universität wien

MASTERARBEIT / MASTER'S THESIS

Titel der Masterarbeit / Title of the Master's Thesis

„Notions of Locality in Quantum Processes“

verfasst von / submitted by

Stefan Lukas Ludescher, BSc

angestrebter akademischer Grad / in partial fulfilment of the requirements for the degree of

Master of Science (MSc)

Wien, 2020 / Vienna, 2020

Studienkennzahl lt. Studienblatt / degree programme code as it appears on the student record sheet:	A 066 876
Studienrichtung lt. Studienblatt / degree programme as it appears on the student record sheet:	Masterstudium Physik
Betreut von / Supervisor:	Univ.-Prof. Mag. Dr. Caslav Brukner

Abstract

In the last few years, the idea of indefinite causal structure has received an enormous amount of attention. The idea is to formulate very general processes that relate local laboratories into a global structure, producing correlations between the local actions. With the formalism of process matrices, a mathematical framework has been found that captures this idea under the assumption that quantum mechanics holds locally. In the definition of process matrices, every local laboratory has well-defined input and output Hilbert spaces. However, in ordinary quantum theory, states do not a priori include any information about the factorization of the global Hilbert space. This raises the question of what were to happen if this assumption was relaxed - that is, if the local laboratories were assumed to be spatially separated, but without fixing the factorization beforehand. Would this lead to a modified set of possible processes? This thesis answers this question, by showing that the only quantum processes compatible with the relaxed formulation are the quantum states.

In the second part of the thesis, a thought experiment involving a Bell-like scenario in a superposition of temporal orders is introduced. At first sight, the scenario seems capable of allowing nonsignalling correlations that are more general than quantum - so-called "almost quantum correlations". However, for several versions of the scenario, it is shown that this initial conjecture does not hold up: all correlations are quantum. Additionally, a theorem is proven which gives a necessary conditions for such Bell-like scenarios to generate almost quantum correlations.

Zusammenfassung

In den letzten Jahren bekam der Gedanke von indefiniten kausalen Strukturen viel Aufmerksamkeit. Die Idee ist es, sehr generelle Prozesse zu beschreiben, welche lokale Labore in eine globale Struktur einbetten und dadurch die Korrelationen zwischen den Laboren bestimmen. Mittels des Prozessmatrizenformalismus wurde ein mathematischer Formalismus gefunden, der genau diese Idee, unter der Annahme, dass die Quantenmechanik lokal Gültigkeit besitzt, verwirklicht. In der Definition von Prozessmatrizen ist bereits jedem lokalen Labor ein Eingangs- und Ausgangshilbertraum zugeordnet. Im Gegenteil dazu, beinhaltet ein Zustand in der normalen Quantenmechanik keine Information bezüglich einer Faktorisierung des globalen Hilbertraumes. Daher stellt sich die Frage was es für Auswirkungen hätte, wenn man diese Annahme lockert und stattdessen bloß fordert, dass alle Labore lokal separiert sind, man jedoch keine Faktorisierung fixiert. Führt dies zu einer modifizierten Menge an erlaubten Prozessen? Diese Arbeit beantwortet diese Frage, indem gezeigt wird, dass Quantenzustände die einzigen Quantenprozesse beschreiben, welche kompatibel mit der gelockerten Annahme sind.
Im zweiten Teil der Arbeit wird ein Gedankenexperiment präsentiert, welches ein Bell-artiges Szenario in Superposition von zeitlichen Abläufen beschreibt. Auf den ersten Blick scheint es, dass dieses Szenario Non-Signaling-Korrelationen zulässt, welche allgemeiner sind als Quanten-Korrelationen - sogenannte Almost-Quantum-Korrelationen. Jedoch wird für verschiedene Konfigurationen des Szenarios gezeigt, dass die ursprüngliche Vermutung nicht hält: alle Korrelationen sind Quanten-Korrelationen. Zusätzlich wird ein Theorem bewiesen, welches eine notwendige Bedingung für Bell-artige Szenarien darstellt um Almost-Quantum-Korrelationen zu generieren.

Acknowledgments

First of all I want to thank Dr. Markus Müller, who introduced me to the topics of indefinite causal structures, process matrices and many more. He always found time for lively discussions and supported me with a lot of ideas and explanations. Thanks to him my understanding of physics and mathematics grew a great deal in the past year.
Furthermore, I want to thank Dr. Flaminia Giacomini from Perimeter Institute of Theoretical Physics for all the thought-provoking discussions we had. She really aided my understanding of process matrices and greatly supported me during the development of my thesis.
I want to thank Univ.-Prof. Mag. Dr. Caslav Brukner not only for supervising my thesis, but also for being the first one to arouse my interests in the field of quantum foundations. Through his lecture "Modern Foundations of Quantum Mechanics" I got in touch with the field for the first time and because of the seminar "Quantum Foundations - (Journal Club)" I decided that I want to pursue a career in this field.
Moreover, I want to thank my parents for giving me the opportunity to study physics.
Last but not least I want to thank Sophie Steinbrenner. She supported me in many aspects throughout my time as a physics student, not only those concerning physics. Without her I would not have been able to pursue my goals (again in- and outside of physics) in the way I did.

Contents

Chapter 1

Introduction

In this master thesis we will investigate different notions of locality in quantum processes. For the simplest case of a quantum processes, namely quantum states, there are three different possibilities to implement local separation of two (or more) experimenters. In quantum information theory local separation is usually implemented by associating a Hilbert space to every experimenter and then these spaces are added together via tensor product [1]. In quantum field theory local separation is typically invoked by linking a subalgebra of a measurement operator to every experimenter and demanding that all the subalgebras commute [1, 2]. More recently a third way of implementing local separation was introduced in [3]; it concerns Bell-like scenarios. Instead of demanding that the subalgebras commute in general, it is just demanded that the commutator of two measurement operators belonging to two different experimenters vanishes, if it acts on the state associated with the experiment at hand.
In this thesis we will explore, if we can find analogies between those three notions of local separation for states and more general quantum processes. Especially, we will focus on process matrices introduced in [4] and processes, where different time orderings are brought in superposition.
We are first going to set the stage by recapitulating some concepts such as channels, operations, CJ-representations etc. Those concepts will accompany us throughout the whole thesis.
Afterwards we will concern ourselves with the concept of processes in general and subsequently we will spend some time on introducing and understanding process matrices. Next, we will analyze Tsirelson's problem (see [5]) in finite dimensions.
After that, all well established concepts, which we will need later on, will be introduced and we will start with the part of the thesis where we are going to tackle some new questions. First we will concern ourselves with the following:
The results of [5] imply that for two locally separated experimenters not the state $\rho \in \mathcal{B}(\mathcal{H})$, but the subalgebras associated with the experimenters give a partition of the Hilbert space $\mathcal{H}$ into $\mathcal{H}^A \otimes \mathcal{H}^B$. In the standard definition of process matrices (see [4]) the situation is different. In the definition of a process matrix it is already assumed that the localization of the laboratories of the experimenters is given and with that the partition of the Hilbert space $\mathcal{H} = \mathcal{H}^{A_1} \otimes \mathcal{H}^{A_1} \otimes \mathcal{H}^{B_1} \otimes \mathcal{H}^{B_2}$ is given as well. This leads to the question: if we do not assume that the localization of the experimenters is given, but only that they all reside in different local laboratories, do all process matrices still produce valid probability distributions for all possible choices of operations held by the experimenters? If not, can we find a subset of process matrices, which produce valid probability distributions under this new assumption?
After answering these questions we will turn our attention to a thought experiment, which is

inspired by the quantum switch (see [6]). For this thought experiment we will utilize two control-qubits to create a superposition of different temporal orders. We will explain the exact experimental setup later. However, the question we will try to answer is, if it is possible to find correlations within our setup, which are outside the set of quantum correlations but inside the set of almost quantum correlations (see [3])? Furthermore, if we use postselection, is it possible to find any other correlations than quantum correlations?
In many parts of the thesis we will use the Dirac formalism. For an introduction to the formalism see [7], for example. Furthermore we will use mathematical methods which are common in physics (see for example [8] or [9]). Additionally, we will also use techniques and notations which are typical in the field of quantum mechanics and quantum information theory (see for example [10] or [11]).

Chapter 2

Preliminaries

We start things off by introducing some mathematical concepts and definitions, which we will need throughout the thesis. We will only consider finite-dimensional Hilbert spaces in the whole thesis. However some of the definitions and statements can be easily generalized to infinite dimensions.

2.1 Operations, Channels and Instruments

Most parts of this section (including definitions and main ideas) are adopted from the book [10] and only extra information will be cited explicitly.

We first define what we mean when we say that a map $\mathcal{M}$ is positive or completely positive:

Definition 1. *[11] We say that that $\mathcal{M} : \mathcal{L}(\mathcal{H}^A) \to \mathcal{L}(\mathcal{H}^B)$ is positive iff $\mathcal{M}(\rho)$ is positive semi-definite for all positive semi-definite $\rho \in \mathcal{L}(\mathcal{H}^A)$.*

Definition 2. *[11] We say that $\mathcal{M} : \mathcal{L}(\mathcal{H}^A) \to \mathcal{L}(\mathcal{H}^B)$ is completely positive iff $(\mathcal{I} \otimes \mathcal{M})(A)$ is positive semi-definite for all positive semi-definite $A \in \mathcal{L}(\mathcal{H}^A \otimes \mathcal{H}^R)$ and all auxiliary systems R with associated finite-dimensional Hilbert space $\mathcal{H}^R$, where $\mathcal{I}$ is the identity map on system R.*

The most general allowed transformation acting on a quantum system is called an operation and is defined in the following way:

Definition 3. *A mapping $\mathcal{M} : \mathcal{L}(\mathcal{H}^A) \to \mathcal{L}(\mathcal{H}^B)$ is called a (quantum-)operation if the following properties hold:*

1. *$\mathcal{M}$ is linear,*
2. *$\mathcal{M}$ is completely positive,*
3. *$\mathcal{M}$ is trace non-increasing.*

Operations are the most general transformations, which take a quantum system as an input and return a quantum system as an output. However these transformations are not deterministic in general. The probability of the appearance of an operation is given by $\mathrm{Tr}(\mathcal{M}(\rho))$. A deterministic operation i.e. a transformation appearing with probability $\mathrm{Tr}(\mathcal{M}(\rho)) = 1$ is called a quantum channel and is defined as follows:

Definition 4. *An operation $\mathcal{M}$ is called a (quantum-)channel if it is trace preserving.*

Before defining what we mean by the notion of an instrument we have to do some preparatory work. In this section, when we talk about effects we think of operators $E \in \mathcal{L}(\mathcal{H})$, which are bounded, self-adjoint operators satisfying $O \leq E \leq I$. Furthermore, we will denote the space of all effects by

$$\mathcal{E}(\mathcal{H}) = \{E \in \mathcal{L}(\mathcal{H}) \,|\, O \leq E \leq I \text{ and } E = E^\dagger\}, \tag{2.1}$$

where $E^\dagger$ is the adjoint operator.
Next we want to introduce the notion of a positive operator-valued measurement (POVM), but before we can do this we have to establish a couple of definitions first:

Definition 5. *A σ-algebra on Ω, where Ω is a non-empty set, is a collection $\mathcal{F}$ of subsets of Ω, which satisfies the following conditions:*

1. *$\emptyset \in \mathcal{F}$ and $\Omega \in \mathcal{F}$,*
2. *if $X \in \mathcal{F}$ then $\Omega \backslash X \in \mathcal{F}$,*
3. *if $X_1, X_2, \ldots \in \mathcal{F}$ then $\bigcup_i X_i \in \mathcal{F}$.*

We call the pair $(\Omega, \mathcal{F})$ a measurable space, the sets $X \in \mathcal{F}$ events and $\mathcal{F}$ the collection of all events.

Definition 6. *A map $p : \mathcal{F} \to [0, 1]$ is called a probability measure if the following is satisfied:*

1. *$p(\emptyset) = 0$,*
2. *$p(\Omega) = 1$,*
3. *$p(\bigcup_i X_i) = \sum_i p(X_i)$ for any sequence $\{X_i\}$, where $X_i \in \mathcal{F}$ and $X_i \cap X_j = \emptyset$, $\forall i \neq j$.*

Now we have the tools to define POVMs:

Definition 7. *A POVM is a map $A : \mathcal{F} \to \mathcal{E}(\mathcal{H})$, for which*

1. *$A(\emptyset) = 0$,*
2. *$A(\Omega) = I$,*
3. *$A(\bigcup_i X_i) = \sum_i A(X_i)$ (in the weak topology) for any sequence $\{X_i\}$, where $X_i \in \mathcal{F}$ and $X_i \cap X_j = \emptyset$, $\forall i \neq j$,*

holds.

The last definition is equivalent to the statement that $A : \mathcal{F} \to \mathcal{E}(\mathcal{H})$ is a POVM if and only if $X \mapsto \langle\psi| A(X) |\psi\rangle =Tr(|\psi\rangle \langle\psi| A(X))$ is a valid probability measure for every unit vector $\psi \in \mathcal{H}$. This statement can be generalized to every state $\rho \in S(\mathcal{H})$, by the observation that every state ρ is a pure state of the form $|\psi\rangle \langle\psi|$ or a convex combination of pure states and thus, $A : \mathcal{F} \to \mathcal{E}(\mathcal{H})$ is a POVM if and only if $X \mapsto$ Tr$(\rho A(X))$ is a valid probability measure for every $\rho \in \mathcal{S}(\mathcal{H})$, where

$$\mathcal{S}(\mathcal{H}) = \{\rho \in \mathcal{L}(\mathcal{H}) \,|\, \rho \geq 0 \text{ and } \mathrm{Tr}(\rho) = 1\}, \tag{2.2}$$

is the state space.
Moreover, the measurable space $(\Omega, \mathcal{F})$, on which the POVM A is defined, will be called the outcome space, while Ω will be called the sample space.
We can now define for every POVM A, with associated outcome space $(\Omega, \mathcal{F})$, a map $\Phi_A : \mathcal{S}(\mathcal{H}) \to Prob(\Omega)$ by

$$\Phi_A(\rho) := \mathsf{Tr}(\rho A(\cdot)), \tag{2.3}$$

where $Prob(\Omega)$ is the space of all probability measures on the outcome space $(\Omega, \mathcal{F})$. The map Φ_A can be considered as a device, which maps states to probability distributions.
This is not the end of the story. We could also think of a more sophisticated device not only capable of returning some measurement outcomes / probability distributions after acting on some input system, but also provides access to the system itself after the measurement process.
We can think of such a measurement process in the following way:
First the system couples to a second system, the probe system. In this step, the system and the probe system become correlated. In a second step the system decouples from the probe system and the probe system alone is measured. By the coupling between the system and the probe in the first place, we also gain information about the initial state of the system by measuring the probe. The whole process blue describes a measurement corresponding to some observable A.
Now let us put our recent considerations into a mathematically rigorous context.

Definition 8. *Let A be a POVM, with outcome space $(\Omega, \mathcal{F})$, acting on a system with an associated Hilbert space $\mathcal{H}$. Furthermore, let $\mathcal{K}$ be the Hilbert space of the probe and ξ the initial state of the probe. Moreover, let $\mathcal{V} : \mathcal{L}(\mathcal{H} \otimes \mathcal{K}) \to \mathcal{L}(\mathcal{H} \otimes \mathcal{K})$ be the channel describing the interaction between the system and the probe and finally, let F be a POVM acting on the probe, having the same outcome space $(\Omega, \mathcal{F})$ as the POVM A. Then, we call the quadruple $\mathcal{N} = \langle \mathcal{K}, \xi, \mathcal{V}, F \rangle$ a measurement model of A iff*

$$Tr(\rho A(X)) = Tr(\mathcal{V}(\rho \otimes \xi)(I \otimes F(X))) \tag{2.4}$$

for all $\rho \in \mathcal{S}(\mathcal{H})$ and $X \in \mathcal{B}(\Omega)$, where $\mathcal{B}(\Omega)$ denotes the Borel algebra.

Equation (2.4) is called the probability reproducibility condition and states that the probabilities obtained for the POVM F acting on the probe are the same as the probabilities we would obtain by performing a direct measurement of the POVM A on the system.
It is also possible not to predetermine a POVM A, but rather a quadruple $\langle \mathcal{K}, \xi, \mathcal{V}, F \rangle$ and then find an unique POVM A via the probability reproducibility condition given by equation (2.4). After the detour of defining and understanding POVMs and general measurement models we have finally acquired all the means we need for defining and understanding the concept of instruments. But before writing down the definition of instruments we will motivate this notion.
As already previously mentioned we could also think of measurement devices, which do not merely return a measurement outcome, but rather in addition some state, which might depend on the input state. In this scenario we could perform another measurement on the system, obtained after the first measurement, to get more information about the initial state.
Moreover, it is possible to prepare a conditional state via the measurement procedure described above.
However, in both scenarios we have to find a way to characterize the influence of the first measurement on the initial state. We will see that the concept of instruments is a compact form of describing such influences.

Let us consider the following description of two consecutive measurements. We characterize the first measurement associated with a POVM A by a measurement model $\mathcal{N} = \langle \mathcal{K}, \xi, \mathcal{V}, F \rangle$, but the second measurement we merely describe by a direct measurement of the POVM B on the system obtained by the first measurement and we do not write down a measurement model for B. Depending on the input state we will find some joint probability distribution for A and B denoted by $p_\rho(a \in X, b \in Y)$, which is the probability of observing measurement outcomes from the sets X and Y for the observable A and B respectively and we can write:

$$p_\rho(a \in X, b \in Y) = \mathsf{Tr}\left(\mathcal{V}(\rho \otimes \xi)(B(Y) \otimes F(X))\right), \tag{2.5}$$

which is valid for arbitrary X and Y. For now we will fix the set X and rearrange

$$\begin{aligned} p_\rho(a \in X, b \in Y) &= \mathsf{Tr}\left(\mathcal{V}(\rho \otimes \xi)(B(Y) \otimes F(X))\right) \\ &= \mathsf{Tr}\left(B(Y)\mathsf{Tr}_{\mathcal{K}}(\mathcal{V}(\rho \otimes \xi)(I \otimes F(X)))\right) \\ &= \mathsf{Tr}(\mathcal{M}_X^{\mathcal{N}}(\rho)B(Y)), \end{aligned} \tag{2.6}$$

where we have defined

$$\mathcal{M}_X^{\mathcal{N}}(\rho) := \mathsf{Tr}_{\mathcal{K}}(\mathcal{V}(\rho \otimes \xi)(I \otimes F(X))). \tag{2.7}$$

The last expression is uniquely determined by requirement that

$$p_\rho(a \in X, b \in Y) = \mathsf{Tr}(\mathcal{M}_X^{\mathcal{N}}(\rho)B(Y)) \tag{2.8}$$

holds for all B and for all Y. From the definition of $\mathcal{M}_X^{\mathcal{N}}(\rho)$ given by equation (2.7) it can be shown that the following properties hold:

i1 $\mathcal{M}_X^{\mathcal{N}}$ is an operation $\forall X$,

i2 $\mathcal{M}_\emptyset^{\mathcal{N}}(\rho) = O$ and $\mathsf{Tr}(\mathcal{M}_\Omega^{\mathcal{N}}(\rho)) = 1 \; \forall \rho \in \mathcal{S}(\mathcal{H})$,

i3 For every sequence of mutually disjoint sets $\{X_i\}$ and $\rho \in \mathcal{S}(\mathcal{H})$ the following holds

$$\mathsf{Tr}(\mathcal{M}_{\cup_i X_i}^{\mathcal{N}}(\rho)) = \sum_i \mathsf{Tr}(\mathcal{M}_{X_i}^{\mathcal{N}}(\rho)).$$

Next, let Ω' be the sample space of B. We observe

$$p_\rho(a \in X) = p_\rho(a \in X, b \in \Omega') = \mathsf{Tr}(\mathcal{M}_X^{\mathcal{N}}(\rho)B(\Omega')) = \mathsf{Tr}(\mathcal{M}_X^{\mathcal{N}}(\rho)). \tag{2.9}$$

Our last observation before defining the notion of an instrument is that for a fixed set X, $p_\rho(a \in X, b \in Y) = \mathsf{Tr}(\mathcal{M}_X^{\mathcal{N}}(\rho)B(Y))$ implies that the expression $\mathcal{M}_X^{\mathcal{N}}(\rho)$ can be considered as the (non-normalized) conditional state after the first measurement, on which B subsequently acts. Hence, the map $X \mapsto \mathcal{M}_X^{\mathcal{N}}(\rho)$ includes not only information about the measurement associated with A, but also about the updated state after the measurement conditioned on the set X. This finally motivates our definition of an instrument:

Definition 9. *We call a mapping $\mathcal{M}$ from the outcome space $(\Omega, \mathcal{F})$ to the set of operations acting on $\mathcal{L}(\mathcal{H})$ an instrument iff it satisfies the relations (i1)-(i3).*

2.2 CJ-Isomorphism

In this section we will introduce the Choi-Jamiołkowsky-isomorphism (CJ-isomorphism). We will present two versions. The first version is only defined for linear operators over pure states and the second one is more generally defined for maps over density matrices [12]. The CJ-isomorphism will be used later when we introduce the process matrix formalism [4, 6, 12].
We start with the version for linear operators over pure states. In this case the CJ-isomorphism for an operator $K \in \mathcal{L}(\mathcal{H}^A, \mathcal{H}^B)$ gives us a vector of the space $\mathcal{H}^A \otimes \mathcal{H}^B$ i.e.

$$\mathsf{CJ} : \mathcal{L}(\mathcal{H}^A, \mathcal{H}^B) \to \mathcal{H}^A \otimes \mathcal{H}^B \tag{2.10}$$

and is defined by

$$\mathsf{CJ}(K) = \mathbb{1}_A \otimes K^* \left|\mathbb{1}\right\rangle\rangle^{AA} = \left|K^*\right\rangle\rangle^{AB}, \tag{2.11}$$

where $\left|\mathbb{1}\right\rangle\rangle^{AA} = \sum_{j=1}^{d_A} \left|jj\right\rangle$. Furthermore, $\{\left|j\right\rangle\}_{j=1}^{d_A}$ is a freely chosen orthonormal basis (ONB) of the Hilbert space $\mathcal{H}^A$. The dimension of $\mathcal{H}^A$ is denoted by d_A and with $K^* \in \mathcal{L}(\mathcal{H}^A, \mathcal{H}^B)$ we identify the complex conjugate as

$$K^* = \sum_{kj} K^*_{kj} \left|k\right\rangle_B \left\langle j\right|_A, \tag{2.12}$$

where $\{\left|j\right\rangle_A\}_{j=1}^{d_A}$ is the same chosen ONB of $\mathcal{H}^A$ as before, $\{\left|k\right\rangle_B\}_{k=1}^{d_B}$ is a freely chosen ONB of $\mathcal{H}^B$ and $K_{kj} \in \mathbb{C}$ are the coefficients of K in the chosen bases [6, 12].
The map

$$\left[\left\langle\psi\right|^A \otimes \mathbb{1}_B \left|K^*\right\rangle\rangle^{AB}\right]^* = \left[\sum_j \underbrace{\left\langle\psi\right| j\rangle}_{=\psi_j^*} K^* \left|j\right\rangle\right]^* = K \sum_j \psi_j \left|j\right\rangle = K\left|\psi\right\rangle \tag{2.13}$$

gives us the inverse CJ-isomorphism [12].
We will now consider the case where the dimensions of $\mathcal{H}^A$ and $\mathcal{H}^B$ coincide i.e. $d_A = d_B = d$. Instead of the non-normalized maximally entangled state $\left|\mathbb{1}\right\rangle\rangle^{AA}$, we will consider the normalized maximally entangled Bell state $\left|\phi^+\right\rangle_{AB} = \frac{1}{d}\sum_i \left|ii\right\rangle_{AB}$ and we will prove the following:

Theorem 1. *[12] The state $\mathbb{1}_A \otimes U_B \left|\phi^+\right\rangle_{AB}$ is maximally entangled iff U is unitary.*

Proof: For the proof we will need the entanglement measure for bipartite pure states

$$E(\left|\varphi\right\rangle\left\langle\varphi\right|) = S(\rho_A) = S(\rho_B), \tag{2.14}$$

where ρ_A is the reduced state for system A, ρ_B is the reduced state for system B and $S(\rho) = -\mathrm{Tr}(\rho \log_2(\rho))$ is the Von Neumann entropy [13]. The entanglement measure has the property

$$0 \leq E(\left|\varphi\right\rangle\left\langle\varphi\right|) \leq \log_2(d), \tag{2.15}$$

where $E(\left|\varphi\right\rangle\left\langle\varphi\right|) = 0$ iff $\left|\varphi\right\rangle$ is separable and $E(\left|\varphi\right\rangle\left\langle\varphi\right|) = \log_2(d)$ iff $\left|\varphi\right\rangle$ is maximally entangled [13]. Moreover, $S(\rho) = \log_2(d)$ iff ρ is the maximally mixed state [11]. Therefore, $\left|\varphi\right\rangle$ is maximally entangled iff $\rho_A = \mathbb{1}_A/d$ and $\rho_B = \mathbb{1}_B/d$.

We will use the last fact to prove theorem 1. We first assume that U is unitary. From this it follows that

$$\rho_A = \mathrm{Tr}_B((\mathbb{1}_A \otimes U_B) |\phi^+\rangle_{AB} \langle\phi^+| (\mathbb{1}_A \otimes U_B^\dagger)) = \frac{1}{d}\sum_{ij} |i\rangle_A \langle j| \cdot \underbrace{\mathrm{Tr}(U_B |i\rangle \langle j| U^\dagger)}_{=\delta_{ij}} = \frac{\mathbb{1}}{d}, \tag{2.16}$$

and

$$\rho_B = \mathrm{Tr}_A((\mathbb{1}_A \otimes U_B) |\phi^+\rangle_{AB} \langle\phi^+| (\mathbb{1}_A \otimes U_B^\dagger)) = \frac{U_B \mathbb{1}_B U_B^\dagger}{d} = \frac{\mathbb{1}_B}{d}. \tag{2.17}$$

Hence, if U is unitary then $\mathbb{1}_A \otimes U_B |\phi^+\rangle_{AB}$ is maximally entangled. To show that the converse is also true, we assume that $\mathbb{1}_A \otimes U_B |\phi^+\rangle_{AB}$ is maximally entangled and calculate

$$\frac{\mathbb{1}_B}{d} = \rho_B = \mathrm{Tr}_A((\mathbb{1}_A \otimes U_B) |\phi^+\rangle_{AB} \langle\phi^+| (\mathbb{1}_A \otimes U_B^\dagger) = \frac{U_B U_B^\dagger}{d}) \tag{2.18}$$

and thus

$$U_B U_B^\dagger = \mathbb{1}_d, \tag{2.19}$$

which implies that U is unitary. $\square$

Obviously, theorem 1 implies that $(\mathbb{1} \otimes U^* |1\rangle\rangle^{AA})$ is maximally entangled iff U is unitary [12], since the only difference between $|1\rangle\rangle^{AA})$ and $|\phi^+\rangle_{AB}$ is that $|1\rangle\rangle^{AA})$ is non-normalized and U^* is unitary iff U is unitary.

Before we turn our attention to the CJ-representation of operators, which cannot only act on pure states but also on mixed states, we will give an example for an operator in its CJ-vector representation. More explicitly, we want to find the CJ-vector-representation of an operator, which describes a measurement in the direction of $|\phi\rangle \in \mathcal{H}^A$ followed by a preparation of a state $|\psi\rangle \in \mathcal{H}^B$. The according operator is given by $|\phi\rangle \langle\psi|$ [12]. We find

$$\begin{aligned} |(|\psi\rangle \langle\phi|)^*\rangle\rangle^{AB} &= \mathbb{1} \otimes (|\psi\rangle \langle\phi|)^* |\mathbb{1}\rangle\rangle = \sum_{jk} |j\rangle \otimes |\psi\rangle^* \phi_k \langle k| j\rangle \\ &= \sum_j \phi_j |j\rangle \otimes |\psi\rangle^* = |\phi\rangle \otimes |\psi\rangle^* \end{aligned} \tag{2.20}$$

by plugging the operator $|\psi\rangle \langle\phi|$ into equation (2.11) [12].

Now, we turn our attention to the CJ-isomorphism for maps acting on density matrices. Hence, we consider linear maps of the form $\mathcal{M} : \mathcal{L}(\mathcal{H}^A) \to \mathcal{L}(\mathcal{H}^B)$ and define the CJ-isomorphism for such maps as follows

$$M^{AB} = (\mathcal{I} \otimes \mathcal{M}(|\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|))^T, \tag{2.21}$$

where $|\mathbb{1}\rangle\rangle \equiv |\mathbb{1}\rangle\rangle^{AA}$, $\mathcal{I} : \mathcal{L}(\mathcal{H}^A) \to \mathcal{L}(\mathcal{H}^A)$ is the identity map and $M^{AB} \in \mathcal{L}(\mathcal{H}^A \otimes \mathcal{H}^B)$[4, 12].

Furthermore, we find that the inverse isomorphism is given by

$$
\begin{aligned}
\left[\mathsf{Tr}_A\left((\rho\otimes\mathbb{1})M^{AB}\right)\right]^T &= \left[\mathsf{Tr}_A\left(\sum_{ij\ldots}\rho_{ij}\left|i\right\rangle\left\langle j\right|\otimes\mathbb{1})(\left|k\right\rangle\left\langle n\right|\otimes\mathcal{M}(\left|k\right\rangle\left\langle n\right|)^T\right)\right]^T \\
&= \sum_{ij\ldots}\left[\mathsf{Tr}_A\left(\rho_{ij}\left|i\right\rangle\left\langle j\right|\otimes\mathbb{1}\cdot(\left|k\right\rangle\left\langle n\right|\otimes\mathcal{E}_s\left|k\right\rangle\left\langle n\right|\mathcal{E}_s^\dagger)^T\right)\right]^T \\
&= \left[\sum_{nks}\rho_{nk}\mathcal{E}_s^*\left|k\right\rangle\left\langle n\right|\mathcal{E}_s^T\right]^T = \left[\sum_s\mathcal{E}_s^*\rho^T\mathcal{E}_s^T\right]^T \\
&= \sum_s\mathcal{E}_s\rho\mathcal{E}_s^\dagger = \mathcal{M}(\rho),
\end{aligned}
\tag{2.22}
$$

where we have used the Kraus representation of $\mathcal{M}(\rho)=\sum_s\mathcal{E}_s\rho\mathcal{E}_s^\dagger$ [12]. Furthermore we have used that the basis $\{\left|i\right\rangle_A\otimes\left|j\right\rangle_B\}$ of $\mathcal{H}^A\otimes\mathcal{H}^B$ was chosen such that every element of the basis is given by a tensor product of an element of a basis $\{\left|i\right\rangle_A\}$ of the Hilbert space $\mathcal{H}^A$ with an element of a basis $\{\left|j\right\rangle_B\}$ and therefore the following properties for the transpose map hold

$$
\begin{aligned}
K^T &= \left(\sum_{ijkl}K_{ij,kl}\left|i\right\rangle\left\langle j\right|\otimes\left|k\right\rangle\left\langle l\right|\right)^T = (K^{T_A})^{T_B} = \left(\sum_{ijkl}K_{ji,kl}\left|i\right\rangle\left\langle j\right|\otimes\left|k\right\rangle\left\langle l\right|\right)^{T_B} \\
&= \sum_{ijkl}K_{ji,lk}\left|i\right\rangle\left\langle j\right|\otimes\left|k\right\rangle\left\langle l\right|,
\end{aligned}
\tag{2.23}
$$

where $K\in\mathcal{B}(\mathcal{H}^A\otimes\mathcal{H}^B)$ is an arbitrary operator.
To find the connection between the two introduced CJ-isomorphisms defined by equation (2.11) and equation (2.21), we consider operators of the form $\mathcal{M}_A(\rho)=A\rho A^\dagger$. A calculation yields

$$
\begin{aligned}
M^{AB} &= (\mathcal{I}\otimes\mathcal{M}_{\mathcal{A}}(|\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|))^T = \sum_{kl}\left(\left|k\right\rangle\left\langle l\right|\otimes A\left|k\right\rangle\left\langle l\right|A^\dagger\right)^T \\
&= \sum_{kl}\left|k\right\rangle\left\langle l\right|\otimes(A^\dagger)^T\left|k\right\rangle\left\langle l\right|(A^T) = \sum_{kl}\left|k\right\rangle\left\langle l\right|\otimes A^*\left|k\right\rangle\left\langle l\right|(A^*)^\dagger \\
&= \left((\mathbb{1}_A\otimes A^*)\sum_l\left|ll\right\rangle\right)\left((\sum_k\left\langle kk\right|)(\mathbb{1}_A\otimes(A^*)^\dagger\right) \\
&= |A^*\rangle\rangle\langle\langle A^*|,
\end{aligned}
\tag{2.24}
$$

and therefore M^{AB} reduces to a projection onto the CJ-vector $|A^*\rangle\rangle$[12].
Next, we want to present a condition, which can be used to identify whether a map $\mathcal{M}$ is completely positive or not by looking at its CJ-matrix defined by equation (2.21). To see if a linear map $\mathcal{M}$ is completely positive we just have to check if the CJ-isomorphism M^{AB} yields a positive semi-definite matrix i.e. if $M^{AB}\geq 0$. This is a necessary and sufficient condition, a fact we will not prove[12, 14].
Working with density matrices, the conservation of probabilities under a transformation is taken care of by the condition that the transformation is trace preserving [11]. For the CJ-isomorphism

this condition translates to the condition that $\mathcal{M}$ is trace preserving iff $\mathsf{Tr}_B(M^{AB}) = \mathbb{1}_A$ [12].
Proof:(if part):

$$\begin{aligned}\mathsf{Tr}_B(\mathcal{M}(\rho)) &= \mathsf{Tr}_B\left(\left[\mathsf{Tr}_A(\rho\otimes\mathbb{1}_B\cdot M^{AB})\right]^T\right) = \mathsf{Tr}_A\left(\mathsf{Tr}_B(\rho\otimes\mathbb{1}_B\cdot M^{AB})\right)\\ &= \mathsf{Tr}_A\left(\rho\mathbb{1}_A\right) = \mathsf{Tr}_A(\rho),\end{aligned} \tag{2.25}$$

where we took the trace on both sides of equation (2.22) [12].
(only if part):

$$\begin{aligned}\mathsf{Tr}_B(M^{AB}) &= \mathsf{Tr}_B((\mathcal{I}\otimes\mathcal{M}(|\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|))^T) = \sum_{jk}|j\rangle\langle k|\overbrace{\mathsf{Tr}_B\left((\mathcal{M}(|j\rangle\langle k|))^T\right)}^{=\delta_{jk}}\\ &= \sum_j |j\rangle\langle j| = \mathbb{1}_A,\end{aligned} \tag{2.26}$$

which completes the proof. $\square$
Similarly, we find that the CJ-matrix of a trace non-increasing map $\mathcal{M}$ has to satisfy

$$\mathbb{1}_A - \mathsf{Tr}_B(M^{AB}) \geq 0, \tag{2.27}$$

which is a necessary and sufficient condition[12].
We will finish this section by giving a couple of examples of CJ-isomorphisms. We begin with an example of a trace preserving map. We define $\mathcal{M}(\sigma) = \mathsf{Tr}(\sigma)\rho$, with $\mathsf{Tr}(\rho) = 1$. This describes just an input independent preparation of the state ρ (with some pre-factor for unnormalized states). The CJ-ismorphism is then given by

$$\begin{aligned}M^{AB} &= (\mathcal{I}\otimes\mathcal{M}(|\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|))^T = \sum_{jk}|k\rangle\langle j|\otimes\rho^T\underbrace{\mathsf{Tr}(|k\rangle\langle j|)}_{=\delta_{kj}}\\ &= \sum_j |j\rangle\langle j|\otimes\rho^T = \mathbb{1}\otimes\rho^T.\end{aligned} \tag{2.28}$$

Our next example is the CJ-isomorphism of an element of a POVM $\mathcal{M}(\rho) = \mathsf{Tr}(\rho E)$, which has a trivial output space i.e. $H^B = \mathbb{C}$,

$$\begin{aligned}M^{AB} &= \left(\sum_{jk}|j\rangle\langle k|\cdot\mathsf{Tr}(E|j\rangle\langle k|)\right)^T = \left(\sum_{ijk}E_{ij}|j\rangle\langle k|\cdot\mathsf{Tr}(|i\rangle\langle k|)\right)^T\\ &= \sum_{jk}E_{kj}|j\rangle\langle k| = E,\end{aligned} \tag{2.29}$$

which is an example of a trace non-increasing map. Finally, we combine our last two examples i.e. we consider a map where we first measure a POVM element and then independently prepare a state ρ. This map is then given by $\mathcal{M}(\sigma) = \rho\mathsf{Tr}(E\sigma)$ and its CJ-matrix

$$M^{AB} = \left(\sum_{ij}|i\rangle\langle j|\otimes\rho\mathsf{Tr}(E|i\rangle\langle j|)\right)^T = \sum_{ij}E_{ij}|i\rangle\langle j|\otimes\rho^T = E\otimes\rho^T, \tag{2.30}$$

which is what we had expected [12].

Chapter 3

Processes and Causal Structures

In this chapter we will elaborate on what we mean by a process and we will give a precise definition of what we mean by a causal structure. Both definitions will be adopted from [15].
It is anticipated that a theory of quantum gravity includes dynamic and probabilistic causal structures [16]. We will see that the definition of a causal process is compatible with the idea of probabilistic causal structures [15].

3.1 Processes

We will study a general framework, in which we can describe local laboratories, which we call parties and we label them by Alice, Bob, Charlie, ... The framework describes situations where quantum mechanics does not necessarily have to hold globally or locally. Furthermore we are not assuming that the local laboratories are embedded in a global space time or any other form of a prefixed global causal structure i.e. we only assume a fixed order of events inside the laboratories, but no global causal order between the laboratories. The only exchange of information between a specific laboratory and the outer world is by means of an input and an output system. Furthermore, we assume that every laboratory only opens twice. Once, when the input system enters the laboratory and once, when the output system exits the laboratory. Every party is allowed to perform some operations on the input system. Since we are dealing with situations where quantum mechanics does not necessarily have to hold, we will give a more general definition of operations than the definition of quantum operations. However, we will come back to the case where we assume that quantum mechanics holds locally and therefore the case where we only consider quantum operations, when we discuss process matrices [4, 15].
We describe every local experiment by a set of inputs $\{x_A\}$ and a set of outputs $\{a_A\}$, where the subscript A indicates that the settings and outputs belong to Alice's experiment. We call the pair of one setting and one output $\{x_A, a_A\}$ an event, and a set of possible events $\{\{x_A, a_A\}\}_{a_A}$ for a given setting x_A an operation. The appearance and circumstances of an experiment might depend on some variable $w^{A,B\dots} \in \Omega^{A,B\dots}$, where $\Omega^{A,B\dots}$ denotes the set of all possible $w^{A,B\dots}$[15]. Now we can define what we mean, when we talk about a process.

Definition 10. *[15] For a given set of measurement parties $\{A, B, \dots\}$ we say that a process $\mathcal{W}$ is a set of conditional probabilities*

$$\mathcal{W} = \{P(a_A, a_B \dots | x_A, x_B \dots, w^{A,B\dots})\}_{a_A,a_B\dots,x_A,x_B\dots}, \tag{3.1}$$

for a given $w^{A,B\cdots}$.

From now on we will assume that for every considered process the variable $w^{A,B\cdots}$ is obtained without any post-selection. Even though every process is conditioned on a variable $w^{A,B\cdots}$ that defines the initial circumstances, henceforth, we will not write the variable explicitly when we write down a process[15].

3.2 Causal Structures

In this section we will introduce the notion of causal structures and we want to clarify what we mean by saying that a process is causal.
Recall that we have not assumed that a process must be embedded in a space time or any other structure that yields a natural causal structure. However, one could ask if a process at hand is compatible with a causal structure. A causal structure is defined via a strict partial order (SPO) on the set of experimenters $\{A, B\ldots\}$. The SPO gives us relations between the experimenters i.e. it tells us if, for example, A is in the absolute past of B (or equivalent, if B is in the absolute future of A). However, for some parties there is no relation between them and then we say that they are in the absolute elsewhere of each other[15]. More precisely:

Definition 11. *[15] A SPO is a binary operation $\prec$ on the set $\{A, B\ldots\}$ that satisfies*

1. *$A \prec A$ is not allowed (irreflexivity)*
2. *if $A \prec B$, then $B \prec A$ is not allowed (anti-symmetry)*
3. *if $A \prec B$ and $B \prec C$ then $A \prec C$ (transitivity).*

For $A \prec B$ we say that A is in the absolute past of B and equivalently we say that B is in the absolute future of A. Furthermore, instead of $A \prec B$ we might write $B \succ A$. Moreover, we introduce the notation $A \npreceq B$, which means $A \neq B$ and $A \nprec B$. Similarly, $A \nsucceq B$ means $A \neq B$ and $A \nsucc B$. The last notation we will use in this context is $A \npreceq\nsucceq B$, which means $A \npreceq B$ and $A \nsucceq B$. In this case we say that A and B are independent of each other or that they are in the absolute elsewhere of each other. If the set $\{A, B\ldots\}$ is equipped with a SPO and two elements of the set A and B are taken, then always one of the four relations $A \prec B, A \succ B, A \npreceq\nsucceq B$ or $A = B$ holds. Therefore, we can equally describe a SPO by a list $\kappa(A, B\ldots)$, which lists all pairwise relations between the elements of $\{A, B, \ldots\}$. Note that we do not have to explicitly quote the relation between an element with itself in the list $\kappa(A, B\ldots)$, since $X = X$ holds for every SPO[15].
Most generally, for the notion of causality we will not impose that the process at hand is compatible with one fixed SPO, but that it is consistent with a random choice of a SPO from a set of plausible SPOs. Causality should cover the idea that the chosen settings for a local experiment are not allowed to influence the occurrence of events in the absolute past or the absolute elsewhere, neither does it effect the experiments or the SPO in the absolute past or absolute elsewhere[15].
The above can be formalized in the following way:

Definition 12. *[15] We call a process $\mathcal{W} = \{P(a_A, a_B\ldots|x_A, x_B\ldots)\}_{a_A,a_B\ldots,x_A,x_B\ldots}$ causal iff there is a probability distribution $P(\kappa(A, B\ldots)a_A, a_B\ldots|x_A, x_B\ldots)$ s.t.*

$P(a_A, a_B \ldots | x_A, x_B \ldots) = \sum_{\kappa(A,B\ldots)} P(\kappa(A, B \ldots) a_A, a_B \ldots | x_A, x_B \ldots)$, *where the sum runs over* $\kappa(A, B \ldots)$*s describing possible SPOs for* $\{A, B \ldots\}$ *and for every local experiment (e.g.* A*)*

$$P(\kappa(A, \mathcal{X}), A \not\preceq \mathcal{X}, a_{\mathcal{X}} | x_A, x_b, \ldots) = P(\kappa(A, \mathcal{X}), A \not\preceq \mathcal{X}, a_{\mathcal{X}} | x_b, \ldots), \tag{3.2}$$

holds for every subset $\mathcal{X} \subseteq \{B, \ldots\}$ *of parties besides A. Furthermore,* $a_{\mathcal{X}}$ *describes all outcomes corresponding to* $\mathcal{X}$*. The condition* $A \not\prec \mathcal{X}$ *states that all events* $of \mathcal{X}$ *are in the absolute past or absolute elsewhere of* A*. Moreover, we receive the probability distribution* $P(\kappa(A, \mathcal{X}), A \not\preceq \mathcal{X}, a_{\mathcal{X}} | x_A, x_b, \ldots)$ *from* $P(\kappa(A, B \ldots) a_A, a_B \ldots | x_A, x_B \ldots)$ *by, on the one hand side, summing over all cases where* $\kappa(A, B \ldots)$ *is compatible with* $A \not\preceq \mathcal{X}$ *and* $\kappa(A, \mathcal{X})$*, and on the other hand side, summing over all possible outputs corresponding to the experiments* $\{A, B \ldots\}/\mathcal{X}$ *.*

Of course it is possible that a process is compatible with a fixed causal order.

Definition 13. *[15] We say that a process* $\mathcal{W} = \{P(a_A, a_B \ldots | x_A, x_B \ldots)\}_{a_A, a_B \ldots, x_A, x_B \ldots}$ *is a fixed-ordered causal process if it is compatible with a deterministic causal configuration, that is, it obeys equation (3.2)for a* $\kappa(A, B \ldots)$ *that satisfies*

$$\begin{array}{rcl} P(\kappa(A, B \ldots) a_A, a_B \ldots | x_A, x_B \ldots) & = & 0 \\ & \textit{iff} & \\ \kappa_*(A, B \ldots) & \neq & \kappa(A, B \ldots), \\ \forall x_A, \forall x_B \ldots \forall a_A, \forall a_B \ldots & & \end{array}$$

which means that $\kappa(A, B \ldots)$ *takes one specific value* $\kappa_*(A, B \ldots)$ *with unit probability for all settings.*

3.2.1 Bipartite Processes

In this section we want to make a connection between the no signaling condition and the notion of a causal bipartite process. We start by defining a no signaling condition for bipartite processes:

Definition 14. *[15] We say that a bipartite process* $\mathcal{W}$ *allows no signaling from Alice to Bob iff the probability distributions seen by Bob do not depend on the settings of Alice i.e.*

$$P(a_B | x_A, x_B) = \sum_{a_A} P(a_A a_B | x_A, x_B) = P(a_B | x_B), \tag{3.3}$$

must hold $\forall x_A$*.*

(A generalization to more than two parties is not straightforward and we refer to [15] for further details.)
In [15] it was shown for a fixed-ordered causal bipartite process that, if $A \prec B$ there is no signaling from B to A, if $B \prec A$ there is no signaling from A to B and if $A \not\preceq\not\succeq B$ there is no signaling in neither direction. Moreover, it was shown that every causal bipartite process $\mathcal{W}_c$ can be written in the form

$$\mathcal{W}_c = p(A \prec B)\mathcal{W}^{A \prec B} + p(B \prec A)\mathcal{W}^{B \prec A} + p(A \not\preceq\not\succeq B)\mathcal{W}^{A \not\preceq\not\succeq B} \tag{3.4}$$

that is, as a probabilistic mixture of three process $\mathcal{W}^{A \prec B}$, $\mathcal{W}^{B \prec A}$ and $\mathcal{W}^{A \npreceq \nsucceq B}$ compatible with the causal structures $A \prec B$, $B \prec A$ and $A \npreceq \nsucceq B$, respectively. This form is not only necessary but also sufficient. Furthermore, it was shown that (3.4) can be rewritten as

$$\mathcal{W}_c = q\mathcal{W}^{A \npreceq B} + (1-q)\mathcal{W}^{B \npreceq A}, \tag{3.5}$$

where $0 \leq q \leq 1$, $\mathcal{W}^{A \npreceq B}$ is a process that allows no signaling from A to B and $\mathcal{W}^{B \npreceq A}$ is a process that allows no signaling from B to A. However, the split up of $\mathcal{W}_c$ into two processes $\mathcal{W}^{A \npreceq B}$ and $\mathcal{W}^{B \npreceq A}$ might not be unique. Process $\mathcal{W}$, which cannot be rewritten in the form (3.5) are non causal. To check if some process is non causal one could construct a causal inequality and check if the process at hand can be used to break the inequality. If the answer is yes, we know that the process is non causal[4, 15]. An example for a causal inequality and a process that breaks this inequality can be found in [4].

Chapter 4

Process Matrices

Now we will come back to the case where we assume that local quantum mechanics holds. This will be implemented by assuming that every laboratory is equipped with an output Hilbert space, an input Hilbert space and a set of quantum mechanical instruments which are mappings from the input to the output space. These instruments will be represented via the CJ-isomorphism[4]. A formalism based on the CJ-ismorphism, which describes maps that take quantum circuits as an input and depending on the input gives back another quantum circuit, was introduced via quantum combs. Furthermore, every quantum circuit board can be associated with a comb. However, quantum combs always have a definite causal order [17, 18].
In [4] the process matrix formalism was introduced, which is capable of describing situations where locally quantum mechanics hold, but a priori no causal order is assumed and even more so it was shown that the framework can be used to break causal inequalities.
Moreover, the formalism can be used to describe situations, where two (or more) processes can be brought in superposition via a control system. An example for a process matrix describing such a situation is given by the quantum switch, which was already experimentally implemented [6, 12, 19].

4.1 Definition of Process Matrices

In the previous chapter we did not specify which operations Alice, Bob, etc. are allowed to perform on their systems. In this section we will assume that local quantum mechanics holds i.e. all operations that Alice, Bob, etc. perform on their system must obey the rules of quantum mechanics[4, 15].
We will assume that Alice, Bob etc. each hold an instrument. Furthermore, we will just consider the case where the sample space Ω^X is finite, where $X = A, B, \dots$ is the label for Alice, Bob,.... If Alice uses her instrument $\mathcal{M}^A$ she denotes here outcome by the label $i = 1, \dots, n$, with $i \in \Omega^A$. Every $i \in \Omega^A$ corresponds to an operation $\mathcal{M}_i^A : \mathcal{L}(\mathcal{H}^{A_1}) \to \mathcal{L}(\mathcal{H}^{A_2})$, where $\mathcal{H}^{A_1}$ is the d_{A_1}-dimensional Hilbert space corresponding to the system which enters the laboratory, while $\mathcal{H}^{A_2}$ is the d_{A_2}-dimensional Hilbert space associated with the system exiting Alice's laboratory. Every operation $\mathcal{M}_i^A$, when acting on a $\rho \in \mathcal{L}(\mathcal{H}^{A_1})$, can be written in its Kraus representation $\mathcal{M}_i^A(\rho) = \sum_{k=1}^m E_{ik}^\dagger \rho E_{ik}$, with $m = d_{A_1} d_{A_2}$ and $\sum_{k=1}^m E_{ik} E_{ik}^\dagger \leq \mathbb{1}_{A_1} \forall i$, where E_{ik} is a $d_{A_2} \times d_{A_1}$- dimensional matrix. For the complete set of operations $\{\mathcal{M}_i^A\}_{i=1}^n$ we notice that $\mathcal{M}_\Omega^A(\rho) = \mathcal{M}_{\cup_i i}^A(\rho) = \sum_i \mathcal{M}_i^A(\rho) = \sum_i \sum_k E_{ik}^\dagger \rho E_{ik}$, is a channel and therefore trace preserving,

which implies $\sum_{ik} E_{ik}^{\dagger} E_{ik} = \mathbb{1}_A$ [4, 10, 11].
Henceforth we will use the CJ-ismorphism defined by equation (2.21) to represent the operations included in the instruments of Alice, Bob,.... The probability for Alice's i-th, Bob's j-th,... outcomes is then given by

$$P(\mathcal{M}_i^A, \mathcal{M}_j^B, \ldots) = \mathrm{Tr}(W^{A^1A^2B^1B^2\cdots}(M^{A^1A^2} \otimes M^{B^1B^2} \otimes \ldots)), \tag{4.1}$$

where $M^{X^1X^2}$ is the CJ-representation of $\mathcal{M}_i^X$ and $W^{A^1A^2B^1B^2\cdots} \in \mathcal{L}(\mathcal{H}^{A^1} \otimes \mathcal{H}^{A^2} \otimes \mathcal{H}^{B^1} \otimes \mathcal{H}^{B^2} \otimes \ldots)$ is a process matrix which describes the causal structure outside of the laboratories [4, 15, 20]. Obviously, we do require that $W^{A^1A^2B^1B^2\cdots}$ only leads to positive probabilities independent of the choice of the set of completely positive maps $\mathcal{M}_i^A$, $\mathcal{M}_j^B \ldots$. Furthermore, we demand that in every laboratory the experimenter is allowed to hold an auxiliary system, which might be entangled with the other auxiliary systems in the other laboratories. Every experimenter can act on the original system plus his auxiliary system. Therefore, we also demand that the probabilities obtained from $\mathrm{Tr}((W^{A^1A^2B^1B^2\cdots} \otimes \rho^{A'^1B'^1\cdots})(M^{A^1A'^1A^2} \otimes M^{B^1B'^1B^2} \otimes \ldots))$ are all positive, independent of the choice $M^{X^1X'^1X^2}\,\forall X$. It can be shown that this requirement leads to the fact that $W^{A^1A^2B^1B^2\cdots}$ has to be positive semi-definite [4, 15]. We have already mentioned in a previous section that channels i.e. completely positive trace preserving (CPTP) maps do correspond to deterministic transformations.[10]. Moreover we have seen that $\mathcal{M}_i^X$ is CPTP iff $M_i^{X^1X^2} \geq 0$ and $\mathrm{Tr}_{X^2}(M_i^{X^1X^2}) = \mathbb{1}_{X^1}$[4, 12]. All this motivates the definition of a valid process matrix:

Definition 15. *[4, 15] We call the matrix $W^{A^1A^2B^1B^2\cdots} \in \mathcal{L}(\mathcal{H}^{A^1} \otimes \mathcal{H}^{A^2} \otimes \mathcal{H}^{B^1} \otimes \mathcal{H}^{B^2} \otimes \ldots)$ a process matrix iff the requirements*

1. $W^{A^1A^2B^1B^2\cdots} \geq 0$,
2. *Tr*$(W^{A^1A^2B^1B^2\cdots}(M^{A^1A^2} \otimes M^{B^1B^2} \otimes \ldots)) = 1$,
   $\forall M^{A^1A^2}, M^{B^1B^2}, \ldots \geq 0$ *and* $Tr_{A_2} M^{A^1A^2} = \mathbb{1}_{A^1}$ *and* $Tr_{B_2} M^{B^1B^2} = \mathbb{1}_{B^1}$ *and* $\ldots$,

are fulfilled.

4.2 Allowed Terms Appearing in a Process Matrix

For now we will focus on a bipartite scenario, where Alice and Bob are located in two separated laboratories. Therefore, we have to consider process matrices of the form $W^{A^1A^2B^1B^2}$, which can be written as

$$W^{A^1A^2B^1B^2} = \sum_{ijkl} w_{ijkl} \cdot \sigma_i^{A_1} \otimes \sigma_j^{A_2} \otimes \sigma_k^{B_1} \otimes \sigma_l^{B_2}, \tag{4.2}$$

where $w_{ijkl} \in \mathbb{R}$ (because $W^{A^1A^2B^1B^2}$ is hermitian) and we demand that the Hilbert-Schmidt-basis $\{\sigma_i^X\}_{i=0}^{d_X^2-1}$ satisfies the properties $\sigma_0^X = \mathbb{1}_X$, $\mathrm{Tr}(\sigma_i^X) = 0$ for $i = 1, \ldots, d_X^2 - 1$ and $\mathrm{Tr}(\sigma_i^X \sigma_j^X) = d_X \delta_{ij}$ [4].
Terms of the form $\sigma_i^{X_p} \otimes \mathbb{1}_{\text{rest}}$ will be called X_p terms, while terms of the form $\sigma_i^{X_p} \otimes \sigma_j^{Y_q} \otimes \mathbb{1}_{\text{rest}}$ will be referred to as X_pY_q terms and so on. For example an A_1 term can be seen as a term where Alice holds some state, while a state shared by Alice and Bob corresponds to an A_1B_1 term. Also signaling terms are allowed, for example an A_2B_1 term can be interpreted as a channel from Alice's output to Bob's input. We will see shortly that right from the definition of process matrices

it follows that not all kinds of terms are allowed. For example A_1A_2 terms are not allowed, these kind of terms could be interpreted as local loops [4].
Before turning our attention to the bipartite case we investigate which terms are allowed for a single-party scenario. First we consider the CJ-Matrix $M^{A_1A_2}$ of an operation written in the Hilbert-Schmidt basis

$$M^{A_1A_2} = \sum_{pq} m_{pq}\sigma_p \otimes \sigma_q, \tag{4.3}$$

where $m_{pq} \in \mathbb{R}$. If we assume that $M^{A_1A_2}$ is the CJ-matrix of an channel, the condition $\text{Tr}_{A_2}(M^{A_1A_2}) = \mathbb{1}_{A_1}$ then yields

$$\text{Tr}_{A_2}(M^{A_1A_2}) = \text{Tr}_{A_2}(\sum_{pq} m_{pq}\sigma_p \otimes \sigma_q) = m_{00}\mathbb{1}_{A_1} + \sum_{i>0} m_{i0}\sigma_i^{A_1} = \mathbb{1}_{A_1}, \tag{4.4}$$

from which it immediately follows that $m_{00} = 1/d_{A_2}$ and $m_{i0} = 0\ \forall i > 0$. Therefore, every CJ-matrix of a CPTP map can be written in the general form

$$M^{A_1A_2} = \frac{1}{d_{A_2}}\left(\mathbb{1}_{A_1A_2} + N_A\left(\sum_{i>0} a_i\mathbb{1}_{A_1}\otimes\sigma_i^{A_2} + \sum_{i,j>0} t_{ij}\sigma_i^{A_1}\otimes\sigma_j^{A_2}\right)\right), \tag{4.5}$$

where $a_i, t_{ij} \in \mathbb{R}$ are arbitrary and $N_A \in \mathbb{R}$ must be chosen such that $M^{A_1A_2} \geq 0$ [4].
Now it follows from the second condition of definition 15:

$$\begin{aligned}
\text{Tr}(W^{A_1A_2}M^{A_1A_2}) &= \text{Tr}\left(\frac{1}{d_{A_2}}\left(\sum_{pq} w_{pq}\sigma_p^{A_1}\otimes\sigma_q^{A_2}\right)\right. \\
&\quad \cdot \left.\left(\mathbb{1}_{A_1A_2} + N_A\left(\sum_{i>0} a_i\mathbb{1}_{A_1}\otimes\sigma_i^{A_2} + \sum_{i,j>0} t_{ij}\sigma_i^{A_1}\otimes\sigma_j^{A_2}\right)\right)\right) \\
&= d_{A_1}\left(w_{00} + N_A\left(\sum_{i>0} w_{0i}a_i + \sum_{ij>0} w_{ij}t_{ij}\right)\right) = 1,
\end{aligned} \tag{4.6}$$

which holds $\forall a_i, t_{ij} \in \mathbb{R}$ and therefore $w_{00} = 1/d_{d_{A_1}}$ and $w_{0i} = 0\ \forall i > 0$ and $w_{ij} = 0\ \forall i > 0$ and $j > 0$. Hence, every process matrix in the single-party case is of the form

$$W^{A_1A_2} = \frac{1}{d_{A_1}}\left(\mathbb{1}_{A_1A_2} + \sum_{i>0} v_i\sigma^{A^1}\otimes\mathbb{1}_{A_2}\right) = \rho^{A_1}\otimes\mathbb{1}_{A_2}, \tag{4.7}$$

which together with the first requirement of definition (15) that $W^{A_1A_2} \geq 0$ can be interpreted as a state [4]. Now we come back to the bipartite scenario, therefore the process matrix $W^{A_1A_2B_1B_2}$ can be written in the form given by equation (4.2). Again Alice and Bob are each acting on their system with a CPTP map. Therefore we write for Alice's CPTP map

$$M^{A_1A_2} = \frac{1}{d_{A_2}}\left(\mathbb{1}_{A_1A_2} + N_A\left(\sum_{i>0} a_i\mathbb{1}_{A_1}\otimes\sigma_i^{A_2} + \sum_{i,j>0} t_{ij}\sigma_i^{A_1}\otimes\sigma_j^{A_2}\right)\right), \tag{4.8}$$

where $a_i, t_{ij} \in \mathbb{R}$ are arbitrary and $N_A \in \mathbb{R}$ must be chosen such that $M^{A_1A_2} \geq 0$ and for Bob's

$$M^{B_1B_2} = \frac{1}{d_{B_2}} \left(\mathbb{1}_{B_1B_2} + N_B \left(\sum_{i>0} b_i \mathbb{1}_{B_1} \otimes \sigma_i^{B_2} + \sum_{i,j>0} s_{ij} \sigma_i^{B_1} \otimes \sigma_j^{B_2} \right) \right), \tag{4.9}$$

where $b_i, s_{ij} \in \mathbb{R}$ are arbitrary and $N_B \in \mathbb{R}$ must be chosen such that $M^{B_1B_2} \geq 0$.
First we set $M^{B_1B_2} = \mathbb{1}_{B_1B_2}/d_{B_2}$ and from this we get the condition

$$\begin{aligned} \mathrm{Tr}\left(W^{A_1A_2B_1B_2}(M^{A_1A_2} \otimes M^{B^1B^2}) \right) &= d_{A_1B_1} \left(w_{0000} + N_A \left(\sum_{i>0} w_{0i00} a_i \right.\right. \\ &+ \left.\left. \sum_{ij>0} w_{ij00} t_{ij} \right) \right) = 1. \end{aligned} \tag{4.10}$$

Similarly, we find

$$\begin{aligned} \mathrm{Tr}\left(W^{A_1A_2B_1B_2}(M^{A_1A_2} \otimes M^{B^1B^2}) \right) &= d_{A_1B_1} \left(w_{0000} + N_B \left(\sum_{i>0} w_{000i} b_i \right.\right. \\ &+ \left.\left. \sum_{ij>0} w_{00ij} s_{ij} \right) \right) = 1, \end{aligned} \tag{4.11}$$

by setting $M^{A_1A_2} = \mathbb{1}_{A_1A_2}/d_{A_2}$. By recognizing that equation (4.10) and equation (4.11) hold for arbitrary $a_i, b_i, t_{ij}, s_{ij} \in \mathbb{R}$ we find that $w_{0000} = 1/(d_{A_1}d_{B_1})$, $w_{0i00} = w_{000i} = 0 \, \forall i \neq 0$ and $w_{00ij} = w_{ij00} = 0 \; \forall i > 0$ and $j > 0$ [4].
Now let us consider any CPTP maps $M^{A^1A^2}$ given by equation (4.8) and $M^{B^1B^2}$ given by equation (4.9) and notice that

$$\begin{aligned} \frac{\mathrm{Tr}(W^{A_1A_2B_1B_2}(M^{A_1A_2} \otimes M^{B_1B_2})) - 1}{d_{A_1}d_{B_1}N_AN_B} &= \sum_{ij>0} a_i b_j w_{0i0j} + \sum_{ikl>0} a_i s_{kl} w_{0ikl} \\ &+ \sum_{kij>0} b_k t_{ij} w_{ij0k} + \sum_{ijkl>0} t_{ij} s_{kl} w_{ijkl} \\ &= 0. \end{aligned} \tag{4.12}$$

Again this is true for arbitrary $a_i, b_i, t_{ij}, s_{ij} \in \mathbb{R}$ and therefore $w_{0i0j} = w_{0ijk} = w_{ij0k} = w_{ijkl} = 0$ $\forall i > 0$ and $\forall j > 0$ and $\forall k > 0$ and $\forall l > 0$ [4].

Hence, every bipartite process matrix is of the form

$$\begin{aligned}
W^{A_1A_2B_1B_2} &= \frac{1}{d_{A_1B_1}}\Bigg(\mathbb{1}_{A_1A_2B_1B_2} + \sum_{i>0} x_i \sigma_i^{A_1} \otimes \mathbb{1}_{A_2} \otimes \mathbb{1}_{B_1} \otimes \mathbb{1}_{B_2} \\
&+ \sum_{i>0} y_i \mathbb{1}_{A_1} \otimes \mathbb{1}_{A_2} \otimes \sigma_j^{B_1} \otimes \mathbb{1}_{B_2} + \sum_{ij>0} z_{ij} \sigma_i^{A_1} \otimes \mathbb{1}_{A_2} \otimes \sigma_j^{B_1} \otimes \mathbb{1}_{B_2} \\
&+ \sum_{ij>0} c_{ij} \mathbb{1}_{A_1} \otimes \sigma_i^{A_2} \otimes \sigma_j^{B_1} \otimes \mathbb{1}_{B_2} + \sum_{ijk>0} e_{ijk} \sigma_i^{A_1} \otimes \sigma_j^{A_2} \otimes \sigma_k^{B_1} \otimes \mathbb{1}_{B_2} \\
&+ \sum_{ij>0} s_{ij} \sigma_i^{A_1} \otimes \mathbb{1}_{A_2} \otimes \mathbb{1}_{B_1} \otimes \sigma_j^{B_2} + \sum_{ijk>0} f_{ijk} \sigma_i^{A_1} \otimes \mathbb{1}_{A_2} \otimes \sigma_j^{B_1} \otimes \sigma_k^{B_2}\Bigg) \\
&= \frac{1}{d_{A_1} d_{B_1}}\left(\mathbb{1}_{A_1A_2B_1B_2} + \sigma^{A\preceq B} + \sigma^{B\preceq A} + \sigma^{A\npreceq\npreceq B}\right), \qquad (4.13)
\end{aligned}$$

where we have defined

$$\begin{aligned}
\sigma^{A\npreceq\npreceq B} &:= \sum_{i>0} x_i \sigma_i^{A_1} \otimes \mathbb{1}_{A_2} \otimes \mathbb{1}_{B_1} \otimes \mathbb{1}_{B_2} + \sum_{i>0} y_i \mathbb{1}_{A_1} \otimes \mathbb{1}_{A_2} \otimes \sigma_j^{B_1} \otimes \mathbb{1}_{B_2} \\
&+ \sum_{ij>0} z_{ij} \sigma_i^{A_1} \otimes \mathbb{1}_{A_2} \otimes \sigma_j^{B_1} \otimes \mathbb{1}_{B_2}, \qquad (4.14) \\
\sigma^{A\preceq B} &:= \sum_{ij>0} c_{ij} \mathbb{1}_{A_1} \otimes \sigma_i^{A_2} \otimes \sigma_j^{B_1} \otimes \mathbb{1}_{B_2} + \sum_{ijk>0} e_{ijk} \sigma_i^{A_1} \otimes \sigma_j^{A_2} \otimes \sigma_k^{B_1} \otimes \mathbb{1}_{B_2}, \qquad (4.15) \\
\sigma^{B\preceq A} &:= \sum_{ij>0} s_{ij} \sigma_i^{A_1} \otimes \mathbb{1}_{A_2} \otimes \mathbb{1}_{B_1} \otimes \sigma_j^{B_2} + \sum_{ijk>0} f_{ijk} \sigma_i^{A_1} \otimes \mathbb{1}_{A_2} \otimes \sigma_j^{B_1} \otimes \sigma_k^{B_2}, \qquad (4.16)
\end{aligned}$$

where $x_i, y_i, z_{ij}, c_{ij}, s_{ij}, e_{ijk}, f_{ijk} \in \mathbb{R}$[4]. Now, let us interpret the allowed terms of a process matrix $W^{A_1A_2B_1B_2}$. First, we can distinguish two classes of allowed terms by their causal order, namely terms with causal order $B \npreceq A$ and $A \npreceq B$. Terms of the form $A_2B_1$and $A_1A_2B_1$ belong to the class $B \npreceq A$, while terms of the form A_1B_2 and $A_1B_1B_2$ belong to $A \npreceq B$. Moreover, terms of the form A_1, B_1, A_1B_1 can be assigned to both classes $B \npreceq A$ and $A \npreceq B$[4].
As already mentioned earlier the terms A_1, B_1 and A_1B_1 do describe states, while terms of the form A_2B_1 and A_1B_2 are channels. Furthermore, terms of the form $A_1A_2B_1$ and $A_1B_1B_2$ are channels with a memory. That is, for example the term $A_1A_2B_1$, describes a process where not only the output system of Alice influences the input system of Bob, but also the input system of Alice influences the input system of Bob [4].
Next let us focus on the not allowed terms of a two partied process matrix. First of all, the process matrix formalism does not allow post selection i.e. terms of the form A_2, B_2 and A_2, B_2. Furthermore, any loops are prohibited. We talk about a loop if the output system influences the input system of the same laboratory. We distinguish between local loops i.e. terms of the form A_1A_2 and B_1, B_2, local loops with a channel i.e. terms of the form $A_1A_2B_2$ and $A_2B_1B_2$ and global loops i.e. terms of the form $A_1A_2B_1B_2$. However, none of these just mentioned loop terms appear in a two partied process matrix[4].

4.3 Quantum Switch

The idea of the quantum switch traces back to [19]. However, in this paper the process matrix formalism was not used. We will discuss the quantum switch and the notion of causal separability based on [12] and only extra information will be cited explicitly.

For the construction of the quantum switch we consider a three-partite scenario. However, we restrict ourselves to the case, where no-signaling from third parity C to the other two parties A and B is allowed and furthermore, we only consider situations where the output space from party C is one-dimensional. Taking all this into account, we call a process matrix causally separable if it can be written in the form

$$\mathcal{W}^{A_1A_2B_1B_2C_1C_2} = q\mathcal{W}^{A\prec B\prec C} + (1-q)\mathcal{W}^{B\prec A\prec C}, \tag{4.17}$$

where $0 \le q \le 1$ and $\mathcal{W}^{X\prec Y\prec C}$ is a process matrix compatible with the causal structure $X \prec Y \prec C$. We should notice that processes matrices, which satisfy the two-way no-signaling condition between A and B, are compatible with the causal structure $A \prec B \prec C$ and the structure $B \prec A \prec C$. This leads to the fact, that the splitting of a causally separable process matrix $\mathcal{W}^{A_1A_2B_1B_2C_1C_2}$ into the process matrices $\mathcal{W}^{A\prec B\prec C}$ and $\mathcal{W}^{B\prec A\prec C}$ is not always unique [4]. (We refer to [15] for an introduction to the topic of signaling and non-signaling process and process matrices.)

Furthermore, we should notice that the causal separability condition (4.17) is not equal to the condition (3.5) that a process is causal. A process matrix might be causally non separable, but still causal. For a causally non-separable process matrix $\mathcal{W}$ there are no two process matrices $\mathcal{W}^{A\prec B\prec C}$ and $\mathcal{W}^{B\prec A\prec C}$ such that $\mathcal{W}$ can be written as a convex combination of $\mathcal{W}^{A\prec B\prec C}$ and $\mathcal{W}^{B\prec A\prec C}$. Since not every process can be written as a process matrix there might be two processes $\tilde{\mathcal{W}}^{A\prec B\prec C}$ and $\tilde{\mathcal{W}}^{B\prec A\prec C}$ such that $\mathcal{W}$ can be written as a convex combination of $\tilde{\mathcal{W}}^{A\prec B\prec C}$ and $\tilde{\mathcal{W}}^{B\prec A\prec C}$[15]. Indeed, there are causally non-separable processes, which cannot be used to break a causal inequality. An example for such a process matrix is the quantum-switch which we will introduce in a moment [15]. However, there are also process matrices, which are not only non-causally separable, but also non-causal. In [4] the biparite process matrix

$$\mathcal{W}_{OCB} = \frac{1}{4}\left(\mathbb{1}^{A_1A_2B_1B_2} + \frac{1}{\sqrt{2}}\left(\mathbb{1}^{A_1}\otimes\sigma_z^{A_2}\otimes\sigma_z^{B_1}\otimes\mathbb{1}^{B_2} + \sigma_x^{A_1}\otimes\mathbb{1}^{A_2}\otimes\mathbb{1}^{B_1}\otimes\sigma_x^{B_2}\right)\right) \tag{4.18}$$

was introduced and it was shown that $\mathcal{W}_{OCB}$ can be used to break a causal inequality.

We want to point out some analogies to entanglement theory. The first one is the analogy between Bell inequalities and causal inequalities [4]. If a Bell inequality can be broken given a specific state, we know that the state must be non-local and that it must be entangled [21, 22]. Likewise, we know that, if a process matrix can be used to break a causal inequality it must be non-causal and therefore non-causally separable [4]. There are situations where we have a Bell inequality and an entangled state at hand and we cannot use the state to break the bell inequality [22]. Similarly, there are causally non-separable process matrices which cannot break causal inequalities. To detect the causally non-separability of these process matrices a causal-witness might be used. A causal witness S is a hermitian operator which has the property that $\mathrm{Tr}(W_{sep}S) \ge 0$, for every causally separable process matrix W_{sep}. Furthermore, it is always possible to find a causal witness S for a given causally non-separable process matrix $\mathcal{W}_{nsep}$, such that $\mathrm{Tr}(W_{nsep}S) \le 0$ holds. An entanglement witness [10] can be considered as the analogy in entanglement theory.

For the quantum-switch we consider a total system containing two qubits, one control qubit and one target qubit. Alice and Bob both hold a unitary operator U_A and U_B, respectively. The state of the control qubit determines the order in which Alice and Bob apply their operators to the target qubit, that is, if the control qubit is in the state $|0\rangle$ Alice applies U_A before Bob applies U_B and for $|1\rangle$ Alice and Bob apply their operations in the reverse order. To formalize this, we first recognize that an identity channel from an input space $\mathcal{H}^{\text{in}}$ to an output space $\mathcal{H}^{\text{out}}$ written in its CJ-vector is given by $|\mathbb{1}\rangle\rangle^{\text{in, out}}$ [6]. Furthermore, we consider the target qubit to be in an arbitrary initial state $|\psi\rangle$.

If the control qubit is in the state $|0\rangle$ we write

$$|w_0\rangle = |\psi\rangle^{A_1} \otimes |\mathbb{1}\rangle\rangle^{A_2B_1} \otimes |\mathbb{1}\rangle^{B_2C_t} \otimes |0\rangle^{C_c}\,, \tag{4.19}$$

where C_c and C_t are the respective labels for the control and the target qubit received by party C. Similarly, we find that for the case where the control qubit is in the state $|1\rangle$

$$|w_1\rangle = |\psi\rangle^{B_1} \otimes |\mathbb{1}\rangle\rangle^{B_2A_1} \otimes |\mathbb{1}\rangle^{A_2C_t} \otimes |1\rangle^{C_c}\,. \tag{4.20}$$

However, concerning the quantum switch we consider the control qubit to be in the state $(|0\rangle + |1\rangle)/\sqrt{2}$. This leads to a superposition of the order of the application of Alice's and Bob's operations. The corresponding process vector is then given by

$$|w\rangle = \frac{|w_0\rangle + |w_1\rangle}{\sqrt{2}}. \tag{4.21}$$

From this we can construct the process matrix $\mathcal{W}$ given by

$$\mathcal{W} = |w\rangle\langle w|\,. \tag{4.22}$$

The process matrix $\mathcal{W}$ is causally non-separable, which follows directly from the fact, that $\mathcal{W}$ is a projection on the vector $|w\rangle$, which is a vector in superposition of the vector $|w_0\rangle$, compatible with the causal structure $A \prec B \prec C$ and the vector $|w_1\rangle$, compatible with the structure $B \prec A \prec C$. Sometimes one is just interested in situations where party C conducts a measurement on the control qubit [6]. In such cases, we consider the reduced density matrix

$$\mathcal{W}_{\text{SWITCH}} = \text{Tr}_{C_t}(\mathcal{W}) = \text{Tr}_{C_t}(|w\rangle\langle w|). \tag{4.23}$$

Just by construction it is not clear if $\mathcal{W}_{\text{SWITCH}}$ is causally separable or not. It can be shown (see section 5 in [12]) that one can construct a causal witness to prove that $\mathcal{W}_{\text{SWITCH}}$ is causally non-separable.

The question arises, whether the third party is needed to obtain a causally non-separable process matrix, or if we can actually discard the control qubit as well. For this purpose we first observe

$$\text{Tr}_{C_t}\left(|\mathbb{1}\rangle\rangle^{XC_t}\langle\langle\mathbb{1}|^{XC_t}\right) = \sum_{jk} |j\rangle\langle k|\,\text{Tr}(|j\rangle\langle k|) = \sum_k |k\rangle\langle k| = \mathbb{1}_X. \tag{4.24}$$

This can be used to calculate

$$\begin{aligned}
\widetilde{\mathcal{W}} \;&:=\; \text{Tr}_C(|w\rangle\langle w|) = \text{Tr}_{C_t}\text{Tr}_{C_c}\left(|w\rangle\langle w|\right)\\
&=\; \frac{1}{2}\Big(|\psi\rangle^{A_1}\langle\psi|^{A_1} \otimes |\mathbb{1}\rangle\rangle^{A_2B_1}\langle\langle\mathbb{1}|^{A_2B_1} \otimes \text{Tr}_{C_t}\left(|\mathbb{1}\rangle\rangle^{B_2C_t}\langle\langle\mathbb{1}|^{B_2C_t}\right)\\
&+\; |\psi\rangle^{B_1}\langle\psi|^{B_1} \otimes |\mathbb{1}\rangle\rangle^{B_2A_1}\langle\langle\mathbb{1}|^{B_2A_1} \otimes \text{Tr}_{C_t}\left(|\mathbb{1}\rangle\rangle^{A_2C_t}\langle\langle\mathbb{1}|^{A_2C_t}\right)\Big)\\
&=\; \frac{1}{2}\Big(|\psi\rangle^{A_1}\langle\psi|^{A_1} \otimes |\mathbb{1}\rangle\rangle^{A_2B_1}\langle\langle\mathbb{1}|^{A_2B_1} \otimes \mathbb{1}_{B_2}\\
&+\; |\psi\rangle^{B_1}\langle\psi|^{B_1} \otimes |\mathbb{1}\rangle\rangle^{B_2A_1}\langle\langle\mathbb{1}|^{B_2A_1} \otimes \mathbb{1}_{A_2}\Big),
\end{aligned} \tag{4.25}$$

which implies that if we discard party C we obtain a causally separable process matrix, because $\widetilde{\mathcal{W}}$ is of the form

$$\widetilde{\mathcal{W}} = \frac{1}{2}\widetilde{\mathcal{W}}^{A\prec B} + \frac{1}{2}\widetilde{\mathcal{W}}^{B\prec A}, \tag{4.26}$$

where

$$\widetilde{\mathcal{W}}^{A\prec B} = |\psi\rangle^{A_1} \langle\psi|^{A_1} \otimes |\mathbb{1}\rangle\rangle^{A_2B_1}\langle\langle\mathbb{1}|^{A_2B_1} \otimes \mathbb{1}_{B_2} \tag{4.27}$$

and

$$\widetilde{\mathcal{W}}^{B\prec A} = |\psi\rangle^{B_1} \langle\psi|^{B_1} \otimes |\mathbb{1}\rangle\rangle^{B_2A_1}\langle\langle\mathbb{1}|^{B_2A_1} \otimes \mathbb{1}_{A_2}. \tag{4.28}$$

4.3.1 An Example for an Application of the Quantum Switch

In [23] it was shown that a quantum process with an indefinite causal structures can be used to distinguish a channel including only commuting Kraus operators from a channel that only includes anti-commuting Kraus operators with a single query. However, we will only concentrate on unitary channels. This restriction was also made in [24]. **We will base our discussion in this section on [24] and only extra information will be cited explicitly.**
In this paper the process matrix formalism is not used. Our goal is to translate the ideas from [24] into the process matrix formalism.
We consider again a three-partite scenario. Alice and Bob both are holding unitary operators U_A and U_B, respectively. We demand that U_A and U_B commute or anti-commute i.e. that $U_AU_B = U_BU_A$ or $U_AU_B = -U_BU_A$, but else U_A and U_B are arbitrary. Our goal is to determine if U_A and U_B commute or anti-commute. Furthermore, we only allow Alice and Bob to use their transformation once. We will see that using the quantum switch it is possible to determine with just one single run of the experiment, whether U_A and U_B commute or anti-commute.
Alice and Bob both apply their transformation to the target qubit, while Charlie acts only on the control qubit. Charlie holds an instrument, which first applies the Hadamard gate to the qubit and then performs a protective measurement of the qubit in the $\{|0\rangle, |1\rangle\}$-basis. The Hadamard has the form

$$H = \frac{1}{\sqrt{2}}\left(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|\right), \tag{4.29}$$

if it is also written in the $\{|0\rangle, |1\rangle\}$-basis [11]. While the projectors P_i are given by

$$P_i = |i\rangle\langle i|, \tag{4.30}$$

for $i \in \{0, 1\}$. Furthermore, we introduce the operators

$$E_0 = P_0H = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1|) \tag{4.31}$$

and

$$E_1 = P_1H = \frac{1}{\sqrt{2}}(|1\rangle\langle 0| - |1\rangle\langle 1|). \tag{4.32}$$

Charlies instrument then consists of the two operations $\mathcal{M}_{E_0}(\rho) = \mathsf{Tr}(E_0 \rho E_0^\dagger)$ and $\mathcal{M}_{E_1}(\rho) = \mathsf{Tr}(E_1 \rho E_1^\dagger)$. For the CJ-matrix representation of $\mathcal{M}_{E_0}$ we calculate

$$\begin{aligned} M_0^{C_c} &= (\mathcal{I} \otimes M_{E_0}(|\mathbb{1}\rangle\rangle\langle\langle \mathbb{1}|))^T = \left(\sum_{ij} |i\rangle\langle j| \cdot \mathsf{Tr}\left(E_0 |i\rangle\langle j| E_0^\dagger\right)\right)^T \\ &= \frac{1}{2}\left(\sum_{ij}(\delta_{i0}\delta_{j0} + \delta_{i0}\delta_{j1} + \delta_{i1}\delta_{j0} + \delta_{i1}\delta_{j1}) |i\rangle\langle j| \cdot \mathsf{Tr}\left(E_0 |0\rangle\langle 0|^\dagger\right)\right) \\ &= \frac{1}{2}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|) = \frac{1}{2}\sum_{\pi\eta} |\pi\rangle\langle\eta| \end{aligned} \tag{4.33}$$

and similarly for $\mathcal{M}_{E_1}$

$$M_1^{C_c} = \frac{1}{2}(|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|) = \frac{1}{2}\sum_{\pi\eta}(-1)^{\pi+\eta} |\pi\rangle\langle\eta| . \tag{4.34}$$

The operation associated with a unitary operator is given by $\mathcal{M}_U(\rho) = U\rho U^\dagger$ [10] and using equation (2.24) we find the CJ-matrix

$$\begin{aligned} M^{X_1X_2} &= |U^*\rangle\rangle\langle\langle U^*| = \sum_{lkstmn} |l\rangle\langle k| \otimes U_{st}^* |s\rangle\langle t| l\rangle\langle k| n\rangle\langle m| U_{mn} \\ &= \sum_{slmk} U_{sl}^* U_{mk} |l\rangle\langle k| \otimes |s\rangle\langle m| . \end{aligned} \tag{4.35}$$

All in all, for the CJ-representation of the product of Alice's, Bob's and Charlie's operations we can write

$$\begin{aligned} M^{A_1A_2} \otimes M^{B_1B_2} \otimes M_0^{C_c} &= \sum_{\alpha\beta\ldots} \frac{1}{2}(U_A)_{\gamma\alpha}^*(U_A)_{\tau\beta}(U_B)_{\xi\sigma}^*(U_B)_{\Omega\omega} \\ &\cdot \quad |\alpha\rangle\langle\beta| \otimes |\gamma\rangle\langle\tau| \otimes |\sigma\rangle\langle\omega| \otimes |\xi\rangle\langle\Omega| \otimes |\pi\rangle\langle\eta| , \end{aligned} \tag{4.36}$$

if Charlie applies $\mathcal{M}_{E_0}$ on his system, and

$$\begin{aligned} M^{A_1A_2} \otimes M^{B_1B_2} \otimes M_1^{C_c} &= \sum_{\alpha\beta\ldots} \frac{1}{2}(-1)^{\pi+\eta}(U_A)_{\gamma\alpha}^*(U_A)_{\tau\beta}(U_B)_{\xi\sigma}^*(U_B)_{\Omega\omega} \\ &\cdot \quad |\alpha\rangle\langle\beta| \otimes |\gamma\rangle\langle\tau| \otimes |\sigma\rangle\langle\omega| \otimes |\xi\rangle\langle\Omega| \otimes |\pi\rangle\langle\eta| , \end{aligned} \tag{4.37}$$

if he applies $\mathcal{M}_{E_1}$.
Since Charlie only acts on the control qubit we will work with the process matrix $\mathcal{W}_{\textsf{SWITCH}}$. For further calculations we will write out $\mathcal{W}_{\textsf{SWITCH}}$ explicitly. For that purpose we first observe

$$\begin{aligned} |w\rangle &= \frac{1}{\sqrt{2}}\left(\sum_{jx\ldots} \psi_j |j\rangle_{A_1} \otimes |x\rangle_{A_2} \otimes |x\rangle_{B_1} \otimes |y\rangle_{B_2} \otimes |y\rangle_{C_t} \otimes |0\rangle_{C_c}\right. \\ &+ \left.\sum_{ia\ldots} \psi_i |a\rangle_{A_1} \otimes |b\rangle_{A_2} \otimes |i\rangle_{B_1} \otimes |a\rangle_{B_2} \otimes |b\rangle_{C_t} \otimes |1\rangle_{C_c}\right) . \end{aligned} \tag{4.38}$$

This can be used to calculate the explicit form of $\mathcal{W}_{\text{SWITCH}} = \text{Tr}_{C_t}(|w\rangle\langle w|)$, namely

$$\begin{aligned}
\mathcal{W}_{\text{SWITCH}} &= \frac{1}{2}\left(\sum_{jk\ldots}\psi_j\psi_k^*\,|j\rangle\langle k|\otimes|x\rangle\langle l|\otimes|x\rangle\langle l|\otimes|y\rangle\langle y|\otimes|0\rangle\langle 0|\right.\\
&+ \sum_{jn\ldots}\psi_j\psi_n^*\,|j\rangle\langle s|\otimes|x\rangle\langle y|\otimes|x\rangle\langle n|\otimes|y\rangle\langle s|\otimes|0\rangle\langle 1|\\
&+ \sum_{ik\ldots}\psi_i\psi_k^*\,|a\rangle\langle k|\otimes|b\rangle\langle l|\otimes|i\rangle\langle l|\otimes|a\rangle\langle b|\otimes|1\rangle\langle 0|\\
&+ \left.\sum_{in\ldots}\psi_i\psi_n^*\,|a\rangle\langle s|\otimes|b\rangle\langle b|\otimes|i\rangle\langle n|\otimes|a\rangle\langle s|\otimes|1\rangle\langle 1|\right). \qquad (4.39)
\end{aligned}$$

Our next goal is to calculate the probabilities $P(\mathcal{M}_{U_A},\mathcal{M}_{U_B},\mathcal{M}_{E_0})$ and $P(\mathcal{M}_{U_A},\mathcal{M}_{U_B},\mathcal{M}_{E_1})$ given by the generalized born rule, which is defined by equation (4.1). To achieve this goal some preparatory work is required.

For each line of $\mathcal{W}_{SWITCH}$ given in equation (4.39) we build the product with $M^{A_1A_2}\otimes M^{B_1B_2}\otimes M_0^{C_c}$ given by equation (4.36), then we take the trace of each independent product. We start with the first line

$$\begin{aligned}
&\text{Tr}\left(\left(\frac{1}{2}\sum_{jk\ldots}\psi_j\psi_k^*\,|j\rangle\langle k|\otimes|x\rangle\langle l|\otimes|x\rangle\langle l|\otimes\mathbb{1}\otimes|0\rangle\langle 0|\right)\left(M^{A_1A_2}\otimes M^{B_1B_2}\otimes M_0^{C_c}\right)\right)\\
&= \text{Tr}\left(\left(\frac{1}{2}\sum_{jk\ldots}\psi_j\psi_k^*\,|j\rangle\langle k|\otimes|x\rangle\langle l|\otimes|x\rangle\langle l|\otimes\mathbb{1}\otimes|0\rangle\langle 0|\right)\right.\\
&\left.\left(\sum_{\alpha\beta\ldots}\frac{1}{2}(U_A)^*_{\gamma\alpha}(U_A)_{\tau\beta}(U_B)^*_{\xi\sigma}(U_B)_{\Omega\omega}\cdot|\alpha\rangle\langle\beta|\otimes|\gamma\rangle\langle\tau|\otimes|\sigma\rangle\langle\omega|\otimes|\xi\rangle\langle\Omega|\otimes|\pi\rangle\langle\eta|\right)\right)\\
&= \frac{1}{4}\text{Tr}\left(\sum_{jk\ldots}\psi_j\psi_k^*(U_A)^*_{\gamma k}(U_A)_{\tau\beta}(U_B)^*_{\xi\gamma}(U_B)_{\Omega\omega}\,|j\rangle\langle\beta|\right.\\
&\left.\otimes|x\rangle\langle\tau|\otimes|x\rangle\langle\omega|\otimes|\xi\rangle\langle\Omega|\otimes|0\rangle\langle\eta|\right)\\
&= \frac{1}{4}\left(\sum_{jk\ldots}\psi_k^*(U_A)^*_{\gamma k}(U_B)^*_{\Omega\gamma}(U_B)_{\Omega\tau}(U_A)_{\tau j}\psi_j\right) = \frac{1}{4}\langle\psi|\,U_A^\dagger U_B^\dagger U_B U_A\,|\psi\rangle\\
&= \frac{1}{4}\langle\psi|\,\psi\rangle = \frac{1}{4}, \qquad (4.40)
\end{aligned}$$

where, in the second step of the last line, we have used that U_A and U_B are unitary. Similarly, we find that for all other lines

$$\begin{aligned}\mathsf{Tr}\left(\frac{1}{2}\left(\sum_{jn\ldots}\psi_j\psi_n^*\left|j\right\rangle\left\langle s\right|\otimes\left|x\right\rangle\left\langle y\right|\otimes\left|x\right\rangle\left\langle n\right|\otimes\left|y\right\rangle\left\langle s\right|\otimes\left|0\right\rangle\left\langle 1\right|\right)\right.\\ \left.\cdot\left(M^{A_1A_2}\otimes M^{B_1B_2}\otimes M_0^{C_c}\right)\right)=\frac{1}{4}\left\langle\psi\right|U_B^\dagger U_A^\dagger U_BU_A\left|\psi\right\rangle,\end{aligned}\tag{4.41}$$

$$\begin{aligned}\mathsf{Tr}\left(\frac{1}{2}\left(\sum_{ik\ldots}\psi_i\psi_k^*\left|a\right\rangle\left\langle k\right|\otimes\left|b\right\rangle\left\langle l\right|\otimes\left|i\right\rangle\left\langle l\right|\otimes\left|a\right\rangle\left\langle b\right|\otimes\left|1\right\rangle\left\langle 0\right|\right)\right.\\ \left.\cdot\left(M^{A_1A_2}\otimes M^{B_1B_2}\otimes M_0^{C_c}\right)\right)=\frac{1}{4}\left\langle\psi\right|U_A^\dagger U_B^\dagger U_AU_B\left|\psi\right\rangle,\end{aligned}\tag{4.42}$$

$$\begin{aligned}\mathsf{Tr}\left(\frac{1}{2}\left(\sum_{in\ldots}\psi_i\psi_n^*\left|a\right\rangle\left\langle s\right|\otimes\mathbb{1}\otimes\left|i\right\rangle\left\langle n\right|\otimes\left|a\right\rangle\left\langle s\right|\otimes\left|1\right\rangle\left\langle 1\right|\right)\left(M^{A_1A_2}\otimes M^{B_1B_2}\otimes M_0^{C_c}\right)\right)\\ =\frac{1}{4}\left\langle\psi\right|U_B^\dagger U_A^\dagger U_AU_B\left|\psi\right\rangle=\frac{1}{4}.\end{aligned}\tag{4.43}$$

In the same manner, we build the product of each line of $\mathcal{W}_{SWITCH}$ with $P(\mathcal{M}_{U_A},\mathcal{M}_{U_B},\mathcal{M}_{E_1})$ given by equation (4.37), then we take the trace of each independent product. Similar calculations as before yield

$$\begin{aligned}\mathsf{Tr}\left(\left(\frac{1}{2}\sum_{jk\ldots}\psi_j\psi_k^*\left|j\right\rangle\left\langle k\right|\otimes\left|x\right\rangle\left\langle l\right|\otimes\left|x\right\rangle\left\langle l\right|\otimes\mathbb{1}\otimes\left|0\right\rangle\left\langle 0\right|\right)\right.\\ \left.\cdot\left(M^{A_1A_2}\otimes M^{B_1B_2}\otimes M_1^{C_c}\right)\right)=\frac{1}{4}\left\langle\psi\right|U_A^\dagger U_B^\dagger U_BU_A\left|\psi\right\rangle=\frac{1}{4},\end{aligned}\tag{4.44}$$

$$\begin{aligned}\mathsf{Tr}\left(\frac{1}{2}\left(\sum_{jn\ldots}\psi_j\psi_n^*\left|j\right\rangle\left\langle s\right|\otimes\left|x\right\rangle\left\langle y\right|\otimes\left|x\right\rangle\left\langle n\right|\otimes\left|y\right\rangle\left\langle s\right|\otimes\left|0\right\rangle\left\langle 1\right|\right)\right.\\ \left.\cdot\left(M^{A_1A_2}\otimes M^{B_1B_2}\otimes M_1^{C_c}\right)\right)=-\frac{1}{4}\left\langle\psi\right|U_B^\dagger U_A^\dagger U_BU_A\left|\psi\right\rangle,\end{aligned}\tag{4.45}$$

$$\begin{aligned}\mathsf{Tr}\left(\frac{1}{2}\left(\sum_{ik\ldots}\psi_i\psi_k^*\left|a\right\rangle\left\langle k\right|\otimes\left|b\right\rangle\left\langle l\right|\otimes\left|i\right\rangle\left\langle l\right|\otimes\left|a\right\rangle\left\langle b\right|\otimes\left|1\right\rangle\left\langle 0\right|\right)\right.\\ \left.\cdot\left(M^{A_1A_2}\otimes M^{B_1B_2}\otimes M_1^{C_c}\right)\right)=-\frac{1}{4}\left\langle\psi\right|U_A^\dagger U_B^\dagger U_AU_B\left|\psi\right\rangle,\end{aligned}\tag{4.46}$$

$$\begin{aligned}\mathsf{Tr}\left(\frac{1}{2}\left(\sum_{in\ldots}\psi_i\psi_n^*\left|a\right\rangle\left\langle s\right|\otimes\mathbb{1}\otimes\left|i\right\rangle\left\langle n\right|\otimes\left|a\right\rangle\left\langle s\right|\otimes\left|1\right\rangle\left\langle 1\right|\right)\left(M^{A_1A_2}\otimes M^{B_1B_2}\otimes M_1^{C_c}\right)\right)\\ =\frac{1}{4}\left\langle\psi\right|U_B^\dagger U_A^\dagger U_AU_B\left|\psi\right\rangle=\frac{1}{4}.\end{aligned}\tag{4.47}$$

From equations (4.40), (4.41), (4.42) and (4.43) we can directly calculate the probability

$$
\begin{aligned}
P(\mathcal{M}_{U_A}, \mathcal{M}_{U_B}, \mathcal{M}_{E_0}) &= \mathrm{Tr}\left(\mathcal{W}_{SWITCH}(M^{A_1A_2} \otimes M^{B_1B_2} \otimes M_0^{C_c})\right) \\
&= \frac{1}{2} + \frac{1}{4}\langle\psi| U_B^\dagger U_A^\dagger U_B U_A |\psi\rangle + \frac{1}{4}\langle\psi| U_A^\dagger U_B^\dagger U_A U_B |\psi\rangle\,, \quad (4.48)
\end{aligned}
$$

and from equations (4.44), (4.45), (4.46) and (4.47) we find

$$
\begin{aligned}
P(\mathcal{M}_{U_A}, \mathcal{M}_{U_B}, \mathcal{M}_{E_1}) &= \mathrm{Tr}\left(\mathcal{W}_{SWITCH}(M^{A_1A_2} \otimes M^{B_1B_2} \otimes M_1^{C_c})\right) \\
&= \frac{1}{2} - \frac{1}{4}\langle\psi| U_B^\dagger U_A^\dagger U_B U_A |\psi\rangle - \frac{1}{4}\langle\psi| U_A^\dagger U_B^\dagger U_A U_B |\psi\rangle\,. \quad (4.49)
\end{aligned}
$$

Up to now we have not used the fact, that we just consider commuting or anti-commuting operators U_A and U_B. We start with the investigation of the case where U_A and U_B commute i.e. when $U_AU_B = U_BU_A$ holds. Hence, we can rewrite equation (4.48):

$$
\begin{aligned}
P(\mathcal{M}_{U_A}, \mathcal{M}_{U_B}, \mathcal{M}_{E_0}) &= \frac{1}{2} + \frac{1}{4}\langle\psi| U_B^\dagger U_A^\dagger U_B U_A |\psi\rangle + \frac{1}{4}\langle\psi| U_A^\dagger U_B^\dagger U_A U_B |\psi\rangle \\
&= \frac{1}{2} + \frac{1}{4}\langle\psi| U_B^\dagger U_A^\dagger U_A U_B |\psi\rangle + \frac{1}{4}\langle\psi| U_A^\dagger U_B^\dagger U_B U_A |\psi\rangle \\
&= \frac{1}{2} + \frac{1}{4}\langle\psi|\psi\rangle + \frac{1}{4}\langle\psi|\psi\rangle = 1.
\end{aligned}
$$

At this point, we can notice that the last result is independent of the initial state of the target qubit.

Similarly, we find that the probability $P(\mathcal{M}_{U_A}, \mathcal{M}_{U_B}, \mathcal{M}_{E_1}) = 0$, if $U_AU_B = U_BU_A$ holds.

The probabilities for the case where U_A and U_B anti-commutes i.e. where $U_AU_B = -U_BU_A$ holds, can be found in a similar fashion as for the case where U_A and U_B commute. The probabilities from both cases can be summarized

$$
P(\mathcal{M}_{U_A}, \mathcal{M}_{U_B}, \mathcal{M}_{E_0}) = \begin{cases} 1 & \text{if } U_AU_B = U_BU_A \\ 0 & \text{if } U_AU_B = -U_BU_A \end{cases}, \quad (4.50)
$$

$$
P(\mathcal{M}_{U_A}, \mathcal{M}_{U_B}, \mathcal{M}_{E_1}) = \begin{cases} 0 & \text{if } U_AU_B = U_BU_A \\ 1 & \text{if } U_AU_B = -U_BU_A \end{cases}. \quad (4.51)
$$

This implies that, if Charlie observes the result 0, associated with $\mathcal{M}_{E_0}$, we can be sure that U_A and U_B commute, while, if Charlie measures 1, associated with $\mathcal{M}_{E_1}$, we know that U_A and U_B anti-commute.

4.4 Non-Signaling Bipartite Process Matrices

In this section we are going to prove that if we constrain a bipartite process matrix to be non-signaling, we end up with normal quantum states.

For this purpose we write down a n-partied process matrix $W^{A_1A_2B_1B_2C_1C_2\dots}$ in a Hilbert-Schmidt basis $\{\sigma_i^X\}_{i=0}^{d_X^2-1}$:

$$
W^{A_1A_2B_1B_2C_1C_2\dots} = \sum_{ijklmn\dots} w_{ijklmn\dots}\sigma_i^{A_1} \otimes \sigma_j^{A_2} \otimes \sigma_k^{B1} \otimes \sigma_l^{B_2} \otimes \sigma_m^{C1} \otimes \sigma_n^{C_2}\dots, \quad (4.52)
$$

where we demand that $\sigma_0^X = \mathbb{1}_X$ and $\mathrm{Tr}(\sigma_i^X\sigma_j^X) = d_X\sigma_{ij}$. Furthermore, we will need the following two theorems, which were introduced and proven in [15].

Theorem 2. *[15] The condition $Tr(W^{A_1A_2B_1B_2C_1C_2\cdots}(M^{A_1A_2}M^{B_1B_2}M^{C_1C_2}\ldots)) = 1$ $\forall M^{A_1A_2}, M^{B_1B_2}, M^{C_1C_2}, \ldots \geq 0$, where $Tr_{A_2}(M^{A_1A_2}) = \mathbb{1}_{A_1}$, $Tr_{B_2}(M^{B_1B_2}) = \mathbb{1}_{B_1}$, $Tr_{C_2}(M^{C_1C_2}) = \mathbb{1}_{C_1}, \ldots$, for an operator $W^{A_1A_2B_1B_2C_1C_2\cdots}$ of the form (4.52) is met iff besides the identity term all other terms are of the form, such that there is at least one party $X \in \{A, B, C\ldots\}$ for which there is a non-trivial $\sigma_i^{X_1}$,with $i \in \{1,2,3\}$ and a trivial $\sigma_0^{X_2} = \mathbb{1}_{X_2}$.*

Theorem 3. *[15] A n-partied process matrix $W^{A_1A_2B_1B_2C_1C_2\cdots}$, with parties $\{1,2,3,\ldots,n\}$ allows no signaling from parties $\{1$ and 2 and $\ldots$ and $k\}$ to $\{k+1$ and $k+2$ and $\ldots$ and $n\}$ iff $W^{A_1A_2B_1B_2C_1C_2\cdots}$ includes only terms, whose restrictions on $1_1 1_2 \ldots k_1 k_2$ are still of the allowed forms for process matrices as specified in Theorem 2. The restriction of a term $\sigma_i^{1_1} \otimes \sigma_j^{1_2} \otimes \ldots \sigma_l^{k_1} \otimes \sigma_s^{k_2} \otimes \sigma_t^{(k+1)_1} \otimes \sigma_p^{(k+1)_2} \otimes \ldots \sigma_q^{n_1} \otimes \sigma_r^{n_2}$ to $1_1 1_2 \ldots k_1 k_2$ is given by $\sigma_i^{1_1} \otimes \sigma_j^{1_2} \otimes \ldots \sigma_l^{k_1} \otimes \sigma_s^{k_2}$.*

We will use the same nomenclature for the different kinds of terms as before. For example we will call a term of the form $\sigma_i^{A_1} \otimes \mathbb{1}_{A_2} \otimes \sigma_j^{B_1} \otimes \sigma_k^{B_2} \otimes \ldots \otimes \mathbb{1}_{N_2}$, with $i,j,k \in \{1,2,3\}$, an $A_1B_1B_2$-term.
To start the proof that a non-signaling bipartite process matrix is just a normal quantum mechanical state, we write down a list of the allowed terms (additionally to the identity term) for a general bipartite process matrix $W^{A_1A_2B_1B_2}$. This list can be easily found by using Theorem 2 or by looking at the general form of a bipartite process matrix (4.13) and is given by terms of the following forms

$$A_1, B_1, A_1B_1, A_1B_2, A_1B_1B_2, A_2B_1, A_1A_2B_1. \tag{4.53}$$

Similarly we find that a single-party process matrix $W^{A_1A_2}$ only includes terms (additionally to the identity term) of the form A_1.
Now using theorem 3, we can easily check that if we assume no signaling from A to B our initial list (4.53) of allowed terms reduces to

$$A_1, B_1, A_1B_1, A_1B_2, A_1B_1B_2. \tag{4.54}$$

Furthermore, we see that this list reduces even more, if we additionally assume no signaling from Bob to Alice. Then our final list is given by

$$A_1, B_1, A_1B_1. \tag{4.55}$$

This implies that a non-signaling bipartite process matrix $W_{ns}^{A_1A_2B_1B2}$ must be of the form

$$\begin{aligned} W_{ns}^{A_1A_2B_1B_2} &= \frac{1}{d_{A_1}d_{B_1}}\Bigg(\mathbb{1}_{A_1A_2B_1B_2} + \sum_{i>0} x_i\sigma_i^{A_1} \otimes \mathbb{1}_{A_2} \otimes \mathbb{1}_{B_1} \otimes \mathbb{1}_{B_2} \\ &+ \sum_{i>0} y_i \mathbb{1}_{A_1} \otimes \mathbb{1}_{A_2} \otimes \sigma_j^{B_1} \otimes \mathbb{1}_{B_2} + \sum_{ij>0} z_{ij}\sigma_i^{A_1} \otimes \mathbb{1}_{A_2} \otimes \sigma_j^{B_1} \otimes \mathbb{1}_{B_2}\Bigg), \end{aligned} \tag{4.56}$$

where $x_i, y_j, z_{ij} \in \mathbb{R}, \forall i,j$. But, $W_{ns}^{A_1A_2B_1B_2}$ being of this form together with the condition $W_{ns}^{A_1A_2B_1B_2} \geq 0$ allows us to interpret $W_{ns}^{A_1A_2B_1B_2}$ as a quantum mechanical state. □

Chapter 5

Tsirelson's Problem

Since the work of Bell in 1964 [21] we know that quantum mechanics cannot be explained by local hidden variable models. He used correlation functions to show this fact. These correlation functions require some notion of local separation of different experimenters. In 1993 Tsirelson stated that the two different ways of implementing local separation in quantum mechanics give the same set of correlations. He stated that the first way of implementing local separation is via a tensor product structure and for the other implementation he required that the measurement operator belonging to two different experimenters does commute [25]. However he gave no proof of the statement. In 2006 Tsirelson revoked his statement and listed it as an open question [26]. At the same time he provided proof that his statement holds in finite dimensions. Eventually, it was shown by Slofstra [27] that in infinite dimensions the set of correlations obtained by the implementation of local separation via commuting algebras is bigger than the set obtained via tensor product structure. Before going into more detail concerning the Tsirelson's Problem in finite dimensions, we will first set the groundwork by giving some mathematical definitions and theorems.

5.1 Preliminaries

In this section we want to explore some basic properties of C^*-algebras. An introduction to this topic can be found in [28] for finite-dimensional C^*-algebras and in [29] for general C^*-algebras. We will concentrate on finite-dimensional C^*-algebras. We start with some basic definitions.

Definition 16. *[28] Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$. We define the commutant $\mathcal{A}'$ by*

$$\mathcal{A}' := \{T \in \mathcal{B}(\mathcal{H}) \,|\, Ta = aT \,\forall a \in \mathcal{A}\}. \tag{5.1}$$

Definition 17. *[28] Let $\mathcal{A}$ be a $*$-algebra. The center of $\mathcal{A}$, denoted by $Z(\mathcal{A})$, is given by*

$$Z(\mathcal{A}) := \{z \in \mathcal{A} \,|\, za = az \;\forall a \in \mathcal{A}\}. \tag{5.2}$$

Definition 18. *[28] Let $\mathcal{A}$ be a $*$-algebra. We say that a map $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a representation of $\mathcal{A}$ if the following requirements are fulfilled:*

1. $\pi(\lambda a + \mu b) = \lambda\pi(a) + \mu\pi(b) \;\forall a, b \in \mathcal{A}$ *and* $\forall \lambda, \mu \in \mathbb{C}$,
2. $\pi(ab) = \pi(a)\pi(b) \;\forall a, b \in \mathcal{A}$ *(multiplication preserving),*

3. $\pi(a^*) = \pi(a)^*\ \forall a \in \mathcal{A}$ *(involution preserving).*

Additionally, we say that π is unital, if $\mathcal{A}$ is unital and $\pi(1) = 1$ holds. Furthermore, we call π a faithful representation, if π is injective.

Definition 19. *[28] A subspace $\mathcal{K} \subseteq \mathcal{H}$ is invariant under $W \in \mathcal{B}(\mathcal{H})$ iff (if and only if) $W\mathcal{K} \subseteq \mathcal{K}$.*

Definition 20. *[29] We say that the subspace $\mathcal{K} \subseteq \mathcal{H}$ is invariant under the action of the representation π of an algebra $\mathcal{A}$ iff $\pi(a)\mathcal{K} \subseteq \mathcal{K}$ for all $a \in \mathcal{A}$.*

Definition 21. *[28] We say that a representation π of $\mathcal{A}$ is irreducible iff the only invariant subspaces are $\{0\}$ and $\mathcal{H}$ itself.*

Definition 22. *[30] Let $\tilde{\mathcal{A}} \subseteq \mathcal{A}$ be a subalgebra. Then we call $\tilde{\mathcal{A}}$ a left ideal iff $\tilde{\mathcal{A}}\mathcal{A} \subseteq \tilde{\mathcal{A}}$ and a right ideal iff $\mathcal{A}\tilde{\mathcal{A}} \subseteq \tilde{\mathcal{A}}$. Furthermore, we call $\tilde{\mathcal{A}}$ a two-sided ideal or just ideal iff $\tilde{\mathcal{A}}$ is a left and right ideal.*

Definition 23. *[31] Let $\mathcal{A}$ be a $*$-algebra.We call $\mathcal{A}$ a simple algebra iff $\mathcal{A}$ has no non-trivial two-sided ideal.*

Lemma 1. *[28] The center of $M_n(\mathbb{C})$ is given by $Z(M_n(\mathbb{C})) = \mathbb{C}\mathbb{1}$, where $M_n(\mathbb{C})$ is the algebra of the complex $n \times n$ matrices.*

Proof: We want to show the following equivalence:

$$n \in Z(M_N(\mathbb{C})) \iff n = c\mathbb{1}, \text{with } c \in \mathbb{C}. \tag{5.3}$$

($\Leftarrow$): trivial.
($\Rightarrow$): Consider $m = e_{ij} \in M_n(\mathbb{C})$, where e_{ij} is the matrix, which has a 1 in the $(i,j)^{\text{th}}$ entry and all other entries are 0. The set $\{e_{ij}\}$ forms a basis of $M_n(\mathbb{C})$ i.e. we can write every element $m \in M_n(\mathbb{C})$ as $m = \sum_{ij} m_{ij}e_{ij}$, where $m_{ij} \in \mathbb{C}$. Furthermore, the relation

$$e_{ij}e_{kl} = \delta_{jk}e_{il} \tag{5.4}$$

holds. Then for $n \in Z(M_n(\mathbb{C}))$

$$e_{ij}n = \sum_{st} n_{st}e_{ij}e_{st} = \sum_{st} n_{st}\delta_{js}e_{it} = \sum_{t} n_{jt}e_{it} = ne_{ij} = \sum_{t} n_{ti}e_{tj}, \tag{5.5}$$

by comparing the coefficients of

$$\sum_{t} n_{jt}e_{it} = \sum_{t} n_{ti}e_{tj}, \tag{5.6}$$

we find that $n_{ti} = 0\ \forall t \neq i$ and $n_{jt} = 0\ \forall t \neq j$. Thus (5.6) reads

$$n_{ii}e_{ij} = n_{jj}e_{ij}, \tag{5.7}$$

which holds $\forall i,j$ and therefore $n = c\mathbb{1}$, with $c \in \mathbb{C}$. □
Next we want to prove the following theorem:

Theorem 4. *Consider a finite-dimensional algebra $\mathcal{A}$, which is the direct sum of two finite-dimensional algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ i.e. $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}$, then also the commutant $\mathcal{A}'$ is given by the direct sum of the commutants $\mathcal{A}_1'$ and $\mathcal{A}_2'$ i.e. $\mathcal{A}' = \mathcal{A}_1' \oplus \mathcal{A}_2'$.*

Proof: ($\supseteq$): Let $A = A_1 \oplus A_2 \in \mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$ and let $X = X_1 \oplus X_2$, with $X_1 \in \mathcal{A}_1'$ and $\mathcal{A}_2'$, then

$$XA = (X_1 \oplus X_2)(A_1 \oplus A_2) = (X_1 A_1 \oplus X_2 A_2) = (A_1 X_1 \oplus A_2 X_2) = (A_1 \oplus A_2)(X_1 \oplus X_2) = AX.$$

($\subseteq$):Let $A = A_1 \oplus A_2 \in \mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$. Thus we can write A in the matrix form

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

where A_1 is a $n \times n$-matrix and A_2 is a $m \times m$-matrix.
Furthermore, let $X \in \mathcal{A}'$. We can write X in matrix form too

$$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix},$$

where X_1 is a $n \times n$-matrix, X_2 is a $n \times m$-matrix, X_3 is a $m \times n$-matrix and X_4 is a $m \times m$-matrix.
Now we can write

$$\begin{aligned} XA &= \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \cdot \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} = \begin{pmatrix} X_1 A_1 & X_2 A_2 \\ X_3 A_1 & X_4 A_2 \end{pmatrix} = \\ AX &= \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \cdot \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} = \begin{pmatrix} A_1 X_1 & A_1 X_2 \\ A_2 X_3 & A_2 X_4 \end{pmatrix}. \end{aligned}$$

Thus,

$$\begin{aligned} X_1 A_1 &= A_1 X_1, \\ X_2 A_2 &= A_1 X_2, \\ X_3 A_1 &= A_2 X_3, \\ X_4 A_2 &= A_2 X_4. \end{aligned}$$

This holds $\forall A_1, A_2$ and therefore $X_2 = X_3 = 0$ and $X \in \mathcal{A}_1' \oplus \mathcal{A}_2'$, which completes the proof. $\square$

Theorem 5. *Consider the direct sum of m-copies of the matrix algebra $M_n(\mathbb{C})$ i.e. $\oplus_{j=1}^m M_n(\mathbb{C}) = \mathbb{1}_m \otimes M_n(\mathbb{C}) \subseteq M_{mn}(\mathbb{C})$, then $\tilde{X} \in (\mathbb{1}_m \otimes M_n(\mathbb{C}))'$ iff $\tilde{X} = X \otimes \mathbb{1}_n$, where $X \in M_m(\mathbb{C})$.*

Proof: ($\Leftarrow$): We observe that

$$\begin{aligned} \tilde{X}(\mathbb{1}_m \otimes M) &= (X \otimes \mathbb{1}_n)(\mathbb{1}_m \otimes M) = X \otimes M \\ &= (\mathbb{1}_m \otimes M)(X \otimes \mathbb{1}_n) = (\mathbb{1}_m \otimes M)\tilde{X}, \end{aligned}$$

holds for every $M \in M_n(\mathbb{C})$.
($\Rightarrow$): Let $M \in M_n(\mathbb{C})$ and $\tilde{X} \in (\mathbb{1}_m \otimes M_n(\mathbb{C}))' \subseteq M_{mn}(\mathbb{C})$. We can write

$$\tilde{X} = \begin{pmatrix} X_{11} & \dots & X_{1m} \\ \vdots & \ddots & \vdots \\ X_{m1} & \dots & X_{mm} \end{pmatrix},$$

where X_{ij} is a $n \times n$-matrix $\forall i, j$. Then we can calculate

$$\begin{aligned}
\tilde{X}(\mathbb{1}_m \otimes M) &= \begin{pmatrix} X_{11} & \dots & X_{1m} \\ \vdots & \ddots & \vdots \\ X_{m1} & \dots & X_{mm} \end{pmatrix} \cdot \begin{pmatrix} M & & \\ & \ddots & \\ & & M \end{pmatrix} = \begin{pmatrix} X_{11}M & \dots & X_{1m}M \\ \vdots & \ddots & \vdots \\ X_{m1}M & \dots & X_{mm}M \end{pmatrix}, \\
(\mathbb{1}_m \otimes M)\tilde{X} &= \begin{pmatrix} M & & \\ & \ddots & \\ & & M \end{pmatrix} \cdot \begin{pmatrix} X_{11} & \dots & X_{1m} \\ \vdots & \ddots & \vdots \\ X_{m1} & \dots & X_{mm} \end{pmatrix} = \begin{pmatrix} MX_{11} & \dots & MX_{1m} \\ \vdots & \ddots & \vdots \\ MX_{m1} & \dots & MX_{mm} \end{pmatrix}.
\end{aligned}$$

From this we find that $MX_{ij} = X_{ij}M \ \forall i, j$ and $\forall M \in M_n(\mathbb{C})$. The matrix X_{ij} itself is an element of $M_n(\mathbb{C})$ and thus, must be in the center of $M_n(\mathbb{C})$. We have already seen that $Z(M_n(\mathbb{C})) = c\mathbb{1}_n$. Therefore $X_{ij} = c_{ij}\mathbb{1}_n$, where $c_{ij} \in \mathbb{C}$. Hence we can write

$$\tilde{X} = \begin{pmatrix} X_{11} & \dots & X_{1m} \\ \vdots & \ddots & \vdots \\ X_{m1} & \dots & X_{mm} \end{pmatrix} = \begin{pmatrix} c_{11}\mathbb{1}_n & \dots & c_{1m}\mathbb{1}_n \\ \vdots & \ddots & \vdots \\ c_{m1}\mathbb{1}_n & \dots & c_{mm}\mathbb{1}_n \end{pmatrix} = \begin{pmatrix} c_{11} & \dots & c_{1m} \\ \vdots & \ddots & \vdots \\ c_{m1} & \dots & c_{mm} \end{pmatrix} \otimes \mathbb{1}_n,$$

where $\begin{pmatrix} c_{11} & \dots & c_{1m} \\ \vdots & \ddots & \vdots \\ c_{m1} & \dots & c_{mm} \end{pmatrix} \in M_m(\mathbb{C})$. $\square$

Lemma 2. *[28]The canonical representation of $M_n(\mathbb{C})$ is irreducible.*

Proof: We denote the canonical representation as well as the abstract algebra by $M_n(\mathbb{C})$ and say explicitly, whether we mean the algebra or the representation, if it is not already clear from the context.
Let $M_n(\mathbb{C})$ be the canonical representation and let $\mathcal{S} \subsetneq \mathcal{H}$ an arbitrary non-empty linear subspace, which is not equal to $\mathcal{H}$ itself. Furthermore, let $|\phi\rangle \in \mathcal{S}$ be a normalized vector and $|\psi\rangle \in \mathcal{H}$, but $|\psi\rangle \notin \mathcal{S}$. We find for the operator $|\psi\rangle\langle\phi| \in M_n(\mathbb{C})$ that

$$(|\psi\rangle\langle\phi|)\,|\phi\rangle = |\psi\rangle, \tag{5.8}$$

which implies that $\mathcal{S}$ is not an invariant subspace. Therefore, the only invariant subspaces of $M_n(\mathbb{C})$ are $\{0\}$ and $\mathcal{H}$ and thus $M_n(\mathbb{C})$ is irreducible. $\square$

Lemma 3. *The canonical representation is the only irreducible representation of $M_n(\mathbb{C})$.*

Proof: For the proof we use the fact that simple algebras possess exactly one irreducible representation [32]. Therefore, we only have to show that $M_n(\mathbb{C})$ is a simple algebra. To show that $M_n(\mathbb{C})$ is simple we follow a proof given in [33].
We assume that there is a two-sided ideal $\tilde{\mathcal{A}}$, which contains at least one non-zero matrix. That is, there is a matrix $\tilde{M} \in \tilde{\mathcal{A}}$, which we can write as

$$\tilde{M} = \sum_{kl} a_{kl} e_{kl}, \tag{5.9}$$

that has at least one non-zero entry a_{ij}. Since e_{ii} and e_{jj} are both in $M_n(\mathbb{C})$ and $\tilde{M}$ is an element of a two-sided ideal, we know that

$$\overbrace{\underbrace{e_{ii}\tilde{M}}_{\in\tilde{\mathcal{A}}\subseteq M_n(\mathbb{C})} \quad e_{jj}}^{\in\tilde{\mathcal{A}}} = e_{ii}(\sum_{kl} a_{kl}e_{kl})e_{jj} = \sum_{kl} a_{kl}\delta_{ik}e_{il}e_{jj} = \sum_{l} a_{il}\delta_{lj}e_{ij} = a_{ij}e_{ij}, \tag{5.10}$$

is also element of $\tilde{\mathcal{A}}$. Similarly, we can use $\sum_s k_{ns}e_{ns} \in M_n(\mathbb{C})$, $\sum_l c_{lm}e_{lm} \in M_n(\mathbb{C})$ and $a_{ij}e_{ij} \in \tilde{\mathcal{A}}$ to find that

$$\begin{aligned}\left(\sum_s k_{ns}e_{ns}\right) a_{ij}e_{ij} \left(\sum_l c_{lm}e_{lm}\right) &= \sum_{sl} k_{ns}a_{ij}c_{lm}\delta_{jl}e_{ns}e_{im} = \sum_s k_{ns}a_{ij}c_{jm}\delta_{si}e_{nm} \\ &= k_{ni}a_{ij}c_{jm}e_{nm},\end{aligned} \tag{5.11}$$

is an element of $\tilde{\mathcal{A}}$ for all $n, m \in \{1, \ldots, d\}$ and $k_{ni}, c_{jm} \in \mathbb{C}$. This implies that $b_{nm}e_{nm}$ is an element of $\tilde{\mathcal{A}}$ for every $n, m \in \{1, \ldots, d\}$ and $b_{nm} \in \mathbb{C}$. Since $\tilde{\mathcal{A}}$ is a linear subspace, every $\sum_{nm} b_{nm}e_{nm}$ is an element of $\tilde{\mathcal{A}}$ for all $b_{nm} \in \mathbb{C}$ and thus $\tilde{\mathcal{A}} = \{\sum_{nm} b_{nm}e_{nm} | b_{nm} \in \mathbb{C}\} = M_n(\mathbb{C})$. To sum up, we have seen that a two-sided ideal $\tilde{\mathcal{A}}$ that includes a non-zero matrix must be $M_n(\mathbb{C})$ itself and therefore the only two-sided ideals are 0 and $M_n(\mathbb{C})$ and thus $M_n(\mathbb{C})$ is a simple algebra and possesses only one irreducible representation. $\square$

Next, we will clarify what we mean when we talk about isomorphisms in the context of algebras. For this purpose we first define what an $*$-homomorphism is:

Definition 24. *[34] Let A and B be $*$-algebras. We call a map $\varphi : \mathcal{A} \to \mathcal{B}$ a $*$-homomorphism if φ is linear and $\varphi(xy) = \varphi(x)\varphi(y)\, \forall x, y \in \mathcal{A}$ and $\varphi(x^*)=\varphi(x)^*$ holds.*

Definition 25. *[34] Let A and B be algebras. We call a map $\varphi : \mathcal{A} \to \mathcal{B}$ a $*$-isomorphism if the following two conditions hold:*

1. *φ is a $*$-homomorphism,*
2. *φ is bijective.*

We will often call a $*$-isomorphism an isomorphism if it is clear from the context what we mean. Furthermore, if a $*$-isomorphism from $\mathcal{A}$ to $\mathcal{B}$ exists, we say that $\mathcal{A}$ is isomorphic to $\mathcal{B}$, which we denote by $\mathcal{A} \cong \mathcal{B}$.

To close this section we will elaborate some properties for isomorphisms. First we notice, that for two unital algebras $\mathcal{A}$ and $\mathcal{B}$ the unit element $1_{\mathcal{A}}$ of $\mathcal{A}$ is mapped to the unit element $1_{\mathcal{B}}$ of $\mathcal{B}$ by any isomorphism $\varphi : \mathcal{A} \to \mathcal{B}$. We observe

$$\varphi(a) = \varphi(1_{\mathcal{A}}a) = \varphi(1_{\mathcal{A}})\varphi(a), \tag{5.12}$$

$\forall a \in \mathcal{A}$, which implies

$$\varphi(1_{\mathcal{A}}) = 1_{\mathcal{B}}. \tag{5.13}$$

Next, we want to show that

$$\varphi(a^{-1}) = \varphi(a)^{-1}, \tag{5.14}$$

for all $a \in \mathcal{A}$ if a^{-1} exists. We calculate

$$1_{\mathcal{B}} = \varphi(1_{\mathcal{A}}) = \varphi(aa^{-1}) = \varphi(a)\varphi(a^{-1}), \tag{5.15}$$

which implies equation (5.14).
Next, we want to show that any element of the spectrum of an operator $a \in \mathcal{A} \subset \mathcal{B}(\mathcal{H}_A)$ is also an element of the spectrum of $\varphi(a) \in \mathcal{B} \subset \mathcal{B}(\mathcal{H}_B)$, where $\varphi : \mathcal{A} \to \mathcal{B}$ is an isomorphism. For that reason we make the following definition:

Definition 26. *[28] Let $a \in \mathcal{B}(\mathcal{H})$. We say that the set*

$$\textit{Spec}(a) = \{\lambda | (a - \lambda \mathbb{1}) \textit{ is not invertible and} \, \lambda \in \mathbb{C}\},$$

is the spectrum of a.

Now we can formulate the previous claim in a more rigorous way:

Lemma 4. *Any $\lambda \in \mathbb{C}$ is element of Spec($\mathcal{A}$) iff λ is also element of Spec($\varphi(a)$), where φ is an algebra-isomorphism, $a \in \mathcal{A} \subset \mathcal{B}(\mathcal{H}_A)$ and $\varphi(a) \in \mathcal{B} \subset \mathcal{B}(\mathcal{H}_B)$.*

Proof: Let $\lambda \in \mathrm{Spec}(a)$. This implies that $a - \lambda \mathbb{1}_{\mathcal{A}}$ is not invertible. From equation (5.14) we find that $\varphi(a - \lambda \mathbb{1}_{\mathcal{A}})$ is also not invertible. Furthermore, $\varphi(a - \lambda \mathbb{1}_{\mathcal{A}})$ can be rewritten as

$$\varphi(a - \lambda \mathbb{1}_{\mathcal{A}}) = \varphi(a) - \lambda \varphi(\mathbb{1}_{\mathcal{A}}) = \varphi(a) - \lambda \mathbb{1}_{\mathcal{B}}, \tag{5.16}$$

and therefore $\varphi(a) - \lambda \mathbb{1}_{\mathcal{B}}$ is not invertible. Thus, λ is also an element of $\mathrm{Spec}(\varphi(a))$.
The other way around, if we assume that $\lambda \in \mathrm{Spec}(\varphi(a))$, we can use equation (5.16) to see that $\varphi(a - \lambda \mathbb{1}_{\mathcal{A}})$ is not invertible, and therefore, again using equation (5.14), we find that $a - \lambda \mathbb{1}_{\mathcal{A}}$ is not invertible and therefore $\lambda \in \mathrm{Spec}(a)$. $\square$
We should be aware of the fact that the multiplicity of the element $\lambda \in \mathrm{Spec}(a)$ is not necessarily conserved under isomorphisms. This can be seen by the following example. Let $\mathcal{A} \subset \mathcal{B}(\mathbb{C}^2)$ be an algebra and let us consider the algebra-isomorphism given by $\varphi : \mathcal{A} \to \mathcal{B} \subset \mathcal{B}(\mathbb{C}^4)$

$$\varphi(a) = \mathbb{1}_2 \otimes a = a \oplus a, \tag{5.17}$$

where we have defined $\mathcal{B}$ as the image of φ. Let us briefly check that φ defines an isomorphism. First we notice that φ is bijective, because $\mathcal{B} = \mathrm{Im}(\varphi(a))$ and the inverse function is given by

$$\varphi^{-1}(\mathbb{1}_2 \otimes a) = a. \tag{5.18}$$

Furthermore, linearity can be easily shown

$$\varphi(\mu_1 a_1 + \mu_2 a_2) = \mathbb{1}_2 \otimes (\mu_1 a_1 + \mu_2 a_2) = \mu_1(\mathbb{1} \otimes a_1) + \mu_2(\mathbb{1}_2 \otimes a_2) = \mu_1 \varphi(a_1) + \mu_2 \varphi(a_2), \tag{5.19}$$

$\forall a_1, a_2 \in \mathcal{A}$ and $\forall \mu_1, \mu_2 \in \mathbb{C}$.
Obviously,

$$\varphi(a_1 a_2) = (\mathbb{1}_2 \otimes (a_1 a_2)) = (\mathbb{1}_2 \otimes a_1)(\mathbb{1}_2 \otimes a_2) = \varphi(a_1)\varphi(a_2), \tag{5.20}$$

$\forall a_1, a_2 \in \mathcal{A}$. Furthermore, it should be clear that $\mathcal{B}$ defines an algebra as well. All together we have shown that φ defines an isomorphism.
Now assume that $\tilde{a}$ given by

$$\tilde{a} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \tag{5.21}$$

is element of $\mathcal{A}$. Obviously, $\tilde{a}$ has the eigenvalues 1 and 2 and both eigenvalues have multiplicity 1, while

$$\varphi(\tilde{a}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \tag{5.22}$$

also possesses the eigenvalues 1 and 2, but this time both eigenvalues have multiplicity 2.
Furthermore, the fact that the multiplicity of the eigenvalues of an element of a matrix algebra is not necessarily conserved under an isomorphism, implies that also the trace and the determinant are not necessarily conserved. Notice that for our example $\mathrm{Tr}(\tilde{a}) = 3$ but $\mathrm{Tr}(\varphi(\tilde{a})) = 6$ and $\det(\tilde{a}) = 2$ and $\det(\varphi(\tilde{a})) = 4$.
The last thing we want to show before we concern ourselves with Tsirelson's problem is the following theorem:

Theorem 6. *Every $*$-automorphism $\phi : \mathcal{B}(\mathbb{C}^d) \to \mathcal{B}(\mathbb{C}^d)$ is of the form $\phi(A) = UAU^\dagger$, if $d > 3$.*

Proof: For the proof we use that any normal preserving linear map $\tilde{\phi}$ on M_n, where $n \geq 3$, has one of two forms:

$$\begin{aligned} \tilde{\phi}(A) &= cUAU^\dagger + f(A)\mathbb{1} \qquad \forall A \in M_n, & (5.23)\\ \tilde{\phi}(A) &= cUA^TU^\dagger + f(A)\mathbb{1} \qquad \forall A \in M_n, & (5.24) \end{aligned}$$

where $c \in \mathbb{C}$, U is a unitary, $f(A)$ is a linear functional, M_n denotes the algebra of $n \times n$ complex matrices and A^T denotes the transpose of A with respect to some a priori fixed basis of $\mathbb{C}^n$[35, 36]. Since ϕ is an automorphism it is linear, thus we only have to check that ϕ is normal preserving to see that ϕ is either of the form (5.23) or (5.24). We assume that $[N, N^\dagger] = 0$, where $N \in \mathcal{B}(\mathbb{C}^d)$ and we calculate

$$\phi(NN^\dagger) = \phi(N)\phi(N^\dagger) = \phi(N)\phi(N)^\dagger, \tag{5.25}$$

and

$$\phi(NN^\dagger) = \phi(N^\dagger N) = \phi(N^\dagger)\phi(N) = \phi(N)^\dagger\phi(N). \tag{5.26}$$

Comparing the last two equations we find $[N, N^\dagger] = 0 \Rightarrow [\phi(N), \phi(N)^\dagger] = 0$. Moreover, by noticing that ϕ^{-1} is an automorphism as well, by a similar calculation as before it can be easily shown that $[\phi(N), \phi(N)^\dagger] = 0 \Rightarrow [N, N^\dagger] = 0$ and therefore $[N, N^\dagger] = 0 \Leftrightarrow [\phi(N), \phi(N)^\dagger] = 0$.
First we will assume that ϕ is of the form (5.23). By setting $c = 0$ we get that

$$\phi(A) = f(A)\mathbb{1}, \tag{5.27}$$

but this is a mapping from a d^2-dimensional space to a 1-dimensional space, which is in conflict with ϕ being an automorphism and therefore $c \neq 0$.
Next we consider the action of ϕ on arbitrary orthogonal projectors P. On the one hand we notice that

$$\phi(P) = cUPU^\dagger + f(P)\mathbb{1} \tag{5.28}$$

and on the other hand we calculate

$$\phi(P) = \phi(PP) = \phi(P)\phi(P) = c^2UPU^\dagger + 2cf(P)UPU^\dagger + f(P)^2\mathbb{1}. \tag{5.29}$$

By comparing (5.28) with (5.29) we find

$$f(P) = f(P)^2 \tag{5.30}$$

and

$$c^2 + 2cf(P) = c. \tag{5.31}$$

Equation (5.30) implies that $f(P) \in \{0, \pm 1\}$ for all projectors P. Moreover we can rewrite (5.31) as

$$f(P) = \frac{1-c}{2}, \tag{5.32}$$

where we have used that $c \neq 0$. Since equation (5.32) holds for all projectors P, we observe that $f(P)$ is constant for all projectors P. Now, we consider a rank two projector $\tilde{P}$, which we can rewrite as a sum

$$\tilde{P} = P_1 + P_2, \tag{5.33}$$

such that

$$P_1 P_2 = P_2 P_1 = 0, \tag{5.34}$$

where P_1 and P_2 are two rank one projectors [10]. Next, we observe

$$\frac{1-c}{2} = f(\tilde{P}) = f(P_1 + P_2) = f(P_1) + f(P_2) = \frac{1-c}{2} + \frac{1-c}{2} \tag{5.35}$$

and therefore we find that $c = 1$ and

$$f(P) = 0 \tag{5.36}$$

for all projectors P. Furthermore, we recall that we can write every hermitian operator A as a weighted sum of projectors

$$A = \sum_i \lambda_i \left|\varphi_i\right\rangle \left\langle\varphi_i\right|, \tag{5.37}$$

where λ_i are the eigenvalues and $|\varphi_i\rangle$ are the corresponding eigenvectors of A[10]. Therefore, we find

$$f(A) = f(\sum_i \lambda_i \left|\varphi_i\right\rangle \left\langle\varphi_i\right|) = \sum_i \lambda_i \overbrace{f(\left|\varphi_i\right\rangle \left\langle\varphi_i\right|)}^{=0} = 0, \tag{5.38}$$

which holds for all hermitian operators A. Finally, we recall that any operator $C \in \mathcal{B}(\mathbb{C}^d)$ can be written as a sum

$$C = A + iB, \tag{5.39}$$

where A and B are both hermitian[37]. Eventually, we find

$$f(C) = f(A + iB) = f(A) + if(B) = 0, \tag{5.40}$$

which holds $\forall C \in \mathcal{B}(\mathbb{C}^d)$ and thus ϕ must be of the form

$$\phi(A) = UAU^\dagger, \tag{5.41}$$

where it is trivial that every ϕ of this form also respects the requirements of being an algebra automorphism.
Next, we want to investigate if ϕ is always of the form (5.41), or if there are also ϕ's of the form

(5.24) . To this end, we assume that ϕ is of the form (5.24). Furthermore, we observe that P^T is again an orthogonal projector if P is an orthogonal

$$P^T = (P)^T = (PP)^T = P^T P^T \tag{5.42}$$

and

$$(P^T)^\dagger = ((P^T)^T)^* = P^* = P^T. \tag{5.43}$$

This being said, one can follow a similar discussion as before to end up with the result that ϕ must be of the form

$$\phi(A) = UA^T U^\dagger, \tag{5.44}$$

but this time $\phi(A)$ does not describe a valid autmorphism, because it does not preserve multiplication. This can be easily seen

$$\phi(AB) = U(AB)^T U^\dagger = UB^T A^T U^\dagger, \tag{5.45}$$

but

$$\phi(A)\phi(B) = UA^T B^T U^\dagger, \tag{5.46}$$

and therefore $\phi(AB) \neq \phi(A)\phi(B)$ in most cases. $\square$

5.2 Tsirelson's Problem in Finite Dimensions

This section is based on section 2 from [5] and only extra information that was added will be cited explicitly.

We consider two parties Alice and Bob. Both are acting on parts of a shared system. Alice possesses a finite collection of devices, which she can use to conduct different measurements on her part of the system. We denote the set of Alice's measurement devices by I and a specific device by i. We will often refer to $i \in I$ as the measurement setting chosen by Alice. For each setting there are a finite number of fixed measurement outcomes and we will denote the set of outcomes for the setting $i \in I$ by A_i and a specific outcome by α. Likewise, Bob holds a finite set of measurement devices, which he uses to act on his part of the system. We will denote his set of devices, his measurement settings, his set of outcomes corresponding to a specific setting and a specific outcome by J, j, B_j and β respectively.

We will just consider situations where Alice's and Bob's measurements are independent of each other i.e. we want them to commute independent of the settings they choose and the results they obtain. Furthermore, we will assume that quantum mechanics hold. The most general way to formulate this in a mathematical framework is as follows: We will associate each of the measurement devices $i \in I$ with a collection of positive operators acting on a finite-dimensional Hilbert space. Every single operator corresponds to a specific outcome. We can write $\{X\}_{i,\alpha} \subset \mathcal{B}(\mathcal{H})$, with $i \in I$, $\alpha \in A_i$ and $X_{i\alpha} \geq 0 \; \forall X_{i\alpha} \in \{X\}_{i,\alpha}$, for the set of Alice's devices. Furthermore, we demand that if we take the sum of the operators corresponding to a specific setting over the outcomes we get the identity, that is, $\sum_\alpha X_{i,\alpha} = \mathbb{1}$. Similarly, we can describe Bob's measurement devices by the set $\{Y\}_{j,\beta}$ with $j \in J$ and $\beta \in B_j$. Again we demand that $\{Y\}_{j,\beta}$ has the same properties as $\{X\}_{i,\alpha}$.

For the next discussion we will first consider just one party (Alice) and then generalize our findings to two parties. We want to describe how to find the possible probability distributions of a specific outcome α given the setting i i.e. we are looking for a way to describe $P(\alpha|i)$. The system Alice is acting on, is in a specific state and this state determines the probability distributions Alice can find for any possible measurement she can conduct. Mathematically a state is a positive, linear functional $\omega : \mathcal{B}(\mathcal{H}) \to \mathbb{R}$, which maps the identity to one i.e. $\omega(\mathbb{1}) = 1$. Positive in this context means that $\omega(\sigma) \geq 0$, if $\sigma \in \mathcal{B}(\mathcal{H})$ is positive semi-definite. In quantum mechanics every state can be associated with a density matrix via the Hilbert-Schmidt-inner product. A density matrix $\rho \in \mathcal{S}(\mathcal{H})$ is a positive semi-definite operator with trace 1. The state ω is then given by $\omega(Z) = \mathrm{tr}(\rho Z)$. We say that a system is in a state ω, if the probability of getting α given the setting i is given by

$$P(\alpha|i) = \omega(X_{i,\alpha}) \quad \forall i \in I, \alpha \in A_i, \tag{5.47}$$

with $X_{i,\alpha} \in \{X\}_{i,\alpha}$.

Now we will generalize the above to two parties. We still want to describe the probability distribution $P(\alpha, \beta|i, j)$ by means of states ω.

According to the postulates of quantum mechanics a system consisting of two subsystems has an associated Hilbert space $\mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B$, where $\mathcal{H}^A$ and $\mathcal{H}^B$ are the Hilbert spaces of the subsystems. Furthermore Alice's measurement operators only act non-trivially on $\mathcal{H}^A$ i.e. $X_{i,\alpha} = \tilde{X}_{i,\alpha} \otimes \mathbb{1}_B$, with $\tilde{X}_{i,\alpha} \in \mathcal{B}(\mathcal{H}^A)$. Similarly, Bob's operators only act non-trivially on $\mathcal{H}^B$ and we write $Y_{j,\beta} = \mathbb{1}_A \otimes \tilde{Y}_{j,\beta}$, with $\tilde{Y}_{j,\beta} \in \mathcal{B}(\mathcal{H}^B)$. Obviously, every $X_{i,\alpha}$ commutes with every $Y_{j,\beta}$ $\forall i, j, \alpha, \beta$. The state ω is then acting on $\mathcal{B}(\mathcal{H}^A \otimes \mathcal{H}^B)$ and returning a real number. The joint probability that Alice obtains outcome α and Bob obtains β given the settings i and j is then described by

$$P(\alpha, \beta|i, j) = \omega(\tilde{X}_{i_\alpha} \otimes \tilde{Y}_{j,\beta}). \tag{5.48}$$

This is not the only imaginable approach to describe the separation of the two measurement parties Alice and Bob. Instead of assuming that we can decompose $\mathcal{H}$ we can also just demand that $[X_{i,\alpha}, Y_{j,\beta}] = 0 \, \forall i, j, \alpha, \beta$. Then the wanted probability distributions are given by

$$P(\alpha, \beta|i, j) = \omega(X_{i,\alpha} \cdot Y_{j,\beta}). \tag{5.49}$$

Naturally the question arises, whether the two approaches are equivalent. It should be clear that the latter approach includes all scenarios of the first approach. The other way around, it is far from trivial that all scenarios covered by the latter approach are included in the first one. It was recently shown in [27] that, if the underlying Hilbert space $\mathcal{H}$ is a general infinite-dimensional Hilbert space the two approaches do not coincide.

However, if we restrict ourselves to finite-dimensional Hilbert spaces it was shown by Tsirelson [26] that the two approaches coincide. In the rest of this section we will explain why the two approaches coincide, if the underlying Hilbert space is finite-dimensional. To this end, we start by proving the following theorem:

Theorem 7. *[5] Let $\{X\}_{i,\alpha} \subset \mathcal{B}(\mathcal{H})$ and $\{Y\}_{j,\beta} \subset \mathcal{B}(\mathcal{H})$ be finite and let them describe sets of measurement operators as described above. Moreover, let the underlying Hilbert space $\mathcal{H}$ be finite-dimensional. Furthermore, we assume that $\{X\}_{i,\alpha}$ and $\{Y\}_{j,\beta}$ commute, that is, $[X_{i,\alpha}, Y_{j,\beta}] = 0$ $\forall i, j, \alpha, \beta$. From these assumptions it follows that a finite-dimensional decomposable Hilbert space $\bar{\mathcal{H}} = \mathcal{H}_A \otimes \mathcal{H}_B$ exists, such that the von Neumann algebra generated by $\{X\}_{i,\alpha}$ can be mapped isomorphically into $\mathcal{B}(\mathcal{H}_A)$ and correspondingly the von Neumann algebra generated by $\{Y\}_{j,\beta}$ can be mapped into $\mathcal{B}(\mathcal{H}_B)$.*

Proof: We denote the Von Neumann algebra generated by $\{X\}_{i,\alpha}$ as $\mathcal{A}_X$ and the Von Neumann algebra generated by $\{Y\}_{j,\beta}$ as $\mathcal{A}_Y$. Since every element of the algebra $\mathcal{A}_Y$ commutes with every element of $\mathcal{A}_X$, the algebra $\mathcal{A}_Y$ lies in the commutant $\mathcal{A'}_X$.
The algebra $\mathcal{A}_X \subset \mathcal{B}(\mathcal{H})$ is isomorphic to the decomposition

$$\mathcal{A}_X \cong M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_r}(\mathbb{C}), \tag{5.50}$$

where the right hand side is a direct sum of abstract $n_k \times n_k$ matrix-subalgebras [28]. Furthermore, the Hilbert space decomposes into a direct sum $\mathcal{H} = \bigoplus_{k=1}^{r} \mathcal{H}_k$ as well. To decompose every $\mathcal{H}_k$ even further, we consider the representation $\pi(M_{n_k}(\mathbb{C}))$ of $M_{n_k}(\mathbb{C})$ on the associated Hilbert space $\mathcal{H}_k$. The representation $\pi(M_{n_k})$ on $\mathcal{H}_k$ is not necessarily irreducible, but every representation can be written as a direct sum of irreducible representations. We have already seen that the only irreducible representation of $M_{n_k}(\mathbb{C})$ is given by the canonical representation also denoted by $M_{n_k}(\mathbb{C})$. Thus, we find that the representation of k-th subalgebra on $\mathcal{H}_k$ is given by m_k copies of the irreducible representation $M_{n_k}(\mathbb{C})$ i.e. it is given by

$$\pi(M_{n_k}(\mathbb{C})) = \bigoplus_{j=1}^{m_k} M_{n_k}(\mathbb{C}) = \mathbb{1}_{m_k} \otimes M_{n_k}(\mathbb{C}). \tag{5.51}$$

Therefore, we have a further decomposition of each Hilbert space $\mathcal{H}_k = \mathcal{H}_k^2 \otimes \mathcal{H}_k^1$, where $\pi(M_{n_k})$ only acts non-trivial on $\mathcal{H}_{n_k}^1$.
Using the fact that $\mathcal{A}_Y$ is in the commutant of $\mathcal{A}_X$, as well as theorem 4 and theorem 5, we find

$$\begin{aligned}
\mathcal{A}_Y \quad &\subseteq \quad \mathcal{A}'_X = \left(\bigoplus_{k=1}^{r} \mathbb{1}_{m_k} \otimes M_{n_k}(\mathbb{C})\right)' = \bigoplus_{k=1}^{r} (\mathbb{1}_{m_k} \otimes M_{n_k}(\mathbb{C}))' \\
&= \quad \bigoplus_{k=1}^{r} M_{m_k}(\mathbb{C}) \otimes \mathbb{1}_{m_k}.
\end{aligned} \tag{5.52}$$

Since every $X \in \mathcal{A}_X = \bigoplus_{k=1}^{r} \mathbb{1}_{m_k} \otimes M_{n_k}(\mathbb{C})$ can be written as

$$X = \bigoplus_{k=1}^{r} \mathbb{1}_{m_k} \otimes \tilde{X}^k, \tag{5.53}$$

we can define the $*$-isomorphism $\varphi_X : \mathcal{A}_X \to \mathcal{B}(\bigoplus \mathcal{H}_k^1)$ in the following way

$$\varphi_X \left(\bigoplus_{k=1}^{r} \mathbb{1}_{m_k} \otimes \tilde{X}^{(k)}\right) = \bigoplus_{k=1}^{r} \tilde{X}^{(k)}. \tag{5.54}$$

Let us briefly check that this defines a $*$-isomorphism. First we show linearity

$$\begin{aligned}
&\varphi_X \left(a \left(\bigoplus_{k=1}^{r} \mathbb{1}_{m_k} \otimes \tilde{X}^{(k)}\right) + b \left(\bigoplus_{l=1}^{r} \mathbb{1}_{m_l} \otimes \tilde{Z}^{(l)}\right)\right) = \varphi_X \left(\left(\bigoplus_{k=1}^{r} \mathbb{1}_{m_k} \otimes a\tilde{X}^{(k)}\right)\right. \\
&+ \left.\left(\bigoplus_{l=1}^{r} \mathbb{1}_{m_l} \otimes b\tilde{Z}^{(l)}\right)\right) = \varphi_X \left(\bigoplus_{k=1}^{r} \mathbb{1}_{m_k} \otimes \left(a\tilde{X}^{(k)} + b\tilde{Z}^{(k)}\right)\right) = \bigoplus_{k=1}^{r} \left(a\tilde{X}^{(k)} + b\tilde{Z}^{(k)}\right) \\
&= a \left(\bigoplus_{k=1}^{r} \tilde{X}^{(k)}\right) + b \left(\bigoplus_{l=1}^{r} \tilde{Z}^{(l)}\right) = a\varphi_X \left(\bigoplus_{k=1}^{r} \mathbb{1}_{m_k} \otimes \tilde{X}^{(k)}\right) + b\varphi_X \left(\bigoplus_{l=1}^{r} \mathbb{1}_{m_l} \otimes \tilde{Z}^{(l)}\right),
\end{aligned} \tag{5.55}$$

where $a, b \in \mathbb{C}$ and $\tilde{X}^k, \tilde{Z}^k \in M_{n_k}(\mathbb{C})$. Next we show that φ_X is product preserving

$$\begin{aligned}
\varphi_X\left(\left(\bigoplus_{k=1}^{r} \mathbb{1}_{m_k} \otimes \tilde{X}^{(k)}\right)\left(\bigoplus_{l=1}^{r} \mathbb{1}_{m_l} \otimes \tilde{Z}^{(l)}\right)\right) &= \varphi_X\left(\bigoplus_{k=1}^{r} \mathbb{1}_{m_k} \otimes \left(\tilde{X}^{(k)}\tilde{Z}^{(k)}\right)\right) \\
&= \bigoplus_{k=1}^{r} \tilde{X}^{(k)}\tilde{Z}^{(k)} = \left(\bigoplus_{k=1}^{r} \tilde{X}^{(k)}\right)\left(\bigoplus_{l=1}^{r} \tilde{Z}^{(l)}\right) \\
&= \varphi_X\left(\bigoplus_{k=1}^{r} \mathbb{1}_{m_k} \otimes \tilde{X}^{(k)}\right)\varphi_X\left(\bigoplus_{l=1}^{r} \mathbb{1}_{m_l} \otimes \tilde{Z}^{(l)}\right). \qquad (5.56)
\end{aligned}$$

Finally we show that φ_X is involution preserving

$$\begin{aligned}
\varphi_X\left(\left(\bigoplus_{k=1}^{r} \mathbb{1}_{m_k} \otimes \tilde{X}^{(k)}\right)^{\dagger}\right) &= \varphi_X\left(\bigoplus_{k=1}^{r} \mathbb{1}_{m_k} \otimes \left(\tilde{X}^{(k)}\right)^{\dagger}\right) = \bigoplus_{k=1}^{r}\left(\tilde{X}^{(k)}\right)^{\dagger} \\
&= \left(\bigoplus_{k=1}^{r} \tilde{X}^{(k)}\right)^{\dagger} = \varphi_X\left(\left(\bigoplus_{k=1}^{r} \mathbb{1}_{m_k} \otimes \tilde{X}^{(k)}\right)\right)^{\dagger}. \qquad (5.57)
\end{aligned}$$

The corresponding map from $\mathcal{A}_Y$ into $\mathcal{B}\left(\bigoplus_{k=1}^{r} \mathcal{H}_k^2\right)$ is given by

$$\varphi_Y\left(\bigoplus_{k=1}^{r} Y^k \otimes \mathbb{1}_{n_k}\right) = \bigoplus_{k=1}^{r} Y^k. \qquad (5.58)$$

In a final step we have to embed the Hilbert space $\mathcal{H} = \bigoplus_k \mathcal{H}_k^2 \otimes \mathcal{H}_k^1$ into the Hilbert space $(\bigoplus_l \mathcal{H}_l^2) \otimes (\bigoplus_k \mathcal{H}_k^1)$ and we identify $\mathcal{H}_A = \bigoplus_k \mathcal{H}_k^1$ and $\mathcal{H}_B = \bigoplus_k \mathcal{H}_k^2$. □

We can also extend the above theorem 7 to more than two parties. For example consider the three sets of operators $\{X\}_{i,\alpha}$, $\{Y\}_{j,\beta}$ and $\{Z\}_{k,\gamma} \subset \mathcal{B}(\mathcal{H})$ all commuting pairwise. In this case we would first look at the two algebras $\mathcal{A}_X$ generated by $\{X\}_{i,\alpha}$ and $\mathcal{A}_{YZ}$ generated by $\{Y\}_{j,\beta} \cup \{Z\}_{k,\gamma}$. Now using Theorem 7 we find that there is a Hilbert space $\bar{\mathcal{H}} = \mathcal{H}_A \otimes \mathcal{H}_{BC}$, such that $\{X\}_{i,\alpha}$ can be isomorphically mapped into $\mathcal{B}(\mathcal{H}_A)$ and respectively $\{Y\}_{j,\beta} \cup \{Z\}_{k,\gamma}$ into $\mathcal{B}(\mathcal{H}_{BC})$.
Next, we can concentrate on $\{Y\}_{j,\beta}$, $\{Z\}_{k,\gamma}$ and $\mathcal{H}_{BC}$. Again, using Theorem 7 we find that there is a Hilbert space $\bar{\mathcal{H}}_{BC} = \mathcal{H}_A \otimes \mathcal{H}_B$, such that $\{Y\}_{j,\beta}$ can be isomorphically mapped into $\mathcal{B}(\mathcal{H}_B)$ and respectively $\{Z\}_{k,\gamma}$ into $\mathcal{B}(\mathcal{H}_C)$. In summary, we have found a Hilbert space $\tilde{\mathcal{H}} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, such that we can isomorphically map $\{X\}_{i,\alpha}$ to $\mathcal{B}(\mathcal{H}_A)$, $\{Y\}_{j,\beta}$ to $\mathcal{B}(\mathcal{H}_B)$ and $\{Z\}_{k,\gamma}$ to $\mathcal{B}(\mathcal{H}_C)$.
From the above, it should be clear how all this can be extended to n-sets of operators.
Next, we will show that both approaches to implement local separation of two measurement parties can be used to describe the same probability distributions.

Theorem 8. *Let $\mathcal{H}$, $\bar{\mathcal{H}} = \mathcal{H}_A \otimes \mathcal{H}_B$, $\{X\}_{i,\alpha} \subset \mathcal{B}(\mathcal{H})$, $\{Y\}_{j,\beta} \subset \mathcal{B}(\mathcal{H})$, $\varphi_X : \mathcal{A}_X \to \mathcal{B}(\mathcal{H}_A)$ and $\varphi_Y : \mathcal{A}_Y \to \mathcal{B}(\mathcal{H}_B)$ be the Hilbert spaces, sets of measurement operators and the maps described in theorem 7 and in its proof.*
For every state $\omega : \mathcal{B}(\mathcal{H}) \to \mathbb{R}$ a state $\bar{\omega} : \mathcal{B}(\bar{\mathcal{H}}) \to \mathbb{R}$ exists, that produces the same probabilities for $\varphi_Y(Y_{j,\beta}) \otimes \varphi_X(X_{i,\alpha}) = \bar{Y}_{j,\beta} \otimes \bar{X}_{i,\alpha}$ as ω produces for $X_{i,\alpha} \cdot Y_{j,\beta}$ i.e.

$$P(\alpha, \beta | i, j) = \omega(X_{i,\alpha} \cdot Y_{j,\beta}) = \bar{\omega}(\bar{Y}_{j,\beta} \otimes \bar{X}_{i,\alpha}), \qquad (5.59)$$

$\forall i, j, \alpha, \beta.$

Before we prove this theorem we will prove the following lemma:

Lemma 5. *Let $\rho \in \mathcal{S}(\mathcal{H})$ be a density matrix given by*

$$\rho = \begin{pmatrix} \rho_{11} & \dots & \rho_{1n} \\ \vdots & \ddots & \vdots \\ \rho_{n1} & \dots & \rho_{nn} \end{pmatrix}, \tag{5.60}$$

where $n \in \mathbb{N}\backslash\{0\}$, and ρ_{jk} is a $l_j \times l_k$- matrix, then

$$\tilde{\rho} = \begin{pmatrix} \rho_{11} & 0 & \dots & \dots & 0 \\ 0 & \rho_{22} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & \rho_{nn} \end{pmatrix} \tag{5.61}$$

is a density matrix as well.

Proof: Since $\tilde{\rho}$ has the same diagonal entries as ρ, it follows that

$$\mathsf{Tr}(\tilde{\rho}) = \mathsf{Tr}(\rho) = 1. \tag{5.62}$$

Next, we will show that the positive semi-definitness of $\tilde{\rho}$ follows from the positive semi-definitness of ρ. For this purpose we define the following vector

$$|\psi_j\rangle = \begin{pmatrix} 0_1 \\ \vdots \\ 0_{j-1} \\ \psi_j \\ 0_{j+1} \\ \vdots \\ 0_n \end{pmatrix}, \tag{5.63}$$

where $\psi_j \in \mathbb{C}^{l_j}$ is a vector with l_j entries and $0_k \in \mathbb{C}^k$ is the 0-vector. Now, we calculate

$$\rho\,|\psi_j\rangle = \begin{pmatrix} \rho_{11} & \dots & \rho_{1j} & \dots & \rho_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \rho_{j1} & \dots & \rho_{jj} & \dots & \rho_{jn} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \rho_{n1} & \dots & \rho_{jn} & \dots & \rho_{nn} \end{pmatrix} \begin{pmatrix} 0_1 \\ \vdots \\ 0_{j-1} \\ \psi_j \\ 0_{j+1} \\ \vdots \\ 0_n \end{pmatrix} = \begin{pmatrix} \rho_{1j}\psi_j \\ \vdots \\ \rho_{jj}\psi_j \\ \vdots \\ \rho_{nj}\psi_j \end{pmatrix} \tag{5.64}$$

and since $\rho \geq 0$, we find

$$\langle\psi_j|\,\rho\,|\psi_j\rangle = \begin{pmatrix} 0_1^\dagger & \dots & 0_{j-1}^\dagger & \psi_j^\dagger & 0_{j+1}^\dagger & \dots & 0_n^\dagger \end{pmatrix} \begin{pmatrix} \rho_{1j}\psi_j \\ \vdots \\ \rho_{jj}\psi_j \\ \vdots \\ \rho_{nj}\psi_j \end{pmatrix} = \psi_j^\dagger \rho_{jj} \psi_j \geq 0, \tag{5.65}$$

which holds for every $j \in \{1, \ldots, n\}$ and every $\psi_j \in \mathbb{C}^{lj}$ and therefore ρ_{jj} is positive semi-definite for every $j \in \{1, \ldots, n\}$. Moreover, $\tilde{\rho} = \bigoplus_{j=1}^{n} \rho_{jj}$ is a direct product of positive semi-definite matrices and therefore itself positive semi-definite. This completes our proof that $\tilde{\rho}$ defines a density matrix as well. $\square$

Proof of theorem 8: We will use that every state $\omega : \mathcal{B}(\mathcal{H}) \to \mathbb{R}$ can be associated with a density matrix $\rho \in \mathcal{B}(\mathcal{H})$ via

$$\omega(K) = \mathsf{Tr}(\rho K), \tag{5.66}$$

where $K \in \mathcal{B}(\mathcal{H})$, vice versa every density matrix $\rho \in \mathcal{B}(\mathcal{H})$ defines a state via (5.66) [10].
We calculate

$$\begin{aligned} P(\alpha, \beta | i, j) &= \omega(X_{i,\alpha} Y_{j,\beta}) = \mathsf{Tr}(\rho X_{i,\alpha} Y_{j,\beta}) \\ &= \mathsf{Tr}\left(\rho \left(\bigoplus_{k=1}^{r} \mathbb{1}_{m_k} \otimes X_{i,\alpha}^{(k)}\right)\left(\bigoplus_{l=1}^{r} Y_{j,\beta}^{(l)} \otimes \mathbb{1}_{n_l}\right)\right) \\ &= \mathsf{Tr}\left(\rho \left(\bigoplus_{k=1}^{r} Y_{j,\beta}^{(k)} \otimes X_{i,\alpha}^{(k)}\right)\right) = \mathsf{Tr}\left(\bigoplus_{l=1}^{r} \rho_{ll} \left(\bigoplus_{k=1}^{r} Y_{j,\beta}^{(k)} \otimes X_{i,\alpha}^{(k)}\right)\right) \\ &= \mathsf{Tr}\left(\bigoplus_{k=1}^{r} \rho_{kk} \left(Y_{j,\beta}^{(k)} \otimes X_{i,\alpha}^{(k)}\right)\right), \end{aligned} \tag{5.67}$$

where we have obtained the density matrix $\tilde{\rho} = \bigoplus_{k=1}^{r} \rho_{kk} \in \mathcal{S}(\mathcal{H})$ from ρ as described in lemma 5. Next, we define the density matrix $\sigma \in \mathcal{S}(\bar{\mathcal{H}})$

$$\sigma = \bigoplus_{k,l=1}^{r} \sigma_{kl}, \tag{5.68}$$

where $\sigma_{kl} \in \mathcal{B}(\mathcal{H}_k^2 \otimes \mathcal{H}_l^1)$ such that

$$\sigma_{kl} = \begin{cases} \rho_{kk} & \text{if} \quad k = l \\ 0 & \text{if} \quad k \neq l \end{cases}. \tag{5.69}$$

We find for the associated state

$$\begin{aligned} \bar{\omega}(\bar{Y}_{j,\beta} \otimes \bar{X}_{i,\alpha}) &= \bar{\omega}(\varphi_Y(Y_{j,\beta}) \otimes \varphi_X(X_{i,\alpha})) = \mathsf{Tr}(\sigma(\varphi_Y(Y_{j,\beta}) \otimes \varphi_X(X_{i,\alpha}))) \\ &= \mathsf{Tr}\left(\bigoplus_{m,n=1}^{r} \sigma_{mn} \left(\left(\bigoplus_{l=1}^{r} Y_{j,\beta}^{(l)}\right) \otimes \left(\bigoplus_{k=1}^{r} X_{i,\alpha}^{(k)}\right)\right)\right) \\ &= \mathsf{Tr}\left(\bigoplus_{l,k=1}^{r} \sigma_{lk} \left(Y_{j,\beta}^{(l)} \otimes X_{i,\alpha}^{(k)}\right)\right) = \mathsf{Tr}\left(\bigoplus_{k=1}^{r} \rho_{kk} \left(Y_{j,\beta}^{(k)} \otimes X_{i,\alpha}^{(k)}\right)\right). \end{aligned} \tag{5.70}$$

By comparing equation (5.67) and equation (5.70) we find

$$P(\alpha, \beta | i, j) = \omega(X_{i,\alpha} Y_{j,\beta}) = \bar{\omega}(\bar{Y}_{j,\beta} \otimes \bar{X}_{i,\alpha}), \tag{5.71}$$

which holds for all i, j, α and β. $\square$

Both approaches to implement local separation can be considered equivalent, since they can be used to describe the same probability distributions and therefore the same correlations.

Chapter 6

Second Kind Process Matrices

In this section we want to introduce the notion of second kind process matrices.
For the definition of process matrices (which we will call first kind process matrices from now on) we assumed that Alice holds a set of instruments all of the form $\{\mathcal{M}_i^A\} : \mathcal{H}^{A_1} \to \mathcal{H}^{A_2}$, while Bob holds a set of instruments all of the form $\{\mathcal{M}_j^B\} : \mathcal{H}^{B_1} \to \mathcal{H}^{B_2}$ and this was how we included the notion of separated local laboratories.
However, for second kind process matrices we will introduce the notion of local laboratories in a different way. Then we will define second kind process matrices in such a way that they yield valid probability distributions for all operations held by Alice and Bob, if Alice's and Bob's laboratories are locally separated in the new sense.

6.1 Local Laboratories

We look at the input space $\mathcal{H}_1 = \mathcal{H}^{A_1} \otimes \mathcal{H}^{B_1}$ and the output space $\mathcal{H}_2 = \mathcal{H}^{A_2} \otimes \mathcal{H}^{B_2}$, where we assume that $\mathcal{H}^{A_1} = \mathcal{H}^{A_2} = \mathcal{H}^{B_1} = \mathcal{H}^{B_2} = \mathbb{C}^d$ and therefore $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{C}^{d^2}$.
We say that Alice's and Bob's laboratories are standardly located, if Alice holds the subalgebra

$$\mathcal{A}_0 = \mathcal{L}(\mathbb{C}^d) \otimes \mathbb{1}_{B_1} = \{A \otimes \mathbb{1}_{B_1} | A \in \mathcal{L}(\mathbb{C}^d)\} \tag{6.1}$$

and Bob's subalgebra is given by

$$\mathcal{B}_0 = \mathbb{1}_{A_1} \otimes \mathcal{L}(\mathbb{C}^d) = \{\mathbb{1}_{A_1} \otimes B | B \in \mathcal{L}(\mathbb{C}^d)\}. \tag{6.2}$$

We will see later on that these subalgebras indeed correspond to the laboratories in the definition of first kind process matrices.
However, for second kind process matrices we do not only consider such subalgebras, but instead we will state more generally that Alice and Bob each hold a d^2-dimensional subalgebra $\mathcal{A} \subset \mathcal{L}(\mathcal{H}_1)$ and $\mathcal{B} \subset \mathcal{L}(\mathcal{H}_1)$, respectively, such that

$$\mathcal{A} = U A_0 U^\dagger \tag{6.3}$$

and

$$\mathcal{B} = U \mathcal{B}_0 U^\dagger, \tag{6.4}$$

where $U \in \mathcal{L}(\mathcal{H}_1)$ is an arbitrary unitary. Obviously, $[\mathcal{A}, \mathcal{B}] = 0$ holds and therefore we can consider Alice and Bob as locally separated. This gives us a connection between unitaries and a more general notion of local laboratories:

Definition 27. *For some given unitary $U \in \mathcal{L}(\mathcal{H}_1)$, $\mathcal{A}_0$ defined in (6.1) and $\mathcal{B}_0$ defined in (6.2), we call the pair $\{\mathcal{A},\mathcal{B}\}$ a pair of local laboratories, where the two subalgebras $\mathcal{A}$ and $\mathcal{B}$ are defined in (6.3) and in (6.4), respectively.*

Definition 28. *Let $\mathcal{A} \in \mathcal{L}(\mathcal{H}_1)$ be a subalgebra of dimension d^2. We call an operation*

$$M_i^A(\rho) = \sum_k A_i^{(k)} \rho A_i^{(k)\dagger}, \tag{6.5}$$

an $\mathcal{A}$-operation if every $A_i^{(k)}$ is element of $\mathcal{A}$. Furthermore, we call an instrument including only $\mathcal{A}$-operations an $\mathcal{A}$-instrument.

6.1.1 CJ-representation for a Pair of Local Operations

We start this section with the calculation of the CJ-representation for a combination of two standard seperatated operations $\mathcal{M}_j^{B_0} \circ \mathcal{M}_i^{A_0}$,where $\mathcal{M}_i^{A_0}$ and $\mathcal{M}_j^{B_0}$ are $\mathcal{A}_0$-operation and $\mathcal{B}_0$-operation, respectively i.e.

$$\mathcal{M}_i^{A_0}(\rho) = \sum_k \bar{M}_i^{(k)} \rho \bar{M}_i^{(k)\dagger}, \tag{6.6}$$

where all $\bar{M}_i^{(k)} \in \mathcal{A}_0$, that is $\bar{M}_i^{(k)} = M_i^{(k)} \otimes \mathbb{1}$ and

$$\mathcal{M}_j^{B_0}(\rho) = \sum_k \bar{N}_j^{(k)} \rho \bar{N}_j^{(k)\dagger}, \tag{6.7}$$

where all $\bar{N}_j^{(k)} \in \mathcal{B}_0$, hence $\bar{N}_j^{(k)} = \mathbb{1} \otimes N_j^{(k)}$.
We calculate

$$\begin{aligned}
M_{ij}^{A_0B_0} &= \left(\mathcal{I} \otimes \mathcal{M}_j^{B_0} \circ \mathcal{M}_i^{A_0} \left(|\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|\right)\right)^T \\
&= \left(\sum_{a,b,c,d,k,s} |ab\rangle_{A_1B_1} \langle cd|_{A_1B_1} \otimes (M_i^{(k)} \otimes N_j^{(s)})\, |ab\rangle_{A_1B_1} \langle cd|_{A_1B_1} (M_i^{(k)\dagger} \otimes N_j^{(s)\dagger}) \right)^T \\
&= \left(\sum_{k,s,} (M_i^{(k)})_{pa} (N_j^{(s)})_{xb} (M_i^{(k)})^*_{\beta c} (N_j^{(s)})^*_{\tau d} \, |abpx\rangle_{A_1B_1A_2B_2} \langle cd\beta\tau|_{A_1B_1A_2B_2} \right)^T \\
&= \left(\sum_{k,s,} (M_i^{(k)})_{pa} (N_j^{(s)})_{xb} (M_i^{(k)})^*_{\beta c} (N_j^{(s)})^*_{\tau d} \, |cd\beta\tau\rangle_{A_1B_1A_2B_2} \langle abpx|_{A_1B_1A_2B_2} \right) \\
&= M_i^{A_1A_2} \otimes M_j^{B_1B_2},
\end{aligned} \tag{6.8}$$

where in the last step we have changed the ordering from $A_1B_1A_2B_2$ to $A_1A_2B_1B_2$ and we introduced the CJ-representations $M_i^{A_1A_2} \in \mathcal{L}(\mathcal{H}^{A_1} \otimes \mathcal{H}^{A_2})$ of the operator $\tilde{\mathcal{M}}_i^{A_0} : \mathcal{H}^{A_1} \to \mathcal{H}^{A_2}$ given by

$$\tilde{\mathcal{M}}_i^{A_0}(\rho) = \sum_k M_i^{(k)} \rho M_i^{(k)\dagger} \tag{6.9}$$

and the CJ-representations $M_j^{B_1B_2} \in \mathcal{L}(\mathcal{H}^{B_1} \otimes \mathcal{H}^{B_2})$ of the operator $\tilde{\mathcal{M}}_j^{B_0} : \mathcal{H}^{B_1} \to \mathcal{H}^{B_2}$ given by

$$\tilde{\mathcal{M}}_j^{B_0}(\rho) = \sum_k N_j^{(k)} \rho N_j^{(k)\dagger}. \tag{6.10}$$

Next, we want to calculate the CJ-representation for a combination of local operations $\mathcal{M}_j^B \circ \mathcal{M}_i^A$, where $\mathcal{M}_i^A$ and $\mathcal{M}_j^B$ are $\mathcal{A}$-operation and $\mathcal{B}$-operation, respectively i.e.

$$\mathcal{M}_i^A(\rho) = \sum_k U \bar{M}_i^{(k)} U^\dagger \rho U \bar{M}_i^{(k)\dagger} U^\dagger, \tag{6.11}$$

where all $\bar{M}_i^{(k)} \in \mathcal{A}_0$, that is $\bar{M}_i^{(k)} = M_i^{(k)} \otimes \mathbb{1}$ and

$$\mathcal{M}_j^B(\rho) = \sum_k U \bar{N}_j^{(k)} U^\dagger \rho U \bar{N}_j^{(k)\dagger} U^\dagger, \tag{6.12}$$

where all $\bar{N}_j^{(k)} \in \mathcal{B}_0$, hence $\bar{N}_j^{(k)} = \mathbb{1} \otimes N_j^{(k)}$ and $U \in \mathcal{L}(\mathcal{H}_1)$ is a unitary.
We find

$$\begin{aligned}
M_{ij}^{AB} &= \left(\mathcal{I} \otimes \mathcal{M}_j^B \circ \mathcal{M}_i^A \left(|\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|\right)\right)^T = \left(\sum_{a,b,c,d,k,s} |ab\rangle_{A_1B_1} \langle cd|_{A_1B_1}\right.\\
&\otimes \left. U(M_i^{(k)} \otimes N_j^{(s)})U^\dagger |ab\rangle_{A_1B_1} \langle cd|_{A_1B_1} U(M_i^{(k)\dagger} \otimes N_j^{(s)\dagger})U^\dagger\right)^T\\
&= \left(\sum_{a,b,c,d,\alpha,\beta,\ldots} U_{\alpha,\beta,\pi\sigma}(M_i^{(k)})_{\pi\chi}(N_j^{(s)})_{\sigma\eta} U^*_{ab,\chi\eta} U_{cd,\lambda\mu} (M_i^{(k)})^*_{\epsilon\lambda} (N_j^{(s)})^*_{\omega\mu} U^*_{\gamma\tau,\epsilon\omega}\right.\\
&\cdot \left. |ab\alpha\beta\rangle_{A_1B_1A_2B_2} \langle cd\gamma\tau|_{A_1B_1A_2B_2}\right)^T = \sum_{a,b,c,d,\alpha,\beta,\ldots} U_{\alpha,\beta,\pi\sigma}(M_i^{(k)})_{\pi\chi}(N_j^{(s)})_{\sigma\eta}\\
&\cdot U^*_{ab,\chi\eta} U_{cd,\lambda\mu} (M_i^{(k)})^*_{\epsilon\lambda} (N_j^{(s)})^*_{\omega\mu} U^*_{\gamma\tau,\epsilon\omega} |cd\gamma\tau\rangle_{A_1B_1A_2B_2} \langle ab\alpha\beta|_{A_1B_1A_2B_2}\\
&= (U^{A_1B_1} \otimes (U^{A_2B_2})^*)(M^{A_1A_2} \otimes M^{B_1B_2})((U^{A_1B_1})^\dagger \otimes ((U^{A_2B_2})^*)^\dagger),
\end{aligned} \tag{6.13}$$

where in the last line we have to be careful with the ordering of the Hilbert spaces, because $M^{A_1A_2} \in \mathcal{L}(\mathcal{H}^{A_1A_2})$ is the CJ-representation of an operator acting on H^{A_1} and having an output space H^{A_2}, $M^{B_1B_2} \in \mathcal{L}(\mathcal{H}^{B_1B_2})$ is the CJ-representation of an operator acting on H^{B_1} and having an output space H^{B_1}, $U^{A_1B_1}$ acts on the input space $\mathcal{H}_1 = \mathcal{H}^{A_1} \otimes \mathcal{H}^{B_1}$ and $(U^{B_1B_2})^*$ is the complex conjugate of $U^{A_1B_1}$, but acts on the output space $\mathcal{H}_2 = \mathcal{H}^{A_2} \otimes \mathcal{H}^{B_2}$. Thus, the ordering of the Hilbert spaces concerning $U^{A_1B_1} \otimes (U^{A_2B_2})^*$ and $(U^{A_1B_1})^\dagger \otimes ((U^{A_2B_2})^*)^\dagger$ is $A_1B_1A_2B_2$, but concerning $(M^{A_1A_2} \otimes M^{B_1B_2})$ it is $A_1A_2B_1B_2$.

6.2 Definition of Second Kind Process Matrices

Now, we are ready to define second kind process matrices.

Definition 29. *A linear functional $\mathcal{W}$ is called a second kind process, if it assigns a real number to every operator $\mathcal{M}$ such that for every pair of local laboratories $(\mathcal{A}, \mathcal{B})$ the following holds:*

1. $\mathcal{W}(\mathcal{M}_i^A \circ \mathcal{M}_j^B) \geq 0$, *for every $\mathcal{A}$-operator $\mathcal{M}_i^A$ and $\mathcal{B}$-operator $\mathcal{M}_j^B$,*
2. $\mathcal{W}(\mathcal{M}_i^A \circ \mathcal{M}_j^B) = 1$, *for every completely positive trace preserving $\mathcal{A}$-operator $\mathcal{M}_i^A$ and every completely positive trace preserving $\mathcal{B}$-operator $\mathcal{M}_j^B$.*

Furthermore, when a second kind process $\mathcal{W}(\mathcal{M}_i^A \circ \mathcal{M}_j^B)$ is rewritten as

$$\mathcal{W}(\mathcal{M}_i^A \circ \mathcal{M}_j^B) = \mathit{Tr}(W M_{ij}^{AB}), \tag{6.14}$$

where M_{ij}^{AB} is the CJ-representation of $\mathcal{M}_i^A \circ \mathcal{M}_j^B$, then W is called a second kind process matrix.

We denote the set of all first kind process matrices by $P_{\otimes}$ and the set of all second kind process matrices P_c.
It is not obvious, whether every second kind process matrix is already positive semi-definite. In the following chapters we are only interested in the set of all positive semi-definite second kind process matrices, which we denote with P_{c+}. Clearly, $P_{c+} \subseteq P_c$ and $P_{c+} \subseteq P_{\otimes}$.

6.3 Continuous SWAP

Before turning our attention to the case where we allow all kinds of pairs of local laboratories $\{\mathcal{A}, \mathcal{B}\}$ and explore which consequences this will have on the set of all first kind process matrices $P_{\otimes}$, we will only consider the local laboratories, which are generated by a continuous swap of the two standardly localized laboratories $\{\mathcal{A}_0, \mathcal{B}_0\}$ and address the question if we can find some restrictions on $P_{\otimes}$, if we assume that $W \in P_{\otimes}$ should still yield valid probability distributions for all laboratories just described.

6.3.1 From the SWAP Operator to the Continuous SWAP

The operator

$$SWAP = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.15}$$

is the so called SWAP-operator acting on two qubits[38]. We immediately observe that $SWAP$ is symmetric and it is easy to check that it is unitary as well. Since $SWAP$ is unitary we can find a one-parameter family of unitaries

$$U_{SWAP}(t) = e^{itH}, \tag{6.16}$$

where H is hermitian and $t \in \mathbb{R}$ [39], such that

$$U_{SWAP}(0) = \mathbb{1} \qquad \text{and} \qquad U_{SWAP}(1) = SWAP.$$

Our first goal in this section is to find the one-parameter family just described.
To this end we first want to find H. We start by diagonalising $SWAP$. The eigenvalues of $SWAP$ are given by $\lambda_{1,3,4} = 1$ and $\lambda_2 = -1$ and the according normed eigenvectors are given by

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{6.17}$$

and

$$v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}. \tag{6.18}$$

We can use this eigenbasis to diagonalise $SWAP$:

$$SWAP_d = T \cdot SWAP \cdot T^\dagger = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{6.19}$$

with

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = T^\dagger. \tag{6.20}$$

Now, we can find the diagonalised version of H

$$SWAP_d = e^{iH_d} = \begin{pmatrix} e^{iE_1} & 0 & 0 & 0 \\ 0 & e^{iE_2} & 0 & 0 \\ 0 & 0 & e^{iE_3} & 0 \\ 0 & 0 & 0 & e^{iE_4} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6.21}$$

This gives us $E_i = 2\pi n_i$, with $n_i \in \mathbb{Z}$, where $i \in \{1, 3, 4\}$ and $E_2 = \pi + 2\pi n_2$, with $n_2 \in \mathbb{Z}$. We are only interested in one particular H_d and therefore we make the choice $n_i = 0\ \forall i \in \{1, 2, 3, 4\}$. This gives us

$$H_d = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \pi & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{6.22}$$

Furthermore, $H = T^\dagger H_d T$ and thus,

$$\begin{aligned} U_{SWAP}(t) &= e^{itH} = e^{itT^\dagger H_d T} = T^\dagger e^{itH_d} T = T^\dagger \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{it\pi} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} T \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1+e^{it\pi}}{2} & \frac{1-e^{it\pi}}{2} & 0 \\ 0 & \frac{1-e^{it\pi}}{2} & \frac{1+e^{it\pi}}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{6.23}$$

For further use of $U_{SWAP}(t)$ we rewrite it as

$$U_{SWAP}(t) = \frac{3 + e^{it\pi}}{4} (\mathbb{1} \otimes \mathbb{1}) + \frac{1 - e^{it\pi}}{4} \sum_{j=1}^{3} \sigma_j \otimes \sigma_j, \tag{6.24}$$

where $\mathbb{1}$ is the 2×2-identity matrix and σ_j are the Pauli matrices.

6.3.2 Continuous SWAP of Alice's and Bob's Laboratories

Now, we are ready to describe a continuous swap between the two standardly localized laboratories of Alice and Bob.
We consider that Alice and Bob each receive a qubit, that is, we assume that the input space $\mathcal{H}_1 = \mathcal{H}^{A_1} \otimes \mathcal{H}^{B_1}$ is four-dimensional and $\mathcal{H}^{A_1}$ and $\mathcal{H}^{B_1}$ are both two-dimensional.
Furthermore, we consider the family of unitaries $U_{SWAP}(t)$, which we found in the last section. This family of unitaries describes exactly the continuous swap of Alice's and Bob's laboratories (see figur (6.1)).

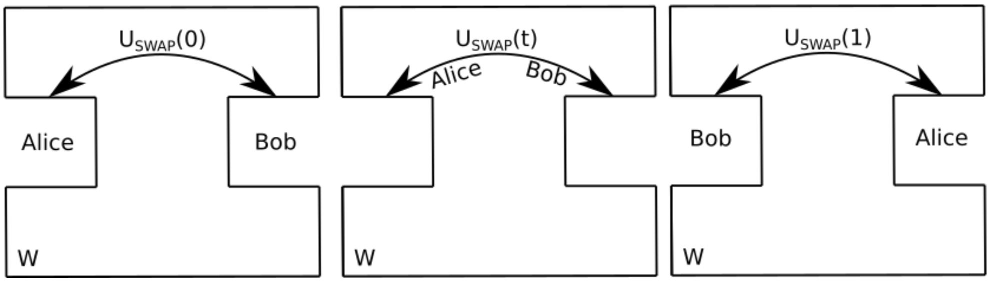

Figure 6.1: Illustration of $U_{SWAP}(t)$. On the left side $t = 0$ and $U(0)$ is just the identity map, therefore Alice and Bob reside in their "normal" laboratories. In the middle, $U(t)$ contentiously swaps Alice's and Bob's laboratories. On the right side $t = 1$ and the swap is completed. Alice's and Bob's laboratories are exchanged i.e. $U(1) = SWAP$.

We will need that

$$U_{SWAP}(t)(\sigma_p \otimes \mathbb{1})U^\dagger_{SWAP}(t) = \frac{1+\cos(t\pi)}{2}\sigma_p \otimes \mathbb{1} + \frac{1-\cos(t\pi)}{2}\mathbb{1} \otimes \sigma_p + \frac{\sin(t\pi)}{2}\sum_{lk}\epsilon_{plk}\sigma_l \otimes \sigma_k \tag{6.25}$$

and

$$U^*_{SWAP}(t)(\sigma_p \otimes \mathbb{1})(U^*)^\dagger_{SWAP}(t) = \frac{1+\cos(t\pi)}{2}\sigma_p \otimes \mathbb{1} + \frac{1-\cos(t\pi)}{2}\mathbb{1} \otimes \sigma_p - \frac{\sin(t\pi)}{2}\sum_{lk}\epsilon_{plk}\sigma_l \otimes \sigma_k. \tag{6.26}$$

The calculations for the last two equations can be found in appendix A. Next, we say that Alice holds a measurement instrument given by the CJ-representation

$$M^{A_1A_2} = \frac{1}{2}(\mathbb{1}^{A_1} \otimes \mathbb{1}^{A_2} + \sigma_p^{A_1} \otimes \sigma_q^{A_2}), \tag{6.27}$$

and Bob holds

$$M^{B_1B_2} = \frac{1}{2}\mathbb{1}^{B_1} \otimes \mathbb{1}^{B_2}, \tag{6.28}$$

if Alice and Bob reside in standardly located laboratories. Thus the CJ-representation $M^{A_0B_0}$ is given by

$$\begin{aligned} M^{A_0B_0} &= M^{A_1A_2} \otimes M^{B_1B_2} = \frac{1}{4}(\mathbb{1}^{A_1} \otimes \mathbb{1}^{B_1} \otimes \mathbb{1}^{A_2} \otimes \mathbb{1}^{B_2} \\ &+ \sigma_p^{A_1} \otimes \mathbb{1}^{B_1} \otimes \sigma_q^{A_2} \otimes \mathbb{1}^{B_2}). \end{aligned} \tag{6.29}$$

Now, we want that Alice's and Bob's laboratories are continuously swapped. The corresponding family of CJ-representations $M^{AB}(t)$ is given by

$$\begin{aligned} M^{AB}(t) &= U_{SWAP}(t) \otimes U^*_{SWAP}(t) M^{A_0B_0} U^\dagger_{SWAP}(t) \otimes (U^*_{SWAP})^\dagger(t) \\ &= \frac{1}{4}[\mathbb{1}^{A_1} \otimes \mathbb{1}^{B_1} \otimes \mathbb{1}^{A_2} \otimes \mathbb{1}^{B_2} \\ &+ U_{SWAP}(t)(\sigma_p^{A_1} \otimes \mathbb{1}^{B_1}) U_{SWAP}(t)^\dagger \otimes U^*_{SWAP}(t)(\sigma_q^{A_2} \otimes \mathbb{1}^{B_2})(U^*_{SWAP})^\dagger(t)] \\ &= \frac{1}{4}\Bigg[\mathbb{1}^{A_1} \otimes \mathbb{1}^{B_1} \otimes \mathbb{1}^{A_2} \otimes \mathbb{1}^{B_2} + \frac{(1+\cos(t\pi))^2}{4}\sigma_p^{A_1} \otimes \mathbb{1}^{B_1} \otimes \sigma_q^{A_2} \otimes \mathbb{1}^{B_2} \\ &+ \frac{(1-\cos(t\pi))^2}{4}\mathbb{1}^{A_1} \otimes \sigma_p^{B_1} \otimes \mathbb{1}^{A_2} \otimes \sigma_q^{B_2} + \frac{1-\cos^2(t\pi)}{4}\big(\sigma_p^{A_1} \otimes \mathbb{1}^{B_1} \otimes \mathbb{1}^{A_2} \otimes \sigma_q^{B_2} \\ &+ \mathbb{1}^{A_1} \otimes \sigma_p^{B_1} \otimes \sigma_q^{A_2} \otimes \mathbb{1}^{B_2}\big) - \frac{\sin(t\pi)}{4}\sum_{lkmn} \epsilon_{plk}\epsilon_{qnm}\sigma_l^{A_1} \otimes \sigma_k^{B_1} \otimes \sigma_n^{A_2} \otimes \sigma_m^{B_2} \\ &+ \frac{(1+\cos(t\pi))\sin(t\pi)}{4}\Bigg(\sum_{lk}\epsilon_{plk}\sigma_l^{A_1} \otimes \sigma_k^{B_1} \otimes \sigma_q^{A_2} \otimes \mathbb{1}^{B_2} \\ &- \sum_{nm}\epsilon_{qnm}\sigma_p^{A_1} \otimes \mathbb{1}^{B_1} \otimes \sigma_n^{A_2} \otimes \sigma_m^{B_2}\Bigg) + \frac{(1-\cos(t\pi))\sin(t\pi)}{4} \\ &\cdot \Bigg(\sum_{lk}\epsilon_{plk}\sigma_l^{A_1} \otimes \sigma_k^{B_1} \otimes \mathbb{1}^{A_2} \otimes \sigma_q^{B_2} - \sum_{nm}\epsilon_{qnm}\mathbb{1}^{A_1} \otimes \sigma_p^{B_1} \otimes \sigma_n^{A_2} \otimes \sigma_m^{B_2}\Bigg)\Bigg]. \end{aligned} \tag{6.30}$$

We have already mentioned that $P_{c+} \subseteq P_\otimes$ and therefore we can write every positive semi definite second kind process $W \in P_{c+}$ in the form

$$\begin{aligned} W &= \frac{1}{d_{A_1}d_{B_1}}\Bigg((\mathbb{1}^{A_1} \otimes \mathbb{1}^{B_1} \otimes \mathbb{1}^{A_2} \otimes \mathbb{1}^{B_2} + \sum_i a_i\sigma_i^{A_1} \otimes \mathbb{1}^{B_1} \otimes \mathbb{1}^{A_2} \otimes \mathbb{1}^{B_2} \\ &+ \sum_i b_i\mathbb{1}^{A_1} \otimes \sigma_i^{B_1} \otimes \mathbb{1}^{A_2} \otimes \mathbb{1}^{B_2} + \sum_{ij} c_{ij}\sigma_i^{A_1} \otimes \sigma_j^{B_1} \otimes \mathbb{1}^{A_2} \otimes \mathbb{1}^{B_2} \\ &+ \sum_{ij} t_{ij}\mathbb{1}^{A_1} \otimes \sigma_i^{B_1} \otimes \sigma_j^{A_2} \otimes \mathbb{1}^{B_2} + \sum_{ij} s_{ij}\sigma_i^{A_1} \otimes \mathbb{1}^{B_1} \otimes \mathbb{1}^{A_2} \otimes \sigma_j^{B_2} \\ &+ \sum_{ijk} g_{ijk}\sigma_i^{A_1} \otimes \sigma_j^{B_1} \otimes \sigma_k^{A_2} \otimes \mathbb{1}^{B_2} + \sum_{ijk} l_{ijk}\sigma_i^{A_1} \otimes \sigma_j^{B_1} \otimes \mathbb{1}^{A_2} \otimes \sigma_k^{B_2}. \end{aligned} \tag{6.31}$$

Since we are dealing with a 4-dimensional input space the prefactor is given by $1/(d_{A_1}d_{B_2}) = 1/4$. Furthermore, every $W \in P_{c+}$ must produce valid probability distributions for $M^{AB}(t)$ i.e.

$\mathsf{Tr}(WM^{AB}(t)) = 1\ \forall t$. Thus, we calculate

$$\begin{aligned}\mathsf{Tr}(WM^{AB}(t)) &= 1 + \frac{1-\cos^2(t\pi)}{4}(s_{pq}+t_{pq}) + \frac{(1+\cos(t\pi))\sin(t\pi)}{4}\sum_{lk}\epsilon_{plk}g_{lkq} \\ &+ \frac{1-\cos(t\pi)\sin(t\pi)}{4}\sum_{lk}\epsilon_{plk}l_{lkq} = 1. \end{aligned} \tag{6.32}$$

The last equation must hold $\forall t$ and therefore we find the following conditions

$$s_{pq} = -t_{pq} \qquad \forall p,q \tag{6.33}$$

$$\sum_{lk}\epsilon_{plk}g_{lkq} = 0 \qquad \forall p,q \tag{6.34}$$

and

$$\sum_{lk}\epsilon_{plk}l_{lkq} = 0 \qquad \forall p,q. \tag{6.35}$$

Condition (6.33) implies (at least in 4×4-dimensions) that process matrices representing a channel from Alice to Bob or a channel from Bob to Alice are not within the set P_{c+}. Furthermore, we note that we have found conditions on terms associated with channels and channels with memories, but not on terms which are associated with states. Until now we only used one specific family of unitaries and one specific set of instruments held by Alice and Bob and we have already got a lot of conditions for the allowed process matrices $W \in P_{c+}$. Therefore we conjecture that P_{c+} consists only of states and even more so we conjecture $P_{c+} \cong S(\mathcal{H}_1)$.

6.4 $P_{c+} \cong S(\mathcal{H}_1)$

In this section we want to prove our last conjecture that $P_{c+} \cong S(\mathcal{H}_1)$. To this end, we will first find a symmetry which the set P_{c+} must satisfy and then, to finish the proof, we will use similar arguments as in [20], where it was shown that the only continuous and reversible process matrix transformations are local unitaries.
To start things off we show that every process matrix $W \in P_{\otimes}$ representing a state i.e.

$$\begin{aligned} W &= \frac{1}{d_{A_1}d_{B_1}}\Bigg((\mathbb{1}^{A_1}\otimes\mathbb{1}^{B_1}\otimes\mathbb{1}^{A_2}\otimes\mathbb{1}^{B_2} + \sum_i a_i\sigma_i^{A_1}\otimes\mathbb{1}^{B_1}\otimes\mathbb{1}^{A_2}\otimes\mathbb{1}^{B_2} \\ &+ \sum_i b_i\mathbb{1}^{A_1}\otimes\sigma_i^{B_1}\otimes\mathbb{1}^{A_2}\otimes\mathbb{1}^{B_2} + \sum_{ij}c_{ij}\sigma_i^{A_1}\otimes\sigma_j^{B_1}\otimes\mathbb{1}^{A_2}\otimes\mathbb{1}^{B_2}\Bigg) \\ &= \rho^{A_1B_1}\otimes\mathbb{1}^{A_2}\otimes\mathbb{1}^{B_2}, \end{aligned} \tag{6.36}$$

where $\rho^{A_1B_1} \in S(\mathcal{H}_1)$ is also in P_{c+}. (Note that obviously every $\rho^{A_1B_1} \in S(\mathcal{H}_1)$ defines a valid process matrix via (6.36).)
From

$$\begin{aligned}\mathsf{Tr}(WM^{AB}) &= \mathsf{Tr}(W(U^{A_1B_1}\otimes(U^{A_2B_2})^*)M^{A_0B_0}((U^{A_1B_1})^\dagger\otimes((U^{A_2B_2})^*)^\dagger)) \\ &= \mathsf{Tr}(((U^{A_1B_1})^\dagger\otimes((U^{A_2B_2})^*)^\dagger)W(U^{A_1B_1}\otimes(U^{A_2B_2})^*)M^{A_0B_0}), \end{aligned} \tag{6.37}$$

we find that, if $W \in P_{c+}$ then $((U^{A_1B_1})^\dagger \otimes (U^{A_2B_2})^*)^\dagger)W(U^{A_1B_1} \otimes (U^{A_2B_2})^*)$ must be in $P_\otimes$ for every $U^{A_1B_1}$, since (6.37) defines a valid probability measure for every $M^{A_0B_0}$ and any unitary transformation preserves the spectrum and therefore the positive definiteness of W. On the other hand, if for a first kind process matrix $W \in P_\otimes$ the product $((U^{A_1B_1})^\dagger \otimes (U^{A_2B_2})^*)^\dagger)W(U^{A_1B_1} \otimes (U^{A_2B_2})^*)$ defines a valid probability measure via (6.37) for any unitary $U^{A_1B_1}$ then $W \in P_{c+}$. This can be used to show that states are in the set P_{c+}:

$$\begin{aligned} ((U^{A_1B_1})^\dagger \otimes (U^{A_2B_2})^*)^\dagger)W(U^{A_1B_1} \otimes (U^{A_2B_2})^*) &= ((U^{A_1B_1})^\dagger \rho^{A_1B_1} U^{A_1B_1}) \otimes \mathbb{1}^{A_2B_2} \\ &= \widetilde{\rho} \otimes \mathbb{1}^{A_1B_1}, \end{aligned} \tag{6.38}$$

where $\widetilde{\rho} = ((U^{A_1B_1})^\dagger \rho^{A_1B_1} U^{A_1B_1}) \in S(\mathcal{H}_1)$ for every $U^{A_1B_1}$ and therefore $\rho \otimes \mathbb{1}^{A_1B_2} \in P_{c+}$ for every $\rho \in S(\mathcal{H}_1)$.

So far we have seen, that every first kind process matrix representing a state is in the set P_{c+}. Now, we want to show that these are the only elements of P_{c+}. Again we will use the fact that $W \in P_{c+}$ iff $((U^{A_1B_1})^\dagger \otimes (U^{A_2B_2})^*)^\dagger)W(U^{A_1B_1} \otimes (U^{A_2B_2})^*) \in P_\otimes$ for every unitary $U^{A_1B_1}$ and moreover we will use similar arguments as in appendix C of [20].

We look at the one parameter family of unitaries

$$U_\lambda = e^{i\lambda H}, \tag{6.39}$$

where $H \in \mathcal{L}(\mathcal{H}_1)$ is an arbitrary hermitian operator and $\lambda \in \mathbb{R}$ can be continuously varied. Furthermore

$$U_\lambda^* = e^{-i\lambda H^*} = e^{-i\lambda H^T}, \tag{6.40}$$

where H^T is again hermitian. From now on we choose λ infinitesimal and we find the expansions

$$U_\lambda = \mathbb{1} + i\lambda H \tag{6.41}$$

and

$$U_\lambda^* = \mathbb{1} - i\lambda H^T. \tag{6.42}$$

From this we get

$$U_\lambda \otimes U_\lambda^* = (\mathbb{1} + i\lambda H) \otimes (\mathbb{1} - i\lambda H^T) = \mathbb{1} \otimes \mathbb{1} + i\lambda H \otimes \mathbb{1} - i\lambda \mathbb{1} \otimes H^T \tag{6.43}$$

and

$$U_\lambda^\dagger \otimes (U_\lambda^*)^\dagger = (\mathbb{1} - i\lambda H) \otimes (\mathbb{1} + i\lambda H^T) = \mathbb{1} \otimes \mathbb{1} - i\lambda H \otimes \mathbb{1} + i\lambda \mathbb{1} \otimes H^T. \tag{6.44}$$

Hence we find

$$\begin{aligned} &((U_\lambda^{A_1B_1})^\dagger \otimes (U_\lambda^{A_2B_2})^*)^\dagger)W(U_\lambda^{A_1B_1} \otimes (U_\lambda^{A_2B_2})^*) = (\mathbb{1}^{A_1B_1} \otimes \mathbb{1}^{A_2B_2} - i\lambda H^{A_1B_1} \otimes \mathbb{1}^{A_2B_2} \\ &+ i\lambda \mathbb{1}^{A_1B_1} \otimes (H^{A_2B_2})^T)W(\mathbb{1}^{A_1B_1} \otimes \mathbb{1}^{A_2B_2} + i\lambda H^{A_1B_2} \otimes \mathbb{1}^{A_2B_2} - i\lambda \mathbb{1}^{A_1B_1} \otimes (H^{A_2B_2})^T) \\ &= W + i\lambda \left([W, H^{A_1B_1} \otimes \mathbb{1}^{A_2B_2}] - [W, \mathbb{1}^{A_1B_2} \otimes (H^{A_2B_2})^T] \right). \end{aligned} \tag{6.45}$$

Moreover, we will need the projector P on $P_\otimes$

$$P(W) = {}_{A2}W + {}_{B_2}W - {}_{A_2B_2}W - {}_{A_1A_2}W - {}_{B_12B_2}W + {}_{A_2B_1B_2}W + {}_{B_2A_1A_2}W, \tag{6.46}$$

where ${}_xW = \mathbb{1}_x \mathrm{Tr}_x(W)$ [12, 20]. We note that $P(T) = T$, if T is an allowed term of a process matrix and $P(F) = 0$, if F is a non-allowed term (see Section 4.2 for which terms are allowed and which are not)[20]. Since $((U_\lambda^{A_1B_1})^\dagger \otimes (U_\lambda^{A_2B_2})^*)^\dagger)W(U_\lambda^{A_1B_1} \otimes (U_\lambda^{A_2B_2})^*) \in P_\otimes$ iff $W \in P_{c+}$ we find that

$$\begin{aligned} P(((U_\lambda^{A_1B_1})^\dagger \otimes (U_\lambda^{A_2B_2})^*)^\dagger)W(U_\lambda^{A_1B_1} \otimes (U_\lambda^{A_2B_2})^*)) &= ((U_\lambda^{A_1B_1})^\dagger \otimes (U_\lambda^{A_2B_2})^*)^\dagger) \\ &\cdot \quad W(U_\lambda^{A_1B_1} \otimes (U_\lambda^{A_2B_2})^*), \end{aligned} \tag{6.47}$$

which can be rewritten as

$$\begin{aligned} & P\left(W + i\lambda\left([W, H^{A_1B_1} \otimes \mathbb{1}^{A_2B_2}] - [W, \mathbb{1}^{A_1B_1} \otimes (H^{A_2B_2})^T]\right)\right) \\ &= P(W) + i\lambda P\left([W, H^{A_1B_1} \otimes \mathbb{1}^{A_2B_2}] - [W, \mathbb{1}^{A_1B_1} \otimes (H^{A_2B_2})^T]\right) \\ &= W + i\lambda\left([W, H^{A_1B_1} \otimes \mathbb{1}^{A_2B_2}] - [W, \mathbb{1}^{A_1B_1} \otimes (H^{A_2B_2})^T]\right) \end{aligned} \tag{6.48}$$

and therefore

$$\begin{aligned} P\left([W, H^{A_1B_1} \otimes \mathbb{1}^{A_2B_2}] - [W, \mathbb{1}^{A_1B_1} \otimes (H^{A_2B_2})^T]\right) &= [W, H^{A_1B_1} \otimes \mathbb{1}^{A_2B_2}] \\ &- [W, \mathbb{1}^{A_1B_1} \otimes (H^{A_2B_2})^T]. \end{aligned} \tag{6.49}$$

Next, we write H as

$$H = h_{\alpha\beta}\sigma_\alpha \otimes \sigma_\beta, \tag{6.50}$$

where $h_{\alpha\beta} \in \mathbb{R}\ \forall \alpha, \beta$ and we have used the Einstein sum convention, that is, if a greek index appears twice we sum over the index from 0 to $d^2 - 1$ and if a roman index appears twice we sum over the index from 1 to $d^2 - 1$. Note that we have not used this convention so far and after the end of the chapter we will not continue to use it unless we explicitly state it. The operators σ_i, with $1 \le i \le d^2 - 1$ are the generators of the traceless hermitian Lie algebra su(d) and $\sigma_0 = \mathbb{1}$. The operators satisfy the relation

$$\sigma_\alpha\sigma_\beta = \delta_{\alpha\beta}\mathbb{1} + d_{\alpha\beta\gamma}\sigma_\gamma + if_{\alpha\beta\gamma}\sigma_\gamma, \tag{6.51}$$

where

$$d_{0\alpha\beta} = d_{\alpha0\beta} = d_{\alpha\beta0} = f_{0\alpha\beta} = f_{\alpha0\beta} = f_{\alpha\beta0} = 0 \qquad \forall \alpha, \beta \tag{6.52}$$

and $d_{\alpha\beta\gamma}$ is traceless (i.e. $d_{\alpha\alpha\beta} = 0$) and totally symmetric, while $f_{\alpha\beta\gamma}$ is totally anti-symmetric[20]. To calculate $(P([W, H^{A_1B_1} \otimes \mathbb{1}^{A_2B_2}]))$ and $[W, H^{A_1B_1} \otimes \mathbb{1}^{A_2B_2}]$ we will need the relations

$$[\sigma_\nu \otimes \mathbb{1}, \sigma_\alpha \otimes \sigma_\beta] = 2if_{\nu\alpha\gamma}\sigma_\gamma \otimes \sigma_\beta, \tag{6.53}$$

where we have used that $\delta_{\alpha\nu}$ and $d_{\alpha\nu\gamma}$ are symmetric in α and ν while $f_{\alpha\nu\gamma}$ is anti-symmetric in α and ν. Similarly we find

$$[\mathbb{1} \otimes \sigma_\eta, \sigma_\alpha \otimes \sigma_\beta] = 2if_{\eta\beta\gamma}\sigma_\alpha \otimes \sigma_\gamma \tag{6.54}$$

and

$$[\sigma_\nu \otimes \sigma_\eta, \sigma_\alpha \otimes \sigma_\beta] = 2i(\delta_{\alpha\nu}f_{\eta\beta\gamma}\mathbb{1} \otimes \sigma_\gamma + \delta_{\beta\eta}f_{\nu\alpha\gamma}\sigma_\gamma \otimes \mathbb{1} + (d_{\alpha\nu\gamma}f_{\eta\beta\tau} + d_{\beta\eta\tau}f_{\nu\alpha\gamma})\sigma_\gamma \otimes \sigma_\tau). \tag{6.55}$$

Moreover we write every $W \in P_{c+}$ in the form

$$\begin{aligned} W &= \frac{1}{d_{A_1}d_{B_1}}\left((\mathbb{1}^{A_1} \otimes \mathbb{1}^{B_1} \otimes \mathbb{1}^{A_2} \otimes \mathbb{1}^{B_2} + a_\nu\sigma_\nu^{A_1} \otimes \mathbb{1}^{B_1} \otimes \mathbb{1}^{A_2} \otimes \mathbb{1}^{B_2}\right. \\ &+ b_\eta\mathbb{1}^{A_1} \otimes \sigma_\eta^{B_1} \otimes \mathbb{1}^{A_2} \otimes \mathbb{1}^{B_2} + c_{\nu\eta}\sigma_\nu^{A_1} \otimes \sigma_\eta^{B_1} \otimes \mathbb{1}^{A_2} \otimes \mathbb{1}^{B_2} \\ &+ t_{\nu\eta}\mathbb{1}^{A_1} \otimes \sigma_\nu^{B_1} \otimes \sigma_\eta^{A_2} \otimes \mathbb{1}^{B_2} + s_{\nu\eta}\sigma_\nu^{A_1} \otimes \mathbb{1}^{B_1} \otimes \mathbb{1}^{A_2} \otimes \sigma_\eta^{B_2} \\ &+ g_{\nu\eta\tau}\sigma_\nu^{A_1} \otimes \sigma_\eta^{B_1} \otimes \sigma_\tau^{A_2} \otimes \mathbb{1}^{B_2} + l_{\nu\eta\tau}\sigma_\nu^{A_1} \otimes \sigma_\eta^{B_1} \otimes \mathbb{1}^{A_2} \otimes \sigma_\tau^{B_2}, \end{aligned} \tag{6.56}$$

where we have defined

$$\begin{aligned} a_0 &= b_0 = c_{0\eta} = c_{\nu 0} = t_{0\eta} = t_{\nu 0} = s_{0\eta} = s_{\nu 0} = g_{0\nu\eta} = g_{\nu 0\eta} = g_{\nu\eta 0} = l_{0\nu\eta} \\ &= l_{\nu 0\eta} = l_{\nu\eta 0} = 0, \end{aligned} \tag{6.57}$$

$\forall \eta, \nu$.

Next, we calculate the commutator of every term of W with $H^{A_1B_1} \otimes \mathbb{1}^{A_2B_2}$ independently. We start with

$$\begin{aligned} [g_{\nu\eta k}\sigma_\nu^{A_1} \otimes \sigma_\eta^{B_1} \otimes \sigma_k^{A_2} \otimes \mathbb{1}^{B_2}, H^{A_1B_1} \otimes \mathbb{1}^{A_2B_2}] &= g_{\nu\eta k} h_{\alpha,\beta} \\ &\cdot [\sigma_\nu^{A_1} \otimes \sigma_\eta^{B_1} \otimes \sigma_k^{A_2} \otimes \mathbb{1}^{B_2}, \sigma_\alpha^{A_1} \otimes \sigma_\beta^{B_1} \otimes \mathbb{1}^{A_2B_2}] \\ &= 2i g_{\nu\eta k} h_{\alpha\beta} \Big(\delta_{\alpha\nu} f_{\eta\beta\tau} \mathbb{1}^{A_1} \otimes \sigma_\tau^{B_1} \otimes \sigma_k^{A_2} \otimes \mathbb{1}^{B_2} \\ &+ d_{\alpha\nu\gamma} f_{\eta\beta\tau} \sigma_\gamma^{A_1} \otimes \sigma_\tau^{B_1} \otimes \sigma_k^{A_2} \otimes \mathbb{1}^{B_2} \\ &+ \delta_{\eta\beta} f_{\nu\alpha\gamma} \sigma_\gamma^{A_1} \otimes \mathbb{1}^{B_1} \otimes \sigma_k^{A_2} \otimes \mathbb{1}^{B_2} \\ &+ f_{\nu\alpha\gamma} d_{\beta\eta\tau} \sigma_\gamma^{A_1} \otimes \sigma_\tau^{B_1} \otimes \sigma_k^{A_2} \otimes \mathbb{1}^{B_2}) \\ &= 2i h_{\alpha\beta} (g_{\nu\eta k} (f_{\nu\alpha\gamma} d_{\beta\eta\tau} + d_{\alpha\nu\gamma} f_{\eta\beta\tau}) \\ &\cdot \sigma_\gamma^{A_1} \otimes \sigma_\tau^{B_1} \otimes \sigma_k^{A_2} \otimes \mathbb{1}^{B_2} \\ &+ g_{\alpha\eta k} f_{\eta\beta\tau} \mathbb{1}^{A_1} \otimes \sigma_\tau^{B_1} \otimes \sigma_k^{A_2} \otimes \mathbb{1}^{B_2} \\ &+ g_{\nu\beta k} f_{\nu\alpha\gamma} \sigma_\gamma^{A_1} \otimes \mathbb{1}^{B_1} \otimes \sigma_k^{A_2} \otimes \mathbb{1}^{B_2}). \end{aligned} \tag{6.58}$$

From this we observe

$$\begin{aligned} P([g_{\nu\eta k}\sigma_\nu^{A_1} \otimes \sigma_\eta^{B_1} \otimes \sigma_k^{A_2} \otimes \mathbb{1}^{B_2}, H^{A_1B_1} \otimes \mathbb{1}^{A_2B_2}]) &= [g_{\nu\eta k}\sigma_\nu^{A_1} \otimes \sigma_\eta^{B_1} \otimes \sigma_k^{A_2} \otimes \mathbb{1}^{B_2}, H^{A_1B_1} \otimes \mathbb{1}^{A_2B_2}] \\ &- 2i h_{\alpha\beta} g_{\nu\beta k} f_{\nu\alpha\gamma} \\ &\cdot (\sigma_\gamma^{A_1} \otimes \mathbb{1}^{B_1} \otimes \sigma_k^{A_2} \otimes \mathbb{1}^{B_2}). \end{aligned} \tag{6.59}$$

Similarly, we find

$$\begin{aligned} P([l_{\nu\eta k}\sigma_\nu^{A_1} \otimes \sigma_\eta^{B_1} \otimes \mathbb{1}^{A_2} \otimes \sigma_k^{B_2}, H^{A_1B_1} \otimes \mathbb{1}^{A_2B_2}]) &= [l_{\nu\eta k}\sigma_\nu^{A_1} \otimes \sigma_\eta^{B_1} \otimes \mathbb{1}^{A_2} \otimes \sigma_k^{B_2}, H^{A_1B_1} \otimes \mathbb{1}^{A_2B_2}] \\ &- 2i h_{\alpha\beta} l_{\alpha\eta k} f_{\eta\beta\tau} \\ &\cdot (\mathbb{1}^{A_1} \otimes \sigma_\tau^{B_1} \otimes \mathbb{1}^{A_2} \otimes \sigma_k^{B_2}), \end{aligned} \tag{6.60}$$

$$P([a_\nu\sigma_\nu^{A_1} \otimes \mathbb{1}^{B_1} \otimes \mathbb{1}^{A_2} \otimes \mathbb{1}^{B_2}, H^{A_1B_1} \otimes \mathbb{1}^{A_2B_2}]) = [a_\nu\sigma_\nu^{A_1} \otimes \mathbb{1}^{B_1} \otimes \mathbb{1}^{A_2} \otimes \mathbb{1}^{B_2}, H^{A_1B_1} \otimes \mathbb{1}^{A_2B_2}] \tag{6.61}$$

$$P([b_\eta\mathbb{1}^{A_1} \otimes \sigma_\eta^{B_1} \otimes \mathbb{1}^{A_2} \otimes \mathbb{1}^{B_2}, H^{A_1B_1} \otimes \mathbb{1}^{A_2B_2}]) = [b_\eta\mathbb{1}^{A_1} \otimes \sigma_\eta^{B_1} \otimes \mathbb{1}^{A_2} \otimes \mathbb{1}^{B_2}, H^{A_1B_1} \otimes \mathbb{1}^{A_2B_2}] \tag{6.62}$$

$$P([c_{\nu\eta}\sigma_\nu^{A_1} \otimes \sigma_\eta^{B_1} \otimes \mathbb{1}^{A_2} \otimes \mathbb{1}^{B_2}, H^{A_1B_1} \otimes \mathbb{1}^{A_2B_2}]) = [c_{\nu\eta}\sigma_\nu^{A_1} \otimes \sigma_\eta^{B_1} \otimes \mathbb{1}^{A_2} \otimes \mathbb{1}^{B_2}, H^{A_1B_1} \otimes \mathbb{1}^{A_2B_2}] \tag{6.63}$$

$$P([s_{\nu k}\sigma_\nu^{A_1} \otimes \mathbb{1}^{B_1} \otimes \mathbb{1}^{A_2} \otimes \sigma_k^{B_2}, H^{A_1B_1} \otimes \mathbb{1}^{A_2B_2}]) = [s_{\nu k}\sigma_\nu^{A_1} \otimes \mathbb{1}^{B_1} \otimes \mathbb{1}^{A_2} \otimes \sigma_k^{B_2}, H^{A_1B_1} \otimes \mathbb{1}^{A_2B_2}] \tag{6.64}$$

$$P([t_{\eta k}\mathbb{1}^{A_1} \otimes \sigma_\eta^{B_1} \otimes \sigma_k^{A_2} \otimes \mathbb{1}^{B_2}, H^{A_1B_1} \otimes \mathbb{1}^{A_2B_2}]) = [t_{\eta k}\mathbb{1}^{A_1} \otimes \sigma_\eta^{B_1} \otimes \sigma_k^{A_2} \otimes \mathbb{1}^{B_2}, H^{A_1B_1} \otimes \mathbb{1}^{A_2B_2}] \tag{6.65}$$

Now, we concern ourselves with the commutator between the terms of W and $\mathbb{1}^{A_1B_1}\otimes(H^{A_2B_2})^T$. For all terms that are associated with states we observer immediately

$$\begin{aligned}[a_\nu\sigma_\nu^{A_1}\otimes\mathbb{1}^{B_1}\otimes\mathbb{1}^{A_2}\otimes\mathbb{1}^{B_2},\mathbb{1}^{A_1B_1}\otimes(H^{A_2B_2})^T]&=[b_\eta\mathbb{1}^{A_1}\otimes\sigma_\eta^{B_1}\otimes\mathbb{1}^{A_2}\otimes\mathbb{1}^{B_2},\mathbb{1}^{A_1B_1}\otimes(H^{A_2B_2})^T]\\&=[c_{\nu\eta}\sigma_\nu^{A_1}\otimes\sigma_\eta^{B_1}\otimes\mathbb{1}^{A_2}\otimes\mathbb{1}^{B_2},\mathbb{1}^{A_1B_1}\otimes(H^{A_2B_2})^T]=0\end{aligned}\tag{6.66}$$

and obviously $P(0)=0$.
For further calculations we recall that H^T is a hermitian operator as well and therefore we can write

$$H^T=\tilde{h}_{\alpha\beta}\sigma_\alpha\otimes\sigma_\beta,\tag{6.67}$$

where $\tilde{h}_{\alpha\beta}\in\mathbb{R}\ \forall\alpha,\beta$ and the $\tilde{h}_{\alpha\beta}$ depend on $h_{\gamma\tau}$. Now, we find in similar fashion as before

$$\begin{aligned}P([t_{i\nu}\mathbb{1}^{A_1}\otimes\sigma_i^{B_1}\otimes\sigma_\nu^{A_2}\otimes\mathbb{1}^{B_2},\mathbb{1}^{A_1B_1}\otimes(H^{A_2B_2})^T])&=[t_{i\nu}\mathbb{1}^{A_1}\otimes\sigma_i^{B_1}\otimes\sigma_\nu^{A_2}\otimes\mathbb{1}^{B_2},\mathbb{1}^{A_1B_1}\otimes(H^{A_2B_2})^T]\\&-2it_{i\nu}\tilde{h}_{\alpha j}f_{\nu\alpha\gamma}\\&\cdot(\mathbb{1}^{A_1}\otimes\sigma_i^{B_1}\otimes\sigma_\gamma^{A_2}\otimes\sigma_j^{B_2}),\end{aligned}\tag{6.68}$$

$$\begin{aligned}P([g_{ij\nu}\sigma_i^{A_1}\otimes\sigma_j^{B_1}\otimes\sigma_\nu^{A_2}\otimes\mathbb{1}^{B_2},\mathbb{1}^{A_1B_1}\otimes(H^{A_2B_2})^T])&=[g_{ij\nu}\sigma_i^{A_1}\otimes\sigma_j^{B_1}\otimes\sigma_\nu^{A_2}\otimes\mathbb{1}^{B_2},\mathbb{1}^{A_1B_1}\otimes(H^{A_2B_2})^T]\\&-2ig_{ij\nu}\tilde{h}_{\alpha p}f_{\nu\alpha\gamma}\\&\cdot(\sigma_i^{A_1}\otimes\sigma_j^{B_1}\otimes\sigma_\gamma^{A_2}\otimes\sigma_p^{B_2}),\end{aligned}\tag{6.69}$$

$$\begin{aligned}P([s_{i\eta}\sigma_i^{A_1}\otimes\mathbb{1}^{B_1}\otimes\mathbb{1}^{A_2}\otimes\sigma_\eta^{B_2},\mathbb{1}^{A_1B_1}\otimes(H^{A_2B_2})^T])&=[s_{i\eta}\sigma_i^{A_1}\otimes\mathbb{1}^{B_1}\otimes\mathbb{1}^{A_2}\otimes\sigma_\eta^{B_2},\mathbb{1}^{A_1B_1}\otimes(H^{A_2B_2})^T]\\&-2is_{i\eta}\tilde{h}_{p\beta}f_{\eta\beta\gamma}\\&\cdot(\sigma_i^{A_1}\otimes\mathbb{1}^{B_1}\otimes\sigma_p^{A_2}\otimes\sigma_\gamma^{B_2}),\end{aligned}\tag{6.70}$$

$$\begin{aligned}P([l_{ij\eta}\sigma_i^{A_1}\otimes\sigma_j^{B_1}\otimes\mathbb{1}^{A_2}\otimes\sigma_\eta^{B_2},\mathbb{1}^{A_1B_1}\otimes(H^{A_2B_2})^T])&=[l_{ij\eta}\sigma_i^{A_1}\otimes\sigma_j^{B_1}\otimes\mathbb{1}^{A_2}\otimes\sigma_\eta^{B_2},\mathbb{1}^{A_1B_1}\otimes(H^{A_2B_2})^T]\\&-2il_{ij\eta}\tilde{h}_{p\beta}f_{\eta\beta\gamma}\\&\cdot(\sigma_i^{A_1}\otimes\sigma_j^{B_1}\otimes\sigma_p^{A_2}\otimes\sigma_\gamma^{B_2}).\end{aligned}\tag{6.71}$$

Putting everything together we observe

$$\begin{aligned}P\left([W,H^{A_1B_1}\otimes\mathbb{1}^{A_2B_2}]-[W,\mathbb{1}^{A_1B_1}\otimes(H^{A_2B_2})^T]\right)&=[W,H^{A_1B_1}\otimes\mathbb{1}^{A_2B_2}]\\-[W,\mathbb{1}^{A_1B_1}\otimes(H^{A_2B_2})^T]-2ih_{\alpha\beta}g_{\nu\beta k}f_{\nu\alpha\gamma}\sigma_\gamma^{A_1}\otimes\mathbb{1}^{B_1}\otimes\sigma_k^{A_2}\otimes\mathbb{1}^{B_2}\\-2ih_{\alpha\beta}l_{\alpha\eta k}f_{\eta\beta\tau}\mathbb{1}^{A_1}\otimes\sigma_\tau^{B_1}\otimes\mathbb{1}^{A_2}\otimes\sigma_k^{B_2}+2it_{i\nu}\tilde{h}_{\alpha j}f_{\nu\alpha\gamma}\mathbb{1}^{A_1}\otimes\sigma_i^{B_1}\otimes\sigma_\gamma^{A_2}\otimes\sigma_j^{B_2}\\+2ig_{ij\nu}\tilde{h}_{\alpha p}f_{\nu\alpha\gamma}\sigma_i^{A_1}\otimes\sigma_j^{B_1}\otimes\sigma_\gamma^{A_2}\otimes\sigma_p^{B_2}+2is_{i\eta}\tilde{h}_{p\beta}f_{\eta\beta\gamma}\sigma_i^{A_1}\otimes\mathbb{1}^{B_1}\otimes\sigma_p^{A_2}\otimes\sigma_\gamma^{B_2}\\+2il_{ij\eta}\tilde{h}_{p\beta}f_{\eta\beta\gamma}\sigma_i^{A_1}\otimes\sigma_j^{B_1}\otimes\sigma_p^{A_2}\otimes\sigma_\gamma^{B_2}&=[W,H^{A_1B_1}\otimes\mathbb{1}^{A_2B_2}]\\&-[W,\mathbb{1}^{A_1B_1}\otimes(H^{A_2B_2})^T]\end{aligned}\tag{6.72}$$

and therefore

$$\begin{aligned}-h_{\alpha\beta}g_{\nu\beta k}f_{\nu\alpha\gamma}\sigma_\gamma^{A_1}\otimes\mathbb{1}^{B_1}\otimes\sigma_k^{A_2}\otimes\mathbb{1}^{B_2}-h_{\alpha\beta}l_{\alpha\eta k}f_{\eta\beta\tau}\mathbb{1}^{A_1}\otimes\sigma_\tau^{B_1}\otimes\mathbb{1}^{A_2}\otimes\sigma_k^{B_2}\\+t_{i\nu}\tilde{h}_{\alpha j}f_{\nu\alpha\gamma}\mathbb{1}^{A_1}\otimes\sigma_i^{B_1}\otimes\sigma_\gamma^{A_2}\otimes\sigma_j^{B_2}+g_{ij\nu}\tilde{h}_{\alpha p}f_{\nu\alpha\gamma}\sigma_i^{A_1}\otimes\sigma_j^{B_1}\otimes\sigma_\gamma^{A_2}\otimes\sigma_p^{B_2}\\+s_{i\eta}\tilde{h}_{p\beta}f_{\eta\beta\gamma}\sigma_i^{A_1}\otimes\mathbb{1}^{B_1}\otimes\sigma_p^{A_2}\otimes\sigma_\gamma^{B_2}+l_{ij\eta}\tilde{h}_{p\beta}f_{\eta\beta\gamma}\sigma_i^{A_1}\otimes\sigma_j^{B_1}\otimes\sigma_p^{A_2}\otimes\sigma_\gamma^{B_2}=0.\end{aligned}\tag{6.73}$$

Since all the $\sigma_\alpha^{A_1} \otimes \sigma_\beta^{B_1} \otimes \sigma_\gamma^{A_2} \otimes \sigma_\tau^{B_2}$'s are linear independent of each other, condition (6.73) implies

$$\begin{aligned} h_{\alpha\beta} g_{\nu\beta k} f_{\nu\alpha\gamma} &= 0 \qquad \forall \gamma, k, & (6.74)\\ h_{\alpha\beta} l_{\alpha\eta k} f_{\eta\beta\tau} &= 0 \qquad \forall \tau, k, & (6.75)\\ t_{i\nu} \tilde{h}_{\alpha j} f_{\nu\alpha\gamma} &= 0 \qquad \forall i, \gamma, j, & (6.76)\\ s_{i\eta} \tilde{h}_{p\beta} f_{\eta\beta\gamma} &= 0 \qquad \forall i, p, \gamma. & (6.77) \end{aligned}$$

These conditions must hold for every choice of H and therefore for every choice of the $h_{\alpha\beta} \in \mathbb{R}$'s, which again implies that the conditions must also hold for every choice of the $\tilde{h}'_{\alpha\beta} s$ and hence

$$\begin{aligned} g_{\nu\beta k} f_{\nu\alpha\gamma} &= 0 \qquad \forall \gamma, k, \alpha, \beta, & (6.78)\\ l_{\alpha\eta k} f_{\eta\beta\tau} &= 0 \qquad \forall \tau, k, \alpha, \beta, & (6.79)\\ t_{i\nu} f_{\nu\alpha\gamma} &= 0 \qquad \forall i, \gamma, \alpha, & (6.80)\\ s_{i\eta} f_{\eta\beta\gamma} &= 0 \qquad \forall i, \gamma, \beta. & (6.81) \end{aligned}$$

However, the conditions (6.78)-(6.81) implies that

$$s_{ij} = t_{ij} = g_{ijk} = l_{ijk} = 0 \qquad \forall i, j, k, \tag{6.82}$$

which implies that every $W \in P_{c+}$ must be of the form $W = \rho^{A_1 B_1} \otimes \mathbb{1}^{A_2 B_2}$, where $\rho^{A_1 B_1} \in S(\mathcal{H}_1)$. This completes our proof that $P_{c+} \cong S(\mathcal{H}_1)$. $\square$

6.5 Conclusion

The results of [20] already implied that the set $P_\otimes$ has no other continuous reversible symmetries than the local symmetries of the form

$$U(t) = U_1^{A_1}(t) \otimes U_2^{B_1}(t) \otimes U_3^{A_2}(t) \otimes U_4^{B_2}(t), \tag{6.83}$$

where $U_i^{X_j}(t)$ are all unitaries. The fact that we have found that $W \in P_{c+}$ iff $((U^{A_1B_1})^\dagger \otimes (U^{A_2B_2})^*)^\dagger) W (U^{A_1B_1} \otimes (U^{A_2B_2})^*) \in P_\otimes$ for every unitary $U^{A_1B_1}$, implies that P_{c+} must have continuous and reversible symmetries of the form

$$U(t) = (U(t)^{A_1B_1})^\dagger \otimes (U(t)^{A_2B_2})^*)^\dagger, \tag{6.84}$$

which is a broader class of symmetry transformations than (6.83); this already implies that $P_{c+} \subsetneq P_\otimes$. However, there was still hope that there was some broader set than the set of states having the symmetries of the the form (6.84). Unfortunately, we found that only states have these symmetries. This implies that we cannot loosen the assumption in the definition of process matrices, that we already know on which sub-spaces Alice and Bob act, in such a way, that we only assume that they are in locally separated laboratories, if we want to find something more general than quantum mechanical states. However, in some cases it might be interesting to restrict oneself to a subset of unitaries which allow one to define local laboratories and then examine which first kind process matrix still yield valid probability distributions for this sub set of local laboratories. Or vice versa, one could start from one specific first kind process matrix and then ask, which unitaries and therefore which local laboratories are allowed, so one still finds valid probability distributions.

Chapter 7

Bell Scenario in a Superposition of Causal Orders

In the last few chapters we concerned ourselves with process matrices and their properties. But it is not always necessary to use the process matrix formalism to describe a superposition of causal orders. For example, in [24] the quantum switch was introduced without using process matrices.
We want to construct a Bell-like scenario, where the arrival time of the systems received by Alice and Bob can be in superposition of arriving early and late. This will be done without using the process matrix formalism.
Especially, we are hoping that our setup implements almost quantum correlations (see [3]) such that the correlations are outside of the set of normal quantum correlations. We will start with an introduction of almost quantum correlations before we deal with the setup of our thought experiment.
Again, we will restrict our discussion to finite-dimensional Hilbert spaces.

7.1 Quantum Correlations vs. Almost Quantum Correlations

So far we have considered two different ways to implement the local separation of different measurement parties. On the one hand side, we implemented the separation via a partition of the Hilbert space by using a tensor product structure, and on the other hand, we implemented the local separation via the conditions that every measurement party holds a subalgebra and all the subalgebras commute with each other. Furthermore, we saw that both these descriptions of local separation are equivalent in finite dimensions in the sense that both descriptions yield the same correlations (see section 5.2). However, in [3] another plausible way of implementing the local separation of different measurement parties was introduced, but this time they were able to show that this notion of locality can be utilized to find a broader class of correlations, the so called almost quantum correlations.
To make things more precise, we give the following two definitions:

Definition 30. *[3] A non signaling n-partite probability distribution $P(a_1, \dots, a_n|x_1, \dots, x_n)$, where a_i denotes the outputs and x_i the inputs, is called almost quantum iff a normalized vector $|\varphi\rangle \in \mathcal{H}$, where $\mathcal{H}$ is some Hilbert space, and a set of orthogonal projectors $\{E_k^{a_k,x_k}\} \subset \mathcal{B}(\mathcal{H})$ exist, such that:*

1. $\sum_{a_k} E_k^{a_k,x_k} = \mathbb{1} \quad \forall x_k, k$

2. $E_1^{a_1,x_1}\dots E_n^{a_n,x_n}\left|\varphi\right\rangle = E_{\pi(1)}^{a_{\pi(1)},x_{\pi_1}}\dots E_{\pi(n)}^{a_{\pi(n)},x_{\pi_n}}\left|\varphi\right\rangle$, *where* $\pi \in S_n$ *is an arbitrary permutation of the n parties,*

3. $P(a_1,\dots,a_n|x_1,\dots,x_n) = \left\langle\varphi\right| E_1^{a_1,x_1}\dots E_n^{a_n,x_n}\left|\varphi\right\rangle$

and we denote the set of all almost quantum correlations with $\tilde{Q}$.

Definition 31. *[3] A non signaling n-partite probability distribution* $P(a_1,\dots,a_n|x_1,\dots,x_n)$, *where* a_i *denotes the outputs and* x_i *the inputs, is called quantum iff a normalized vector* $\left|\varphi\right\rangle \in \bigotimes_{k=1}^n \mathcal{H}_k$, *where* $\bigotimes_{k=1}^n \mathcal{H}_k$ *is some Hilbert space, and* n *sets of orthogonal projectors* $\{E_k^{a_k,x_k}\} \subset \mathcal{B}(\mathcal{H}_k)$ *exist, such that:*

1. $\sum_{a_k} E_k^{a_k,x_k} = \mathbb{1} \quad \forall x_k, k$,

2. $P(a_1,\dots,a_n|x_1,\dots,x_n) = \left\langle\varphi\right| \bigotimes_{k=1}^n E_k^{a_k,x_k}\dots\left|\varphi\right\rangle$

and we denote the set of all quantum correlations with Q.

The set of quantum correlations Q is exactly the set of correlations which can be found, if we implement the local separation of the different measurement parties via a tensor product structure on the Hilbert space. From these definitions it is obvious that $Q \subseteq \tilde{Q}$. However, it is not trivial that $Q \subsetneq \tilde{Q}$. For an explicit example of $P \in \tilde{Q}$ but $P \notin Q$ see [3].

7.2 A Generalized Bell-Like Scenario

In this section we wanted to find a physically realizable generalization of a bipartite Bell-like scenario, where the local separation of the experimenters is implemented via almost quantum correlations. Furthermore, we asked ourselves if the correlations are outside of the set of quantum correlations. Unfortunately the answer to this question is no.

We consider the following set-up. Alice and Bob are located in two different laboratories. Each of them receives a particle, on which they can perform measurements i.e. Alice and Bob hold a set of projective measurement devices $\{X\}_{i,\alpha} \subset \mathcal{B}(\mathcal{H}_{AB})$ and $\{Y\}_{j,\beta} \subset \mathcal{B}(\mathcal{H}_{AB})$ respectively, and they satisfy $X_{i\alpha}^2 = X_{i\alpha} = X_{i\alpha}^\dagger$ and $Y_{j\beta}^2 = Y_{j\beta} = Y_{j\beta}^\dagger$. Moreover we demand $\sum_\alpha X_{i,\alpha} = \sum_\beta Y_{j,\beta} = \mathbb{1}_{AB}\,\forall i,j$.

The two particles are produced at the same location and thus they can be entangled. Demanding that the events of Alice and Bob performing their measurements are space-like separated would give us just a standard Bell-scenario, but at this point we will include one more degree of freedom. We allow both particles to arrive earlier or later at Alice's and Bob's laboratories respectively. The arrival time of the two particles will be correlated later.

In the case that both particles arrive early or both particles arrive late, we demand that the events of Alice's and Bob's measurements are space-like separated and this gives us just a standard Bell scenario. We say that this is the case where Alice's and Bob's particles arrive at the same time (with respect to a coordinate system having its origin outside of Alice's and Bob's laboratories).

Moreover, if Alice's particle arrives early and Bob's particle arrives late signaling from Alice to Bob is possible. The other way around, if Bob's particle arrives early and Alice's particle arrives late, signaling from Bob to Alice is allowed. (For an illustration of the setup see figure (7.1))

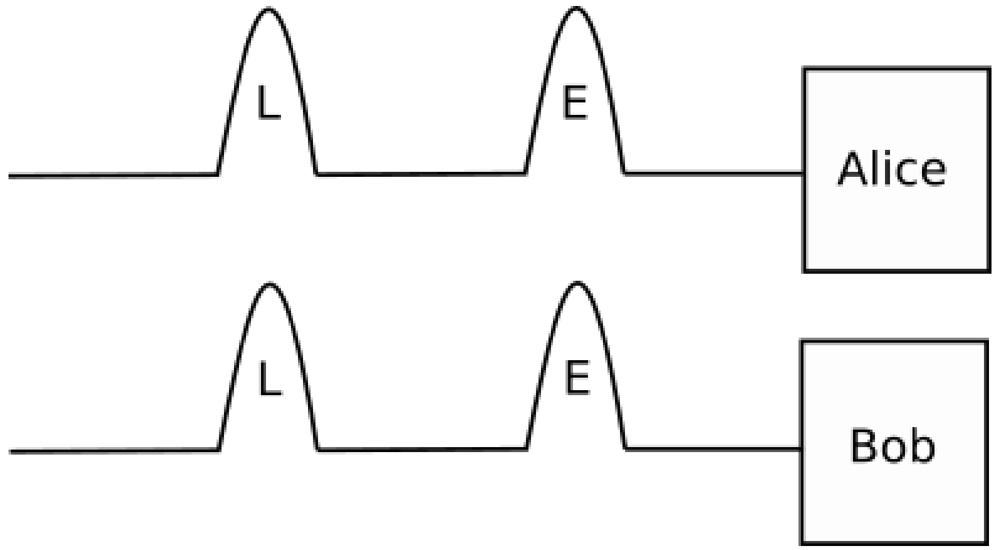

Figure 7.1: Illustration of the experimental setup. Alice and Bob each receive a particle. Both their particles are allowed to arrive at an earlier or a later instance or in a superposition of early and late. If Alice's and Bob's particles are arriving at the same time, no signaling is allowed. If one party receives its particle early and the other party late, signaling from the former to the later party is possible.

The Hilbert space associated with the complete system is given by

$$\mathcal{H} = \mathcal{H}_{AB} \otimes \mathbb{C}^2_{\bar{A}} \otimes \mathbb{C}^2_{\bar{B}} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathbb{C}^2_{\bar{A}} \otimes \mathbb{C}^2_{\bar{B}}, \tag{7.1}$$

where $\mathcal{H}_A$ and $\mathcal{H}_B$ are the Hilbert spaces on which Alice and Bob respectively are acting with their measurement devices. Furthermore,

$$\mathbb{C}^2_X = \text{span}(|E_X\rangle, |L_X\rangle) \tag{7.2}$$

with $X = \bar{A}, \bar{B}$. The ket-vector $|E_X\rangle$ is associated with the case, where the particle arrives early and similarly $|L_X\rangle$ is associated with the case, where the particle arrives late.
Moreover we are working in the Heisenberg picture, that is, the whole time evolution is included in the measurement operators and the wave functions of the particles are time independent. In more detail, consider the case where Alice's particle arrives early and she, for example, performs the measurement

$$X_{i,\alpha} = \tilde{X}_{i,\alpha} \otimes \mathbb{1}_B, \tag{7.3}$$

then, if she wants to perform the same measurement on a late arriving particle, her associated measurement operator is then given by

$$U^\dagger_{AB} X_{i,\alpha} U_{AB} = U^\dagger_{AB} (\tilde{X}_{i,\alpha} \otimes \mathbb{1}) U_{AB}, \tag{7.4}$$

where U_{AB} is a unitary operator, which we can consider as the global time evolution of the whole system between the earlier arrival time and the later arrival time. Similarly, we find for Bob

$$Y_{j,\beta} = \mathbb{1}_A \otimes \tilde{Y}_{j,\beta} \in \mathcal{B}(\mathcal{H}_{AB}), \tag{7.5}$$

and

$$U_{AB}^\dagger Y_{j,\beta} U_{AB} = U_{AB}^\dagger (\mathbb{1}_A \otimes \tilde{Y}_{j,\beta}) U_{AB} \in \mathcal{B}(\mathcal{H}_{AB}). \tag{7.6}$$

For the set $\{U_{AB}^\dagger X_{i,\alpha} U_{AB}\}_{i,\alpha}$ the property

$$\sum_\alpha U_{AB}^\dagger X_{i,\alpha} U_{AB} = U_{AB}^\dagger \overbrace{\sum_\alpha X_{i,\alpha}}^{=\mathbb{1}_{AB}} U_{AB} = U_{AB}^\dagger U_{AB} = \mathbb{1}_{AB} \quad \forall i, \tag{7.7}$$

still holds. Similarly, we find

$$\sum_\beta U_{AB}^\dagger Y_{j,\beta} U_{AB} = \mathbb{1}_{AB} \quad \forall j. \tag{7.8}$$

It is trivial that $X_{i,\alpha}$ given by equation (7.3) and $Y_{j,\beta}$ given by equation (7.5) do commute. Also in the case where both particles arrive late, the measurement operators given by equation (7.4) and equation (7.6) do commute

$$\begin{aligned} U_{AB}^\dagger X_{i,\alpha} \overbrace{U_{AB} U_{AB}^\dagger}^{=\mathbb{1}_{AB}} Y_{j,\beta} U_{AB} &= U_{AB}^\dagger X_{i,\alpha} Y_{j,\beta} U_{AB} = U_{AB}^\dagger Y_{j,\beta} X_{i,\alpha} U_{AB} \\ &= U_{AB}^\dagger Y_{j,\beta} \mathbb{1}_{AB} X_{i,\alpha} U_{AB} = U_{AB}^\dagger Y_{j,\beta} U_{AB} U_{AB}^\dagger X_{i,\alpha} U_{AB}. \end{aligned}$$

However, $X_{i,\alpha}$ does not commute with $U_{AB}^\dagger Y_{j,\beta} U_{AB}$ in general, this reflects the fact that signaling from Alice to Bob is possible, if her particle arrives early and his particle arrives late. Similarly, $Y_{j,\beta}$ does not commute with $U_{AB}^\dagger X_{i,\alpha} U_{AB}$ in general.
Up until now we have just discussed operators acting on the Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. Now we want to consider the whole Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathbb{C}^2_{\bar{A}} \otimes \mathbb{C}^2_{\bar{B}}$ and we want to consider operators acting on the complete Hilbert space.
If we consider $\mathcal{H}$ we do not only distinguish between the cases in which the particle arrives early or late, but we also allow a superposition of the two cases. Furthermore, if we consider for example that Alice's particle is in a superposition of early and late, then we want Alice to measure, whether her particle arrives early or late and depending on that she applies $X_{i,\alpha}$, if she measures that her particle arrives early and $U_{AB}^\dagger X_{i,\alpha} U_A B$ if her particle arrives late. This motivates the following definition of Alice's measurement operators

$$\hat{X}_{i,\alpha} = \tilde{X}_{i,\alpha} \otimes \mathbb{1}_B \otimes |E_{\bar{A}}\rangle \langle E_{\bar{A}}| \otimes \mathbb{1}_{\bar{B}} + U_{AB}^\dagger (\tilde{X}_{i,\alpha} \otimes \mathbb{1}_B) U_{AB} \otimes |L_{\bar{A}}\rangle \langle L_{\bar{A}}| \otimes \mathbb{1}_{\bar{B}} \in \mathcal{B}(\mathcal{H}), \tag{7.9}$$

and similarly for Bob's measurement operators,

$$\hat{Y}_{j,\beta} = \mathbb{1}_A \otimes \tilde{Y}_{j,\beta} \otimes \mathbb{1}_{\bar{A}} \otimes |E_{\bar{B}}\rangle \langle E_{\bar{B}}| + U_{AB}^\dagger (\mathbb{1}_A \otimes \tilde{Y}_{j,\beta}) U_{AB} \otimes \mathbb{1}_{\bar{A}} \otimes |L_{\bar{B}}\rangle \langle L_{\bar{B}}| \in \mathcal{B}(\mathcal{H}). \tag{7.10}$$

Still, the property

$$\begin{aligned} \sum_\alpha \hat{X}_{i,\alpha} &= \sum_\alpha \left(X_{i,\alpha} \otimes |E_{\bar{A}}\rangle \langle E_{\bar{A}}| \otimes \mathbb{1}_{\bar{B}} + U_{AB}^\dagger X_{i,\alpha} U_{AB} \otimes |L_{\bar{A}}\rangle \langle L_{\bar{A}}| \otimes \mathbb{1}_{\bar{B}} \right) \\ &= \sum_\alpha X_{i,\alpha} \otimes |E_{\bar{A}}\rangle \langle E_{\bar{A}}| \otimes \mathbb{1}_{\bar{B}} + U_{AB}^\dagger \sum_\alpha X_{i,\alpha} U_{AB} \otimes |L_{\bar{A}}\rangle \langle L_{\bar{A}}| \otimes \mathbb{1}_{\bar{B}} \\ &= \mathbb{1}_{AB} \otimes \underbrace{(|E_{\bar{A}}\rangle \langle E_{\bar{A}}| + |L_{\bar{A}}\rangle \langle L_{\bar{A}}|)}_{=\mathbb{1}_{\bar{A}}} \otimes \mathbb{1}_{\bar{B}} = \mathbb{1}_{\mathcal{H}} \quad \forall i, \end{aligned} \tag{7.11}$$

as well as

$$\sum_{\beta} \hat{Y}_{j,\beta} = \mathbb{1}_{\mathcal{H}} \quad \forall j, \tag{7.12}$$

holds.
A short calculation shows that the two operators $\hat{X}_{i,\alpha}$ and $\hat{Y}_{j,\beta}$ do not commute unless U_{AB} is chosen very specifically

$$\begin{aligned}
[\hat{X}_{i,\alpha}, \hat{Y}_{j,\beta}] &= \Big((\tilde{X}_{i,\alpha} \otimes \mathbb{1}_B)(U_{AB}^{\dagger}(\mathbb{1}_A \otimes \tilde{Y}_{j,\beta})U_{AB}) - (U_{AB}^{\dagger}(\mathbb{1}_A \otimes \tilde{Y}_{j,\beta})U_{AB})(\tilde{X}_{i,\alpha} \otimes \mathbb{1}_B)\Big) \\
&\otimes \ |E_{\bar{A}}\rangle \langle E_{\bar{A}}| \otimes |L_{\bar{B}}\rangle \langle L_{\bar{B}}| + \Big((U_{AB}^{\dagger}(\tilde{X}_{i,\alpha} \otimes \mathbb{1}_B)U_{AB})(\mathbb{1}_A \otimes \tilde{Y}_{j,\beta}) \\
&- \ (\mathbb{1}_A \otimes \tilde{Y}_{j,\beta})(U_{AB}^{\dagger}(\tilde{X}_{i,\alpha} \otimes \mathbb{1}_B)U_{AB})\Big) \otimes |L_{\bar{A}}\rangle \langle L_{\bar{A}}| \otimes |E_{\bar{B}}\rangle \langle E_{\bar{B}}| \\
&= \ [X_{i,\alpha}, U_{AB}^{\dagger} Y_{j,\beta} U_{AB}] \otimes |E_{\bar{A}}\rangle \langle E_{\bar{A}}| \otimes |L_{\bar{B}}\rangle \langle L_{\bar{B}}| \\
&+ \ [U_{AB}^{\dagger} X_{i,\alpha} U_{AB}, Y_{j,\beta}] \otimes |L_{\bar{A}}\rangle \langle L_{\bar{A}}| \otimes |E_{\bar{B}}\rangle \langle E_{\bar{B}}| \,.
\end{aligned} \tag{7.13}$$

Since the operators of the set $\{\hat{X}\}_{i,\alpha}$ do not commute with the operators of the set $\{\hat{Y}\}_{j,\beta}$, we cannot consider Alice and Bob to be space-like separated in general. However, the space-like separation of the two measurement parties is a necessary condition for a Bell-like experiment and therefore we must implement the separation in a different way than the usual way via commuting measurement operators. We recall from the last section that there is another way of implementation, namely rather than assuming that the operators of Alice and Bob commute in general, we demand that the operators of the different measurement parties commute with each other, if they act on the state which is associated with the experimental setup at hand.
By looking at equation (7.13) we see that the terms where signaling from Alice to Bob or vice versa is possible i.e. terms where the projector of Alice's particle being early is combined with the projector of Bob's being late or vice versa, are accountable for the non-commutativity of $\hat{X}_{i,\alpha}$ and $\hat{Y}_{j,\beta}$.
Therefore $\hat{X}_{i,\alpha}$ and $\hat{Y}_{j,\beta}$ do commute, if they are acting on wave functions which do not include terms where Alice's particle arrives early and Bob's particle arrives late or vice versa. To state the above more precisely, we define the following projector

$$\Pi_{EL,LE} = \mathbb{1}_{AB} \otimes (|E_{\bar{A}} L_{\bar{B}}\rangle \langle E_{\bar{A}} L_{\bar{B}}| + |L_{\bar{A}} E_{\bar{B}}\rangle \langle L_{\bar{A}} E_{\bar{B}}|) \,. \tag{7.14}$$

Obviously, $\Pi_{EL,LE}^2 = \Pi_{EL,LE}$ and $\Pi_{EL,LE}^{\dagger} = \Pi_{EL,LE}$ and therefore $\Pi_{EL,LE}$ is an orthogonal projector. Now, if we restrict ourselves to wave functions $|\varphi\rangle \in \mathcal{H}$ satisfying

$$\Pi_{EL,LE} |\varphi\rangle = 0 \tag{7.15}$$

then

$$[\hat{X}_{i,\alpha}, \hat{Y}_{j,\beta}] |\varphi\rangle = 0 \tag{7.16}$$

follows. In this case, it makes sense to consider A and B as acting in a space-like separated manner, at least in some generalized sense. Thus, we can conclude that our scenario defines a meaningful generalization of the Bell scenario.
Naturally the question arises, whether this yields almost quantum correlations which are outside of the set of quantum correlations?. Unfortunately the answer is no. To understand this some

preparatory work is required.
Equation (7.15) restricts the Hilbert space $\mathcal{H}$ to $\mathcal{H}_{\text{permitted}}$ given by

$$\mathcal{H}_{\text{permitted}} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \text{span}(|E_{\bar{A}}E_{\bar{B}}\rangle, |L_{\bar{A}}L_{\bar{B}}\rangle). \tag{7.17}$$

Easily the projector $\Pi_{EE,LL}$ on $\mathcal{H}_{\text{permitted}}$ can be constructed as follows

$$\Pi_{EE,LL} = \mathbb{1}_{AB} \otimes (|E_{\bar{A}}E_{\bar{B}}\rangle \langle E_{\bar{A}}E_{\bar{B}}| + |L_{\bar{A}}L_{\bar{B}}\rangle \langle L_{\bar{A}}L_{\bar{B}}|) \tag{7.18}$$

and again $\Pi^2_{EE,LL} = \Pi_{EE,LL}$ and $\Pi^\dagger_{EE,LL} = \Pi_{EE,LL}$ holds and hence, $\Pi_{EE,LL}$ defines an orthogonal projector, which also implies that $\mathcal{H}_{\text{permitted}}$ is a linear subspace of $\mathcal{H}$ [28].
Using this projector we can find an equivalent condition to (7.15), namely

$$\Pi_{EE,LL} |\varphi\rangle = (\mathbb{1} - \Pi_{EL,LE}) |\varphi\rangle = |\varphi\rangle, \tag{7.19}$$

which holds for every $|\varphi\rangle \in \mathcal{H}_{\text{permitted}}$. Thus, we can write every state $|\varphi\rangle \in \mathcal{H}_{\text{permitted}}$ in the general form

$$|\varphi\rangle = \sum_{kl} \varphi^E_{kl} |klE_{\bar{A}}E_{\bar{B}}\rangle + \sum_{kl} \varphi^L_{kl} |klL_{\bar{A}}L_{\bar{B}}\rangle, \tag{7.20}$$

where $\varphi^E_{kl}, \varphi^L_{kl} \in \mathbb{C}$ and $\sum_{kl} |\varphi^E_{kl}|^2 + \sum_{kl} |\varphi^L_{kl}|^2 = 1$. Furthermore, let us define the projector

$$\Pi_{EE} = \mathbb{1}_{AB} \otimes |E_{\bar{A}}E_{\bar{B}}\rangle \langle E_{\bar{A}}E_{\bar{B}}|. \tag{7.21}$$

Using this projector, we can calculate the probability that both particles arrive early

$$P(EE) = \langle\varphi| \Pi_{EE} |\varphi\rangle = \sum_{sk} |\varphi^E_{sk}|^2. \tag{7.22}$$

Moreover, we define the state

$$|\varphi_{|EE}\rangle = \frac{\sum_{jk} \varphi^E_{sk} |sk\rangle}{\sqrt{P(EE)}}, \tag{7.23}$$

which is an element of $\mathcal{H}_{AB}$. Similarly, we define the projector

$$\Pi_{LL} = \mathbb{1}_{AB} \otimes |L_{\bar{A}}L_{\bar{B}}\rangle \langle L_{\bar{A}}L_{\bar{B}}|. \tag{7.24}$$

We can calculate the probability that both particles arrive late

$$P(LL) = \langle\varphi| \Pi_{LL} |\varphi\rangle = \sum_{sk} |\varphi^L_{sk}|^2. \tag{7.25}$$

We define the state

$$|\varphi_{|LL}\rangle = \frac{\sum_{jk} \varphi^L_{sk} |sk\rangle}{\sqrt{P(LL)}}, \tag{7.26}$$

which is again an element of $\mathcal{H}_{AB}$. Furthermore,

$$P(EE) + P(LL) = 1, \tag{7.27}$$

for every $|\varphi\rangle \in \mathcal{H}_{\text{permitted}}$. Now we are ready to show that $\{\hat{X}\}_{i,\alpha}$ and $\{Y\}_{j,\beta}$ together with $\mathcal{H}_{\text{permitted}}$ yield normal quantum mechanical probability distributions

$$
\begin{aligned}
P(\alpha,\beta|i,j) &= \langle\varphi|\,\hat{X}_{i,\alpha}\hat{Y}_{j,\beta}\,|\varphi\rangle = \sum_{sklm}\varphi^E_{lm}\left(\varphi^E_{sk}\right)^*\langle sk|\,\widetilde{X}_{i,\alpha}\otimes\widetilde{Y}_{j,\beta}\,|lm\rangle \\
&+ \sum_{sklm}\varphi^L_{lm}\left(\varphi^L_{sk}\right)^*\langle sk|\,\widetilde{X}_{i,\alpha}\otimes\widetilde{Y}_{j,\beta}\,|lm\rangle \\
&= P(EE)\,\langle\varphi_{|EE}|\,\widetilde{X}_{i,\alpha}\otimes\widetilde{Y}_{j,\beta}\,|\varphi_{|EE}\rangle + P(LL)\,\langle\varphi_{|LL}|\,\widetilde{X}_{i,\alpha}\otimes\widetilde{Y}_{j,\beta}\,|\varphi_{|LL}\rangle \\
&= P(EE)P_{|EE}(\alpha,\beta|i,j) + P(LL)P_{|LL}(\alpha,\beta|i,j), \qquad (7.28)
\end{aligned}
$$

which holds for every $|\varphi\rangle \in \mathcal{H}_{\text{permitted}}$. The probability distributions $P_{|EE}(\alpha,\beta|i,j) = \langle\varphi_{|EE}|\,\widetilde{X}_{i,\alpha}\otimes\widetilde{Y}_{j,\beta}\,|\varphi_{|EE}\rangle$ and $P_{|LL}(\alpha,\beta|i,j) = \langle\varphi_{|LL}|\,\widetilde{X}_{i,\alpha}\otimes\widetilde{Y}_{j,\beta}\,|\varphi_{|LL}\rangle$ are both in Q. This together with equation (7.27) implies that $P(\alpha,\beta|i,j)$ is a convex combination of quantum mechanical probability distributions and therefore it is also an element of Q.

So far we have seen that our construction does not yield any correlations outside of the set of quantum correlations. However, we want to develop a better understanding why this is the case and we will see that we can formulate a necessary condition for two sets of operators and for a subspace to yield almost quantum correlations outside of the set of quantum correlations.

We start with the observation

$$
\begin{aligned}
\Pi_{EE,LL}\hat{X}_{i,\alpha} &= X_{i,\alpha}\otimes|E_{\bar{A}}E_{\bar{B}}\rangle\langle E_{\bar{A}}E_{\bar{B}}| + U^\dagger_{AB}X_{i,\alpha}U_{AB}\otimes|L_{\bar{A}}L_{\bar{B}}\rangle\langle L_{\bar{A}}L_{\bar{B}}| \\
&= \hat{X}_{i,\alpha}\Pi_{EE,LL}, \qquad (7.29)
\end{aligned}
$$

and thus, $\hat{X}_{i,\alpha}$ commutes with $\Pi_{EE,LL}$. This is equivalent to the statement that $\hat{X}_{i,\alpha}$ leaves $\mathcal{H}_{permitted}$ invariant, that is, $\hat{X}_{i,\alpha}\mathcal{H}_{permitted} \subset \mathcal{H}_{permitted}$ [28]. The same holds for $\hat{Y}_{j,\beta}$ i.e. $\hat{Y}_{j,\beta}\Pi_{EE,LL} = \Pi_{EE,LL}\hat{Y}_{j,\beta}$. The fact that the operators $\hat{X}_{i,\alpha}$ leave the subspace $\mathcal{H}_{\text{permitted}}$ invariant is already the reason why we only found normal quantum correlations. Before we formalize this in a more rigorous way, we will prove the following lemma, which will help us to prove the last claim.

Lemma 6. *Let $\mathcal{H}$ be a Hilbert space and $\widetilde{\mathcal{H}}$ a subspace of $\mathcal{H}$ i.e. $\widetilde{\mathcal{H}} \subseteq \mathcal{H}$. Let $X, Y \in \mathcal{B}(\mathcal{H})$ be two operators acting on $\mathcal{H}$. At least one of them, say X, leaves the subspace $\widetilde{\mathcal{H}}$ invariant i.e. $X\widetilde{\mathcal{H}} \subseteq \widetilde{\mathcal{H}}$. Moreover, let Π be the orthogonal projector on the subspace $\widetilde{\mathcal{H}}$.*
Furthermore, assume that $[X,Y]\,|\widetilde{\varphi}\rangle = 0$, $\forall\,|\widetilde{\varphi}\rangle \in \widetilde{\mathcal{H}}$. From these assumptions $[\Pi X\Pi^\dagger, \Pi Y\Pi^\dagger] = 0$ follows.

Proof:
In the following we will use the fact that, if for the operators $S, T \in \mathcal{B}(\mathcal{H})$ the relation $\langle\varphi|\,T\,|\varphi\rangle = \langle\varphi|\,S\,|\varphi\rangle$ holds $\forall\,|\varphi\rangle \in \mathcal{H}$, then $T = S$ [10], in the case of the 0-operator this condition reads

$$
\langle\varphi|\,M\,|\varphi\rangle = 0,\ \forall\,|\varphi\rangle \in \mathcal{H} \quad \Leftrightarrow \quad M = 0, \qquad (7.30)
$$

where $M \in \mathcal{B}(\mathcal{H})$.
For our proof we start with the observations that $\Pi\,|\widetilde{\varphi}\rangle = |\widetilde{\varphi}\rangle$, $\forall\,|\widetilde{\varphi}\rangle \in \widetilde{\mathcal{H}}$, and that $[X,Y]\,|\widetilde{\varphi}\rangle = 0$, $\forall\,|\widetilde{\varphi}\rangle \in \widetilde{\mathcal{H}}$ obviously implies $\langle\widetilde{\varphi}|\,[X,Y]\,|\widetilde{\varphi}\rangle = 0$, $\forall\,|\widetilde{\varphi}\rangle \in \widetilde{\mathcal{H}}$. From this we find

$$
\begin{aligned}
0 &= \langle\widetilde{\varphi}|\,[X,Y]\,|\widetilde{\varphi}\rangle = \langle\widetilde{\varphi}|\,\Pi^\dagger[X,Y]\Pi\,|\widetilde{\varphi}\rangle = \langle\widetilde{\varphi}|\,\Pi^2[X,Y]\Pi^2\,|\widetilde{\varphi}\rangle \\
&= \langle\widetilde{\varphi}|\,(\Pi^2XY\Pi^2 - \Pi^2YX\Pi^2)\,|\widetilde{\varphi}\rangle = \langle\widetilde{\varphi}|\,(\Pi X\Pi Y\Pi^2 - \Pi^2Y\Pi X\Pi)\,|\widetilde{\varphi}\rangle \\
&= \langle\widetilde{\varphi}|\,(\Pi X\Pi^\dagger\Pi Y\Pi^\dagger - \Pi Y\Pi^\dagger\Pi X\Pi^\dagger)\,|\widetilde{\varphi}\rangle = \langle\widetilde{\varphi}|\,(\bar{X}\bar{Y} - \bar{Y}\bar{X})\,|\widetilde{\varphi}\rangle \\
&= \langle\widetilde{\varphi}|\,[\bar{X},\bar{Y}]\,|\widetilde{\varphi}\rangle, \qquad (7.31)
\end{aligned}
$$

where we have defined $\bar{X} = \Pi X \Pi^\dagger$, $\bar{Y} = \Pi Y \Pi^\dagger$ and we have used that X leaves the subspace $\widetilde{\mathcal{H}}$ invariant, which is equivalent to $[X, \Pi] = 0$ [28]. We recognize that from equation (7.31) it follows that

$$\begin{aligned}
\langle\varphi| [\bar{X}, \bar{Y}] |\varphi\rangle &= (\langle\widetilde{\varphi}| + \langle\psi|)[\bar{X}, \bar{Y}](|\widetilde{\varphi}\rangle + |\psi\rangle) \\
&= \overbrace{\langle\widetilde{\varphi}| [\bar{X}, \bar{Y}] |\widetilde{\varphi}\rangle}^{=0} + \langle\widetilde{\varphi}| [\bar{X}, \bar{Y}] |\psi\rangle + \langle\psi| [\bar{X}, \bar{Y}] |\widetilde{\varphi}\rangle + \langle\psi| [\bar{X}, \bar{Y}] |\psi\rangle \\
&= \langle\widetilde{\varphi}| \Pi X \Pi Y \overbrace{\Pi |\psi\rangle}^{=0} - \langle\widetilde{\varphi}| \Pi Y \Pi X \overbrace{\Pi |\psi\rangle}^{=0} + \overbrace{\langle\psi| \Pi}^{=0} X \Pi Y \Pi |\widetilde{\varphi}\rangle \\
&- \overbrace{\langle\psi| \Pi}^{=0} Y \Pi X \Pi |\widetilde{\varphi}\rangle + \overbrace{\langle\psi| \Pi}^{=0} X \Pi Y \overbrace{\Pi |\psi\rangle}^{=0} - \overbrace{\langle\psi| \Pi}^{=0} Y \Pi X \overbrace{\Pi |\psi\rangle}^{=0} \\
&= 0,
\end{aligned} \tag{7.32}$$

$\forall \varphi \in \mathcal{H}$, where we have written $|\varphi\rangle$ as $|\varphi\rangle = |\widetilde{\varphi}\rangle + |\psi\rangle$, with $|\widetilde{\varphi}\rangle \in \widetilde{\mathcal{H}}$ and $|\psi\rangle \in \widetilde{\mathcal{H}}^\perp$. Finally, we notize that equation (7.32) implies

$$[\bar{X}, \bar{Y}] = 0. \qquad \square \tag{7.33}$$

Theorem 9. *Let $\mathcal{H}$ be a Hilbert space and let $\widetilde{\mathcal{H}} \subseteq \mathcal{H}$. Furthermore, let $\{X\}_{i,\alpha} \subset B(\mathcal{H})$ and $\{Y\}_{j,\beta} \subset \mathcal{B}(\mathcal{H})$ be two sets of protective measurement operators, which fulfill the properties described before in this section. Moreover, assume that at least on of the sets, say $\{X\}_{i,\alpha}$, leaves $\widetilde{\mathcal{H}}$ invariant.*
Then $[X_{i,\alpha}, Y_{j,\beta}] |\widetilde{\varphi}\rangle = 0$, $\forall i, \forall j, \forall \alpha, \forall \beta$ and $\forall |\widetilde{\varphi}\rangle \in \widetilde{\mathcal{H}}$ implies that all correlations obtained from $\{X\}_{i,\alpha}$,$\{Y\}_{j,\beta}$ and $\widetilde{\mathcal{H}}$ are within the set of quantum correlations Q.

Proof: From a calculation similar to that in (7.31) we find that

$$P(\alpha, \beta|i, j) = \langle\widetilde{\varphi}| X_{i\alpha} Y_{j,\beta} |\widetilde{\varphi}\rangle = \langle\widetilde{\varphi}| (\Pi X_{i\alpha} \Pi^\dagger)(\Pi Y_{j,\beta} \Pi^\dagger) |\widetilde{\varphi}\rangle, \tag{7.34}$$

where $|\widetilde{\varphi}\rangle \in \widetilde{\mathcal{H}}$ and Π is the orthogonal projector on $\widetilde{\mathcal{H}}$. Thus, the correlations, which are obtained from $\{X\}_{i,\alpha}$,$\{Y\}_{j,\beta}$ and $\widetilde{\mathcal{H}}$ are the same as the correlations obtained from $\{\Pi X \Pi^\dagger\}_{i,\alpha}$,$\{\Pi Y \Pi^\dagger\}_{j,\beta}$ and $\widetilde{\mathcal{H}}$. However, from lemma 6 we know that

$$[\Pi X_{i\alpha} \Pi^\dagger, \Pi Y_{j,\beta} \Pi^\dagger] = 0, \tag{7.35}$$

for all i, j, β and α. We have seen in section 5.2 that all correlations obtained from the approach of implementing local separation via commuting subalgebras can also be obtained by the approach of using a tensor product structure. This completes our proof that all correlations must be in Q. $\square$
The last theorem tells us that, if we want to implement almost quantum correlations, which are outside of the set of quantum correlations, it is necessary to construct both sets of measurement operators $\{X\}_{i,\alpha}$ and $\{Y\}_{j,\beta}$ in such a way that the operators of both sets map at least some of the states $|\widetilde{\varphi}\rangle \in \widetilde{\mathcal{H}}$ into $\mathcal{H} \backslash \widetilde{\mathcal{H}}$.

7.3 Measuring the Control-Qubit

In this section we introduce two additional control qubits. These control qubits are entangled with the early/late degrees of freedom of the subsytems respectively. One control qubit is responsible

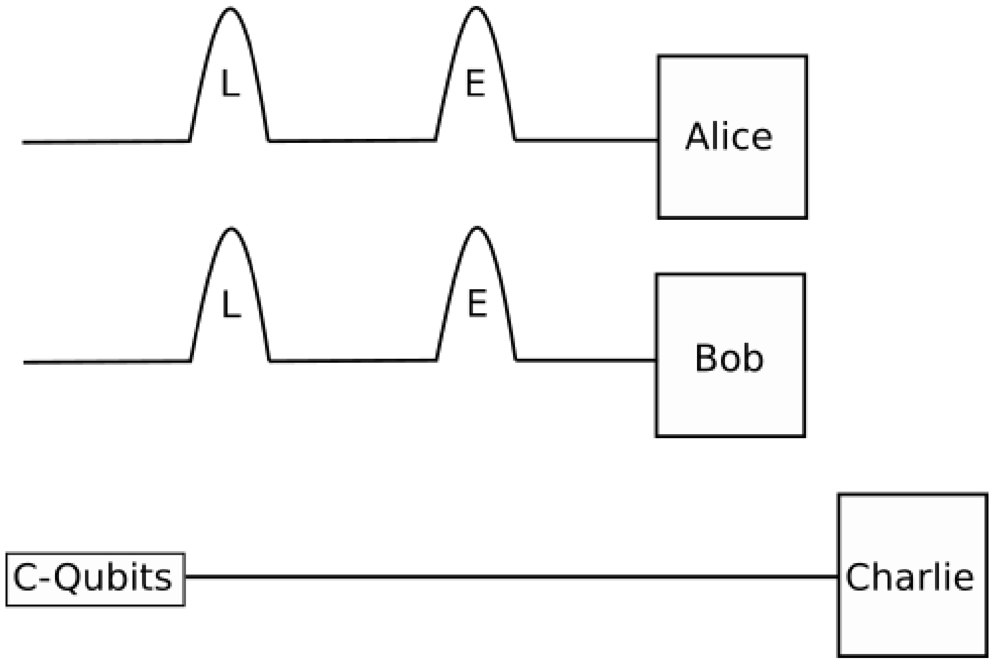

Figure 7.2: Illustration of updated experimental setup. Additionally to the previous setup a third party (Charlie) is introduced. Charlie can perform measurements on the control-qubits after Alice and Bob are finished with their measurements.

for the arrival time of Alice's particle and the other one is accountable for the arrival time of Bob's particle. To formalize this, we have to extend the Hilbert space $\mathcal{H}$ to

$$\bar{\mathcal{H}} = \mathcal{H} \otimes \mathbb{C}^2_{C_A} \otimes \mathbb{C}^2_{C_B} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathbb{C}^2_{\bar{A}} \otimes \mathbb{C}^2_{\bar{B}} \otimes \mathbb{C}^2_{C_A} \otimes \mathbb{C}^2_{C_B}, \tag{7.36}$$

where

$$\mathbb{C}_{C_X} = \mathsf{span}(|E_X\rangle, |L_X\rangle), \tag{7.37}$$

with $X \in A, B$. Furthermore, we restrict ourselves to wave functions $|\varphi\rangle$ of the form

$$|\varphi\rangle = \sum_{i=0}^{d_A-1} \sum_{j=0}^{d_B-1} \sum_{k,l \in \{E,L\}} \varphi_{ijkl} |ijklkl\rangle, \tag{7.38}$$

where $\varphi_{ijkl} \in \mathbb{C}$ with $\sum_{ijkl} |\varphi_{ijkl}|^2 = 1$ and $\{|i\rangle\}_{i=0}^{d_A-1}$ and $\{|j\rangle\}_{j=0}^{d_B-1}$ are ONB of $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. Moreover, we dropped the labels for $|E\rangle = |E_X\rangle$ and $|L\rangle = |L_X\rangle$, but instead we will always keep the same ordering.

The inclusion of control-qubits allows us to introduce a third party, say Charlie, who can perform a final measurement on the control-qubits after Alice and Bob are finished with their measurements. In the following two sections we will investigate two different scenarios, where Charlie performs a final measurement. (For an illustration of the setup see figure (7.2)) In both cases we will utilize postselection and we will scrutinize, if this leads to more general correlations than quantum-correlations.

7.3.1 Utilizing Postselection First Attempt

In our first setup we demand that the state $|\varphi\rangle \in \bar{\mathcal{H}}$ of the form (7.38), if reduced to the subsystem associated with $\mathcal{H}$, yields a pure state, which is element of $\mathcal{H}_{\text{permitted}}$ or a mixed state, which is element of $\mathcal{B}(\mathcal{H}_{\text{permitted}})$.
For this purpose we calculate the reduced state

$$\begin{aligned}\rho_r &= \mathrm{Tr}_{C_A C_B}\left(|\varphi\rangle\langle\varphi|\right) = \sum_{i,j\ldots} \varphi_{ijkl}\varphi^*_{abcd}\,|ijkl\rangle\langle abcd|\,\mathrm{Tr}(|kl\rangle\langle cd|)\\ &= \sum_{i,j\ldots} \varphi_{ijkl}\varphi^*_{abkl}\,|ijkl\rangle\langle abkl|\,, \end{aligned}\tag{7.39}$$

where Tr_{C_X} is the partial trace over $\mathbb{C}^2_{C_X}$. The condition $\rho_r \in \mathcal{B}(\mathcal{H}_{\text{permitted}})$ translates to

$$\Pi_{EE,LL}\rho_r\Pi_{EE,LL} = \rho_r \tag{7.40}$$

and therefore

$$\begin{aligned}\Pi_{EE,LL}\rho_r\Pi_{EE,LL} &= \sum_{i,j\ldots} \varphi_{ijEE}\varphi^*_{abEE}\,|ijEE\rangle\langle abEE|\\ &+ \sum_{i,j\ldots} \varphi_{ijLL}\varphi^*_{abLL}\,|ijLL\rangle\langle abLL|\\ &= \sum_{i,j\ldots} \varphi_{ijkk}\varphi^*_{abkk}\,|ijkk\rangle\langle abkk|\,. \end{aligned}\tag{7.41}$$

This implies

$$\varphi_{ijEL}\varphi^*_{abEL} = \varphi_{ijLE}\varphi^*_{abLE} = 0 \qquad \forall i,j,a,b, \tag{7.42}$$

therefore

$$|\varphi_{ijEL}| = |\varphi_{ijLE}| = 0 \qquad \forall i,j, \tag{7.43}$$

and thus

$$\varphi_{ijEL} = \varphi_{ijLE} = 0 \qquad \forall i,j. \tag{7.44}$$

We conclude that restriction (7.40) leads to wave functions of the form

$$|\varphi\rangle = \sum_{i,j} \varphi_{ijEE}\,|ijEEEE\rangle + \sum_{ij} \varphi_{ijLL}\,|ijLLLL\rangle\,. \tag{7.45}$$

Now, let us consider that Charlie holds the set of the following measurement operators

$$\Pi_1 = |EE\rangle\langle EE|\,, \tag{7.46}$$
$$\Pi_2 = |LL\rangle\langle LL|\,, \tag{7.47}$$
$$\Pi_3 = |LE\rangle\langle LE|\,, \tag{7.48}$$

and

$$\Pi_4 = |EL\rangle\langle EL|\,. \tag{7.49}$$

In the following we will only concentrate our discussion on Π_1 and Π_2, because Charlie's probabilities for the measurement outcomes associated with Π_3 and Π_4 vanish, if we only allow wave

functions of the form (7.45).
Furthermore, we can discuss Π_1 and Π_2 at the same time by writing

$$\Pi_m = |mm\rangle \langle mm| \,, \tag{7.50}$$

where $m \in \{E, L\}$.
As mentioned in the introduction to this section we want to explore what kind of correlations between Alice and Bob we are dealing with, if we postselect on Charlie's measurement outcomes. To this end we want to calculate the conditional state

$$\rho_{|m} = \frac{\mathrm{Tr}_{C_A C_B}((\mathbb{1} \otimes \Pi_m) \cdot \rho)}{\mathrm{Tr}((\mathbb{1} \otimes \Pi_m) \cdot \rho)}, \tag{7.51}$$

with $\rho = |\varphi\rangle \langle\varphi|$, where $|\varphi\rangle$ is of the form (7.45). First we notice

$$\langle mm| \, \Pi_m \, |mm\rangle = 1 \tag{7.52}$$

and

$$\langle ij| \, \Pi_m \, |kl\rangle = 0, \tag{7.53}$$

if $i \neq m$ or $j \neq m$ or $k \neq m$ or $l \neq m$. Utilizing this, we find

$$\begin{aligned} \mathrm{Tr}_{C_A C_B}((\mathbb{1} \otimes \Pi_m) \cdot \rho)) &= \sum_{i,j,a,b,k,c} \varphi_{ijkk} \varphi^*_{abcc} \, |ijkk\rangle \langle abcc| \cdot \langle kk| \, \Pi_m \, |cc\rangle \\ &= \sum_{i,j,a,b} \varphi_{ijmm} \varphi^*_{abmm} \, |ijmm\rangle \langle abmm| \end{aligned} \tag{7.54}$$

and

$$\mathrm{Tr}((\mathbb{1} \otimes \Pi_m) \cdot \rho)) = \sum_{i,j} \varphi_{ijmm} \varphi^*_{ijmm} = N, \tag{7.55}$$

where $N \in \mathbb{R}^+ \backslash \{0\}$. (The case $N = 0$ would be associated with the case, where Charlie never observes the outcome associated with Π_m. Then $\rho_{|m} = 0$ and all associated probabilities for Alice's and Bob's measurements would vanish. Therefore we exclude this case for our further discussion.)
From the last two equations we find

$$\begin{aligned} \rho_{|m} &= \sum_{i,j,a,b} \frac{\varphi_{ijmm}}{\sqrt{N}} \cdot \frac{\varphi^*_{abmm}}{\sqrt{N}} \, |ijmm\rangle \langle abmm| \\ &= \left(\sum_{i,j} \varphi^m_{ij} \, |ijmm\rangle \right) \left(\sum_{a,b} (\varphi)^*_{ab} \, \langle abmm| \right) = |\varphi^m\rangle \langle \varphi^m| \,, \end{aligned} \tag{7.56}$$

where $\varphi^m_{ij} = \varphi_{ijmm}/\sqrt{N}$ and $|\varphi^m\rangle = \sum_{i,j} \varphi^m_{ij} \, |ijmm\rangle \ \in \mathcal{H}_{\text{permitted}}$. We found that the conditional density matrix $\rho_{|m}$ describes a pure state $|\varphi^m\rangle \in \mathcal{H}_{\text{permitted}}$, but we have already seen that Alice's measurement operators $\{\hat{X}_{i,\alpha}\}$ and Bob's measurement operators $\{\hat{Y}_{j,\beta}\}$ together with a state, which is element of $\mathcal{H}_{\text{permitted}}$ produce normal quantum mechanical probability distributions and therefore we have not found any more general correlations.

7.3.2 Utilizing Postselection Second Attempt

In contrast to the last setup, we allow $|\varphi\rangle \in \bar{\mathcal{H}}$ to be an arbitrary wave function of the from (7.38). Moreover, we consider Charlie to perform a measurement described by two rank-two projectors $\Pi_1 \in \mathcal{B}(\mathbb{C}_A \otimes \mathbb{C}_B)$ and $\Pi_2 \in \mathcal{B}(\mathbb{C}_A \otimes \mathbb{C}_B)$, which are given by

$$\Pi_1 = |EE\rangle \langle EE| + |LL\rangle \langle LL| \tag{7.57}$$

and

$$\Pi_2 = |EL\rangle \langle EL| + |LE\rangle \langle LE| . \tag{7.58}$$

The situation can be interpreted as follows. We do not include the restriction of Alice and Bob being space-like separated in the wave function $|\varphi\rangle$. Rather, we allow wave functions which describe situations where Alice and Bob are space-like separated, time-like separated or in some kind of "superposition" of being space-like and time-like separated. After Alice and Bob are done with their measurements, we learn from Charlie's subsequent measurement, if Alice and Bob have been space-like separated, which is associated with Π_1, or time-like separated, which is associated with Π_2. At this point we should notice that even for cases, where $|\varphi\rangle$ includes non-zero terms of the form $\varphi_{ijEL} |ijELEL\rangle$ or $\varphi_{ijLE} |ijLELE\rangle$, but not only terms of this form, the probability for Charlie's measurement outcome, which is associated with Π_1 does not vanish. This implies that even for such wave functions, Charlie observes from time to time, that Alice and Bob have been space-like separated.

This suggests that we concentrate on the statistics for Alice's and Bob's measurements, for the cases where Charlie's measurement outcome is associated with Π_1 i.e. we postselect on Π_1.

To achieve this, we start by looking at the density matrix ρ associated with the wave function $|\varphi\rangle$. As usual this density matrix is defined as

$$\rho = |\varphi\rangle \langle \varphi| . \tag{7.59}$$

Now we condition the density matrix ρ on Π_1, which gives us the conditional densitiy matrix

$$\rho_{|1} = \frac{\mathrm{Tr}_{C_A C_B}((\mathbb{1} \otimes \Pi_1) \cdot \rho)}{\mathrm{Tr}((\mathbb{1} \otimes \Pi_1) \cdot \rho)}, \tag{7.60}$$

where $\mathrm{Tr}_{C_A C_B}$ is the partial trace over the two control-systems C_A and C_B. Before we actually calculate $\rho_{|1}$ we notice that

$$\langle ii| \Pi_1 |ii\rangle = 1 \tag{7.61}$$

for $i \in \{E, L\}$ and

$$\langle ij| \Pi_1 |kl\rangle = 0, \tag{7.62}$$

unless $i = j = k = l$, where $i, j, k, l \in \{E, L\}$. Using this, it is easy to calculate

$$\begin{aligned}
\mathrm{Tr}_{C_A C_B}((\mathbb{1} \otimes \Pi_1) \cdot \rho) &= \mathrm{Tr}_{C_A C_B} \left((\mathbb{1} \otimes \Pi_1) \cdot \sum_{ij\ldots} \varphi_{ijkl} \varphi^*_{abcd} |ijklkl\rangle \langle abcdcd| \right) \\
&= \sum_{ij\ldots} \varphi_{ijkl} \varphi^*_{abcd} |ijkl\rangle \langle abcd| \langle cd| \Pi_1 |kl\rangle \\
&= \sum_{ij\ldots} \varphi_{ijkk} \varphi^*_{abkk} |ijkk\rangle \langle abkk|
\end{aligned} \tag{7.63}$$

and

$$\text{Tr}((\mathbb{1} \otimes \Pi_1) \cdot \rho) = \sum_{ij\ldots} \varphi_{ijkk}\varphi^*_{ijkk} = N, \tag{7.64}$$

where $N \in \mathbb{R}^+ \backslash \{0\}$. (We have excluded $N = 0$ for similar reasons as in setup 1)
From Equation (7.63) and Equation (7.64) we find

$$\rho_{|1} = \sum_{ij\ldots} \frac{\varphi_{ijkk}\varphi^*_{abkk}}{N} |ijkk\rangle \langle abkk| = \sum_{ij\ldots} \Psi^k_{ijab} |ijkk\rangle \langle abkk| \,, \tag{7.65}$$

where we have defined $\Psi^k_{ijab} = (\varphi_{ijkk}\varphi^*_{abkk})/(N)$.
Let us halt the discussion of our setup for a moment, to make some general considerations which will be useful when we proceed with our discussion.
Let us introduce an arbitrary density matrix $\varrho \in \mathcal{B}(\hat{\mathcal{H}})$ and two sets of protective measurement operators $\{A_{i\alpha}\} \subset \mathcal{B}(\hat{\mathcal{H}})$ and $\{B_{j\beta}\} \subset \mathcal{B}(\hat{\mathcal{H}})$, with $\sum_\alpha A_{i\alpha} = \sum_\beta B_{j\beta} = \mathbb{1} \;\forall i,j$. We denote the probability to get outcome α given the setting i and the state ϱ by

$$P_\varrho(\alpha|i) = \text{Tr}(A_{i\alpha}\varrho). \tag{7.66}$$

After a measurement where we have chosen setting i and received outcome α we find

$$\varrho_{i\alpha} = \frac{A_{i\alpha}\varrho A_{i\alpha}}{\text{Tr}(A_{i\alpha}\varrho)} = \frac{A_{i\alpha}\varrho A_{i\alpha}}{P_\varrho(\alpha|i)} \tag{7.67}$$

being the updated state [11]. For a subsequent measurement we find that the probability for getting first outcome α and then outcome β given the settings i and j respectively, is given by

$$P_{\text{A before B}}(\alpha,\beta|i,j) = P_\varrho(\alpha|i)P_{\varrho_{i\alpha}}(\beta|j) = \text{Tr}(B_{j\beta}A_{i\alpha}\varrho A_{i\alpha}) = \text{Tr}(A_{i\alpha}B_{j\beta}A_{i\alpha}\varrho). \tag{7.68}$$

And similarly we find for the case where the ordering of the two measurements is interchanged

$$P_{\text{B before A}}(\alpha,\beta|i,j) = \text{Tr}(B_{j\beta}A_{i\alpha}B_{j\beta}\varrho). \tag{7.69}$$

Now, we will assume that the following conditions hold

$$[A_{i,\alpha}, B_{j,\beta}]\varrho = 0 \qquad \forall i,j,\alpha,\beta. \tag{7.70}$$

It follows that

$$\begin{aligned} P_{\text{A before B}}(\alpha,\beta|i,j) &= \text{Tr}(A_{i\alpha}B_{j\beta}A_{i\alpha}\varrho) = \text{Tr}(A_{i\alpha}A_{i\alpha}B_{j\beta}\varrho) = \text{Tr}(A_{i\alpha}B_{j\beta}\varrho) \\ &= \text{Tr}(B_{j\beta}A_{i\alpha}\varrho) = \text{Tr}(B_{j\beta}B_{j\beta}A_{i\alpha}\varrho) = \text{Tr}(B_{j\beta}A_{i\alpha}B_{j\beta}\varrho) \\ &= P_{\text{B before A}}(\alpha,\beta|i,j), \end{aligned} \tag{7.71}$$

and we can write

$$P(\alpha,\beta|i,j) = P_{\text{A before B}}(\alpha,\beta|i,j) = P_{\text{B before A}}(\alpha,\beta|i,j) = \text{Tr}(A_{i\alpha}B_{j\beta}\varrho). \tag{7.72}$$

The last result states that the ordering of the two measurements does not influence the statistics of the observed outcomes. Furthermore, it is easy to check that $P(\alpha,\beta|i,j)$ is non-signaling i.e.

$$\begin{aligned} P(\alpha|i,j) &= \sum_\beta P(\alpha,\beta|i,j) = \sum_\beta \text{Tr}(A_{i\alpha}B_{j\beta}\varrho) = \text{Tr}\left(A_{i\alpha}\left(\sum_\beta B_{j\beta}\right)\varrho\right) \\ &= \text{Tr}\left(A_{i\alpha}\varrho\right) = P(\alpha|i), \end{aligned} \tag{7.73}$$

and similarly

$$P(\beta|i,j)P(\beta|j). \tag{7.74}$$

Altogether, if the sets $\{A_{i,\alpha}\}, \{B_{j,\beta}\}$ and the density matrix ϱ obey the conditions (7.70), we can consider the setup compatible with a space-time configuration, where the two measurements associated with $\{A_{i,\alpha}\}$ and $\{B_{j,\beta}\}$ are space-like separated and produce non-signaling statistics.
Now, let us get back to our experimental setup. In the following we show that $\{\hat{X}_{i,\alpha}\}, \{\hat{Y}_{j,\beta}\}$ and $\rho_{|1}$ obey condition (7.70). For this purpose we recall that $\hat{X}_{i,\alpha}$ and $\hat{Y}_{j,\beta}$ are all projectors $\forall i,j,\alpha,\beta$. Then, we calculate

$$\begin{aligned}
\hat{X}_{i,\alpha}\hat{Y}_{j,\beta}\rho_{|1} &= \Big(\tilde{X}_{i,\alpha}\otimes \mathbb{1}\otimes |E\rangle\langle E|\otimes \mathbb{1} + U_{AB}^{\dagger}(\tilde{X}_{i,\alpha}\otimes \mathbb{1})U_{AB}\otimes |L\rangle\langle L|\otimes \mathbb{1}\Big)\\
&\cdot \Big(\mathbb{1}\otimes \tilde{Y}_{j,\beta}\otimes \mathbb{1}\otimes |E\rangle\langle E| + U_{AB}^{\dagger}(\mathbb{1}\otimes \tilde{Y}_{j,\beta})U_{AB}\otimes \mathbb{1}\otimes |L\rangle\langle L|\Big)\\
&\cdot \left(\sum_{i,j,a,b}\Psi^{E}_{ijab}\,|ijEE\rangle\langle abEE| + \sum_{i,j,a,b}\Psi^{L}_{ijab}\,|ijLL\rangle\langle abLL|\right)\\
&= \sum_{i,j,a,b}\Psi^{E}_{ijab}((\tilde{X}_{i,\alpha}\otimes\tilde{Y}_{j,\beta})\cdot|ij\rangle\langle ab|)\otimes|EE\rangle\langle EE|\\
&+ \sum_{i,j,a,b}\Psi^{L}_{ijab}(U_{AB}^{\dagger}(\tilde{X}_{i,\alpha}\otimes\tilde{Y}_{j,\beta})U_{AB}\cdot|ij\rangle\langle ab|)\otimes|LL\rangle\langle LL|\\
&= \Big(\mathbb{1}\otimes \tilde{Y}_{j,\beta}\otimes \mathbb{1}\otimes |E\rangle\langle E| + U_{AB}^{\dagger}(\mathbb{1}\otimes \tilde{Y}_{j,\beta})U_{AB}\otimes \mathbb{1}\otimes |L\rangle\langle L|\Big)\\
&\cdot \Big(\tilde{X}_{i,\alpha}\otimes \mathbb{1}\otimes |E\rangle\langle E|\otimes \mathbb{1} + U_{AB}^{\dagger}(\tilde{X}_{i,\alpha}\otimes \mathbb{1})U_{AB}\otimes |L\rangle\langle L|\otimes \mathbb{1}\Big)\\
&\cdot \left(\sum_{i,j,a,b}\Psi^{E}_{ijab}\,|ijEE\rangle\langle abEE| + \sum_{i,j,a,b}\Psi^{L}_{ijab}\,|ijLL\rangle\langle abLL|\right)\\
&= \hat{Y}_{j,\beta}\hat{X}_{i,\alpha}\rho_{|1},
\end{aligned} \tag{7.75}$$

which is equivalent to

$$[\hat{X}_{i,\alpha},\hat{Y}_{j,\beta}]\rho_{|1} = 0. \tag{7.76}$$

Eventually, for the on Π_1 postselected probability distributions we can write

$$P^1(\alpha,\beta|i,j) = \mathsf{Tr}(\hat{X}_{i,\alpha}\hat{Y}_{j,\beta}\rho_{|1}), \tag{7.77}$$

and we know from Equation (7.76) that $P^1(\alpha,\beta|i,j)$ is non-signaling.
Naturally the question arises, if $P^1(\alpha,\beta|i,j)$ produces more general correlations than standard quantum ones?
To this end, we will look at the restriction of $\rho_{|1}\in\mathcal{B}(\mathcal{H})$ to $\bar{\rho}_{|1}\in\mathcal{B}(\mathcal{H}_{\text{permitted}})$. Thus, we calculate

$$\begin{aligned}
\bar{\rho}_{|1} &= \Pi_{EE,LL}\rho_{|1}\Pi_{EE,LL} = (\mathbb{1}\otimes(|EE\rangle\langle EE| + |LL\rangle\langle LL|))\\
&\cdot \left(\sum_{i,j,a,b}\Psi^{E}_{ijab}\,|ijEE\rangle\langle abEE| + \sum_{i,j,a,b}\Psi^{L}_{ijab}\,|ijLL\rangle\langle abLL|\right)\\
&\cdot (\mathbb{1}\otimes(|EE\rangle\langle EE| + |LL\rangle\langle LL|))\\
&= \sum_{i,j,a,b}\Psi^{E}_{ijab}\,|ijEE\rangle\langle abEE| + \sum_{i,j,a,b}\Psi^{L}_{ijab}\,|ijLL\rangle\langle abLL|\\
&= \rho_{|1},
\end{aligned} \tag{7.78}$$

therefore we can consider $\rho_{|1}$ as an operator acting on the Hilbert space $\mathcal{H}_{\text{permitted}}$ i.e. $\rho_{|1} \in \mathcal{B}(\mathcal{H}_{\text{permitted}}) \subset \mathcal{B}(\mathcal{H})$. This already suggests that $P^1(\alpha, \beta|i, j)$ produces only normal quantum mechanical correlations, which is easy to check. First, we calculate

$$\begin{aligned} P^1(\alpha,\beta|i,j) &= \mathsf{Tr}(\hat{X}_{i,\alpha}\hat{Y}_{j,\beta}\rho_{|1}) = \mathsf{Tr}(\hat{X}_{i,\alpha}\hat{Y}_{j,\beta}\Pi_{EE,LL}\rho_{|1}\Pi_{EE,LL}) \\ &= \mathsf{Tr}(\Pi_{EE,LL}\hat{X}_{i,\alpha}\hat{Y}_{j,\beta}\Pi_{EE,LL}\rho_{|1}) \\ &= \mathsf{Tr}(\Pi_{EE,LL}\hat{X}_{i,\alpha}\Pi_{EE,LL}\Pi_{EE,LL}\hat{Y}_{j,\beta}\Pi_{EE,LL}\rho_{|1}) \\ &= \mathsf{Tr}(\bar{X}_{i,\alpha}\bar{Y}_{j,\beta}\rho_{|1}), \end{aligned} \tag{7.79}$$

where $\bar{X}_{i,\alpha} = \Pi_{EE,LL}\hat{X}_{i,\alpha}\Pi^{\dagger}_{EE,LL}$, $\bar{Y}_{j,\beta} = \Pi_{EE,LL}\hat{Y}_{j,\beta}\Pi^{\dagger}_{EE,LL}$ and we have used that $[\hat{X}_{i,\alpha}, \Pi_{EE,LL}] = 0$. Furthermore, we recall that $[\bar{X}_{i,\alpha}, \bar{Y}_{j,\beta}] = 0$, which completes our proof that $P^1(\alpha, \beta|i, j)$ is a quantum mechanical probability distribution. $\square$

Let us interpret $\rho_{|1}$. In our original setup, where only two parties Alice and Bob were included, we restrained ourselves to wave functions $|\varphi\rangle$ living in the Hilbert $\mathcal{H}_{\text{permitted}}$. Such wave functions are only able to describe pure states. In contrast, in our recent setup we ended up with the operator $\rho_{|1} \in \mathcal{B}(\mathcal{H}_{\text{permitted}})$. In general $\rho_{|1}$ will describe a mixed state, which can be decomposed into pure states i.e.

$$\sum_i \lambda_i \left|\varphi_i\right\rangle \left\langle\varphi_i\right|, \tag{7.80}$$

where $0 \le \lambda_i \le 1$, $\sum_i \lambda_i = 1$ and $|\varphi_i\rangle \in \mathcal{H}_{\text{permitted}}$[40].

Finally, we draw the conclusion that we might not have been careful enough of how to implement that Charlie is the last one to measure. If we consider that Charlie measures first, then Alice and Bob would still end up with the conditional state (7.65). This can be traced back to the fact that there is a tensor product structure on the Hilbert space and Charlie only acts on $\mathbb{C}^2_{C_A} \otimes \mathbb{C}^2_{B}$, but this is a subspace on which Alice and Bob do not act. Therefore, we can consider Charlie as locally separated.

In the interpretation that Charlie is the first one to measure, he destroys the superposition of temporal order completely in the first attempt (section (7.3.1)), and in the second attempt (this section) Charlie's measurement influences the setup in such a way, that after the postselection we end up in a similar scenario as in section (7.2).

7.4 Measuring the Control-Qubit (Revisited)

In this section we will be more careful of how to implement that Charlie is the last one who measures. To this end, we discard the entangled control qubits and rather consider the systems associated to $\mathbb{C}^2_{\bar{A}}$ and $\mathbb{C}^2_{\bar{B}}$ themselves as control qubits. Furthermore, we assume that Charlie acts on $\mathbb{C}^2_{\bar{A}} \otimes \mathbb{C}^2_{\bar{B}}$, but this is a subspace on which Alice and Bob act as well. Again we will post-select our outcomes on Charlie's outcomes.

7.4.1 Utilizing Postselection Third Attempt

In the first setup in this section we allow our total system to be initially in a state described by a general wave function $|\varphi\rangle \in \mathcal{H}$. Furthermore we want to build a meaningful POVM out of the operators $\widetilde{X}_{i,\alpha}$, $\widetilde{Y}_{j,\beta}$ and U_{AB}. Including U_{AB} into the POVM means that we are working again

in the Heisenberg picture. Again we will assume that $\widetilde{X}_{i,\alpha}$ and $\widetilde{Y}_{j,\beta}$ are projectors.
Recall that we have assumed, that for wave functions $|\widetilde{\varphi}\rangle \otimes |EE\rangle$ and $|\widetilde{\varphi}\rangle \otimes |LL\rangle$ Alice and Bob are space-like separated and furthermore, that for wave-functions of the form $|\hat{\varphi}\rangle \otimes |EL\rangle$ Alice measures before Bob and for wave-functions of the form $|\hat{\varphi}\rangle \otimes |LE\rangle$ Bob measures before Alice. This motivates the definition of the following POVM elements

$$\begin{aligned} E_{i\alpha,j\beta} &= (\widetilde{X}_{i,\alpha} \otimes \widetilde{Y}_{j,\beta}) \otimes |EE\rangle\langle EE| + U_{AB}^{\dagger}(\widetilde{X}_{i,\alpha} \otimes \widetilde{Y}_{j,\beta})U_{AB} \otimes |LL\rangle\langle LL| \\ &+ (\widetilde{X}_{i,\alpha} \otimes \mathbb{1})U_{AB}^{\dagger}(\mathbb{1} \otimes \widetilde{Y}_{j,\beta})U_{AB}(\widetilde{X}_{i,\alpha} \otimes \mathbb{1}) \otimes |EL\rangle\langle EL| \\ &+ (\mathbb{1} \otimes \widetilde{Y}_{j,\beta})U_{AB}^{\dagger}(\widetilde{X}_{i,\alpha} \otimes \mathbb{1})U_{AB}(\mathbb{1} \otimes \widetilde{Y}_{j,\beta}) \otimes |LE\rangle\langle LE|. \end{aligned} \tag{7.81}$$

We are not only interested in the statistics between Alice and Bob but also in the final state after their measurement. To find the final state we use the fact that every POVM element is the product of the adjoint of an operator with itself, that is

$$E_{i\alpha,j\beta} = M_{i\alpha,j\beta}^{\dagger} M_{i\alpha,j\beta} \tag{7.82}$$

and then

$$\frac{M_{i\alpha,j\beta}|\varphi\rangle}{\sqrt{\langle\varphi| M_{i\alpha,j\beta}^{\dagger} M_{i\alpha,j\beta}|\varphi\rangle}} = \frac{M_{i\alpha,j\beta}|\varphi\rangle}{\sqrt{E_{i\alpha,j\beta}}} = \frac{M_{i\alpha,j\beta}|\varphi\rangle}{\sqrt{P(\alpha,\beta|i,j)}} \tag{7.83}$$

is the state after the measurement associated with $E_{i\alpha,j\beta}$ [11]. To describe our scenario, it makes sense to choose

$$\begin{aligned} M_{i\alpha,j\beta} &= (\widetilde{X}_{i,\alpha} \otimes \widetilde{Y}_{j,\beta}) \otimes |EE\rangle\langle EE| + (\widetilde{X}_{i,\alpha} \otimes \widetilde{Y}_{j,\beta})U_{AB} \otimes |LL\rangle\langle LL| \\ &+ (\mathbb{1} \otimes \widetilde{Y}_{j,\beta})U_{AB}(\widetilde{X}_{i,\alpha} \otimes \mathbb{1}) \otimes |EL\rangle\langle EL| \\ &+ (\widetilde{X}_{i,\alpha} \otimes \mathbb{1})U_{AB}(\mathbb{1} \otimes \widetilde{Y}_{j,\beta}) \otimes |LE\rangle\langle LE|. \end{aligned} \tag{7.84}$$

We denote the final state as

$$|\varphi\rangle_{\text{final}} = \frac{M_{i\alpha,j\beta}|\varphi\rangle}{\sqrt{(P(\alpha,\beta|i,j))}}, \tag{7.85}$$

and we write the initial state in the following way

$$|\varphi\rangle = \sum_{ab} \sum_{s,l\in\{E,L\}} \varphi_{absl}|absl\rangle, \tag{7.86}$$

where $\sum_{absl}|\varphi_{absl}|^2 = 1$ and $\{|a\rangle\}_a$ and $\{|b\rangle\}_b$ are ONBs of $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively.
After Alice's and Bob's measurement, Charlie performs his measurement using the following operators

$$\Pi_1 = \mathbb{1} \otimes (|EE\rangle\langle EE| + |LL\rangle\langle LL|) \tag{7.87}$$

and

$$\Pi_2 = \mathbb{1} \otimes (|EL\rangle\langle EL| + |LE\rangle\langle LE|). \tag{7.88}$$

We want to post-select Alice's and Bob's probability distributions on the measurement outcome 1 of Charlie's measurement, which is associated with Π_1. To this end, we start by calculating the probability distribution that Alice measures α Bob β and Charlie 1, given the settings i,j:

$$\begin{aligned}
P(\alpha,\beta,1|i,j) &= P_{|\varphi\rangle}(\alpha,\beta|i,j)P_{|\varphi\rangle_{\text{final}}}(1) = P_{|\varphi\rangle}(\alpha,\beta|i,j)\frac{\langle\varphi| M^{\dagger}_{i\alpha,j\beta}\Pi_1 M_{i\alpha,j\beta}|\varphi\rangle}{P_{|\varphi\rangle}(\alpha,\beta|i,j)} \\
&= \sum_{a,b,s,l}\varphi_{abEE}\varphi^*_{slEE}\langle sl|(\widetilde{X}_{i,\alpha}\otimes\widetilde{Y}_{j,\beta})|ab\rangle \\
&+ \sum_{a,b,s,l}\varphi_{abLL}\varphi^*_{slLL}\langle sl|U^{\dagger}_{AB}(\widetilde{X}_{i,\alpha}\otimes\widetilde{Y}_{j,\beta})U_{AB}|ab\rangle\,. \qquad (7.89)
\end{aligned}$$

The on outcome 1 post-selected probability distribution is then given by

$$P(\alpha,\beta|i,j,1) = \frac{P(\alpha,\beta,1|i,j)}{P(1|i,j)}. \qquad (7.90)$$

Thus, we calculate

$$\begin{aligned}
P(1|i,j) &= \sum_{\alpha,\beta}P(\alpha,\beta,1|i,j) = \sum_{a,b,s,l}\varphi_{abEE}\varphi^*_{slEE}\langle sl|\overbrace{\sum_{\alpha,\beta}(\widetilde{X}_{i,\alpha}\otimes\widetilde{Y}_{j,\beta})}^{=\mathbb{1}}|ab\rangle \\
&+ \sum_{a,b,s,l}\varphi_{abLL}\varphi^*_{slLL}\langle\varphi_{sl}|\overbrace{\sum_{\alpha,\beta}U^{\dagger}_{AB}(\widetilde{X}_{i,\alpha}\otimes\widetilde{Y}_{j,\beta})U_{AB}}^{\mathbb{1}}|\varphi_{ab}\rangle \\
&= \sum_{ab}\sum_{k\in\{E,L\}}|\varphi_{abkk}|^2 = N, \qquad (7.91)
\end{aligned}$$

where $N \in \mathbb{R}^+\backslash\{0\}$. We excluded the case $N = 0$ for similar reasons as before. Plugging equation (7.89) and (7.91) into equation (7.90) gives us

$$\begin{aligned}
P(\alpha,\beta|i,j,1) &= \sum_{a,b,s,l}\frac{\varphi_{abEE}\varphi^*_{slEE}}{N}\langle sl|(\widetilde{X}_{i,\alpha}\otimes\widetilde{Y}_{j,\beta})|ab\rangle \\
&+ \sum_{a,b,s,l}\frac{\varphi_{abLL}\varphi^*_{slLL}}{N}\langle sl|U^{\dagger}_{AB}(\widetilde{X}_{i,\alpha}\otimes\widetilde{Y}_{j,\beta})U_{AB}|ab\rangle\,, \qquad (7.92)
\end{aligned}$$

but these are the same probabilities as produced by the mixed state

$$\rho = \sum_{a,b,s,l}\frac{\varphi_{abEE}\varphi^*_{slEE}}{N}|ab\rangle\langle sl| + \sum_{a,b,s,l}\frac{\varphi_{abLL}\varphi^*_{slLL}}{N}|ab\rangle\langle sl|\,, \qquad (7.93)$$

together with Alice's measurement operators $\widetilde{X}_{i,\alpha}\otimes\mathbb{1}$ and Bob's measurement operators $\mathbb{1}\otimes\widetilde{Y}_{j,\beta}$. To see that ρ is really a quantum mechanical state, we will rewrite it as a convex combination of two pure quantum states. For this purpose we notice that

$$\sum_{ab}\varphi_{abEE}|ab\rangle \qquad (7.94)$$

is a sub-normalized state. To find the corresponding proper normalized state we just have to normalize (7.94) and we find

$$|\varphi_{EE}\rangle = \sum_{ab}\frac{\varphi_{abEE}}{\sqrt{N_1}}|ab\rangle \qquad (7.95)$$

where $N_1 = \sum_{ab} |\varphi_{abEE}|^2 \in \mathbb{R}^+\backslash\{0\}$. (For $N_1 = 0$ ρ would already be a pure-state, which will be clear in a moment.)
Similarly, we get

$$|\varphi_{LL}\rangle = \sum_{ab} \frac{\varphi_{abLL}}{\sqrt{N_2}} |ab\rangle , \tag{7.96}$$

where $N_2 = \sum_{ab} |\varphi_{abLL}|^2 \in \mathbb{R}^+\backslash\{0\}$.(Again, for $N_2 = 0$ the density matrix ρ would define a pure-state.) Now we rewrite

$$\rho = \frac{N_1}{N} |\varphi_{EE}\rangle \langle\varphi_{EE}| + \frac{N_2}{N} U_{AB} |\varphi_{LL}\rangle \langle\varphi_{LL}| U_{AB}^{\dagger} \tag{7.97}$$

and we notice that $N_1/N > 0$, $N_2/N > 0$ and $N = N_1 + N_2$. Therefore

$$\frac{N_1}{N} + \frac{N_2}{N} = \frac{N_1 + N_2}{N_1 + N_2} = 1, \tag{7.98}$$

which completes the proof that ρ can be considered as a convex combination of the two pure-states $|\varphi_{EE}\rangle$ and $U_{AB} |\varphi_{LL}\rangle$.
Again, we want to understand why we have not found anything new. For this reason we point out that

$$[\Pi_1, E_{i\alpha,j\beta}] = [\Pi_1, M_{i\alpha,j\beta}] = [\Pi_2, E_{i\alpha,j\beta}] = [\Pi_2, M_{i\alpha,j\beta}] = 0, \tag{7.99}$$

holds. Hence, all of Charlie's measurement operators commute with all measurement operators that describe Alice's and Bob's measurements. Like before, this implies that alternatively we could also describe the situation in this section by saying that Charlie measures first and all the probabilites would stay the same. In this interpretation Alice and Bob would receive the conditional state

$$|\varphi_{|1}\rangle = \frac{\Pi_1 |\varphi\rangle}{\sqrt{\langle\varphi| \Pi_1 |\varphi\rangle}} = \sum_{ij} \sum_{k\in\{E,L\}} \frac{\varphi_{ijkk}}{\sqrt{N}} |ijkk\rangle , \tag{7.100}$$

where $N = \sum_{ij} \sum_{k\in\{E,L\}} \varphi_{ijkk}\varphi^*_{ijkk}$. However, $|\varphi_{|1}\rangle$ is an element of $\mathcal{H}_{\text{permitted}}$ and therefore we ended up in a similar situation as in section 7.2 .
Also other postselection scenarios lead to convex combinations of standard quantum correlations.

7.5 Conclusion

Even though the initial setup in section 7.2 seemed like it had good chances of yielding almost quantum correlations, which are more general than quantum correlations, we ended up with mere quantum correlations. Still we were capable of detecting what went wrong and this led to theorem 9. From this theorem we learned that it is necessary to construct the sets of measurement operators in such a way, that they do not leave $\mathcal{H}_{\text{permitted}}$ invariant, if we want to find correlations in the set $\widetilde{Q}\backslash Q$.
We adapted the initial set up by including a third party, which is capable of performing a final measurement. Even after looking at different scenarios, where we used post selection, we always ended up with standard quantum mechanical correlations. Again, we saw that even after postselection, the main problem was that Alice's and Bob's measurement operators leave $\mathcal{H}_{\text{permitted}}$ invariant. However, it is hard to imagine that it is possible to construct physically meaningful measurement operators that map wave functions living in the subspace $\mathcal{H}_{\text{permitted}}$ into $\mathcal{H}\backslash\mathcal{H}_{\text{permitted}}$.

Bibliography

[1] Miguel Navascues and David Perez-Garcia. Quantum steering and space-like separation. *arXiv:1204.6220*, 2012.

[2] Hans Halvorson and Michael Mueger. Algebraic quantum field theory. *arXiv:math-ph/0602036*, 2006.

[3] Miguel Navascués, Yelena Guryanova, Matty J. Hoban, and Antonio Acín. Almost quantum correlations. *arXiv:1403.4621v1*, 2014.

[4] Ognyan Oreshkov, Fabio Costa, and Caslav Brukner. Quantum correlations with no causal order. *arXiv:1105.4464*, 2013.

[5] V. B. Scholz and R. F. Werner. Tsirelson's problem. *arXiv:0812.4305v1*, 2008.

[6] Giulia Rubino, Lee A. Rozema, Adrien Feix, Mateus Araújo, Jonas M. Zeuner, M. Procopio, Lorenzo, Časlav Brukner, and Philip Walther. Experimental verification of an indefinite causal order. *arXiv:1608.01683v2*, 2016.

[7] Wolfgang Nolting. *Grundkurs Theoretische Physik : 5/1 : Quantenmechanik - Grundlagen*. Springer Spektrum, Berlin [u.a.], 8. aufl.. edition, 2013.

[8] Hans Kerner. *Mathematik für Physiker*. Springer-Lehrbuch. Springer Berlin Heidelberg, Berlin, Heidelberg, 3., überarb. aufl. 2013. edition, 2013.

[9] Christian B Lang. *Mathematische Methoden in der Physik*. Lehrbuch. Springer Spektrum, Berlin [u.a.], 2. aufl.. edition, 2005.

[10] Teiko Heinosaari and Mario Ziman. *The Mathematical Language of Quantum Theory : From Uncertainty to Entanglement*. Cambridge Univ. Pr., Cambridge, 1. publ.. edition, 2012.

[11] Michael A. Nielsen and Isaac L. Chunang. *Quantum computation and quantum information*. 10th anniversary ed.. edition, 2010.

[12] Mateus Araújo, Cyril Branciard, Fabio Costa, Adrien Feix, Christina Giarmatzi, and Časlav Brukner. Witnessing causal nonseparability. *New Journal of Physics*, 17(10):102001, oct 2015.

[13] Charles H. Bennett, David P. DiVincenzo, John A. Smolin, and William K. Wootters. Mixed-state entanglement and quantum error correction. *Phys. Rev. A*, 54:3824–3851, Nov 1996.

[14] Choi and Man-Duen. Completely positive linear maps on complex matrices. *Linear Algebra and Its Applications*, 10(3):285–290, 1975.

[15] Ognyan Oreshkov and Christina Giarmatzi. Causal and causally separable processes. *arXiv:1506.05449v3*, 2016.

[16] Lucien Hardy. Probability theories with dynamic causal structure: A new framework for quantum gravity. *arXiv:gr-qc/0509120v1*, 2005.

[17] G. Chiribella, G. M. D'Ariano, and P. Perinotti. Quantum circuit architecture. *Phys. Rev. Lett.*, 101:060401, Aug 2008.

[18] Giulio Chiribella, Giacomo Mauro D'Ariano, and Paolo Perinotti. Theoretical framework for quantum networks. *Phys. Rev. A*, 80:022339, Aug 2009.

[19] G Chiribella, G D'Ariano, P Perinotti, and B Valiron. Quantum computations without definite causal structure. *arXiv:0912.0195v4*, 2013.

[20] Esteban Castro-Ruiz, Flaminia Giacomini, and Časlav Brukner. Dynamics of quantum causal structures. *Physical Review X*, 8:011047, 2018.

[21] J. S. Bell. On the einstein podolsky rosen paradox. *Physics Physique Fizika*, 1:195–200, Nov 1964.

[22] Christoph Spengler, Marcus Huber, and Beatrix C Hiesmayr. A geometric comparison of entanglement and quantum nonlocality in discrete systems. *Journal of Physics A: Mathematical and Theoretical*, 44(6):065304, jan 2011.

[23] G Chiribella. Perfect discrimination of no-signalling channels via quantum superposition of causal structures. *arXiv:1109.5154v3*, 2012.

[24] Lorenzo M Procopio, Amir Moqanaki, Mateus Araújo, Fabio Costa, Irati Alonso Calafell, Emma G Dowd, Deny R Hamel, Lee A Rozema, Časlav Brukner, and Philip Walther. Experimental superposition of orders of quantum gates. *Nature communications*, 6(1):7913–7913, 2015.

[25] B. S. Tsirelson. Some results and problems on quantum bell-type inequalities. *Hadronic Journal Supplement*, 8(4):329–345, 1993.

[26] B.S. Tsirelson. Bell inequalities and operator algebras, 2006. `http://web.archive.org/web/20090414083019/http://www.imaph.tu-bs.de/qi/problems/33.html`.

[27] William Slofstra. Tsirelson's problem and an embedding theorem for groups arising from non-local games. *arXiv:1606.03140v2*, 2016.

[28] Sundar S. Finite dimensional c^*-algebras. -, 2012. `https://www.imsc.res.in/~knr/mathasp14/ss_notes.pdf`.

[29] Ian F. Putnam. Lecture notes on c^*-algebras, January 2019. `www.math.uvic.ca/faculty/putnam/ln/C*-algebras.pdf`.

[30] Peter Szekeres. *A course in modern mathematical physics : groups, Hilbert space and differential geometry*. Cambridge Univ. Pr., Cambridge, 1. publ.. edition, 2004.

[31] Peter Šemrl. Maps on matrix spaces. *Linear Algebra and Its Applications*, 413(2):364–393, 2006.

[32] H. Blaine Lawson. *Spin Geometry (PMS-38)*. Princeton Mathematical Series ; 38. 2016.

[33] Arturo Magidin (https://math.stackexchange.com/users/742/arturo magidin). Why is the ring of matrices over a field simple? Mathematics Stack Exchange. `https://math.stackexchange.com/questions/22629/why-is-the-ring-of-matrices-over-a-field-simple` (version: 2011-02-18).

[34] Dirk Werner. *Funktionalanalysis*. Springer-Lehrbuch. Springer Spektrum, Berlin, 8., vollständig überarbeitete auflage. edition, 2018.

[35] Catherine M Kunicki and Richard D Hill. Normal-preserving linear transformations. *Linear Algebra and Its Applications*, 170(C):107–115, 1992.

[36] Matej Brešar and Peter Šemrl. Normal-preserving linear mappings. *Canadian Mathematical Bulletin*, 37(3):306–309, 1994.

[37] Roger A. Horn and Charles R. Johnson. *Matrix Analysis.*, volume 2nd ed. Cambridge eText, 2012.

[38] Farrokh Vatan and Colin Williams. Optimal quantum circuits for general two-qubit gates. *arXiv:quant-ph/0308006v3*, 2004.

[39] M. H. Stone. On one-parameter unitary groups in hilbert space. *Annals of Mathematics*, 33(3):643–648, 1932.

[40] Albert Messiah. *Quantenmechanik : 1*. de Gruyter, 1991.

[41] Tilo Arens. *Mathematik*. Springer Berlin Heidelberg Imprint: Springer Spektrum, Berlin, Heidelberg, 3. aufl. 2015. edition, 2015.

[42] Wolfgang Nolting. *Grundkurs Theoretische Physik 5/2: Quantenmechanik - Methoden und Anwendungen*. Springer-Lehrbuch. Springer Berlin Heidelberg, Berlin, Heidelberg, 2015.

Appendix A

Action of the Continuous Swap on $\sigma_p \otimes \mathbb{1}$

To show that equation (6.25) holds we will need the relation

$$\sum_i \epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}, \tag{A.1}$$

for the total anti-symmetric tensor [41] and the relation

$$\sigma_i\sigma_j = \delta_{ij}\mathbb{1} + i\sum_n \epsilon_{ijn}\sigma_n, \tag{A.2}$$

where σ_i are the Pauli matrices[42]. From the last two equation we find

$$\begin{aligned}
\sigma_i\sigma_j\sigma_l &= \left(\delta_{ij}\mathbb{1} + i\sum_n \epsilon_{ijn}\sigma_n\right)\sigma_l = \delta_{ij}\sigma_l + i\sum_k \epsilon_{ijl}\mathbb{1} - \sum_{mk} \epsilon_{ijk}\epsilon_{klm}\sigma_m \\
&= \delta_{ij}\sigma_l + i\sum_k \epsilon_{ijl}\mathbb{1} - \sum_m (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\sigma_m \\
&= \delta_{ij}\sigma_l + \delta_{jl}\sigma_i - \delta_{il}\sigma_j + i\sum_k \epsilon_{ijl}\mathbb{1}.
\end{aligned} \tag{A.3}$$

For convenience, we will write U instead of $U_{SWAP}(t)$. Now, we start with the actual calculation of equation (6.25):

$$\begin{aligned}
U(\sigma_p \otimes \mathbb{1})U^\dagger &= \left(\frac{3+e^{it\pi}}{4}(\mathbb{1}\otimes\mathbb{1}) + \frac{1-e^{it\pi}}{4}\left(\sum_k \sigma_k \otimes \sigma_k\right)\right)(\sigma_p \otimes \mathbb{1}) \\
&\cdot \left(\frac{3+e^{-it\pi}}{4}(\mathbb{1}\otimes\mathbb{1}) + \frac{1-e^{-it\pi}}{4}\left(\sum_j \sigma_j \otimes \sigma_j\right)\right) \\
&= \frac{3+e^{it\pi}}{4}\cdot\frac{3+e^{-it\pi}}{4}(\sigma_p\otimes\mathbb{1}) \\
&+ \frac{3+e^{it\pi}}{4}\cdot\frac{1-e^{-it\pi}}{4}\underbrace{\sum_j(\sigma_p\otimes\mathbb{1})(\sigma_j\otimes\sigma_j)}_{=a} \\
&+ \frac{1-e^{it\pi}}{4}\cdot\frac{3+e^{-it\pi}}{4}\underbrace{\sum_k(\sigma_k\otimes\sigma_k)(\sigma_p\otimes\mathbb{1})}_{=b} \\
&+ \frac{1-e^{it\pi}}{4}\cdot\frac{1-e^{-it\pi}}{4}\underbrace{\sum_{kj}(\sigma_k\otimes\sigma_k)(\sigma_p\otimes\mathbb{1})(\sigma_j\otimes\sigma_j)}_{=c}\,. \qquad \text{(A.4)}
\end{aligned}$$

We calculate a, b and c independently. Let us start with a:

$$\begin{aligned}
a &= \sum_j(\sigma_p\otimes\mathbb{1})(\sigma_j\otimes\sigma_j) = \sum_j(\sigma_p\sigma_j\otimes\sigma_j) = \sum_j\left(\delta_{pj}\mathbb{1}\otimes\sigma_j + i\sum_n \epsilon_{pjn}\sigma_n\otimes\sigma_j\right) \\
&= \mathbb{1}\otimes\sigma_p - i\sum_{jn}\epsilon_{pnj}\sigma_n\otimes\sigma_j. \qquad \text{(A.5)}
\end{aligned}$$

Similarly, we find for b:

$$b = \mathbb{1}\otimes\sigma_p + i\sum_{jn}\epsilon_{pnj}\sigma_n\otimes\sigma_j. \qquad \text{(A.6)}$$

Next, we calculate c:

$$\begin{aligned}
\sum_{jk}(\sigma_k\sigma_p\sigma_j)\otimes(\sigma_k\sigma_j) &= \sum_{jk}\left(\delta_{kp}\sigma_j + \delta_{pj}\sigma_k - \delta_{kj}\sigma_p + i\epsilon_{kpj}\right)\otimes\left(\delta_{kj}\mathbb{1} + i\sum_n\epsilon_{kjn}\sigma_n\right) \\
&= \underbrace{\sum_{jk}\left(\delta_{kp}\sigma_j + \delta_{pj}\sigma_k - \delta_{kj}\sigma_p\right)\otimes\delta_{kj}\mathbb{1}}_{=x} \\
&+ \underbrace{i\sum_{jkn}\epsilon_{kjn}\left(\delta_{kp}\sigma_j + \delta_{pj}\sigma_k - \delta_{kj}\sigma_p\right)\otimes\sigma_n}_{=y} \\
&+ \underbrace{\sum_{jk} i\epsilon_{kpj}\delta_{kj}\mathbb{1}\otimes\mathbb{1}}_{=0} + \underbrace{i^2\sum_{jkn}\epsilon_{kpj}\epsilon_{kjn}\mathbb{1}\otimes\sigma_n}_{=z}\,. \qquad \text{(A.7)}
\end{aligned}$$

We calculate w, x and z independently. We start with x:

$$\begin{aligned} x &= \sum_{jk} (\delta_{kp}\delta_{kj}\sigma_j \otimes \mathbb{1} + \delta_{pj}\delta_{kj}\sigma_k \otimes \mathbb{1} - \delta_{kj}\delta_{kj}\sigma_p \otimes \mathbb{1}) = 2\sigma_p \otimes \mathbb{1} - 3\sigma_p \otimes \mathbb{1} \\ &= -\sigma_p \otimes \mathbb{1}. \end{aligned} \tag{A.8}$$

Next, we calculate

$$\begin{aligned} y &= i\sum_{jkn}\left(\delta_{kp}\epsilon_{kjn}\sigma_j \otimes \sigma_n + \delta_{pj}\epsilon_{kjn}\sigma_k \otimes \sigma_n - \underbrace{\delta_{kj}\epsilon_{kjn}}_{=0}\sigma_p \otimes \sigma_n\right) \\ &= i\sum_{kn}\left(\epsilon_{pkn}\sigma_k \otimes \sigma_n + \underbrace{\epsilon_{kpn}}_{=-\epsilon_{pkn}}\sigma_k \otimes \sigma_n\right) = 0. \end{aligned} \tag{A.9}$$

Furthermore, we derive

$$\begin{aligned} z &= \sum_{kjn} (\epsilon_{kjn}\epsilon_{kjp}\mathbb{1} \otimes \sigma_n) = \sum_{kjn} ((\delta_{jj}\delta_{np} - \delta_{jp}\delta_{nj})(\mathbb{1} \otimes \sigma_n)) = 3\mathbb{1} \otimes \sigma_p - \mathbb{1} \otimes \sigma_p \\ &= 2\mathbb{1} \otimes \sigma_p. \end{aligned} \tag{A.10}$$

Now, we plug (A.8),(A.9) and (A.10) into (A.7)

$$z = -\sigma_p \otimes \mathbb{1} + 2\mathbb{1} \otimes \sigma_p. \tag{A.11}$$

Finally, we can plug (A.5),(A.6) and (A.11) into (A.4):

$$\begin{aligned} U(\sigma_p \otimes \mathbb{1})U^\dagger &= \frac{10 + 3e^{it\pi} + 3e^{-it\pi}}{16}\sigma_p \otimes \mathbb{1} \\ &+ \frac{2 + e^{it\pi} - 3e^{-it\pi}}{16}\left(\mathbb{1} \otimes \sigma_p - i\sum_{jn}\epsilon_{pnj}\sigma_n \otimes \sigma_j\right) \\ &+ \frac{2 - 3e^{it\pi} + e^{-it\pi}}{16}\left(\mathbb{1} \otimes \sigma_p + i\sum_{jn}\epsilon_{pnj}\sigma_n \otimes \sigma_j\right) \\ &+ \frac{2 - e^{it\pi} - e^{-it\pi}}{16}(-\sigma_p \otimes \mathbb{1} + 2\mathbb{1} \otimes \sigma_p) \\ &= \frac{8 + 4(e^{it\pi} + e^{-it\pi})}{16}\sigma_p \otimes \mathbb{1} + \frac{8 - 4(e^{it\pi} + e^{-it\pi})}{16}\mathbb{1} \otimes \sigma_p \\ &+ i\frac{4(e^{-it\pi} - e^{it\pi})}{16}\sum_{jn}\epsilon_{pnj}\sigma_n \otimes \sigma_j \\ &= \frac{1 + \cos(t\pi)}{2}\sigma_p \otimes \mathbb{1} + \frac{1 - \cos(t\pi)}{2}\mathbb{1} \otimes \sigma_p \\ &+ i\frac{\sin(t\pi)}{2}\sum_{jn}\epsilon_{pnj}\sigma_n \otimes \sigma_j, \end{aligned} \tag{A.12}$$

which completes our proof that equation (6.25) holds. $\square$

To show that (6.26) holds as well, we first observe

$$U^*_{SWAP}(t) = \frac{3+e^{-it\pi}}{4}\mathbb{1}\otimes 1 + \frac{1-e^{-it\pi}}{4}\sum_j \sigma_j\otimes\sigma_j = U_{SWAP}(-t), \tag{A.13}$$

where we have used that $(\sigma_j\otimes\sigma_j)^* = \sigma_j\otimes\sigma_j$ holds $\forall j$. This immediately gives us

$$\begin{aligned}
U^*_{SWAP}(t)(\sigma_p\otimes\mathbb{1})(U^*_{SWAP}(t))^\dagger &= U_{SWAP}(-t)(\sigma_p\otimes\mathbb{1})(U^\dagger_{SWAP}(-t)) \\
&= \frac{1+\cos(-t\pi)}{2}\sigma_p\otimes\mathbb{1} + \frac{1-\cos(-t\pi)}{2}\mathbb{1}\otimes\sigma_p \\
&+ i\frac{\sin(-t\pi)}{2}\sum_{jn}\epsilon_{pnj}\sigma_n\otimes\sigma_j \\
&= \frac{1+\cos(t\pi)}{2}\sigma_p\otimes\mathbb{1} + \frac{1-\cos(t\pi)}{2}\mathbb{1}\otimes\sigma_p \\
&- i\frac{\sin(t\pi)}{2}\sum_{jn}\epsilon_{pnj}\sigma_n\otimes\sigma_j,
\end{aligned} \tag{A.14}$$

and this ends our proof that (6.26) holds. $\square$