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„The Quantum Harmonic Oscillator Under the Influence of Gravity"

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Abstract

In this work we study the influence of gravity on the quantum harmonic oscillator. We discuss a geometric model of a (special) relativistic quantum harmonic oscillator (RHO) which is based on Quantum Field Theory in Curved Spacetime, and extend it to include relativistic effects of gravity in form of a weak gravitational background field. A consistent way to include relativistic effects of gravity is proposed by treating the Schwarzschild metric as a small perturbation to the anti-de Sitter metric of the RHO. We obtain the corresponding Klein-Gordon equation for the perturbed metric. An expansion of the Klein-Gordon equation in terms of c^{-2} is carried out. The zeroth order yields the Schrödinger equation of an harmonic oscillator with additional Newtonian gravitational potential. The first order leads to relativistic corrections due to the original RHO model, and the relativistic treatment of gravity. We find a coupling of the harmonic oscillator potential and the gravitational potential. A detailed analysis of the zeroth order non-relativistic equation is given. We carefully analyze the influence of the Newtonian gravitational potential beyond its linear approximation. The dominating effects are a constant shift in the oscillator's position, its ground state energy and in its frequency. Beyond that, we find an energy level dependent change in the spectrum and the wavefunctions which we discuss physically. We consider the time evolution of a coherent state in the presence of the Newtonian potential, observing a change in its oscillatory behavior. The main result is a time-dependent damping of the oscillation. A discussion on the size of the effects is given in an experiment related context.

Zusammenfassung

In dieser Arbeit untersuchen wir den Einfluss der Gravitation auf den quantenmechanischen harmonischen Oszillator. Wir diskutieren ein geometrisches Modell eines (speziell-) relativistischen quantenmechanischen harmonischen Oszillators (RHO), das auf der Quantenfeldtheorie in gekrümmter Raum-Zeit basiert, und erweitern dieses, um relativistische Gravitationseffekte in Form eines schwachen Hintergrundfeldes zu berücksichtigen. Indem wir die Schwarzschildmetrik als eine kleine Störung der Anti-de-Sitter-Metrik des RHO betrachten, schlagen wir eine konsistente Methode vor, relativistische Gravitationseffekte miteinzubeziehen. Wir erhalten die Klein-Gordon-Gleichung für die Metrik mit der Störung. Eine Reihenentwicklung der Klein-Gordon-Gleichung in c^{-2} wird durchgeführt. Aus der nullten Ordnung folgt die Schrödinger-Gleichung eines harmonischen Oszillators mit zusätzlichem Newtonschen Gravitationspotential. Die erste Ordnung liefert relativistische Korrekturen, die von dem ursprünglichen RHO-Modell und der relativistischen Betrachtung der Gravitation herrühren. Resultierend aus der relativistischen Betrachtung der Gravitation, beobachten wir eine Kopplung zwischen dem harmonischen Oszillatorpotential und dem Gravitationspotential. Außerdem analysieren wir die nicht-relativistische Schrödinger-Gleichung im Detail: Der Einfluss des Newtonschen Gravitationspotentials wird über die lineare Näherung hinaus sorgfältig untersucht. Die dominierenden Effekte sind eine konstante Verschiebung der Oszillatorposition, seiner Grundzustandsenergie und seiner Frequenz. Darüber hinaus finden wir eine vom Energieniveau abhängige Änderung des Spektrums und der Wellenfunktionen und diskutieren diese physikalisch. Wir betrachten die Zeitentwicklung eines kohärenten Zustands in der Anwesenheit des Newtonschen Gravitationspotentials. Dabei stellen wir eine Änderung in seinem Schwingungsverhalten fest, wobei der Haupteffekt eine zeitabhängige Dämpfung der Schwingung ist. Wir diskutieren die Größe der Effekte in einem Experiment orientierten Zusammenhang.

Contents

1 Introduction

Although general relativity (GR) and quantum mechanics (QM) are certainly the most fundamental and best tested theories in physics, finding a way to combine them is still one of the most important and at the same time one of the most challenging problems in modern physics. Beside attempts to formulate a unified theory of quantum gravity, the question of how quantum systems are affected by gravity is of particular interest.

In standard non-relativistic quantum mechanics, gravity can be included by simply adding the Newtonian gravitational potential to the Schrödinger equation of the quantum system under consideration. In this way, space and time are understood in the Newtonian sense: Time is considered to be a universal background parameter, and space is considered to be flat. Even within this framework of Newtonian gravity, testing gravitational effects on quantum systems is a quite challenging task. However, there have been great improvements in the experiments over the last years. One example is the *qBounce* project [1, 2], where a neutron falls down a small step due to gravity and bounces off a mirror. Beside the idea of testing the Newtonian law of gravity on short distances, the future aim of this project is to search for effects beyond Newtonian gravity [3]. Another example are atom interferometers. Atoms are sent through an interferometer in which the strength of the gravitational potential differs in the two paths, leading to a measurable phase difference. In this way, one is able to measure gravitational acceleration [4] and gravity gradients [5] with high precision. With further improvements in the technology, the aim is to test for effects of GR in the near future [6].

There are already experiments using quantum systems to probe relativistic effects. For example an optical atomic clock measuring gravitational time-dilation[1], like in the famous experiment done by the Wineland group [7]. They could observe time-dilation caused by the Earth's gravitational field at height differences of less than one meter. Another example is the detection of gravitational waves in the LIGO experiment [8] where they used a laser interferometer to measure spacetime distortions. But in these cases the dynamics of the quantum systems are still described by the Schrödinger equation, and are therefore inherently non-relativistic. Relativistic effects enter classically as *ad hoc* modifications (through modifications of time and/or space variables). For a more detailed review of recent advances in experiments we refer to [9].

In fact, the current experiments are all based on non-relativistic QM, i.e.

[1]We would like to mention, however, that QM enters only by using a two-level system. There is no interaction between GR and QM

the dynamics of the systems is described by the Schrödinger equation. But, with quantum experiments reaching precision where one is able to see effects beyond Newtonian gravity, it will be important to have an adequate description of how relativity affects quantum systems. For a consistent description of relativistic effects, one has to take into account that, in contrast to QM, time is not absolute in GR. This can be done by using Quantum Field Theory in Curved Spactime (QFTCS), which is a generalization of Quantum Field Theory to curved spacetime. In contrast to the descriptions mentioned above, where relativistic effects like time-dilation are simply added to the Schrödinger equation classically, QFTCS allows to include the concepts of GR on a more fundamental level[2]. Examples for interesting effects predicted by QFTCS are Hawking radiation [10, 11], the Unruh effect [12, 13, 14] and particle creation of an expanding universe [15]. Unfortunately, an experimental verification of these effects is far from being reached at the moment. However, indirect experimental verification of the Hawking radiation has been given recently [16, 17] within the framework of *analogue gravity*. More laboratory related effects have also been discussed recently, e.g. effects on the precision of quantum clocks [18] and a proposal for gravitational wave detection with the help of Bose-Einstein condensates [19].

However, it is still not fully understood how to describe arbitrary quantum systems within the framework of QFTCS. In particular, there is no canonical way to include external potentials, except for the case of the minimally coupled electromagnetic potential. Even in the simple case of a quantum harmonic oscillator, a consistent relativistic treatment is not well-established. As a first step towards a better understanding of how quantum systems are affected by gravity, the aim of this work is to give a careful analysis of how gravity can affect one of the simplest quantum systems, namely the quantum harmonic oscillator. For this purpose, we consider a one-dimensional quantum harmonic oscillator in the gravitational field of a nearby source. We propose a consistent relativistic description involving the Schwarzschild metric within the framework of QFTCS, which for small c^{-2} yields the Schrödinger equation of a the quantum harmonic oscillator with relativistic corrections. Additionally, we give a detailed analysis of the quantum harmonic oscillator in a Newtonian gravitational potential, which will be important to distinguish between effects coming from relativity and effects coming from the Newtonian potential when going beyond its linear approximation.

In Chapter 2 we start with a short introduction to some concepts of

[2]We would like to remark, however, that QFTCS is not a full theory of quantum gravity, since it does neither quantize gravity nor "gravitize" quantum mechanics. Instead gravity is treated as a classical background field.

QFTCS. In particular, we introduce the Klein-Gordon theory in a curved spacetime, and discuss a one-particle description within this framework. We argue that a meaningful notion of a particle can be obtained within the Newton-Wigner interpretation.

In Chapter 3 we discuss a model of a special relativistic harmonic oscillator (RHO), which is based on the Klein-Gordon theory of a free particle in anti-de Sitter space [20]. In particular, the model is obtained by using the one-particle sector of a free field in anti-de Sitter space. We obtain relativistic Schrödinger wavefunctions and eigenenergies from the corresponding Klein-Gordon equation within the Newton-Wigner interpretation, and we finally show that the quantum harmonic oscillator is naturally recovered in the non-relativistic limit.

In Chapter 4 we extend the RHO model to include relativistic effects of gravity in form of a gravitational background field. This is done by a modification of the anti-de Sitter metric of the RHO. In particular, we treat the Schwarzschild metric as a small perturbation to the anti-de Sitter metric. We obtain a Klein-Gordon equation that yields the RHO in the limit of a vanishing source of gravity. Expanding the Klein-Gordon equation up to first relativistic order, i.e. up to c^{-2}, yields the Schrödinger equation of the quantum harmonic oscillator, but with additional Newtonian potential and relativistic correction terms.

In Chapter 5 we analyze in detail the effect of the Newtonian potential on the quantum harmonic oscillator. For this purpose, we consider the Newtonian potential beyond its linear approximation. We find that the dominating effects are a constant shift in the oscillator's position, its ground state energy and its frequency. Using time-independent perturbation theory, we additionally find changes in the wavefunctions and the spectrum that depend on the energy level. Furthermore, we analyze the time evolution of an initially coherent state in the presence of a Newtonian gravitational potential. We find that the oscillation of the coherent state gets damped, but recurs at a later time. Finally, we discuss the order of magnitude of the effects in a more experiment related context.

2 Klein-Gordon Theory in Curved Spacetime

As already mentioned, part of this work is to find a consistent relativistic description of the quantum harmonic oscillator based on QFTCS. For this reason, we want to start with a short introduction to some concepts of the theory. In particular, we introduce the Klein-Gordon theory in curved spacetime, providing a consistent description of spinless free particles in a curved spacetime. Since we are only interested in the low-energy regime, i.e. assuming that no particle creation occurs, we consider a one-particle description within the framework of the Newton-Wigner interpretation.

2.1 Klein-Gordon Equation in Curved Spacetime

Consider a spacetime metric in $(1+1)$ dimensions (sign convention $(-+)$)

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \ . \tag{2.1}$$

The Klein-Gordon equation, describing spinless free particles of mass m in a curved spacetime with metric (2.1), is given by (in the minimal coupling case) [21]

$$\left(\Box_g - m^2\right)\phi = \left(g^{\mu\nu}\nabla_\mu \nabla_\nu - \frac{m^2 c^2}{\hbar^2}\right)\phi = 0 \ , \tag{2.2}$$

where ∇_μ denotes the covariant derivative operator. Defining

$$g \equiv det(g_{\mu\nu}) \ , \tag{2.3}$$

it is convenient to rewrite the Klein-Gordon equation as

$$\frac{1}{\sqrt{-g}}\partial_\mu \left(\sqrt{-g} g^{\mu\nu}\partial_\nu \phi\right) - \frac{m^2 c^2}{\hbar^2}\phi = 0 \ . \tag{2.4}$$

The corresponding Klein-Gordon inner product can be defined by [21]

$$\langle \phi | \phi' \rangle = i \int_\Sigma d\sigma_\nu \sqrt{-g}\ g^{\mu\nu} \phi \overleftrightarrow{\partial_\mu} \phi'^* \ , \tag{2.5}$$

where $\phi \overleftrightarrow{\partial_\mu} \phi'^* \equiv \phi \partial_\mu \phi'^* - \phi'^* \partial_\mu \phi$, and Σ is an arbitrary spacelike hypersurface. In this work we will only consider (1+1) static, diagonal spacetime metrics, which take the simple form

$$ds^2 = g_{00}(x) c^2 dt^2 + g_{11}(x) dx^2 \ . \tag{2.6}$$

The Klein-Gordon equation (2.4) can then be rewritten as

$$-\frac{1}{c^2}\partial_t^2\ \phi(t,x) = \frac{-g_{00}}{\sqrt{-g}}\left[-\partial_x \left(\frac{\sqrt{-g}}{g_{11}}\ \partial_x\right) + \frac{m^2 c^2}{\hbar^2}\sqrt{-g}\right]\ \phi(t,x) \ , \tag{2.7}$$

where we have used $g_{00} = 1/g^{00}$ and $g_{11} = 1/g^{11}$. Since the inner product (2.5) is independent of the chosen spacelike hypersurface Σ [22], we can choose Σ as $t = \text{const}$, and (2.5) becomes

$$\langle \phi | \phi' \rangle = i \int_\Sigma dx \frac{\sqrt{-g}}{g_{00}}\ \phi \overleftrightarrow{\partial_t} \phi'^* \ . \tag{2.8}$$

Since the metric (2.6) is independent of t, there exists a timelike Killing vector field ∂_t. With respect to this Killing vector field we can define the set of positive (respectively negative) frequency solutions of the Klein-Gordon equation. They are of the the form

$$\phi_k^+(t,x) = \frac{1}{\sqrt{2\omega_k}}\, e^{-i\omega_k t}\psi_k(x) \;, \quad \left(\phi_k^- = (\phi_k^+)^*\right) \;. \tag{2.9}$$

Inserting this into the Klein-Gordon equation, one is left with a Sturm-Liouville type problem for the spatial part $\psi_k(x)$

$$\frac{\omega_k^2}{c^2}\, \psi_k(x) = \frac{-g_{00}}{\sqrt{-g}} \left[-\partial_x \left(\frac{\sqrt{-g}}{g_{11}}\, \partial_x \right) + \frac{m^2 c^2}{\hbar^2} \sqrt{-g} \right] \psi_k(x) \;. \tag{2.10}$$

It follows that the spatial solutions $\psi_k(x)$ can be chosen as orthonormal with respect to the inner product

$$\langle \psi_k | \psi_j \rangle_\rho = \int_\Sigma dx\; \rho(x)\; \psi_k^*(x)\psi_j(x) \;, \tag{2.11}$$

with the weight function

$$\rho(x) = -\frac{\sqrt{-g}}{g_{00}} \; > 0 \;. \tag{2.12}$$

The spatial functions $\psi_k(x)$ provide a basis for the Hilbert space $L^2(\Sigma, \rho(x)dx)$, and the positive frequency solutions $\phi_k^+(t,x)$ are orthonormal with respect to the inner product (2.8), i.e. $\langle \phi_k^+ | \phi_j^+ \rangle = \delta_{kj}$ ($\langle \phi_k^- | \phi_j^- \rangle = -\delta_{kj}$ for the negative frequency solutions ϕ_k^-).

This treatment (with the consequence of the existence of negative frequency solutions) does not directly allow for a probability interpretation, in the sense of a spatial probability density $|\psi|^2$. It follows that the notion of a particle remains unclear in this context. The physical meaning is obtained by quantizing the solutions, leading to the quantum field description where particles are interpreted as excitations of the resulting quantum field. In this work, however, we want to study the behavior of a quantum system in a low-energy regime, i.e. assuming no particle creation. For this reason we will discuss a one-particle description in the next section.

2.2 One-Particle Description

The question whether to use the particle or the field interpretation, i.e. whether particles or fields are more fundamental, is still subject of many discussions (for an interesting philosophical discussion on this topic we highly recommend [23] and references therein). The field interpretation follows from the quantization of the classical field ϕ by promoting it to an operator $\hat{\phi}$ that acts on the Fock space. In this way a particle can be interpreted to be an excitation of the field in the following sense: Given a state $|\psi\rangle$, the quantity $u(t,x) = \langle 0|\hat{\phi}|\psi\rangle$ turns out to be the single particle "wavefunction" given below (2.13). It is therefore often said that $\hat{\phi}$ creates a particle from the vacuum at (t,x). However, as we have discussed before, (2.13) cannot be properly interpreted as a wavefunction in the sense of quantum mechanics, and therefore the notion of a single particle remains unclear (for an exhaustive discussion on these problems we refer to [24]). One of the main reasons for this issue is that position as an observable is problematic in quantum field theory in general, even for a free relativistic particle [24, 25]. For this reason, we will introduce the Newton-Wigner interpretation, which provides a notion of a position observable, and therefore allows for a particle interpretation.

A quantum state of a single particle can be defined as the general positive frequency solution [24] (assuming a discrete spectrum)

$$\phi(t,x) = \sum_k c_k \; \phi_k^+(t,x) \; , \tag{2.13}$$

with the complex numbers $c_k \in l^2$. The norm of these states is induced by (2.8), and is given by

$$||\phi||^2 = \sum_k |c_k|^2 \; , \tag{2.14}$$

which, as already mentioned, does not allow for a probability interpretation of $|\phi(t,x)|^2$. To obtain a one-particle description, including a notion of a position observable, one can define the Newton-Wigner field [24]

$$\phi_{NW}(t,x) = \sum_k c_k \; \psi_k(x) e^{-i\omega_k \, t} \; . \tag{2.15}$$

The Newton-Wigner field also solves (2.7). It is just a different representation of the same abstract state. The norm with respect to the Newton-Wigner field turns out to be

$$||\phi_{NW}(t,x)||^2 = \int_\Sigma \rho(x) \; |\phi_{NW}(t,x)|^2 \; dx \; . \tag{2.16}$$

Then, in a mathematical sense, $|\phi_{NW}(t,x)|^2$ can be associated with the probability density for observing x at time t, and the Newton-Wigner position operator [24] acts on the ϕ_{NW}'s as usual

$$\hat{x} \; \phi_{NW} = x \; \phi_{NW} \; . \tag{2.17}$$

In this way we have defined a position operator, allowing for a one-particle interpretation. However, it suffers from some problems as well. For example, a particle localized in some region Ω_0 at some time t_0 would have a nonzero probability to be found outside of its lightcone at a any later time $t_0 + \delta$, and therefore traveling faster than light. Therefore, interpreting ϕ_{NW} as a particle wavefunction still remains problematic. As a one-particle sector of a multiparticle point of view, however, this problem seems to be less severe [24]. Within the scope of this work, i.e. working in the one-particle sector, we will use the Newton-Wigner one-particle description, and the results given above will be sufficient.

3 Relativistic Harmonic Oscillator (RHO)

Despite the great success of the Klein-Gordon theory in describing free fields, there is no canonical way of dealing with external potentials. Even for the simple case of the quantum harmonic oscillator, a special relativistic generalization is not well-established. The first proposals [26, 27] were all based on the naive covariant generalization $x^2 \to x_\mu x^\mu$, where the meaning of the time-component remains unclear. A different approach is to introduce an equal vector and scalar potential coupling to the Klein-Gordon equation (i.e. $\partial_\mu \to \partial_\mu + A_\mu$ and $m \to m + S$, choosing $A_\mu = (\pm S, 0)$), which also leads to some problems [28]. A third approach are the so called *geometric models* discussed in [20, 29, 30], where one uses certain spacetime metrics to simulate the oscillatory behavior. The special relativistic models are then built from Klein-Gordon theory in curved spacetime. Within the scope of this work, we will consider these geometric models.

We demand from an adequate model the following reasonable properties:

- When quantized, it should provide the correct non-relativistic quantum limit, i.e. we should be able to recover the quantum harmonic oscillator, with corresponding wavefunctions and energies.

- The classical behavior of a free particle in the metric should consist of amplitude independent oscillations, and the Newtonian equations of motion for a harmonic oscillator should be recovered in the non-relativistic limit.

Among the different generalized geometric models in [29], all of which satisfy the first property, only the one considered also in [20] satisfies both properties. This model of the relativistic harmonic oscillator (RHO) consists of the one-particle sector of a free field in the anti-de Sitter (AdS) metric. Moreover, the model is motivated by a group theoretical approach which consists of a generalization of the algebra of a free relativistic particle and the algebra of the non-relativistic harmonic oscillator [31, 32]. In fact, the model resulting from the group theoretical approach is shown to be equivalent to considering the one-particle sector of a free field in AdS space. In this way, the AdS metric can be used to simulate the harmonic oscillator interaction in Minkowski space.

In the following, we will give a short introduction to this RHO model. For this purpose, we will first discuss the AdS space and determine the classical behavior, including the non-relativistic limit. Then, we will give the quantum description, using Klein-Gordon theory, and we will finally show that the quantum harmonic oscillator is recovered in the limit $c \to \infty$.

3.1 Anti-de Sitter Metric and Geodesics

The AdS metric used for the construction of the RHO is given by [20]:

$$ds^2 = -\left(1 + \frac{\omega^2 x^2}{c^2}\right) c^2 dt^2 + \frac{dx^2}{1 + \frac{\omega^2 x^2}{c^2}} \ . \tag{3.1}$$

To show the correct classical behavior, let us first determine the geodesics of the AdS metric. They can be obtained from the geodesic equations, given by:

$$\ddot{x} = -\,\omega^2 x \tag{3.2}$$

$$\ddot{t} = \frac{2\omega^2 x}{c^2} \frac{E}{mc^2\left(1 + \frac{\omega^2 x^2}{c^2}\right)^2}\, \dot{x} \ , \tag{3.3}$$

where $\dot{x} \equiv \frac{dx}{d\tau}$, with τ being the proper time. Additionally, we have used $\dot{x}_\mu \dot{x}^\mu = -c^2$ and the constant of motion associated with the Killing vector ∂_t:

$$mc^2 \left(1 + \frac{\omega^2 x^2}{c^2}\right) \dot{t} = \text{const} \equiv E \ . \tag{3.4}$$

We find for the timelike geodesics of the AdS metric the oscillatory behavior

$$x(t) = A \sin\left(\arctan\left(\frac{\tan(\omega t)}{\sqrt{1 + \frac{\omega^2 A^2}{c^2}}}\right)\right) \ , \tag{3.5}$$

where the amplitude A is related to the energy E through

$$E = mc^2 \sqrt{1 + \frac{\omega^2 A^2}{c^2}} \, . \tag{3.6}$$

To illustrate the geodesic behavior, we plotted geodesics for certain values of A in Figure 1.

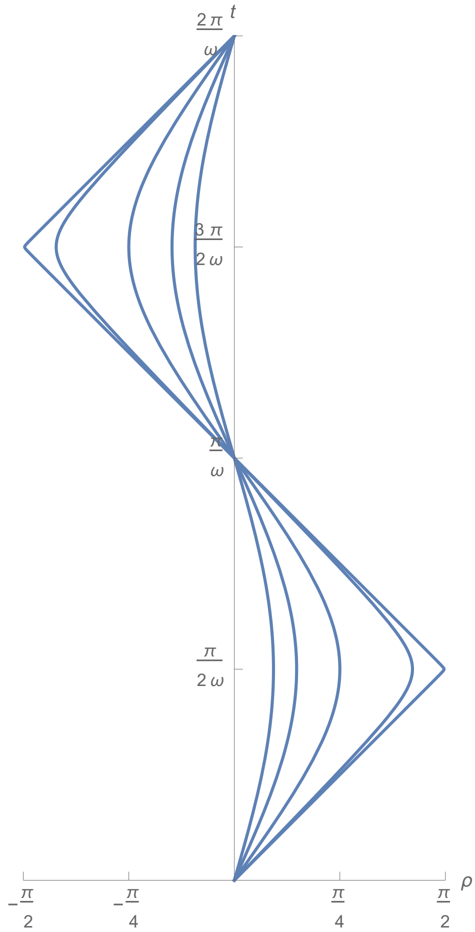

Figure 1: Geodesics of the AdS metric in conformal coordinates (i.e., the spatial coordinate is compactified according to $dx = (1 + \frac{\omega^2 x^2}{c^2})\, d\rho$). The straight lines correspond to null-geodesics.

Obviously, the geodesics are periodic with period $2\pi/\omega$, which is independent of the amplitude. This is one of the desired properties to use the metric as a model for a harmonic oscillator. Furthermore, we can easily see that it provides the correct non-relativistic limit, i.e. the Newtonian equations of motion for a harmonic oscillator are recovered from (3.2) in the limit $c \to \infty$:

$$\ddot{x} = -\,\omega^2 x \ , \tag{3.7}$$

$$t = \tau \ , \tag{3.8}$$

and the energy (3.6) becomes

$$E = mc^2 + \tfrac{1}{2} m\omega^2 A^2 \ . \tag{3.9}$$

As a result, we have shown that the AdS metric (3.1) provides the desired classical behavior, including the non-relativistic limits. We can now use this metric to construct the RHO using Klein-Gordon theory, which will be part of the next section.

3.2 Klein-Gordon Theory of the RHO

The RHO, according to [20], can be defined as the one-particle sector of the Klein-Gordon field theory on AdS space. In the following, we will use Klein-Gordon theory to obtain the relativistic energy spectrum and the corresponding relativistic wavefunctions for the RHO. For this purpose, we will follow the steps of [20]. Let us begin by determining the Klein-Gordon equation on AdS space. Using the metric (3.1) in (2.7), the resulting Klein-Gordon equation is

$$-\frac{1}{c^2}\partial_t^2\ \phi(t,x) = \left(1+\frac{\omega^2x^2}{c^2}\right)\left[-2\frac{\omega^2}{c^2}x\ \partial_x - \left(1+\frac{\omega^2x^2}{c^2}\right)\partial_x^2 + \frac{m^2c^2}{\hbar^2}\right]\phi(t,x)\ . \tag{3.10}$$

According to (2.8), the corresponding inner product is

$$\langle\phi|\phi'\rangle = -i\int dx\,\frac{1}{1+\frac{\omega^2x^2}{c^2}}\,\phi\overleftrightarrow{\partial_t}\phi'^{\,*}\ . \tag{3.11}$$

It is worth mentioning that the Klein-Gordon equation (3.10) could also be obtained starting from the Hamiltonian of a free particle in the AdS metric (3.1)

$$H^2 = m^2c^4 + p^2c^2 + m^2\omega^2c^2x^2 + 2\,\omega^2x^2p^2 + \frac{\omega^4}{c^2}x^4p^2\ . \tag{3.12}$$

This Hamiltonian differs from the naive definition $\left(H - \frac{1}{2}m\omega^2x^2\right)^2 = m^2c^4 + p^2c^2$, where one simply adds the harmonic oscillator potential to the free Hamiltonian in Minkowski spacetime, which, in contrast to the present approach, would not lead to square-integrable quantum wavefunctions [20].

According to the considerations in the previous chapter, we can use the ansatz $\phi(t,x) = e^{-iE\,t/\hbar}\psi(x)$ in (3.10) to obtain the eigenvalue equation for the spatial part $\psi(x)$:

$$\frac{E^2}{\hbar^2 c^2}\psi(x) = \left(1+\frac{\omega^2 x^2}{c^2}\right)\left[-2\frac{\omega^2}{c^2}x\,\partial_x - \left(1+\frac{\omega^2 x^2}{c^2}\right)\partial_x^2 + \frac{m^2c^2}{\hbar^2}\right]\psi(x)\ . \tag{3.13}$$

The corresponding inner product follows from (2.11) and is given by

$$\langle\psi\,|\,\psi'\rangle_\rho = \int \frac{1}{\left(1+\frac{\omega^2x^2}{c^2}\right)}\,\psi^*\,\psi'\,dx\ , \tag{3.14}$$

and the spatial solutions $\psi_n(x)$ will be chosen as orthonormal with respect to this inner product.

Imposing $\psi \to 0$ for $x \to \pm\infty$, the solutions are given by [20]

$$\psi_n(x) = N_n^\gamma\left(1+\frac{\omega^2x^2}{c^2}\right)^{-\lambda_n/2} C_n^{-(n+\gamma)}\left(-i\,\frac{\omega}{c}\,x\right)\ , \tag{3.15}$$

where $C_n^\alpha(z)$ are the Gegenbauer polynomials [33] with $n = 0,1,2,\ldots$. N_n^γ is a normalization constant with

$$(N_n^\gamma)^{-2} = \sqrt{\pi}\frac{c}{\omega}\,\frac{4^n}{n!}\,\frac{(2\,\gamma)!}{(2\,\gamma+n)!}\left(\frac{(\gamma+n)!}{(\gamma)!}\right)^2\frac{1}{\gamma+n+\frac{1}{2}}\,\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\frac{1}{2})}\ , \tag{3.16}$$

and the parameters λ_n and γ are given by:

$$\lambda_n \equiv \frac{E_n}{\hbar\omega} = \frac{1}{2}+n+\gamma\ , \tag{3.17}$$

$$\gamma = \frac{1}{2}\sqrt{1+4\,\frac{m^2c^4}{\hbar^2\omega^2}}\ . \tag{3.18}$$

The corresponding energy spectrum follows then from the definition of λ_n (3.17) and is therefore given by

$$E_n = \left(\frac{1}{2}+n+\frac{1}{2}\sqrt{1+4\,\frac{m^2c^4}{\hbar^2\omega^2}}\right)\hbar\,\omega\ . \tag{3.19}$$

Notice that the energy spacing does not differ from the one in the non-relativistic case. However, there is a nontrivial mixing of the non-relativistic harmonic oscillator zero-point energy $E_0 = \frac{\hbar\omega}{2}$ and the relativistic rest mass energy mc^2.

Finally, the normalized Klein-Gordon energy eigenstates (with positive frequency) are given by

$$\phi_n(t,x) = \frac{1}{\sqrt{2E_n/\hbar}}\, N_n^\gamma \, e^{-iE_n t/\hbar} \left(1 + \frac{\omega^2 x^2}{c^2}\right)^{-\lambda_n/2} C_n^{-(n+\gamma)} \left(-i\,\frac{\omega}{c}\,x\right) \ . \tag{3.20}$$

They are orthonormal with respect to the inner product (3.11), which is obviously not the standard Hilbert space product. As we have discussed before, it follows that the Klein-Gordon wavefunctions (3.20) cannot be interpreted as probability amplitudes. However, we can define the corresponding Newton-Wigner field according to (2.15)

$$\phi_{NW}(t,x) = \sum_k \, c_k \, \psi_k(x)\, e^{-iE_k t/\hbar} \ , \tag{3.21}$$

and the norm becomes

$$||\phi_{NW}||^2 = \int_{-\infty}^{\infty} \frac{1}{1 + \frac{\omega^2 x^2}{c^2}} \, |\phi_{NW}|^2 \, dx \ , \tag{3.22}$$

which allows for a probability interpretation, at least in a mathematical sense. To obtain “Schrödinger”-type wavefunctions, with the corresponding standard Hilbert space product, another transformation can be done. In fact, we can define [20]

$$\psi_S(t,x) = \frac{1}{\sqrt{1 + \frac{\omega^2 x^2}{c^2}}} \, \phi_{NW}(t,x) \ , \tag{3.23}$$

for which the norm takes the standard form

$$||\psi_S||^2 = \int_{-\infty}^{\infty} |\psi_S(t,x)|^2 \, dx \ . \tag{3.24}$$

The meaning of the transformation can be understood in the following way: The Schrödinger-type wavefunctions $\psi_S(t,x)$ are solutions of the wave equation, resulting from the Hamiltonian (3.12) by promoting H, p and x to operators, while accounting for ordering ambiguities [20]. This wave equation is equivalent to the Klein-Gordon equation (3.10) through the transformation (3.23).

Finally, the energy eigenstates in Schrödinger representation, providing the usual probability interpretation, are

$$\Psi_n(t,x) = e^{-iE_nt/\hbar}\left(1+\frac{\omega^2x^2}{c^2}\right)^{-1/2}\psi_n(x) \tag{3.25}$$

$$= N_n^\gamma\, e^{-iE_nt/\hbar}\left(1+\frac{\omega^2x^2}{c^2}\right)^{-(\lambda_n+1)/2} C_n^{-(n+\gamma)}\left(-i\,\frac{\omega}{c}\,x\right)\,. \tag{3.26}$$

They are orthonormal with respect to the inner product (3.24), and therefore provide an orthonormal basis for the Hilbert space $L^2(\mathbb{R}, dx)$. In this representation the position operator acts as usual, i.e. $\hat{x}\,\Psi_n(t,x) = x\,\Psi_n(t,x)$.

To illustrate the behavior of the relativistic Schrödinger-type wavefunctions, we plotted the probability density for the states $\Psi_0(t,x)$ and $\Psi_4(t,x)$ in Figure 2 and Figure 3, respectively, for different values of $N = m\,c^2/\hbar\,\omega$. The non-relativistic counterpart is included in each plot to see the respective differences. Length is measured in units of the typical length $l = \sqrt{\hbar/m\omega}$.

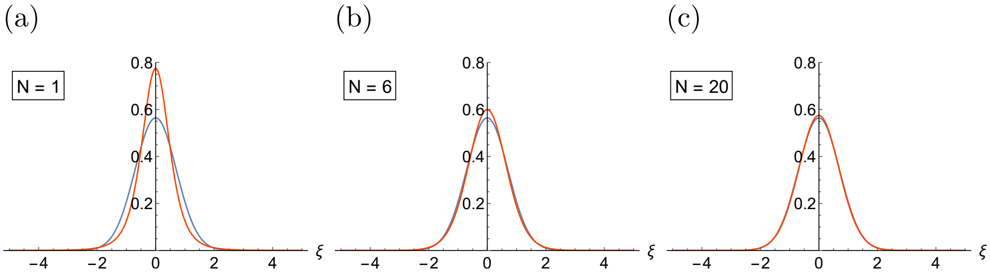

Figure 2: Probability density of the ground state: For the relativistic wavefunction (red line), and for the non-relativistic counterpart (blue line), for different values of $N = m\,c^2/\hbar\,\omega$. Length is measured in units of $l = \sqrt{\hbar/m\omega}$, i.e. $\xi = x/l$.

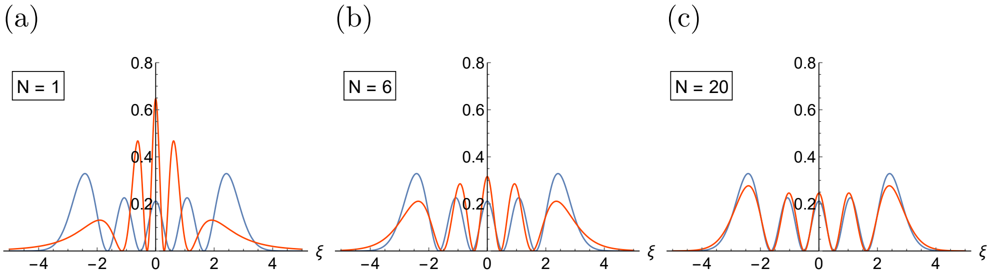

Figure 3: Probability density of the state corresponding to $n = 4$: For the relativistic wavefunction (red line), and for its non-relativistic counterpart (blue line), for different values of $N = m\,c^2/\hbar\,\omega$. Length is measured in units of $l = \sqrt{\hbar/m\omega}$, i.e. $\xi = x/l$.

We can observe that the probability density differs significantly for small N's (corresponding to highly relativistic systems). For larger N's, the probability density approaches the one corresponding to the usual harmonic oscillator, as it should be the case.

The difference will become even clearer if we consider the expectation value $\langle\Psi_n|\hat{x}|\Psi_n\rangle$ and its standard deviation. We find that

$$\langle\Psi_n\,|\,\hat{x}\,|\,\Psi_n\rangle = \int_{-\infty}^{\infty} x\,|\Psi_n|^2\,dx = 0\;, \tag{3.27}$$

$$(\Delta x)^2 = \frac{N}{\sqrt{1+4N^2}}\,(2n+1)\,l^2\;. \tag{3.28}$$

The expectation value is equivalent to the one in the non-relativistic case. But the standard deviation is smaller. However, the probability to find the particle outside the classically forbidden region of the quantum harmonic oscillator is higher in the relativistic case (see Figures 2 and 3).

To conclude, the model provides an energy spectrum with the same energy spacing as the quantum harmonic oscillator, but with a nontrivial change in the zero point energy. Furthermore, we have obtained relativistic Schrödinger-type energy eigenstates providing a significantly different probability density. Finally, it remains to show that we can recover the quantum harmonic oscillator in the non-relativistic limit.

3.3 Non-Relativistic Limit

We demanded from a good model of a RHO that we can recover the quantum harmonic oscillator in the non-relativistic limit. To see that this is indeed the case for the model presented above, let us first show that the Klein-Gordon equation (3.10) provides the Schrödinger equation for the harmonic oscillator in the limit $c\to\infty$. For that purpose, following the method in [34, 35], we can insert the ansatz

$$\phi(t,x) = e^{\frac{i}{\hbar}\left(c^2\,S_0+S_1+\frac{1}{c^2}\,S_2+\ldots\right)} \tag{3.29}$$

into (3.10), and compare terms with the same power in c. The first orders, c^4 and c^2, provide that $S_0=-m\,t$ (where the minus sign has been chosen for consistency, i.e. to obtain positive frequency solutions). The next order (c^0) yields

$$-2\,m\,\partial_t S_1 = m^2\omega^2 x^2 + (\partial_x S_1)^2 - i\,\hbar\,\partial_x^2 S_1\;, \tag{3.30}$$

which, by introducing $\varphi(t,x) \equiv e^{i\,S_1/\hbar}$, turns out to be the Schrödinger equation for the harmonic oscillator

$$i\,\hbar\,\partial_t\,\varphi = -\tfrac{\hbar^2}{2\,m}\,\partial_x^2\,\varphi + \tfrac{1}{2}\,m\,\omega^2\,x^2\,\varphi\ , \tag{3.31}$$

hence, providing the correct non-relativistic limit.

First relativistic correction terms follow from the next order. In fact, with the definition $\chi \equiv \varphi\, e^{i\,S_2/\hbar\,c^2}$ we obtain

$$\begin{aligned} i\,\hbar\,\partial_t\,\chi = &-\frac{\hbar^2}{2\,m}\,\partial_x^2\,\chi + \frac{1}{2}\,m\,\omega^2 x^2\,\chi \\ &-\frac{\hbar^4}{8\,m^3 c^2}\partial_x^4\,\chi - \frac{3\,\hbar^2\omega^2 x^2}{4\,m\,c^2}\,\partial_x^2\,\chi - \frac{\hbar^2\omega^2 x}{2\,m\,c^2}\,\partial_x\,\chi - \frac{m\,\omega^4 x^4}{8\,c^2}\,\chi + \frac{\hbar^2\omega^2}{4\,m\,c^2}\,\chi\ , \end{aligned} \tag{3.32}$$

where we have dropped terms of order c^{-4} for consistency. Let us mention that this result is in complete accordance with the one obtained in [36] using a different method. We would like to remark that the perturbative part, the second line in (3.32), is not Hermitian (with respect to the standard Hilbert space inner product of the harmonic oscillator). Thus, a perturbative treatment would require a modification of the scalar product, which will not be within the scope of this work. For a discussion on this issue we refer to [35, 36].

Finally, let us consider the non-relativistic limit of the energy spectrum (3.19), and of the relativistic wavefunctions (3.25). The energy spectrum in the limit $c \to \infty$ becomes the usual harmonic oscillator energy spectrum plus the rest mass energy:

$$E_n = \hbar\,\omega\left(n + \frac{1}{2}\right) + m\,c^2\ . \tag{3.33}$$

For the wavefunctions (3.25) we find that (notice that $\lambda_n = \lambda_n(c)$)

$$\lim_{c\to\infty}\left(1 + \frac{\omega^2 x^2}{c^2}\right)^{-(\lambda_n+1)/2} = e^{-\frac{m\,\omega}{\hbar}\,\frac{x^2}{2}}\ , \tag{3.34}$$

which is exactly the Gaussian, appearing in the non-relativistic harmonic oscillator eigenstates. Now, using the relation between Gegenbauer polynomials and Hermite polynomials [33]

$$\lim_{\alpha\to\infty}\frac{n!}{\alpha^{n/2}}\,C_n^{\alpha}\left(\frac{x}{\sqrt{\alpha}}\right) = H_n(x)\ , \tag{3.35}$$

we have that

$$\lim_{c\to\infty} \frac{(-i)^n\, n!}{N^{n/2}}\, C_n^{-(n+\gamma)}\left(-i\,\frac{\xi}{\sqrt{N}}\right) = H_n(\xi)\,, \tag{3.36}$$

where $\xi = \sqrt{m\,\omega/\hbar}\,x$. Together with the asymptotic behavior of the normalization constant

$$\lim_{c\to\infty} \frac{N^n}{(n!)^2}(N_n^\gamma)^2 = \sqrt{\frac{m\,\omega}{\hbar}}\frac{1}{\sqrt{\pi}\,n!\,2^n}\;, \tag{3.37}$$

we finally recover the well known eigenstates of the harmonic oscillator, i.e.

$$\lim_{c\to\infty} \Psi_n = e^{-i\,E_n\,t/\hbar}\left(\frac{m\,\omega}{\pi\,\hbar}\right)^{\frac{1}{4}} \frac{1}{\sqrt{n!\,2^n}}\; e^{-\frac{m\,\omega}{\hbar}\frac{x^2}{2}}\, H_n\left(\sqrt{\frac{m\,\omega}{\hbar}}\,x\right)\;, \tag{3.38}$$

in complete accordance with what one would expect from (3.31).

In conclusion, we have presented a relativistic generalization of the quantum harmonic oscillator, using Klein-Gordon theory for the AdS metric. We have shown that the model provides the correct non-relativistic limits, i.e. in the limit $c \to \infty$ the AdS metric provides the classical harmonic oscillator, and the quantum harmonic oscillator is recovered from Klein-Gordon theory. We will use this RHO model as a starting point for the construction of a relativistic harmonic oscillator under the influence of gravity in the next chapter.

4 RHO with Gravitational Perturbation

In this chapter we use the RHO model presented above to study the influence of gravity on the RHO. We propose a consistent way of how gravitational effects can be included. We restrict our considerations to the effects of an external source of gravity, not considering the effects of self-gravity. In particular, we use the model of the RHO presented above and add a gravitational background by using the Schwarzschild metric as a perturbation to the AdS metric. The result is a combined metric, which we use to derive the Klein-Gordon equation. Lastly, we calculate the non-relativistic limit, yielding the Schrödinger equation for the non-relativistic harmonic oscillator with Newtonian gravitational potential, and discuss first relativistic corrections.

4.1 AdS Metric with Gravitational Background

The RHO, introduced in [20] and presented in the previous chapter, is "built" over a Minkowski flat spacetime in coordinates (t, x). It is important to notice that these coordinates are the ones with physical meaning of time and space in the static reference frame where the RHO is built, and not any proper length or time that one could compute from the AdS metric. The AdS metric is introduced only for acting as the trapping potential giving rise to the RHO.

We wish now the background metric to be the radial sector of the Schwarzschild metric. In particular, we want the coordinates (t, x) to correspond to a static reference frame around some point at a radius r_0. The coordinate t will be the proper time at the radial position r_0, and the coordinate x the proper distance measured by static observers from r_0 ($x(r_0) = 0$) along the radial direction.

The Schwarzschild metric, written in Schwarzschild coordinates (t_{S}, r), is given by

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) c^2 dt_{\mathrm{S}}^2 + \frac{dr^2}{1 - \frac{r_s}{r}} \ , \tag{4.1}$$

with the Schwarzschild radius $r_s \equiv 2MG/c^2$, where M is the mass of the source of gravity and G the gravitational constant. To use the Schwarzschild metric as a background for the RHO, we have to move to coordinates (t, x) with the same physical meaning of space and time as they have in the RHO model. These coordinates are given by the proper time at r_0 and the proper distance to r_0:

$$dt := \sqrt{1 - \frac{r_s}{r_0}}\, dt_{\mathrm{S}} \ , \quad dx := \frac{1}{\sqrt{1 - \frac{r_s}{r}}}\, dr \ . \tag{4.2}$$

In the new coordinates, we can write the Schwarzschild metric as

$$\left(g^S_{\mu\nu}\right) = (\eta_{\mu\nu}) + \begin{pmatrix} 1 - \frac{1 - \frac{r_s}{r(x)}}{1 - \frac{r_s}{r_0}} & 0 \\ 0 & 0 \end{pmatrix} \ , \tag{4.3}$$

with $\eta_{\mu\nu}$ being the Minkowski metric. In this way, we can treat the Schwarzschild metric as a small perturbation to the underlying Minkowski metric of the RHO (small as compared to the AdS metric), and we obtain the combined metric:

$$ds^2 = -\left(\frac{1 - \frac{r_s}{r(x)}}{1 - \frac{r_s}{r_0}} + \frac{\omega^2 x^2}{c^2}\right) c^2 dt^2 + \frac{dx^2}{1 + \frac{\omega^2 x^2}{c^2}} \ . \tag{4.4}$$

Notice that we recover the AdS metric in the absence of a source of gravity, i.e. in the limit $M \to 0$. On the other hand, the Schwarzschild metric is recovered in the absence of the oscillator, i.e. in the limit $\omega \to 0$. In the absence of both, we recover the Minkowski metric.

Let us now consider the geodesic equations resulting from this metric. The geodesics are given by

$$\ddot{x} = \frac{\omega^2 x}{c^2} \frac{1}{1+\frac{\omega^2 x^2}{c^2}} \dot{x}^2 - \left(1+\frac{\omega^2 x^2}{c^2}\right)\left(\frac{\omega^2 x}{c^2} + \frac{1}{2}\frac{r_s}{r^2(x)}\frac{r'(x)}{\left(1-\frac{r_s}{r_0}\right)}\right) c^2\, \dot{t}^2 \ , \quad (4.5)$$

$$\ddot{t} = -\frac{\left(1-\frac{r_s}{r_0}\right)\frac{\omega^2}{c^2}\, x + \frac{1}{2}\frac{r_s}{r^2(x)} r'(x)}{\left(1-\frac{r_s}{r}\right) + \frac{\omega^2 x^2}{c^2}\left(1-\frac{r_s}{r_0}\right)}\, \dot{x}\, \dot{t} \ , \quad (4.6)$$

which in the limit $c \to \infty$ become

$$\ddot{x} = -\omega^2 x - \frac{M\, G}{(x+r_0)^2} \ , \quad (4.7)$$

$$t = \tau \ . \quad (4.8)$$

Hence, in the non-relativistic limit we recover the Newtonian equation of motion of a harmonic oscillator with an underlying Newtonian gravitational potential. This shows that the metric (4.4) provides the correct classical non-relativistic limit.

4.2 Klein-Gordon Theory of the RHO with Gravitational Background

We can now use Klein-Gordon theory to construct the corresponding quantum model. For this purpose we can insert the metric (4.4) into (2.7) to obtain the corresponding Klein-Gordon equation:

$$\begin{aligned} -\frac{1}{c^2}\,\partial_t^2\,\phi(t,x) = & \left(1+\frac{\omega^2 x^2}{c^2}\right)\left[-2\,\frac{\omega^2 x}{c^2}\,\partial_x - \left(1+\frac{\omega^2 x^2}{c^2}\right)\partial_x^2 + \frac{m^2 c^2}{\hbar^2}\right]\phi(t,x) \\ & + \frac{\alpha}{1-\alpha}\left[-a_1(x)\partial_x - a_2(x)\partial_x^2 + \left(1-\frac{1}{\zeta(x)}\right)\frac{m^2c^2}{\hbar^2}\right]\phi(t,x) \ , \end{aligned} \quad (4.9)$$

where we have introduced $\alpha \equiv r_s/r_0$ and $\zeta(x) \equiv r(x)/r_0$, and the coefficients a_1 and a_2 are given by

$$a_1(x) = \left(1 + \frac{\omega^2 x^2}{c^2}\right) \frac{\zeta'(x)}{2\,\zeta^2(x)} + \frac{\omega^2 x}{c^2}\left(1 - \frac{1}{\zeta(x)}\right) , \tag{4.10}$$

$$a_2(x) = \left(1 + \frac{\omega^2 x^2}{c^2}\right)\left(1 - \frac{1}{\zeta(x)}\right) . \tag{4.11}$$

Notice that the first line of (4.9) is the Klein-Gordon equation of the RHO (3.10), and the second line consists of additional terms appearing due to gravity. Thus, the parameter α, which is the ratio between the Schwarzschild radius and the center of the harmonic oscillator, controls the contribution of the gravitational effects. In the limit $\alpha \to 0$, i.e. for the center of the harmonic oscillator being far away from the source of gravity, the second line of (4.9) vanishes and we recover the Klein-Gordon equation of the RHO.

The corresponding inner product, according to (2.8), is given by

$$\langle \phi \,|\, \phi' \rangle = -i \int_\Omega dx \, \frac{\phi \overleftrightarrow{\partial_t} \phi'^*}{\sqrt{\left(1 + \frac{\omega^2 x^2}{c^2}\right)\left(1 + \frac{\omega^2 x^2}{c^2} + \frac{\alpha}{1-\alpha}\left(1 - \frac{1}{\zeta(x)}\right)\right)}} , \tag{4.12}$$

where the domain is restricted to a sufficiently small[3] region Ω.

Let us mention that the Klein-Gordon equation (4.9) could also be obtained from the squared Hamiltonian [37] resulting from the metric (4.4)

$$\begin{aligned} H^2 =& m^2c^4 + 2\omega^2x^2p^2 + c^2p^2 + m^2c^2\omega^2x^2 + \frac{\omega^4x^4}{c^2}p^2 \\ &+ \frac{\alpha}{1-\alpha}\left(1 - \frac{1}{\zeta(x)}\right)\left(m^2c^4 + \omega^2x^2p^2 + c^2p^2\right) . \end{aligned} \tag{4.13}$$

Promoting H, x and p to operators, and choosing a suitable ordering (for terms like $x^m p^n$) yields the Klein-Gordon equation (4.9). However, this does not fix the ordering ambiguities. To solve the problem of ordering, one would have to consider techniques like *geometric quantization* or similar ones (for further reading on this topic, see for example [38, 39]). For our purpose, however, it will be sufficient to use the Klein-Gordon equation obtained directly from the metric.

[3]The RHO is placed at a sufficiently large distance to the source of the gravitational background field. In this case, the solutions ϕ will only be significantly different from zero in a region Ω that does not contain the singularity (not even the non-physical coordinate singularity). Therefore, taking the integral over Ω will be sufficient.

Since the metric (4.4) is static, we can use again the ansatz $\phi(t,x) = e^{-iE\,t/\hbar}\,\psi(x)$ in (4.9), and we are left with the eigenvalue equation for the spatial part:

$$\frac{E^2}{\hbar^2 c^2}\psi = \left(1+\frac{\omega^2 x^2}{c^2}\right)\left[-2\,\frac{\omega^2 x}{c^2}\,\partial_x - \left(1+\frac{\omega^2 x^2}{c^2}\right)\partial_x^2 + \frac{m^2 c^2}{\hbar^2}\right]\psi$$
$$+\frac{\alpha}{1-\alpha}\left[-a_1(x)\partial_x - a_2(x)\partial_x^2 + \left(1-\frac{1}{\zeta(x)}\right)\frac{m^2c^2}{\hbar^2}\right]\psi\,. \qquad (4.14)$$

The corresponding inner product follows from (2.11) and reads

$$\langle\psi\,|\,\psi'\rangle = \int_\Omega dx\,\frac{\psi^*(x)\,\psi'(x)}{\sqrt{\left(1+\frac{\omega^2x^2}{c^2}\right)\left(1+\frac{\omega^2x^2}{c^2}+\frac{\alpha}{1-\alpha}\left(1-\frac{1}{\zeta(x)}\right)\right)}}\,. \qquad (4.15)$$

Now, to obtain the one-particle description with corresponding wavefunctions, one would have to solve (4.14), and continue by defining the corresponding Newton-Wigner field and following the same steps as in the previous chapter. Unfortunately, since $\zeta(x)$ is only given implicitly through the coordinate transformation (4.2), equation (4.14) does not have an analytic solution. Thus, one would have to use approximate methods or deploy numerical calculations to find approximate solutions of the Klein-Gordon equation. We will leave a discussion on these problems for the Appendix A. Instead, we will continue by calculating the non-relativistic limit of our model, showing that we will recover the Schrödinger equation of the harmonic oscillator plus the Newtonian gravitational potential. Additionally, we will discuss first relativistic corrections to this Schrödinger equation.

4.3 Non-Relativistic Limit and Relativistic Corrections

Let us finally consider the non-relativistic quantum limit, and see if the Klein-Gordon equation (4.9) provides the correct Schrödinger equation in the limit $c\to\infty$. For that purpose we can again use the ansatz $\phi(t,x) = e^{\frac{i}{\hbar}\left(c^2\,S_0+S_1+\frac{1}{c^2}\,S_2+\dots\right)}$ in (4.9), and compare terms with equal power in c. The first orders (c^4 and c^2) provide $S_0=-m\,t$. The order c^0, using the definition $\varphi\equiv e^{i\,S_1/\hbar}$, yields the Schrödinger equation

$$i\hbar\,\partial_t\,\varphi = \frac{1}{2}\,m\,\omega^2x^2\,\varphi + \frac{m\,M\,G}{r_0}\,V_G(x)\,\varphi - \frac{\hbar^2}{2\,m}\,\partial_x^2\,\varphi\,, \qquad (4.16)$$

where we have introduced $V_G(x) \equiv 1 - \frac{1}{x/r_0+1}$. This is obviously the Schrödinger equation for a harmonic oscillator with the additional Newtonian gravitational potential. Hence, our model provides indeed the correct non-relativistic limit.

We are now interested in the first relativistic corrections. For this reason we have to consider the next order, i.e. terms of the order c^{-2}. Defining $\chi \equiv \varphi\, e^{i\,S_2/\hbar c^2}$, and using (4.16) for φ, we find

$$\begin{aligned}
i\hbar\,\partial_t\,\chi =& -\frac{\hbar^2}{2m}\partial_x^2\,\chi + \frac{1}{2}m\omega^2 x^2\,\chi + \frac{m\,M\,G}{r_0}V_G(x)\,\chi \\
& - \frac{\hbar^4}{8\,m^3 c^2}\partial_x^4\,\chi - \frac{3\,\hbar^2\omega^2 x^2}{4\,m\,c^2}\,\partial_x^2\,\chi - \frac{\hbar^2\omega^2 x}{2\,m\,c^2}\,\partial_x\,\chi - \frac{m\,\omega^4 x^4}{8\,c^2}\,\chi + \frac{\hbar^2\omega^2}{4\,m\,c^2}\,\chi \\
& + \alpha\left[-\frac{\hbar^2}{4m}\,V_G(x)\,\partial_x^2\,\chi - \frac{m}{4}\,V_G(x)\,\omega^2 x^2\,\chi - \frac{m\,M\,G}{4\,r_0}\,V_G^2(x)\,\chi\right. \\
& \left. + \frac{m\,M\,G}{r_0}\,V_G(x)\,\chi - \frac{m\,M\,G}{2}\,{V_G}'(x)\,\log\left(\frac{x}{r_0}+1\right)\,\chi + \frac{\hbar^2}{8\,m}\,{V_G}''(x)\,\chi\right],
\end{aligned} \tag{4.17}$$

where we have neglected terms with power higher than c^{-2} for consistency. Notice that the first line in (4.17) is the Schrödinger equation for the function χ. The second line corresponds to the relativistic correction resulting from the RHO (see (3.32)), and the third and fourth line are relativistic corrections appearing due to gravity. Their contribution is controlled by the parameter α. For this reason it makes sense to consider (4.17) to be the Schrödinger equation of a harmonic oscillator with Newtonian gravitational potential, plus relativistic corrections resulting from treating the oscillator relativistically, with a general relativistic treatment of gravity.

However, we have to mention that the Hamiltonian corresponding to (4.17) is not Hermitian with respect to the Schrödinger scalar product. The reason is that the relativistic correction is not Hermitian (which is a result of the construction, i.e. starting from Klein-Gordon equation with Klein-Gordon inner product). Therefore, to make it Hermitian, one would have to perform a transformation of the Hamiltonian and the corresponding wave-function. For a discussion on this issue we refer to [35]. Despite this, we can still observe an interesting effect. The relativistic correction provides a coupling of the gravitational potential and the harmonic oscillator potential, which can be seen in the third line of (4.17). This is clearly a feature of the relativistic treatment, since such coupling does not occur in the non-relativistic Schrödinger equation.

Finally, we would like to mention that it is common to approximate the

Newtonian potential in the Schrödinger equation through $V_G \approx mgx$. However, if one wants to observe the effects of the relativistic corrections, one will have to distinguish between effects resulting from using mgx instead of the full Newtonian potential and effects resulting from the relativistic treatment. In fact, it could happen that the error resulting from using mgx is of the same order as the relativistic corrections. For this reason, we will give a careful analysis of the Schrödinger equation for a harmonic oscillator with the full Newtonian gravitational potential in the next chapter.

5 Quantum Harmonic Oscillator with Newtonian Gravity

We have argued in the previous chapter that if we wish to understand the influence of a (relativistic) gravitational background on the quantum harmonic oscillator, we will first have to work out in some detail the effect of the Newtonian gravitational potential on the quantum harmonic oscillator. For that purpose, we study the effect of the gravitational potential beyond the linear approximation in this chapter. We treat the Newtonian potential as a small perturbation to the Schrödinger equation of the harmonic oscillator, allowing us to use time-independent perturbation theory to calculate approximate solutions. Lastly, we consider the time evolution of an initially coherent state under the influence of the perturbation.

5.1 Analysis of the Full Potential

Consider a particle with mass m in an Harmonic oscillator centered at $x = 0$, with potential

$$V_0 = \frac{1}{2} m \, \omega^2 x^2 \; . \tag{5.1}$$

Additionally, we have an Newtonian gravitational potential, produced by a pointlike source at $x = -r_0$. The gravitational potential is then given by

$$V_1 = -\frac{m \, MG}{x + r_0} \; , \tag{5.2}$$

with M being the mass of the source and G is the gravitational constant. Obviously, the combined full potential V of this problem is given by

$$V = V_0 + V_1 \; , \tag{5.3}$$

and the resulting Hamiltonian can be written as

$$\mathbf{H} = -\frac{\hbar^2}{2m}\partial_x^2 + V(x) = -\frac{\hbar^2}{2m}\partial_x^2 + \frac{1}{2}m\,\omega^2 x^2 - \frac{m\,MG}{x + r_0}\,, \tag{5.4}$$

leading to a Schrödinger equation similar to (4.16).

For the following, it will be convenient to introduce the dimensionless parameter β, which combines the frequency of the harmonic oscillator ω, the position of the source of gravity r_0 and its mass M, in the following way:

$$\beta = \frac{MG}{\omega^2 r_0^3}\,. \tag{5.5}$$

Then, the potential (5.3) can be written as

$$V = \frac{1}{2}m\,\omega^2 x^2 - m\,\beta\frac{w^2 r_0^3}{x + r_0}\,. \tag{5.6}$$

The shape of the potential can be seen in Figure 4, where we measured length and energy in units of the typical length $l = \sqrt{\frac{\hbar}{m\omega}}$ and the typical energy $\hbar\omega$ of the harmonic oscillator.

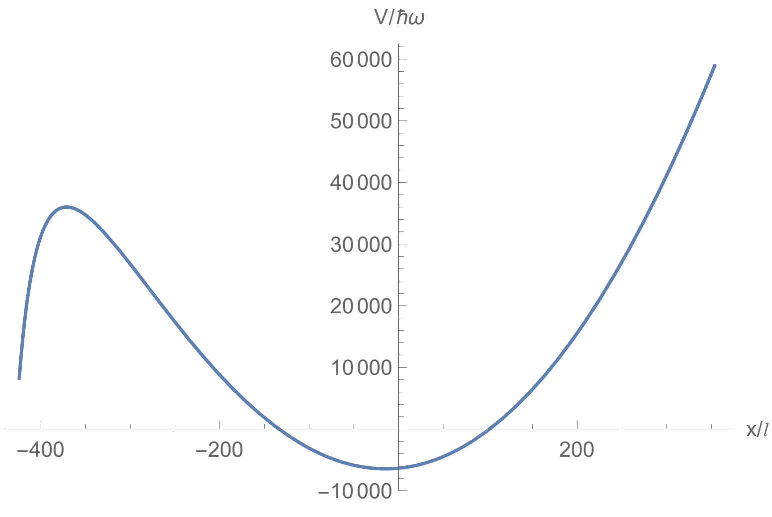

Figure 4: Shape of the full potential V. $\beta = 0.03$, $r_0 = 460\,l$.

Now, our aim is to analyze the effect of the gravitational potential on the harmonic oscillator potential. Compared with the harmonic oscillator potential (5.1), we observe that the minimum of the full potential (5.3), is shifted in both position and energy. Furthermore, the potential now has a maximum on the side closer to the source of gravity and a singularity at $x = -r_0$. However, this singularity is not physical, since it only occurs when assuming a pointlike mass as the gravitational source. If one wants to avoid the singularity, one actually will have to use a piecewise function to describe the potential around $x = -r_0$, taking away the singularity. But, since we are only interested in the behavior near the minimum of the potential (5.3), it will be sufficient to simply ignore the singularity, assuming that it is smoothed by a mass distribution.

Sufficiently close to the new minimum, the potential can still be approximated by a harmonic oscillator potential, but with shifted frequency. Thus, we will describe the potential (5.3) as a shifted harmonic oscillator potential, denoted by $\tilde{V}_0$, plus some remaining "small" perturbation part, denoted by V_{per}, collecting the remaining effects:

$$V = \tilde{V}_0 + V_{per} \ . \tag{5.7}$$

The task will be now to determine the new harmonic oscillator potential and the perturbing potential. Let us begin by determining the shifted potential

$$\tilde{V}_0 = \frac{1}{2} m\tilde{\omega}^2 (x - x_0)^2 + \mathcal{E} \ , \tag{5.8}$$

where $\tilde{\omega}$ denotes the shifted frequency, x_0 denotes the shift in the position and $\mathcal{E}$ denotes the constant shift in the energy. The shifts can be obtained analytically from the potential V (5.3), by imposing the following conditions:

$$x_0 : \qquad \left.\frac{\partial V}{\partial x}\right|_{x=x_0} = 0 \ , \tag{5.9}$$

$$\mathcal{E} : \qquad V(x_0) = \mathcal{E} \ , \tag{5.10}$$

$$\tilde{\omega} : \qquad \left.\frac{\partial^2 V}{\partial x^2}\right|_{x=x_0} = m\tilde{\omega}^2 \ . \tag{5.11}$$

The first and second condition (5.9),(5.10) ensure that the local minima of $\tilde{V}_0$ and V coincide. The third condition (5.11) determines the shifted frequency such that $\tilde{V}_0$ and V coincide up to second order in terms of their series expansions around x_0. Thus, it follows by construction that the perturbing potential V_{per} is small with respect to the potential $\tilde{V}_0$ in a certain region around x_0 (i.e. $V_{per}/\tilde{V}_0$ is of the order x close to x_0), which is exactly what we demanded. Now, we can explicitly determine $\tilde{V}_0$ by calculating the shifts. Evaluating (5.9),(5.10) and (5.11) yields

$$
\begin{aligned}
x_0 &= r_0\,\gamma(\beta)\ , \\
\tilde{\omega} &= \omega\,\sqrt{\delta(\beta)}\ , \\
\mathcal{E} &= \frac{1}{2}m\omega^2 r_0^2\left(\gamma^2(\beta)-\beta\frac{2}{\gamma(\beta)+1}\right)\ ,
\end{aligned}
\tag{5.12}
$$

where we have introduced the shift functions $\gamma(\beta)$ and $\delta(\beta)$, defined through

$$
\gamma(\beta) = \frac{1}{3}\left(-2+F(\beta)+\frac{1}{F(\beta)}\right)\ , \tag{5.13}
$$

$$
\delta(\beta) = 1-54\beta\left(1+\frac{1}{F(\beta)}+F(\beta)\right)^{-3}\ , \tag{5.14}
$$

with the function $F(\beta)$ being the principal third root[4] (the root closest to the real axis)

$$
F(\beta) = \left(-\frac{27}{2}\beta+1+\frac{3}{2}\sqrt{3}\sqrt{\beta\,(27\beta-4)}\right)^{1/3}\ . \tag{5.15}
$$

To ensure that the full potential V has a minimum at $x = x_0$, β has to be restricted to $0 \leq \beta < \frac{4}{27}$, which implies constraints for $\gamma(\beta)$ and $\delta(\beta)$. It follows that $0 \geq \gamma(\beta) > -\frac{1}{3}$ and $1 \geq \delta(\beta) > 0$.

Now, the meaning of the parameter β has become clear. It completely controls the shifts through the shift functions $\gamma(\beta)$ and $\delta(\beta)$, and therefore determines the shape of the new harmonic oscillator potential. Compared with the harmonic oscillator potential V_0, one can see that with growing β the minimum is shifted towards the source of gravity, and the minimal energy and the frequency are both lowered, whereas the potential V_0 is recovered for $\beta \to 0$.

Since we have determined the new harmonic oscillator potential $\tilde{V}_0$, we are now able to determine the perturbing potential V_{per}. According to (5.7), the perturbing potential is given by

$$
V_{per} = V - \tilde{V}_0\ , \tag{5.16}
$$

[4]Notice that $\gamma(\beta)$ and $\delta(\beta)$ are real. In fact, one can show that $\gamma(\beta) = -\frac{4}{3}\sin^2(\frac{\phi(\beta)}{2})$ and $\delta(\beta) = 1-54\beta\,(1+2\cos(\phi(\beta))^{-3}$, where $\phi(\beta)$ is the phase of $F(\beta)$.

and, using (5.8) together with the conditions given in (5.9), (5.10) and (5.11), we obtain (a detailed calculation can be found in Appendix B)

$$V_{per} = m\beta \frac{\omega^2 r_0^3}{(x_0 + r_0)^4} \frac{(x - x_0)^3}{\frac{x - x_0}{x_0 + r_0} + 1} \,. \tag{5.17}$$

For further simplification, it is convenient to introduce the dimensionless coordinate ξ defined through

$$x - x_0 \equiv \frac{\tilde{l}}{\sqrt{2}} \, \xi \,, \tag{5.18}$$

where $\tilde{l} = \sqrt{\frac{\hbar}{\tilde{\omega} m}}$ is the typical length of the new harmonic oscillator with potential $\tilde{V}_0$. The meaning of the new coordinate is obvious: the shift x_0 is implemented such that the potential's minimum is centered at $\xi = 0$ and lengths are measured in units of the typical length $\tilde{l}$. Finally, in new coordinates, and with the help of the relations given in (5.12), we can write the perturbing potential in the following simple way:

$$V_{per} = \lambda \, \hbar \tilde{\omega} \left(\frac{\xi^3}{\eta \, \xi + 1} \right) \,, \tag{5.19}$$

where (remember that δ and γ are functions of β):

$$\lambda \left(\beta, \frac{l}{r_0} \right) \equiv \frac{\beta}{2 \sqrt{2} \, \delta \, (1 + \gamma)^4} \frac{\tilde{l}}{r_0} = \frac{\beta}{2 \sqrt{2} \, \sqrt[4]{\delta} \, \delta \, (1 + \gamma)^4} \frac{l}{r_0} \,, \tag{5.20}$$

$$\eta \left(\beta, \frac{l}{r_0} \right) \equiv \frac{1}{\sqrt{2} \, (1 + \gamma)} \frac{\tilde{l}}{r_0} = \frac{1}{\sqrt{2} \, \sqrt[4]{\delta} \, (1 + \gamma)} \frac{l}{r_0} \,. \tag{5.21}$$

The last equality in (5.20) and (5.21) is obtained by using the relation $\tilde{l} = \frac{1}{\sqrt[4]{\delta}} \, l$, which follows directly from the relation between ω and $\tilde{\omega}$, given in (5.12). The dimensionless parameter functions λ and η were introduced to write down the perturbing potential in a more compact way. The whole dependence of V_{per} on the parameters is now collected in λ and η.

After having taken all this steps, we can finally write down the full potential in the very simple way

$$V = \hbar \tilde{\omega} \left(\frac{1}{4} \, \xi^2 + \lambda \frac{\xi^3}{\eta \xi + 1} \right) . \tag{5.22}$$

As a result, we have achieved a form where it is quite obvious that the first part in (5.22) is essentially a (position, zero point energy and frequency shifted) harmonic oscillator potential. The effects other than simple shifts are collected in the second part of (5.22) and the strength of its contribution is controlled by the parameter functions λ and η. More precisely, λ controls whether there will be a contribution at all, whereas η determines how much the perturbation differs from a cubic potential. These parameters themselves are controlled through (5.20) and (5.21) by the ratio $\frac{l}{r_0}$, which compares the size of the oscillator with its distance to the source of gravity, and the parameter β. In more detail, λ and η are both proportional to the ratio $\frac{l}{r_0}$. However, their dependence on β differs. In fact, for $\beta \ll 1$ it follows that $\lambda \to \frac{l}{2r_0\sqrt{2}}\beta$, whereas $\eta \to \frac{l}{r_0\sqrt{2}}$. On the other hand, for growing β, both λ and η increase, i.e. the stronger the contribution, the more the perturbing potential will differ from a cubic one. For the potential (5.22) this means that for small λ it is quite easy to see that the oscillator will be the dominating part in a region close to $\xi = 0$. However, taking into account the effect of the perturbation, η will be responsible for the fact that, in total, the potential will differ more from a harmonic oscillator for $\xi < 0$ then for $\xi > 0$. In addition, for $\xi < 0$ the potential will be lower compared to the harmonic oscillator potential and for $\xi > 0$ it will be slightly larger. To illustrate the behavior of the full potential V compared with the harmonic oscillator potential $\tilde{V}_0$, a plot is given in Figure 5.

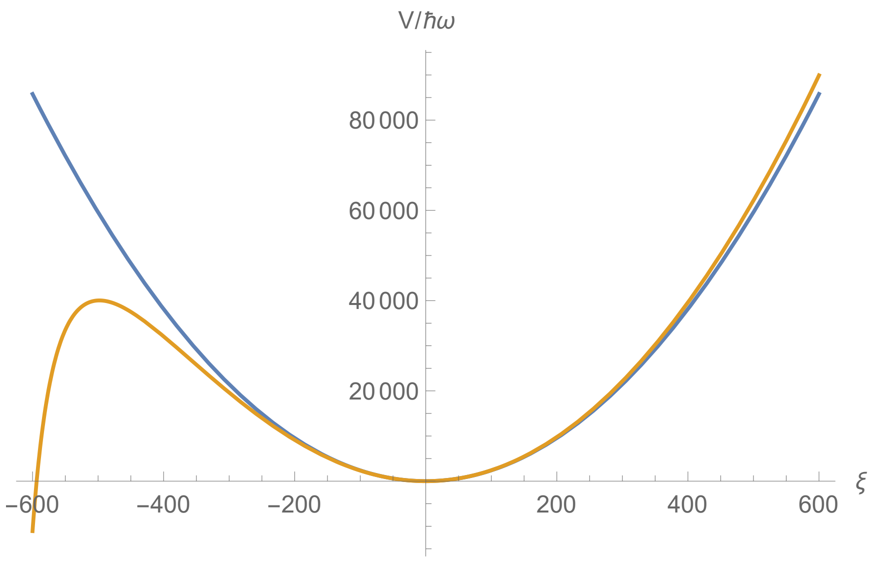

Figure 5: Comparison of the shifted harmonic oscillator potential $\tilde{V}_0$ (blue line) and the full potential V (orange line). $\beta = 0.04$, $r_0 = 700\frac{l}{\sqrt{2}}$.

Given this qualitative analysis of the potential, it is easy to see that it is reasonable to treat the potential as a harmonic oscillator with a small perturbation. It follows that the solutions of the corresponding Schrödinger equation with potential V will not differ much from the well known solutions of the harmonic oscillator with potential $\tilde{V}_0$, at least for sufficiently low energies (i.e. close to the minimum). Thus, it is legitimate to apply time-independent perturbation theory to calculate approximate solutions. However, before continuing, it will be convenient to do some reasonable approximations of the perturbing potential, in order to further simplify the problem.

5.2 Approximations

Although we want to postpone a detailed treatment in perturbation theory until the next section, it is necessary to discuss some properties right here to see that we can avoid dealing with the singularity of the Newtonian potential when calculating the approximate solutions. Moreover, we show that it will be sufficient to use an approximation of the perturbing potential.

In time-independent perturbation theory, the crucial ingredient will be the matrix elements $(V_{per})_{nm}$ of V_{per} in the representation of the harmonic oscillator eigenstates ψ_n (solutions to the unperturbed problem), which are given by

$$(V_{per})_{nm} = \int_{-\infty}^{\infty} \psi_n^* \, \psi_m \, V_{per} \, d\xi \; , \tag{5.23}$$

where the ψ_n's are

$$\psi_n = \left(\frac{1}{\sqrt[4]{2\pi}} \right) \frac{1}{\sqrt{2^n n!}} \, e^{-\frac{\xi^2}{4}} \, H_n \left(\frac{\xi}{\sqrt{2}} \right) \; , \tag{5.24}$$

and the H_n's are the Hermite polynomials. It is easy to see that the ψ_n's decrease exponentially outside several times their typical length $\sqrt{2n+1}\, l$. Therefore, there will be no significant contribution to the integral (5.23) outside of $-k\sqrt{2\,\max(n,m)+1} < \xi < k\sqrt{2\,\max(n,m)+1}$ for some sufficiently large k, and it will then be sufficient to replace the integral limits in (5.23) by $-k\sqrt{2\,\max(n,m)+1}$ and $k\sqrt{2\,\max(n,m)+1}$ (a more detailed discussion can be found in Appendix C.1). In that way, we can avoid the singularity, at least for sufficiently small m, n's. However, the integral does not provide analytic solutions and an additional approximation will be useful.

Since η is proportional to $\frac{l}{r_0}$ (see (5.21)), it is easy to see that η will be small. Thus, we can expand the potential V_{per} (5.19), with respect to $\eta\xi$ to obtain the approximate perturbation potential $V_{\approx}$, defined by

$$V_{per} \approx \lambda\, \hbar\tilde{\omega} \left(\xi^3 - \eta\xi^4\right) \equiv V_{\approx} \; . \tag{5.25}$$

This potential will be a sufficient approximation close to the minimum. More explicitly, $V_{\approx}$ will be a good approximation as long as $\frac{V_{\approx}}{V_{per}} = 1 - \eta^2\xi^2$ is sufficiently close to 1 (implying that $\eta^2\xi^2 \ll 1$). However, as mentioned before (and discussed in Appendix C.1), the integral in (5.23) has no significant contribution outside several times the typical length for each respective wavefunction. Thus, it will be sufficient that $\eta^2\xi^2 \ll 1$ in this region. It turns out, that $V_{\approx}$ will be a good approximation for values of n with $n \ll \frac{1}{5\eta^2}$ (a detailed explanation can be found in Appendix C.2).

Given the considerations above it will be reasonable to use $V_{per} = V_{\approx}$ from now on, and the Hamiltonian of the problem finally reads

$$\mathbf{H} = \hbar\tilde{\omega} \left(-\partial_\xi^2 + \frac{1}{4}\,\xi^2 + \frac{\mathcal{E}}{\hbar\tilde{\omega}} + \lambda\, \left(\xi^3 - \eta\xi^4\right)\right) \; , \tag{5.26}$$

where the first three terms correspond to the harmonic oscillator and the last part represents the perturbation. It is now convenient to switch to the ladder operator representation. For this purpose, we can use the relations

$$\xi = a + a^\dagger \; , \tag{5.27}$$

$$\partial_\xi = \frac{1}{2}\left(a - a^\dagger\right) \; , \tag{5.28}$$

between the ladder operators a and $a^\dagger$ and ξ and ∂_ξ. Inserting (5.27) and (5.28) in (5.26) provides the Hamiltonian in ladder operator representation:

$$\mathbf{H} = \hbar\tilde{\omega}\left(a^\dagger a + \frac{1}{2}\right) + \mathcal{E} + \lambda\hbar\tilde{\omega}\left(\left(a + a^\dagger\right)^3 - \eta\left(a + a^\dagger\right)^4\right) \; . \tag{5.29}$$

We have finally arrived at a Hamiltonian which we can further analyze. The true merit of this Hamiltonian is that time-independent perturbation theory will now provide analytic expressions for the approximate solutions. This enables us to estimate the effect of the perturbation potential, i.e. to estimate the change in the energies and the change in the eigenstates due to the presence of the gravitational potential. Furthermore, using the ladder operator representation will simplify the following required calculations. This will become clear in the next section, where we explicitly calculate the changes in the energy and in the eigenstates to estimate the effect of the gravitational perturbation.

For the sake of completeness, let us mention that the approximation implies a constraint on the parameters λ and η. In fact, to ensure that second order perturbation is much smaller than first order, we have to impose $\frac{\lambda}{\eta} \ll 1$, which is definitely true for β being small.

5.3 Solutions in Perturbation Theory

Our aim is now to use time-independent perturbation theory to find solutions for the problem

$$\mathbf{H} \left|\varphi_n\right\rangle = E'_n \left|\varphi_n\right\rangle \ , \tag{5.30}$$

with the Hamiltonian $\mathbf{H}$ being the one in (5.29). To do so, we split the Hamiltonian (5.29) into two parts:

$$\mathbf{H} = \mathbf{H}_0 + V_{per} \ , \tag{5.31}$$

where the dominating harmonic oscillator Hamiltonian, denoted by $\mathbf{H}_0$, and a small perturbation part, denoted by V_{per}, are given by

$$\mathbf{H}_0 = \hbar\tilde{\omega}\left(a^\dagger a + \frac{1}{2}\right) + \mathcal{E} \ , \tag{5.32}$$

$$V_{per} = \lambda\hbar\tilde{\omega}\left(\left(a + a^\dagger\right)^3 - \eta\left(a + a^\dagger\right)^4\right) \ . \tag{5.33}$$

The solutions for the unperturbed harmonic oscillator are given by the energy eigenstates $|n\rangle$:

$$\mathbf{H}_0 \left|n\right\rangle = E_n \left|n\right\rangle \ , \tag{5.34}$$

with the corresponding energy spectrum

$$E_n = \hbar\tilde{\omega}\left(n + \frac{1}{2}\right) + \mathcal{E} \ . \tag{5.35}$$

Given the solutions $(\left|n\right\rangle, E_n)$, time-independent perturbation theory will then provide approximate solutions to the full problem stated in (5.30). Therefore, noticing that the strength of V_{per}'s contribution to $\mathbf{H}$ is controlled by the parameter λ, we will expand the energy eigenvalues E'_n and energy eigenstates $\left|\varphi_n\right\rangle$ in powers of λ. This expansions can then be used to construct approximate solutions, in principle of arbitrary order (it would take more and more effort for each consecutive order). Up to first order, the energy change ΔE_n and the energy eigenstates $|\varphi_n\rangle$ are given by (a detailed discussion about time-independent perturbation theory can be found in [40]):

$$\Delta E_n \equiv E'_n - E_n = (V_{per})_{nn} \ , \tag{5.36}$$

$$|\varphi_n\rangle = |n\rangle + \sum_{k\neq n} |k\rangle \frac{(V_{per})_{kn}}{E_n - E_k} \ , \tag{5.37}$$

where the matrix elements

$$(V_{per})_{kn} \equiv \langle k|V_{per}|n\rangle \tag{5.38}$$

are taken with respect to the unperturbed harmonic oscillator eigenstates $|n\rangle$. Notice, that the energy change up to first order is completely determined by the diagonal matrix elements, whereas the first order contribution of the eigenstate is determined by the off-diagonal elements.

We can now use the explicit expressions for V_{per}, given in (5.33), to calculate the energy changes and the eigenstates. Let us begin by determining the energy change ΔE_n, i.e. determining the diagonal matrix elements $(V_{per})_{nn}$. For that purpose, we insert (5.33) into (5.36), obtaining

$$\Delta E_n = (V_{per})_{nn} = \lambda\hbar\tilde{\omega}\left(\langle n|\left(a + a^\dagger\right)^3|n\rangle - \eta\,\langle n|\left(a + a^\dagger\right)^4|n\rangle\right) \ , \tag{5.39}$$

which makes clear the true merit of using ladder operator representation: Instead of dealing with integrals, we can use the known properties of the ladder operators to calculate the scalar products given in (5.39). One can easily see that the first term in (5.39) vanishes. Hence, we are left with the second term in (5.39) only, which yields (a detailed calculation can be found in Appendix D):

$$\Delta E_n = -\lambda\eta\;\hbar\tilde{\omega}\,\langle n|\left(a + a^\dagger\right)^4|n\rangle = -\lambda\eta\;\hbar\tilde{\omega}\,(6n(n+1)+3) \ . \tag{5.40}$$

Finally, according to (5.36), the spectrum of the full problem can be approximated up to first order by:

$$E'_n = E_n + \Delta E_n = \hbar\tilde{\omega}\left(n + \frac{1}{2} - \lambda\eta\,(6n(n+1)+3)\right) + \mathcal{E} \ . \tag{5.41}$$

First, from (5.41) we can notice that, in the presence of V_{per}, the harmonic oscillator energies E_n get lowered, and the lowering being stronger with larger

n. The reason is that for higher n the wavefunctions explore a larger region, where the full potential differs more significantly from a harmonic oscillator. In fact, the energy gaps of two consecutive energies, say E'_n and E'_{n+1}, are given by

$$E'_{n+1} - E'_n = \hbar\tilde{\omega}\left(1 - \lambda\eta\, 12\,(n+1)\right)\ , \tag{5.42}$$

which is not constant anymore, in contrast to the unperturbed harmonic oscillator. Thus, the presence of V_{per} obviously provides a non-trivial effect (i.e. not only a constant shift) on the harmonic oscillator spectrum.

To illustrate the situation, we compare the harmonic oscillator spectrum with the spectrum of the full potential in Figure 6.

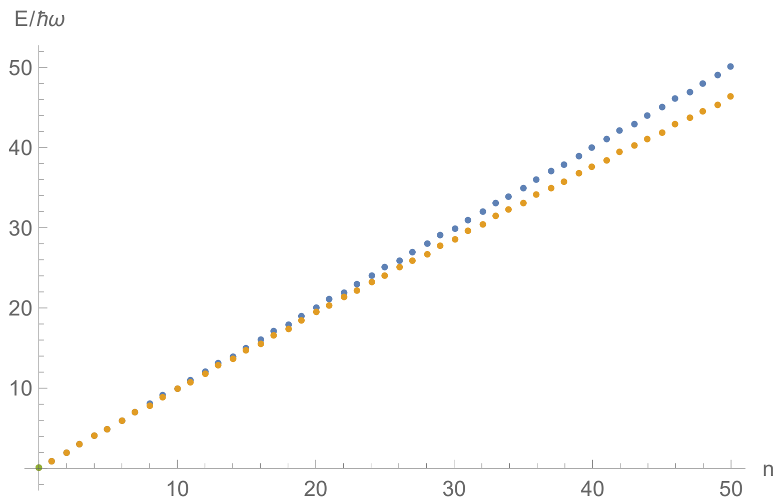

Figure 6: Comparison of the harmonic oscillator energies E_n (blue dots) and the perturbed energies E'_n (orange dots), using the parameters $\beta = 0.08$, $r_0 = 20\frac{l}{\sqrt{2}}$

Having calculated the change in the energy spectrum, we are now interested in the approximate eigenstates $|\varphi_n\rangle$. In contrast to the energy change, according to (5.37), the crucial terms are now the off-diagonal elements $(V_{per})_{kn}$. Using the explicit expression for V_{per} (5.33), we find that

$$(V_{per})_{kn} = \lambda\hbar\tilde{\omega}\left(\langle k|\left(a+a^\dagger\right)^3|n\rangle - \eta\,\langle k|\left(a+a^\dagger\right)^4|n\rangle\right)\ . \tag{5.43}$$

The first scalar product, $\langle k|\left(a+a^\dagger\right)^3|n\rangle$, is only non-zero for $k \in \{n-3, n-1, n+1, n+3\}$ and the second one, $\langle k|\left(a+a^\dagger\right)^4|n\rangle$, is non-zero for

$k \in \{n-4, n-2, n+2, n+4\}$ (further details can be found in Appendix D). It follows, that we can write the approximate eigenstates, according to (5.37), as

$$|\varphi_n\rangle = |n\rangle + \lambda \left[\sum_{k=0}^{3} c_{k,n} |n+2k-3\rangle - \eta \sum_{k=0;\ k\neq 2}^{4} d_{k,n} |n+2k-4\rangle \right] \tag{5.44}$$

$$\equiv |n\rangle + \lambda |\phi_n\rangle \ , \tag{5.45}$$

where we have introduced $|\phi_n\rangle$ to collect all first order contributions. The coefficients in (5.44) are given by

$$c_{k,n} = \frac{\langle n+2k-3| \left(a+a^\dagger\right)^3 |n\rangle}{-2k+3} \ , \tag{5.46}$$

$$d_{k,n} = \frac{\langle n+2k-4| \left(a+a^\dagger\right)^4 |n\rangle}{-2k+4} \ , \tag{5.47}$$

and they explicitly read:

$$\begin{aligned}
c_{0,n} &= \frac{\sqrt{(n-2)(n-1)n}}{3} & d_{0,n} &= \frac{\sqrt{n(n-1)(n-2)(n-3)}}{4} \\
c_{1,n} &= 3n\sqrt{n} & d_{1,n} &= \frac{\sqrt{n(n-1)}(4n-2)}{2} \\
c_{2,n} &= -3\sqrt{n+1}(n+1) & d_{3,n} &= \frac{-\sqrt{(n+2)(n+1)}(4n+6)}{2} \\
c_{3,n} &= \frac{-\sqrt{(n+1)(n+2)(n+3)}}{3} & d_{4,n} &= \frac{-\sqrt{(n+4)(n+3)(n+2)(n+1)}}{4}
\end{aligned} \tag{5.48}$$

We can observe from (5.44) that in the presence of V_{per}, up to first order, each harmonic oscillator eigenstate $|n\rangle$ turns into a superposition of the harmonic oscillator eigenstates $|k\rangle$, with k running from $n-4$ to $n+4$. However, for small λ and η, the contribution of the states $|n \pm 1\rangle$ and $|n \pm 3\rangle$ will be stronger than the one of the states $|n \pm 2\rangle$ and $|n \pm 4\rangle$. In spite of the superposition, the approximate states are orthonormal up to first order, i.e.

$$\langle \varphi_n | \varphi_k \rangle = \delta_{n,k} + \mathcal{O}(\lambda^2) \ , \tag{5.49}$$

which follows from V_{per} being Hermitian[5].

Finally, it will be instructive to look at the expectation value $\langle\varphi_n|a + a^\dagger|\varphi_n\rangle$, related to the expectation value of the position operator through (5.27). We find that

$$\langle\varphi_n|a + a^\dagger|\varphi_n\rangle = \underbrace{\langle n|a + a^\dagger|n\rangle}_{=0} + 2\lambda\,\langle\phi_n|a + a^\dagger|n\rangle + \mathcal{O}(\lambda^2) \tag{5.50}$$

$$= 2\lambda\left(\sqrt{n}\;c_{1,n} + \sqrt{n+1}\;c_{2,n}\right) + \mathcal{O}(\lambda^2) \tag{5.51}$$

$$= -6\lambda\,(2n+1) + \mathcal{O}(\lambda^2)\;, \tag{5.52}$$

which is non-zero, in contrast to the harmonic oscillator. Even more, it is not only non-zero but also grows linearly with n. This is because the slope of the full potential increases slower than the slope of the harmonic oscillator in the direction of the source of the potential V_{per}, whereas it increases faster in the opposite direction. Thus, in the presence of V_{per}, the expectation value of the position operator gets slightly shifted from the (already shifted) minimum towards the source of V_{per}, implying a higher chance to find a particle in a state $|\varphi_n\rangle$ in the region $\xi < 0$ than in $\xi > 0$.

After having calculated the approximate spectrum E'_n (5.41) and the approximate eigenstates $|\varphi_n\rangle$ (5.44) of the Hamiltonian $\mathbf{H}$ (5.31), we could show that the presence of the perturbation V_{per} causes non-trivial effects on the harmonic oscillator. We could observe a n-dependent change in the spectrum, and a change in the behavior of the eigenfunctions. We will study some further properties in the next section.

5.4 Time Evolution of a Coherent State

In this section we will use the results obtained in the previous section to further study the influence of the gravitational potential on the behavior of the harmonic oscillator. In particular, we will examine the time evolution of a coherent state under the influence of the Hamiltonian $\mathbf{H}$ given in (5.31). A coherent state represents the closest approximation to the classical harmonic oscillator, in the sense that its time evolution in a harmonic oscillator potential can be described by a Gaussian wave packet with constant width, oscillating around the potential's minimum. Our aim is now to see how this oscillatory behavior will be disturbed due to the presence of V_{per}.

Let us begin by recalling some properties of the coherent state. The coherent state $|\alpha\rangle$ is defined as the right eigenstate of the annihilation operator,

[5] It's an inherent property of time-independent perturbation theory for Hermitian perturbations. For further details, like renormalization, etc., we refer to [40].

i.e. $a\,|\alpha\rangle = \alpha\,|\alpha\rangle$ and can be written in terms of harmonic oscillator energy eigenstates $|n\rangle$ as

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\,|n\rangle \ , \tag{5.53}$$

where $\alpha \equiv |\alpha|e^{i\varphi}$ is a complex number. Then, the time evolution of a coherent state with respect to the harmonic oscillator Hamiltonian $\mathbf{H}_0$, is given by

$$|\alpha(t)\rangle = e^{-i\mathbf{H}_0 t/\hbar}\,|\alpha\rangle = e^{-\frac{|\alpha|^2+i\tilde{\omega}}{2}} \sum_{n=0}^{\infty} \frac{\alpha(t)^n}{\sqrt{n!}}\,|n\rangle \ , \tag{5.54}$$

where we have defined $\alpha(t) \equiv |\alpha|e^{-i(\tilde{\omega}t-\varphi)}$. It follows that the probability density of the corresponding wavefunction $\psi_\alpha(\xi,t) = \langle\xi|\alpha(t)\rangle$ is a Gaussian

$$|\psi_\alpha(\xi,t)|^2 = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{(\xi-\langle\xi(t)\rangle)^2}{2}} \ , \tag{5.55}$$

where the oscillating mean $\langle\xi(t)\rangle$ and the constant width $\Delta\xi$ are given by

$$\langle\xi(t)\rangle = 2\,|\alpha|\cos(\tilde{\omega}t-\varphi) \ , \tag{5.56}$$

$$\Delta\xi = 1 \ . \tag{5.57}$$

This means that the Gaussian oscillates with frequency $\tilde{\omega}$ and amplitude $2|\alpha|$ around the potential's minimum at $\xi = 0$, while keeping its shape constant. This behavior shows explicitly why the time evolution of a coherent state can be considered as the closest approximation to a classical harmonic oscillator.

Given this brief summary of the coherent state's properties, we are now interested in how the behavior of a coherent state changes in the presence of the perturbation given by V_{per} (5.33). More precisely, we want to study the time evolution, its expectation value $\langle\xi(t)\rangle$ and its standard deviation $\Delta\xi(t) = \sqrt{\langle\xi^2(t)\rangle - \langle\xi(t)\rangle^2}$, where

$$\langle\xi(t)\rangle = \langle\tilde{\alpha}(t)|\,\xi\,|\tilde{\alpha}(t)\rangle \ , \tag{5.58}$$

$$\langle\xi^2(t)\rangle = \langle\tilde{\alpha}(t)|\,\xi^2\,|\tilde{\alpha}(t)\rangle \ , \tag{5.59}$$

and the expectation values are now taken with respect to $|\tilde{\alpha}(t)\rangle$, which represents the time evolution of an initially coherent state with respect to the Hamiltonian $\mathbf{H}$. Explicitly, the time evolved state $|\tilde{\alpha}(t)\rangle$ is given by

$$|\tilde{\alpha}(t)\rangle = e^{-i\mathbf{H}t/\hbar}\,|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n,l} e^{-iE_n' t/\hbar} \frac{\alpha^l}{\sqrt{l!}}\,\langle\varphi_n\,|\,l\rangle\ \ |\varphi_n\rangle \ . \tag{5.60}$$

The last equality is obtained by using the time evolution operator

$$e^{-i\mathbf{H}t/\hbar} = \sum_n e^{-iE'_n t/\hbar} \left|\varphi_n\right\rangle \left\langle\varphi_n\right| \ , \tag{5.61}$$

where E'_n and $|\varphi_n\rangle$ are the approximate energies and eigenstates obtained in the previous section, given by (5.41) and (5.44).

Before we continue, let us make some comments on the time evolution operator given in (5.61). In fact, by using (5.61) we assume that we would know $|\varphi_n\rangle$ as well as E'_n up to arbitrary n. However, recalling the discussion on the approximations given in Section 5.2, this is not the case. The results obtained in the previous section are only valid for $n \ll \frac{1}{5\eta^2}$. Nevertheless, the factor $1/\sqrt{l!}$ in (5.60) ensures that terms with large n will not contribute significantly. This becomes clear when calculating the scalar product $\langle\varphi_n \,|\, l\rangle$, which yields

$$\langle\varphi_n \,|\, l\rangle = \langle n \,|\, l\rangle + \lambda \,\langle\phi_n \,|\, l\rangle \tag{5.62}$$

$$= \delta_{n,\,l} \ + \ \lambda \left[\sum_{k=0}^{3} c_{k,n}\, \delta_{n+2k-3,\,l} - \eta \sum_{k=0,k\neq 2}^{4} d_{k,n}\, \delta_{n+2k-4,\,l}\right] \ , \tag{5.63}$$

where $|\phi_n\rangle$ is given in (5.44). Thus, for each n, (5.63) shows that the scalar product $\langle\varphi_n \,|\, l\rangle$ is only nonzero for $l = n, n+2k-3, n+2k-4$. This means that the terms appearing in (5.60) are suppressed at least by the factor $1/\sqrt{(n-4)!}$ for each n, respectively. This justifies to use the time evolution operator as it is given in (5.61).

In the following it will be convenient to make use of (5.63) to rewrite $|\tilde{\alpha}(t)\rangle$ by taking the sum over l in (5.60). We obtain

$$|\tilde{\alpha}(t)\rangle = e^{-\frac{|\alpha|^2}{2}} \Big(|\psi_0\rangle + \lambda\,|\psi_1\rangle + \lambda\,|\psi_2\rangle - \lambda\eta\,|\psi_3\rangle \Big) \ , \tag{5.64}$$

where we have defined:

$$|\psi_0\rangle \equiv \sum_n e^{-iE'_n t/\hbar} \frac{\alpha^n}{\sqrt{n!}} \ |n\rangle \ , \tag{5.65}$$

$$|\psi_1\rangle \equiv \sum_n e^{-iE'_n t/\hbar} \frac{\alpha^n}{\sqrt{n!}} \ |\phi_n\rangle \ , \tag{5.66}$$

$$|\psi_2\rangle \equiv \sum_n \sum_{k=0}^{3} e^{-iE'_n t/\hbar} \frac{\alpha^{n+2k-3}}{\sqrt{(n+2k-3)!}} \, c_{k,n} \ |n\rangle \ , \tag{5.67}$$

$$|\psi_3\rangle \equiv \sum_n \sum_{k=0,k\neq 2}^{4} e^{-iE'_n t/\hbar} \frac{\alpha^{n+2k-4}}{\sqrt{(n+2k-4)!}} \, d_{k,n} \ |n\rangle \ . \tag{5.68}$$

Notice that we have dropped second order terms (i.e. second order in the parameter λ) in (5.67) and (5.68). It is worth mentioning, that $e^{-\frac{|\alpha|^2}{2}} |\psi_0\rangle$ looks similar to $|\alpha(t)\rangle$, but with different energies, whereas $|\psi_1\rangle$, $|\psi_2\rangle$ and $|\psi_3\rangle$ are a result of both the energy change and the change in the states. The true merit of the representation given in (5.64) is that we can now calculate the expectation values $\langle\xi(t)\rangle$ and $\langle\xi^2(t)\rangle$ term by term, keeping track of the parts multiplied by λ. In fact, using (5.64), we immediately obtain the expression for $\langle\xi(t)\rangle$, which is

$$\begin{aligned}\langle\tilde{\alpha}(t)|\,\xi\,|\tilde{\alpha}(t)\rangle = e^{-|\alpha|^2}\Big(&\langle\psi_0\,|\,\xi\,|\,\psi_0\rangle + \lambda\; 2\mathrm{Re}\left[\langle\psi_0\,|\,\xi\,|\,\psi_1\rangle\right] \\ &+\lambda\; 2\mathrm{Re}\left[\langle\psi_0\,|\,\xi\,|\,\psi_2\rangle\right] - \lambda\eta\; 2\mathrm{Re}\left[\langle\psi_0\,|\,\xi\,|\,\psi_3\rangle\right]\Big)\,,\end{aligned} \tag{5.69}$$

where we have dropped again second order terms. The scalar products appearing in (5.69) can then be obtained by going back to the explicit expressions for the states $|\psi_0\rangle$, $|\psi_1\rangle$, $|\psi_2\rangle$ and $|\psi_3\rangle$, given in (5.65)-(5.68), respectively. We find:

$$\langle\psi_0\,|\,\xi\,|\,\psi_0\rangle = \sum_{n,m}\frac{\alpha^n\alpha^{*m}}{\sqrt{n!\;m!}}e^{-it/\hbar(E'_n-E'_m)}\,\langle m\,|\,\xi\,|\,n\rangle\;, \tag{5.70}$$

$$\langle\psi_0\,|\,\xi\,|\,\psi_1\rangle = \sum_{n,m}\frac{\alpha^n\alpha^{*m}}{\sqrt{n!\;m!}}e^{-it/\hbar(E'_n-E'_m)}\,\langle m\,|\,\xi\,|\,\phi_n\rangle\;, \tag{5.71}$$

$$\langle\psi_0\,|\,\xi\,|\,\psi_2\rangle = \sum_{n,m}\sum_{k=0}^{3}\frac{\alpha^{n+2k-3}\alpha^{*m}}{\sqrt{(n+2k-3)!\;m!}}e^{-it/\hbar(E'_n-E'_m)}c_{k,n}\,\langle m\,|\,\xi\,|\,n\rangle\;, \tag{5.72}$$

$$\langle\psi_0\,|\,\xi\,|\,\psi_3\rangle = \sum_{n,m}\sum_{k=0,k\neq 2}^{4}\frac{\alpha^{n+2k-4}\alpha^{*m}}{\sqrt{(n+2k-4)!\;m!}}e^{-it/\hbar(E'_n-E'_m)}d_{k,n}\,\langle m\,|\,\xi\,|\,n\rangle\;. \tag{5.73}$$

Thus, the remaining task is to calculate the scalar products $\langle m\,|\,\xi\,|\,n\rangle$ and $\langle m\,|\,\xi\,|\,\phi_n\rangle$. This can be done, using the ladder operator representation of ξ, given in (5.27). The results are

$$\langle m|a+a^\dagger|n\rangle = \sqrt{m+1}\quad\delta_{m+1,n} + \sqrt{n+1}\quad\delta_{n+1,m} \tag{5.74}$$

and

$$\begin{aligned}\langle m|a+a^\dagger|\phi_n\rangle =&\sqrt{m+1}\ \ \langle m+1|\phi_n\rangle + \sqrt{m}\ \ \langle m-1|\phi_n\rangle && (5.75)\\ =&\sqrt{m+1}\left[\sum_{k=0}^{3} c_{k,n}\ \delta_{m+1,\ n+2k-3} - \eta \sum_{k=0,k\neq 2}^{4} d_{k,n}\ \delta_{m+1,\ n+2k-4}\right]\\ &+\sqrt{m}\left[\sum_{k=0}^{3} c_{k,n}\ \delta_{m-1,\ n+2k-3} - \eta \sum_{k=0,k\neq 2}^{4} d_{k,n}\ \delta_{m-1,\ n+2k-4}\right] . && (5.76)\end{aligned}$$

This enables us to evaluate the expressions given in (5.70)-(5.73) and therefore determine $\langle \xi(t)\rangle$ by using (5.69). However, we can see in (5.69) that, due to the multiplication with λ, the term $\langle \psi_0 \,|\, \xi \,|\, \psi_0\rangle$ will be more important than the other ones. For that reason, we will leave the detailed expressions for the terms other than $\langle \psi_0 \,|\, \xi \,|\, \psi_0\rangle$ for the Appendix E. The explicit calculation of these terms would be analogous to the calculation of $\langle \psi_0 \,|\, \xi \,|\, \psi_0\rangle$, but rather cumbersome, and not giving further insight. Instead, we will collect the remaining terms of (5.69) in the function $F(t,\lambda)$ for convenience:

$$F(t,\lambda) \equiv\ 2\mathrm{Re}\left[\langle \psi_0 \,|\, \xi \,|\, \psi_1\rangle\right] + 2\mathrm{Re}\left[\langle \psi_0 \,|\, \xi \,|\, \psi_2\rangle\right] - \eta\ 2\mathrm{Re}\left[\langle \psi_0 \,|\, \xi \,|\, \psi_3\rangle\right]\ . \quad (5.77)$$

Now, to determine the expression for $\langle \psi_0 \,|\, \xi \,|\, \psi_0\rangle$, we can use (5.74) to obtain

$$\begin{aligned}\langle \psi_0 \,|\, \xi \,|\, \psi_0\rangle =&\sum_{n,m} \frac{\alpha^n {\alpha^*}^m}{\sqrt{n!\ m!}} e^{-it/\hbar(E'_n-E'_m)}\sqrt{m+1}\quad \delta_{m+1,n}\\ &+\sum_{n,m} \frac{\alpha^n {\alpha^*}^m}{\sqrt{n!\ m!}} e^{-it/\hbar(E'_n-E'_m)}\sqrt{n+1}\quad \delta_{n+1,m} && (5.78)\\ =&\sum_{n}\left[\alpha\ \frac{|\alpha|^{2n}}{n!} e^{-it/\hbar(E'_{n+1}-E'_n)} + c.c.\right] && (5.79)\\ =&2|\alpha|\ \ e^{|\alpha|^2\cos(12\tilde{\omega}t\lambda\eta)}\cos\left[\tilde{\omega}t\,(1-12\lambda\eta) - |\alpha|^2\sin\left(12\tilde{\omega}t\lambda\eta\right) - \varphi\right] , && (5.80)\end{aligned}$$

where the second equality is obtained by summation over n in the first line and summation over m in the second line, and the last equality follows from the summation over n, while using the expression for the energies E'_n, given in (5.41).

Finally, using (5.80) together with (5.77), we obtain from (5.69) the following expression for the expectation value:

$$\langle \xi(t) \rangle = 2\,|\alpha|\ \ e^{|\alpha|^2(\cos(12\tilde{\omega} t\lambda\eta)-1)} \cos\left[\tilde{\omega} t\,(1-12\lambda\eta) - |\alpha|^2 \sin(12\tilde{\omega} t\lambda\eta) - \varphi\right] \tag{5.81}$$
$$+ \lambda\, e^{-|\alpha|^2} F(t,\lambda)\ .$$

In this form it is easy to see that, in the limit $\lambda \to 0$, we recover the oscillating expectation value given in (5.56), corresponding to the unperturbed harmonic oscillator. However, for $\lambda \neq 0$, we can observe a few important differences. First, the factor $e^{|\alpha|^2(\cos(12\tilde{\omega} t\lambda\eta)-1)}$ in the first line of (5.81) will cause a damping of the amplitude that oscillates itself with period $T = \frac{\pi}{6\tilde{\omega}\lambda\eta}$. Second, the frequency will become dependent on the amplitude. And third, taking into account the second line of (5.81), there will be other slight changes from which the most important ones are a change in the maximum of the amplitude and a shift of the oscillation's center (see Appendix E). To illustrate the effects, a plot of $\langle \xi(t) \rangle$ is given in Figure 7.

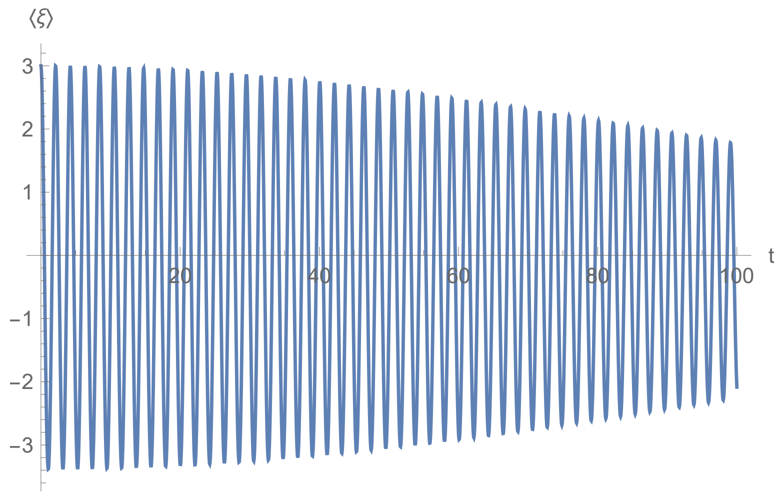

Figure 7: Oscillation of the expectation value. $\lambda = 0.008$, $\eta = 0.02$, $\alpha = 1.5$, $\tilde{\omega} = 3\frac{1}{s}$

Let us consider now the standard deviation $\Delta\xi(t)$. Having already calculated $\langle \xi(t) \rangle$, we just have to determine the expectation value $\langle \xi^2(t) \rangle$. This can be done in a completely analogous way, by following the steps of the previous calculation, but replacing ξ by ξ^2. Thus, we can start from

$$\langle\tilde{\alpha}(t)|\,\xi^2\,|\tilde{\alpha}(t)\rangle = e^{-|\alpha|^2}\Big(\left\langle\psi_0\,\middle|\,\xi^2\,\middle|\,\psi_0\right\rangle + \lambda\;2\mathrm{Re}\left[\left\langle\psi_0\,\middle|\,\xi^2\,\middle|\,\psi_1\right\rangle\right] \tag{5.82}$$

$$+\lambda\;2\mathrm{Re}\left[\left\langle\psi_0\,\middle|\,\xi^2\,\middle|\,\psi_2\right\rangle\right] - \lambda\eta\;2\mathrm{Re}\left[\left\langle\psi_0\,\middle|\,\xi^2\,\middle|\,\psi_3\right\rangle\right]\Big)\;.$$

Again, we only want to give the term $\langle\psi_0\,|\,\xi^2\,|\,\psi_0\rangle$ explicitly, while collecting the remaining λ-terms in a function $G(t,\lambda)$ (see Appendix E), defined by:

$$G(t,\lambda) \equiv\; 2\mathrm{Re}\left[\left\langle\psi_0\,\middle|\,\xi^2\,\middle|\,\psi_1\right\rangle\right] + 2\mathrm{Re}\left[\left\langle\psi_0\,\middle|\,\xi^2\,\middle|\,\psi_2\right\rangle\right] - \eta\;2\mathrm{Re}\left[\left\langle\psi_0\,\middle|\,\xi^2\,\middle|\,\psi_3\right\rangle\right]\;, \tag{5.83}$$

Using

$$\langle m|\left(a+a^\dagger\right)^2|n\rangle = \sqrt{(m+1)(m+2)}\;\delta_{m+2,n} + \sqrt{(n+1)(n+2)}\;\delta_{n+2,m} + (2n+1)\;\delta_{m,n}\;, \tag{5.84}$$

we find that

$$\langle\psi_0|\xi^2|\psi_0\rangle = \sum_n\left[\alpha^2\,\frac{|\alpha|^{2n}}{n!}e^{-it/\hbar(E'_{n+2}-E'_n)} + c.c.\right] + \sum_n\left[\frac{|\alpha|^{2n}}{n!}\,(2n+1)\right]\;. \tag{5.85}$$

Then, carrying out the sums in (5.85), and using the definition of $G(t,\lambda)$ (5.83), the expectation value $\langle\xi^2(t)\rangle$ (5.82) turns out to be:

$$\begin{aligned}\langle\alpha(t)|\,\xi^2\,|\alpha(t)\rangle = {}& 1+2|\alpha|^2\\ &+2\;|\alpha|^2\;e^{|\alpha|^2(\cos(24\tilde{\omega}t\lambda\eta)-1)}\\ &\quad\times\cos\left[2\tilde{\omega}t\,(18\lambda\eta-1)+|\alpha|^2\sin\left(24\tilde{\omega}t\lambda\eta\right)-2\varphi\right]\\ &+\lambda\;e^{-|\alpha|^2}\;G(t,\lambda)\;.\end{aligned} \tag{5.86}$$

Hence, the standard deviation reads

$$
\begin{aligned}
\Delta\xi(t) =& \sqrt{\langle \xi^2(t)\rangle - \langle \xi(t)\rangle^2} \\
=& \Big[1 + 2|\alpha|^2 + 2\,|\alpha|^2\ e^{|\alpha|^2(\cos(24\tilde{\omega}t\lambda\eta)-1)} \\
& \quad \times \cos\left[2\tilde{\omega}t\,(18\lambda\eta - 1) + |\alpha|^2 \sin\left(24\tilde{\omega}t\lambda\eta\right) - 2\varphi \right] \\
& + \lambda\, e^{-|\alpha|^2}\ G(t,\lambda) - 4\,|\alpha|^2\ \ e^{2\,|\alpha|^2(\cos(12\tilde{\omega}t\lambda\eta)-1)} \\
& \quad \times \cos^2\left[\tilde{\omega}t\,(1 - 12\lambda\eta) - |\alpha|^2 \sin\left(12\tilde{\omega}t\lambda\eta\right) - \varphi \right] \\
& - 2\,\lambda\, e^{-2\,|\alpha|^2}\ \langle \psi_0 \,|\, \xi \,|\, \psi_0 \rangle\ F(t,\lambda) \Big]^{1/2}, \qquad (5.87)
\end{aligned}
$$

where we have again neglected second order terms. Then, for $\lambda \to 0$, we recover $\Delta\xi(t) = 1$, in complete accordance with the standard deviation for the harmonic oscillator. On the other hand, for $\lambda \neq 0$ we can notice that the standard deviation will not only differ from one, but will also become time-dependent, which can be seen in Figure 8. We can see there that the standard deviation oscillates in time, but there is also an overall increase of the standard deviation on a larger time scale. Compared with Figure 7, one can observe, that the decrease in the amplitude of the expectation value over time, appears together with the overall increase of the standard deviation. However, both effects have a maximum at some time, and the initial situation will be recovered at some later time, i.e. both effects are periodic on a larger time scale, too.

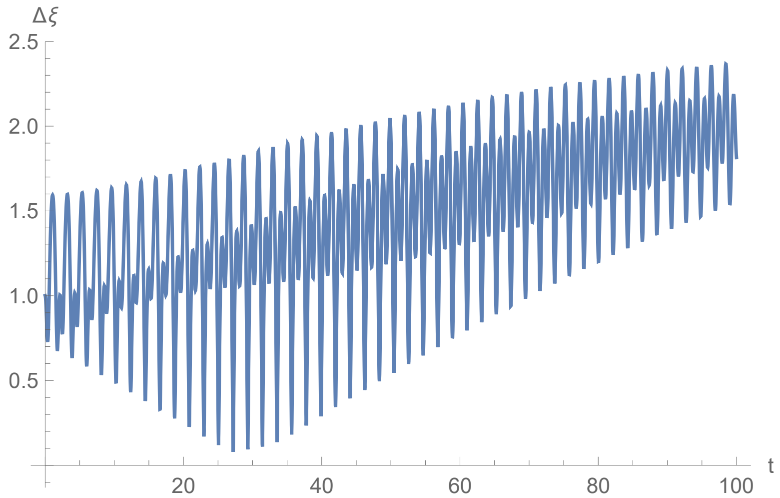

Figure 8: Behavior of the standard deviation. $\lambda = 0.008$, $\eta = 0.02$, $\alpha = 1.5$, $\tilde{\omega} = 3\frac{1}{s}$

Finally, it will be instructive to look at probability distribution and how it evolves in time. For that purpose, we have to replace $|n\rangle$ in (5.60) by $\psi_n = \langle\xi|n\rangle$, given in (5.24), to obtain the time evolved wavefunction $\langle\xi|\tilde{\alpha}(t)\rangle$ of the coherent state. The probability distribution is then given by:

$$p(\xi, t) = |\langle\xi|\tilde{\alpha}(t)\rangle|^2 \ . \tag{5.88}$$

To illustrate the difference between the oscillating Gaussian probability distribution (5.55) and the time evolution of the probability distribution corresponding to (5.88), a plot is given in Figure 9 and 10, where the behavior during one period is shown at two clearly distinct times, respectively.

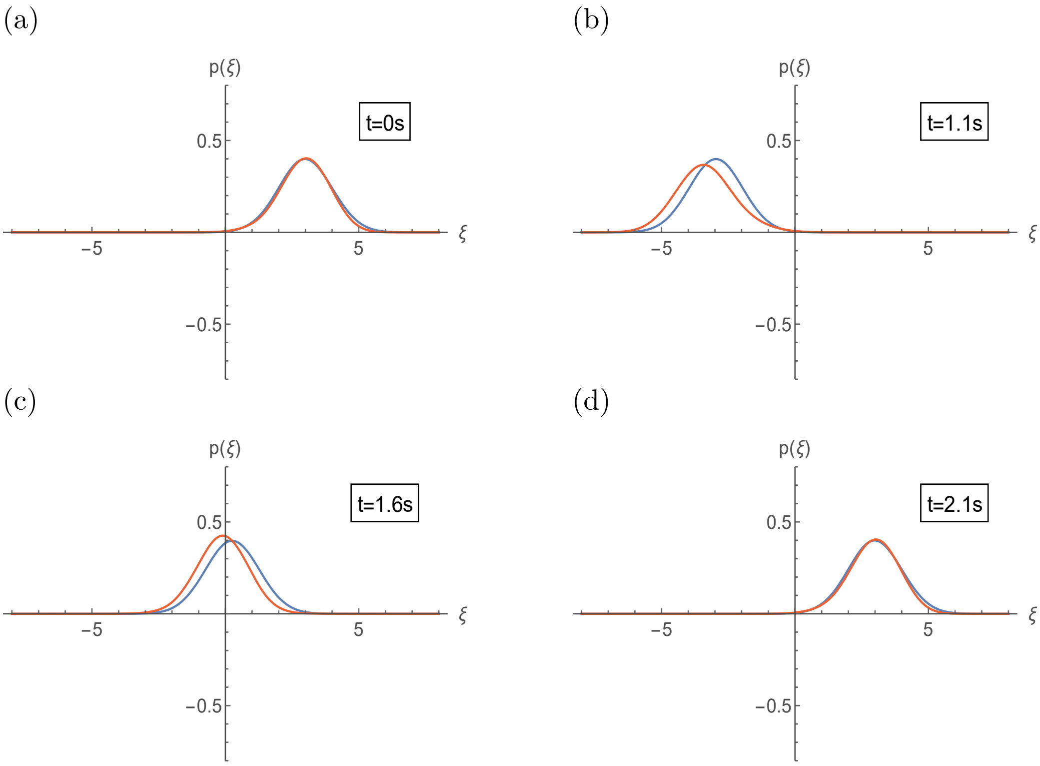

Figure 9: Time evolution of the Gaussian distribution (blue line) and the distribution $p(\xi, t)$ (red line), during one period (approximately). The parameters are chosen to be $\lambda = 0.008$, $\eta = 0.02$, $\alpha = 1.5$ and the frequency is $\tilde{\omega} = 3\ \frac{1}{s}$. The probability distributions start at $t = 0s$ (a), where $p(\xi, t)$ is still quite similar to the Gaussian. They evolve to the left up to some turning point and move backwards again (b). One can see that $p(\xi, t)$ has already changed its shape. While moving further to the right (c), the distributions become closer end they end up in (d), being quite similar again.

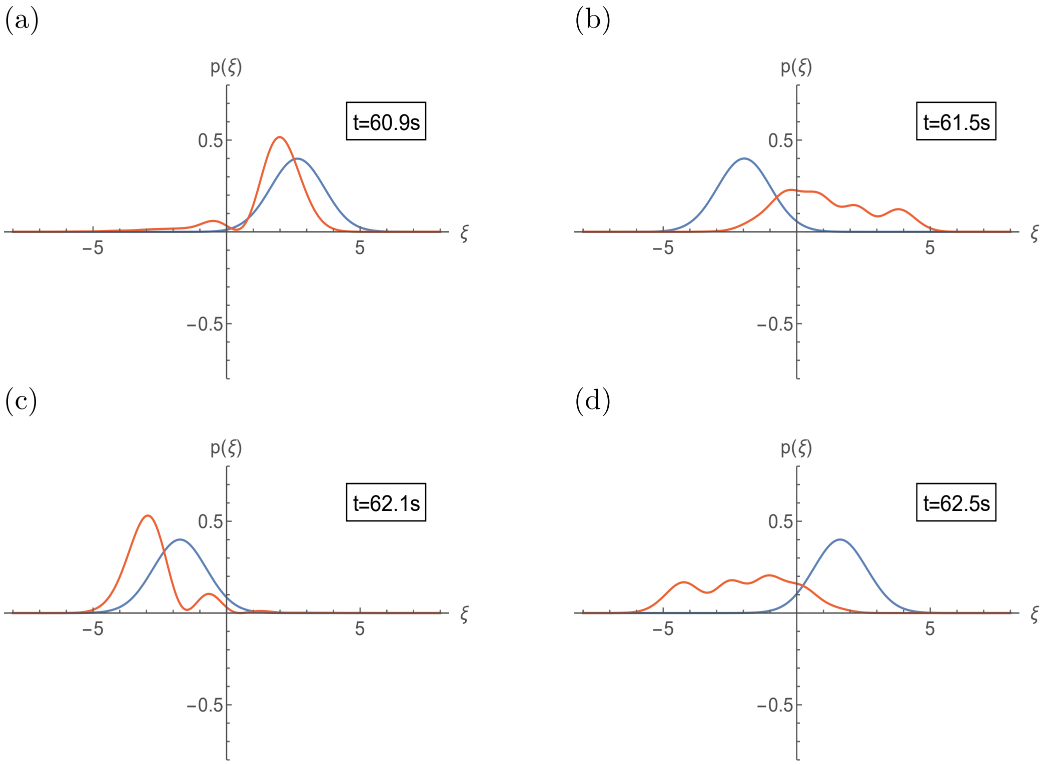

Figure 10: Time evolution of the Gaussian distribution (blue line) and the distribution $p(\xi, t)$ (red line), during one period (approximately). The parameters are chosen to be $\lambda = 0.008$, $\eta = 0.02$, $\alpha = 1.5$, $\tilde{\omega} = 3\ \frac{1}{s}$. The probability distributions start at $t = 60.9s$ (a), where $p(\xi, t)$ has already changed its shape, compared to the Gaussian. They evolve to the left (b) while $p(\xi, t)$ has started to become broader. Shortly after reaching the turning point (c), $p(\xi, t)$ is narrower again, while still not having recovered Gaussian shape. Moving further to the right (d), $p(\xi, t)$ becomes broadened again and at the end of the oscillation a situation close to (a) will recur.

As a result, we can see from the calculations and from Figures 9 and 10, that the time evolution of a coherent state, in the presence of the perturbation, clearly differs from the time evolution caused by a harmonic oscillator potential. Although the probability distribution $p(\xi, t)$ still oscillates, i.e. providing an oscillating expectation value, its shape changes within one period. This change is small in the beginning, where $p(\xi, t)$ could still be approximated by an oscillating Gaussian, but becomes crucial at some later time, leading to a damping in the amplitude and a broadening of the probability distribution, i.e. an overall increase in the standard deviation. However, this effect is reversible and the initial situation will be recovered at a much later point of time. Let us finally mention that the time evolution of the coherent state provides a good example to illustrate the effect of the perturbation, since the originally small effects can be amplified by time and the changes become visible.

5.5 Size of the Parameters and Discussion of the Effects

We have given the theoretical discussion of the effects of the gravitational potential beyond its linear approximation. In this section, we want to put it in an experiment related context. That is, we want to estimate the size of the effects with respect to the parameter values in concrete physical situations. For that purpose, we consider a particle of mass m trapped in a harmonic oscillator potential with trapping frequency ω. Additionally, the particle is subject to the gravitational potential produced by an object of mass M placed at a distance of r_0. It is then clear that the size of the effects depends on the behavior of the gravitational potential with respect to the length scale of the harmonic oscillator. In particular, we expect the size of the effects to increase with increasing M, decreasing r_0, and increasing typical length of the oscillator $l = \sqrt{\hbar/m\,\omega}$.

We have found in the previous sections that the size of the first non-trivial effects, i.e. effects other than constant shifts, is determined by the quantities λ and $\lambda\eta$. Their explicit form is given in (5.20) and (5.21), and they are related to the parameters of the setup given above as follows:

$$\lambda = \lambda\,(\,\beta,\; l/r_0)\ , \quad \eta = \eta\,(\,\beta,\; l/r_0) \tag{5.89}$$

$$\beta = \frac{MG}{\omega^2 r_0^3}\,, \qquad\qquad l = \sqrt{\frac{\hbar}{m\,\omega}} \tag{5.90}$$

We would like to remark that the physical meaning of β is given by the ratio of the gravity gradient to the oscillator gradient around the center of the oscillator, typically being very small. Now, λ and η are monotonously increasing functions of β and they are proportional to l/r_0. Thus, to make the effects as large as possible, we have to make β and l/r_0 as large as possible. One way to achieve this is to make the distance r_0 between the source of gravity and the oscillator as small as possible. However, since any realistic source of gravity is not pointlike, there is a minimal possible distance which is given by placing the oscillator on the surface of the source. For a sphere of radius R this corresponds to $r_0 = R$. In this case, the largest possible β is given by

$$\beta_{max} = \frac{MG}{\omega^2 R^3} = \frac{4\pi\,G}{3}\,\frac{\rho}{\omega^2}\,, \tag{5.91}$$

where $R^3 = 3M/4\pi\rho$, assuming the sphere to have a constant density ρ, which is the optimal situation. This is remarkable, since it shows that β_{max} is completely determined by the density ρ of the source of gravity and the trapping frequency ω, independent of the mass M. This means that it does not make a difference for β_{max} whether to place the oscillator next to a massive object like the Earth or next to a small sphere with the same density.

A further limitation of the size of λ and η is given by an additional reasonable constraint on r_0: While reducing the distance r_0, one has to keep some distance to the surface, i.e. $r_0 \geq R+l > R$, to avoid undesired effects of the surface. We can express the distance to the surface in terms of multiples of the typical length l, obtaining

$$r_0 = R + k\,l, \text{ with } \quad k \in [1, \infty]\ , \tag{5.92}$$

from which follows that we now have

$$\beta = \frac{\beta_{max}}{\left(1 + k\frac{l}{R}\right)^3}\ , \quad \frac{l}{r_0} = \frac{l}{R}\,\frac{1}{1 + k\frac{l}{R}}\ , \tag{5.93}$$

with $k = 1$ corresponding to the case where the oscillator's distance to the surface is exactly its typical length. In this way, λ and η will now merely dependent on l/R if we assume ω (or equivalently β_{max}) to be fixed. Their behavior can be studied for β being very small, which is typically the case, by using the approximations

$$\lambda \approx \beta\,\frac{l}{2\sqrt{2}\,r_0} = \frac{\beta_{max}}{2\sqrt{2}\,\left(1 + k\frac{l}{R}\right)^4}\,\frac{l}{R}\,, \quad \eta \approx \frac{l}{\sqrt{2}\,r_0} = \frac{1}{\sqrt{2}\left(1 + k\frac{l}{R}\right)}\,\frac{l}{R}\ . \tag{5.94}$$

To illustrate the dependence of the quantities λ and $\lambda\eta$ on l/R within the approximation (5.94), we have plotted them in Figure 11 for different values of k, respectively. We can observe that for each k there exists an optimal ratio for which λ, respectively $\lambda\eta$, become maximal:

$$\lambda_{max} = \lambda\left(\frac{l}{R} = \frac{1}{3k}\right) = \frac{27}{512\sqrt{2}}\,\frac{\beta_{max}}{k}\ , \tag{5.95}$$

$$\lambda\eta_{max} = \lambda\left(\frac{2}{3k}\right)\cdot\eta\left(\frac{2}{3k}\right) = \frac{108}{3125\sqrt{2}}\,\frac{\beta_{max}}{k^2}\ . \tag{5.96}$$

It follows that the optimal situation is obtained for $k = 1$, and $l/R = 1/3$ for λ or $1/R = 2/3$ for $\lambda\eta$. This would correspond to $l/r_0 = 1/4$ and $l/r_0 = 2/5$. A more realistic situation would require to keep l/r_0 sufficiently small to avoid undesirable interactions (that is interactions other then gravitational

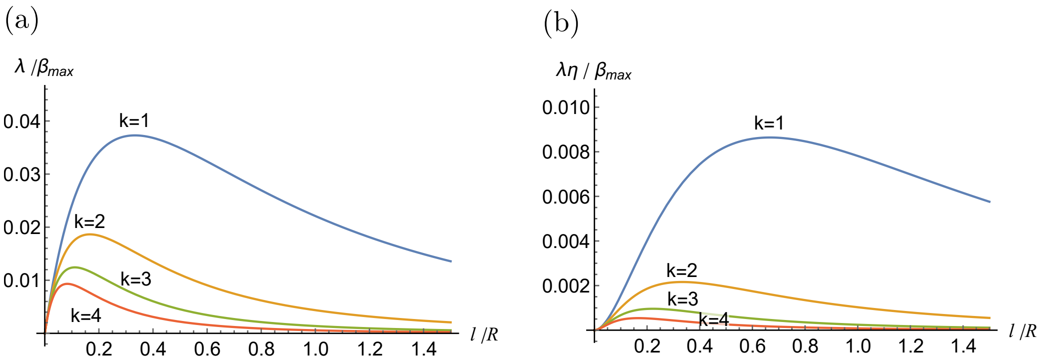

Figure 11: Dependence of the quantities (a) λ and (b) $\lambda\eta$ on the ratio l/R for different values of k, corresponding to distances to the surface of k multiples of l.

ones) between the source of gravity and the oscillator. In such case, we can use the following approximations close to the surface of the source of gravity:

$$\beta \approx \beta_{max}\,, \quad \frac{l}{r_0} \approx \frac{l}{R}\,. \tag{5.97}$$

Let us now determine the size of the effects in a concrete physical setup. For that purpose, we first have to fix the dimensions of the setup. Let us make the reasonable assumption $l/r_0 = 10^{-2} \approx l/R$. Further assume that we could trap a hydrogen atom with a frequency of 1 Hz, providing a typical length $l = 2.3 \times 10^{-4}$m. Then, this would correspond to using a sphere with radius $R = 2.3 \times 10^{-2}$m. If we now choose the sphere to consist of Osmium, which has a density of $\rho = 2.3 \times 10^4\,\mathrm{kg/m^3}$, (5.20) and (5.21) together with (5.97) will then yield for the parameters:

$$\lambda = 2.3 \times 10^{-8}\,, \quad \eta = 0.007, \quad \lambda\,\eta = 1.6 \times 10^{-10}\,. \tag{5.98}$$

For the energy difference between two consecutive energy levels (5.42), this yields

$$E'_{n+1} - E'_n = \hbar\tilde{\omega}\,(1 - 12\lambda\eta\,(n+1)) = \hbar\tilde{\omega}\left(1 - 1.9 \times 10^{-9}\,(n+1)\right)\,, \tag{5.99}$$

from which one can see that the relative change in the transition energy is $1.9 \times 10^{-9}(n+1)$, corresponding to a change in the transition frequency on the nanohertz scale for small n. The position expectation values will be shifted from the center of the harmonic oscillator according to (5.52). Using the parameters given above, the shifts will be of the order of picometers for small n:

$$\langle\varphi_n|x|\varphi_n\rangle = -6\sqrt{2}\,\lambda\,l\,(n+1/2) = -(n+1/2)\cdot 45 \times 10^{-12}\ \mathrm{m}\,. \tag{5.100}$$

This shows that the effects are very small, in accordance with what one would expect due to the smallness of the gravitational constant.

To conclude, we have shown that the increase of the parameters is limited by the density of the source of the gravitational potential, the size of the trapping frequency and the ratio between the typical length of the oscillator and its distance to the source of the gravitational potential. It follows that if we assume the density and the trapping frequency to be fixed, the strength of the effects will be controlled only by the ratio l/r_0. This means that the effect of a nearby Earth would be even weaker, since the ratio l/r_0 would be much smaller in that case. This can be seen in Table 1, where we compare λ, η and $\lambda\eta$ of the setup given above with the optimal case (using the same density and trapping frequency) and the case of the oscillator being close to the surface of the Earth (using the same trapping frequency but the density of the Earth).

	λ	η	$\lambda\eta$
Concrete Setup	2.3×10^{-8}	0.007	1.6×10^{-10}
Optimal Case	2.4×10^{-7}	0.18	4.2×10^{-8}
Earth	2.0×10^{-17}	2.5×10^{-11}	5.0×10^{-28}

Table 1: Values of λ, η and $\lambda\eta$ for the concrete physical setup, for the optimal case, and the case of an oscillator near the surface of the Earth.

6 Conclusion

In this work we have analyzed the quantum harmonic oscillator under the influence of gravity. We have presented a consistent way to construct a model of a relativistic harmonic oscillator with an underlying gravitational background field, which in the non-relativistic limit provides the Schrödinger equation of a harmonic oscillator plus Newtonian gravitational potential, in complete accordance with what one would expect from non-relativistic quantum mechanics.

We would like to emphasize, that the construction of our model is based on adding a gravitational background field to the AdS metric, which has been shown is suitable to simulate a RHO. In this way, the model is based on the more general concept of quantum field theory in curved spacetime and does not require any *ad hoc* modifications (in contrast to other approaches).

We could observe that the effect of the gravitational background field leads to additional terms in the Klein-Gordon equation of the RHO (3.10), and the strength of their contribution is controlled by the ratio between the Schwarzschild radius and the center of the harmonic oscillator r_s/r_0. The full quantum model could, in principle, be obtained from the resulting Klein-Gordon equation (4.9), in the same way as it has been shown for the RHO, i.e. using the Newton-Wigner interpretation to obtain the relativistic wavefunctions with corresponding energy spectrum. However, exactly solving the Klein-Gordon equation was not possible, but some ideas to solve it approximately are discussed in the Appendix A.

Instead, we have shown that we can expand the Klein-Gordon equation of our model up to the order c^{-2} to obtain the Schrödinger equation of a harmonic oscillator, plus the Newtonian gravitational potential and relativistic corrections, resulting from the RHO and the gravitational background field (4.17). It is worth mentioning that the corrections corresponding to the RHO are equivalent to the ones obtained in [36], where they are calculated with a completely different method.

The effect of the Newtonian potential on the quantum harmonic oscillator has been studied beyond the linear approximation of the potential. Up to the quadratic order of the Newtonian potential, the effects have been shown to be simple shifts in the position, the frequency and the zero-point energy, and can therefore be absorbed analytically in a shifted harmonic oscillator potential. We could observe non-trivial effects resulting from the higher orders of the Newtonian potential by treating it perturbatively. In particular, the presence of the gravitational potential leads to a change in the energy spectrum. That is, a n-dependent lowering of the energy spectrum E_n, resulting in an energy spacing that linearly grows with n. Furthermore, for a given eigenstate the probability to find a particle at some particular position x becomes higher on the side closer to the gravitational potential. Additionally, we have analyzed how the behavior of a coherent state in a harmonic oscillator changes due to the influence of the gravitational potential. The main effects are a damping in the oscillation of the coherent state's position expectation value together with a loss of coherence (which will, however, be recovered at a later time, due to unitarity).

As expected, the effects have been found to be small and will be most likely dominated by other effects in today's experiments. However, we could at least shown that the most promising scenario is the case of a harmonic oscillator being very close to a "small" object with a high density, in contrast to the case of putting the oscillator in the neighborhood of a big object like the Earth for which the ratio l/r_0 would be unfavorably small.

Additionally taking into account the relativistic corrections turns out to be more subtle. The reason is that ordinary perturbation theory, i.e. treating the problem as a perturbed quantum harmonic oscillator, is not directly applicable because the full Hamiltonian is not Hermitian with respect to the standard scalar product, due to the appearance of non-Hermitian terms like $-\frac{\hbar^2}{4m}\, V_G(x)\, \partial_x^2$. Therefore, a perturbative treatment would require a modification of the scalar product and/or the Hamiltonian themselves. However, since the Hamiltonian has been derived from higher principles, naively adding terms to the Hamiltonian, with the aim of restoring hermiticity, seems to be inconsistent. Further analyzing these issues will be an interesting task for future work.

Despite this, we can at least qualitatively discuss some effects. First of all, we can see that the relativistic correction resulting from the gravitational background field will become more and more important when placing the oscillator near the Schwarzschild radius. In fact, it could reach a non-negligible order of magnitude compared to the effects resulting from the Newtonian potential beyond its linear approximation. This would be the case when $\alpha = r_s/r_0$ becomes similar to β (on the surface of the Earth one would have $\alpha/\beta \approx 10^{-4}$, assuming a frequency of 1 Hz). Unfortunately, the parameters themselves will be unfavorably tiny in such case. However, there is an interesting qualitative effect of the relativistic correction. The term $-\alpha\, \frac{m\, V_G(x)}{4}\, \omega^2 x^2$ in the second line of (4.17) provides a coupling of the gravitational field to the harmonic oscillator. Such coupling does not appear in the non-relativistic case and is clearly a feature of the relativistic treatment. In a work by Pikovski et al.[41], such coupling has been shown to be responsible for decoherence, in principle even at the surface of the Earth, when considering a spacelike superposition of two oscillators (where the oscillators serve as the inner degrees of freedom of a superposed quantum object).

Finally, we would like to point out that the model presented in this work does neither describe a superposition like it is mentioned above nor does it consider effects like self-gravity. Rather, the model describes just a harmonic oscillator in the presence of a gravitational background field. That is, we have presented a consistent way to include gravitational effects in a quantum harmonic oscillator, leading to the Schrödinger equation of a quantum harmonic oscillator, including corrections up to the first relativistic order that could lead to interesting physical effects. In that sense, the present work can be viewed as a step towards a better understanding of how gravity could affect quantum systems.

A Approaches to Solve the RHO with Underlying Gravitational Background

Although the Klein-Gordon equation (4.9) for the combined metric has no analytic solution, we would like to discuss possible approaches and approximations.

Let us start by assuming that α is small, i.e. the harmonic oscillator being far away from the Schwarzschild radius, which will allow us to simplify equations (4.9) and (4.14). Expanding the gravitational part of (4.9) up to first order in α yields

$$\frac{\alpha}{1-\alpha}\, a_1(x) \approx \alpha \left[\left(1 + \frac{\omega^2 x^2}{c^2}\right) \frac{1}{2\left(\frac{x}{r_0}+1\right)^2} + \frac{\omega^2 x}{c^2}\left(1 - \frac{1}{\left(\frac{x}{r_0}+1\right)}\right)\right] \tag{A.1}$$

$$\frac{\alpha}{1-\alpha}\, a_2(x) \approx \alpha \left(1 + \frac{\omega^2 x^2}{c^2}\right)\left(1 - \frac{1}{\left(\frac{x}{r_0}+1\right)}\right) \tag{A.2}$$

$$\frac{\alpha}{1-\alpha}\left(1 - \frac{1}{\zeta(x)}\right) \approx \alpha \left(1 - \frac{1}{\left(\frac{x}{r_0}+1\right)}\right) , \tag{A.3}$$

where we have used that $\zeta(x)$ depends on α through the coordinate transformation (4.2). It follows that within this approximation $\zeta(x)$ takes the explicit form $\zeta(x) = x/r_0 + 1$. This means that we can approximate the gravitational part, i.e. the second line of (4.9) (and to be consistent also the weight function in the scalar products), through the substitutions $\frac{\alpha}{1-\alpha} \to \alpha$ and $\zeta(x) \to \zeta(x) = x/r_0 + 1$.

We would like to remark that the same result could have been obtained by directly approximating the Schwarzschild metric (4.1) through

$$ds^2 = -\left(1 - \frac{2\,M\,G}{c^2\,r}\right) c^2 dt_S^2 + dr^2 \tag{A.4}$$

which would correspond to the Newtonian approximation of the Schwarzschild metric with potential $U = -\frac{M\,G}{r}$.

Although we have achieved a significant simplification, there is still no known analytic solution to the resulting Klein-Gordon equation. However, we can rewrite (4.14) within the approximation in the following form

$$\frac{E^2}{\hbar^2 c^2}\,\psi = \mathcal{D}_0\,\psi + \alpha\,\mathcal{D}_G\,\psi , \tag{A.5}$$

where we have introduced the operators $\mathcal{D}_0$ and $\mathcal{D}_G$, corresponding to the RHO and the gravity part, respectively. They are given by

$$\mathcal{D}_0 = \left(1 + \frac{\omega^2 x^2}{c^2}\right)\left[-2\,\frac{\omega^2 x}{c^2}\,\partial_x - \left(1 + \frac{\omega^2 x^2}{c^2}\right)\partial_x^2 + \frac{m^2c^2}{\hbar^2}\right] \tag{A.6}$$

$$\mathcal{D}_G = -\,a_1(x)\,\partial_x - a_2(x)\,\partial_x^2 + \left(1 - \frac{1}{\frac{x}{r_0}+1}\right)\frac{m^2c^2}{\hbar^2}\,. \tag{A.7}$$

In this way we can treat the gravitational part as a small perturbation to the Klein-Gordon equation of the RHO, with perturbation parameter α. One could then try to use the known solutions of $\mathcal{D}_0$ (which are the spatial modes of the RHO model) to construct approximate solutions for the full problem $\mathcal{D}_0 + \alpha\,\mathcal{D}_G$, analogous to Schrödinger perturbation theory. However, this turns out to be more subtle, since the Hilbert spaces corresponding to the full problem and to the RHO, respectively, do not coincide. In particular, the Hilbert space corresponding to the full problem has a different domain and a different scalar product (due to the different weight function). Therefore, one would have to find a generalization of the standard Schrödinger perturbation theory for Sturm-Liouville problems.

A different approach is using conformal coordinates x', in which the metric (4.4) takes the form

$$ds^2 = A^2(x')\left(-c^2dt^2 + dx'^2\right)\,, \tag{A.8}$$

where $A^2(x') = -g_{00}(x(x'))$. In this way, the corresponding Klein-Gordon equation is given by

$$\left[\frac{1}{c^2}\partial_t^2 - \partial_{x'}^2 + A^2(x')\,\frac{m^2c^2}{\hbar^2}\right]\phi(t,x') = 0\,, \tag{A.9}$$

from which, by using the ansatz $\phi(t,x') = e^{-i\,E\,t/\hbar}\,\psi(x')$, we obtain the equation for the spatial part

$$\left[-\frac{\hbar^2}{2m}\partial_{x'}^2 + V_R(x')\right]\psi(x') = \mathcal{E}\,\psi(x')\,, \tag{A.10}$$

where we have introduced the "relativistic potential" $V_R \equiv \frac{mc^2}{2}\,(A^2-1)$ and the "energy" $\mathcal{E} \equiv \frac{E^2}{2mc^2} - \frac{mc^2}{2}$ to explicitly see that the equation for $\psi(x')$ is now formally equivalent to a time-independent Schrödinger equation. It follows that the corresponding inner product is just the standard Hilbert space product (i.e. the weight function is $\rho = 1$). The true merit of writing down the problem in this way is that one could now apply all the techniques developed for solving Schrödinger equations. Although this looks like a significant simplification, we would like to stress that, due to the coordinate transformation $x = x(x')$, the potential V_R will become a complicated expression. Thus, trying to solve (A.10) could be at least as hard as trying to solve the Klein-Gordon equation in (t, x) coordinates. However, applying approximate methods could be easier in conformal coordinates, since the inner product is the standard one there.

B Calculation of the Perturbation Potential

The perturbing potential is given by

$$V_{per} = V - \tilde{V}_0 = \underbrace{\frac{1}{2}m\omega^2 x^2}_{V_0} \underbrace{-\beta m \frac{\omega^2 r_0^3}{x + r_0}}_{V_1} - \underbrace{\left(\frac{1}{2}m\tilde{\omega}^2(x - x_0)^2 + V(x_0)\right)}_{\tilde{V}_0} \tag{B.1}$$

Defining $\tilde{x} \equiv x - x_0$, we obtain

$$V_{per} = \frac{1}{2}m\omega^2(\tilde{x} + x_0)^2 - \beta m \frac{\omega^2 r_0^3}{\tilde{x} + x_0 + r_0} - \frac{1}{2}m\tilde{\omega}^2\tilde{x}^2 - V(x_0) \tag{B.2}$$

$$= \frac{1}{2} \underbrace{m\left(\omega^2 - \tilde{\omega}^2\right)}_{\frac{\partial^2}{\partial x^2}(V_0 - V)\big|_{x=x_0}} \tilde{x}^2 + \underbrace{\frac{1}{2}m\omega^2 x_0^2}_{V_0(x_0)} + \underbrace{m\omega^2 x_0}_{\frac{\partial V_0}{\partial x}\big|_{x=x_0}} \tilde{x} + V_1(\tilde{x} + x_0) - V(x_0) \tag{B.3}$$

$$= -\frac{1}{2} \left.\frac{\partial^2 V_1}{\partial x^2}\right|_{x=x_0} \tilde{x}^2 - \left.\frac{\partial V_1}{\partial x}\right|_{x=x_0} \tilde{x} + V_1(\tilde{x} + x_0) - V_1(x_0) \tag{B.4}$$

Where we have used the conditions $\left.\frac{\partial^2 V}{\partial x^2}\right|_{x=x_0} = m\tilde{\omega}^2$ and $\left.\frac{\partial V}{\partial x}\right|_{x=x_0} = \left.\frac{\partial V_0}{\partial x}\right|_{x=x_0} + \left.\frac{\partial V_1}{\partial x}\right|_{x=x_0} = 0$ to obtain the third line. The derivatives are given by

$$\left.\frac{\partial^2 V_1}{\partial x^2}\right|_{x=x_0} = 2\,\frac{V_1(x_0)}{(x_0 + r_0)^2} \tag{B.5}$$

$$\left.\frac{\partial V_1}{\partial x}\right|_{x=x_0} = -\frac{V_1(x_0)}{x_0 + r_0} \tag{B.6}$$

and we obtain

$$V_{per} = V_1(x_0)\left(-\frac{\tilde{x}^2}{(x_0+r_0)^2} + \frac{\tilde{x}}{x_0+r_0} - 1 + \frac{1}{\frac{\tilde{x}}{x_0+r_0}+1}\right) \tag{B.7}$$

$$= -\frac{V_1(x_0)}{(x_0+r_0)^3}\frac{\tilde{x}^3}{\left(\frac{\tilde{x}}{(x_0+r_0)}+1\right)} \tag{B.8}$$

C Approximations of V_{per}

Here, we want to explain in more detail, what approximations we have done and how far they can be used. In C.1 we will discuss the convergence behavior of the integrals and in C.2 we will discuss for what range of n the approximation will hold.

C.1 Convergence of the Integrals

The exponential decrease of the harmonic oscillator's wavefunctions ψ_n can be approximated outside of several times the typical length by

$$\psi_n \approx \frac{1}{\sqrt[4]{2\pi(2n+1)}}\, e^{-\frac{\xi^2}{4(2n+1)}} \, . \tag{C.1}$$

Therefore, the product of two wavefunctions can be written as

$$\psi_n\psi_m \approx \frac{1}{\sqrt{2\pi}\,\sqrt[4]{(2n+1)(2m+1)}} e^{-\frac{\xi^2(m+n+1)}{2(2n+1)(2m+1)}} \tag{C.2}$$

$$< \frac{1}{\sqrt{2\pi(2\max(n,m)+1)}} e^{-\frac{\xi^2}{2(2\max(n,m)+1)}} \tag{C.3}$$

where the inequality is true for $\xi^2 > k^2\sqrt{2\max(n,m)+1}$ with k being large enough. It follows that the product $\psi_n\psi_m$ can be approximated by a Gaussian with standard deviation $\sigma = \sqrt{2\max(n,m)+1}$. Thus, it is quite obvious that there won't be a significant contribution to the integral in (5.23) outside several times σ. This means that the integral is convergent and it will be sufficient to integrate between $-k\ \sigma$ and $k\ \sigma$, for a suitable k, to calculate its value. We can therefore avoid to deal with piecewise functions (at least as long as the range of the integration does not contain the "singularity" of V_{per}). The convergence behavior is illustrated in Figure 12, where we have plotted the integral $\frac{1}{\lambda}\int_{-k\sqrt{2n+1}}^{k\sqrt{2n+1}} d\xi\ \ \psi_n^*\,\psi_n V_{per}$, dependent on the integration range k with different values for η and for different n's.

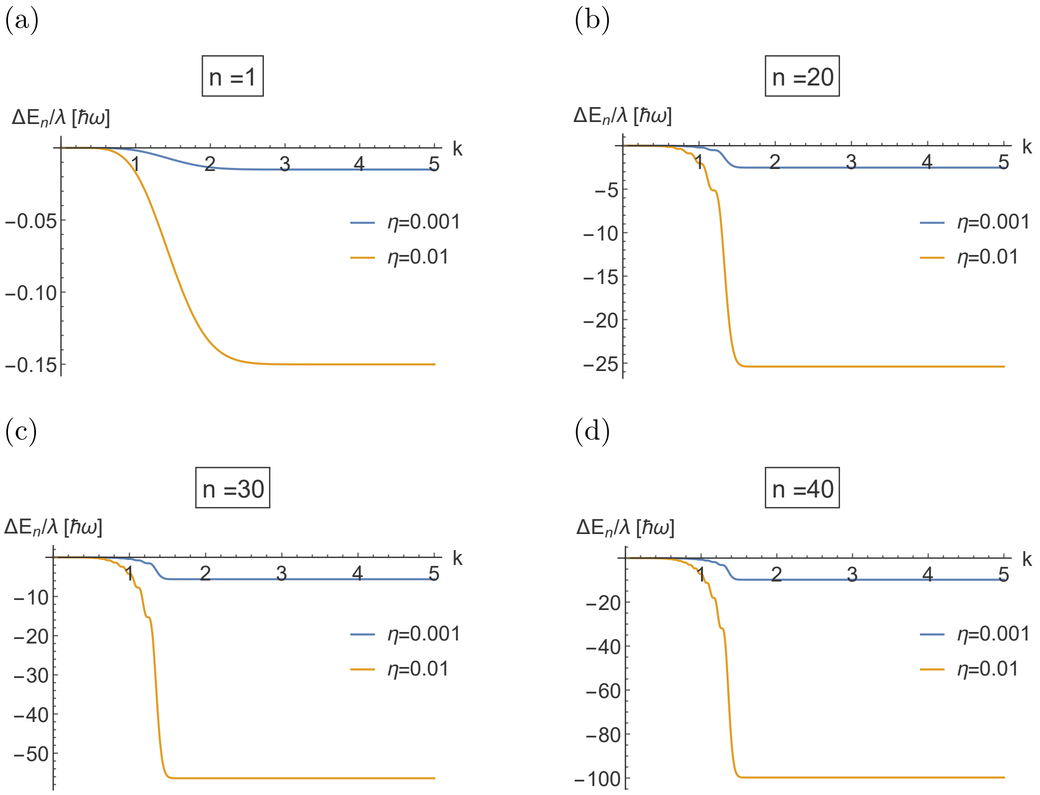

Figure 12: Convergence of the integral dependent on the integral limits $(-k\,\sigma, k\,\sigma)$ for different values of η and different n's.

The plots show explicitly, that the integrals will converge quite fast, and using piecewise functions for V_{per} from a certain k on would therefore not significantly change the convergence behavior.

C.2 Range of the Approximation

The quality of the approximation of V_{per} can be quantified by the quotient $\frac{V_{\approx}}{V_{per}} = 1 - \eta^2\xi^2$ ranging from 0 to 1. However, given the considerations in AppendixC.1, it will be sufficient to use $1 - \eta^2\xi^2 \geq 1 - \eta^2 k^2(2n+1)$ in order to integrate between $-k\sqrt{2n+1}$ and $k\sqrt{2n+1}$. This means that $\frac{V_{\approx}}{V_{per}} \geq 1 - \eta^2 k^2(2n+1)$ within the integration range. Furthermore, we can then use the mean value of $1 - \eta^2 k^2(2n+1)$, weighted by a Gaussian, to account for the differently strong contribution along k:

$$\overline{1-\eta^2k^2(2n+1)} = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\Big(1-\eta^2k^2(2n+1)\Big)\,e^{-\frac{k^2}{2}}\;dk = 1-\eta^2(2n+1)\;, \tag{C.4}$$

with the standard deviation

$$\Delta = \eta^2\sqrt{2}(2n+1) \tag{C.5}$$

Finally, we can estimate the quality of the approximation dependent on n by using the mean value given in (C.4) and additionally subtracting the standard deviation (C.5) to account for the worst case. Thus, we obtain

$$1-\eta^2(2n+1)-\eta^2\sqrt{2}(2n+1) = 1-\eta^2(1+\sqrt{2})(2n+1) \tag{C.6}$$

as a reasonable estimation for the accuracy of the approximation. Now, this allows us to determine the range for the n's, for which the approximation will definitely hold. If we wish the accuracy to be for example 0.9, we can use n's up to $n = \frac{1}{2}\left(\frac{0.1}{\eta^2(1+\sqrt{2})}-1\right)$. As an example, this would give us $n = 200$ for $\eta = 0.01$. More general, it follows from (C.6) that the accuracy of the approximation will be sufficiently good up to a n_{max}, given by

$$n_{max} \ll \frac{1}{\eta^2\;2(1+\sqrt{2})}-\frac{1}{2}\;. \tag{C.7}$$

D Matrix Elements $(V_{per})_{kn}$

We want to determine the matrix elements $(V_{per})_{kn} = \langle k|V_{per}|n\rangle$. For that reason we have to examine how $\left(a+a^\dagger\right)^3$ and $\left(a+a^\dagger\right)^3$ act on $|n\rangle$. We find that

$$\left(a+a^\dagger\right)^3 = a^3+{a^\dagger}^3 \qquad \to |n\mp 3\rangle \tag{D.1}$$

$$+\,{a^\dagger}^2a+a^\dagger aa^\dagger+a{a^\dagger}^2 \qquad \to |n+1\rangle \tag{D.2}$$

$$+\,a^2a^\dagger+a^\dagger a^2+aa^\dagger a \qquad \to |n-1\rangle \tag{D.3}$$

and

$$\left(a+a^\dagger\right)^4 = a^4 + a^{\dagger 4} \qquad \rightarrow |n \mp 4\rangle \tag{D.4}$$

$$+ a^3a^\dagger + a^\dagger a^3 + a^2a^\dagger a + aa^\dagger a^2 \qquad \rightarrow |n-2\rangle \tag{D.5}$$

$$+ a^{\dagger 3}a + aa^{\dagger 3} + a^{\dagger 2}aa^\dagger + a^\dagger aa^{\dagger 2} \qquad \rightarrow |n+2\rangle \tag{D.6}$$

$$+ a^{\dagger 2}a^2 + a^\dagger aa^\dagger a + aa^{\dagger 2}a + a^2a^{\dagger 2} + a^\dagger a^2 \quad a^\dagger + aa^\dagger aa^\dagger \rightarrow |n\rangle \tag{D.7}$$

where each line, when acting on $|n\rangle$, provides a state, proportional to the state given on the right-hand side of each particular line. Thus, for the diagonal matrix elements $(V_{per})_{nn}$, only the last line (D.7) of $\left(a+a^\dagger\right)^4$ provides a contribution and we have

$$\begin{aligned}
(V_{per})_{nn} &= -\hbar\tilde{\omega}\lambda\eta \langle n| \left(a+a^\dagger\right)^4 |n\rangle && \text{(D.8)}\\
&= -\hbar\tilde{\omega}\lambda\eta \langle n|a^{\dagger 2}a^2 + a^\dagger aa^\dagger a + aa^{\dagger 2}a + a^2a^{\dagger 2} + a^\dagger a^2a^\dagger + aa^\dagger aa^\dagger|n\rangle && \text{(D.9)}\\
&= -\hbar\tilde{\omega}\lambda\eta \Big(n(n-1) + n^2 + n(n+1) + (n+1)(n+2) \\
&\qquad\qquad + n(n+1) + (n+1)^2\Big) && \text{(D.10)}\\
&= -\hbar\tilde{\omega}\lambda\eta\, 3\left(2n(n+1)+1\right) \ . && \text{(D.11)}
\end{aligned}$$

For the off-diagonal matrix elements $(V_{per})_{kn}$ we have to examine $\langle k| \left(a+a^\dagger\right)^3 |n\rangle$ and $\langle k| \left(a+a^\dagger\right)^4 |n\rangle$, respectively. First, we notice, that in contrast to the diagonal elements, all parts of $\left(a+a^\dagger\right)^3$ and $\left(a+a^\dagger\right)^4$, except the line (D.7), may contribute for different k's. In fact, we have

$$(V_{per})_{kn} \begin{cases} \neq 0 & k \in \{n-4, n-3, n-2, n-1, n+1, n+2, n+3, n+4\} \\ = 0 & \text{the rest} \end{cases} , \tag{D.12}$$

where for each different k, only the corresponding line in either $\left(a+a^\dagger\right)^3$ or $\left(a+a^\dagger\right)^4$ will have a contribution. For example, for $k = n+1$, the only contribution will come from line (D.2) and the matrix element will be

$$
\begin{aligned}
(V_{per})_{(n+1)n} &= \hbar\tilde{\omega}\lambda \langle n+1| \left(a+a^\dagger\right)^3 |n\rangle & \text{(D.13)}\\
&= \hbar\tilde{\omega}\lambda \langle n+1|{a^\dagger}^2 a + a^\dagger a a^\dagger + a{a^\dagger}^2|n\rangle & \text{(D.14)}\\
&= \hbar\tilde{\omega}\lambda \left(3(n+1)\sqrt{n+1}\right) \langle n+1|n+1\rangle \ . & \text{(D.15)}
\end{aligned}
$$

In this way we can continue and determine the remaining matrix elements.

E Functions F(t) and G(t)

The function F is given by

$$
F(t,\lambda) \equiv \ 2\mathrm{Re}\left[\langle\psi_0\,|\,\xi\,|\,\psi_1\rangle\right] + 2\mathrm{Re}\left[\langle\psi_0\,|\,\xi\,|\,\psi_2\rangle\right] - \eta\ 2\mathrm{Re}\left[\langle\psi_0\,|\,\xi\,|\,\psi_3\rangle\right]\ , \quad \text{(E.1)}
$$

with

$$
\begin{aligned}
Re[\langle\psi_0\,|\,\xi\,|\,\psi_1\rangle] =& \mathbf{F1}\,, & \text{(E.2)}\\
Re[\langle\psi_0\,|\,\xi\,|\,\psi_2\rangle] =& \mathbf{F3}\,, & \text{(E.3)}\\
Re[\langle\psi_0\,|\,\xi\,|\,\psi_3\rangle] =& \mathbf{F2}+\mathbf{F4}\,. & \text{(E.4)}
\end{aligned}
$$

The functions $\mathbf{F1}-\mathbf{F4}$ are given in the *Mathematica* code on page 61.

The function G is given by

$$
G(t,\lambda) \equiv \ 2\mathrm{Re}\left[\langle\psi_0|\xi^2|\psi_1\rangle\right] + 2\mathrm{Re}\left[\langle\psi_0|\xi^2|\psi_2\rangle\right] - \eta\ 2\mathrm{Re}\left[\langle\psi_0|\xi^2|\psi_3\rangle\right]\ , \quad \text{(E.5)}
$$

with

$$
\begin{aligned}
Re[\left\langle\psi_0\,\middle|\,\xi^2\,\middle|\,\psi_1\right\rangle] =& \mathbf{G1}\,, & \text{(E.6)}\\
Re[\left\langle\psi_0\,\middle|\,\xi^2\,\middle|\,\psi_2\right\rangle] =& \mathbf{G3}\,, & \text{(E.7)}\\
Re[\left\langle\psi_0\,\middle|\,\xi^2\,\middle|\,\psi_3\right\rangle] =& \mathbf{G2}+\mathbf{G4}\,. & \text{(E.8)}
\end{aligned}
$$

The functions $\mathbf{G1}-\mathbf{G4}$ are given in the *Mathematica* code on page 62.

$$F1 := -3\, e^{\alpha^2} \left(1 + 2\,\alpha^2\right) + 2\, e^{\alpha^2 \mathrm{Cos}[24\, t\, w\, \eta\, \lambda]}\, \alpha^2\, \mathrm{Cos}\left[2\, (t\, w\, (-1 + 18\, \eta\, \lambda) + \phi) + \alpha^2\, \mathrm{Sin}[24\, t\, w\, \eta\, \lambda]\right]$$

$$\begin{aligned}
F2 :=\; & -6\, e^{\alpha^2 \mathrm{Cos}[12\, t\, w\, \eta\, \lambda]}\, \alpha \\
& \left(\alpha^2\, \mathrm{Cos}\left[t\, w - 24\, t\, w\, \eta\, \lambda - \phi - \alpha^2\, \mathrm{Sin}[12\, t\, w\, \eta\, \lambda]\right] + \right. \\
& \quad \left. \mathrm{Cos}\left[t\, w - 12\, t\, w\, \eta\, \lambda - \phi - \alpha^2\, \mathrm{Sin}[12\, t\, w\, \eta\, \lambda]\right]\right) + \\
& \frac{1}{4}\, e^{\alpha^2 \mathrm{Cos}[36\, t\, w\, \eta\, \lambda]}\, \alpha^3 \\
& \left(3\, \mathrm{Cos}\left[3\, (t\, w\, (-1 + 24\, \eta\, \lambda) + \phi) + \alpha^2\, \mathrm{Sin}[36\, t\, w\, \eta\, \lambda]\right] + \right. \\
& \quad \left. \alpha^2\, \mathrm{Cos}\left[3\, (t\, w\, (-1 + 36\, \eta\, \lambda) + \phi) + \alpha^2\, \mathrm{Sin}[36\, t\, w\, \eta\, \lambda]\right]\right)
\end{aligned}$$

$$\begin{aligned}
F3 :=\; & \frac{1}{3}\, e^{\alpha^2 \mathrm{Cos}[12\, t\, w\, \eta\, \lambda]} \\
& \Big(9\, \mathrm{Cos}\left[t\, w\, (-1 + 12\, \eta\, \lambda) + \alpha^2\, \mathrm{Sin}[12\, t\, w\, \eta\, \lambda]\right] + \\
& \quad \alpha^2\, \Big(-9\, \mathrm{Cos}\left[t\, w\, (1 - 12\, \eta\, \lambda) - \alpha^2\, \mathrm{Sin}[12\, t\, w\, \eta\, \lambda]\right] + \\
& \qquad 3\, \Big(3\, \mathrm{Cos}\left[t\, w - 24\, t\, w\, \eta\, \lambda - 2\, \phi - \alpha^2\, \mathrm{Sin}[12\, t\, w\, \eta\, \lambda]\right] + \\
& \qquad\quad 9\, \mathrm{Cos}\left[t\, w\, (-1 + 24\, \eta\, \lambda) + \alpha^2\, \mathrm{Sin}[12\, t\, w\, \eta\, \lambda]\right] + \\
& \qquad\quad \mathrm{Cos}\left[t\, w\, (-1 + 36\, \eta\, \lambda) - 2\, \phi + \alpha^2\, \mathrm{Sin}[12\, t\, w\, \eta\, \lambda]\right] - \\
& \qquad\quad 6\, \mathrm{Cos}\left[t\, w\, (-1 + 12\, \eta\, \lambda) + 2\, \phi + \alpha^2\, \mathrm{Sin}[12\, t\, w\, \eta\, \lambda]\right]\Big) + \\
& \qquad \alpha^2\, \Big(-9\, \mathrm{Cos}\left[t\, w\, (1 - 24\, \eta\, \lambda) - \alpha^2\, \mathrm{Sin}[12\, t\, w\, \eta\, \lambda]\right] + \\
& \qquad\quad \mathrm{Cos}\left[t\, w - 48\, t\, w\, \eta\, \lambda - 4\, \phi - \alpha^2\, \mathrm{Sin}[12\, t\, w\, \eta\, \lambda]\right] + \\
& \qquad\quad 9\, \mathrm{Cos}\left[t\, w - 36\, t\, w\, \eta\, \lambda - 2\, \phi - \alpha^2\, \mathrm{Sin}[12\, t\, w\, \eta\, \lambda]\right] - \\
& \qquad\quad \mathrm{Cos}\left[t\, w - 12\, t\, w\, \eta\, \lambda + 2\, \phi - \alpha^2\, \mathrm{Sin}[12\, t\, w\, \eta\, \lambda]\right] + \\
& \qquad\quad 9\, \mathrm{Cos}\left[t\, w\, (-1 + 36\, \eta\, \lambda) + \alpha^2\, \mathrm{Sin}[12\, t\, w\, \eta\, \lambda]\right] + \\
& \qquad\quad \mathrm{Cos}\left[t\, w\, (-1 + 48\, \eta\, \lambda) - 2\, \phi + \alpha^2\, \mathrm{Sin}[12\, t\, w\, \eta\, \lambda]\right] - \\
& \qquad\quad 9\, \mathrm{Cos}\left[t\, w\, (-1 + 24\, \eta\, \lambda) + 2\, \phi + \alpha^2\, \mathrm{Sin}[12\, t\, w\, \eta\, \lambda]\right] - \\
& \qquad\quad \mathrm{Cos}\left[t\, w\, (-1 + 12\, \eta\, \lambda) + 4\, \phi + \alpha^2\, \mathrm{Sin}[12\, t\, w\, \eta\, \lambda]\right]\Big)\Big)\Big)
\end{aligned}$$

$$\begin{aligned}
F4 :=\; & \frac{1}{4}\, e^{\alpha^2 \mathrm{Cos}[12\, t\, w\, \eta\, \lambda]}\, \alpha \\
& \Big(24\, \mathrm{Cos}\left[t\, (w - 24\, w\, \eta\, \lambda) + \phi - \alpha^2\, \mathrm{Sin}[12\, t\, w\, \eta\, \lambda]\right] + \\
& \quad \alpha^2 \\
& \quad \Big(4\, \Big(3\, \mathrm{Cos}\left[t\, w - 36\, t\, w\, \eta\, \lambda - 3\, \phi - \alpha^2\, \mathrm{Sin}[12\, t\, w\, \eta\, \lambda]\right] + \\
& \qquad\quad 9\, \mathrm{Cos}\left[t\, w - 36\, t\, w\, \eta\, \lambda + \phi - \alpha^2\, \mathrm{Sin}[12\, t\, w\, \eta\, \lambda]\right] - \\
& \qquad\quad 3\, \mathrm{Cos}\left[t\, w - 12\, t\, w\, \eta\, \lambda + \phi - \alpha^2\, \mathrm{Sin}[12\, t\, w\, \eta\, \lambda]\right] + \\
& \qquad\quad \mathrm{Cos}\left[t\, w\, (-1 + 48\, \eta\, \lambda) - 3\, \phi + \alpha^2\, \mathrm{Sin}[12\, t\, w\, \eta\, \lambda]\right] - \\
& \qquad\quad 5\, \mathrm{Cos}\left[t\, w\, (-1 + 12\, \eta\, \lambda) + 3\, \phi + \alpha^2\, \mathrm{Sin}[12\, t\, w\, \eta\, \lambda]\right]\Big) + \\
& \qquad \alpha^2\, \Big(\mathrm{Cos}\left[t\, w - 60\, t\, w\, \eta\, \lambda - 5\, \phi - \alpha^2\, \mathrm{Sin}[12\, t\, w\, \eta\, \lambda]\right] + \\
& \qquad\quad 8\, \mathrm{Cos}\left[t\, w - 48\, t\, w\, \eta\, \lambda - 3\, \phi - \alpha^2\, \mathrm{Sin}[12\, t\, w\, \eta\, \lambda]\right] + \\
& \qquad\quad 8\, \mathrm{Cos}\left[t\, w - 48\, t\, w\, \eta\, \lambda + \phi - \alpha^2\, \mathrm{Sin}[12\, t\, w\, \eta\, \lambda]\right] - \\
& \qquad\quad 8\, \mathrm{Cos}\left[t\, w - 24\, t\, w\, \eta\, \lambda + \phi - \alpha^2\, \mathrm{Sin}[12\, t\, w\, \eta\, \lambda]\right] - \\
& \qquad\quad \mathrm{Cos}\left[t\, w - 12\, t\, w\, \eta\, \lambda + 3\, \phi - \alpha^2\, \mathrm{Sin}[12\, t\, w\, \eta\, \lambda]\right] + \\
& \qquad\quad \mathrm{Cos}\left[t\, w\, (-1 + 60\, \eta\, \lambda) - 3\, \phi + \alpha^2\, \mathrm{Sin}[12\, t\, w\, \eta\, \lambda]\right] - \\
& \qquad\quad 8\, \mathrm{Cos}\left[t\, w\, (-1 + 24\, \eta\, \lambda) + 3\, \phi + \alpha^2\, \mathrm{Sin}[12\, t\, w\, \eta\, \lambda]\right] - \\
& \qquad\quad \mathrm{Cos}\left[t\, w\, (-1 + 12\, \eta\, \lambda) + 5\, \phi + \alpha^2\, \mathrm{Sin}[12\, t\, w\, \eta\, \lambda]\right]\Big)\Big)\Big)
\end{aligned}$$

```
G1 :=
  -20 ⅇ^(α^2 Cos[12 t w η λ]) α
      (Cos[t w - 12 t w η λ - ϕ - α^2 Sin[12 t w η λ]] +
        α^2 Cos[t w (-1 + 24 η λ) + ϕ + α^2 Sin[12 t w η λ]]) +
    4 ⅇ^(α^2 Cos[36 t w η λ]) α^3 Cos[3 (t w (-1 + 24 η λ) + ϕ) + α^2 Sin[36 t w η λ]] /.
   ϕ → -ϕ
G2 :=
  -6 ⅇ^(α^2) (1 + 4 α^2 + 2 α^4) -
    5 ⅇ^(α^2 Cos[24 t w η λ]) α^2
     (3 Cos[2 (t w (-1 + 18 η λ) + ϕ) + α^2 Sin[24 t w η λ]] +
       2 α^2 Cos[2 (t w (-1 + 30 η λ) + ϕ) + α^2 Sin[24 t w η λ]]) +
    2 ⅇ^(α^2 Cos[48 t w η λ]) α^4 Cos[4 (t w (-1 + 30 η λ) + ϕ) + α^2 Sin[48 t w η λ]] /.
   ϕ → -ϕ
G3 :=
  1/3 (6 ⅇ^(α^2) α Cos[ϕ] (3 + 2 α^2 + 2 α^2 Cos[2 ϕ]) +
       ⅇ^(α^2 Cos[24 t w η λ]) α
        (6 (Cos[2 t w - 60 t w η λ + ϕ - α^2 Sin[24 t w η λ]] +
              6 Cos[2 t w (-1 + 18 η λ) + ϕ + α^2 Sin[24 t w η λ]]) +
          α^2 (9 Cos[2 t w - 60 t w η λ - 3 ϕ - α^2 Sin[24 t w η λ]] -
              9 Cos[2 t w - 36 t w η λ - ϕ - α^2 Sin[24 t w η λ]] +
              6 Cos[2 t w - 84 t w η λ + ϕ - α^2 Sin[24 t w η λ]] +
              45 Cos[2 t w (-1 + 30 η λ) + ϕ + α^2 Sin[24 t w η λ]] -
              27 Cos[2 t w (-1 + 18 η λ) + 3 ϕ + α^2 Sin[24 t w η λ]] +
              α^2 (Cos[2 t w (1 - 54 η λ) - 5 ϕ - α^2 Sin[24 t w η λ]] +
                  9 Cos[2 t w - 84 t w η λ - 3 ϕ - α^2 Sin[24 t w η λ]] -
                  9 Cos[2 t w - 60 t w η λ - ϕ - α^2 Sin[24 t w η λ]] -
                  Cos[2 t w - 36 t w η λ + ϕ - α^2 Sin[24 t w η λ]] +
                  Cos[2 t w (1 - 54 η λ) + ϕ - α^2 Sin[24 t w η λ]] +
                  9 Cos[2 t w (-1 + 42 η λ) + ϕ + α^2 Sin[24 t w η λ]] -
                  9 Cos[2 t w (-1 + 30 η λ) + 3 ϕ + α^2 Sin[24 t w η λ]] -
                  Cos[2 t w (-1 + 18 η λ) + 5 ϕ + α^2 Sin[24 t w η λ]])))) /.
   ϕ → -ϕ
G4 :=
  1/4 (8 ⅇ^(α^2) α^2 ((6 + 4 α^2) Cos[2 ϕ] + α^2 Cos[4 ϕ]) +
       ⅇ^(α^2 Cos[24 t w η λ])
        (24 Cos[2 t w (1 - 18 η λ) - α^2 Sin[24 t w η λ]] +
          α^2 (96 Cos[2 t w (1 - 30 η λ) - α^2 Sin[24 t w η λ]] +
              12 Cos[2 (t (w - 42 w η λ) + ϕ) - α^2 Sin[24 t w η λ]] +
              α^2 (60 Cos[2 t w (1 - 42 η λ) - α^2 Sin[24 t w η λ]] +
                  4 (-7 Cos[2 t w - 36 t w η λ - 4 ϕ - α^2 Sin[24 t w η λ]] +
                      2 Cos[2 t (w - 54 w η λ) + 2 ϕ - α^2 Sin[24 t w η λ]] -
                      3 Cos[2 t w (-1 + 18 η λ) + α^2 Sin[24 t w η λ]] +
                      3 Cos[2 t w (-1 + 42 η λ) + 4 ϕ + α^2 Sin[24 t w η λ]]) +
                  α^2 (8 Cos[2 t w (1 - 54 η λ) - α^2 Sin[24 t w η λ]] -
                      Cos[2 t w - 36 t w η λ - 6 ϕ - α^2 Sin[24 t w η λ]] +
                      Cos[2 t w (1 - 66 η λ) - 6 ϕ - α^2 Sin[24 t w η λ]] -
                      8 Cos[2 t w - 60 t w η λ - 4 ϕ - α^2 Sin[24 t w η λ]] +
                      Cos[2 t w (1 - 66 η λ) + 2 ϕ - α^2 Sin[24 t w η λ]] -
                      8 Cos[2 t w (-1 + 30 η λ) + α^2 Sin[24 t w η λ]] -
                      Cos[2 t w (-1 + 18 η λ) - 2 ϕ + α^2 Sin[24 t w η λ]] +
                      8 Cos[2 t w (-1 + 54 η λ) + 4 ϕ +
                          α^2 Sin[24 t w η λ]]))))) /. ϕ → -ϕ
```

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