DISSERTATION / DOCTORAL THESIS

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„Stability Analysis of Infinite-Dimensional Phase Retrieval Problems"

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The present thesis contains results that have been obtained by the author and his collaborators. Parts of this thesis can be found verbatim in original research papers by the author and his collaborators, that is in particular in the articles by Rathmair [Rat19], by Grohs and Rathmair [GR19a, GR19b] and by Grohs, Koppensteiner and Rathmair [GKR19].
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Contents

Basic notation and definitions

$\mathbb{N}$	natural numbers
$\mathbb{Z}$	integers
$\mathbb{R}$	real numbers
$\mathbb{C}$	complex numbers
$\operatorname{Re} z$, $\operatorname{Im} z$	real and imaginary part of a complex number z
$\bar{z}$	complex conjugate of a complex number z
$\log f$	natural or complex logarithm of f, depending on the context
E^c	complement of a set E
$E \times F$	cartesian product of sets E and F
∂E	topological boundary of a set E
$\bar{E}$	closure of a set E
E°	interior of a set E
$f\|_E$	restriction of a function f to a subset of its domain E
$\operatorname{supp} f$	support of a function f
χ_E	characteristic function of a set E
$\|x\|$	euclidean length of a vector $x \in \mathbb{R}^d$
$\|E\|$	Lebesgue measure of a measurable set $E \subset \mathbb{R}^d$
$\operatorname{dist}(x, y)$	distance between two points x, y in a metric space
$B_r(x)$	ball of radius r centered at x in a metric space
$\operatorname{diam}(E)$	diameter of a subset E of a metric space
$g \circ f$	composition of two functions or operators g and f
$T\|_V$	restriction of an operator T to a subspace V
$\operatorname{ran} T$	range of an operator T
$\operatorname{span} \Phi$	linear span of a family of vectors Φ
$L^p(D)$	Banach space of p-integrable functions on D
$C^k(D)$	space of k-times continuously differentiable functions on $D \subset \mathbb{R}^d$
$Lip(D)$	space of Lipschitz continuous functions on D
$C^k_c(D)$	space of functions in $C^k(D)$ with compact support
$Lip_c(D)$	space of functions in $Lip_c(D)$ with compact support
f_x	partial derivative of a function f w.r.t. the variable x
∇f	gradient of a function f
$\hat{f}$	Fourier transform of $f \in L^2(\mathbb{R}^d)$
$\mathcal{F}$	Fourier transform operator on $L^2(\mathbb{R}^d)$
$f(x) \lesssim g(x)$	there exists $c > 0$ such that $f(x) \leq cg(x)$ for all x
$f(x) \asymp g(x)$	iff $f(x) \lesssim g(x)$ and $g(x) \lesssim f(x)$
$A \ll B$	A is considerably smaller than B

Chapter 1

Introduction

1.1 Motivation

The problem of phase retrieval, i.e., the problem of recovering a function from the magnitudes of its Fourier transform, naturally arises in various fields of physics, such as astronomy [CDF87], radar [Jam99], speech recognition [RJ93] and quantum mechanics [Pau46]. The most prominent example, however, is diffraction imaging, where in a basic experiment an object is placed in front of a laser which emits coherent electromagnetic radiation. The object interacts with the incident wave in a manner creating a new wave front, which is described by Kirchhoff's diffraction equation. An adequate approximation of the resulting wave front in the far field is given by the Fraunhofer diffraction equation, which essentially states that the wave front in a plane in a sufficiently large distance from the object is given by the Fourier transform (with appropriate spatial scaling) of the function representing the object, cf. [Goo05] for an introduction to diffraction theory.

The aim in diffractive imaging is to determine the object from measurements of the diffracted wave. This objective is seriously impeded by the fact that measurement devices usually are only capable of capturing the intensities and a loss of phase information takes place. Reconstructing the object from the far field diffraction intensities, the so-called diffraction pattern, therefore requires to solve the *Fourier phase retrieval* problem

$$\text{given } |\hat{f}|, \text{ find } f \text{ (up to trivial ambiguities).}$$

The name 'phase retrieval' accounts for the fact that recovery of the phase of $\hat{f}$ is equivalent to recovering f itself.

In microscopy a lens is employed to essentially invert the Fourier transformation and create the image of the object. While this is possible in case of visible light, which has a wavelength of approximately $10^{-7}m$, lenses which perform this task are not available for waves of much shorter wavelength (e.g. for x-rays with a wavelength in the range between $10^{-8}m$ and $10^{-11}m$). Since the spatial resolution of the optical system is proportional to the wavelength of light the direct approach using lenses can only achieve a certain level of resolution. In order to obtain high resolution it is necessary to compute the image from the diffraction pattern.

Determining objects from diffraction patterns – and therefore the question of phase retrieval – for the first time became relevant when Max von Laue discovered in 1912 that x-rays are diffracted when interacting with crystals, an insight for which he was awarded the Nobel prize in Physics two years later. The discovery of this phenomenon launched the field of x-ray crystallography. Crystallography seeks to determine the atomic and molecular structure of a crystal, i.e., a material whose atoms are arranged in a periodic fashion. In the diffraction pattern the periodicity of the crystalline sample manifests itself in form of strong peaks (Bragg peaks) lying on the so-called reciprocal lattice, cf. [Mil90]. From the position and the intensities of these peaks chrystallographers can deduce the electron density of the crystal. Over the course of the past century the methods of x-ray crystallography have developed into the most powerful tool for analyzing the atomic structure of various materials and have enabled scientists to achieve breakthrough results in different fields such as chemistry, medicine, biology, physics and material sciences. This is highlighted by the fact that more than a dozen Nobel prizes have been awarded for work involving x-ray crystallography, the discovery of the double helix structure of DNA [WC53] being just one example. For an exhaustive introduction to x-ray crystallography the interested reader may have a look at [HoC01, LP14].

In 1980 it was proposed by David Sayre [SFG+80] to extend the approach of x-ray crystallography to non-crystalline specimens. Almost twenty years later, facilitated by the development of new powerful x-ray sources, Sayre et al.[MCKS99] for the first time successfully reconstructed the image of a sample with resolution at nanometer scale from its x-ray diffraction pattern, see Figure 1.1.
This approach is nowadays known under name of Coherent Diffraction Imaging(CDI). The process consists of two principal steps. Firstly, the acquisition of one or multiple diffraction patterns and secondly, processing the diffraction patterns in order to obtain the image of the sample, which is usually done by applying iterative phase retrieval algorithms. Plenty of CDI methods have been developed over recent years and have been employed to great success in physics, biology and chemistry. See [MIRM15, SCEC+15] for very recent overviews on CDI methods, their limitations and their achievements in various applications and for algorithmic phase retrieval methods in diffraction imaging.

1.2 General problem formulation

Phase retrieval in the most general formulation is concerned with reconstructing a function f in a space $\mathcal{X}$ from the phaseless information of some transform of f. The operator describing the transform, which will be denoted by T, is mapping elements of $\mathcal{X}$ into another space $\mathcal{Y}$ of either real- or complex valued functions and is usually linear, i.e.,

$$T : \mathcal{X} \to \mathcal{Y}.$$

Furthermore, T is usually nicely invertible, which means that $T : \mathcal{X} \to \operatorname{ran} T$ has a bounded inverse. In order to have a concrete example in mind one may think of $\mathcal{X} = \mathcal{Y} = L^2(\mathbb{R}^d)$ and $T = \mathcal{F}$, the Fourier transform operator. In this case it is well-known that T is a unitary map. Under the above assumptions the linear measurement process does not introduce a loss of information. However, the situation changes significantly if the phase information of the transform is absent. The problem arises of studying the obviously nonlinear mapping

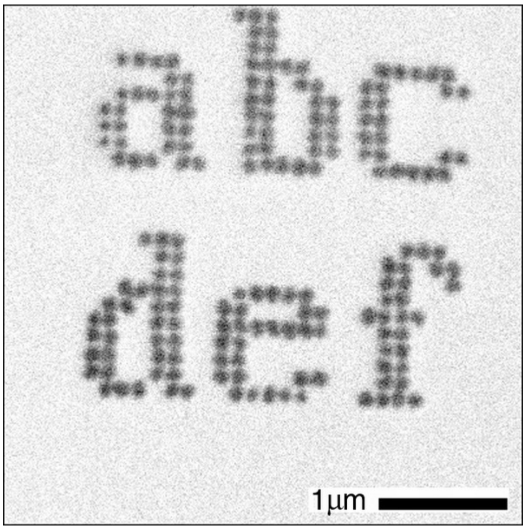
$$\mathcal{A} : f \mapsto |Tf|, \quad f \in \mathcal{X}$$

and its invertibility properties. Well posedness in the sense of Hadamard of an inverse problem associated with $f \mapsto \mathcal{A}f$ requires

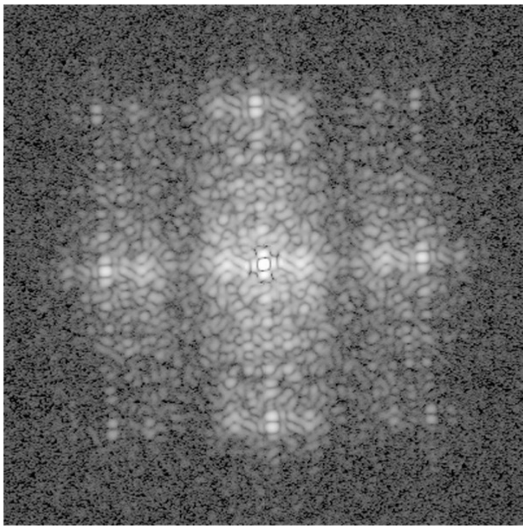
(1) *existence* of a solution, i.e., $\mathcal{A}$ to be surjective

(2) *uniqueness*, i.e., $\mathcal{A}$ to be injective and

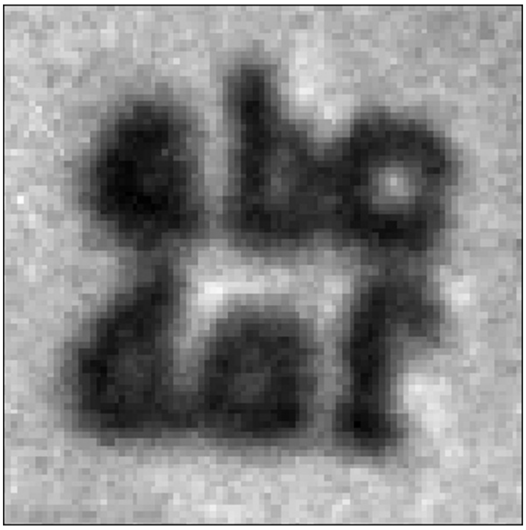
(3) *stability*, meaning that the solution continuously depends on the data, i.e., $\mathcal{A}^{-1}$ to be continous.



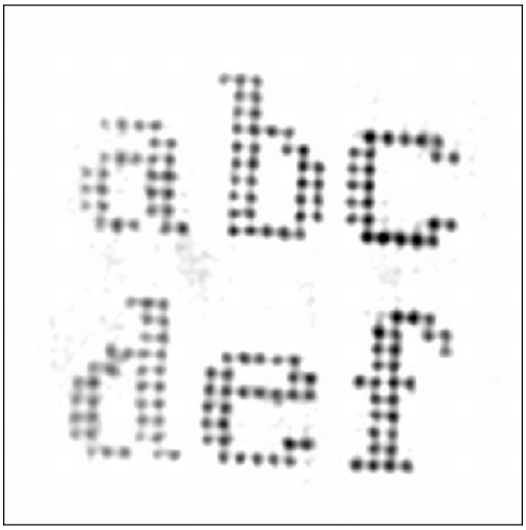
(a) Scanning electron microscope image of the specimen.



(b) Diffraction pattern of the specimen.



(c) Image obtained by optical microscope.



(d) Image of the specimen as reconstructed from the diffraction pattern by an iterative algorithm.

Figure 1.1: The non-crystalline specimen in the experiments carried out in [MCKS99] consists of gold dots, each of diameter $\approx 100nm$, placed on a silicon nitride membrane.

For the problem of phase retrieval, Condition (1) amounts to identifying the image of the operator $\mathcal{A}$. The question is often of minor importance compared to (2) and (3) as it is simply assumed that the input data arise from the measurement process described by $\mathcal{A}$.
Provided that $\mathcal{X}$ is a vector space – exluding trivial cases – $\mathcal{A}$ is not injective due to the simple observation that

$$\mathcal{A}f = \mathcal{A}(cf), \quad f \in \mathcal{X},\ |c| = 1. \tag{1.1}$$

Further ambiguities may occur, such as translations in the Fourier example but also less trivial ones. The first key question in the mathematical analysis of a phase retrieval problem is to

identify all ambiguities. Depending on the context a particular source of ambiguities is either classified as trivial or as severe. If there exist severe ambiguities the phase retrieval problem is hopeless as there exist different objects yielding identical measurements. If on the other hand all occurring ambiguities are considered trivial, f and g may be identified ($f \sim g$) whenever $\mathcal{A}f = \mathcal{A}g$. Let $\tilde{\mathcal{X}} = \mathcal{X}/\sim$ denote the quotient set, then – by definition – $\mathcal{A}$ is injective as mapping acting on $\tilde{\mathcal{X}}$ and uniqueness is in this new sense is ensured.
In order to study stability $\tilde{\mathcal{X}}$ has to be endowed with a reasonable topology first. In case $\mathcal{X}$ is a normed space and the only ambiguities occurring are of the type as in (1.1) usually the pseudometric

$$d([f]_\sim, [g]_\sim) := \inf_{|c|=1} \|f - cg\|$$

is used. If there are other ambiguities a suitable choice may be less obvious.
Beyond determining whether the mapping $\mathcal{A}$ on $\tilde{\mathcal{X}}$ is continuously invertible further continuity properties of the inverse are often studied such as (local) Lipschitz continuity.

If there are nontrivial ambiguities, i.e., if injectivity is not attained after identifying all trivial ambiguities or if the inverse is not continuous, one or both of the following measures may be taken in order to render the phase retrieval problem well posed:

- Restriction of $\mathcal{A}$: The restriction $\mathcal{A} : \tilde{\mathcal{X}}' \to \mathcal{A}(\tilde{\mathcal{X}}')$, where $\tilde{\mathcal{X}}' \subset \tilde{\mathcal{X}}$ is eventually injective (has a continuous inverse) if $\tilde{\mathcal{X}}'$ is choosen sufficiently small, $\tilde{\mathcal{X}}'$ consisting of a single element being the extremal, trivial example.
  Restriction of $\mathcal{A}$ to a smaller domain can be understood as imposing additional a priori constraints on the function f to be determined. In applications of the phase retrieval problem from Fourier measurements, for instance, it is typically sensible to demand that f is nonnegative, as other functions do not hold a physical meaning.

- Modification of T: The idea is to introduce additional redundancy in the linear transform in order to soften the setback which is suffered by the subsequent removal of the phase information.
  In case of the Fourier phase retrieval problem this can be achieved by applying several different manipulations of f before computation of the Fourier transform, e.g. using

  $$T'f := \left(\hat{f}, \widehat{fg_1}, \ldots, \widehat{fg_m}\right), \tag{1.2}$$

  for known functions $g_1, \ldots, g_m$ instead of $Tf = \hat{f}$. In the context of diffraction imaging this approach is common practice as a physical system which produces measurements $|T'f| = \left(|\hat{f}|, |\widehat{fg_1}|, \ldots\right)$ can often be implemented. In ptychography [NMR95] different sections of an object are illuminated one after another and the object is to be reconstructed from several diffraction patterns. For suitable, localized window functions $g_1, g_2, \ldots$ Equation (1.2) serves as a reasonable mathematical model.
  As a second example of a redundancy introducing modification in diffraction imaging let us mention interferometry [Gab48], where the diffracted waves interfere with the wave field of a known object. This idea amounts to an additive distortion of the wave field:

  $$T'f := \left(\hat{f}, \widehat{f + g_1}, \ldots, \widehat{f + g_m}\right)$$

These ideas have been systematically implemented for the first time by Jaming, see for instance [Jam99, Jam14].

When studying a concrete phase retrieval problem with an application in the background it is useful to keep in mind that often there is a certain degree of freedom in the way how the measurements are acquired. For instance, in diffraction imaging there is the fundamental observation that the wave in the object plane and the wave in the far field are connected in terms of the Fourier transform. However there are many different options in how to generate one or several diffraction patterns. Instead of viewing a phase retrieval problem as the analysis of a fixed operator $\mathcal{A}$ one may aswell include the question of how to design the measurement process in order to get a well posed problem.

Even though the problem of recovering lost phase information has been omnipresent in various disciplines for more than a century most of the activity by the mathematics community on this topic is quite recent. Notable exceptions include the work by Hauptman [Hau86] and by Rosenblatt [Ros84].
Phase retrieval problems have been studied in a rich variety of shapes. It can be distinguished between finite- and infinite- dimensional as well as between discrete and continuous phase retrieval problems. Furthermore phase retrieval problems differ in what kind of measurements are considered, i.e., the choice of the operator T. The most common choice is that T involves some sort of Fourier transform [Aku56, Wal63, Hof64, ADGY19, MW15]. Moreover there is a huge body of research in the more abstract setting of frames, where it is assumed that T is induced by a frame [BCE06, BCMN14, CCD16, AG17]. Phase retrieval problems where the quantity of interest is assumed to arise as the solution of certain differential equations have also been studied [JPE17, Jam17].

Beyond the question of well-posedness it is desirable to provide a method that recovers a function f (at least the equivalence class $[f]_\sim$) from the observed measurements $\mathcal{A}f$. Such a method could be an explicit expression of the inverse of $\mathcal{A}$. Mostly the aim of coming up with an explicit expression is rather hopeless. In practice iterative algorithms are employed, which serve as approximate inverses of the measurement mapping $\mathcal{A}$. A framework based on iterative projections which is often simple to implement and proved to be very flexible was introduced by Gerchberg and Saxton [GS71] and was later extended by Fienup [Fie82]. These methods have been employed to great empirical success but due to the absence of convexity there is no guarantee of convergence. Recently – based on pioneering work by Candes et al.[JSV13] – randomized measurement setups have appeared to be very beneficial in order to prove convergence guarantees for various phase retrieval algorithms.

1.3 Outline

This thesis deals predominantly with the study of phase retrieval problems in infinite-dimensional function spaces, with the involved operator being the Fourier transform or a variant thereof. The focus lies on establishing quantitative Lipschitz-type bounds in order to gain insight in the stability properties of the reconstruction process.

In Chapter 2 we review some related results on phase retrieval in infinite-dimensional spaces, both from a very general and abstract frame theoretic point of view as well as concrete Fourier type examples.
Chapter 3 discusses the question of uniqueness for the problem of phase retrieval from phaseless short-time Fourier transform measurements.
In Chapter 4 we discuss the problem of phase retrieval for analytic measurements and establish the first quantitative results. In Chapter 5 we study the logarithmic derivative of entire functions satisfying certain growth restrictions and derive uniform bounds for its growth. Chapter 6 deals with the topic of weighted Poincaré inequalities. The results from Chapters 4, 5 and 6 combined will eventually enable us to obtain a complete understanding of phase retrieval for the so-called Gabor transform.

In Chapter 7 the 1d Fourier phase retrieval problem is considered. In certain measurement scenarios Lipschitz-type stability results can be established.

Chapter 2

Related results and extensions

We shall now discuss results that are closely related with the topic of this thesis. Besides providing an overview of the relevant literature, the main purpose of this chapter is to convince the reader that

- mapping $f \mapsto |\mathcal{F}f|$ comes with a significant loss of information; hence in order to recover f either a priori information is required or the measurement mapping needs to be altered.
- the question of stability is a hard one for any infinite-dimensional phase retrieval problem.

2.1 Phase retrieval from Fourier type measurements

First observe that since the Fourier transform operator $\mathcal{F}$ is unitary on $L^2(\mathbb{R}^d)$ an arbitrary change of phase in the Fourier domain yields an ambiguous solution, i.e., for all $f \in L^2(\mathbb{R}^d)$ and $\theta : \mathbb{R}^d \to \mathbb{R}$ measurable we have that $|\mathcal{F}f| = |\mathcal{F}f_\theta|$, where

$$f_\theta := \mathcal{F}^{-1}\left(e^{i\theta}\mathcal{F}f\right).$$

One possible attempt to remove (some) ambiguity of the reconstruction problem is to restrict to compactly supported functions giving rise to what is often called the *classical Fourier phase retrieval problem*:

> Recover a compactly supported function $f \in L^2(\mathbb{R}^d)$ from the modulus of its Fourier transform $|\mathcal{F}f|$.

This assumption often is meaningful, for example in diffraction imaging where the object of interest typically is of finite extent. Note that ambiguities such as multiplication by a unimodular constant and translation are clearly not removed by restriction to compactly supported functions.
Uniqueness results for the classical Fourier phase retrieval problem oftentimes rely on the fact that the Fourier transform of a compactly supported L^2-function f extends to an entire function of exponential type. The extension is precisely the Fourier-Laplace F transform of f

$$F(z) := \int_{\mathbb{R}^d} f(t)e^{-2\pi i z\cdot t}\,dt, \quad z \in \mathbb{C}^d.$$

By the Paley-Wiener theorem also the converse is true, i.e., every entire function of exponential type is the Fourier-Laplace transform of a compactly supported function.

Careful examination of the Fourier-Laplace transform reveals that nontrivial ambiguities can only exist if the Fourier-Laplace transform is reducible [SH84].

In the one-dimensional case the classical results due to Akutowicz [Aku56, Aku57], Hofstetter [Hof64] and Walther [Wal63] give a complete characterization of all ambiguities. All nontrivial ambiguities are obtained by flipping a subset of the zeros of the Fourier-Laplace transform across the real axis.

Theorem 2.1 (Akutzowicz-Hofstetter-Walther)**.** *Suppose $f, g \in L^2(\mathbb{R})$ are compactly supported. Let F denote the Fourier-Laplace transform of f with Hadamard factorization*

$$F(z) = e^{az+b} z^m \prod_{\zeta \in \mathcal{Z}} (1 - z/\zeta) e^{z/\zeta},$$

where $\mathcal{Z}$ denotes the multiset of nonzero roots of F (each root appears according to its multiplicity).
Then we have that $|\mathcal{F}g| = |\mathcal{F}f|$ if and only if there exist $\alpha \in \mathbb{R}$, $\tau \in \mathbb{R}^d$ and $J \subset \mathcal{Z}$ such that

$$G(z) = e^{i(\alpha + \tau \cdot z)} e^{az+b} z^m \prod_{\zeta \in J} (1 - z/\bar{\zeta}) e^{z/\bar{\zeta}} \cdot \prod_{\zeta \in \mathcal{Z} \setminus J} (1 - z/\zeta) e^{z/\zeta}.$$

Consequently a compactly supported univariate function is uniquely determined by the modulus of its Fourier transform up to multiplication by a unimodular factor and/or translation if and only if its Fourier-Laplace transform has all its zeros located on the real axis.

Thakur [Tha11] studied the question of reconstructing a realvalued bandlimited function given its modulus. Note that this setting corresponds to the classical Fourier phase retrieval problem under the additional restriction to functions with realvalued Fourier transform. Thakur proofs that such functions can be recovered from samples of the modulus at more than twice the Nyquist rate and moreover provides an algorithm to fulfill this task.
Restrictions formulated in terms of f rather than of $\hat{f}$ which render the Fourier phase retrieval problem unique up to trivial ambiguities include for example monotonicity or some sort of symmetry [KST95] or gaps of significant length in the support [CF81].

So far we discussed mostly the one-dimensional Fourier phase retrieval problem. Let us briefly comment on the case where $d \geq 2$. The picture here is conjectured to be completely different. While in 1d polynomials are never irreducible (except for trivial cases) according to the fundamental theorem of algebra and univariate functions are rarely irreducible (Hadamard's factorization theorem), the set of polynomials of several variables which are reducible is of measure zero [HM82]. This observation suggests to conjecture that a generic entire function of several variables is irreducible, see for example [BN84] where it is stated that

> "Irreducibilty extends to general functions of two variables with infinite sets of zeros, so that exact alternative solutions are most unlikely in 2-D phase retrieval."

However, we are not aware of a rigorous argument of this claim.

Using similar techniques from complex analysis as outlined above (Paley-Wiener Theorem, Hadamard factorization, etc.) Jaming proves in [Jam14] that a univariate entire function of exponential type is uniquely determined (up to a sign factor) given its modulus on two (not completely arbitrary) lines. This fact is then used to derive a number of uniqueness results for various phase retrieval problems such as reconstruction from fractional Fourier transform magnitudes and from certain windowed Fourier and wavelet transform magnitudes.

If, in the Fourier phase retrieval problem the assumption on f to be compactly supported is dropped one can no longer expect that its Fourier-Laplace transform F extends to an entire function. However, under suitable assumptions on the decay of f, $\hat{f}$ can still be analytically extended to a subdomain of $\mathbb{C}$, such as a strip. Blaschke products and inner-outer factorization can be used to take over the role of Hadamard's factorization and allow for characterization of ambiguous sulutions in this more general case [JKPi19], [PLB17].

Mallat and Waldspurger [MW15] consider the problem of reconstruction from phaseless Cauchy wavelet transform measurements. This reconstruction problem is in fact closely related to the question whether a function which is holomorphic on the upper half plane can be recovered given its phaseless values on a certain infinite set of lines parallel to the real axis. Besides giving a positive answer to the question of uniqueness also the stability properties of the reconstruction problem are studied in depth. It is pointed out that all instabilities that were observed experimentally share a common qualitative behaviour:

> "All instabilities [...] that we were able to observe in practice were of the form we described: each time, the wavelet transforms of g_1 and g_2 were equal up to a phase whose variation was slow [...], except at the points where $g_1 * \psi_j$ was small."

Furthermore, estimates are provided which confirm that this is the case for all instabilities.

Discussion of the articles [ADGY19] and [AG19] concerning phase retrieval from the modulus of the Gabor transform would certainly not be out of place here but will be postponed to Section 4.2 where we will have the necessary preliminary material at hand.

2.2 Phase retrieval in the frame theoretic setting

The most general formulation of phase retrieval is concerned with the problem of recovering a vector from the absolute values of a bunch of linear measurements. Thus, it has appeared to be very useful to study such kind of problems in the context of frames, as first suggested by Balan, Casazza and Edidin [BCE06].
This section is concerned with summarizing Alaifari and Grohs' article on "Phase retrieval in the general setting of continuous frames for Banach spaces" [AG17] as well as extending some of their results. Note that parts of the results that will be presented here have been established prior, albeit in a less general context; see [BCE06] for the finite-dimensional and [CCD16] for the infinite-dimensional Hilbert space setting.

To make the setup concrete suppose that $\mathcal{B}$ denotes a Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $\mathcal{B}'$ its dual. Any family of measurement vectors $\Phi = (\varphi_\lambda)_{\lambda \in \Lambda} \subset \mathcal{B}'$, where the index set Λ may well be uncountable, gives rise to the mappings

$$\begin{aligned} T_\Phi : \mathcal{B} \longrightarrow & \mathbb{R}^\Lambda \\ f \longmapsto & (\langle f, \varphi_\lambda \rangle)_{\lambda \in \Lambda} \end{aligned}$$

and

$$\begin{aligned} \mathcal{A}_\Phi : \mathcal{B} \longrightarrow & \mathbb{R}_+^\Lambda \\ f \longmapsto & (|\langle f, \varphi_\lambda \rangle|)_{\lambda \in \Lambda} \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the dual pairing.
Φ is said to *do phase retrieval* if any $f \in \mathcal{B}$ is uniquely determined by $\mathcal{A}_\Phi f$ up to a unimodular constant, or in other words, if $\mathcal{A}_\Phi$ as a mapping on $\mathcal{B}/_\sim$ is injective, where

$$g \sim f \quad :\Leftrightarrow \quad \exists c \in \mathbb{K}, \, |c| = 1 \text{ such that } g = cf.$$

For a family of bounded linear functionals $\Phi = (\varphi_\lambda)_{\lambda \in \Lambda} \subset \mathcal{B}'$ and $S \subset \Lambda$ we define $\Phi_S := (\varphi_\lambda)_{\lambda \in S}$. For a linear subspace $V \subset \mathcal{B}'$ let $V_\perp$ denote the *annihilator* of V, i.e.,

$$V_\perp = \{f \in \mathcal{B} : \langle f, v \rangle = 0 \; \forall v \in V\}.$$

Definition 2.2. The family $\Phi = (\varphi_\lambda)_{\lambda\in\Lambda} \subset \mathcal{B}'$ satisfies the *complement property* in $\mathcal{B}$ if for every $S \subset \Lambda$ it holds that either $(\operatorname{span}\Phi_S)_\perp = \{0\}$ or that $(\operatorname{span}\Phi_{\Lambda\setminus S})_\perp = \{0\}$.

The complement property is a necessary condition for $\mathcal{A}_\Phi$ to be injective. In the real case it is also sufficient.

Theorem 2.3 ([AG17, Lemma 2.2, Lemma 2.3])**.** *Let $\mathcal{B}$ be a Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $\Phi \subset \mathcal{B}'$ a family of bounded linear functionals. Then the following holds:*

(i) Suppose that $\mathcal{A}_\Phi$ is injective, then Φ satisfies the complement property.

(ii) If $\mathbb{K} = \mathbb{R}$ and Φ satisfies the complement property, then $\mathcal{A}_\Phi$ is injective.

We proceed with discussing the question of stability, i.e., continuity properties of the inverse of the measurement mapping $\mathcal{A}_\Phi$. Before we can talk about continuity we have to settle the topological structure on both the domain of $\mathcal{A}_\Phi$ and its image. A sensible metric on $\mathcal{B}$ should respect that $f \sim cf$ for unimodular c. Thus, a natural choice is to define

$$d_{\mathcal{B}}(g,f) := \inf_{c\in\mathbb{K}:|c|=1} \|g - cf\|_{\mathcal{B}}.$$

We assume that there is a fixed Banach space $\mathfrak{B}$ which contains all possible measurements, i.e., $\operatorname{ran}\mathcal{A}_\Phi \subset \mathfrak{B} \subset \mathbb{K}^\Lambda$. The following weak assumptions are imposed on $\mathfrak{B}$:

Definition 2.4 (Admissible Banach space)**.** Let Λ be a σ-compact topological space. A Banach space $\mathfrak{B} \subset \mathbb{K}^\Lambda$ over $\mathbb{K}$ is called admissible if it satisfies the following properties:

(i) For every compact set $K \subset \Lambda$ it holds that $\|\chi_K\|_{\mathfrak{B}} < +\infty$.

(ii) $\mathfrak{B}$ is solid, meaning that for all $z \in \mathfrak{B}$ with $|z(\lambda)| \geq |w(\lambda)|$ for all $\lambda \in \Lambda$ it holds that

$$\|w\|_{\mathfrak{B}} = \||w|\|_{\mathfrak{B}} \leq \|z\|_{\mathfrak{B}}.$$

(iii) The set of elements in $\mathfrak{B}$ with compact support is dense in $\mathfrak{B}$.

Note that for example L^p-spaces for finite p satisfy the assumptions of Definition 2.4. Furthermore, we will require that the measurement system constitutes a so-called Banach frame which is defined as follows.

Definition 2.5 (Banach frame)**.** Let $\Phi = (\varphi_\lambda)_{\lambda\in\Lambda} \subset \mathcal{B}'$ be such that $\lambda \mapsto \varphi_\lambda$ is continuous. Suppose there is an admissible Banach space $\mathfrak{B} \subset \mathbb{K}^\Lambda$ satisfying the following properties:

(i) There are $B \geq A > 0$ such that

$$A\|f\|_{\mathcal{B}} \leq \|T_\Phi f\|_{\mathfrak{B}} \leq B\|f\|_{\mathcal{B}} \qquad \forall f \in \mathcal{B}.$$

(ii) There exists a continuous operator $R : \mathfrak{B} \to \mathcal{B}$ such that

$$R(T_\Phi f) = f \qquad \forall f \in \mathcal{B}.$$

Then Φ is called a frame for $\mathcal{B}$. The best possible such constants are denoted by A_Φ and B_Φ and called lower and upper frame bound of Φ respectively.

Note that for any $\Lambda' \subset \Lambda$ it makes sense to consider $T_{\Phi_{\Lambda'}} f$ and $\mathcal{A}_{\Phi_{\Lambda'}} f$ as elements of $\mathfrak{B}$ by identifying it with $(T_\Phi f)\chi_{\Lambda'}$ and $(\mathcal{A}_\Phi f)\chi_{\Lambda'}$.
Again the assumption that Φ constitutes a frame is very weak; in fact it is necessary for stable recovery even if the phase information was available.
Injectivity of $\mathcal{A}_\Phi$ always entails weak stability, i.e., continuity of $\mathcal{A}_\Phi^{-1}$, cf. [AG17, Theorem 3.3]. In view of practical applications it is desirable to establish quantitative bounds.

Definition 2.6. Given a measurement system $\Phi \subset \mathcal{B}'$ and an associated Banach space $\mathfrak{B}$ we say that the corresponding phase retrieval problem is *uniformly stable* if there exists a finite C such that

$$d_{\mathcal{B}}(f,g) \leq C\|\mathcal{A}_\Phi f - \mathcal{A}_\Phi g\|_{\mathfrak{B}} \qquad \forall f,g \in \mathcal{B}.$$

The smallest possible such number C is called the *global stability constant* and is denoted by C_{glob}.
For fixed $f \in \mathcal{B}$ the *local stability constant* $C_{\text{loc}}(f)$ is defined as the smallest number C such that

$$d_{\mathcal{B}}(f,g) \leq C\|\mathcal{A}_\Phi f - \mathcal{A}_\Phi g\|_{\mathfrak{B}} \qquad \forall g \in \mathcal{B}.$$

Note that a given phase retrieval problem is uniformly stable if and only if the local stability constant is uniformly bounded, i.e., if

$$\sup_{f\in\mathcal{B}} C_{\text{loc}}(f) = C_{\text{glob}} < +\infty.$$

Next quantitative versions of the complement property are introduced.

Definition 2.7 (Strong complement property)**.** The system $\Phi = (\varphi_\lambda)_{\lambda\in\Lambda} \subset \mathcal{B}'$ satisfies the σ-strong complement property if $\sigma_\Phi > 0$, where

$$\sigma_\Phi := \inf_{S\subset\Lambda} \max\left\{A_{\Phi_S}, A_{\Phi_{\Lambda\setminus S}}\right\}.$$

For fixed $f \in \mathcal{B}$ we define

$$h_\Phi(f) := \inf_{h\in\mathcal{B}, S\subset\Lambda} \frac{\max\{\|\mathcal{A}_{\Phi_S} h\|_{\mathfrak{B}}, \|\mathcal{A}_{\Phi_{\Lambda\setminus S}}(f-h)\|_{\mathfrak{B}}\}}{\min\{\|h\|_{\mathcal{B}}, \|f-h\|_{\mathcal{B}}\}}$$

Remark 2.8. Assuming $\sigma_\Phi > 0$, from the definition of σ_Φ and from

$$A_{\Phi_{\Lambda'}} = \inf_{f\in\mathcal{B}:\|f\|_{\mathcal{B}}=1} \|T_{\Phi_{\Lambda'}} f\|_{\mathfrak{B}}, \qquad \Lambda' \subset \Lambda$$

it follows that there exist normalized functions $f_1, f_2 \in \mathcal{B}$ and $S \subset \Lambda$ such that

$$\|T_{\Phi_S} f_1\|_{\mathfrak{B}} < 2\sigma_\Phi \qquad \text{and} \qquad \|T_{\Phi_{\Lambda\setminus S}} f_2\|_{\mathfrak{B}} < 2\sigma_\Phi.$$

Thus, if the quantity σ_Φ is small there exists a partition of the measurement system $\Phi = \Phi_S \cup \Phi_{\Lambda\setminus S} =: \Phi_1 \cup \Phi_2$ and (at least one-dimensional) subspaces $V_1, V_2 \subset \mathcal{B}$ such that the restrictions $T_{\Phi_1}|_{V_1}$ and $T_{\Phi_2}|_{V_2}$ have small operator norm.
Similarly, if $h_\Phi(f)$ is small, there exists a partition $\Phi = \Phi_1 \cup \Phi_2$ and a decomposition $f = f_1 + f_2$ such that

$$\|T_{\Phi_1} f_1\|_{\mathfrak{B}} \ll \|f_1\|_{\mathcal{B}} \qquad \text{and} \qquad \|T_{\Phi_2} f_2\|_{\mathfrak{B}} \ll \|f_2\|_{\mathcal{B}}.$$

According to Theorem 2.3 the complement property is necessary for uniqueness. Similarly, its quantitative counterpart determines the stability behaviour.

Theorem 2.9 ([AG17, Theorem 3.9])**.** *Let $\Phi = (\varphi_\lambda)_{\lambda\in\Lambda} \subset \mathcal{B}'$ constitute a frame for $\mathcal{B}$. Then*

$$C_{glob} \gtrsim \sigma_\Phi^{-1},$$

where the implicit constant depends on the ratio B_Φ/A_Φ only.

By slightly modifying the proof of Theorem 2.9 we obtain a similar estimate for the local stability constant in terms of $h_\Phi(f)$.

Theorem 2.10. *Let $\Phi = (\varphi_\lambda)_{\lambda\in\Lambda} \subset \mathcal{B}'$ constitute a frame for $\mathcal{B}$. Let $f \in \mathcal{B}$ with $h_\Phi(f) < A_\Phi/16$, then*

$$C_{loc}(f) \gtrsim h_\Phi(f)^{-1},$$

where the implicit constant depends on the ratio B_Φ/A_Φ only.

Proof. Let $h > h_\Phi(f)$. Then there exist $f_1, f_2 \in \mathcal{B}$ and $S \subset \Lambda$ such that $f = f_1 + f_2$ and

$$\max\{\|T_{\Phi_S} f_1\|_{\mathfrak{B}}, \|T_{\Phi_{\Lambda\setminus S}} f_2\|_{\mathfrak{B}}\} \leq h \min\{\|f_1\|_{\mathcal{B}}, \|f_2\|_{\mathcal{B}}\}. \tag{2.1}$$

We set $g := f_1 - f_2$. By definition we have that

$$C_{\mathrm{loc}}(f) \geq \frac{d_{\mathcal{B}}(f,g)}{\|\mathcal{A}_\Phi f - \mathcal{A}_\Phi g\|_{\mathfrak{B}}}, \tag{2.2}$$

therefore, in order to obtain a lower bound we will estimate the nominator and the denominator from below and above respectively.

For fixed $\lambda \in \Lambda$ we have by the inverse triangle inequality and linearity of φ_λ that

$$||\langle f, \varphi_\lambda\rangle| - |\langle g, \varphi_\lambda\rangle|| \leq |\langle f \pm g, \varphi_\lambda\rangle|.$$

Using that $\mathfrak{B}$ is solid we can therefore estimate

$$\|\mathcal{A}_{\Phi_S} f - \mathcal{A}_{\Phi_S} g\|_{\mathfrak{B}} \leq 2\|\mathcal{A}_{\Phi_S} f_1\|_{\mathfrak{B}} = 2\|T_{\Phi_S} f_1\|_{\mathfrak{B}},$$

and similarly that

$$\|\mathcal{A}_{\Phi_{\Lambda\setminus S}} f - \mathcal{A}_{\Phi_{\Lambda\setminus S}} g\|_{\mathfrak{B}} \leq 2\|\mathcal{A}_{\Phi_{\Lambda\setminus S}} f_2\|_{\mathfrak{B}} = 2\|T_{\Phi_{\Lambda\setminus S}} f_2\|_{\mathfrak{B}}.$$

Together with (2.1) it follows that

$$\begin{aligned} \|\mathcal{A}_\Phi f - \mathcal{A}_\Phi g\|_{\mathfrak{B}} &= \|\mathcal{A}_{\Phi_S} f - \mathcal{A}_{\Phi_S} g + \mathcal{A}_{\Phi_{\Lambda\setminus S}} f - \mathcal{A}_{\Phi_{\Lambda\setminus S}} g\|_{\mathfrak{B}} \\ &\leq \|\mathcal{A}_{\Phi_S} f - \mathcal{A}_{\Phi_S} g\|_{\mathfrak{B}} + \|\mathcal{A}_{\Phi_{\Lambda\setminus S}} f - \mathcal{A}_{\Phi_{\Lambda\setminus S}} g\|_{\mathfrak{B}} \\ &\leq 4h \min\{\|f_1\|_{\mathcal{B}}, \|f_2\|_{\mathcal{B}}\}. \end{aligned} \tag{2.3}$$

We proceed with estimating $d_{\mathcal{B}}(f,g) = \inf_{|c|=1} \|f - cg\|_{\mathcal{B}}$ from below, i.e., to lower bound $\|f - cg\|_{\mathcal{B}}$ independently from $|c| = 1$. By the upper frame inequality we get that

$$\begin{aligned} B_\Phi \|f - cg\|_{\mathcal{B}} &\geq \|T_\Phi(f - cg)\|_{\mathfrak{B}} \\ &\geq \frac{1}{2}\|T_{\Phi_S}(f - cg)\|_{\mathfrak{B}} + \frac{1}{2}\|T_{\Phi_{\Lambda\setminus S}}(f - cg)\|_{\mathfrak{B}} \\ &= \frac{1}{2}\|T_{\Phi_S}[(1-c)f_1 + (1+c)f_2]\|_{\mathfrak{B}} + \frac{1}{2}\|T_{\Phi_{\Lambda\setminus S}}[(1-c)f_1 + (1+c)f_2]\|_{\mathfrak{B}}. \end{aligned}$$

Applying the reverse triangle inequality twice, by the lower frame inequality and by (2.1) we get that

$$\begin{aligned} &\|T_{\Phi_S}[(1-c)f_1 + (1+c)f_2]\|_{\mathfrak{B}} \\ &\qquad \geq |1+c|\|T_{\Phi_S} f_2\|_{\mathfrak{B}} - |1-c|\|T_{\Phi_S} f_1\|_{\mathfrak{B}} \\ &\qquad = |1+c|\|T_\Phi f_2 - T_{\Phi_{\Lambda\setminus S}} f_2\|_{\mathfrak{B}} - |1-c|\|T_{\Phi_S} f_1\|_{\mathfrak{B}} \\ &\qquad \geq |1+c|\|T_\Phi f_2\|_{\mathfrak{B}} - |1+c|\|T_{\Phi_{\Lambda\setminus S}} f_2\|_{\mathfrak{B}} - |1-c|\|T_{\Phi_S} f_1\|_{\mathfrak{B}} \\ &\qquad \geq |1+c| A_\Phi \|f_2\|_{\mathcal{B}} - 2\max\{|1+c|, |1-c|\} h \min\{\|f_1\|_{\mathcal{B}}, \|f_2\|_{\mathcal{B}}\}. \end{aligned}$$

An analogous argument gives

$$\begin{aligned}
&\|T_{\Phi_{\Lambda\setminus S}}[(1-c)f_1+(1+c)f_2]\|_{\mathfrak{B}} \\
&\qquad\qquad \geq |1-c|A_\Phi\|f_1\|_{\mathcal{B}} - 2\max\{|1+c|,|1-c|\}h\min\{\|f_1\|_{\mathcal{B}},\|f_2\|_{\mathcal{B}}\}.
\end{aligned}$$

Combining these estimates and using that $\max\{|1+c|,|1-c|\}\geq 1$ yields

$$\begin{aligned}
B_\Phi\|f-cg\|_{\mathcal{B}} &\geq \frac{A_\Phi}{2}\max\{|1+c|,|1-c|\}\min\{\|f_1\|_{\mathcal{B}},\|f_2\|_{\mathcal{B}}\} \\
&\quad - 4h\max\{|1+c|,|1-c|\}\min\{\|f_1\|_{\mathcal{B}},\|f_2\|_{\mathcal{B}}\} \\
&\geq \left(\frac{A_\Phi}{2}-4h\right)\min\{\|f_1\|_{\mathcal{B}},\|f_2\|_{\mathcal{B}}\}.
\end{aligned}$$

Thus, since $h\leq A_\Phi/16$, we have that

$$d_{\mathcal{B}}(f,g)\geq \frac{A_\Phi}{4B_\Phi}\min\{\|f_1\|_{\mathcal{B}},\|f_2\|_{\mathcal{B}}\}, \tag{2.4}$$

and therefore, by combining (2.2),(2.3) and (2.4) we get that $C_{\mathrm{loc}}(f)\geq \frac{A_\Phi}{16B_\Phi}h^{-1}$. The statement now follows since $h>h_\Phi(f)$ was arbitrary. □

It follows from a simple compactness argument that finite-dimensional phase retrieval problems are always uniformly stable, cf. [AG17, Theorem 3.11]. In infinite dimensions the situation is drastically different.

Theorem 2.11 ([AG17, Theorem 3.13])**.** *Let $\mathcal{B}$ be an infinite-dimensional Banach space and let $\Phi=(\varphi_\lambda)_{\lambda\in\Lambda}\subset\mathcal{B}'$ constitute a frame for $\mathcal{B}$ such that the upper frame bound B_Φ is finite. Then $\sigma_\Phi=0$ and, consequently by Theorem 2.9 the associated phase retrieval problem is not globally stable.*

By modifying the proof of the preceeding theorem we can even construct a function whose local stability constant is infinite.

Theorem 2.12. *Let $\mathcal{B}$ be an infinite-dimensional Banach space and let $\Phi=(\varphi_\lambda)_{\lambda\in\Lambda}\subset\mathcal{B}'$ constitute a frame for $\mathcal{B}$ such that the upper frame bound B_Φ is finite.*
Then there exists $f\in\mathcal{B}$ such that $C_{loc}(f)=+\infty$.

In order to prove Theorem 2.12 we require two auxiliary results:

Lemma 2.13. *Suppose $\mathcal{B}$ is an infinite-dimensional Banach space and $\Phi=(\varphi_\lambda)_{\lambda\in\Lambda}\subset\mathcal{B}'$ constitutes a frame for $\mathcal{B}$ with $\mathfrak{B}$ an admissible Banach space.*
Then if $K\subset\Lambda$ is compact and $\varepsilon>0$ there exists $h\in\mathcal{B}$, $\|h\|_{\mathcal{B}}=1$ such that

$$\|T_{\Phi_K}h\|_{\mathfrak{B}}<\varepsilon.$$

Proof. This is an immediate consequence of [AG17, Theorem 3.12]. □

Lemma 2.14. *Suppose $\Phi=(\varphi_\lambda)_{\lambda\in\Lambda}$ is a frame for $\mathcal{B}$ with admissible Banach space $\mathfrak{B}$. Let $\varepsilon>0$ and $f\in\mathcal{B}$. Then there exists a compact set $K\subset\Lambda$ such that*

$$\left\|T_{\Phi_{\Lambda\setminus K}}f\right\|_{\mathfrak{B}}<\varepsilon.$$

Proof. By the defining property (iii) of admissibility there exists $w \in \mathfrak{B}$ supported on a compact set K such that

$$\|T_\Phi f - w\|_{\mathfrak{B}} < \varepsilon/2.$$

Thus, using solidity of $\mathfrak{B}$, we get that

$$\begin{aligned}\left\|T_{\Phi_{\Lambda\setminus K}} f\right\|_{\mathfrak{B}} &= \|T_\Phi f \cdot (1-\chi_K)\|_{\mathfrak{B}} \\ &\le \|T_\Phi f - w\|_{\mathfrak{B}} + \|w - T_\Phi f \cdot \chi_K\|_{\mathfrak{B}} \\ &\le 2\,\|T_\Phi f - w\|_{\mathfrak{B}} \\ &< \varepsilon.\end{aligned}$$

□

We resume with the proof of Theorem 2.12.

Proof of Theorem 2.12. In order to simplify notation we write T_A instead of T_{Φ_A} for $A \subset \Lambda$. Choose a sequence $(h_n)_{n\in\mathbb{N}} \subset \{h \in \mathcal{B} : \|h\|_{\mathcal{B}} = 1\}$ and an increasing sequence of compact sets

$$\emptyset = K_0 \subset K_1 \subset K_2 \subset \ldots \subset \Lambda$$

such that for all $n \in \mathbb{N}$ it holds that (a) $\left\|T_{K_{n-1}} h_n\right\|_{\mathfrak{B}} < 2^{-n}$ and (b) $\left\|T_{\Lambda\setminus K_n} h_j\right\|_{\mathfrak{B}} < 2^{-n}$, for $1 \le j \le n$.
We show by induction that such sequences actually exist. For $n = 1$ we can, due to Lemma 2.13, first pick h_1 such that (a) is satisfied and then, by Lemma 2.14 there exists K_1 such that (b) holds.
Assume now that $h_1, \ldots, h_{N-1}$ and $K_1, \ldots, K_{N-1}$ have been chosen such that the properties (a) and (b) are fulfilled for $n \le N-1$. Again, according to Lemma 2.13 there exists h_N, with $\|h_N\|_{\mathcal{B}} = 1$ satisfying (a). By Lemma 2.14 for every $j \le N$ there is a compact set $K_{j,N}$ such that

$$\left\|T_{\Lambda\setminus K_{j,N}} h_j\right\|_{\mathfrak{B}} < 2^{-N};$$

consequently the compact set $K_N := K_{N-1} \cup \bigcup_{j=1}^{N} K_{j,N}$ has the desired properties.
Let $f \in \mathcal{B}$ be defined by $f := \sum_{j\in\mathbb{N}} j^{-2} h_j$, and note that f is well defined since the elements of $(h_j)_{j\in\mathbb{N}}$ are normalized. Moreover, for $n \in \mathbb{N}$ we define $f_n := \sum_{j\le n} j^{-2} h_j$. According to Theorem 2.10 it suffices to show that the ratio

$$\max\{\|T_{\Lambda\setminus K_n} f_n\|_{\mathfrak{B}}, \|T_{K_n}(f-f_n)\|_{\mathfrak{B}}\}\big/\min\{\|f_n\|_{\mathcal{B}}, \|f-f_n\|_{\mathcal{B}}\} \tag{2.5}$$

converges to zero for $n \to \infty$. The norm of f can be estimated from below as follows

$$\|f\|_{\mathcal{B}} = \left\|\sum_{j\in\mathbb{N}} j^{-2} h_j\right\|_{\mathcal{B}} \ge \left|1 - \left\|\sum_{j>1} j^{-2} h_j\right\|_{\mathcal{B}}\right| \ge 1 - \sum_{j>1} j^{-2} > 1/3.$$

A similar argument yields that $\|f_n\|_{\mathcal{B}} > 1/3$. Moreover, we have that

$$\|f-f_n\|_{\mathcal{B}} = \left\|\sum_{j>n} j^{-2} h_j\right\|_{\mathcal{B}} \gtrsim \left\|T_\Lambda \sum_{j>n} j^{-2} h_j\right\|_{\mathfrak{B}} \ge \left\|T_{K_{n+1}\setminus K_n} \sum_{j>n} j^{-2} h_j\right\|_{\mathfrak{B}}$$

where the implicit constant depends on the frame bounds of Φ only. Apply inverse triangle inequality and use solidity to further estimate

$$
\begin{aligned}
\|f - f_n\|_{\mathcal{B}} &\gtrsim (n+1)^{-2} \left\|T_{K_{n+1}\setminus K_n} h_{n+1}\right\|_{\mathfrak{B}} - \sum_{j \geq n+2} j^{-2} \left\|T_{K_{n+1}\setminus K_n} h_j\right\|_{\mathfrak{B}} \\
&\geq (n+1)^{-2} \left(\|T_{\Lambda} h_{n+1}\|_{\mathfrak{B}} - \left\|T_{\Lambda\setminus K_{n+1}} h_{n+1}\right\|_{\mathfrak{B}} - \|T_{K_n} h_{n+1}\|_{\mathfrak{B}}\right) \\
&- \sum_{j \geq n+2} j^{-2} \left\|T_{K_{n+1}} h_j\right\|_{\mathfrak{B}} \\
&\geq (n+1)^{-2} A_{\Phi} - 2(n+1)^{-2} 2^{-(n+1)} - \sum_{j \geq n+2} j^{-2} 2^{-(n+1)} .
\end{aligned}
$$

Thus, for n sufficiently large, we have that

$$
\min\{\|f_n\|_{\mathcal{B}}, \|f - f_n\|_{\mathcal{B}}\} \gtrsim (n+1)^{-2}. \tag{2.6}
$$

To bound the nominator of (2.5) we estimate using (b) that

$$
\left\|T_{\Lambda\setminus K_n} f_n\right\|_{\mathfrak{B}} = \left\|\sum_{j \leq n} j^{-2} T_{\Lambda\setminus K_n} h_j\right\|_{\mathfrak{B}} \leq \sum_{j \leq n} j^{-2} \left\|T_{\Lambda\setminus K_n} h_j\right\|_{\mathfrak{B}} \leq \sum_{j \leq n} j^{-2} 2^{-n+1} \lesssim 2^{-n}
$$

and that

$$
\|T_{K_n}(f - f_n)\|_{\mathfrak{B}} = \left\|\sum_{j > n} j^{-2} T_{K_n} h_j\right\|_{\mathfrak{B}} \leq \sum_{j > n} j^{-2} \left\|T_{K_{j-1}} h_j\right\|_{\mathfrak{B}} \leq \sum_{j > n} j^{-2} 2^{-j} \lesssim 2^{-n},
$$

where we used (a). In particular we get that

$$
\max\{\|T_{\Lambda\setminus K_n} f_n\|_{\mathfrak{B}}, \|T_{K_n}(f - f_n)\|_{\mathfrak{B}}\} \lesssim 2^{-n}. \tag{2.7}
$$

Now (2.6) and (2.7) imply that the ratio (2.5) indeed converges to zero. $\square$

Chapter 3

STFT Phase Retrieval

This chapter deals with the problem of phase retrieval from short-time Fourier transform magnitudes. We only address the question of uniqueness (that is up to a global phase factor) here. Before we can start to pose and discuss the problem in Section 3.2 we settle some notation and preliminaries in Section 3.1. In Section 3.3 we present a formula which allows to reconstruct a function explicitly from its spectrogram. This formula enables us to give a sufficient condition on the window function for phase retrievability.

3.1 Schwartz functions and tempered distributions

For an extensive introduction to distribution theory the reader may have a look at [Hör90a, Fri98]. For our purposes a few basic facts will be sufficient.
To keep notation as simple as possible we will make use of multiindices. For $\alpha \in \mathbb{N}_0^d$ and $x \in \mathbb{R}^d$ we employ the notation

$$D^\alpha = \frac{\partial^{\alpha_1+\ldots+\alpha_d}}{\partial^{\alpha_1}x_1 \cdots \partial^{\alpha_d}x_d}, \quad x^\alpha = \prod_{k=1}^{d} x_k^{\alpha_k}.$$

For a multiindex α we define $|\alpha| = \sum_{k=1}^{d} \alpha_k$. Recall that the *Schwartz space* $\mathcal{S}(\mathbb{R}^d)$ consists of all $\varphi \in C^\infty(\mathbb{R}^d)$ which are rapidly decreasing, i.e., for all $\alpha, \beta \in \mathbb{N}_0^d$ it holds that

$$\|\varphi\|_{\alpha,\beta} := \sup_{x\in\mathbb{R}^d} \left|D^\alpha \left(x \mapsto x^\beta \varphi(x)\right)\right| < +\infty.$$

Convergence in $\mathcal{S}(\mathbb{R}^d)$ is defined by

$$\varphi_n \to \varphi \quad \Longleftrightarrow \quad \lim_{n\in\mathbb{N}} \|\varphi_n - \varphi\|_{\alpha,\beta} = 0 \quad \forall \alpha, \beta \in \mathbb{N}_0^d.$$

The dual of $\mathcal{S}(\mathbb{R}^d)$ is denoted by $\mathcal{S}'(\mathbb{R}^d)$ and called the space of *tempered distributions*. For the dual pairing in $\mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ we use brackets $\langle \cdot, \cdot \rangle_{\mathcal{S}'(\mathbb{R}^d)\times\mathcal{S}(\mathbb{R}^d)}$; occasionally, provided that the context rules out confusion, we will drop the reference to the spaces and just write $\langle \cdot, \cdot \rangle$.
For $1 \le p \le \infty$, any function $f \in L^p(\mathbb{R}^d)$ can be understood as tempered distribution by virtue of

$$f(\varphi) := \langle f, \varphi \rangle := \int_{\mathbb{R}^d} f(x)\varphi(x)\, dx, \qquad \varphi \in \mathcal{S}(\mathbb{R}^d).$$

A linear functional f on $\mathcal{S}(\mathbb{R}^d)$ is bounded, i.e., an element of $\mathcal{S}'(\mathbb{R}^d)$ if and only if there exist $C > 0, M \in \mathbb{N}$ such that

$$|f(\varphi)| \le C \sum_{|\alpha|,|\beta|\le M} \|\varphi\|_{\alpha,\beta}\,, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^d).$$

A sequence $(f_n)_{n\in\mathbb{N}}$ converges to f in $\mathcal{S}'(\mathbb{R}^d)$ if and only if

$$\langle f_n, \varphi\rangle \to \langle f, \varphi\rangle \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^d).$$

The Fourier transform

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} f(t)e^{-2\pi i t\cdot\xi}\, dt, \qquad \xi \in \mathbb{R}^d$$

is an isomorphism on $\mathcal{S}(\mathbb{R}^d)$ and its inverse is given by

$$\mathcal{F}^{-1}f(t) = \check{f}(t) := \int_{\mathbb{R}^d} f(\xi)e^{2\pi i t\cdot\xi}\, d\xi.$$

Recall also the following fundamental identities of the Fourier transform for $f, g \in \mathcal{S}(\mathbb{R}^d)$:

$$\langle \hat{f}, g\rangle = \langle f, \hat{g}\rangle, \quad \langle f, \bar{g}\rangle = \langle \hat{f}, \bar{\hat{g}}\rangle, \quad \widehat{f * g} = \hat{f}\cdot\hat{g}, \quad \widehat{f\cdot g} = \hat{f} * \hat{g}.$$

By means of duality the definition of the Fourier transform can be extended to tempered distributions: For $f \in \mathcal{S}'(\mathbb{R}^d)$ its Fourier transform is explained by

$$\hat{f}(\varphi) = \langle \hat{f}, \varphi\rangle := \langle f, \hat{\varphi}\rangle, \qquad \varphi \in \mathcal{S}(\mathbb{R}^d).$$

3.2 Problem description

We continue with defining the object that is in the focus of this chapter. To that end we denote time-frequency shifts by $\pi(x, y)$, i.e., for a function g on $\mathbb{R}^d$ and $x, y \in \mathbb{R}^d$ the function $\pi(x, y)g$ is defined by

$$(\pi(x, y)g)(t) = e^{2\pi i y\cdot t}g(t - x), \qquad t \in \mathbb{R}^d. \tag{3.1}$$

Definition 3.1. Let $g \in \mathcal{S}(\mathbb{R}^d)$ and $f \in \mathcal{S}'(\mathbb{R}^d)$. Then the *short-time Fourier transform* (STFT) of f (w.r.t. the window function g) is defined by

$$V_g f(x, y) := \langle f, \overline{\pi(x, y)g}\rangle, \quad x, y \in \mathbb{R}^d.^1 \tag{3.2}$$

The modulus of the STFT (or its squared modulus) is called the *spectrogram*. For a fixed location $x \in \mathbb{R}^d$ the short time Fourier transform $V_g f(x, \cdot)$ coincides with the Fourier transform of $\overline{g(\cdot - x)}f$. Now, if g is nicely localized at 0 and f represents a two dimensional object, the spectrogram $|V_g f(x, \cdot)|$ describes the diffraction pattern of the localization of f at x. Hence the STFT perfectly mimics the concept of *ptychography*, a popular and highly successful approach in coherent diffraction imaging based on the idea that multiple diffraction patterns of one and the same object are generated by illuminating different sections of the object seperately in order to introduce redundancy, cf. Figure 3.1.

Whether any f is uniquely up to global sign determined by its spectrogram $|V_g f|$ or not depends on the window function g. Counterexamples can be easily constructed:

[1] For a regular tempered distribution f definition (3.2) agrees with the usual definition of the STFT via the integral

$$V_g f(x, y) = \int_{\mathbb{R}^d} f(t)\overline{g(t - x)}e^{-2\pi i t\cdot y} dt.$$

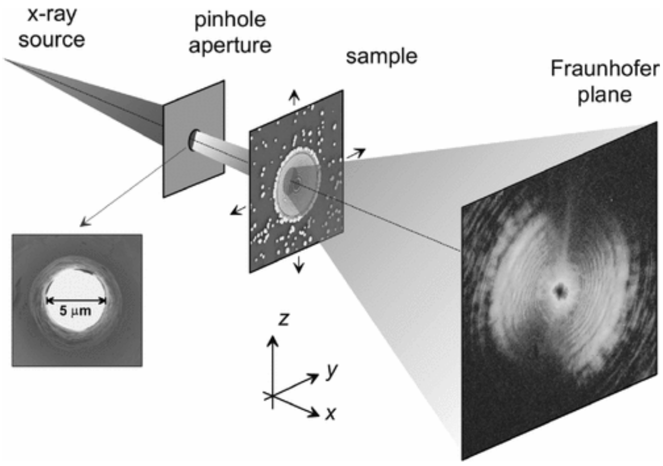


Figure 3.1: Schematic setup of ptychographical experiment. Image taken from [RCHC+07].

Example 3.2. Suppose the window function g is compactly supported. Let us fix $h \in C_c^\infty(\mathbb{R}^d)$ and let $R > 0$ be such that both the support of g and h is contained in $B_R(0)$. For $\tau \in \mathbb{R}^d$ we define $h_\tau := h(\cdot - \tau)$; then $\operatorname{supp} h_\tau \subset B_R(\tau)$. The STFT of h_τ is given by

$$\begin{aligned} V_g h_\tau(x,y) &= \int_{\mathbb{R}^d} h_\tau(t)\overline{g(t-x)}e^{-2\pi i y\cdot t}dt \\ &= \int\limits_{B_R(\tau)\cap B_R(x)} h_\tau(t)\overline{g(t-x)}e^{-2\pi i y\cdot t}dt, \end{aligned}$$

which implies that $\operatorname{supp} V_g h_\tau \subset B_{2R}(\tau) \times \mathbb{R}^d$. For $f_\pm := h_\tau \pm h_{-\tau}$ we compute

$$|V_g f_\pm|^2 = |V_g h_\tau + V_g h_{-\tau}|^2 = |V_g h_\tau|^2 \pm 2\operatorname{Re}\left(V_g h_\tau \cdot \overline{V_g h_\tau}\right) + |V_g h_{-\tau}|^2.$$

For $\tau > 2R$ the cross term vanishes since the two factors have disjoint support; thus we obviously have $|V_g f_-|^2 = |V_g f_+|^2$ in that case. We therefore have constructed a pair of functions (which do not coincide up to a global phase factor) that yield identical spectrograms. In other words, for all $f \in \mathcal{S}'(\mathbb{R}^d)$ to be uniquely determined by their spectrograms w.r.t. a fixed window g it is neccessary that g is not compactly supported.

The main aim of this chapter is to identify those window functions which allow for phase retrieval. This goal is achieved by establishing a factorization formula which gives rise to an explicit inversion formula for the mapping $f \mapsto |V_g f|$ in case of uniqueness.

3.3 A uniqueness result

The main result of this section, Theorem 3.4, states that the Fourier transform of the spectrogram $|V_g f|^2$ is the product of the so-called ambiguity functions of f and g. Before we present this result we have to make sense of the ambiguity function of a tempered distribution.

In order to do that we introduce some notation. Let operators S, T and U acting on functions on $\mathbb{R}^d \times \mathbb{R}^d$ be defined by

$$\begin{aligned} TF(x,y) &= F(x + y/2, x - y/2), \\ SF(x,y) &= F(y,x), \\ UF(x,y) &= F(-y,x). \end{aligned}$$

One can easily show that for F and Φ in a space of suitable test functions (for example in $\mathcal{S}(\mathbb{R}^{2d})$) the adjoint of T is given by $T^*\Phi(x,y) = \Phi(\frac{x+y}{2}, x-y)$, i.e.,

$$\langle TF, \Phi\rangle = \langle F, T^*\Phi\rangle. \tag{3.3}$$

Therefore T extends via a density argument to the space of tempered distributions by using (3.3) as a definition for TF for $F \in \mathcal{S}'(\mathbb{R}^{2d})$. Similarly for S and U it is natural to define their respective extensions via

$$\begin{aligned}\langle SF, \Phi\rangle &= \langle F, S\Phi\rangle,\\ \langle UF, \Phi\rangle &= \langle F, U^*\Phi\rangle,\end{aligned}$$

where the adjoint of U is given by $U^*\Phi(x,y) = \Phi(y,-x)$.

For $f \in \mathcal{S}'(\mathbb{R}^m), g \in \mathcal{S}'(\mathbb{R}^n)$ the tensor product $f \otimes g \in \mathcal{S}'(\mathbb{R}^{m+n})$ is defined by

$$\langle f\otimes g, \varphi\rangle = \langle g, y \mapsto \langle f, \varphi(\cdot, y)\rangle_{\mathcal{S}'(\mathbb{R}^m)\times\mathcal{S}(\mathbb{R}^m)}\rangle_{\mathcal{S}'(\mathbb{R}^n)\times\mathcal{S}(\mathbb{R}^n)}, \quad \varphi \in \mathcal{S}(\mathbb{R}^{m+n}).$$

A property we will require is that the operation $\otimes$ is sequentially continuous from $\mathcal{S}'(\mathbb{R}^m) \times \mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^{m+n})$ [DK10, Chapter 11], meaning that

$$f_n \xrightarrow{\mathcal{S}'(\mathbb{R}^m)} f, \quad g_n \xrightarrow{\mathcal{S}'(\mathbb{R}^n)} g \quad \Rightarrow \quad f_n \otimes g_n \xrightarrow{\mathcal{S}'(\mathbb{R}^{m+n})} f\otimes g. \tag{3.4}$$

For a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ we define its conjugate via

$$\langle \overline{f}, \varphi\rangle = \overline{\langle f, \overline{\varphi}\rangle}, \quad \varphi \in \mathcal{S}(\mathbb{R}^d).$$

The ambiguity function of a tempered distribution is now defined as follows.

Definition 3.3. For $f \in \mathcal{S}'(\mathbb{R}^d)$ the *ambiguity function* of f is defined as

$$\mathcal{A}f := S \circ \mathcal{F}_1 \circ T(f \otimes \bar{f}),^2$$

where $\mathcal{F}_1$ denotes the Fourier transform w.r.t. the first variable.

Now, having the ambiguity function defined for tempered distributions, we can state the main result of this section.

Theorem 3.4. *Suppose $f \in \mathcal{S}'(\mathbb{R}^d)$ and $g \in \mathcal{S}(\mathbb{R}^d)$. Then $\mathcal{F}\left(|V_g f|^2\right) = U(\mathcal{A}f\overline{\mathcal{A}g})$, i.e., for all $\Phi \in \mathcal{S}(\mathbb{R}^{2d})$ it holds that*

$$\langle \mathcal{F}\left(|V_g f|^2\right), \Phi\rangle_{\mathcal{S}'(\mathbb{R}^{2d})\times\mathcal{S}(\mathbb{R}^{2d})} = \langle U(\mathcal{A}f\overline{\mathcal{A}g}), \Phi\rangle_{\mathcal{S}'(\mathbb{R}^{2d})\times\mathcal{S}(\mathbb{R}^{2d})}. \tag{3.5}$$

That the Fourier transform of the spectrogram of a tempered distribution factorizes into two ambiguity functions is essentially folklore. For sufficiently decaying functions the statement can be found in [CM80, Coh89, Grö14]. We conclude this section with the proof in our very general setting.

Proof of Theorem 3.4. The proof is split into two parts. First we show that the statement holds true for nice functions. Then we argue by density that (3.5) holds true for arbitrary tempered distributions.

[2]For $f \in L^2(\mathbb{R}^d)$ this coincides with the usual definition of the ambiguity function, i.e., with

$$\mathcal{A}f(x,y) = \int f(t + x/2)\overline{f(t-x/2)}e^{-2\pi i t\cdot y}dt.$$

Step 1: Suppose first that $f \in \mathcal{S}(\mathbb{R}^d)$. Then we compute

$$\mathcal{F}\left(|V_g f|^2\right)(\xi,\eta) = \iiiint_{\mathbb{R}^{4d}} f(t)\overline{f(s)} \cdot \overline{g(t-x)} g(s-x) e^{-2\pi i y\cdot(t-s)} e^{-2\pi i(x\cdot\xi+y\cdot\eta)}\,dt\,ds\,dx\,dy.$$

The substitutions $u = t-s$ and $v = s-x$ give that

$$\begin{aligned}
&\mathcal{F}\left(|V_g f|^2\right)(\xi,\eta) \\
&= \iiiint_{\mathbb{R}^{4d}} f(t)\overline{f(t-u)} g(v)\overline{g(v+u)} e^{-2\pi i y\cdot u} e^{-2\pi i(t-u-v)\cdot\xi} e^{-2\pi i y\cdot\eta}\,dt\,du\,dv\,dy \\
&= \iint_{\mathbb{R}^{2d}} V_f f(u,\xi) V_g g(-u,-\xi) e^{2\pi i u\cdot\xi} e^{-2\pi i u\cdot y}\ e^{-2\pi i y\cdot\eta}\,du\,dy.
\end{aligned}$$

For any $\varphi \in \mathcal{S}(\mathbb{R}^d)$ it holds that [Grö01, Chapter 4.2]

$$\mathcal{A}\varphi(x,y) = e^{\pi i x\cdot y} V_\varphi \varphi(x,y) \quad \text{and} \quad \mathcal{A}\varphi(-x,-y) = \overline{\mathcal{A}\varphi(x,y)}.$$

Therefore the Fourier transform of the spectrogram can be computed according to

$$\begin{aligned}
\mathcal{F}\left(|V_g f|^2\right)(\xi,\eta) &= \iint_{\mathbb{R}^{2d}} \mathcal{A}f(u,\xi)\mathcal{A}g(-u,-\xi) e^{-2\pi i u\cdot y} du\ e^{-2\pi i y\cdot\eta} dy \\
&= \int_{\mathbb{R}^d} \mathcal{F}\left(\mathcal{A}f(\cdot,\xi)\overline{\mathcal{A}g(\cdot,\xi)}\right)(y)\cdot e^{-2\pi i y\cdot\eta} dy \\
&= \mathcal{F}^{-1}\left(\mathcal{F}\left(\mathcal{A}f(\cdot,\xi)\overline{\mathcal{A}g(\cdot,\xi)}\right)\right)(-\eta) \\
&= \mathcal{A}f(-\eta,\xi)\overline{\mathcal{A}g(-\eta,\xi)},
\end{aligned}$$

hence (3.5) holds true for $f \in \mathcal{S}(\mathbb{R}^d)$.

Step 2: Let $f \in \mathcal{S}'(\mathbb{R}^d)$ be arbitrary and define two operators $\mathcal{Q}_1, \mathcal{Q}_2 : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^{2d})$ by

$$\mathcal{Q}_1 h := \mathcal{F}\left(|V_g h|^2\right) \quad \text{and} \quad \mathcal{Q}_2 h := U\left(\mathcal{A}h \cdot \overline{\mathcal{A}g}\right).$$

To conclude the proof we have to show that $\mathcal{Q}_1 f = \mathcal{Q}_2 f$. For this to hold true it is sufficient that $\mathcal{Q}_1 f$ and $\mathcal{Q}_2 f$ coincide on the dense subset $\mathcal{D} := \mathcal{F}C_0^\infty(\mathbb{R}^{2d}) \subset \mathcal{S}(\mathbb{R}^{2d})$, i.e.,

$$\langle \mathcal{Q}_1 f, \varphi\rangle = \langle \mathcal{Q}_2 f, \varphi\rangle \quad \forall \varphi \in \mathcal{D}.$$

By density there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^d)$ converging to f in $\mathcal{S}'(\mathbb{R}^d)$. For fixed $x, y \in \mathbb{R}^d$ we have that

$$\lim_n V_g f_n(x,y) = \lim_n \langle f_n, \overline{\pi(x,y)g}\rangle = \langle f, \overline{\pi(x,y)g}\rangle = V_g f(x,y)$$

i.e., $V_g f_n$ converges to $V_g f$ pointwise.
Consider the family of bounded functionals $(T_n)_{n\in\mathbb{N}}$ with

$$T_n(\varphi) := \langle f_n, \varphi\rangle, \quad \varphi \in \mathcal{S}(\mathbb{R}^d).$$

Since $(f_n)_{n\in\mathbb{N}}$ converges to f in $\mathcal{S}'(\mathbb{R}^d)$ we have for every $\varphi \in \mathcal{S}(\mathbb{R}^d)$ that $\sup_{n\in\mathbb{N}} |T_n(\varphi)| < +\infty$. Therefore, by a version of Banach-Steinhaus theorem [Die70, 12.16.4], $(T_n)_{n\in\mathbb{N}}$ is equicontinous; thus, there exist $C > 0, M \in \mathbb{N}$ such that for all $n \in \mathbb{N}$

$$|T_n(\varphi)| \le C \sum_{\alpha,\beta\le M} \|\varphi\|_{\alpha,\beta}, \qquad \forall \varphi \in \mathcal{S}(\mathbb{R}^d).$$

In particular for $\varphi = \overline{\pi(x,y)g}$ we get that

$$|V_g f_n(x,y)| = \left|T_n(\overline{\pi(x,y)g})\right| \le C \sum_{\alpha,\beta\le M} \|\pi(x,y)g\|_{\alpha,\beta}.$$

Now, since $(x,y)\mapsto \|\pi(x,y)g\|_{\alpha,\beta}$ is continuous, we get that for every compact $K\subset\mathbb{R}^{2d}$ there exists a constant c such that

$$|V_g f_n(x,y)| \le c, \qquad \forall (x,y)\in K, n\in\mathbb{N}.$$

Therefore, since $\mathcal{F}\varphi$ is compactly supported for $\varphi\in\mathcal{D}$, we can justify interchanging integration and limiting opererations to obtain

$$\begin{aligned}
\lim_{n\in\mathbb{N}}\langle \mathcal{Q}_1 f_n,\varphi\rangle &= \lim_{n\in\mathbb{N}}\langle |V_g f_n|^2, \mathcal{F}\varphi\rangle \\
&= \lim_{n\in\mathbb{N}} \iint_{\mathbb{R}^{2d}} |V_g f_n(x,y)|^2\, \mathcal{F}\varphi(x,y) dx\ dy \\
&= \iint_{\mathbb{R}^{2d}} |V_g f(x,y)|^2\, \mathcal{F}\varphi(x,y) dx\ dy \\
&= \langle \mathcal{Q}_1 f,\varphi\rangle
\end{aligned}$$

For $\mathcal{Q}_2$ we compute

$$\begin{aligned}
\langle \mathcal{Q}_2 f_n,\varphi\rangle &= \langle \mathcal{A} f_n, \overline{\mathcal{A}g}\cdot U\varphi\rangle \\
&= \langle S\circ\mathcal{F}_1\circ T(f_n\otimes\overline{f_n}), \overline{\mathcal{A}g}\cdot U\varphi\rangle \\
&= \langle f_n\otimes\overline{f_n}, S\circ\mathcal{F}_1\circ T^*(\overline{\mathcal{A}g}\cdot U\varphi)\rangle
\end{aligned}$$

which by (3.4) converges to

$$\langle f\otimes\overline{f}, S\circ\mathcal{F}_1\circ T^*(\overline{\mathcal{A}g}\cdot U\varphi)\rangle = \langle \mathcal{Q}_2 f,\varphi\rangle.$$

Altogether we obtain that

$$\langle \mathcal{Q}_1 f,\varphi\rangle = \lim_{n\in\mathbb{N}}\langle \mathcal{Q}_1 f_n,\varphi\rangle = \lim_{n\in\mathbb{N}}\langle \mathcal{Q}_2 f_n,\varphi\rangle = \langle \mathcal{Q}_2 f,\varphi\rangle,$$

which concludes the proof. □

With Theorem 3.4 at hand we can give sufficient conditions on g for the spectrogram $|V_g f|$ to uniquely determine f up to a global sign.

Theorem 3.5. *Let $g\in\mathcal{S}(\mathbb{R}^d)$.*

(i) Suppose that $\mathcal{A}g$ has no zeros and suppose that $f_1,f_2\in\mathcal{S}'(\mathbb{R}^d)$ are such that $|V_g f_1| = |V_g f_2|$. Then there exists $\alpha\in\mathbb{R}$ such that $f_2 = e^{i\alpha} f_1$.

(ii) Supppose that $\mathcal{A}g$ vanishes at most on a set of measure zero and suppose that $f_1,f_2\in L^2(\mathbb{R}^d)$ are such that $|V_g f_1| = |V_g f_2|$. Then there exists $\alpha\in\mathbb{R}$ such that $f_2 = e^{i\alpha} f_1$. Furthermore, if $f\in L^2(\mathbb{R}^d)$ is continuous it holds that

$$f(t)\cdot\overline{f(0)} = \int_{\mathbb{R}^d} \frac{U^*\mathcal{F}|V_g f|^2(t,\xi)}{\overline{\mathcal{A}g(t,\xi)}} e^{\pi i t\cdot\xi} d\xi. \tag{3.6}$$

Proof. (i) Let us fix $\Psi\in C_0^\infty(\mathbb{R}^{2d})$ arbitrary. Since by assumption $\mathcal{A}g$ has no zeros, we have that $\Phi := U\left(\Psi/\overline{\mathcal{A}g}\right)\in C_0^\infty(\mathbb{R}^{2d})$. By definition it holds that $\overline{\mathcal{A}g}\cdot U^*\Psi = \Phi$. Thus, using (3.5) we get for $i=1,2$

$$\langle \mathcal{A}f_i,\Psi\rangle = \langle \mathcal{A}f_i, \overline{\mathcal{A}g}\cdot U^*\Phi\rangle = \langle U(\mathcal{A}f_i\overline{\mathcal{A}g}),\Phi\rangle = \langle \mathcal{F}|V_g f_i|^2,\Phi\rangle.$$

By the assumption that the spectrograms $|V_g f_1|, |V_g f_2|$ coincide and since $C_0^\infty(\mathbb{R}^{2d})$ is a dense subspace of $\mathcal{S}(\mathbb{R}^{2d})$ it follows that $\mathcal{A}f_1 = \mathcal{A}f_2$. Since $S, \mathcal{F}_1$ and T are bijections on $\mathcal{S}'(\mathbb{R}^{2d})$ we have that $f_1 \otimes \overline{f_1} = f_2 \otimes \overline{f_2}$. Next, by evaluating against $\varphi \otimes \bar{\psi}$ for arbitrary $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$ we obtain

$$\langle f_i \otimes \overline{f_i}, \varphi \otimes \bar{\psi} \rangle = \langle f_i, \varphi \rangle \cdot \overline{\langle f_2, \psi \rangle}, \quad i = 1, 2. \tag{3.7}$$

Choosing $\varphi = \psi$ it follows that $|\langle f_1, \psi \rangle| = |\langle f_2, \psi \rangle|$ for all ψ. Unless f_1, f_2 are trivial there exists a ψ such that $|\langle f_1, \psi \rangle| = |\langle f_2, \psi \rangle|$ is non zero. Let $\alpha \in \mathbb{R}$ be such that $e^{i\alpha} = \overline{\langle f_1, \psi \rangle / \langle f_2, \psi \rangle}$. Then by (3.7) for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$ it holds that

$$\langle f_2, \varphi \rangle = \overline{\frac{\langle f_1, \psi \rangle}{\langle f_2, \psi \rangle}} \langle f_1, \varphi \rangle = e^{i\alpha} \langle f_1, \varphi \rangle,$$

i.e., $f_2 = e^{i\alpha} f_1$.

(ii) If $f \in L^2(\mathbb{R}^d)$ we have that $|V_g f|$ and $\mathcal{A}f$ are continuous functions in $L^2(\mathbb{R}^d)$. Equation (3.5) implies that $\mathcal{A}f \cdot \overline{\mathcal{A}g} = U^* \mathcal{F} |V_g f|^2$ pointwise and since $\mathcal{A}g$ vanishes at most on a set of measure zero that $\mathcal{A}f = U^* \mathcal{F} |V_g f|^2 / \overline{\mathcal{A}g}$ almost everywhere. Following the argument in (i) gives that f is uniquely determined by $\mathcal{A}f$ and therefore by $|V_g f|$.
Suppose now that $f \in L^2(\mathbb{R}^d)$ is continuous. From the definition of the ambiguity function, Equation (3.3) and since $S^{-1} = S$ it follows that $T(f \otimes \bar{f}) = \mathcal{F}_1^{-1} \circ S \left(U^* \mathcal{F} |V_g f|^2 / \overline{\mathcal{A}g} \right)$, i.e.,

$$f(x + y/2) \cdot \overline{f(x - y/2)} = T(f \otimes \bar{f})(x, y) = \int_{\mathbb{R}^d} \frac{U^* \mathcal{F} |V_g f|^2 (y, \xi)}{\overline{\mathcal{A}g(y, \xi)}} e^{2\pi i x \cdot \xi} d\xi$$

By setting $x = t/2$ and $y = t$ we obtain (3.6).

□

Infact for uniqueness it suffices that the set of points where $\mathcal{A}g$ does not vanish is dense, as recently shown by Luef and Skrettingland [LS19].

Theorem 3.6 ([LS19])**.** *Let $g, f_1, f_2 \in L^2(\mathbb{R}^d)$. Suppose that the zero set of $\mathcal{A}g(x, y)$ has dense complement and that $|V_g f_1| = |V_g f_2|$. Then there exists $\alpha \in \mathbb{R}$ such that $f_2 = e^{i\alpha} f_1$.*

Chapter 4

Quantitative phase retrieval from analytic measurements

This chapter contains one of the main parts of the thesis. In the preceeding Chapter 3 those windows which are suited for STFT phase retrieval were identified. Here we consider one particular instance of the STFT and study the stability properties of the corresponding phase retrieval problem. The particular choice of the window essentially yields analytic functions meaning that we have complex analysis at our disposal to study the objects of interest.

In Section 4.1 we define the object which is in the center of attention for the most part of this thesis, the so-called Gabor transform. Section 4.2 is concerned with reviewing some related results. In the Sections 4.3 and 4.4 some preliminary material on analytic functions of several complex variables is collected and the precise connection to the Gabor transform is recalled. In Section 4.5 we establish a first quantitative result regarding phase retrieval from (generalized) analytic functions.

4.1 Problem description

The problem of recovering the phase of the so-called Gabor transform when only its magnitude is observed is discussed. Here, by Gabor transform we mean the short-time Fourier transform with Gaussian window:

Definition 4.1. Let $\varphi := 2^{d/4}e^{-\pi|\cdot|^2}$ denote the normalized Gaussian on $\mathbb{R}^d$. The *Gabor* transform of $f \in \mathcal{S}'(\mathbb{R}^d)$ is defined as $\mathcal{G}f := V_\varphi f$.

If f is sufficiently regular (for example $f \in L^2(\mathbb{R}^d)$ suffices) its Gabor transform can be computed according to

$$\mathcal{G}f(x,y) = 2^{d/4}\int_{\mathbb{R}^d} f(t)e^{-\pi|t-x|^2}e^{-2\pi i y\cdot t}dt, \qquad x,y \in \mathbb{R}^d.$$

A standard calculation reveals that the ambiguity function $\mathcal{A}\varphi$, see Definition 3.3, of the Gaussian φ is again Gaussian. In particular $\mathcal{A}\varphi$ is nonvanishing, and by Theorem 3.5 we have that any $f \in \mathcal{S}'(\mathbb{R}^d)$ is (up to a global sign) uniquely determined by $\mathcal{G}f$.
Note that moreover Theorem 3.5 provides an explicit reconstruction formula for $\mathcal{A}f$ and therefore also for $f \sim e^{i\alpha}f$. However, this formula includes division by a the Gaussian $\mathcal{A}\varphi$, which obviously constitutes a rather instable operation. In the following we will not use the reconstruction formula in order to study the stability of the Gabor phase retrieval problem, but develop other tools and techniques.

Due to Theorem 2.11 infinite-dimensional phase retrieval problems cannot be uniformly stable in the sense of Definition 2.6. The focus lies on studying the behaviour of the local stability constant $C_{\mathrm{loc}}(f)$ which denotes the smallest positive number C such that

$$\inf_{|c|=1} \|g - cf\|_{\mathcal{B}} \leq C \||\mathcal{G}g| - |\mathcal{G}f|\|_{\mathfrak{B}}, \qquad \forall g \in \mathcal{B},$$

where the choice of the spaces $\mathcal{B}$ and $\mathfrak{B}$ will be specified later.
In fact we will consider a more general formulation of the problem: Suppose we cannot observe $|\mathcal{G}f|$ on the full space $\mathbb{R}^{2d}$ but only on a domain $D \subset \mathbb{R}^{2d}$. Is it then possible to recover $\mathcal{G}f(x,y)$ for $(x,y) \in D$ up to a unimodular factor?

4.2 Related work

In this section we will briefly summarize and discuss two recent articles concerned with stability of Gabor phase retrieval:

[ADGY19] *Stable Phase Retrieval in Infinite Dimensions.*
Alaifari, Daubechies, Grohs and Yin. FOUND COMPUT MATH (2018).

[AG19] *Gabor phase retrieval is severely ill-posed.*
Alaifari and Grohs. APPL COMPUT HARMON A (2018).

4.2.1 About [ADGY19]

The results that will be presented in this chapter are very much inspired by and built upon work in [ADGY19]. Leaving out some rather technical conditions and assumptions the main results from [ADGY19] can be informally summarized as follows:

Theorem 4.2 ([ADGY19], informal version of main result)**.** *Let $\delta > 0$, let $f \in L^2(\mathbb{R})$ and let $D \subset \mathbb{R}^2$ be a domain, i.e., open and connected, and suppose that $|\mathcal{G}f(x,y)| \geq \delta$, for all $(x,y) \in D$. Then $\mathcal{G}f$ can be stabily reconstructed on D given $|\mathcal{G}f(x,y)|, (x,y) \in D$.*

The bound for the stability constant grows like δ^{-2} for $\delta \to 0$. Thus, enlarging the domain D potentially results in degenerating stability properties of the problem.
In fact a more general class of functions and domains, consisting of so-called *atoll* functions and domains, is considered: Suppose that D and f are as in Theorem 4.2 and that D is not simply connected. Then D has holes $D_1, \ldots, D_l$. Provided that the holes behave geometrically nicely (think of discs for example), the results in [ADGY19] even allow for reconstruction on $D \cup \bigcup_j D_j$.
The Gabor transform yields (almost) entire functions (cf. Proposition 4.5) and, therefore, puts us right into the realm of complex analysis. Both the proofs in [ADGY19] and our methods rely crucially on this strong structural property.

4.2.2 About [AG19]

The second article mentioned above takes another path. Instead of looking for functions that are stabily determined by the modulus of their Gabor transform explicit instabilities are constructed. Typically it is significantly easier to construct instabilities than to show that certain functions can be stabily recovered at least for the following two reasons.

- The abstract theory in Section 2.2 provides instabilities of a certain type. Translating the abstract setting to the concrete Gabor transform case, Theorem 2.10 indicates that a good candidate for an instability can be constructed by choosing two functions $f_1, f_2 \in L^2(\mathbb{R})$ with $\|f_1\|_{L^2(\mathbb{R})}, \|f_2\|_{L^2(\mathbb{R})} \asymp 1$, such that

$$\exists S \subset \mathbb{R}^2: \quad \mathcal{G}f_1 \approx 0 \text{ on } S \qquad \text{and} \qquad \mathcal{G}f_2 \approx 0 \text{ on } \mathbb{R}^2 \setminus S.$$

Then $f_+ := f_1 + f_2$ and $f_- := f_1 - f_2$ are not close, i.e., $\inf_{|c|=1} \|f_+ - cf_-\|_{L^2} \asymp 1$ while the magnitudes of their respective Gabor transforms are very similar since

$$\begin{aligned} |\mathcal{G}f_\pm|^2 &= |\mathcal{G}(f_1 \pm f_2)|^2 \\ &= |\mathcal{G}f_1|^2 + |\mathcal{G}f_2|^2 \pm 2\,\mathrm{Re}\left(\mathcal{G}f_1 \cdot \overline{\mathcal{G}f_2}\right) \\ &\approx |\mathcal{G}f_1|^2 + |\mathcal{G}f_2|^2 . \end{aligned}$$

- In order to show that $f \in L^2(\mathbb{R})$ is an instability it suffices to choose one g and provide a lower bound (which is large) for the ratio

$$\inf_{|c|=1} \|f - cg\|_{L^2(\mathbb{R})} / \||\mathcal{G}f| - |\mathcal{G}g|\|_{L^2(\mathbb{R}^2)}.^1 \tag{4.1}$$

  For $f = f_+ = f_1 + f_2$ obviously $g = f_-$ is a good choice.
  On the other hand, to establish local stability for fixed f one needs to bound the quantity in (4.1) independently from $g \in L^2(\mathbb{R})$ from above (by a moderately sized constant).

The following construction is considered in [AG19]: For $a \in \mathbb{R}$ let φ_a denote the Gaussian translated by a, i.e., $\varphi_a = 2^{1/4} e^{-\pi(\cdot - a)^2}$. The roles of f_1 and f_2 are played by φ_a and φ_{-a} respectively, and thus we set for $a > 0$

$$f^{(a)} := \varphi_a + \varphi_{-a} \quad \text{and} \quad g^{(a)} := \varphi_a - \varphi_{-a}.$$

The main result states that the stability constant degenerates at least exponentially in the parameter a and therefore that the phase retrieval problem is severely ill-posed.

Theorem 4.3 ([AG19], main theorem)**.** *For all $a > 0$ and $k \in (0, \pi/2)$ it holds that*

$$\inf_{|c|=1} \left\| f^{(a)} - cg^{(a)} \right\|_{L^2(\mathbb{R})} \Big/ \left\| \left| \mathcal{G}f^{(a)} \right| - \left| \mathcal{G}g^{(a)} \right| \right\|_{W^{1,2}(\mathbb{R}^2)} \gtrsim e^{ka^2},$$

where $\|\cdot\|_{W^{1,2}(\mathbb{R}^2)} = \|\cdot\|_{L^2(\mathbb{R}^2)} + \|\nabla \cdot\|_{L^2(\mathbb{R}^2)}$ denotes the Sobolev norm on $\mathbb{R}^2$ and the implicit constant is independent from a and k.

Moreover, the construction shows that the Gabor phase retrieval problem can not be regularized using typical priors such as sparsity or smoothness.

4.3 Some preliminary complex analysis

We shall now collect some definitions, notation and basic facts for holomorphic functions of several complex variables. For $x, y \in \mathbb{C}^d$ we write $x \cdot y = \sum_{j=1}^d x_j y_j$ and $x^2 = x \cdot x$.
By virtue of the identification

$$(x, y) \in \mathbb{R}^{2d} \quad \leftrightarrow \quad z = (z_1, \ldots, z_d) = (x_1 + iy_1, \ldots, x_d + iy_d) \in \mathbb{C}^d$$

we identify $\mathbb{R}^{2d}$ with $\mathbb{C}^d$ and consider a domain $D \subset \mathbb{R}^{2d}$ as a domain in $\mathbb{C}^d$ simultaneously. Note that the Euclidean length of (x, y) and z agree; we denote the ball of radius $r > 0$ centered at $z_0 \in \mathbb{C}^d$ by

$$B_r(z_0) = \left\{ z \in \mathbb{C}^d : |z - z_0| < r \right\}.$$

[1] The choice of the L^2-norms here is completely arbitrary; any pair of norms could be used in principle.

The area measure on $\mathbb{C}^d$ will be denoted by A; for a measurable function f on $\mathbb{R}^{2d} \simeq \mathbb{C}^d$ we have that

$$\iint_{\mathbb{R}^{2d}} f \, dx \, dy = \int_{\mathbb{C}^d} f \, dA(z).$$

For $1 \leq p < \infty$ and a nonnegative measurable function w on D we define the weighted L^p-space by

$$L^p(D, w) := \left\{ f : D \to \mathbb{C} \, \text{measurable} : \|f\|_{L^p(D,w)} < +\infty \right\},$$

where

$$\|f\|_{L^p(D,w)} := \left(\int_D |f|^p w \, dA(z) \right)^{1/p}.$$

For $1 \leq j \leq d$, the *Wirtinger* derivatives are defined by

$$\frac{\partial}{\partial z_j} := \frac{1}{2} \left(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z_j}} := \frac{1}{2} \left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right).$$

For a smooth function $f : D \to \mathbb{C}$ we use the notation

$$f' = \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_d} \right)^T.$$

By $\mathcal{O}(D)$ we denote the space of all analytic functions on D consisting of all functions $f \in C^1(D)$ which satisfy the Cauch-Riemann equations

$$\frac{\partial}{\partial \overline{z_j}} f = 0, \quad 1 \leq j \leq d.$$

Due to Hartogs [Hör90b, Theorem 2.2.8] for $f : D \to \mathbb{C}$ to be analytic in D it is sufficient that for all $1 \leq j \leq d$ and $z \in D$

$$\zeta \in \mathbb{C} \mapsto f(z + \zeta e_j)$$

is analytic at 0 in the univariate sense, where $e_j \in \mathbb{C}^d$ denotes that j-th unit vector. If $D = \mathbb{C}^d$ we get $\mathcal{O}(\mathbb{C}^d)$, the space of entire functions.

We say a function f is meromorphic on D if it is locally represented by the quotient of two holomorphic functions, i.e., if for all $z_0 \in D$ there exists $r > 0$ and $p, q \in \mathcal{O}(B_r(z_0))$ with $q \neq 0$ such that

$$f = p/q \qquad \text{on } B_r(z_0),$$

and denote the space of meromorphic functions on D by $\mathcal{M}(D)$.

Note that if $f \in \mathcal{O}(D)$ we have for all $1 \leq j \leq d$ that

$$\frac{\partial f}{\partial x_j} = \left(\frac{\partial}{\partial z_j} + \frac{\partial}{\partial \overline{z_j}} \right) f = \frac{\partial f}{\partial z_j}$$

and, simlarly that

$$\frac{\partial f}{\partial y_j} = i \left(\frac{\partial}{\partial z_j} - \frac{\partial}{\partial \overline{z_j}} \right) f = i \frac{\partial f}{\partial z_j}.$$

4.4 Bargmann transform

As previously indicated the Gabor transform is essentially a holomorphic function. What that means precisely will be elucidated in the present section. To this end we closely follow and summarize the exposition in [Grö01, Chapter 3.4].

Definition 4.4. The *Bargmann* transform of $f \in \mathcal{S}'(\mathbb{R}^d)$ is defined by

$$\mathcal{B}f(z) := 2^{d/4} e^{-\frac{\pi}{2}z^2} \langle f, t \mapsto e^{2\pi t\cdot z - \pi t^2} \rangle_{\mathcal{S}'(\mathbb{R}^d)\times\mathcal{S}(\mathbb{R}^d)}, \qquad z \in \mathbb{C}^d.$$

The Bargmann transform produces entire functions.

Proposition 4.5. *The Bargmann transform $\mathcal{B}f$ is an entire function for all $f \in \mathcal{S}'(\mathbb{R}^d)$, i.e., $\mathcal{B}f \in \mathcal{O}(\mathbb{C}^d)$.*

Proof. It suffices to check that $\mathcal{B}f$ is holomorphic in the univariate sense w.r.t. all of $z_1, \dots, z_d$ seperately. We only give a proof for the first variable; the remaining $d-1$ variables can be treated analogously.
Let $z' \in \mathbb{C}^{d-1}$ be arbitrary and fixed. We shall proof that the function h defined by

$$h : \zeta \mapsto \mathcal{B}f(\zeta, z'), \qquad \zeta \in \mathbb{C}.$$

is entire. The proof is split up into two steps.

Step 1: Suppose first that $f \in \mathcal{S}(\mathbb{R}^d)$. By Morera's theorem it suffices to check that for all triangles $\Delta \subset \mathbb{C}$ it holds that

$$\int_{\partial\Delta} h(\zeta)\, d\zeta = 0. \tag{4.2}$$

By interchanging order of integration, we get that

$$\begin{aligned}\int_{\partial\Delta} h(\zeta)\, d\zeta &= \int_{\partial\Delta} \mathcal{B}f(\zeta, z')\, d\zeta \\ &= 2^{d/4} \int_{\mathbb{R}^d} f(t) \int_{\partial\Delta} e^{-\frac{\pi}{2}(\zeta^2 + z'^2) + 2\pi t\cdot(\zeta, z')^T - \pi t^2}\, d\zeta\, dt,\end{aligned}$$

and, since the integrand is holomorphic w.r.t. ζ that the inner integral vanishes, i.e., (4.2) indeed holds true.

Step 2: For $f \in \mathcal{S}'(\mathbb{R}^d)$ by density we can pick a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^d)$ converging to f in $\mathcal{S}'(\mathbb{R}^d)$. We define $h_n(\zeta) := \mathcal{B}f_n(\zeta, z'), \zeta \in \mathbb{C}$. By the first part $(h_n)_{n\in\mathbb{N}}$ is a family of entire functions which converges pointwise to h since $(f_n)_{n\in\mathbb{N}}$ converges to f in $\mathcal{S}'(\mathbb{R}^d)$. By considering the family of functionals $(T_n)_{n\in\mathbb{N}}$ defined by

$$T_n(\varphi) := \langle f_n, \varphi \rangle, \quad \varphi \in \mathcal{S}(\mathbb{R}^d)$$

a similar argument as in the proof of Theorem 3.4 yields that $(h_n)_{n\in\mathbb{N}}$ is uniformly bounded on compacta, in particular for all $r > 0$ there exists $C_r > 0$ such that

$$|h_n(z)| \le C_r \qquad \forall z \in B_r(0), n \in \mathbb{N}.$$

Using Cauchy's integral formula we get a uniform bound for h_n' on compacta K: Suppose that $R > 0$ such that $K \subset B_R(0)$, then for $z \in K$

$$|h_n'(z)| = \left| \frac{1}{2\pi i} \int_{\partial B_{2R}(0)} \frac{h_n(\zeta)}{(\zeta - z)^2}\, d\zeta \right| \le \frac{1}{2\pi} C_{2R} \frac{4\pi R}{R^2}.$$

Thus $(h_n)_{n\in\mathbb{N}}$ is equicontinuous on compacta. As a consequence of the Arzelá-Ascoli theorem we obtain that $(h_n)_{n\in\mathbb{N}}$ converges uniformly on compacta (to h that is). Consequently, as the uniform limit of holomorphic functions, h must be holomorphic itself. $\square$

With $z = x + iy$ we have that

$$-\frac{\pi}{2}z^2 + 2\pi t \cdot z - \pi t^2 = -\frac{\pi}{2}x^2 + \pi i x \cdot y + \frac{\pi}{2}y^2 + 2\pi t \cdot x + 2\pi i t \cdot y - \pi t^2$$
$$= -\pi(t-x)^2 + 2\pi i t \cdot y + \frac{\pi}{2}(x^2 + 2ix \cdot y + y^2).$$

Thus the Bargmann transform and the Gabor transform are related according to

$$\begin{aligned}\mathcal{B}f(x,y) &= 2^{d/4}\langle f, e^{-\frac{\pi}{2}z^2 + 2\pi t\cdot z - \pi t^2}\rangle \\ &= 2^{d/4}\langle f, e^{-\pi(t-x)^2 + 2\pi i t \cdot y}\rangle \cdot e^{\frac{\pi}{2}(x^2 + 2ix\cdot y + y^2)} \\ &= \mathcal{G}f(x,-y) \cdot e^{\frac{\pi}{2}(x^2+2ix\cdot y + y^2)}.\end{aligned}$$

Hence, we proved the following statement:

Theorem 4.6. *Let $\eta(x,y) := \exp(\pi(x^2 + 2ix \cdot y + y^2)/2)$. Then for all $f \in \mathcal{S}'(\mathbb{R}^d)$ it holds that*

$$z = x + iy \mapsto \mathcal{G}f(x,-y)\eta(x,y)$$

coincides with the Bargmann transform $\mathcal{B}f$ of f and in particular constitutes an entire function on $\mathbb{C}^d$.

4.5 Stable phase retrieval from holomorphic measurements

Theorem 4.6 suggests that

(a) reconstructing the phase of the Gabor transform $\mathcal{G}f$ from its modulus $|\mathcal{G}f|$ and

(b) reconstructing the phase of a Bergmann transform $\mathcal{B}f$ from its modulus $|\mathcal{B}f|$

are essentially equivalent problems and intimately related with the problem of

(c) reconstructing an analytic function F from $|F|$.

In order to cover (a),(b) and (c) all simultaneously we introduce the following notion of admissibility.

Definition 4.7. Let D be a domain in $\mathbb{C}^d$. A subspace $\mathcal{X}$ of $C^1(D)$ is called *admissible* if it satisfies the property

$$F, G \in \mathcal{X},\ F \neq 0 \quad \Rightarrow \quad G/F \in \mathcal{M}(D).$$

Obviously, in the cases (b) and (c) we deal with admissible spaces, i.e., when

$$\mathcal{X} = \left\{F = \mathcal{B}f|_D\,,\ f \in \mathcal{S}'(\mathbb{R}^d)\right\}$$

or $\mathcal{X} = \mathcal{O}(D)$ respectively. Moreover due to Theorem 4.6 also case (a) can be made admissible by means of a simple reflection, meaning that

$$\{F|_D :\ F(x,y) = \mathcal{G}f(x,-y),\ f \in \mathcal{S}'(\mathbb{R}^d)\}$$

is admissible. Within an admissible space phase retrieval is possible up to global sign:

Theorem 4.8. *Suppose that $\mathcal{X} \subset C^1(D)$ is admissible and that $F, G \in \mathcal{X}$ are such that $|G| = |F|$. Then there exists a unimodular constant c such that $G = cF$.*

In order to prove the statement we require the following fact.

Lemma 4.9. *Let $D \subset \mathbb{C}^n$ be a domain, i.e., open and connected and $f \in \mathcal{O}(D)$. Then it holds that $D \setminus \mathcal{N}_f$ is connected, with $\mathcal{N}_f$ denoting the zero set of f.*

Proof. For a proof we refer to [KW11, Theorem 4.6.1]. $\square$

We proceed with the proof of the uniqueness result.

Proof of Theorem 4.8. We may assume w.l.o.g. that F does not vanish identically since then also G vanishes identically. Then, by the admissibility of $\mathcal{X}$ the quotient G/F is meromorphic. The zero set $\mathcal{N}_F := \{z \in D : F(z) = 0\}$ agrees with the zero set of G. By Lemma 4.9 we have that $D' := D \setminus \mathcal{N}_F$ is connected.
Now for arbitrary $z = (z_1, \ldots, z_d) \in D'$ there exists $r > 0$ such that $B_r(z) \subset D'$. By assumption G/F has its values on the unit circle. Applying the open mapping theorem for each variable seperately gives that

$$\zeta \in B_r(z_j) \subset \mathbb{C} \mapsto (G/F)(z_1, \ldots, z_{j-1}, \zeta, z_{j+1}, \ldots, z_d)$$

is constant for each $1 \leq j \leq d$, and thus, that G/F is locally constant. Since D' is connected there must be a global constant such that $G/F = c$ on D'. Moreover $|c| = 1$ since $|G/F| = 1$ in D'. $\square$

The aim of this section is to turn the uniqueness result for admissible spaces, Theorem 4.8 into a quantitative result. The key observation in the proof of Theorem 4.8 is that the meromorphic function $q = G/F$ – under the assumption that $|q| = 1$ – must be constant due to the open mapping theorem. The following lemma can be regarded as a quantitative version of that statement and will play a central role in our analysis. Note also that the methods developed in [ADGY19] crucially rely on the univariate version of the lemma.

Lemma 4.10. *Suppose that F is analytic at $w \in \mathbb{C}^d$ and that $F(w) \neq 0$. Then it holds that $|\nabla|F|(w)| = 2^{-1/2}|\nabla F(w)|$.*

Proof. Let us first assume that $d = 1$. Pick $\delta > 0$ such that F does not vanish on $B_\delta(w)$. Then

$$\log F = \log|F| + i\psi =: u + iv$$

is holomorphic, where the phase function ψ is harmonic on $B_\delta(w)$ and unique up to addition of a multiple of 2π. Thus, with the help of Cauchy-Riemann, we get that

$$F'/F = \frac{\partial}{\partial z}\log F = \frac{\partial}{\partial x}\log F = u_x + iv_x = u_x - iu_y,$$

and hence that $|F'/F| = |\nabla u|$. Moreover, since $|F'| = 2^{-1/2}\,|\nabla F|$ it follows that

$$|\nabla\,|F|| = |\nabla u| \cdot |F| = |F'| = 2^{-1/2}\,|\nabla F|\,.$$

The general case $d > 1$ follows from the univariate case: Note that for each $1 \leq j \leq d$ the mapping $\mathbb{C} \ni z \mapsto F(w_1, \ldots, w_{j-1}, z, w_{j+1}, \ldots, w_d)$ is holomorphic at w_j. Then by the first part it holds that

$$\begin{aligned} |\nabla|F|(w)|^2 &= \sum_{j=1}^d \left[\left(\frac{\partial}{\partial x_j}|F|(w)\right)^2 + \left(\frac{\partial}{\partial y_j}|F|(w)\right)^2\right] \\ &= 2^{-1}\sum_{j=1}^d \left[\left(\frac{\partial}{\partial x_j}F(w)\right)^2 + \left(\frac{\partial}{\partial y_j}F(w)\right)^2\right] = 2^{-1}|\nabla F(w)|^2. \end{aligned}$$

$\square$

Moreover, we will require the concept of weighted Poincaré inequalities. At this point we are content with the definition but we will devote the entire Chapter 6 to an in-depth study of this topic.

Definition 4.11. Let $D \subset \mathbb{C}^d$ be a domain equipped with a nonnegative, integrable weight w and let $1 \leq p < +\infty$. We say that D supports a *(meromorphic) Poincaré inequality* (w.r.t. w) if there exists a finite constant C such that

$$\inf_{c\in\mathbb{C}} \|F - c\|_{L^p(D,w)} \leq C\|\nabla F\|_{L^p(D,w)} \qquad \forall F \in \mathcal{M}(D) \cap L^p(D, w). \tag{4.3}$$

The smallest possible number C in (4.3) is called the *(meromorphic) Poincaré constant* of D and denoted by $C_P^{\mathcal{M}}(D, w, p)$.

A moderately sized Poincaré constant signifies that the size of the gradient $\|\nabla F\|_{L^p(D,w)}$ serves as a useful indicator whether F is close to a constant or not w.r.t. $L^p(D, w)$-norm.
Note that in the defining inequality of Definition 4.11 functions are restricted to be meromorphic. This is certainly non-standard but precisely the right concept for our purposes.

We are now well-equipped to develop the key mechanism that allows us to establish quantitative phase retrieval results. This mechanism is compromised in the following theorem which constitutes one of the main results of this thesis.

Theorem 4.12. *Let $1 \leq p < \infty$, let $D \subset \mathbb{C}^d$ be a domain equipped with a weight w. Suppose that $\mathcal{X} \subset C^1(D)$ is admissible and that $\mathcal{X} \subset L^p(D, w)$. Then for all $F, G \in \mathcal{X}$ it holds that*

$$\begin{aligned}\inf_{|c|=1} \|G - cF\|_{L^p(D,w)} &\leq \||G| - |F|\|_{L^p(D,w)} \\ &+ 2^{3/2} C_P^{\mathcal{M}}(D, w|F|^p, p) \left(\|\nabla|G| - \nabla|F|\|_{L^p(D,w)} + \left\| \frac{\nabla|F|}{|F|} (|G| - |F|) \right\|_{L^p(D,w)} \right)\end{aligned} \tag{4.4}$$

Proof. If F vanishes identically there is nothing to prove. We assume therefore that this is not the case.

In a first, preperatory step we show that the constraint $|c| = 1$ in the distance

$$\inf_{|c|=1} \|G - cF\|_{L^p(D,w)}$$

can effectively be dropped if the unsigned measurements are close, cf. Inequality (4.8). The distance term without the constraint on c will be controlled in the second step by making use of Poincaré's inequality as well as Lemma 4.10.

Step 1: Getting rid of the constraint
For $\varepsilon > 0$ let $c_\varepsilon \in \mathbb{C}$ be such that

$$\|G - c_\varepsilon F\|_{L^p(D,w)} \leq \inf_{c\in\mathbb{C}} \|G - cF\|_{L^p(D,w)} + \varepsilon. \tag{4.5}$$

Note that by continuity c_ε can always be chosen to be nonzero. By triangle inequality we estimate

$$\inf_{|c|=1} \|G - cF\|_{L^p(D,w)} \leq \|G - \frac{c_\varepsilon}{|c_\varepsilon|} F\|_{L^p(D,w)} \leq \|G - c_\varepsilon F\|_{L^p(D,w)} + |1 - |c_\varepsilon|| \cdot \|F\|_{L^p(D,w)}. \tag{4.6}$$

Furthermore the last term can be bounded by

$$
\begin{aligned}
|1-|c_\varepsilon||\cdot\|F\|_{L^p(D,w)} &= \||F|-|c_\varepsilon F|\|_{L^p(D,w)}\\
&\le \||F|-|G|\|_{L^p(D,w)}+\||G|-|c_\varepsilon F|\|_{L^p(D,w)}\\
&\le \||F|-|G|\|_{L^p(D,w)}+\|G-c_\varepsilon F\|_{L^p(D,w)}.
\end{aligned}
\tag{4.7}
$$

Combining (4.6) and (4.7), together with (4.5) yields that

$$
\inf_{|c|=1}\|G-cF\|_{L^p(D,w)} \le 2\inf_{c\in\mathbb{C}}\|G-cF\|_{L^p(D,w)}+2\varepsilon+\||F|-|G|\|_{L^p(D,w)}.
$$

Since $\varepsilon>0$ was arbitrary it holds that

$$
\inf_{|c|=1}\|G-cF\|_{L^p(D,w)} \le 2\inf_{c\in\mathbb{C}}\|G-cF\|_{L^p(D,w)}+\||F|-|G|\|_{L^p(D,w)}.
\tag{4.8}
$$

Step 2: Bound for the unconstrained distance
First we rewrite for arbitrary $c\in\mathbb{C}$

$$
\|G-cF\|_{L^p(D,w)}=\|G/F-c\|_{L^p(D,w|F|^p)}.
$$

Note that

- $G/F\in\mathcal{M}(D)$ due to $F\neq 0$ and the admissibility of $\mathcal{X}$,
- $G/F\in L^p(D,w|F|^p)$ since $\|G/F\|_{L^p(D,w|F|^p)}=\|G\|_{L^p(D,w)}<+\infty$,
- $|F|^p\in L^1(D,w)$ since $F\in L^p(D,w)$.

Therefore Poincaré's inequality can be applied and we obtain

$$
\begin{aligned}
\inf_{c\in\mathbb{C}}\|G-cF\|_{L^p(D,w)} &= \inf_{c\in\mathbb{C}}\|G/F-c\|_{L^p(D,w|F|^p)}\\
&\le C_P^{\mathcal{M}}(D,w|F|^p,p)\,\|\nabla\,(G/F)\|_{L^p(D,w|F|^p)}\\
&= 2^{1/2}C_P^{\mathcal{M}}(D,w|F|^p,p)\,\|\nabla\,|G/F|\|_{L^p(D,w|F|^p)}\ ,
\end{aligned}
\tag{4.9}
$$

where the last equality follows from Lemma 4.10, since G/F is holomorphic in D with the exception of a set of measure zero. Thus it holds almost everywhere in D that

$$
\nabla\,|G/F|=|F|^{-2}\cdot(|F|\nabla|G|-|G|\nabla|F|)\ ,
$$

and hence, we can estimate

$$
\begin{aligned}
\|\nabla\,|G/F|\|_{L^p(D,w|F|^p)} &= \left\||F|^{-2}\cdot(|F|\nabla|G|-|G|\nabla|F|)\right\|_{L^p(D,w|F|^p)}\\
&\le \left\|\frac{|F|\nabla|G|-|F|\nabla|F|}{|F|^2}\right\|_{L^p(D,w|F|^p)}\\
&+\left\|\frac{|F|\nabla|F|-|G|\nabla|F|}{|F|^2}\right\|_{L^p(D,w|F|^p)}\\
&= \|\nabla|F|-\nabla|G|\|_{L^p(D)}+\left\|\frac{\nabla|F|}{|F|}\,(|F|-|G|)\right\|_{L^p(D,w)}.
\end{aligned}
\tag{4.10}
$$

By combining the Estimates (4.8),(4.9) and (4.10) we arrive at the desired bound (4.4). □

The stability estimate of Theorem 4.12 remains to have two major obscurities:

- The behaviour of the Poincaré constant $C_P^{\mathcal{M}}(D, |F|^p, p)$, qualitatively as well as quantitatively, needs to be studied. This will be done in Chapter 6.
- If the logarithmic derivative $\nabla|F|/|F|$ that appears in the final term on the right hand side of (4.4) was not there we would have a bound in terms of the difference of the moduli of F and G (in the Sobolev norm) which would be perfectly fine. In general the logarithmic derivative does behave far from nicely as it has singularities at the zeros of F. As we shall see in the subsequent Chapter 5 the logarithmic derivative can be controlled (in a certain sense uniformly in F) under suitable assumptions on the space $\mathcal{X}$.

Chapter 5

Growth estimates for the logarithmic derivative of entire functions

We turn our attention now towards studying the logarithmic derivative

$$\nabla |F| \,/\, |F| = \nabla(\log |F|).$$

The underlying motivation originates from the phase retrieval stability estimate of Theorem 4.12, where F resides in an admissible space. Recall that we are mainly interested in the phase retrieval problems where the measurements are either the modulus of an analytic function or of a Gabor transform.
We shall initially restrict our attention to analytic functions. Ultimately we wish to control the size of the logarithmic derivative (whatever that may mean) of a preferably large class of entire functions in a uniform way. The following simple example shows that some sort of restrictions are required in order to even hope for uniform bounds.

Example 5.1. Let us consider $F_n(z) := e^{nz^2}$, $z \in \mathbb{C}$, $n \in \mathbb{N}$. Then $|\nabla \log |F_n|| \asymp n\,|z|^2$ and therefore we get that

$$\|\nabla \log |F_n|\|_{L^p(B_r(0))} \asymp nr^{(p+2)/p}.$$

Generally speaking we cannot allow function classes that permit dilations of arbitrary order of its elements. We will shut out such phenomenons by further imposing a growth restriction: More precisely we will assume that there exist $\alpha, \beta > 0$ such that

$$|F(z)| \leq |F(0)|\, e^{\alpha |z|^\beta} \qquad \forall z \in \mathbb{C}^d, \tag{5.1}$$

or, equivalently, in terms of $u = \log |F|$ that

$$u(z) \leq u(0) + \alpha\, |z|^\beta \qquad \forall z \in \mathbb{C}^d. \tag{5.2}$$

We are now ready to state the main result of this section.

Theorem 5.2. *Let $\alpha, \beta > 0$ and $1 \leq p < 1 + 1/(2d-1)$. Suppose that $F \in \mathcal{O}(\mathbb{C}^d)$ satisfies (5.1). Then there exists a constant $c > 0$ depending on d, p, α, β (but independent from F and r) such that for all $r > 1$ it holds that*

$$\|\nabla \log |F|\|_{L^p(B_r)} \leq cr^{2d+\beta-1}.$$

The proof of Theorem 5.2 is postponed to Section 5.2. Before that we need to collect quite some preliminary material first (Section 5.1). In Section 5.3 we will discuss implications of the obtained results for the problem of Gabor phase retrieval.

5.1 Preliminaries on (log-)subharmonic functions

First, recall that a function $u : D \subset \mathbb{R}^k \to [-\infty, \infty)$ is subharmonic by definition if

(i) u is upper-semicontinuous; meaning that the sublevelsets $\{x \in D : u(x) < \lambda\}$ are open for all $\lambda \in \mathbb{R}$ and

(ii) for every $x_0 \in D$ there exists $\varepsilon > 0$ such that for all $0 < r < \varepsilon$

$$u(x_0) \leq \frac{1}{\sigma(\partial B_r(x_0))} \int_{\partial B_r(x_0)} u(\xi)\, d\sigma(\xi), \tag{5.3}$$

where σ denotes the surface measure on $\partial B_r(x_0)$.

Unsurprisingly a log-subharmonic function is defined as follows.

Definition 5.3. A nonnegative function f on D is called *log-subharmonic* (on D) if $\log f$ is subharmonic on D.

Indeed the phaseless measurements we are interested in fall into the class of log-subharmonic functions:

Proposition 5.4. *$|F|$ is log-subharmonic for all $F \in \mathcal{O}(D)$ and $|\mathcal{G}f|$ is log-subharmonic for all $f \in \mathcal{S}'(\mathbb{R}^d)$.*

Proof. It is a well-known fact that $\log |F|$ is subharmonic for $F \in \mathcal{O}(D)$ in the one dimensional case where $D \subset \mathbb{C}$, cf. [HK76]. The proof readily extends to the multivariate setting:
First observe that for every λ the sublevelsets $\{x \in D : \log |F(x)| < \lambda\}$ coincides with the set

$$\{x \in D : |F(x)| < e^{\lambda}\}$$

which is open due to $|F|$ being continuous.
The second defining property for log-subharmonic functions is trivially fulfilled if $F(x_0) = 0$. If $F(x_0) \neq 0$ choose $\varepsilon > 0$ such that F does not vanish on $B_\varepsilon(x_0)$. Then locally $\log |F|$ is harmonic as it is the real part of $\log F \in \mathcal{O}(B_\varepsilon(x_0))$. By the mean value property of harmonic functions $u = \log |F|$ satisfies (5.3) with equality for all $0 < r < \varepsilon$.

To show the statement for $|\mathcal{G}f|$ observe that by Theorem 4.6

$$\log |\mathcal{G}f(x, -y)| = \log |\mathcal{B}f(x, y)| - \frac{\pi}{2}(x^2 + y^2)$$

and thus, $(x, y) \mapsto \log |\mathcal{G}f(x, -y)|$ is – by Proposition 4.5 and by the first part of the proof – the sum of a subharmonic and a harmonic function, and therefore subharmonic itself. Since the property of being subharmonic does not change under reflection we conclude that $|\mathcal{G}f|$ is log-subharmonic. □

The purpose of the remainder of this section is to present and discuss the formula of Poisson-Jensen which serves as the basis of our further analysis on derivatives of subharmonic functions.

5.1.1 The classical version of Poisson-Jensen's formula

In the context of entire functions of one complex variable the formula of Poisson-Jensen is well-known, see for example [Ahl78], and can be stated as follows.

Theorem 5.5 (Poisson-Jensen, classical version)**.** *Let $F \in \mathcal{O}(\mathbb{C})$ and let $r > 0$. Let $z_1, \ldots, z_k$ denote the zeros of F in $B_r(0)$ repeated according to multiplicity. Then for all $|z| < r$ it holds that*

$$\log|F(z)| = -\sum_{j=1}^{k} \log\left|\frac{r^2 - \bar{z}_j z}{r(z - z_j)}\right| + \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\frac{re^{i\theta} + z}{re^{i\theta} - z} \log\left|F(re^{i\theta})\right| d\theta. \tag{5.4}$$

If we set $z = 0$ we recover Jensen's formula *as a special case:*

$$\log|F(0)| = -\sum_{j=1}^{k} \log\frac{r}{|z_j|} + \frac{1}{2\pi}\int_0^{2\pi} \log\left|F(re^{i\theta})\right| d\theta. \tag{5.5}$$

Jensen's formula relates the value of $|F|$ at 0 to the position of its zeros inside $B_r(0)$ and the mean value of $\log|F|$ on $\partial B_r(0)$. Suppose for a moment that $|F(0)| = 1$ then the left hand side in (5.5) vanishes. We have then that

$$\sum_{j=1}^{k} \log\frac{r}{|z_j|} = \frac{1}{2\pi}\int_0^{2\pi} \log\left|F(re^{i\theta})\right| d\theta.$$

Therefore if $\log|F|$ has small average on $\partial B_r(0)$ then either the number of zeros in $B_r(0)$ is small or all zeros are close to the boundary. If on the other hand $\log|F|$ has large average on $\partial B_r(0)$ then there must be either zeros close to the origin or a large number of zeros (or both). Poisson-Jensen's formula can be interpreted similarly.
In that sense equations (5.4) and (5.5) may be regarded as relations between growth of entire functions and the distributions of their zeros.

5.1.2 A general version of Poisson-Jensen's formula

A rather general variant of Poisson-Jensen's formula can be encountered in the theory of subharmonic functions. Before the result can be stated we make some preparatory comments on subharmonic functions. We refer to [HK76] for a detailed and extensive treatment of this topic.

We consider now functions on a domain $D \subset \mathbb{R}^m$, with $m \geq 2$. Depending on the dimension m we define a function K on $\mathbb{R}^m$ by

$$K(x) := \begin{cases} \log|x|, & \text{if } m = 2, \\ -|x|^{2-m}, & \text{if } m > 2. \end{cases}$$

We continue with one of the most fundamental results of the theory of subharmonic functions.

Theorem 5.6 (Riesz representation theorem)**.** *Suppose u is subharmonic on D and not identically $-\infty$. Then there exists a unique Borel measure μ on D such that for any $E \subset\subset D$*

$$u(x) = \int_E K(x - \xi)\, d\mu(\xi) + h(x), \quad x \in E, \tag{5.6}$$

where h is harmonic in the interior of E.

Under the assumptions of Theorem 5.6 the measure μ associated to u is called the Riesz measure of u.

Remark 5.7. In other words, Riesz representation theorem states that a subharmonic function u can always be made harmonic (locally) by subtracting a so-called potential p, where

$$p(x) = \int_E K(x - \xi)\, d\mu(\xi). \tag{5.7}$$

The correct potential is given in terms of its Riesz measure μ.
Note that also the reverse of Theorem 5.6 holds true in the sense that if μ is a Borel measure on D and E a compact subset of D then the function p defined by (5.7) is subharmonic.

In general the function h depends not only on u but also on the choice of E. The role of h in Theorem 5.6 is rather inexplicit. However in special cases – meaning particular choices of E – h is known explicitly (in terms of u). For example if we consider E to be balls $B_r(0)$ of varying radii centered at the origin and slightly modify the kernel K we get the following representation result for subharmonic functions.

Theorem 5.8 (Poisson-Jensen, general version)**.** *Suppose u is subharmonic on $\mathbb{R}^m$. Let μ denote the Riesz measure of u. Then for all $r > 0$ it holds that*

$$u(x) = -\int\limits_{B_r(0)} K_r(x,\xi)\, d\mu(\xi) + \int\limits_{\partial B_r(0)} P_r(x,\zeta) u(\zeta)\, d\sigma(\zeta), \qquad x \in B_r(0),$$

where

$$K_r(x,\xi) := \begin{cases} -\log \frac{r|x-\xi|}{|\xi||x-\xi'|}, & \text{if } m = 2, \\ |x-\xi|^{-m+2} - \left(\frac{|\xi||x-\xi'|}{r}\right)^{-m+2}, & \text{if } m > 2, \end{cases} \tag{5.8}$$

$$P_r(x,\zeta) := \omega_{m-1}^{-1} \frac{r^2 - |x|^2}{r\,|x-\zeta|^m}, \tag{5.9}$$

where σ denotes the surface measure on $\partial B_r(0) \subset \mathbb{R}^m$, ω_{m-1} denotes the surface area of the unit sphere $\partial B_1(0) \subset \mathbb{R}^m$ and where $\xi' := r^2\xi/|\xi|^2$.

5.2 Bounds for the logarithmic derivative of entire functions

In the present section we will derive various bounds for the logarithmic derivative of entire functions with the main objective being to prove Theorem 5.2. Throughout this section we consider numbers $\alpha, \beta > 0$, F is supposed to satisfy the assumptions of Theorem 5.2, i.e.,

$$F \in \mathcal{O}(\mathbb{C}^d) \quad \text{and} \quad |F(z)| \le |F(0)|\, e^{\alpha|z|^\beta}, \quad \forall z \in \mathbb{C}^d.$$

Moreover, let u denote the subharmonic function $u = \log|F|$, which satisfies the growth restriction (5.2).
To simplify notation we denote balls centered at the origin by B_r instead of $B_r(0)$. We denote the dimension of the underlying (real) space by m, i.e. $m = 2d$ and $\mathbb{C}^d \backsimeq \mathbb{R}^m$.
The Riesz measure associated to u will be denoted by μ and we define a function ν by

$$\nu(r) := \int_{B_r} |x|^{-m+2}\, d\mu(x), \qquad r > 0.$$

5.2.1 Preliminary estimates

As a first step we show that μ and ν grow in a controlled manner.

Proposition 5.9. *For all $r > 0$ we have that*

$$\nu(r) \le \alpha 2^{\beta+1} r^\beta \qquad \text{and} \qquad \mu(B_r) \le \alpha 2^{\beta+1} r^{\beta+m-2}.$$

Proof. Using Poisson-Jensen's representation formula at $x = 0$ we get a generalization of Jensen's formula, cf. Equation (5.5):

$$u(0) = -\int_{B_r} K_r(0,\xi)\, d\mu(\xi) + \int_{\partial B_r} P_r(0,\zeta)u(\zeta)\, d\sigma(\zeta), \tag{5.10}$$

where K_r and P_r are defined by (5.8) and (5.9) respectively. Note that P_r is the Poisson kernel on B_r and therefore reproduces harmonic functions, and in particular the constant function $u(0)$. Thus we can rewrite Equation (5.10)

$$\int_{B_r} K_r(0,\xi)\, d\mu(\xi) = \int_{\partial B_r} P_r(0,\zeta)[u(\zeta) - u(0)]\, d\sigma(\zeta)\,. \tag{5.11}$$

Note that we have for $\xi' = r^2\xi/|\xi|^2$ that $|\xi'| = r^2/|\xi|$, and hence, that $|\xi|\,|\xi'| = r^2$. Let now $|\xi| < r/2$. In the case $m = 2$ we get that

$$K_r(0,\xi) = \log(|\xi'|/r) \geq \log 2,$$

since $|\xi'| = r^2/|\xi| > 2r$. Similarly for $m > 2$ we obtain

$$K_r(0,\xi) = |\xi|^{-m+2} - r^{-m+2} \geq \frac{1}{2}|\xi|^{-m+2}\,.$$

Using (5.11) and that K_r is nonnegative on $B_r \times B_r$, we estimate

$$\begin{aligned}
\nu(r/2) &= \int_{B_{r/2}} |\xi|^{-m+2}\, d\mu(\xi) \\
&\leq 2\int_{B_{r/2}} K_r(0,\xi)\, d\mu(\xi) \\
&\leq 2\int_{B_r} K_r(0,\xi)\, d\mu(\xi) \\
&= 2\int_{\partial B_r} P_r(0,\zeta)[u(\zeta) - u(0)]\, d\sigma(\zeta)\,.
\end{aligned}$$

Now, for $|\zeta| = 1$ we have that

$$P_r(0,\zeta) = \omega_{m-1}^{-1} r^{-m+1}$$

and by assumption (5.2) that $u(\zeta) - u(0) \leq \alpha r^\beta$. Thus it follows that

$$\nu(r/2) \leq 2\omega_{m-1}^{-1} r^{-m+1} \cdot \alpha r^\beta \cdot \sigma(B_r) = 2\alpha r^\beta.$$

Now by replacing $r/2$ by r we obtain indeed that $\nu(r) \leq \alpha 2^{\beta+1} r^\beta$.
Finally, the estimate for μ follows from the observation that $\nu(r) \geq r^{-m+2}\mu(B_r)$:

$$\mu(B_r) \leq r^{m-2}\nu(r) \leq \alpha 2^{\beta+1} r^{\beta+m-2}\,.$$

$\square$

As a second step we will derive pointwise estimates for $|\nabla u|$. To that end we require the derivatives of the kernels K_r and P_r w.r.t. the first variable.

Lemma 5.10. *Let K_r be defined by* (5.8)*, then*

$$\nabla K_r(\cdot,\xi)(x) = \begin{cases} \frac{x-\xi'}{|x-\xi'|^2} - \frac{x-\xi}{|x-\xi|^2}\,, & m = 2, \\ (m-2)\left((r/|\xi|)^{m-2}\frac{x-\xi'}{|x-\xi'|^m} - \frac{x-\xi}{|x-\xi|^m}\right)\,, & m > 2. \end{cases}$$

Proof. We consider the case $m = 2$ first. Rewrite K_r according to

$$K_r(x,\xi) = -\log(r/|\xi|) - \frac{1}{2}\log|x-\xi|^2 + \frac{1}{2}\log|x-\xi'|^2 .$$

Differentiation w.r.t. x yields

$$\nabla K_r(\cdot,\xi)(x) = \frac{1}{2}\left(-\frac{2(x-\xi)}{|x-\xi|^2} + \frac{2(x-\xi')}{|x-\xi'|^2}\right) = \frac{x-\xi'}{|x-\xi'|^2} - \frac{x-\xi}{|x-\xi|^2} .$$

For $m > 2$ setting $l = m-2$ we have that

$$K_r(x,\xi) = (|x-\xi|^2)^{-l/2} - (|\xi|/r)^{-l}\,(|x-\xi'|^2)^{-l/2} ,$$

and therefore

$$\begin{aligned}\nabla K_r(\cdot,\xi)(x) &= -\frac{l}{2}\frac{2(x-\xi)}{|x-\xi|^{2(l/2+1)}} + (|\xi|/r)^{-l}\,\frac{l}{2}\frac{2(x-\xi')}{|x-\xi'|^{2(l/2+1)}} \\ &= (m-2)\left((r/|\xi|)^{m-2}\frac{x-\xi'}{|x-\xi'|^m} - \frac{x-\xi}{|x-\xi|^m}\right) .\end{aligned}$$

□

Lemma 5.11. *Let P_r be defined by* (5.9), *then*

$$\nabla P_r(\cdot,\zeta)(x) = \omega_{m-1}^{-1}\frac{-2|x-\zeta|^2 x - (r^2-|x|^2)m(x-\zeta)}{r|x-\zeta|^{m+2}} . \tag{5.12}$$

Proof. Let us first compute

$$\nabla |\cdot-\zeta|^m(x) = \nabla\left((|\cdot-\zeta|^2)^{m/2}\right)(x) = \frac{m}{2}|x-\zeta|^{2(m/2-1)}\,2(x-\zeta).$$

Therefore the derivative of P_r w.r.t. the first variable can be computed according to

$$\begin{aligned}\nabla P_r(\cdot,\zeta)(x) &= \nabla\left(\omega_{m-1}^{-1}\frac{r^2-|\cdot|^2}{r|\cdot-\zeta|^m}\right)(x) \\ &= \omega_{m-1}^{-1}\frac{-2r|x-\zeta|^m x - (r^2-|x|^2)rm|x-\zeta|^{m-2}(x-\zeta)}{r^2|x-\zeta|^{2m}} \\ &= \omega_{m-1}^{-1}\frac{-2|x-\zeta|^2 x - (r^2-|x|^2)m(x-\zeta)}{r|x-\zeta|^{m+2}} .\end{aligned}$$

□

Before we get to deriving a pointwise estimate for the logarithmic derivative, we observe that $u = \log|F|$ cannot be differentiable at the zeros of F.

Proposition 5.12. *There exists a constant c which depends on d, α and β (but not on F and r!) such that for all $r > 0$ and for all $|x| < r/2$ with $F(x) \neq 0$ the following inequality holds true.*

$$|\nabla\log|F|(x)| \le c\left(r^{\beta-1} + \int_{B_r}|x-\xi|^{-2d+1}\,d\mu(\xi)\right) . \tag{5.13}$$

Proof. In the case where F vanishes identically there is nothing to show. If F does not vanish identically it follows from the growth assumption (5.1) that $F(0) \neq 0$. Now since the logarithmic derivative is invariant under multiplication of the function by a nonzero constant we may as well normalize, i.e., assume that $|F(0)| = 1$.

We consider now a fixed x with $F(x) \neq 0$ and $|x| < r/2$. Then, differentiating Poisson-Jensen's representation formula gives

$$\begin{aligned}\nabla u(x) &= -\int\limits_{B_r} \nabla_x K_r(x,\xi)\, d\mu(\xi) + \int\limits_{\partial B_r} \nabla_x P_r(x,\zeta) u(\zeta)\, d\sigma(\zeta) \\ &=: v_1(x) + v_2(x),\end{aligned} \tag{5.14}$$

where integration of the vector-valued integrands is to be understood componentwise. We will justify in a moment that integration and differentiation can indeed be interchanged. After that the functions v_1 and v_2 will be estimated one after another.

Estimate for the kernel of v_2: First we argue for the second integral in (5.14). The function $u = \log|F|$ is clearly bounded from above on any compact set since F is continuous. Since $F(0) \neq 0$ it follows from the the mean value property for subharmonic functions that

$$-\infty < \log|F(0)| = u(0) \leq \omega_{2d-1}^{-1} r^{-2d+1} \int_{\partial B_r} u(\zeta)\, d\sigma(\zeta).$$

Thus u is absolutely integrable on ∂B_r w.r.t. σ for all $r > 0$
Next, by inspecting (5.12), observe that $\nabla_x P_r(x,\zeta)$ is continuous on $B_{r/2} \times \partial B_r$ and thus there exists a constant $c' > 0$ such that

$$|\nabla_x P_r(x,\zeta)| \leq c', \quad \forall (x,\zeta) \in B_{r/2} \times B_r.$$

Consequently $c'\,|u(\zeta)|$ is an integrable majorant of the integrand which justifies interchanging integration and differentiaton for the second term. Note that a closer examinations gives that

$$|\nabla_x P_r(x,\zeta)| \lesssim_m r^{-m}, \qquad \forall (x,\zeta) \in B_{r/2} \times \partial B_r, \tag{5.15}$$

an estimate which we shall require later on.

Estimate for the kernel of v_1: We focus now on the first integral expression on the right hand side of (5.14). Note that by continuity we can find $\delta > 0$ such that F does not vanish on $B_\delta(x)$ and such that $B_\delta(x) \subset B_{r/2}$. It suffices to prove that

$$h := \sup_{y \in B_{\delta/2}(x)} |\nabla_y K_r(y,\cdot)| \in L^1(B_r, d\mu).$$

We claim that $\mu(B_\delta(x)) = 0$. Observe that $u = \log|F|$ is harmonic in $B_\delta(x)$ since F does not vanish by the choice of δ. By the uniqueness property of the Riesz measure we can deduce that $\mu|_{B_\delta(x)}$ is the Riesz measure of $u|_{B_\delta(x)}$. Let $\tilde{\mu}$ denote the zero measure on $B_\delta(x)$. Now $\tilde{\mu}$ obviously satisfies the defining property of the Riesz measure, Equation (5.6). Again, by uniqueness of the Riesz measure, we obtain that

$$\mu|_{B_\delta(x)} = \tilde{\mu} = 0.$$

For $y \in B_{r/2}$ and $|\xi| < r$ we estimate using Lemma 5.10

$$\begin{aligned}|\nabla_y K_r(y,\xi)| &\lesssim_m (r/|\xi|)^{m-2}\, |y - \xi'|^{-m+1} + |y-\xi|^{-m+1} \\ &\lesssim_m r^{-1}\, |\xi|^{-m+2} + |y-\xi|^{-m+1}\end{aligned} \tag{5.16}$$

where we used that $|\xi' - y| > r/2$. For $\xi \in B_r \setminus B_\delta(x)$ and $y \in B_{\delta/2}(x)$ we get that $|y - \xi| > \delta/2$ and therefore that

$$h(\xi) \lesssim_m r^{-1} |\xi|^{-m+2} + \delta^{-m+1}, \quad \xi \in B_r \setminus B_\delta(x).$$

Now, since μ is supported on the complement of $B_\delta(x)$ we obtain

$$\begin{aligned}
\int_{B_r} h(\xi)\, d\mu(\xi) &= \int_{B_r \setminus B_\delta(x)} h(\xi)\, d\mu(\xi) \\
&\lesssim_m \int_{B_r} r^{-1} |\xi|^{-m+2} + \delta^{-m+1}\, d\mu(\xi) \\
&\leq r^{-1}\nu(r) + \delta^{-m+1}\mu(B_r)\,,
\end{aligned}$$

which is finite according to Proposition 5.9.

Bound for v_1: Using (5.16) and Proposition 5.9 we obtain

$$\begin{aligned}
|v_1(x)| &\leq \int_{B_r} |\nabla K_r(x,\xi)|\, d\mu(\xi) \\
&\lesssim_m \int_{B_r} r^{-1} |\xi|^{-m+2} + |x - \xi|^{-m+1}\, d\mu(\xi) \\
&\leq r^{-1}\nu(r) + \int_{B_r} |x - \xi|^{-m+1}\, d\mu(\xi) \\
&\lesssim_{m,\alpha,\beta} r^{\beta-1} + \int_{B_r} |x - \xi|^{-m+1}\, d\mu(\xi)\,.
\end{aligned} \tag{5.17}$$

Bound for v_2: By (5.15) we have that

$$|v_2(x)| \leq \int_{\partial B_r} |\nabla P_r(x,\zeta)|\, |u(\zeta)|\, d\sigma(\zeta) \lesssim_m r^{-m} \int_{\partial B_r} |u(\zeta)|\, d\sigma(\zeta)\,. \tag{5.18}$$

For all $r > 0$ we have that the number $\tau(r) := \sup_{|\zeta|=r} |F(\zeta)|$ is positive, due to the maximum modulus principle (recall F is assumed not to vanish identically). Moreover by the growth assumption (5.1) we have that

$$\log \tau(r) \leq \log |F(0)| + \alpha r^\beta = \alpha r^\beta.$$

By making use of triangle inequality we further estimate

$$\begin{aligned}
\int_{\partial B_r} |u(\zeta)|\, d\sigma(\zeta) &\leq \int_{\partial B_r} |\log \tau(r) - \log |F(\zeta)||\, d\sigma(\zeta) + \int_{\partial B_r} \log \tau(r)\, d\sigma(\zeta) \\
&= \int_{\partial B_r} \log \tau(r) - \log |F(\zeta)|\, d\sigma(\zeta) + \int_{\partial B_r} \log \tau(r)\, d\sigma(\zeta) \\
&= 2\int_{\partial B_r} \log \tau(r)\, d\sigma(\zeta) - \int_{\partial B_r} \log |F(\zeta)|\, d\sigma(\zeta) \\
&\lesssim_m \alpha r^\beta r^{m-1} - \int_{\partial B_r} \log |F(\zeta)|\, d\sigma(\zeta)
\end{aligned} \tag{5.19}$$

Again, by the mean value property for subharmonic functions we have that

$$0 = \log |F(0)| \leq \sigma(\partial B_r)^{-1} \int_{\partial B_r} \log |F(\zeta)|\, d\sigma(\zeta),$$

and thus that the final integral in (5.19) is nonnegative. Therefore combining (5.18) and (5.19) yields that

$$|v_2(x)| \lesssim_m \alpha r^{\beta-1}. \tag{5.20}$$

The desired estimate, Inequality (5.13) follows now from (5.14), (5.17) and (5.20). □

5.2.2 Proof of the main result of this chapter

We are now well-equipped to derive uniform bounds of the L^p-norm of the logarithmic derivative.

Proof of Theorem 5.2. Let us remind ourselves that we want to control the L^p-norm of $\nabla \log |F|$ on balls centered at the origin. Recall, that in Proposition 5.12 we derived the pointwise bound

$$|\nabla \log |F|(x)| \lesssim_{m,\alpha,\beta} r^{\beta-1} + \int_{B_r} |x-\xi|^{-2d+1} \, d\mu(\xi)$$

for $|x| < r/2$ and $F(x) \neq 0$. Since the zero set of a holomorphic function that does not vanish identically is of measure zero we have by triangle inequality that

$$\|\nabla \log |F|\|_{L^p(B_{r/2})} \lesssim_{d,\alpha,\beta} r^{\beta-1} \|x \mapsto 1\|_{L^p(B_{r/2})} + \left\| \int_{B_r} |\cdot - \xi|^{-2d+1} \, d\mu(\xi) \right\|_{L^p(B_{r/2})} . \tag{5.21}$$

We continue to estimate the second term on the right hand side. By making use of Jensen's inequality and changing order of integration we have that

$$\begin{aligned} \left\| \int_{B_r} |\cdot - \xi|^{-2d+1} \, d\mu(\xi) \right\|_{L^p(B_{r/2})}^p &= \int_{B_{r/2}} \left| \int_{B_r} |x-\xi|^{-2d+1} \, d\mu(\xi) \right|^p dx \\ &= \mu(B_r)^p \int_{B_{r/2}} \left| \int_{B_r} |x-\xi|^{-2d+1} \, \frac{d\mu(\xi)}{\mu(B_r)} \right|^p dx \\ &\leq \mu(B_r)^{p-1} \int_{B_r} \int_{B_{r/2}} |x-\xi|^{-(2d-1)p} \, dx \, d\mu(\xi) \, . \end{aligned} \tag{5.22}$$

We further estimate the inner integral: Observe that for $|\xi| < r$ and $|x| < r/2$ we have that $B_{2r}(\xi) \supset B_{r/2}$ and henceforth that

$$\int_{B_{r/2}} |x-\xi|^{-(2d-1)p} \, dx \leq \int_{B_{2r}(\xi)} |x-\xi|^{-(2d-1)p} \, dx = \int_{B_{2r}} |x|^{-(2d-1)p} \, dx. \tag{5.23}$$

Note that by assumption we have that $p < 1 + 1/(2d-1)$, or equivalently that

$$(p-1)(2d-1) < 1. \tag{5.24}$$

This is precisely the condition for which the integral of $x \mapsto |x|^{-(2d-1)p}$ converges: Changing to spherical coordinates gives

$$\int_{B_{2r}} |x|^{-(2d-1)p} \, dx \asymp_d \int_0^{2r} s^{-(p-1)(2d-1)} \, ds = \frac{(2r)^{1-(p-1)(2d-1)}}{1-(p-1)(2d-1)} \, . \tag{5.25}$$

Combining (5.22),(5.23) and (5.25) yields

$$\begin{aligned} \left\| \int_{B_r} |\cdot - \xi|^{-2d+1} \, d\mu(\xi) \right\|_{L^p(B_{r/2})}^p &\lesssim_{d,p} r^{1-(p-1)(2d-1)} \mu(B_r)^p \\ &\lesssim_{\alpha,\beta} r^{1+p(\beta+2d-2)}, \end{aligned} \tag{5.26}$$

where we used that $r > 1$, Proposition 5.9 and (5.24) for the final estimate.
Finally since $\|1\|_{L^p(B_{r/2})} \asymp_{d,p} r^{2d/p}$ it follows from (5.21) and (5.26) that

$$\|\nabla \log |F|\|_{L^p(B_{r/2})} \lesssim_{d,p,\alpha,\beta} r^{2d+\beta-1},$$

which concludes the proof. □

5.3 Implications for Gabor phase retrieval

In the following we shall see that the bounds we derived in the preceeding section enable us to turn the main result of the preceeding chapter, Theorem 4.12 in a 'proper' stability result by absorbing the logarithmic derivative in a weight which grows like a lower order polynomial.
To that end we introduce modulations spaces.

Definition 5.13. For $1 \leq p \leq \infty$ the *modulation* spaces are defined by

$$\mathcal{M}^p := \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \mathcal{G}f \in L^p(\mathbb{R}^{2d}) \right\},$$

with induced norm $\|f\|_{\mathcal{M}^p} := \|\mathcal{G}f\|_{L^p(\mathbb{R}^2d)}$.
We shall also define a local notion of the modulation space norm: For a domain $D \subset \mathbb{R}^{2d}$ let

$$\|f\|_{\mathcal{M}^p(D)} := \|\mathcal{G}f\|_{L^p(D)}.$$

We turn now towards the logarithmic derivative of the spectrogram $|\mathcal{G}f|$.

Proposition 5.14. *Let $D \subset \mathbb{R}^{2d}$ and let $1 \leq p < 2d/(2d-1)$ and $q > p/(1 - p\frac{2d-1}{2d})$. Suppose that $f \in \mathcal{M}^\infty$ such that $|\mathcal{G}f|$ has a global maximum at z_0. Then there exists a constant c which only depends on d, p and q such that for all measurable functions H it holds that*

$$\|(\nabla \log |\mathcal{G}f|)H\|_{L^p(\Omega)} \leq c \left\|(1 + |\cdot - z_0|)^{2d+2} H\right\|_{L^q(\Omega)}.$$

Proof. We assume w.l.o.g. that $\Omega = \mathbb{R}^{2d}$ and that $z_0 = 0$(otherwise translate and modulate f). The proof is split up into two parts: First we derive uniform bounds of the norm of the logarithmic derivative on balls centered at the origin. In the second part the logarithmic derivative is absorbed in a polynomial weight by making use of Hölder's inequality and Part 1.
Part 1: Let us set $G(x, y) := \mathcal{G}f(x, -y)$. From the assumptions that $f \in \mathcal{M}^\infty$ and that $|\mathcal{G}f|$ has a maximum at the origin it follows that the Bargman transform $\mathcal{B}f$ satisfies

$$|\mathcal{B}f(z)| \leq |\mathcal{B}f(0)| \, e^{\frac{\pi}{2}|z|^2},$$

Thus $F := \mathcal{B}f$ satisfies the assumptions of Theorem 5.2 with $\alpha = \pi/2$ and $\beta = 2$. Moreover it holds that

$$|\nabla \log |G|| \leq |\nabla \log |F|| + \left|\nabla \log e^{\frac{\pi}{2}|\cdot|^2}\right| = |\nabla \log |F|| + \pi \, |\cdot| \, .$$

Applying Theorem 5.2 gives for all $r > 1$ and $1 \leq s < 1 + 1/(2d-1)$ that

$$\|\nabla \log |G|\|_{L^s(B_r)} \leq \|\nabla \log |F|\|_{L^s(B_r)} + \pi \, \|\cdot\|_{L^s(B_r)} \lesssim_{d,s} r^{2d+1} \tag{5.27}$$

Part 2: We define s by the equation $1/p = 1/q + 1/s$. One can elementary verify that the assumptions on p and q imply that $1 \leq s < 1 + 1/(2d-1)$. Thus the L^s-norm of the logarithmic derivative can be bounded as in Part 1, see Equation (5.27).
Let $D_0 := B_1$ and $D_j := B_{2^j} \setminus B_{2^{j-1}}$ for $j \in \mathbb{N}$. Then we have that

$$\|\nabla \log |G| \cdot H\|^p_{L^p(\mathbb{R}^{2d})} = \sum_{j \geq 0} \|\nabla \log |G| \cdot H\|^p_{L^p(D_j)}.$$

We apply now Hölder's inequality on every D_j to obtain

$$\|\nabla \log |\mathcal{G}f| \cdot H\|_{L^p(D_j)} \leq \|\nabla \log |\mathcal{G}f|\|_{L^s(D_j)} \|H\|_{L^q(D_j)} \lesssim_{d,p,q} 2^{j(2d+1)} \|H\|_{L^q(D_j)},$$

where we used Estimate (5.27) from Part 1. Let $r := q/p > 1$ and r' its Hölder conjugate, i.e., $1/r + 1/r' = 1$. By applying Hölder's inequality for sums we estimate further

$$\begin{aligned}
\|\nabla \log|G| \cdot H\|^p_{L^p(\mathbb{R}^{2d})} &\lesssim_{d,p,q} \sum_{j\geq 0} 2^{j(2d+1)p} \|H\|^p_{L^q(D_j)} \\
&= \sum_{j\geq 0} 2^{-j/r'} \cdot 2^{j(2dp+p+1/r')} \|H\|^p_{L^q(D_j)} \\
&\leq \left(\sum_{j\geq 0} 2^{-j}\right)^{1/r'} \cdot \left(\sum_{j\geq 0} 2^{j(2dp+p+1/r')r} \|H\|^q_{L^q(D_j)}\right)^{p/q}.
\end{aligned}$$

The first factor is a finite constant depending on r' and therefore ultimately on p and q only. The second factor is estimated in the following way:

$$\begin{aligned}
\sum_{j\geq 0} 2^{j(2dp+p+1/r')r} \|H\|^q_{L^q(D_j)} &\leq \sum_{j\geq 0} \int_{D_j} 2^{j(2d+2)pr} |H(z)|^q \, dz \\
&= \sum_{j\geq 0} \int_{D_j} 2^{j(2d+2)q} |H(z)|^q \, dz \\
&\lesssim \|(1 + |\cdot|^{2d+2})H\|^q_{L^q(\mathbb{R}^{2d})},
\end{aligned}$$

where we used that for $z \in D_j$ it holds that $2^{j(2d+2)} \lesssim 1 + |z|^{2d+2}$. Thus we get that

$$\|\nabla \log|G| \cdot H\|_{L^p(\mathbb{R}^{2d})} \lesssim_{d,p,q} \|(1 + |\cdot|^{2d+2})H\|_{L^q(\mathbb{R}^{2d})}.$$

To obtain the desired estimate for the spectrogram set $H^*(x, y) := H(x, -y)$; then by change of variables we get that

$$\begin{aligned}
\|\nabla \log|\mathcal{G}f| \cdot H\|_{L^p(\mathbb{R}^{2d})} &= \|\nabla \log|G| \cdot H^*\|_{L^p(\mathbb{R}^{2d})} \\
&\lesssim_{d,p,q} \|(1 + |\cdot|^{2d+2})H^*\|_{L^q(\mathbb{R}^{2d})} \\
&= \|(1 + |\cdot|^{2d+2})H\|_{L^q(\mathbb{R}^{2d})}.
\end{aligned}$$

□

Now we pick up on the stability result of the preceeding chapter, Theorem 4.12, again to obtain the following improvement.

Theorem 5.15. *Let $D \subset \mathbb{R}^{2d}$ be a domain and let $1 \leq p < 2d/(2d-1)$ and $q > p/(1 - p\frac{2d-1}{2d})$. Suppose $f \in \mathcal{M}^p$.*
Then there exists a finite constant c depending on d, p and q (independent from f and D!) such that

$$\begin{aligned}
\inf_{|c|=1} \|g - cf\|_{\mathcal{M}^p(D)} &\leq c(1 + C_P^{\mathcal{M}}(D, |\mathcal{G}f|^p, p)) \\
&\quad \cdot \left(\||\mathcal{G}g| - |\mathcal{G}f|\|_{W^{1,p}(D)} + \left\|(1 + |\cdot - z_0|)^{2d+2}(|\mathcal{G}g| - |\mathcal{G}f|)\right\|_{L^q(D)} \right) \quad \forall g \in \mathcal{M}^p,
\end{aligned}$$

where z_0 denotes a global maximum of $|\mathcal{G}f|$.

Proof. First note that for all p we have that $\mathcal{M}^p \subset \mathcal{M}^\infty$, due to $\mathcal{G}f$ being continuous. The statement now follows from Theorem 4.12 with $w = 1$ by applying Proposition 5.4 for $H = |\mathcal{G}g| - |\mathcal{G}f|$. □

Chapter 6

Weighted Poincaré inequalities

This chapter is devoted to the study of weighted Poincaré inequalities. The main emphasis lies on establishing quantitative estimates for Poincaré constants in order to understand their qualitative behaviour.
The underlying motivation of course comes from the preceeding analysis of phase retrieval from phaseless Gabor measurements. To recap, Theorem 5.15 can informally be summarized as follows:

> Given $D \subset \mathbb{R}^{2d}$ we denote $\mathcal{L} := \{\mathcal{G}g|_D : g \in \mathcal{M}^p\}$. Provided that the Poincaré constant $C_P^{\mathcal{M}}(D, |\mathcal{G}f|^p, p)$ is of moderate size, we have that the function $\mathcal{G}f|_D$ is stabily determined by its modulus within $\mathcal{L}$ (up to a unimodular factor).

Thus, understanding the behaviour of the Poincaré constant is of fundamental interest in view of comprehending the stability properties of the Gabor phase retrieval problem.

This chapter will be organized as follows. In Section 6.1 we describe the setting and give an overview on related results. In Section 6.2 we define the function spaces we shall consider and collect some properties which we will require later. In Section 6.3 the Poincaré constant is related to an isoperimetric quantity, the so-called Cheeger constant. Implications for Gabor phase retrieval are discussed in the subsequent Section 6.4. In Section 6.5 the Poincaré constant w.r.t. a weight is related to unweighted Poincaré constants on certain domains, a concept that is in general better understood.

6.1 Introduction

6.1.1 Setup

In Section 4.5 we have defined weighted Poincaré inequalities and constants with the underlying function space consisting exclusively of meromorphic functions, cf. Definition 4.11. In order to treat the topic in greater generality we work in this chapter with the following notion.

Definition 6.1. Let $D \subset \mathbb{R}^m$ be a domain and $\mathcal{L}$ a space of differentiable (at least in a weak sense) functions on D. Let $1 \le p, q < \infty$ and let w, v be two continuous and nonnegative weights on D.
The Poincaré constant $C_{v,w}^{q,p}(D, \mathcal{L})$ is defined as the smallest number $C > 0$ such that

$$\|f\|_{L^q(D,v)} \le C \|\nabla f\|_{L^p(D,w)} \qquad \forall f \in \mathcal{L}. \tag{6.1}$$

In case $q = p$ and $v = w$ we just write $C_w^p(D, \mathcal{L})$ instead. If $w = v = const.$ we drop the reference to the weight and write $C^{q,p}(D, \mathcal{L})$, or $C^p(D, \mathcal{L})$ if $q = p$.

Note that, excluding trivial examples, the Poincaré constant can only be finite if the zero function is the only constant function residing in the space $\mathcal{L}$. In the literature there are two classical choices for $\mathcal{L}$:

(a) vanishing on the boundary: $f|_{\partial D} = 0$,[1] $\quad \forall f \in \mathcal{L}$.

(b) vanishing mean: $\int_D f w \, dx = 0, \quad \forall f \in \mathcal{L}$.

Poincaré type inequalities play an important role in the theory of PDEs. For instance, for elliptic problems existence results as well as a-apriori estimates of solutions can be given in terms of certain weighted Poincaré constants. In the context of elliptic PDEs (a) and (b) correspond to the type of boundary conditions, i.e., Dirichlet or Neumann respectively. For an excellent introduction to this topic we refer to the book by Evans [Eva10].

To include the function space in the definition of the Poincaré constant gives great flexibility into two directions. Firstly, it allows to incorporate any type of restriction, way more general than (a) and (b). However, we will only consider restriction of type (b) here. Secondly, one can specify the type of regularity of the functions considered, i.e., in which sense they are differentiable.

Remark 6.2. When deriving Poincaré type estimates a typical approach is to start with nice functions, such as $\mathcal{L} = C^\infty(D)$. Suppose a Poincaré inequality can be established for $\mathcal{L}$ then by a density argument the inequality (6.1) extends to $\mathrm{cl}(\mathcal{L})$ which denotes the closure of $\mathcal{L}$ in an appropriate weighted Sobolev space.
While in the unweighted case $v = w = 1$ and if $q = p < \infty$ it is always true that smooth functions are dense in the Sobolev space according to Meyers and Serrin's famous "$H = W$" paper [MS64], this is in general not true anymore if weighted spaces are considered, cf. [CPSC94].

Recall that we are particularly interested in weighted Poincaré estimates for meromorphic functions. Since in the literature typically smooth functions are considered, such results are not necessarily applicable in our situation due to the phenomenon outlined in Remark 6.2.

Recall that in Section 4.5 for a Poincaré inequality to hold we required that

$$\inf_c \|f - c\|_{L^p(D,w)} \lesssim \|\nabla f\|_{L^p(D,w)}$$

where the implicit constant is independent from f. The following lemma states that assuming vanishing mean is essentially an equivalent concept. A proof is given in [DK13].

Lemma 6.3. *Suppose that w is nonnegative and integrable on D. Then for all real-valued $f \in L^p(D, w)$ it holds that*

$$\inf_{c \in \mathbb{R}} \|f - c\|_{L^p(D,w)} \leq \|f - f_{D,w}\|_{L^p(D,w)} \leq 2 \inf_{c \in \mathbb{R}} \|f - c\|_{L^p(D,w)},$$

where $f_{D,w} := \left(\int_D f w \, dx / \int_D w \, dx \right)$.

6.1.2 Related results

The subject of Poincaré inequalities is a huge topic and has been studied in various fields such as PDEs, probability theory and spectral geometry. In the present paragraph we summarize an incomplete selection of results on the subject.

[1] Note that this also requires that functions $f \in \mathcal{L}$ can be evaluated on the boundary of D in a meaningful way.

Within the following paragraph we shall, for $q, p \geq 1$ and w a weight on D, use the notation $C_w^{q,p}(D) := C_w^{q,p}(D, \mathcal{L}_w)$, where $\mathcal{L}_w$ denotes the space of smooth functions on D satisfying the vanishing mean condintion (b). If w is constant the reference to the weight is again dropped and if $q = p$ we just write $C_w^p(D)$ instead.

In the unweighted case $v = w = 1$ a Poincaré inequality is known to hold, i.e., $C^{q,p}(D) < +\infty$, under very general conditions, provided that D is bounded with regular boundary (Lipschitz boundary suffices): In case $q = p$ this is due to the *Deny-Lions* lemma [DL54], the case $p < d$ and $q \leq dp/(d-p)$ is proved in [Boj88].
Explicit upper bounds of the Poincaré constant are only known under rather specific assumptions on the geometry of the domain, for instance if D is convex it holds that

$$C^p(D) \leq \begin{cases} \operatorname{diam}(D)/2, & p = 1, \\ \operatorname{diam}(D)/\pi_p & p \geq 2, \end{cases}$$

where $\pi_p := 2\pi(p-1)^{1/p}/(p\sin(\pi/p))$. The bound for $p = 2$ was first established in [PW60]. The extension to the cases $p = 1$ and $p \geq 2$ were achieved in [AD04] and [ENT13]. Moreover, explicit upper bounds are known in the case of starshaped domains [FR16].

Let us now pass over to Poincaré constants with weights. A widely studied class of weights are so called *Muckenhoupt* weights, for which Poincaré inequalities were shown to hold in [FKS82]. A class of weights that has experienced extensive interest in probability theory are *log-concave* weights, i.e., weights of the form e^{-V}, where V is a convex function. Bobkov [Bob99] proved that for log-concave probability measures w on the full space the Poincaré constant $C_w^2(\mathbb{R}^n)$ can be controlled in terms of the quantity

$$\|x - x_0\|^2_{L^2(\mathbb{R}^n, w)} \tag{6.2}$$

where $x_0 = \int x w(x)\, dx$ denotes the barycenter of $w(x)dx$. The bound in (6.2) captures how far the measure is spread out. Thus if the log-concave weight w is strongly concentrated around its barycenter the Poincaré constant is rather small.
In the onedimensional situation a bound on the Poincaré constant for arbitrary p, q and w is provided in [CW00]. Computing the bound amounts to determining the supremum of a possibly complicated expression and therefore may not be feasible depending on w. In [CW06] the same authors give explicit bounds on the Poincaré constant when $q \leq p$, D is bounded and convex and w is a positive power of a concave function.

6.2 Preliminaries on weighted BV spaces

There are many relaxed notions of differentiability. We will discuss here one in particular, namely functions of bounded variations. Functions of bounded variation are classical spaces and widely known, see for example [EG15] for an introduction.
However, their weighted versions seem to be less understood and there are quite some subtle issues that appear if, for example, the weight has zeros. Throughout this section D denotes a domain in $\mathbb{R}^n$ and w a weight on D.
Recall that the Sobolev space, denoted by $W^{1,p}(D, w)$ consists of all weakly differentiable functions u on D of finite Sobolev norm

$$\|u\|_{W^{1,p}(D,w)} := \left(\|u\|^p_{L^p(D,w)} + \|\nabla u\|^p_{L^p(D,w)}\right)^{1/p}.$$

Note that we do not claim that $W^{1,p}(D, w)$ is complete. Indeed this is not the case in general, see [KO84].

The space $H^{1,p}(D,w)$ is defined as the closure of $C^\infty(D)$ w.r.t. the Sobolev norm $\|\cdot\|_{W^{1,p}(D,w)}$.

We shall say a real-valued function $u \in L^1(D,w)$ is of bounded variation w.r.t. w if it has finite total variation $V_w(u,D)$, where

$$V_w(u,D) := \sup_{\substack{\varphi \in Lip_c(D,\mathbb{R}^n) \\ |\varphi| \leq w}} \int_D u \operatorname{div} \varphi \, dx,$$

with $Lip_c(D,\mathbb{R}^n)$ denoting the space of vector-valued Lipschitz continuous functions with compact support in D. The space of all such functions u will be denoted by $BV(D,w)$. For $u \in BV(D,w)$ we will write

$$\|u\|_{BV(D,w)} := \|u\|_{L^1(D,w)} + V_w(u,D).$$

The weighted spaces $BV(D,w)$ were introduced by Baldi [Bal01] and have also been investigated in detail by Camfield [Cam08, Cam10].

Furthermore, we shall define the *weighted perimeter* of a set $E \subset D$ by

$$P_w(E,D) := V_w(\chi_E, D).$$

We will denote the space of functions that are locally Lipschitz on D by $Lip_{loc}(D)$.
For nice functions and sets the total variation and the weighted perimeter, respectively, measure familiar quantities.

Proposition 6.4. *Let $D \subset \mathbb{R}^n$ with a weight $w \in L^1(D) \cap Lip_{loc}(D)$. Then for all real-valued $u \in Lip_{loc}(D)$ with $\|\nabla u\|_{L^1(D,w)} < +\infty$ we have that*

$$V_w(u,D) = \|\nabla u\|_{L^1(D,w)}.$$

Proof. For arbitrary $\varphi \in Lip_c(D,\mathbb{R}^n)$ with $|\varphi| \leq w$ it holds that

$$\int_D u \operatorname{div} \varphi \, dx = -\int_D \nabla u \cdot \varphi \, dx \leq \int_D |\nabla u| \, w \, dx,$$

and therefore $V_w(u,D) \leq \int_D |\nabla u| \, w \, dx$.
For the reverse inequality let $\varepsilon > 0$ be arbitrary. Choose $K \subset D$ compact such that

$$\|\nabla u\|_{L^1(D\setminus K,w)} \leq \varepsilon. \tag{6.3}$$

Define a vector-valued function $v : D \to \mathbb{R}^n$ by

$$v(x) := \begin{cases} -\frac{\nabla u(x)}{|\nabla u(x)|} w(x), & \text{if } |\nabla u(x)| \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Since $w \in L^1(D)$ we have that also $v \in L^1(D)$, and as such can be approximated by a vector-valued function $v_\varepsilon \in C_c^\infty(D,\mathbb{R}^n)$ with

$$\|v - v_\varepsilon\|_{L^1(D)} \leq \varepsilon / L_K, \tag{6.4}$$

where $L_K := \|\nabla u\|_{L^\infty(K)}$. Note that L_K is finite since $u \in Lip_{loc}(D)$ and due to K being compact. To obtain $\varphi \in Lip_c(D,\mathbb{R}^n)$ with $|\varphi| \leq w$ set

$$\varphi(x) := \begin{cases} v_\varepsilon(x), & |v_\varepsilon(x)| \leq w(x), \\ \frac{w(x)}{|v_\varepsilon(x)|} v_\varepsilon(x), & |v_\varepsilon(x)| > w(x). \end{cases}$$

Then, with $E := \{x \in D : |v_\varepsilon(x)| > w(x)\}$, φ satisfies

$$\begin{aligned}
\|\varphi - v\|_{L^1(D)} &\le \int_E \left| \frac{w(x)}{|v_\varepsilon(x)|} v_\varepsilon(x) - v(x) \right| dx + \|v - v_\varepsilon\|_{L^1(D)} \\
&\le \int_E \left| \frac{w(x)}{|v_\varepsilon(x)|} v_\varepsilon(x) - v_\varepsilon(x) \right| dx + 2\,\|v - v_\varepsilon\|_{L^1(D)} \\
&= \int_E \big| w(x) - |v_\varepsilon(x)| \big| \, dx + 2\,\|v - v_\varepsilon\|_{L^1(D)} \\
&\le 3\,\|v - v_\varepsilon\|_{L^1(D)} \\
&\le 3\varepsilon/L_K.
\end{aligned}$$

Thus, we get that

$$\begin{aligned}
\int_D u \operatorname{div} \varphi \, dx &= -\int_D \nabla u \cdot \varphi \, dx \\
&= -\int_D \nabla u \cdot (v + \varphi - v) \, dx \\
&= \int_D |\nabla u| \, w \, dx - \int_D \nabla u \cdot (\varphi - v) \, dx.
\end{aligned}$$

By making use of (6.3) and (6.4) the last integral can be estimated as follows

$$\begin{aligned}
\left| \int_D \nabla u \cdot (\varphi - v) \, dx \right| &\le \int_K |\nabla u| \, |\varphi - v| \, dx + 2 \int_{D \setminus K} |\nabla u| \, w \, dx \\
&\le L_K \, \|\varphi - v\|_{L^1(D)} + 2\,\|\nabla u\|_{L^1(D\setminus K, w)} \\
&\le 5\varepsilon
\end{aligned}$$

Thus we have shown that there exists a $\varphi \in Lip_c(D, \mathbb{R}^n)$ with $|\varphi| \le w$ and

$$\|\nabla u\|_{L^1(D,w)} \le \int_D u \operatorname{div} \varphi \, dx + 5\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary the assertion follows. □

Given a domain D and $E \subset D$ we will denote the reduced boundary of E in D by $\partial^* E := \partial E \cap D$. Moreover, the $(n-1)$-dimensional Hausdorff measure will be denoted by $\mathcal{H}^{n-1}$.

Proposition 6.5. *Let D be a domain with a weight $w \in Lip_{loc}(D)$. Suppose that $E \subset D$ is such that $\partial^* E$ is smooth(C^2 suffices). Then it holds that*

$$P_w(E, D) = \int_{\partial^* E} w(s) \, d\mathcal{H}^{n-1}(s).$$

In particular E is of finite weighted perimeter if and only if $w \in L^1(\partial^* E, \mathcal{H}^{n-1})$.

Proof. Let $\varphi \in Lip_c(D, \mathbb{R}^n)$ with $|\varphi| \le w$ and choose a compact $K \subset D$ with smooth boundary such that $\operatorname{supp} \varphi \subset K$. Apply the divergence theorem to obtain

$$\begin{aligned}
\int_D \chi_E \operatorname{div} \varphi \, dx &= \int_{E \cap K} \operatorname{div} \varphi \, dx \\
&= \int_{\partial(E \cap K)} \varphi \cdot \nu \, d\mathcal{H}^{n-1} \\
&= \int_{\partial E \cap K} \varphi \cdot \nu \, d\mathcal{H}^{n-1} \\
&= \int_{\partial^* E} \varphi \cdot \nu \, d\mathcal{H}^{n-1}
\end{aligned}$$

where ν denotes the outer normal unit vector. Since $|\varphi| \le w$ we obtain

$$\int_D \chi_E \operatorname{div} \varphi \, dx \le \int_{\partial^* E} |\varphi \cdot \nu| \, d\mathcal{H}^{n-1} \le \int_{\partial^* E} w \, d\mathcal{H}^{n-1}$$

and therefore that $P_w(E, D) \le \int_{\partial^* E} w \, d\mathcal{H}^{n-1}$.
Next, we prove the existence of a sequence $(\varphi_n)_{n\in\mathbb{N}} \subset Lip_c(D)$ with $|\varphi_n| \le w$ such that

$$-\varphi_n(s) \cdot \nu(s) \nearrow w(s), \qquad s \in \partial^* E.$$

Then, by monotone convergence we get that

$$\begin{aligned}
\int_{\partial^* E} w \, d\mathcal{H}^{n-1} &= \int_{\partial^* E} \lim_{n\in\mathbb{N}} (-\varphi_n \cdot \nu) \, d\mathcal{H}^{n-1} \\
&= \lim_{n\in\mathbb{N}} \int_{\partial^* E} (-\varphi_n \cdot \nu) \, d\mathcal{H}^{n-1} \\
&= \int_E \operatorname{div} \varphi_n \, dx \le P_w(E, D)
\end{aligned}$$

which yields the assertion.
To construct $(\varphi_n)_{n\in\mathbb{N}}$ choose a sequence $K_1, K_2, \ldots \subset D$ of compact sets with

$$K_n \subset K_{n+1}^\circ, \quad n \ge 1$$

where E° denotes the interior of a set E. Note that for every n the restriction $v_n := -\nu|_{\partial^* E \cap K_{n+1}}$ is Lipschitz and has values on the sphere, i.e., $|v_n| \equiv 1$. Thus v_n has a Lipschitz continuous extension to all of K_{n+1} which still will be denoted by v_n, see for examle [McS34]. By continuity there exists an open set U_n with $\partial^* E \cap K_n \subset U_n \subset K_{n+1}^\circ$ such that

$$|v_n(x)| \ge 1/2, \quad \forall x \in U_n.$$

Choose $\psi_n \in C_c^\infty(U_n)$ with $0 \le \psi_n \le 1$ such that

$$\psi_n(x) = 1, \qquad \forall x \in \partial^* E \cap K_n$$

and define

$$\varphi_n := \frac{v_n}{|v_n|} \psi_n w.$$

Clearly φ_n is compactly supported and satisfies $|\varphi_n| \le w$. Moreover we have that

$$-\varphi_n(s) \cdot \nu(s) = w(s)\psi_n(s), \qquad s \in \partial^* E,$$

which implies that $-\varphi_n \cdot \nu \nearrow w$ on $\partial^* E$.
Thus, it remains to show that φ_n is Lipschitz continuous. Since the product of bounded Lipschitz functions is again Lipschitz and since $\psi_n w$ is such a function it suffices to show that $v_n / |v_n|$ is Lipschitz on U_n: For $x, y \in U_n$ we estimate

$$\begin{aligned}
\left| \frac{v_n(x)}{|v_n(x)|} - \frac{v_n(y)}{|v_n(y)|} \right| &\le \frac{|v_n(x) - v_n(y)|}{|v_n(x)|} + |v_n(y)| \left| \frac{1}{|v_n(x)|} - \frac{1}{|v_n(y)|} \right| \\
&\le 2L_n |x - y| + \|v_n\|_{L^\infty(K_{n+1})} \frac{||v_n(x)| - |v_n(y)||}{|v_n(x)| \, |v_n(y)|} \\
&\le \left(2 + 4 \|v_n\|_{L^\infty(K_{n+1})}\right) L_n |x - y|,
\end{aligned}$$

where L_n denotes the Lipschitz constant of v_n. □

6.3 Cheeger's isoperimetric constant

In this section we relate the Poincaré constant to an isoperimetric constant. The method we pick up is due to *Cheeger* [Che69], who developed the ideas in order to study the connection between the spectral gap of the Laplacian on a Riemannian manifold and the geometry of the manifold.
The aim is to establish the results that appear in [Che69] in a general setting which covers the Poincaré constant that appears in our analysis of Gabor phase retrieval.

6.3.1 Coarea formula

The key ingredient for Cheeger's inequality is the coarea formula. In the setting of functions of bounded variations with weights the coarea formula reads as follows; a proof can be found in [Cam10].

Theorem 6.6 (Coarea formula)**.** *Let $f \in L^1_{loc}(D, w)$ be real-valued. For $t \in \mathbb{R}$ set $E_t := \{x \in D : f(x) > t\}$.*
Then it holds that

$$V_w(f, D) = V_w(f, D \setminus E) = \int_{\mathbb{R}} P_w(E_t, D)\, dt,$$

where $E = \{x \in D : w(x) = 0\}$.

6.3.2 Cheeger's constant and inequality

In the following for a set $E \subset D$ we shall use the notation $w(E) = \int_E w\, dx$.

Definition 6.7. Let $D \subset \mathbb{R}^n$ be a domain equipped with a continuous weight $w \in L^1(D)$. Then for a set $E \subset D$ we define the *Cheeger ratio* by

$$h_{D,w}(E) := \frac{P_w(E, D)}{w(E)}.$$

The *Cheeger constant* is defined as

$$\underline{h}_{D,w} := \inf_{E \in \mathcal{C}(D,w)} h_{D,w}(E),$$

where we set

$$\mathcal{C}(D, w) := \{E \subset D \,\text{open} : \partial^* E \,\text{is smooth and}\, w(E) \le w(D)/2\}.$$

Remark 6.8. If $w \in Lip_{loc}(D)$ it follows from Propostion 6.5 that we can rewrite the Cheeger constant

$$\underline{h}_{D,w} = \inf_{E \in \mathcal{C}(D,w)} \frac{\int_{\partial^* E} w\, d\mathcal{H}^{n-1}}{w(E)} = \inf_{\substack{E \subset D \text{ open} \\ \partial^* E \text{ smooth}}} \frac{\int_{\partial^* E} w\, d\mathcal{H}^{n-1}}{\min\{w(E), w(D \setminus E)\}}$$

which admits the following interpretation of the Cheeger constant: Suppose that $\underline{h}_{D,w}$ is small, then there exists a partition of the domain $D = E \cup (D \setminus E)$ of roughly equal measure such that the weight w is small along the seperating boundary of the two components. In that sense the weight w may be regarded as 'disconnected' on D.
Conversely, if w is concentrated on a single connected component, the Gaussian being a prime example there does not exist a partition that satisfies both objectives at the same time, i.e., either the numerator of the Cheeger ratio

$$\frac{\int_{\partial^* E} w\, d\mathcal{H}^{n-1}}{\min\{w(E), w(D \setminus E)\}}$$

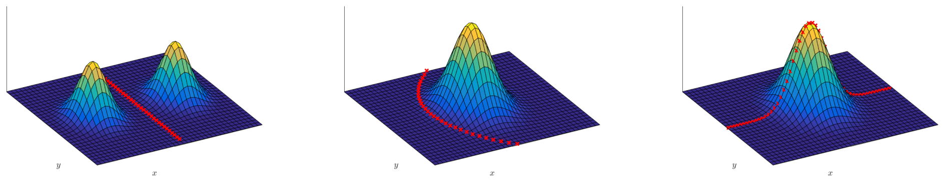


Figure 6.1: Comparison of possible partitions of the domain in the disconnected (left) and connected case (center and right).

is large or its denumerator is small, see Figure 6.1 where two concrete examples are considered. A discrete version of the Cheeger constant can be defined on (weighted) graphs and is studied in the context of spectral graph theory, where for example one seeks to find a partition of a graph which is optimal in terms of the Cheeger ratio. As in the continuous case, a close connection between Laplacian eigenvalues, Poincaré constants and Cheeger constants is encountered in the discrete, graph theoretic setting. For an introduction on spectral graph theory we refer to [Chu97].

We are ready to state the main result of the present section.

Theorem 6.9. *Let $p \geq 1$ and $D \subset \mathbb{R}^n$ be a domain equipped with a weight $w \in L^1(D) \cap Lip_{loc}(D)$. Then for all real-valued $u \in C^\infty(D) \cap L^p(D,w)$ it holds that*

$$\inf_{t\in\mathbb{R}} \|u-t\|_{L^p(D,w)} \leq p\underline{h}_{D,w}^{-1} \|\nabla u\|_{L^p(D,w)} .$$

In particular $C_w^p(D,\mathcal{L}) \leq p\underline{h}_{D,w}^{-1}$, where

$$\mathcal{L} := \{u \in H^{1,p}(D,w) : \int_D uw\, dx = 0\}.$$

Before we prove the statement we need one more auxiliary result.

Lemma 6.10. *Let $D \subset \mathbb{R}^n$ be a domain and $f \in C^\infty(D)$ real-valued. Set $E_t := \{x \in D : f(x) > t\}$ for $t \in \mathbb{R}$. Then for almost all $t \in \mathbb{R}$ it holds that $\partial^* E_t$ is smooth.*

Proof. It follows from a theorem due to Sard [Sar42] that the set

$$J := \{t \in \mathbb{R} : \exists x \in D : f(x) = t, \nabla f(x) = 0\}$$

is of measure zero.
Let $t \in \mathbb{R} \setminus J$ and define $A_t := \{x \in D : f(x) = t\}$. We claim that $\partial^* E_t = A_t$. Suppose first that $x \in \partial^* E_t$. Then in every neighborhood U of x there exist $y, y' \in U$ such that $f(y) > t$ and $f(y') \leq t$. Since f is continuous it follows that $f(x) = t$ and thus that $\partial^* E_t \subset A_t$.
Let now $x \in A_t$ and assume that $x \notin \partial^* E_t$. Then it follows that $x \notin \overline{E_t}$ and thus there exists a neighborhood U of x such that $U \cap \overline{E_t} = \emptyset$. This implies that x is a local maximum of f and therefore $\nabla f(x) = 0$ which contradicts $t \in \mathbb{R} \setminus J$. Consequently we have indeed that $\partial^* E_t = A_t$. The assertion now follows by application of the implicit function theorem on $A_t = \partial^* E_t$. $\square$

We resume to prove Cheeger's inequality.

Proof of Theorem 6.9. W.l.o.g. we may assume that $\|\nabla u\|_{L^p(D,w)} < +\infty$. Again we set $E_t := \{x \in D : u(x) > t\}$. Note that since

$$t \in \mathbb{R} \mapsto w(E_t)$$

is continuous and attains values in $(0, w(D))$ there exists t_0 such that $w(E_{t_0}) = w(D)/2$. Moreover, also $w(D \setminus E_{t_0}) = w(D)/2$. We will assume w.l.o.g. that $t_0 = 0$ to estimate

$$\inf_{t \in \mathbb{R}} \|u - t\|_{L^p(D,w)} \leq \|u - t_0\|_{L^p(D,w)} = \|u\|_{L^p(D,w)}.$$

Step 1: Let u_+ and u_- denote positive and negative parts of u, repectively. We shall first show that

$$\left\|u_\pm^p\right\|_{L^1(D,w)} \leq \underline{h}_{D,w}^{-1} \left\|\nabla u_\pm^p\right\|_{L^1(D,w)}. \tag{6.5}$$

We only prove (6.5) for u_+; for u_- one can argue similarly. To that end let F_t denote the superlevelsets of u_+^p, i.e.,

$$F_t := \{x \in D : \, u_+^p(x) > t\}.$$

Using Proposition 6.4 and the coarea formula (Theorem 6.6) and the fact that for $t < 0$ we have that $F_t = D$ and therefore $P_w(F_t, D) = 0$, it follows that

$$\begin{aligned}
\left\|\nabla u_+^p\right\|_{L^1(D,w)} &= V_w(u_+^p, D) \\
&= \int\limits_{[0,\infty)} P_w(F_t, D)\, dt \\
&\geq \int\limits_{[0,\infty)\cap\{t: w(F_t)>0\}} \frac{P_w(F_t, D)}{w(F_t)} w(F_t)\, dt \\
&= \int\limits_{[0,\infty)\cap\{t: w(F_t)>0\}} h_{D,w}(F_t) w(F_t)\, dt.
\end{aligned}$$

Since by Lemma 6.10 the reduced boundary $\partial^* F_t$ is smooth for almost all t and since for $t \geq 0$ we have that

$$w(F_t) \leq w(F_0) = w(E_0) = w(D)/2$$

we can conclude that

$$\left\|\nabla u_+^p\right\|_{L^1(D,w)} \geq \underline{h}_{D,w} \int_{[0,\infty)} w(F_t)\, dt.$$

On the other hand, by Fubini's theorem we can rewrite

$$\begin{aligned}
\left\|u_+^p\right\|_{L^1(D,w)} &= \int_D u_+^p(x) w(x)\, dx \\
&= \int_D \int_{\mathbb{R}} \chi_{[0, u_+^p(x))}(t)\, dt\, w(x)\, dx \\
&= \int_{[0,\infty)} \int_{F_t} w(x)\, dx\, dt \\
&= \int_{[0,\infty)} w(F_t)\, dt,
\end{aligned}$$

which gives (6.5).

Step 2: By making use of (6.5) we get that

$$\begin{aligned}
\|u\|_{L^p(D,w)}^p &= \|u_+\|_{L^p(D,w)}^p + \|u_-\|_{L^p(D,w)}^p \\
&= \left\|u_+^p\right\|_{L^1(D,w)} + \left\|u_-^p\right\|_{L^1(D,w)} \\
&\le \underline{h}_{D,w}^{-1} \left(\left\|\nabla u_+^p\right\|_{L^1(D,w)} + \left\|\nabla u_-^p\right\|_{L^1(D,w)} \right) \\
&= p\underline{h}_{D,w}^{-1} \left\| |u|^{p-1} \nabla u \right\|_{L^1(D,w)} \\
&\le p\underline{h}_{D,w}^{-1} \|u\|_{L^p(D,w)}^{p-1} \|\nabla u\|_{L^p(D,w)},
\end{aligned}$$

where for the last estimate we used Hölder's inequality and

$$\left\| |u|^{p-1} \right\|_{L^{p'}(D,w)} = \left\| |u|^{(p-1)p'} \right\|_{L^1(D,w)}^{1/p'} = \|u\|_{L^p(D,w)}^{p-1}.$$

Therefore, we get indeed that

$$\|u\|_{L^p(D,w)} \le p\underline{h}_{D,w}^{-1} \|\nabla u\|_{L^p(D,w)}.$$

□

So far we considered the Poincaré constant for real-valued functions only. However, in the next section we shall also require a version for meromorphic functions:

Corollary 6.11. *Let $p \ge 1$, $D \subset \mathbb{C}^d \simeq \mathbb{R}^{2d}$ be a domain and $w \in L^1(D) \cap Lip_{loc}(D)$ a nonnegative weight. Then it holds that $C_P^{\mathcal{M}}(D,w) \le 2p\underline{h}_{D,w}^{-1}$, which means that*

$$\inf_{c\in\mathbb{C}} \|f-c\|_{L^p(D,w)} \le 2p\underline{h}_{D,w}^{-1} \|\nabla f\|_{L^p(D,w)}, \qquad \forall f \in \mathcal{M}(D) \cap L^p(D,w).$$

Proof. We want to apply Theorem 6.9 to both u and v, where $f = u + iv$. In order to do this let E denote the set of singular points of f, which is a complex analytic set of dimension $d-1$ (except for the trivial case $f \equiv 0$). Moreover $u, v \in C^\infty(D \setminus E) \cap L^p(D \setminus E, w)$ which allows to apply Theorem 6.9.

$$\begin{aligned}
\inf_{c\in\mathbb{C}} \|f-c\|_{L^p(D,w)} &= \inf_{c\in\mathbb{C}} \left(\int_D |u - \operatorname{Re} c|^p \, w \, dx + \int_D |v - \operatorname{Im} c|^p \, w \, dx \right)^{1/p} \\
&= \left(\inf_{\xi\in\mathbb{R}} \int_{D\setminus E} |u-\xi|^p \, w \, dx + \inf_{\eta\in\mathbb{R}} \int_{D\setminus E} |v-\eta|^p \, w \, dx \right)^{1/p} \\
&\le p\underline{h}_{D\setminus E,w}^{-1} \left(\|\nabla u\|_{L^p(D\setminus E,w)}^p + \|\nabla v\|_{L^p(D\setminus E,w)}^p \right)^{1/p} \\
&\le 2p\underline{h}_{D\setminus E,w}^{-1} \|\nabla f\|_{L^p(D,w)}
\end{aligned}$$

The assertion now follows from the observation that $\underline{h}_{D\setminus E,w} \ge \underline{h}_{D,w}$ since $\mathcal{C}(D \setminus E, w) \subset \mathcal{C}(D,w)$. □

6.4 Implications for Gabor phase retrieval

In the following we seek to relate the Cheeger constant to the problem of Gabor phase retrieval. By making use of Corollary 6.11 we may rewrite Theorem 5.15 as follows.

Theorem 6.12. *Let $D \subset \mathbb{R}^{2d}$ be a domain and let $1 \leq p < 2d/(2d-1)$ and $q > p/(1-p\frac{2d-1}{2d})$. Suppose $f \in \mathcal{M}^p$.*
Then there exists a finite constant c depending on d, p and q (independent from f and D!) such that

$$\inf_{|c|=1} \|g - cf\|_{\mathcal{M}^p(D)} \leq c(1 + h^{-1})$$
$$\cdot \left(\||\mathcal{G}g| - |\mathcal{G}f|\|_{W^{1,p}(D)} + \left\|(1 + |\cdot - z_0|)^{2d+2}(|\mathcal{G}g| - |\mathcal{G}f|)\right\|_{L^q(D)} \right) \quad \forall g \in \mathcal{M}^p,$$

where z_0 denotes a global maximum of $|\mathcal{G}f|$ and where $h := \underline{h}_{D,|\mathcal{G}f|^p}$ denotes the Cheeger constant, see Definition 6.7.

Theorem 6.12 may well be regarded as the main statement of this thesis. It reveals that stable recovery of $\mathcal{G}f(z)$, $z \in D$ (up to global phase factor) given $|\mathcal{G}f|$ on D is possible provided that the spectrogram $|\mathcal{G}f|$ is connected as measured in terms of the Cheeger constant.
In order to put this insight into relation let us briefly reconsider the abstract results of Section 2.2 in the context of Gabor phase retrieval. Applying Theorem 2.10 to

$$\Phi = (\pi(x,y)\varphi)_{(x,y)\in D},$$

where $\varphi = e^{-\pi|\cdot|^2}$ and π denotes the time-frequency shift, cf. Definition (3.1), and to $\mathcal{B} = \mathcal{M}^p(D)$ and $\mathfrak{B} = L^p(D)$ yields that the stability constant of Gabor phase retrieval problem can be estimated from below in terms of the reciprocal of

$$h_\Phi(f) = \inf_{h \in \mathcal{M}^p(D),\, E \subset D} \frac{\max\{\|\mathcal{G}h\|_{L^p(E)}, \|\mathcal{G}(f-h)\|_{L^p(D\setminus E)}\}}{\min\{\|h\|_{\mathcal{M}^p(D)}, \|f-h\|_{\mathcal{M}^p(D)}\}}.$$

This quantity strongly resembles the Cheeger constant

$$\underline{h}_{D,|\mathcal{G}f|^p} = \inf_{\substack{E \subset D \text{ open} \\ \partial^* E \text{ smooth}}} \frac{\int_{\partial^* E} |\mathcal{G}f|^p \, d\mathcal{H}^{n-1}}{\min\{\int_C |\mathcal{G}f|^p \, dx, \int_{D\setminus E} |\mathcal{G}f|^p \, dx\}},$$

for we expect in the disconnected case, i.e., if $\underline{h}_{D,|\mathcal{G}f|^p}$ is small that there is an underlying decomposition of f

$$f = h + (f - h)$$

such that $\mathcal{G}h$ is essentially vanishing on E and $\mathcal{G}(f-h)$ is essentially vanishing on $D \setminus E$ while both components h and $f - h$ contain a considerable portion of the $\mathcal{M}^p(D)$-norm of f. These intuitive arguments will be made more rigorous in Section 6.4.4.
Once one accepts that $h_\Phi(f)$ and $\underline{h}_{D,|\mathcal{G}f|^p}$ are indicators of one and the same qualitative behavior – (dis-)connectedness that is – the discussion above permits to informally conclude that

> **Gabor phase retrieval is stable if and only if the measurements are connected.**

An important point that has not yet been discussed is the question whether the Cheeger constant gives a finite bound of the Poincaré constant, for if $\underline{h}_{D,|\mathcal{G}f|^p} = 0$ the stability result, Theorem 6.12 is empty. In Section 6.4.2 we discuss the concrete example of $f = \varphi$ being the Gaussian. In Section 6.4.3 we shall see that the Cheeger constant $\underline{h}_{D,w}$ is indeed always positive if D is bounded and w arises from Gabor measurements. Before we study the Cheeger constant for weights arising from Gabor magnitudes we state two corollaries of Theorem 6.12.

6.4.1 Noise and multicomponent stability

In virtually any practical situation the measurements are corrupted by noise, i.e., one is faced with the problem of reconstructing f from $|\mathcal{G}f|+\gamma$ instead of $|\mathcal{G}f|$. As we will see next, Theorem 6.12 implies a stability result for reconstruction from noisy Gabor magnitudes.

Corollary 6.13. *Let $D \subset \mathbb{R}^{2d}$, let $1 \leq p < 1 + 1/(2d-1)$ and $q > p/(1 - p\frac{2d-1}{2d})$. Suppose that $f \in \mathcal{M}^p$ is such that its spectrogram $|\mathcal{G}f|$ has a global maximum at z_0. Suppose that γ is a smooth function on D and suppose that $g \in \mathcal{M}^p(\mathbb{R}^d)$ is such that*

$$\||\mathcal{G}f| + \gamma - |\mathcal{G}g|\|_{\mathcal{D}} \leq \varepsilon$$

where

$$\|F\|_{\mathcal{D}} := \|F\|_{W^{1,p}(D)} + \|(1 + |\cdot - z_0|^{2d+2})F\|_{L^q(D)}.$$

Then it holds that

$$\inf_{|a|=1} \|\mathcal{G}g - a\mathcal{G}f\|_{L^p(D)} \lesssim (1 + \underline{h}_{D,|\mathcal{G}f|^p}^{-1})(\varepsilon + \|\gamma\|_{\mathcal{D}}),$$

where the implicit constant depends on d, p and q only.

Proof. By Theorem 6.12 we have that

$$\inf_{|a|=1} \|\mathcal{G}g - a\mathcal{G}f\|_{L^p(D)} \lesssim (1 + \underline{h}_{D,|\mathcal{G}f|^p}^{-1})\||\mathcal{G}g| - |\mathcal{G}f|\|_{\mathcal{D}}.$$

The statement then follows from the estimate

$$\||\mathcal{G}g| - |\mathcal{G}f|\|_{\mathcal{D}} \leq \||\mathcal{G}g| - (|\mathcal{G}f| + \gamma)\|_{\mathcal{D}} + \|\gamma\|_{\mathcal{D}} \leq \varepsilon + \|\gamma\|_{\mathcal{D}}.$$

□

Next we discuss yet another consequence of the main stability result, Theorem 6.12, which tells us that instabilities for Gabor phase retrieval must be of disconnected type. In other words reconstruction of the Gabor transform is stable on domains D where $|\mathcal{G}f|$ is connected.

We now want to pick up on the multicomponent paradigm, which was introduced in [ADGY19]: Suppose that the phase retrieval problem is relaxed as we require no longer that $\mathcal{G}f$ is to be reconstructed up to a global phase factor but instead only demand that the phase factor is constant on each component but may take different values on different components. A component is here a subdomain D_i of D on which $|\mathcal{G}f|$ is connected, i.e., stable recovery on D_i is possible.

The multicomponent paradigm – i.e., to identify $F = \sum_{i=1}^k F_i$, where F_i is concentrated on D_i with $\sum_{i=1}^k a_i F_i$ whenever $|a_1|, \ldots, |a_k| = 1$ – is especially meaningful for applications in audio as a change of phase on individual components is usually imperceptible for the human ear.

The relaxation accomplishes that Gabor phase retrieval becomes stable. By applying Theorem 6.12 on every single component $D_i \subset D$ we obtain the following result.

Corollary 6.14. *Let $D \subset \mathbb{R}^{2d}$, let $1 \leq p < 1 + 1/(2d-1)$ and $q > p/(1 - p\frac{2d-1}{2d})$. Suppose that $f \in \mathcal{M}^p$ is such that its spectrogram $|\mathcal{G}f|$ has a global maximum at z_0. Suppose that D is partitioned into subdomains $D_1, D_2, \ldots, D_k$, i.e., $D_i \cap D_j = \emptyset$, $\forall i \neq j$ and $\bigcup_{i=1}^k \overline{D_i} = \overline{D}$. Then it holds for all $g \in \mathcal{M}^p$ that*

$$\inf_{|a_1|,\ldots,|a_k|=1} \sum_{i=1}^k \|\mathcal{G}g - a_i\mathcal{G}f\|_{L^p(D_i)}$$
$$\lesssim (1 + h_*^{-1}) \left(\||\mathcal{G}f| - |\mathcal{G}g|\|_{W^{1,p}(D)} + \|(1 + |\cdot - z_0|^{2d+2}) (|\mathcal{G}f| - |\mathcal{G}g|) \|_{L^q(D)}\right),$$

where $h_* := \min_{1 \leq i \leq k} \underline{h}_{D_i,|\mathcal{G}f|^p}$ and where the implicit constant depends on d, p and q only.

6.4.2 Global stability for the Gaussian

We shall now discuss in detail the quantity $\underline{h}_{\mathbb{R}^{2d},|\mathcal{G}\varphi|^p}$, where $\varphi = 2^{d/4}e^{-\pi|\cdot|^2}$ denotes the Gaussian. It is a simple integration exercise to calculate

$$\int_{\mathbb{R}} e^{-\pi t^2} e^{-\pi(t-x)^2} e^{-2\pi i y t}\, dt = 2^{-1/2} \exp\Big(-\frac{\pi}{2}(x^2 + 2ixy + y^2)\Big) \tag{6.6}$$

Since the Gabor transform of the d-dimensional Gaussian coincides with the tensor product of d copies of (6.6) up to the constant $2^{d/4}$ we notice that

$$|\mathcal{G}\varphi(x,y)| = 2^{-d/4} e^{-\pi/2(|x|^2+|y|^2)}. \tag{6.7}$$

Moreover, since the Cheeger ratio is invariant w.r.t. multiplication by a constant we may as well consider the weight

$$w(z) = e^{-\pi p|z|^2/2}, \qquad z \in \mathbb{R}^{2d}.$$

We will, in the following draw upon the *Gaussian isoperimetric inequality* which was established idependently in [Bor75] and [SC74]. We will require the following version due to Ehrhard [Ehr84].

Theorem 6.15 (Gaussian isoperimetric inequailty, [Ehr84])**.** *Suppose that $E \subset \mathbb{R}^n$ has smooth boundary (rectifiable suffices) then it holds that*

$$P_\gamma(E, \mathbb{R}^n) \geq I(\gamma(E)),$$

where $\gamma(x) := (2\pi)^{-n/2} e^{-|x|^2/2}$ and

$$I(s) := (2\pi)^{-1/2} \exp\left(-(\Phi^{-1}(s))^2/2\right), \quad s \in (0,1)$$

with $\Phi(x) := (2\pi)^{-1/2} \int_{-\infty}^{x} e^{-t^2/2}\, dt$.

As a consequence we get a bound for the Cheeger constant in Gauss space.

Corollary 6.16. *For γ defined as in Theorem 6.15 the Cheeger constant satisfies*

$$\underline{h}_{\mathbb{R}^n,\gamma} \geq \sqrt{2/\pi}.$$

Proof. Let Φ and I be defined as in Theorem 6.15. Since γ is normalized every $E \in \mathcal{C}(\mathbb{R}^n, \gamma)$ satisfies $0 < \gamma(E) \leq 1/2$.
Using $\Phi'(x) = (2\pi)^{-1/2} e^{-x^2/2}$ we notice that $I(s) = \Phi'(\Phi^{-1}(s))$ and that for $s \in (0,1)$

$$(\Phi^{-1})'(s) = \frac{1}{\Phi'(\Phi^{-1}(s))} = \frac{1}{I(s)} > 0.$$

Now we compute

$$I'(s) = -I(s)\Phi^{-1}(s)(\Phi^{-1})'(s) = -\frac{I(s)\Phi^{-1}(s)}{\Phi'(\Phi^{-1}(s))} = -\Phi^{-1}(s).$$

In particular we have that I is concave in $(0, 1/2]$. Note that I can be continuously extended by setting $I(0) = 0$. Theorem 6.15 implies that

$$P_\gamma(E, \mathbb{R}^n) \geq I(\gamma(E)) \geq 2I(1/2)\gamma(E) = 2(2\pi)^{-1/2}\gamma(E).$$

Since $E \in \mathcal{C}(\mathbb{R}^n, \gamma)$ was arbitrary we conclude that $\underline{h}_{\mathbb{R}^n,\gamma} \geq \sqrt{2/\pi}$. $\square$

Furthermore a bound for the Cheeger constant of $|\mathcal{G}f|^p$ can be established.

Theorem 6.17. *For $\alpha > 0$ define the weight $w := e^{-\alpha|\cdot|^2}$. Then it holds that*

$$\underline{h}_{\mathbb{R}^n,w} > \sqrt{4\alpha/\pi}.$$

Proof. We denote $T : x \mapsto (2\alpha)^{-1/2}x$, $x \in \mathbb{R}^n$. Hence

$$\gamma(x) = (2\pi)^{n/2}e^{-|x|^2/2} = (2\pi)^{n/2}w(T(x)).$$

For measurable $E \subset \mathbb{R}^n$ define

$$\tilde{E} := T^{-1}(E) = \{x \in \mathbb{R}^n : (2\alpha)^{1/2}x \in E\}.$$

By a change of variables we obtain

$$w(E) = \int_E w(x)\,dx = \int_{\tilde{E}} w(T(u))\,|\det dT(u)|\,du = (2\alpha)^{-n/2}\int_{\tilde{E}} e^{-|u|^2/2}\,du = (\pi/\alpha)^{n/2}\gamma(\tilde{E})$$

In particular we have that $E \in \mathcal{C}(\mathbb{R}^n, w)$ if and only if $\tilde{E} \in \mathcal{C}(\mathbb{R}^n, \gamma)$.
To compute the weighted perimeter use that $\partial\tilde{E} = T^{-1}(\partial E)$ and $\mathcal{H}^{n-1}(\lambda A) = \lambda^{n-1}\mathcal{H}^{n-1}(A)$ for $\lambda > 0$ (see for example [EG15, Chapter 2]). Therefore the pushforward measure $T_*\mu = \mu \circ T^{-1}$ of

$$\mu(A) := \mathcal{H}^{n-1}(A), \quad A \subset \partial E$$

coincides with $(2\alpha)^{(n-1)/2}\mathcal{H}^{n-1}$. Using again change of variables formula we get that

$$\begin{aligned}
P_\gamma(\tilde{E}, \mathbb{R}^n) &= \int_{\partial\tilde{E}} \gamma(s)\,d\mathcal{H}^{n-1}(s) \\
&= (2\pi)^{-n/2}\int_{\partial\tilde{E}} w(T(s))\,d\mathcal{H}^{n-1}(s) \\
&= (2\pi)^{-n/2}\int_{\partial E} w(s)\,d(T_*\mu)(s) \\
&= (2\pi)^{-n/2}(2\alpha)^{(n-1)/2}\int_{\partial E} w(s)\,d\mathcal{H}^{n-1}(s) \\
&= (2\pi)^{-n/2}(2\alpha)^{(n-1)/2}P_w(E, \mathbb{R}^n).
\end{aligned}$$

We can now compare the respective Cheeger ratios

$$\begin{aligned}
\underline{h}_{\mathbb{R}^n,w}(E) &= \frac{P_w(E, \mathbb{R}^n)}{w(E)} \\
&= \frac{P_\gamma(\tilde{E}, \mathbb{R}^n)}{\gamma(\tilde{E})} \cdot \frac{(2\pi)^{n/2}(2\alpha)^{-(n-1)/2}}{(\pi/\alpha)^{n/2}} \\
&= h_{\mathbb{R}^n,\gamma}(\tilde{E}) \cdot (2\alpha)^{1/2}\,,
\end{aligned}$$

and get by Corollary 6.16 that $\underline{h}_{\mathbb{R}^n,w} \geq \sqrt{4\alpha/\pi}$. $\square$

As a consequence we obtain that if $f = \varphi$ the phase of $\mathcal{G}f$ is globally stabily determined by $|\mathcal{G}f|$ (up to a unimodular factor). More precisely Theorem 6.12 and Theorem 6.17 imply the following result.

Corollary 6.18. *Let $\varphi = 2^{-d/4}e^{-\pi|\cdot|^2}$ denote the Gaussian. Let $1 \leq p < 2d/(2d-1)$ and $q > p/(1-p\frac{2d-1}{2d})$. Then there exists a finite constant c depending on d, p and q such that*

$$\inf_{|c|=1} \|g - cf\|_{\mathcal{M}^p} \leq$$
$$c \cdot \left(\||\mathcal{G}g| - |\mathcal{G}f|\|_{W^{1,p}(\mathbb{C}^d)} + \left\|(1+|\cdot|)^{2d+2}(|\mathcal{G}g| - |\mathcal{G}f|)\right\|_{L^q(\mathbb{C}^d)} \right) \quad \forall g \in \mathcal{M}^p.$$

Remark 6.19. Let us point out that one could even get a tighter bound for the Gaussian; by having a look at (6.7) we notize that $F = \mathcal{G}\varphi$ has no zeros and the logarithmic derivative is given by

$$|\nabla \log |F| (z)| = \pi |z| .$$

Therefore we can directly apply Theorem 4.12 to obtain that there exists a finite constant c such that

$$\inf_{|c|=1} \|g - c\varphi\|_{\mathcal{M}^p} \leq c \cdot \left(\|(1+|\cdot|)(|\mathcal{G}g| - |\mathcal{G}\varphi|)\|_{L^p(\mathbb{C}^d)} + \|\nabla (|\mathcal{G}g| - |\mathcal{G}\varphi|)\|_{L^p(\mathbb{C}^d)} \right)$$

for all $g \in \mathcal{M}^p$.

6.4.3 Reconstruction on finite domains

The aim of this paragraph is to establish the following result.

Theorem 6.20. *Let $p \geq 1$ and suppose that $D \subset \mathbb{C}$ is a bounded domain with Lipschitz boundary. Then for all $f \in \mathcal{S}'(\mathbb{R})$ such that $\mathcal{G}f$ does not vanish on the boundary of D it holds that*

$$\underline{h}_{D,|\mathcal{G}f|^p} > 0.$$

Remark 6.21. The statement of Theorem 6.20, together with Theorem 6.12 can be seen as a complementary result to Theorem 2.12. If the phase is to be recovered on the full space, Theorem 2.12 states that there exist functions for which the local stability constant is infinite. However, if one is only interested in local reconstruction, i.e., on a finite domain, this cannot happen in the Gabor case.

To prove the statement we will emulate the exposition presented in [ILR05]. The basic idea is to consider a variational problem, namely to minimize the functional

$$\mathcal{J}(u) := V_w(u, D), \tag{6.8}$$

over the set

$$U := \{\chi_E / w(E) : \, E \in \mathcal{C}(D, w)\}, \tag{6.9}$$

where $w = |\mathcal{G}f|^p$. In order to infer that a minimizing sequence of $\mathcal{J}$ has an accumulation point u^* in $L^1(D, w)$ we have to derive an adequate compactness result. Careful examination of u^* will then allow to conclude hat $\underline{h}_{D,|\mathcal{G}f|^p}$ is positive.
Note that in order to do so we cannot apply the respective results in [ILR05] as this would require that w is bounded from below by a positive constant, which is not the case in general since the Gabor transform $\mathcal{G}f$ may possess zeros in D.
Before we prove Theorem 6.20 we collect some preliminary material.

Some more properties of $BV(D,w)$

Following the exposition in [Cam10] we define

$$\tilde{V}_w(u,D) := \inf_{\substack{(u_k)_{k\in\mathbb{N}}\subset Lip_{loc}(D)\\ u_k\to u \text{ in} L^1(D,w)}} \left\{ \liminf_k \int_D |\nabla u_k|\, w\, dx \right\}, \quad u \in L^1_{loc}(D,w).$$

Remark 6.22. Note that if $\tilde{V}_w(u,D)$ is finite, then there exists a sequence $(u_k)_{k\in\mathbb{N}} \subset Lip_{loc}(D) \cap W^{1,1}(D,w)$ with $u_k \to u$ in $L^1(D,w)$ and such that

$$\lim_k \int_D |\nabla u_k|\, w\, dx = \tilde{V}_w(u,D).$$

The following lemma provides us with a sufficient condition under which $V_w(u,D)$ and $\tilde{V}_w(u,D)$ can be used interchangebly.

Lemma 6.23 ([Cam10, Theorem 5.3])**.** *Let $D \subset \mathbb{R}^n$ be a domain equipped with a continuous weight $w > 0$. Then for all real-valued $u \in L^1_{loc}(D,w)$ it holds that*

$$\tilde{V}_w(u,D) = V_w(u,D).$$

Under certain assumptions the zeros of the weight can be neglected:

Lemma 6.24 ([Cam10, Theorem 4.2])**.** *Let $D \subset \mathbb{R}^n$ be a domain equipped with a lower semicontinuous, nonnegative weight w. Let $E := \{x \in D :\ w(x) = 0\}$ and suppose that*

$$\lim_{\varepsilon\to 0} w(D_\varepsilon)/\varepsilon = 0 \tag{6.10}$$

where $D_\varepsilon := \{x \in D : \operatorname{dist}(x,E) < \varepsilon\}$. Then for every real-valued $u \in L^1_{loc}(u,D)$ it holds that

$$\tilde{V}_w(u,D) = \tilde{V}_w(u, D\setminus E).$$

Moreover we will require that $V_w(\cdot, D)$ is lower semicontinuous.

Lemma 6.25 ([Cam10, Prop. 2.3], [Bal01, Thm. 3.2])**.** *Let $(u_k)_{k\in\mathbb{N}} \subset BV(D,w)$ such that $u_k \to u$ in $L^1(D,w)$. Then*

$$V_w(u,D) \le \liminf_k V_w(u_k,D).$$

Compact embeddings

We shall now establish a compactness result which will play a crucial role in the proof of the main theorem of this section.

Proposition 6.26. *Let $0 < R < 1$ and let $p \ge 1$. Let $B_R \subset \mathbb{C} \simeq \mathbb{R}^2$ denote the disc of radius R centered at the origin and*

$$w(x) := |x|^p, \quad x \in B_R.$$

Suppose $(u_n)_{n\in\mathbb{N}} \subset Lip_{loc}(B_R)$ is a sequence of real-valued functions satisfying

$$\sup_n \|u_n\|_{W^{1,1}(D,w)} < +\infty.$$

Then $(u_n)_{n\in\mathbb{N}}$ is precompact in $L^1(D,w)$, i.e., there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ and $u \in L^1(D,w)$ such that

$$\lim_k \|u_{n_k} - u\|_{L^1(D,w)} = 0.$$

In order to prove Proposition 6.26 we require the following estimate.

Lemma 6.27. *Let B_R and w be defined as in Proposition 6.26. Then for all $0 < r < R < 1$ and real-valued $f \in Lip_{loc}(B_R)$ it holds that*

$$\|f\|_{L^1(B_r,w)} \leq \frac{r}{R-r} \|f\|_{W^{1,1}(B_R,w)} .$$

Proof. Since f and $|f|$ are both Lipschitz these two functions are differentiable almost everywhere. It follows from the triangle inequality that $|\nabla|f|(x)| \leq |\nabla f(x)|$ for almost all x. Thus we may assume w.l.o.g. that f is nonnegative.
For $\theta \in [0, 2\pi)$ and $0 \leq s, t \leq R$ let $x = se^{i\theta}$ and $y = te^{i\theta}$. Then it holds that

$$s^{p+1} f(se^{i\theta}) = t^{p+1} f(te^{i\theta}) + \int_t^s (p+1)u^p f(ue^{i\theta}) + u^{p+1}\nabla f(ue^{i\theta}) \cdot (\cos\theta, \sin\theta)^T \, du.$$

Assuming additionally that $s \leq t$ it follows, since f is nonnegative that

$$s^{p+1} f(se^{i\theta}) \leq t^{p+1} f(te^{i\theta}) + \int_s^t u^{p+1} \left|\nabla f(ue^{i\theta})\right| \, du,$$

and thus that

$$s^{p+1} f(se^{i\theta}) \leq t^{p+1} f(te^{i\theta}) + \int_0^R u^{p+1} \left|\nabla f(ue^{i\theta})\right| \, du. \tag{6.11}$$

Next, integrate both sides of (6.11), i.e., apply $\int_0^r ds$, $\int_r^R dt$ and $\int_0^{2\pi} d\theta$ to obtain

$$(R-r)\|f\|_{L^1(B_r,w)} \leq r\|f\|_{L^1(B_R\setminus B_r,w)} + r(R-r)\|\nabla f\|_{L^1(B_R,w)} ,$$

which implies the claim. □

To prove Proposition 6.26 we argue as in [OK90, Lemma 17.3].

Proof of Proposition 6.26. We will use that in the unweighted case the embedding of $W^{1,1}(D)$ into $L^1(D)$ is compact provided that D is a bounded Lipschitz domain, cf. [EG15, Chapter 4.6]. For $0 < r < R$ let us define the annulus $A_{r,R} := B_R \setminus \overline{B_r}$. Since for every such r we have that there exist positive constants c_r and C_r such that

$$c_r \leq w(x) \leq C_r, \quad \forall x \in A_{r,R}$$

it follows that $W^{1,1}(A_{r,R}, w)$ is compactly embedded into $L^1(A_{r,R}, w)$. Thus we can inductively construct a sequence of subsequences $(u_n^{(l)})_{n\in\mathbb{N}}$, $l \in \mathbb{N}$ such that for all $l \in \mathbb{N}$ we have that $(u_n^{(l)})_{n\in\mathbb{N}}$ is a subsequence of $(u_n^{(l-1)})_{n\in\mathbb{N}}$, where $(u_n^{(0)})_{n\in\mathbb{N}} = (u_n)_{n\in\mathbb{N}}$ and such that for all $l \in \mathbb{N}$ it holds that

$$\lim_n \left\|u_n^{(l)} - v_l\right\|_{L^1(A_{1/l,R},w)} = 0.$$

for some $v_l \in L^1(A_{1/l,R}, w)$.

To conclude the proof it suffices to show that the diagonal sequence $(u_l^{(l)})_{l\in\mathbb{N}}$ is a Cauchy sequence since $L^1(B_R, w)$ is complete. Let $\varepsilon > 0$. Then for $k \in \mathbb{N}$ with $1/k < R$ we have that

$$\left\|u_l^{(l)} - u_m^{(m)}\right\|_{L^1(B_R,w)} \leq \left\|u_l^{(l)} - u_m^{(m)}\right\|_{L^1(A_{1/k,R},w)} + \left\|u_l^{(l)}\right\|_{L^1(B_{1/k},w)} + \left\|u_m^{(m)}\right\|_{L^1(B_{1/k},w)}$$

According to Lemma 6.27 we can choose k such that the sum of the last two terms is bounded by $\varepsilon/2$. Since $(u_l^{(l)})_{l\in\mathbb{N}}$ is a subsequence of $(u_n^{(k)})_{n\in\mathbb{N}}$ (up to finitely many elements) it is in particular a Cauchy sequence in $L^1(A_{1/k,R}, w)$. Thus for l, m sufficiently large we have that

$$\left\|u_l^{(l)} - u_m^{(m)}\right\|_{L^1(A_{1/k,R},w)} < \varepsilon/2,$$

which yields the claim. □

Proposition 6.26 enables us to derive the following compactness result for domains weights weights arising from analytic functions.

Theorem 6.28. *Let $p \geq 1$, let $D \subset \mathbb{C}$ be a Lipschitz domain and suppose F is analytic on a neighboorhood of D such that F has no zeros on the boundary of D. Define a weight by $w(z) := |F(z)|^p$ on D. Suppose $(u_n)_{n\in\mathbb{N}} \subset Lip_{loc}(D)$ is bounded in $W^{1,1}(D,w)$, then $(u_n)_{n\in\mathbb{N}}$ is precompact in $L^1(D,w)$.*

Proof. Let $z_1,\ldots,z_k$ denote the zeros of F in D and let $R>0$ be such that $\operatorname{dist}(z_j,z_{j'}) > 3R$, $\forall j \neq j'$ and

$$\operatorname{dist}(B_R(z_j),\partial D) > 0, \quad \forall j = 1,\ldots,k$$

and decompose D according to

$$D = B_R(z_1) \cup \ldots \cup B_R(z_k) \cup \left(D \setminus \bigcup_j B_R(z_j) \right).$$

Clearly it suffices to prove the assertion for each of the (finitely many) components above.
Consider first the disc $B_R(z_j)$ for fixed j. Let $m \geq 1$ denote the order to which F vanishes at z_j, then there exists a nonvanishing analytic function H in a neighboorhood of $B_R(z_j)$ such that $F(z) = z^m H(z)$, and thus there exist positive constants c, C such that

$$c\,|z|^{pm} \leq w(z) \leq C\,|z|^{pm}, \quad \forall z \in B_R(z_j).$$

This together with Proposition 6.26 proves the assertion for the disc $B_j(z_j)$.
For the remaining component

$$D' := \left(D \setminus \bigcup_j B_R(z_j) \right)$$

argue similarly to see that there exist positive constants c', C' such that

$$c' \leq w(z) \leq C', \quad \forall z \in D'.$$

Since D' has Lipschitz boundary we can draw on the corresponding result in the unweighted setting to obtain the assertion for D'. ☐

Proof of the main result of this section

Proof of Theorem 6.20. Since D is bounded we have that there exist positive constants c, C and an entire function F such that

$$c\,|\mathcal{G}f(z)| \leq |F(z)| \leq C\,|\mathcal{G}f(z)|, \quad \forall z \in D.$$

Thus it suffices to consider $w = |F|^p$. For this choice of w let $\mathcal{J}$ and U be defined as in (6.8) and (6.9) respectively. Furthermore let $(u_n)_{n\in\mathbb{N}} \subset U$ be a minimizing sequence of $\mathcal{J}$, i.e.,

$$\lim_n \mathcal{J}(u_n) = \inf_{u\in U} \mathcal{J}(U).$$

w **satisfies** (6.10)**:** Let $z_1,\ldots,z_k$ denote the zeros of w in D. For δ small enough there exists a constant M such that

$$w(z) \leq M\,|z - z_j|^{pm_j}, \quad \forall 1 \leq j \leq k,\ z \in B_\delta(z_j),$$

where m_j denotes the multiplicity of the zero of F at z_j. Thus for all $\varepsilon < \delta$ it holds that

$$w(B_\varepsilon(z_j)) = \int_{B_\varepsilon(z_j)} w(z)\, dA(z) \leq M\varepsilon^{pm_j} \cdot \pi\varepsilon^2.$$

The estimate

$$w(E_\varepsilon) \leq \sum_{j=1}^{k} w(B_\varepsilon(z_j)) \leq kM\pi\varepsilon^{p+2}$$

implies (6.10).

Approximation by Lipschitz functions: Now the coarea formula (Theorem 6.6) together with Lemma 6.23 and Lemma 6.24 implies that for all real-valued $u \in L^1(D,w)$ it holds that

$$V_w(u,D) = V_w(u, D \setminus E) = \tilde{V}_w(u, D \setminus E) = \tilde{V}_w(u,D),$$

where $E = \{z \in D : w(z) = 0\}$.
In particular there exists a sequence $(v_n)_{n\in\mathbb{N}} \subset Lip_{loc}(D) \cap W^{1,1}(D,w)$ such that for all $n \in \mathbb{N}$

$$\|u_n - v_n\|_{L^1(D,w)} < 1/n \quad \text{and} \quad \left| V_w(u_n, D) - \int_D |\nabla v_n|\, w\, dx \right| < 1/n. \tag{6.12}$$

Existence of accumulation point: Since $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^1(D,w)$ and $\sup_n V_w(u_n, D)$ is finite it follows from (6.12) that $(v_n)_{n\in\mathbb{N}}$ is bounded in $W^{1,1}(D,w)$. By Theorem 6.28 there exists a subsequence $(v_{n_l})_{l\in\mathbb{N}}$ and $u^* \in L^1(D,w)$ such that

$$\lim_l \|v_{n_l} - u^*\|_{L^1(D,w)} = 0,$$

and furthermore

$$\lim_l \|u_{n_l} - u^*\|_{L^1(D,w)} \leq \lim_l \Big(\|u_{n_l} - v_{n_l}\|_{L^1(D,w)} + \|v_{n_l} - u^*\|_{L^1(D,w)} \Big) = 0, \tag{6.13}$$

i.e., u^* is also an accumulation point of $(u_n)_{n\in\mathbb{N}}$. By semicontinuity of the total variation (Lemma 6.25) we have that

$$V_w(u^*, D) \leq \liminf_l V_w(u_{n_l}, D) = \inf_{u\in U} \mathcal{J}(u),$$

thus it suffices to show that $V_w(u^*, D) > 0$. The proof will be concluded by proving

(i) $V_w(u,D) = 0$ implies that u is constant (i.e., there exists a constant representative) and

(ii) u^* cannot be constant.

Proof of (i): Since E is a set of measure zero and since $D \setminus E$ is connected it suffices to prove the assertion locally in $D \setminus E$. Let $z \in D \setminus E$ and $\varepsilon > 0$ such that $B_\varepsilon(z) \subset D \setminus E$. Recall, that in the unweighted case BV spaces are usually defined using smooth test functions, cf. [EG15, Chapter 5]. For example in our case, $u \in L^1(B_\varepsilon(z))$ is of bounded variation if it has finite total variation

$$V(u, B_\varepsilon(z)) := \sup_{\substack{\varphi \in C_c^\infty(B_\varepsilon(z),\mathbb{R}^2) \\ |\varphi|\leq 1}} \int_{B_\varepsilon(z)} u \operatorname{div} \varphi\, dx.$$

Suppose $V(u, B_\varepsilon(z))$ is positive. Then there exists $\varphi \in C_c^\infty(B_\varepsilon(z), \mathbb{R}^2)$ with $|\varphi| \leq 1$ such that

$$\int_{B_\varepsilon(z)} u \operatorname{div} \varphi\, dx > 0.$$

Let $c := \inf_{z\in B_\delta(z)} w(z) > 0$, then $|c\varphi| \le w$ pointwise and, therefore we have that

$$0 = V_w(u, D) \ge \int_D u \operatorname{div}(c\varphi) = c \int_{B_\varepsilon(z)} u \operatorname{div} \varphi \, dx > 0,$$

which is a contradiction and hence $V(u, B_\varepsilon(z)) = 0$.
It remains now to draw onto Poincaré inequality for functions of (unweighted) bounded variation, see [EG15, Theorem 5.10], which implies that u must be constant in $B_\varepsilon(z)$.

Proof of (ii): Assume now that u^* is constant. Since u^* is the limit of a normalized (in $L^1(D, w)$) sequence it follows that $u^* = 1/w(D)$ (up to a set of measure zero). For arbitrary $u \in U$, i.e., $u = \chi_G/w(G)$ with $G \in \mathcal{C}(D, w)$ we estimate

$$\begin{aligned}
\|u^* - u\|_{L^1(D,w)} &= \|u^* - u\|_{L^1(G,w)} + \|u^*\|_{L^1(D\setminus G,w)} \\
&= \left|\frac{1}{w(D)} - \frac{1}{w(G)}\right| w(G) + \frac{w(D \setminus G)}{w(D)} \\
&= (1 - w(G)/w(D)) + (1 - w(G)/w(D)) \\
&\ge 1.
\end{aligned}$$

Thus the assumption that u^* is constant contradicts (6.13). □

6.4.4 A converse result

The stability result, Theorem 6.12, in a nutshell, states that recovery of the phase from phaseless Gabor measurements can only be instable if the Cheeger constant is small. In this section we aim at establishing a complementary result, i.e., to show that the assumption that the Cheeger constant $\underline{h}_{D,|\mathcal{G}f|^p}$ is small implies that f is an instability.

We shall only discuss the case where $D = \mathbb{C}^d \simeq \mathbb{R}^{2d}$ is the full space. Morever, in order to have the convenient structure of a Hilbert space present, we choose $p = 2$. We consider a fixed $f \in \mathcal{M}^2 = L^2(\mathbb{R}^d)$ which gives rise to the weight

$$w(x) := |\mathcal{G}f(x)|^2, \quad x \in \mathbb{R}^{2d}.$$

A sufficient condition

Recall the definition of the Cheeger constant

$$\underline{h}_{\mathbb{R}^{2d},w} = \inf_{E\in\mathcal{C}(D,w)} P_w(E, \mathbb{R}^{2d})/w(E).$$

We will here consider a modified version of the Cheeger ratio. Assume in the following that ρ is a nonnegative and nonincreasing function on $[0, \infty)$. For $E \subset \mathbb{R}^{2d}$ with smooth boundary we define the quantity

$$\mathfrak{h}_{w,\rho}(E) := \frac{\int_{\mathbb{R}^{2d}} w(x)\rho(\operatorname{dist}(x, \partial E))\, dx}{w(E)}.$$

Remark 6.29. The quantity $\mathfrak{h}_{w,\rho}(E)$ can be understood as a smoothed variant of the the Cheeger ratio $h_{\mathbb{R}^{2d},w}(E)$: While the numerator of the Cheeger ratio is computed by integrating the weight over the boundary, the numerator of $\mathfrak{h}_{w,\rho}(E)$ is an average of all values of w with weighting depending on the distance to the boundary ∂E.
More particularly, while the Cheeger ratio only takes into account the values of w on the hypersurface ∂E, for $\mathfrak{h}_{w,\rho}(E)$ to be small w necessarily has to be small on a neighboorhood of ∂E.

We spend the remainder of this paragraph to establish the following complementary (to Theorem 6.12) result.

Theorem 6.30. *Let $f \in L^2(\mathbb{R}^d)$ and $w = |\mathcal{G}f|^2$. Let $\rho(t) := (4/\pi)^d \Gamma(d, \pi t^2/2)$, where Γ denotes the incomplete Gamma function*

$$\Gamma(s,x) := \int_x^\infty t^{s-1} e^{-t}\, dt.$$

Suppose that there exists $E \in \mathcal{C}(\mathbb{R}^{2d}, w)$ such that

$$\mathfrak{h}_{w,\rho}(E) \le (4\pi)^{-2} \frac{\|F\|^2_{L^2(E)}}{\|F\|^2_{L^2(\mathbb{R}^{2d})}}. \tag{6.14}$$

Then there exists $g \in L^2(\mathbb{R}^d)$ such that

$$\inf_{|c|=1} \|g - cf\|_{L^2(\mathbb{R}^d)} \ge (2\pi^2)^{-1/2} \mathfrak{h}_{w,\rho}(E)^{-1/2}\, \| |\mathcal{G}g| - |\mathcal{G}f| \|_{L^2(\mathbb{R}^{2d})}$$

Assumption (6.14) expresses that $\mathfrak{h}_{w,\rho}(E)$ must be small compared to the ratio $w(E)/w(\mathbb{R}^{2d}) < 1/2$. Hence, the restriction in mild if E and E^c carry roughly the same mass ($w(E) \approx w(E^c)$) and is strong if $w(E)$ becomes small.
Before we proceed to prove Theorem 6.30 we collect some preliminaries. Recall that the inverse of the Gabor transform operator $\mathcal{G} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d})$ can be written in terms of an integral operator, cf. [Grö01, Corollary 3.2.3]:

$$f = T_{\mathbb{R}^{2d}} \mathcal{G} f, \quad \forall f \in L^2(\mathbb{R}^d), \tag{6.15}$$

where we define for measurable $E \subset \mathbb{R}^{2d}$

$$(T_E F)(t) := 2^{d/4} \iint_E F(x,y) e^{2\pi i y\cdot t} e^{-\pi|t-x|^2}\, dxdy, \quad F \in L^2(E).$$

Moreover, define

$$P := \mathcal{G} \circ T_{\mathbb{R}^{2d}}.$$

Note that the inversion formula (6.15) implies that P reproduces elements in $\mathcal{G}(L^2(\mathbb{R}^d))$, i.e.,

$$P(\mathcal{G}f) = \mathcal{G}f, \quad \forall f \in L^2(\mathbb{R}^d).$$

The operator P can also be described in terms of an integral operator.

Lemma 6.31. *For all $F \in L^2(\mathbb{R}^{2d})$ it holds that*

$$PF(x,y) = \iint_{\mathbb{R}^{2d}} F(\xi,\eta) \Psi(x,y,\xi,\eta)\, d\xi d\eta,$$

where

$$\Psi(x,y,\xi,\eta) := \exp\left\{ -\pi |x-\xi|^2/2 - \pi|y-\eta|^2/2 - \pi i (x+\xi)\cdot(y-\eta) \right\}. \tag{6.16}$$

Proof. For $F \in L^2(\mathbb{R}^{2d})$ and $x, y \in \mathbb{R}^d$ we have that

$$\begin{aligned}
PF(x,y) &= \mathcal{G}(T_{\mathbb{R}^{2d}} F)(x,y) \\
&= 2^{d/4} \int_{\mathbb{R}^d} (T_{\mathbb{R}^{2d}} F)(t) e^{-2\pi i y\cdot t} e^{-\pi|t-x|^2}\, dt \\
&= 2^{d/2} \iiint_{\mathbb{R}^{3d}} F(\xi,\eta) e^{2\pi i \eta\cdot t} e^{-\pi|t-\xi|^2} e^{-2\pi i y\cdot t} e^{-\pi|t-x|^2}\, d\xi d\eta dt \\
&= 2^{d/2} \iint_{\mathbb{R}^{2d}} F(\xi,\eta) \left[\int_{\mathbb{R}^d} e^{2\pi i \eta\cdot t} e^{-\pi|t-\xi|^2} e^{-2\pi i y\cdot t} e^{-\pi|t-x|^2}\, dt \right] d\xi d\eta
\end{aligned}$$

To compute the integral inside the brackets observe that the integrand is a tensor product of d copies of the function

$$h(t) := \exp\left\{-\pi(t-x)^2 - \pi(t-\xi)^2 - 2\pi i(y-\eta)t\right\}, \quad x,\xi,y,\eta \in \mathbb{R}.$$

Now a standard calculation shows that

$$\begin{aligned}\int_{\mathbb{R}} h(t)\,dt &= \left(e^{-\pi(\cdot-x)^2-\pi(\cdot-\xi)^2}\right)^{\wedge}(y-\eta)\\ &= 2^{-1/2}e^{-\pi(y-\eta)^2/2-\pi(x-\xi)^2/2-\pi i(x+\xi)(y-\eta)}.\end{aligned}$$

Putting everything together yields that

$$PF(x,y) = \iint_{\mathbb{R}^{2d}} F(\xi,\eta)\Psi(x,y,\xi,\eta)\,d\xi d\eta,$$

with Ψ defined as in (6.16). □

Another property that will come in handy is the following.

Lemma 6.32. *P is the orthogonal projection on* $\operatorname{ran}\mathcal{G}$.

Proof. First note that P is a bounded linear operator on $L^2(\mathbb{R}^{2d})$ as a composition of two such operators. Moreover it follows from (6.15) that $\operatorname{ran}P \supset \mathcal{G}(L^2(\mathbb{R}^d))$. The reverse inclusion $\operatorname{ran}P \subset \mathcal{G}(L^2(\mathbb{R}^d))$ follows directly from the definition of P.
It remains to show that $P^2 = P$. Since $T_{\mathbb{R}^{2d}}\mathcal{G}$ coincides with the identity on $L^2(\mathbb{R}^d)$ we get for arbitrary $F \in L^2(\mathbb{R}^{2d})$ that

$$P^2F = \mathcal{G}T_{\mathbb{R}^{2d}}\mathcal{G}T_{\mathbb{R}^{2d}}F = \mathcal{G}(T_{\mathbb{R}^{2d}}\mathcal{G})T_{\mathbb{R}^{2d}}F = \mathcal{G}T_{\mathbb{R}^{2d}}F = PF,$$

which completes the proof. □

The idea of the proof of Theorem 6.30 is to mimic the approach that has already been discussed in Section 2.2 in the abstract frame theoretic setting. More concretely, if $\mathfrak{h}_{w,\rho}(E)$ is small we expect that f consists of two components $f = f_1+f_2$ such that $\mathcal{G}f_1 \approx \mathcal{G}f\cdot\chi_E$ and $\mathcal{G}f_2 \approx \mathcal{G}f\cdot\chi_{E^c}$.

Proof of Theorem 6.30. Let $F := \mathcal{G}f$ and define

$$H := F\left(\chi_E - \chi_{E^c}\right).$$

Furthermore set $G := PH$ and note that $G \in \mathcal{G}(L^2(\mathbb{R}^d))$, i.e., there exists $g \in L^2(\mathbb{R}^d)$ such that $\mathcal{G}g = G$. We seek to estimate the quantity

$$\frac{\inf_{|c|=1}\|g-cf\|_{L^2(\mathbb{R}^d)}}{\||G|-|F|\|_{L^2(\mathbb{R}^{2d})}} = \frac{\inf_{|c|=1}\|G-cF\|_{L^2(\mathbb{R}^{2d})}}{\||G|-|F|\|_{L^2(\mathbb{R}^{2d})}} \tag{6.17}$$

from below. We split the proof into three steps.

Step 1: Let δ denote the numerator of $\mathfrak{h}_{w,\rho}(E)$

$$\delta := \int_{\mathbb{R}^{2d}} w(x)\rho(\operatorname{dist}(x,\partial E))\,dx.$$

We claim that $\|G-H\|_{L^2(\mathbb{R}^{2d})} \le 2\pi\delta^{1/2}$. First split

$$\|G-H\|^2_{L^2(\mathbb{R}^{2d})} = \|G-H\|^2_{L^2(E)} + \|G-H\|^2_{L^2(E^c)}.$$

Let Ψ be defined as in Lemma 6.31. For $z \in E$ we have that

$$H(z) = F(z) = \int_{\mathbb{R}^{2d}} F(z')\Psi(z,z')\,dz' = \int_E F(z')\Psi(z,z')\,dz' + \int_{E^c} F(z')\Psi(z,z')\,dz'$$

and similarly

$$G(z) = PH(z) = \int_E F(z')\Psi(z,z')\,dz' - \int_{E^c} F(z')\Psi(z,z')\,dz'.$$

Therefore we obtain

$$\begin{aligned}
\|G-H\|_{L^2(E)}^2 &= 4\int_E \left|\int_{E^c} F(z')\Psi(z,z')\,dz'\right|^2 dz \\
&\le 4\int_E \left(\int_{E^c} |F(z')|\, e^{-\pi|z-z'|^2/2}\,dz'\right)^2 dz \\
&= 4\int_E \left(\int_{\mathbb{R}^{2d}} |F(z')|\,\chi_{E^c}(z')e^{-\pi|z-z'|^2/2}\,dz'\right)^2 dz
\end{aligned}$$

Next, note that $C_d e^{-\pi|z-z'|^2/2}dz'$ is a probability measure on $\mathbb{R}^{2d}$ for fixed z, where we set

$$C_d := 2^{-1}\frac{(\pi/2)^{d-1}}{(d-1)!}.$$

Application of Jensen's inequality therefore yields that

$$\begin{aligned}
\|G-H\|_{L^2(E)}^2 &\le 4C_d^{-2}\int_E\int_{\mathbb{R}^{2d}} |F(z')|^2\,\chi_{E^c}(z')C_d e^{-\pi|z-z'|^2/2}\,dz'dz \\
&= 4C_d^{-1}\int_E\int_{E^c} w(z')e^{-\pi|z-z'|^2/2}\,dz'dz \\
&= 4C_d^{-1}\int_{E^c} w(z')\left[\int_E e^{-\pi|z-z'|^2/2}\,dz\right] dz'
\end{aligned}$$

For fixed $z' \in E^c$ let us denote $B := B_{\operatorname{dist}(z',E)}(z')$. Moreover let $\omega_{2d-1} = 2\pi^d/(d-1)!$ denote the surface area of the unit sphere. Use change of variables $s = \pi r^2/2$ to further estimate

$$\begin{aligned}
\int_E e^{-\pi|z-z'|^2/2}\,dz &\le \int_{\mathbb{R}^{2d}\setminus B} e^{-\pi|z-z'|^2/2}\,dz \\
&= \int\limits_{\{|z|>\operatorname{dist}(z',E)\}} e^{-\pi|z|^2/2}\,dz \\
&= \omega_{2d-1}\int\limits_{\operatorname{dist}(z',E)}^{\infty} e^{-\pi r^2/2}r^{2d-1}\,dr \\
&= \omega_{2d-1}(2/\pi)^{d-1}\int\limits_{\pi\operatorname{dist}(z',E)^2/2}^{\infty} e^{-s}s^{d-1}\,ds \\
&= \omega_{2d-1}(2/\pi)^{d-1}\Gamma(d,\pi\operatorname{dist}(z',E)^2/2)\,,
\end{aligned}$$

and since $\operatorname{dist}(z',E) = \operatorname{dist}(z',\partial E)$ that

$$\begin{aligned}
\|G-H\|_{L^2(E)}^2 &\le 4C_d^{-1}\omega_{2d-1}(2/\pi)^{d-1}\int_{E^c} w(z')\Gamma(d,\pi\operatorname{dist}(z',\partial E)^2/2)\,dz' \\
&= 4\pi^2(4/\pi)^d\int_{E^c} w(z')\Gamma(d,\pi\operatorname{dist}(z',\partial E)^2/2)\,dz'\,.
\end{aligned} \tag{6.18}$$

Analogously one obtains the estimate

$$\|G-H\|^2_{L^2(E^c)} \le 4\pi^2(4/\pi)^d \int_E w(z')\Gamma(d,\pi\,\mathrm{dist}(z',\partial E)^2/2)\,dz'\,. \tag{6.19}$$

Finally, by adding (6.18) and (6.19) and taking square roots we get that

$$\begin{aligned}\|G-H\|_{L^2(\mathbb{R}^{2d})} &\le 2\pi(4/\pi)^{d/2}\left(\int_{\mathbb{R}^{2d}} w(z')\Gamma(d,\pi\,\mathrm{dist}(z',\partial E)^2/2)\,dz'\right)^{1/2}\\ &= 2\pi\left(\int_{\mathbb{R}^{2d}} w(z')\rho(\mathrm{dist}(x,\partial E))\,dz'\right)^{1/2}.\end{aligned}$$

Step 2: Next we estimate the numerator of (6.17) from below. Let $\langle\cdot,\cdot\rangle$ denote the inner product in $L^2(\mathbb{R}^{2d})$. To simplify notation we shall write $\|\cdot\|$ instead of $\|\cdot\|_{L^2(\mathbb{R}^{2d})}$.
Observe that the numerator can be rewritten in the following way

$$\begin{aligned}\inf_{|c|=1}\|G-cF\|^2 &= \|G\|^2+\|F\|^2-2\sup_{|c|=1}\mathrm{Re}\langle G,cF\rangle\\ &= \|G\|^2+\|F\|^2-2\,|\langle G,F\rangle|\,.\end{aligned}$$

Note that $\langle H,F\rangle = \|F\|^2_{L^2(E)} - \|F\|^2_{L^2(E^c)}$ which is nonpositive since $E\in\mathcal{C}(\mathbb{R}^{2d},w)$. For the inner product we estimate

$$\begin{aligned}|\langle G,F\rangle| &\le |\langle G-H,F\rangle| + |\langle H,F\rangle|\\ &\le \|G-H\|\,\|F\| + \|F\|^2_{L^2(E^c)} - \|F\|^2_{L^2(E)}\,.\end{aligned}$$

Thus we get that

$$\begin{aligned}&\|G\|^2+\|F\|^2-2\,|\langle G,F\rangle|\\ &\quad\ge (\|G-H\|-\|H\|)^2+\|F\|^2\\ &\qquad -2\left(\|G-H\|\,\|F\|+\|F\|^2_{L^2(E^c)}-\|F\|^2_{L^2(E)}\right)\\ &\quad\ge \|G-H\|^2-2\,\|G-H\|\,\|H\|+\|H\|^2+\|F\|^2\\ &\qquad -2\,\|G-H\|\,\|F\|-2\,\|F\|^2_{L^2(E^c)}+2\,\|F\|^2_{L^2(E)}\\ &\quad= 4\,\|F\|^2_{L^2(E)}+\|G-H\|^2-4\,\|G-H\|\,\|F\|\\ &\quad\ge 4\,\|F\|^2_{L^2(E)}-8\pi\delta^{1/2}\,\|F\|\,,\end{aligned} \tag{6.20}$$

where we used Step 1 in the final estimate.

Step 3: In order to estimate the denominator of (6.17) note that $|H|=|F|$. Thus we get that

$$\||G|-|F|\|^2 = \||G|-|H|\|^2 \le \|G-H\|^2 \le 4\pi^2\delta\,. \tag{6.21}$$

Putting everything together: It follows from (6.17), (6.20) and (6.21) that

$$\begin{aligned}\frac{\inf_{|c|=1}\|g-cf\|^2_{L^2(\mathbb{R}^d)}}{\||G|-|F|\|^2_{L^2(\mathbb{R}^{2d})}} &\ge \frac{4\,\|F\|^2_{L^2(E)}-8\pi\delta^{1/2}\,\|F\|_{L^2(\mathbb{R}^{2d})}}{4\pi^2\delta}\\ &= \frac{\|F\|^2_{L^2(E)}-2\pi\delta^{-1/2}\,\|F\|_{L^2(\mathbb{R}^{2d})}}{\pi^2\delta}\,.\end{aligned}$$

An elementary calculation shows that assumption (6.14) is equivalent to

$$\|F\|_{L^2(E)}^2 - 2\pi\delta^{-1/2}\|F\|_{L^2(\mathbb{R}^{2d})} \geq \|F\|_{L^2(E)}^2/2.$$

Therefore it further follows that

$$\frac{\inf_{|c|=1}\|g-cf\|_{L^2(\mathbb{R}^d)}^2}{\||G|-|F|\|_{L^2(\mathbb{R}^{2d})}^2} \geq (2\pi^2)^{-1}\frac{\|F\|_{L^2(E)}^2}{\delta} = (2\pi^2)^{-1}\mathfrak{h}_{w,\rho}(E)$$

which after taking square roots proves the assertion. □

When E is a halfspace

In the previous paragraph we replaced the Cheeger ratio $h_{\mathbb{R}^{2d},w}(E)$ by $\mathfrak{h}_{w,\rho}(E)$ and proved, provided that $w = |\mathcal{G}f|^2$ is disconnected in terms of this new notion, the existence of an approximately ambiguous function g, meaning that $|\mathcal{G}g| \approx |\mathcal{G}f|$, see Theorem 6.30.
As mentioned above, for $\mathfrak{h}_{w,\rho}(E)$ to be small we require that w is small in a neighboorhood of ∂E. A natural question to ask is whether disconnectedness in terms of the Cheeger ratio $h_{\mathbb{R}^{2d},w}(E)$ is already sufficient for $\mathfrak{h}_{w,\rho}(E)$ to be small.
If we allow for arbitrary weights this certainly cannot be the case since $h_{\mathbb{R}^{2d},w}(E)$ only takes into account the values of w on the hypersurface ∂E. However, as we shall see, weights arising from the Gabor transform act in a sufficiently well-behaved manner which allows for such a conclusion, at least in the special case where E is a halfspace.

For the remainder of this section we assume that $E \subset \mathbb{C}^d \simeq \mathbb{R}^{2d}$ is an open halfspace. In particular there exists a vector $\nu \in \mathbb{R}^{2d}$ of unit length and a number $b \in \mathbb{R}$ such that

$$E = \{z \in \mathbb{R}^{2d} : z \cdot \nu > b\}.$$

or equivalently, by identifying $z = (x, y)$ with $x + iy \in \mathbb{C}^d$ and similarly for ν that

$$E = \{z \in \mathbb{C}^d : \mathrm{Re}\langle z, \nu\rangle_{\mathbb{C}^d} > b\},$$

where $\langle\cdot,\cdot\rangle_{\mathbb{C}^d}$ denotes the standard inner product in $\mathbb{C}^d$.

Theorem 6.33. *Let $q > 5$. Then there exist positive constants c_1, c_2 (depending on d and q) such that for all $f \in L^2(\mathbb{R}^d)$ satisfying that*

$$h_{\mathbb{R}^{2d},w}(E) \leq c_1\frac{w(E)}{w(\mathbb{R}^{2d})},$$

where we set $w = |\mathcal{G}f|^2$ and where $E \subset \mathbb{R}^{2d}$ is an open halfspace, there exists $g \in L^2(\mathbb{R}^d)$ such that

$$\inf_{|c|=1}\|g-cf\|_{L^2(\mathbb{R}^d)} \geq c_2 h_{\mathbb{R}^{2d},w}(E)^{-1/(2q)}\left(\frac{w(E)}{w(\mathbb{R}^{2d})}\right)^{(q-1)/(2q)}\||\mathcal{G}g|-|\mathcal{G}f|\|_{L^2(\mathbb{R}^{2d})}.$$

Proof. The proof is split up into several steps.

Preperatory step: We claim that we may assume w.l.o.g. that $E = \mathbb{H} := \{z = (x,y) \in \mathbb{R}^{2d} : x_1 > 0\}$.
First observe that since

$$\left\{|\mathcal{G}f| : f \in L^2(\mathbb{R}^d)\right\}$$

is translation invariant we may assume w.l.o.g. that $b = 0$, i.e., that

$$E = \{z \in \mathbb{C}^d : \operatorname{Re}\langle z, \nu\rangle_{\mathbb{C}^d} > 0\}.$$

Next choose a unitary matrix $U \in \mathbb{C}^{d\times d}$ such that $Ue_1 = \nu$. It is a simple linear algebra exercise to show that any unitary matrix in $\mathbb{C}^{d\times d}$ is a rotation in $\mathbb{R}^{2d}$ after the usual identification

$$\mathbb{C}^d \ni x + iy \simeq (x, y) \in \mathbb{R}^{2d}.$$

From the choice of U it follows that $U(\mathbb{H}) = E$. Using change of variables one can verify that $h_{\mathbb{R}^{2d},w}(E) = h_{\mathbb{R}^{2d},w\circ U}(\mathbb{H})$ and that $\mathfrak{h}_{w,\rho}(E) = \mathfrak{h}_{w\circ U,\rho}(\mathbb{H})$.
The Bargmann transform is a unitary operator from $L^2(\mathbb{R}^d)$ onto the Fock space

$$\mathcal{F}^2(\mathbb{C}^d) := \left\{F \in \mathcal{O}(\mathbb{C}^d) : \int_{\mathbb{C}^d} |F(z)|^2 \, e^{-\pi|z|^2} \, dA(z)\right\},$$

cf. [Grö01, Theorem 3.4.3]. Since the Bargmann transform $\mathcal{B}f$ and the Gabor transform $\mathcal{G}f$ coincide up to reflection and a weight factor it follows that the range of the Gabor transform $\mathcal{G}(L^2(\mathbb{R}^d))$ is closed under rotations. In particular there exists $\tilde{f} \in L^2(\mathbb{R}^d)$ such that for all $z \in \mathbb{R}^{2d}$

$$w \circ U(z) = |\mathcal{G}f(Uz)|^2 = |\mathcal{G}f \circ U(z)|^2 = \left|\mathcal{G}\tilde{f}(z)\right|^2,$$

which proves the claim.

Computing the respective numerators: We define ρ and Γ as in Theorem 6.30, i.e.,

$$\Gamma(d, x) = \int_x^\infty e^{-s} s^{d-1} \, ds$$

and $\rho(x) = (4/\pi)^d \Gamma(d, \pi x^2/2)$. Let δ denote the numerator of $\mathfrak{h}_{w,\rho}(\mathbb{H})$

$$\delta = \int_{\mathbb{R}^{2d}} w(z)\rho(\operatorname{dist}(z, \partial\mathbb{H})) \, dz = \iint_{\mathbb{R}^{2d}} w(x, y)\rho(|x_1|) \, dx \, dy.$$

Since $\mathcal{G}f(x, \cdot)$ coincides with the Fourier transform of $2^{d/4} f e^{-\pi|\cdot - x|^2}$ it follows from Plancherel's formula that

$$\int_{\mathbb{R}^d} w(x, y) \, dy = \int_{\mathbb{R}^d} |\mathcal{G}f(x, y)|^2 \, dy = \|\mathcal{G}f(x, \cdot)\|^2_{L^2(\mathbb{R}^d)} = 2^{d/2} \left\| f e^{-\pi|\cdot - x|^2} \right\|^2_{L^2(\mathbb{R}^d)}. \tag{6.22}$$

Therefore we obtain that

$$\begin{aligned}
\delta &= 2^{d/2} \iint_{\mathbb{R}^{2d}} |f(t)|^2 \, e^{-2\pi|t-x|^2} \rho(|x_1|) \, dx \, dt \\
&= 2^{d/2} \int_{\mathbb{R}^d} |f(t)|^2 \left[\int_{\mathbb{R}^d} e^{-2\pi|t-x|^2} \rho(|x_1|) \, dx\right] dt \\
&= 2^{d/2} \int_{\mathbb{R}^d} |f(t)|^2 \left[2^{-(d-1)/2} \int_{\mathbb{R}} e^{-2\pi(t_1-x_1)^2} \rho(|x_1|) \, dx_1\right] dt \\
&= 2^{1/2} \int_{\mathbb{R}^d} |f(t)|^2 \int_{\mathbb{R}} e^{-2\pi(t_1-x_1)^2} \rho(|x_1|) \, dx_1 \, dt \,.
\end{aligned} \tag{6.23}$$

Similarly, for the numerator of the Cheeger ratio $h_{\mathbb{R}^{2d},w}(\mathbb{H})$ we compute with the help of (6.22) that

$$\begin{aligned}
\int_{\partial\mathbb{H}} w(s)\, d\mathcal{H}^{2d-1}(s) &= \int_{\mathbb{R}^{2d-1}} w(0,z')\, dz' \\
&= \underset{\mathbb{R}^{d-1}}{\int\cdots\int} \left[\int_{\mathbb{R}^d} w(0,x_2,\ldots,x_d,y)\, dy\right] dx_2\ldots dx_d \\
&= 2^{d/2} \underset{\mathbb{R}^{d-1}}{\int\cdots\int} \left[\int_{\mathbb{R}^d} |f(t)|^2\, e^{-2\pi\left[t_1^2+\sum_{k=2}^d (t_k-x_k)^2\right]}\, dt\right] dx_2\ldots dx_d \qquad (6.24)\\
&= 2^{d/2}\int_{\mathbb{R}^d} |f(t)|^2\, e^{-2\pi t_1^2} \left[\int_{\mathbb{R}^{d-1}} e^{-2\pi|u|^2}\, du\right] dt \\
&= 2^{1/2}\int_{\mathbb{R}^d} |f(t)|^2\, e^{-2\pi t_1^2}\, dt\,.
\end{aligned}$$

Relate $\mathfrak{h}_{w,\rho}(\mathbb{H})$ to $h_{\mathbb{R}^{2d},w}(\mathbb{H})$: Pick $r\in(0,1)$ arbitrary. The function ρ can be estimated as follows

$$\begin{aligned}
\rho(|x_1|) &= (4/\pi)^d \int_{\pi x_1^2/2}^{\infty} e^{-s} s^{d-1}\, ds \\
&\le (4/\pi)^d e^{-\pi r x_1^2/2} \int_0^\infty e^{-(1-r)s} s^{d-1}\, ds \\
&= c_{d,r} e^{-\pi r x_1^2/2}.
\end{aligned}$$

Hence for the inner integral on the right hand side of (6.23) it holds that

$$\begin{aligned}
\int_{\mathbb{R}} e^{-2\pi(t_1-x_1)^2} \rho(|x_1|)\, dx_1 &\le c'_{d,r}\int_{\mathbb{R}} e^{-2\pi(t_1-x_1)^2} e^{-\pi r x_1^2/2}\, dx_1 \\
&= c''_{d,r} e^{-2\pi\frac{r}{4+r}t^2}.
\end{aligned}$$

Define $q:=(4+r)/r = 1+4/r$ and let q' denote its Hölder conjugate, i.e., $q'=1+r/4$. Make use of (6.23) and Hölder's inequality to further estimate

$$\begin{aligned}
\delta &\le c'''_{d,r}\int_{\mathbb{R}^d} |f(t)|^2\, e^{-2\pi t_1^2/q}\, dt \\
&= c'''_{d,r}\int_{\mathbb{R}^d} |f(t)|^{2/q}\, e^{-2\pi t_1^2/q}\cdot |f(t)|^{2/q'}\, dt \\
&\le c'''_{d,r}\left(\int_{\mathbb{R}^d} |f(t)|^2\, e^{-2\pi t_1^2}\, dt\right)^{1/q}\cdot\left(\int_{\mathbb{R}^d} |f(t)|^2\, dt\right)^{1/q'} \\
&= c'''_{d,r} w(\mathbb{R}^{2d})^{1/q'}\left(\int_{\mathbb{R}^d} |f(t)|^2\, e^{-2\pi t_1^2}\, dt\right)^{1/q}.
\end{aligned}$$

Use (6.24) to get that

$$\mathfrak{h}_{w,\rho}(\mathbb{H}) = \delta/w(\mathbb{H}) \le c'''_{d,r}\left(\frac{w(\mathbb{R}^{2d})}{w(\mathbb{H})}\right)^{1/q'} h_{\mathbb{R}^{2d},w}(\mathbb{H})^{1/q}\,.$$

Apply Theorem 6.30: First note that for Assumption (6.14) to be satisfied it is sufficient that

$$c'''_{d,r}\left(\frac{w(\mathbb{R}^{2d})}{w(\mathbb{H})}\right)^{1/q'} h_{\mathbb{R}^{2d},w}(\mathbb{H})^{1/q} \le (4\pi)^{-2}\frac{w(\mathbb{H})}{w(\mathbb{R}^{2d})},$$

or equivalently that

$$h_{\mathbb{R}^{2d},w}(\mathbb{H}) \leq (4\pi)^{-2q}(c'''_{d,r})^{-q}\frac{w(\mathbb{H})}{w(\mathbb{R}^{2d})} =: c_1\frac{w(\mathbb{H})}{w(\mathbb{R}^{2d})}. \tag{6.25}$$

Thus, provided that (6.25) is fulfilled, according to Theorem 6.30 there exists $g \in L^2(\mathbb{R}^d)$ such that

$$\begin{aligned}\inf_{|c|=1} \|g - cf\|_{L^2(\mathbb{R}^d)} &\geq (2\pi^2)^{-1/2}\mathfrak{h}_{w,\rho}(\mathbb{H})^{-1/2} \, \||\mathcal{G}g| - |\mathcal{G}f|\|_{L^2(\mathbb{R}^{2d})} \\ &\geq (2\pi^2)^{-1/2}(c'''_{d,r})^{-1/2}\left(\frac{w(\mathbb{H})}{w(\mathbb{R}^{2d})}\right)^{1/(2q')} h_{\mathbb{R}^{2d},w}(\mathbb{H})^{-1/(2q)} \\ &\quad \cdot \||\mathcal{G}g| - |\mathcal{G}f|\|_{L^2(\mathbb{R}^{2d})} .\end{aligned}$$

The assertion finally follows from the observation that $q = 1+4/r$ can be choosen in $(5, \infty)$. $\square$

6.5 On how Poincaré inequalities imply weighted ones

In this section we present yet another method on how to derive upper bounds of Poincaré constants on domains with weights. While in the weighted case explicit estimates are in general difficult to obtain the situation is often different if the weight is constant. The main result of this section relates the Poincaré constant w.r.t. a weight w on a domain $D \subset \mathbb{R}^n$ to unweighted Poincaré constants for the superlevel sets of the weight, i.e. the sets

$$D_t^w := \{x \in D : \, w(x) > t\}, \quad t \geq 0.$$

The underlying ideas to do so stem from a paper by Dyda and Kassmann [DK13], where radially decreasing weights are considered.

We shall consider real-valued Lipschitz functions. More precisely, given a domain D with a weight w and $q \geq 1$, let us define

$$\mathcal{L} = \mathcal{L}(D, w, q) := \{f \in Lip_{loc}(D) \cap L^q(D, w) : \, f \text{ real-valued}\}. \tag{6.26}$$

The space of functions with zero mean in $\mathcal{L}$ will be denoted by $\mathcal{L}_0$, i.e.,

$$\mathcal{L}_0 = \mathcal{L}_0(D, w, q) := \left\{f \in \mathcal{L}(D, w, q) : \, \int_D f(x)w(x) = 0\right\}.$$

If $E \subset D$ and v a weight on D we shall write $C^{q,p}_{w,v}(E)$ instead of $C^{q,p}_{w,v}(E, \mathcal{L}_0(E, w, q))$ in order to keep notation simple. Moreover, in the unweighted case $w = v = 1$ we drop the reference to the weights.
The main theorem of this section reads as follows.

Theorem 6.34. *Let $1 \leq q \leq p$, let w be a bounded and integrable weight on a domain $D \subset \mathbb{R}^n$ and ρ another weight on D. Then it holds that*

$$\begin{aligned}C^{q,p}_{w,w\rho}(D) \leq 8 \cdot \inf_{\substack{0 \leq \tau \leq \|w\|_{L^\infty(D)} \\ q \leq s \leq p}} &\left(\frac{\|w\|_{L^\infty(D)}}{\tau} \cdot \frac{|D|}{|D_\tau^w|}\right)^{1/q} \cdot (w\rho)(D)^{\frac{1}{q} - \frac{s}{pq}} \\ &\cdot \left\|t \mapsto C^{q,p}_{1,\rho}(D_t^w)\right\|_{L^{\frac{sq}{s-q}}(0,\tau)} .\end{aligned} \tag{6.27}$$

Assuming that there is a number τ such that the quantity

$$\left(\frac{\|w\|_{L^\infty(D)}}{\tau} \cdot \frac{|D|}{|D_\tau^w|} \right)^{1/q}$$

is of moderate size, Theorem 6.34 reveals – by choosing $\rho \equiv 1$ – that the weighted Poincaré constant $C_{w,w}^{q,p}(D)$ can be essentially controlled by the norm of $t \mapsto C^{q,s}(D_t^w)$. The Poincaré constant $C^{q,s}(D_t^w)$ is infinite whenever the underlying domain consists of more than one connected component. Thus application of Theorem 6.34 only makes sense if all the superlevel sets D_t^w are connected (at least for $t \in [0, \tau]$).
In previous sections of this chapter we studied Poincaré constants using the concept of Cheeger constants. Theorem 6.9 states that – loosely speaking – the Poincaré constant is rather large in the presence of a bottleneck, meaning that the domain can be partitionated into two subdomains of roughly equal measure w.r.t. w such that the weight is small on the seperating boundary of the two subdamains. In this spirit Theorem 6.34 can be qualitatively read as follows:

(i) If none of the superlevel sets D_t^w possesses a bottleneck w.r.t. to the constant weight 1 then neither will D w.r.t. w. In that case $C_{w,w}^{q,p}(D)$ is small.

(ii) However if $C_{w,w}^{q,p}(D)$ is large then some of the superlevel sets D_t^w will have large Poincaré constants w.r.t. the constant weight 1 and therefore a bottleneck, see Figure 6.2.

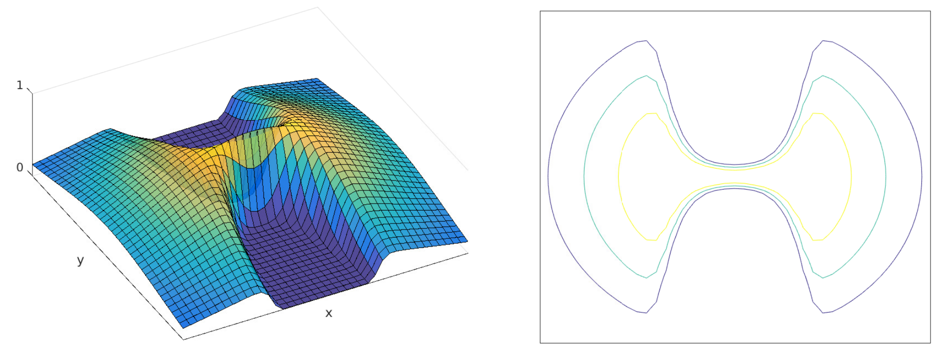


Figure 6.2: The left figure shows a typical example of a bottlenecked weight on a plane domain. The two main parts of its superlevel sets – as sketched on the right – are connected by a thin bridge and therefore we observe the presence of a bottleneck.

Two auxiliary results are needed before we proceed to prove the main result of this section. The first of which concerns how an integral w.r.t. a weight can be rewritten as a double integral.

Lemma 6.35. *Let $D \subset \mathbb{R}^n$, let w be an integrable weight on D and let $g \in L^1(D, w)$ then*

$$\int_D g(x)w(x)\ dx = \int_0^\infty \int_{D_t^w} g(x)\ dx\ dt.$$

Proof. First, rewrite the weighted integral of g

$$\int_D g(x)w(x)\ dx = \int_D g(x) \int_0^{w(x)} dt\ dx = \int_D \int_0^\infty g(x)\chi_{[0,w(x))}(t)\ dt\ dx. \tag{6.28}$$

Obviously Equation (6.28) still holds true when g is replaced by $|g|$. Since $g \in L^1(D, w)$ we can apply Fubini's theorem and change order of integration. Using $\chi_{[0,w(x))}(t) = \chi_{D_t^w}(x)$ we obtain

$$\int_D \int_0^\infty g(x)\chi_{[0,w(x))}(t)\ dt\ dx = \int_0^\infty \int_D g(x)\chi_{D_t^w}(x)\ dx\ dt = \int_0^\infty \int_{D_t^w} g(x)\ dx\ dt.$$

□

Secondly, we establish a simple relation between spaces as defined in (6.26).

Lemma 6.36. *Let w be a weight on a domain D and $q \geq 1$. Then for all $t > 0$ it holds that*

$$\mathcal{L}(D, w, q) \subset \mathcal{L}(D_t^w, 1, q).$$

Proof. Suppose $f \in \mathcal{L}(D, w, q) = Lip_{loc}(D) \cap L^q(D, w)$. In particular we have that $f \in Lip_{loc}(D_t^w)$ for all $t > 0$ since $D_t^w \subset D$. It remains to show that $f \in L^q(D)$, which is verified by the following estimate:

$$\|f\|^q_{L^q(D_t^w)} = \int_{D_t^w} |f(x)|^q\ dx \leq \int |f(x)|^q \frac{w(x)}{t}\ dx \leq t^{-1} \|f\|^q_{L^q(D,w)} < +\infty.$$

□

We finally turn towards the proof of the main result.

Proof of Theorem 6.34. For $\tau \in [0, \|w\|_{L^\infty(D)}]$ let w_τ be the weight w cut off at level τ, i.e.,

$$w_\tau(x) := \min\{w(x), \tau\}.$$

Then obviously

$$w_\tau \leq w \leq \frac{\|w\|_{L^\infty(D)}}{\tau} w_\tau$$

and therefore by Lemma 6.3 we obtain for any $f \in L^q(D, w)$

$$\begin{aligned} \|f - f_{D,w}\|^q_{L^q(D,w)} &\leq 2^q \inf_{c\in\mathbb{R}} \|f - c\|^q_{L^q(D,w)} \\ &\leq 2^q \frac{\|w\|_{L^\infty(D)}}{\tau} \inf_{c\in\mathbb{R}} \|f - c\|^q_{L^q(D,w_\tau)} \\ &\leq 2^q \frac{\|w\|_{L^\infty(D)}}{\tau} \|f - f_{D,w_\tau}\|^q_{L^q(D,w_\tau)} \end{aligned} \tag{6.29}$$

We set $g := f - f_{D,w_\tau}$. Since $D_t^{w_\tau}$ coincides with $D_t := D_t^w$ for $t \leq \tau$ and is empty for $t > \tau$, using Lemma 6.35 we obtain that

$$\begin{aligned} 0 = \int_D g \cdot w_\tau\ dx &= \int_0^\infty \int_{D_t^{w_\tau}} g(x)\ dx\ dt \\ &= \int_0^\tau \int_{D_t} g(x)\ dx\ dt \\ &= \int_0^\tau g_{D_t,1} \cdot |D_t|\ dt \end{aligned}$$

and observe that

$$t \mapsto g_{D_t,1} =: g_{D_t} \text{ has integral mean 0 w.r.t. to the measure } |D_t|\, dt \text{ on } [0,\tau]. \tag{6.30}$$

Next make use of the inequality $|a+b|^q \leq 2^{q-1}\left(|a|^q + |b|^q\right)$ to estimate

$$\begin{aligned}\|g\|^q_{L^q(D,w_\tau)} &= \int_0^\tau \int_{D_t} |g(x)|^q \; dx \; dt \\ &\leq 2^{q-1} \int_0^\tau \int_{D_t} \left(|g(x) - g_{D_t}|^q + |g_{D_t}|^q\right) \; dx \; dt =: 2^{q-1} \cdot (I + II)\,.\end{aligned} \tag{6.31}$$

The inner integral of I is already set up to apply Poincaré's inequality. The key trick lies in how to bound II in terms of I. First II can be rewritten in the following way:

$$II = \int_0^\tau |g_{D_t}|^q \cdot |D_t| \; dt = |D_\tau|^{-1} \int_{D_\tau} \int_0^\tau |g_{D_t}|^q \cdot |D_t| \; dt \; dx.$$

Using observation (6.30) together with Lemma 6.3 guarantees that the inner integral of the right hand side is bounded by

$$2^q \int_0^\tau |g_{D_t} - c|^q \cdot |D_t| \; dt,$$

where c is an arbitrary real number. The choice $c = g(x)$ yields

$$II \leq |D_\tau|^{-1} \cdot 2^q \int_{D_\tau} \int_0^\tau |g_{D_t} - g(x)|^q \cdot |D_t| \; dt \; dx\,.$$

Changing order of integration and since for $t \in [0,\tau]$ the inclusions $D_\tau \subset D_t \subset D$ hold we arrive at

$$II \leq 2^q \frac{|D|}{|D_\tau|} \cdot I. \tag{6.32}$$

Collecting the estimates from (6.29), (6.31) and (6.32) gives

$$\|f - f_{D,w}\|^q_{L^q(D,w)} \leq 2^{3q} \cdot \frac{\|w\|_{L^\infty(D)}}{\tau} \cdot \frac{|D|}{|D_\tau|} \cdot I. \tag{6.33}$$

We proceed with bounding the expression I by applying Poincaré's inequality on each of the superlevel sets D_t. Let $s \in [q,p]$ and set $r := s/q$ and $1/r + 1/r' = 1$, then

$$\begin{aligned} I &= \int_0^\tau \int_{D_t} |g(x) - g_{D_t}|^q \; dx \; dt \\ &\leq \int_0^\tau C^{q,s}_{1,\rho}(D_t)^q \cdot \left(\int_{D_t} |\nabla g(x)|^s \rho(x) \; dx\right)^{q/s} \; dt \\ &\leq \left\| t \mapsto C^{q,s}_{1,\rho}(D_t)^q \right\|_{L^{r'}([0,\tau])} \cdot \left(\int_0^\tau \int_{D_t} |\nabla g(x)|^s \rho(x) \; dx \; dt\right)^{1/r} \\ &\leq \left\| t \mapsto C^{q,s}_{1,\rho}(D_t) \right\|^q_{L^{\frac{sq}{s-q}}([0,\tau])} \cdot \|\nabla g\|^q_{L^s(D,w\rho)} \end{aligned} \tag{6.34}$$

Again making use of Hölder's inequality we can estimate the last term in the following way

$$\|\nabla g\|_{L^s(D,w\rho)} \leq \|\nabla g\|_{L^p(D,w\rho)} \cdot w\rho(D)^{1-\frac{s}{p}}. \tag{6.35}$$

Since $\nabla g = \nabla f$ we obtain by combining (6.33), (6.34) and (6.35) that

$$\begin{aligned}\|f - f^w_D\|_{L^q(D,w)} \leq & 8 \left(\frac{\|w\|_{L^\infty(D)}}{\tau} \cdot \frac{|D|}{|D_\tau|}\right)^{1/q} \cdot w\rho(D)^{\frac{1}{q} - \frac{s}{pq}} \\ & \cdot \left\| t \mapsto C^{q,s}_{1,\rho}(D_t) \right\|_{L^{\frac{sq}{s-q}}([0,\tau])} \cdot \|\nabla f\|_{L^p(D,w\rho)}\,.\end{aligned}$$

Since this holds for arbitrary $\tau \in [0, \|w\|_{L^\infty(D)}]$ and $s \in [q,p]$ we can conclude that (6.27) holds true. □

Chapter 7

Stable phase retrieval from 1d Fourier transform magnitudes

This chapter is devoted to the study of phase retrieval from 1d Fourier transform magnitudes. As discussed in Chapter 2.1 in the onedimensional case its rather hopeless to recover a function from the modulus of its Fourier transform: Most functions $f \in L^2(\mathbb{R})$ are not uniquely determined by the modulus of their Fourier transform, even if we restrict ourselves to compactly supported functions. Thus, the measurement process needs to be tweaked in order to obtain a well-posed problem.

We will discuss two particular instances for which quantitative results can be established. Firstly, in Section 7.1 we make use of the techniques that served us so well for the problem of phase retrieval from Gabor magnitudes. In order to do so we will assume that besides $|\hat{f}|$ also the modulus of $\hat{f}'$ can be observed.

In Section 7.2 we consider so called interferometric measurements, i.e., the problem of recovering f from $|\hat{f} + \hat{h}|$ where h is a suitably chosen and fixed reference function.

7.1 Recovery from two Fourier measurements

Let us briefly recall the main ingredients of our analysis for Gabor phase retrieval:

- Right at the beginning stands the observation that the quotient of two Gabor transforms $q = \mathcal{G}g/\mathcal{G}f$ constitutes a meromorphic function.
- According to Lemma 4.10 it holds that

$$|\nabla |q|| = 2^{-1/2} |\nabla q| \quad a.e. \tag{7.1}$$

- Applying Poincaré's inequality together with (7.1) yields that

$$\inf_{|c|=1} \|\mathcal{G}g - c\mathcal{G}f\|_{L^p(D)} \lesssim \||\mathcal{G}g| - |\mathcal{G}f|\|_{L^p(D)} + C_P^{\mathcal{M}}(D, |\mathcal{G}f|^p, p) \|\nabla |q|\|_{L^p(D,|\mathcal{G}f|^p)} .$$

- The study of the logarithmic derivative in Chapter 5 allows us to control $\|\nabla |q|\|$:

$$\|\nabla |q|\|_{L^p(D,|\mathcal{G}f|^p)} \leq \||\mathcal{G}g| - |\mathcal{G}f|\|_{\mathcal{W}(D)} ,$$

  where $\|\cdot\|_{\mathcal{W}(D)}$ denotes a weighted, first order Sobolev norm on D which is essentially independent from f and g.

- The main result can then be summarized as follows: Provided that $C_P^{\mathcal{M}}(D, |\mathcal{G}f|^p, p)$ is of moderate size the phase of $\mathcal{G}f$ on D is stabily determined by

$$|\mathcal{G}f| \quad \text{and} \quad \nabla |\mathcal{G}f| \quad \text{on } D.$$

The aim is now to apply similar arguments to $q = \hat{g}/\hat{f}$. Provided that both f and g are sufficiently well-behaved their Fourier transforms can be extended to holomorphic functions F and G on a domain $D \subset \mathbb{C}$ that contains the real line. For example if f has exponential decay, then its Fourier-Laplace transform F is an entire function.

In order to immitate the proof techniques outlined above there are still two major issues to overcome: Firstly, it is well-known that f is in general not even uniquely determined up to global sign by the modulus of its Fourier transform, see Theorem 2.1. Secondly, given $|\hat{f}|$ we can also access $|F|_x = |\hat{f}|'$ but not the derivative of $|F|$ w.r.t. y.
To deal with these issues we will assume that we are given a second phaseless measurement besides $|\hat{f}|$, namely the function $|\hat{f}'|$. Note that since

$$\hat{f}'(x) = (t \mapsto -2\pi i t f(t))^{\wedge}(x)$$

this kind of measurement can be realized as the modulus of the Fourier transform of $tf(t)$. Thus, in the context of diffraction imaging for example, it is reasonable to assume that $|\hat{f}'|$ can be acquired.

For $I \subset \mathbb{R}$ an interval we shall denote by $\mathcal{O}(I)$ the space of functions on I which have a holomorphic extension to a neighbourhood of I, i.e.,

$$\mathcal{O}(I) := \left\{ f : I \to \mathbb{C} : \exists D \subset \mathbb{C}\,\text{open}, F \in \mathcal{O}(D)\,\text{s.t.}\, I \subset D,\ F\big|_I = f \right\}.$$

Since the analytic extension is unique (if it exists) we will for the remainder of the section identify $f \in O(I)$ with its extension $F \in \mathcal{O}(D)$.

Lemma 7.1. *Let I be a nonempty interval contained in the real axis $\mathbb{R} \subset \mathbb{C}$. Suppose $F, G \in \mathcal{O}(I)$ are such that for all $z \in I$ it holds that*

$$|F(z)| = |G(z)| \quad \textit{and} \quad |F'(z)| = |G'(z)|.$$

Then there exists $c \in \mathbb{C}$ with $|c| = 1$ such that $G = cF$ or $G = c\overline{F(\bar{\cdot})}$.

Proof. We shall identify $\mathbb{C}$ with $\mathbb{R}^2$ and in particular consider functions on $\mathbb{C}$ depending on $z = x + iy$ as functions of (x, y).
If F vanishes identically the statement is trivial, we may therefore omit this case. Moreover, we may assume that F (and therefore also G) do not possess any zeros on I (otherwise pass over to a smaller interval).
Since $F = |F| e^{i\varphi}$ is assumed to be holomorphic we get that

$$F' = F_x = |F|_x e^{i\varphi} + i\varphi_x |F| e^{i\varphi}$$

and thus that

$$|F'|^2 = (|F|_x)^2 + (\varphi_x)^2 |F|^2. \tag{7.2}$$

Similarly it holds for $G = |G| e^{i\psi}$ that

$$|G'|^2 = (|G|_x)^2 + (\psi_x)^2 |G|^2.$$

From the assumptions it follows that for all $z \in I$ it holds that $\varphi_x(z)^2 = \psi_x(z)^2$. Since φ and ψ are real analytic (away from the zeros of F and G that is) it follows that either $\psi_x(z) = \varphi_x(z)$ or $\psi_x(z) = -\varphi_x(z)$ for $z \in I$. In the first case it follows that there exists $\alpha \in \mathbb{R}$ such that

$$\psi(z) = \varphi(z) + \alpha, \quad z \in I$$

and hence that the following identity holds in I:

$$G = |G|\, e^{i\psi} = |F|\, e^{i(\varphi+\alpha)} = e^{i\alpha} F.$$

Similarly, in the second case we have that in I

$$G = |G|\, e^{i\psi} = |F|\, e^{i(-\varphi+\alpha)} = e^{i\alpha}\overline{F(\bar{\cdot})}.$$

The statement now follows from the identity theorem. □

As an immediate consequence of Lemma 7.1 we get the following uniqueness result.

Theorem 7.2. *Suppose that $f, g \in L^2(\mathbb{R})$ are such that their Fourier transforms lie in $\mathcal{O}(\mathbb{R})$ and suppose that for all $x \in \mathbb{R}$ it holds that*

$$|\hat{f}(x)| = |\hat{g}(x)| \quad \textit{and} \quad |\hat{f}'(x)| = |\hat{g}'(x)|.$$

Then there exists $c \in \mathbb{C}$ with $|c| = 1$ such that either $g = cf$ or $g = c\overline{f}(-\cdot)$.

For $p \geq 1$ and an interval $I \subset \mathbb{R}$ equipped with an integrable weight w the Poincaré constant, i.e., the smallest positive constant C such that

$$\inf_{c\in\mathbb{R}} \|f - c\|_{L^p(I,w)} \leq C\, \|f'\|_{L^p(I,w)}\,, \quad \forall f \in L^p(I,w) \cap C^1(I),$$

will be denoted by $C_P(I, w, p)$.

Proposition 7.3. *Let $p \geq 1$ and $I \subset \mathbb{R}$ an interval. Suppose $F \in \mathcal{O}(I)$ such that F is nonvanishing on I and $F\big|_I \in L^p(I)$. Then for all $G \in \mathcal{O}(I)$ it holds that*

$$\inf_{|c|=1} \|G - cF\|_{L^p(I)} \lesssim \||G| - |F|\|_{L^p(I)} + C_P(I, |F|^p, p) \left(\|\nabla|G| - \nabla|F|\|_{L^p(I)} + \left\| \frac{\nabla|F|}{|F|} (|G| - |F|) \right\|_{L^p(I)} \right) \quad (7.3)$$

Proof. We may assume that $G\big|_I \in L^p(I)$ since otherwise the right hand side of (7.3) is infinite. Since F is assumed not to vanish on I it follows that $q = G/F$ is holomorphic in a neighbourhood of I and thus continuously differentiable on I.
Note that Theorem 4.12 cannot be applied directly because I is not a domain in $\mathbb{C}$. However, replicating the proof of Theorem 4.12 word by word together with the observation that for every holomorphic q

$$\left|\frac{\partial q}{\partial x}\right| = |q'| = 2^{-1/2}\, |\nabla q|$$

leads to Inequality (7.3). □

If the estimate in Proposition 7.3 is to be used to derive a stability result we need to get our hands onto $|F|_y$, where F is the holomorphic extension of $\hat{f}$. Naively, that would require that F is known on a neighbourhood of I. As we shall see next, in fact it almost suffices to have knowledge of $|\hat{f}|$ and $|\hat{f}'|$ due to the analyticity of the functions involved.

Lemma 7.4. *Let $D \subset \mathbb{C}$ be a domain and let $F \in \mathcal{O}(D)$. Then for all $z \in D$ with $F(z) \neq 0$ it holds that*

$$(|F|_y(z))^2 = |F'(z)|^2 - (|F|_x(z))^2.$$

Proof. Since $F = |F| e^{i\varphi}$ does not vanish in z we have that

$$\log F = \log|F| + i\varphi$$

is holomorphic in a neighbourhood of z. It follows from Cauchy-Riemann that

$$\varphi_x = -(\log|F|)_y = -|F|_y / |F|,$$

which implies that $(|F|_y)^2 = |F|^2 (\varphi_x)^2$. According to (7.2) it holds that

$$|F|^2 (\varphi_x)^2 = |F'|^2 - (|F|_x)^2,$$

which concludes the proof. □

Unfortunately, Lemma 7.4 only allows us to relate $|\,|F|_y\,|$ to $|\hat{f}|$ and $|\hat{f}'|$ but not $|F|_y$ itself. In the special case where f is a Gaussian the situation simplifies significantly as the Fourier transform of f then is Gaussian as well and in particular strictly positive. The following quantitative result can be established.

Theorem 7.5. *Let $f := e^{-\alpha \cdot^2}$ with $\alpha > 0$. Then there exists a constant c depending on α only such that for all $g \in L^2(\mathbb{R})$ with $\hat{g} \in \mathcal{O}(\mathbb{R})$ it holds that*

$$\inf_{|c|=1} \|g - cf\|_{L^2(\mathbb{R})} \leq$$
$$c\left(\left\| (1+|\cdot|)\left(|\hat{g}| - |\hat{f}|\right) \right\|_{L^2(\mathbb{R})} + \left\| |\hat{g}'|^2 - |\hat{f}'|^2 \right\|_{L^1(\mathbb{R})} + \left\| (|\hat{g}|_x)^2 - (|\hat{f}|_x)^2 \right\|_{L^1(\mathbb{R})} \right)$$

Proof. We denote again by F and G the analytic extension of $\hat{f}$ and $\hat{g}$, respecively. First apply Proposition 7.3 to obtain

$$\begin{aligned}\inf_{|c|=1} \|g - cf\|_{L^2(\mathbb{R})} &= \inf_{|c|=1} \left\| \hat{g} - c\hat{f} \right\|_{L^2(\mathbb{R})} \\ &\lesssim \left\| |\hat{g}| - |\hat{f}| \right\|_{L^2(\mathbb{R})} + \| \nabla |G| - \nabla |F| \|_{L^2(\mathbb{R})} + \left\| \nabla(\log|F|)(|\hat{g}| - |\hat{f}|) \right\|_{L^2(\mathbb{R})}, \end{aligned} \tag{7.4}$$

where the implicit constant depends on the L^2-Poincaré constant, which – since $\hat{f}$ is a Gaussian – can be found to be finite. We refer to the well-known *Gaussian Poincaré inequality*, which states that

$$C_P(\mathbb{R}, e^{-\cdot^2/2}, 2) = 1,$$

see for example [Bec89] and the references therein.
To estimate the second term on the right hand side of (7.4) observe that since F is the analytic extension of $\hat{f}$ which is a Gaussian we have that

$$F(z) = ce^{-\beta z^2},$$

for suitable constants $c, \beta > 0$. Hence,

$$|F(z)| = ce^{-\beta(x^2 - y^2)},$$

which implies that $|F|$ is even w.r.t. the real axis and therefore $|F|_y(z) = 0$ for all $z \in \mathbb{R}$. Consequently, it holds that

$$\|\nabla |G| - \nabla |F|\|_{L^2(\mathbb{R})}^2 = \||G|_x - |F|_x\|_{L^2(\mathbb{R})}^2 + \left\||G|_y\right\|_{L^2(\mathbb{R})}^2 .$$

Use Lemma 7.4 and that $|F|_y = 0$ to estimate

$$\begin{aligned}
\left\||G|_y\right\|_{L^2(\mathbb{R})}^2 &= \int_{\mathbb{R}} \left(|G'|^2 - (|G|_x)^2\right) dx \\
&= \left\||G'|^2 - (|G|_x)^2 - |F'|^2 + (|F|_x)^2\right\|_{L^1(\mathbb{R})} \\
&\leq \left\||G'|^2 - |F'|^2\right\|_{L^1(\mathbb{R})} + \|(|G|_x)^2 - (|F|_x)^2\|_{L^1(\mathbb{R})} \\
&= \left\||\hat{g}'|^2 - |\hat{f}'|^2\right\|_{L^1(\mathbb{R})} + \left\|(|\hat{g}|_x)^2 - (|\hat{f}|_x)^2\right\|_{L^1(\mathbb{R})} .
\end{aligned}$$

For the logarithmic derivative of $|F|$ it holds that

$$\nabla(\log |F|)(z) \asymp |z| ,$$

which results in the desired estimate

$$\begin{aligned}
&\inf_{|c|=1} \|g - cf\|_{L^2(\mathbb{R})} \lesssim \\
&\qquad \left\|(1 + |\cdot|)\left(|\hat{g}| - |\hat{f}|\right)\right\|_{L^2(\mathbb{R})} + \left\||\hat{g}'|^2 - |\hat{f}'|^2\right\|_{L^1(\mathbb{R})} + \left\|(|\hat{g}|_x)^2 - (|\hat{f}|_x)^2\right\|_{L^1(\mathbb{R})} .
\end{aligned}$$

□

7.2 Recovery from a single interferometric measurement

In this section we shall see that, if we allow for additive distortions, that f is stabily determined by $|\hat{f} + \hat{g}|$ if g is chosen carefully. More precisely, if the additive distortion g is choosen to be a suitable multiple of the delta distribution, f can be recovered from $|\hat{f} + c|$. The interference of f with such a g pushes all the zeros of the analytic extension of the Fourier transform to the upper half plane. In this case the relation between phase and magnitude is described by the Hilbert transform [BFG+76, BFGR74] and remarkably, phase retrieval renders not only unique but also stable.

Theorem 7.6. *For $a, b > 0$ let*

$$\mathcal{X}_{a,b} := \{f \in L^2(\mathbb{R}) : \|f\|_{L^\infty(\mathbb{R})} < a, \operatorname{supp} f \subset [0, b]\}$$

and for $c \in \mathbb{C}$ let $L_c^2(\mathbb{R}) := \{f + c : f \in L^2(\mathbb{R})\}$ endowed with the metric

$$d_{L_c^2(\mathbb{R})}(f, g) := \|f - g\|_{L^2(\mathbb{R})}.$$

Suppose $c > ab$. Then $\mathcal{A} : f \mapsto |\hat{f} + c|$ is an injective mapping from $\mathcal{X}_{a,b}$ to $L_c^2(\mathbb{R})$ and $\mathcal{A}^{-1} : \mathcal{A}(\mathcal{X}_{a,b}) \subset L_c^2(\mathbb{R}) \to \mathcal{X}_{a,b}$ is uniformly continuous, i.e., there exists a constant $C > 0$ such that

$$\|f_1 - f_2\|_{L^2(\mathbb{R})} \leq C \cdot d_{L_c^2(\mathbb{R})}(|\hat{f}_1 + c|, |\hat{f}_2 + c|), \quad f_1, f_2 \in \mathcal{X}_{a,b}.$$

Proof. In order to show that $\mathcal{A}$ maps from $\mathcal{X}_{a,b}$ to $L^2_c(\mathbb{R})$ let $f \in \mathcal{X}_{a,b} \subset L^2(\mathbb{R})$ be arbitrary. We have to verify that $\mathcal{A}f - c \in L^2(\mathbb{R})$. By the reverse triangle inequality we have that

$$|\mathcal{A}f - c| = ||\hat{f} + c| - c| \leq |\hat{f} + c - c| = |\hat{f}|.$$

Since $\hat{f} \in L^2(\mathbb{R})$, it follows that also $\mathcal{A}f - c \in L^2(\mathbb{R})$.

Let us denote by g the analytic extension of $\hat{f} + c$, i.e.,

$$g(z) = \int_{\mathbb{R}} f(t)e^{-2\pi izt}dt + c, \quad z \in \mathbb{C}.$$

Then – provided that g has all its zeros in the upper (or lower) halfplane – phase and magnitude of g are related via the Hilbert transform [BFGR74], i.e.,

$$\varphi(x) := H(\log|g|)(x) := -\frac{1}{\pi}P.V.\int_{\mathbb{R}} \frac{\log|g(t)|}{t-x}dt, \quad x \in \mathbb{R}, \tag{7.5}$$

satisfies $g = |g|e^{i\varphi}$.
In order to make use of this identity we check that g has no zeros in the lower halfplane: For $\operatorname{Im} z \leq 0$ it holds that

$$|\int_{\mathbb{R}} f(t)e^{-2\pi izt}dt| \leq \|f\|_{L^1(\mathbb{R})} \leq ab,$$

therefore – since $c > ab$ – it holds that $|g(z)| \geq ||f(z)| - |c|| \geq c - ab > 0$ in the lower halfplane.

For $f_1, f_2 \in \mathcal{X}_{a,b}$ let $g_k := \hat{f}_k + c$ and let $\varphi_k := H(\log|g_k|)$. Then we have for $k = 1, 2$ that

$$|g_k| = |\hat{f}_k + c| \geq c - |\hat{f}_k| \geq c - ab > 0$$

and similarly that $|g_k| \leq c + ab$. It follows that there exists a constant d (depending on a, b, c) such that

$$|\log|g_1| - \log|g_2|| \leq d \cdot ||g_1| - |g_2||, \tag{7.6}$$

which implies that the difference $\log|g_1| - \log|g_2|$ is an element of $L^2(\mathbb{R})$. According to (7.5) the phase difference $\delta := \varphi_1 - \varphi_2$ can be computed by $\delta = H(\log|g_1| - \log|g_2|)$. By using the well-known fact that the Hilbert transform is an isometry on $L^2(\mathbb{R})$ (see for example [Tit37]) and (7.6) it follows that there exists a constant d' (depending on a, b, c) such that

$$\|\delta\|_{L^2(\mathbb{R})} \leq d' \cdot \||g_1| - |g_2|\|_{L^2(\mathbb{R})}.$$

Thus we obtain by using the elementary inequality $|1 - e^{it}| \leq |t|, t \in \mathbb{R}$ that

$$\begin{aligned}
\|f_1 - f_2\|_{L^2(\mathbb{R})} &= \|\hat{f}_1 - \hat{f}_2\|_{L^2(\mathbb{R})} \\
&= \|g_1 - g_2\|_{L^2(\mathbb{R})} \\
&= \||g_1|e^{i\varphi_1} - |g_2|e^{i\varphi_2}\|_{L^2(\mathbb{R})} \\
&\leq \|g_1 \cdot (1 - e^{-i\delta})\|_{L^2(\mathbb{R})} + \||g_1| - |g_2|\|_{L^2(\mathbb{R})} \\
&\leq \|g_1\|_{L^\infty(\mathbb{R})} \cdot \|\delta\|_{L^2(\mathbb{R})} + \||g_1| - |g_2|\|_{L^2(\mathbb{R})} \\
&\leq d''\||g_1| - |g_2|\|_{L^2(\mathbb{R})},
\end{aligned}$$

for suitable constant d''. □

Remark 7.7. Note that the assumption $\operatorname{supp} f \subset [0, b]$ implies not only that $\hat{f}$ is bandlimited but also $|\hat{f}|^2$ and $\operatorname{Re} \hat{f}$. Therefore also the function

$$|\hat{f} + c|^2 - c^2 = |\hat{f}|^2 + 2c \operatorname{Re} \hat{f}$$

is bandlimited and $|\hat{f} + c|$ can be uniquely and stably determined from samples. Together with Theorem 7.6 this implies that any $f \in \mathcal{X}_{a,b}$ can be recovered stably from the samples of $|\hat{f} + c|$ on a suitable discrete set.

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Deutsche Zusammenfassung

In zahlreichen Anwendungen kann zwar die Intensität eines wellenförmigen Signals, nicht aber dessen Phase gemessen werden. Beispiele dafür sind etwa bildgebende Verfahren, deren Aufgabe es ist Strukturen im Nanometerbereich sichtbar zu machen.
Da die Phaseninformationen essentiell für die Rekunstruktion des zugrundeliegenden Objekts ist, gilt es diese aus den vorliegenden Intensitätsdaten zu bestimmen.

Die mathematische Formulierung solcher Experimente führt zu dem so genannten "Phase Retrieval" Problem, eine Funktion zu bestimmen, von der lediglich die Intensität ihrer Fouriertransformierten bekannt ist. Wie unlängst in großer Allgemeinheit gezeigt verhalten sich Rekunstruktionsprobleme dieser Art stets instabil. Konkret ist der Lösungsoperator, so dieser existiert, niemals gleichmäßig stetig für "Phase Retrieval" in unendlichdimensionalen Räumen.

Zentraler Gegenstand dieser Arbeit ist es eingehendes Verständnis über die qualitative Beschaffenheit von Stabilitäten sowie Instabilitäten diverser "Phase Retrieval" Probleme zu erlangen. Das Hauptaugenmerk gilt dem Problem der Phasenrekonstruktion für die Kurzzeit-Fourier-Transformation, oder genauer gesagt für die Gabor Transformation. Die Hauptresultate der vorliegenden Arbeit offenbaren, dass die Phaseninformationen der Gabor Transformierten stabil rekonstruiert werden kann, vorausgesetzt die Messungen sind konzentriert auf einer zusammenhängenden Menge.
Diese anschauliche Beschreibung wird durch die Cheeger Konstante konkretisiert, ein Konzept, das auch für das Clustering von Graphen eine wichtige Rolle spielt. Die verwendeten Beweismethoden bedienen sich tiefliegender Resultate aus unterschiedlichen mathematischen Gebieten wie Komplexer Analysis, Funktionalanalysis, Spektralgeometrie und Potentialtheorie. Als Zwischenresultate werden neue Ergebnisse zu gewichteten Poincare Ungleichungen, Kompaktheit von Einbettung in gewichteten Sobolevräumen sowie über das Wachstum der logarithmischen Ableitung ganzer Funktionen in mehreren Veränderlichen bewiesen.