiener-Ikehara Theorem, leading to the Prime Number Theorem as a corollary.

3.1 Tauberian theorems

In many contexts in mathematics, it is helpful to be able to assign a value to an infinite series $\sum_{n\geq 1} a_n$ with $a_i \in \mathbb{C}$. The easiest case is when this sum converges in the usual sense. Then, by definition, the infinite sum equals the limit of the partial sums.
A different method of assigning a value to an infinite series is to consider the limit of the averages of the partial sums, i.e. if s_n is the sum of a_i for $1 \leq i \leq n$, then one computes

$$\lim_{n\to\infty} \frac{s_1 + ... + s_n}{n} = \lim_{n\to\infty} \left(a_1 \cdot \frac{n}{n} + a_2 \cdot \frac{n-1}{n} + \cdots + a_n \cdot \frac{1}{n} \right).$$

If such a limit exists, it is called *Cesàro sum* of $\sum_{n\geq 1} a_n$ and the series is said to be *Cesàro summable*. Loosely speaking, the additional weights $\frac{n+1-i}{n}$ on the a_i "smoothen" the expression, even though as $n \to \infty$ the weight of any fixed a_i approaches 1.

This gives an intuitive understanding of a result of Cauchy from the year 1821 that any series that converges ordinary is also Cesàro summable and both limits equal one another [Kor04, p. 3]. The converse is not true: The series $1 - 1 + 1 - 1 + \ldots$ has Cesàro sum $1/2$, while it does not converge in the conventional sense.

Another possible approach for assigning a value to $\sum_{n\geq 1} a_n$ is to consider the power series

$$f(x) = \sum_{n\geq 1} a_n x^n$$

as x tends to 1. The series $\sum_{n\geq 1} a_n$ is then called *Abel summable*, if f converges for $|x| < 1$ and $f(x)$ tends to a (finite) limit as $x \to 1^-$.
A result of Abel from 1826 is the following [Kor04, p. 4]:

Theorem 3.1.1. *Any ordinary convergent sum is also Abel summable with the same limit.*

Even more general, Frobenius was able to prove that Cesàro summability implies Abel summability. Hence

Ordinary convergence $\Rightarrow$ *Cesàro summability* $\Rightarrow$ *Abel summability*

while the converse does not hold in either case. Consider the series $1 - 2 + 3 - 4 + \cdots$. Its corresponding power series is

$$f(x) = \sum_{n\geq 1} (-1)^{n+1} n x^n = \frac{x}{(x+1)^2}$$

for $|x| < 1$ and converges to $1/4$ as $x \to 1^-$. Still, the averages of the partial sums are not converging, hence the series is not Cesàro summable.
In conclusion, there are different definitions of the value of infinite series, where sometimes one summability implies the other. The study under what conditions the converse is true, is known as **Tauberian theory**. The first such result was proven by Austrian mathematician Alfred Tauber in 1897. His statement gives conditions under which Abel summability implies ordinary convergence [Kor04, p. 10]:

Theorem 3.1.2. *If a series $\sum_{n\geq 1} a_n$ is Abel summable and $a_n = o(1/n)$, then the series converges.*

In 1914, Hardy and Littlewood where able to relax the condition on the sequence a_n. Specifically they show that the result still holds true, if one replaces the bound $a_n = o(1/n)$ by the one-sided condition $a_n \geq c/n$ for some $c \in \mathbb{R}^-$. [Kor04, p. 15]

To connect Tauberian theory with the Prime Number Theorem, consider the function

$$L_\Lambda(s) - \zeta(s) = -\frac{\zeta'(s)}{\zeta(s)} - \zeta(s) = \sum_{n\geq 1} \frac{\Lambda(n) - 1}{n^s}.$$

We established that this series converges for $\mathrm{Re}(s) > 1$ and can be extended meromorphically to $\mathrm{Re}(s) > 0$. This extension has no pole at $s = 1$ since the poles of ζ

and L_Λ cancel. Hence, the limit

$$\lim_{s\to 1^+} \sum_{n\geq 1} \frac{\Lambda(n)-1}{n^s}$$

exists and is finite. On the other hand, we are for now unable to tell, whether the series converges to a finite limit if we set $s=1$. To establish such a result, one would need a Tauberian theorem. The Prime Number Theorem is then a rather straightforward consequence of the convergence of this sequence, as will be shown in Chapter 4.3.

The next and final example of a Tauberian theorem will involve integrals rather than sums [Kor04, p. 135].

Theorem 3.1.3. *Let F be a bounded function, such that its Laplace transform*

$$G(s) = \int_0^\infty F(t)e^{-st}\mathrm{d}t$$

exists when $\mathrm{Re}(s) > 0$*. If G can be extended continuously to a function on* $\mathrm{Re}(s) \geq 0$*, then*

$$\int_0^\infty F(t)\mathrm{d}t$$

exists and equals $G(0)$.

Hence the theorem allows under certain condition to set $s = 0$ in the integral $\int_0^\infty F(t)e^{-st}\mathrm{d}t$. Our main goal of this chapter is to prove the so-called *Theorem of Wiener and Ikehara*, from which the Prime Number Theorem follows in a straightforward way as we will see.

Theorem 3.1.4 (Wiener, Ikehara)**.** *Let $\varphi : [0,\infty) \to \mathbb{R}^+$ be a non-decreasing function. Furthermore, let*

$$F(s) := \int_0^\infty \varphi(u)e^{-su}\mathrm{d}u - \frac{1}{s-1}$$

be a continuous function when $\mathrm{Re}(s) > 1$ *that has a continuous extension on the closed half-plane* $\{s : \mathrm{Re}(s) \geq 1\}$*. Then*

$$\lim_{x\to\infty} \varphi(x)e^{-x} = 1.$$

This theorem is usually also referred to as a Tauberian theorem, even though it is not of the usual form. However it stands in close relation to Theorem 3.1.3, in fact one has:

Proposition 3.1.5. *The Wiener-Ikehara Theorem is a consequence of Theorem 3.1.3.*

Proof. Let φ and F be given as in the theorem of Wiener and Ikehara. Using

$$\frac{1}{s-1} = \int_0^\infty e^{-(s-1)u}\mathrm{d}u,$$

F can also be written as

$$F(s) = \int_0^\infty (\varphi(u)e^{-u} - 1)e^{-(s-1)u}\mathrm{d}u.$$

Now we can apply Theorem 3.1.3 to the function $u \mapsto \varphi(u)e^{-u} - 1$ and substitute $s - 1 \mapsto s$, which implies that the integral

$$\int_0^\infty \left(\varphi(u)e^{-u} - 1\right) \mathrm{d}u$$

converges to a finite value. We claim that $\varphi(u)e^{-u} \to 1$.
Assume $\limsup_{u\to\infty}(\varphi(u)e^{-u}) > 1 + 2\varepsilon$ for some positive ε. Then, there exists a sequence $(u_i)_{i\in\mathbb{N}}$, $u_i \to \infty$, such that $\varphi(u_i)e^{-u_i} > 1 + 2\varepsilon$. Since φ is non-decreasing, there exists some $\delta > 0$ such that

$$\varphi(u)e^{-u} > 1 + \varepsilon \textit{ for } u \in [u_i, u_i + \delta]$$

for all $i \in \mathbb{N}$. Then one has $\int_{u_i}^{u_i+\delta}(\varphi(u)e^{-u} - 1)\mathrm{d}u > \delta\epsilon$ for every i, hence the integral $\int_0^\infty (\varphi(u)e^{-u} - 1)\,\mathrm{d}u$ cannot converge.
In a similar way, one excludes the case $\liminf_{u\to\infty}(\varphi(u)e^{-u}) < 1$, by considering a sequence $(u_i)_{i\in\mathbb{N}}$ with $\varphi(u_i)e^{-u_i} < 1 - 2\varepsilon$ and then examines the behaviour when $u \in [u_i - \delta, u_i]$ for some fixed δ.

□

The motivation for using the Laplace transform to prove the Prime Number Theorem, is that every L-series can be interpreted in this way: If f is an arithmetic function, define $\varphi(x) = \sum_{n\le e^x} f(n)$. Disregarding convergence for the moment, the integral $\int_0^\infty \varphi(u)e^{-su}\mathrm{d}u$ can be evaluated as follows:

$$\begin{aligned}\int_0^\infty \varphi(u)e^{-su}\mathrm{d}u &= \int_0^\infty \sum_{n\le e^u} f(n)e^{-su}\mathrm{d}u = \sum_{n\ge1} f(n)\int_{\log(n)}^\infty e^{-su}\mathrm{d}u \\ &= \frac{1}{s}\sum_{n\ge1}\frac{f(n)}{n^s} = \frac{1}{s}L_f(s).\end{aligned}$$

Note that $\varphi(x)$ is just an alternate "summation function" of f.

3.2 The Fejér kernel and its properties

Our proof of the Wiener-Ikehara Theorem will be using the so-called Fejér kernel to a great extent.

Definition. *The Fejér kernel is defined as*

$$\begin{aligned}K_\lambda(x) :=& \frac{1}{2}\int_{-2\lambda}^{2\lambda}\left(1 - \frac{|t|}{2\lambda}\right)e^{itx}\mathrm{d}t \\ =& \frac{\lambda}{2}\int_{-2}^{2}\left(1 - \frac{|t|}{2}\right)e^{itx\lambda}\mathrm{d}t.\end{aligned}$$

Note that it can be viewed as the Fourier-transform of $(1 - \frac{|t|}{2\lambda})$ *where* $t \in [-2\lambda, 2\lambda]$.

In order to get a closed form for $K_\lambda(x)$ which will enable us to get some useful results, we first show

Proposition 3.2.1. *Let $x \in \mathbb{R}$ and $\lambda \in \mathbb{R}^+$. Then:*

a)

$$K_\lambda(x) = \lambda \left(\frac{\sin(\lambda x)}{\lambda x} \right)^2 ,$$

where the case $x = 0$ is covered by $\lim_{x \to 0} \lambda \left(\frac{\sin^2(\lambda x)}{(\lambda x)^2} \right) = \lambda$. Hence for a fixed $\lambda > 0$, $K_\lambda(x)$ is a continuous bounded function.

b) The inequality $0 \leq K_\lambda(x) \leq \min(\lambda, \frac{1}{\lambda x^2})$ holds true.

c) The integral

$$\int_{-\infty}^{\infty} K_\lambda(u) \mathrm{d}u =: \rho$$

exists and is independent of λ. We will define its value as ρ.

d) For any fixed ϵ, as λ increases, the weight of $\int K_\lambda(u)\mathrm{d}u$ outside the interval $[-\epsilon, \epsilon]$ decreases. More specifically we have

$$\int_{|u|>\epsilon} K_\lambda(u) \mathrm{d}u \leq \frac{2}{\lambda \epsilon}.$$

Remark 3.2.2. *The constant ρ can actually be computed, in fact $\rho = \pi$, as described in Appendix A.3. However, as the exact value is not important, we will continue denoting it by ρ.*

Proof. a) Notice that the substitution $t \mapsto -t$ gives

$$\frac{\lambda}{2} \int_{-2}^{2} \left(1 - \frac{|t|}{2} \right) e^{it\lambda u} \mathrm{d}t = \frac{\lambda}{2} \int_{-2}^{2} \left(1 - \frac{|t|}{2} \right) e^{-it\lambda u} \mathrm{d}t.$$

Since the left and right integrand are conjugate, the integral is real valued. Therefore, we can focus on evaluating its real part:

$$\begin{aligned} \frac{\lambda}{2} \int_{-2}^{2} \left(1 - \frac{|t|}{2}\right) e^{it\lambda u} \mathrm{d}t &= \frac{\lambda}{2} \int_{-2}^{2} \mathrm{Re}\left(\left(1 - \frac{|t|}{2}\right) e^{it\lambda u} \right) \mathrm{d}t = \\ &= \frac{\lambda}{2} \int_{-2}^{2} \left(1 - \frac{|t|}{2}\right) \cos(t\lambda u) \mathrm{d}t = \\ &= \lambda \int_{0}^{2} \left(1 - \frac{t}{2}\right) \cos(t\lambda u) \mathrm{d}t. \end{aligned}$$

Using partial integration in the case $u \neq 0$ yields

$$\underbrace{\lambda\Big(1-\frac{t}{2}\Big)\frac{\sin(t\lambda u)}{\lambda u}\bigg|_0^2}_{=0}+\frac{\lambda}{2}\int_0^2\frac{\sin(t\lambda u)}{\lambda u}\mathrm{d}t=\frac{\lambda}{2}\Big(\frac{-\cos(t\lambda u)}{\lambda^2u^2}\Big)\bigg|_0^2$$

$$=\frac{1-\cos(2\lambda u)}{2\lambda u^2}=\frac{\sin^2(\lambda u)}{\lambda u^2}.$$

Considering the case $u=0$ one gets

$$\lambda\int_0^2\Big(1-\frac{t}{2}\Big)\mathrm{d}t=\lambda\Big(t-\frac{1}{4}t^2\Big)\bigg|_0^2=\lambda$$

which agrees with the fact that $\lim_{u\to 0}\frac{\sin^2(\lambda u)}{\lambda u^2}=\lambda$ by L'Hopital's rule.

b) The inequality $0\le K_\lambda(x)$ holds by inspection. Further since $|\sin(\lambda x)|\le 1$, we get the estimate

$$K_\lambda(x)=\lambda\Big(\frac{\sin(\lambda x)}{\lambda x}\Big)^2\le\frac{1}{\lambda x^2}.$$

On the other hand, if we use the well known inequality $|\sin(\lambda x)|\le|\lambda x|$ we get $K_\lambda(x)\le\lambda$.

c) The boundedness of this integral follows from the inequalities above. Afterwards, the substitution $\lambda u=y$ yields

$$\int_{\mathbb{R}}K_\lambda(u)\mathrm{d}u=\int_{\mathbb{R}}\frac{\sin^2(\lambda u)}{\lambda u^2}\mathrm{d}u=\int_{\mathbb{R}}\frac{\sin^2(y)}{\lambda(y/\lambda)^2}\cdot\frac{1}{\lambda}\mathrm{d}y=\int_{\mathbb{R}}K_1(y)\mathrm{d}y=\rho.$$

d) Integrating the estimate $K_\lambda(x)\le\frac{1}{\lambda x^2}$ gives precisely the statement. □

In the following figure the Fejér kernel $K_\lambda(x)$ is plotted for $\lambda=1,2,4,6$.

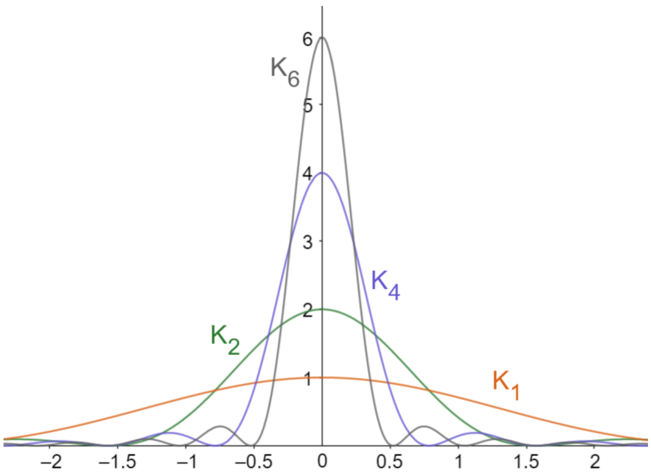

Figure 3.1: The Fejér Kernel

3.3 One key lemma

We will now prove a key lemma on our way to the Wiener-Ikehara Theorem. It will in a first step only contain the Fejér kernel in its closed form rather than its integral representation.

Lemma 3.3.1. *Let $\varphi : [0, \infty) \to \mathbb{R}$ be a monotone increasing function and $\varphi \geq 0$. If*

$$\lim_{x\to\infty} \int_0^\infty \varphi(u)e^{-u} K_\lambda(u-x)\mathrm{d}u = \rho$$

holds for every $\lambda > 0$, then

$$\lim_{x\to\infty} \varphi(x)e^{-x} = 1.$$

Proof. The proof will consist of two parts. We will start by showing the inequality $\limsup_{x\to\infty} \varphi(x)e^{-x} \leq 1$ followed by $\liminf_{x\to\infty} \varphi(x)e^{-x} \geq 1$. Both parts will use the key fact that $\varphi(x)e^{-x}$ cannot decrease arbitrarily fast, in the sense that

$$\frac{\varphi(x+\epsilon)e^{-(x+\epsilon)}}{\varphi(x)e^{-x}} \geq e^{-\epsilon}$$

using the monotonicity of φ. Following this thought, if $\varphi(y)e^{-y} > 1+\epsilon$ for some y, then there exists an δ_ϵ dependent on ϵ only, such that $\varphi(x)e^{-x} > 1$ when x is in the interval $[y, y+\delta_\epsilon]$. A similar statement can be made in the case $\varphi(y)e^{-y} < 1-\epsilon$. The parameters λ and x then allow us to centre the weight of the kernel $K_\lambda(u-x)$ in the interval $u \in [y, y+\delta_\epsilon]$, which leads to a contradiction with the assumption of the lemma.

Now for the rigorous proof:

- **Claim:** $\limsup_{x\to\infty} \varphi(x)e^{-x} \leq 1$. *Proof by contradiction:*

  Assume there exists an $\epsilon > 0$ and a monotone increasing sequence $(x_i)_{i\geq 1}$, $x_i \to \infty$, such that
  $$\varphi(x_i)e^{-x_i} \geq 1+\epsilon.$$
  Then we have for $u \in [x_i, x_i + \log(1+\epsilon/2)]$
  $$\varphi(u)e^{-u} \geq \varphi(x_i)e^{-x_i-\log(1+\epsilon/2)} \geq \frac{1+\epsilon}{1+\epsilon/2} =: \delta > 1.$$

  Now by Proposition 3.2.1 c) and d) we can choose λ sufficiently large, such that $\int_{|u|\leq\log(1+\epsilon/2)/2} K_\lambda(u)\mathrm{d}u > \rho\frac{1}{\sqrt{\delta}}$.

  Finally, we consider the sequence $y_i = x_i + \log(1+\epsilon/2)/2$ of midpoints of the described intervals. Then we have $y_i \to \infty$ as well as

$$\begin{aligned}\int_0^\infty \varphi(u)e^{-u}K_\lambda(u-y_i)\mathrm{d}u &\geq \int_{|u-y_i|\leq \log(1+\epsilon/2)/2} \varphi(u)e^{-u}K_\lambda(u-y_i)\mathrm{d}u \\ &\geq \delta \cdot \int_{|u|\leq \log(1+\epsilon/2)/2} K_\lambda(u)\mathrm{d}u \\ &\geq \delta\cdot\rho\frac{1}{\sqrt{\delta}} > \rho.\end{aligned}$$

Hence $\lim_{x\to\infty}\int_0^\infty \varphi(u)e^{-u}K_\lambda(u-x)\mathrm{d}u \neq \rho$.

- **Claim:** $\liminf_{x\to\infty}\varphi(x)e^{-x}\geq 1$. *Proof by contradiction:*

  Assume the existence of an $\epsilon\in(0,1)$ and of a sequence $(x_i)_{i\geq 1}$, $x_i\to\infty$, such that

  $$\varphi(x_i)e^{-x_i}\leq 1-\epsilon.$$

  Again we get a sequence of intervals, such that for $u\in[x_i-\log(1+\epsilon),x_i]$

  $$\varphi(u)e^{-u}\leq \varphi(x_i)e^{-(x_i-\log(1+\epsilon))}\leq \frac{1-\epsilon}{1+\epsilon} =: \delta < 1.$$

  Note that we can replace the sequence x_i by a subsequence if necessary, to ensure that $x_i-\log(1+\epsilon)>0$ for all $i\geq 1$. Now we need to estimate the integral from above. Luckily, the proof of the first claim implies the existence of a $K\geq 1$ such that

  $$\varphi(x)e^{-x}\leq K \text{ for every } x\in\mathbb{R}^+.$$

  Now choose λ sufficiently large, such that

  $$\int_{|u|\geq\log(1+\epsilon)/2} K_\lambda(u)\mathrm{d}u < \rho\cdot\underbrace{\frac{1}{K}\cdot\frac{1-\delta}{2}}_{\in(0,1)}.$$

  As before, we consider the sequence of midpoints y_i of these intervals, i.e. $y_i = x_i - \log(1+\epsilon)/2$. We are now ready to estimate the integrals from above:

  $$\begin{aligned}&\int_0^\infty \varphi(u)e^{-u}K_\lambda(u-y_i)\mathrm{d}u\\ \leq&\int_{|u-y_i|\leq\log(1+\epsilon)/2}+\int_{|u-y_i|\geq\log(1+\epsilon)/2}\varphi(u)e^{-u}K_\lambda(u-y_i)\mathrm{d}u\\ \leq&\delta\cdot\rho+K\cdot\left(\rho\cdot\frac{1}{K}\cdot\frac{1-\delta}{2}\right)=\rho\cdot\frac{1+\delta}{2}<\rho\end{aligned}$$

  This again prohibits $\lim_{x\to\infty}\int_0^\infty\varphi(u)e^{-u}K_\lambda(u-x)\mathrm{d}u=\rho$ and hence finishes the proof.

□

3.4 Proving the Wiener-Ikehara Theorem

We are now ready to prove the Wiener-Ikehara Theorem.

Theorem (Wiener, Ikehara)**.** *Let $\varphi : [0,\infty) \to \mathbb{R}^+$ be a non-decreasing function. Furthermore, let*

$$F(s) := \int_0^\infty \varphi(u)e^{-su}\mathrm{d}u - \frac{1}{s-1}$$

be a continuous function for $\mathrm{Re}(s) > 1$ *that can be extended to a continuous function on* $\{s : \mathrm{Re}(s) \geq 1\}$*. Then*

$$\lim_{x\to\infty} \varphi(x)e^{-x} = 1.$$

The proof consists of 3 steps.

Step 1. *For every $\epsilon > 0$ there exists a K_ϵ, such that $\varphi(x)e^{-x} \leq K_\epsilon e^{\epsilon x}$ for all $x > 0$.*

Proof. For any ϵ and $x > 0$ we have by setting $s = 1 + \epsilon$

$$\begin{aligned}
F(1+\epsilon) + \frac{1}{\epsilon} &= \int_0^\infty \varphi(u)e^{-(1+\epsilon)u}\mathrm{d}u \geq \int_x^\infty \varphi(u)e^{-(1+\epsilon)u}\mathrm{d}u \\
&\geq \varphi(x)\int_x^\infty e^{-(1+\epsilon)u}\mathrm{d}u = -\varphi(x)\frac{e^{-(1+\epsilon)u}}{1+\epsilon}\Big|_x^\infty \\
&= \varphi(x)\frac{e^{-(1+\epsilon)x}}{1+\epsilon}
\end{aligned}$$

or alternatively

$$\varphi(x)e^{-x} \leq (1+\epsilon)\Big(F(1+\epsilon) + \frac{1}{\epsilon}\Big)e^{\epsilon x}.$$

Therefore the statement holds for $K_\epsilon := (1+\epsilon)\left(F(1+\epsilon) + \frac{1}{\epsilon}\right)$.

□

Step 2. *For $\epsilon, \lambda, x > 0$ we have*

$$\int_0^\infty \left(\varphi(u)e^{-u} - 1\right)e^{-\epsilon u}K_\lambda(u-x)\mathrm{d}u = \frac{\lambda}{2}\int_{-2}^{2}\left(1 - \frac{|t|}{2}\right)e^{i\lambda x}F(1+\epsilon-i\lambda t)\mathrm{d}t \quad (3.1)$$

and as a consequence

$$\lim_{x\to\infty}\lim_{\epsilon\to 0}\int_0^\infty \left(\varphi(u)e^{-u} - 1\right)e^{-\epsilon u}K_\lambda(u-x)\mathrm{d}u = 0. \quad (3.2)$$

Before proceeding with the proof of this step, we briefly want to examine the wonderful identity (3.1). The left-hand side is a real integral, which already looks similar to the one in Lemma 3.3.1 which will ultimately be used to show $\lim_{x\to\infty}\varphi(x)e^{-x} = 1$. The right integral is complex and heavily depends on the value of F near the line $\{s : \mathrm{Re}(s) = 1\}$. The transition between the two sides can be made via the different methods of expressing the Fejér kernel.

Proof. First, we note that

$$\begin{aligned}\int_0^\infty \left(\varphi(u)e^{-u}-1\right)e^{-(s-1)u}\mathrm{d}u &= \int_0^\infty \varphi(u)e^{-su}\mathrm{d}u - \int_0^\infty e^{-(s-1)u}\mathrm{d}u\\ &= \int_0^\infty \varphi(u)e^{-su}\mathrm{d}u - \frac{1}{s-1} = F(s).\end{aligned}$$

The left-hand side of (3.1) can be written differently using the initial definition of K_λ as

$$\begin{aligned}&\int_0^\infty \left(\varphi(u)e^{-u}-1\right)e^{-\epsilon u}K_\lambda(u-x)\mathrm{d}u\\ =&\int_0^\infty \left(\varphi(u)e^{-u}-1\right)e^{-\epsilon u}\frac{\lambda}{2}\int_{-2}^2\left(1-\frac{|t|}{2}\right)e^{i\lambda(u-x)}\mathrm{d}t\mathrm{d}u\\ =&\frac{\lambda}{2}\int_{-2}^2\left(1-\frac{|t|}{2}\right)e^{-i\lambda xt}\int_0^\infty\left(\varphi(u)e^{-u}-1\right)e^{-u(\epsilon-i\lambda tu)}\mathrm{d}u\mathrm{d}t\\ =&\frac{\lambda}{2}\int_{-2}^2\left(1-\frac{|t|}{2}\right)e^{-i\lambda xt}F(1+\epsilon-i\lambda t)\mathrm{d}t.\end{aligned}$$

Here we were allowed to change the order of the integrals since the integrals converge absolutely by Step 1, as

$$\begin{aligned}\left|\left(1-\frac{|t|}{2}\right)e^{-i\lambda xt}\left(\varphi(u)e^{-u}-1\right)e^{-u(\epsilon-i\lambda tu)}\right| &\le |\varphi(u)e^{-u}-1|e^{-\epsilon u}\\ &\le \left(K_{\frac{\epsilon}{2}}e^{\frac{\epsilon}{2}u}+1\right)e^{-\epsilon u}\\ &\le 2K_{\frac{\epsilon}{2}}e^{\frac{\epsilon}{2}u}e^{-\epsilon u}\\ &= 2K_{\frac{\epsilon}{2}}e^{-\frac{\epsilon}{2}u}\end{aligned}$$

and $\int_{-2}^2\int_0^\infty 2K_{\frac{\epsilon}{2}}e^{-\frac{\epsilon}{2}u}\mathrm{d}u\mathrm{d}t < \infty$. This proves the first part of Step 2.

We will now focus on the right-hand side of the equation (3.1). Here it is easy to take the limit $\epsilon \to 0$ followed by $x \to \infty$. To do so, we will need the following theorem and the Riemann-Lebesgue Lemma respectively:

Theorem 3.4.1 (Dominated convergence for binary functions)**.** *Let I be an interval, $f : (0,\infty) \times I \to \mathbb{C}$, and $g : I \to \mathbb{R}$. For any fixed $t > 0$ let $f(t,.)$ and g be integrable as well as $|f(t,x)| \le g(x)$ for $x \in I$. Finally, let the limiting function*

$$f(x) := \lim_{t\to 0} f(t,x) \text{ for } x \in I$$

exist. Then

$$\lim_{t\to 0}\int_I f(t,x)\mathrm{d}x = \int_I f(x)\mathrm{d}x.$$

How this theorem can be deduced from the *theorem of dominated convergence* is outlined in Appendix A.4.1.

Continuing the proof of Step 2, we note that $F(1 + a + bi)$ is continuous when $(a, b) \in [0, 1] \times [-2\lambda, 2\lambda]$, hence there exists some constant C_λ such that the integrand is bounded by

$$\left| \left(1 - \frac{|t|}{2}\right) e^{i\lambda x} F(1 + \epsilon - i\lambda t) \right| \leq C_\lambda$$

for $t \in [-2, 2]$. Therefore we can take the limit $\epsilon \to 0$ by Theorem 3.4.1 which yields for any fixed x

$$\lim_{\epsilon \to 0} \frac{\lambda}{2} \int_{-2}^{2} \left(1 - \frac{|t|}{2}\right) e^{i\lambda x} F(1 + \epsilon - i\lambda t) \mathrm{d}t = \frac{\lambda}{2} \int_{-2}^{2} \left(1 - \frac{|t|}{2}\right) e^{i\lambda x} F(1 - i\lambda t) \mathrm{d}t.$$

Here we used the fact that F has a continuous extension to $\{s : \mathrm{Re}(s) = 1\}$, i.e. does not have a pole on this line. Now we can apply the Riemann-Lebesgue Lemma to the right-hand side to get

$$\lim_{x \to 0} \frac{\lambda}{2} \int_{-2}^{2} \left(1 - \frac{|t|}{2}\right) e^{i\lambda x} F(1 - i\lambda t) \mathrm{d}t = 0.$$

To recap, we can apply $\lim_{\epsilon \to 0}$ followed by $\lim_{x \to \infty}$ to the right-hand side of (3.1) and get 0. An analogous treatment of the left-hand side proves the second part of Step 2. □

As already mentioned, equation (3.2) looks already close to the condition needed to apply Lemma 3.3.1. In the last step, we will separate the -1 term from the parenthesis which will pave the way for proving the Wiener-Ikehara Theorem.

Step 3. *The Wiener-Ikehara Theorem holds true.*

Proof. In Step 2 we established the formula (3.2), saying

$$\lim_{x \to \infty} \lim_{\epsilon \to 0} \int_0^\infty \left(\varphi(u) e^{-u} - 1\right) e^{-\epsilon u} K_\lambda(u - x) \mathrm{d}u = 0.$$

We will for now focus on evaluating just a part of this equation, namely $\lim_{x \to \infty} \lim_{\epsilon \to 0} \int_0^\infty e^{-\epsilon u} K_\lambda(u - x) \mathrm{d}u$. For that we will need the monotone convergence theorem for binary functions:

Theorem 3.4.2 (Monotone convergence for binary functions)**.** *Let I be an interval, $f : (0, \infty) \times I \to \mathbb{C}$, such that for $t > 0$ the function $f(t, .)$ is integrable and let f be monotone decreasing in t. Again, let the limiting function*

$$f(x) := \lim_{t \to 0} f(t, x) \text{ for } x \in I$$

exist. Then

$$\lim_{t \to 0} \int_I f(t, x) \mathrm{d}x = \int_I f(x) \mathrm{d}x.$$

A proof of this theorem is given in Appendix A.4.2.

As $\epsilon \mapsto e^{-\epsilon u} K_\lambda(u-x)$ is monotone decreasing, we may apply Theorem 3.4.2 to get

$$\lim_{\epsilon \to 0} \int_0^\infty e^{-\epsilon u} K_\lambda(u-x) \mathrm{d}u = \int_0^\infty K_\lambda(u-x) \mathrm{d}u$$

for any fixed $x \in \mathbb{R}$. We further have

$$\begin{aligned} \lim_{x\to\infty} \lim_{\epsilon \to 0} \int_0^\infty e^{-\epsilon u} K_\lambda(u-x) \mathrm{d}u &= \lim_{x\to\infty} \int_0^\infty K_\lambda(u-x) \mathrm{d}u \\ &= \lim_{x\to\infty} \int_{-x}^\infty K_\lambda(u) \mathrm{d}u \\ &= \int_{-\infty}^\infty K_\lambda(u) \mathrm{d}u = \rho. \end{aligned}$$

Adding this expression to (3.2) gives

$$\lim_{x\to\infty} \lim_{\epsilon \to 0} \int_0^\infty \varphi(u) e^{-(u+\epsilon)} K_\lambda(u-x) \mathrm{d}u = \rho.$$

As the integrand is greater than 0 and monotone decreasing in ε, we can once again use Theorem 3.4.2 to get

$$\lim_{x\to\infty} \lim_{\epsilon \to 0} \int_0^\infty \varphi(u) e^{-(u+\epsilon)} K_\lambda(u-x) \mathrm{d}u = \lim_{x\to\infty} \int_0^\infty \varphi(u) e^{-u} K_\lambda(u-x) \mathrm{d}u = \rho.$$

This is precisely the condition needed to apply Lemma 3.3.1 which implies

$$\lim_{x\to\infty} \varphi(x) e^{-x} = 1$$

and thereby finishes the proof of the Wiener-Ikehara Theorem.

□

3.5 Applying the Wiener-Ikehara Theorem to Dirichlet series

We now show how the Wiener-Ikehara Theorem can be applied to prove asymptotic behaviour of the partial sums $\sum_{n\le x} f(n)$ given certain conditions on the corresponding L-function $L_f(s)$.

Corollary 3.5.1. *Let f be a non-negative arithmetic function and assume that its associated L-series $L_f(s)$ converges when $\mathrm{Re}(s) > 1$. Further, let $L_f(s) - \frac{1}{s-1}$ be extendable to a continuous function on $\mathrm{Re}(s) \ge 1$, i.e. $L_f(s)$ has only one pole on $\{s : \mathrm{Re}(s) \ge 1\}$, namely at $s = 1$ with residue 1. Then*

$$\sum_{n \le x} f(n) \sim x.$$

Proof. Define the function φ to be the sum $\varphi(x) = \sum_{n \le e^x} f(n)$. Note that φ is monotone increasing since $f \ge 0$. Then, as already shown, the integral $\int_0^\infty \varphi(u) e^{-su} \mathrm{d}u$ is precisely $L_f(s)/s$:

$$\int_0^\infty \varphi(u)e^{-su}\mathrm{d}u = \int_0^\infty \sum_{n\leq e^u} f(n)e^{-su}\mathrm{d}u = \sum_{n\geq 1} f(n)\int_{\log(n)}^\infty e^{-su}\mathrm{d}u$$
$$= \frac{1}{s}\sum_{n\geq 1}\frac{f(n)}{n^s} = \frac{1}{s}L_f(s).$$

This equation holds true when $\mathrm{Re}(s) > 1$ which allows switching the order of integration and summation due to absolute convergence. Hence, $\int_0^\infty \varphi(u)e^{-su}\mathrm{d}u - \frac{1}{s-1}$ can be extended to a continuous function on $\{s : \mathrm{Re}(s) \geq 1\}$. This allows us to apply the theorem of Wiener and Ikehara which implies

$$\lim_{x\to\infty} \varphi(x)e^{-x} = 1 \quad \Leftrightarrow \quad \sum_{n\leq x} f(n) \sim x.$$

□

Corollary 3.5.2. *The Prime Number Theorem holds.*

Proof. Apply the previous corollary to $f \equiv \Lambda$. □

This proof of the Prime Number Theorem is very different to the previous one using complex analysis. Note that here no growth conditions on L_Λ needed to be established and used. Hence the method can be applied in a relatively simple manner to similar problems.
In the following chapter we will introduce generalizations of Corollary 3.5.1 to cover the case when L_f has poles of higher order and not necessarily at $s = 1$.

Chapter 4

A deeper understanding of the Prime Number Theorem

Previously, we established how singularities of L-functions L_f translate to asymptotic behaviour of $F(x) = \sum_{n \leq x} f(n)$. For the Prime Number Theorem we used the series L_Λ to conclude $\psi(x) \sim x$. While we were able to prove this result in two different ways, we did not capture the relation in its entire form. It was sufficient to consider the case of a L-function that has a simple pole at $s = 1$ and no other poles on $\{s : \mathrm{Re}(s) \geq 1\}$. It was already pointed out, what crucial part the non-existence of poles on the line $\mathrm{Re}(s) = 1$ played.
In this chapter, we will establish a much deeper relation between the singularities of L_f and properties of F. In fact, they often stand in a one-to-one correspondence to one another. Hence, we will look at L-functions which have a pole of higher order or more than one pole on its line of convergence and how the asymptotic behaviour of F is influenced then.
Coming back to the Prime Number Theorem, our goal is to eventually understand why the poles on the line of convergence of L_Λ had to be used in both proofs. With this in mind, we will discuss the different ideas and methods of the two preceding proofs in detail.

The outline of this chapter is the following:

(1) We will start by introducing the Mellin-transform, which generalizes the notion of L-series. The reason is that many results are more natural to describe using Mellin-transforms. It will be explained in detail how Dirichlet series fit this new description.

(2) Mellin transforms often allow to continue L-series beyond their initial domain of convergence.

(3) We study the connection between poles of L_f and the asymptotic behaviour of F. As an additional result, we will be able to describe the domain of convergence of Mellin-transforms and show how the Prime Number Theorem can be deduced from the convergence of the series $\sum_{n \geq 1} (\Lambda(n) - 1)/n$.

(4) Given the results in (3), we will be able to examine cases where L-functions have multiple poles, possibly of higher order and how they relate to F. Further,

we will show that the Prime Number Theorem is in some sense equivalent to the non-existence of poles besides the one at $s = 1$. Last, we will mention results which can be applied to estimate the growth of F when multiple poles are present.

(5) We are then able to reflect on the two given proofs, discuss in detail how they worked and compare them to one another. As it turns out, their structure is remarkably similar.

(6) This chapter is concluded by a description of the error term in the equation $\pi(x) = \frac{x}{\ln(x)} + E(x)$. It will be discussed what the current best estimation of this error term is and how it connects to the Riemann Hypothesis.

Remark 4.0.1. *Throughout this chapter we will use f to indicate an arithmetic function, $F(x) := \sum_{n \leq x} f(n)$ as its summatory function and L_f as the corresponding L-function. We will always write $s = \sigma + it$.*

4.1 Dirichlet series and Mellin transforms

Previously we showed how properties of L_Λ can be used to deduce the Prime Number Theorem. As our goal now is to discuss L-functions in greater detail, we will turn to a generalization of them, namely Mellin transforms. They are not only a natural extension of Dirichlet series, but also allow to write many results in a cleaner and more elegant way.

We will define the Mellin transform on a specific class of functions:

Definition. *Let $\mathcal{V}$ be the set of complex valued functions on $\mathbb{R}$, such that for all $F \in \mathcal{V}$*

(1) $F(x) = 0$ for $x < 1$,

(2) F is continuous from the right and

(3) F is locally of bounded variation.

Recall that the *variation* of a function $F : \mathbb{R} \to \mathbb{R}$ in some compact interval $[a, b]$ is defined as

$$\sup_{P \in \mathcal{P}} \sum_{i=0}^{n_P - 1} \left| F(x_{i+1}) - F(x_i) \right|$$

where $\mathcal{P}$ is the set of all partitions $P = (x_0, ..., x_{n_P})$ of the interval $[a, b]$, i.e. $a = x_0 \leq ... \leq x_{n_P} = b$. Further, F is said to be *locally of bounded variation*, if its variation is bounded on every compact interval. We will mostly consider functions for which it can be easily checked that they are locally of bounded variation.
Finally, for a function $F : \mathbb{R} \to \mathbb{R}$ which satisfies $F(x) = 0$ for all $x < 1$ we define its **total variation** F_v by $F_v(x) = 0$ for $x < 1$ and otherwise as the variation of F in the interval $[1^-, x]$.

Definition. *For $F \in \mathcal{V}$ consider the improper Riemann-Stieltjes integral*

$$\widehat{F}(s) = \int x^{-s} \mathrm{d}F(x) := \lim_{X \to \infty} \int_{1^-}^{X} x^{-s} \mathrm{d}F(x).$$

This is called the Mellin transform of F. We say that $\widehat{F}$ converges absolutely at s, if the integral $\int x^{-s}\mathrm{d}F(x)$ does so, or put differently if

$$\int x^{-\sigma}\mathrm{d}F_v(x)$$

converges, where $\sigma = \mathrm{Re}(s)$ and F_v is the total variation of F. Also, define the abscissa of absolute convergence $\sigma_a = \sigma_a(\widehat{F}) := \inf\{\sigma : \int x^{-\sigma}\mathrm{d}F_v(x) < \infty\}$. Similarly, the abscissa of convergence $\sigma_c = \sigma_c(\widehat{F})$ is defined to be the infimum over all σ, such that $\widehat{F}(s)$ converges for some $s \in \mathbb{C}$ with $\mathrm{Re}(s) = \sigma$.

Remark 4.1.1. *In the literature, the Mellin transform is usually defined by $\int x^{s-1}\mathrm{d}F(x)$, i.e. they replace s by $-s+1$. Any result can be changed back by reversing the substitution on s. We will use our way since it is far better in line with results of Dirichlet series, as also used in [BD04]. Specifically, we use this notation in order to get the following nice-looking proposition.*

Proposition 4.1.2. *Let f be an arithmetic function and let F be the corresponding summatory function*

$$F(x) := \sum_{n \le x} f(n).$$

Then the Dirichlet series of f is the Mellin transform of F, i.e.

$$L_f(s) = \widehat{F}(s).$$

Proof. Note that F is a right-continuous piecewise constant function that has jumps of size $f(n)$ when n is a positive integer. In the integral $\int x^{-s}\mathrm{d}F(x)$ these jumps get multiplied with precisely n^{-s}, resulting in $\sum f(n)/n^s$.

□

Example. We now give some examples of Mellin transforms:

- Define $N(x)$ to be the summatory function of the constant one function, i.e. $N(x) = \sum_{n \le x} 1 = \lfloor x \rfloor$. We already encountered its Mellin transform, which is

$$\int x^{-s}\mathrm{d}N(x) = \sum_{n \ge 1} n^{-s} = \zeta(s).$$

- If we replace $N(x)$ with $id(x) := x$ we have

$$\int x^{-s}\mathrm{d}id(x) = \int x^{-s}\mathrm{d}x = \frac{1}{s-1}.$$

  Note that on the one hand we have $N(x) \sim id(x)$ and on the other hand the two Mellin transforms $\zeta(s)$ and $\frac{1}{s-1}$ share the same singularities on $\{s : \mathrm{Re}(s) \ge 1\}$.

- As a final example, the L-series corresponding to the von Mangoldt function Λ can be written as

$$L_\Lambda(s) = \int x^{-s}\mathrm{d}\psi(x) = \widehat{\psi}(s).$$

All the above mentioned functions are meromorphic as we have shown. More general, the following theorem holds:

Theorem 4.1.3. *Let F be a polynomially bounded function. Then $\widehat{F}(s)$ converges and is analytic on the open half-plane $\{s : \mathrm{Re}(s) > \sigma_c(\widehat{F})\}$.*

Later we will prove the first part of this theorem, i.e. the convergence statement. We will omit the proof of analyticity and refer the interested reader to [BD04, p. 125].

4.2 Extending Mellin transforms

In the two proofs of the Prime Number Theorem we needed that L_Λ can be extended to a meromorphic function including the region $\{s : \mathrm{Re}(s) \geq 1\}$. The following theorem gives a condition under which Mellin transforms can be extended beyond their initial domain of convergence as a meromorphic function.

Theorem 4.2.1. *Let $F \in \mathcal{V}$. Further let F be approximated by*

$$F(x) = \sum_{j=1}^{r} \sum_{k=1}^{m_j} c_{jk} x^{s_j} \log^{k-1}(x) + E(x),$$

where $s_1, ..., s_r \in \mathbb{C}$, $m_1, ..., m_r \in \mathbb{Z}^+$, $c_{jk} \in \mathbb{C}$ and E an error term that is bounded by $E(x) = O(x^{\theta+\epsilon})$, for a fixed θ and every positive ϵ, where $\theta < \mathrm{Re}(s_1), ..., \mathrm{Re}(s_r)$.
Then $\widehat{F}$ has a meromorphic continuation to the half-plane $\{s : \mathrm{Re}(s) > \theta\}$, given by the formula

$$\widehat{F}(s) = s \sum_{j=1}^{r} \sum_{k=1}^{m_j} c_{jk} \frac{(k-1)!}{(s-s_j)^k} + s \int_1^\infty E(x) x^{-s-1} \mathrm{d}x.$$

Proof. We again refer the reader to [BD04, p. 127] for the proof. □

Alternatively speaking, if $F(x)$ is the linear combination of terms of the form $cx^a \log^b(x)$ and some error term which is $O(x^{\theta+\epsilon})$, then $\widehat{F}(s)$ can be extended and is well-understood on $\{s : \mathrm{Re}(s) > \theta\}$. This also means that it can be beneficial, if one can approximate some function F with a “comparison function” of the described type.

Example. We already mentioned the two functions $id(x) = x$ and $N(x) = \lfloor x \rfloor$ with their corresponding Mellin transforms $\frac{1}{s-1}$ and $\zeta(s)$ respectively. As both $id(x)$ and $N(x)$ are of linear growth, their Mellin transforms converge in the half-plane $\{s : \mathrm{Re}(s) > 1\}$ as we will show in Theorem 4.3.1.
The Mellin transform of N is $\widehat{N}(s) = \zeta(s)$, which is in the first place only defined for $\{s : \mathrm{Re}(s) > 1\}$. However, finding or even proving that an extension exists is harder. Here one can apply the previous theorem: As $N(x) = x + O(1)$, Theorem 4.2.1 shows the existence of a continuation of the zeta function to $\{s : \mathrm{Re}(s) > 0\}$, with the same singularities as $\widehat{id}$ in this region.

In general, say we are given two functions F and G, where it is easy to describe the poles of $\widehat{G}$ and easy to show that a meromorphic extensions beyond its the line of convergence exist. Additionally, let the two functions be of the same growth, i.e.

their difference is some relatively small error term. Then $\widehat{F}$ can often be extended by $\widehat{F} = \widehat{F - G} + \widehat{G}$. Here the previous theorem gives useful condition for the existence of such extensions.

4.3 Two important theorems

In this section, we will prove two main theorems. Both relate the set of points where a Mellin transform $\widehat{F}$ converges with the growth of F. They will build the foundation for understanding the duality between growth of F and the positions and type of poles of $\widehat{F}$.

In Corollary 1.3.2 we proved that for a multiplicative arithmetic function g which is bounded by $g(n) = O(n^c)$, its L-series converges on $\{s : \mathrm{Re}(s) > c + 1\}$. This result can also be extended to Mellin-transforms. Additionally, a slightly weaker converse holds true as well:

Theorem 4.3.1. *Let $F \in \mathcal{V}$ and suppose that $F(x) = O(x^c)$ for some $c \geq 0$. Then $\int x^{-s} \mathrm{d}F(x)$ converges when $\mathrm{Re}(s) > c$. Conversely, if $\int x^{-s} \mathrm{d}F(x)$ converges for some s with $\mathrm{Re}(s) = \sigma > 0$, then $F(x) = o(x^\sigma)$.*

Proof. First assume $F(x) = O(x^c)$. Then for $s \in \mathbb{C}$, $\mathrm{Re}(s) > c$ partial integration yields

$$\lim_{X\to\infty} \int_{1^-}^{X} x^{-s} \mathrm{d}F(x) = \lim_{X\to\infty} \left(x^{-s} F(x) \Big|_{1^-}^{X} + s \int_{1^-}^{X} x^{-s-1} F(x) \mathrm{d}x \right).$$

The limit of both terms on the right exists, since $F(x) = O(x^c)$ and $\mathrm{Re}(s) > c$.

Similar, the proof of the second part is rather short. Assume that $\int x^{-s} \mathrm{d}F(x)$ converges for some fixed s with $\mathrm{Re}(s) > 0$. We now shorten the domain of integration from $[1^-, \infty)$ to $[y, \infty)$ and define

$$\varphi(y) := -\int_{y}^{\infty} x^{-s} \mathrm{d}F(x) = o(1).$$

Since $dF(y) = y^s d\varphi(y)$ one can express F in terms of φ as

$$F(x) = \int_{1^-}^{x} \mathrm{d}F = \int_{1^-}^{x} y^s \mathrm{d}\varphi(y).$$

Applying partial integration to the last integral yields further

$$y^s \varphi(y)\big|_{1^-}^{x} - \int_{1}^{x} s y^{s-1} \varphi(y) \mathrm{d}y = x^s \varphi(x) + \int_{1^-}^{\infty} y^{-s} \mathrm{d}F(y) - s \int_{1}^{x} y^{s-1} \varphi(y) \mathrm{d}y.$$

The first and third summand are indeed of order $o(x^\sigma)$ while the second is constant. Hence we get $F(x) = o(x^\sigma)$ as claimed.

□

This theorem, even though it might not seem impressive at first, has far-reaching consequences. For instance, we mentioned in the beginning of Chapter 3 that the Prime Number Theorem can be deduced from the fact that the infinite sum

$$\sum_{n \geq 1} \frac{\Lambda(n) - 1}{n}$$

converges. We have now the tools for proving this claim:
If the above sum converges, then the L-series

$$L_f(s) = \sum_{n \geq 1} \frac{\Lambda(n) - 1}{n^s} = L_\Lambda(s) - \zeta(s)$$

converges at $s = 1$, where $f(n) = \Lambda(n) - 1$. We can apply Theorem 4.3.1 and get $\sum_{n \leq x} f(n) = F(x) = \psi(x) - N(x) = o(x)$ where $N(x) = \lfloor x \rfloor$. Rearranging gives then $\psi(x) = x + o(x)$, implying the Prime Number Theorem. In fact, analogously one can show that the Prime Number Theorem is implied by the convergence of $L_f(s)$ for any s with $\mathrm{Re}(s) = 1$.

Theorem 4.3.1 can also be used to address the general shape of the set of points for which a Mellin transform converges.

Corollary 4.3.2 (Points of convergence). *Let $F \in \mathcal{V}$ and $s_0 = \sigma_0 + it_0$, $\sigma_0 \geq 1$ be some complex number, such that the Mellin transform $\int x^{-s_0} \mathrm{d}F(x)$ converges. Then it also converges on the open half-plane $\{s : \mathrm{Re}(s) > s_0\}$.*
*As a consequence, for any given Mellin transform, the set of points for which it converges is some open half-plane $\{s : \mathrm{Re}(s) > r\}$ and possibly some points on the line $\sigma = r$. We call the set $\{s : \mathrm{Re}(s) = r\}$ the **line of convergence**.*

Proof. This is a direct consequence of the two parts of Theorem 4.3.1: First, we know from the second part that $F(x) = o(x^{\sigma_0})$. Hence we can apply the first part of the theorem which proves that $\int x^{-s} \mathrm{d}F(x)$ converges for $\{s : \mathrm{Re}(s) > \sigma_0\}$. □

Remark 4.3.3. *The condition $\sigma_0 \geq 1$ in Corollary 4.3.2 can actually be dropped. See for instance [BD04, p. 118].*

This result on the points of convergence of Mellin-transforms seems similar to a corresponding theorem for power series. For the latter one considers the circle of convergence and it is known that at some point on its radius of convergence the series diverges. However, the analogue cannot be said about Mellin transforms in general:
Take for example the arithmetic function $f(n) = (-1)^{n+1}$ and its corresponding L-series given by $L_f(s) = \sum_{n \geq 1} (-1)^{n+1}/n^s$. This series converges when $s \in \mathbb{R}^+$ by the Leibniz criterion and hence by Theorem 4.3.1 for $\{s \in \mathbb{C}, \mathrm{Re}(s) > 0\}$. However, it is obviously not convergent when $\mathrm{Re}(s) \leq 0$. Hence its line of convergence is

the imaginary axis. Still, it can be written as

$$\begin{aligned}L_f(s) &= \sum_{n\geq 1}\frac{(-1)^{n+1}}{n^s} = \sum_{n\geq 1}\frac{1}{n^s} - 2\cdot\sum_{n\geq 1}\frac{1}{(2n)^s}\\ &= \zeta(s) - 2\zeta(s)\frac{1}{2^s} = \zeta(s)\left(1-\frac{1}{2^{s-1}}\right).\end{aligned}$$

Recall that $\zeta(s)$ is holomorphic on $\mathbb{C}$ except for a simple pole at $s=1$. This pole is perfectly cancelled by the simple zero of $(1-2^{1-s})$ at $s=1$, which means that $L_f(s)$ can be extended to an analytic function on $\mathbb{C}$. Hence $L_f(s)$ does not have a pole on its line of convergence.

We want to finish this section with the following theorem:

Theorem 4.3.4. *Let $F \in \mathcal{V}$ with $F(x) \sim x$. Theorem 4.3.1 already showed that $\widehat{F}(s)$ converges when* $\mathrm{Re}(s) > 1$. *If furthermore $\widehat{F}$ has a meromorphic extension to a region containing $\{s : \mathrm{Re}(s) \geq 1\}$, then $\widehat{F}$ has no pole in the half-plane $\{s : \mathrm{Re}(s) \geq 1\}$, except for a simple pole at $s = 1$ with residue 1.*

Proof. Integration by parts and using $F(x) = x\big(1 + o(1)\big)$ yields

$$\widehat{F}(s) = s\int x^{-s}\big(1+o(1)\big)\mathrm{d}x = 1 + \frac{1}{s-1} + |s|o\left(\frac{1}{\sigma-1}\right) \tag{4.1}$$

uniformly on the half-plane $\mathrm{Re}(s) > 1$ as $\sigma \to 1^+$. We now consider the function

$$\tilde{F}(s) := \widehat{F}(s) - \frac{1}{s-1} = o\left(\frac{1}{\sigma-1}\right). \tag{4.2}$$

Fix some $t \in \mathbb{R}$ and consider the behaviour of $\tilde{F}$ at the point $1+it$. If $\tilde{F}$ had a pole at $1+it$, it would have "order less than 1" by (4.2). This is not possible considering the Laurent expansion at the point $1+it$: at this point the expansion is of the form

$$\tilde{F}(s) = \sum_{n\geq N} c_n(s-1-it)^n$$

for some $N \in \mathbb{Z}$ and $c_N \neq 0$. In addition, c_N can be computed by

$$c_N = \lim_{\sigma\to 1^+}\tilde{F}(\sigma+it)(\sigma-1)^{-N}.$$

This last limit equals zero when $N < 0$ since $\tilde{F}(\sigma+it) = o(\frac{1}{\sigma-1})$, hence $c_N = 0$ which is a not possible. As a result, we have $N \geq 0$, i.e. $\tilde{F}$ does not have a pole at $1+it$. Consequently, $\widehat{F}$ has a simple pole at $s=1$ with residue 1 and is otherwise analytic when $\mathrm{Re}(s) \geq 1$. This is yet another wonderful proof due to the miracles of complex analysis.

□

Corollary 4.3.5. *Let F be as described in the above theorem, with the condition $F(x) \sim x$ replaced by $F(x) = o(x)$. Then $\widehat{F}$ does not have poles in the closed half-plane $\{s : \mathrm{Re}(s) \geq 1\}$.*

Proof. The proof is analogue to the previous one. □

4.4 Understanding poles of L-functions

The previous two theorems gave us a foundation for understanding the connection between L-functions and asymptotic behaviour of summatory functions. Hence we are ready to tackle some more elaborate problems. Namely we will answer the following 3 questions:

(1) What can be said when L_f has poles of higher order or which are not at $s = 1$.

(2) Are there necessary and sufficient conditions for a L-function L_f that imply that $F(x) \sim x$, especially in connection to the Prime Number Theorem.

(3) What can be said about F, when L_f has poles on its line of convergence. Can one still establish bounds on the growth of F?

4.4.1 Poles of higher order

Previously we focused on L-functions with poles on the line $\{s : \mathrm{Re}(s) = 1\}$ and of order 1. Here we will discuss briefly what can be said if those conditions are not met. The following theorem extends Corollary 3.5.1 of the Wiener-Ikehara Theorem, which directly implied the Prime Number Theorem. If a L-function has only one pole on its line of convergence, which additionally lies on the real axis, then the theorem describes how the growth of F is influenced by the real part of the position of the pole, the residue and the order.

Theorem 4.4.1. *Let $F \in \mathcal{V}$ be a real valued monotone increasing function with $\sigma_c(\widehat{F}) = \alpha > 0$. For s with $\mathrm{Re}(s) > \alpha$, let its Mellin transform be expressible in the form*

$$\int x^{-s} \mathrm{d}F(x) = (s-\alpha)^{-\gamma} g(s) + h(s)$$

where g, h are analytic on the closed half-plane $\{s : \mathrm{Re}(s) \geq \alpha\}$, $g(\alpha) \neq 0$ and γ a positive real number. Then

$$F(x) \sim x^{\alpha} \big(\log(x)\big)^{\gamma-1} \frac{g(\alpha)}{\alpha \Gamma(\gamma)}$$

where Γ denotes the Euler gamma function.

Proof. The proof can be found in [BD04, p. 154]. □

Example. Define the divisor function τ by mapping each positive integer to its number of positive divisors, i.e. $\tau(n) := \sum_{d|n} 1$. As it turns out, for s with real part greater than 1 we have

$$\zeta^2(s) = \Big(\sum_{n\geq 1} \frac{1}{n^s}\Big)^2 = \sum_{a,b\geq 1} \frac{1}{(ab)^s} = \sum_{m\geq 1} \frac{\tau(m)}{m^s} = L_\tau(s).$$

This identity not only links $L_\tau(s)$ with the zeta function but also allows us to extend it on and to the left of the line $\{s : \mathrm{Re}(s) = 1\}$ by $\zeta^2(s)$.
We already know that ζ has precisely one pole, namely at $s = 1$ with residue 1 and of first order. Hence

$$\zeta(s) = \frac{1}{s-1} + h(s)$$

where $h(s)$ is an analytic function. Squaring this equation yields

$$L_\tau(s) = \frac{1}{(s-1)^2}\big(1 + 2(s-1)h(s)\big) + h^2(s).$$

This allows us to apply the above theorem to get

$$F(x) = \sum_{n \le x} \tau(n) \sim x \log(x).$$

4.4.2 Necessary and sufficient conditions for the Prime Number Theorem

In this section we examine the two proofs of the Prime Number Theorem again. We will try to find out which properties of L_Λ where needed and if they could be dropped in general. This will lead to a result which describes the necessary and sufficient conditions completely.

As mentioned several times, both proofs of the Prime Number Theorem used very little information about $\widehat{\psi} = L_\Lambda$ to deduce that $\psi(x) \sim x$. In fact, both approaches used 3 conditions for the continuation of L_Λ:

(1) L_Λ converges on $\{s : \mathrm{Re}(s) > 1\}$ and has a simple pole at $s = 1$ with residue 1.

(2) It has no other pole on its line of convergence $\{s : \mathrm{Re}(s) = 1\}$.

(3) The arithmetic function Λ is non-negative, hence ψ is monotone increasing.

Additionally, the proof using complex analysis needed some growth conditions of $L_\Lambda(s)$ near and on the line $\{s : \mathrm{Re}(s) = 1\}$ in order to shift the path of integration of a specific integral.

If we focus on non-negative arithmetic functions, i.e. take the third point for granted, then the first 2 conditions L_f not only imply that $\sum_{n \le x} f(n) \sim x$ by Corollary 3.5.1, but rather are equivalent to it:
If $F(x) \sim x$, applying Theorem 4.3.4 shows the existence of a simple pole with residue 1 at $s = 1$ and proves that we cannot have another pole in the closed half-plane $\{s : \mathrm{Re}(s) \ge 1\}$. Hence, we have proven

Theorem 4.4.2. *Let f be a non-negative arithmetic function and F its summatory function. Let $L_f(s)$ converge for* $\mathrm{Re}(s) > 1$ *and admit a meromorphic continuation to a region containing* $\mathrm{Re}(s) \ge 1$. *Then the following are equivalent:*
(1) $F(x) \sim x$.
(2) $L_f(s)$ has a pole of order 1 *at $s = 1$ with residue* 1 *and no other poles when* $\mathrm{Re}(s) \ge 1$.

An example, where $L_f(s)$ has a pole at $s = 1$, converges when $\mathrm{Re}(s) > 1$ but does have additional poles at $\mathrm{Re}(s) = 1$ is the following:

Example. Consider the arithmetic function

$$f(n) = \begin{cases} 2^k & \text{if } n = 2^k, k \geq 0, \\ 0 & \text{else.} \end{cases}$$

As $F(x) = 1 + 2 + 4 + \cdots + 2^{\lfloor \log_2(x) \rfloor} \leq 2 \cdot 2^{\lfloor \log_2(x) \rfloor} < 2x$ and therefore $F(x) = O(x)$, we know that $L_f(s)$ converges when $\mathrm{Re}(s) > 1$. Further, it can be expressed as a geometric series and thereby be extended to a meromorphic function on $\mathbb{C}$ as

$$L_f(s) = \sum_{k \geq 0} \frac{2^k}{(2^k)^s} = \sum_{k \geq 0} 2^{k(1-s)} = \frac{1}{1 - 2^{1-s}}.$$

This function has poles precisely when

$$\begin{aligned} & & 2^{1-s} = e^{\log(2)(1-s)} &= 1 & \\ \Leftrightarrow & & \log(2)(1-s) &= n \cdot (2\pi i) & \text{for some } n \in \mathbb{Z} \\ \Leftrightarrow & & s &= 1 - \frac{2n\pi i}{\log(2)} & \text{for some } n \in \mathbb{Z} \end{aligned}$$

i.e. along the line $\mathrm{Re}(s) = 1$ in regular intervals of length $2\pi/\log(2)$. At these points the function has simple poles, each of which with residue $1/\log(2) \approx 1.44$, as can be computed by L'Hopital's rule.
Hence, by Theorem 4.4.2 we do not have $F(x) \sim \frac{x}{\log(2)}$. This can also be checked easily, since for instance at $x = 2^k - 1$ we have

$$F(x) = 1 + 2 + ... + 2^{k-1} = 2^k - 1 = x,$$

therefore $\liminf_{x \to \infty} \frac{F(x)}{x} \leq 1 < \frac{1}{\log(2)}$.

Theorem 4.4.2 already gives us a very clear picture of the connection between poles of L-functions and asymptotic growth of the summatory function of f. It also explains why our two proofs had to use these conditions for L_Λ and therefore the non-existence of zeros on the line with real part 1 of the zeta function. Indeed, all subsequent proofs of the Prime Number Theorem use this fact. However, the one exception is the famous "elementary" proof by Selberg from 1949, which circumvents using L-series in general.

4.4.3 Poles on the line of convergence

We now know for f a non-negative arithmetic function that L_f having no poles on $\{s : \mathrm{Re}(s) \geq 1\}$ except for a simple pole at $s = 1$ is equivalent to $F(x) \sim x$. Now we turn to the question, which condition on F lead to poles on its line of convergence.

Consider the function $F(x) = x^\alpha \sin(\beta \log(x))$ with $\alpha, \beta \in \mathbb{R}^+$. This function oscillates with increasingly larger period and amplitude. Since $F(x) = O(x^\alpha)$, we know that its Mellin transform converges on $\{s : \mathrm{Re}(s) > \alpha\}$. Notice that $F(1) = 0$, hence its Mellin transform can be computed when $\mathrm{Re}(s) > \alpha$ as

$$
\begin{aligned}
\int_{1^-}^{\infty} x^{-s}\mathrm{d}F(x) &= \int_1^{\infty} x^{-s}F'(x)\mathrm{d}x \\
&= \underbrace{x^{-s}F(x)\Big|_1^{\infty}}_{=0} + s\int_1^{\infty} x^{-s-1}F(x)\mathrm{d}x \\
&= s\int_1^{\infty} x^{-s-1+\alpha}\sin\big(\beta\log(x)\big)\mathrm{d}x
\end{aligned}
$$

where we used the fact that $F(x) = O(x^{\alpha})$. Using the identity $\sin(y) = \frac{1}{2i}(e^{iy} - e^{-iy})$ one further obtains

$$
\begin{aligned}
& s\int_1^{\infty} x^{-s-1+\alpha}\frac{1}{2i}\big(e^{i\beta\log(x)} - e^{-i\beta\log(x)}\big)\mathrm{d}x \\
=& \frac{s}{2i}\int_1^{\infty} x^{-s-1+\alpha}\big(x^{i\beta} - x^{-i\beta}\big)\mathrm{d}x \\
=& \frac{-s}{2i}\Big(\frac{1}{-s+\alpha+i\beta} - \frac{1}{-s+\alpha-i\beta}\Big) \\
=& \frac{s\beta}{(s-\alpha)^2+\beta^2}.
\end{aligned}
$$

This function has poles precisely at $s = \alpha \pm i\beta$. Therefore, poles which do not lie on the real line encapsulate a certain oscillation of the summatory function. In detail, the imaginary part β tells us how fast the function is oscillating. Similarly, as $L_{f+g} = L_f + L_g$, if F is the sum of such oscillations, the corresponding L-function has multiple poles reflecting the individual frequencies.

There are results, which bound the behaviour of F based on the non-existence of poles only on the finite line segment $\{1 + it : t \in [-B, B]\}$. If the next theorem is applied to L-functions, it gives a quantitative statement, which allows to describe the growth of F, when its corresponding Mellin transform has poles with non-zero imaginary part.

Theorem 4.4.3. *Let $\varphi : \mathbb{R}^+ \to \mathbb{R}$ be a bounded measurable function. Further let its Laplace transform*

$$
g(s) := \int_0^{\infty} \varphi(t)e^{-st}\mathrm{d}t
$$

be well-defined and analytic on the open half-plane $\{s : \mathrm{Re}(s) > 0\}$, admitting a continuous extension to the domain $\{1 + it : t \in [-B, B]\}$. Then

$$
\limsup_{T\to\infty}\left|\int_0^{T} \varphi(t)\mathrm{d}t - g(0)\right| \le \frac{2M}{B},
$$

where $M = \sup_{t>0}\varphi(t)$.

Proof. The proof can be found in [Kor03]. □

Remark 4.4.4. *Theorem 3.1.3 and therefore also the Wiener-Ikehara Theorem can be proven as a corollary of the above theorem.*

Let us now apply this theorem to L-functions. Consider some arithmetic function f and assume that its corresponding L-function has a simple pole with residue 1 at $s = 1$ and no further pole on $\{1 + it : t \in [-B, B]\}$. Then consider

$$\varphi(t) = \frac{F(e^t) - \lfloor e^t \rfloor}{e^t},$$

where F is once again the summatory function of f. One gets

$$\begin{aligned} g(s) :=& \int_0^\infty \varphi(t) e^{-st} \mathrm{d}t = \int_0^\infty \big(F(e^t) - \lfloor e^t \rfloor\big) e^{-(s+1)t} \mathrm{d}t \\ =& \int_0^\infty \sum_{n \le e^t} \big(f(n) - 1\big) e^{-(s+1)t} \mathrm{d}t = \sum_{n \ge 1} \big(f(n) - 1\big) \int_{\log(n)}^\infty e^{-(s+1)t} \mathrm{d}t \\ =& \frac{1}{s+1} \sum_{n \ge 1} \frac{f(n) - 1}{n^{s+1}} = \frac{1}{s+1} \big(L_f(s+1) - \zeta(s+1)\big). \end{aligned}$$

If we additionally assume that φ is bounded, we can apply Theorem 4.4.3. Note that in the case of $f \equiv \Lambda$, we can in fact bound φ by the methods described in Theorem 1.1.2 and hence find an upper bound M. Finally, the expression in the statement of the theorem can be evaluated to

$$\begin{aligned} \int_0^T \varphi(t) \mathrm{d}t - g(0) &= \int_0^T \frac{F(e^t) - \lfloor e^t \rfloor}{e^t} \mathrm{d}t - g(0) \\ &= \int_0^T \frac{F(e^t)}{e^t} \mathrm{d}t - T + \int_0^T \frac{\{e^t\}}{e^t} \mathrm{d}t - g(0) \\ &= \int_0^T \frac{F(e^t)}{e^t} \mathrm{d}t - T + c - r(T) - g(0) \end{aligned}$$

where $c := \int_0^\infty \frac{\{e^t\}}{e^t} \mathrm{d}t < \infty$ and

$$r(T) = \int_T^\infty \frac{\{e^t\}}{e^t} \mathrm{d}t < \int_T^\infty \frac{1}{e^t} \mathrm{d}t = e^{-T}.$$

Hence, Theorem 4.4.3 says

$$\limsup_{T \to \infty} \left| \int_0^T \frac{F(e^t)}{e^t} \mathrm{d}t - T + c - r(T) - g(0) \right| \le \frac{2M}{B}.$$

If one replaces T by S and takes differences, one arrives at

$$\limsup_{S,T \to \infty} \left| \int_S^T \frac{F(e^t)}{e^t} \mathrm{d}t - (T - S) \right| \le \frac{4M}{B}. \tag{4.3}$$

As a result, we get $\int_S^T F(e^t)/e^t \mathrm{d}t \approx T - S$, where the error term can be made explicit in terms of M and B. Hence the theorem implies that $F(e^t)/e^t$ is on average "close" to 1.

4.5 Comparison of the two proofs

Now with all the additional knowledge and the understanding of the connection between poles of L_Λ and the asymptotic of ψ and hence π, we will compare the two proofs of the Prime Number Theorem. For instance we now know that the lack of poles on the line $\{s : \mathrm{Re}(s) = 1\}$ is the key ingredient needed.
Recall that we have proven in Theorem 4.3.4 that if $F(x) \sim x$, then $\widehat{F}$ does not have poles in the closed half-plane $\{s : \mathrm{Re}(s) \geq 1\}$ except for a simple pole at $s = 1$. This fact was remarkably easy to show. The two proofs basically show the converse, which is much harder to establish.
As we will see, there are major parallels between the proofs. In fact, nearly all procedures of one have a counterpart in the other and even mostly in the same order. Hence, we will introduce the individual steps, followed by how they are executed as well as a short explanation how they stand in relation to our understanding of L-functions.

The 5 steps common in both proofs are:

- **Step 1: Pick some identity involving L_Λ and ψ.**

The analytic proof uses directly

$$L_\Lambda(s) = \sum_{n\geq 1} \frac{\Lambda(n)}{n^s} = \int x^{-s}\mathrm{d}\psi(x)$$

while the other starts with

$$\frac{L_\Lambda(s)}{s} = \int_0^\infty \varphi(u)e^{-su}\mathrm{d}u$$

where $\varphi(u) = \psi(e^u) = \sum_{n\leq e^u} \Lambda(n)$.
As already elaborated, both identities involve a weighted sum of values of Λ, where the weights decay rather slowly when s is near 1. On the contrary, we are after $\psi(x)$, which assigns the weight 1 only to the first $\lfloor x \rfloor$ terms, i.e, $\Lambda(1), ..., \Lambda(\lfloor x \rfloor)$. Hence, the task is to make this transition of weighting.
At this point, the two proofs differ slightly in the order of procedure. Hence, we will split Step 2 in two parts:

- **Step 2.1: "Subtract" the pole at $s = 1$, i.e. pull the pole out of the sum/integral**.
- **Step 2.2: Introduce a convolution with a kernel. Then change the order of summation/integration.** Justify this change by absolute convergence, as the real part of the complex variable is strictly greater than 1. Finally use properties of the kernel to get an expression, where one side is a weighted (purely real) expression of the values of Λ and the other is a complex integral.

  Here a variable x will be introduced which controls how the weight is distributed.

2.1 in Tauberian: In the Tauberian proof the steps are executed in the order listed. The representation one starts with makes it very easy to subtract the pole by

$$F(s) := \int_0^\infty \left(\varphi(u)e^{-u} - 1\right) e^{-(s-1)u}\mathrm{d}u = \frac{1}{s}L_\Lambda(s) - \frac{1}{s-1}.$$

2.2 in Tauberian: After this step, one introduces the kernel K_λ and considers

$$\int_0^\infty \left(\varphi(u)e^{-u} - 1\right) e^{-(s-1)u} K_\lambda(u-x)\mathrm{d}u.$$

The weighting of the values of Λ is now controlled by x and λ. First, the function $u \mapsto K_\lambda(u-x)$ is maximized at $u = x$. This means that $\varphi(x)$ will have the highest weight, i.e. the sum $\sum_{n \le e^x} \Lambda(n)$. The decay of $K_\lambda(u-x)$ is controlled by λ.
After rewriting the Fejér kernel as a complex integral and changing the order of integration, one arrives at the miraculous formula

$$\int_0^\infty \left(\varphi(u)e^{-u} - 1\right) e^{-\epsilon u} K_\lambda(u-x)\mathrm{d}u = \frac{\lambda}{2}\int_{-2}^{2}\left(1 - \frac{|t|}{2}\right) e^{i\lambda x} F(1+\epsilon - i\lambda t)\mathrm{d}t. \quad (4.4)$$

Note that the argument of F has real part $1 + \epsilon > 1$.
Also, the higher λ is, the more precise the weighting of $\varphi(u)$ on the left-hand side of (4.4) is. On the other hand, this corresponds to taking the integral of F on a longer line segment. This is reflected in the bound (4.3) established in Chapter 4.4.3.

2.2 in Analytic: The two parts are performed in reverse order in the analytic proof. First, the kernel

$$\frac{x^s}{s(s+1)}$$

is introduced for some $x \in \mathbb{R}^+$. Now integration along the path from $c - i\infty$ to $c + i\infty$ with $c > 1$ yields

$$\frac{1}{x}\sum_{n \le x}\left(1 - \frac{n}{x}\right)\Lambda(n) = \frac{\psi_1(x)}{x^2} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^{s-1}}{s(s+1)}\mathrm{d}s.$$

Here again switching the order of summation and integration is justified since $\mathrm{Re}(s) = c > 1$. The left-hand side already consists of only a partial sum of the values of Λ, but not all with the same weight. Again x controls which values of Λ are weighted.

2.1 in Analytic: Then the pole is subtracted from the integral by again using Perron's formula to arrive at

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{s-1}h(s)\mathrm{d}s = \frac{\psi_1(x)}{x^2} - \frac{1}{2}\left(1 - \frac{1}{x}\right)^2$$

where h is an analytic function.

One should note that both Perron's formula and the integral representation of the Feyér kernel can be derived from (inverse-)Mellin-transformation, adding one more similarity between the proofs.
The remaining steps are again very similar: One has to consider the real part of s (which is denoted by $1 + \epsilon$ in the Tauberian proof) tending to 1. The previous steps all took place when $\mathrm{Re}(s) > 1$, hence we were not majorly concerned with any convergence issues. This is about to change in

- **Step 3: Let $\mathrm{Re}(s)$ tend to 1.**

As we have shown, at some point we have to use the fact that L_Λ does not have additional poles on the line $\{s : \mathrm{Re}(s) = 1\}$ in order to rule out an oscillation of ψ as discussed in 4.4.3.
In the analytic proof, we are indeed able to set the real value of s to be equal to 1. This transition is heavily reliant on bounding the integrand and therefore L_Λ. After several transformations one gets to the desired expression for $x \geq 1$

$$\begin{aligned}\frac{\psi_1(x)}{x^2} - \frac{1}{2}\Big(1 - \frac{1}{x}\Big)^2 =& \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} x^{s-1}h(s)\mathrm{d}s = \\ =& \frac{1}{2\pi}\int_{-\infty}^{\infty} h(1+it)e^{it\log(x)}\mathrm{d}t\end{aligned}$$

where $h(s)$ is a continuous function on $\{s : \mathrm{Re}(s) \geq 1\}$. This is the part which actually demanded most of the work.
In the Tauberian proof, this step is similarly not as straightforward. It corresponds to taking the limit $\epsilon \to 0$ in (4.4). In contrast to the analytic proof, we are able to set $\epsilon = 0$ only on the right-hand side. Here one has to use the theorem of dominated convergence for binary functions.
Note that the reason, why establishing growth conditions in the Tauberian proof is not necessary, is that here only an integral over a finite line segment of L_Λ is considered. As a disadvantage, one does not get to work with the function ψ_1.

- **Step 4: Let x tend to infinity.**

In the analytic proof, this step is easy to do, as one can apply the Riemann-Lebesgue Lemma to the right-hand side of the above equation. This implies $\psi_1(x) \sim x^2/2$.
In the other proof, we get similarly by the Riemann-Lebesgue Lemma

$$\lim_{x\to\infty}\lim_{\epsilon\to 0}\int_0^\infty \big(\varphi(u)e^{-u} - 1\big)\, e^{-\epsilon u}K_\lambda(u-x)\mathrm{d}u = 0.$$

The -1 term can then be extracted to get

$$\lim_{x\to\infty}\int_0^\infty \varphi(u)e^{-u}K_\lambda(u-x)\mathrm{d}u = \rho. \tag{4.5}$$

Note that both expressions we ended up with in the two proofs after Step 4 are some weighted sum/integral of the von Mangoldt function. In order to get an estimate for the expression $\sum_{n\leq x}\Lambda(n)$ we use

- **Step 5: Translate the results to ψ.**

In Step 4 one proves the asymptotic behaviour of a weighted average of the values of Λ. In the analytic proof $\Lambda(n)$ has the weight $(1 - n/x)\cdot \mathbf{1}_{n\leq x}$ while the Tauberian proof yielded a similar weighted integral (4.5). The missing piece is Proposition 2.1.1 which proves that $\psi_1(x) \sim x^2/2$ implies $\psi(x) \sim x$ and the fact that $\varphi(u)/e^u \to 1$ is a consequence of (4.5). Note that in both proofs the emphasis lies on Λ being a non-negative function, as the two reasonings would be wrong without that assertion.

In conclusion, while the two proofs at first glance seem to be rather uncorrelated, closer inspection uncovers their similarities. In fact, nearly every step in one proof has a perfect counterpart in the other. The main difference simply seems to be their choice of a kernel, how it is incorporated with L_Λ and also the additional parameter λ. This results in the trade between the convenience of not having to establish bounds for L_Λ with the appealing and easy to interpret function ψ_1. Still, the execution seems to run in parallel.

4.6 Riemann-Hypothesis and error estimates

We now briefly mention how the error term in the relation $\psi(x) \sim x$ can be estimated. As it turns out, it depends directly on the position of poles of L_Λ. Recall that we proved in Theorem 1.4.2 that any poles of L in the strip $\{s : 0 < \mathrm{Re}(s) < 1\}$ are precisely at the zeros of the zeta function.
We begin with a first observation:

Proposition 4.6.1. *Let ρ be a zero of the zeta function with $c := \mathrm{Re}(\rho) \in (0,1)$. Then the relation*

$$\psi(x) - x = o(x^c)$$

does not hold.

Proof. First we note that $\psi(x)-x = o(x^c)$ if and only if $\psi(x^{1/c})-x^{1/c} = o(x)$. Hence, we define the function $F \in \mathcal{V}$ by

$$F(x) := \begin{cases} \psi(x^{1/c}) - x^{1/c} + 1 & \text{if } x \geq 1, \\ 0 & \text{else} \end{cases}$$

and one gets

$$\widehat{F}(s) = \int x^{-s}\big(d\psi(x^{1/c}) - d(x^{1/c})\big) = L_\Lambda(s \cdot c) - \frac{1}{sc-1}.$$

Assume now that $F(x) = o(x)$. Theorem 4.3.4 implies the non-existence of poles of $\widehat{F}$ in the region $\{s : \mathrm{Re}(s) \geq 1\}$, i.e. L_Λ has no poles on $\{s : \mathrm{Re}(s) \geq c\}$ and the zeta function has no zeros there, a contradiction to $\zeta(\rho) = 0$. □

As it turns out, the converse is also true. Specifically, one has [Ing90, p. 84]

Theorem 4.6.2. *Let $0 < c < 1$, then the following are equivalent:*

- *The Riemann zeta function has no zeros in the half-plane $\{s : \mathrm{Re}(s) > c\}$.*
- *The estimate*

  $$\psi(x) = x + O_\epsilon(x^{c+\epsilon})$$

  holds for every fixed $\epsilon > 0$.

This motivates studying the zeros of the zeta function in the critical strip. As can be shown, there are infinitely many such zeros on the vertical line $\{s : \mathrm{Re}(s) = 1/2\}$. This implies by the above theorem that the optimal bound on the error term is of the form

$$\psi(x) = x + O(x^{1/2+\epsilon}).$$

Remarkably, no other zeros were found with a different real part in the critical strip. In fact, the famous **Riemann Hypothesis** states that no such zeros exist. This conjecture is one of the most famous unsolved problems in mathematics.
Currently, the best-known results for zero-free regions are essentially of the form

$$\zeta(s+it) \neq 0 \textit{ when } s > 1 - \frac{c}{\log^{\alpha}(|t|+1)}$$

where $\alpha \in (0,1)$ [For02, pp. 565-566]. These are known to be equivalent to an error term of the form

$$\psi(x) = x + O(xe^{-d\log^{\beta}(x)}) \text{ where } \beta = \frac{1}{1+\alpha}$$

[Pin06, p. 191] Making the error term explicit in the two presented proofs also leads to a similar error term. Note that knowing that the zeta function is zero-free in a region as described in Theorem 4.6.2 would improve the error term immensely compared to the best known results.
Results on the growth of ψ can again be translated to the growth of π. As it turns out, instead of the usual $x/\ln(x)$ for the asymptotic growth, one should turn to the **offset logarithmic integral function** or **Eulerian logarithmic integral**

$$\mathrm{Li}(x) := \int_2^x \frac{1}{\ln(t)}\mathrm{d}t.$$

This function satisfies $\mathrm{Li}(x) \sim x/\ln(x)$ and is in fact a better approximation for $\pi(x)$. Hence, any precise error term estimation will use this main term. One reason, why $x/\ln(x)$ might not be the perfect leading term, can be found in the proof of Theorem 1.1.3. While the relation $\theta(x) \sim \psi(x)$ is sharp, in the sense that one has the bound $\psi(x) - \theta(x) = O\big(\sqrt{x}\log(x)\big)$, the relation between $\pi(x)$ and $\theta(x)/\log(x)$ is not, as we only showed $\pi(x) - \theta(x)/\log(x) = O\big(x/\log^2(x)\big)$. This can be improved to $O\big(x/\log^n(x)\big)$ for any $n \geq 1$ but not to an error term such as $O(x^{\alpha})$ for $\alpha \in (0,1)$ which would be needed. Such sharp estimates are only possible when using $\mathrm{Li}(x)$.
If β is an upper bound on the real part of every zero of the zeta function in the critical strip, it can be shown that

$$\pi(x) - \mathrm{Li}(x) = O\big(x^{\beta}\log(x)\big)$$

[Ing90, p. 83]. However, even though many attempts of proving the Riemann Hypothesis were made, a complete proof does not seem to be in sight.

Appendix A

In the previous chapters we used a few theorems whose proofs did not belong in the main thesis. They will be addressed here in the order of encountering to provide additional information.

A.1 Partial summation

In the first chapter, the following theorem proved to be a very useful tool.

Theorem (Partial summation)**.** *Let f be an arbitrary arithmetic function. Define for $x \geq 1$ the sum*

$$F(x) = \sum_{n \leq x} f(n).$$

Further, let $g : [1, \infty) \to \mathbb{R}$ be a differentiable function. Then for $x \geq 2$ the identity

$$\sum_{n \leq x} f(n)g(n) = F(x)g(x) - \int_1^x F(t)g'(t)\mathrm{d}t$$

holds.

Proof. Note that the left-hand side is a right-continuous step-function which has jumps when x is an integer. Therefore, it suffices to prove the identity when $x \in \mathbb{N}_{\geq 2}$ and to show that the right-hand side stays constant when x is in between two consecutive integers.

First, in the case $x = m \geq 2$ where m is an integer, the right-hand side can be evaluated as

$$\begin{aligned}
&F(m)g(m) - \int_1^m F(t)g'(t)\mathrm{d}t \\
=&F(m)g(m) - \sum_{k=1}^{m-1} \int_k^{k+1} F(t)g'(t)\mathrm{d}t \\
=&F(m)g(m) - \sum_{k=1}^{m-1} F(k)\left(g(k+1) - g(k)\right).
\end{aligned}$$

Collecting all the $g(k)$ terms yields further

$$\begin{aligned}
&g(1)F(1) + \sum_{k=2}^{m-1} g(k)\Big(F(k) - F(k-1)\Big) + g(m)\Big(F(m) - F(m-1)\Big) \\
=&g(1)F(1) + \sum_{k=2}^{m} g(k)f(k) = \sum_{k=1}^{m} f(k)g(k)
\end{aligned}$$

as was to be shown. Last, we have to prove that the right side stays constant, when replacing m by $m+r$ where $0 < r < 1$:

$$\begin{aligned}
&F(m+r)g(m+r) - \int_1^{m+r} F(t)g'(t)\mathrm{d}t \\
=&F(m)g(m+r) - \int_1^{m} F(t)g'(t)\mathrm{d}t - \int_m^{m+r} F(t)g'(t)\mathrm{d}t \\
=&F(m)g(m+r) - \int_1^{m} F(t)g'(t)\mathrm{d}t - F(m)\int_m^{m+r} g'(t)\mathrm{d}t \\
=&F(m)g(m) - \int_1^{m} F(t)g'(t)\mathrm{d}t
\end{aligned}$$

This finishes the proof.

□

A.2 The Riemann-Lebesgue Lemma

Used in both proofs of the Prime Number Theorem, the following result should be proved as well.

Lemma (Riemann-Lebesgue)**.** *Let $f : \mathbb{R} \to \mathbb{C}$ be an L^1-integrable function on $\mathbb{R}$, then*

$$\lim_{|n|\to\infty} \int_{-\infty}^{\infty} f(x)e^{nxi}\mathrm{d}x = 0.$$

That is, the Fourier transform of f vanishes at infinity.

Proof. First suppose that $\chi_{(a,b)}$ is the characteristic function of an open interval (a, b). Then we indeed have

$$\int_{-\infty}^{\infty} \chi_{(a,b)}(x)e^{nxi}\mathrm{d}x = \int_a^b e^{nxi}\mathrm{d}x = \frac{e^{ibn} - e^{ian}}{in} \to 0$$

as $|n| \to \infty$. Next, the statement is also true if f is a simple function, i.e. f is of the form

$$f = \sum_{k=1}^{N} c_k \chi_{a_k,b_k}$$

for $c_i \in \mathbb{C}, a_i < b_i \in \mathbb{R}$, because of the additivity of the integrals.

Finally, let $f \in L^1$ be arbitrary and fix some $\epsilon > 0$. Since simple functions are dense in L^1, there exists a simple function g such that $\int_{-\infty}^{\infty} |f(x) - g(x)| \mathrm{d}x < \epsilon$. Applying the triangle inequality to the equation

$$\int_{-\infty}^{\infty} f(x)e^{nxi}\mathrm{d}x = \int_{-\infty}^{\infty} \big(f(x) - g(x)\big)e^{nxi}\mathrm{d}x + \int_{-\infty}^{\infty} g(x)e^{nxi}\mathrm{d}x$$

yields

$$\left|\int_{-\infty}^{\infty} f(x)e^{nxi}\mathrm{d}x\right| \leq \int_{-\infty}^{\infty} \big|f(x) - g(x)\big|\mathrm{d}x + \left|\int_{-\infty}^{\infty} g(x)e^{nxi}\mathrm{d}x\right|$$

and hence

$$\limsup_{|n|\to\infty} \left|\int_{-\infty}^{\infty} f(x)e^{nxi}\mathrm{d}x\right| \leq \epsilon.$$

Recalling that ϵ can be chosen arbitrarily small proves the statement.

□

A.3 The weight of the Fejér kernel

In the proof of the Wiener-Ikehara Theorem, we used repeatedly the Fejér kernel $K_\lambda(x) = \lambda\left(\frac{\sin(\lambda x)}{\lambda x}\right)^2$. We showed that the integral over this kernel is independent of λ and denoted it by ρ. Its precise value was not of interest, but we would still like to add the following result:

Proposition A.3.1. *For any $\lambda > 0$ we have*

$$\int_{-\infty}^{\infty} K_\lambda(x)\mathrm{d}x = \int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2}\mathrm{d}x = \pi.$$

Proof. The first equation was already shown in Proposition 3.2.1. Regarding the second equation, partial integration links the integral to

$$\begin{aligned}\lim_{R\to\infty} \int_{-R}^{R} \frac{\sin^2(x)}{x^2}\mathrm{d}x &= \lim_{R\to\infty} \underbrace{-\frac{\sin(x)^2}{x}\Big|_{-R}^{R}}_{\to 0} + \int_{-R}^{R} \frac{\sin(2x)}{x}\mathrm{d}x \\ &= \lim_{R\to\infty} \int_{-R}^{R} \frac{\sin(y)}{y}\mathrm{d}y\end{aligned}$$

using $\big(\sin^2(x)\big)' = \sin(2x)$ and the substitution $y = 2x$.

In order to calculate the last integral, we will use the residue theorem. Specifically we will compute the integral of the complex function $f(z) = \frac{e^{iz}}{z}$ along the path given in the following figure. The arcs Γ_R and γ_ϵ are semicircles with centre 0 and radius R and ϵ respectively. We chose f specifically, such that $\mathrm{Im}\big(\frac{e^{iz}}{z}\big) = \mathrm{Im}\big(\frac{\cos(z)+i\sin(z)}{z}\big) = \frac{\sin(z)}{z}$.

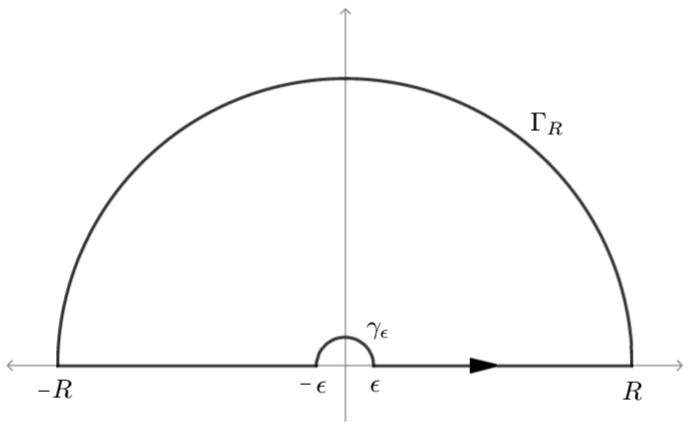

Figure A.1: Path of integration

Eventually we will consider the limit $R \to \infty$ as well as $\epsilon \to 0$.

As f does only have a pole at $z = 0$, which is not inside the path, we know that the integral equals zero along the whole path, hence

$$\int_{-R}^{-\epsilon} + \int_{\gamma_\epsilon} + \int_{\epsilon}^{R} + \int_{\Gamma_\epsilon} f(z)\mathrm{d}z = 0.$$

We will now evaluate these four integrals separately.

First up is the integral along Γ_R: using the substitution $z = R\left(\cos(\theta) + i\sin(\theta)\right) = \mathrm{Re}^{i\theta}$ we have

$$\begin{aligned}
\left|\int_{\Gamma_R} f(z)\mathrm{d}z\right| &= \left|\int_0^\pi \frac{e^{iR(\cos(\theta)+i\sin(\theta))}}{\mathrm{Re}^{i\theta}}\left(Rie^{i\theta}\right)\mathrm{d}\theta\right| \\
&\leq \int_0^\pi \left|e^{iR(\cos(\theta)+i\sin(\theta))}\right|\mathrm{d}\theta \\
&\leq \int_0^\pi e^{-R\sin(\theta)}\mathrm{d}\theta.
\end{aligned}$$

Since the integrand is in the interval $[0, 1]$ and converges 0 for $\theta \in (0, \pi)$, we conclude that the integral converges to 0.

Next, we consider the integral along γ_ϵ. For small z, say $|z| < 1$, we have

$$f(z) = \frac{e^{iz}}{z} = \frac{1}{z} + B(z)$$

where $B(z)$ is bounded. Therefore, we have

$$\lim_{\epsilon\to 0}\int_{\gamma_\epsilon} f(z)\mathrm{d}z = \lim_{\epsilon\to 0}\int_{\gamma_\epsilon}\frac{1}{z}\mathrm{d}z + \underbrace{\lim_{\epsilon\to 0}\int_{\gamma_\epsilon} B(z)\mathrm{d}z}_{=0}$$
$$= \int_{-\pi}^{0}\frac{1}{\epsilon e^{-i\theta}}\left(-i\epsilon e^{-i\theta}\right)\mathrm{d}\theta = \int_{-\pi}^{0} -i\mathrm{d}\theta = -i\pi$$

using the parametrization $z = \epsilon e^{-i\theta}$ with $\theta \in [-\pi, 0]$.

This leaves us to compute the two integrals in the interval $[-R, -\epsilon]$ and $[\epsilon, R]$. Here we have

$$\lim_{\substack{R\to\infty\\ \epsilon\to 0}} \operatorname{Im}\left(\int_{-R}^{-\epsilon} + \int_{\epsilon}^{R} f(z)\mathrm{d}z\right) = \lim_{\substack{R\to\infty\\ \epsilon\to 0}} \int_{-R}^{-\epsilon} + \int_{\epsilon}^{R}\frac{\sin(z)}{z}\mathrm{d}z = \lim_{R\to\infty}\int_{-R}^{R}\frac{\sin(z)}{z}\mathrm{d}z$$

as $\sin(z)/z$ is bounded near $z = 0$.

Putting the segments together and only considering the imaginary part indeed yields

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{\sin(z)}{z}\mathrm{d}z = \pi$$

□

A.4 Preparatory lemmas

In this section we address two specific theorems, which were helpful in Chapter 3 in the last steps (namely Step 2 and 3) of the proof of the Wiener-Ikehara Theorem.
To be specific, we used different variations of the theorem of dominated convergence and the theorem of monotone convergence. Since the versions we used are applied to functions in two variables rather than a sequence of unary functions, we labelled them as being "for binary functions". Here we will show how these can be inferred from the underlying unary ones.

A.4.1 Theorem of dominated convergence

Theorem (Dominated convergence). *Let I be an interval, $f_n : I \to \mathbb{C}$ and $g : I \to \mathbb{R}$. Further let every f_n and g be integrable and $|f_n(x)| \leq g(x)$ for every $n \in \mathbb{N}$ and $x \in I$. Moreover, let the limit $\lim_{n\to\infty} f_n(x)$ exist for every $x \in I$ and define*

$$f(x) := \lim_{n\to\infty} f_n(x).$$

Then f is integrable and

$$\lim_{n\to\infty}\int_I f_n(x)\mathrm{d}x = \int_I f(x)\mathrm{d}x.$$

See [Bog07, p. 130] for a proof.

We are now interested in the case, where the family f_n is not countable, i.e. we have for every $t > 0$ a function $f_t(x) =: f(t, x)$. Here a similar theorem holds:

Theorem (Dominated convergence for binary functions). *Let I be an interval, $f : (0, \infty) \times I \to \mathbb{C}$, and $g : I \to \mathbb{R}$. For any fixed $t > 0$ let $f(t, .)$ and g be integrable as well as $|f(t, x)| \leq g(x)$ for $x \in I$. Last, let the limiting function*

$$f(x) := \lim_{t \to 0} f(t, x) \text{ for } x \in I$$

exist. Then

$$\lim_{t \to 0} \int_I f(t, x) \mathrm{d}x = \int_I f(x) \mathrm{d}x.$$

We will show, how this theorem can be deduced from the previous:

Proof. Let $(t_n)_{n \in \mathbb{N}}$ be a sequence of positive real numbers converging to 0. Defining $f_n(x) := f(t_n, x)$ gives $f(x) = \lim_{n \to \infty} f_n(x)$. Applying now the previous theorem of dominated convergence gives

$$\lim_{n \to \infty} \int_I f(t_n, x) \mathrm{d}x = \lim_{n \to \infty} \int_I f_n(x) \mathrm{d}x = \int_I f(x) \mathrm{d}x.$$

As $(t_n)_{n \in \mathbb{N}}$ was chosen arbitrarily, we get

$$\lim_{t \to 0} \int_I f(t, x) \mathrm{d}x = \int_I f(x) \mathrm{d}x.$$

□

A.4.2 Monotone convergence theorem

Theorem (Monotone convergence). *Let I be an interval, $f_n : I \to \mathbb{R}$ for every $n \in \mathbb{N}$ and $0 \leq f_n(x) \leq f_{n+1}(x)$ for every $x \in I$. Let the limiting function*

$$f(x) := \lim_{n \to \infty} f_n(x)$$

exist for every $x \in I$. Then

$$\lim_{n \to \infty} \int_I f_n(x) \mathrm{d}x = \int_I f(x) \mathrm{d}x.$$

See [Bog07, pp. 130-131] for a proof.

Similar as before, a more general statement can be made as follows:

Theorem (Monotone convergence for binary functions). *Let I be an interval, $f : (0, \infty) \times I \to \mathbb{C}$, such that for $t > 0$ the function $f(t, .)$ is integrable and let f be monotone decreasing in t. Again, let the limiting function*

$$f(x) := \lim_{t \to 0} f(t, x) \text{ for } x \in I$$

exist, then

$$\lim_{t\to 0}\int_I f(t,x)\mathrm{d}x = \int_I f(x)\mathrm{d}x.$$

Proof. Choose an arbitrary monotone decreasing sequence $(t_n)_{n\in\mathbb{N}}$ with limit 0. Define $f_n(x) := f(t_n, x)$ and apply the theorem of monotone convergence to get

$$\lim_{n\to\infty}\int_I f(t_n,x)\mathrm{d}x = \lim_{n\to\infty}\int_I f_n(x)\mathrm{d}x = \int_I f(x)\mathrm{d}x.$$

This then implies

$$\lim_{t\to 0}\int_I f(t,x)\mathrm{d}x = \int_I f(x)\mathrm{d}x.$$

□

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