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Schrödinger equation with generalized initial value

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1 C_0-semigroups

This section is a summary of the subject of strongly continuous semi groups. Our aim in this chapter is to develop the theory as far as to define the unique mild solution for the abstract Cauchy problem. Furthermore we will state and prove the important theorem of Stone, which is a statement about strongly continuous unitary groups and their infinitesimal generators. This will be very useful to prove the existence of a mild solution for the Schrödinger equation with zero potential. The main references for this chapter are Dirk Werner [21, Chapter VII.4] and the book by Pazy [11].

In what follows X stands for a complex Banach space and $L(X)$ for the space of linear continuous maps from X to itself. Let $1 : X \to X$, $x \mapsto x$ denote the identity on X. Furthermore $\mathbb{R}^+ = [0, \infty)$ is defined to be the set of non negative real numbers with zero included.

Definition 1.1. A *C_0-semigroup* or *strongly continuous semigroup* is a function $T : \mathbb{R}^+ \to L(X)$ with the following properties:

(i). $T(0) = 1$,

(ii). $\forall s, t \in \mathbb{R}^+ : T(s + t) = T(s) + T(t)$,

(iii). $\forall x \in X : T(t)x \to x$ as $t \to 0$.

If the stronger condition $\lim_{t\to 0} ||T(t) - 1|| = 0$ holds, the semigroup is called *norm continuous*. Moreover we call T a *C_0-group* if we may replace above the positive real numbers $\mathbb{R}^+$ by $\mathbb{R}$.

That norm continuity implies (iii) in the definition follows immediately from the inequality $||T(t)x - x|| \leq ||T(t) - 1||\ ||x||$. Furthermore (ii) implies that the operators of the C_0-semigroup commute. The term of strong continuity comes from the fact that condition (iii) means continuity of T in the strong operator topology on $L(X)$ defined by the family of seminorms $p_x(T) = ||T(x)||$ where $x \in X$ and $T \in L(X)$ see Werner [21, ChapterVIII] .

Example. Let A be a bounded operator on X and define the map:

$$T : \mathbb{R}^+ \to L(X), \quad t \mapsto e^{tA} := \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}.$$

Since A is bounded the series converges absolutely by the simple observation

$$\sum_{k=0}^{\infty} \left|\left| \frac{t^k A^k}{k!} \right|\right| \leq \sum_{k=0}^{\infty} \frac{t^k ||A||^k}{k!} = \exp(t||A||) < \infty.$$

Thus $T(t)$ defines a bounded operator in $L(X)$. We claim that T is a norm continuous semigroup. By the definition of T we see that $T(0)$ is the identity. Condition (ii) from Definition 1.1 is shown like the proof of the functional equation of the exponential function $\exp : \mathbb{R} \to \mathbb{R}$ via the Cauchy product of series

$$\begin{aligned} T(s+t) &= \sum_{k=0}^{\infty} \frac{(s+t)^k A^k}{k!} = \sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{s^l A^l t^{k-l} A^{k-l}}{l!(k-l)!} \\ &= \sum_{l=0}^{\infty} \frac{s^l A^l}{l!} \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} = T(s)T(t). \end{aligned}$$

In the last equation we have used the absolutely convergence of the series as noted before. To see that the map is norm continuous we consider

$$||T(t) - 1|| \leq \sum_{k=1}^{\infty} \frac{t^k ||A||^k}{k!} \leq \left(e^{t||A||} - 1 \right) \to 0 \quad \text{as } t \to 0.$$

This proves the desired statement.

In what follows we use for the strongly continuous semi group $T(t)$ the shorter notation T_t. We want to assign a linear operator A on the Banach space X to a C_0-semigroup, which will be unbounded in general.

Definition 1.2. Let T be a C_0-semigroup we define the operator

$$Ax \coloneqq \lim_{h \to 0} \frac{T_h x - x}{h}$$

on the domain

$$\operatorname{dom}(A) = \{x \in X | \lim_{h \to 0} \frac{T_h x - x}{h} \text{ exists}\}$$

and call it the *infinitesimal generator* or short *generator* of T.

By the definition of the generator it is not at all clear that $\operatorname{dom}(A)$ contains more than the zero vector in X. The next theorem states indeed, that the generator is densely defined and closed in X.

Theorem 1.3. *Let A be a generator of a C_0-semigroup on X, then A is densely defined thus $\overline{\operatorname{dom}(A)} = X$ and closed.*

Our aim is to find a unique solution to the *abstract Cauchy problem* for a generator A of a C_0-semigroup:

$$\frac{du(t)}{dt} = Au(t),\ t > 0 \quad u(0) = x_0 \quad \text{for} \quad x_0 \in X. \tag{1.1}$$

Where by a solution of the abstract Cauchy problem we mean a continuously differentiable X-valued function with $u(t) \in \mathrm{dom}\,(A)$ for $t > 0$, such that (1.1) is satisfied.

Theorem 1.4. *Let A be a generator of a C_0-semigroup T on X and $x_0 \in \mathrm{dom}\,(A)$, then the function $u : [0, \infty) \to X$, $t \mapsto T_t(x_0)$ is the unique solution to the abstract Cauchy problem* (1.1). *Furthermore the solution depends continuously on the initial value x_0.*

Definition 1.5. Let $U : \mathbb{R} \to L(H)$ be a C_0-group on a complex Hilbert space H, such that U_t is unitary for all $t \in \mathbb{R}$, i.e. $U_t^* = U_t^{-1}$. Then U is called a strongly continuous unitary group

We recall the spectral theorem of self-adjoint operators, because we need to define the exponential function for unbounded self adjoint operators. See e.g. Werner [21, ChapterVII.3].

For a self-adjoint operator $A : H \supseteq \mathrm{dom}\,(A) \to H$ there exists a unique spectral measure E such that for all $x \in \mathrm{dom}\,(A)$: $\langle Ax, y\rangle = \int_{\mathbb{R}} \lambda \; d\langle E_\lambda x, y\rangle$, $y \in H$. For a measurable function $h : \mathbb{R} \to \mathbb{R}$ the associated functional calculus

$$\begin{aligned} \langle h(A)x, y\rangle &= \int_{\mathbb{R}} h(\lambda) \; d\langle E_\lambda x, y\rangle, \\ x \in D_h &= \{x \in X : \int_{\mathbb{R}} |h(\lambda)|^2 \; d\langle E_\lambda x, y\rangle < \infty\}, \end{aligned}$$

defines a self adjoint operator $h(A) : D_h \to H$, with dense domain D_h.

The next theorem will be used later on, to prove that the Laplace operator is self adjoint. This empowers us to define the exponential $e^{it\Delta}$, in the sense of the previously remarked functional calculus.

Theorem 1.6 (Stone)**.** *A densely defined operator A on a Hilbert space H is the infinitesimal generator of a strongly continuous unitary group U if and only if iA is self-adjoint. In this context the strongly continuous unitary group is given by $U_t = \exp{(tA)}$.*

Since we want to deal with generalized initial values in Schrödinger's equation, which do not lie in the natural domain of the Schrödinger Operator, namely the Sobolev space H^2, we have to define a different solution concept to an abstract Cauchy problem.

Definition 1.7. Let X be a Banach space and A an infinitesimal generator of a strongly continuous semigroup T_t then the *mild solution* of the abstract Cauchy problem (1.1) is the function $u : \mathbb{R}^+ \to X$ with $u(t) = T_t x_0$ with initial value $x_0 \in X$.

Lemma 1.8. *If T_t is a C_0-semigroup on the Banach space X, then the function $X \times [0,\infty) \to X$ with $(x,t) \mapsto T_t x$ is continuous furthermore it is uniformly continuous with respect to the variable t on compact subsets of $[0,\infty)$. In particular for all $x \in X$ fixed the map $u(t) := T_t x$ is continuous, hence $u \in C([0,\infty), X)$.*

Since the domain of a generator A is dense, there exists for every $x_0 \in X \setminus \operatorname{dom}(A)$ a sequence $x_n \in \operatorname{dom}(A)$, $n \in \mathbb{N}$ for every $x_0 \in X \setminus \operatorname{dom}(A)$, such that the classical continuously differentiable solutions $T_t x_n$ converge uniformly in time on compact intervals to the mild solution $T_t x_0$. This means that the mild solution is arbitrarily close to a classical solution for all times.

2 Schrödinger equation

We will follow Pazy [11, Chapter VII]. Consider the abstract Cauchy problem

$$\partial_t u = i\Delta u, \quad u(0) = x_0 \quad \text{for } x_0 \in L^2(\mathbb{R}^n), \tag{2.1}$$

with the differential operator $A_0 = i\Delta$ on the domain $\operatorname{dom}(A_0) = H^2(\mathbb{R}^n)$.

To solve the Schrödinger equation (2.1) we proceed in two different ways. First, we show that the operator iA_0 is self adjoint in $L^2(\mathbb{R}^n)$ and therefore by Stone's theorem prove that there exists a unique mild solution.

The other approach is to define a classical solution in terms of the Fourier transform with initial value in $\mathcal{S}(\mathbb{R}^n)$ and extend it to $L^2(\mathbb{R}^n)$ and view the differential operator A_0 in the distributional setting. After that we prove that these two solutions agree indeed.

For the rest of this thesis we will use the following convention for the Fourier transform of a function $f \in L^1(\mathbb{R}^n)$:

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, \xi \rangle} dx. \tag{2.2}$$

We recall, that the Sobolev spaces H^m, $m \in \mathbb{N}$ are the functions φ in $L^2(\mathbb{R}^n)$ such that the distributional derivatives up to order m are in $L^2(\mathbb{R}^n)$. Thus

$$H^2(\mathbb{R}^n) = \{\varphi \in L^2(\mathbb{R}^n) : \partial^\alpha \varphi \in L^2(\mathbb{R}^n), |\alpha| \leq m\}.$$

Lemma 2.1. *Let $A_0 = i\Delta$ and $\operatorname{dom}(iA_0) = H^2(\mathbb{R}^n)$. Then the densely defined operator $iA_0 : L^2(\mathbb{R}^n) \supset \operatorname{dom}(iA_0) \to L^2(\mathbb{R}^n)$ is selfadjoint on the Hilbert space $L^2(\mathbb{R}^n)$.*

Proof. Let $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, $\mathcal{F}(\varphi) = \hat{\varphi}$ denote the Fourier transform. Then the well known formula $\mathcal{F}(\partial^\alpha \varphi)(\xi) = (2\pi i \xi)^\alpha \hat{\varphi}(\xi)$ holds in the distributional sense for all $\varphi \in L^2(\mathbb{R}^n) \supset H^2(\mathbb{R}^n)$. Hence the equation

$$\mathcal{F}\left[-\Delta \mathcal{F}^{-1}(\varphi)\right](x) = 4\pi^2 |x|^2 \varphi(x)$$

is valid for every $x \in \mathbb{R}^n$, $\varphi \in H^2(\mathbb{R}^n)$. The multiplication operator $M_f(\varphi) := f \cdot \varphi$ with the function $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) = 4\pi^2|x|^2$ is self adjoint on the domain $\operatorname{dom}(M_f) = \{\varphi \in L^2(\mathbb{R}^n) : f\varphi \in L^2(\mathbb{R}^n)\}$ (see e.g. Werner [21]). Since the Fourier image of $\operatorname{dom}(M_f)$ is equal to the space $H^2(\mathbb{R}^n)$, we conclude that the operator iA_0 is unitarily equivalent to M_f. Thus iA_0 is self adjoint on $L^2(\mathbb{R}^n)$. □

With Stone's Theorem 1.6 we directly get the mild solution for the Schrödinger equation.

Corollary 2.2. *The operator A_0 is the infinitesimal generator of a strongly continuous unitary group. Therefore the unique unitary mild solution of the Cauchy problem* (2.1) *with initial value $x_0 \in L^2(\mathbb{R}^n)$ is the function defined via the spectral measure $u(t) = \exp(tA_0)x_0$.*

To solve the Schrödinger equation in the setting of Fourier analysis, we recall Young's inequality from real analysis. Since we will apply a variation of this theorem in Section 4.3, we attach this statement here. Our reference is Folland [6].

Theorem 2.3 (Young's inequality)**.** *Let $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$ for $p \in [1, \infty]$, then $f * g(x)$ exists for Lebesque almost every x and*

$$\|f * g\|_{L^p} \leq \|f\|_{L^1} \|g\|_{L^p}.$$

This implies that $f * g \in L^p(\mathbb{R}^n)$.

The proof is an easy consequence of Minkowski's inequality see [6, Chapter 8].

The other fact we want to show, depends on topological statements and therefore we change from the euclidean space $\mathbb{R}^n$ to general locally compact Hausdorff spaces.

We recall that a topological space X is locally compact, if every point $x \in X$ has a compact neighbourhood.

Definition 2.4. A function $f \in C(X)$ is said to be *vanishing at infinity*, if for every $\varepsilon > 0$ the set $\{x \in X : |x| \geq \varepsilon\}$ is compact in X. We define $C_0(X)$ to be the space of functions vanishing at infinity.

We state without proof the following

Lemma 2.5 (Urysohn)**.** *Let X be a locally compact Hausdorff space and $K \subset U \subset X$ with K compact and U open in X, then there exists a function $f \in C(X, [0, 1])$ and a compact set $K_1 \subset U$ such that $f|_K \equiv 1$ and $f|_{X \setminus K_1} \equiv 0$.*

The proof can be found in Folland [6, Chapter 4].

Proposition 2.6. *If X is a locally compact Hausdorff space then $C_0(X)$ is the uniform closure of $C_c(X)$, i.e. the space of continuous, compactly supported functions on X.*

Proof. We show that if $f_n \to f$ uniformly for $f_n \in C_c(X)$, then $f \in C_0(X)$. Note that the function f has to be continuous, since the uniform limit of continuous functions is continuous. Uniform convergence of the sequence $(f_n)_{n \in \mathbb{N}}$ means

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \quad \text{such that } \|f_n - f\|_\infty < \varepsilon \quad \forall n \geq N.$$

For $x \notin \operatorname{supp}(f_n)$ we have that $|f(x)| = |f(x) - f_n(x)| < \varepsilon$. Hence the set $A := \{x : |f(x)| \geq \varepsilon\}$ is contained in the compact set $\operatorname{supp}(f_n) \subset X$. Since A is closed this implies that A is compact. Thus we have proved $f \in C_0(X)$.

Let $f \in C_0(X)$ given. We have to construct a sequence $f_n \in C_c(X)$ such that $f_n \to f$ uniformly. To this end define for every $n \in \mathbb{N}$ the set $K_n = \{x : |f(x)| \geq \frac{1}{n}\}$. By the definition of $C_0(X)$, the sets K_n are compact $\forall n \in \mathbb{N}$. Owing to Lemma 2.5, for every $n \in \mathbb{N}$ there exists a function $g_n \in C_c(X, [0, 1])$ such that $g_n|_{K_n} \equiv 1$. So if we define the functions $f_n := g_n \cdot f$, we see that $f_n \in C_c(X)$ for every $n \in \mathbb{N}$ and by definition of the sets K_n and the sequence f_n we get

$$\|f_n - f\|_\infty = \sup_{x \in X} |(f_n - f)(x)| = \sup_{x \notin K_n} |(f_n - f)(x)| < \frac{1}{n}.$$

Thus $f_n \to f$ uniformly, which completes the proof of the corollary. $\square$

Notation. Let $1 \leq p \leq \infty$, then q is said to be a *conjugate exponent to p* if $p^{-1} + q^{-1} = 1$. Here we use the convention $\frac{1}{\infty} := 0$. The translation of a function $f : \mathbb{R}^n \to \mathbb{C}$ by $y \in \mathbb{R}^n$ will be denoted as $T_y f := f(. - y)$. Furthermore the reflection of the function f is defined by $Rf(x) := f(-x)$.

Proposition 2.7. *Let $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$ with conjugate exponents $p, q \in [1, \infty]$, then the convolution $(f * g)(x)$ exists for every $x \in \mathbb{R}^n$, is uniformly continuous and furthermore $\|f * g\|_\infty \leq \|f\|_{L^p} \|g\|_{L^q}$. If $1 < p < \infty$, then $f * g \in C_0(\mathbb{R}^n)$.*

Proof. The existence for every $x \in \mathbb{R}^n$ follows by applying Hölder's inequality to obtain

$$|(f * g)(x)| \leq \int_{\mathbb{R}^n} |f(x-y)g(y)|dy = \| \left(T_x[Rf](y)\right) g\|_{L^1} \leq \|f\|_{L^p}\|g\|_{L^q}.$$

This also proves the inequality $\|fg\|_\infty \leq \|f\|_{L^p}\|g\|_{L^q}$. A direct calculation yields $T_y(f*g) = (T_y f) * g$. For the proof of the next assumption, we can assume $1 \leq p < \infty$ without loss of generality , since for the case $p = \infty$ we just have to interchange the roles of f and g. Hence

$$\|T_y(f*g) - f*g\|_\infty = \|(T_y f) * g - f * g\|_\infty \leq \|T_y f - f\|_{L^p}\|g\|_{L^q}$$

implies uniform continuity of $f*g$. Indeed, the last term in the equation converges to zero, because the translation operator is continuous on $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$ (see Folland [6, Proposition 8.5]).

We recall, that in general for two measurable functions $v, w : \mathbb{R}^n \to \mathbb{R}$ we have

$$\operatorname{supp}(v*w) \subset \overline{\{x+y : x \in \operatorname{supp}(v), y \in \operatorname{supp}(w)\}}.$$

(see e.g.Folland [6, Proposition 8.6]. Here of course the domain of the convolution $v*w$ consists of all $x \in \mathbb{R}^n$, such that the integral $\int_{\mathbb{R}^n} v(x-y)w(y)dy$ exists.

Now let $1 < p, q < \infty$ then we can choose sequences $f_n, g_n \in C_c(\mathbb{R}^n)$ such that $\|f_n - f\|_{L^p} \to 0$ and $\|g_n - g\|_{L^q} \to 0$. For further considerations we declare the sets

$$A_n := \{x+y : x \in \operatorname{supp}(f_n), y \in \operatorname{supp}(g_n)\} \quad \text{for} \quad n \in \mathbb{N}.$$

It is clear, that for every $n \in \mathbb{N}$ the set $\operatorname{supp}(f_n) \times \operatorname{supp}(g_n)$ is compact since f_n and g_n have compact support. Moreover, because addition is continuous on the product $\mathbb{R}^n \times \mathbb{R}^n$, the set A_n is compact in $\mathbb{R}^n$. Therefore it follows that the convolution $f_n * g_n$ is compactly supported for every $n \in \mathbb{N}$, since

$$\operatorname{supp}(f_n * g_n) \subset \overline{A_n} = A_n.$$

With the estimation

$$\|f_n * g_n - f * g\|_\infty = \|(f_n - f) * g_n + f * (g_n - g)\|_\infty \leq \\ \|f_n - f\|_{L^p}\|g\|_{L^q} + \|f\|_{L^p}\|g_n - g\|_{L^q},$$

it follows that the functions $f_n * g_n \in C_c(\mathbb{R}^n)$ converge uniformly to $f * g$ as $n \to \infty$. This implies that $f * g$ lies in the uniform closure of $C_c(\mathbb{R}^n)$. Thus by Proposition 2.6 $f * g \in C_0(\mathbb{R}^n)$, which proves the proposition. $\square$

After this short digression we come back to the Fourier analytic method in solving the Schrödinger equation. The following two propositions are from Stein [17, Chapter 8.6.1].

A natural suggestion for a solution to the Problem (2.1), after a formal calculation, would be the following definition.

Definition 2.8. Let $f \in L^2(\mathbb{R}^n)$ and define for all $t \in \mathbb{R}$ the operators

$$\mathcal{F}\left(e^{it\Delta} f\right)(\xi) := e^{-it4\pi^2|\xi|^2} \hat{f}(\xi)$$

Proposition 2.9. *For all $t \in \mathbb{R}$ and $f \in \mathcal{S}(\mathbb{R}^n)$ the following properties hold:*

(i). $e^{it\Delta}(f) \in \mathcal{S}(\mathbb{R}^n)$.

(ii). Define the function $u(x,t) = e^{it\Delta}(f)(x)$, then $u \in C^\infty(\mathbb{R}^n \times \mathbb{R})$ and it solves the abstract Cauchy problem (2.1) for the initial value $u(x,0) = f(x)$.

(iii). Let $t \neq 0$, $K_t(x) = (4\pi it)^{-n/2} e^{-|x|^2/(4it)}$ then the solution can be written as $u(x,t) = f * K_t(x)$.

(iv). $\forall t \neq 0$, $\|e^{it\Delta}(f)\|_{L^\infty} \leq (4\pi|t|)^{-n/2}\|f\|_{L^1}$.

Proof. Since the function f is in $\mathcal{S}(\mathbb{R}^n)$ the same is true for $\hat{f}$. For all multi-indices $\alpha, \beta \in \mathbb{N}^n$ one calculates by usage of the generalized Leibniz Formula the following inequality:

$$\begin{aligned}\sup_{\xi \in \mathbb{R}^n} |\xi^\alpha \partial^\beta \left(e^{-it4\pi^2|\xi|} \hat{f}(\xi)\right)| &\leq \sup_{\xi \in \mathbb{R}^n} |\xi^\alpha \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} \partial^\gamma e^{-it4\pi^2|\xi|} \partial^{\beta-\gamma} \hat{f}(\xi)| \\ &\leq \sup_{\xi \in \mathbb{R}^n} \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} |\xi^\alpha \operatorname{Pol}(\xi,\gamma) \partial^{\beta-\gamma} \hat{f}(\xi)|,\end{aligned}$$

where $\operatorname{Pol}(\xi,\gamma)$ denotes a polynomial in the variable $\xi \in \mathbb{R}^n$ of order $|\gamma|$. Therefore we have shown that $\mathcal{F}\left(e^{it\Delta} f\right)$ is in $\mathcal{S}(\mathbb{R}^n)$ which implies that the inverse Fourier transform is also in the Schwartz space, which proves (i).

With the inverse Fourier transform we write

$$u(x,t) = \int_{\mathbb{R}^n} e^{-it4\pi^2|\xi|^2} \hat{f}(\xi) e^{2\pi i\xi x} d\xi. \tag{2.3}$$

To show that the function u is differentiable in (x,t), we check the conditions for differentiability of the parameter dependent integral like in Forster[7, Chapter 11].

First to show that u is differentiable in the t- variable we inspect the integrand $h(t,\xi) = e^{-it4\pi^2|\xi|^2} \hat{f}(\xi) e^{2\pi\xi x}$ and consider the x-variable fixed. For every fixed $t \in \mathbb{R}$ we need to show that the function $h(t,\xi)$ is integrable in $\mathbb{R}^n$. But this is an easy consequence of the fact that $\hat{f}$ is in $\mathcal{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$ and because $|h(t,\xi)| = |\hat{f}(\xi)|$.

It is clear that $h(t,\xi)$ is differentiable with respect to t. The last condition to prove differentiability of u in t, is to find an integrable positive function $F : \mathbb{R}^n \mapsto \mathbb{R}^+$ such that $|\partial_t h(t,\xi)| \leq F(\xi)$ for all (t,ξ) in $\mathbb{R} \times \mathbb{R}^n$. If we just calculate $|\partial_t h(t,\xi)| = 4\pi^2|\xi|^2|\hat{f}|$ we see that we can use the absolute value of the derivative of h, since this function is clearly integrable, again due to rapid decay of $\hat{f}$. Recursive application of this argument, shows that u is smooth with respect to t.

To prove that $u(t,.) \in C^\infty(\mathbb{R}^n)$ one has to check in a similarly manner the latter conditions of parameter dependent integrals for each x_i, $i \in \{1,\ldots,n\}$.

With these preliminaries we are allowed to differentiate under the integral sign with respect to t and x. It follows that $\partial_t u = i\Delta u$ and if we insert $t = 0$ in (2.3) we conclude that $u(x,0) = f(x)$ by the Fourier inversion theorem. Thus (ii) is proven.

To prove (iii) we have to show that the identity

$$\widehat{K_t}(\xi) = e^{-it4\pi^2|\xi|^2} \quad \forall t \neq 0 \tag{2.4}$$

holds. Where we consider $x \mapsto K_t(x)$ as tempered distribution in $\mathcal{S}'(\mathbb{R}^n)$ for every $t \neq 0$. Because then by equation (2.3) we see that $u(x,t) = \mathcal{F}^{-1}\left(\hat{f}\hat{K}_t\right)$ and the assertion follows by an application of the convolution theorem for tempered distributions e.g. Stein [17, Chapter 3 Prop. 1.5]. For our considerations we recall the formula

$$\mathcal{F}\left(g(\delta x)\right)(\xi) = \delta^{-n}\hat{g}(\delta^{-1}\xi) \quad \text{for } \delta > 0. \tag{2.5}$$

If $g(x) := e^{-\pi|x|^2}$ and since the Fourier transform of the Gaussian function is again the Gaussian function see Stein [16, Chapter 6], we obtain $\hat{g}(\xi) = g(\xi)$. Therefore with $\delta := \sqrt{k}$, $k > 0$ we can write

$$k^{-\frac{n}{2}} e^{\frac{-\pi|x|^2}{k}} = \frac{1}{\left(\sqrt{k}\right)^n} e^{-\pi|x\frac{1}{\sqrt{k}}|^2} = \frac{1}{\delta^n}\hat{g}(\delta^{-1}x).$$

Together with (2.5) this shows

$$\begin{aligned} \mathcal{F}\left(k^{-\frac{n}{2}} e^{\frac{-\pi|x|^2}{k}}\right)(\xi) &= \mathcal{F}\left(\frac{1}{\delta^n}\hat{g}(\delta^{-1}x)\right) \\ &= \mathcal{F}\left(\hat{g}(\delta x)\right)(\xi) = g(\delta\xi) = e^{-\pi|\xi|^2 k}. \end{aligned} \tag{2.6}$$

Because the identity (2.6) holds true for the open interval $(0,\infty)$, we can extend it by analytic continuation to complex $k = 4\pi(\sigma + it)$, $\sigma > 0$. Now for every

$\sigma > 0$ we have by (2.6)

$$\mathcal{F}\left((4\pi(\sigma+it))^{-\frac{n}{2}} e^{\frac{-|x|^2}{4(\sigma+it)}}\right)(\xi) = e^{-4\pi^2|\xi|^2(\sigma+it)}. \tag{2.7}$$

Finally if $t \neq 0$ is fixed, then for $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and some $\sigma > 0$

$$\lim_{\sigma\to 0}\int_{\mathbb{R}^n} \mathcal{F}\left((4\pi(\sigma+it))^{-\frac{n}{2}} e^{\frac{-|x|^2}{4(\sigma+it)}}\right)(\xi)\varphi(\xi)d\xi = \int_{\mathbb{R}^n} \widehat{K_t}(\xi)\varphi(\xi)d\xi. \tag{2.8}$$

This follows by an application of the dominated convergence theorem, because the integrands in the sequence of (2.8) are clearly bounded by a positive integrable function and converge pointwise to $\widehat{K_t}(\xi)\varphi(\xi)$.

Again by an application of dominated convergence one shows

$$\lim_{\sigma\to 0}\int_{\mathbb{R}^n} e^{-4\pi^2|\xi|^2(\sigma+it)}\varphi(\xi)d\xi = \int_{\mathbb{R}^n} e^{-it4\pi^2|\xi|^2}\varphi(\xi)d\xi. \tag{2.9}$$

Thus the left side and the right side of (2.7) converge in the sense of tempered distributions to $\widehat{K_t}(\xi)$ and $e^{-it4\pi^2|\xi|^2}$ respectively, which establishes the identity (2.4) and therefore the assumption (iii) is proved.

Assertion (iv) follows from Young's inequality. Since K_t is bounded and $f \in L^1(\mathbb{R}^n)$ we have

$$\|f * K_t\|_{L^\infty} \leq \|K_t\|_{L^\infty}\|f\|_{L^1} = (4\pi|t|)^{-n/2}\|f\|_{L^1}, \quad \forall t \neq 0.$$

□

Proposition 2.10. *For every $t \in \mathbb{R}$ and $f \in L^2(\mathbb{R}^n)$ the following properties hold:*

(i). The operator $U_t := e^{it\Delta} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is unitary, in particular $\|U_t f\|_{L^2} = \|f\|_{L^2}$.

(ii). For every $f \in L^2(\mathbb{R}^n)$, the mapping $t \mapsto U_t(f)$ is continuous in the L^2-norm.

(iii). The function $u(x,t) := (U_t f)(x)$ solves the Schrödinger equation on $\mathbb{R}^n \times \mathbb{R}$ in the sense of distributions for initial value $u(x,0) = f(x)$.

Proof. Since the factor $e^{-it4\pi^2|\xi|^2}$ has absolute value one, we get

$$\int |e^{-it4\pi^2|\xi|^2}\hat{f}(\xi)|^2 d\xi = \|\hat{f}\|_{L^2}^2 < \infty$$

and observe that $\|\widehat{(U_t f)}\|_{L^2} = \|\hat{f}\|_{L^2}$. Thus (i) follows by an application of Plancherel's theorem.

Because $\hat{f} \in L^2(\mathbb{R}^n)$ we see that $e^{-it4\pi^2|\xi|^2}\hat{f}$ converges to $e^{-it_04\pi^2|\xi|^2}\hat{f}$ in the L^2 norm as $t \to t_0$. Thus using Plancherel's theorem again, part (ii) follows.

We define the operator $\mathcal{L} = \partial_t - i\Delta$ and consider its adjoint in the distributional sense $\mathcal{L}' = -\partial_t - i\Delta$. To prove (iii) we have to show, that the following equation

$$\begin{aligned} &\langle \mathcal{L}\left(U_t f\right), \phi\rangle = \langle (U_t f)\,, \mathcal{L}'(\phi)\rangle \\ &= \iint_{\mathbb{R}^n \times \mathbb{R}} \mathcal{L}'(\phi)(x,t)(U_t f)(x)dxdt = 0, \end{aligned} \tag{2.10}$$

holds for all ϕ in $\mathcal{D}(\mathbb{R}^n \times \mathbb{R})$ and $f \in L^2(\mathbb{R}^n)$.

By Proposition 2.9 it is clear, that (2.10) holds for all $f \in \mathcal{S}(\mathbb{R}^n)$ in the classical sense hence clearly in the distributional sense. Now for $f \in L^2(\mathbb{R}^n)$ we can find a sequence $(f_n)_{n\in\mathbb{N}}$ with $f_n \in \mathcal{S}(\mathbb{R}^n)$ and such that f_n converges to f in L^2. This is true because $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$. By assertion (i), $e^{it\Delta}$ is unitary and hence we obtain:

$$\|\mathcal{L}'(\phi)(x,t)e^{it\Delta}f_n(x)\|_{L^2} \leq \|\mathcal{L}'(\phi)(x,t)\|_{L^2}\|f_n\|_{L^2} \quad \forall n \in \mathbb{N}.$$

So we can apply Lebesque's dominated convergence theorem and the assertion is proved. $\square$

Remark. We want to emphasize, that for $f \in \mathcal{S}(\mathbb{R}^n)$ the formula

$$e^{it\Delta}(f) = f * K_t \quad \text{for} \quad t \neq 0 \tag{2.11}$$

from Proposition 2.9 (iii) can be extended to functions $f \in L^2(\mathbb{R}^n)$. To this end we sketch a generalization of convolution of distributions. In Stein [17, Chapter 3] convolution is defined for a pair of distributions, if at least one of them has compact support. But in (2.11) the kernel K_t viewed as distribution is not compactly supported and $f \in L^2(\mathbb{R}^n)$ is also not compactly supported in general. Therefore this definition does not apply in this context.

We denote by $O(\mathbb{R}^n)$ the space of functions $\psi \in C^\infty(\mathbb{R}^n)$ such that $\partial^\beta\psi$ is of polynomial growth for each multi-index $\beta \in \mathbb{N}^n$. The fact that the product $\psi \cdot v$ is a tempered distribution whenever $\psi \in O(\mathbb{R}^n)$ and $v \in \mathcal{S}'(\mathbb{R}^n)$, enables us to define

$$v * g := \mathcal{F}^{-1}(\hat{v}\hat{g}) \in \mathcal{S}'(\mathbb{R}^n) \quad \text{for} \quad v \in \mathcal{S}'(\mathbb{R}^n), g \in \mathcal{F}^{-1}(O(\mathbb{R}^n)). \tag{2.12}$$

Because $O(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ it is clear that also $\mathcal{F}^{-1}(O(\mathbb{R}^n)) \subset \mathcal{S}'(\mathbb{R}^n)$, hence the Fourier transform of the function g in the equation is well defined. Let $\mathcal{E}'(\mathbb{R}^n)$ be the space of compactly supported distributions, we recall that for $w \in \mathcal{E}'(\mathbb{R}^n) \subset$

$\mathcal{S}'(\mathbb{R}^n)$ it follows that $\hat{w} \in O(\mathbb{R}^n)$. Therefore $\mathcal{E}'(\mathbb{R}^n) \subset \mathcal{F}^{-1}(O(\mathbb{R}^n))$ and in this case $v * w$ defined as in (2.12), boils down to the classical convolution of distributions. By definition, the convolution from (2.12) satisfies

$$v * g = g * v \quad \text{and} \quad \mathcal{F}(v * g) = \hat{v}\hat{g} \quad \text{for} \quad v \in \mathcal{S}'(\mathbb{R}^n), g \in \mathcal{F}^{-1}(O(\mathbb{R}^n)).$$

Furthermore associativity is fulfilled whenever two of the three tempered distributions from the convolutional product $g * v * w$ are in $\mathcal{F}^{-1}(O(\mathbb{R}^n))$. For further details and proofs that have been omitted in this remark, we refer to Constantin [3, Chapter 6.1].

With these preparations we can finaly extend Proposition 2.9 (iii) to functions in $L^2(\mathbb{R}^n)$.

Corollary 2.11. *Let $t \neq 0$, $K_t(x) = (4\pi i t)^{-n/2} e^{-|x|^2/(4it)}$ then the unitary operator $U_t : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ from Proposition 2.10 can be written in terms of a spatial convolution $U_t(f) = f * K_t$ for $f \in L^2(\mathbb{R}^n)$.*

Proof. We have that the function f is a tempered distribution, since $L^2(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$. Observe that in the proof of Proposition 2.9 (iii) we showed $\widehat{K_t}(\xi) = e^{-it4\pi^2|\xi|^2}$ for every $t \neq 0$. Thus $\widehat{K_t} \in L^\infty(\mathbb{R}^n) \cap O(\mathbb{R}^n)$ which implies that $K_t \in \mathcal{F}^{-1}(O(\mathbb{R}^n))$, so the convolution of $f * K_t$ is defined by (2.12) in the remark above. By Definition 2.8 we see that $\mathcal{F}(U_t(f)) = \hat{f}\widehat{K_t}$, combined with Proposition 2.10(i) this shows that $\hat{f}\widehat{K_t}$ is in $L^2(\mathbb{R}^n)$. Putting these results together we obtain for every $f \in L^2(\mathbb{R}^n)$

$$f * K_t = \mathcal{F}^{-1}(\hat{f}\widehat{K_t}) = \mathcal{F}^{-1}(\mathcal{F}(U_t(f))) = U_t(f) \in L^2(\mathbb{R}^n).$$

Thus the corollary is proved. $\square$

If f is additionally required to be in the space $L^1(\mathbb{R}^n)$, the convolution $f * K_t$ is defined in the classical sense, since $K_t \in L^\infty(\mathbb{R}^n)$. Thus applying Young's inequality, generalizes Proposition 2.9 (iv) to $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, which we will state separately.

Corollary 2.12. *If $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and $u(t, x) = e^{it\Delta}(f)(x)$, then for every $t \neq 0$*

$$\|u(t, x)\|_{L^\infty} \leq (4\pi|t|)^{-n/2}\|u(0, x)\|_{L^1} = (4\pi|t|)^{-n/2}\|f\|_{L^1}.$$

3 Square root of a probability measure

This chapter is from the article of Hörmann [9, Section 3.1]. Recall that a positive Borel probability measure μ on $\mathbb{R}^n$ satisfies $\mu(\mathbb{R}^n) = 1$. We want to define the square root of such a measure by a convolution with a general mollifier.

In what follows we define the mollifier $\rho : \mathbb{R}^n \to \mathbb{R}^+$, which is a normalized Lebesgue-integrable function such that the square root of ρ is integrable too. Explicitly, we demand

$$\rho \in L^1(\mathbb{R}^n), \rho \geq 0, \sqrt{\rho} \in L^1(\mathbb{R}^n), \int \rho(x)dx = 1. \tag{3.1}$$

We define the net $\rho_\varepsilon(x) := \frac{1}{\varepsilon^n}\rho(\frac{x}{\varepsilon^n})$ for $\varepsilon \in]0,1]$ and observe that

$$\|\rho_\varepsilon\|_{L^1} = \|\rho\|_{L^1} = 1 \quad \text{for every } \varepsilon > 0.$$

Definition 3.1. Let $M(\mathbb{R}^n)$ be the space of complex Borel measures on $\mathbb{R}^n$. Then for $\mu \in M(\mathbb{R}^n)$ and $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$ we define the convolution of the measure μ with the function f by

$$\mu * f(x) \coloneqq \int_{\mathbb{R}^n} f(x-y)d\mu(y).$$

That the convolution of the measure μ with ρ is well defined, follows from the next theorem, which is the Young inequality for measures (see Folland [6, Chap 8]). We denote the total variation of a complex measure μ by $||\mu||_{\mathrm{var}}$. The next theorem is Young's inequality for measures.

Theorem 3.2. *Let $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$ and $\mu \in M(\mathbb{R}^n)$. Then the convolution $\mu * f(x)$ exists for almost every x in the Lebesgue sense and the inequality $||\mu * f||_p \leq ||f||_p||\mu||_{var}$ holds.*

Since the variation norm of a complex Borel measure is finite , it follows that $\mu * f \in L^p(\mathbb{R}^n)$. Note that $L^1(\mathbb{R}^n)$ can be embedded in $M(\mathbb{R}^n)$ via the map $f \mapsto \mu_f$ with $\mu_f(B) \coloneqq \int_{\mathbb{R}^n} 1_B(x)f(x)dx$ for every Borel set $B \subseteq \mathbb{R}^n$.

We want to emphasize that $\sqrt{\rho}$ is in $L^2(\mathbb{R}^n)$, if ρ is a mollifier as above. This is a consequence of positivity and integrability of ρ. In the next Proposition we show that the net ρ_ε forms a delta regularization.

Proposition 3.3. *Let $\varepsilon \in]0,1]$, then the net $(\rho_\varepsilon)_{\varepsilon>0} \in L^1(\mathbb{R}^n)$ with ρ_ε defined as above, converges in the sense of tempered distributions in $\mathcal{S}'(\mathbb{R}^n)$ to the delta distribution.*

Proof. We need to show that for all $\phi \in \mathcal{S}(\mathbb{R}^n)$

$$\lim_{\varepsilon\to 0}\langle \rho_\varepsilon, \phi\rangle = \langle \delta, \phi\rangle = \phi(0).$$

By a change of variables we obtain

$$\int_{\mathbb{R}^n} \rho_\varepsilon(x)\phi(x)dx = \int_{\mathbb{R}^n} \frac{1}{\varepsilon^n}\rho(\frac{x}{\varepsilon})\phi(x)dx = \int_{\mathbb{R}^n} \rho(x)\phi(x\varepsilon)dx.$$

Next we define $f_\varepsilon(x) := \rho(x)\phi(x\varepsilon) \in L^1(\mathbb{R}^n)$ and it is obvious that

$$\lim_{\varepsilon\to 0} f_\varepsilon(x) = \rho(x)\phi(0) \quad \text{for almost every } x \in \mathbb{R}^n.$$

Furthermore we note that the inequality

$$|f_\varepsilon(x)| \leq M\rho(x) \in L^1(\mathbb{R}^n) \quad \text{with } M := ||\phi||_\infty$$

holds for all $\varepsilon > 0$. Thus the functions f_ε are dominated by the Lebesgue integrable positive function $M\rho$ and we can apply Lebesgue's dominated convergence theorem to derive

$$\langle \rho_\varepsilon, \phi\rangle = \int_{\mathbb{R}^n} f_\varepsilon(x)dx \to \int_{\mathbb{R}^n} \rho(x)\phi(0)dx = \phi(0) = \langle \delta, \phi\rangle$$

as $\varepsilon \to 0$, which proves the proposition. □

Based on this general observation we define a kind of positive square root of the positive Borel measure μ by the net $\left(\sqrt{\mu * \rho_\varepsilon}\right)_{\varepsilon>0}$. In this case clearly $(\mu * \rho_\varepsilon)(x) \geq 0$ for every $x \in \mathbb{R}^n$.

The next property is mentioned in Hörmann [9, 3.1.(R)].

Proposition 3.4. *Let μ be a positive Borel probability measure on $\mathbb{R}^n$ and $\rho \in L^1(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \rho(x)dx = 1$. For $\varepsilon \in]0,1]$ define the delta regularisation $\rho_\varepsilon(x) = \frac{1}{\varepsilon^n}\rho(\frac{x}{\varepsilon})$ as above. Then the convolution $\mu * \rho_\varepsilon$ converges to μ in the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ and weakly in the space of finite complex Borel measures $M(\mathbb{R}^n)$, i.e.*

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} f(x)(\mu * \rho_\varepsilon)(x)dx = \int_{\mathbb{R}^n} f d\mu \quad \forall f \in C_b(\mathbb{R}^n).$$

Here $C_b(\mathbb{R}^n)$ is the space of bounded continuous functions on $\mathbb{R}^n$.

Proof. We just show weak convergence in $M(\mathbb{R}^n)$, since the Schwarz space $\mathcal{S}(\mathbb{R}^n)$ is a subset of $C_b(\mathbb{R}^n)$ and convergence in $\mathcal{S}'(\mathbb{R}^n)$ follows. Furthermore convergence in $\mathcal{S}'(\mathbb{R}^n)$ can also be observed directly by Proposition 3.3, since there we proved that ρ_ε converges to the Dirac delta in tempered distributions, thus $\mu * \rho_\varepsilon \to \mu * \delta = \mu$ in $\mathcal{S}'(\mathbb{R}^n)$.

Let $f \in C_b(\mathbb{R}^n)$ and plug in the definition of the convolution:

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} f(x)(\mu * \rho_\varepsilon)(x)dx = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \rho_\varepsilon(x-y)f(x)d\mu(y)dx. \tag{3.2}$$

In order to reverse the order of integration in (3.2) we first show that the absolute value of the integrand is $\mu \otimes \lambda$ integrable, where λ is the Lebesgue measure on $\mathbb{R}^n$. This will be done by using Tonelli's theorem to evaluate the integral as an iterated integral. Then we can apply Fubini's theorem to establish the reverse order see [6, Chapter 2.5]. To this end we calculate

$$\begin{aligned}\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\rho_\varepsilon(x-y)f(x)|dxd\mu(y) &\leq ||f||_\infty \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\rho_\varepsilon(x-y)|dxd\mu(y) \\ &= ||f||_\infty \int_{\mathbb{R}^n} 1d\mu(y) = ||f||_\infty < \infty.\end{aligned} \tag{3.3}$$

Since this integral is finite, Tonelli's theorem guarantees that the function $\rho_\varepsilon(x-y)f(x)$ is $\mu \otimes \lambda$ integrable. Thus the requirements for the change of the order of integrations are fulfilled.

In the next step we want to apply the dominated convergence theorem to $\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} h_\varepsilon(y)d\mu(y)$, where we define

$$h_\varepsilon(y) := \int_{\mathbb{R}^n} \rho_\varepsilon(x-y)f(x)dx = \int_{\mathbb{R}^n} \rho(x)f(x\varepsilon + y)dx.$$

It follows by Equation (3.3), that the functions h_ε are in $L^1(\mu)$ and are dominated by the constant function $||f||_\infty \in L^1(\mu)$. By another simple dominated convergence argument with respect to the Lebesgue measure λ it follows that

$$\lim_{\varepsilon \to 0} h_\varepsilon(y) = \int_{\mathbb{R}^n} \rho(x)f(y)dx \quad \forall y \in \mathbb{R}^n.$$

Putting it all together we get the desired result:

$$\begin{aligned}\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} f(x)(\mu * \rho_\varepsilon)(x)dx &= \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \rho_\varepsilon(x-y)f(x)dxd\mu(y) \\ = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \rho(x)f(\varepsilon x + y)dxd\mu(y) &= \int_{\mathbb{R}^n} f(y)d\mu(y).\end{aligned}$$

□

We consider again the properties (3.1) for the mollifier ρ. Then for every $\varepsilon \in]0,1]$ the function $\sqrt{\mu * \rho_\varepsilon}$ is in $L^2(\mathbb{R}^n)$, which follows immediately on the one hand by applying Theorem 3.2, because

$$\|\sqrt{\mu * \rho_\varepsilon}\|_{L^2}^2 = \int_{\mathbb{R}^n} |(\mu * \rho_\varepsilon)(x)|dx = \|\mu * \rho_\varepsilon\|_{L^1} \leq \|\mu\|_{\mathrm{var}} \|\rho_\varepsilon\|_{L^1} = \|\mu\|_{\mathrm{var}}.$$

Alternatively, an application of Fubini's theorem yields the L^2-norm of the function $\sqrt{\mu * \rho_\varepsilon}$ directly

$$\begin{aligned} \|\sqrt{\mu * \rho_\varepsilon}\|_{L^2}^2 &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \rho_\varepsilon(x-y)d\mu(y)dx \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \rho_\varepsilon(x-y)dxd\mu(y) = \int_{\mathbb{R}^n} 1d\mu(y) = 1. \end{aligned} \tag{3.4}$$

Now consider the Cauchy problem (2.1) with initial value $u_\varepsilon|_{t=0} = \sqrt{\mu * \rho_\varepsilon}$, explicitly written as

$$\partial_t u_\varepsilon = i\Delta u_\varepsilon, \quad u_\varepsilon|_{t=0} = \sqrt{\mu * \rho_\varepsilon} \quad \in L^2(\mathbb{R}^n). \tag{3.5}$$

The solution to (3.5) in dependence on $\varepsilon > 0$ is given by Corollary 2.2 in the sense of the strongly continuous unitary group $U_t \coloneqq \exp(it\Delta)$ $(t \in \mathbb{R})$ of operators on $L^2(\mathbb{R}^n)$ by

$$u_\varepsilon(t,x) = U_t(\sqrt{\mu * \rho_\varepsilon}) \quad \forall t \in \mathbb{R}, \varepsilon \in \,]0,1]. \tag{3.6}$$

By the remark to Proposition 2.10, for $t \neq 0$ the solution of (3.5) can be represented in terms of a spatial convolution

$$u_\varepsilon(t,x) = (K_t * \sqrt{\mu * \rho_\varepsilon})(x), \quad \text{where } K_t(x) = (4\pi i t)^{-n/2} e^{-|x|^2/(4it)}. \tag{3.7}$$

Definition 3.5. For any $t \in \mathbb{R}$ let μ_ε^t denote the positive measure on $\mathbb{R}^n$ having density $|u_\varepsilon(t,.)|^2$ with respect to the Lebesgue measure (see Bauer [1]). This means that for a Lebesgue measurable set $A \subset \mathbb{R}^n$ the measure is given by

$$\mu_\varepsilon^t(A) = \int_A |u_\varepsilon(t,x)|^2 dx.$$

Proposition 3.6. *For every $\varepsilon \in \,]0,1]$ and $t \in \mathbb{R}$ the positive measure μ_ε^t on $\mathbb{R}^n$ having density $|u_\varepsilon(t,.)|^2$ is a probability measure, thus $\mu_\varepsilon^t(\mathbb{R}^n) = 1$.*

Proof. To prove the proposition we need to show that $\mu_\varepsilon^t(\mathbb{R}^n) = 1$ for all $t \in \mathbb{R}$ and $\varepsilon \in \,]0,1]$. The representation of $u_\varepsilon(t,x)$ in (3.6) together with the unitarity of the operators U_t yields

$$\begin{aligned} \mu_\varepsilon^t(\mathbb{R}^n) &= \int_{\mathbb{R}^n} |u_\varepsilon(t,x)|^2 dx = \int_{\mathbb{R}^n} |U_t\sqrt{\mu * \rho_\varepsilon}(x)|^2 dx \\ &= \|U_t\sqrt{\mu * \rho_\varepsilon}\|_{L^2}^2 = \|\sqrt{\mu * \rho_\varepsilon}\|_{L^2}^2. \end{aligned}$$

By equation (3.4) we conclude that

$$\mu_\varepsilon^t(\mathbb{R}^n) = \|\sqrt{\mu * \rho_\varepsilon}\|_{L^2}^2 = 1,$$

which completes the proof. $\square$

We recall the fact, that every positive distribution is a positive measure (see Hörmander [8, Theorem 2.1.7]). Therefore the delta distribution can be considered as the Borel probability measure $\delta \in M(\mathbb{R}^n)$, which assigns for every Borel set $A \subset \mathbb{R}^n$

$$\delta(A) = \begin{cases} 0, & \text{if } 0 \notin A \\ 1 & \text{if } 0 \in A \end{cases}.$$

The measure δ is also called the Dirac measure concentrated at zero.

If we set in the Schrödinger equation (3.5) the initial Borel probability measure μ to be the Dirac measure δ, then

$$u_\varepsilon|_{t=0}(x) = u_\varepsilon(0,x) = \sqrt{\delta * \rho_\varepsilon(x)} = \sqrt{\rho_\varepsilon(x)}. \tag{3.8}$$

In this setting it follows by the representation of $u_\varepsilon(t,x)$ in terms of a spatial convolution (3.7), that for every $t \neq 0$

$$\begin{aligned} u_\varepsilon(t,x) = \left(K_t * \sqrt{\delta * \rho_\varepsilon}\right)(x) &= K_t(x) * \sqrt{\int_{\mathbb{R}^n} \rho_\varepsilon(x-y) d\delta(y)} \\ &= \left(K_t * \sqrt{\rho_\varepsilon}\right)(x). \end{aligned} \tag{3.9}$$

With these preliminaries we obtain the following bounds of the solutions u_ε.

Theorem 3.7. *Let $\mu \in M(\mathbb{R}^n)$ be the Dirac-measure δ concentrated at zero, $\rho \in L^1(\mathbb{R}^n)$ defined as in (3.1) and $\rho_\varepsilon(x) := \frac{1}{\varepsilon^n}\rho(\frac{x}{\varepsilon^n})$ for $\varepsilon \in]0,1]$. Hence by (3.9) $u_\varepsilon(t,x) = (K_t * \sqrt{\rho_\varepsilon})(x)$ for every $t \neq 0$ and $u_\varepsilon(0,x) = \sqrt{\rho_\varepsilon(x)}$. Then for every $t \neq 0$*

$$\|u_\varepsilon(t,x)\|_{L^\infty} \leq \frac{\|\sqrt{\rho}\|_{L^1}}{(4\pi|t|)^{\frac{n}{2}}}\varepsilon^{\frac{n}{2}}. \tag{3.10}$$

Proof. The assumptions of Corollary 2.12 are clearly fulfilled and therefore

$$\|u_\varepsilon(t,x)\|_{L^\infty} \leq \frac{\|u_\varepsilon(0,x)\|_{L^1}}{(4\pi|t|)^{\frac{n}{2}}}. \tag{3.11}$$

Since we have as initial probability measure the Dirac measure and thus $u_\varepsilon(0,x) = \sqrt{\rho_\varepsilon(x)}$, we calculate

$$\begin{aligned}\|u_\varepsilon(0,x)\|_{L^1} &= \int_{\mathbb{R}^n} \sqrt{\rho_\varepsilon(x)}dx = \int_{\mathbb{R}^n} \sqrt{\frac{1}{\varepsilon^n}\rho\left(\frac{x}{\varepsilon}\right)}dx \\ &= \int_{\mathbb{R}^n} \varepsilon^n \sqrt{\frac{1}{\varepsilon^n}\rho\,(u)}du = \varepsilon^{\frac{n}{2}}\|\sqrt{\rho}\|_{L^1}.\end{aligned} \tag{3.12}$$

The proof of the theorem is completed, if we insert (3.12) into the inequality (3.11). □

Remark (quantum mechanical interpretation). Equation (2.1) is the Schrödinger equation for a particle of mass equal to $\frac{1}{2}$ with zero potential. This means, that the particle is not exposed to an external force. In quantum mechanics the solution to the Schrödinger equation $u(t,x)$ is interpreted in the following way:

The probability μ of finding the particle in a Lebesque set $E \subset \mathbb{R}^n$ at time $t \in \mathbb{R}$, is given by $\mu^t(E) = \int_E |u(t,x)|^2 dx$. Hence as described in Proposition 3.6, the function $|u(t,.)|^2$ is the density for the probability measure μ. We note, that the density of the probability, that the momentum of a particle lies in E at time t is given by $|\hat{u}(t,.)|^2$ (see Rauch [13, Section 3.2]).

In this setting the initial value $u(0) \in L^2(\mathbb{R}^n)$ in equation (2.1) is given and $|u(0)|^2$ is an initial density of a probability measure μ.

In the article of Hörmann [9, Section 1] the starting point was the reverse assumption. Let an initial probability measure μ be given, which should describe the probability that the particle is in E at time zero by $\mu(E)$. Thus with the regularization $u_\varepsilon(0) := \sqrt{\mu * \rho_\varepsilon} \in L^2(\mathbb{R}^n)$ in equation (3.5) we constructed initial values for the Schrödinger equation, such that the solution at $t = 0$ as ε tends to zero fulfils $\mu(E) = \int_E |u(0)(x)|^2 dx$. For the rest of this thesis we analyse convergence properties of the probability measures μ_ε^t as $\varepsilon \to 0$ for $t \neq 0$ in different topologies on the space of complex Borel measures $M(\mathbb{R}^n)$.

4 Convergence properties of the measures μ_ε^t

Recall that we denote by $C_b(\mathbb{R}^n)$ the space of bounded continuous functions $\mathbb{R}^n \to \mathbb{C}$, respectively by $C_0(\mathbb{R}^n)$ the space of continuous functions $\mathbb{R}^n \to \mathbb{C}$ vanishing at infinity and by $C_{\mathcal{C}}(\mathbb{R}^n)$ the space of continuous functions $\mathbb{R}^n \to \mathbb{C}$ with compact support. It is clear by definition that $C_{\mathcal{C}}(\mathbb{R}^n) \subset C_0(\mathbb{R}^n) \subset C_b(\mathbb{R}^n)$.

Let X and Y be vector spaces, and $(x,y) \mapsto \langle x,y\rangle \in \mathbb{C}$ a bilinear map on $X \times Y$ then $(X, Y, \langle .,.\rangle)$ is called a dual pairing if

$$\begin{aligned}&\forall x \in X \setminus \{0\}\ \exists y \in Y \quad \text{such that } \langle x,y\rangle \neq 0 \quad \text{and} \\ &\forall y \in Y \setminus \{0\}\ \exists x \in X \quad \text{such that } \langle x,y\rangle \neq 0 \quad .\end{aligned}$$

Now let $(X, Y, \langle .,. \rangle)$ be a dual pairing then $\sigma(X, Y)$ denotes the locally convex topology on X generated by the set of seminorms $\{p_y : y \in Y\}$, where $p_y(x) := |\langle x, y\rangle|$ (see Werner [21, Chapter 8]).

Let I be a directed set and recall that the net $(x_i)_{i\in I}$ converges in $(X, \sigma(X, Y))$ to 0 if and only if the net $\langle x_i, y\rangle_{i\in I}$ converges to 0 in $\mathbb{C}$ for every $y \in Y$.

Furthermore let $(M(\mathbb{R}^n), C_b(\mathbb{R}^n), \langle .,. \rangle)$ denote the dual pairing via the bilinear map $\langle .,. \rangle : M(\mathbb{R}^n) \times C_b(\mathbb{R}^n) \to \mathbb{C}$, $\langle \mu, f\rangle := \int_{\mathbb{R}^n} f(x)d\mu(x)$.

Definition 4.1. Let $(\tau_k)_{k\in\mathbb{N}} \in M(\mathbb{R}^n)$ and $\tau \in M(\mathbb{R}^n)$, then we say that the measures τ_k converge *vaguely* or with respect to the *vague topology* to τ, if $(\tau_k)_{k\in\mathbb{N}}$ converges in the locally convex topology $\sigma(M(\mathbb{R}^n), C_c(\mathbb{R}^n))$ to τ, thus

$$\tau_k \to \tau \quad \text{vaguely,} \quad \text{if} \quad \forall f \in C_c(\mathbb{R}^n)) \quad \langle \tau_k, f\rangle \to \langle \tau, f\rangle \quad \text{as } k \to \infty.$$

Moreover we say that $(\tau_k)_{k\in\mathbb{N}}$ converges *weakly in the sense of probability theory* to τ, if $\tau_k \to \tau$ in the topology $\sigma(M(\mathbb{R}^n), C_b(\mathbb{R}^n))$ for $k \to \infty$ (see Bauer [1]). As usual a similar definition applies in the situation of nets τ_ε, $\varepsilon > 0$.

In what follows, we want to investigate convergence properties of the net of measures μ^t_ε, again having density $|u_\varepsilon(t, x)|^2 = |U_t\sqrt{\mu * \rho_\varepsilon}|^2$ with respect to the Lebesgue measure. In order to use the inequality of Theorem 3.7, we set the initial probability measure μ to be the Dirac measure δ on $\mathbb{R}^n$.

We will consider the net μ^t_ε for every $t \neq 0$ as $\varepsilon \to 0$, with respect to $\mathcal{S}'(\mathbb{R}^n)$, the vague topology, the topology $\sigma(M(\mathbb{R}^n), C_0(\mathbb{R}^n))$ and the weak topology in the sense of probability theory. The results can be found in Hörmann [9, Section 3.2].

Let $\varepsilon \in]0, 1]$, first we note that by Theorem 3.7 it is clear that the net $u_\varepsilon(t, x)$ converges uniformly to 0. Since a function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is in $L^1(\mathbb{R}^n)$, we obtain by uniform convergence of $u_\varepsilon(t, x)$ to 0, that for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$$\begin{aligned}\langle \mu^t_\varepsilon, \varphi\rangle &= \int_{\mathbb{R}^n} \varphi(x)|u_\varepsilon(t, x)|^2 dx \\ &\leq \|u_\varepsilon(t, .)\|^2_{L^\infty(\mathbb{R}^n)}\|\varphi\|_{L^1(\mathbb{R}^n)} \to 0 \quad \text{as} \quad \varepsilon \to 0.\end{aligned}$$

Hence this implies that $\mu^t_\varepsilon \to 0$ as $\varepsilon \to 0$ in $\mathcal{S}'(\mathbb{R}^n)$.

A similar calculation can be applied to $\langle \mu^t_\varepsilon, f\rangle$, where $f \in C_c(\mathbb{R}^n)$ and therefore μ^t_ε converges vaguely to 0.

To prove that $\mu^t_\varepsilon \to 0$ as $\varepsilon \to 0$ in the weak* topology $\sigma(M(\mathbb{R}^n), C_0(\mathbb{R}^n))$, we split the integral

$$\langle \mu^t_\varepsilon, f\rangle = \int_{\mathbb{R}^n} f(x)|u_\varepsilon(t, x)|^2 dx \quad f \in C_0(\mathbb{R}^n),$$

into a sum of an integral over a compact set $K \subset \mathbb{R}^n$ and an integral over the complementary set $\mathbb{R}^n \setminus K$. Thus we obtain the inequality

$$|\langle \mu_\varepsilon^t, f \rangle| \leq \int_K |f(x)||u_\varepsilon(t,x)|^2 dx + \int_{\mathbb{R}^n \setminus K} |f(x)||u_\varepsilon(t,x)|^2 dx. \tag{4.1}$$

By Theorem 3.7 we can estimate the first integral over the compact set K in (4.1) as follows:

$$\begin{aligned} \int_K |f(x)||u_\varepsilon(t,x)|^2 dx &\leq \|u_\varepsilon(t,.)\|_{L^\infty}^2 \int_K |f(x)| dx \\ &\leq \frac{\|\sqrt{\rho}\|_{L^1}^2}{(4\pi|t|)^n} \varepsilon^n \int_K |f(x)| dx. \end{aligned} \tag{4.2}$$

If we apply Proposition 3.6 we conclude for the second integral in (4.1) over the complementary set $\mathbb{R}^n \setminus K$ that

$$\begin{aligned} \int_{\mathbb{R}^n \setminus K} |f(x)||u_\varepsilon(t,x)|^2 dx &\leq \|f|_{\mathbb{R}^n \setminus K}\|_\infty \int_{\mathbb{R}^n} |u_\varepsilon(t,x)|^2 dx \\ &= \|f|_{\mathbb{R}^n \setminus K}\|_\infty 1. \end{aligned} \tag{4.3}$$

Where we denote by $f|_{\mathbb{R}^n \setminus K}$ the restriction of the function f to the set $\mathbb{R}^n \setminus K$.

For $\eta > 0$ arbitrary we can choose a compact set $K \subset \mathbb{R}^n$ such that $\|f|_{\mathbb{R}^n \setminus K}\|_\infty < \frac{\eta}{2}$, since f is a continuous function vanishing at infinity. Furthermore let $I_{K,f} := \int_K |f(x)| dx < \infty$, then there exists an $\varepsilon_1 \in]0,1]$ with the property that

$$\varepsilon_1 < \frac{4\pi|t|}{\|\sqrt{\rho}\|_{L^1}^{2/n}} \left(\frac{\eta}{2I_{K,f}} \right)^{1/n}.$$

Hence we obtain for $f \in C_0(\mathbb{R}^n)$ by inserting the inequalities (4.2) and (4.3) in (4.1) for every $\varepsilon < \varepsilon_1$ the inequality

$$\begin{aligned} |\langle \mu_\varepsilon^t, f \rangle| &\leq I_{K,f} \|u_\varepsilon(t,.)\|_{L^\infty}^2 + \|f|_{\mathbb{R}^n \setminus K}\|_\infty \\ &< \frac{\eta}{2} + \frac{\eta}{2} = \eta, \end{aligned}$$

which proves $\mu_\varepsilon^t \to 0$ in $\sigma(M(\mathbb{R}^n), C_0(\mathbb{R}^n))$.

This can also be proven by a more general Lemma from Storch,Wiebe [19, Chapter 17.D].

Lemma 4.2. *Let $\mu_k, \mu \in M(\mathbb{R}^n)$, $k \in \mathbb{N}$ such that the norms $\|\mu_k\|_{var}$ and $\|\mu\|_{var}$ are bounded by a fixed finite constant $M \in [0, \infty[$. Furthermore let $F \subseteq C_c(\mathbb{R}^n)$*

be a dense subset of $C_c(\mathbb{R}^n)$*. If*

$$\forall f \in F \quad \lim_{k\to\infty} \langle \mu_k, f\rangle = \langle \mu, f\rangle$$

then the sequence μ_k *converges vaguely and in the topology of* $\sigma(M(\mathbb{R}^n), C_0(\mathbb{R}^n))$*.*

Proof. Let $g \in C_0(\mathbb{R}^n)$ and $\varepsilon > 0$. Then there exists a function $f \in F$ such that $\|g - f\|_\infty \leq \varepsilon$. Therefore,

$$\begin{aligned}
|\langle \mu_k, g\rangle - \langle \mu, g\rangle| &\leq |\langle \mu_k, g\rangle - \langle \mu_k, f\rangle| + |\langle \mu_k, f\rangle - \langle \mu, f\rangle| + |\langle \mu, f\rangle - \langle \mu, g\rangle| \\
&\leq \|g - f\|_\infty \|\mu_k\|_{\mathrm{var}} + |\langle \mu_k, f\rangle - \langle \mu, f\rangle| + \|g - f\|_\infty \|\mu\|_{\mathrm{var}} \\
&\leq 2\varepsilon M + |\langle \mu_k, f\rangle - \langle \mu, f\rangle|.
\end{aligned}$$

Together with the assumption $\langle \mu_k, f\rangle \to \langle \mu, f\rangle$ for $f \in F$ this completes the proof of the lemma. □

Until now we have shown $\mu_\varepsilon^t \to 0$ in $\mathcal{S}'(\mathbb{R}^n)$, vaguely and in the topology $\sigma(M(\mathbb{R}^n), C_0(\mathbb{R}^n))$.

On the other hand we will now prove, that μ_ε^t does not converge weakly in the sense of probability theory. This is easily shown, because if we assume that μ_ε^t converges to some Borel measure τ in $\sigma(M(\mathbb{R}^n), C_b(\mathbb{R}^n))$ then the limit τ would have to be equal to the vague limit. This leads to the contradiction

$$\langle \mu_\varepsilon^t, 1\rangle = \mu_\varepsilon^t(\mathbb{R}^n) = 1 \not\to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Thus μ_ε^t does not converge in $\sigma(M(\mathbb{R}^n), C_b(\mathbb{R}^n))$.

4.1 Example in one spatial dimension

We now want to focus on the failure of weak convergence in probability of the net μ_ε^t where $\varepsilon \in]0, 1]$ and give an example of a function $f \in C_b(\mathbb{R})$, such that there are uncountably many cluster points of the net $\langle \mu_\varepsilon^t, f\rangle$. This means that there exists an uncountable set $V \subset \mathbb{C}$ such that for every $v \in V$ we can find a null sequence $(\varepsilon_n)_{n\in\mathbb{N}} \subset]0, 1]$ with the property $\lim_{\varepsilon_n\to 0}\langle \mu_{\varepsilon_n}^t, f\rangle = v$.

The aim for the construction of such an example is that we hope to find a subspace $X \subset C_b(\mathbb{R})$ of the continuous bounded functions, for which the net μ_ε^t converges in the topology $\sigma(M(\mathbb{R}), X)$.

Example Let $f \in C_b(\mathbb{R})$ be given by $f(x) = e^{i\log(1+|x|)}$ for $x \in \mathbb{R}$. Furthermore let again the Dirac-measure be the initial probability measure in the Schrödinger equation with the Gaussian as mollifier. Hence

$$\rho(x) := e^{-\pi x^2} \quad \text{and } \rho_\varepsilon(x) = \frac{1}{\varepsilon} e^{\frac{-\pi x^2}{\varepsilon^2}}.$$

Therefore by (3.8) we can write the Schrödinger equation as

$$\partial_t u_\varepsilon = i\Delta u_\varepsilon, \quad u_\varepsilon(0,x) = \sqrt{\rho_\varepsilon(x)} = \sqrt{\frac{1}{\varepsilon} e^{\frac{-\pi x^2}{\varepsilon^2}}} = \frac{1}{\sqrt{\varepsilon}} e^{\frac{-\pi x^2}{2\varepsilon^2}}. \tag{4.4}$$

Proposition 2.10 states the solution $u_\varepsilon(x,t)$ is given by the application of the operator $e^{it\Delta}$ to the function $\sqrt{\rho_\varepsilon(x)}$. Since we want to calculate a concrete formula for $u_\varepsilon(x,t)$ we first consider its Fourier transform and obtain by Definition 2.8,

$$\widehat{u_\varepsilon}(t,\xi) = \mathcal{F}\left(e^{it\Delta}\sqrt{\rho_\varepsilon}\right)(\xi) = e^{-it4\pi^2\xi^2}\mathcal{F}\left(\frac{1}{\sqrt{\varepsilon}} e^{\frac{-\pi x^2}{2\varepsilon^2}}\right)(\xi).$$

To calculate the Fourier transform of $\sqrt{\rho_\varepsilon}$ we define for $a > 0$ the scaled Gaussian $G_a(x) := \frac{1}{\sqrt{a}} e^{\frac{-\pi x^2}{a}}$ and recall that in this case $\widehat{G}_a(x) = e^{-\pi a x^2}$ see e.g. Stein [16]. Since the function $\sqrt{\rho_\varepsilon}$ is again a Gaussian, we can rewrite it in the following way:

$$\sqrt{\rho_\varepsilon(x)} = \frac{1}{\sqrt{\varepsilon}} e^{\frac{-\pi x^2}{2\varepsilon^2}} = \sqrt{2\varepsilon}\frac{1}{\sqrt{2\varepsilon^2}} e^{\frac{-\pi x^2}{2\varepsilon^2}} = \sqrt{2\varepsilon}G_{2\varepsilon^2}(x).$$

This leads us, by using the formula for the Fourier transform of a scaled Gaussian, to the representation

$$\begin{aligned}\widehat{u_\varepsilon}(t,\xi) &= e^{-it4\pi^2\xi^2}\mathcal{F}\left(\sqrt{2\varepsilon}G_{2\varepsilon^2}(x)\right) = \sqrt{2\varepsilon}e^{-it4\pi^2\xi^2}e^{-2\pi\varepsilon^2\xi^2}\\ &= \sqrt{2\varepsilon}e^{-\pi\xi^2\left(4i\pi t+2\varepsilon^2\right)} = \sqrt{2\varepsilon}\widehat{G}_{(4i\pi t+2\varepsilon^2)}(\xi).\end{aligned}$$

We are now able to write the solutions to (4.4) in an explicit formula for every $\varepsilon \in \,]0,1]$, as

$$\begin{aligned}u_\varepsilon(t,x) &= \mathcal{F}^{-1}(\widehat{u_\varepsilon}(t,\xi))(x) = \sqrt{2\varepsilon}\mathcal{F}^{-1}(\widehat{G}_{(4i\pi t+2\varepsilon^2)})\\ &= \sqrt{2\varepsilon}G_{(4i\pi t+2\varepsilon^2)}(x) = \sqrt{\frac{\varepsilon}{2i\pi t+\varepsilon^2}}\exp\left(\frac{-\pi x^2}{4i\pi t+2\varepsilon^2}\right)\end{aligned}$$

Since we are interested in the net $\langle \mu_\varepsilon^t, f\rangle = \int_{-\infty}^{\infty} f(x)|u_\varepsilon(t,x)|^2 dx$, we need to investigate the term $|u_\varepsilon(t,x)|^2$. If we use the basic rules about complex conjugation, especially the fact that for every $z \in \mathbb{C}$ we have $\overline{\sqrt{z}} = \sqrt{\overline{z}}$, one obtains

$$\begin{aligned}
|u_\varepsilon(t,x)|^2 &= u_\varepsilon(t,x)\overline{u_\varepsilon(t,x)} \\
&= \varepsilon(2i\pi t+\varepsilon^2)^{-1/2}(-2i\pi t+\varepsilon^2)^{-1/2}\left|\exp\left(\frac{-\pi x^2}{4i\pi t+2\varepsilon^2}\right)\right|^2 \\
&= \frac{\varepsilon}{\sqrt{\varepsilon^4+4\pi^2t^2}}\left(\exp\left(\frac{-\pi x^2\varepsilon^2}{2\varepsilon^4+8\pi^2t^2}\right)\right)^2 \\
&= \frac{\varepsilon}{\sqrt{\varepsilon^4+4\pi^2t^2}}\exp\left(\frac{-\pi x^2\varepsilon^2}{\varepsilon^4+4\pi^2t^2}\right) \\
&= \frac{\varepsilon}{\sqrt{\varepsilon^4+4\pi^2t^2}}\rho\left(\frac{\varepsilon}{\sqrt{\varepsilon^4+4\pi^2t^2}}x\right).
\end{aligned}$$

This yields the suitable form

$$|u_\varepsilon(t,x)|^2 = c_\varepsilon(t)\rho(c_\varepsilon(t)x), \quad \text{with} \quad c_\varepsilon(t) := \frac{\varepsilon}{\sqrt{\varepsilon^4+4\pi^2t^2}}.$$

Furthermore we note that the net $c_\varepsilon(t)$ converges to zero as $\varepsilon \to 0$.

We remark that the continuous bounded function $f(x) = e^{i\log(1+|x|)}$ and the mollifier $\rho(x) = e^{-\pi x^2}$ are clearly symmetric. Therefore we can convert the integral of the product of these two functions over $\mathbb{R}$ into an integral over the nonnegative real numbers $[0,\infty[$, hence

$$\begin{aligned}
\langle \mu_\varepsilon^t, f\rangle &= \int_{-\infty}^{\infty} f(x)|u_\varepsilon(t,x)|^2dx = \int_{-\infty}^{\infty} f(x)c_\varepsilon(t)\rho(c_\varepsilon(t)x)dx \\
&= 2\int_0^{\infty} f(x)c_\varepsilon(t)\rho(c_\varepsilon(t)x)dx = 2c_\varepsilon(t)\int_0^{\infty} e^{i\log(1+x)}\rho(c_\varepsilon(t)x)dx.
\end{aligned}$$

By a change of variables in the last integral, setting $y = c_\varepsilon(t)x$ gives

$$\langle \mu_\varepsilon^t, f\rangle = 2\int_0^{\infty} e^{i\log\left(\frac{c_\varepsilon(t)+y}{c_\varepsilon(t)}\right)}\rho(y)dy = 2e^{-i\log c_\varepsilon(t)}\int_0^{\infty} e^{i\log(c_\varepsilon(t)+y)}\rho(y)dy. \tag{4.5}$$

Furthermore the last integrand $e^{i\log(c_\varepsilon(t)+y)}\rho(y)$ is bounded by $\rho(y)$ and converges pointwise to $e^{i\log y}\rho(y)$ as $\varepsilon \to 0$, this yields by dominated convergence the integral limit

$$\lim_{\varepsilon\to 0}\int_0^{\infty} e^{i\log(c_\varepsilon(t)+y)}\rho(y)dy = \int_0^{\infty} e^{i\log y}\rho(y)dy =: \gamma. \tag{4.6}$$

In order to calculate the different cluster points of the net $\langle \mu_\varepsilon^t, f\rangle_{\varepsilon\in\ ]0,1]}$, we need to show that the integral $\gamma \in \mathbb{C}$ is non zero. To this end we recall that the

Γ-function is defined as

$$\Gamma : \{z \in \mathbb{C} | \operatorname{Re}(z) > 0\} \longrightarrow \mathbb{C} \quad \text{with} \quad \Gamma(z) = \int_0^\infty y^{z-1} e^{-y} dy,$$
$$\text{with} \quad y^{(z-1)} := e^{(z-1)\log y}, \quad \log y \in \mathbb{R}.$$

Since the Gamma-function can be represented in the product formula

$$\Gamma(z) = \lim_{n\to\infty} \frac{n^z n!}{z(z+1)\cdots(z+n)} = \frac{1}{z}\prod_{k=1}^{\infty} \frac{\left(1+\frac{1}{k}\right)^z}{\left(1+\frac{z}{k}\right)},$$

the so called Gauss-product representation, it follows that the Gamma-function possesses no zeros. For this result and further properties of the Γ-function we refer to Storch,Wiebe [18, Chapter 17.B].

We proceed with equation (4.6) and want to prove that

$$\gamma = \frac{1}{2}\pi^{-\frac{i+1}{2}}\Gamma\left(\frac{i+1}{2}\right).$$

This will also show that $\gamma \neq 0$. To obtain the equality we make a change of variables in (4.6), by setting $y = \frac{\sqrt{v}}{\sqrt{\pi}}$. Thus by noting that ρ is the Gaussian mollifier we calculate

$$\begin{aligned}
\gamma &= \int_0^\infty e^{i\log y} e^{-\pi y^2} dy = \int_0^\infty \frac{1}{2\sqrt{v\pi}} e^{i\log\left(\frac{\sqrt{v}}{\sqrt{\pi}}\right)} e^{-v} dv \\
&= \int_0^\infty \frac{1}{2\sqrt{\pi}} v^{-\frac{1}{2}} e^{i\log\sqrt{v}} e^{-i\log\sqrt{\pi}} e^{-v} dv = \frac{1}{2}\pi^{-\frac{i+1}{2}} \int_0^\infty \frac{1}{2} e^{\left(\frac{i-1}{2}\right)\log v} e^{-v} dv \\
&= \frac{1}{2}\pi^{-\frac{i+1}{2}} \int_0^\infty v^{\frac{i+1}{2}-1} e^{-v} dv = \frac{1}{2}\pi^{-\frac{i+1}{2}}\Gamma\left(\frac{i+1}{2}\right).
\end{aligned}$$

With these preparations we are now able to find an uncountable set of cluster points. Let $\alpha \in [0, 2\pi[$ then there exists an integer $n_\alpha \in \mathbb{N}$ and a positive null sequence $(\varepsilon_n)_{n>n_\alpha}$, where $\varepsilon_n \in]0,1]$ for every $n \in \{n_\alpha, n_\alpha+1, \ldots\}$, such that $c_{\varepsilon_n}(t) = e^{-\alpha-2\pi n}$. This is justified by the fact that the sequence $e^{-\alpha-2\pi n}$ and the net $c_\varepsilon(t) = \frac{\varepsilon}{\sqrt{\varepsilon^4+4\pi^2t^2}}$ converge to zero and furthermore since $c_\varepsilon(t)$ is continuous in ε. Thus by the intermediate value theorem there exists such a sequence.

Together with (4.5) and (4.6)we infer

$$\begin{aligned}
\lim_{\substack{n\to\infty\\ n\geq n_\alpha}} \langle \mu^t_{\varepsilon_n}, f\rangle &= \lim_{\substack{n\to\infty\\ n\geq n_\alpha}} \int_0^\infty e^{i\log(c_{\varepsilon_n}(t)+y)}\rho(y)dy \\
&= 2\gamma \lim_{\substack{n\to\infty\\ n\geq n_\alpha}} e^{-i\log\left(e^{-\alpha-2\pi n}\right)} = 2\gamma e^{i\alpha}.
\end{aligned}$$

Since $\gamma \neq 0$, $\alpha \neq \beta \in [0, 2\pi[$ implies $2\gamma e^{i\alpha} \neq 2\gamma e^{i\beta}$. Thus, we have found uncountably many clusterpoints of the net $\langle \mu^t_\varepsilon, f\rangle_{\varepsilon\in]0,1]}$.

4.2 Convergence on bounded functions possessing limits at infinity

Our next goal is to find a subspace $\tilde{C} \subset \mathbb{C}_b(\mathbb{R})$, such that a limit of the net $\langle \mu_\varepsilon^t, f\rangle_{\varepsilon\in\ ]0,1]}$ exists for every function f in the space $\tilde{C}$. From the construction of clusterpoints in the example of the last subsection we expect, that functions which have limits at $\pm\infty$ may fulfil the desired property. Thus we specify for further considerations the spaces

$$\begin{aligned}
C_\pm(\mathbb{R}) &= \{f \in C_b(\mathbb{R}) | \exists \mathsf{L}_-(f) \text{ and } \exists \mathsf{L}_+(f)\} \\
L^\infty_\pm(\mathbb{R}) &= \{f \in L^\infty(\mathbb{R}) | \exists \mathsf{L}_-(f) \text{ and } \exists \mathsf{L}_+(f)\} \quad \text{where} \\
\mathsf{L}_\pm(f) &= \lim_{x\to\pm\infty} f(x).
\end{aligned}$$

It is clear that by the above definition we have $C_\pm(\mathbb{R}) \subset L^\infty_\pm(\mathbb{R})$. With these remarks we now prove the following

Proposition 4.3. *If $f \in L^\infty_\pm(\mathbb{R})$ and μ_ε^t is given by the Dirac-measure as initial probability measure, then*

$$\lim_{\varepsilon\to 0} \langle \mu_\varepsilon^t, f\rangle = \frac{\mathsf{L}_-(f) + \mathsf{L}_+(f)}{2}.$$

Proof. Let f be as in the hypothesis. We split the following integral into three parts

$$\begin{aligned}
\langle \mu_\varepsilon^t, f\rangle &= \int_{-\infty}^{-1} f(x)|u_\varepsilon(t,x)|^2 dx + \int_{-1}^{1} f(x)|u_\varepsilon(t,x)|^2 dx + \int_{1}^{\infty} f(x)|u_\varepsilon(t,x)|^2 dx \\
&=: A_\varepsilon + B_\varepsilon + C_\varepsilon.
\end{aligned}$$

An application of Theorem 3.7 to the term B_ε yields the estimate

$$|B_\varepsilon| \le \frac{\|\sqrt{\rho}\|^2_{L^1}}{(4\pi|t|)} \varepsilon \int_{-1}^{1} f(x) dx.$$

Thus the term B_ε converges to zero as $\varepsilon \to 0$.

Remember that by the representation (3.7) of the solutions $u_\varepsilon(t,x) = \sqrt{\rho_\varepsilon} * K_t$ as a spatial convolution we write by a change of variable, where we set $\frac{y}{\varepsilon} = z$

$$\begin{aligned}
|u_\varepsilon(t,x)|^2 &= \frac{1}{4\pi|t|} \left| \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4it}} \sqrt{\frac{1}{\varepsilon}\rho(\frac{y}{\varepsilon})} dy \right|^2 \\
&= \frac{\varepsilon}{4\pi|t|} \left| \int_{-\infty}^{\infty} e^{\frac{iz^2\varepsilon^2}{4t}} e^{-\frac{i\varepsilon z x}{2t}} \sqrt{\rho(z)} dz \right|^2.
\end{aligned}$$

As we proceed we will show that $C_\varepsilon \to \mathsf{L}_+(f)/2$ as $\varepsilon \to 0$. This will proof the proposition, since the calculation of $\lim_{\varepsilon\to 0} A_\varepsilon = \mathsf{L}_-(f)/2$ follows by an analogous calculation.

If we use again a change of variables in the representation of the last equation with $x = \frac{r}{\varepsilon}$, we infer

$$
\begin{aligned}
C_\varepsilon &= \int_1^\infty \frac{\varepsilon}{4\pi|t|} \left| \int_{-\infty}^\infty e^{\frac{iz^2\varepsilon^2}{4t}} e^{-\frac{i\varepsilon zx}{2t}} \sqrt{\rho(z)} dz \right|^2 f(x) dx \\
&= \frac{1}{4\pi|t|} \int_\varepsilon^\infty \left| \int_{-\infty}^\infty e^{\frac{iz^2\varepsilon^2}{4t}} e^{-\frac{izr}{2t}} \sqrt{\rho(z)} dz \right|^2 f(\frac{r}{\varepsilon}) dr \\
&=: \frac{1}{4\pi|t|} \int_\varepsilon^\infty |h_\varepsilon(r)|^2 f(\frac{r}{\varepsilon}) dr.
\end{aligned} \tag{4.7}
$$

Where we introduced $h_\varepsilon(r)$ for simplification of notation. In the next step we apply another change of variables with $z = 4\pi tv$ to write $h_\varepsilon(r)$ as a scaled Fourier transform of $\sqrt{\rho}$, thus

$$
\begin{aligned}
h_\varepsilon(r) &= \int_{-\infty}^\infty 4|t|\pi e^{4i\varepsilon^2\pi^2 tv} \sqrt{\rho(4\pi tv)} e^{-2\pi irv} dv \\
&= 4|t|\pi \mathcal{F}_{v\to r} \left(e^{4i\varepsilon^2\pi^2 tv} \sqrt{\rho(4\pi tv)} \right)(r).
\end{aligned}
$$

With this equation it follows that as $\varepsilon \to 0$ the net of functions $h_\varepsilon(r)$ converges to $h(r) := 4|t|\pi\mathcal{F}\left(\sqrt{\rho(4\pi tv)}\right)(r)$ in $L^2(\mathbb{R})$. To see this we use Parseval's theorem and derive

$$
\begin{aligned}
&\frac{1}{16|t|\pi^2} \int_{-\infty}^\infty |h_\varepsilon(r) - h(r)|^2 dr = \left\| \mathcal{F}\left(\sqrt{\rho(4\pi tv)} \left(e^{4i\varepsilon^2\pi^2 tv} - 1 \right) \right)(r) \right\|_{L^2}^2 \\
&= \left\| \sqrt{\rho(4\pi tv)} \left(e^{4i\varepsilon^2\pi^2 tv} - 1 \right) \right\|_{L^2}^2 = \int_{-\infty}^\infty \rho(4\pi tv) \left| e^{4i\varepsilon^2\pi^2 tv} - 1 \right|^2 dv.
\end{aligned}
$$

Where the last integral converges to zero by dominated convergence, since by definition the mollifier ρ is in $L^1(\mathbb{R})$ and the integrand converges pointwise to zero. Thus we have proved $h_\varepsilon \to h$ in $L^2(\mathbb{R})$ as $\varepsilon \to 0$.

The next task will be to show that

$$
\lim_{\varepsilon\to 0} C_\varepsilon = \frac{\mathsf{L}_+(f)}{4|t|\pi} \int_0^\infty |h(r)|^2 dr. \tag{4.8}
$$

To this end we insert (4.7) for C_ε and estimate

$$\begin{aligned}
&4\pi|t|\left|C_\varepsilon - \frac{\mathsf{L}_+(f)}{4|t|\pi}\int_0^\infty |h(r)|^2dr\right| \\
&= \left|\int_\varepsilon^\infty |h_\varepsilon(r)|^2 f(\frac{r}{\varepsilon})dr - \mathsf{L}_+(f)\int_0^\infty |h(r)|^2dr\right| \\
&= \left|-\int_0^\varepsilon |h_\varepsilon(r)|^2 f(\frac{r}{\varepsilon})dr + \int_0^\infty |h_\varepsilon(r)|^2 f(\frac{r}{\varepsilon}) - \mathsf{L}_+(f)|h(r)|^2dr\right| \\
&\le \varepsilon\|f\|_\infty\|h_\varepsilon\|_\infty^2 + \int_0^\infty \left||h_\varepsilon(r)|^2 - |h(r)|^2\right| |f(\frac{r}{\varepsilon})| + \left|f(\frac{r}{\varepsilon}) - \mathsf{L}_+(f)\right| |h(r)|^2dr \\
&\le \varepsilon\|f\|_\infty\|\sqrt{\rho}\|_{L^1}^2 + \|f\|_\infty\left|\|h_\varepsilon\|_{L^2}^2 - \|h\|_{L^2}^2\right| + \int_0^\infty \left|f(\frac{r}{\varepsilon}) - \mathsf{L}_+(f)\right| |h(r)|^2dr
\end{aligned}$$

Now we consider the terms in the last upper bound and show that all of them converge to zero as $\varepsilon \to 0$. This statement is clearly true for the first term. To show convergence to zero for the second term, we recall that we proved $\lim_{\varepsilon\to 0} h_\varepsilon = h$ in $L^2(\mathbb{R})$. Finally the third term can be seen to converge to zero by dominated convergence, because $f(\frac{r}{\varepsilon})$ converges pointwise to $\mathsf{L}_+(f)$ as $\varepsilon \to 0$.

To complete the prove, it remains to show that $\int_0^\infty |h(r)|^2dr = 2\pi|t|$. Since plugging this formula in equation (4.8) would yield the desired result $\lim_{\varepsilon\to 0} C_\varepsilon = \mathsf{L}_+(f)/2$.

So we continue by noting that $h(-\varepsilon) = \overline{h(\varepsilon)}$, which shows

$$\|h\|_{L^2}^2 = \int_{-\infty}^\infty |h(r)|^2dr = 2\int_0^\infty |h(r)|^2dr.$$

We recall that for a positive constant $a > 0$ and for a square integrable function $f \in L^2(\mathbb{R})$, the Fourier transform satisfies the formula $\mathcal{F}(g(ax))(\xi) = \frac{1}{a}\hat{g}(\frac{\xi}{a})$. With this identity and Parseval's theorem one observes by a routine calculation

$$\begin{aligned}
\int_0^\infty |h(r)|^2dr &= \frac{1}{2}\|h\|_{L^2}^2 = \frac{1}{2}\int_{-\infty}^\infty \left|4|t|\pi\mathcal{F}\left(\sqrt{\rho(4\pi tv)}\right)(r)\right|^2 dr \\
&= \frac{1}{2}\int_{-\infty}^\infty \left|\sqrt{\rho(\frac{v}{4\pi t})}\right|^2 dv = 2\pi|t|\int_{-\infty}^\infty |\sqrt{\rho(w)}|^2dw = 2\pi|t|1.
\end{aligned}$$

In the last line of the equation we have used the fact that a mollifier is normalized. Hence the proof of the proposition is completed. $\square$

We want to state this convergence from the last proposition in terms of the Dirac delta concentrated at infinity. The following is from Bauer [1].We define the extended number line or the two point compactification of $\mathbb{R}$ in the common sense to be the set $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$.

Let $\mathcal{B}(\mathbb{R})$ be the σ-algebra generated by Borel subsets on $\mathbb{R}$. We say that $A \subset \overline{\mathbb{R}}$ is Borel in the extended numberline, if $A \cap \mathbb{R} \in \mathcal{B}(\mathbb{R})$. From a constructive point of view, this means that Borel subsets of $\overline{\mathbb{R}}$ are all sets B, $B \cup \{+\infty\}$, $B \cup \{-\infty\}$ and $B \cup \{-\infty, +\infty\}$ whenever $B \in \mathcal{B}(\mathbb{R})$. The σ-algebra generated by Borel sets on $\overline{\mathbb{R}}$ is denoted by $\mathcal{B}(\overline{\mathbb{R}})$. Recall that $M(\overline{\mathbb{R}})$ is the normed vector space of complex Borel measures on $\overline{\mathbb{R}}$.

Consider $\overline{\mathbb{R}}$ with the usual neighbourhood basis for points $x \in \mathbb{R}$ and define a neighbourhood basis for $+\infty$, by the sets of the form $\{x \mid x > a\}$ for every real number a. Similar sets serve as a neighbourhood basis of $-\infty$. With this construction $\overline{\mathbb{R}}$ becomes a compact topological space. That the extended number line is indeed compact, follows by the order preserving homeomorphism

$$h : \overline{R} \to [-1, 1], \quad h(x) = \begin{cases} \frac{x}{1+|x|} & \text{if } x \in \mathbb{R} \\ 1 & \text{if } x = +\infty \\ -1 & \text{if } x = -\infty \end{cases}.$$

Our next task is to prove that the dual of $C_{\pm}(\mathbb{R})$ is isometrically isomorphic to $M(\overline{\mathbb{R}})$, termed as $C_{\pm}(\mathbb{R})' \cong M(\overline{\mathbb{R}})$. If we declare for $x \in \overline{\mathbb{R}}$ the continuous function function

$$T(f)(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{R} \\ \mathsf{L}_{+}(f) & \text{if } x = +\infty \\ \mathsf{L}_{-}(f) & \text{if } x = -\infty, \end{cases}$$

then the map $T : C_{\pm}(\mathbb{R}) \to C(\overline{\mathbb{R}})$, $f \mapsto T(f)$ defines an isometric isomorphism. Indeed, since surjectivity is clear whereas injectivity and continuity follows from

$$\|T(f)\|_{\infty} = \sup_{x \in \overline{\mathbb{R}}} |T(f)(x)| = \sup_{x \in \mathbb{R}} |f(x)| = \|f\|_{\infty}.$$

This implies $C_{\pm}(\mathbb{R}) \cong C(\overline{\mathbb{R}})$. We recall the Riesz representation theorem, which states the following (see Werner [21]):
If K is a compact topological Hausdorff space, then $C(K)' \cong M(K)$. The isomorphism is given by

$$M(K) \to C(K)', \quad T(\alpha)(f) = \int_K f d\alpha \quad \text{for } \alpha \in M(K), f \in C(K).$$

This proves the desired property $C_{\pm}(\mathbb{R})' \cong C(\overline{\mathbb{R}})' \cong M(\overline{\mathbb{R}})$. Together with Proposition 4.3 we have for $f \in C(\overline{\mathbb{R}})$ and denoting by $\delta_{+\infty}$, $\delta_{-\infty} \in M(\overline{\mathbb{R}})$ the Dirac measure concentrated at $\pm\infty$, that

$$\lim_{\varepsilon \to 0} \langle \mu_{\varepsilon}^{t}, f \rangle = \frac{\mathsf{L}_{-}(f) + \mathsf{L}_{+}(f)}{2} = \frac{f(+\infty) + f(-\infty)}{2} = \int_{\overline{\mathbb{R}}} f(x) \frac{\delta_{+\infty} + \delta_{-\infty}}{2}(x)$$

To summarize we obtain, considering by abuse of notation the measures μ_{ε}^{t} as elements in $C(\overline{\mathbb{R}})'$, the following

Corollary 4.4. *The net of measures $(\mu_\varepsilon^t)_{\varepsilon\in]0,1]}$ converges in the weak* topology $\sigma(M(\overline{\mathbb{R}}), C(\overline{\mathbb{R}}))$, to the measure $\frac{1}{2}(\delta_{+\infty} + \delta_{-\infty}) \in M(\overline{\mathbb{R}})$.*

4.3 Convergence on trigonometric polynomials

We require again $\mu = \delta$ and $u_\varepsilon(0, x) = \sqrt{\mu * \rho_\varepsilon}$ as initial value in the Schrödinger equation, where the mollifier $\rho \in L^1(\mathbb{R}^n)$ fulfils (3.1). Furthermore we consider the net of measures $(\mu_\varepsilon^t)_{\varepsilon\in]0,1]}$ for $t \neq 0$. The aim is to find a subclass of functions $f \in C_b(\mathbb{R})$ which is different from $C_\pm(\mathbb{R})$ but also having the property that $\lim_{\varepsilon\to 0}\langle \mu_\varepsilon^t, f\rangle$ exists. We will follow Hörmann [9] and show that this limit exists if f is a trigonometric polynomial.

In order to accomplish the convergence, we investigate the Fourier transforms of the measures μ_ε^t in the n-dimensional Euclidean space.

Lemma 4.5. *If the net of measures $(\mu_\varepsilon^t)_{\varepsilon\in]0,1]}$ with the initial value regularization $\rho \in L^1(\mathbb{R}^n)$ satisfying (3.1) and $\mu = \delta_a \in M(\mathbb{R}^n)$ is the Dirac-measure concentrated at $a \in \mathbb{R}^n$, then $\lim_{\varepsilon\to 0} \mathcal{F}(\mu_\varepsilon^t)(\xi) = 0$ for every $t \neq 0$ and $\xi \neq 0$ and $\lim_{\varepsilon\to 0} \mathcal{F}(\mu_\varepsilon^t)(0) = 1$.*

Proof. We obtain for the initial value regularization:

$$\sqrt{\mu * \rho_\varepsilon(x)} = \sqrt{\delta_a * \rho_\varepsilon(x)} = \sqrt{\rho_\varepsilon(x-a)} = \sqrt{T_a\rho_\varepsilon(x)}.$$

Let $t \neq 0$, since $u_\varepsilon(t, x) = e^{it\Delta}(\sqrt{\mu * \rho_\varepsilon})(x)$ we may write according to Definition 2.8

$$\mathcal{F}(u_\varepsilon(t, .))(\xi) = e^{-it4\pi^2|\xi|^2}\mathcal{F}\left(\sqrt{T_a\rho_\varepsilon}\right)(\xi). \tag{4.9}$$

Thus with the definition of the measure μ_ε^t, we can calculate its Fourier transform as

$$\begin{aligned}\mathcal{F}\left(\mu_\varepsilon^t\right)(\xi) &= \int_{\mathbb{R}^n} e^{-2\pi i\langle x,\xi\rangle} d\mu_\varepsilon^t(x) = \int_{\mathbb{R}^n} e^{-2\pi i\langle x,\xi\rangle}|u_\varepsilon(t, x)|^2 dx \\ &= \mathcal{F}\left(|u_\varepsilon(t, .)|^2\right)(\xi) = \mathcal{F}\left(u_\varepsilon(t, .)\overline{u_\varepsilon(t, .)}\right)(\xi).\end{aligned} \tag{4.10}$$

Since μ_ε^t are probability measures it is clear that

$$\lim_{\varepsilon\to 0} \mathcal{F}\left(\mu_\varepsilon^t\right)(0) = \int_{\mathbb{R}^n} d\mu_\varepsilon^t(x) = 1.$$

We want to emphasize that similarly to the convolution theorem $\mathcal{F}(g * h) = \hat{g}\hat{h}$ for functions $g, h \in L^2(\mathbb{R}^n)$, one proves the convolution theorem for the inverse Fourier transform, thus $\mathcal{F}^{-1}(g * h) = \mathcal{F}^{-1}(g)\mathcal{F}^{-1}(h)$. This yields the formula

$gh = \mathcal{F}^{-1}(\hat{g} * \hat{h})$, from which we obtain, resuming (4.10) and using (4.9) the equation

$$\begin{aligned}\mathcal{F}\left(u_\varepsilon(t,.)\overline{u_\varepsilon(t,.)}\right) &= \mathcal{F}\left(u_\varepsilon(t,.)\right) * \mathcal{F}\left(\overline{u_\varepsilon(t,.)}\right)(\xi)\\ &= \left(e^{-it4\pi^2|.|^2}\mathcal{F}\left(\sqrt{T_a\rho_\varepsilon}\right)\right) * \left(e^{it4\pi^2|.|^2}R\overline{\mathcal{F}\left(\sqrt{T_a\rho_\varepsilon}\right)}\right)(\xi)\\ &= \int_{\mathbb{R}^n} e^{-it4\pi^2|y|^2}e^{it4\pi^2|\xi-y|^2}\mathcal{F}\left(\sqrt{T_a\rho_\varepsilon}\right)(y)\overline{\mathcal{F}\left(\sqrt{T_a\rho_\varepsilon}\right)(y-\xi)}dy. \quad (4.11)\end{aligned}$$

By the identity $|\xi - y|^2 = |\xi|^2 - 2\langle\xi, y\rangle + |y|^2$ and the fact that ρ_ε is real valued we obtain further from (4.11),

$$\begin{aligned}\mathcal{F}\left(\mu_\varepsilon^t\right)(\xi) &= e^{it4\pi^2|\xi|^2}\int_{\mathbb{R}^n} e^{-it8\pi^2\langle y,\xi\rangle}\mathcal{F}\left(\sqrt{T_a\rho_\varepsilon}\right)(y)\mathcal{F}\left(\sqrt{T_a\rho_\varepsilon}\right)(\xi-y)dy\\ &= e^{it4\pi^2|\xi|^2}\int_{\mathbb{R}^n} e^{2\pi i\langle -4\pi t y,\xi\rangle}\mathcal{F}\left(\sqrt{T_a\rho_\varepsilon}\right)(y)T_\xi\left[R\mathcal{F}\left(\sqrt{T_a\rho_\varepsilon}\right)\right](y)dy\\ &= e^{it4\pi^2|\xi|^2}\mathcal{F}^{-1}\left[\mathcal{F}\left(\sqrt{T_a\rho_\varepsilon}\right)\mathcal{F}\left(e^{2\pi i\langle\xi,.\rangle}R\sqrt{T_a\rho_\varepsilon}\right)\right](-4\pi t\xi)\\ &= e^{it4\pi^2|\xi|^2}\left[\sqrt{T_a\rho_\varepsilon} * \left(e^{2\pi i\langle\xi,.\rangle}R\sqrt{T_a\rho_\varepsilon}\right)\right](-4\pi t\xi).\end{aligned}$$

In the third line of the equation above we used the formula

$$T_\eta\mathcal{F}(g) = \mathcal{F}\left(e^{2\pi i\langle\eta,.\rangle}g(.)\right) \quad \text{for } g \in L^1(\mathbb{R}^n), \eta \in \mathbb{R}^n.$$

See e.g. [6, Chapter 8.2]. This leads to the following inequality,

$$\begin{aligned}|\mathcal{F}\left(\mu_\varepsilon^t\right)(\xi)| &= \left|\sqrt{T_a\rho_\varepsilon} * \left(e^{2\pi i\langle\xi,.\rangle}R\sqrt{T_a\rho_\varepsilon}\right)(-4\pi t\xi)\right|\\ &\leq \int_{\mathbb{R}^n}\sqrt{T_a\rho_\varepsilon(x)}\sqrt{T_a\rho_\varepsilon(4\pi t\xi + x)}dx\\ &= \frac{1}{\varepsilon^n}\int_{\mathbb{R}^n}\sqrt{\rho\left(\frac{x-a}{\varepsilon}\right)}\sqrt{\rho\left(\frac{x+4\pi t\xi - a}{\varepsilon}\right)}dx.\end{aligned}$$

Thus by a change of variables with $y = \frac{x-a}{\varepsilon}$, we get

$$|\mathcal{F}\left(\mu_\varepsilon^t\right)(\xi)| \leq \int_{\mathbb{R}^n}\sqrt{\rho(y)}\sqrt{\rho\left(\frac{4\pi t\xi}{\varepsilon} + y\right)}dy = (\sqrt{\rho} * R\sqrt{\rho})\left(-\frac{4\pi t\xi}{\varepsilon}\right).$$

The last term converges to zero for $\xi \neq 0$ as $\varepsilon \to 0$. This follows because $\sqrt{\rho}$ and $R\sqrt{\rho}$ are clearly in $L^2(\mathbb{R}^n)$ by the definition of ρ, hence $(\sqrt{\rho} * R\sqrt{\rho}) \in C_0(\mathbb{R}^n)$ by Proposition 2.7. Therefore the lemma is proved. $\square$

Now we come back to the one dimensional situation.

Corollary 4.6. *If $f : \mathbb{R} \to \mathbb{C}$ is a trigonometric polynomial on the real line, this is a function of the form*

$$f(x) = \sum_{j=0}^{m} a_j e^{2\pi i x \xi_j} \quad \textit{where } a_j \in \mathbb{C},\ \xi_j \in \mathbb{R} \quad \textit{for } j \in \{0, \dots, m\},$$

then the limit $\lim_{\varepsilon\to 0}\langle \mu_\varepsilon^t, f\rangle$ exists. Suppose furthermore that ξ_0 is zero and $\xi_j \neq 0$ for $j \neq 0$, then the identity

$$\lim_{\varepsilon\to 0}\langle \mu_\varepsilon^t, f\rangle = a_0 = \lim_{R\to\infty} \frac{1}{2R}\int_{-R}^{R} f(x)dx,$$

is valid for all such trigonometric polynomials.

Proof. By changing the order of integration and summation we get

$$\begin{aligned}
\langle \mu_\varepsilon^t, f\rangle &= \int_{-\infty}^{\infty} \sum_{j=0}^{m} a_j e^{2\pi i x \xi_j} |u_\varepsilon(t,x)|^2 dx \\
&= \sum_{j=0}^{m} a_j \int_{-\infty}^{\infty} e^{2\pi i x \xi_j} |u_\varepsilon(t,x)|^2 dx \\
&= a_0 \mu_\varepsilon^t(\mathbb{R}) + \sum_{j=1}^{m} a_j \mathcal{F}(\mu_\varepsilon^t)(-\xi_j).
\end{aligned}$$

Thus Lemma 4.5 is applicable and implies

$$\lim_{\varepsilon\to 0}\langle \mu_\varepsilon^t, f\rangle = a_0 \cdot 1 + \sum_{j=1}^{m} a_j \lim_{\varepsilon\to 0} \mathcal{F}(\mu_\varepsilon^t)(-\xi_j) = a_0. \tag{4.12}$$

To infer the representation of the limit as mean value, we calculate

$$\begin{aligned}
\lim_{R\to\infty} \frac{1}{2R}\int_{-R}^{R} f(x)dx &= \lim_{R\to\infty} \frac{1}{2R}\left(2R\cdot a_0 + \sum_{j=1}^{m} a_j \int_{-R}^{R} e^{2\pi i x \xi_j} dx\right) \\
&= a_0 + \lim_{R\to\infty} \frac{1}{2R} \sum_{j=1}^{m} a_j \frac{e^{2\pi i R \xi_j} - e^{-2\pi i R \xi_j}}{2\pi i \xi_j} \qquad (4.13) \\
&= a_0 + \lim_{R\to\infty} \frac{1}{2R} \sum_{j=1}^{m} a_j \frac{\sin(2\pi R \xi_j)}{\pi \xi_j} = a_0.
\end{aligned}$$

Hence combining (4.12) with (4.13) finishes the proof of the corollary. ☐

5 Almost periodic functions on the real line

We now introduce almost-periodic functions, since we will see that they are the uniform limits of trigonometric polynomials. In this section we follow the book of Katznelson [10, Section VI.5], where proofs which we omit and many more facts can be found.

Definition 5.1. Let $\varepsilon > 0$, for a function $f : \mathbb{R} \to \mathbb{C}$ we call $\tau \in \mathbb{R}$ an *ε-almost-period of f* if

$$\sup_{x\in\mathbb{R}} |f(x-\tau) - f(x)| < \varepsilon.$$

Furthermore let f be continuous, we say that f is *almost-periodic* on $\mathbb{R}$, if for every $\varepsilon > 0$ there exists a positive real number $\lambda = \lambda(\varepsilon, f)$, such that every interval $I \subset \mathbb{R}$ of length λ contains and ε-almost-period of f. We denote the set of all almost-periodic functions on the real line by $AP(\mathbb{R})$.

Example. (i). Continuous periodic functions on the real line are almost-periodic.

(ii). Let $a \in \mathbb{C}$ and $r \in \mathbb{R}$. If $f \in AP(\mathbb{R})$, then $|f|$, af, $f(rx)$ and $\overline{f}$ are almost-periodic as well.

The goal considering almost-periodic functions in our situation, is to find a subspace of $C_b(\mathbb{R})$ where the weak*-limit of the measures μ_ε^t exists. That we obtain indeed $AP(\mathbb{R}) \subset C_b(\mathbb{R})$, shows the next

Lemma 5.2. *Every almost-periodic function is bounded.*

Proof. For $f \in AP(\mathbb{R})$ choose $\varepsilon = 1$ and let $\lambda = \lambda(1, f)$. Then for every $x \in \mathbb{R}$ the interval $[x - \lambda, x]$ contains a 1-almost-period τ. Which implies that $|f(x-\tau) - f(x)| < 1$. The obvious fact that $(x - \tau) \in [0, \lambda]$ shows the inequality

$$|f(x)| \leq |f(x-\tau) - f(x)| + |f(x-\tau)| < 1 + |f(x-\tau)| \leq 1 + \sup_{y\in[0,\lambda]} |f(y)|.$$

The function f is by definition continuous and therefore the image under f of the compact interval $[0, \lambda]$ is compact, hence the supremum on the right side of the inequality above is finite. Since $x \in \mathbb{R}$ was arbitrary, we have shown that f is bounded. $\square$

As an immediate consequence of the boundedness property we obtain the following

Corollary 5.3. *If $f \in AP(\mathbb{R})$, then $f^2 \in AP(\mathbb{R})$.*

Proof. For $f \in AP(\mathbb{R})$ we may assume $|f(x)| < \frac{1}{2}$ for every real number x without loss of generality. This follows by boundedness of f and example (ii) above. If τ is an ε-almost-period for f, we calculate

$$\begin{aligned}|f^2(x-\tau) - f^2(x)| &= |f(x-\tau) - f(x)||f(x-\tau) + f(x)| \\ &\leq |f(x-\tau) - f(x)||f(x-\tau)| + |f(x-\tau) - f(x)||f(x)| \\ &< \varepsilon|f(x-\tau)| + \varepsilon|f(x)| = \varepsilon.\end{aligned}$$

Thus τ is also an ε-almost-period for f^2. ☐

The next lemma states, that the set of almost-periodic functions is substantially different from the space $C_\pm(\mathbb{R})$ in section 4.2.

Lemma 5.4. *If f is an almost-periodic function possessing limits at infinity, then f is constant. Hence $C_\pm(\mathbb{R}) \cap AP(\mathbb{R}) = \operatorname{span}\{1\}$.*

Proof. We recall the notation $\mathsf{L}_\pm(f) = \lim_{x\to\pm\infty} f(x)$ for a function $f \in C_\pm(\mathbb{R})$.

Let $f \in C_\pm(\mathbb{R}) \cap AP(\mathbb{R})$ and $\varepsilon > 0$. Then there exists a number $y_\varepsilon > 0$ such that $|f(y) - \mathsf{L}_+(f)| < \frac{\varepsilon}{2}$ for every $y \geq y_\varepsilon$. Since f is also almost-periodic, we can choose a $\lambda = \lambda(\frac{\varepsilon}{2}, f)$ such that every interval of length λ contains an $\frac{\varepsilon}{2}$-almost-period. For every $x \in \mathbb{R}$ the interval $[x - y_\varepsilon - \lambda, x - y_\varepsilon]$ contains such a period τ. Furthermore by the choice of the interval we have $x - \tau \geq y_\varepsilon$. This implies the estimate

$$|f(x) - \mathsf{L}_+(f)| \leq |f(x) - f(x-\tau)| + |f(x-\tau) - \mathsf{L}_+(f)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ can be chosen arbitrarily small this shows, that the function f is constant with the value $\mathsf{L}_+(f)$. ☐

We state another interesting fact about $AP(\mathbb{R})$, where the proof uses a standard $\frac{\varepsilon}{3}$-argument (see [10, Chapter VI, Lemma 5.4]).

Lemma 5.5. *Every almost-periodic function is uniformly continuous.*

We want to recall the following topological notion. Let X be a topological space, a set $A \subset X$ is called precompact, if its closure $\overline{A}$ is compact in X.

Now we can state an important theorem, which will enable us to extend the definition of almost-periodic functions on the real line to locally compact abelian groups in the next chapter.

Theorem 5.6. *A bounded function $f \in L^\infty(\mathbb{R})$ is almost-periodic if and only if the set of translates $W_0(f) = \{T_y f |\ y \in \mathbb{R}\}$ is precompact in $L^\infty(\mathbb{R})$ with respect to the usual topology generated by the supremum norm.*

The proof of this theorem in [10, Chapter VI, Theorem 5.5] relies on the fact, that in a complete metric space X, a set $A \subset X$ is precompact if and only if for every $\varepsilon > 0$, the set A can be covered by finitely many balls of radius ε. This last property is called total boundedness (see Werner [21, Theorem B.1.7]).

Definition 5.7. For a bounded function $f \in L^\infty(\mathbb{R})$ we define the *translation convex hull* $W(f)$ to be the uniformly closed absolute convex hull of the set $\bigcup_{|a|\leq 1} W_0(af)$. Hence it is the set of uniform limits of functions

$$\sum_{k=1}^{m} a_k T_{x_k} f \quad m \in \mathbb{N}, \quad x_k \in \mathbb{R}, \quad \sum |a_k| \leq 1. \tag{5.1}$$

The next lemma shows, that in the case of almost-periodic functions, there is an alternative representation of $W(f)$.

Lemma 5.8. *Let f be a bounded uniformly continuous function, then the translation convex hull $W(f)$ is the uniform closure of functions of the form*

$$f * \phi, \quad \phi \in L^1(\mathbb{R}), \quad \|\phi\|_{L^1} \leq 1. \tag{5.2}$$

Proof. Let $\varphi \in L^1(\mathbb{R})$ and $\int_{-\infty}^{\infty} \varphi dx = 1$, furthermore we recall $\varphi_\eta := \eta^{-1}\varphi(\eta^{-1}x)$ for $\eta > 0$. Then for a bounded uniformly continuous function f, the sequence $f * \varphi_\eta$ converges uniformly to f as $\eta \to 0$ (see Folland [6, Theorem 8.14]). This implies that the translation $T_y f$ for $y \in \mathbb{R}$ can be uniformly approximated by the sequence $f * T_y\varphi_\eta$ if $\varphi \in L^1(\mathbb{R})$. This shows, that for every function of the form $h := \sum_{k=1}^{n} a_k T_{x_k} f$ as in (5.1) we have

$$\lim_{\eta \to 0} \left\| h - f * \left(\sum_{k=1}^{n} a_k T_{x_k} \varphi_\eta \right) \right\|_\infty = 0.$$

Because clearly the inequality $\| \sum_{k=1}^{n} a_k T_{x_k} \varphi_\eta \| \leq 1$ holds, h is the uniform limit of functions as in (5.2).

Let $g \in L^1(\mathbb{R})$ with $\|g\|_{L^1} \leq 1$. By assumption f is bounded and uniformly continuous, therefore the convolution $f * g$ is well defined by Young's inequality 2.3. Since the space $C_c(\mathbb{R})$ is dense in $L^1(\mathbb{R})$, there exists a sequence $\{g_n\}_{n\in\mathbb{N}}$ such that $g_n \to g$ in the L^1-norm as n tends to infinity. If we apply Theorem 2.7 we have for every $n \in \mathbb{N}$ the inequality

$$\|f * g - f * g_n\|_\infty = \|f * (g - g_n)\|_\infty \leq \|f\|_\infty \|g - g_n\|_{L^1}.$$

This shows that the sequence $\{f * g_n\}_{n\in\mathbb{N}}$ converges uniformly to $f * g$. To complete the proof of this lemma, we will show that for every $n \in \mathbb{N}$ the convolution $f * g_n$ can be uniformly approximated by sums of the form in

(5.1). To this end let $n \in \mathbb{N}$ be fixed. We define for every $x \in \mathbb{R}$ the function $s_x : \mathbb{R} \to \mathbb{C}$, $y \mapsto f(x-y)g_n(y)$. Since the functions g_n and f are both uniformly continuous and bounded, it follows by a routine argument, that the family $\{s_x\}_{x\in\mathbb{R}}$ is uniformly equicontinuous. This means, that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\sup_{x\in\mathbb{R}} |s_x(y) - s_x(y')| \leq \varepsilon$ whenever $|y - y'| < \delta$. Recall, that $g_n \in C_c(\mathbb{R})$, hence we can assume that $\operatorname{supp} g_n \subset [a, b]$, where $-\infty < a < b < \infty$. Therefore we can approximate $f * g_n(x) = \int_a^b s_x(y)dy$ by Riemann sums $\sum_{k=1}^m s_x(y_k)\Delta y_k$, where $\Delta y_k := y_k - y_{k-1}$. Let $\delta > 0$, such that $\sup_{x\in\mathbb{R}} |s_x(y) - s_x(y')| \leq \frac{\varepsilon}{b-a}$. Then for a partition $a = y_0 < \ldots < y_m = b$, with $\sup_{k=1}^m \Delta y_k < \delta$ we have

$$\sup_{x\in\mathbb{R}} \left| \int_a^b s_x(y)dy - \sum_{k=1}^m s_x(y_k)\Delta y_k \right| \leq$$
$$(b-a) \sup_{x\in\mathbb{R}} \left\{ \sup_{k=1}^m \{|s_x(y) - s_x(y_k)| : y \in (y_{k-1}, y_k)\} \right\} \leq (b-a)\frac{\varepsilon}{b-a} = \varepsilon$$

Therefore we obtain

$$\sup_{x\in\mathbb{R}} \left| f * g_n - \sum_{k=1}^m f(x-y_k)g(y_k)\Delta y_k \right| \leq \varepsilon.$$

Hence the convolution $f * g_n$ is the uniform limit of sums of the form (5.1), which completes the proof. □

In the following lemma we obtain another identification of almost-periodic functions.

Lemma 5.9. *A function f is almost-periodic if and only if the translation convex hull $W(f)$ is compact in the topology generated by the norm in $L^\infty(\mathbb{R})$.*

If $f, g \in AP(\mathbb{R})$, we clearly have the relation $W(f + g) \subset W(f) + W(g)$. Because addition is continuous and both sets $W(f)$ and $W(g)$ are compact it follows that $W(f) + W(g)$ is compact in $C_b(\mathbb{R})$. Furthermore the set $W(f + g)$ is closed by definition, hence being a closed subset of a compact set it is compact. Thus by Lemma 5.9 $f + g \in AP(\mathbb{R})$. With the above example (ii) in this section shows that $AP(\mathbb{R})$ is a subspace of $C_b(\mathbb{R})$. Writing the product of two almost-periodic functions f, g as $fg = (\frac{1}{2}((f + g)^2 - f^2 - g^2)$, the Corollary 5.3 implies that $AP(\mathbb{R})$ is a subalgebra of $L^\infty(\mathbb{R})$. By a routine $\frac{\varepsilon}{3}$-argument one shows that the space of almost-periodic functions is closed. Since $C_b(\mathbb{R})$ is an Abelian unital C^*-algebra, we obtain the following

Theorem 5.10. *The space $AP(\mathbb{R})$ is a closed Abelian unital C^*-subalgebra of $C_b(\mathbb{R})$.*

Another important concept of almost-periodic functions is the invariant mean, which will turn out to be the limit of the net $\langle \mu_\varepsilon^t, f\rangle$ for $f \in AP(\mathbb{R})$. To this end we make the following

Definition 5.11. For a bounded function $h \in L^\infty(\mathbb{R})$ we define the norm spectrum of h as the set

$$\sigma(h) := \{\xi \in \mathbb{R} | ae^{2\pi i \xi x} \in W(h) \quad \text{for some } a \neq 0\}.$$

The norm spectrum can clearly be the empty set for nontrivial bounded functions. For example, let $f \in C_0(\mathbb{R})$, then $W(f) \subset C_0(\mathbb{R})$ and because $e^{2\pi i \xi \cdot} \notin C_0(\mathbb{R})$ for every $\xi \in \mathbb{R}$, the norm spectrum $\sigma(f)$ is the empty set.

Lemma 5.12. *Let f be an almost-periodic function. If $0 \notin \sigma(f)$, then for every $F \in L^1(\mathbb{R})$*

$$\lim_{\omega \to 0} \|\omega F(\omega .) * f\|_{L^\infty} = 0$$

For a proof we refer to [10, Lemma 5.11].

Remark. We note that without loss of generality, we can assume $\|F\|_{L^1} \leq 1$ for $F \in L^1(\mathbb{R})$. Therefore we have $\omega F(\omega x) * f \in W(f)$, if we use the alternative description of $W(f)$ in (5.2). Since zero is the uniform limit of these functions, the constant map zero is contained in $W(f)$. Because of the assumption $0 \notin \sigma(f)$ it also is the only constant function in $W(f)$.

We now state an important fact in the theory of almost-periodic functions, namely the existence and uniqueness of the invariant mean.

Theorem 5.13. *To every almost-periodic function f, there corresponds a unique real number $M(f)$ such that $0 \notin \sigma(f - M(f))$.*

Definition 5.14. The unique number $M(f)$ for the function $f \in AP(\mathbb{R})$ is called the invariant mean of f.

The next Corollary delivers an explicit representation of the invariant mean in terms of a uniform limit of convolutions.

Corollary 5.15. *If $f \in AP(\mathbb{R})$, then for every $F \in L^1(\mathbb{R})$ the following holds:*

$$\lim_{\omega \to 0} \|\omega F(\omega .) * f\|_{L^\infty} = \hat{F}(0) M(f).$$

If we define the integrable function

$$F(x) := \begin{cases} \frac{1}{2} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1, \end{cases} \quad \text{we clearly have} \quad \hat{F}(0) = 1.$$

Let $\omega \in \mathbb{R}$ and set $R := \omega^{-1}$, then we calculate the convolution

$$\begin{aligned}(\omega F(\omega.) * f)(x) &= \int_{-\infty}^{\infty} \omega F(\omega y) f(x-y) dy = \frac{1}{2} \int_{-\frac{1}{\omega}}^{\frac{1}{\omega}} \omega f(x-y) dy \\ &= \frac{1}{2R} \int_{-R}^{R} f(x-y) dy.\end{aligned} \tag{5.3}$$

By Corollary 5.15 the convolution converges uniformly to the constant value $M(f)$. Therefore it is enough to calculate the pointwise limit at 0 and we obtain from equation (5.3)

$$M(f) = \lim_{\omega \to 0} (\omega F(\omega.) * f)(0) = \lim_{R \to \infty} \frac{1}{2R} \int_{-R}^{R} f(y) dy. \tag{5.4}$$

This shows that the invariant mean $M : AP(\mathbb{R}) \to \mathbb{C}$, is a linear functional on the space of almost-periodic functions, which is translation invariant. Hence $M(T_y f) = M(f)$ for every $y \in \mathbb{R}$. This follows from Corollary 5.15, since $F * T_y f = T_y(F * f)$ for $f \in AP(\mathbb{R})$ and $F \in L^1(\mathbb{R})$.

Since clearly trigonometric polynomials are almost-periodic functions and $AP(\mathbb{R})$ is closed, it follows that uniform limits of trigonometric polynomials are almost-periodic too. The next theorem states that the converse is true as well (see [10, Theorem 5.21]).

Theorem 5.16. *The space $AP(\mathbb{R})$ is the uniform closure of the subspace of trigonometric polynomials on the line.*

6 Locally compact Abelian groups

In this section we gather the facts from locally compact abelian groups, needed to continue with the pointwise convergence of the measures μ_ε^t on the space of almost-periodic functions on $\mathbb{R}^n$.

As a general reference for this guideline we used the summarized presentation in Katznelson [10, Chapter VII.].

We will follow the convention in Folland [5, Chapter 4] and thus for an abelian group G, we denote the group operation by $\cdot : G \times G \to G$, where the neutral element will be written as 1 and the inverse element of $g \in G$ as g^{-1}.

Definition 6.1. Let G be an abelian group, equipped with a locally compact Hausdorff topology τ. We call G a *locally compact abelian group* (short LCA group), if the map $(x, y) \mapsto xy^{-1}$ is continuous from the product space $G \times G$ onto G with respect to the topology τ.

We recall from Folland [6, Chapter 7], that a Borel measure λ on a locally compact Hausdorff space X is called *Radon*, if it is finite on all compact sets of X and if it fulfils for every Borel set $B \subset X$ and open set $V \subset X$

$$\lambda(B) = \inf\{\lambda(U) | U \supset B,\ U \text{ open}\}, \quad \lambda(V) = \sup\{\lambda(K) | K \subset V,\ K \text{ compact}\}.$$

Definition 6.2. Let G be a LCA group. A Radon measure λ on G is called *Haar measure*, if it is not identically zero and if it is *translation invariant*. This means that if $B \subset G$ is a Borel set and if $g \in G$, then we have $\lambda(Bg) = \lambda(B)$.

The next theorem guarantees the existence of a Haar measure, which is unique up to multiplication by a positive constant on a LCA group.

Theorem 6.3. *Let G be a LCA group and $\mathcal{B}_G$ be the Borel σ-algebra on G. Then there exists a Haar measure $\lambda : \mathcal{B}_G \to [0, \infty]$. Furthermore if ν is another Haar measure on G, there exists a real number $\alpha > 0$ such that $\nu = \alpha\lambda$.*

We can therefore assume, that each LCA group is equipped with a fixed Haar measure and denote the integral of a complex valued function f with respect to this measure as $\int_G f(x)dx$. Furthermore for $1 \leq p \leq \infty$ we denote by $L^p(G)$ the usual L^p-space on G, with respect to the Haar measure. We define the convolution as $f * g := \int_G f(xy^{-1})g(y)dy$ whenever it makes sense and one proves that $\|f * g\|_{L^1(G)} \leq \|f\|_{L^1(G)}\|g\|_{L^1(G)}$ for functions $f, g \in L^1(G)$.

When G is a compact LCA group, one normalizes the Haar measure λ such that $\lambda(G) = 1$. If G is discrete, then we normalize the Haar measure such that it assigns to each element $g \in G$ the mass one.

Definition 6.4. Let G be a LCA group. A *character* on G is a continuous homomorphism $\xi : G \to \mathbb{T} \subset \mathbb{C}$, where $\mathbb{T}$ denotes the multiplicative group of complex numbers having absolute value 1. Thus a character is a continuous function $\xi : G \to \mathbb{C}$ satisfying

$$|\xi(x)| = 1 \quad \text{and } \xi(xy) = \xi(x)\xi(y) \quad x, y \in G.$$

The set of all characters on G will be denoted by the symbol $\widehat{G}$. For $\xi_1, \xi_2 \in \widehat{G}$ we define multiplication by $(\xi_1 \cdot \xi_2)(x) := \xi_1(x)\xi_2(x)$. Therefore $(\widehat{G}, \cdot)$is a commutative group, called the *dual group* of G.

Remark. The neutral element $1 \in \widehat{G}$ is the constant function $1(x) = 1$ for every $x \in G$. Because of the duality theorem of Pontryagin, which we will state later, between the LCA group G and $\widehat{G}$, we write $\langle x, \xi \rangle$ instead of $\xi(x)$.

We equip the dual group $\widehat{G}$ with the topology of compact convergence (see Querenburg [12, Chapter 9]). In this topology a neighbourhood basis at $1 \in \widehat{G}$ is given by sets of the form $U_{K,\varepsilon} = \{\xi \in \widehat{G} : |\langle x, \xi \rangle - 1| < \varepsilon \quad \forall x \in K\}$, where

$K \subset G$ compact and $\varepsilon > 0$. Hence a neighbourhood basis for an element $\gamma \in \widehat{G}$ is given by the family of translates $\gamma \cdot U_{K,\varepsilon}$. Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence in the dual group $\widehat{G}$ of the LCA group G. For $\xi \in \widehat{G}$, the sequence ξ_n converges to the character ξ as n tends to infinity, if for all compact sets $K \subset G$:

$$\sup_{x\in K} |\langle x, \xi_n\rangle - \langle x, \xi\rangle| \to 0 \quad \text{as} \quad n \to \infty.$$

Now that a topology is defined on the dual group $\widehat{G}$, one can prove the following theorem (see Rudin [14, Theorem 1.2.6]).

Theorem 6.5. *If G is a LCA group, then the dual group $\widehat{G}$ of G is also a LCA group with respect to the topology of compact convergence.*

We remark that for a LCA group G, every element $x \in G$ defines a character on the dual group $\widehat{G}$, by the function

$$h_x : \widehat{G} \to \mathbb{T}, \quad h_x(\xi) = \langle x, \xi\rangle \tag{6.1}$$

The dual of $\widehat{G}$ is denoted by $\widehat{\widehat{G}}$. The next theorem states that, in fact, every character on $\widehat{G}$ is of this form.

Theorem 6.6 (Pontryagin)**.** *Let G be a LCA group and define for every $x \in G$ the function h_x as in* (6.1)*. Then the map $\Phi : G \to \widehat{\widehat{G}}$, $\Phi(x) := h_x$ is a group isomorphism and a topological homeomorphism from G onto $\widehat{\widehat{G}}$.*

For a proof see Rudin [14, Theorem 1.7.2].

Definition 6.7. Let G be a LCA group, for $f \in L^1(G)$ the function $\hat{f} : \widehat{G} \to \mathbb{C}$ defined by

$$\hat{f}(\xi) := \int_G \overline{\langle x, \xi\rangle} f(x)dx,$$

is called the *Fourier transform* of f. We define $A(\widehat{G})$ to be the space of all Fourier transforms of functions in $L^1(G)$.

We note that since $\langle x^{-1}, \xi\rangle\langle x, \xi\rangle = 1$, it follows that $\overline{\langle x, \xi\rangle} = \langle x^{-1}, \xi\rangle$. By an application of Fubini's theorem, we observe

$$\begin{aligned}\widehat{f * g}(\xi) = \int_G \langle x^{-1}, \xi\rangle \int_G f(xy^{-1})g(y)dydx \\ = \int_G \langle y^{-1}, \xi\rangle g(y) \int_G f(xy^{-1})\langle x^{-1}y, \xi\rangle dxdy = \hat{f}\hat{g}(\xi).\end{aligned}$$

Thus the convolution theorem holds also in this abstract setting. Since clearly $\widehat{f+g} = \hat{f} + \hat{g}$, the space $A(\widehat{G})$ is an algebra.

We will need the following two theorems, which can be found in Rudin [14, Chapter 1.2].

Theorem 6.8. *If G is a LCA group, then the space $A(\widehat{G})$ is dense in $C_0(\widehat{G})$ and for the linear map $\mathcal{F} : L^1(G) \to C_0(\widehat{G})$, $\mathcal{F}(f) := \hat{f}$ we have $\|\hat{f}\|_\infty \leq \|f\|_{L^1(G)}$. Hence $\mathcal{F}$ is bounded.*

Theorem 6.9. *Let G be a LCA group, then the following hods:*

(i) If G is discrete, then the dual $\widehat{G}$ is compact.

(ii) If G is compact, then the dual $\widehat{G}$ is discrete.

Only the proof of (ii) will be given below, since it contains an important fact, which will be applied in the next section. Whereas for a proof of (i) we refer to Rudin [14, Theorem 1.2.5].

Proof of (ii). Since G is compact, we emphasize that the trivial character $1 \in \widehat{G}$ is also integrable, thus $1 \in L^1(G)$. Furthermore the Haar measure is considered to be normalized, as mentioned above. Therefore we obtain $\hat{1}(1) = \int_G \langle x^{-1}, 1\rangle dx = 1$. If $\xi \in \widehat{G}$ and $\xi \neq 1$, then there exists an element $x_0 \in G$ such that $\langle x_0, \xi\rangle \neq 1$. By using the translation invariance of the Haar measure, we calculate

$$\int_G \langle x^{-1}, \xi\rangle dx = \langle x_0, \xi\rangle \int_G \langle x^{-1}x_0^{-1}, \xi\rangle dx = \langle x_0, \xi\rangle \int_G \langle x^{-1}, \xi\rangle dx,$$

which shows that the integral has to be zero in this case. Thus we summarize

$$\hat{1}(\xi) = \int_G \langle x, \xi\rangle dx := \begin{cases} 1 & \text{if } \xi = 1 \\ 0 & \text{if } \xi \neq 1. \end{cases} \tag{6.2}$$

By Theorem 6.8 we also have that the function $\hat{1}$ is continuous on $\widehat{G}$ and therefore the singleton $\{1\} = \hat{f}^{-1}(\mathbb{C} \setminus \{0\})$ is open in $\widehat{G}$. This implies by translation, that the dual group $\widehat{G}$ is discrete. $\square$

Remark. Let $M(G)$ denote the space of finite regular Borel measures on the LCA group G. Then the convolution of two measures $\omega, \nu \in M(G)$ is defined by $\nu * \omega(E) := \int_E \nu(Ex^{-1})d\omega(x)$, where E ranges over all Borel sets in G. One can show, that the convolution is again a measure, i.e., $\nu * \omega \in M(G)$. Furthermore convolution is commutative, associative and we have $\|\nu * \omega\|_{\text{var}} \leq \|\nu\|_{\text{var}}\|\omega\|_{\text{var}}$. Therefore $M(G)$ is a Banach algebra. For a proof we refer to [14, Theorem 1.3.2].

A generalization of the Riesz representation theorem states, that for a locally compact Hausdorff space X there is an isometric isomorphism from $M(X)$ to the dual space $C_0(X)'$. Thus in the LCA group setting we have $M(G) \cong C_0(G)'$ (see Folland [6, Theorem 7.17])

Definition 6.10. Let G be a LCA group, then the *Fourier-Stieltjes transform* of a measure $\omega \in M(G)$ is defined by

$$\widehat{\omega} : \widehat{G} \to \mathbb{C}, \quad \widehat{\omega}(\xi) = \int_G \overline{\langle x, \xi \rangle} d\omega(x).$$

For $\omega \in M(G)$ the function $\widehat{\omega}$ is bounded and uniformly continuous. If $\omega = f dx$ for a function $f \in L^1(G)$, then the Fourier-Stieltjes transform coincides with the Fourier transform, i.e. $\widehat{\omega}(\xi) = \hat{f}(\xi)$. Furthermore if $\omega, \nu \in M(G)$, then $\widehat{\nu * \omega} = \widehat{\nu}\widehat{\omega}$ (see [14, Theorem 1.3.3]).

The following is from Folland [5, Theorem 4.7]:
If $G_1, \dots, G_n$ are LCA groups and ξ_j are characters in the dual groups $\widehat{G_j}$ for each $j \in \{1 \dots n\}$, then the tuple $\xi = (\xi_j)_{j=1}^n \in \prod_{j=1}^n \widehat{G_j}$ defines a charachter on the product $\prod_{j=1}^n G_j$ by

$$\langle x, \xi \rangle = \langle x_1, \xi_1 \rangle \cdots \langle x_n, \xi_n \rangle, \quad \text{for} \quad x = (x_1, \dots, x_n). \tag{6.3}$$

Moreover let χ be an arbitrary character on $\prod_{j=1}^n G_j$, then it can be written in the form (6.3) as follows

$$\begin{aligned} \langle (x_1, \dots, x_n), \chi \rangle = \langle x_1, \chi_1 \rangle \cdots \langle x_n, \chi_n \rangle \quad \text{where} \\ \langle x_j, \chi_j \rangle := \langle (1, \dots, 1, x_j, 1, \dots, 1), \chi \rangle \end{aligned}$$

Thus we have proven

Proposition 6.11. *If $G_1, \dots, G_n$ are LCA groups, then*

$$(G_1 \times \cdots \times G_n)\widehat{\ } \cong \widehat{G_1} \times \cdots \times \widehat{G_n}.$$

Example. If we consider the real line $G = \mathbb{R}$ as a LCA group, one can show that the dual group $\widehat{\mathbb{R}}$ can be identified with the real line itself. This is shown by proving that characters on G are of the form $x \mapsto e^{2\pi i \xi x}$ with $\xi \in \mathbb{R}$. Thus $\mathbb{R} \cong \widehat{\mathbb{R}}$ via the dual pairing $\langle x, \xi \rangle = e^{2\pi i \xi x}$ (see Folland [5, Theorem 4.6]). With Proposition 6.11 we obtain $\widehat{\mathbb{R}^n} = \mathbb{R}^n$ and the characters on $\mathbb{R}^n$ are given by $\langle x, \xi \rangle = e^{2\pi i \xi \cdot x}$ for $x, \xi \in \mathbb{R}^n$, where $\xi \cdot x$ is the $\mathbb{R}^n$-scalar product of ξ and x.

6.1 Almost-periodic functions on a LCA group

Definition 6.12. Let G be a noncompact LCA group and define $\widehat{G}_d$ to be the dual group of G equipped with the discrete topology. Then by Theorem 6.9 the dual group of $\widehat{G}_d$ is compact and will be called the *Bohr compactification of G*. We denote the Bohr compactification by bG.

By the Pontryagin Theorem 6.6 an element in G can be considered as a continuous homomorphism from $\widehat{G}$ to $\mathbb{T}$. Whereas an element in bG is a not necessarily continuous homomorphism from $\widehat{G}$ to the multiplicative group $\mathbb{T}$, since $\widehat{G}_d$ is discrete. This shows that $G \subset bG$. The next theorem states an important property of the embedding from G into bG.

Theorem 6.13. *Let G be a noncompact LCA group, then the canonical embedding $\iota : G \hookrightarrow bG$ is continuous and the image $\iota(G)$ is dense in bG.*

For a proof, we refer to Folland [5, Chapter 4.7]. Nevertheless continuity of the embedding ι is easily observed as follows. By the remark after Definition 6.4 the topology of the dual group is that of convergence on compact sets. Since $\widehat{G}_d$ is discrete, the topology on bG is equivalent to that of pointwise convergence. Clearly the topology of pointwise convergence is weaker than that of convergence on compact sets on G. This proves the desired continuity.

We remark that bG is not a compactification in the usual sense, since one can show that the embedding ι is not a homeomorphism onto its range (see Folland [5]).

In view of Theorem 5.6 we generalize the concept of almost-periodic functions. Because we have written the group operation on a LCA group G as multiplication, the translation of a function $f : G \to \mathbb{C}$ by $y \in G$ is defined as $T_y f(x) = f(xy^{-1})$.

Definition 6.14. Let G be a noncompact LCA group, then we call a function $f \in L^\infty(G)$ *almost-periodic* if the set of translates of f, i.e. $W_0(f) = \{T_y f | y \in G\}$, is precompact in $L^\infty(G)$ with respect to the supremum norm topology. The space of almost-periodic functions on G, will be denoted by $AP(G)$.

One can show, that almost-periodic functions are uniformly continuous. Hence $AP(G) \subset C_b(G)$. Since the topology of a LCA group is translation invariant, a function f of a subset $H \subset G$ to $\mathbb{C}$ is uniformly continuous, if and only if for every $\varepsilon > 0$, there exists a neighbourhood U of zero such that $x, y \in H$ implies $|f(x) - f(y)| < \varepsilon$ whenever $xy^{-1} \in U$ (see Rudin [14, Appendix B]).

In order to get a more convenient characterisation of $AP(G)$, we define *trigonometric polynamials* to be finite linear combinations of characters on G. This means the function $f : G \to \mathbb{C}$ is a trigonometric polynomial on the LCA group if it is of the form

$$f(x) = \sum_{j=0}^{m} a_j \langle x, \xi_j \rangle, \quad a_j \in \mathbb{C},\ \xi_j \in \widehat{G} \quad \text{for } j \in \{0, \dots, m\}.$$

We now come to the announced characterization of $AP(G)$ (see Folland [5, Chapter 4.7]).

Theorem 6.15. *Let G be a noncompact LCA group and $f \in C_b(G)$, then the following are equivalent:*

(i) $f \in AP(G)$.

(ii) *f is the uniform limit of trigonometric polynomials in G.*

(iii) *f is the restriction to G of some function $\tilde{f} \in C(bG)$.*

As on the real line, one can prove the existence of an invariant mean of a function in AP(G). We refer to Dixmier [4].

Theorem 6.16. *Let G be a noncompact LCA group and $f \in AP(G)$. Then the uniform closure of the convex hull of the set $W_0(f)$, contains exactly one constant function, denoted by $M(f)$. Moreover, let $\tilde{f} \in C(bG)$ denote the continuous extension of f from Theorem 6.15 (ii), then $M(f) = \int_{bG} \tilde{f}(x)dx$. Thus M(f) is the integral of $\tilde{f}$ with respect to the normalized Haar measure on bG.*

The constant function $M(f)$ is called the *invariant mean* of f. The mapping $T : AP(G) \to \mathbb{C}$ with $f \mapsto M(f)$ is a translation invariant linear functional. If the function $f \in AP(G)$ is positive i.e. $f \geq 0$, then $M(f) \geq 0$. Moreover for $f \equiv 1$ we obtain $M(f) = 1$.

7 Convergence on $AP(\mathbb{R}^n)$

In this section we focus on the LCA group $\mathbb{R}^n$. Our aim is to prove convergence of the measures μ_ε^t with respect to the weak* topology $\sigma(M(b\mathbb{R}^n), AP(\mathbb{R}^n))$, see Hörmann [9]. The next lemma connects the space of measures on $\mathbb{R}^n$ with the space of measures on the Bohr compactification $b\mathbb{R}^n$.

Lemma 7.1. *Every measure $\omega \in M(\mathbb{R}^n)$ can be extended to a measure $\tilde{\omega} \in M(b\mathbb{R}^n)$.*

Proof. The Riesz representation theorem enables us, by abuse of notation, to view the measure $\omega \in M(\mathbb{R}^n)$ as an element in $C_0(\mathbb{R}^n)'$. Furthermore a routine argument shows that a function $f \in C_0(\mathbb{R}^n)$ is uniformly continuous. Note that since $b\mathbb{R}^n$ is compact we have $C_0(b\mathbb{R}^n) = C(b\mathbb{R}^n)$. Since $\mathbb{R}^n$ is dense in the Bohr compactification $b\mathbb{R}^n$, a function $f \in C_0(\mathbb{R}^n)$ can be continuously extended to $C(b\mathbb{R}^n)$. This follows from a general metric space theorem, which states that every uniformly continuous real function on a dense subset can be continuously extended to its closure (see Rudin [15, Chapter 4, Ex. 13]). Therefore the measure ω can be considered as element of the dual space $C(b\mathbb{R}^n)'$. Thus again by the Riesz representation theorem ω can be considered as element in $M(b\mathbb{R}^n)$. □

In what follows, it will be crucial to note, that the Fourier-Stieltjes transform of the normalized Haar measure λ of a compact abelian group G is given by

$$\widehat{\lambda}(\xi) = \int_G \langle x, \xi \rangle d\lambda(x) := \begin{cases} 1 & \text{if } \xi = 1 \\ 0 & \text{if } \xi \neq 1. \end{cases} \tag{7.1}$$

This was shown implicitly in the proof of Theorem 6.9 in Equation (6.2).

The next theorem will be stated for general compact LCA groups. Recall the trivial fact, that on a compact LCA group K we have $C_c(K) = C_0(K) = C_b(K) = C(K)$.

Theorem 7.2 (Levy). *Let K be a compact LCA group and $\omega_k \in M(K)$ be probability measures for $k \in \mathbb{N}$. Then $(\omega_k)_{k\in\mathbb{N}}$ converges to the normalized Haar measure λ in the weak* topology $\sigma(M(K), C(K))$ if and only if $\widehat{\omega}_k(\xi) \to \widehat{\lambda}(\xi)$ for every character $\xi \in \widehat{K}$.*

Proof. First consider that the measures ω_k converge weak* to the normalized Haar measure λ. This means that for every $f \in C_b(K) = C(K)$ we have

$$\lim_{k\to\infty} \langle \omega_k, f \rangle = \lim_{k\to\infty} \int_K f d\omega_k = \int_K f d\lambda.$$

By definition, every character $\xi : K \to \mathbb{T}$ is continuous, hence by the above equation we have

$$\lim_{k\to\infty} \widehat{\omega_k}(\xi) = \lim_{k\to\infty} \int_K \overline{\langle x, \xi \rangle} d\omega_k(x) = \int_K \overline{\langle x, \xi \rangle} d\lambda(x) = \widehat{\lambda}(\xi).$$

Thus the Fourier-Stieltjets transforms of the measures ω_k converge to the Fourier-Sieltjes transform of the normalized Haar measure.

On the other hand, let $\widehat{\omega_k}(\xi) \to \widehat{\lambda}(\xi)$ for every $\xi \in \widehat{K}$. By Theorem 6.8 $A(\widehat{K})$ the space of all Fourier transforms of functions in $L^1(\widehat{K})$ is dense in $C(\widehat{K})$. Therefore by Pontryagins theorem $A(K) = A\left(\widehat{\widehat{K}}\right)$ is dense in $C_0(K)$. Furthermore by discreteness of $\widehat{K}$ we obtain $L^1(\widehat{K}) \subset C(\widehat{K})$. Let $f \in C(K)$, the latter considerations enable us to find for every $\varepsilon > 0$ a function $g \in C(\widehat{K})$ with $\|f - \mathcal{F}_{\widehat{K}} g\|_\infty < \varepsilon$, where we define $\mathcal{F}_{\widehat{K}} g := \int_{\widehat{K}} \overline{\langle x, \xi \rangle} g(\xi) d\xi$. Here $d\xi$ is the normalized Haar measure on $\widehat{K}$. Before we proceed with the proof, we need the following identity, which we infer by application of Fubini's theorem:

$$\begin{aligned} \int_K \mathcal{F}_{\widehat{K}}(g)(x) dx &= \int_K \int_{\widehat{K}} \overline{\langle \xi, x \rangle} g(\xi) d\xi dx = \int_{\widehat{K}} g(\xi) \int_K \overline{\langle x, \xi \rangle} dx d\xi \\ &= \int_{\widehat{K}} g(\xi) d\widehat{\lambda}(\xi) \end{aligned}$$

Thus with the above identity and remembering, that the ω_k are probability measures, we can now calculate

$$
\begin{aligned}
|\langle \omega_k, f\rangle - \langle \lambda, f\rangle| &= \left| \int_K f d\omega_k - \int_K f d\lambda \right| \\
&\leq |\langle \omega_k, f - \mathcal{F}_{\widehat{K}} g\rangle| + |\langle \omega_k, \mathcal{F}_{\widehat{K}} g\rangle - \langle \lambda, \mathcal{F}_{\widehat{K}} g\rangle| + |\langle \lambda, \mathcal{F}_{\widehat{K}} g - f\rangle| \\
&\leq \|f - \mathcal{F}_{\widehat{K}} g\|_\infty + \left| \int_K \mathcal{F}_{\widehat{K}} g(x) d\omega_k(x) - \int_K \mathcal{F}_{\widehat{K}} g(x) d\lambda(x) \right| + \|f - \mathcal{F}_{\widehat{K}} g\|_\infty \\
&< 2\varepsilon + \int_{\widehat{K}} |g(\xi)| |\widehat{\omega}_k(\xi) - \widehat{\lambda}(\xi)| d\xi.
\end{aligned}
$$

Because the Fourier-Stieltjes transform of a finite regular Borel measure is bounded and furthermore the function g is in $L^1(\widehat{K})$, we can apply Lebesque's dominated convergence theorem. Hence the right hand side of the above inequality converges to zero, which shows that ω_k converges to the normalized Haar measure λ in the weak* topology $\sigma(M(K), C(K))$ as $k \to \infty$. □

The above proof is a modification from Forst [20, Theorem 3.14], which states Levy's theorem for general LCA groups and bounded measures.

The next theorem was shown by Blum and Eisenberg [2]. We note, that by the example in Section 6 we have $\widehat{\mathbb{R}^n} = \mathbb{R}^n$ and therefore, as usual, the neutral element of $\widehat{\mathbb{R}^n}$ is 0.

Theorem 7.3. *Let $\omega_k \in M(\mathbb{R}^n)$ be probability measures for every $k \in \mathbb{N}$. Consider ω_k as elements of $M(b\mathbb{R}^n)$, then the sequence $(\omega_k)_{k\in\mathbb{N}}$ converges in the weak* topology $\sigma(M(b\mathbb{R}^n), C(b\mathbb{R}^n))$ to the normalized Haar measure on $b\mathbb{R}^n$, if and only if the Fourier-Stieltjes transforms $\widehat{\omega_k} : \mathbb{R}^n \to \mathbb{C}$ fulfil*

$$
\lim_{k\to\infty} \widehat{\omega_k}(\xi) = \begin{cases} 0 & \text{if } \xi \neq 0 \\ 1 & \text{if } \xi = 0. \end{cases} \tag{7.2}
$$

Proof. The n-dimensional euclidean space equipped with the discrete topology, will be denoted by $(\mathbb{R}^n)_d$. Thus by Pontryagin's theorem, the group $(\mathbb{R}^n)_d = (\widehat{\mathbb{R}^n})_d$ is the dual group of the Bohr compactification $b\mathbb{R}^n$. This implies that $\mathbb{R}^n = \widehat{b\mathbb{R}^n}$. Since we consider $\omega_k \in M(b\mathbb{R}^n)$ and the Fourier-Stieltjes transform $\widehat{\omega_k}$ is defined for every character on $\widehat{b\mathbb{R}^n}$, we may apply Levy's theorem by noting that (7.2) describes the Fourier transform of the Haar measure on $b\mathbb{R}^n$ thanks to (7.1). □

With this theorem and the example in Section 6, we can reformulate Lemma 4.5 into the setting of the Bohr compactification of the euclidean space $\mathbb{R}^n$.

Proposition 7.4. *The net of measures $(\mu_\varepsilon^t)_{\varepsilon\in\,]0,1]}$ where $\mu = \delta_a$, $a \in \mathbb{R}^n$ with the initial value regularization $\rho \in L^1(\mathbb{R}^n)$ satisfying* (3.1)*, converges in the* *weak* topology $\sigma(M(b\mathbb{R}^n), C_b(b\mathbb{R}^n))$ to the normalized Haar measure on the Bohr compactification $b\mathbb{R}^n$.*

We note that by Theorem 6.15 and again with the example in Section 6 the space of almost-periodic functions $AP(\mathbb{R}^n)$ is indeed the uniform closure of trigonometric polynomials on $\mathbb{R}^n$. This is the reason why we are dealing with almost-periodic functions, since in Subsection 4.3 we worked out the weak* convergence of the probability measures μ_ε^t with respect to trigonometric polynomials.

Furthermore Theorem 6.15 characterizes $AP(\mathbb{R}^n) \cong C(b\mathbb{R}^n)$ and since $b\mathbb{R}^n$ is compact we have by the Riesz representation theorem that $AP(\mathbb{R}^n)' \cong M(b\mathbb{R}^n)$. Therefore we can consider the measures μ_ε^t as elements in $AP(\mathbb{R}^n)'$. Now with Proposition 7.4 and the properties of the invariant mean in Theorem 6.16 we have for $f \in AP(\mathbb{R}^n)$

$$\lim_{\varepsilon \to 0} \int_{b\mathbb{R}^n} \tilde{f} \mu_\varepsilon^t = \int_{b\mathbb{R}^n} \tilde{f} d\lambda = m(f),$$

where $\tilde{f} \in C(b\mathbb{R}^n)$ is the extension of f from Theorem 6.15 and λ is the normalized Haar measure on $b\mathbb{R}^n$.

We summerize our conclusions in the following

Theorem 7.5. *Let $\mu = \delta_a \in M(\mathbb{R}^n)$ with $a \in \mathbb{R}^n$ and the initial value regularization $\rho \in L^1(\mathbb{R}^n)$ with the property* (3.1)*, then for every almost-periodic function $f \in AP(\mathbb{R}^n)$ the net $\langle \mu_\varepsilon^t, f \rangle_{\varepsilon \in]0,1]}$ converges to the invariant mean $m(f)$ as $\varepsilon \to 0$.*

The representation of the invariant mean of an almost-periodic function on the real line in (5.4), yields an explicit formula for the weak* limit of μ_ε^t.

Corollary 7.6. *If δ_a is the Dirac-measure on the real line, then with $\rho \in L^1(\mathbb{R})$ satisfying property* (3.1)*, the net of probability measures $(\mu_\varepsilon^t)_{\varepsilon \in]0,1]}$ fulfils*

$$\lim_{\varepsilon \to 0} \langle \mu_\varepsilon^t, f \rangle = \lim_{R \to \infty} \frac{1}{2R} \int_{-R}^{R} f(y) dy.$$

A Summaries

Abstract

We recapitulate some properties about C_0-semigroups to formulate Stone's theorem, which will be applied to solve the Schrödinger equation with zero potential. Then we continue with the definition of the square root of a measure in terms of regularizations and use them as initial values in the Schrödinger equation. We focus on the 'square root' of the Dirac measure as initial value and study the convergence properties of the time evolved probability measures. After a summary of locally compact abelian groups and almost-periodic functions we prove that these solutions converge to the normalized Haar measure on the Bohr compactification.

Zusammenfassung

Wir wiederholen einige Eigenschaften der C_0-Halbgruppen um den Satz von Stone zu formulieren. Dieser Satz wird verwendet, um die Schrödinger Gleichung ohne Potenzial zu lösen. Danach kommen wir zur Definition von der „Wurzel“ eines Maßes im Sinne einer Regularisierung. Jenes Konstrukt verwenden wir als Anfangswert in der Schrödinger Gleichung. Wir konzentrieren uns dabei auf die „Wurzel“ des Dirac Maßes als Anfangswert und untersuchen die Konvergenzeigenschaften der in der Zeit fortgeschrittenen Wahrscheinlichkeitsmaße. Nach einem Überblick über lokalkompakte abelsche Gruppen und fast-periodische Funktionen werden wir beweisen, dass jene erwähnten Wahrscheinlichkeitsmaße auf der Bohr Kompaktifizierung gegen das normierte Haar Maß konvergieren.

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